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"Asymptotically holomorphic extensions and their application to Denjoy-Carleman classes"

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Abstract

The aim of this thesis is to characterize certain smooth function classes defined on $\mathbb{R}$ by the existence of extensions to $\mathbb{C}$ whose derivative with respect to $\overline{z}$ vanishes on $\mathbb{R}$ and decreases rapidly near $\mathbb{R}$. Such extensions are called asymptotically holomorphic extensions. The speed of decay of the derivative with respect to $\overline{z}$ of such an extension then determines the regularity of the given function defined on $\mathbb{R}$. We focus on a characterization of Denjoy-Carleman classes. Those are classes of smooth functions which satisfy a growth condition on all their derivatives in terms of a weight sequence, i.e. the n-th derivative can be dominated by the n-th weight.

Most of the proofs of the major theorems use the so-called two constants theorem. In order to formulate the theorem, we define in chapter 2 the concept of harmonic measure for Jordan domains and for certain unbounded domains. Apart from proving the two constants theorem, we also give a geometric description of the harmonic measure.

In chapter 3 we characterize Denjoy-Carleman classes in terms of the existence of asymptotically holomorphic extensions. We use this characterization to derive an alternative proof of closedness under composition of such classes.

In chapter 4 we give an alternative proof of the Denjoy-Carleman theorem which also uses this characterization. We are actually able to show a quantitative result in the case of non-quasianalytic weight sequences.

In chapter 5 we present two results of Borichev and Volberg that can be used to prove that under certain conditions a system of ODEs defined by a quasianalytic function only admits finitely many limit cycles.

Zusammenfassung

Das Ziel dieser Arbeit ist es, gewisse Klassen glatter auf $\mathbb{R}$ definierter Funktionen durch die Existenz von Ausdehnungen nach $\mathbb{C}$, deren Ableitung nach $\overline{z}$ auf $\mathbb{R}$ verschwindet und nahe $\mathbb{R}$ rasch abfällt, zu charakterisieren. Solche Ausdehnungen werden asymptotisch holomorphe Ausdehnungen genannt. Das Abfallverhalten der Ableitung nach $\overline{z}$ solch einer Ausdehnung bestimmt dann die Regularität der gegebenen, auf $\mathbb{R}$ definierten, Funktion. Wir konzentrieren uns auf eine Charakterisierung von Denjoy-Carleman Klassen. Dies sind Klassen glatter Funktionen, die eine Wachstumsbedingung an alle Ableitungen bezüglich einer Gewichtsfolge erfüllen, d.h. die n-te Ableitung kann durch das n-te Gewicht dominiert werden.

Die meisten Beweise der Haupttheoreme verwenden das so genannte Zwei-Konstanten-Theorem. Um dieses formulieren zu können, definieren wir in Kapitel 2 das Konzept des harmonischen Maßes für Jordan Gebiete und für gewisse unbeschränkte Gebiete. Neben einem Beweis des Zwei-Konstanten-Theorems erarbeiten wir auch eine geometrische Beschreibung des harmonischen Maßes.

In Kapitel 3 charakterisieren wir Denjoy-Carleman Klassen durch die Existenz asymptotisch holomorpher Ausdehnungen. Wir verwenden diese Charakterisierung um einen alternativen Beweis der Abgeschlossenheit solcher Klassen unter Komposition herzuleiten.

In Kapitel 4 geben wir einen alternativen Beweis des Denjoy-Carleman Theorems, der auch diese Chrakterisierung verwendet. Im Falle einer nicht-quasianalytischen Gewichtsfolge ist es uns sogar möglich ein quantitatives Resultat zu zeigen.

In Kapitel 5 präsentieren wir zwei Resultate von Borichev und Volberg, die verwendet werden können um zu zeigen, dass ein System gewöhnlicher Differentialgleichungen, das durch eine quasianalytische Funktion definiert ist, unter gewissen Voraussetzungen nur endlich viele Limes Zyklen zulässt.

1 Prerequisites

This chapter represents a collection of several definitions and theorems being used throughout the following chapters. The majority of the theorems deals with subharmonic, harmonic and analytic functions. Before treating those theorems we fix notation:

There are some specific subsets of $\mathbb{C}$ that will appear quite frequently:

As it is common in the literature

$$\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$$

denotes the upper half-plane. We will write

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \Re(z) > 0\}$$

for the right half-plane and

$$\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$$

for the positive reals. In addition for $a \in \mathbb{C}$ and $r > 0$,

$$B(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$$

denotes the ball with center a and radius r. For the unit disc we will also use a different notation, namely

$$\mathbb{D} := B(0, 1).$$

In the following z and ζ always denote complex numbers. If not otherwise defined we will write $x = \Re(z)$ ($\xi = \Re(\zeta)$) and $y = \Im(z)$ ($\eta = \Im(\zeta)$).

By *smooth* functions we understand infinitely many times continuously differentiable functions (i.e. C^∞-functions).

For a continuously differentiable function f defined on some open set $U \subseteq \mathbb{C}$ (i.e. $f \in C^1(U)$) we define the so-called *Wirtinger derivatives*:

$$\partial f(z) := \frac{1}{2}\left(\frac{\partial f}{\partial x}(z) - i\frac{\partial f}{\partial y}(z)\right),$$
$$\overline{\partial} f(z) := \frac{1}{2}\left(\frac{\partial f}{\partial x}(z) + i\frac{\partial f}{\partial y}(z)\right).$$

We will also write $\frac{\partial f}{\partial z}$ instead of ∂f (resp. $\frac{\partial f}{\partial \overline{z}}$ instead of $\overline{\partial} f$). $\overline{\partial} f$ is also referred to as d-bar derivative or derivative with respect to $\overline{z}$.

It is a classical result of one variable function theory that $\overline{\partial} f \equiv 0$ on an open set $U \subseteq \mathbb{C}$ is equivalent to f being holomorphic on U. In this case ∂f is the ordinary complex derivative.

The following lemma gathers elementary properties of the Wirtinger derivatives.

Lemma 1.0.1. *Let $\alpha, \beta \in \mathbb{C}$ and $f, g \in C^1(U)$ for some open $U \subseteq \mathbb{C}$. Let $h \in C^1(V)$ where $f(U) \subseteq V$. Then*

$$\overline{\partial}(\alpha f + \beta g) = \alpha\overline{\partial} f + \beta\overline{\partial} g, \tag{1.0.1}$$

$$\overline{\partial}(fg) = (\overline{\partial} f)g + f(\overline{\partial} g), \tag{1.0.2}$$

$$\partial(h \circ f)(z) = \partial h(f(z))\partial f(z) + \overline{\partial} h(f(z))\partial\overline{f}(z), \tag{1.0.3}$$

$$\overline{\partial}(h \circ f)(z) = \partial h(f(z))\overline{\partial} f(z) + \overline{\partial} h(f(z))\overline{\partial f}(z). \tag{1.0.4}$$

(1.0.1) *and* (1.0.2) *hold with $\overline{\partial}$ replaced by ∂ as well.*

Proof. Observe that $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ fulfil (1.0.1) and (1.0.2) (with $\overline{\partial}$ replaced by $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$). Since ∂ and $\overline{\partial}$ are linear combinations of these operators, (1.0.1) and (1.0.2) follow. For (1.0.3) and (1.0.4), see [12, 1.4.4 (4), p. 62]. □

We will write $d\xi \wedge d\eta$ for the standard base element of 2-forms on $\mathbb{R}^2$. Quite frequently we will integrate with respect to $d\zeta \wedge d\overline{\zeta}$. Those 2-forms are connected as follows

$$d\zeta \wedge d\overline{\zeta} = (d\xi + id\eta) \wedge (d\xi - id\eta) = -2id\xi \wedge d\eta.$$

Definition Let $\Omega \subseteq \mathbb{C}$ be open. A function $f : \Omega \to \mathbb{R} \cup \{-\infty\}$ is *upper-semicontinuous* iff for all $z_0 \in \Omega$

$$\limsup_{z \to z_0} f(z) \leq f(z_0).$$

An upper-semicontinuous function f is *subharmonic* iff for all $z \in \Omega$ and $r > 0$ such that $\overline{B(z, r)} \subseteq \Omega$

$$f(z) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it})dt.$$

This means, that $f(z)$ is smaller than the arithmetic mean of f over each circle lying in Ω.

Theorem 1.0.2 (Jensen's formula)**.** *Let f be analytic in an open neighbourhood of $\overline{B(0, r)}$ and assume $f(0) \neq 0$. Then*

$$\log|f(0)| + \int_0^r \frac{n(t)}{t} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(re^{it})|dt$$

where $n(t) := |\{z \in B(0, t) : f(z) = 0\}|$.

