



universität
wien

MASTERARBEIT

Titel der Masterarbeit

“Lens Spaces”

verfasst von

Julius Grüning, BSc

angestrebter akademischer Grad

Master of Science (MSc)

Wien, 2015

Studienkennzahl lt. Studienblatt: A 066 821

Studienrichtung lt. Studienblatt: Masterstudium Mathematik

Betreut von: ao. Univ.-Prof. tit. Univ.-Prof. Dr. Andreas Kriegl

Acknowledgements

I want to express my gratitude towards my advisor Andreas Kriegl for choosing a great topic for my thesis that I learned a lot from, and for his useful comments, remarks and his engagement.

Above all, I want to thank my parents for their support throughout the years.

Contents

Introduction	1
1 Important Notions	5
1.1 Singular (Co)Homology	5
1.1.1 Cellular Homology	8
1.1.2 The Universal Coefficient Theorem	9
1.1.3 Bockstein Homomorphisms	9
1.2 Homotopy Groups	10
1.2.1 Whitehead's theorem	13
1.2.2 The Hurewicz Homomorphism	15
1.3 Simple-Homotopy Theory	17
1.3.1 Whitehead Groups	17
1.3.2 (R, G) -complexes	20
1.3.3 The Torsion of an Acyclic Complex	24
1.3.4 Changing Rings	27
1.3.5 The Torsion of a CW Pair	27
1.3.6 Properties of the Torsion of a CW Pair	31
1.3.7 The Torsion of a Homotopy Equivalence	31
2 Constructions of Lens Spaces	37
2.1 Construction via Tori	37
2.2 Construction from a 3-Ball	39
2.3 Construction as an Orbit Space of S^3	41
2.4 Higher-Dimensional Lens Spaces	42
3 (Co)Homology of Lens Spaces	43
3.1 A Cell Decomposition of Lens Spaces	43
4 The Homotopy Classification	49
5 The Homeomorphy Classification	55
5.1 Franz' Theorem	55
5.2 The Classification Result	60
Appendix	64
Bibliography	67

Introduction

Lens spaces are classical examples of manifolds. Especially the three-dimensional lens spaces played an important role in the history of algebraic topology. The goal of this thesis is to give an overview of how lens spaces can be constructed and to explain a proof of the known homotopy and homeomorphism classification results of lens spaces.

According to [1] the name "lens spaces" (Linsenräume) appeared for the first time in the paper [2, p. 58] from 1931 by Threlfall and Seifert. The classical definition of a lens space used by Threlfall and Seifert, also later in their book [3, p. 210], was first made by Tietze in 1908 (see [4, p. 110]): A three-dimensional lens space $L_{l/m}$ is a quotient space of the 3-ball, where the top hemisphere gets identified with the bottom hemisphere after a twist of $2\pi l/m$, for some $m \in \mathbb{Z}$, $m \geq 2$ and some $l \in \mathbb{Z}$ relatively prime to m . The 3-ball used in this definition is oftentimes drawn in the shape of a lens (see for example [3, p. 210], [2, p. 58], [5, p. 29]). The lens shape naturally occurs in another definition of lens spaces, as orbit spaces of an action of \mathbb{Z}_m on S^3 , whose fundamental domain (in the case $m = 3$) looks like a lens (see 2.8). We will discuss these constructions and their equivalence in chapter 2, as well as a third construction: Historically, the first mentioning of examples of lens spaces was through Heegard diagrams by Dyck (reference: [1]). Lens spaces are exactly the three-dimensional manifolds of Heegard genus 1, which means a lens space can be obtained by gluing together two full tori via a homeomorphism of their boundaries. At the end of chapter 2, we explain a definition for higher-dimensional lens spaces - our main subjects of further study.

The question Tietze posed was in which cases two given lens spaces are homeomorphic - this has been an open question until 1935, when first Reidemeister developed a concept to answer this question(, which is today known as Reidemeister torsion): two lens spaces $L_{l/m}$ and $L_{l'/m}$ are homeomorphic, if and only if $l \equiv \pm l'$ or $l \cdot l' \equiv \pm 1 \pmod{m}$.

In chapter 5 we will see a proof of this statement, but with a different approach, which is due to Whitehead (the concept - geometrical deformations - was first described in his paper [6]) and which also answers the question of homeomorphism for higher-dimensional lens spaces. We will see that two lens spaces are homeomorphic if and only if they are

simple-homotopy equivalent (in theorem 5.15) and we will briefly go over some important notions of simple-homotopy theory in the section 1.3.

The theory of simple-homotopy has a close connection to developments in modern topology, such as the s-cobordism theorem (see [7, p. 83]), and is very interesting in its own right; the homeomorphism classification of lens spaces is a mere application of some of the very strong theorems one can prove within simple-homotopy theory; still, there is some tough algebra involved (in particular Franz' theorem, which is stated in 5.1).

In their second paper [8, p. 548] from 1933, Threlfall and Seifert gave a necessary condition for lens spaces to be homeomorphic: If $L_{l/m} \cong L_{l'/m}$, then the congruence $l \equiv \pm l' \cdot k^2 \pmod{m}$ has a solution. Today, this congruence is known to be the classifying condition for two lens spaces to be homotopy equivalent, which was first proven by Whitehead in 1941 (in the paper [9]). We will see a proof of this statement(, which is generalized to higher-dimensional lens spaces) in chapter 4. Our proof uses Bockstein homomorphisms and more classical notions of algebraic topology. For a given $m \in \mathbb{Z}$, all homology and homotopy groups of two lens spaces $L_{l/m}$ and $L_{l'/m}$ are the same (which we will see in chapter 3), so it may seem surprising that not much more theory is needed to tell their homotopy equivalence classes apart. Although historically, the analysis of the two lens spaces $L_{1/5}$ and $L_{2/5}$ (which are not homotopy equivalent) has led to the development of "Verschlingungszahlen" (described for example in [5, chapter 14.7]) by Alexander. (To see Alexander's proof of how $L_{1/5}$ and $L_{2/5}$ are not homeomorphic go to [10, p. 258-260].) As one can easily see from the two classification results, that there are not only examples of lens spaces, which share the same groups and are not homotopy equivalent, but that there are lens spaces, which are not homeomorphic, but are homotopy equivalent - one example of this phenomenon are the two spaces $L_{1/7}$ and $L_{2/7}$ (see example 5.1).

We will briefly review the important concepts of homology groups and homotopy groups as well as some important theorems used in the proof of the homotopy classification in the sections 1.1 and 1.2. We will compute the homology of lens spaces in chapter 3 using a CW decomposition, which will also help us in the proof of the homeomorphism classification.

Lens spaces also played an important role in the construction of the first counter example to the "Hauptvermutung der kombinatorischen Topologie" by Milnor (in [11]): If L denotes the lens space $L_{1/7}$, then $M := L \times \Delta^3$ is a 6-manifold with border $\partial M \cong L \times S^2$. We define $Y := (\partial M \times I)/(\partial M \times \{1\})$ to be the cone over ∂M , and identify ∂M with $\partial M \times \{0\} \subset Y$ to set $X := M \cup Y$. Milnor proved that there are two simplicial complexes K_1 and K_2 , such that $|K_1| \cong |K_2| \cong X$ and that there is no subdivision of K_1 which is isomorphic to any subdivision of K_2 . However, we will not go into any details of this

within this thesis. An explanation on how to construct more counter-examples to the Hauptvermutung can be found in [7, p. 82-84].

Also of interest are infinite-dimensional lens spaces, which we will briefly discuss in 2.10. Infinite-dimensional lens spaces are examples of Eilenberg-McLane spaces $K(\mathbb{Z}_m, 1)$, and their homotopy type hence only depends on the number $m \in \mathbb{Z}$, much different than in the finite-dimensional case. We will make use of infinite-dimensional lens spaces to calculate the cup-product structure on the cohomology ring of a lens space in 3.8.

For the entire thesis we will suppose knowledge of elementary notions of topology, as well as homological algebra and category theory. Good references are [12], [5], [13].

Chapter 1

Important Notions

We start by introducing some important notions - mostly of algebraic topology. The first one: Singular homology groups are of great use in homotopy theory and have led to many different developments in many fields of mathematics.

For our proof of the homotopy classification result of lens spaces, we will need to compute their homology groups.

1.1 Singular (Co)Homology

Definition 1.1 (Singular Homology). Denote by Δ^n the n -simplex, i.e. the convex hull of $\{e_0, \dots, e_n\} \subseteq \mathbb{R}^{n+1}$, where the $(e_i)_{i \in \{0 \dots n\}}$ are the standard basis for \mathbb{R}^{n+1} . For $q \geq 1$ and $0 \leq i \leq n$ define $\delta_{n-1}^i : \Delta^{n-1} \rightarrow \Delta^n$ to be the affine map defined by

$$\delta_{n-1}^i(e_j) = \begin{cases} e_j & \text{if } j < i \\ e_{j+1} & \text{if } i \leq j \leq n \end{cases},$$

i.e. the embedding of Δ^{n-1} 's i th facet into Δ^n .

For any topological space X , define $C_n(X)$ to be the free \mathbb{Z} -module generated by the set of all *singular simplices*, i.e. all continuous mappings $\Delta^n \rightarrow X$. If (X, A) is a topological pair, we have a canonical injective linear map $C_n(A) \hookrightarrow C_n(X)$ and we define $C_n(X, A) := C_n(X)/C_n(A)$.

For any topological space X , define the *boundary operator* $\partial = \partial_n : C_n(X) \rightarrow C_{n-1}(X)$ by $\partial(\sigma) = \sum_{i=0}^n (-1)^i (\sigma \circ \delta_{n-1}^i)$. Obviously this also induces a map $\partial : C_n(X, A) \rightarrow C_{n-1}(X, A)$. One may show ([5, p. 216]) that $\partial\partial = \partial_n \circ \partial_{n-1} = 0$, so that $C_*(X, A)$ becomes a chain complex, called the *singular chain complex*.

Any continuous map $f : (X, A) \rightarrow (Y, B)$ between topological pairs induces a chain map $f_{\#} : C_n(X, A) \rightarrow C_n(Y, B)$ (that means $\partial f_{\#} = f_{\#} \partial$), by $\sum_{\sigma} n_{\sigma} \sigma \mapsto \sum_{\sigma} (f \circ \sigma)$.

We define the *singular homology* of a pair (X, A) to be the homology of the singular chain complex, denoted by $H_*(X, A)$. And by $f_* : H_n(X, A) \rightarrow H_n(Y, B)$ we denote the induced map from the chain map $f_{\#}$ for any continuous mapping $f : (X, A) \rightarrow (Y, B)$.

Definition 1.2 ((Co)Homology with coefficients in G). If G is any abelian group, we may define $C^n(X, A; G) := \text{Hom}(C_n(X, A), G)$, the *nth cochain group with coefficients in G* of the pair (X, A) . From $\partial : C_n(X, A) \rightarrow C_{n-1}(X, A)$ we obtain a coboundary map $\delta : C^{n-1}(X, A; G) \rightarrow C^n(X, A; G)$. Taking cohomology groups, we define the *nth cohomology group with coefficients in G* by $H^n(X, A; G) := H^n(C^*(X, A; G))$.

Similarly, we may define *homology with coefficients in G* , by taking the tensor product of $C_n(X, A)$ with G . That means we define $H_n(X, A; G) := H_n(C_*(X, A; G))$, where $C_*(X, A; G)$ is given in each degree n by $C_n(X, A; G) := C_n(X, A) \otimes_{\mathbb{Z}} G$.

Definition 1.3 (Cup Product). In definition 1.2, if we choose instead of an abelian group G , a commutative ring with unity R , then we may define a product on $C^*(X; R)$ via

$$\cup : C^p(X; R) \times C^q(X; R) \rightarrow C^{p+q}(X; R)$$

$$(\phi \cup \psi)(\sigma) := \phi(\sigma|_{[e_0, \dots, e_p]}) \cdot \psi(\sigma|_{[e_p, \dots, e_{p+q}]})$$

where $\Delta^{j-i} = [e_i, \dots, e_j] \subseteq \Delta^n = [e_0, \dots, e_n]$ is the $j-i$ simplex spanned by $\{e_i, \dots, e_j\}$.

One may show ([12, p. 206]) that

$$\delta(\phi \cup \psi) = (\delta\phi) \cup \psi + (-1)^p \phi \cup \delta\psi,$$

so it follows that there is an induced product structure on cohomology:

$$\cup : H^p(X; R) \times H^q(X; R) \rightarrow H^{p+q}(X; R)$$

$$([\phi], [\psi]) \mapsto [\phi \cup \psi].$$

Proposition 1.4. *The product \cup is associative and has unity $1 \in H^0(X; R)$.*

If $f : Y \rightarrow X$ is continuous, then $f^(\alpha \cup \beta) = (f^*\alpha) \cup (f^*\beta)$ for all $\alpha, \beta \in H^*(X; R)$.*

Also \cup satisfies the graded commutativity property:

$$\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha$$

for any $\alpha \in H^p(X; R)$ and $\beta \in H^q(X; R)$.

Reference: [12, p. 210]. □

The most important properties of homology groups are summarized in the following theorem.

Theorem 1.5 (Eilenberg-Steenrod Axioms). *Singular homology satisfies the Eilenberg-Steenrod Axioms:*

1. Homotopy invariance: *If $f : (X, A) \rightarrow (Y, B)$ is homotopic to g , then $f_* = g_* : H_n(X, A) \rightarrow H_n(Y, B)$.*
2. Long exact sequence: *For any topological triple (X, A, B) and every degree q there is a map $\partial = \partial_q : H_q(X, A) \rightarrow H_{q-1}(A, B)$, such that the following sequence is exact:*

$$\cdots \longrightarrow H_q(A, B) \xrightarrow{i_*} H_q(X, B) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A, B) \longrightarrow \cdots,$$

where $i : (A, B) \hookrightarrow (X, B)$ and $j : (X, B) \hookrightarrow (X, A)$ are the inclusion mappings.

3. Excision: *If (X, A) is a topological pair, and $U \subseteq X$ such that $\bar{U} \subseteq \mathring{A}$, then the inclusion map $i : (X \setminus U, A \setminus U) \rightarrow (X, A)$ induces an isomorphism i_* in homology.*
4. Additivity: *Inclusion mappings induce an isomorphism*

$$\bigoplus_{\alpha} H_q(X_{\alpha}, A_{\alpha}) \rightarrow H_q\left(\bigsqcup_{\alpha} X_{\alpha}, \bigsqcup_{\alpha} A_{\alpha}\right).$$

5. Dimension: *$H_q(\{*\}) = 0$ for all $q \neq 0$ and $H_0(\{*\}) = \mathbb{Z}$.*

Reference: [5, p. 221-229]. □

Corollary 1.6 (Suspension). *For a topological space X , define the suspension of X as*

$$\Sigma X := \underbrace{X \times [0, 1]/X \times \{1\}}_{=: C_+} \cup \underbrace{X \times [-1, 0]/X \times \{-1\}}_{=: C_-}.$$

Then $H_q(X, *) \cong H_{q+1}(\Sigma X, *)$ naturally for all q .

Proof (see [14, p. 158]). Looking at the long exact sequence

$$0 = H_{q+1}(C_+X, *) \longrightarrow H_{q+1}(C_+X, X) \xrightarrow{\partial} H_q(X, *) \longrightarrow H_q(C_+X, *) = 0,$$

we see that ∂ must be an isomorphism (C_+X is contractible). But $H_{q+1}(C_+X, X) \cong H_{q+1}(\Sigma X, *)$, with excision (for more details, see [14, p. 157 Corollary IV.9.2]). □

1.1.1 Cellular Homology

An important way of computing homology groups of a space - in particular a CW complex - is with the help of a CW decomposition of it.

Definition 1.7. For any CW pair (X, A) with characteristic mappings $\phi_\alpha : (D^n, \partial D^n) \rightarrow (X^n, X^{n-1})$, respectively $\varphi_\alpha : S^{n-1} = \partial D^n \rightarrow X^{n-1}$ for any cell α , define the *cellular chain complex* of (X, A) by $C_n^{\text{cell}}(X, A) := \mathbb{Z}[\mathcal{E}_n(X \setminus A)] = \bigoplus_{\alpha \in \mathcal{E}_n(X \setminus A)} \mathbb{Z}$, where $\mathcal{E}_n(X \setminus A)$ denotes the set of all n -cells of $X \setminus A$. Define the boundary map by

$$(\partial_n^{\text{cell}})_{\beta, \alpha} := \deg(S^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \longrightarrow \frac{X^{n-1}}{X^{n-1} \setminus \beta} \xrightarrow{\phi_\beta^{-1}} \frac{D^{n-1}}{\partial D^{n-1}} \xrightarrow{\cong} S^{n-1}).$$

For a cellular map $f : (X, A) \rightarrow (Y, B)$ define $f_* : C_n^{\text{cell}}(X, A) \rightarrow C_n^{\text{cell}}(Y, B)$ by

$$(f_*)_{\beta, \alpha} := \deg(S^n \xrightarrow{\phi_\alpha} \frac{X^n}{X^{n-1}} \xrightarrow{f} \frac{Y^n}{Y^{n-1}} \longrightarrow \frac{Y^n}{Y^{n-1} \setminus \beta} \xrightarrow{\phi_\beta^{-1}} \frac{D^n}{\partial D^n} \xrightarrow{\cong} S^n).$$

Define *cellular homology* to be the homology of the cellular chain complex.

Theorem 1.8. *There is a natural isomorphism between singular homology and cellular homology.*

Reference: [12, p. 137-140]. □

For our later purposes of defining Whitehead torsion of a CW pair (X, A) (in section 1.3.5) we will need an enriched algebraic structure on $C_*(\tilde{X}, \tilde{A})$ for a universal covering (\tilde{X}, \tilde{A}) of (X, A) , which is explained in the following definition.

Definition 1.9. Let (X, A) be a CW pair, $p : \tilde{X} \rightarrow X$ the universal covering of X and let $G := \text{Deck}(p)$ its group of covering automorphisms. Let $\tilde{A} = p^{-1}(A)$. We define an action of G on $C_*(\tilde{X}, \tilde{A})$ by $g \cdot \sigma = g_\#(\sigma)$. This way $C_*(\tilde{X}, \tilde{A})$ becomes a $\mathbb{Z}[G]$ -complex if we define

$$\left(\sum_i n_i g_i \right) \cdot \sigma := \sum_i n_i (g_i \cdot \sigma) = \sum_i n_i (g_i)_\#(\sigma).$$

Proposition 1.10. *Let (X, A) be a CW pair, $p : \tilde{X} \rightarrow X$ the universal covering of X and let $G := \text{Deck}(p)$ its group of covering automorphisms. Let $\tilde{A} = p^{-1}(A)$. Then $C_*(\tilde{X}, \tilde{A})$ is a free $\mathbb{Z}[G]$ -complex with basis $\{\tilde{\alpha} : \alpha \in \mathcal{E}(X \setminus A)\}$. Where $\tilde{\alpha}$ is the cell of $\tilde{X} \setminus \tilde{A}$ that has characteristic map $\tilde{\phi}_\alpha$, which is a specifically chosen lift of ϕ_α (see 1.20 for existence of lifts).*

Reference: [7, p. 11]. □

1.1.2 The Universal Coefficient Theorem

Another useful tool to compute (co)homology are the universal coefficient theorems. For an explanation of the Tor- and Ext-functors, go to [13, p. 29].

Theorem 1.11 (Universal Coefficient Theorem). *Let (X, A) be a topological pair, G an abelian group.*

Then there is a natural short exact sequence

$$0 \longrightarrow H_n(X, A) \otimes G \longrightarrow H_n(X, A; G) \longrightarrow \text{Tor}(H_{n-1}(X, A), G) \longrightarrow 0.$$

This sequence splits (unnaturally).

Reference: [5, p. 263].

□

Theorem 1.12 (Universal Coefficient Theorem). *Let (X, A) be a topological pair, G an abelian group.*

Then there is a natural short exact sequence

$$0 \longrightarrow \text{Ext}(H_{n-1}(X, A), G) \longrightarrow H^n(X, A; G) \longrightarrow \text{Hom}(H_n(X, A), G) \longrightarrow 0.$$

This sequence splits (unnaturally).

Reference: [5, p. 333].

□

1.1.3 Bockstein Homomorphisms

We will make good use of Bockstein homomorphisms in the proof of the homotopy classification of lens spaces, especially of the Bockstein homomorphism β associated to the sequence $0 \longrightarrow \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_{m^2} \longrightarrow \mathbb{Z}_m \longrightarrow 0$.

Definition 1.13. Let $0 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 0$ be a short exact sequence of abelian groups. Apply the covariant functor $\text{Hom}(C_n(X), -)$ and obtain

$$0 \longrightarrow C^n(X; G) \longrightarrow C^n(X; H) \longrightarrow C^n(X; K) \longrightarrow 0$$

which is as well exact, because $C_n(X)$ is free.

Since we have a short exact sequence of chain complexes, we get an associated long exact sequence in homology:

$$\cdots \longrightarrow H^n(X; G) \longrightarrow H^n(X; H) \longrightarrow H^n(X; K) \longrightarrow H^{n+1}(X; G) \longrightarrow \cdots$$

The boundary map $H^n(X; K) \rightarrow H^{n+1}(X; G)$ of this sequence is called a *Bockstein homomorphism*.

For our purposes the most important Bockstein homomorphism will be the one associated to the sequence $0 \longrightarrow \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_{m^2} \longrightarrow \mathbb{Z}_m \longrightarrow 0$. This β satisfies a useful property:

Lemma 1.14. *Let β be as stated above. Then β satisfies the derivation property*

$$\beta(a \cup b) = \beta(a) \cup b + (-1)^{|a|} a \cup \beta(b).$$

Also β satisfies $\beta^2 = 0$.

Proof (see [12, p. 304, 305]). Let ϕ and ψ be \mathbb{Z}_m cocycles representing a and b , and let $\tilde{\phi}$ and $\tilde{\psi}$ be lifts of these to \mathbb{Z}_{m^2} cochains (regarding ϕ and ψ as functions on singular simplices to \mathbb{Z}_m , one can take $\tilde{\phi}$ and $\tilde{\psi}$ to be the same functions when regarding \mathbb{Z}_m to be a subset of \mathbb{Z}_{m^2}). We then have $\delta\tilde{\phi} = m\eta$ and $\delta\tilde{\psi} = m\mu$ for \mathbb{Z}_m cocycles η and μ representing $\beta(a)$ and $\beta(b)$ by the definition of β . Now $\tilde{\phi} \cup \tilde{\psi}$ is a \mathbb{Z}_{m^2} cochain lifting the \mathbb{Z}_m cocycle $\phi \cup \psi$ and we have

$$\delta(\tilde{\phi} \cup \tilde{\psi}) = \delta\tilde{\phi} \cup \tilde{\psi} + (-1)^{|\tilde{\phi}|} \tilde{\phi} \cup \delta\tilde{\psi} = m\eta \cup \tilde{\psi} + (-1)^{|\tilde{\phi}|} \tilde{\phi} \cup m\mu = m(\eta \cup \tilde{\psi} + (-1)^{|\tilde{\phi}|} \tilde{\phi} \cup \mu)$$

because of the derivation property of δ . We get that $\eta \cup \tilde{\psi} + (-1)^{|\tilde{\phi}|} \tilde{\phi} \cup \mu$ represents $\beta(a \cup b)$ and the claim follows.

