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Dimension Theory of Commutative Rings

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Abstract

In this thesis, we are concerned with the dimension theory of commutative rings with 1. We will investigate the properties of dimension in polynomial rings, arbitrary direct products and power series rings. We will also analyze the algebraic structure of 0-dimensional rings.

Zusammenfassung

Diese Arbeit befasst sich mit der Dimension kommutativer Ringe mit 1. Wir untersuchen die Eigenschaften der Dimension für Polynomringe, für ein direktes Produkt beliebig vieler Ringe, und für formale Potenzreihen. Wir analysieren die algebraische Struktur 0-dimensionaler Ringe.

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1 Introduction

The concept of dimension is ubiquitous in mathematics, appearing in a number of specializations. One of the most common and well-known notions of dimension lies in the theory of vector spaces. Namely, for a vector space V, the dimension of V is the cardinality of a basis of V. For finite-dimensional vector spaces, this is also the maximal length of a chain of subspaces of V (partially ordered with respect to set theoretic inclusion). In our case, we will be interested in the dimension of a ring.[1] Analogous to the case for vector spaces, the Krull dimension of a ring R is defined to be the supremum of lengths of all finite chains of prime ideals of R. If the supremum has no finite bound we say R is infinite-dimensional. Note that for for vector spaces the only invariant is dimension. That is, two vector spaces are isomorphic if and only if they have the same dimension. As we shall see, this is most certainly not the case with rings.

The Krull dimension was first considered for affine k-domains where k is a field ([16], p. 215). An affine k-domain is an integral domain which is finitely generated as a k-algebra. Thus if A is a k-domain then $A \cong R/P$ where $R = k[X_1, \ldots, X_n]$ is the polynomial ring over k in n variables and P is a prime ideal of R. It is taken as an axiom that R should have dimension n. This is also the transcendence degree of $\operatorname{Quot}(R)$ over k. Up to at least 1935 the dimension of A was defined as the transcendence degree of A over k ([16], p. 216). This definition agrees with the one in the previous paragraph, but it cannot be generalized to arbitrary rings. The definition for affine k-domains does agree with the one defined in terms of chains of prime ideals (Proposition 3.1.25). The current definition was defined in 1937 by Wofgang Krull ([16], p. 217). According to ([16], pp. 218-220), this definition satisfies certain axioms which correspond to geometric properties. Furthermore, these axioms determine the definition of dimension uniquely for the class of Noetherian rings. That is, if a function from the class of Noetherian rings to $\mathbb{N} \cup \{\infty\}$ satisfies these axioms then it is equal to the Krull dimension.

In most courses in commutative algebra and algebraic geometry, material on Krull dimension is usually restricted to the case of Noetherian rings. For instance, a Noetherian ring R satisfies $\dim(R[X]) = \dim(R[[X]]) = \dim(R) + 1$ (Theorem 3.1.32 and Theorem 5.1.4). The general situation is much more complicated, and it is our aim to study the properties of dimension in arbitrary rings. We will be naturally led to study rings which have infinite–even uncountable chains of prime ideals. According to the classical definition of Krull dimension, each of these rings are considered to be infinite-dimensional. However, this does not differentiate between rings with at most countably infinite chains of prime ideals and rings with uncountable chains of prime ideals. Therefore we will need to generalize the definition of dimension so that we can consider arbitrary chains of prime ideals. This will be the topic of the first part of section 2. The remaining part of section 2 will be a review of the necessary ring-theoretic and set-theoretic prerequisites for understanding the rest of the text.

In section 3, we study the dimension theory of polynomial rings. R is finite-dimensional if and only if $R[X]$ is finite-dimensional (Corollary 3.1.2). If R is infinite-dimensional, then $\dim(R) = \dim(R[X])$ (Theorem 3.2.1). For a finite-dimensional ring R, we have $1 + \dim(R) \leq \dim(R[X]) \leq 2\dim(R) + 1$ (Proposition 3.1.4). Furthermore, for every n and m with $1 + n \leq m \leq 2n + 1$, we give an example of a ring with $\dim(R) = n$ and $\dim(R[X]) = m$ (Theorem 3.1.17). There exist other rings besides Noetherian rings that satisfy the dimension formula $\dim(R) + 1 = \dim(R[X])$. These include Prüfer rings, semi-hereditary rings, and 0-dimensional rings, as we shall see. When R is an integral domain, there is a subring of $\operatorname{Quot}(R)[X]$ called the ring of integer-valued polynomials, denoted $\operatorname{Int}(R)$. Just like for polynomial rings, for every n and m with $1+n \leq m \leq 2n+1$, there exists an integral domain R with $\operatorname{Int}(R) \neq R$, $\dim(R) = n$ and $\dim(\operatorname{Int}(R)) = 2n + 1$ ([18], Theorem 3.4). We investigate the relationship between $\dim(R)$,

[1] All rings that we consider will be commutative with 1.

$\dim(R[X])$, and $\dim(\mathrm{Int}(R))$. There are a couple of open questions regarding the dimension of $\mathrm{Int}(R)$.

In section 4, we look at 0-dimensional rings and arbitrary direct products. It is straightforward to give a complete description of the prime ideals of a finite direct product of rings. This is not possible in general for infinite direct products. An infinite direct product of rings is either 0-dimensional or infinite-dimensional ([23], Theorem 3.4), and it is possible even for a product of 0-dimensional rings to be infinite-dimensional. In the infinite-dimensional case we obtain lower bounds for the dimension (Theorem 4.2.10). However, the only upper bounds we are aware of are in terms of the cardinality of the power set of the direct product. Besides these results, a number of different ring-theoretic characterizations of 0-dimensional rings are given (Theorem 4.1.8). Certainly, an integral domain is 0-dimensional if and only if it is field. There are many examples of such rings that are not fields. One can view a 0-dimensional ring as a generalization of a field. We give necessary and sufficient conditions for a ring to be embeddable into a 0-dimensional ring (Theorem 4.3.10).

In section 5, we study the dimension theory of power series rings. The behavior of power series rings with respect to dimension is much different than polynomial rings. An important concept in this section is that of an SFT ring, which can be viewed as a generalization of a Noetherian ring. If R is not an SFT ring, then the dimension of R is uncountably infinite ([27], Theorem 13). As in the case for the arbitrary direct product, the only known general upper bounds for the dimension is in terms of the cardinality of the power set of $R[[X]]$. There exist 0-dimensional rings that are not SFT rings and SFT rings that are infinite-dimensional as well. We end this section with a summary on the dimension theory of power series rings over almost Dedekind rings.

2 Preliminaries

Throughout this paper, all rings considered are commutative with 1. We also assume that the axiom of choice and the continuum hypothesis are true.

Definition 2.0.1. *Let R and S be rings with $R \subset S$. Let Q be a prime ideal of S and P a prime ideal of R. The contraction of Q to R is the prime ideal $R \cap Q$ of R. If $R \cap Q = P$, then we say that Q lies over P. The extension of P to S is the ideal of S generated by P, denoted by PS. It is not necessarily prime.*

Definition 2.0.2. *Let R be a ring and I a prime ideal of R. P is called a minimal prime ideal over I if it is minimal among all prime ideals containing I. P is called a minimal prime ideal of R if it is a minimal prime ideal over $\langle 0 \rangle$. Note that any prime ideal is minimal over itself.*

Definition 2.0.3. *Let D be an integral domain. Then $\langle 0 \rangle$ is the only minimal prime of D. A nonzero prime ideal of D minimal with respect to containing $\langle 0 \rangle$ is called a minimal nonzero prime of D. If D is finite-dimensional, then there exists a minimal nonzero prime ideal of D.*

2.1 Krull Dimension

Definition 2.1.1. *Let $(S, \leq)$ be a partially ordered set. A chain $\mathfrak{C}$ of S is a totally ordered subset of S.*

Let R be a ring. The set of prime ideals of R is a partially ordered set with respect to set theoretic inclusion. In the classical case one writes $\dim(R) = \infty$ for any ring in which is not finite-dimensional. However, this does not differentiate between rings with uncountable chains of prime ideals and infinite-dimensional rings with at most countable chains of prime ideals. In the literature ([30],[32]), the following generalization has been made which does differentiate these two cases:

Definition 2.1.2. *Let R be a ring and $\mathfrak{C}$ be an arbitrary chain of prime ideals in R. Then length of $\mathfrak{C}$ is defined by $|\mathfrak{C}| - 1$. The cardinal Krull dimension of R is the largest cardinal number α (if any) such that there exists a chain of prime ideals in R whose length is equal to α.*

This definition agrees with the classical one for finite-dimensional rings. However, we give an example of a ring whose cardinal Krull dimension does not exist. We first need some preliminary material.

Definition 2.1.3. *An rng is an algebraic structure that satisfies all the axioms of a ring, but does not have an identity.*

All rngs considered will be commutative. We can define prime ideals and cardinal Krull dimension for rngs the same way as for rings.

Example 2.1.4. *Any ideal of a ring is an rng.*

Example 2.1.5. *Let $\{R_i\}_{i \in I}$ be a collection of rings. Then $R = \bigoplus_{i \in I} R_i$ is a rng with componentwise addition and multiplication.*

Proposition 2.1.6. *Let R be as in the previous example. The proper prime ideals of R are of the form $\bigoplus_{i \in I} P_i$ where for some j, P_j is a proper prime ideal of R_j and for $i \neq j$, $P_i = R_i$.*

Proof. It is clear that any ideal of R of the form stated above is prime. Let P be a prime ideal of R. For $k \in I$, let e_k be the element of R whose kth coordinate is 1 and all other coordinates are 0. Now $e_j \notin P$ for some $j \in I$. Since $e_j e_i = 0 \in P$, $e_i \in P$ for $i \neq j$. Thus $\bigoplus_{i\neq j} R_i \subseteq P$. Let $\pi_j : S \to R_j$ be the canonical projection. Then $\pi_j(P)$ is a prime ideal of R_j and $P = \pi_j(P) \bigoplus (\bigoplus_{i\neq j} R_i)$. □

Proposition 2.1.7 ([5], p. 209)**.** *Any rng can be embedded in a ring with unity.*

Proof. Let R be a rng and $R^* = R \times \mathbb{Z}$ For (r_1, n_1) and $(r_2, n_2) \in R^*$, define the following:

$$(r_1, n_1) + (r_2, n_2) = (r_1 + r_2, n_1 + n_2).$$

and

$$(r_1, n_1) \cdot (r_2, n_2) = (r_1 r_2 + n_2 r_1 + n_1 r_2, n_1 n_2).$$

These operations make R^* into a ring. The identity of R^* is $(0, 1)$. The map $\phi \colon R \to R^*$ defined by $\phi(r) = (r, 0)$ is an embedding. □

Note that $\phi(R)$ is an ideal of R^* and $R/\phi(R) \cong \mathbb{Z}$. Also, R^* is commutative. It is called the unitalization of R.

Definition 2.1.8. *Let A, B and C be subsets of a ring R. Then*

$$[A : B]_C = \{x \in C : xa \in B \ \forall \ a \in A\}.$$

Let A be a proper ideal of the ring R and P a prime ideal of A. It is straightforward to prove the following:

1. $[P : A]_R$ is the unique prime ideal of R lying over P.
2. If $P_1 \subset P_2$, then $[P_1 : A]_R \subset [P_2 : A]_R$.
3. Suppose that Q is a prime ideal of R that does not contain A (so $Q \cap A$ is a prime ideal of A). If Q' is a prime ideal of R with $Q' \subset Q$, then $Q' \cap A \subset Q \cap A$.

Theorem 2.1.9. *There exists a ring whose cardinal Krull dimension does not exist.*

Proof. Let K be a field and for each $n > 0$ let K_n be the polynomial ring over K in n variables. We will prove in Corollary 3.1.21 that the classical dimension of K_n is n. Let $R = \bigoplus_{i=1}^{\infty} K_i$. Since for every n there exists a chain of prime ideals of R of length n and there are no infinite chains, the cardinal Krull dimension of R does not exist. Let R^* and ϕ be as in Proposition 7. By point 2, every chain of prime ideals of R lifts to a chain of prime ideals of R^* of the same length. Let $\mathfrak{C}$ be a chain of prime ideals of R^* of length α. Since $R^*/\phi(R) \cong \mathbb{Z}$, at most two prime ideals of $\mathfrak{C}$ can contain R. It follows from point 3 that α is finite and that the cardinal Krull dimension of R^* does not exist. □

We want a generalization of classical dimension that is defined for all rings and agrees with the classical one for finite-dimensional rings. Taking into consideration a suggestion from mathoverflow,[2] we make the following definition of Krull dimension:

Definition 2.1.10. *Let R be a ring. The Krull dimension of R is the smallest cardinality α, such that there are no chains of prime ideals of R of length $\geq \alpha$. We write* $\dim(R) = \alpha$. *If $\alpha = n$ is finite, we also say that R is n-dimensional.*

[2] http://mathoverflow.net/questions/208424/

Any set of cardinals is well-ordered ([13], p. 210). Thus the Krull dimension always exists. Note that this definition agrees with the classical dimension for finite-dimensional rings and it agrees with the cardinal Krull dimension when it exists. From now on we will simply refer to Krull dimension as dimension.

Example 2.1.11. *Let R be the rng of example 5. Then* $\dim(R^*) = \aleph_0$.

2.2 Integral Ring Extensions

Definition 2.2.1. *Let $R \subseteq S$ be a ring extension.*

1. *An element $s \in S$ is called integral over R if there exists a monic polynomial $f \in R[X]$ such that $f(s) = 0$. If every $s \in S$ is integral over R, then S is said to be integral over R.*

2. *The integral closure of R in S is the set of elements of S that are integral over R.*

3. *The ring R is said to be integrally closed in S if R is equal to its integral closure in S. An integral domain is called integrally closed if if it is integrally closed in its field of fractions.*

It can be shown that the integral closure of R in S is a subring of S.

Theorem 2.2.2 (Incomparability Theorem)**.** *Let S be integral over R and let Q_1,Q_2 be prime ideals of S that lie over the prime ideal P of R. If $Q_1 \subseteq Q_2$, then $Q_1 = Q_2$.*

A proof of this result can be found in [16] on page 131.

Theorem 2.2.3 (Going Up Theorem)**.** *Let S be integral over R and let $\{P_i\}_{i\in I}$ be an arbitrary chain of prime ideals of R. Then there exists a chain $\{Q_i\}_{i\in I}$ of prime ideals of S such that $Q_i \cap R = P_i$ for all $i \in I$.*

A proof of this can be found in [14] on page 3892. Note that this theorem is more general than the Going Up Theorem usually encountered in commutative algebra texts, in which only finite chains of prime ideals are considered.

Theorem 2.2.4. *Let S be integral over R. Then* $\dim(S) = \dim(R)$.

Proof. Let $\{Q_i\}_{i\in I}$ be a chain of prime ideals of S. By the Incomparability Theorem, $\{Q_i \cap R\}_{i\in I}$ is a chain of prime ideals with the same cardinality as I. Thus $\dim(S) \leq \dim(R)$. The reverse inequality follows from the Going Up Theorem. □

2.3 Localization

Let R be a ring and S a multiplicatively closed subset of R (we will always assume $1 \in S$ and $0 \notin S$). Define an equivalence relation on $R \times S$ by $(a, s) \sim (b, t)$ if and only if $u(at - bs) = 0$ for some $u \in S$. Denote the set of equivalence classes of $\sim$ by $S^{-1}R$. Then $S^{-1}R$ is a ring with multiplication and addition defined just like multiplication and addition in the field of fractions of an integral domain. The equivalence class determined by (a, s) will be denoted by a/s.

Definition 2.3.1. *The ring $S^{-1}R$ is called the localization of R at S.*

Note that in some older texts, such rings are called quotient rings. When we refer to quotient rings, it will be the usual current definition and not localizations.

There is a natural map $\pi \colon R \to S^{-1}R$ defined by $\pi(r) = (r/1)$. Unlike the case of the field of fractions of an integral domain, π is not necessarily an injection. In general for a ring R and multiplicatively closed subset S, the natural map $\pi\colon R \to S^{-1}R$ is an injection if and only if S contains no zero divisors.

The proper prime ideals of $S^{-1}R$ are precisely the extensions of prime ideals of R which do not intersect S. For such a prime ideal Q, we have

$$QS^{-1}R = \{q/s : q \in Q\}.$$

Let P be a prime ideal of R. Then $S = R - P$ is multiplicatively closed. We denote $S^{-1}R$ by R_P and say it is the localization of R at P (instead of the localization of R at $R - P$). R_P is a local ring with maximal ideal PR_P. In a local ring, the set of units are the elements that are not in the maximal ideal. For a particular example of a localization at a prime, consider the ring $\mathbb{Z}$ and any prime number p. Then $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} : p \nmid b\}$.

Definition 2.3.2. *Let R be a ring and P a prime ideal of R. The height of P, denoted by* $\mathrm{ht}(P)$, *is the dimension of R_P.*

Localization is an important concept in commutative algebra and we will be drawing on it heavily.

2.4 Valuation Rings

Totally Ordered Abelian Groups and Valuation Rings

Definition 2.4.1. *An Abelian group Γ that is a totally ordered set under a binary relation $\leq$ is called a totally ordered Abelian group provided it satisfies the following condition: $\alpha \leq \beta$ implies $\alpha + \gamma \leq \beta + \gamma$ for all α,β,$\gamma \in \Gamma$.*

Example 2.4.2. *$\mathbb{Z}$ with the usual ordering is a totally ordered Abelian group.*

Example 2.4.3. *([19], p. 62) Let $\{\Gamma_i : i \in I\}$ be a family of totally ordered Abelian groups. In the Cartesian product $\prod_{i\in I} \Gamma_i$, consider all vectors $\alpha = (\dots, \alpha_i, \dots)$ such that the support of α ($\{j \in I : \alpha_j \neq 0\}$) is well-ordered in the ordering of I. These vectors form a subgroup of the Cartesian product called the Hahn product $H\Gamma_i$ of the family $\{\Gamma_i : i \in I\}$. $H\Gamma_i$ becomes a totally ordered Abelian group if one defines $\alpha = (\dots, \alpha_i, \dots) > 0$ whenever for the first element j in the well-ordering of the support of α, one has $\alpha_j > 0$ in Γ_j. If I is well-ordered, the Hahn product is often called the lexicographic product.*

Proposition 2.4.4. *Let K be a field, $K^\times$ be the multiplicative group of nonzero elements of K, and R a subring of K. The following are equivalent:*

1. *For every $x \in K^\times$, either $x \in R$ or $x^{-1} \in R$.*
2. *The ideals of R are totally ordered by inclusion.*
3. *The principal ideals of R are totally ordered by inclusion.*
4. *There is a totally ordered Abelian group G (called the value group) and a surjective group homomorphism (called the valuation) $v\colon K^\times \to G$ such that $v(x+y) \geq \min(v(x), v(y))$ for $x \neq -y$ and*
$$R = \{x \in K^\times : v(x) \geq 0\} \cup \{0\}.$$

Proof. (1) $\implies$ (2) Suppose that I and J are ideals of R with $I \not\subseteq J$ and $J \not\subseteq I$. Let $x \in J - I$ and $y \in I - J$. Then $x/y \notin D$ and $y/x \notin D$.