Proof. See [9, I, p. 1-3]. □

There are some important facts on subharmonic functions that will be used:

Lemma 1.0.3. *(i) Subharmonic functions are closed under taking sums and maximums, i.e. for f, g subharmonic also $f + g$ and $\max(f, g)$ are subharmonic.*

(ii) Maximum principle for subharmonic functions:

Let $\Omega \subseteq \mathbb{C}$ be a domain and f subharmonic on Ω. If there exists $a \in \Omega$ such that $f(a) \geq f(z)$ for all $z \in \Omega$, then f is constant on Ω.

(iii) Let f be analytic, then

$$z \mapsto \log |f(z)|$$

is subharmonic.

Proof. (i): Follows directly from the definition.

(ii): See [4, 3.2, p. 264].

(iii): If z_0 is a zero of f, $\log |f(z_0)| = -\infty$ trivially fulfils the defining inequality for subharmonic functions. If not, apply Jensen's formula (w.l.o.g. $z_0 = 0$), which leads to

$$\log |f(0)| + \underbrace{\int_0^r \frac{n(t)}{t} dt}_{\geq 0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{it})| dt$$

implying subharmonicity of $\log |f(z)|$.

□

Theorem 1.0.4. *Let $\Omega \subseteq \mathbb{C}$ be open and u harmonic on Ω. Then for all $z \in \Omega$ and $r > 0$ such that $\overline{B(z,r)} \subseteq \Omega$*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it}) dt. \tag{1.0.5}$$

That means u has the mean value property for all $z \in \Omega$.

Proof. See [4, 1.4 Mean value theorem, p. 253]. □

Remark 1.0.5. *This shows that harmonic functions are subharmonic.*

Theorem 1.0.6 (Maximum principle)**.** *Let $\Omega \subseteq \mathbb{C}$ and u a continuous real valued function on Ω having the mean value property for all $z \in \Omega$ (e.g. u harmonic). If there exists $a \in \Omega$ such that*

$$u(a) \geq u(z)$$

for all $z \in \Omega$. Then u is constant.

Proof. Follows from lemma 1.0.3 (ii). □

Remark 1.0.7. *From the maximum principle it follows immediately that harmonic functions on bounded domains, extending continuously to the boundary, attain their maximum on the boundary.*

The following is a generalisation of the fact that harmonic functions on bounded domains attain their maximum on the boundary. If we drop the assumption of boundedness of the domain, it is clear that the maximum need not be attained at the boundary. This is simply due to the fact that there are unbounded harmonic functions. Take for example $f(z) = \Im(z)$, then f is bounded on the real line by 0 but certainly this is not the maximum of the function on the upper half plane. However, we have the following lemma whose proof is taken from [7, Lemma 1.1, p. 2]:

Lemma 1.0.8. *Let u be harmonic and bounded on some open $\Omega \subseteq \mathbb{C}$ with the additional property $\overline{\Omega} \neq \mathbb{C}$. Suppose*

$$\limsup_{z \to \zeta} u(z) \leq 0 \tag{1.0.6}$$

only fails for finitely many $\zeta \in \partial\Omega$. Then

$$u(z) \leq 0 \text{ on } \Omega.$$

Proof. Due to the assumption $\overline{\Omega} \neq \mathbb{C}$, it is possible to assume w.l.o.g. that Ω is bounded:

There exists some $z_0 \in \mathbb{C} \backslash \overline{\Omega}$. Because $\mathbb{C} \backslash \overline{\Omega}$ is open, there is some $r > 0$, such that $B(z_0, r) \subseteq \mathbb{C} \backslash \overline{\Omega}$. Therefore the absolute value of $\psi(z) := 1/(z - z_0)$ is bounded by $\frac{1}{r}$ on Ω. As ψ is analytic and not locally-constant on Ω, it follows that $\psi(\Omega)$ is open and contained in $B(0, 1/r)$. As ψ' does not vanish on Ω (and ψ is one-to-one), it follows that $\psi^{-1} : \psi(\Omega) \to \Omega$ is analytic. As u is harmonic, it can be written as the real part of some analytic function $\tilde{u}$. Therefore $u \circ \psi^{-1} = \Re(\tilde{u} \circ \psi^{-1})$. As real and imaginary parts of an analytic function are harmonic, one immediately gets that $u \circ \psi^{-1}$ is harmonic.

So it is possible to reduce the proof for unbounded Ω to the bounded case via an application of the function ψ^{-1}.

Now suppose Ω is bounded. Let $F = \{\zeta_1, \ldots, \zeta_n\}$ be the finite set where condition (1.0.6) fails. Then for every $\varepsilon > 0, 1 \leq j \leq n$, the function f_j^ε defined for $z \in \Omega$ by

$$z \mapsto \varepsilon \log \left(\frac{\operatorname{diam}(\Omega)}{|z - \zeta_j|} \right)$$

is positive (as $\frac{\operatorname{diam}(\Omega)}{|z - \zeta_j|} > 1$ for $z \in \Omega$) and a direct computation shows $\Delta f_j^\varepsilon \equiv 0$. Now define

$$u_\varepsilon(z) := u(z) - \sum_{j=1}^{n} f_j^\varepsilon(z).$$

Since $f_j^\varepsilon(z) \to +\infty$ as $z \to \zeta_j$, $\limsup_{z \to \zeta} u_\varepsilon(z)$ is non-positive for all boundary values ζ. So by applying the ordinary maximum principle for harmonic functions, it follows that $u_\varepsilon \leq 0$ on Ω. This implies $u(z) \leq \sum_{j=1}^{n} f_j^\varepsilon(z)$ for all $\varepsilon > 0$. But as the right hand side tends to zero, as $\varepsilon \to 0$, one gets $u(z) \leq 0$. Thus the proof is completed. ☐

Theorem 1.0.9. *Let $\Omega \subseteq \mathbb{C}$ be a domain and f analytic on Ω. Then for all $z \in \Omega$ and $r > 0$ such that $\overline{B(z, r)} \subseteq \Omega$*

$$|f(z)| \leq \max_{\zeta \in \partial B(z,r)} |f(\zeta)|.$$

If Ω is in addition bounded and f continuously extends to the boundary, f attains it's maximum on the boundary.

Proof. See [13, 10.24 The maximum modulus theorem, p. 212]. $\square$

In analogy to the above generalisation for harmonic functions, there is a similar generalisation for analytic functions.

Theorem 1.0.10. *Let $\Omega \subseteq \mathbb{C}$ be a domain and $\Omega \neq \mathbb{C}$. Let f be analytic and bounded on Ω and assume that $\limsup_{z \to \zeta} |f(z)| \leq m$ for all $\zeta \in \partial\Omega$, then $|f(z)| \leq m$ for all $z \in \Omega$.*

Proof. See [9, III B, p. 23]. $\square$

A very important formula, with numerous applications, consists of the inhomogeneous Cauchy integral formula:

Theorem 1.0.11. *Let $\Omega \subseteq \mathbb{C}$ be bounded and have piecewise smooth boundary $\partial\Omega$. In addition let $\partial\Omega$ be positively oriented. If f is continuously differentiable on an open set $V \supset \overline{\Omega}$, then for all $z \in \Omega$*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\Omega} \frac{\overline{\partial} f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta} \tag{1.0.7}$$

Proof. See [8, Theorem 1.2.1, p. 3]. $\square$

The following simple observation will be used in many proofs in which the non-holomorphic summand of the representation from theorem 1.0.11 is estimated.

Lemma 1.0.12. *For $z_0 \in \mathbb{C}$ and $r > 0$*

$$\int_{B(z_0,r)} \frac{1}{|\zeta - z_0|} d\xi \wedge d\eta = 2\pi r.$$

Proof. A change to polar coordinates yields

$$\int_{B(z_0,r)} \frac{1}{|\zeta - z_0|} d\xi \wedge d\eta = \int_0^{2\pi} \int_0^r \frac{1}{s} s ds d\phi = 2\pi r.$$

$\square$

Lemma 1.0.13. *Let $g : \Omega \to \mathbb{C}$ be a continuously differentiable function, where $\Omega \subseteq \mathbb{C}$ is open and has smooth boundary. If $\overline{\partial} g$ is absolutely integrable on Ω, then*

$$h(z) := g(z) - \frac{1}{2\pi i} \int_{\Omega} \frac{\overline{\partial} g(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}$$

is analytic on Ω. If g is continuous up to the boundary, h admits a continuous extension to $\overline{\Omega}$.