To prove $\beta^2 = 0$, note that if $\tilde{\beta}$ denotes the Bockstein associated to the sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}_m \longrightarrow 0$ and if ρ denotes the homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}_m$ reducing coefficients mod m , then $\beta = \rho\tilde{\beta}$. Now $\beta^2 = \rho\tilde{\beta}\rho\tilde{\beta}$, but $\tilde{\beta}\rho = 0$, since $\text{im } \rho = \ker \tilde{\beta}$ in the long exact sequence containing $\tilde{\beta}$. \square

1.2 Homotopy Groups

Definition 1.15. Let $n \geq 0$, X a topological space, $x_0 \in X$.

Define the *n*th homotopy group of (X, x_0) to be $\pi_n(X, x_0) := [(I^n, \partial I^n), (X, x_0)]$ the set of all homotopy classes of maps $(I^n, \partial I^n), (X, x_0)$ together with the algebraic structure:

- (a) of a pointed set for $n = 0$,
- (b) of a group for $n = 1$, via $[f][g] := [fg]$, where fg is the concatenation of paths,

(c) of an abelian group for $n > 1$ via $[f] + [g] := [f + g]$, where

$$f + g : (I^n, \partial I^n) \rightarrow (X, x_0), (s_1, \dots, s_n) \mapsto \begin{cases} f(2s_1, s_2, \dots, s_n) & 0 \leq s \leq 1/2 \\ g(2s - 1, s_2, \dots, s_n) & 1/2 \leq s \leq 1 \end{cases}$$

(concatenation in the first variable).

Definition 1.16. Let $n \geq 1$, (X, A) a topological pair, $x_0 \in A$.

Define the n th relative homotopy group to be $\pi_n(X, A, x_0) := [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)]$, where $J^{n-1} := \overline{\partial I^n \setminus I^{n-1}}$, with the algebraic structure defined with the same formulas as in 1.15 (c). Note that this is now only a group for $n \geq 2$ and only abelian for $n \geq 3$.

For a proof that these structures are indeed well-defined groups, respectively abelian groups, see [12, p. 25, p. 340-342, p. 343].

Remark 1.17. Note that

$$\begin{aligned} [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)] &= [(I^n/J^{n-1}, \partial I^n/J^{n-1}, *), (X, A, x_0)] \\ &= [(D^n, S^{n-1}, s_0), (X, A, x_0)]. \end{aligned}$$

Proposition 1.18 (Compression Criterion). *A map $f : (D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$ represents zero in $\pi_n(X, A, x_0)$ if and only if it is homotopic rel S^{n-1} to a map with image contained in A .*

Reference: [12, p. 343]. □

Theorem 1.19 (Long Exact Sequence of Homotopy Groups). *Let $x_0 \in B \subseteq A \subseteq X$, then the sequence*

$$\begin{aligned} \cdots \longrightarrow \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \longrightarrow \cdots \\ \cdots \longrightarrow \pi_1(X, A, x_0) \end{aligned}$$

is natural and exact, where $i : (A, B) \hookrightarrow (X, B)$ and $j : (X, B) \hookrightarrow (X, A)$ are the inclusion mappings and $\partial[f] = [f|_{S^{n-1}}]$.

Reference: [12, p. 344-345]. □

Now the fundamental group $\pi_1(X, x_0)$ of a space (X, x_0) is known to have a close connection to its covering theory and we will state some of the most important results. For the higher homotopy groups $\pi_n(X, x_0)$, $n \geq 2$, we have 1.22.

Theorem 1.20. *Let $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ a covering of pointed spaces and let (Y, y_0) be a connected, locally path connected pointed space. Then a map $f : (Y, y_0) \rightarrow (X, x_0)$ has a lift $\tilde{f} : (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ if and only if $f_*(\pi_1(Y, y_0))$ is contained in the characteristic subgroup of p , that means $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. In this case the lift \tilde{f} is unique.*

Reference: [14, p. 81, Theorem II.4.5]. □

Theorem 1.21. *Let $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ be a universal covering of a locally path connected space X . Then $\text{Deck}(p) \cong \pi_1(X, x_0)$.*

For future references, we call this isomorphism $\theta = \theta(x_0, \tilde{x}_0) : \pi_1(X, x_0) \rightarrow \text{Deck}(p)$; for any $[\alpha] \in \pi_1(X, x_0)$ it is given by

$$\theta_{[\alpha]}(y) = \widetilde{\alpha \cdot p\omega}(1),$$

where $y \in \tilde{X}$, ω is a path from \tilde{x}_0 to y , and $\widetilde{\alpha \cdot p\omega}$ is the unique lift of $\alpha \cdot p\omega$ with $\widetilde{\alpha \cdot p\omega}(0) = \tilde{x}_0$.

Reference: [14, p. 84, Corollary II.4.11 and p. 86, Example II.5.4]. □

Proposition 1.22. *Let $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ be a covering map.*

Then $p_ : \pi_n(\tilde{X}, \tilde{x}_0) \rightarrow \pi_n(X, x_0)$ is an isomorphism for every $n \geq 2$ and injective for $n = 1$.*

Proof (see [12, p. 342]). Let $[f] \in \pi_n(X, x_0)$, i.e. $f : (S^n, s_0) \rightarrow (X, x_0)$. Since $\pi_1(S^n) = 0$ for $n \geq 2$, we may (by 1.20) lift f over p to an $\tilde{f} : (S^n, s_0) \rightarrow (\tilde{X}, \tilde{x}_0)$. Since $p \circ \tilde{f} = f$ we get $p_*([\tilde{f}]) = [f]$, so surjectivity is shown.

Now let $\tilde{f}, \tilde{g} : (S^n, s_0) \rightarrow (\tilde{X}, \tilde{x}_0)$, s.t. $p_*([\tilde{f}]) = p_*([\tilde{g}])$.

This means that there is a homotopy $H : S^n \times I \rightarrow X$ from $H_0 = p \circ \tilde{f}$ to $H_1 = p \circ \tilde{g}$, that fixes basepoints, i.e. $H_t(s_0) = x_0$ for all $t \in I$.

Such a homotopy may be lifted to a $\tilde{H} : S^n \times I \rightarrow \tilde{X}$, with $\tilde{H}_0 = \tilde{f}$ and $p \circ \tilde{H} = H$ (see [14, p. 76, Theorem II.3.3]). Since $\tilde{H}_0(s_0) = \tilde{f}(s_0) = \tilde{x}_0$ and $p(\tilde{H}_t(s_0)) = H_t(s_0) = x_0$, we get that $\tilde{H}_t(s_0) = \tilde{x}_0$ for all $t \in I$. But since $p \circ \tilde{H}_1 = H_1 = g$, our \tilde{H}_1 is as well a lift of g with $\tilde{H}_1(s_0) = \tilde{x}_0 = \tilde{g}(s_0)$ and it follows $\tilde{H}_1 = \tilde{g}$ since such a lift is unique. We have shown $\tilde{f} \simeq \tilde{g}$, and injectivity of p_* follows. □

1.2.1 Whitehead's theorem

Lemma 1.23 (Compression Lemma). *Let (X, A) be a CW pair and (Y, B) a topological pair, s.t. $B \neq \emptyset$ and that for all n : If $X \setminus A$ has n -cells, then $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f : (X, A) \rightarrow (Y, B)$ is homotopic rel A to a map $X \rightarrow B$.*

Proof (see [12, p. 347]). Assume inductively that f has already been homotoped to take the skeleton X^{k-1} to B . Let ϕ be the characteristic map of a cell e^k of $X \setminus A$. Then the composition $f \circ \phi : (D^k, \partial D^k) \rightarrow (Y, B)$ can be homotoped into B rel ∂D^k , since $\pi_k(Y, B, y_0) = 0$ as we supposed (see 1.18). This homotopy of $f \circ \phi$ now induces a homotopy rel X^{k-1} of f on the quotient space $X^{k-1} \cup e^k$ of $X^{k-1} \sqcup D^k$. Doing this for all the k -cells of $X \setminus A$ simultaneously, and taking the constant homotopy on A , we obtain a homotopy of $f|_{X^k \cup A}$ to a map into B . Since every CW pair has the homotopy extension property (see [12, p. 15]), this homotopy extends to a homotopy defined on all of X , and we completed the induction step.

If the dimension of the cells of $X \setminus A$ is bounded, then we are already done. If not, we may define a homotopy, by performing the homotopy of the k th induction step during the interval $[1 - 1/2^k, 1 - 1/2^{k+1}]$. \square

Theorem 1.24 (Whitehead's theorem). *Let X, Y be connected CW complexes and $f : X \rightarrow Y$, s.t. $f_* : \pi_n(X) \rightarrow \pi_n(Y)$ is an isomorphism for all n . Then*

(a) *f is a homotopy equivalence.*

(b) *If $f : X \hookrightarrow Y$ is the inclusion of a subcomplex, then X is a strong deformation retract of Y .*

Proof (see [12, p. 347-348]). We first prove (b):

From our assumptions it follows, that all relative homotopy groups are trivial (via the long exact sequence of homotopy groups):

$$\pi_n(Y, X) = 0 \text{ for all } n.$$

Now from the compression lemma we get that $\text{id} : (Y, X) \rightarrow (Y, X)$ is homotopic rel X to a mapping $r : Y \rightarrow X$.

To prove (a), note that X and Y embed into the mapping cylinder $M_f := (X \times I \sqcup Y) / \sim$ (where $(x, 1) \sim f(x)$), which deformation retracts onto Y . Now our assumptions imply

$$\pi_n(M_f, X) = 0 \text{ for all } n.$$

Note that if $f : X \rightarrow Y$ was a cellular map, then (M_f, X) would be a CW pair and we were done by (b), since $f : X \rightarrow Y$ is the composition of the inclusion $X \hookrightarrow M_f$ and the retraction $M_f \rightarrow Y$.

Otherwise, we first obtain a homotopy rel X of the inclusion $(X \cup Y, X) \hookrightarrow (M_f, X)$ to a map $X \cup Y \rightarrow X$. This homotopy then extends to a homotopy from $\text{id} : M_f \rightarrow M_f$ to a map $g : M_f \rightarrow M_f$, which takes $X \cup Y$ to X .

Now applying the compression lemma again to the composition

$$(X \times I \sqcup Y, X \times \partial I \sqcup Y) \longrightarrow (M_f, X \cup Y) \xrightarrow{g} (M_f, X)$$

we obtain a deformation retraction of M_f onto X . \square

We will now discuss a few applications of Whitehead's theorem that will come in handy, when we come to the definition of Whitehead torsion in section 1.3.

Proposition 1.25. *Let (K, L) be a pair of connected CW complexes, and let $p : \tilde{K} \rightarrow K$ be the universal covering of K . Let $\tilde{L} := p^{-1}L$. If $i_* : \pi_1 L \hookrightarrow \pi_1 K$ is an isomorphism, then $p|_{\tilde{L}} : \tilde{L} \rightarrow L$ is the universal covering of L . Further, if L is a strong deformation retract of K then \tilde{L} is a strong deformation retract of \tilde{K} .*

Proof (see [7, p. 9]). \tilde{L} consists of the lifts of the cells of L and is hence a subcomplex of \tilde{K} . Obviously $p|_{\tilde{L}}$ is a covering of L . Note that by 1.22 we have that $p_* : \pi_n(\tilde{K}, \tilde{L}) \rightarrow \pi_n(K, L)$ is an isomorphism for all $n \geq 1$. To see that \tilde{L} is connected, notice that $\pi_1(K, L) = 0$, since we have exactness in the sequence

$$\pi_1(L) \xrightarrow{\cong} \pi_1(K) \longrightarrow \pi_1(K, L) \longrightarrow \pi_0(L) \xrightarrow{\cong} \pi_0(K).$$

Thus $\pi_1(\tilde{K}, \tilde{L}) = 0$. So since \tilde{K} is connected and the sequence

$$0 = \pi_1(\tilde{K}, \tilde{L}) \longrightarrow \pi_0(\tilde{L}) \longrightarrow \pi_0(\tilde{K})$$

is exact, it follows that \tilde{L} is connected. \tilde{L} is simply connected, because of the commutativity of the diagram

$$\begin{array}{ccc} \pi_1 \tilde{L} & \xrightarrow{\subseteq} & \pi_1 \tilde{K} = 0 \\ \downarrow p_* & & \downarrow 0 \\ \pi_1 L & \xrightarrow{\cong} & \pi_1 K \end{array}$$

So $p : \tilde{L} \rightarrow L$ is the universal covering. Now if L is a strong deformation retract of K , we have $\pi_n(K, L) = 0$ for all n , and hence $\pi_n(\tilde{K}, \tilde{L}) = 0$ for all n . So by Whitehead's theorem \tilde{L} is a strong deformation retract of \tilde{K} . \square

Proposition 1.26. *Let $f : K \rightarrow L$ be a cellular map between connected CW complexes, such that $f_* : \pi_1 K \rightarrow \pi_1 L$ is an isomorphism. If \tilde{K}, \tilde{L} are universal covering spaces of K, L and $\tilde{f} : \tilde{K} \rightarrow \tilde{L}$ is a lift of f , then $M_{\tilde{f}}$ is a universal covering space of M_f .*

Proof (see [7, p. 10]). Since \tilde{f} is cellular and \tilde{L} is a strong deformation retract of $M_{\tilde{f}}$ we know that $M_{\tilde{f}}$ is a simply connected CW complex. Now let $p : \tilde{K} \rightarrow K$ and $p' : \tilde{L} \rightarrow L$ be the covering maps. Define $\alpha : M_{\tilde{f}} \rightarrow M_f$ by

$$\begin{aligned}\alpha[w, t] &:= [p(w), t], \text{ if } 0 \leq t \leq 1, w \in \tilde{K} \\ \alpha[z] &:= [p'(z)], \text{ if } z \in \tilde{L}\end{aligned}$$

If $[w, 1] = [z]$ then $\tilde{f}(w) = z$, so $\alpha[w, 1] = [p(w), 1] = [fp(w), 1] = [p'\tilde{f}(w)] = [p'(z)]$. Hence α is well-defined. Note that $\alpha|_{M_{\tilde{f}} \setminus \tilde{L}} = \alpha|_{\tilde{K} \times [0, 1]} = p \times \text{id}_{[0, 1]}$ and $\alpha|_{\tilde{L}} = p'$. That means $\alpha|_{M_{\tilde{f}} \setminus \tilde{L}}$ and $\alpha|_{\tilde{L}}$ are covering maps, and α takes cells homeomorphically onto cells.

Let $\beta : \hat{M}_f \rightarrow M_f$ be the universal cover of M_f , and $\hat{K} := \beta^{-1}(K), \hat{L} := \beta^{-1}(L)$. By 1.25, $\beta|_{\hat{L}} : \hat{L} \rightarrow L$ is a universal covering. We supposed that $f_* : \pi_1 K \rightarrow \pi_1 L$ is an isomorphism, so $i_* : \pi_1 K \rightarrow \pi_1 M_f$ is as well an isomorphism. Hence, again by 1.25, \hat{K} is simply connected. Now clearly $\beta|_{\hat{M}_f \setminus \hat{L}} : \hat{M}_f \setminus \hat{L} \rightarrow M_f \setminus K$ is a covering and since $\pi_i(M_f \setminus L, K) = \pi_i(\hat{M}_f \setminus \hat{L}, \hat{K}) = 0$ for all i , we may conclude that $\hat{M}_f \setminus \hat{L}$ is simply connected and $\beta|_{\hat{M}_f \setminus \hat{L}}$ is as well a universal covering.

Now let $\hat{\alpha} : M_{\tilde{f}} \rightarrow \hat{M}_f$ be a lift of α . By the uniqueness of the universal covering spaces of $M_f \setminus L$ and L , $\hat{\alpha}$ must take $M_{\tilde{f}} \setminus \tilde{L}$ homeomorphically onto $\hat{M}_f \setminus \hat{L}$ and \tilde{L} homeomorphically onto \hat{L} . Thus $\hat{\alpha}$ is a continuous bijection. But $\hat{\alpha}$ takes each cell e homeomorphically onto a cell $\hat{\alpha}(e)$ and hence takes \bar{e} homeomorphically to $\hat{\alpha}(\bar{e}) = \overline{\hat{\alpha}(e)}$. Since $M_{\tilde{f}}$ and \hat{M}_f carry the weak topology with respect to closed cells, it follows that $\hat{\alpha}$ is a homeomorphism. Since $\beta\hat{\alpha} = \alpha$ it follows that α is a covering map. \square

1.2.2 The Hurewicz Homomorphism

In this section, we review the Hurewicz homomorphism, which makes an important connection between the homology groups and the homotopy groups of a pair (X, A) .

Definition 1.27. Fix a generator $\tilde{\alpha}_n$ of $H_n(S^n) \cong \mathbb{Z}$. Let (X, x_0) be a pointed space and $n \geq 1$. For any $f : (S^n, s_0) \rightarrow (X, x_0)$ representing an element of $\pi_n(X, x_0)$ define $h_n^{(X, x_0)}([f]) := f_*(\tilde{\alpha}_n)$, where $f_* : H_n(S^n) \rightarrow H_n(X)$ is the induced map on homology.

$$h_n^{(X, x_0)} : \pi_n(X, x_0) \rightarrow H_n(X), [f] \mapsto f_*([\tilde{\alpha}_n])$$

is called the *Hurewicz homomorphism*.

Now fix a generator α_n of $H_n(D^n, \partial D^n) \cong \mathbb{Z}$. If (X, A, x_0) is a pointed topological pair, then define the *relative Hurewicz homomorphism* by

$$h_n^{(X,A,x_0)} : \pi_n(X, A, x_0) \rightarrow H_n(X, A), [f] \mapsto f_*(\alpha_n)$$

Again $f_* : H_n(D^n, \partial D^n) \rightarrow H_n(X, A)$ is the induced map on homology.

Theorem 1.28 (Hurewicz). *For any topological space X the Hurewicz homomorphism induces an isomorphism*

$$\pi_1(X, x_0)_{ab} \xrightarrow{\cong} H_1(X)$$

between the abelized fundamental group of X and the first homology group of X .

Reference: [14, p. 169-172]. □

Theorem 1.29 (Hurewicz). *Let X be $(n-1)$ -connected, $n \geq 2$. Then $\tilde{H}_i(X) = 0$ for every $i < n$ and $h : \pi_n(X) \rightarrow H_n(X)$ is an isomorphism.*

Reference: [12, p. 366-367]. □

Corollary 1.30. *If X is simply connected, then the smallest number $q \in \mathbb{Z}$, where $\pi_q(X) \neq 0$ is also the smallest number $q \in \mathbb{Z}$, where $H_q(X) \neq 0$. And for this q , we have: $\pi_q(X) \cong H_q(X)$.*

Proof. This is obvious from 1.29. □

Corollary 1.31 (Hurewicz, relative version). *Let (X, A) be $(n-1)$ -connected, $n \geq 2$. Let $A \neq \emptyset$ be simply connected. Then $H_i(X, A) = 0$ for every $i < n$ and $h : \pi_n(X, A, x_0) \rightarrow H_n(X, A)$ is an isomorphism.*

Reference: [12, p. 366-367]. □

Corollary 1.32. *Let X and Y be simply connected and $f : X \rightarrow Y$, such that $f_* : H_n(X) \rightarrow H_n(Y)$ is an isomorphism for all n . Then $f_* : \pi_n(X) \rightarrow \pi_n(Y)$ is as well an isomorphism for all n . If in addition X and Y are CW complexes, then f is a homotopy equivalence.*

Proof (see [12, p. 367]). Without loss of generality we may assume, that $X \subseteq Y$ and $f : X \hookrightarrow Y$ is the inclusion (because X embeds into the mapping cylinder M_f , which

deformation retracts onto Y and the composition $r \circ i$ of the inclusion and the retraction is equal to f : $X \xrightarrow{i} M_f \xrightarrow{r} Y$).

The requirement that $f_* : H_n(X) \rightarrow H_n(Y)$ is an isomorphism for all n now reads:

$$H_n(Y, X) = 0 \text{ for all } n.$$

Since (Y, X) is simply connected, with 1.30 and 1.31 we conclude $\pi_2(Y, X) = 0$. And, inductively, that $\pi_n(Y, X) = 0$ for all n . Via the long exact sequence of homotopy groups, we see that $f_* : \pi_n(X) \rightarrow \pi_n(Y)$ is an isomorphism for all n .

If X and Y are CW complexes, we conclude via Whitehead's theorem, that f is a homotopy equivalence. \square

1.3 Simple-Homotopy Theory

In this section, we will review some important notions of simple-homotopy theory. These will be crucial for our proof of the homeomorphy classification of lens spaces. Our main reference will be [7].

Throughout this section we will always assume R to be a ring with unity that satisfies the following property:

If M is any finitely generated free module over R then any two bases of M have the same cardinality.

This is not much of a restriction for us, since our examples of rings will restrict to those of the form $R = \mathbb{Z}[G]$, i.e. group rings for some group G , which all of those satisfy this condition (see [7, p. 36]). Also any appearing free R -module will be assumed to be finitely generated.

Furthermore, any appearing subgroup G of the group of units of a ring R will be assumed to contain (-1) .

1.3.1 Whitehead Groups

Definition 1.33. Let R be a ring. Let $\text{GL}_n(R)$ be the group of invertible matrices with entries in R . Note that there is a natural injection of $\text{GL}_n(R)$ into $\text{GL}_{n+1}(R)$ given by

$$A \mapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}.$$

s.t. $f(G) \subseteq G'$ to the category of abelian groups and group homomorphisms.

$$\begin{aligned} (R, G) &\mapsto K_G(R) \\ (f : G \rightarrow G') &\mapsto (f_* : K_G(R) \rightarrow K_{G'}(R')). \end{aligned}$$

This then gives rise to a covariant functor from the category of groups to the category of abelian groups and group homomorphisms.

$$\begin{aligned} G &\mapsto \text{Wh}(G) \\ (f : G \rightarrow G') &\mapsto (f_* : \text{Wh}(G) \rightarrow \text{Wh}(G')). \end{aligned}$$

Lemma 1.37. *If $g \in G$ and if $f : G \rightarrow G$ is the group homomorphism s.t. $f(x) = gxg^{-1}$ for all x then $f_* : \text{Wh}(G) \rightarrow \text{Wh}(G)$ is the identity map.*

Proof. Clearly, f_* is conjugation with a sufficiently large matrix with only gs on the diagonal. Since such a matrix has zero torsion, $f_* : \text{Wh}(G) \rightarrow \text{Wh}(G)$ must be the identity map. \square

Lemma 1.38. *If A, B and X are $n \times n, m \times m$ and $n \times m$ matrices respectively and if A has a right inverse or B has a left inverse then*

1. $\begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ is non-singular $\Leftrightarrow A$ and B are non-singular.
2. If A and B are non-singular then

$$\tau \left(\begin{pmatrix} A & X \\ 0 & B \end{pmatrix} \right) = \tau(A) + \tau(B).$$

Reference: [7, p. 40]. \square

Lemma 1.39. *Let R be a commutative ring and G a subgroup of the group U of all units of R . Let $\text{SK}_1(R) := \tau_G(\text{SL}(R))$ where $\tau_G : \text{GL}(R) \rightarrow K_G(R)$ and $\text{SL}(R)$ is the subgroup of $\text{GL}(R)$ of matrices of determinant 1. Then $\text{SK}_1(R)$ is independent of the group chosen and there is a split short exact sequence*

$$0 \longrightarrow \text{SK}_1(R) \xrightarrow{\subseteq} K_G(R) \begin{array}{c} \xrightarrow{[\det]} \\ \xleftarrow{s} \end{array} U/G \longrightarrow 0$$

where $[\det](\tau_G A) := (\det A) \cdot G$ and $s(u \cdot G) := \tau_G(u)$. In particular, if R is a field, $[\det]$ is an isomorphism.