(2) $\implies$ (3) is obvious.

(3) $\implies$ (4) Let U be the group of units of R. Consider the factor group $G = K^\times/U$. Define a binary relation $\leq$ on G as follows: $xU \leq yU$ if and only if $y/x \in R$. If (3) holds, then so does (1). It follows that $\leq$ is well-defined. It is clear that $\leq$ is reflexive, antisymmetric, and transitive. It follows again from (1) that $\leq$ is a total order. Furthermore, $aU \leq bU$ implies $aUcU \leq bUcU$. Let $v\colon K^\times \to G$ be the natural epimorphism. It is clear that $v(x+y) \geq \min(v(x), v(y))$ for $x \neq -y$ and that $R = \{x \in K^\times : v(x) \geq 0\} \cup \{0\}$.

(4) $\implies$ (1) Let $x \in K^\times - R$. Then $v(x^{-1}) = -v(x) > 0$. Hence $x^{-1} \in R$. □

Definition 2.4.5. *If R satisfies any of the above we say that R is a valuation ring. Note that* $\mathrm{Quot}(R) = K$.

In [19], a valuation ring is defined as a ring (not necessarily an integral domain) that satisfies condition (2) of the proposition, and a valuation domain is a valuation ring that is an integral domain. Other authors use valuation ring for both cases. We will use the two terms interchangeably and a ring that satisfies condition (2) will be referred to as a chained ring (following [36]). Some authors extend the definition of a valuation and define $v(0) = \infty$ but we will not use this convention.

Example 2.4.6. *One way of constructing a valuation ring is to define a function v on an integral domain R satisfying the conditions above and then extending v to* $\mathrm{Quot}(R)$ *in a natural way: $v(a/b) = v(a) - v(b)$. Then v is a valuation on* $\mathrm{Quot}(R)$ *and R is a valuation ring. Given a field K, consider $K[[X]]$, the ring of formal power series over K. Let $g(X) = \sum_{i=0}^{\infty} a_i X^i \in K[[X]]$. If a_j is the least integer with $a_j \neq 0$, let $v(g) = j$. Then $v\colon K((X))^\times \to \mathbb{Z}$ is a valuation. Note that $K[[X]] = K + \langle X \rangle$ and that $\langle X \rangle$ is the maximal ideal of $K[[X]]$.*

Example 2.4.7. *Let R be a valuation ring and I a ideal of R that is not prime. Then R/I is a chained ring that is not a valuation ring.*

Definition 2.4.8. *Let R be a local ring with maximal ideal M. The field R/M is called the residue field of R.*

Note that if R is a valuation ring, then $M = \{x \in R : v(x) > 0\}$.

Theorem 2.4.9 (Krull)**.** *Given a field K and a totally ordered Abelian group Γ, there exists a valuation domain R with residue field isomorphic to K and with value group order-isomorphic to Γ.*

Proof. One way of proving this is by generalizing the example of $K[[X]]$. See [19], Theorem 3.8 for the details. □

Convex Subgroups

Definition 2.4.10. *Let G be a totally ordered Abelian group. A subgroup H of G is said to be convex (or isolated) if $a < b < c$ with $a, c \in H$, $b \in G \implies b \in H$.*

Example 2.4.11 ([19], Example 2.3)**.** *Consider the lexicographic product $\Gamma_n = H_{i=1}^n G_i$ where $G_i = \mathbb{Z}$ for all i. For $1 \leq i \leq n$ define*

$$H_i = \{(0, \ldots, 0, k_i, \ldots, k_n) : k_j \in \mathbb{Z}\ i \leq j \leq n\}.$$

Then H_i is a convex subgroup of Γ_n. In fact these are the only nonzero convex subgroups of Γ_n.

Let R be a valuation ring with value group G. Note that the set of proper convex subgroups of G are totally ordered. The cardinality of the set of proper convex subgroups of G is called the rank of G. In [10], Theorem 7.1 (ii), it is shown that there is an inclusion reversing bijection between the nonzero prime ideals of R and the convex subgroups of G, which is defined as follows: For a prime ideal P of R, let $P^* = \{g \in G : -v(x) < g < v(x) \ \forall \ x \in P\}$. Then P^* is convex. If H is convex, let $H^* = \{x \in R : v(x) > \max(h, -h) \ \forall \ h \in H\}$. Then H^* is prime. Thus the rank of G is equal to the dimension of R.

Consider the previous example. The rank of Γ_n is n. Thus the dimension of any valuation ring with value group Γ_n is n.

Example 2.4.12. *It was pointed out on mathoverflow*[3] *that there exist rings with countably infinite chains of prime ideals but no uncountable chains: Let* $\Gamma = H_{i=1}^{\infty} G_i$ *where* $G_i = \mathbb{Z}$ *for all* i*. Any valuation ring* V *with value group* Γ *has a countable chain of prime ideals but does not have an uncountable chain. This can be easily seen by using convex subgroups and generalizing the previous example. For* $i \geq 1$ *let* $H_i = \{(0, \dots, 0, k_i, k_{i+1}, \dots) : k_j \in \mathbb{Z} \ i \leq j\}$*. Then the* H_i *are precisely the convex subgroups of* Γ*. The result follows.*

Discrete Rank 1 Valuation Rings

Theorem 2.4.13. *The following properties of a ring are equivalent:*

1. R is a valuation ring with value group $\mathbb{Z}$.
2. R is a PID with a unique maximal ideal $P \neq 0$.
3. R is a Noetherian integral domain that is also a local ring whose unique maximal ideal is nonzero and principal.
4. R is a Noetherian, integrally closed, integral domain that is a local ring of dimension 1.

A proof of this theorem can be found in [15] on pages 757–758.

Definition 2.4.14. *A ring* R *satisfying any of the above properties is called a discrete rank* 1 *valuation ring. We will write* R *is a rank* 1 *DVR.*

In many algebra texts, these rings are simply called discrete valuation rings. However, we will use later on a more general definition of discrete valuation rings. Examples of rank 1 DVRs include $\mathbb{Z}_{(p)}$ and $K[[X]]$.

2.5 Dedekind Domains

Throughout this subsection, R is an integral domain with fraction field $K \neq R$.

Definition 2.5.1. *A fractional ideal of* R *is an* R*-submodule* A *of* K *such that* $dA \subseteq R$ *for some nonzero nonzero* $d \in R$*.*

If A and B are fractional ideals then the product AB is defined to be the set of all finite sums of elements of the form ab where $a \in A$ and $b \in B$. It is straightforward to show that the product of fractional ideals is a fractional ideal.

Definition 2.5.2. *A fractional ideal* A *is said to be invertible if there exists a fractional ideal* B *with* $AB = R$*. We say* B *is the inverse of* A *and write* $B = A^{-1}$*.*

[3] http://mathoverflow.net/questions/217651/

If A is invertible, then $A^{-1} = \{x \in K : xA \subseteq R\}$. It follows easily from this fact that an invertible ideal is finitely generated.

Theorem 2.5.3. *The following are equivalent:*

1. The ring R is Noetherian, integrally closed, and has dimension 1.
2. The ring R is Noetherian and for each nonzero prime P of R the localization R_P is a rank 1 DVR.
3. Every nonzero fractional ideal of R in K is invertible.
4. Any nonzero ideal I of R has a unique factorization $I = P_1^{r_1} \cdots P_n^{r_n}$ where the P_i are distinct prime ideals and the r_i are nonzero integers.

A proof of this theorem can be found in [15] on pages 765–766.

Definition 2.5.4. *A ring satisfying any of the conditions above is called a Dedekind domain or a Dedekind ring.*

Definition 2.5.5. *Let L be an extension field of $\mathbb{Q}$.*

1. *An element $k \in L$ is called an algebraic integer if k is integral over $\mathbb{Z}$.*
2. *The integral closure of $\mathbb{Z}$ in L is called the ring of integers of L.*
3. *L is said to be an algebraic number field if it is of finite degree over $\mathbb{Q}$.*

Any PID is a Dedekind domain. The ring of integers in an algebraic number field is a Dedekind domain.

2.6 Ultrafilters

Some methods of producing uncountable chains of prime ideals use ultrafilters.

Definition 2.6.1. *Let X be a set. A filter on X is $F \subseteq P(X)$ such that:*

1. *$X \in F$.*
2. *$\emptyset \notin F$.*
3. *If $A \in F$ and $A \subseteq B$, then $B \in F$.*
4. *If $A, B \in F$, then $A \cap B \in F$.*

Example 2.6.2. *Let X be an infinite set and $\mathcal{F} = \{A \subseteq X : X - A$ is finite$\}$. Then $\mathcal{F}$ is a filter, called the cofinite filter.*

Definition 2.6.3. *A filter F on X is an ultrafilter if for any $A \subseteq X$, $A \in F$ or $X - A \in F$.*

Example 2.6.4. *Let $x \in X$. Let $\mathcal{F} = \{A \subseteq X : x \in A\}$. This is an ultrafilter on X, called the principal filter generated by x.*

The cofinite filter on an infinite set is not an ultrafilter.

Proposition 2.6.5. *Every filter is contained in an ultrafilter.*

This can be proved using a Zorn's Lemma argument.

Corollary 2.6.6. *Any infinite set has a nonprincipal ultrafilter.*

Proof. The cofinite filter on an infinite set is not an ultrafilter and is not contained in any principal ultrafilter. □

Theorem 2.6.7. *Let $\{K_i : i \in I\}$ be a collection of fields. The prime ideals in the ring $\prod_{i\in I} K_i$ are in bijection with the ultrafilters on I. The ultrafilter F corresponds to the prime ideal consisting of elements of the form (a_i), where the set of indices i such that $a_i = 0$ is in F.*[4]

Theorem 2.6.8. *The set of ultrafilters on an infinite set is uncountable.*[5]

Corollary 2.6.9. *The set of prime ideals of an infinite direct product of fields is uncountable.*

Ultraproducts

Definition 2.6.10. *Let $\{A_i\}_{i\in I}$ be a collection of sets and $\mathcal{F}$ an ultrafilter on I. Let $f,g \in A = \prod_{i\in I} A_i$. Define $f \sim g$ if the set $\{i \in I : f(i) = g(i)\} \in \mathcal{F}$. Then $\sim$ is an equivalence relation on A. The ultraproduct $\prod_{\mathcal{F}} A_i$ is defined as the set of equivalence classes of $\sim$.*

Definition 2.6.11. *An ultrafilter $\mathcal{F}$ is countably incomplete if $\mathcal{F}$ has a countable subset G such that $\bigcap G = \emptyset$.*

Example 2.6.12. *Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Each set $A_i = \{i, i+1, \dots\}$ is in $\mathcal{U}$ and $\bigcap_{i\in\mathbb{N}} A_i = \emptyset$. Thus $\mathcal{U}$ is countably incomplete.*

Proposition 2.6.13. *Let $\mathcal{F}$ be a countably incomplete ultrafilter on I. Then $\prod_{\mathcal{F}} A_i$ is either finite or $\mathrm{Card}(\prod_{\mathcal{F}} A_i) \geq \aleph_1$.*[6]

Example 2.6.14. *Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Let $A_n = \{0, 1, \dots, n\}$ and $A = \prod_{n\in\mathbb{N}} A_n$. An element of A is given by a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(i) \leq i$ for all $i \in \mathbb{N}$. We claim that $\prod_{\mathcal{U}} A_i$ is not finite. Arguing by contradiction, suppose that $f_1, \dots, f_n$ are representatives for the distinct equivalence classes of $\sim$. For $t > n$ let $a_t \in A_t - \{f_1(t), \dots, f_n(t)\}$. Define $g \in A$ by $g(t) = 0$ for $t \leq n$ and $g(t) = a_t$ for $t > n$. Then g is not in the equivalence class of any f_i. This is a contradiction. By the previous proposition, we have $\mathrm{Card}(\prod_{\mathcal{U}} A_i) \geq \aleph_1$.*

[4] https://math.berkeley.edu/~kruckman/ultrafilters.pdf
[5] http://math.stackexchange.com/questions/83526/
[6] https://www.math.wisc.edu/~keisler/ultraproducts-web-final.pdf

3 Polynomial Rings

Notation

1. Let R and S be rings with $R \subseteq S$. Let A be a subset of S. Then we denote the subring of S generated by A and R by $R[S]$. When $A = \{a_1, \ldots, a_n\}$, we write $R[a_1, \ldots, a_n]$. Capital letters will denote variables, so that $R[X_1, \ldots, X_n]$ is the polynomial ring in n variables over R. More generally, $R[\{X_i\}_{i \in I}]$ is the polynomial ring over R with set of variables $\{X_i\}_{i \in I}$.

2. Let P be a prime ideal of R. Then $PR[X]$ is a prime ideal of $R[X]$. It consists of the polynomials which have all coefficients in P. For that reason we will denote the ideal by $P[X]$.

3. Let T be a subset of a ring S and P a prime ideal of S. Let $a \in R$. Denote the class of a in R/P by $\overline{a}$ and let $\{\overline{t} : t \in T\} = \overline{T}$.

3.1 Finite-dimensional Rings

In this section, we will prove many results in detail, often to a greater extent than in the papers from which they were obtained. Where necessary, we will replace older terminology with current terminology.

Dimension of $R[X]$ in terms of Dimension of R

Proposition 3.1.1 ([34], Theorem 1). *Let R be ring and let P_1, P_2, P_3 be distinct ideals in $R[X]$. If $P_1 \subset P_2 \subset P_3$ and P_2 is prime, then P_1, P_2, P_3 cannot all have the same contraction to R.*

Proof. If $P_1 \cap R \neq P_2 \cap R$ then there is nothing to prove. So let $P_1 \cap R = P_2 \cap R = p$ and consider $R[X]/P_2$. Let $f \in R[X]$. Then $f(x) = \sum a_i x^i + \sum b_j x^j$ where $a_i \notin P$ for all i and $b_j \in p$ for all j. So $\overline{f(x)} = \sum \overline{a_i} \cdot \overline{x}^i$. Thus $R[X]/P_2 = \overline{R}[\overline{x}]$.

We claim that $\overline{x}$ is algebraic over $\overline{R}$. Let $h(x) = \sum a_k x_k \in P_2$, $h \notin p[X]$ ($p[X] \subseteq P_1 \subset P_2$). Consider $l(x) = \sum \overline{a_k} x^k$. Now l is nonzero and $l(\overline{x}) = \overline{l(x)} = 0$. Thus $\overline{x}$ is algebraic over $\overline{R}$. Equivalently, $\overline{R}[\overline{x}]$ is an algebraic extension of $\overline{R}$.

Now let $g \in P_3$, $g \notin P_2$. Then $\overline{g} \in \overline{P_3}$, $\overline{g} \neq 0$. For some $c_i \in R$, $\sum_{i=0}^{n} \overline{c_i} \cdot \overline{g}^i = 0$. Since $R[X]/P_2$ is an integral domain, we can assume $\overline{c_0} \neq 0$. Note that $\overline{c_0} \in \overline{P_3} \cap \overline{R}$. Thus $c_0 \in P_3 \cap R$, $c_0 \notin p$ and hence $P_3 \cap R \neq p$. ☐

Corollary 3.1.2. *R is finite-dimensional if and only if $R[X_1, \ldots, X_n]$ is finite dimensional for all $n > 0$.*

For the rest of this section, all rings will be finite-dimensional.

Corollary 3.1.3. *If R is 1-dimensional, and P_1, P_2, P_3 are distinct prime ideals in $R[X]$ different from $\langle 0 \rangle$ with $P_1 \subset P_2 \subset P_3$, then $P_1 \cap R = \langle 0 \rangle$, P_2 is the extension of its contraction to R, and P_3 is maximal.*

Proposition 3.1.4 ([34], Theorem 2). *If R is n-dimensional, then $R[X]$ is at least $(n+1)$-dimensional and at most $(2n+1)$-dimensional.*

Proof. Let $P_0 \subset \cdots \subset P_n$ be a chain of prime ideals in $R[X]$. Then $P_0[X] \subset \cdots \subset P_n[X] \subset \langle P_n[X], X \rangle$ is a chain of prime ideals in $R[X]$. Hence $R[X]$ is at least $(n+1)$-dimensional.

Now let R be n-dimensional and let $R[X]$ be $2m$ or $2m+1$ dimensional. Let $Q_0 \subset \cdots \subset Q_{2m}(\subset Q_{2m+1})$ be a maximal chain of prime ideals in $R[X]$. By Proposition 1, $\{Q_{2i}\}_{i=0}^{m}$ is a chain of prime ideals of R of length m. Thus $R[X]$ is at most $(2n+1)$-dimensional. □

Corollary 3.1.5. *If R is 1-dimensional, then $R[X]$ is either 2 or 3-dimensional.*

Definition 3.1.6. *A 1-dimensional ring R is a B-ring if $R[X]$ is 3-dimensional.*

Theorem 3.1.7 ([34], Theorem 3)**.** *If R is an n-dimensional integral domain but $R[X]$ is not $(n+1)$-dimensional, then for at least one minimal nonzero prime ideal p of R either the localization R_p is a B-ring or R/p is m-dimensional and $R/p[X]$ is not $(m+1)$-dimensional, and $m < n$.*

Proof. Suppose for some minimal nonzero prime p of R, $p[X]$ is not minimal nonzero in $R[X]$. Then there exists a prime ideal P such that $\langle 0\rangle \subset P \subset p[X]$. I claim that $\langle 0\rangle \subset PR_p[X] \subset pR_p[X]$ is a chain of prime ideals in $R_p[X]$. Let $f \in p[X]$, $f \notin P$. Then $f \in pR_p[X]$. If $f \in PR_p[X]$, then for some $s \in R - P$, $sf \in P$. Since P is prime and $f \notin P$, $s \in P$. But s is a unit in R_p, and this contradicts the fact that P is proper. Thus $f \notin PR_p[X]$ and $PR_p[X] \subset pR_p[X]$. Observe that $pR_p[X]$ is prime because pR_p is a prime ideal of R_p. Now suppose $f \cdot g \in PR_p[X]$. Then for some s,$t \in R-p$ we have sf and $tg \in R[X]$ and $sf \cdot tg \in P$. Then sf or $tg \in P$. So $sf \in PR_p[X]$ or $tg \in PR_p[X]$. Since s and t are units, $f \in PR_p[X]$ or $g \in PR_p[X]$. Thus $PR_p[X]$ is prime. It follows that

$$\langle 0\rangle \subset PR_p[X] \subset pR_p[X] \subset \langle pR_p[X], X\rangle$$

is a chain of prime ideals in $R_p[X]$. This shows that R_p is a B-ring.