Proof. We first observe that $\int_\Omega \frac{\overline{\partial} g(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}$ exists for all $z \in \Omega$. This follows since $\zeta \mapsto \overline{\partial} g(\zeta)$ is bounded on some ball $B(z, \varepsilon) \subseteq \Omega$ (due to continuity) and $\zeta \mapsto \frac{1}{\zeta - z}$ is integrable on $B(z, \varepsilon)$. Outside this small ball $\zeta \mapsto \frac{1}{\zeta - z}$ is bounded and $\zeta \mapsto \overline{\partial} g(\zeta)$ is integrable.

Let $(K_n)_{n \in \mathbb{N}}$ be a compact exhaustion of Ω, where each K_n is smoothly bounded. The inhomogeneous Cauchy-Riemann integral formula provides the following equality for $z \in K_n^\circ$

$$g(z) = \frac{1}{2\pi i} \int_{\partial K_n} \frac{g(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{K_n} \frac{\overline{\partial} g(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}.$$

Define for $z \in K_n^\circ$

$$g_n(z) := \frac{1}{2\pi i} \int_{\partial K_n} \frac{g(\zeta)}{\zeta - z} d\zeta,$$

g_n is clearly analytic on it's domain.

Now let $n_0 \in \mathbb{N}$ be fixed and $z \in K_{n_0}^\circ$ arbitrary, then it holds for $m > n_0$ (observe that since $(K_n)_n$ is a compact exhaustion, there is some $r > 0$, such that $\operatorname{dist}(K_{n_0}, \Omega \backslash K_m) \geq r$)

$$\begin{aligned} |h(z) - g_m(z)| &\leq \frac{1}{\pi} \int_{\Omega \backslash K_m} \left| \frac{\overline{\partial} g(\zeta)}{z - \zeta} \right| d\xi \wedge d\eta \\ &\leq \underbrace{\frac{1}{\pi r} \int_{\Omega \backslash K_m} |\overline{\partial} g(\zeta)| d\xi \wedge d\eta}_{\to 0 \text{ as } m \to \infty}. \end{aligned}$$

Therefore $h|_{K_{n_0}^\circ}$ is the uniform limit of a sequence of analytic functions. As the argument holds for all $n_0 \in \mathbb{N}$, it follows that h is analytic on Ω.

Since the map $z \mapsto \frac{1}{2\pi i} \int_\Omega \frac{\overline{\partial} g(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}$ is continuous for $z \in \overline{\Omega}$ and g is by assumption continuous on $\overline{\Omega}$ it follows that h is continuous on $\overline{\Omega}$. □

Lemma 1.0.14. *Let $\Omega \subseteq \mathbb{C}$ be open and (X, μ) a measure space. Let $f : \Omega \times X \to \mathbb{C}$ be a function such that for all compact discs $K \subseteq \Omega$, there exists an absolutely integrable function $g_K : X \to \mathbb{R}$, such that $|f(z, x)| \leq g_K(x)$ for all $z \in K$. In addition assume*

$$x \mapsto f(z, x) \text{ is absolutely integrable for all } z \in \Omega,$$
$$z \mapsto f(z, x) \text{ is analytic on } \Omega \text{ for all } x \in X.$$

Then the function F defined for $z \in \Omega$ as

$$F(z) := \int_X f(z, x) d\mu(x)$$

is analytic on Ω.

Proof. Simple consequence of the dominated convergence theorem. □

Theorem 1.0.15 (Riemann mapping theorem)**.** *Let* $\Omega \subseteq \mathbb{C}$ *be a simply connected domain with* $\Omega \neq \mathbb{C}$*. Then for all* $a \in \Omega$ *there exists a unique biholomorphic (i.e. bijective and analytic with analytic inverse) function* $f : \Omega \to \mathbb{D}$ *with*

$$f(a) = 0, f'(a) > 0.$$

Proof. See [4, 4.2 Riemann mapping theorem, p. 160]. □

2 The harmonic measure

Here we define an important concept, which is a tool in some of the proofs later on. Its main strength, and the reason why we deal with it, is outlined in section 2.4. Generally speaking we need two ingredients to define a harmonic measure, namely a domain $\Omega \subseteq \mathbb{C}$ (with non-empty boundary) and a point $z \in \Omega$. Then the *harmonic measure for* Ω *as seen from* z is a Borel measure on the boundary $\partial\Omega$ with certain properties related to harmonic functions on Ω. To make the dependence clear, it is written as

$$\omega(z, E, \Omega)$$

where E is a Borel-measurable subset of $\partial\Omega$.

The defining property of the harmonic measure is that it recovers the value of a harmonic function at z when only the boundary values are known. In other words, knowledge of the harmonic measure in every point gives a solution of the Dirichlet problem for Ω. Although the harmonic measure can be defined for more general subsets, we only deal with Jordan domains as it suffices for our purposes. Those domains are considerably easier to handle as they are by definition simply connected. The main reason is that the Riemann mapping theorem 1.0.15 is available for such domains.

2.1 Explicit construction for $\mathbb{D}$

We will use the well-known solution of the Dirichlet problem for $\mathbb{D}$:

Theorem 2.1.1. *Let $f \in C(\partial\mathbb{D})$. Then there exists a unique function u_f defined on $\overline{\mathbb{D}}$ having the following properties*

(i) u_f is harmonic on $\mathbb{D}$,

(ii) u_f is continuous on $\overline{\mathbb{D}}$,

(iii) $u_f|_{\partial\mathbb{D}} = f$.

u_f can be written as

$$u_f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}dt = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta}\frac{1-|z|^2}{|\zeta-z|^2}d\zeta. \tag{2.1.1}$$

Proof. See [7, Theorem 1.3, p. 5]. □

It makes sense to define for an arbitrary Borel set $E \subseteq \partial\mathbb{D}$

$$\omega(z, E, \mathbb{D}) := \frac{1}{2\pi i}\int_E \frac{1-|z|^2}{\zeta|\zeta-z|^2}d\zeta. \tag{2.1.2}$$

This notion defines a Borel measure on $\partial\mathbb{D}$ and we call it the *harmonic measure for* $\mathbb{D}$ *as seen from* z. Now we can reformulate (2.1.1):

$$u_f(z) = \int_{\partial\mathbb{D}} f(\zeta) d\omega(z, \zeta, \mathbb{D}). \tag{2.1.3}$$

Observe that since constant functions are harmonic, it follows immediately that the harmonic measure for $\mathbb{D}$ is a probability measure for each $z \in \mathbb{D}$. Due to the Riesz representation theorem (see [13, 6.19 Theorem, p. 130] for details) it is also possible to use (2.1.3) as definition for the harmonic measure:

Fix $z \in \mathbb{D}$. Then theorem 2.1.1 makes it possible to define $\phi(f) := u_f(z)$ for all $f \in C(\partial\mathbb{D})$. Lemma 1.0.8 immediately implies

$$-||f||_\infty \leq \phi(f) \leq ||f||_\infty$$

which shows that ϕ is a bounded linear functional. As positive boundary conditions correspond to positive harmonic functions, we also get that ϕ is positive. Thus an application of Riesz representation theorem implies the existence of a unique Borel measure μ_z on $\partial\mathbb{D}$ such that

$$\phi(f) = u_f(z) = \int_{\partial\mathbb{D}} f(\zeta) d\mu_z(\zeta) \tag{2.1.4}$$

for all $f \in C(\partial\mathbb{D})$. And then we may define for a Borel set $E \subseteq \partial\mathbb{D}$

$$\omega(z, E, \mathbb{D}) := \mu_z(E) \tag{2.1.5}$$

which yields the same measure as above.

It is worth noting that the harmonic measure on $\mathbb{D}$ is invariant under biholomorphic functions. That means, given a biholomorphic function ϕ from $\mathbb{D}$ to $\mathbb{D}$ which extends continuously to a homeomorphism of $\overline{\mathbb{D}}$, for all $z \in \mathbb{D}$ and Borel sets $E \subseteq \partial\mathbb{D}$

$$\omega(z, E, \mathbb{D}) = \omega(\phi(z), \phi(E), \mathbb{D}). \tag{2.1.6}$$

This invariance property follows from theorem 2.1.1: Take $f \in C(\partial\mathbb{D})$ and ϕ a biholomorphic function as above. Then $f \circ \phi \in C(\partial\mathbb{D})$, so we can apply theorem 2.1.1 to the boundary condition $f \circ \phi$ and get a uniquely defined harmonic function $u_{f\circ\phi}$ on $\mathbb{D}$. On the other hand, since a composition of a harmonic and a holomorphic function is again harmonic, we get that $u_f \circ \phi$ is also a solution for the boundary condition $f \circ \phi$. The uniqueness result from theorem 2.1.1 implies $u_{f\circ\phi}(z) = u_f(\phi(z))$ for all $z \in \mathbb{D}$. As this equality holds for all $f \in C(\partial\mathbb{D})$, we get $\omega(z, \phi^{-1}(E), \mathbb{D}) = \omega(\phi(z), E, \mathbb{D})$ for all Borel sets $E \subseteq \partial\mathbb{D}$ which implies the desired result.