Proof. Let $\tau_1 : \mathrm{GL}(R) \rightarrow \mathrm{K}_1(R)$. We have that

$$\mathrm{GL}(R) \begin{array}{c} \xrightarrow{\tau_1} \mathrm{K}_1(R) \xrightarrow{\pi} \mathrm{K}_G(R), \\ \xrightarrow{\tau_G} \end{array}$$

where $\pi : \mathrm{K}_1(R) \rightarrow \mathrm{K}_G(R)$ is the natural map. Now $\pi|_{\tau_1(\mathrm{SL}(R))} : \tau_1(\mathrm{SL}(R)) \rightarrow \tau_G(\mathrm{SL}(R))$ is obviously surjective and we wish to show the injectivity of $\pi|_{\tau_1(\mathrm{SL}(R))}$, so that $\tau_G(\mathrm{SL}(R)) \cong \tau_1(\mathrm{SL}(R))$ and $\mathrm{SK}_1(R)$ is independent of the group G . For that it suffices to show that if $\tau_G A = 0$ then $\tau_1 A = 0$ for any matrix $A \in \mathrm{SL}(R)$. But if $\tau_G A = 0$ then A is a product of some matrices A_p of the form $I_n + aE_{i,j}^n$ (for some $n, i, j \in \mathbb{N}$, $a \in R$ dependent on p) and some matrices B_q of the form

$$B_q = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & g_q & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}, g_q \in G.$$

And since $\det A = 1$, all appearing g s, say $g_1, \dots, g_k \in G$, must multiply to 1, $\prod_{q=1}^k g_i = 1$.

But now $\tau_1 A$ may be computed as $\tau_1(\prod_{q=1}^k B_q)$, so we get with 1.38 that

$$\tau_1(A) = \tau_1\left(\prod_{q=1}^k B_q\right) = \sum_{q=1}^k \tau_1(g_q) = \tau_1\left(\prod_{q=1}^k g_q\right) = \tau_1(1) = 0.$$

It is obvious that $[\det] \circ s = \mathrm{id}$ and the exactness of the sequence follows immediately from the definitions of $[\det]$ and $\mathrm{SK}_1(R)$. \square

1.3.2 (R, G) -complexes

Definition 1.40. For a module homomorphism $f : M_1 \rightarrow M_2$ where M_1 and M_2 have bases $x = \{x_1, \dots, x_p\}$ and $y = \{y_1, \dots, y_q\}$ respectively, we write $\langle f \rangle_{x,y}$ for the matrix representing f , i.e. the matrix (a_{ij}) where $f(x_i) = \sum_j a_{ij}y_j$.

If $x = \{x_1, \dots, x_p\}$ and $y = \{y_1, \dots, y_p\}$ are two bases of the same module M , we write $\langle x/y \rangle$ for the change-of-base matrix, i.e. the matrix (a_{ij}) where $x_i = \sum_j a_{ij}y_j$.

Definition 1.41. An (R, G) -module is defined to be a free R -module M along with a distinguished family B of bases which satisfy:

If b and b' are bases of M and if $b \in B$ then

$$b' \in B \Leftrightarrow \tau(\langle b/b' \rangle) = 0 \in K_G(R).$$

If M_1 and M_2 are (R, G) -modules and if $f : M_1 \rightarrow M_2$ is a module isomorphism then the *torsion of f* - in symbols $\tau(f)$ - is defined to be $\tau(A) \in K_G(R)$, where A is the matrix of f with respect to any distinguished bases of M_1 and M_2 . If $\tau(f) = 0$ then f is called a *simple isomorphism of (R, G) -modules*. In this case we write $f : M_1 \stackrel{\Sigma}{\approx} M_2$.

Now an (R, G) -complex is defined to be a free chain complex over R

$$C : 0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0$$

such that each C_i is an (R, G) -module. A *distinguished basis of C* is then a basis $c = \bigcup c_i$ where each c_i is a distinguished basis of C_i .

A *simple isomorphism of (R, G) -complexes* $f : C \rightarrow C'$ is a chain mapping such that $f|_{C_i} : C_i \stackrel{\Sigma}{\approx} C'_i$ for all i .

For a group G , we define a $\text{Wh}(G)$ -complex to be a (R, T) -complex, where $R = \mathbb{Z}[G]$ is the group ring and $T = G \cup -G$ is the group of trivial units.

Definition 1.42. An (R, G) -complex C is called an *elementary trivial complex* if it is of the form

$$C : 0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \longrightarrow 0$$

where d is a simple isomorphism of (R, G) -modules.

An (R, G) -complex is called *trivial* if it is the direct sum, in the category of (R, G) -complexes, of elementary trivial complexes.

Two (R, G) -complexes C and C' are called *stably equivalent*, in symbols $C \stackrel{s}{\approx} C'$, if there are trivial complexes T and T' such that $C \oplus T \stackrel{\Sigma}{\approx} C' \oplus T'$.

To prove the uniqueness of the torsion of an acyclic (R, G) -complex in 1.50, we need to be able to simplify such a complex (see 1.47). For the definition of torsion, we need to have some knowledge about chain contractions. The following lemmas provide the needed technicalities used in 1.49 and 1.50.

Definition 1.43. An R -module M is called *stably free* if there exist free R -modules F_1 and F_2 such that $M \oplus F_1 = F_2$ (note that we only speak of free modules with finite basis).

Definition 1.44. Let C be a free chain complex over R with boundary operator d . A degree-one homomorphism $\delta : C \rightarrow C$ is called a *chain contraction* if $\delta d + d\delta = 1$.

Lemma 1.45. If C is a free acyclic chain complex over R with boundary operator d , and if we denote $B_i := dC_{i+1}$, then

1. B_i is stably free for all i
2. There is chain contraction $\delta : C \rightarrow C$.
3. Any chain contraction $\delta : C \rightarrow C$ satisfies: $d\delta|_{B_{i-1}} = 1$, $C_i = B_i \oplus \delta B_{i-1}$ for all i .

Reference: [7, p. 47]. □

Lemma 1.46. Suppose that C is an acyclic (R, G) -complex with chain contractions δ and $\bar{\delta}$. For fixed i , let $1 \oplus \delta d : C_i \rightarrow C_i$ be defined by

$$1 \oplus \delta d : B_i \oplus \bar{\delta} B_{i-1} = C_i \rightarrow B_i \oplus \delta B_{i-1} = C_i.$$

Then

1. $1 \oplus \delta d$ is a simple isomorphism.
2. If B_i and B_{i-1} are free modules with bases b_i and b_{i-1} and if c_i is a basis of C_i , then $\tau\langle b_i \cup \delta b_{i-1} / c_i \rangle = \tau\langle b_i \cup \bar{\delta} b_{i-1} / c_i \rangle$.

Reference: [7, p. 49]. □

Lemma 1.47 (Simplifying a Complex). If C is an acyclic (R, G) -complex of the form

$$C : 0 \longrightarrow C_n \xrightarrow{d_n} \cdots \xrightarrow{d_{i+3}} C_{i+2} \xrightarrow{d_{i+2}} C_{i+1} \xrightarrow{d_{i+1}} C_i \longrightarrow 0 \quad (n \geq i+1)$$

and if $\delta : C \rightarrow C$ is a chain contraction, then $C \stackrel{s}{\sim} C_\delta$ where C_δ is the complex

$$C_\delta : 0 \longrightarrow C_n \xrightarrow{d_n} \cdots \xrightarrow{d_{i+3}} C_{i+2} \xrightarrow{d_{i+2}} C_{i+1} \longrightarrow 0$$

$$\begin{array}{c} \oplus \\ \nearrow \delta_{i+1} \\ C_i \end{array}$$

Proof (see [7, p. 50-51]). We may assume without loss of generality that $i = 0$.

Let T be the trivial complex with $T_1 = T_2 = C_0$, $T_i = 0$ for $i \neq 1, 2$, and $\partial_2 = \text{id} : T_2 \rightarrow T_1$. Let T' be the trivial complex with $T'_0 = T'_1 = C_0$, $T'_i = 0$ if $i \neq 0, 1$, and

$\partial'_1 = \text{id} : T'_1 \rightarrow T'_0$. We claim that $C \oplus T \xrightarrow{\Sigma} C_\delta \oplus T'$. These complexes look like

$$\begin{aligned} C \oplus T : \cdots &\longrightarrow C_3 \longrightarrow C_2 \oplus C_0 \xrightarrow{d_2 \oplus 1} C_1 \oplus C_0 \xrightarrow{d_1 \oplus 0} C_0 \longrightarrow 0 \\ C_\delta \oplus T' : \cdots &\longrightarrow C_3 \longrightarrow C_2 \oplus C_0 \xrightarrow{d_2 \oplus \delta_1} C_1 \oplus C_0 \xrightarrow{0 \oplus 1} C_0 \longrightarrow 0 \end{aligned}$$

And we define $f : C \oplus T \rightarrow C_\delta \oplus T'$ by

$$\begin{aligned} f_i &= 1, \text{ if } i \neq 1 \\ f_1(c_0 + c_1) &= \delta_1 c_0 + (c_1 + d_1 c_1), \text{ if } c_0 \in C_0 \text{ and } c_1 \in C_1. \end{aligned}$$

By computing

$$\begin{aligned} (d_2 \oplus \delta_1 \circ f_2)(c_2 + c_0) &= d_2 \oplus \delta_1(c_2 + c_0) \\ &= \delta_1 c_0 + d_2 c_2 \\ &= \delta_1 c_0 + (d_2 c_2 + \underbrace{d_1 d_2 c_2}_{=0}) \\ &= f_1(d_2(c_2) + c_0) \\ &= f_1 \circ (d_2 \oplus 1)(c_2 + c_0) \end{aligned}$$

and

$$\begin{aligned} (0 \oplus 1) \circ f_1(c_1 + c_0) &= (0 \oplus 1)(\delta_1 c_0 + (c_1 + d_1 c_1)) \\ &= d_1 c_1 \\ &= (d_1 \oplus 0)(c_1 + c_0) \\ &= f_0 \circ (d_1 \oplus 0)(c_1 + c_0), \end{aligned}$$

we see that f is a chain map. And it remains to show that f is a simple isomorphism. This is obvious except for $i = 1$. But

$$\langle f_1 \rangle = \begin{array}{c} C_0 \quad C_1 \\ C_0 \\ C_1 \end{array} \begin{pmatrix} 0 & \langle \delta_1 \rangle \\ \langle d_1 \rangle & I \end{pmatrix} = \begin{pmatrix} -I & \langle \delta_1 \rangle \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ \langle d_1 \rangle & I \end{pmatrix}.$$

Since $\langle \delta_1 \rangle \langle d_1 \rangle = \langle d_1 \delta_1 \rangle = \langle d_1 \delta_1 + \delta_0 d_0 \rangle = I$. So by 1.38, $\tau(f_1) = 0$. □

1.3.3 The Torsion of an Acyclic Complex

Definition 1.48. Let C be an acyclic (R, G) -complex with boundary operator d . Let δ be any chain contraction of C . Set

$$\begin{aligned} C_{\text{odd}} &:= C_1 \oplus C_3 \oplus \dots \\ C_{\text{even}} &:= C_0 \oplus C_2 \oplus \dots \\ (d + \delta)_{\text{odd}} &:= (d + \delta)|_{C_{\text{odd}}} : C_{\text{odd}} \rightarrow C_{\text{even}}. \end{aligned}$$

And define $\tau(C) := \tau(\langle (d + \delta)_{\text{odd}} \rangle) \in K_G(R)$.

Lemma 1.49. $\tau(C)$ is well-defined, i.e. $\langle d + \delta \rangle$ is non-singular and $\tau(d + \delta)$ is independent of the distinguished basis chosen and independent of which chain contraction δ is chosen.

Proof (see [7, p. 52-54]). To show that $\langle d + \delta \rangle$ is non-singular we fix a distinguished basis c for C . Since $(d + \delta)_{\text{even}} \circ (d + \delta)_{\text{odd}} = (d^2 + d\delta + \delta d + \delta^2)|_{C_{\text{odd}}} = (1 + \delta^2)|_{C_{\text{odd}}}$ we have

$$\langle (d + \delta)_{\text{odd}} \rangle_c \langle (d + \delta)_{\text{even}} \rangle_c = \begin{matrix} & C_1 & C_3 & C_5 & \dots \\ C_1 & \left(\begin{array}{cccc} I & \langle \delta_3 \delta_2 \rangle & & \\ & I & \langle \delta_5 \delta_4 \rangle & \\ & & I & \ddots \\ \vdots & & & \ddots \end{array} \right) & & & \end{matrix}$$

But now by 1.38 this matrix is non-singular and has zero torsion. Similarly, this holds for $\langle (d + \delta)_{\text{even}} \rangle_c \langle (d + \delta)_{\text{odd}} \rangle_c$. Note that we also have $\tau(\langle (d + \delta)_{\text{odd}} \rangle_c) = -\tau(\langle (d + \delta)_{\text{even}} \rangle_c)$.

Now let $c = \bigcup c_i$ and $c' = \bigcup c'_i$ be basis of C where the c_i and c'_i are distinguished bases of C_i . Then

$$\begin{aligned} \langle d + \delta \rangle_c &= \langle c_{\text{odd}}/c'_{\text{odd}} \rangle \langle d + \delta \rangle_{c'} \langle c'_{\text{even}}/c_{\text{even}} \rangle \\ &= \begin{pmatrix} \langle c_1/c'_1 \rangle & & & \\ & \langle c_3/c'_3 \rangle & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \langle d + \delta \rangle_{c'} \begin{pmatrix} \langle c'_0/c_0 \rangle & & & \\ & \langle c'_2/c_2 \rangle & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \end{aligned}$$

So with 1.38 we get

$$\tau(\langle d + \delta \rangle_c) = \sum_i \tau(\langle c_{2i+1}/c'_{2i+1} \rangle) + \tau(\langle d + \delta \rangle_{c'}) + \sum_i \tau(\langle c'_{2i}/c_{2i} \rangle) = \tau(\langle d + \delta \rangle_{c'}).$$

Lastly suppose that δ and $\bar{\delta}$ are two different chain contractions, and we wish to show that $\tau(d + \delta) = \tau(d + \bar{\delta})$. For that we compute

$$\begin{aligned}
\tau(d + \bar{\delta}) - \tau(d + \delta) &= \tau\langle (d + \bar{\delta})_{\text{odd}} \rangle + \tau\langle (d + \delta)_{\text{even}} \rangle \\
&= \tau\langle (d + \delta) \circ (d + \bar{\delta})|_{C_{\text{odd}}} \rangle \\
&= \tau\langle (\delta d + d\bar{\delta} + \delta\bar{\delta})|_{C_{\text{odd}}} \rangle \\
&= \tau \begin{pmatrix} C_1 & C_3 & C_5 & \dots \\ C_1 \left(\begin{array}{cccc} \langle \delta d + d\bar{\delta} \rangle & \langle \delta\bar{\delta} \rangle & & \\ & \langle \delta d + d\bar{\delta} \rangle & \langle \delta\bar{\delta} \rangle & \\ & & \langle \delta d + d\bar{\delta} \rangle & \ddots \\ \vdots & & & \ddots \end{array} \right) \\ C_3 & \\ C_5 & \\ \vdots & \end{pmatrix} \\
&= \sum_i \tau\langle (\delta d + d\bar{\delta})|_{C_{2i+1}} \rangle, \text{ by 1.38 if each } (\delta d + d\bar{\delta})|_{C_{2i+1}} \text{ is non-singular.}
\end{aligned}$$

However, $(\delta d + d\bar{\delta})|_{C_j} = 1 \oplus \delta d : B_j \oplus \bar{\delta} B_{j-1} \rightarrow B_j \oplus \delta B_{j-1}$, since

$$\begin{aligned}
(\delta d + d\bar{\delta})(b_j) &= b_j \\
(\delta d + d\bar{\delta})(\bar{\delta} b_{j-1}) &= (\delta d)(\bar{\delta} b_{j-1}) + (1 - \bar{\delta} d)(\bar{\delta} b_{j-1}) \\
&= \delta d(\bar{\delta} b_{j-1}) + \bar{\delta} b_{j-1} - \bar{\delta} b_{j-1} \\
&= \delta d(\bar{\delta} b_{j-1}).
\end{aligned}$$

So we may apply 1.46 and get that $(\delta d + d\bar{\delta})|_{C_{2i+1}}$ is a simple isomorphism. Conclusion: $\tau(d + \bar{\delta}) = \tau(d + \delta)$. \square

Theorem 1.50. *Let R be a ring, G a subgroup of the units of R containing (-1) and let \mathcal{C} be the class of acyclic (R, G) -complexes, then the torsion map $\tau : \mathcal{C} \rightarrow K_G(R)$ defined in 1.48 satisfies*

$$(P1) \ C \overset{\Sigma}{\approx} C' \Rightarrow \tau(C) = \tau(C')$$

$$(P2) \ \tau(C' \oplus C'') = \tau(C') + \tau(C'')$$

$$(P3) \ \tau(0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \longrightarrow 0) = (-1)^{n-1} \tau(d).$$

And τ is uniquely determined by these properties.

Proof (see [7, p. 56-57]). Let d, d', d'' denote boundary operators for C, C', C'' respectively.

For (P1), suppose that $f : C \xrightarrow{\cong} C'$. This means that there are distinguished bases of C and C' , such that $\langle d_n \rangle = \langle d'_n \rangle$ and $\langle f_n \rangle = I$, for all n . Choose a chain contraction $\delta : C \rightarrow C$ and define $\delta' := f\delta f^{-1}$. Then $\langle d + \delta \rangle = \langle d' + \delta' \rangle$, and hence $\tau(C) = \tau(C')$.

For (P2), suppose that $C = C' \oplus C''$ (i.e. $d = d' \oplus d''$) and that δ', δ'' are chain contractions for C' and C'' . Then obviously $\delta' \oplus \delta''$ is a chain contraction for C . Since permutation of rows and columns does not change the torsion of a matrix, we get

$$\begin{aligned} \tau(C) &= \tau\langle d + \delta \rangle = \tau\langle (d' \oplus d'') + (\delta' \oplus \delta'') \rangle \\ &= \tau \begin{pmatrix} \langle d' + \delta' \rangle & \\ & \langle d'' + \delta'' \rangle \end{pmatrix} \\ &= \tau(C') + \tau(C''). \end{aligned}$$

For (P3), suppose that C is the complex

$$0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \longrightarrow 0.$$

Set $\delta_j := 0$ if $j \neq n$, and $\delta_n := d_n^{-1} : C_{n-1} \rightarrow C_n$. If n is odd, then $\delta|_{C_{\text{odd}}} = 0$, so $\tau(C) = \tau(d) = (-1)^{n-1}\tau(d)$. If n is even, then $\delta|_{C_{\text{even}}} = 0$, so $\tau(C) = -\tau(d) = -(-1)^n\tau(d) = (-1)^{n-1}\tau(d)$.

Finally, we wish to show uniqueness of $\tau : \mathcal{C} \rightarrow K_G(R)$ with these properties, so suppose that $\mu : \mathcal{C} \rightarrow K_G(R)$ also satisfies (P1)-(P3). Inductively applying 1.47, we get that any (R, G) -complex C is stably equivalent to a complex C' , where C' is of the form $0 \longrightarrow C'_m \xrightarrow{d'} C'_{m-1} \longrightarrow 0$. By definition this means $C \oplus T \xrightarrow{\cong} C' \oplus T'$ for some trivial complexes T, T' . But now the properties (P1)-(P3) imply that $\tau(C) = \tau(C) + \tau(T) = \tau(C') + \tau(T') = \tau(C')$. Similarly $\mu(C) = \mu(C')$. But then by (P3) $\tau(C') = (-1)^{m-1}\tau(d') = \mu(C')$. That means $\tau(C) = \mu(C)$. \square

Lemma 1.51. *Let $0 \longrightarrow C' \xrightarrow{i} C \xrightarrow{j} C'' \longrightarrow 0$ be a short exact sequence of acyclic chain complexes and that $\sigma : C'' \rightarrow C$ is a degree-zero section (not necessarily a chain map). Assume further that C, C' and C'' are (R, G) -complexes with preferred bases c, c' and c'' . Then*

$$\tau(C) = \tau(C') + \tau(C'') + \sum (-1)^k \tau\langle c'_k c''_k / c_k \rangle$$

where $c'_k c''_k := i(c'_k) \cup \sigma(c''_k)$. In particular, if $i(c'_k) \cup \sigma(c''_k)$ is a distinguished basis of C_k for all k , then $\tau(C) = \tau(C') + \tau(C'')$.

Reference: [7, p. 57]. \square

1.3.4 Changing Rings

Definition 1.52. Let C be an (R, G) -complex and $h : R \rightarrow R'$ a ring homomorphism with $h(G) \subseteq G'$. Construct the (R', G') -complex C_h as follows: Choose a distinguished basis $c = \{c_k^i\}$ for C , and let C_h be the free graded R' -module generated by the set c . We denote $c = \hat{c}$ when c is being thought of as a subset of C_h . Define $\hat{d} : C_h \rightarrow C_h$ by setting $\hat{d}(\hat{c}_k^i) = \sum_j h(a_{kj})\hat{c}_j^{i-1}$ if $d(c_k^i) = \sum_j a_{kj}c_j^{i-1}$. By stipulating that \hat{c} is a distinguished basis for C_h we make C_h into an (R', G') -complex.

Define further

$$\tau_h(C) := \tau(C_h) \in K_{G'}(R').$$

Alternatively, one may define C_h as the (R', G') -complex $(R' \otimes_h C, 1 \otimes d)$, where R' is a right R -module via $r' \cdot r := r'h(r)$ for all $r \in R, r' \in R'$, so $R' \otimes_h C := R' \otimes_R C$ is a left R' -module via $\rho(r' \otimes x) := \rho r' \otimes x$ for all $(\rho, r', x) \in R' \times R' \times C$.