Now consider the case in which $p[X]$ is minimal nonzero for every minimal nonzero prime ideal p of R. Note that for a minimal nonzero prime ideal p, R/p is at most $(n-1)$-dimensional. Let $\langle 0\rangle \subset P_1 \subset \ldots P_{n+2}$ be a chain of prime ideals in $R[X]$. Suppose that $P_1 \cap R = p \neq \langle 0\rangle$. Without loss of generality, we can assume p is minimal nonzero. We have

$$\langle 0\rangle \subset p[X] \subset P_2 \subset \cdots \subset P_{n+2}.$$

$R/p[X] \cong R[X]/p[X]$, so $R/p[X]$ is at least $(n+1)$-dimensional. Suppose that $P_1 \cap R = \langle 0\rangle$. Then $P_2 \cap R = p_2 \neq \langle 0\rangle$. Let p be a minimal nonzero prime contained in p_2. Then $\langle 0\rangle \subset p[X] \subset P_2$ since $p[X]$ is minimal nonzero but P_2 is not. Replacing P_1 by $p[X]$, we reduce to the previous case and the result follows. □

Corollary 3.1.8. *If R is an integral domain that is a B-ring, then so is some localization R_p of R.*

Proof. Let p be as in the statement of Theorem 7. Then if R/p is 0-dimensional, $R/p[X]$ is 1-dimensional. Since the second condition of the theorem does not hold, we must have that R_p is a B-ring. □

Proposition 3.1.9 ([34], Theorem 4)**.** *If R is a 1-dimensional valuation ring, then $R[X]$ is 2-dimensional.*

Proof. Arguing by contradiction, suppose that $R[X]$ is 3-dimensional. Then there is a chain of prime ideals

$$\langle 0\rangle \subset P_1 \subset P_2 \subset P_3.$$

Since R is 1-dimensional, by Corollary 3 $P_2 = q[X]$ for some prime ideal q of R and $P_1 \cap R = \langle 0\rangle$. Let $f \in P_1$, $f \neq 0$. Let c be a coefficient of f of least value. Then $f = cg$ where g has at least one coefficient equal to 1. Thus $g \notin P_1$ and so we must have $c \in P_1$. It follows that $P_1 \cap R \neq \langle 0\rangle$. This is a contradiction. Thus $R[X]$ is 2-dimensional. □

Theorem 3.1.10 ([34], Theorem 6)**.** *If R is integrally closed with only one maximal ideal p, α an element of the quotient field of R, and $\alpha^{-1} \notin R$, then $pR[\alpha]$ is prime. If also $\alpha \notin R$, then $pR[\alpha]$ is not maximal.*

Proof. If $\alpha \in R$, then $R[\alpha] = R$ and so $pR[\alpha] = R$. Now assume $\alpha \notin R$. Consider the epimorphism $\phi\colon R[X] \to R[\alpha]$ defined by $\phi(f) = f(a)$. Note that $\phi(p[X]) \subseteq pR[\alpha]$. Thus ϕ induces an epimorphism

$$\overline{\phi}\colon R[X]/p[X] \to R[\alpha]/pR[\alpha].$$

To show that $\overline{\phi}$ is an isomorphism, it suffices to prove that $\phi(p[X]) = pR[\alpha]$.

First, note that $\langle pR[\alpha], \alpha \rangle \neq R[\alpha]$, since equality would imply that α^{-1} is integral over R. Let $g \in R[X]$ be a monic polynomial of positive degree. Then $g(\alpha) = c \in R$ is impossible, as $g(\alpha) - c$ would be an equation of integral dependence for α over R. In particular, $g(\alpha) \neq 0$. Also, $g(\alpha)^{-1} \notin R$, since otherwise it would be a nonunit in R and hence an element of p. This would imply that $1 \in pR[\alpha]$ and α^{-1} is integral over R, a contradiction. By what we have already shown, $\langle pR[g(\alpha)], g(\alpha) \rangle \neq R[\alpha]$. Since α satisfies $g(x) - g(\alpha) = 0$, $R[\alpha]$ is integral over $R[g(\alpha)]$. We have $\langle pR[g(\alpha)], g(\alpha) \rangle \subseteq q$ for some prime ideal q of $R[g(\alpha)]$. By the Going up Theorem, there is a prime ideal Q of $R[\alpha]$ that lies over q. This implies that $p \subseteq Q$ and $g(\alpha) \in Q$. Thus $\langle pR[\alpha], \alpha \rangle \subseteq Q \subset R[\alpha]$. Since $1 + g(\alpha)$ is monic of positive degree, $\langle pR[\alpha], 1 + g(\alpha) \rangle$ is also proper. Thus $g(\alpha) \notin pR[\alpha]$, for otherwise

$$1 = 1 + g(\alpha) - g(\alpha) \in \langle pR[\alpha], 1 + g(\alpha) \rangle.$$

Also, $g(\alpha) \notin pR[\alpha]$ for $g = 1$.

Now let $g \in R[X]$ be any polynomial not in $p[X]$. Then $g = g_1 + g_2$ where g_2 is in $p[X]$ and no coefficient of g_1 is in p. In particular, this is true for the leading coefficient c of g_1. Note that c is a unit of R and $c^{-1}g_1$ is a monic polynomial in $R[X]$. Thus $c^{-1}g_1(\alpha)$ is not in $pR[\alpha]$. It follows that $g(\alpha)$ is not in $pR[\alpha]$. Thus $\phi(p[X]) = pR[\alpha]$ and $\overline{\phi}$ is an isomorphism. Since $p[X]$ is prime but not maximal, the same is true for $pR[\alpha]$. □

Proposition 3.1.11 ([34], Theorem 7)**.** *Let R be an integrally closed integral domain, p a proper ideal therein, a an element in the quotient field of R, but a and a^{-1} not in R_p. Then $pR[a]$ is prime but not maximal.*

Proof. Let $\phi\colon R_p[X] \to R_p[a]$ be defined as in the previous theorem. Then $\phi(pR_p[X]) = pR_p[a]$. Suppose that $g \in R[X]$ and $g(a) \in pR[a]$. Then there exists $f \in pR_p[X]$ such that $f(a) = g(a)$. Then $\phi(f - g) = (f - g)(a) = 0 \in pR_p[a]$. Thus $f - g \in pR_p[X]$ and so $g \in pR_p[X] \cap R[X] = p[X]$. It follows that $\phi\restriction_{R[X]}(p[X]) = pR[a]$. Hence $pR[a]$ is prime but not maximal. □

Theorem 3.1.12 ([34], Theorem 8)**.** *If R is a 1-dimensional integral domain, then $R[X]$ is 2-dimensional if and only if every localization R_p of the integral closure of R is a valuation ring.*

Proof. By Theorem 2.2.4, we may assume that R is integrally closed. If R is a B-ring, then by Corollary 8, so is R_p for some prime ideal p. By Proposition 9, R_p is not a valuation ring.

Conversely, suppose some R_p is not a valuation ring. Then for some a in the quotient field of R_p, $a \notin R_p$ and $a^{-1} \notin R_p$. Then, by Theorem 11, $p \cdot R[a]$ is prime but not maximal. Thus $R[a]$ is at least 2-dimensional. Consider $\phi\colon R[X] \to R[a]$ defined as in Theorem 10. Then $R[X]/\ker(\phi) \cong R[a]$ and $\ker(\phi)$ is a nonzero prime ideal of $R[X]$. Thus $R[X]$ is 3-dimensional and hence R is a B-ring. □

Definition 3.1.13. *A ring R is of type (n, m) if R is n-dimensional and $R[X]$ is m-dimensional.*

The proofs of the next three theorems are inspired from a combination of ideas from [35], Theorems 1 and 2, and from [20], Proposition 30.15. Theorem 1 and 2 of [35] may be hard to follow for the modern reader and the proof of Proposition 30.15 of [20] uses certain results from exercises in previous chapters.

Theorem 3.1.14. *Let D be an integral domain of type (n, m) with quotient field K and L a proper field extension of K. If V is a DVR containing L and M is the maximal ideal of V, then $D' = D + M$ is of type $(n + 1, m + 2)$.*

Proof. Let v be the valuation on V. Then $M = \{x \in D' : v(x) > 0\}$. M is a prime ideal of D', and is in fact the unique nonzero prime of D'. To see this, let P be a prime ideal of D'. Let $p \in P$ and $m \in M$. Then $v(p) = s \geq 0$ and

$$v(m^{s+1}) = (s + 1) \cdot v(m) > v(p).$$

It follows that $m^{s+1} = rp$ where $r \in M$. Thus $M \subseteq P$. Now $D'/M \cong D$, so D' is $(n + 1)$-dimensional.

The localization D'_M is not a valuation ring. To see this, let $a \in L - K$. Then $a = \frac{am}{m} \in \mathrm{Quot}(D')$. Suppose $a \in D'_M$. Then $a = \frac{d+m}{t+q}$ where $t \neq 0 \in D$ and $q \in M$. We have $at - d \in M$, so $at = d$ and this implies that $a \in K$, a contradiction. The same argument holds for a^{-1} as well, so D'_M is not a valuation ring. By Proposition 9, $D'_M[X]$ is 3-dimensional. It follows from Corollary 3 that $MD'_M[X]$ is not minimal in $D'_M[X]$. Thus $M[X]$ is not minimal in $D'[X]$. Note that

$$D'[X]/M[X] \cong D'/M[X] \cong D[X].$$

Hence $D'[X]$ is at least $(m + 2)$-dimensional.

Let $\langle 0\rangle \subset P_1 \subset \cdots \subset P_s$ be a chain of prime ideals of $D'[X]$ with P_1 minimal nonzero. Then $M[X] \nsubseteq P_1$, and therefore $P_1 \cap R = \langle 0\rangle$. It follows that $M \subseteq P_2 \cap R$ and therefore $M[X] \subseteq P_2$. Hence $s \leq m + 2$ and so $D'[X]$ is $(m + 2)$-dimensional. □

Theorem 3.1.15. *Let D be an integrally closed domain of type (n, m) with quotient field K, and let L be a purely transcendental field extension of K. If V is a DVR of the form $L + M$, then $D' = D + M$ is an integrally closed domain of type $(n + 1, m + 2)$.*

Proof. By the previous theorem, we only need to show that D' is integrally closed. Let $\alpha \in \mathrm{Quot}(V)$ and suppose that α is integral over D'. Let

$$\alpha^s + a_{s-1}\alpha^{s-1} + \cdots + a_0 = 0$$

be an equation of integral dependence. Suppose that $v(\alpha^{-1}) > 0$. Dividing the equation by α^{-1}, we get that

$$0 = v(1) = v(-(a_{s-1}\alpha^{-1} + \cdots + a_0\alpha^{-s})) > 0.$$

Thus $v(\alpha) \geq 0$ and $\alpha \in V$. Since $V/M \cong L$, $\overline{\alpha}$ is integral over K and therefore $\overline{\alpha} \in K$. Since D is integrally closed, $\overline{\alpha} \in D$. Thus $\alpha \in D'$ and D' is integrally closed. □

Theorem 3.1.16. *Let D be an integrally closed domain of type (n, m) with quotient field K. If V is a rank one discrete valuation ring of the form $K + M$, then $D' = D + M$ is an integrally closed domain of type $(n + 1, m + 1)$*

Proof. An argument similar to the proofs of the previous two theorems shows that D' is integrally closed, M is the unique minimal prime ideal of D', D' is $(n + 1)$-dimensional, and $D'[X]$ is at least $(m+1)$-dimensional. We claim that D'_M is a valuation ring. To see this, note that $V \subseteq D'_M$

and $\text{Quot}(V) = \text{Quot}(D') = \text{Quot}(D'_M)$. Since V is a valuation ring, so is D'_M. It follows that $MD'_M[X]$ is minimal in $D'_M[X]$. Thus $M[X]$ is minimal in $D'[X]$. Let $\langle 0\rangle \subset P_1 \subset \cdots \subset P_s$ be a chain of prime ideals of $D'[X]$ with P_1 minimal. If $P_1 \cap D' = M$ then $M[X] = P_1$ and $s \leq m+1$. If $P_1 \cap R = \langle 0\rangle$, then $M \subseteq P_2 \cap R$. Since $M[X]$ is minimal, $M[X] \subset P_2$. Thus $s \leq m+1$ and so $D'[X]$ is $(m+1)$-dimensional. □

The next theorem appears in both [20] and [35], where it is noted that from previous theorems, the result follows by induction. It is possible to give explicit examples of rings with the desired properties. Although they are not "nice" rings to look at, they are not too pathological either.

Theorem 3.1.17. *For every n and m such that $n+1 \leq m \leq 2n+1$ there exist integrally closed rings of type (n, m).*

Proof. Let K_0 be a field and consider $K_0[[X]]$, the valuation ring of Example 2.4.6. For the sake of clarity, denote the maximal ideal of $K_0[[X]]$ by $XK_0[[X]]$. Fix a positive integer n. For each k with $0 \leq k \leq n$, define a tower of fields as follows: Let $K_{k,0} = K_0$ and for $1 \leq j \leq k$ let

$$K_{k,j} = \text{Quot}(K_0 + X_1 K_{k,1}[[X_1]] + \cdots + X_{k,j-1} K_{k,j-1}[[X_{j-1}]]).$$

For $k+1 \leq j \leq n$, let $K_{k,j}$ be a purely transcendental extension of

$$\text{Quot}(K_0 + X_1 K_{k,1}[[X_1]] + \cdots + X_{k,j-1} K_{k,j-1}[[X_{j-1}]]).$$

Let $D_{k,0} = K_0$. For $1 \leq j \leq n$ let $V_{k,j} = K_{k,j}[[X_j]]$, $M_{k,j} = X_j K_{k,j}[[X_j]]$, and

$$D_{k,j} = K_0 + M_{k,1} + \cdots + M_{k,j}.$$

Then $D_{k,j} = D_{k,j-1} + M_{k,j}$. For $1 \leq j \leq k$, $\text{Quot}(D_{k,j-1}) = K_{k,j}$. By the previous theorem,

$$\dim(D_{k,j}) = 1 + \dim(D_{k,j-1}) \text{ and } \dim(D_{k,j}[Y]) = 1 + \dim(D_{k,j-1}[Y]).$$

$D_{k,0}$ is of type $(0,1)$, so $D_{k,k}$ is of type $(k, k+1)$. For $k+1 \leq j \leq n$, $K_{k,j}$ is purely transcendental over $\text{Quot}(D_{k,j-1})$. By Theorem 15,

$$\dim(D_{k,j}) = 1 + \dim(D_{k,j-1})$$

and $\dim(D_{k,j}[Y]) = 2 + \dim(D_{k,j-1}[Y])$. Thus $D_{k,n}$ is of type $(n, 2n-k+1)$. □

Simple Algebraic Extensions

Theorem 3.1.18. *Given a simple algebraic extension $R[a]$ of an t-dimensional ring R, we have $0 \leq \dim R[a] \leq 2t$. For every s and t such that $0 < s \leq 2t$, there exists an t-dimensional integral domain R and an element $a \in \text{Quot}(R)$ such that $R[a]$ is s-dimensional.*

Proof. The result is trivial for 0-dimensional rings and for $s = t$ so let $t > 0$, $s \neq t$. Since $\dim R = t$, $\dim R[X] \leq 2t+1$ and thus $\dim R[a] \leq 2t$. Let $t+1 \leq s \leq 2t$. Then

$$(t-1) + 1 \leq s - 1 \leq 2(t-1) + 1.$$

By Theorem 17, there exists an integrally closed domain D of type $(t-1, s-1)$. Now let D' and M be defined as in Theorem 14. Then D' is of type $(t, s+1)$. Let $a \in \text{Quot}(D')$, but $a \notin D'_M$, $a^{-1} \notin D'_M$. Then by Proposition 11,

$$D'[a]/MD'[a] \cong D'/M[X] \cong D[X].$$

Thus $D'[a]$ is s-dimensional. Hence the result is true for $t+1 \leq s \leq 2t$.

Now consider s with $0 < s < t$. First, we note some general observations. Let S be an integral domain, $q \in S$, and Q a proper nonzero prime ideal of S. Then the following hold:

1. $QS[q^{-1}]$ is a prime ideal of $S[q^{-1}]$. We have $q \in Q$ iff $QS[q^{-1}] = S[q^{-1}]$.
2. Let $Q_1 \subset Q_2$ be prime ideals of S with $q \notin Q_2$. Then $Q_1S[q^{-1}] \subset Q_2S[q^{-1}]$.
3. If P is a prime ideal of $S[q^{-1}]$, then P is the extension of its contraction to S.

Now let R be a valuation ring of dimension t and $\langle 0 \rangle \subset P_1 \subset \cdots \subset P_t$ be the chain of prime ideals in R. Let $c_0 \in P_1$ and for $0 < s < t$ let $c_s \in P_{s+1}$, $c_s \notin P_s$. It follows from the above observations that $\dim(R[c_s^{-1}]) = s$. □

Height and Special Chains

Proposition 3.1.19 ([6], Lemma 1)**.** *Let Q be a prime ideal of $R[X]$ and let $P = Q \cap R$. If $P[X] \subset Q$, then $\mathrm{ht}(Q) = \mathrm{ht}(P[X]) + 1$, and for each integer $n > 1$, $\mathrm{ht}(Q[X_2, \ldots, X_n]) = \mathrm{ht}(P[X_1, \ldots, X_n]) + 1$.*

Proof. To prove the first assertion, we induct on $\mathrm{ht}(P)$. If $\mathrm{ht}(P) = 0$, then $\mathrm{ht}(P[X]) = 0$. Let $P' \subset Q$. We have $P' \cap R \subseteq P$, so $P' \cap R = P$. Then

$$P[X] = P' \cap R[X] \subseteq P' \subset Q,$$

so $P[X] = P'$. Thus $P[X]$ is the unique prime ideal properly contained in Q. Hence $\mathrm{ht}(Q) = 1 = \mathrm{ht}(P[X]) + 1$. Now let $\mathrm{ht}\, P = m$ and assume the result is true for all $k < m$. For any prime ideal $Q' \subset Q$, $\mathrm{ht}(Q') \leq \mathrm{ht}(P[X])$. To see this, first note that if $Q' \cap R = P$, then $Q' = P[X]$. Now suppose that $Q' \cap R \subset P$. Then $\mathrm{ht}(Q') < m$ and by the induction hypothesis, $\mathrm{ht}(Q') = \mathrm{ht}((Q' \cap R)[X]) + 1$. Since $(Q' \cap R)[X] \subset P[X]$, $\mathrm{ht}(Q') \leq \mathrm{ht}(P[X])$. If there are no prime ideals between Q' and Q, then $\mathrm{ht}(Q) = \mathrm{ht}(Q') + 1 \leq \mathrm{ht}(P[X]) + 1$. Hence $\mathrm{ht}(Q) = \mathrm{ht}(P[X]) + 1$.