Due to the abstract nature of definition (2.1.5), it is certainly easier to derive estimates for the harmonic measure by using (2.1.2). But also the more constructive definition (2.1.2) is lacking of an easy geometric description. So the next goal is to derive a geometric description of $\omega(z, E, \mathbb{D})$.

2.2 Geometric description

The ideas in this section are taken from [7, exercises and further results, p. 26].

First we fix a notation for subarcs of the boundary of a ball $B(a, r)$ where $a \in \mathbb{C}$ and $r > 0$. Given two points $\zeta_1, \zeta_2 \in \partial B(a, r)$, let $[\zeta_1, \zeta_2]_{\partial B(a,r)}$ be the positively oriented arc between ζ_1 and ζ_2 on $\partial B(a, r)$. $[\zeta_1, \zeta_1]_{\partial B(a,r)}$ shall denote the set $\{\zeta_1\}$ (and not $\partial B(a, r)$). For $a = 0$ and $r = 1$ we write $[\zeta_1, \zeta_2]_{\partial\mathbb{D}}$. E.g. $[-i, i]_{\partial\mathbb{D}}$ denotes the right half of the unit circle whereas $[i, -i]_{\partial\mathbb{D}}$ denotes the left half.

If we write $[\zeta_1, \zeta_2]_{\partial B(a,r)} = [a + re^{i\theta_1}, a + re^{i\theta_2}]_{\partial B(a,r)}$, we always assume that θ_1, θ_2 are chosen in such a way that the length of the arc equals $r(\theta_2 - \theta_1)$.

Later on, we will need certain biholomorphic maps from the unit disc onto itself (automorphisms) which extend to homeomorphisms of the closed unit disc. This is taken care of in the following lemma.

Lemma 2.2.1. *For $a \in \mathbb{D}$, the map*

$$z \mapsto \frac{z - a}{1 - \overline{a}z} =: \phi_a(z)$$

is a homeomorphism of $\overline{\mathbb{D}}$ and an automorphism of $\mathbb{D}$.

Proof. First observe that ϕ_a is clearly analytic on $\mathbb{D}$ as it is the quotient of analytic functions with non-vanishing denominator. Due to the same reason it is continuous on $\overline{\mathbb{D}}$. Next observe that, for $z = e^{i\theta} \in \partial\mathbb{D}$

$$\phi_a(z) = e^{-i\theta} \frac{e^{i\theta} - a}{e^{-i\theta} - \overline{a}} = e^{-i\theta} \frac{e^{i\theta} - a}{\overline{e^{i\theta} - a}}$$

has absolute value 1. This shows that $\phi_a(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$. Due to the maximum principle for analytic functions, see theorem 1.0.9, and as non-constant analytic functions map open sets to open sets, we get $\phi_a(\mathbb{D}) \subseteq \mathbb{D}$ and $\phi_a(\overline{\mathbb{D}}) \subseteq \overline{\mathbb{D}}$.

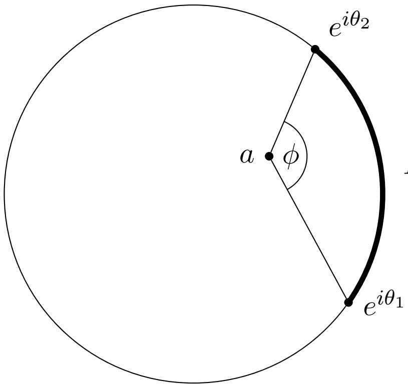
A direct computation shows $\phi_a^{-1} = \phi_{-a}$ which implies that ϕ_a is bijective on $\overline{\mathbb{D}}$. This completes the proof. □

Using the above lemma we can give a first geometric description of the harmonic measure.

Lemma 2.2.2. *Let $E = [e^{i\theta_1}, e^{i\theta_2}]_{\partial\mathbb{D}}$ be an arc in $\partial\mathbb{D}$. Let $a \in \mathbb{D}$ and let ϕ be the interior angle at a of $S(a, E)$, which consists of E and the line segments connecting a with $e^{i\theta_j}$ for $j = 1, 2$. Then*

$$\omega(a, E, \mathbb{D}) = \frac{1}{2\pi}(2\phi - (\theta_2 - \theta_1)). \tag{2.2.7}$$

The picture shows the situation in lemma 2.2.2:



Proof. Observe that

$$\omega(0, E, \mathbb{D}) = \frac{1}{2\pi}(\theta_2 - \theta_1)$$

which can be verified by a direct computation of the integral in (2.1.2). Combining this with (2.1.6) for the function ϕ_a from lemma 2.2.1, we can reduce the problem of finding the value of $\omega(a, E, \mathbb{D})$ to simply computing $\phi_a(E)$. Thus we have to show that the length of the arc $\phi_a(E)$ equals $2\phi - (\theta_2 - \theta_1)$. To this end we compute $\phi_a(e^{i\theta_j})$ for $j = 1, 2$. Observe that $\phi_a(e^{i\theta_j}) = e^{-i\theta_j}\frac{e^{i\theta_j}-a}{\overline{e^{i\theta_j}-a}}$. Denoting $\phi_j = \operatorname{Arg}(e^{i\theta_j} - a)$ and using $\frac{z}{\bar{z}} = e^{i2\operatorname{Arg}(z)}$, we end up with

$$\phi_a(e^{i\theta_j}) = e^{i(2\phi_j - \theta_j)}.$$

Thus it follows $\phi_a(E) = [e^{i(2\phi_1-\theta_1)}, e^{i(2\phi_2-\theta_2)}]_{\partial\mathbb{D}}$ whose length is $2\phi - (\theta_2 - \theta_1)$. □

The next theorem gives a very useful geometric description of the harmonic measure of an arc as a certain angle.

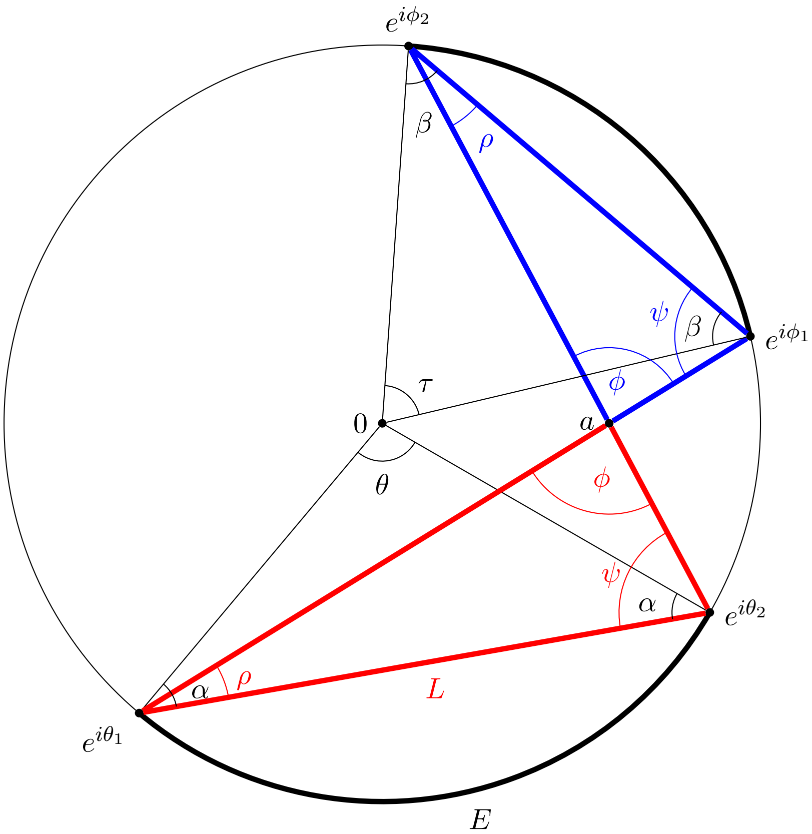
Theorem 2.2.3. *Let a, ϕ and E be as in lemma 2.2.2. Let $e^{i\phi_j}$ be the second intersection with $\partial\mathbb{D}$ of the line l_j passing through the points $e^{i\theta_j}$ and a ($e^{i\theta_j}$ is the first intersection). Let τ be the length of $[e^{i\phi_1}, e^{i\phi_2}]_{\partial\mathbb{D}}$ (i.e. $\tau = \phi_2 - \phi_1$) or equivalently the interior angle at 0 of $S(0, [e^{i\phi_1}, e^{i\phi_2}]_{\partial\mathbb{D}})$). Then*

$$\tau = 2\phi - (\theta_2 - \theta_1). \tag{2.2.8}$$

Together with lemma 2.2.2 this implies

$$\frac{\tau}{2\pi} = \omega(a, E, \mathbb{D}). \tag{2.2.9}$$

Proof. Let L be the line segment connecting $e^{i\theta_1}$ with $e^{i\theta_2}$. Then $\overline{\mathbb{D}}\backslash L$ has two connected components. Assume a does not lie in the component of E. This is implies $\phi < \pi$. Assume in addition $\theta := \theta_2 - \theta_1 < \pi$ and $\tau < \pi$.
The picture shows the situation under these assumptions