Lemma 1.53. *If C is an acyclic (R, G) -complex and $h : (R, G) \rightarrow (R', G')$ is a ring homomorphism, then C_h is acyclic and $\tau_h(C) = h_*\tau(C)$ where $h_* : K_G(R) \rightarrow K_{G'}(R')$ is the induced map.*

Proof (see [7, p. 60]). Choose a chain contraction δ of C and suppose that $\delta(c_k^i) = \sum_j b_{kj}c_j^{i+1}$. Define $\hat{\delta} : C_h \rightarrow C_h$ by $\hat{\delta}(\hat{c}_k^i) = \sum_j h(b_{kj})\hat{c}_j^{i+1}$. Since $h(G) \subseteq G'$ we have $h(1) = 1$, so that: $\langle \hat{d}\hat{\delta} + \hat{\delta}\hat{d} \rangle = h_*\langle d\delta + \delta d \rangle = h_*(I) = I$. So $\hat{\delta}$ is a chain contraction and $\tau_h(C) = \tau(C_h) = h_*\tau(C)$. \square

Lemma 1.54. *If $0 \longrightarrow C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \longrightarrow 0$ is a split exact sequence of (R, G) -complexes with distinguished bases c', c and c'' respectively, such that $c = \alpha(c') \cup b$, where $\beta(b) = c''$, then*

$$0 \longrightarrow C'_h = R' \otimes_h C' \xrightarrow{1 \otimes \alpha} C_h = R' \otimes_h C \xrightarrow{1 \otimes \beta} C''_h = R' \otimes_h C'' \longrightarrow 0$$

is an exact sequence of (R', G') -complexes whose preferred bases have the analogous property.

Reference: [7, p. 61]. \square

1.3.5 The Torsion of a CW Pair

Definition 1.55. If (K, L) is a pair of finite, connected CW complexes, such that L is a strong deformation retract of K , then we may define the *torsion of (K, L)* - in symbols

$\tau(K, L)$ - by

$$\tau(K, L) := \tau(C(\tilde{K}, \tilde{L})) \in \text{Wh}(\pi_1 L)$$

where (\tilde{K}, \tilde{L}) is the universal covering of (K, L) .

To explain this definition, we must first define how $C(\tilde{K}, \tilde{L})$ is a $\text{Wh}(\pi_1 L)$ complex. For that, let $p : \tilde{K} \rightarrow K$ the universal covering of K and $G := \text{Deck}(p)$ the group of covering homeomorphisms. Then $p|_{\tilde{L}} : \tilde{L} \rightarrow L$ is the universal covering of L and \tilde{L} is a strong deformation retract of \tilde{K} , by 1.25. By 1.10, $C(\tilde{K}, \tilde{L})$ is a free $\mathbb{Z}[G]$ -complex, where a basis can be constructed by choosing for each cell $e_\alpha \in \mathcal{E}(K \setminus L)$ a specific lift $\tilde{\phi}_\alpha$ of ϕ_α . If \mathcal{B} denotes the set of all bases constructed in this fashion, then this determines the structure of a $\text{Wh}(G)$ -complex on $C(\tilde{K}, \tilde{L})$:

Lemma 1.56. *The complex $C(\tilde{K}, \tilde{L})$, along with the family of bases \mathcal{B} , determines an acyclic $\text{Wh}(G)$ -complex.*

Proof (see [7, p. 62-63]). $C(\tilde{K}, \tilde{L})$ is acyclic, since \tilde{L} is a strong deformation retract of \tilde{K} .

Suppose that $c, c' \in \mathcal{B}$ restrict to bases $c_n = \{\langle \tilde{\phi}_1 \rangle, \dots, \langle \tilde{\phi}_q \rangle\}$ and $c'_n = \{\langle \tilde{\psi}_1 \rangle, \dots, \langle \tilde{\psi}_q \rangle\}$ of $C_n(\tilde{K}, \tilde{L})$. We wish to show that $\tau(\langle c_n/c'_n \rangle) = 0$. Now

$$\langle \tilde{\psi}_j \rangle = \sum_k a_{jk} \langle \tilde{\phi}_k \rangle = \sum_{i,k} n_i^{jk} \langle g_i \tilde{\phi}_k \rangle$$

for some $a_{jk} = \sum_i n_i^{jk} g_i \in \mathbb{Z}[G]$. But the cell $\tilde{\psi}_j(\tilde{I}^n)$, as a lift of e_j is equal to one of the cells $g_{i_j} \tilde{\phi}_j(\tilde{I}^n)$ and is disjoint from all the others. Hence all the coefficients in the last sum are 0 except for $N = n_{i_j}^{jj}$. But then $\tilde{\phi}_j(\tilde{I}^n) = g_{i_j}^{-1} \tilde{\psi}_j(\tilde{I}^n)$, so by the same argument, $\langle \tilde{\phi}_j \rangle = N' \langle g_{i_j}^{-1} \tilde{\psi}_j \rangle$. Hence $\langle \tilde{\psi}_j \rangle = NN' \langle \tilde{\psi}_j \rangle$. So $N = \pm 1$ and $\langle \tilde{\psi}_j \rangle = \pm g_{i_j} \langle \tilde{\phi}_j \rangle$. Hence

$$\langle c_n/c'_n \rangle = \begin{pmatrix} \pm g_{i_1} & & \\ & \ddots & \\ & & \pm g_{i_q} \end{pmatrix},$$

so $\tau(\langle c_n/c'_n \rangle) = 0$. □

Furthermore, we need to clarify how the definition 1.55 is independent of the choice of base points.

Lemma 1.57. *Let X be path connected and let $x, y \in X$. Then all of the paths from x to y induce the same isomorphism $f_{x,y}$ of $\text{Wh}(\pi_1(X, x))$ onto $\text{Wh}(\pi_1(X, y))$. Also $f_{y,x} \circ f_{x,y} = f_{x,z}$.*

Proof (see [7, p. 63]). If $\gamma : I \rightarrow X$ is a path from x to y , let $f_\gamma : \pi_1(X, x) \rightarrow \pi_1(X, y)$ denote the usual isomorphism given by $f_\gamma[\omega] = [\bar{\gamma}\omega\gamma]$. Then, if α, β are two such paths, $f_\beta^{-1}f_\alpha([\omega]) = [\beta\bar{\alpha}][\omega][\beta\bar{\alpha}]^{-1}$ for all $[\omega] \in \pi_1(X, x)$. Hence $f_\beta^{-1}f_\alpha$ is an inner automorphism, so by 1.37, $(f_\beta^{-1})_*(f_\alpha)_* = (f_\beta^{-1}f_\alpha)_* = \text{id}_{\text{Wh}(\pi_1(X, x))}$. Thus $(f_\alpha)_* = (f_\beta)_*$ for all such α, β and we may set $f_{x,y} := (f_\alpha)_*$. After this, it is now obvious that $f_{y,z} \circ f_{x,y} = f_{x,z}$. \square

So there is a canonical isomorphism between the groups $\text{Wh}(\pi_1(K, x))$ and $\text{Wh}(\pi_1(K, y))$ for any $x, y \in K$ and we may therefore identify all these groups and call the resulting group $\text{Wh}(\pi_1 K)$. Also, for any continuous $f : K \rightarrow K'$ inducing a homomorphism between fundamental groups $f_* : \pi_1(K, x) \rightarrow \pi_1(K', x')$, and hence between Whitehead groups $f_* : \text{Wh}(\pi_1(K, x)) \rightarrow \text{Wh}(\pi_1(K', x'))$, we would have $f_* \circ f_{x,y} = f_{x',y'} \circ f_*$, for $x, y \in K$ and $x', y' \in K'$ satisfying $f(x) = x'$ and $f(y) = y'$, so that we can speak of $f_* : \text{Wh}(\pi_1 K) \rightarrow \text{Wh}(\pi_1 K')$, dropping the base points. This is because for any $[\alpha] \in \pi_1(K, x)$ and any path $\gamma : I \rightarrow K$ from x to y , we have $f_*[\bar{\gamma}\alpha\gamma] = [(\overline{f \circ \gamma})f_\alpha(f \circ \gamma)]$. So since $f \circ \gamma : I \rightarrow K'$ is a path from x' to y' , we get $f_* \circ f_{x,y} = f_{x',y'} \circ f_*$.

Lastly, we wish to identify $G = \text{Deck}(p)$ with the fundamental group of K . If we choose base points $x \in K$ and $\tilde{x} \in p^{-1}(x)$ then we may do so via the isomorphism (see 1.21) $\theta = \theta(x, \tilde{x}) : \pi_1(K, x) \rightarrow G$, for any $[\alpha] \in \pi_1(K, x)$ given by

$$\theta_{[\alpha]}(y) = \widetilde{\alpha \cdot p\omega}(1),$$

where $y \in \tilde{K}$, ω is a path from \tilde{x} to y , and $\widetilde{\alpha \cdot p\omega}$ is the unique lift of $\alpha \cdot p\omega$ with $\widetilde{\alpha \cdot p\omega}(0) = \tilde{x}$. So this isomorphism - or rather its inverse $\psi := \theta^{-1}$ - induces a ring isomorphism $\psi : \mathbb{Z}[G] \rightarrow \mathbb{Z}[\pi_1(K, x)]$ and we can now construct from $C(\tilde{K}, \tilde{L})$ the acyclic $\text{Wh}(\pi_1(K, x))$ -complex $C(\tilde{K}, \tilde{L})_\psi$. The following lemma will assure the independence of all the choices made:

Lemma 1.58. *Let $\tilde{p} : \tilde{K} \rightarrow K$ and $\hat{p} : \hat{K} \rightarrow K$ be universal coverings of the connected complex K . Let $\tilde{G} := \text{Deck}(\tilde{p})$, $\hat{G} := \text{Deck}(\hat{p})$ the groups of covering homeomorphisms. Let $x, y \in K$, $\tilde{x} \in \tilde{p}^{-1}(x)$ and $\hat{y} \in \hat{p}^{-1}(y)$. Let $\tilde{\psi} : \tilde{G} \rightarrow \pi_1(K, x)$ and $\hat{\psi} : \hat{G} \rightarrow \pi_1(K, y)$ the group isomorphisms determined by (x, \tilde{x}) and (y, \hat{y}) . Then $\tau(C(\hat{K}, \hat{L})_\psi) = f_{x,y}(\tau(C(\tilde{K}, \tilde{L})_{\tilde{\psi}}))$.*

Proof (see [7, p. 64-65]). Let $h : \tilde{K} \rightarrow \hat{K}$ be a homeomorphism covering $\text{id} : K \rightarrow K$ (using 1.20). Let $H : \tilde{G} \rightarrow \hat{G}$ be defined by $H(g) := hgh^{-1}$. We first claim that $\tau C(\hat{K}, \hat{L}) = H_*(\tau C(\tilde{K}, \tilde{L}))$. For that, let $\{\tilde{\phi}_k^i\}$ be a basis for $C_i(\tilde{K}, \tilde{L})$ and let $\hat{\phi}_k^i := h \circ \tilde{\phi}_k^i$ for all i, k ; and we may compute the torsions of $C(\tilde{K}, \tilde{L})$ and $C(\hat{K}, \hat{L})$ with these bases. Now h induces a chain isomorphism of \mathbb{Z} -complexes $C(\tilde{K}, \tilde{L}) \rightarrow C(\hat{K}, \hat{L})$ and one

now immediately verifies that, if the matrix of \tilde{d}_i is $(a_{kj}) \in \text{GL}(\mathbb{Z}[\tilde{G}])$, then the matrix of \hat{d}_i is $(H(a_{kj}))$, where $H : \mathbb{Z}[\tilde{G}] \rightarrow \mathbb{Z}[\hat{G}]$ is induced from $H : \tilde{G} \rightarrow \hat{G}$. So by (the proof of) 1.53, $\tau C(\hat{K}, \hat{L}) = H_*(\tau C(\tilde{K}, \tilde{L}))$.

Now let $\hat{x} = h(\tilde{x})$ and choose a path $\Omega : I \rightarrow \hat{K}$ from \hat{x} to \hat{y} . Define $\omega := \hat{p}\Omega$ and let $f_\omega : \pi_1(K, x) \rightarrow \pi_1(K, y)$ the usual isomorphism. Denote $\tilde{\theta} = \theta(x, \tilde{x}) = \tilde{\psi}^{-1}$ and $\hat{\theta} = \theta(y, \hat{y}) = \hat{\psi}^{-1}$. Then the following diagram commutes

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{H} & \hat{G} \\ \tilde{\theta} \uparrow & & \uparrow \hat{\theta} \\ \pi_1(K, x) & \xrightarrow{f_\omega} & \pi_1(K, y) \end{array}$$

since if $[\alpha] \in \pi_1(K, x)$, we have

$$\begin{aligned} (\hat{\theta} f_\omega[\alpha])(\hat{y}) &= \hat{\theta}_{[\tilde{\omega}\alpha]}(\hat{y}) \\ &= \widehat{\alpha\omega}(1) \text{ where } \widehat{\alpha\omega}(0) = \bar{\Omega}(1) = \hat{x} \\ &= h(\widetilde{\alpha\omega}(1)) \text{ because } h(\widetilde{\alpha\omega}(0)) = \hat{x} \\ &= h(\alpha \cdot \widetilde{\hat{p}h^{-1}\Omega}(1)) \text{ because } \omega = \hat{p}\Omega = \hat{p}h^{-1}\Omega \\ &= h(\tilde{\theta}_{[\alpha]}h^{-1}\Omega(1)) \\ &= (h\tilde{\theta}_{[\alpha]}h^{-1})(\hat{y}) \\ &= (H\tilde{\theta}[\alpha])(\hat{y}). \end{aligned}$$

Now because $\hat{\theta} f_\omega[\alpha]$ and $H\tilde{\theta}[\alpha]$ agree at one point, they need to agree everywhere. Since $[\alpha]$ was arbitrary, $\hat{\theta} f_\omega = H\tilde{\theta}$. Finally, we get with 1.53 that

$$\begin{aligned} f_{x,y}\tau(C(\tilde{K}, \tilde{L})_{\tilde{\psi}}) &= f_{\omega*}\tilde{\psi}_*(\tau C(\tilde{K}, \tilde{L})) \\ &= \hat{\psi}_*H_*(\tau C(\tilde{K}, \tilde{L})) \\ &= \hat{\psi}_*(\tau C(\hat{K}, \hat{L})) \\ &= \tau(C(\hat{K}, \hat{L})_{\hat{\psi}}). \end{aligned}$$

□

To conclude the explanation of the definition 1.55, if $i : L \hookrightarrow K$ denotes the inclusion map, we define $\tau(K, L)$ to be $i_*^{-1}(\tau(C(\tilde{K}, \tilde{L})_{\psi})) \in \text{Wh}(\pi_1 L)$.

Definition 1.59. Let (K, L) be a pair of finite CW-complexes (not necessarily connected), such that L is a strong deformation retract of K . Let K_1, \dots, K_q and L_1, \dots, L_q be the components of K and L respectively, ordered so that L_j is a strong deformation

retract of K_j for all j . We define

$$\tau(K, L) := \sum_{j=1}^q \tau(K_j, L_j) \in \bigoplus_{j=1}^q \text{Wh}(\pi_1 L_j) =: \text{Wh}(L).$$

1.3.6 Properties of the Torsion of a CW Pair

Lemma 1.60. *If $K \supseteq L \supseteq M$ are CW (sub)complexes, where L is a strong deformation retract of K and M is a strong deformation retract of L , then*

$$\tau(K, M) = \tau(L, M) + i_*^{-1} \tau(K, L)$$

where $i : M \hookrightarrow L$ is the inclusion map.

Reference: [7, p. 67]. □

Lemma 1.61 (Excision). *Let K, L and M be subcomplexes of the complex $K \cup L$, with $M = K \cap L$. If M is a strong deformation retract of K , then $\tau(K \cup L, L) = j_* \tau(K, M)$ where $j : M \hookrightarrow L$ is the inclusion map.*

Reference: [7, p. 68]. □

1.3.7 The Torsion of a Homotopy Equivalence

Definition 1.62. Suppose that $f : K \rightarrow L$ is a cellular homotopy equivalence between finite CW complexes. Then K is a strong deformation retract of the mapping cylinder M_f and $f_* : \text{Wh}(K) \rightarrow \text{Wh}(L)$ is an isomorphism. Define the *torsion of the homotopy equivalence f* by

$$\tau(f) := f_* \tau(M_f, K) \in \text{Wh}(L).$$

Lemma 1.63 (Homotopy Invariance). *If $f, g : K \rightarrow L$ are homotopic cellular homotopy equivalences, then $\tau(f) = \tau(g)$.*

Reference: [7, p. 72]. □

Remark 1.64. The proof in [7, p. 72] takes up only one line. But it uses the equivalence of the functor $\text{Wh}(L)$ as described in this chapter to another functor, described in [7, Chapter II, p.14-23]. The equivalence of the two functors is proven in [7, p. 70-71].

Definition 1.65. A cellular homotopy equivalence $f : K \rightarrow L$ is called a *simple-homotopy equivalence* if $\tau(f) = 0$.

Lemma 1.66. *If $L \subseteq K$ is a subcomplex and a strong deformation retract of K , then $\tau(i) = i_*\tau(K, L)$, where $i : L \hookrightarrow K$ is the inclusion map.*

Reference: [7, p. 72]. □

Lemma 1.67. *Let $f : K \rightarrow L$ and $g : L \rightarrow M$ be cellular homotopy equivalences, then*

$$\tau(gf) = \tau(g) + g_*\tau(f).$$

Reference: [7, p. 72-73]. □

Corollary 1.68. *If $f : K \rightarrow L$ and $g : L \rightarrow K$ are cellular homotopy equivalences which are homotopy inverses of each other, then $\tau(g) = -g_*\tau(f)$.*

Proof (see [7, p. 73]). This is obvious from the preceding lemmata, since, if we have $gf \simeq \text{id}_K$ then $0 = \tau(gf) = \tau(g) + g_*\tau(f)$. □

Theorem 1.69. *Let X and Y be finite CW complexes. If $f : X \rightarrow Y$ is a homeomorphism, then f is a simple-homotopy equivalence.*

Reference: [7, p. 102-106]. □

Remark 1.70. The proof of 1.69 in [7, p. 102-106] uses results of infinite-dimensional topology. In fact, a stronger result is proven: If X and Y are finite CW complexes, then $f : X \rightarrow Y$ is a simple-homotopy equivalence if and only if $f \times \text{id}_Q : X \times Q \rightarrow Y \times Q$ is homotopic to a homeomorphism $X \times Q \rightarrow Y \times Q$. Here $Q := \prod_{j=1}^{\infty} [-1, 1]$ denotes the Hilbert cube.

Theorem 1.71 (Computing Torsion). *Let $f : K \rightarrow L$ be a cellular homotopy equivalence between connected spaces and let $\tilde{f} : \tilde{K} \rightarrow \tilde{L}$ be a lift of f to their universal covering spaces, inducing $\tilde{f}_* : C(\tilde{K}) \rightarrow C(\tilde{L})$. Let G_K and G_L be the groups of covering homeomorphisms of \tilde{K} and \tilde{L} respectively and let $C(\tilde{K}) \rightarrow C(\tilde{L})$ be viewed as $\text{Wh}(G_K)$ - and $\text{Wh}(G_L)$ -complexes with boundary operators d and d' respectively. Choose base points x, y and points \tilde{x}, \tilde{y} covering them, so that $f(x) = y$ and $\tilde{f}(\tilde{x}) = \tilde{y}$. And let $f_* : G_K \rightarrow G_L$ be induced from $f_* : \pi_1(K, x) \rightarrow \pi_1(L, y)$.*

Then $\tau(f) \in \text{Wh}(G_L)$ is the torsion of the $\text{Wh}(G_L)$ -complex \mathcal{C} which is given by

1. $\mathcal{C}_n = [C(\tilde{K})_{f_*}]_{n-1} \oplus C_n(\tilde{L})$

2. $\partial_n : \mathcal{C}_n \rightarrow \mathcal{C}_{n-1}$ has the matrix

$$\begin{array}{c} [C(\tilde{K})_{f_*}]_{n-2} \quad C_{n-1}(\tilde{L}) \\ [C(\tilde{K})_{f_*}]_{n-1} \left(\begin{array}{cc} -f_* \langle d_{n-1} \rangle & \langle \tilde{f}_* \rangle \\ 0 & \langle d'_n \rangle \end{array} \right) \\ C_n(\tilde{L}) \end{array}$$

In particular if we let $\bar{C}(\tilde{K})$ be the $\text{Wh}(G_L)$ -complex with $\bar{C}_n(\tilde{K}) = [C(\tilde{K})_{f_*}]_{n-1}$ and the boundary operator \bar{d} given by $\langle \bar{d}_n \rangle = -f_* \langle d_{n-1} \rangle$ then $\tau(f) = \tau(\mathcal{C})$ where there is a basis-preserving short exact sequence of $\text{Wh}(G_L)$ -complexes

$$0 \longrightarrow C(\tilde{L}) \longrightarrow \mathcal{C} \longrightarrow \bar{C}(\tilde{K}) \longrightarrow 0.$$

Proof (see [7, p. 74-76]). To compute $\tau(M_f, K)$ we use the universal covering of M_f given in 1.26. So let $\alpha : M_{\tilde{f}} \rightarrow M_f$ be the natural projection such that $\alpha|_{\tilde{K}}$ and $\alpha|_{\tilde{L}}$ are the universal coverings of K and L . Let $G := \text{Deck}(\alpha)$ the group of covering homeomorphisms of $M_{\tilde{f}}$. If $g \in G_K$ define $E(g) \in G$ to be the unique extension of g to $M_{\tilde{f}}$. If $h \in G$ define $R(h) \in G_L$ to be the restriction of h to \tilde{L} . Now in the following commutative diagram

$$\begin{array}{ccccc} G_K & \xrightarrow{i_*} & G & \xrightarrow{p_*} & G_L \\ \theta(x, \tilde{x}) \uparrow & & \theta(x, \tilde{x}) \uparrow & & \theta(y, \tilde{y}) \uparrow \\ \pi_1(K, x) & \xrightarrow{i_*} & \pi_1(M_f, x) & \xrightarrow{p_*} & \pi_1(L, y) \\ & & \searrow & \nearrow & \\ & & & f_* & \end{array}$$

we have that $E = i_* : G_K \rightarrow G$ (using the fact that the inclusion $\tilde{i} : \tilde{K} \hookrightarrow M_{\tilde{f}}$ covers $i : K \hookrightarrow M_f$ and that $\tilde{i} \circ g = i_*(g) \circ \tilde{i}$ for any $g \in G_K$ (see [7, p. 12])) and that $R = p_* : G \rightarrow G_L$ (because $\tilde{f}(\tilde{x}) = \tilde{y}$ and the fact that the projection $\tilde{p} : M_{\tilde{f}} \rightarrow \tilde{L}$ covers $p : M_f \rightarrow L$, this follows again with the formula $\tilde{p} \circ h = p_*(h) \circ \tilde{p}$, for all $h \in G$). Hence $f_* = RE : G_K \rightarrow G_L$.

We view $C(M_{\tilde{f}}, \tilde{K})$ as a $\text{Wh}(G)$ -complex, so that $E_*^{-1}(\tau C(M_{\tilde{f}}, \tilde{K})) = \tau(M_f, K) \in \text{Wh}(G_K)$ and $\tau(f) = f_* \tau(M_f, K) = R_*(\tau C(M_{\tilde{f}}, \tilde{K}))$. So by 1.53, $\tau(f) = \tau(\mathcal{C})$ where \mathcal{C} is the $\text{Wh}(G_L)$ -complex $[C(M_{\tilde{f}}, \tilde{K})]_R$. To show that \mathcal{C} satisfies the conclusion of the theorem, we first study $C(M_{\tilde{f}}, \tilde{K})$.