For the second assertion, we first view $R[X_1, \ldots, X_n]$ as $R[X_1][X_2, \ldots, X_n]$. Then it is not hard to see that $R[X_2, \ldots, X_n] \cap Q[X_2, \ldots, X_n] = P[X_2, \ldots, X_n]$. Next, we view $R[X_1, \ldots, X_n]$ as $R[X_2, \ldots, X_n][X_1]$. Since $P[X] \subset Q$,

$$P[X_2, \ldots, X_n][X_1] \subset Q[X_2, \ldots, X_n].$$

From the first part of the assertion,

$$\mathrm{ht}(Q[X_2, \ldots, X_n]) = \mathrm{ht}(P[X_2, \ldots, X_n][X_1]) + 1.$$

Finally, since $P[X_2, \ldots, X_n][X_1] = P[X_1, \ldots, X_n]$, the second assertion follows. □

Theorem 3.1.20 ([6], Theorem 1)**.** *Let Q be a prime ideal of $R[X_1, \ldots, X_n]$ and $P = Q \cap R$. Then*

$$\begin{aligned} \mathrm{ht}(Q) &= \mathrm{ht}(P[X_1, \ldots, X_n]) + \mathrm{ht}(Q/P[X_1, \ldots, X_n]) \\ &\leq \mathrm{ht}(P[X_1, \ldots, X_n]) + n. \end{aligned}$$

Proof. We induct on the number of variables n. Consider the case $n = 1$. If $Q = P[X]$, then the result is trivial. If $P[X] \subset Q$, then the result follows from the previous theorem. Now suppose $n > 1$ and that the result is true for $n-1$. As in the previous theorem, we consider $R[X_1, \ldots, X_n]$ as $R[X_1][X_2, \ldots, X_n]$. Let $Q_1 = Q \cap R[X_1]$. By the induction hypothesis, we have

$$\begin{aligned} \mathrm{ht}(Q) &= \mathrm{ht}(Q_1[X_2, \ldots, X_n]) + \mathrm{ht}(Q/Q_1[X_2, \ldots, X_n]) \\ &\leq \mathrm{ht}(Q_1[X_2, \ldots, X_n]) + n - 1. \end{aligned}$$

If $P[X_1] = Q_1$, then the result follows immediately. If $P[X_1] \subset Q_1$, then by the previous proposition we have

$$\mathrm{ht}(Q_1[X_2, \ldots, X_n] = \mathrm{ht}(P[X_1, \ldots, X_n]) + 1.$$

It follows that $\mathrm{ht}(Q) \leq \mathrm{ht}(P[X_1, \ldots, X_n]) + n$. Since

$$P[X_1, \ldots, X_n] \subset Q_1[X_2, \ldots, X_n],$$

we have

$$\mathrm{ht}(Q/Q_1[X_2, \ldots, X_n]) + 1 \leq \mathrm{ht}(Q/P[X_1, \ldots, X_n]).$$

Thus

$$\mathrm{ht}(Q) \leq \mathrm{ht}(P[X_1, \ldots, X_n]) + \mathrm{ht}(Q/P[X_1, \ldots, X_n]).$$

The reverse of this last inequality is true in general, so the proof is complete. □

Corollary 3.1.21. *Let K be a field. Then* $\dim(K[X_1, \ldots, X_n]) = n$.

Proof. It is clear that $\dim(K[X_1, \ldots, X_n]) \geq n$. If Q is any proper ideal of $K[X_1, \ldots, X_n]$, then $Q \cap K = \langle 0 \rangle$. Thus, by the previous theorem,

$$\dim(K[X_1, \ldots, X_n]) \leq n.$$

□

Definition 3.1.22. *A chain $\mathcal{C} = \{P_0 \subset \cdots \subset P_n\}$ of prime ideals of*

$$R[X_1, \ldots, X_n]$$

is called a special chain if for each P_i, the ideal $(P_i \cap R)[X_1, \ldots, X_n]$ belongs to $\mathcal{C}$.

Theorem 3.1.23 (Jaffard's Special Chain Theorem)**.** *If Q is a prime ideal of $R[X_1, \ldots, X_n]$ of finite height, then $\mathrm{ht}(Q)$ can be realized as the length of a special chain of prime ideals of $R[X_1, \ldots, X_n]$.*

Proof. We use induction on $\mathrm{ht}(Q)$. If $\mathrm{ht}(Q) = 0$, then Q is the extension of its contraction to R and therefore the singleton $\{Q\}$ is a special chain. Now suppose $\mathrm{ht}(Q) = n > 0$ and that the result is true for $n - 1$. By Theorem 19, there exists a maximal chain $Q_0 \subset \cdots \subset Q_n = Q$ with $Q \cap R = Q_j$ for some j. Note that $\mathrm{ht}(Q_{n-1}) = n - 1$. By the induction hypothesis, $Q_i \cap R$ belongs to the chain $\{Q_0 \subset \cdots \subset Q_{n-1}\}$ for $1 \leq i \leq n - 1$. Thus this maximal chain of Q is a special chain. □

Jaffard's Special Chain Theorem also implies the first part of Theorem 19. This follows from the fact that a special chain terminating at Q contains

$$(Q \cap R)[X].$$

Dimension and Transcendence Degree

Let R and S be integral domains with $R \subset S$. The transcendence degree of S over R is defined as the transcendence degree of $\mathrm{Quot}(S)$ over $\mathrm{Quot}(R)$. Denote it by $\mathrm{trdeg}(S/R)$. Using a Zorn's lemma argument, one can prove that there exists a maximal algebraically independent subset of S over R. It is straightforward to prove that such a subset is also a transcendence base of $\mathrm{Quot}(S)$ over $\mathrm{Quot}(R)$.[7] This will simplify arguments involving transcendence degrees.

[7] http://math.stackexchange.com/questions/1010815/

Theorem 3.1.24 (Noether's Normalization Lemma)**.** *Let K be a field and suppose that $A = K[r_1, \dots, r_m]$ is a finitely generated K-algebra. Then for some q, $0 \le q \le m$, there are algebraically independent elements $y_1, \dots, y_q \in A$ such that A is integral over $K[y_1, \dots, y_q]$.*

For a proof, see [15], pages 699–700.

Proposition 3.1.25. *Let K be a field and suppose that $A = K[r_1, \dots, r_m]$ is an affine K-domain. Then* $\dim(A) = \operatorname{trdeg}(A/K)$.

Proof. Let y_q, $0 \le q \le m$ be as in the previous theorem. Then $\{y_1, \dots, y_m\}$ is a transcendence base of A over K. Thus $\operatorname{trdeg}(A/K) = m$. Since A is integral over $K[y_1, \dots, y_m]$, $\dim(A) = m$. □

Proposition 3.1.26. *Let K be a field an A be a K-algebra that is an integral domain. Then* $\dim(A) \le \operatorname{trdeg}(A/K)$.

Proof. This proof is taken from mathoverflow.[8] Let $P_0 \subset \cdots \subset P_m$ be a maximal chain of prime ideals of A. For $1 \le i \le m$, let $x_i \in P_i - P_{i-1}$. Consider $B = K[x_1, \dots, x_m]$. Since B is finitely generated over K, by the previous proposition $\dim(B) = \operatorname{trdeg}(B/K) \le \operatorname{trdeg}(A/K)$. The primes $Q_i = P_i \cap B$ form a chain of length m in B because $x_i \in Q_i - Q_{i-1}$. Thus $m \le \operatorname{trdeg}(A/K)$. □

It is possible that $\dim(A) < \operatorname{trdeg}(A/K)$. For example, $\dim(K(X)) = 0$ but $\operatorname{trdeg}(K(X)/K) = 1$. Therefore $K(X)$ is not a finitely generated K-algebra.

Polynomial Rings over B-rings

Theorem 3.1.27. *If R is a 1-dimensional integral domain, then $R[X_1, \dots, X_n]$ is at most $2n+1$-dimensional.*

For a proof, see [35], Theorem 6.

Theorem 3.1.28 ([35], Theorem 7)**.** *If R is an integral domain and an B-ring, then $R[X_1, \dots, X_n]$ is at least $(n+2)$-dimensional and at most $(2n+1)$-dimensional. For any N, $n+2 \le N \le 2n+1$, there is a B-ring R such that $R[X_1, \dots, X_n]$ is N-dimensional.*

Proof. By the previous theorem, $R[X_1, \dots, X_n]$ is at most $(2n+1)$-dimensional. $R[X_1, \dots, X_n]$ is also at least $(n+2)$-dimensional, because $R[X_1, \dots, X_n] \cong R[X_1][X_2, \dots, X_n]$ and $R[X_1]$ is 3-dimensional.

Let K be a field and $K' = K(Y_1, \dots, Y_m)$. Let $g(X) = \sum_{i=0}^{t} a_i X^i \in K'[X]$. If a_j is the least integer with $a_j \neq 0$, let $v(g) = j$. Then v extends to a valuation on $K'(X)$. Let V be the corresponding rank 1 DVR. Then $M = \langle X \rangle$ is the maximal ideal of V. Consider $D = K + M$. Since K is properly contained in K' and V contains K', by Theorem 14 D is a B-ring.

Now we will show that for $m \le n$, $D[X_1, \dots, X_n]$ is $(m+n+1)$-dimensional. Let P_m be the prime ideal of $D[X_1, \dots, X_n]$ consisting of the polynomials that vanish for $X_i = Y_i$, $1 \le i \le m$. We claim that $P_m \subset M[X_1, \dots, X_n]$. Let $f = \sum a_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n} \in P_m$. Then

$$f = \sum c_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n} + \sum m_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n}$$

where $c_{i_1 \cdots i_n} \in K$ and $m_{i_1 \cdots i_n} \in M$. Since f vanishes for $X_i = Y_i$, $1 \le i \le m$ it also vanishes for $X_i = Y_i$, $1 \le i \le m$, $X = 0$. Thus $\sum c_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n}$ vanishes for $X_i = Y_i$, $1 \le i \le m$. Hence $P_m \subseteq M[X_1, \dots, X_n]$. This containment is proper, since for $m \neq 0 \in M$, $m \notin P_m$.

[8] http://mathoverflow.net/questions/79959/

For $1 \leq j < m$ Let P_j be the prime ideal of $D[X_1, \ldots, X_n]$ consisting of the polynomials that vanish for $X_i = Y_i$, $1 \leq i \leq j$. Then

$$\langle 0 \rangle \subset P_1 \subset \cdots \subset P_m \subset M[X_1, \ldots, X_n].$$

Thus $D[X_1, \ldots, X_n]$ is at least $(m + n + 1)$-dimensional. Since the transcendence degree of $D[X_1, \ldots, X_n]$ over K is $m + n + 1$, $D[X_1, \ldots, X_n]$ is at most $(m + n + 1)$-dimensional. □

Noetherian, Semi-hereditary and Prüfer Rings

Proposition 3.1.29 ([6], Corollary 2)**.** *Let $\mathcal{S}$ be a class of rings which is closed under localizations. Suppose that whenever $R \in \mathcal{S}$ and R is a local ring with maximal ideal M, it follows that* $\text{ht}(M) = \text{ht}(M[X_1, \ldots, X_n])$. *Then if $S \in \mathcal{S}$,* $\dim S[X_1, \ldots, X_n] = n + \dim S$.

Proof. Let $R \in \mathcal{S}$ and Q be a maximal ideal of $R[X_1, \ldots, X_n]$ such that $\text{ht}(Q) = \dim(R[X_1, \ldots, X_n])$. Consider the ring $T = R[X_1, \ldots, X_n]_P$, where $P = Q \cap R$. There is a one-to-one correspondence between the prime ideals of T and the prime ideals of $R[X_1, \ldots, X_n]$ whose contraction to R is contained in P. It follows that $\text{ht}(Q) = \text{ht}(QT)$. Since $T \cong R_P[X_1, \ldots, X_n]$,

$$\text{ht}(Q) \leq \dim R_P[X_1, \ldots, X_n] = \dim R_P + n \leq \dim R + n.$$

It is true in general that $\dim R + n \leq \dim R[X_1, \ldots, X_n]$, so the result follows. □

Theorem 3.1.30 (Krull's Height Theorem)**.** *Let R be a Noetherian ring. Let $I = (a_1, \ldots, a_k)$ be a proper ideal of R generated by k elements. Let P be a prime ideal of R that is minimal over I. Then* $\text{ht}(P) \leq k$.

A proof of this theorem can be found in [16], page 233.

Corollary 3.1.31. *Let R be a Noetherian ring, and let P be a prime ideal of R. Let* $\text{ht}(P) = n$. *Then, there are elements $a_1, \ldots, a_n$ such that P is minimal over $(a_1, \ldots, a_n)$.*

For a proof, see [16], page 233.

Theorem 3.1.32. *Let R be a Noetherian ring. Then* $\dim R[X_1, \ldots, X_n] = n + \dim R$.

Proof. If R is Noetherian, so is $R[X]$. Thus it suffices to prove the case $n = 1$. Since Noetherian rings are closed under localization, by Proposition 29, it suffices to prove that $\text{ht}(P) = \text{ht}(P[X])$ for a prime ideal P of R. Note that we do not need to assume that R is a local ring and that P is maximal for the proof. It is clear that $\text{ht}(P) \leq \text{ht}(P[X])$. Let $\text{ht}(P) = k$. By Corollary 31, there is a prime ideal $I = (a_1, \ldots, a_k)$ of R such that P is minimal over I. Then $P[X]$ is minimal over $I[X]$. Now $I[X]$ is generated by $a_1, \ldots, a_k$, so by Krull's Height Theorem $\text{ht}(P[X]) \leq k$. □

Definition 3.1.33. *An R-module is projective if it is a direct summand of a free R-module.*

Definition 3.1.34. *A ring in which every finitely generated ideal is projective is called a semi-hereditary ring.*

An example of a non-Noetherian semi-hereditary ring is a finite direct product of valuation rings, where at least one factor is non-Noetherian.

Definition 3.1.35. *A Prüfer domain is a semi-hereditary integral domain. Equivalently, a Prüfer domain is an integral domain in which every R_p is a valuation ring.*

Let $R = K[X, X^{1/2}, X^{1/3}, \dots, Y, Y^{1/2}, Y^{1/3}, \dots]$. Let $P_1 = \langle X, X^{1/2}, X^{1/3}, \dots \rangle$ and $P_2 = \langle Y, Y^{1/2}, Y^{1/3}, \dots \rangle$. Let $S = R - P_1 \cup P_2$. Then R_S is a 1-dimensional Prüfer domain that is neither Noetherian nor a valuation ring. The only maximal ideals of R_S are $P_1 R_S$ and $P_2 R_S$ ([26], Example 38).

Theorem 3.1.36. *If R is semi-hereditary, then R_p is a valuation ring.*

Proof. Let R be a semi-hereditary ring. We claim that R_p is semi-hereditary. If I is a finitely generated ideal of R_p, then I is the extension of a finitely generated ideal J of R. J is a direct summand of a free R-module. Since localization commutes with direct products, it follows that I is a direct summand of a free R_p-module. Thus I is projective and R_p is semi-hereditary. Every projective module over a local ring is free ([31], Theorem 2). A free ideal is a principal ideal generated by a non-zero divisor. This is because by commutativity, any set of two elements of R is linearly dependent. Thus R_p is an integral domain and hence a Prüfer domain. The localization at the maximal ideal of R_p is isomorphic to R_p, so R_p is a valuation ring. □

Theorem 3.1.37. *Let R be a semi-hereditary ring. Then* $\dim(R[X_1, \dots, X_n]) = n + \dim(R)$.

Proof. Let R be a semi-hereditary ring. By the previous theorem, R_p is a valuation ring. Since the class of semi-hereditary rings is closed under localizations, by Proposition 29 it suffices to prove that if V is a valuation ring with maximal ideal M, then $\mathrm{ht}(M) = \mathrm{ht}(M[X_1, \dots, X_n])$. Suppose that $f \in Q \subset M[X_1, \dots, X_n]$ and f is nonzero. Then factoring out a coefficient of least value, $f = cg$ where $c \in V$ and some coefficient of g is 1. Then $g \notin Q$ and since Q is prime, $c \in Q \cap V$. Thus $f \in (Q \cap V)[X_1, \dots, X_n]$ and so $Q = (Q \cap V)[X_1, \dots, X_n]$. Thus distinct ideals contained in $M[X_1, \dots, X_n]$ have distinct contractions to R and therefore $\mathrm{ht}(M) = \mathrm{ht}(M[X_1, \dots, X_n])$. □

Theorem 3.1.38 ([35], Theorem 4)**.** *Let R be an m-dimensional Prüfer ring and $S = R[a_1, \dots, a_n]$ a finitely generated algebra over R that is an integral domain. Then*

$$\dim(S) \leq \mathrm{trdeg}(S/R) + m.$$

Proof. Let P be a prime ideal of S and $p = R \cap P$. We define the dimension of P relative to R as $\dim(P) = \mathrm{trdeg}((S/P)/(R/p)) + \dim(R/p)$. The following holds for general integral domains:

1. Let $\phi\colon S \to \phi(S)$ be a homomorphism and $ker(\phi) \subseteq P$. Then $\phi(P)$ is a prime ideal of $\phi(S)$ and relative to $\phi(R)$, $\dim(\phi(P)) = \dim(P)$.
2. A subset of S/P that is algebraically independent over R/p is determined by a subset T of S satisfying the following:
   (a) $t \notin P$ for all $t \in T$ and $t - u \notin P$ for all $t,u \in T$ with $t \neq u$.
   (b) Let $t_1, \dots, t_k$ be distinct elements of T. Then for all $f \in R[X_1, \dots, X_k]$, if $f(t_1, \dots, t_k) \in P$ then $f \in p[X_1, \dots, X_k]$.
3. Let M be a nonempty multiplicatively closed subset of R not meeting $R \cap P$, R_M the localization of R at M, and S_M the localization of S at M. Then

$$\mathrm{trdeg}((S/P)/(R/p)) = \mathrm{trdeg}((S_M/PS_M)/(R_M/pR_M)).$$

The proof of points 1 and 2 are straightforward. For point 3, first note that $R_M \cap PS_M = pR_M$. Let T be a subset of S that satisfies the conditions of point 2. As a subset of S_M, T satisfies both of these conditions with respect to PS_M and R_M. It follows that

$$\mathrm{trdeg}((S/P)/(R/p)) \leq \mathrm{trdeg}((S_M/PS_M)/(R_M/pR_M)).$$

Let T' be a subset of S_M satisfying the two conditions with respect to PS_M and R_M. Let $s/m \in T'$. Then $s \notin P$. Let $u/w \in T'$ and $s/m \neq u/w$. Then $s - u \notin P$. (If $s - u \in P$, consider

$$f(X,Y) = mX - wY \in R_M[X,Y] - (pR_M)[X,Y].$$

Then $f(s/m, u/w) = 0$, a contradiction). Let $N(T') \subset S$ be the set of numerators of the elements of T'. Then $N(T')$ has the same number of elements as T' and it satisfies both conditions with respect to P and R. Thus $\operatorname{trdeg}((S_M/PS_M)/(R_M/pR_M)) \leq \operatorname{trdeg}((S/P)/(R/p))$.