Observe that the blue and red triangle are similar, thus the labelling of the angles, appearing in the coloured triangles, is correct. In addition we have

$$\alpha = \frac{\pi}{2} - \frac{\theta}{2}, \tag{2.2.10}$$

$$\beta = \frac{\pi}{2} - \frac{\tau}{2}, \tag{2.2.11}$$

$$\psi = \alpha + \beta - \rho, \tag{2.2.12}$$

$$\rho = \pi - \phi - \psi. \tag{2.2.13}$$

Equations (2.2.10) and (2.2.11) are clear, as the defining triangles are isosceles. Similarly (2.2.12) follows since the triangle defined by the points $e^{i\phi_2}, 0, e^{i\theta_2}$ is isosceles. Finally (2.2.13) just uses the fact that the angular sum in a triangle is π.
Using these equations we may derive

$$\begin{aligned} &\rho = \pi - \phi - \alpha - \beta + \rho && \text{use } (2.2.12), (2.2.13) \\ \Leftrightarrow\ &0 = \pi - \phi - \frac{\pi}{2} + \frac{\theta}{2} - \frac{\pi}{2} + \frac{\tau}{2} && (2.2.10), (2.2.11) \\ \Leftrightarrow\ &\tau = 2\phi - \theta \end{aligned}$$

If some of the angles that were assumed to be less than π are actually larger, an analogous argumentation using their respective complementary angles leads to the same result. If $\phi = \pi$, which happens if and only if $e^{i\theta_j}$ and a lie on the same line, we get $e^{i\phi_1} = e^{i\theta_2}$ and $e^{i\phi_2} = e^{i\theta_1}$. Thus $\tau = 2\pi - (\theta_2 - \theta_1) = 2\phi - (\theta_2 - \theta_1)$ in this case as well. □

2.3 Construction for Jordan domains

In this section we will construct harmonic measures for more general domains Ω. The construction is based upon the existence of biholomorphic maps from Ω to $\mathbb{D}$ which extend to homeomorphisms of the respective closures. In view of the Riemann mapping theorem 1.0.15, it is necessary to assume Ω to be simply connected to ensure the existence of biholomorphic maps to the unit disc. In order to be able to extend such mappings to homeomorphisms of the closures we have to make sure that the boundary is "sufficiently nice". The next definition tells what is meant by that.

Definition A simply connected domain in $\mathbb{C}$ whose boundary is a Jordan curve (i.e. homeomorphic to S^1) is called a *Jordan domain.*

The next theorem shows that it is possible to derive the aspired result for Jordan domains. The respective proof of the theorem as well as the subsequent construction are taken from [7, I.3. Carathéodory's theorem, p. 13-16].

Theorem 2.3.1 (Charathéodory's theorem)**.** *Let ϕ be a biholomorphic map from the unit disc $\mathbb{D}$ onto a Jordan domain Ω. Then ϕ has a continuous extension to $\overline{\mathbb{D}}$, and the extension is bijective from $\overline{\mathbb{D}}$ to $\overline{\Omega}$.*

Proof. Let $\zeta \in \partial\mathbb{D}$. First a continuous extension at ζ is constructed. For $0 < \delta < 1$ define $\gamma_\delta := (\partial B(\zeta,\delta)) \cap \mathbb{D}$. Then $\phi(\gamma_\delta)$ is a Jordan arc, i.e. homeomorphic to $(0,1)$, with length

$$L(\delta) = \int_{\gamma_\delta} |\phi'(z)| ds = \int_{t_1^\delta}^{t_2^\delta} |\phi'(\zeta + \delta e^{it})| \delta dt,$$

where $0 \leq t_1^\delta < t_2^\delta < 2\pi$ are the solutions of $|\zeta + \delta e^{it}| = 1$. An application of the Cauchy-Schwarz inequality yields

$$\begin{aligned} L(\delta)^2 &= \left(\int_{t_1^\delta}^{t_2^\delta} |\phi'(\zeta + \delta e^{it})| \delta dt \right)^2 \\ &\leq \left(\int_{t_1^\delta}^{t_2^\delta} |\phi'(\zeta + \delta e^{it})|^2 \delta^2 dt \right) \left(\int_{t_1^\delta}^{t_2^\delta} dt \right) \\ &\leq \pi\delta \int_{\gamma_\delta} |\phi'(z)|^2 ds. \end{aligned}$$

Therefore $\rho < 1$,

$$\int_0^\rho \frac{L(\delta)^2}{\delta} d\delta \leq \pi \int_0^\rho \int_{t_1^\delta}^{t_2^\delta} |\phi'(\zeta + \delta e^{it})|^2 \delta dt d\delta$$
$$= \pi \int_{B(\zeta,\rho)\cap\mathbb{D}} |\phi'(z)|^2 dxdy$$
$$= \pi \text{Area}(\phi(B(\zeta,\rho)\cap\mathbb{D})) < \infty.$$

But this implies the existence of a null sequence δ_n with $L(\delta_n) \to 0$ as $n \to \infty$. Otherwise $L(\delta) \geq C > 0$ for sufficiently small δ, but then $\int_0^\rho \frac{L(\delta)^2}{\delta} d\delta \geq \int_0^\rho \frac{C^2}{\delta} d\delta = \infty$ being a contradiction to the above integrability condition.

Let α_n and β_n be the endpoints of $\phi(\gamma_{\delta_n})$, then $\alpha_n, \beta_n \in \overline{\Omega}$. They lie in $\partial\Omega$. Note that

$$|\alpha_n - \beta_n| \leq L(\delta_n). \tag{2.3.14}$$

Define for a subset U of $\mathbb{C}$ the diameter of U as $\text{diam}(U) := \sup\{|x-y| : x, y \in U\}$. Denote by σ_n the closed subarc of $\partial\Omega$ connecting the two points α_n, β_n having smaller diameter (there exist exactly two subarcs connecting the two points). Since $\partial\Omega$ is homeomorphic to $\partial\mathbb{D}$ by assumption and by (2.3.14) it follows that $\text{diam}(\sigma_n)$ tends to 0. Define U_n to be the bounded connected component of $\mathbb{C}\backslash(\phi(\gamma_{\delta_n}) \cup \sigma_n)$; this notion is well defined because of the Jordan curve theorem. It immediately follows that $\text{diam}(\partial U_n)$ tends to 0 and therefore also

$$\text{diam}(U_n) \to 0 \tag{2.3.15}$$

for $n \to \infty$. Since U_n is connected, either $\phi(\mathbb{D}\backslash\overline{B(\zeta,\delta_n)}) = U_n$ or $\phi(B(\zeta,\delta_n)\cap\mathbb{D}) = U_n$. But the first possibility cannot hold for large n since in this case $\text{diam}(U_n) \geq \text{diam}(\phi(B(0,\frac{1}{2}))) > 0$ contradicting (2.3.15). Thus $\text{diam}(\phi(B(\zeta,\delta_n)\cap\mathbb{D})) \to 0$. Since in addition $\phi(B(\zeta,\delta_{n+1})\cap\mathbb{D}) \subseteq \phi(B(\zeta,\delta_n)\cap\mathbb{D})$ for all n, it follows

$$\left|\bigcap_{n\in\mathbb{N}} \overline{\phi(B(\zeta,\delta_n)\cap\mathbb{D})}\right| = 1 \tag{2.3.16}$$

and we define that one point to be $\phi(\zeta)$. As by (2.3.15) $\{U_n : n \in \mathbb{N}\}$ forms a neighbourhood basis it thus follows that the extension of ϕ to $\mathbb{D} \cup \{\zeta\}$ is continuous. Use the same construction for all points $\zeta \in \partial\mathbb{D}$. This procedure defines a continuous function on $\overline{\mathbb{D}}$.