Now $C(M_{\tilde{f}}, \tilde{K})$ is isomorphic as a \mathbb{Z} -complex to the algebraic mapping cone of \tilde{f}_* (see [7, p. 8]) which is given by

$$\begin{aligned} C_n &= C_{n-1}(\tilde{K}) \oplus C_n(\tilde{L}) \\ \partial_n(a) &= -d_{n-1}(a) + \tilde{f}_*(a), \text{ for } a \in C_{n-1}(\tilde{K}) \\ \partial_n(b) &= d'_n(b), \text{ for } b \in C_n(\tilde{L}) \end{aligned}$$

A typical cell e^{n-1} of K gives rise, upon choosing a fixed lift, to an element $\langle \tilde{e}^{n-1} \rangle$ of $C_{n-1}(\tilde{K})$. The image of $\langle \tilde{e}^{n-1} \rangle$ under our isomorphism is the element $\langle \tilde{e}^{n-1} \times (0, 1) \rangle \in C_n(M_{\tilde{f}}, \tilde{K})$. Suppose that, when $d\langle \tilde{e}^{n-1} \rangle$ is written as a linear combination in $\mathbb{Z}\langle G_K \rangle$ we get

$$d\langle \tilde{e}^{n-1} \rangle = \sum_{i,j} n_{i,j} g_i \langle \tilde{e}_j^{n-2} \rangle, \text{ for some } g_i \in G_K.$$

Then, over \mathbb{Z} , we get

$$d\langle \tilde{e}^{n-1} \rangle = \sum_{i,j} n_{i,j} \langle g_i \tilde{e}_j^{n-2} \rangle.$$

Applying the isomorphism, the corresponding boundary in $C(M_{\tilde{f}}, \tilde{K})$ is

$$\begin{aligned} \partial\langle \tilde{e}^{n-1} \times (0, 1) \rangle &= - \left(\sum_{i,j} n_{i,j} \langle g_i \tilde{e}_j^{n-2} \times (0, 1) \rangle \right) + \tilde{f}_* \langle \tilde{e}^{n-1} \rangle \\ &= - \left(\sum_{i,j} n_{i,j} E(g_i) \langle \tilde{e}_j^{n-2} \times (0, 1) \rangle \right) + \tilde{f}_* \langle \tilde{e}^{n-1} \rangle. \end{aligned}$$

The last equation gives the boundary with $\mathbb{Z}[G]$ -coefficients and it holds because $E(g_i)|_{M_{\tilde{f}} \setminus \tilde{L}} = E(g_i)|_{\tilde{K} \times [0,1]} = g_i \times \text{id}_{[0,1]}$.

Similarly, if we write $\tilde{f}_* \langle \tilde{e}^{n-1} \rangle$ as a linear combination with coefficients in $\mathbb{Z}[G_L]$, denoting cells in L with u 's, we get

$$\begin{aligned} \tilde{f}_* \langle \tilde{e}^{n-1} \rangle &= \sum_{p,q} m_{p,q} \langle h_p \tilde{u}_q^{n-1} \rangle, \text{ for some } h_p \in G_L \\ &= \sum_{p,q} m_{p,q} \langle R^{-1} h_p \tilde{u}_q^{n-1} \rangle, \text{ since } h_p|_{\tilde{L}} = (R^{-1} h_p)|_{\tilde{L}} \\ &= \sum_{p,q} m_{p,q} \langle R^{-1} h_p \rangle \langle \tilde{u}_q^{n-1} \rangle. \end{aligned}$$

We can compute $\partial_n \langle \tilde{u}_n \rangle$ in an analogous way and we conclude that the matrix of $\partial_n : C_n(M_{\tilde{f}}, \tilde{K}) \rightarrow C_{n-1}(M_{\tilde{f}}, \tilde{K})$, when $C(M_{\tilde{f}}, \tilde{K})$ is considered as a $\mathbb{Z}[G]$ -module, is given

by

$$\langle \partial_n \rangle = \begin{pmatrix} -E \langle d_{n-1} \rangle & R^{-1} \langle \tilde{f}_* \rangle \\ & R^{-1} \langle d'_n \rangle \end{pmatrix}$$

where $\langle d_{n-1} \rangle$ is a $\mathbb{Z}[G_K]$ matrix and $\langle \tilde{f}_* \rangle$ and $\langle d'_n \rangle$ are $\mathbb{Z}[G_L]$ matrices. From the equation $f_* = RE$ it now follows that the complex $\mathcal{C} = C(M_{\tilde{f}}, \tilde{K})_R$, with $\tau(f) = \tau(\mathcal{C})$ fulfills the claimed properties.

The assertion that the sequence $0 \longrightarrow C(\tilde{L}) \longrightarrow \mathcal{C} \longrightarrow \bar{C}(\tilde{K}) \longrightarrow 0$ is exact and basis-preserving follows immediately from the first part of the theorem. \square

Chapter 2

Constructions of Lens Spaces

There are several constructions that lead to what we call lens spaces. In this chapter we will consider the popular constructions and see how they are equivalent. At first we will deal with 3-dimensional lens spaces and then see how one can generalize to higher dimensions.

2.1 Construction via Tori

Consider two solid tori $T_1 = D^2 \times S^1$ and $T_2 = S^1 \times D^2$ and a homeomorphism of their boundaries $\partial T_i \cong S^1 \times S^1$, $i \in \{1, 2\}$, given by $f : S^1 \times S^1 \rightarrow S^1 \times S^1$, $(z, w) \mapsto (z^a w^c, z^b w^d)$, where $ad - bc = \pm 1$, $a, b, c, d \in \mathbb{Z}$. When gluing T_1 and T_2 together along their borders via f , we arrive at a closed 3-manifold $M = M\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := T_1 \cup_f T_2$ and call this a *lens space*. To visualize, take a look at the meridian curve $t \mapsto (e^{2\pi i t}, 1)$, which gets transformed to the $(|a|, |b|)$ torus curve $t \mapsto (e^{2\pi i a t}, e^{2\pi i b t})$.

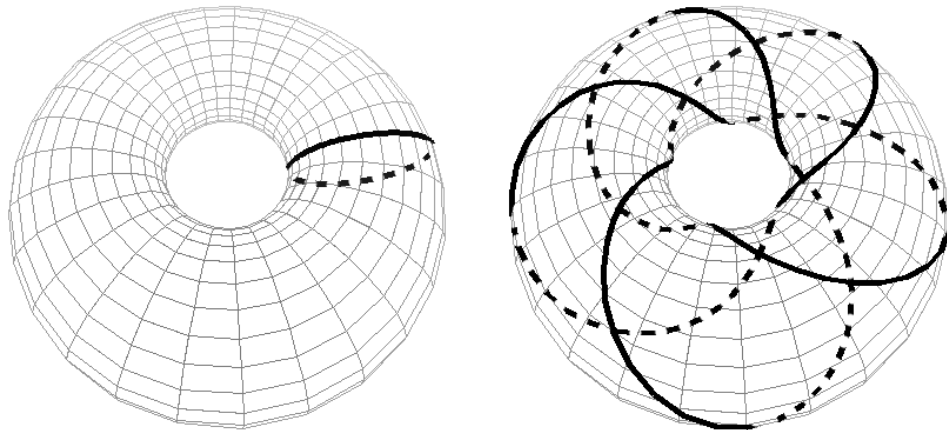


FIGURE 2.1: f maps a meridian curve to a $(5,3)$ torus curve, when $|a| = 5$, $|b| = 3$.

Proposition 2.1. $M = M\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is uniquely defined up to homeomorphism by the numbers a and b . If $a = 0$ then $M \cong S^2 \times S^1$. If $a = 1$ then $M \cong S^3$. Otherwise $M \cong M(a, b)$ where a, b are coprime and satisfy $1 \leq b < a$.

Proof (see [15, p. 16]). For other a', b', c', d' , such that $a'd' - b'c' = \pm 1$, if there are numbers $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$ and $n, m \in \mathbb{Z}$, such that

$$\begin{pmatrix} \gamma & 0 \\ n & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \alpha & m \\ 0 & \beta \end{pmatrix},$$

that is

$$a' = \alpha\gamma a, b' = \beta\gamma b - m\alpha\beta\gamma a, c' = n\alpha a + \alpha\delta c, d' = n\beta b + \beta\delta d - mn\alpha\beta a - m\alpha\beta\delta c,$$

then by defining the two homeomorphisms

$$F : D^2 \times S^1 \rightarrow D^2 \times S^1, (z, w) \mapsto (z^\alpha w^m, w^\beta)$$

$$G : S^1 \times D^2 \rightarrow S^1 \times D^2, (w, z) \mapsto (w^\gamma, z^\delta w^n),$$

we would get $(G|_{S^1 \times S^1}) \circ f = f' \circ (F|_{S^1 \times S^1})$ and therefore $M \cong M'$, where f and f' are the gluing maps of M and $M' := M\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ respectively (using [15, Proposition 1.49]).

In the case ($a \leq 0$) we set $\alpha = -1, \beta = \gamma = \delta = 1$ and $m, n = 0$ and get $a' \geq 0$, so from now on, w.l.o.g. $a \geq 0$.

If ($ad - bc < 0$) we set $\alpha = \beta = \gamma = 1, \delta = -1, m = n = 0$ and arrive at $a' = a \geq 0, b' = b$ and $a'd' - b'c' = \alpha\beta\gamma\delta(ad - bc) > 0$. So w.l.o.g. $ad - bc = 1$.

In the case ($a = 0$), we get $bc = -1$ and with $\alpha = c, \beta = b, \gamma = \delta = 1, n = 0, m = d$ we arrive at $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so $M \cong M\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cong (D^2 \cup_{\text{id}_{S^1}} D^2) \times S^1 \cong S^2 \times S^1$.

Now suppose that $a\tilde{d} - b\tilde{c} = 1$. It follows, that $-b\tilde{c} \equiv 1 \equiv -bc \pmod{a}$, i.e. $c \equiv \tilde{c} \pmod{a}$ so there is an $n \in \mathbb{Z}$, s.t. $\tilde{c} = c + na$. Analogously there exists an $\tilde{n} \in \mathbb{Z}$, s.t. $\tilde{d} = d + \tilde{n}b$. Plugging this into the equation $a\tilde{d} - b\tilde{c} = 1$, we see, that actually $n = \tilde{n}$.

Now with $\alpha = \beta = \gamma = \delta = 1, m = 0$ and n as above, we get $M \cong M\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, so M is uniquely determined by (a, b) , and we may solely write $M(a, b)$ instead of $M\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

With $\alpha = \beta = \gamma = \delta = 1$ and $n = 0$, we get $M(a, b) \cong M(a, b - ma)$ for any $m \in \mathbb{Z}$, i.e. b may be reduced mod a .

For $a = 1$ this means that $M(1, b) \cong M(1, 0) \cong M\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

In the case $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we get $f = \text{id}_{S^1} \times \text{id}_{S^1}$, so $M \cong D^2 \times S^1 \cup S^1 \times D^2 = \partial(D^2 \times D^2) \cong S^3$. \square

Remark 2.2. In fact, we could just use any auto-homeomorphism f of the torus (not only the ones given by $(z, w) \mapsto (z^a w^c, z^b w^d)$) for this construction and the homeomorphism type of the resulting space will only depend on what f does on a meridian. We outline a proof of this, as described in [10, p. 254]. For, let $f, f' : S^1 \times S^1 \rightarrow S^1 \times S^1$ be two homeomorphisms such that $f(m) = f'(m)$, where $m := S^1 \times \{1\} \subset T_1$ is a meridian. We wish to extend the identity map $\text{id}_{T_2} : T_2 \rightarrow T_2$ to a homeomorphism $F : T_1 \cup_f T_2 \rightarrow T_1 \cup_{f'} T_2$. Let $D := D^2 \times \{1\} \subset T_1$ be a disc that spans m . We may already extend id_{T_2} to a map $G : D \cup_f|_m T_2 \rightarrow D \cup_{f'}|_m T_2$, since $f(m) = f'(m)$. But now the complement $(T_1 \cup_f T_2) \setminus (D \cup T_2) = \hat{T}_1 \setminus D \cong B^2 \times (0, 1)$ is topologically a ball and we may hence further extend G to the homeomorphism F which we desire, since the homeomorphism is already given on its boundary.

2.2 Construction from a 3-Ball

Let $m, l \in \mathbb{Z}$, where $1 \leq l < m$ and m, l are coprime. Consider a closed 3-ball $B \subset \mathbb{R}^3$ and divide its equator $C = S^1 \times 0 \subset B$ into m equal segments $[v_j, v_{j+1}] \subset C$, where $v_j = (\cos(2\pi j/m), \sin(2\pi j/m), 0) \in C$ (oftentimes B is drawn in the shape of a lens, see 2.8). Joining each point of a segment $[v_j, v_{j+1}]$ to either the north pole $N \in S^2$ or the south pole $S \in S^2$ via arcs of great circles on $\partial B = S^2$ gives $2p$ triangles $\Delta_j := [v_j, v_{j+1}, N]$, $\Delta'_j := [v_j, v_{j+1}, S]$ of S^2 . Now identify the top hemisphere with the lower hemisphere via reflection after a twist by an angle of $2\pi l/m$, so that each triangle Δ_j gets identified with the triangle Δ'_{j+l} . We arrive at a closed 3-manifold called the *lens space* $L_{l/m}$.

Of course this construction doesn't make any sense for $m = 0$, but otherwise it is equivalent to the construction in the previous section.

Proposition 2.3. *The manifolds $L_{l/m}$, constructed in this section, and $M(m, l)$, constructed in the previous section, are homeomorphic, when $1 \leq l < m$ and m, l are coprime.*

Proof (see [10, p. 256-257]). Cut out a core cylinder Z from B , which connects the north and south pole of S^2 . Draw m equally spaced lines on the cylinder, connecting the ends of the remainder of each triangle $\Delta_j \cap Z$ with the remainder $\Delta'_j \cap Z$, $j \in \{1 \dots m\}$.

After the identification of the top and bottom faces of Z via the rotation of angle $2\pi/m$, these lines join to form an (m, l) torus curve c .

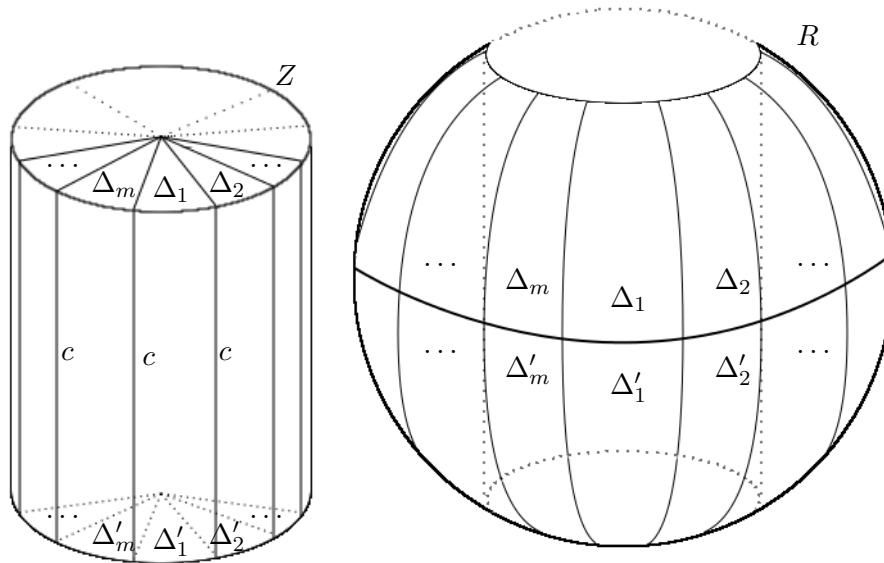


FIGURE 2.2: The removed cylinder Z and the rest R .

To see what happens to the rest $R := \overline{B^3} \setminus Z$, we divide it into m equal parts, by vertical cuts through the boundaries of the Δ_j , $j \in \{1 \dots m\}$. After the identification of the $\Delta_j \cap R$ with $\Delta'_{j+l} \cap R$, the result is a cylinder with the curve c as the edge at each end. Now when the m equal parts are rejoined, the ends of the cylinder close to form a solid torus, with c as a meridian curve.

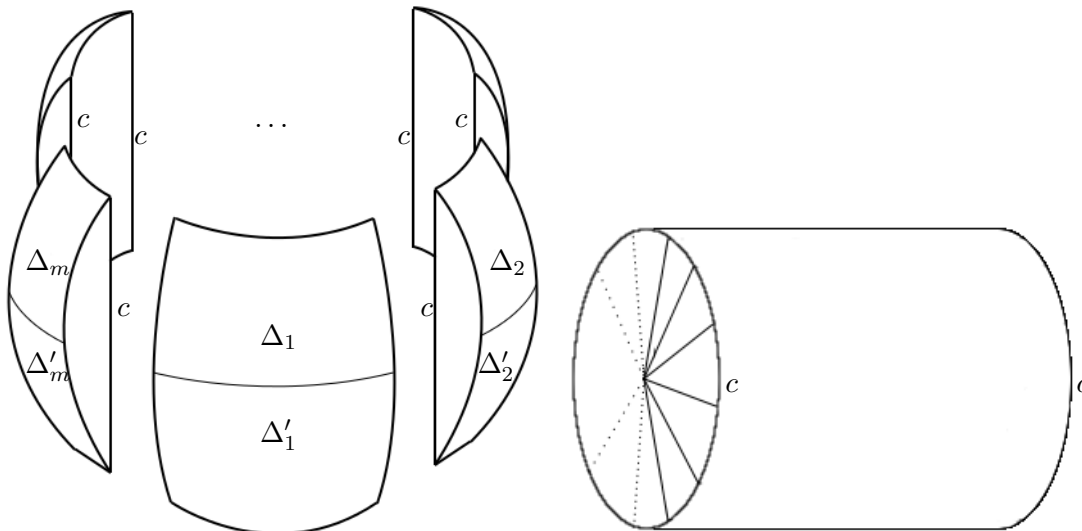


FIGURE 2.3: R divided into m equal parts and then $\Delta_j \cap R$ identified with $\Delta'_{j+l} \cap R$.

So $L_{l/m}$ is obtained by glueing two solid tori together via a homeomorphism of their boundaries, which transforms a meridian curve to a (m, l) torus curve, so the desired result now follows from 2.2. □

Remark 2.4. A *Heegard splitting* of an orientable 3-dimensional manifold M is a decomposition of M into two handlebodies, $M = V \cup_f W$. Here $f : \partial V \rightarrow \partial W$ is an orientation reversing homeomorphism of the boundaries of V and W . Every orientable 3-manifold has a Heegard splitting and the lens spaces are exactly the 3-dimensional manifolds of *Heegard genus 1*.

2.3 Construction as an Orbit Space of S^3

Let $m \in \mathbb{Z}$, $1 \leq l_n < m$, l_n, m coprime for $n \in \{1, 2\}$. Consider the action of \mathbb{Z}_m on $S^3 \subset \mathbb{C}^2$ generated by $\rho : S^3 \rightarrow S^3, (z_1, z_2) \mapsto (e^{2\pi i l_1/m} z_1, e^{2\pi i l_2/m} z_2)$, for l_1, l_2 coprime to m . Define the *lens space* $L_m(l_1, l_2) := S^3/\mathbb{Z}_m$ to be the orbit space of this action on S^3 .

We wouldn't call this again a lens space if this construction wasn't equivalent to the previous ones.

Proposition 2.5. *Let $m \in \mathbb{Z}$, $1 \leq l_n < m$, and let l_n, m be coprime for $n \in \{1, 2\}$. Then $L_m(l_1, l_2) \cong L_m(l, 1) \cong L_{l/m}$, where $l := l_1 r$ and $r \in \mathbb{Z}$ is chosen to satisfy $l_2 r \equiv 1 \pmod{m}$.*

Proof (see [12, p. 145]). Divide the unit circle $C := 0 \times S^1$ in the second \mathbb{C} factor of \mathbb{C}^2 into m segments $[v_j, v_{j+1}]$, with $v_j := (0, e^{2\pi i j/m})$, subscripts taken mod m . Join the $v_j \in C$ to the circle $S^1 = S^1 \times 0$ in the first \mathbb{C} factor of \mathbb{C}^2 via arcs of great circles in S^3 and get a 2-dimensional ball B_j^2 that is bounded by S^1 , i.e. $B_j^2 := \{\cos \theta \cdot v_j + \sin \theta \cdot (z, 0) : z \in S^1, 0 \leq \theta \leq \frac{\pi}{2}\}$. Joining each point of the segment $[v_j, v_{j+1}]$ to S^1 gives a 3-dimensional ball $B_j^3 := \{\cos \theta \cdot v + \sin \theta \cdot (z, 0) : z \in S^1, 0 \leq \theta \leq \frac{\pi}{2}, v \in [v_j, v_{j+1}]\}$ bounded by $B_j^2 \cup B_{j+1}^2$. Now ρ maps S^1 to itself (rotates it by an angle of $2\pi l_1$) and rotates C by an angle of $2\pi l_2$, that means it permutes the B_j^2 's and the B_j^3 's. We may choose an r , such that $rl_2 \equiv 1 \pmod{m}$, since m and l_2 are coprime. Now ρ^r will as well generate the action of \mathbb{Z}_m , but will also take each B_j^2 and B_j^3 to the next one. So we may obtain L as the quotient of one B_j^3 by identifying its two faces B_j^2 and B_{j+1}^2 via ρ^r , but this is exactly the construction described in the previous section for the lens space $L_{l/m}$, where $l := rl_1$. \square

Remark 2.6. From the construction given in this section, we immediately see, that

$$\pi_1(L_{l/m}) \cong \mathbb{Z}_m,$$

since $L_{l/m}$ is the orbit space of a free action from \mathbb{Z}_m on S^3 , which is simply connected and therefore its universal cover (using covering theory [14, Proposition II.2.3, Proposition II.5.1]).

Remark 2.7. We have $L_{1/2} \cong \mathbb{RP}^3$, since in this case ρ is the antipodal map.

Remark 2.8. In the case $m = 3$, the sets $B_j^3 \subset S^3$ in the proof of 2.5 look, after a stereographic projection $S^3 \rightarrow \mathbb{R}^3$, like lenses.

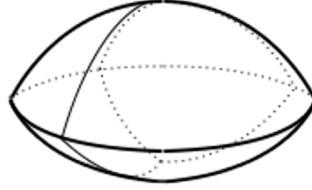


FIGURE 2.4: One of the sets B_j^3 in the case $m = 3$ after stereographic projection.

2.4 Higher-Dimensional Lens Spaces

The construction used in the last section can easily be generalized to higher dimensions. We will define higher dimensional lens spaces to be the orbit space of an action on S^{2n-1} generated by \mathbb{Z}_m .

Definition 2.9. Let $m > 1$ be an integer and l_1, \dots, l_n positive integers relatively prime to m . Define the *lens space* $L := L_m(l_1, \dots, l_n) := S^{2n-1}/\mathbb{Z}_m$, where the action of \mathbb{Z}_m on S^{2n-1} is generated by $\rho : S^{2n-1} \rightarrow S^{2n-1}, (z_1, \dots, z_n) \mapsto (e^{2\pi i l_1/m} z_1, \dots, e^{2\pi i l_n/m} z_n)$.