Let P and Q be prime ideals of S with $P \subset Q$. Then $\dim(P) > \dim(Q)$. Let $p = R \cap P$ and $q = R \cap Q$. To prove the statement, first consider the case in which $p = q$. By point 1, we can assume that $p = q = \langle 0 \rangle$. Letting $M = R - \{0\}$ and considering point 3, we may assume that $R = K$ is a field and $S = K[a_1, \ldots, a_n]$. Let $T = \{t_1, \ldots, t_k\}$ be a subset of S that determines a subset of S/Q that is algebraically independent over K. Let $q \in Q - P$. Then $T \cup \{q\}$ determines a subset of S/P that is algebraically independent over K. $T \cup \{q\}$ clearly satisfies the first condition. For the second condition, let $f \in K[X_1, \ldots, X_k, Y]$ and $f(t_1, \ldots, t_k, q) \in P$. View f as an element of $K[X_1, \ldots, X_k][Y]$. Then $f = \sum_{i=0}^{n} f_i Y^i$. Since $q \in Q$, $f_0(t_1, \ldots, t_k) \in Q$ and therefore $f_0 = 0$. Then $f = Y \cdot \sum_{i=1}^{n} f_i Y^{i-1}$. Since $q \notin P$, $\sum_{i=1}^{n} f_i(t_1, \ldots, t_k) q^{i-1} \in P$. It follows that $f_1(t_1, \ldots, t_k) \in Q$ and therefore $f_1 = 0$. Continuing like we conclude that $f = 0$. Thus given an algebraically independent subset of S/Q over K with k elements, there is an algebraically independent subset of S/P over K with $k+1$ elements. Hence $\dim(P) > \dim(Q)$.

Now consider the case in which $p \subset q$. We only need to show that

$$\operatorname{trdeg}((S/P)/(R/p)) \geq \operatorname{trdeg}((S/Q)/(R/q)).$$

Letting $M = R - q$ and considering point 3, we may assume that R is a valuation ring with unique maximal ideal q. Let $T = \{t_1, \ldots, t_k\}$ be a subset of S that determines an algebraically independent subset of S/Q over R/q. Then T clearly satisfies the first condition with respect to S/P and R/p. For the second condition, let $f \in K[X_1, \ldots, X_k]$ and $f(t_1, \ldots, t_k) \in P$. Let c be a coefficient of f of least value. Then $f = cg$ where at least one of the coefficients of g is 1. Note that $g(t_1, \ldots, t_k) \notin P$. (Otherwise $g(t_1, \ldots, t_k) \in Q$ and therefore $g \in q[X_1, \ldots, X_k]$, which is a contradiction). Thus $c \in p$ and so $f \in p[X_1, \ldots, X_k]$. Hence T satisfies the second condition and the result follows.

Let $d = \dim(S)$ and let $\langle 0 \rangle \subset P_1 \subset \cdots \subset P_d$ be a maximal chain of prime ideals in S. Then $d - i \leq \dim(P_i)$. Thus

$$d \leq \dim(\langle 0 \rangle) = \operatorname{trdeg}(S/R) + m.$$

□

Corollary 3.1.39. *Let R be an integral domain of type $(1,2)$. Then*

$$\dim(R[a_1, \ldots, a_n]) \leq 1 + \operatorname{trdeg}(R[a_1, \ldots, a_n]/R).$$

In particular, $\dim(R[X_1, \ldots, X_n]) = 1 + n$.

Proof. Let S be the integral closure of R in $\operatorname{Quot}(R)$. Then S is also of type $(1,2)$. By Theorem 12, S is a Prüfer ring. Thus the first assertion is true for S. Since $S[a_1, \ldots, a_n]$ is integral over $R[a_1, \ldots, a_n]$, $\dim(R[a_1, \ldots, a_n]) \leq 1 + \operatorname{trdeg}(R[a_1, \ldots, a_n]/R)$. The second assertion is clear. □

Polynomial Rings over 0-dimensional Rings

Proposition 3.1.40. *Let R be a 0-dimensional ring. Then*

$$\dim(R[X_1, \ldots, X_n]) = n.$$

Proof. Let Q be a prime ideal of $R[X_1, \ldots, X_n]$ and let $P = Q \cap R$. Since R is 0-dimensional, we have $\operatorname{ht}(P[X_1, \ldots, X_n]) = 0$. By Theorem 3.1.20, $\operatorname{ht}(Q) \leq n$. Thus $\dim(R[X_1, \ldots, X_n]) = n$. □

The Dimension Sequence of a Ring

Let R be a ring of finite Krull dimension n_0. Let

$$\dim(R[X_1,\ldots,X_m]) = n_m.$$

The sequence $\{n_i\}_{i=0}^{\infty}$ is called the dimension sequence for R. Let $d_i = n_i - n_{i-1}$. The sequence $\{d_i\}_{i=1}^{\infty}$ is called the difference sequence for R. More generally, if $s = \{s_i\}_{i=0}^{\infty}$ is a sequence of nonnegative integers, then the sequence $\{s_i - s_{i-1}\}_{i=1}^{\infty}$ is called the difference sequence for s. The sequences of nonnegative integers that can be realized as a dimension sequence of a ring have been determined in [4].

Let S be the set of all sequences $\{n_i\}_{i=0}^{\infty}$ of nonnegative integers with the property that for all i, $1 \leq d_{i+1} \leq d_i \leq n_0 + 1$. If $s_1,\ldots,s_n \in \mathcal{S}$ where $s_k = \{n_i^{(k)}\}_{i=0}^{\infty}$, then we define $\sup_{1\leq k\leq n}\{s_k\}$ to be the sequence $\{\max_{1\leq k\leq n}\{n_i^{(k)}\}\}_{i=0}^{\infty}$. Let $\mathcal{D} = \{\sup_{1\leq k\leq n}\{s_k\} : s_1,\ldots,s_n \in \mathcal{S}\}$. Then every dimension sequence is an element of $\mathcal{D}$ ([4], Theorem 4.4).

Definition 3.1.41. *A semi-quasi-local domain is an integral domain with a finite number of maximal ideals.*

Theorem 3.1.42 ([4], Theorem 4.11)**.** *Let F be a field, let $\{X_i\}_{i=0}^{\infty}$ be a countably infinite set of indeterminates over F, and let $s_1,\ldots,s_n \in \mathcal{S}$. There is an integrally closed semi-quasilocal domain D such that D has quotient field $F(\{X_i\}_{i=0}^{\infty})$, D has at most n maximal ideals and $\sup\{s_i\}_{i=0}^{\infty}$ is the dimension sequence for D. Moreover, D can be chosen so that each proper prime ideal of D is contained in a unique maximal ideal of D and so that the set of prime ideals of D is linearly ordered by inclusion.*

Given a nondecreasing sequence of nonnegative integers, it is difficult to determine whether or not it is a dimension sequence using the above characterization. In order that a sequence be a dimension sequence it is necessary and sufficient that the following inequality is satisfied for every $n > 0$: $na_n \leq (n+1)a_{n-1} + 1$ ([33], Theorem 2).

3.2 Infinite-dimensional Rings

Theorem 3.2.1. *Let R be a ring of infinite dimension α. Then* $\dim(R) = \dim(R[X])$.

Proof. It is clear that $\dim(R[X]) \geq \alpha$. Let $\mathcal{C}$ be an arbitrary chain of prime ideals in $R[X]$. Consider $\mathcal{C}_{\mathcal{R}} = \{Q \cap R : Q \in \mathcal{C}\}$, a chain of prime ideals in R. By Proposition 3.1.1 there are at most two prime ideals in $\mathcal{C}$ lying over a prime ideal of $\mathcal{C}_{\mathcal{R}}$. Thus there exists an embedding of $\mathcal{C}$ into $\mathcal{C}_{\mathcal{R}} \times \{1,2\}$. It follows that $\mathcal{C}$ and $\mathcal{C}_{\mathcal{R}}$ have the same cardinality. Hence if R is infinite-dimensional, then $\dim(R) = \dim(R[X])$. □

Example 3.2.2. *Let R be an integral domain and $D = R[\{X_i\}_{i\in\mathbb{N}}]$ the polynomial ring over R in countably many indeterminates. For each positive real number a, let P_a be the prime ideal generated by all X_n with $n < a$. Then $P_a \subset P_b$ if and only if $a < b$, so $\{P_a\}_{a\in\mathbb{R}^+}$ is a chain of prime ideals of D.*[9] *Thus* $\dim(D)$ *is uncountable. We also have* $|D| = \max(|R|, \mathbb{N})$.[10] *If R is at most countable, then $|D| = \aleph_0$. Thus in this case* $\dim(D) = 2^{\aleph_0} = \aleph_1$. *Note that* $\operatorname{trdeg}(D/R) = \aleph_0$, *so in general it is not true that* $\dim(A) \leq tr(A/K)$ *for a K-algebra A that is an integral domain. As we have already seen, this is true in the case in which A is finite-dimensional.*

[9] http://mathoverflow.net/questions/146157/

[10] See example 3 in http://math.uga.edu/~pete/settheorypart4.pdf

3.3 Integer-Valued Polynomials

Definition 3.3.1. *Let D be an integral domain with quotient field K. The ring of integer-valued polynomials of D is the ring* $\mathrm{Int}(D) = \{f \in K[X] : f(D) \subseteq D\}$*.*

Example 3.3.2. $\mathrm{Int}(\mathbb{Z})$ *is a free $\mathbb{Z}$-module with basis*

$$\{f_n(X) = X(X-1)\cdots(X-n+1)/n! : n \in \mathbb{N}\}$$

([8], Proposition I.1.1). For each $n \in \mathbb{Z}$, f_n has degree n. Such a basis of $\mathrm{Int}(\mathbb{Z})$ *is called a regular basis.*

Definition 3.3.3. *Let P be a prime ideal of D and $a \in D$. We define $\mathcal{Q}_{P,a}$ to be the set $\{f \in \mathrm{Int}(D) : f(a) \in P\}$. It is straightforward to show that $\mathcal{Q}_{P,a}$ is a nonzero prime ideal of* $\mathrm{Int}(D)$*.*

Proposition 3.3.4 ([8], Proposition V.1.5)**.** *Let D be a finite-dimensional domain. Then* $\dim(\mathrm{Int}(D)) \geq 1 + \dim(D)$*.*

Proof. Let P_1 and P_2 be prime ideals of D. If $P_1 \subset P_2$, then for any $a \in D$ we have $\mathcal{Q}_{P_1,a} \subset \mathcal{Q}_{P_2,a}$. The result follows. □

Definition 3.3.5. *Let R be a ring and P a prime ideal of R. The field* $\mathrm{Quot}(R/P)$ *is called the residue field of P.*

Since a finite integral domain is a field, any nonmaximal prime ideal has an infinite residue field.

Proposition 3.3.6. *Let D be an integral domain. If p is a prime ideal of D with infinite residue field, then* $\mathrm{Int}(D)_p = D_p[X]$ *([8], Proposition I.3.4).*

It is possible that $\mathrm{Int}(D) = D[X]$. For example, suppose that D is a domain for which every prime ideal has an infinite residue field. It is straightforward to show that $\mathrm{Int}(D) \subseteq \mathrm{Int}(D)_p$. Thus by the previous proposition, $\mathrm{Int}(D) \subseteq \bigcap_p D_p[X]$. Since $D = \bigcap_p D_p$, the result follows.

Proposition 3.3.7 ([8], Proposition V.1.6)**.** *Let D be a finite-dimensional domain. Then* $\dim(\mathrm{Int}(D)) \geq \dim(D[X]) - 1$*.*

Proof. Let $P_0 \subset \cdots \subset P_n$ be a maximal chain of prime ideals of $D[X]$. Then $P_{n-1} = M[X]$ where M is a maximal ideal of D. Let $Q = P_{n-2} \cap D$. Then by Jaffard's Special Chain Theorem $Q[X]$ is an element of the chain and either $Q[X] = P_{n-2}$ or $Q[X] = P_{n-3}$. Since the residue field of Q is infinite, we have $\mathrm{Int}(D)_Q = D_Q[X]$. Let Q' be prime ideal of $\mathrm{Int}(D)$ that corresponds to the extension of $Q[X]$ to $D_p[X]$. Then $\mathrm{ht}(Q') \geq \mathrm{ht}(Q)$. Furthermore, for any $a \in D$ we have $Q' \subset \mathcal{Q}_{Q,a} \subset \mathcal{Q}_{M,a}$. Thus $\dim(\mathrm{Int}(D)) \geq n - 1$. □

There is an analogue of Theorem 3.1.17 for integer-valued polynomials:

Theorem 3.3.8 ([18], Theorem 3.4)**.** *Let $n \geq 1$. For each h, with $n+1 \leq h \leq 2n+1$, there exists an integral domain D such that $\dim(D) = n$,* $\dim(\mathrm{Int}(D)) = h$*, and* $\mathrm{Int}(D) \neq D[X]$*.*

It is possible to have $\dim(\mathrm{Int}(D)) = \dim(D[X]) - 1$. We first need some preliminary definitions and results:

1. The nonzero prime ideals of $\mathrm{Int}(D)$ that contract to $\langle 0 \rangle$ in D are in one-to-one correspondence with the monic irreducible polynomials of K[X]. If q is a monic irreducible, then q corresponds to the prime ideal $qK[X] \cap \mathrm{Int}(D)$ ([8], Corollary V.1.2).

2. If I is an ideal of D, let $\mathrm{Int}(I) = \{f \in \mathrm{Int}(D) : f(D) \subseteq I\}$. Note that $\mathrm{Int}(I)$ is an ideal of $\mathrm{Int}(D)$.

3. A maximal ideal M of D is said to be pseudo-principal if there is a positive integer n and an element $t \in M$ such that $M^n \subseteq \langle t \rangle$.

4. The prime ideals of $\mathrm{Int}(D)$ that contract to a pseudo-principal maximal M in D with finite residue field contain the ideal $\mathrm{Int}(M)$ and are maximal ideals ([8], Proposition V.1.11).

Now we can proceed to the example.

Example 3.3.9. *This is a slight generalization of [8], Example V.1.12. Let k be a finite field and K a transcendental extension of k. Let $V = K + M$ be a rank 1 DVR and $D = k + M$ It follows from Theorem 3.1.14 that $\dim(D) = 1$ and $\dim(D[X]) = 3$. Note that M is a pseudo-principal maximal ideal of D. The contraction of a prime ideal of $\mathrm{Int}(D)$ to D is either $\langle 0 \rangle$ or M (since these are the only prime ideals of R). By point 1, distinct prime ideals that contract to $\langle 0 \rangle$ are incomparable. The residue field of M is finite (viewed as a maximal ideal of D), so by point 4 distinct prime ideals of $\mathrm{Int}(D)$ that contract to M are incomparable. Thus we have $\dim(\mathrm{Int}(D)) \leq 2$. By the previous proposition, $\dim(\mathrm{Int}(D)) \geq 2$. Thus $\dim(\mathrm{Int}(D)) = \dim(D[X]) - 1$.*

No example is known in which $\dim(\mathrm{Int}(D)) > \dim(D[X])$. We will describe three classes of rings in which $\dim(\mathrm{Int}(D)) \leq \dim(D[X])$.

Valuative Dimension and Jaffard Domains

Definition 3.3.10. *Let A be an integral domain and $K = \mathrm{Quot}(A)$. A valuation ring V with $A \subseteq V \subseteq K$ is called a valuation overring of A.*

Definition 3.3.11. *Let D be an integral domain. D is said to have valuative dimension n, $\dim_v(D) = n$ if each valuation overring of D has dimension at most n and if there exists a valuation overring of D of dimension n. If no such integer n exists, we say that the valuative dimension of D is infinite.*

The valuative dimension of a finite-dimensional integral domain D satisfies $\dim(D) \leq \dim_v(D)$ ([20], Corollary 19.7) and $\dim_v(D[X_1, \ldots, X_n]) = \dim_v(D) + n$ ([20], Theorem 30.11).

Proposition 3.3.12. *Let A be a finite-dimensional integral domain. Then*

$$\dim(\mathrm{Int}(A)) \leq 1 + \dim_v(A).$$

Proof. We have $\dim(Int(A)) \leq \dim_v(\mathrm{Int}(A)) \leq \dim_v(A[X]) = \dim_v(A) + 1$. □

Note that this proposition does not imply that if A is finite-dimensional, then $\mathrm{Int}(A)$ is finite-dimensional. In fact, for any $n \geq 0$ and m in $\mathbb{Z}^+ \cup \{\infty\}$ there is an integrally closed domain D such that $\dim(D) = n$ and $\dim_v(D) = m$ ([20], p. 360). It is currently not known if D finite-dimensional implies $\mathrm{Int}(D)$ is finite-dimensional.

Proposition 3.3.13. *Let D be a finite-dimensional integral domain. The following are equivalent:*

1. $\dim(D[X_1, \ldots, X_n]) = \dim(D) + n$ *for all* $n \geq 1$.

2. $\dim(D) = \dim_v(D)$.

Proof. (1) $\implies$ (2) By [20], Theorem 30.9 $\dim_v(D) = m$ if and only if $\dim(D[X_1, \ldots, X_m]) = 2m$. Thus $\dim(D) = \dim_v(D)$.

(2) $\implies$ (1) We have $\dim(D[X_1, \ldots, X_n]) \leq \dim_v(D[X_1, \ldots, X_n]) = \dim_v(D) + n = \dim(D) + n$. The reverse inequality is always true. □

Definition 3.3.14. *Any finite-dimensional integral domain that satisfies either of the above properties is called a Jaffard domain.*

If D is a Jaffard domain, then $\dim(\text{Int}(D)) = \dim(D[X]) = 1 + \dim(D)$. We have already seen a number of examples of Jaffard domains: Noetherian rings, Prüfer rings, semi-hereditary rings, and 0-dimensional rings.

Locally Essential domains

Definition 3.3.15. *Let R be an integral domain.*

1. *R is said to be essential if R is an intersection of valuation rings that are localizations of R.*

2. *R is said to be locally essential if R_P is essential for any prime ideal P of R.*

Locally essential domains satisfy $\dim(\text{Int}(D)) = \dim(D[X])$ ([17], p. 3).