It is still left to show the bijectivity of the extended function. We show surjectivity first: Since ϕ is continuous and $\phi(\mathbb{D}) = \Omega$, it immediately follows $\phi(\overline{\mathbb{D}}) \subseteq \overline{\phi(\mathbb{D})} = \overline{\Omega}$. But since $\overline{\mathbb{D}}$ is compact it also follows $\phi(\overline{\mathbb{D}}) \supseteq \overline{\phi(\mathbb{D})}$; let $y_\infty \in \overline{\phi(\mathbb{D})}$ then y_∞ is the limit of some convergent sequence $y_n = \phi(x_n) \in \phi(\mathbb{D})$, the sequence (x_n) has a convergent subsequence (with limit $x_\infty \in \overline{\mathbb{D}}$) due to compactness, which shows that $\phi(x_\infty) = y_\infty$.

It is still left to prove injectivity: Assume the contrary, i.e. there are distinct points $\zeta_1, \zeta_2 \in \partial\mathbb{D}$ with $\phi(\zeta_1) = \phi(\zeta_2)$. Denote by W the interior of the Jordan curve

$$\{\phi(r\zeta_1) : 0 \leq r \leq 1\} \cup \{\phi(r\zeta_2) : 0 \leq r \leq 1\}$$

which is connected. Thus $\phi^{-1}(W)$ is one of the two connected components of

$$\mathbb{D}\backslash\left(\{r\zeta_1 : 0 \le r \le 1\} \cup \{r\zeta_2 : 0 \le r \le 1\}\right).$$

Note that $\phi(\partial\mathbb{D} \cap \partial\phi^{-1}(W)) \subseteq \partial W \cap \partial\Omega = \{\phi(\zeta_1)\}$. This shows that ϕ is actually constant on a subarc of $\partial\mathbb{D}$. The Schwarz reflection principle thus implies that ϕ is constant, contradicting the assumption that ϕ is biholomorphic. Therefore ϕ has to be injective on $\overline{\mathbb{D}}$. □

So let now Ω be a Jordan domain and $\phi : \overline{\mathbb{D}} \to \overline{\Omega}$ be a homeomorphism that is biholomorphic between $\mathbb{D}$ and Ω. Having these ingredients, it is possible to derive a solution of the Dirichlet problem for Ω:

Take $f \in C(\partial\Omega)$, then $f \circ \phi \in C(\partial\mathbb{D})$. By the considerations of section 2.1, we find a solution

$$u_{f\circ\phi}(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f\circ\phi(\zeta)}{\zeta}\frac{1-|z|^2}{|\zeta - z|^2}d\zeta.$$

Thus a solution for the initial boundary condition f is given by

$$u_f(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f\circ\phi(\zeta)}{\zeta}\frac{1-|\phi^{-1}(z)|^2}{|\zeta - \phi^{-1}(z)|^2}d\zeta = u_{f\circ\phi}(\phi^{-1}(z)). \tag{2.3.17}$$

Then for $z \in \Omega$ and a Borel set $E \subseteq \partial\Omega$ we set

$$\omega(z, E, \Omega) := \omega(\phi^{-1}(z), \phi^{-1}(E), \mathbb{D}) = \frac{1}{2\pi i}\int_{\phi^{-1}(E)} \frac{1-|\phi^{-1}(z)|^2}{\zeta|\zeta - \phi^{-1}(z)|^2}d\zeta. \tag{2.3.18}$$

For fixed z the above notion (2.3.18) is a Borel measure on $\partial\Omega$. With this definition we can reformulate (2.3.17) as follows

$$u_f(z) = \int_{\partial\Omega} f(\zeta)d\omega(z,\zeta,\Omega). \tag{2.3.19}$$

Thus we have found a solution of the Dirichlet problem for Ω by the means of a measure on $\partial\Omega$. By the maximum principle it follows that the measure is actually independent of the chosen map ϕ.

2.4 Two constants theorem

The aspired result of this section consists of a sharpened maximum principle for analytic functions. Take for example an analytic function f on $\mathbb{D}$ that has a continuous extension to $\overline{\mathbb{D}}$. Suppose $|f| \le m$ on some large subset E of the boundary and $|f| \le M$ on the whole boundary, where m is small and M is large. The maximum principle 1.0.9 tells us that $|f|$ is bounded by M on the whole disc. The intuition suggests that for z close to E, $|f(z)|$ should be affected mostly by the the boundary behaviour on E, i.e. $|f(z)| \le m+\varepsilon$ for some small ε. The two constants theorem makes that intuition precise. The following theorems and their proofs are taken from [9, VII B, p. 256-257].

The two constants theorem is a corollary of the following theorem.

Theorem 2.4.1. *Let Ω be a Jordan domain and f a continuous function on $\overline{\Omega}$ that is analytic on Ω. Then for $z \in \Omega$*

$$\log|f(z)| \leq \int_{\partial\Omega} \log|f(\zeta)| d\omega(z, \zeta, \Omega). \tag{2.4.20}$$

Proof. Define for $M > 0$ and $z \in \overline{\Omega}$

$$V_M(z) := \max(\log|f(z)|, -M).$$

Since constant functions are clearly subharmonic, $\log|f|$ is subharmonic by lemma 1.0.3 and max preserves subharmonicity, we get that V_M is subharmonic on Ω. Since $\log|f|$ is continuous on $\overline{\Omega}\backslash N_f$, where N_f is the set of zeros of f, we immediately get that V_M is continuous on $\overline{\Omega}$. Since the harmonic measure serves for solving the Dirichlet problem for Ω and V_M is continuous on the boundary, we have that

$$z \mapsto \int_{\partial\Omega} V_M(\zeta) d\omega(z, \zeta, \Omega)$$

is harmonic on Ω and coincides with V_M on the boundary by theorem 2.1.1. Therefore

$$g_M(z) := V_M(z) - \int_{\partial\Omega} V_M(\zeta) d\omega(z, \zeta, \Omega)$$

is subharmonic on Ω by lemma 1.0.3 and continuous up to the boundary. But by construction $g_M(\zeta) = 0$ for all $\zeta \in \partial\Omega$. By the maximum principle, see lemma 1.0.3, we therefore conclude that $g_M \leq 0$ on Ω. Thus we have for $z \in \Omega$

$$\log|f(z)| \leq V_M(z) \leq \int_{\partial\Omega} V_M(\zeta) d\omega(z, \zeta, \Omega).$$

Now let $M \to \infty$, then the right hand side tends to $\int_{\partial\Omega} \log|f(\zeta)| d\omega(z, \zeta, \Omega)$. This can be seen as follows:

For a function f with values in the extended reals (i.e. $\pm\infty$ are allowed as values), we set $f^+ := \max(f, 0), f^- := -\min(f, 0)$. For all $\zeta \in \partial\Omega$, we have $V_M^+(\zeta) = (\log|f(\zeta)|)^+$ for all $M \geq 0$. And V_M^+ is bounded on $\partial\Omega$ by continuity of V_M^+ and compactness of the boundary of a Jordan domain. Thus V_M^+ is integrable on $\partial\Omega$ and the integral equals $\int_{\partial\Omega} (\log|f(\zeta)|)^+ d\omega(z, \zeta, \Omega)$. In addition $V_M^-(\zeta)$ is increasing in M for each fixed $\zeta \in \partial\Omega$ and its pointwise limit is $(\log|f(\zeta)|)^-$. Thus an application of the monotone convergence theorem (see [13, 1.26, p. 26] for details) yields that $\int_{\partial\Omega} V_M^-(\zeta) d\omega(z, \zeta, \Omega)$ tends to $\int_{\partial\Omega} (\log|f(\zeta)|)^- d\omega(z, \zeta, \Omega)$ as $M \to \infty$. This implies the statement.

□

We can finally prove the two constants theorem, which is now an easy task as we have already shown theorem 2.4.1.