Remark 2.10. This construction also gives us the opportunity to define *infinite-dimensional lens spaces*. Given a sequence of integers l_1, l_2, \dots coprime to m we can define the space $L_m(l_1, l_2, \dots)$ to be the orbit space S^∞/\mathbb{Z}_m .

$L_m(l_1, l_2, \dots)$ is a CW complex, since it is the union of an increasing sequence of finite-dimensional complexes, namely the lens spaces $L_m(l_1, l_2) \subset L_m(l_1, l_2, l_3) \subset \dots$, each $L_m(l_1, \dots, l_n)$ being a subcomplex of $L_m(l_1, \dots, l_n, l_{n+1})$ (see 3.1).

Since the universal cover of such a space is S^∞ (see [14, Proposition II.2.3, Proposition II.5.1]), which is contractible, $L_m(l_1, l_2, \dots)$ is an Eilenberg-MacLane space $K(\mathbb{Z}_m, 1)$ (for a description, see [12, p. 365]). Hence its homotopy type does only depend on m (see [12, p. 366, Proposition 4.30]). Not so in the finite-dimensional case, which we will see in 4.1.

Chapter 3

(Co)Homology of Lens Spaces

3.1 A Cell Decomposition of Lens Spaces

We will now describe a CW decomposition of lens spaces (as in [12, p. 145]).

Let $C := 0 \times \cdots \times 0 \times S^1 \subset \mathbb{C}^n$ be the unit circle of the n th \mathbb{C} factor of \mathbb{C}^n . Divide C into m segments $[v_j, v_{j+1}]$, where the j th vertex $v_j := (0, \dots, 0, e^{2\pi i j/m})$, subscripts taken mod m . Joining $v_j \in C$ by arcs of great circles in S^{2n-1} to the unit sphere $S^{2n-3} \subset \mathbb{C}^{n-1} \times 0$ gives a $(2n-2)$ -dimensional ball B_j^{2n-2} that is bounded by S^{2n-3} . That is $B_j^{2n-2} := \{\cos \theta \cdot v_j + \sin \theta \cdot (z, 0) : z \in S^{2n-3}, 0 \leq \theta \leq \frac{\pi}{2}\}$. Similarly, joining each point of the j th segment $[v_j, v_{j+1}] \subset C$ to S^{2n-3} gives a ball B_j^{2n-1} bounded by $B_j^{2n-2} \cup B_{j+1}^{2n-2}$, again every subscript taken mod m . Now ρ (from 2.9) maps S^{2n-3} to itself and rotates C by an angle of $2\pi l_n/m$, so it permutes the B_j^{2n-2} 's and the B_j^{2n-1} 's. Since l_n and m are coprime, we may choose an r , such that $rl_n \equiv 1 \pmod{m}$. Now ρ^r will as well generate the action of \mathbb{Z}_m , but will also take each B_j^{2n-2} and B_j^{2n-1} to the next one. Thus we may obtain L as the quotient of one B_j^{2n-1} by identifying its two faces B_j^{2n-2} and B_{j+1}^{2n-2} via ρ^r .

Since $L_m(l_1, \dots, l_{n-1})$ is the quotient of $S^{2n-3} \subset S^{2n-1}$ it is naturally a subspace of $L_m(l_1, \dots, l_n)$, since S^{2n-3} is compact. Since $L_m(l_1, \dots, l_n)$ can be obtained from this subspace by attaching two cells, of dimensions $2n-1$ and $2n-2$, that is, the interior of B_j^{2n-1} and the interiors of its two identified faces B_j^{2n-2} and B_{j+1}^{2n-2} , we can apply induction to arrive at a CW decomposition of $L_m(l_1, \dots, l_n)$ with one cell e^k in each dimension $k \leq 2n-1$.

The CW decomposition now allows us to compute the homology groups of the lens space.

Proposition 3.1. *By setting $e_j^k := \mathring{B}_j^k$ with the B_j^k , $j \in \{0 \dots m-1\}$, $k \in \{0 \dots 2n-1\}$ as explained above, we obtain a CW structure on S^{2n-1} . Denoting $\tilde{e}_k := e_0^k$, the*

cellular chain complex $C_k(S^{2n-1})$ determined by this CW structure satisfies the following properties:

1. $e_j^k = \rho^t \tilde{e}_k$, where $tl_j \equiv j \pmod{m}$
2. $d(\tilde{e}_{2i+1}) = \rho^{r_{i+1}} \tilde{e}_{2i} - \tilde{e}_{2i}$, where $r_{i+1}l_{i+1} \equiv 1 \pmod{m}$
3. $d(\tilde{e}_{2i}) = \tilde{e}_{2i-1} + \rho \tilde{e}_{2i-1} + \cdots + \rho^{m-1} \tilde{e}_{2i-1}$
4. $d\rho = \rho d : C_k(S^{2n-1}) \rightarrow C_{k-1}(S^{2n-1})$

Proof (see [7, p. 89-90]). We have already explained how ρ permutes the B_j^{2n-2} 's and the B_j^{2n-1} 's, so (1) is evident.

Denoting the inclusion $\partial B_0^{2i+1} = B_0^{2i} \cup B_1^{2i} \hookrightarrow S^{i+1}$ by ι , we compute

$$\begin{aligned} d(\tilde{e}_{2i+1}) &= \sum_{j=0}^m e_j^{2i} \cdot \deg(S^{2i} \cong B_0^{2i} \cup B_1^{2i} \xrightarrow{\iota} S^{2i+1} \longrightarrow \frac{S^{2i+1}}{S^{2i+1} \setminus e_j^{2i}} \cong S^{2i}) \\ &= e_0^{2i} \cdot \deg(\dots) + e_1^{2i} \cdot \deg(\dots), \text{ since all other maps aren't surjective} \\ &= \epsilon(\rho^{r_{i+1}} \tilde{e}_{2i} - \tilde{e}_{2i}), \end{aligned}$$

since the first map is the same as the second after a reflection, which has degree -1 and a rotation, which has degree $+1$. Now we may choose $\epsilon := 1$ as to how we orient the cells. This gives us (2).

Now, by a similar computation, (3) follows from the fact that $\partial B_j^{2i} = S^{2i-1} = B_0^{2i-1} \cup B_1^{2i-1} \cup \cdots \cup B_{m-1}^{2i-1}$.

Finally (4) follows easily from (1), (2) and (3). \square

Proposition 3.2. *The homology groups of the lens space $L_m(l_1, \dots, l_n)$ are given by*

$$H_k(L_m(l_1, \dots, l_n)) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2n - 1 \\ \mathbb{Z}_m & \text{if } k \text{ is odd and } 0 < k < 2n - 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. For the odd-dimensional cells e^{2i+1} we get from property (2) of 3.1, that

$$d(e^{2i+1}) = d(p_*(\tilde{e}_{2i+1})) = p_*(d(\tilde{e}_{2i+1})) = p_*(\rho^{r_{i+1}} \tilde{e}_{2i} - \tilde{e}_{2i}) = e^{2i} - e^{2i} = 0,$$

where $p : S^{2n-1} \rightarrow L(l_1, \dots, l_n)$ denotes the projection.

Similarly, for even-dimensional cells, we get from property (3) of 3.1, that

$$d(e^{2i}) = m \cdot e^{2i-1}.$$

So the resulting cellular chain complex is given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

And the desired result is an easy computation. \square

Remark 3.3. Note that $p_* : \mathbb{Z} \cong H_{2n-1}(S^{2n-1}) \rightarrow H_{2n-1}(L_m(l_1, \dots, l_n)) \cong \mathbb{Z}$ is multiplication by m , since this is the only map that fits into a commutative square

$$\begin{array}{ccc} C_{2n-1}(S^{2n-1}) \cong \mathbb{Z}^m & \xrightarrow{p_*=(1,1,\dots,1)} & \mathbb{Z} \cong C_{2n-1}(L) \\ \downarrow & & \downarrow \cong \\ H_{2n-1}(S^{2n-1}) \cong \mathbb{Z} & \xrightarrow{p_*} & \mathbb{Z} \cong H_{2n-1}(L) \end{array}$$

Remark 3.4. Analogously, the infinite-dimensional lens space $L_m(l_1, l_2, \dots)$ has a cell decomposition with one cell in each dimension and the homology groups are:

$$H_k(L_m(l_1, l_2, \dots)) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}_m & \text{if } k > 0 \text{ is odd} \\ 0 & \text{if } k > 0 \text{ is even} \end{cases}$$

Corollary 3.5. *The cohomology groups of the lens space $L_m(l_1, \dots, l_n)$ are given by:*

$$H^k(L_m(l_1, \dots, l_n)) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2n - 1 \\ \mathbb{Z}_m & \text{if } k \text{ is odd and } 0 < k < 2n - 1 \\ 0 & \text{otherwise} \end{cases}$$

And the (co)homology groups with coefficients in \mathbb{Z}_m are given by:

$$H_k(L_m(l_1, \dots, l_n); \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & \text{if } 0 \leq k \leq 2n - 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(L_m(l_1, \dots, l_n); \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & \text{if } 0 \leq k \leq 2n - 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. These are straightforward applications of the universal coefficient theorem: From [13, p. 29-30] we know that $\text{Tor}(\mathbb{Z}, \mathbb{Z}_m) = 0$ and $\text{Tor}(\mathbb{Z}_m, \mathbb{Z}_m) = \mathbb{Z}_m$. Denoting $L =$

$L(l_1, \dots, l_n)$ we get

$$H_k(L; \mathbb{Z}_m) \cong \begin{cases} (\mathbb{Z} \otimes \mathbb{Z}_m) \oplus \text{Tor}(\mathbf{0}, \mathbb{Z}_m) & \text{if } k = 0, 2n - 1 \\ (\mathbb{Z}_m \otimes \mathbb{Z}_m) \oplus \text{Tor}(\mathbb{Z}, \mathbb{Z}_m) & \text{if } k = 1 \\ (\mathbb{Z}_m \otimes \mathbb{Z}_m) \oplus \text{Tor}(\mathbf{0}, \mathbb{Z}_m) & \text{if } k \text{ is odd and } 1 < k < 2n - 1 \\ (0 \otimes \mathbb{Z}_m) \oplus \text{Tor}(\mathbb{Z}_m, \mathbb{Z}_m) & \text{if } k \text{ is even and } 0 < k < 2n - 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\cong \begin{cases} \mathbb{Z}_m & \text{if } 0 \leq k \leq 2n - 1 \\ 0 & \text{otherwise} \end{cases}$$

The results for $H^k(L)$ and $H^k(L; \mathbb{Z}_m)$ follow from the facts that $\text{Ext}(\mathbb{Z}, \mathbb{Z}_m) = 0$ and $\text{Ext}(\mathbb{Z}_m, \mathbb{Z}_m) = \mathbb{Z}_m$ (see [13, p. 30]). \square

The following lemma will prove to be useful in the computation of the cup product structure on the cohomology ring of lens spaces.

Lemma 3.6. *Let $L = L(l_1, \dots, l_n)$. Let β be the Bockstein homomorphism $\beta : H^q(L; \mathbb{Z}_m) \rightarrow H^{q+1}(L; \mathbb{Z}_m)$ associated to the sequence $0 \rightarrow \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_{m^2} \rightarrow \mathbb{Z}_m \rightarrow 0$. Let $0 \leq q < 2n - 1$. Then β is an isomorphism if q is odd and the trivial homomorphism if q is even.*

Proof (see [12, p. 303-304]). Let $\tilde{\beta}$ be the Bockstein associated to the sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}_m \rightarrow 0.$$

From the natural map from this sequence to the sequence

$$0 \rightarrow \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_{m^2} \rightarrow \mathbb{Z}_m \rightarrow 0,$$

we get the relationship $\beta = \rho \tilde{\beta}$, where ρ denotes the homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}_m$ reducing coefficients mod m . Hence we get the following commutative diagram, whose upper row is the exact sequence containing $\tilde{\beta}$:

$$\begin{array}{ccccccc} H^q(L; \mathbb{Z}) & \xrightarrow{\rho} & H^q(L; \mathbb{Z}_m) & \xrightarrow{\tilde{\beta}} & H^{q+1}(L; \mathbb{Z}) & \xrightarrow{m} & H^{q+1}(L; \mathbb{Z}) \\ & & & \searrow \beta & \downarrow \rho & & \\ & & & & H^{q+1}(L; \mathbb{Z}_m) & & \end{array}$$

If q is even, the vertical map ρ in the diagram is surjective (see 3.5). And so is $\tilde{\beta}$ since m is trivial (again 3.5). That means $\beta = \rho \circ \tilde{\beta} : \mathbb{Z}_m \rightarrow \mathbb{Z}_m$ is surjective and hence

an isomorphism. If q is odd, the horizontal map ρ in this diagram is surjective, so by exactness $\tilde{\beta} = 0$, hence $\beta = 0$. \square

Remark 3.7. Note that for the infinite-dimensional lens space $L_m(l_1, l_2, \dots)$ the same arguments in the proof of 3.6 apply for any $q \geq 0$. So we have that for any $q \geq 0$ that β is an isomorphism for q odd and 0 for q even.

Proposition 3.8. *The cup product structure on $H^*(L; \mathbb{Z}_m)$ where $L = L_m(l_1, \dots, l_n)$ may be described in the following way:*

There is a generator x of $H^1(L; \mathbb{Z}_m)$ and y of $H^2(L; \mathbb{Z}_m)$, such that $H^q(L; \mathbb{Z}_m)$ is generated by y^k if $q = 2k$ or xy^k if $q = 2k + 1$. When m is odd $x^2 = 0$ and when m is even x^2 is the unique element of order 2 in $H^2(L; \mathbb{Z}_m) \cong \mathbb{Z}_m$.

Proof (see [12, p. 208-209, 304-305]). We first consider the infinite dimensional lens space $L^\infty = L_m(1, 1, \dots)$ i.e. $L^\infty = S^\infty / \mathbb{Z}_m$ where the action of \mathbb{Z}_m on S^∞ is generated by the rotation $v \mapsto e^{2\pi i/m} v$. We will derive the cup product structure of L^∞ from the cup product structure on $\mathbb{C}P^\infty$. First note that the quotient map $S^\infty \rightarrow \mathbb{C}P^\infty$ factors through L^∞ , so we have a projection $L^\infty \rightarrow \mathbb{C}P^\infty$. Looking at the cell structure on L^∞ (3.4), we see that each even-dimensional cell of L^∞ projects homeomorphically onto the corresponding cell of $\mathbb{C}P^\infty$ (for a description of the cell structure of $\mathbb{C}P^\infty$, see [12, p. 7]). From cellular (co)homology it then follows that the map $L^\infty \rightarrow \mathbb{C}P^\infty$ induces isomorphisms on even-dimensional cohomology with \mathbb{Z}_m coefficients.

Since $H^*(\mathbb{C}P^\infty; \mathbb{Z}_m)$ is a polynomial ring (see [12, p. 220]), we deduce that if $y \in H^2(L^\infty; \mathbb{Z}_m)$ is a generator, then y^k generates $H^{2k}(L^\infty; \mathbb{Z}_m)$ for all k .

Now by 3.7 there is a generator $x \in H^1(L^\infty; \mathbb{Z}_m)$ with $\beta(x) = y$ so we get

$$\beta(xy^k) \stackrel{1.14}{=} \beta(x)y^k - x\underbrace{\beta(y^k)}_{\in H^{2k}(L; \mathbb{Z}_m)} \stackrel{3.7}{=} y^{k+1}.$$

So β takes xy^k to a generator, so xy^k must have been a generator of $H^{2k+1}(L^\infty; \mathbb{Z}_m)$ in the first place.

If m is odd we have now completely described the cup product structure on $H^*(L^\infty; \mathbb{Z}_m)$, since in this case the graded commutativity of $H^*(L^\infty; \mathbb{Z}_m)$ implies that $x^2 = 0$. When m is even, then we get from the commutativity property that either $x^2 = 0$ or that x^2 is the unique element of order 2 in $H^2(L^\infty; \mathbb{Z}_m)$. By [12, p. 202, p. 207] we may use a triangulation to compute the cup-product structure on $H^*(L^\infty; \mathbb{Z}_m)$ to decide which case holds: We triangulate the 2-skeleton of L^∞ , which is a 2-cell attached to the circle S^1 by a map of degree m , the following way: Take a regular m -gon subdivided into m triangles T_i by edges e_i around a central vertex and then identify all the outer edges by

rotations of the m -gon to one edge e . Then the T_i are bounded by edges e , e_i and e_{i+1} , subscripts taken mod m . Now a generator x of $H^1(L^\infty; \mathbb{Z}_m)$ is represented by a cocycle φ assigning the value 1 to the edge e , which generates $H_1(L^\infty)$. That φ is a cocycle means that $\varphi(e_i) + \varphi(e) - \varphi(e_{i+1}) = 0$ for all $i \in \mathbb{Z}_m$. So we may take $\varphi(e_i) = i \in \mathbb{Z}_m$. Then $(\varphi \cup \varphi)(T_i) = \varphi(e_i)\varphi(e) = i$. Now $(\varphi \cup \varphi)(\sum_i T_i) = 0 + 1 + \cdots + (m-1) =: k \in \mathbb{Z}_m$, that means $x^2 = [\varphi \cup \varphi] = k \cdot y$. In the case m even, it follows that $k = \frac{m}{2}$, since all the terms 1 and $m-1$ cancel, 2 and $m-2$ cancel, and so on, except for the term $\frac{m}{2}$. Hence x^2 is the unique element of order 2 in $H^2(L^\infty; \mathbb{Z}_m)$.

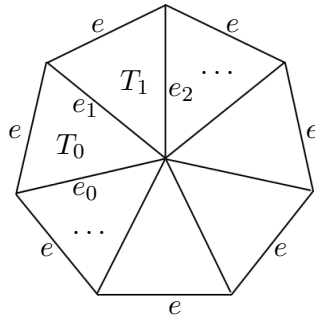


FIGURE 3.1: The triangulation of the 2-skeleton

Since the homotopy type of any infinite-dimensional lens space is already determined only by m (see 2.10) we may choose any sequence l_1, l_2, \dots of numbers coprime to m to get an infinite-dimensional lens space in which an arbitrary finite-dimensional lens space $L = L_m(l_1, \dots, l_n)$ embeds as the $(2n-1)$ -skeleton. Since the cell structure for cells of dimension $\leq 2n-1$ is then the same as that of $L(l_1, l_2, \dots) \simeq L^\infty$ and there are no more cells of dimension $\geq 2n$, it follows that the cup product structure on $H^*(L; \mathbb{Z}_m)$ can be described in the same way as that of $H^*(L^\infty; \mathbb{Z}_m)$ but modulo the relation $y^n = 0$. \square

Chapter 4

The Homotopy Classification

In this chapter, we will see a proof of the homotopy classification of lens spaces. The concepts needed in this proof are dealt with in sections 1.1 and 1.2.

Proposition 4.1. *Let $L_m(l_1, \dots, l_n) \simeq L_m(l'_1, \dots, l'_n)$ be two homotopy equivalent lens spaces. Then $l_1 \cdots l_n \equiv \pm l'_1 \cdots l'_n \cdot k^n$ for some $k \in \mathbb{Z}$ coprime to m .*

Proof. We follow the approach suggested in [12, Exercise 2, p.310].

Let $L := L_m(l_1, \dots, l_n)$, $L' := L_m(l'_1, \dots, l'_n)$. Let β be the Bockstein homomorphism $\beta : H^q(L; \mathbb{Z}_m) \rightarrow H^{q+1}(L; \mathbb{Z}_m)$ associated to the sequence $0 \rightarrow \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_{m^2} \rightarrow \mathbb{Z}_m \rightarrow 0$. Note that $\beta : H^1(L; \mathbb{Z}_m) \rightarrow H^2(L; \mathbb{Z}_m)$ is an isomorphism by 3.6.

Let x be a generator of $H^1(L; \mathbb{Z}_m)$, $y := \beta(x) \in H^2(L; \mathbb{Z}_m)$ and let $z \in H^{2n-1}(L; \mathbb{Z}_m)$ be the image of a generator of $H^{2n-1}(L; \mathbb{Z}) \cong \mathbb{Z}$. Note that z generates $H^{2n-1}(L; \mathbb{Z}_m)$, since the maps $H^{2n-1}(L; \mathbb{Z}) \rightarrow \text{Hom}(H_{2n-1}(L), \mathbb{Z})$ and $H^{2n-1}(L; \mathbb{Z}_m) \rightarrow \text{Hom}(H_{2n-1}(L), \mathbb{Z}_m)$ of the universal coefficient theorem are isomorphisms, that means z can be thought of as the mod m reduction of a generator of $H^{2n-1}(L; \mathbb{Z})$.

Define $t \in \mathbb{Z}_m$ via the equation

$$xy^{n-1} = tz. \quad (*)$$

This is well defined since for another t' to satisfy this equation, we would get from $tz = xy^{n-1} = t'z$ that $tz = t'z$, so $t = t'$ since z is a generator of $H^{2n-1}(L; \mathbb{Z}_m)$.

Also from (the proof of) 3.8 it follows that xy^{n-1} is a generator of $H^{2n-1}(L; \mathbb{Z}_m)$, so it follows that t actually lies in \mathbb{Z}_m^* , the group of units of \mathbb{Z}_m . This allows us to define $\tau(L)$ to be the image of t in $\mathbb{Z}_m^*/\pm(\mathbb{Z}_m^*)^n$, where $\pm(\mathbb{Z}_m^*)^n$ denotes the subgroup $\{\pm a^n, a \in \mathbb{Z}_m^*\}$ of \mathbb{Z}_m^* .

We will now show that $\tau(L)$ does not depend on the choices of x and z and is therefore a homotopy invariant, since homology and the Bockstein are homotopy invariants. For, let x', z' be different choices. Since $H^1(L; \mathbb{Z}_m) \cong \mathbb{Z}_m$, we have $x' = e \cdot x$, for some $e \in \mathbb{Z}_m^*$ and since $H^{2n-1}(L; \mathbb{Z}) \cong \mathbb{Z}$, we have $z' = \pm z$.

Plugging this into the equation $*$ yields

$$\begin{aligned} x'(y')^{n-1} &= tz' \\ \Rightarrow ex \cdot e^{n-1}y^{n-1} &= \pm t'z \\ \Rightarrow xy^{n-1} &= (\pm e^{-n}t')z \\ \Rightarrow \pm e^nt &= t' \end{aligned}$$

So t is unique in $\mathbb{Z}_m^*/\pm(\mathbb{Z}_m^*)^n$.