Definition 3.3.16. *Let D be an integrally closed domain with $\text{Quot}(D) = K$. Suppose there exists a family $\mathcal{F} = \{V_i\}_{i \in I}$ of valuation overrings of D for which the following hold:*

1. *$D = \bigcap_{i \in I} V_i$.*

2. *Each V_i is a rank 1 DVR.*

3. *The family $\mathcal{F}$ has finite character—that is, if $x \in K$, $x \neq 0$, then x is a nonunit in only finitely many of the valuation rings in the family $\mathcal{F}$.*

4. *Each V_i is essential for D—that is, V_i is a localization of D (By [20], Corollary 5.2 we have $V_i = V_{D \cap M_i}$ where M_i is the maximal ideal of V_i).*

Then we say that D is a Krull domain and that $\mathcal{F}$ is a defining family for D.

A Krull domain is a locally essential domain. Here are some examples of Krull domains:

1. Let D be a UFD and let $\mathcal{F} = \{p_i\}_{i \in I}$ be a complete set of nonassociate nonunit prime elements of D. Then D is a Krull domain and $\mathcal{F}$ is a defining family for D ([20], Proposition 43.2).

2. Let D be an integrally closed Noetherian domain and let $\mathcal{F} = \{P_i\}_{i \in I}$ be the set of minimal primes of D. Then D is a Krull domain and $\mathcal{F}$ is a defining family for D ([20], Proposition 43.4)

3. If D is a Krull domain, then so is $D[X]$ ([20], Theorem 43.11) and $D[[X]]$ ([20], Corollary 44.11).

There exist locally essential domains that are not Krull domains (see [17], p. 2). For any integer $r \geq 2$, there exists an r-dimensional essential Jaffard domain D that is not locally essential ([17], Example 2.3). Bouvier's conjecture states that finite-dimensional Krull domains need not be Jaffard domains ([17], p. 2).

Pseudo-valuation Domains of type n

Before defining a pseudo-valuation ring of type n, we will review the definition of a pullback. Let R, S, T, and X be rings and $f\colon R \to T$ and $g\colon S \to T$ be homomorphisms. We say that X is a pullback of f and g if the following conditions are satisfied:

1. There exist homomorphisms $p_1\colon X \to R$ and $p_2\colon X \to S$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{p_1} & R \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{g} & T \end{array}$$

2. X satisfies the following universal property: If Y is a ring and $q_1\colon Y \to R$ and $q_2\colon Y \to S$ are homomorphisms making the diagram

$$\begin{array}{ccc} Y & \xrightarrow{q_1} & R \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{g} & T \end{array}$$

   commute, then there exists a unique homomorphism $u\colon Y \to X$ such that $p_1 \circ u = q_1$ and $p_2 \circ u = q_2$.

A pullback of two homomorphisms is unique up to isomorphism, so therefore we can refer to it as "the" pullback. Define $R \times_T S = \{(a, b) \in R \times S : f(a) = g(b)\}$. Then $R \times_T S$ is a ring that satisfies the two conditions above and is therefore the pullback of f and g.

Let V be a valuation ring with maximal ideal M and residue field k. Let k_0 be a subfield of k. Let $\pi\colon V \to k$ be the canonical epimorphism and $i\colon k_o \to k$ the canonical injection. The pullback of π and i is called a pseudo-valuation domain or PVD. Clearly, $\pi^{-1}(k_0)$ is the pullback of π and i. Note that $\pi^{-1}(k_0)$ is a local ring with the same maximal ideal M. Clearly, any valuation ring is a pseudo-valuation domain. However, there are pseudo-valuation domains that are not valuation rings: the ring D defined in Example 9.

A pseudo-valuation domain of type n or P^nVD is defined inductively as follows: Let S_{n-1} be a P^{n-1}VD with maximal ideal M_{n-1} and residue field k_{n-1}. Let R_{n-1} be a PVD with $\mathrm{Quot}(R_{n-1}) = k_{n-1}$. Let $\pi\colon S_{n-1} \to k_{n-1}$ be the canonical epimorphism and $i\colon R_{n-1} \to k_{n-1}$ the canonical embedding. The pullback of π and i is $\pi^{-1}(R_{n-1})$ and is called a pseudo-valuation ring of type n.

Let D be a PVD determined by a valuation ring V with residue field k and a subfield $k_0 \subseteq k$. One can associate to D a pair of parameters $(a : b)$ where $a = \dim(V)$ and $d = \mathrm{trdeg}(k/k_0)$. Let D' be a P^nVD determined by a P^{n-1}VD T and a PVD R. By induction one can associate to D' a family of parameters $(a_0, \ldots, a_n : d_0, \ldots, d_n)$ where $(a_0, \ldots, a_{n-1} : d_0, \ldots, d_{n-1})$ are the parameters associated to T and $(a_n : d_n)$ are the parameters associated to R.

Proposition 3.3.17 ([18], Lemma 3.1)**.** *Let D be a P^n VD with parameters $(a_0, \ldots, a_n : d_0, \ldots, d_n)$. Then we have the following:*

1. $\dim(D) = \sum_{i=0}^{n} a_i$.

2. $\dim(D[X]) = \sum_{i=0}^{n} a_i + 1 + \sum_{i=0}^{n} \inf\{1, d_i\}$.

3. $\dim(\text{Int}(D)) = \sum_{i=0}^{n} a_i + 1 + \sum_{i=0}^{n} \inf\{1, d_i\}$ *if* $\text{Int}(D) \neq D[X]$.

It follows that if D is a P^nVD, then $\dim(\text{Int}(D)) \leq \dim(D[X])$. Since it is possible for the inequality to be strict, a P^nVD is not necessarily a Jaffard or locally essential domain. A Jaffard domain or locally essential domain need not be a P^nVD, since P^nVDs are local rings. It is conjectured in [17] that any finite-dimensional domain R satisfies $\dim(\text{Int}(R)) \leq \dim(R[X])$.

4 0-Dimensional Rings and Arbitrary Direct Products

4.1 von Neumann Regular Rings

Definition 4.1.1. *A ring R is von Neumann regular if for every $a \in R$ there exists an $x \in R$ such that $a = axa$.*

Here are some examples of von Neumann regular rings:

1. Any field is von Neumann regular.

2. An arbitrary direct product $S = \prod_{i=1}^{\infty} R_i$ is von Neumann regular if and only if each R_i is von Neumann regular.

3. A ring in which $a^{n(a)} = a$ for each $a \in R$ is called an exp-ring.[11] Here $n(a)$ may depend on a. Such a ring is clearly von Neumann regular. A special case of an exp-ring is a Boolean ring, in which $a^2 = a$ for all $a \in R$. For an example of a Boolean ring, let A be any set and consider the powerset of A, $\mathcal{P}(A)$. Addition is symmetric difference and multiplication is intersection.

4. ([26], Example 179) Let K be a field and viewing elements of $\prod_{i=1}^{\infty} K$ as sequences, let R be the subring consisting of eventually constant sequences. Then R is von Neumann regular.

Definition 4.1.2. *Let R be a ring. An element $r \in R$ is said to be nilpotent if $r^k = 0$ for some $k \in \mathbb{N}$. The set of nilpotent elements forms an ideal of R called the nilradical of R. It is denoted by $N(R)$. $N(R)$ is also the intersection of the prime ideals of R.*

Definition 4.1.3. *Let I be an ideal of R. Then the set $\{x \in R : x^k \in I$ for some k in $\mathbb{N}\}$ is called the radical of I and is denoted by $\sqrt{I}$. This is an ideal as it corresponds to $N(R/I)$ under the natural epimorphism $\pi \colon R \to R/I$. The radical of I is also the intersection of the prime ideals of R containing I.*

Definition 4.1.4. *Let R be a ring. The Jacobson radical of R is the intersection of the maximal ideals of R. It is denoted by $J(R)$. For $x \in R$, $x \in J(R)$ if and only if $1 - xy$ is a unit for all $y \in R$.*

Definition 4.1.5. *A ring is said to be reduced if it has no nonzero nilpotent elements.*

It is not hard to show that $R/N(R)$ is reduced for any ring R.

Definition 4.1.6. *Let R be a ring and I an ideal of R. I is said to be idempotent if $I^2 = I$.*

Theorem 4.1.7 ([21], Theorem 3.1)**.** *The following are equivalent for a ring R:*

1. *R is 0-dimensional and reduced.*

2. *R_p is a field for each prime ideal p of R.*

3. *Each ideal of R is a radical ideal.*

4. *Each ideal of R is idempotent.*

5. *Each principal ideal of R is idempotent.*

[11] http://www.scm.org.co/aplicaciones/revista/Articulos/1078.pdf

6. *R is von Neumann regular.*

Any von Neumann regular ring can be embedded into a direct product of fields: Let R be von Neumann regular and $\{M_i\}_{i\in I}$ be the set of maximal ideals of R. Since $J(R) = \langle 0 \rangle$, the natural map $\pi\colon R \to \prod_{i\in I} R/M_i$ is injective. However, not every von Neumann regular ring is a direct product of fields. Let R denote the fourth example given above. Let $M_n = \{(a_i) \in R : a_n = 0\}$. Then each M_n is a prime ideal of R. Let M be the set of all sequences which are eventually zero. Then M is also a prime ideal of R, and M together with the M_n are the only prime ideals of R. Thus the set of prime ideals of R is countable. By Corollary 2.6.9, R is not a direct product of fields.

Using the previous structure theorem for von Neumann regular rings, a structure theorem for general 0-dimensional rings can be deduced. The statement of the following theorem is taken from mathoverflow.[12]

Theorem 4.1.8. *For a ring R, the following are equivalent:*

1. *Every prime ideal is maximal.*
2. $\dim(R) = 0$.
3. *$R/N(R)$ is von Neumann regular.*
4. *For all $a \in R$ there is some $n \in N$ such that a^{n+1} divides a^n.*

Proof. (1) $\implies$ (2) is obvious. (2) $\implies$ (3) was proven in the previous theorem.

(3) $\implies$ (4) Since $R/N(R)$ is von Neumann regular, for some $x \in R$ and $k \in \mathbb{N}$ we have $(a - a^2x)^k = 0$. Expanding this using the binomial theorem, the result follows.

(4) $\implies$ (1) Let $r \in R$ and $n \in \mathbb{N}$ be such that $a^n = ra^{n+1}$. Then $(a(1 - ra))^n{=}a^n(1 - ra)(1 - ra)^{n-1} = 0$. Thus $a - ara \in N(R)$, so $R/N(R)$ is von Neumann regular. ☐

Artinian Rings

Definition 4.1.9. *A ring R is said to be Artinian if there is no infinite decreasing chain of ideals in R.*

Theorem 4.1.10. *Let R be an Artinian ring.*

1. *There are only finitely many maximal ideals of R.*
2. *Let $M_1, \dots, M_n$ be the maximal ideals of R and $K_i = R/M_i$. Then $M/J(R) \cong \prod_{i=1}^{n} K_i$.*
3. $J(R) = N(R)$.
4. *The ring R is isomorphic to the direct product of a finite number of Artinian local rings.*

These statements are proved in [15], pages 752–753.

It is not at all expected that a condition on descending chains of ideals should imply anything about ascending chains of ideals, yet we have the following corollary:

Corollary 4.1.11 ([15], p. 753)**.** *A ring is Artinian if and only if it is Noetherian and 0-dimensional.*

[12] http://math.stackexchange.com/q/636008

Examples of Artinian rings

1. Any finite ring is Artinian.

2. Let R be a PID and r a nonzero element of R. Then $R/\langle r\rangle$ is Artinian.

3. ([15], p. 753) For any field K, a K-algebra R that is finite-dimensional as a vector space over K is Artinian because ideals in R are in particular K-subspaces of R, so the length of any chain of ideals of R is bounded by the vector space dimension of R over K.

Here is an example of a 0-dimensional local ring that is neither Artinian nor von Neumann regular: Consider the ring

$$R = K[t_1, \ldots, t_k, \ldots]/\langle t_1^n, \ldots, t_k^n, \ldots\rangle$$

where $n > 1$. The only prime ideal of R is $M = \langle t_1, \ldots, t_k, \ldots\rangle/\langle t_1^n, \ldots, t_k^n, \ldots\rangle$. Since M is not finitely generated, R is not Noetherian and hence not Artinian. R is not reduced, so it is not von Neumann regular either.

4.2 Arbitrary Direct Products

Proposition 4.2.1. *Let $S = \prod_{i=1}^n R_i$ where each R_i is a commutative ring with identity. The proper prime ideals of S are of the form $\prod_{i=1}^n P_i$ where for some j, P_j is a proper prime ideal of R_j and for $i \neq j$, $P_i = R_i$.*

The proof of this proposition is the same as that of Proposition 2.1.6.

Corollary 4.2.2. $\dim(S) = \max_{1\le i\le n} \dim(R_i)$.

For infinite products, the prime ideal structure is much more complicated. Let I be an infinite set. First of all, not every prime ideal of $S = \prod_{i\in I} R_i$ is of the form described in the previous proposition. Let π_i be projection onto R_i and P a prime ideal of R. Then $P \subseteq \prod_{i\in I} \pi_i(P)$. P is of the above form if and only if equality holds. Consider $J = \bigoplus_{i\in I} R_i$. Since $\pi_i(j) = R_i$, no prime ideal containing J is of the above form.

Proposition 4.2.3 (Maroscia, [23], Proposition 3.1). *If $S = \prod_{i\in I} R_i$ is a product of 0-dimensional rings R_i, then the following are equivalent:*

1. *S is 0-dimensional.*

2. *$J(S) = N(S)$.*

3. *$N(S) = \prod_{i\in I} N(R_i)$.*

Proof. (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (3) It is always true that $N(S) \subseteq \prod_{i\in I} N(R_i)$. For the reverse inclusion, we have $\prod_{i\in I} N(R_i) \subseteq \prod_{i\in I} J(R_i) \subseteq J(S) = N(S)$.

(3) $\Longrightarrow$ (1) $S/N(S) = \prod_{i\in I} R_i/N(R_i)$ Since R_i is 0-dimensional, $R_i/N(R_i)$ is von Neumann regular. Thus $S/N(S)$ is von Neumann regular and S is 0-dimensional. ☐

Proposition 4.2.4 ([23], Proposition 3.2). *Suppose $\{R_a\}_{a\in A}$ is a family of rings, $\dim(R_a) \geq k$ where $k \geq 1$ and A is infinite. If $R = \prod_{a\in A} R_a$, then $\dim(R) \geq k+1$.*

Using the previous proposition and the fact that the integers can be partitioned into infinitely many infinite subsets, one can prove by induction the following theorem:

Theorem 4.2.5 ([23], Theorem 3.3). *Suppose $\{R_a\}_{a\in A}$ is a family of rings and $R = \prod_{a\in A} R_a$. If infinitely many of the rings R_A have positive dimension, then R is infinite-dimensional.*

Definition 4.2.6. *Let $x \in N(R)$. The index of nilpotency of x, denoted $\eta(x)$, is the least positive integer k with $x^k = 0$. Define $\eta(R)$ to be $\sup\{\eta(x) : x \in N(S)\}$.*

Theorem 4.2.7 ([23], Theorem 3.4). *Suppose $\{R_a\}_{a\in A}$ is a family of 0-dimensional rings and let $R = \prod_{a\in A} R_a$. The following conditions are equivalent:*

1. $\dim(R) = 0$.
2. *There exists $k \in \mathbb{Z}$ such that $\{a \in A : \eta(R_a) > k\}$ is finite.*
3. $\dim(R)$ *is finite.*

Example 4.2.8. *Let $p \in \mathbb{Z}$ be prime. Then $R_n = \mathbb{Z}/\langle p^n\rangle$ is 0-dimensional for $n > 0$. Consider $S = \prod_{n=1}^{\infty} R_n$. Let $p_n = p \mod \langle p^n\rangle$ and $x = (p_n)_{n=1}^{\infty}$. Then $x \in \prod_{n=1}^{\infty} N(R_n) - N(S)$. Thus S is infinite-dimensional.*

Theorem 4.2.9 ([23], Theorem 3.5). *Let $\{R_a\}_{a\in A}$ be a family of finite-dimensional rings, let $B = \{a \in A : \dim(R_a) > 0\}$, let $C = A - B$, and let $R = \prod_{a\in A} R_a$.*

1. *R is finite-dimensional if and only if B is finite and there exists $k \in \mathbb{Z}^+$ such that $\{y \in C : \eta(R_y) > k\}$ is finite.*
2. *If $\dim(R)$ is finite, then $\dim(R) = \sup\{\dim(R_a) : a \in A\}$.*

Proof. (1) The forward direction is clear. For the reverse, view R as

$$\prod_{b\in B} R_b \times \prod_{c\in C} R_c.$$

This is a product of two rings, each of which is finite-dimensional. Thus R is finite-dimensional.

(2) If $\dim(R)$ is finite, then by the previous theorem $\dim(\prod_{c\in C} R_c) = 0$. By what we proved about finite products, $\dim(\prod_{b\in B} R_b) = \sup\{\dim(R_b) : b \in B\}$. Hence $\dim(R) = \sup\{\dim(R_a) : a \in A\}$. □

Theorem 4.2.10. *Let $R = \prod_{i\in I}(R_i)$ where I is infinite and $\dim(R)$ is not finite. Then $\dim(R) \geq \aleph_1$.*

Proof. This proof is from mathoverflow.[13] Without loss of generality we can assume $I = \mathbb{N}$. Using the fact that the integers can be partitioned into infinitely many infinite subsets together with Theorems 5 and 7, we can assume that $\dim(R_i) \geq i$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$, $A_n = \{0, 1, \ldots, n\}$ and $A = \prod_{n\in\mathbb{N}} A_n$. Let $S = \prod_{\mathcal{U}} A_n$ be the ultraproduct of Example 2.6.12. Let $[f],[g] \in S$ and define $[f] < [g]$ if $\{n \in \mathbb{N} : f(n) < g(n)\} \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, $<$ is a well-defined total ordering on S.

For each $n \in \mathbb{N}$, let $P_0^n \subset P_1^n \subset \cdots \subset P_n^n$ be a chain of prime ideals of R_n. For any $f \in A$, let P_f be the set of all sequences $(x_n) \in R$ such that $\{n : x_n \in P_{f(n)}^n\} \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, P_f is a prime ideal of R. If $g \in A$ and $f < g$, then $P_f \subset P_g$. It was shown in Example 2.6.12 that $\mathrm{Card}(S) \geq \aleph_1$. Thus $\dim(R) \geq \aleph_1$. □

[13] http://mathoverflow.net/questions/200803/

This theorem is an improvement of [36], Theorem 2.3, in which it is shown that an infinite product of 0-dimensional rings contains a countable chain of prime ideals.