Theorem 2.4.2 (Two constants theorem). *Let Ω be a Jordan domain and f a continuous function on $\overline{\Omega}$ that is analytic on Ω. Suppose $|f(\zeta)| \leq m$ for $\zeta \in E \subseteq \partial\Omega$, where E is Borel, and $|f(\zeta)| \leq M$ for $\zeta \in \partial\Omega$. Then for $z \in \Omega$*

$$|f(z)| \leq m^{\omega(z,E,\Omega)} M^{1-\omega(z,E,\Omega)} \tag{2.4.21}$$

Proof. Theorem 2.4.1 implies

$$\begin{aligned}
|f(z)| &\leq \exp\left(\int_{\partial\Omega} \log|f(\zeta)| d\omega(z,\zeta,\Omega)\right) \\
&\leq \exp\left(\int_{\partial\Omega} (\log m)\chi_E + (\log M)\chi_{\partial\Omega\setminus E} d\omega(z,\zeta,\Omega)\right) \\
&= \exp\left((\log m)\omega(z,E,\Omega) + (\log M)(1-\omega(z,E,\Omega))\right) \\
&= m^{\omega(z,E,\Omega)} M^{1-\omega(z,E,\Omega)}
\end{aligned}$$

where χ_E denotes the characteristic function of E. □

Remark 2.4.3. *It is clear from the proof that the two constants theorem can easily be generalized to an "n constants theorem":*

Given a partition of $\partial\Omega$ into Borel sets E_j for $j = 1, \ldots, n$ and respective bounds m_j of $|f|$, we get

$$|f(z)| \leq \prod_{j=1}^{n} m_j^{\omega(z,E_j,\Omega)}.$$

2.5 Special unbounded domains

Although all considerations concerning the harmonic measure up to this point only dealt with bounded domains, it is also possible to derive similar results for many unbounded domains. The goal of this chapter is to show that the two constants theorem can be applied to special unbounded domains and (special) analytic functions.

Lemma 2.5.1. *Let $\Omega \subseteq \mathbb{C}$ be a (possibly unbounded) domain. Suppose there exists a biholomorphic map $\phi : \Omega \to \mathbb{D}$ which extends to a homeomorphism from $\overline{\Omega}$ to $\overline{\mathbb{D}}\setminus\{\zeta_1, \ldots, \zeta_n\}$ for some $\zeta_j \in \partial\mathbb{D}$. Then for each $f \in C_0(\overline{\Omega})$ that is analytic on Ω, let $\tilde{f}$ be defined as*

$$\tilde{f}(z) := \begin{cases} f \circ \phi^{-1}(z) & , z \in \overline{\mathbb{D}}\setminus\{\zeta_1, \ldots, \zeta_n\} \\ 0 & , z \in \{\zeta_1, \ldots, \zeta_n\}. \end{cases}$$

Then $\tilde{f} \in C(\overline{\mathbb{D}})$ and is analytic on $\mathbb{D}$.

Proof. Continuity on $\overline{\mathbb{D}}\setminus\{\zeta_1, \ldots, \zeta_n\}$ and analyticity on $\mathbb{D}$ of $\tilde{f}$ are clear from the definition. So it is left to show continuity at the points ζ_j:

As the continuous image of a compact set is compact, it follows that $\phi(\overline{\Omega} \cap \overline{B(0,m)})$ is compactly contained in $\overline{\mathbb{D}} \backslash \{\zeta_1, \ldots, \zeta_n\}$ for all $m \in \mathbb{N}$. Thus there is $r_m > 0$ such that

$$\phi(\overline{\Omega} \cap \overline{B(0,m)}) \subseteq \overline{\mathbb{D}} \backslash \bigcup_{i=1}^{n} B(\zeta_i, r_m). \tag{2.5.22}$$

Let $(z_n)_n$ be a sequence in $\overline{\mathbb{D}} \backslash \{\zeta_1, \ldots, \zeta_n\}$ converging to ζ_j for some j. This implies that $|\phi^{-1}(z_n)|$ has to eventually leave every compact set due to (2.5.22), in other words $|\phi^{-1}(z_n)| \to \infty$. This immediately gives continuity of $\tilde{f}$ at the points ζ_j, since $f \in C_0(\overline{\Omega})$. $\square$

Therefore we may apply the two constants theorem 2.4.2 to the function $\tilde{f}$ and derive

$$|\tilde{f}(z)| \leq m^{\omega(z, \tilde{E}, \mathbb{D})} M^{1-\omega(z, \tilde{E}, \mathbb{D})}$$

where m and M bound $|\tilde{f}|$ on $\tilde{E}$ and $\partial\mathbb{D}$ respectively. If bounds m, M of the initial function f are known on some set $E \subseteq \partial\Omega$, these bounds are valid for $\tilde{f}$ on $\phi(E)$. Thus it follows

$$|f(z)| = |f \circ \phi^{-1}(\phi(z))| \leq m^{\omega(\phi(z), \phi(E), \mathbb{D})} M^{1-\omega(\phi(z), \phi(E), \mathbb{D})}.$$

Therefore we have shown

Lemma 2.5.2 (Two constants theorem for unbounded domains)**.** *Let f, Ω, ϕ be as in lemma 2.5.1. Let $E \subseteq \partial\Omega$ be a Borel set with*

$$|f(z)| \leq m \quad \forall z \in E, \quad |f(z)| \leq M \quad \forall z \in \partial\Omega.$$

Then

$$|f(z)| \leq m^{\omega(\phi(z), \phi(E), \mathbb{D})} M^{1-\omega(\phi(z), \phi(E), \mathbb{D})}.$$

As we want to apply the two constants theorem for unbounded domains, we have to examine whether a given Ω has the necessary properties to apply lemma 2.5.2. The prototypical examples are simply connected domains with smooth boundary such that the number of connected components of $\Omega \backslash B(0,m)$ tends to some $n \in \mathbb{N}$, where n determines the number of ζ_j needed to construct a homeomorphism from $\overline{\Omega}$ to $\overline{\mathbb{D}} \backslash \{\zeta_1, \ldots, \zeta_n\}$.

The easiest examples for such domains, and the only ones needed later on, are the upper half plane $\mathbb{H}$ and open strips. To see this we need to construct suitable functions ϕ in these cases. For the upper half plane $\mathbb{H}$ we define (the so-called Cayley mapping)

$$\phi(z) := \frac{z-i}{z+i}. \tag{2.5.23}$$

Then ϕ is biholomorphic from $\mathbb{H}$ to $\mathbb{D}$ and extends (by using the same definition for $z \in \partial\mathbb{H} = \mathbb{R}$) to a homeomorphism from $\overline{\mathbb{H}}$ to $\overline{\mathbb{D}} \backslash \{1\}$:

ϕ is clearly analytic as the quotient of analytic functions with non vanishing denominator. Its inverse is given by $\phi^{-1}(z) := i\frac{1+z}{1-z}$, which is analytic due to the same reason. By using these definitions for functions defined on $\overline{\mathbb{H}}$ and $\overline{\mathbb{D}} \backslash \{1\}$ respectively, it immediately follows that ϕ extends to a homeomorphism.

Using this function we can also define a suitable function $\tilde{\phi} : S \to \mathbb{D}$, where $S := \{z \in \mathbb{C} : 0 < \Im(z) < h\}$ for some $h > 0$. Observe that $\psi : S \to \mathbb{H}$ defined as $z \mapsto e^{\pi \frac{z}{h}}$ is a biholomorphic map from S to $\mathbb{H}$ which extends to a homeomorphism from $\overline{S}$ to $\overline{\mathbb{H}} \backslash \{0\}$. As the restriction to $\overline{\mathbb{H}} \backslash \{0\}$ of the above ϕ is a homeomorphism from $\overline{\mathbb{H}} \backslash \{0\}$ to $\overline{\mathbb{D}} \backslash \{-1, 1\}$, the composition $\phi \circ \psi$ defines a suitable homeomorphism for S. Thus we have found a function

$$\tilde{\phi}(z) := \phi \circ \psi(z) = \frac{e^{\pi \frac{z}{h}} - i}{e^{\pi \frac{z}{h}} + i}$$

which is biholomorphic between S and $\mathbb{D}$ and a homeomorphism between $\overline{S}$ and $\overline{\mathbb{D}} \backslash \{-1, 1\}$.

Therefore we can apply the lemma 2.5.2 to analytic functions, vanishing at infinity, defined on $\mathbb{H}$ and S.

3 Characterization of smooth function classes by asymptotically holomorphic extensions

This chapter gives a first and important example for the strength of the technique of asymptotically holomorphic extensions:

We characterize the class of smooth functions ($C^\infty(\mathbb{R})$) and ultradifferentiable function classes defined by a regular weight sequence ($C\{M_n\}(\mathbb{R})$) (compare the corresponding section 3.2 for the definition and elementary properties) by the existence of extensions to $\mathbb{C}$ with rapidly decreasing derivative with respect to $\overline{z}$. The speed of decay of the derivative with respect to $\overline{z}$ of such an extension then determines the regularity of the given function defined on $\mathbb{R}$.