Now define $\tilde{f} : S^{2n-1} \rightarrow S^{2n-1}$, $(r_1e^{i\theta_1}, \dots, r_n e^{i\theta_n}) \mapsto (r_1e^{ik_1\theta_1}, \dots, r_n e^{ik_n\theta_n})$ for some $k_1, \dots, k_n \in \mathbb{Z}$. If we choose k_1, \dots, k_n to satisfy $k_j l_j \equiv l'_j \pmod{m}$ for all $j \in \{1 \dots n\}$, the following computation shows, that \tilde{f} induces a map $f : L \rightarrow L'$.

$$\begin{aligned} &\text{pr}_j((\rho' \circ \tilde{f})(r_1e^{t\theta_1}, \dots, r_n e^{i\theta_n})) \\ &= e^{2\pi l'_j/m} \cdot r_j e^{ik_j\theta_j} \\ &= e^{2\pi k_j l_j/m} \cdot e^{2\pi i a} \cdot r_j e^{ik_j\theta_j} \\ &= e^{2\pi k_j l_j/m} \cdot r_j e^{ik_j\theta_j} \\ &= \text{pr}_j((\tilde{f} \circ \rho)(r_1e^{t\theta_1}, \dots, r_n e^{i\theta_n})). \end{aligned}$$

Here pr_j is the projection to the j th component, ρ and ρ' are the maps that induce the group actions on S^{2n-1} and we have shown that $\tilde{f} \circ \rho = \rho' \circ \tilde{f}$. So we have that the following diagram commutes:

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\tilde{f}} & S^{2n-1} \\ p \downarrow & & \downarrow p' \\ L & \xrightarrow{f} & L' \end{array}$$

where p and p' are the projections.

We will now show by induction that \tilde{f} has mapping degree $k_1 \cdots k_n$.

($n = 1$) Obviously $\tilde{f} : S^1 \rightarrow S^1, e^{i\theta} \mapsto e^{i\theta \cdot k_1}$ has degree k_1 .

($n \rightarrow n + 1$) Suppose that $\tilde{f} : S^{2n-3} \rightarrow S^{2n-3}$ has degree $k_1 \cdots k_{n-1}$. Now using 1.6, $\Sigma^2 \tilde{f} = g_1$ has degree $k_1 \cdots k_{n-1}$ as well, where Σ is suspension and $g_1(r_1e^{t\theta_1}, \dots, r_n e^{i\theta_n}) = (r_1e^{i\theta_1 k_1}, \dots, r_{n-1}e^{i\theta_{n-1} k_{n-1}}, r_n e^{i\theta_n})$; after a suitable identification of S^{2n-1} with $\Sigma^2 S^{2n-3}$. The map $g : S^1 \rightarrow S^1, e^{i\theta} \mapsto e^{i\theta \cdot k_n}$ has degree k_n and so has $\Sigma^{2n-2} g = g_2$, where $g_2(r_1e^{t\theta_1}, \dots, r_n e^{i\theta_n}) = (r_1e^{i\theta_1}, \dots, r_{n-1}e^{i\theta_{n-1}}, r_n e^{i\theta_n \cdot k_n})$; after a suitable identification

of S^{2n-1} with $\Sigma^{2n-2}S^1$. But $\tilde{f} : S^{2n-1} \rightarrow S^{2n-1}$ is the composition $g_1 \circ g_2$, so it has degree $k_1 \cdots k_n$.

Now f has degree $k_1 \cdots k_n$ as well, meaning its induced homomorphism $f_* : \mathbb{Z} \cong H_{2n-1}(L) \rightarrow H_{2n-1}(L') \cong \mathbb{Z}$ is multiplication by $k_1 \cdots k_n$. (To see this, observe that $(f_* \circ p_*)(z) = (p'_* \circ \tilde{f}_*)(z) = k_1 \cdots k_n \cdot p'_*(z)$. Because p_*, p'_* are both multiplication by $m \neq 0$ (see 3.3), the assertion follows).

Finally we will show that the map $f : L \rightarrow L'$ induces an isomorphism $f_* : \pi_1(L) \rightarrow \pi_1(L')$ of fundamental groups and hence an isomorphism $f^* : H^1(L') \rightarrow H^1(L)$ of cohomology groups by Hurewicz's theorem and the universal coefficient theorem:

Let $\gamma_k : I \rightarrow S^{2n-1}, t \mapsto (e^{2\pi i t k/m}, 0, \dots, 0)$ and $p : S^{2n-1} \rightarrow L$ the projection. We have $\pi_1(L) \cong \mathbb{Z}_m \cong \pi_1(L')$ and the map $e := p \circ \gamma_1$ is a representative of a generator of $\pi_1(L)$, because $e^k \simeq p \circ \gamma_k \not\approx 0$ for any $1 \leq k < m$ (note that every interval $[\frac{j}{k}, \frac{j+1}{k}] \subseteq I$, $0 \leq j < k$ gets mapped to $e(I)$ by $p \circ \gamma_k$). And since $f \circ e = f \circ p \circ \gamma_1 = p' \circ \tilde{f} \circ \gamma_1 = p' \circ \gamma_{k_1}$, we have that $f_*([e]) \in \pi_1(L')$ is a generator.

The proof of our claim is now only a matter of computation. From

$$\begin{aligned} x'y'^{n-1} &= t'z' \\ \Rightarrow f^*(x'y'^{n-1}) &= f^*(t'z') \\ \Rightarrow \pm k^n \cdot xy^{n-1} &= k_1 \cdots k_n \cdot t'z, \text{ for some } 0 \neq k \in \mathbb{Z} \end{aligned}$$

we see, that $\tau(L) = k_1 \cdots k_n \cdot \tau(L')$.

And when $L \simeq L'$, we also have $\tau(L) = \tau(L')$. And hence $\tau(L') = k_1 \cdots k_n \cdot \tau(L)$, so $k_1 \cdots k_n = 1$ in $\mathbb{Z}_m^*/\pm(\mathbb{Z}_m^*)^n$. So $k_1 \cdots k_n \cdot l_1 \cdots l_n = l_1 \cdots l_n$ in $\mathbb{Z}_m^*/\pm(\mathbb{Z}_m^*)^n$. By definition of the $k_j, j \in \{1 \dots n\}$ we get $l'_1 \cdots l'_n = l_1 \cdots l_n$ in $\mathbb{Z}_m^*/\pm(\mathbb{Z}_m^*)^n$. So we have $l_1 \cdots l_n \equiv \pm l'_1 \cdots l'_n \cdot k^n \pmod{m}$ for some $k \in \mathbb{Z}$. \square

Proposition 4.2. *If $m \in \mathbb{Z}$ and $l_1, \dots, l_n, l'_1, \dots, l'_n \in \mathbb{Z}$ are relatively prime to m , and $l_1 \cdots l_n \equiv \pm l'_1 \cdots l'_n \cdot k^n \pmod{m}$ for some $k \in \mathbb{Z}$, then $L_m(l_1, \dots, l_n) \simeq L_m(l'_1, \dots, l'_n)$.*

Proof. We follow the approach suggested in [12, Exercise 29, p.391].

As seen in the section on the cell decomposition of lens spaces, having $\tilde{k} \in \mathbb{Z}$ coprime to m , $\rho^{\tilde{k}}$ will as well generate the action of \mathbb{Z}_m on S^{2n-1} and we get $L_m(l'_1, \dots, l'_n) = L_m(\tilde{k}l'_1, \dots, \tilde{k}l'_n)$. So without loss of generality we may assume that $k = 1$, i.e. $l_1 \cdots l_n \equiv \pm l'_1 \cdots l'_n \pmod{m}$

Let $f : L \rightarrow L'$ be a map of degree $k_1 \cdots k_n$, as constructed in the proof of 4.1. In the following, we will now construct a map $g : L \rightarrow L'$ of degree $k_1 \cdots k_n + d \cdot m$: Let $c : L \rightarrow L \vee S^{2n-1}$ collapse the border of a small ball to a point. Let $\text{id}_L \vee d : L \vee S^{2n-1} \rightarrow L \vee S^{2n-1}$

be the identity map on L on the first component and some map $d : S^{2n-1} \rightarrow S^{2n-1}$ of degree $d \in \mathbb{Z}$ on the second. Let $(f, p') : L \vee S^{2n-1} \rightarrow L'$, where $f : L \rightarrow L'$ is the map of degree $k_1 \cdots k_n$ mentioned before on the first component and the projection $p' : S^{2n-1} \rightarrow L'$ on the second. Define $g : L \rightarrow L'$ as the composition $g := (f, p') \circ (\text{id}_L \vee d) \circ c$. Since we have a natural isomorphism $H_{2n-1}(L \vee S^{2n-1}) \cong H_{2n-1}(L) \oplus H_{2n-1}(S^{2n-1})$, we see that $(\text{id}_L \vee d)_*(z_1, z_2) = (z_1, d \cdot z_2)$ for any $(z_1, z_2) \in H_{2n-1}(L \vee S^{2n-1})$ and that $(f, p')_*(z_1, z_2) = k_1 \cdots k_n \cdot z_1 + m \cdot z_2$. The map c_* maps a generator $e \in H_{2n-1}(L)$ to (e, \tilde{e}) where \tilde{e} is a generator of $H_{2n-1}(S^{2n-1})$, since $L \vee S^{2n-1}$ has two $(2n-1)$ -cells e^{2n-1} and \tilde{e}^{2n-1} both of which yield an entry of 1 in the matrix of c_* .

As the composition of these maps, g has degree $k_1 \cdots k_n + d \cdot m$.

Since $k_1, \dots, k_n \in \mathbb{Z}$ were chosen to satisfy $k_j l_j \equiv l'_j \pmod{m}$ for all $j \in \{1 \dots n\}$ it follows, together with our requirement that $l_1 \cdots l_n \equiv \pm l'_1 \cdots l'_n \pmod{m}$, that $k_1 \cdots k_n \equiv \pm 1 \pmod{m}$. We may hence choose a $d \in \mathbb{Z}$ s.t. $k_1 \cdots k_n + d \cdot m = \pm 1$. We will now show that in this case, the map g of degree ± 1 induces isomorphisms between all homotopy groups $g_* : \pi_k(L) \rightarrow \pi_k(L')$ and is therefore a homotopy equivalence by Whitehead's theorem.

First we show that g induces an isomorphism between the fundamental groups $\pi_1(L) \rightarrow \pi_1(L')$: Observe that any path $\gamma : I \rightarrow L \vee S^{2n-1}$ can be homotoped to a path $\gamma' : I \rightarrow L \subset L \vee S^{2n-1}$, since S^{2n-1} is simply connected for $n > 1$. That means, for any $\gamma \in \pi_1(L)$, that $g_*(\gamma) = f_*(\gamma)$, but we have already seen in the proof of 4.1, that f_* is an isomorphism $\pi_1(L) \rightarrow \pi_1(L')$.

For the higher homotopy groups $\pi_q, q > 1$, consider a lift $\tilde{g} : S^{2n-1} \rightarrow S^{2n-1}$ of $g : L \rightarrow L'$ to the universal covering spaces.

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\tilde{g}} & S^{2n-1} \\ p \downarrow & & \downarrow p' \\ L & \xrightarrow{g} & L' \end{array}$$

Note that $m \cdot \tilde{g}_* = p'_* \circ \tilde{g}_* = g_* \circ p_* = \pm p_* = \pm m : H_{2n-1}(S^{2n-1}) \rightarrow H_{2n-1}(L')$, where multiplication by m is assumed to be a map between the groups $H_{2n-1}(S^{2n-1})$ and $H_{2n-1}(L), H_{2n-1}(L')$ respectively. So we see that $\tilde{g}_* : H_{2n-1}(S^{2n-1}) \rightarrow H_{2n-1}(S^{2n-1})$ is an isomorphism. So $\tilde{g} : S^{2n-1} \rightarrow S^{2n-1}$ induces isomorphisms on every homology group of S^{2n-1} and therefore on every homotopy group of S^{2n-1} (see 1.32). But since $p_* : \pi_q(S^{2n-1}) \rightarrow \pi_q(L)$ resp. $p'_* : \pi_q(S^{2n-1}) \rightarrow \pi_q(L')$ is an isomorphism for every $q > 1$ (see 1.22), so is $g_* = p'_* \circ \tilde{g}_* \circ p_*^{-1} : \pi_q(L) \rightarrow \pi_q(L')$ for every $q > 1$.

So $g : L \rightarrow L'$ induces isomorphisms on all homotopy groups and is therefore a homotopy equivalence by Whitehead's theorem. \square

Combining 4.1 and 4.2 we arrive at the homotopy classification of lens spaces.

Theorem 4.3. *Two lens spaces $L_m(l_1, \dots, l_n)$ and $L_m(l'_1, \dots, l'_n)$ are homotopy equivalent if and only if there is a $k \in \mathbb{Z}$ such that $l_1 \cdots l_n \equiv \pm l'_1 \cdots l'_n \cdot k^n \pmod{m}$.*

Example 4.1. *The lens spaces $L_{1/5} = L_5(1, 1)$ and $L_{2/5} = L_5(2, 1)$ are not homotopy equivalent.*

Example 4.2. *For $m \in \{2, 3, 4, 6\}$ and any given $n \geq 2$, any two lens spaces $L_m(l_1, \dots, l_n)$ and $L_m(l'_1, \dots, l'_n)$ are homotopy equivalent. For, in these cases, the only congruence classes of numbers coprime to m are those of 1 and -1 respectively (in the case $m = 2$ even these aren't distinct), i.e. $l_i \equiv \pm 1 \pmod{m}$. Hence $l_1 \cdots l_n \equiv \pm 1 \equiv l'_1 \cdots l'_n \pmod{m}$ and we may apply 4.3.*

Example 4.3. *The two lens spaces $L_{1/7} = L_7(1, 1)$ and $L_{2/7} = L_7(2, 1)$ are homotopy equivalent, since $1 \cdot 1 \equiv (2 \cdot 1) \cdot 2^2 \pmod{7}$ so by 4.3 we have $L_{1/7} \simeq L_{2/7}$.*

Chapter 5

The Homeomorphy Classification

In this chapter, we will see a proof of the homeomorphy classification of lens spaces. Most of the needed concepts were dealt with in section 1.3, though, we will make use of a number theoretic result (Franz' theorem) which we will explain in the first section of this chapter.

5.1 Franz' Theorem

In the proof of 5.14, we will need to compare two expressions with roots of unities, coming from the torsions of acyclic complexes as computed in 5.13. To arrive at our desired result, we will make use of the following theorem.

Theorem 5.1 (Franz' Theorem). *Let $S := \{j \in \mathbb{Z} : 0 < j < m, \gcd(j, m) = 1\}$. Let $\{a_j\}_{j \in S}$ be a sequence of integers, indexed by S , satisfying*

1. $\sum_{j \in S} a_j = 0$
2. $a_j = a_{m-j}$
3. $\prod_{j \in S} (\xi^j - 1)^{a_j} = 1$ for every m th root of unity $\xi \neq 1$.

Then $a_j = 0$ for all $j \in S$.

The purpose of this section is to explain a proof of Franz' Theorem. Our main source for this is [16].

Definition 5.2. A map $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ is called a *Dirichlet character* or *short character* (mod m) if it satisfies the following conditions

1. $\chi(a) = \chi(b)$ if $a \equiv b \pmod{m}$
2. $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{Z}$
3. $\chi(a) = 0$ if and only if $\gcd(a, m) \neq 1$.

The character χ_0 that suffices $\chi_0(a) = 1$ if $\gcd(a, m) = 1$ is called the *trivial character*.

Lemma 5.3 (Orthogonality Relations). *There are $\phi(m)$ characters (mod m), where ϕ is Euler's totient function, and the following equations hold:*

$$1. \sum_{n=0}^{m-1} \chi(n) = \begin{cases} \phi(m) & \text{if } \chi = \chi_0 \\ 0 & \text{if } \chi \neq \chi_0 \end{cases}$$

$$2. \sum_{\chi} \chi(n) = \begin{cases} \phi(m) & \text{if } n \equiv 1 \pmod{m} \\ 0 & \text{if } n \not\equiv 1 \pmod{m} \end{cases}.$$

Proof (see [17, p. 179-180]). We refer to [17, p. 176, Lemma 6.6] for a proof of why there are exactly $\phi(m)$ characters.

To prove (1), let S denote the sum on the left side of the equation to prove. If $\chi = \chi_0$, then $\chi(a) = 1$ if $\gcd(a, q) = 1$ and $\chi(a) = 0$ otherwise. Hence $S = \phi(m)$.

If $\chi \neq \chi_0$, there exists a number a_1 with $\gcd(a_1, m) = 1$, such that $\chi(a_1) \neq 1$. But then

$$\chi(a_1) \cdot S = \sum_{n=0}^{m-1} \chi(a_1)\chi(n) = \sum_{n=0}^{m-1} \chi(a_1n) = \sum_{n=0}^{m-1} \chi(n) = S,$$

since a_1n runs through all residues (mod m) in the same time as n does. Hence $S = 0$.

To prove (2), let S again denote the sum on the left side of the equation to prove. Now if $\gcd(n, m) = 1$ and $n \equiv 1 \pmod{m}$, then $\chi(n) = \chi(1) = 1$ for all characters $\chi \pmod{m}$, and since there are $\phi(m)$ of them, $S = \phi(m)$.

If $\gcd(n, m) > 1$, then all terms in the sum are zero, hence $S = 0$.

Now suppose $\gcd(n, m) = 1$ and $n \not\equiv 1 \pmod{m}$. That means there must be a character $\chi_1 \pmod{m}$ with $\chi_1(n) \neq 1$ (see [17, p. 176, Lemma 6.6]). But then

$$\chi_1(n) \cdot S = \sum_{\chi} \chi_1(n)\chi(n) = \sum_{\chi} (\chi_1\chi)(n) = \sum_{\chi} \chi(n) = S,$$

and hence $S = 0$. □

Proposition 5.4. Let $\{a_j\}_{j \in S}$ be a sequence of integers, indexed by S (as stated in Franz' Theorem), satisfying $\sum_j a_j \chi(j) = 0$ for all characters $\chi \pmod{m}$. Then $a_n = 0$ for all $n \in S$.

Proof (see [16, p. 2]). Let $n \in S$. Choose $n' \in S$, such that $nn' \equiv 1 \pmod{m}$. Then

$$0 = \sum_x \chi(n') \sum_j a_j \chi(j) = \sum_j a_j \sum_x \chi(jn') = a_n \phi(m),$$

because of 5.4. Hence $a_n = 0$. □

Definition 5.5. The *conductor* of a character $\chi \pmod{m}$ is the smallest number $f > 0$ such that $\chi(a) = 1$ for all $a \in \mathbb{Z}$ with $\gcd(a, m) = 1$ and $a \equiv 1 \pmod{f}$.

If $f = m$, we say that the character χ is *primitive*.

Proposition 5.6. Let χ be a character \pmod{m} and let f be its conductor. Then f divides m .

Proof (see [16, p. 2]). Let $\gcd(f, m) = f_0$. Let a be an integer, such that $\gcd(a, m) = 1$ and $a \equiv 1 \pmod{f_0}$. We may choose integers y and z such that $a - 1 = ym + zf$. By defining $x := a - ym = 1 + zf$, we get $x \equiv a \pmod{m}$ and $x \equiv 1 \pmod{f}$. That means $\chi(x) = \chi(a)$ and $\chi(x) = 1$, hence $\chi(a) = 1$. We conclude that $f_0 \geq f$ and further $f_0 = f$. But that means f divides m . □

Proposition 5.7. We may associate to any character $\chi \pmod{m}$ with conductor f a primitive character $\chi' \pmod{f}$, such that $\chi = \chi_0 \chi'$, where χ_0 is the trivial character \pmod{m} .

Proof (see [16, p. 3]). For any integer a satisfying $\gcd(a, f) = 1$, we define $\chi'(a) := \chi(x)$, where x is chosen to satisfy $\gcd(x, m) = 1$ and $x \equiv a \pmod{f}$. This definition does not depend on the choice of x , since if $\gcd(x, m) = \gcd(y, m) = 1$ and $x \equiv y \pmod{f}$, we get that $y \equiv xx' \pmod{m}$ with $x' \equiv 1 \pmod{f}$ and hence $\chi(x') = 1$ and $\chi(y) = \chi(x)$. □

Definition 5.8. A *Gaussian sum* is an expression of the form

$$G(\chi, \xi) := \sum_{l=0}^{m-1} \chi(l) \xi^l,$$

where χ is a character \pmod{m} and ξ is a m th root of unity.

Proposition 5.9. If χ is a primitive character \pmod{m} and ξ is a primitive m th root of unity, we have that $G(\chi, \xi^k) = \bar{\chi}(k) G(\chi, \xi)$ for any integer k . Also $|G(\chi, \xi)| = \sqrt{m} \neq 0$.

Proof (see [16, p. 4-5]). First suppose that $\gcd(k, m) = 1$. Then $\chi(l) = \chi(kl)\bar{\chi}(k)$, so that

$$G(\chi, \xi^k) = \bar{\chi}(k) \sum_{l=0}^{m-1} \chi(kl)\xi^{kl} = \bar{\chi}(k)G(\chi, \xi),$$

since kl runs through all residues $(\text{mod } m)$ in the same time as l does.

Note that we haven't used yet that ξ is a *primitive* root of unity. Now if ξ wasn't primitive and we had $\xi^d = 1$ for a proper divisor d of m , then, since χ is primitive, we may choose an integer k with $\gcd(k, m) = 1$, $k \equiv 1 \pmod{d}$ and $\chi(k) \neq 1$. Then $\xi^k = \xi$ and we get $G(\chi, \xi) = 0$ from the equation above.

Now let $\gcd(k, m) > 1$. Then ξ^k isn't primitive and hence $G(\chi, \xi^k) = 0$, as we have just seen. But in this case we also have $\chi(k) = 0$.

To prove $|G(\chi, \xi)| = \sqrt{m}$, remark that

$$\sum_{n=0}^{m-1} |G(\chi, \xi^n)|^2 = \phi(m)|G(\chi, \xi)|^2,$$

for, from the first part, all the terms that are not zero are equal and there are $\phi(m)$ of them.

We proceed to calculate:

$$\begin{aligned} \sum_{n=0}^{m-1} |G(\chi, \xi^n)|^2 &= \sum_{n=0}^{m-1} \sum_{l,k} \chi(l)\bar{\chi}(k)\xi^{n(l-k)} \\ &= \sum_{l,k} \chi(l)\bar{\chi}(k) \sum_{n=0}^{m-1} \xi^{n(l-k)} \\ &= m \sum_{l \equiv k \pmod{m}} \chi(l)\bar{\chi}(k) = m\phi(m). \end{aligned}$$

We conclude $|G(\chi, \xi)| = \sqrt{m}$. □

Corollary 5.10. *Let χ be a character $(\text{mod } m)$ with conductor f and let ξ be a primitive f th root of unity. If χ' denotes the associated primitive character $(\text{mod } f)$ of χ , then $G(\chi, \xi^k) = \bar{\chi}'(k)G(\chi, \xi)$ for all integers k . Also $G(\chi, \xi) \neq 0$.*

Proof (see [16, p. 5]). From 5.9, we get $G(\chi', \xi^k) = \bar{\chi}'(k)G(\chi', \xi)$. But for any m th root of unity η , we have

$$\begin{aligned} G(\chi, \eta) &= \sum_{l=0}^{m-1} \chi(l)\eta^l = \sum_{l=0}^{m-1} \chi_0(l)\chi'(l)\eta^l \\ &= \sum_{\substack{0 \leq l < m \\ \gcd(l, m)=1}} \chi'(l)\eta^l \\ &= \sum_{l=0}^{f-1} \chi'(l)\eta^l = G(\chi', \eta). \end{aligned}$$

Also from 5.9 we now conclude that $|G(\chi, \xi)| = \sqrt{f} \neq 0$ and hence $G(\chi, \xi) \neq 0$. \square

Definition 5.11. A *Dirichlet L-series* is a function given by

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

for any Dirichlet character χ and $s \in \mathbb{C}$.