If all but finitely many R_i's are chained rings, this bound can be improved:

Theorem 4.2.11 ([36], Theorem 2.13)**.** *Let $R = \prod_{i\in I} R_i$ and suppose that each R_i has dimension zero. Assume that all but finitely many of the R_i's are chained rings. Then every maximal chain of prime ideals has length either 0 or at least $2^{\aleph_1}$. If I is countably infinite, then the maximal infinite chains have length exactly $2^{\aleph_1}$.*

Example 4.2.12. *Let S be the ring of Example 8. Since $\mathbb{Z}/\langle p^n\rangle$ is a 0-dimensional chained ring for each $n > 0$, we have $\dim(S) = 2^{\aleph_1}$. Now let I be an infinite set S and $R_i = \mathbb{Z}$ for all i in I. Let $T = \prod_{i\in I} R_i$. Since T maps surjectively onto S, we have $\dim(T) \geq \dim(S) = 2^{\aleph_1}$. If I is countably infinite, then $\dim(T) = 2^{\aleph_1}$.*

According to [36], it is an open question if in general a maximal chain of prime ideals in an infinite product of rings has length either 0 or $2^{\aleph_1}$.

4.3 Subrings of 0-dimensional Rings

In this section we consider necessary and sufficient conditions to embed a ring into a 0-dimensional ring and also into a direct product of 0-dimensional rings. Before we characterize rings that can be embedded into 0-dimensional rings, we begin with some preliminary results about primary ideals and total quotient rings.

Definition 4.3.1. *A ideal Q of R is said to be primary if whenever $xy \in Q$, either $x \in Q$ or $y^n \in Q$ for some $n > 0$.*

Note that the radical of a primary ideal is prime.

Example 4.3.2. *For any prime p of $\mathbb{Z}$ and $n > 0$, $\langle p^n\rangle$ is a primary ideal.*

Definition 4.3.3. *An element of a ring is said to be regular if it is not a zero-divisor.*

Definition 4.3.4. *Let R be a ring and S be the set of regular elements of R. S is multiplicatively closed. The total quotient ring of R, denoted $T(R)$ is the localization $S^{-1}R$.*

Since S contains no zero-divisors, the natural map $\pi\colon R \to T(R)$ is injective. Every regular element of R is a unit in $T(R)$ and every zero-divisor of R is nilpotent in $T(R)$.

We can already prove a special case in which a ring can be embedded into a 0-dimensional ring. Suppose that R is a ring for which $\langle 0\rangle = \bigcap_{i=1}^n Q_i$ where each Q_i is a primary ideal. Let $\sqrt{Q_i} = P_i$. Then consider $\alpha\colon R \to \prod_{i=1}^n (R/Q_i)$ and $\beta\colon \prod_{i=1}^n (R/Q_i) \to R' = \prod_{i=1}^n (R/Q_i)_{P_i/Q_i}$ where α and β are the natural homomorphisms. It is clear that α is an injection. To prove that β is an injection, we show that $R/Q_i - P_i/Q_i$ has no zero divisors. Suppose that $\overline{x}$ is a zero divisor of R/Q_i. Then $xy \in Q_i$ for some $y \notin Q_i$. Thus for some $n > 0$ we have $x^n \in P_i$. Thus $\overline{x} \in P_i/Q_i$. So β is injective and therefore $\beta \circ \alpha\colon R \to R'$ is an embedding. The prime ideals of $(R/Q_i)_{P_i/Q_i}$ correspond to the prime ideals of R that contain Q_i and are contained in P_i. The only prime ideal with this property is P_i, so $(R/Q_i)_{P_i/Q_i}$ is 0-dimensional. Thus R' is 0-dimensional.

Definition 4.3.5. *A Laskerian ring is a ring in which every ideal is a finite intersection of primary ideals.*

By the previous paragraph, it follows that any Laskerian ring can be embedded into a 0-dimensional ring.

Theorem 4.3.6 (Lasker-Noether)**.** *A Noetherian ring is Laskerian.*

This is proved in [15], page 684.

Corollary 4.3.7. *A Noetherian ring can be embedded into an Artinian ring.*

There exist non-Noetherian Laskerian rings: Let V be a rank 1 valuation ring that is not a DVR. Clearly, V is not Noetherian. We claim that every ideal of V is primary. Certainly, $\langle 0 \rangle$ is primary. Let M be the maximal ideal of V and I a nonzero ideal of R. Since $\dim(V) = 1$, we have $\sqrt{I} = M$. An ideal whose radical is a maximal ideal must be primary ([15], p. 682). Thus I is primary and V is Laskerian.

We now move on to embeddings into arbitrary products of 0-dimensional rings. In [7], the Gilmer radical of a ring R is defined as the intersection of the primary ideals of R. It is denoted by $G(R)$.

Theorem 4.3.8 ([7], Theorem 5)**.** *The following ideals of R are equal:*

1. *$G(R)$.*
2. *The intersection of the kernels of all maps of R into local 0-dimensional rings.*
3. *The intersection of the kernels of all maps of R into 0-dimensional rings.*

Corollary 4.3.9. *$G(R) = \langle 0 \rangle$ if and only if R is a subring of a product of 0-dimensional rings.*

Proof. $\Longrightarrow$ Let $G(R) = \bigcap_{i \in I} Q_i$. Then consider $S = \prod_{i \in I} (R/Q_i)_{P_i/Q_i}$. S is an arbitrary product of 0-dimensional rings. Let $f\colon R \to S$ be the natural map. As in the case of a finite product, f is an embedding.

$\Longleftarrow$ Let $f\colon R \to \prod_{i \in I} (R_i)$ be an embedding of R into a product of 0-dimensional rings. Let $\pi_i\colon \prod_{i \in I} (R_i) \to R_i$ be the natural projection. Then $\pi_i \circ f$ is a map of R into a 0-dimensional ring. The intersection of the kernels of the π_i is the zero ideal, so the intersection of the kernels of all maps of R into 0-dimensional rings is also the zero ideal. Thus $G(R) = \langle 0 \rangle$. ☐

While an arbitrary product of 0-dimensional rings is not always 0-dimensional, it is a reasonable to ask if any such ring can be embedded in a 0-dimensional ring. This is not the case. In [7], it is proved that if S is an arbitrary product of 0-dimensional rings than S can be embedded into a 0-dimensional ring if and only if S is 0-dimensional. We showed in a previous example that $\prod_{n=1}^{\infty} \mathbb{Z}/\langle p^n \rangle$ is not 0-dimensional. Thus this ring cannot be embedded into a 0-dimensional ring. This is not necessarily true for an arbitrary product of rings which are not all 0-dimensional. For example, any arbitrary direct product of integral domains can be embedded in a 0-dimensional ring.

Theorem 4.3.10 (Arapović, [2], Theorem 7)**.** *A ring R is embeddable in a 0-dimensional ring if and only if R has a family of primary ideals $\{Q_i\}_{i \in I}$ such that the following two conditions are met:*

1. $\bigcap_{i \in I} Q_i = \langle 0 \rangle$.
2. For each $a \in R$, there is $n \in \mathbb{N}$ such that for all $i \in I$ if $a \in \sqrt{Q_i}$, then $a^n \in Q_i$.

Proof. $\Longrightarrow$ If R is embeddable in a 0-dimensional ring S, then by the previous corollary $G(S) = \langle 0 \rangle$. Let $\{P_i\}_{i \in I}$ be the family of primary ideals of S. Preimages of primary ideals under homomorphisms are primary, so $\{P_i \cap R\}_{i \in I}$ is a family of primary ideals of R. Since $G(S) = \langle 0 \rangle$, condition 1 holds for $\{P_i \cap R\}_{i \in I}$.

We will now use another characterization of 0-dimensional rings. Namely, a ring T is 0-dimensional if and only if condition 2 holds for the family of all primary ideals of T ([7], Theorem 4). Thus condition 2 holds for $\{P_i\}_{i \in I}$, so it also holds for $\{P_i \cap R\}_{i \in I}$.

$\Longleftarrow$ We follow [1], Theorem 7. Let $S = \prod_{i \in I} (R/Q_i)_{P_i/Q_i}$ and $\phi\colon R \to S$ the diagonal embedding. Then ϕ is injective. Let $a \in R$. Given $i \in I$ let $a_i = 1$ if $a \notin P_i$, let $a_i = 0$ otherwise. Let $e_a = (a_i)_{i \in I}$. For each $r \in R$, identify $r \in R$ with its image $\phi(r)$. It is straightforward to verify that e_r is idempotent, $r + (1 - e_r)$ is regular, and $r(1 - e_r)$ is nilpotent. Let R_1 be the subring of S generated by R and $\{e_r : r \in R\}$. Then we claim that $T(R_1)$ is 0-dimensional. Arguing by contradiction, suppose $P \subset P'$ where P and P' are prime ideals of $T(R_1)$. Then $P \cap R_1 \subset P' \cap R_1$. It is straightforward to show that R_1 is integral over R. By the Incomparability Theorem, $P \cap R \subset P' \cap R$. Let $b \in P' \cap R - P \cap R$. Then since $b(1 - e_b)$ is nilpotent and $b \notin P$, we have $(1 - e_b) \in P$. Thus $b + (1 - e_b) \in P'$ and $b + (1 - e_b)$ is a unit in $T(R_1)$, a contradiction. □

Theorem 4.3.11. *If R can be embedded in a 0-dimensional ring, then $R[\{X_i\}_{i \in I}]$ can be embedded in a 0-dimensional ring.*

This can be proved using Theorem 10, as is done in [7], Theorem 13.

In a chained ring, an arbitrary intersection of primary ideals is primary ([7], Lemma 3). Thus a chained ring V is a subring of a 0-dimensional ring if and only if $\langle 0 \rangle$ is a primary ideal of V. V/I is a chained ring for any ideal I of V. Thus V/I is a subring of a 0-dimensional ring if and only if I is a primary ideal of V.

5 Power Series Rings

Let P be a prime ideal of R. We define

$$P[[X]] = \{f = \sum_{i=0}^{\infty} r_i X^i \in R[[X]] : r_i \in P \; \forall \; i \geq 0\}$$

and $P + \langle X \rangle = \{f = \sum_0^\infty r_i X^i \in R[[X]] : r_0 \in P\}$. These are both prime ideals of $R[[X]]$. Hence as in the case of polynomials, $\dim(R[[X]]) \geq \dim(R) + 1$. We have $PR[[X]] \subseteq P[[X]]$, but equality does not hold in general.

5.1 SFT Rings

Definition 5.1.1. *Let R be a ring. An ideal I of R is an SFT ideal if there exists a finitely generated ideal $J \subseteq I$ and $n \in \mathbb{N}$ such that $x^n \in J$ for each $x \in I$. A ring R is said to be an SFT ring if every ideal of R is an SFT ideal.*

Proposition 5.1.2. *R is an SFT ring if and only if each prime ideal of R is an SFT ideal.*

This proposition can be proved by a Zorn's lemma argument on the set of non-SFT ideals of R, as is done in [3], Proposition 2.2.

Theorem 5.1.3 ([27], Theorem 13)**.** *If R is not an SFT ring, then* $\dim(R[[X]]) \geq \aleph_1$.

In [30], it is conjectured that if R is a non-SFT ring, then $\dim(R[[X]]) \geq 2^{\aleph_1}$. It is also not currently known whether $\dim(R[[X]])$ finite implies $\dim(R[[X,Y]])$ finite ([22], p. 407). An example of a ring R such that $R[[X]]$ is finite-dimensional but not SFT would show that it is not true in general.

Noetherian Rings

Clearly, any Noetherian ring is an SFT ring.

Theorem 5.1.4. *Let R be a Noetherian ring of finite dimension m. Then*

$$\dim(R[[X_1, \ldots, X_n]]) = m + n.$$

Proof. If R is Noetherian, then so is $R[[X]]$. Thus it suffices to prove the theorem for the case $n = 1$. It is straightforward to show that the maximal ideals of $R[[X]]$ are of the form $M + \langle X \rangle$ where M is a maximal ideal of R. Now $\mathrm{ht}(M) \leq m$. There are elements $r_1, \ldots, r_m$ such that M is minimal over $\langle r_1, \ldots, r_m \rangle$. Then $M' + \langle X \rangle$ is minimal over $\langle r_1, \ldots, r_m, X \rangle$. By Krull's Height Theorem, $\mathrm{ht}(M' + \langle X \rangle) \leq m + 1$. Thus $\dim(R[[X]]) = m + 1$. □

5.2 Prüfer Rings

Unlike in the case of polynomials, the dimension theory of power series rings over Prüfer rings is more complicated than Noetherian rings.

Theorem 5.2.1 ([3], Proposition 3.1)**.** *In order that the Prüfer domain D be an SFT-ring, it is necessary and sufficient that for each nonzero prime ideal P of D, there exists a finitely generated ideal A such that $P^2 \subseteq A \subseteq P$.*

Definition 5.2.2. *Let R be a ring. We define a mixed extension of R as follows: $R[X_1]]$ is either $R[X_1]$ or $R[[X_1]]$ and for each $n > 1$, $R[X_1]] \cdots [X_n]]$ is either $(R[X_1]] \cdots [X_{n-1}]])[X_n]$ or $(R[X_1]] \cdots [X_{n-1}]])[[X_j]]$.*

Theorem 5.2.3 ([28], Theorem 14)**.** *Let D be a finite-dimensional SFT Prüfer domain. If at least one of $[X_i]]$ is $[[X_i]]$, then $\dim(D[X_1]] \cdots [X_n]]) = n \dim(D) + 1$.*

Corollary 5.2.4 ([28], Corollary 15)**.** *Let R be a finite-dimensional ring. Then $\dim(R[[X]]) < \infty$ does not imply $\dim(R[[X]]) \leq 2 \dim(R) + 1$.*

Proof. Let D be a finite-dimensional SFT Prüfer domain and

$$D' = D[X_1, \ldots, X_{n-1}].$$

Then $\dim(D') = \dim(D) + n - 1$. If $\dim(D) > 2(n-1)/(n-2)$, then

$$\dim(D'[[X]]) > 2 \dim(D') + 1.$$

□

Dimension Sequences

Just as for polynomial rings we can consider the sequence

$$A_R = \{\dim(R), \dim(R[[X]], \ldots\}.$$

In [22], it is noted that there is no current method to determine whether or not a sequence of nonnegative integers is equal to A_R for some ring R. We know that if R is finite, $R[[X]]$ is not necessarily finite. Therefore we have to restrict to those rings R for which $\dim(R[[X_1, \ldots, X_n]]) < \infty$ for all n. By Corollary 3, not every A_R is an element of $\mathcal{D}$ (as defined in the section on polynomial rings). Let S be an SFT Prüfer domain of finite dimension m. Using the inequality $na_n \leq (n+1)a_{n-1} + 1$ and Theorem 2, we see that A_R is an element of $\mathcal{D}$ if and only if $m \leq 2$.

Valuation Rings

Definition 5.2.5. *A valuation ring V is discrete if each primary ideal of V is a power of its radical.*

The following proposition is proved in [20], pages 192–193.

Proposition 5.2.6. *Let V be a valuation ring whose value group is an additive subgroup of $\mathbb{R}$. Then V is discrete if and and only if the value group of V is discrete as a subspace of $\mathbb{R}$ in the ordinary topology.*

Thus any discrete rank 1 valuation ring is discrete. There are no nontrivial proper convex subgroups of a totally ordered additive subgroup G of $\mathbb{R}$. Thus G has rank 1. Hence having rank 1 is not sufficient to be discrete. In fact, a finite-dimensional valuation ring is discrete if and only if its value group is isomorphic to the lexicographic product of finitely many copies of $\mathbb{Z}$ ([20], p. 205).

Definition 5.2.7. *Let P be a prime ideal of R. P is branched if there exists a P-primary ideal distinct from P. If P is the only P-primary ideal of R, then P is said to be unbranched.*

By [20], Theorem 17.3(b), a valuation ring V is discrete if and only if each branched prime ideal of V is not idempotent. Let V' be a finite-dimensional valuation ring. Then by [20], Theorem 17.3(e), each nonzero prime ideal of V' is branched. It follows that V' is discrete if and only if the only idempotent prime ideal of V' is $\langle 0 \rangle$. By Theorem 1, V' is discrete if and only if it is an SFT ring. Thus we obtain the following result:

Theorem 5.2.8. *Let V be a finite-dimensional valuation ring. Then if V is discrete, then*

$$\dim(V[[X]]) = \dim(V) + 1.$$

If V is not discrete, then $\dim(V[[X]]) \geq \aleph_1$.

If V is an arbitrary valuation ring that is not discrete, then $\dim(V[[X]]) \geq \aleph_1$ ([29], Corollary 18).

An example of an SFT ring V' such that $\dim(V'[[X]]) \geq \aleph_1$

Definition 5.2.9. *Let G be a totally ordered Abelian group. We say G is Archimedean if a, $b \in G$ with $a > 0$ implies that there is a positive integer n such that $na > b$.*

The following example is a straightforward generalization of the example given in [12], page 86. Let V be a rank one nondiscrete valuation ring with maximal ideal M and residue field K. Let $x \neq 0 \in M$ and $V' = K + xV$. It is straightforward to prove that a valuation ring has rank one if and only if its value group is Archimedean. It follows that V is an integral ring extension of V' and $xV = M \cap V'$ is the unique nonzero prime ideal of V'. To show that V' is an SFT ring it suffices to show that xV is an SFT ideal. So let $y \in xV$. Then $y = xv$ for some $v \in V$. We have $y^2 = x(xv^2) \in xV' \subseteq xV$. Hence xV is an SFT ideal. There exists an uncountable chain $\{Q_a\}_{a \in R^+}$ of prime ideals inside $M[[X]]$ such that each $Q_a \cap V = \langle 0 \rangle$ ([29], Theorem 16). Suppose that $Q_r \cap V'[[X]] = Q_s \cap V'[[X]]$ with $r \neq s$. Let $Q_r \subset Q_s$ and $z \in Q_s - Q_r$. Then $xz \in Q_r \cap V'[[X]]$. Now $z \notin Q_r$, so we must have $x \in Q_r$. This is a contradiction, since $Q_r \cap V = \langle 0 \rangle$. Thus $\{Q_a \cap V'[[X]]\}_{a \in R^+}$ is an uncountable chain of prime ideals of $V'[[X]]$.

5.3 0-dimensional Rings

Theorem 5.3.1. *The following are equivalent for a 0-dimensional ring R:*

1. *R satisfies the SFT-property.*
2. $\dim(R[[X_1, \ldots, X_n]]) = n$.
3. $\dim(R[[X]]) < \infty$.

Proof. (1) $\Longrightarrow$ (2) Let S be an SFT ring. Then the minimal prime ideals of $S[[X_1, \ldots, X_n]]$ are of the form $P[[X_1, \ldots, X_n]]$ where P is a minimal prime ideal of S ([11], Theorem 2). Thus the minimal prime ideals of $A = R[[X_1, \ldots, X_n]]$ are of the form $M[[X_1, \ldots, X_n]]$ where M is a maximal ideal of R. Then $A/M[[X_1, \ldots, X_n]] \cong (R/M)[[X_1, \ldots, X_n]]$ and $\dim(A/M[[X_1, \ldots, X_n]]) = n$. Since this holds for all minimal prime ideals of A, it follows that $\dim(A) = n$.

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1) If R does not satisfy the SFT property, then $\dim(R[[X]]) = \infty$. □

von Neumann Regular Rings

If R is a SFT ring, then R satisfies the ascending chain condition for radical ideals ([3], Proposition 2.5). Every ideal of a von Neumann regular ring is a radical ideal. So if a von Neumann regular ring is SFT, then it is Artinian. Thus a von Neumann regular ring is SFT if and only if it is a finite direct product of fields. A von Neumann regular ring R that is not a finite direct product of fields therefore has $\dim(R) \geq \aleph_1$.

5.4 Dimension of $R[[X]]$ in terms of Dimension of $R[X]$

In all of the examples we have shown in which $\dim(R[[X]])$ is finite,

$$\dim(R[X]) \leq \dim(R[[X]]).$$

It is a reasonable question to ask if this always is the case. In [3], section 4 it is shown that this is not true in general. Let $V = K + M$ be a rank one discrete valuation ring and $D = k + M$. It follows from Theorem 3.1.14 that $\dim(D) = 1$ and $\dim(D[X]) = 3$. Any prime ideal P of $D[[X]]$ with $P \cap D = \langle 0 \rangle$ is minimal ([3], p. 10). Furthermore, no such prime ideal is contained in $M[[X]]$. Suppose Q is a nonzero prime ideal of $D[[X]]$ with $Q \subseteq M[[X]]$. Then we must have $Q \cap D = M$. Since M is a principal ideal of R, it follows that $Q = M[[X]]$. Thus $M[[X]]$ is minimal. Now let $\langle 0 \rangle \subset Q_1 \subset Q_2$ be a chain of prime ideals of $D[[X]]$. Then $M[[X]] \subset Q_2$. Let $f = \sum_0^\infty a_i X^i \in Q_2 - M$. A power series is invertible if and only if its constant term is invertible. Thus $a_0 \in M$. Since Q_2 is prime, it follows that $X \in Q_2$ and therefore $Q_2 = M + \langle X \rangle$ is maximal. Hence $\dim(D[[X]]) = 2$.

5.5 Entire and Analytic Functions

A complex-valued function that is holomorphic over the whole complex plane is called an entire function. Throughout this section, R denotes the ring of entire functions.

Definition 5.5.1. *Let f be an entire function and I an ideal of R.*

1. *We define $A(f) = \{z \in \mathbb{C}\}$ and $A^*(f)$ as the sequence of zeros of f, arranged in order of increasing modulus. (It is a basic result of complex analysis that the zero set of a nonzero holomorphic function is at most countable). Note that if z is a zero of multiplicity m of f, then z appears m times in $A(f)$.*

2. *If $A^*(f) = \{a_n\}$, we define $O_n(f)$ as the multiplicity of a_n as a zero of f. If A is a subset of $A^*(f)$ we define $O_n(f : A)$ as the function $O_n(f)$ with domain A.*

3. *Let M be a maximal ideal of R and $f, g \in M$. If there exists $e \in M$ with $A^*(e) \subseteq A^*(f) \cap A^*(g)$ and such that $O_n(f : A * (e)) \geq O_n(g : A * (e))$ then we write $f \geq g$.*

4. *If $f \geq g^N$ for all positive integers N or if $f = 0$ we write $f \gg g$.*

Every prime ideal of R is contained in an unique maximal ideal ([25], Theorem 2). Suppose that M is a maximal ideal of R and $\Omega \subseteq M$. Let P_M be the set of prime ideals contained in M and $P_\Omega = \{f \in M : f \gg g \text{ for all } g \in \Omega\}$. Then P_Ω is a prime ideal. If P is a prime ideal in P_M then $P = P_\Omega$ where $\Omega = M - P$ ([25], Theorem 4). Thus P_M is linearly ordered under inclusion. In [25],Theorem 5 it is shown that $|P_M| \geq 2^{\aleph_1}$. Every holomorphic function can be given in the form of a power series that converges on the whole complex plane. Hence $|P_M| = 2^{\aleph_1}$.

In [32], page 358 it is noted that this result is true for the ring of holomorphic functions on a region of the complex plane. (Recall that a region is an open and connected subset of the complex plane).

The previous result is generalized to two real-valued cases. Let $f\colon \mathbb{R} \to \mathbb{R}$. We say that f is real analytic if for each $x_0 \in R$ there exists an open neighborhood V of x_0 such that for all $x \in V$ $f(x)$ is the sum of an absolutely convergent power series in powers of $x - x_0$. We say f is entire if f is given by a power series $\sum_{n=0}^{\infty} a_n x^n$ with $\lim_{x\to\infty} |a_n|^{1/n} = 0$. Denote the ring of real analytic functions by $\mathcal{A}(\mathbb{R})$ and the ring of real entire functions by $\mathcal{E}(\mathbb{R})$. In [24], Theorem 4.7 it is shown that $\dim(\mathcal{R}) \geq 2^{\aleph_1}$ where $\mathcal{R}$ is either $\mathcal{A}(\mathbb{R})$ or $\mathcal{E}(\mathbb{R})$. Since both of these rings are contained in the ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, we have $\dim(\mathcal{R}) = 2^{\aleph_1}$.

5.6 Almost Dedekind Domains

Definition 5.6.1. *An integral domain D is called almost Dedekind provided D_M is a rank 1 DVR for each maximal ideal M of D.*

Clearly any Dedekind ring is almost Dedekind. Since Dedekind rings are Noetherian, it follows that $\dim(D[[X]]) = \dim(D) + 1 = 2$. There are almost Dedekind rings that are not Dedekind, as the following example shows:

Example 5.6.2. *Let $\{X_i\}_{i\in I}$ be a nonempty set of indeterminates and $D_n = \mathbb{Q}[\{X_i^{1/2^n}\}_{i\in I}]$ for $n \geq 0$. Set $D = \bigcup_{n=0}^{\infty} D_n$ and $S = \mathbb{Q}[\{X_i\}_{i\in I}] - \bigcup_{i\in I}(1 - X_i)\mathbb{Q}[\{X_i\}_{i\in I}]$. Then D_S is an almost Dedekind domain that is not Dedekind. Let $Q_i = \bigcup_{n=0}^{\infty}(1 - X_i^{1/2^n})D_n$. Then the maximal ideals of D_S are of the form $(Q_i)_S$ and $(1 + X_i^{1/2^n})D_S$. We have $\mathrm{ht}((Q_i)_S[[X]]/(Q_i)_S D_S[[X]]) \geq 2^{\aleph_1}$ ([9], Theorem 3.7). Note that if I is countable, then $\mathrm{ht}((Q_i)_S[[X]]/(Q_i)_S D_S[[X]]) = \dim(D_S) = 2^{\aleph_1}$.*

If D is almost Dedekind but not Dedekind, then $\dim(D[[X]]) \geq 2^{\aleph_1}$. Thus if D is countable, then $\dim(D[[X]]) = 2^{\aleph_1}$. In the next few paragraphs, we will summarize some preliminary results in [9] in order to prove this.

Definition 5.6.3. *Let $(A, \gg)$ be a totally ordered set and B, C be subsets of A. We say $B \gg C$ if $b \gg c$ for each $b \in B$ and $c \in C$. A totally ordered set $(A, \gg)$ is called an η_1-set if for any two countable subsets B, C such that $B \gg C$, there exists an element $a \in A$ such that $B \gg a \gg C$.*

By Corollary 2.6.6, there exists a nonprincipal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Let Φ be the set of all functions $\phi\colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$. Let ϕ, $\psi \in \Phi$. If for each positive integer k, there exists a set $F_k \in \mathcal{F}$ with $\phi(n) > k\psi(n)$ for all $n \in F_k$ then we write $\phi \gg \psi$. Note that $\gg$ is transitive. If there is a positive integer k and a set $F \in \mathcal{F}$ such that $k\phi(n) \geq \psi(n)$ and $k\psi(n) \geq \phi(n)$ then we write $\phi \sim \psi$. Note that $\sim$ is an equivalence relation. For any ϕ, ψ in Φ, exactly one of the following holds: $\phi \gg \psi$, $\phi \sim \psi$, or $\phi \gg \psi$ ([9], Proposition 1.2).

Let $[\Phi] = \Phi/\sim$ be the set of equivalence classes of elements in Φ. For $\phi \in \Phi$ let $[\phi]$ be the equivalence class of ϕ. By abusing notation, define $[\phi] \gg [\psi]$ if $\phi \gg \psi$. Then $\gg$ is well-defined and by [9], Proposition 1.2, $([\Phi], \gg)$ is a totally ordered set. Let

$$\Psi = \{\psi \in \Phi : \psi(n) < \infty \text{ for all } n \geq 0 \text{ and } \lim_{n\to\infty} \psi(n) = \infty\}.$$

Let $[\Psi]$ be the set of equivalence classes of elements of Ψ. Then $[\Psi]$ is an η_1-set and hence $[\Phi]$ contains an η_1-set ([9], Theorem 1.11).

Since D is not Dedekind, there exists some noninvertible maximal ideal M of D. Let $p \in M - 0$. Then one can construct two countably infinite sets $\{M_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ (where each

M_n is a maximal ideal and each $p_n \in D$) such that $p \in \bigcap_{n=1}^{\infty} M_n$ and each $p_n \in M \cap M_m$ except for $m = n$. Let $M_0 = M$ and for each $f = \sum_{i=0}^{\infty} d_n X^n \in D[[X]]$ define ϕ_f by

$$\phi_f(n) = \begin{cases} \min\{i : f_i \notin M_n\} & \text{if } f \notin M_n[[X]] \\ \infty & \text{if } f \in M_n[[X]] \end{cases}.$$

Then $\phi_f \in \Phi$. For f,$g \in D[[X]]$, define $f \sim g$ if $\phi_f \sim \phi_g$ and $f \gg g$ if $\phi_f \gg \phi_g$. Let $P_f = \{h \in D[[X]] : h \gg f\}$. Then either P_f is the empty set or P_f is a prime ideal of $D[[X]]$. Furthermore, $P_f \subset P_g$ if and only if $f \gg g$ ([9], Lemma 2.3).

Let $\phi \in \Psi$. Then there exists $f \in M[[X]]$ such that $\phi_f(n) = \phi(n)$ for all $n \geq 1$ ([9], Lemma 2.4). For each ψ in Ψ, let $f_\psi \in M[[X]]$ be such that $\phi_{f_\psi}(n) = \psi(n)$. Then $\phi \gg \psi$ if and only if $f_\phi \gg f_\psi$. Let Ψ_0 be a set of coset representatives of Ψ and $\Omega = \{f_\psi : \psi \in \Psi_0\}$. Then $(\Omega, \gg)$ is an η_1-set. It follows that $\mathcal{P} = \{P_{f_\psi} : f_\psi \in \Omega\}$ is also an η_1-set.

Theorem 5.6.4 ([9], Theorem 1.12)**.** *Let X be a nonempty set and let $\mathcal{B} = \{A_i\}_{i \in I}$ be a nonempty family of subsets of X. If $\mathcal{B}$ is totally ordered (under inclusion), then so is the set $\mathcal{B}^* = \{\bigcup_{j \in J} A_j : \emptyset \neq J \subseteq I\}$. Furthermore, if $\mathcal{B}^*$ contains an η_1-set, then the cardinality of $\mathcal{B}^*$ is at least $2^{\aleph_1}$.*

Theorem 5.6.5 ([9], Theorem 2.6)**.** *If D is almost Dedekind but not Dedekind, then* $\dim(D[[X]]) \geq 2^{\aleph_1}$.

Proof. Let $\mathcal{P}^* = \{\bigcup_{A \in \mathcal{A}} : \emptyset \neq \mathcal{A} \subseteq \mathcal{P}\}$. Then $\mathcal{P}^*$ contains the η_1-set $\mathcal{P}$, so by the previous theorem the cardinality of $\mathcal{P}^*$ is at least $2^{\aleph_1}$. Since $\mathcal{P}$ is totally ordered, every element of $\mathcal{P}^*$ is a prime ideal of $D[[X]]$ and therefore $\dim(D[[X]]) \geq 2^{\aleph_1}$. □

References

[1] M. Arapović. The minimal 0-dimensional overrings of commutative rings. Glas. Mat. **18** (1983), 47–52.

[2] M. Arapović. On the embedding of a commutative ring into a 0-dimensional ring. Glas. Mat. **18** (1983), 53–59.

[3] J.T. Arnold. Power series rings over Prüfer domains. Pacific J. Math. **44** (1973), no. 1, 1–11.

[4] J.T. Arnold and R. Gilmer. On the dimension sequence of a commutative ring. Amer. J. Math. **90** (1974), 385–408.

[5] J.T. Arnold and R. Gilmer. Dimension theory of commutative rings without identity. J. Pure Appl. Algebra **5** (1974), 209–231.

[6] J.W. Brewer, P.R. Montgomery, E.A. Rutter, and W.J. Heinzer. Krull dimension of polynomial rings, in: A. Dold and B. Eckmann (Eds), Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, **311** (1973), 26–45.

[7] J.W. Brewer, F. Richman. Subrings of zero-dimensional rings, in: J.W. Brewer, S. Glaz, W.J. Heinzer, and B.M. Olberding (Eds), Multiplicative Ideal Theory in Commutative Algebra: A Tribute to the Work of Robert Gilmer. Springer, New York, (2006), 73–88.

[8] P.-J. Cahen and J.-L. Chabert. Integer-Valued Polynomials. Amer. Math. Soc. Surveys and Monographs, Providence, **48** (1997).

[9] G.W. Chang, B.G. Kang, and P.T. Toan. The Krull dimension of power series rings over almost Dedekind domains. J. Algebra **438** (2015) 170–187.

[10] P. M. Cohn. Algebraic numbers and Algebraic Functions. Chapman and Hall Mathematics Series. Chapman & Hall, London, (1991).

[11] J.T. Condo, J. Coykendall, and D.E. Dobbs. Formal power series rings over zero-dimensional SFT-rings. Comm. Algebra **24** (1996), 2687–2698.

[12] J. Coykendall. The SFT property does not imply finite dimension for power series rings. J. Algebra **256** (1) (2002), 85–96.

[13] A. Dasgupta. Set Theory: With an Introduction to Real Point Sets. Springer, New York, (2014).

[14] D. E. Dobbs. A going-up theorem for arbitrary chains of prime ideals. Comm. Algebra **27** (1999), 3887–3894.

[15] D.S. Dummit and R.M. Foote. Abstract Algebra (3rd ed). Wiley, Hoboken, (2004).

[16] D. Eisenbud. Commutative Algebra: With a View Toward Algebraic Geometry. Graduate Texts in Mathematics. Springer-Verlag, New York, **150** (1995).

[17] M. Fontana and S.-E. Kabbaj. Essential domains and two conjectures in dimension theory. Proc. Amer. Math. Soc. **132** (2004), 2529–2535.

[18] M. Fontana, L. Izelgue, S.-E. Kabbaj, and F. Tartarone. On the Krull dimension of domains of integer-valued polynomials. Exposition. Math. **15** (1997), 433–465.

[19] L. Fuchs and L. Salce. Modules over non-Noetherian Domains. Amer. Math. Soc. Surveys and Monographs, Providence, **84** (2001).

[20] R. Gilmer. Multiplicative Ideal Theory. Pure and Applied Mathematics: A Series of Monographs and Textbooks. Marcel Dekker, New York, **12** (1972).

[21] R. Gilmer. Background and preliminaries on zero-dimensional rings. Zero- Dimensional Commutative Rings, in: D.F. Anderson and D.E. Dobbs (Eds), Proc. 1994 John H. Barrett Memorial Lecture and Conference on Commutative Ring Theory. Marcel Dekker, New York, **171** (1995), 1–13.

[22] R. Gilmer. Some questions for further research, in: J.W. Brewer, S. Glaz, W.J. Heinzer, and B.M. Olberding (Eds), Multiplicative Ideal Theory in Commutative Algebra: A Tribute to the Work of Robert Gilmer. Springer, New York, (2006), 405–415.

[23] R. Gilmer and W.J. Heinzer. Products of commutative rings and zero-dimensionality. Trans. Amer. Math. Soc. **331** (1992), 663–680.

[24] M. Golasiński and M. Henriksen. Residue class rings of real-analytic and entire functions. Colloq. Math. **104** (1) (2006), 85–97.

[25] M. Henriksen. On the prime ideals of the ring of entire functions. Pacific J. Math. **3** (1953), 711–720.

[26] H.C. Hutchins. Examples of Commutative Rings. Polygonal Publishing House, Passaic, N.J. (1981).

[27] B.G. Kang, K.A. Loper, T.G. Lucas, M.H. Park, and P.T. Toan. The Krull dimension of power series rings over non-SFT rings. J. Pure Appl. Algebra **217** (2) (2013), 254–258.

[28] B.G. Kang and M.H. Park. Krull dimension of mixed extensions. J. Pure Appl. Algebra **213** (10) (2009), 1911–1915.

[29] B.G. Kang. Krull dimension of the power series ring over a nondiscrete valuation domain is uncountable. J. Algebra **378** (2013), 12–21.

[30] B.G. Kang. How to construct huge chains of prime ideals in power series rings, in: M. Fontana, S. Frisch, and S. Glaz (Eds), Commutative Algebra: Recent Advances in Commutative Rings, Integer-Valued Polynomials, and Polynomial Functions. Springer, New York, (2014), 225–238.

[31] I. Kaplansky. Projective modules. Ann. of Math. **68** (2) (1958), 372–377.

[32] M.S. Osborne. Basic Homological Algebra. Graduate Texts in Mathematics. Springer-Verlag, New York, **196** (2000).

[33] T. Parker. A number-theoretic characterization of dimension sequences. Amer. J. Math. **97** (1975), 308–311.

[34] A. Seidenberg. A note on the dimension theory of rings. Pacific J. Math. **3** (1953), 505–512.

[35] A. Seidenberg. On the dimension theory of rings (II). Pacific J. Math. **4** (1954), 603-614.

[36] J. Shapiro. The prime spectrum of an infinite product of zero-dimensional rings. Zero-Dimensional Commutative Rings, in: D. F. Anderson and D. E. Dobbs (Eds), Proc. 1994 John H. Barrett Memorial Lecture and Conference on Commutative Ring Theory. Marcel Dekker, New York, **171** (1995), 347–356.

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