Theorem 5.12. For any non-trivial character χ , we have $L(1, \chi) \neq 0$.

Reference: [16, p. 8-9]. \square

Proof of Franz' theorem (see [16, p. 11-12]). After 5.4 it suffices to show, that $\sum_j \chi(j) = 0$ for all characters $\chi \pmod{m}$. For the trivial character, this follows from (1). For any character χ satisfying $\chi(-1) = -1$ this follows from (2). So we may from now on suppose that χ is a non-trivial character that satisfies $\chi(-1) = 1$.

From (3), we get

$$\log 1 = \sum_j \log(1 - \xi^j) = - \sum_j a_j \sum_{k=1}^{\infty} \frac{\xi^{kj}}{k}.$$

From $a_n = a_{-n}$, we see that the last sum is real and hence 0, the principal value of the logarithm.

We may also replace ξ with ξ^n and get

$$\sum_j a_j \sum_{k=1}^{\infty} \frac{\xi^{knj}}{k} = 0.$$

By multiplying with $\bar{\chi}(n) = \bar{\chi}(nj)\chi(j)$ and summing up, we get

$$\sum_{k=1}^{\infty} \frac{1}{k} \sum_{j, n} a_j \chi(j) \bar{\chi}(nj) \xi^{kjn} = 0. \quad (*)$$

But $\sum_n \bar{\chi}(nj)\xi^{kjn} = \sum_n \bar{\chi}(n)\xi^{kn} = G(\bar{\chi}, \xi^k)$ and hence by 5.10, if f denotes the conductor of χ and if ξ is a primitive f th root of unity, we have for all integers k :

$$G(\bar{\chi}, \xi^k) = \chi'(K)G(\bar{\chi}, \xi),$$

where χ' denotes the primitive character (mod f) associated to χ . Now equation * becomes

$$\sum_j a_j \chi(j) \cdot G(\bar{\chi}, \xi) \cdot L(1, \chi') = 0.$$

$L(1, \chi)$ is not zero because of 5.12 and $G(\bar{\chi}, \xi)$ is not zero as well because of 5.10. That means $\sum_j a_j \chi(j) = 0$. \square

5.2 The Classification Result

We will deal with a certain acyclic complex in our proof of the homeomorphy classification and need to compute its torsion. This is the purpose of the following proposition.

Proposition 5.13. *Write $\mathbb{Z}_m = \{1, t, \dots, t^{m-1}\}$. Let $R := \mathbb{Z}[\mathbb{Z}_m]$ and let $G := \{\pm t^j : j \in \mathbb{Z}\} \subset R$ be the group of trivial units of R . Let $R' := \mathbb{C}$, and let $\xi \neq 1$ be a m 'th root of unity and let $G' := \{\pm \xi^j : j \in \mathbb{Z}\} \subset R'$. Let (r_1, \dots, r_n) be a sequence of integers coprime to m and let $\Sigma(t) := 1 + t + \dots + t^{m-1} \in R$.*

Define $h : R \rightarrow R'$ by $h(\sum_j n_j t^j) = \sum_j n_j \xi^j$.

Also let C be the (R, G) -complex

$$0 \rightarrow C_{2n-1} \xrightarrow{\langle t^{r_n-1} \rangle} C_{2n-2} \xrightarrow{\langle \Sigma(t) \rangle} C_{2n-3} \xrightarrow{\langle t^{r_{n-1}-1} \rangle} C_{2n-4} \xrightarrow{\langle \Sigma(t) \rangle} \dots \xrightarrow{\langle t^{r_1-1} \rangle} C_0 \rightarrow 0.$$

where each C_j has rank 1 and the 1×1 matrix of $d_j : C_j \rightarrow C_{j-1}$ is written above the arrow.

Then C is not acyclic while C_h (as defined in 1.52) is acyclic and $\tau(C_h) \in K_{G'}(\mathbb{C})$ is equal to the torsion of the 1×1 matrix $\langle \prod_{j=1}^n (\xi^{r_j} - 1) \rangle$.

Proof (see [7, p. 59-60]). C is a chain complex, since

$$\Sigma(t) \cdot (t^{r_j} - 1) = \Sigma(t) \cdot (t - 1)(1 + t + \dots + t^{r_j-1}) = (t^m - 1)(1 + \dots + t^{r_j-1}) = 0.$$

C is not acyclic, since, if $\{c_i\}$ is a basis for C_i , then $\Sigma(t) \cdot c_{2n-1}$ is not a boundary, while $d(\Sigma(t) \cdot c_{2n-1}) = \Sigma(t) \cdot (t^{r_n} - 1)c_{2n-2} = 0$, so it's a cycle. However C_h is acyclic. Since

$(1 + \xi + \dots + \xi^{m-1}) = \frac{\xi^m - 1}{\xi - 1} = 0$ we have that C_h is of the form

$$0 \rightarrow C_{2n-1} \xrightarrow{\langle \xi^{r_{n-1}} \rangle} C_{2n-2} \xrightarrow{0} C_{2n-3} \xrightarrow{\langle \xi^{r_{n-1}-1} \rangle} C_{2n-4} \xrightarrow{0} \dots$$

Since every r_j is coprime to m , we have $\xi^{r_j} \neq 1$ and hence every matrix $\langle \xi^{r_j} - 1 \rangle$ is non-singular and C_h is acyclic.

A chain contraction is $\delta : C_h \rightarrow C_h$, given by $\delta_q = \begin{cases} 0 & \text{if } q \text{ odd} \\ \langle \xi^{r_{k+1}} - 1 \rangle^{-1} & \text{if } q = 2k \text{ even} \end{cases} : C_q \rightarrow C_{q+1}$. Hence we get from 1.39

$$\tau(C_h) = \tau \left(\begin{pmatrix} \xi^{r_{n-1}} & & & \\ & \xi^{r_{n-1}} & & \\ & & \ddots & \\ & & & \xi^{r_1-1} \end{pmatrix} \right) = \tau \left(\langle \prod_{j=1}^n (\xi^{r_j} - 1) \rangle \right).$$

□

Theorem 5.14. *Let $L = L_m(l_1, \dots, l_n)$ and $L' = L_m(l'_1, \dots, l'_n)$ and suppose that $f : L \rightarrow L'$ is a simple-homotopy equivalence. If $f_* : G := \text{Deck}(p) \rightarrow \text{Deck}(p') =: G'$ is the map between the groups of covering automorphisms of L, L' (generated by ρ, ρ') respectively, coming from the induced map between fundamental groups, and if $f_*(\rho) = (\rho')^a$, then there are numbers $\epsilon_i \in \{\pm 1\}$ such that (l_1, \dots, l_n) is equal (mod m) to some permutation of $(\epsilon_1 a l'_1, \epsilon_2 a l'_2, \dots, \epsilon_n a l'_n)$.*

Proof (see [7, p. 97-99]). With the cell structure on $S^{2n-1} = \tilde{L} = \tilde{L}'$ as explained in 3.1, we get two $\mathbb{Z}[G']$ -complexes $C(\tilde{L}')$ and $C(\tilde{L})_{f_*}$ with basis $\{\tilde{e}^k\}$ in dimension k and boundary operators gotten from the described properties in 3.1. Write $\Sigma(x) := 1 + x + \dots + x^{m-1}$, $C'_j := C_j(\tilde{L}')$ and $C_j := [C_j(\tilde{L})]_{f_*}$, then these complexes look like

$$C(\tilde{L}') : 0 \rightarrow C'_{2n-1} \xrightarrow{(\rho')^{r'_{n-1}-1}} C'_{2n-2} \xrightarrow{\Sigma(\rho')} C'_{2n-3} \xrightarrow{(\rho')^{r'_{n-1}-1}} \dots \xrightarrow{\Sigma(\rho')} C'_1 \xrightarrow{(\rho')^{r'_1-1}} C'_0 \rightarrow 0$$

$$C(\tilde{L})_{f_*} : 0 \rightarrow C_{2n-1} \xrightarrow{(\rho')^{ar_n-1}} C_{2n-2} \xrightarrow{\Sigma(\rho'^a)} C_{2n-3} \xrightarrow{(\rho')^{ar_{n-1}-1}} \dots \xrightarrow{\Sigma(\rho'^a)} C_1 \xrightarrow{(\rho')^{ar_1-1}} C_0 \rightarrow 0$$

Now from 1.71, we get that $0 = \tau(f) = \tau(\mathcal{C})$, where \mathcal{C} is an acyclic $\text{Wh}(G')$ -complex which fits into a based short exact sequence of $\text{Wh}(G')$ -complexes

$$0 \longrightarrow C(\tilde{L}') \longrightarrow \mathcal{C} \longrightarrow \bar{C}(\tilde{L}) \longrightarrow 0.$$

Here $\bar{C}(\tilde{L})$ is the complex $C(\tilde{L})_{f_*}$ shifted in dimension by one and with boundary operator multiplied by (-1) . Now these complexes - apart from \mathcal{C} - aren't acyclic, but by 5.13, if we change rings via $h : \mathbb{Z}[G'] \rightarrow \mathbb{C}$, $h(\sum_j n_j (\rho')^j) := \sum_j n_j \xi^j$, with a m th root of

unity $\xi \neq 1$, then we get a based short exact sequence (see 1.54):

$$0 \longrightarrow C(\tilde{L}')_h \longrightarrow \mathcal{C}_h \longrightarrow \bar{C}(\tilde{L})_h \longrightarrow 0,$$

where $C(\tilde{L}')_h$ and $\bar{C}(\tilde{L})_h$ are now acyclic (\mathbb{C}, \bar{G}) -complexes, with $\bar{G} = \{\pm \xi^j : j \in \mathbb{Z}\}$. And from 5.13 we also get

$$\begin{aligned} \tau(C(\tilde{L}')_h) &= \tau(\langle \prod_{k=1}^n (\xi^{r'_k} - 1) \rangle) \in K_{\bar{G}}(\mathbb{C}) \\ \tau(\bar{C}(\tilde{L})_h) &= -\tau(\langle \prod_{k=1}^n (\xi^{ar_k} - 1) \rangle) \in K_{\bar{G}}(\mathbb{C}). \end{aligned}$$

We now get from 1.51 and 1.53 that

$$\begin{aligned} 0 &= h_*(\tau \mathcal{C}) = \tau(\mathcal{C}_h) \\ &= \tau(C(\tilde{L}')_h) + \tau(\bar{C}(\tilde{L})_h) \\ &= \tau(\langle \prod_{k=1}^n (\xi^{r'_k} - 1) \rangle) - \tau(\langle \prod_{k=1}^n (\xi^{ar_k} - 1) \rangle). \end{aligned}$$

We conclude (using 1.39) that the determinants of these two 1×1 matrices can only differ by a factor lying in \bar{G} :

$$\prod_{k=1}^n (\xi^{r'_k} - 1) = \pm \xi^s \prod_{k=1}^n (\xi^{ar_k} - 1) \quad (*)$$

and since there were no restrictions on ξ , this equation holds for every m th root of unity $\xi \neq 1$.

By setting $s_k := r'_k$ and $t_k := ar_k$ we directly get from *, that

$$\left| \prod_{k=1}^n (\xi^{s_k} - 1) \right|^2 = \left| \prod_{k=1}^n (\xi^{t_k} - 1) \right|^2$$

or equivalently

$$\prod_{k=1}^n (\xi^{s_k} - 1)(\xi^{-s_k} - 1) = \prod_{k=1}^n (\xi^{t_k} - 1)(\xi^{-t_k} - 1). \quad (**)$$

Now let S be as described in Franz' Theorem (5.1) and for any $j \in S$ let S_j be the subsequence of $(s_1, -s_1, s_2, -s_2, \dots, s_n, -s_n)$ consisting of those terms x such that $x \equiv j \pmod{m}$. Let b_j be the length of S_j . Similarly define the sequence T_j with length b'_j from the sequence $(t_1, -t_1, t_2, -t_2, \dots, t_n, -t_n)$. Since $\gcd(s_k, m) = 1$, we have that $\pm s_k \equiv j \pmod{m}$ for some $j \in S$, and since $i \neq j$ implies $S_i \cap S_j = \emptyset$, the sequence $(s_1, -s_1, s_2, -s_2, \dots, s_n, -s_n)$ is the disjoint union of the S_j . Hence $\sum_{j \in S} b_j = 2n$. Also

$S_j \rightarrow S_{m-j}, x \mapsto -x$ is obviously a bijection, so $b_j = b_{m-j}$. Analogously we have that $\sum_{j \in S} b'_j = 2n$ and $b'_j = b'_{m-j}$ for all $j \in S$. Now by defining $a_j := b_j - b'_j$ we get

1. $\sum_{j \in S} a_j = 2n - 2n = 0$
2. $a_j = b_j - b'_j = b_{m-j} - b'_{m-j} = a_{m-j}$
3. If $\xi \neq 1$ is any m th root of unity and if $S_j = (\epsilon_{j1}s_{j1}, \dots, \epsilon_{jb_j}s_{jb_j}), T_j = (\delta_{j1}t_{j1}, \dots, \delta_{jb'_j}t_{jb'_j})$ with $\epsilon_{j\alpha}, \delta_{j\beta} \in \{\pm 1\}$, then

$$\begin{aligned} \prod_{j \in S} (\xi^j - 1)^{a_j} &= \prod_{j \in S} (\xi^j - 1)^{b_j} (\xi^j - 1)^{-b'_j} \\ &= \prod_{j \in S} \left[\left(\prod_{i=1}^{b_j} (\xi^{\epsilon_{ji}s_{ji}} - 1) \right) \left(\prod_{i=1}^{b'_j} (\xi^{\delta_{ji}t_{ji}} - 1) \right)^{-1} \right] \\ &= \prod_{k=1}^n (\xi^{s_k} - 1)(\xi^{-s_k} - 1)(\xi^{t_k} - 1)^{-1}(\xi^{-t_k} - 1)^{-1} \\ &= 1, \text{ because of **}. \end{aligned}$$

So we may now apply Franz' Theorem (5.1) and get $a_j = 0$ for all j , and hence $b_j = b'_j$. But any b_j is just the number of terms among (r'_1, \dots, r'_n) which are congruent to $\pm j \pmod m$. And the b'_j behave analogously with respect to the (ar_1, \dots, ar_n) . Hence, under some reordering $r'_{i_1}, \dots, r'_{i_n}$ we get

$$\epsilon_{i_k} r'_{i_k} \equiv ar_k \pmod m, \epsilon_{i_k} \in \{\pm 1\}, k \in \{1, \dots, n\}.$$

So $\epsilon_{i_k} l'_{i_k} \equiv a^{-1}l_k \pmod m$ and $\epsilon_{i_k} al'_{i_k} \equiv l_k \pmod m$. □

Theorem 5.15. *If $L = L_m(l_1, \dots, l_n)$ and $L' = L_m(l'_1, \dots, l'_n)$, the following assertions are equivalent:*

- (a) *There is a number $a \in \mathbb{Z}$ and there are numbers $\epsilon_j \in \{\pm 1\}$ such that (l_1, \dots, l_n) is a permutation of $(\epsilon_1 al'_1, \dots, \epsilon_n al'_n)$.*
- (b) *L is simple-homotopy equivalent to L'*
- (c) *L is homeomorphic to L' .*

Proof (see [7, p. 100-101]). (c \Rightarrow b) follows generally from 1.69.

(b \Rightarrow a) is the subject of 5.14.

For (a \Rightarrow c), suppose that for all $i \in \{1, \dots, n\}$ we have $l_i = \epsilon_{\pi(i)} a l'_{\pi(i)}$, where π is a permutation of $\{1, \dots, n\}$. Let $T_i : \mathbb{C} \rightarrow \mathbb{C}$ be the identical map if $\epsilon_i = 1$ and let $T_i : \mathbb{C} \rightarrow \mathbb{C}$ map z to its complex conjugate \bar{z} whenever $\epsilon_i = -1$. Define $T : S^{2n-1} \rightarrow S^{2n-1}$, $(z_1, \dots, z_n) \mapsto (T_{\pi^{-1}(1)}(z_{\pi^{-1}(1)}), \dots, T_{\pi^{-1}(n)}(z_{\pi^{-1}(n)}))$. Since our assumption implies $(\rho')^a \circ T = T \circ \rho$ we get an induced map $h : L \rightarrow L'$.

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{T} & S^{2n-1} \\ p \downarrow & & \downarrow p' \\ L & \xrightarrow{h} & L' \end{array}$$

But such an $h : L \rightarrow L'$ must be a homeomorphism, since T is (using [15, Proposition 1.49]). \square

Corollary 5.16. *Two three dimensional lens spaces $L_{l/m} = L_m(l, 1)$ and $L_{l'/m} = L_m(l', 1)$ are homeomorphic if and only if $l \equiv \pm l'$ or $l \cdot l' \equiv \pm 1 \pmod{m}$.*

Proof. If the permutation spoken of in 5.15(a) is the identity, we get $l \equiv \pm a l'$ and $1 \equiv \pm a \pmod{m}$, for some $a \in \mathbb{Z}$. This is equivalent to the statement that $l \equiv \pm l' \pmod{m}$.

In the other case, we get that $l \equiv \pm a$ and $1 \equiv \pm a l' \pmod{m}$, for some $a \in \mathbb{Z}$. But this is equivalent to $l \cdot l' \equiv \pm 1 \pmod{m}$ \square

Example 5.1. *The two lens spaces $L_{1/7} = L_7(1, 1)$ and $L_{2/7} = L_7(2, 1)$ are homotopy equivalent but not homeomorphic. We have seen in example 4.3 that they are homotopy equivalent, but from $L_{1/7} \cong L_{2/7}$ would follow by 5.16 that either $1 \equiv \pm 2 \pmod{7}$ or $1 \cdot 2 \equiv \pm 1 \pmod{7}$, both of which is absurd.*

Example 5.2. *For $m \in \{2, 3, 4, 6\}$ and any given $n \geq 2$, any two lens spaces $L_m(l_1, \dots, l_n)$ and $L_m(l'_1, \dots, l'_n)$ are homeomorphic.*

Example 5.3. *Any two lens spaces $L_{l/5}$ and $L_{l'/5}$ are homeomorphic if and only if they are homotopy equivalent.*

Appendix

Zusammenfassung

Diese Masterarbeit beschäftigt sich mit den Mannigfaltigkeiten, die Linsenräume genannt werden. Ein Linsenraum $L_m(l_1, \dots, l_n)$, für $m \in \mathbb{Z}$, $m \geq 2$ und $l_1, \dots, l_n \in \mathbb{Z}$ teilerfremd zu m , entsteht durch eine Wirkung der Gruppe \mathbb{Z}_m auf eine Sphäre S^{2n-1} ungerader Dimension, die von dem Homöomorphismus $\rho : S^{2n-1} \rightarrow S^{2n-1}$, $(z_1, \dots, z_n) \mapsto (e^{2\pi i l_1/m} z_1, \dots, e^{2\pi i l_n/m})$ erzeugt wird. In Kapitel 2 wird näher auf verschiedene Konstruktionen von Linsenräumen eingegangen und die Äquivalenz dieser Konstruktionen bewiesen. Kapitel 4 beschäftigt sich mit der Homotopie-Klassifikation von Linsenräumen, also mit der Fragestellungen, wann sind zwei verschiedene Linsenräume $L_m(l_1, \dots, l_n)$ und $L_m(l'_1, \dots, l'_n)$ homotopie-äquivalent? Dies ist der Fall, wenn es eine Zahl $k \in \mathbb{Z}$ gibt, sodass $l_1 \cdots l_n \equiv \pm l'_1 \cdots l'_n \cdot k^n \pmod{m}$. Die nötigen Vorbereitungen, das heißt die Berechnung der Homologiegruppen und des Cohomologierings von Linsenräumen, werden in Kapitel 3 besprochen. Diese Berechnungen sind auch wichtig für die Homöomorphie-Klassifikation, welche in Kapitel 5 besprochen wird. Zwei Linsenräume $L_m(l_1, \dots, l_n)$ und $L_m(l'_1, \dots, l'_n)$ sind homöomorph genau dann, wenn es eine Zahl $a \in \mathbb{Z}$ und Zahlen $\epsilon_j \in \{\pm 1\}$, $j = 1 \dots n$, gibt, sodass (l_1, \dots, l_n) eine Permutation von $(\epsilon_1 a l'_1, \dots, \epsilon_n a l'_n)$ ist. Der Beweis der Homöomorphie-Klassifikation verwendet ein Resultat der Zahlentheorie, was im ersten Teil von Kapitel 5 besprochen wird.

*Curriculum Vitae***Julius Grüning**

2002-2009	Gymnasium zu St. Katharinen, Oppenheim
2009-2012	BSc Mathematics, Justus-Liebig-Universität, Gießen
since 2012	MSc Mathematics, Universität Wien

Wien, 25th of August, 2015.

Bibliography

- [1] URL http://www.map.mpin-bonn.mpg.de/Lens_spaces_in_dimension_three:_a_history.
- [2] W. Threlfall and H. Seifert. Topologische Untersuchungen der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes. *Mathematische Annalen*, pages 1–70, 1931.
- [3] W. Threlfall and H. Seifert. *Lehrbuch der Topologie*. B.G. Teubner, 1934.
- [4] H. Tietze. Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten. *Monatsheft für Mathematik und Physik*, pages 1–118, 1908.
- [5] Ralph Stöcker and Heiner Zieschang. *Algebraische Topologie*. B.G. Teubner, 1994.
- [6] J. H. C. Whitehead. Simplicial Spaces, Nuclei and m -groups. *Proceedings of the London Mathematical Society*, pages 243–327, 1938.
- [7] Marshall M. Cohen. *A Course in Simple-Homotopy Theory*. Springer, 1973.
- [8] W. Threlfall and H. Seifert. Topologische Untersuchungen der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes (Schluß). *Mathematische Annalen*, pages 543–586, 1933.
- [9] J. H. C. Whitehead. On Incidence Matrices, Nuclei and Homotopy Types. *Annals of Mathematics*, pages 1197–1239, 1941.
- [10] John Stilwell. *Classical Topology and Combinatorial Group Theory -2nd ed.* 1993.
- [11] John Milnor. Two Complexes Which Are Homeomorphic But Combinatorially Distinct. *Annals of Mathematics*, pages 575–590, 1961.
- [12] Allen Hatcher. *Algebraic Topology*. 2001.
- [13] Tilman Bauer. Homologische Algebra und Gruppenkohomologie WS2004.
- [14] Stefan Haller. Algebraische Topologie Vorlesungsskript SS2013.

- [15] Andreas Kriegel. Algebraic Topology Vorlesungsskript SS2006.
- [16] G. de Rham, S. Maumary, and M.A. Kervaire. *Torsion et Type Simple d'Homotopie*. Springer-Verlag, 1967.
- [17] A. J. Hildebrand. Introduction to Analytic Number Theory, Lecture Notes Fall 2005.