3.1 Characterization of C^∞

The following considerations are taken from [6, p. 46-47]. Let $f \in C^\infty(\mathbb{R})$ and $z = x+iy$. Then the function

$$F_n(z) := T_x^n(f, z) = \sum_{k=0}^{n} f^{(k)}(x)\frac{(iy)^k}{k!}$$

defines a smooth extension of f to $\mathbb{C}$ with

$$\overline{\partial} F_n(z) = \frac{1}{2} f^{(n+1)}(x)\frac{(iy)^n}{n!}.$$

Therefore we get for each compact set $K \subseteq \mathbb{R}$ the existence of $C_{K,n} > 0$ such that

$$|\overline{\partial} F_n(z)| \leq C_{K,n}|y|^n \quad \text{for all } x \in K. \tag{3.1.1}$$

Having an extension F_n with property (3.1.1) for all n already characterizes smoothness of a function:

Suppose we have for a given $n \in \mathbb{N}$ an extension F_n with property (3.1.1). Let I be a bounded open interval in $\mathbb{R}$. For $x \in I$ we thus have $C > 0$ such that $|\overline{\partial} F_n(x + iy)| \leq C|y|^n$. Cauchy's integral formula (see theorem 1.0.11) implies for $x \in I$

$$f(x) = \frac{1}{2\pi i}\int_{\partial(I+i[-1,1])} \frac{F_n(\zeta)}{\zeta - x} d\zeta + \frac{1}{2\pi i}\int_{I+i[-1,1]} \frac{\overline{\partial} F_n(\zeta)}{\zeta - x} d\zeta \wedge d\overline{\zeta}.$$

Now we show that both summands in the above representation are $(n-1)$-times differentiable on I:

For the first summand this is clear by differentiating under the integral sign with respect to x; apply the dominated convergence theorem to see that this is legitimate.

For the second summand, we call it $g_n(x)$, we have for $x, x_0 \in I$ with $x \neq x_0$

$$\frac{g_n(x) - g_n(x_0)}{x - x_0} = \frac{1}{2\pi i} \int_{I+i[-1,1]} \frac{\overline{\partial} F_n(\zeta)}{(\zeta - x)(\zeta - x_0)} d\zeta \wedge d\overline{\zeta}. \tag{3.1.2}$$

Observe that $|\zeta - x| \geq |\eta|$, where we wrote $\eta = \Im(\zeta)$. By assumption $\left|\frac{\overline{\partial} F_n(\zeta)}{(\zeta - x)(\zeta - x_0)}\right| \leq \frac{C|\eta|^n}{|\eta|^2}$ for all $\zeta \in I + i[-1,1]$. As $\frac{C|\eta|^n}{|\eta|^2} = C|\eta|^{n-2}$ is integrable on $I + i[-1,1]$ (for $n \geq 2$), we can apply the dominated convergence theorem to derive that we can interchange $\lim_{x \to x_0}$ and integration in equation (3.1.2). Thus we get for $x \in I$

$$g_n'(x) = \frac{1}{2\pi i} \int_{I+i[-1,1]} \frac{\overline{\partial} F_n(\zeta)}{(\zeta - x)^2} d\zeta \wedge d\overline{\zeta}.$$

This argument can be iterated another $(n-2)$ times, showing that f is actually $(n-1)$ times differentiable if an extension F_n of the above form exists; see the proof of theorem 3.2.4 for details.

Remark 3.1.1. *In [10, Lemma 0, p. 116] a single extension is constructed whose derivative with respect to $\overline{z}$ vanishes to infinite order on $\mathbb{R}$, i.e. an extension F which fulfils* (3.1.1) *for all $n \in \mathbb{N}$.*

3.2 Characterization of $C\{M_n\}$

First we need to define the classes under consideration:

Given an increasing sequence of positive numbers $(M_n)_{n=0}^{\infty}$, called *weight sequence*, and an open set $U \subseteq \mathbb{R}$ we will denote

$$C\{M_n\}(U) := \left\{ f \in C^{\infty}(U) \;:\; \forall K \subset\subset U \;\exists B > 0 : ||f||_{K,B} := \sup_{n \geq 0,\; x \in K} \frac{|f^{(n)}(x)|}{B^n M_n} < \infty \right\}$$

and call it a *Denjoy-Carleman class*; we will write DC class. We will also need a similar concept defined as follows

$$C_{\mathrm{gl}}\{M_n\}(U) := \left\{ f \in C^{\infty}(U) \;:\; \exists B > 0 : ||f||_{U,B} := \sup_{n \geq 0,\; x \in U} \frac{|f^{(n)}(x)|}{B^n M_n} < \infty \right\}$$

and call it a *global Denjoy-Carleman class.*

In addition we will write $m_n := \frac{M_n}{n!}$. In order to derive some useful properties of the respective DC class we will only deal with so called *regular* weight sequences. Those are weight sequences with the following additional assumptions

$$m_n^{1/n} \overset{n \to \infty}{\to} \infty, \tag{3.2.3}$$

$$\sup_{n \in \mathbb{N}} \left(\frac{m_{n+1}}{m_n} \right)^{1/n} < \infty, \tag{3.2.4}$$

$$m_n^2 \leq m_{n-1} m_{n+1} \quad \forall n \in \mathbb{N}. \tag{3.2.5}$$

A DC class defined by a regular weight sequence (M_n) is called a *regular* DC class. Condition (3.2.3) ensures that the analytic functions are a proper subset of a regular class, which follows easily from [13, Theorem 19.9, p.378]. Condition (3.2.4) clearly gives $C\{M_{n+1}\} = C\{M_n\}$ which implies that regular classes are derivation closed. Condition (3.2.5) is referred to as *strong logarithmic convexity* of the sequence (M_n), i.e. logarithmic convexity of (m_n). Strong logarithmic convexity implies *logarithmic convexity* which states

$$M_n^2 \leq M_{n-1}M_{n+1} \quad \forall n \in \mathbb{N}.$$

From now on we assume that conditions (3.2.3), (3.2.4) and (3.2.5) are satisfied.

For a given weight sequence we define an associated weight function h by

$$h(r) := \inf\{m_n r^n : n \geq 0\} \quad \text{for } r \geq 0.$$

As h is by definition the infimum of a family of increasing functions, h is increasing. In addition we have $h(0) = 0$.

As the sequence $\left(\frac{m_n}{m_{n+1}}\right)$ is decreasing due to (3.2.5) and tending to 0 due to (3.2.3), we can define

$$K(r) := \max\left\{n \in \mathbb{N} : r \leq \frac{m_n}{m_{n+1}}\right\} \quad \text{for } 0 < r \leq \frac{m_0}{m_1}.$$

For $n \leq K(r)$ we have by definition $\frac{m_n}{m_{n+1}} \geq r$. This is equivalent to $m_n r^n \geq m_{n+1} r^{n+1}$; moreover $m_n r^n > m_{n+1} r^{n+1}$ for $r < \frac{m_n}{m_{n+1}}$. For $n > K(r)$ we have again by definition $\frac{m_n}{m_{n+1}} < r$ which is equivalent to $m_n r^n < m_{n+1} r^{n+1}$. This shows that the sequence $(r^n m_n)_n$ is decreasing for $n \leq K(r)$ and increasing for $n > K(r)$, thus it attains its minimum at $n = K(r) + 1$. This now shows that for $r \in \left(\frac{m_{n+1}}{m_{n+2}}, \frac{m_n}{m_{n+1}}\right]$

$$h(r) = m_{n+1} r^{n+1}, \tag{3.2.6}$$

and for $r > \frac{m_0}{m_1}$

$$h(r) = m_0. \tag{3.2.7}$$

In particular, h is smooth except points of the form $r_n = \frac{m_n}{m_{n+1}}$. In addition a direct computation shows that h is continuous.

Since $h(r) \leq m_n r^n$ for all r and all n by definition and $h(r_n) = m_n r_n^n$, we get

$$m_n = \sup_{r>0} \frac{h(r)}{r^n} \quad \text{for all } n \geq 0. \tag{3.2.8}$$

In addition we have for $0 < b \leq 1$ and $B > 1$ and sufficiently small t

$$bh(t) \geq h(bt), \tag{3.2.9}$$

$$Bh(t) \leq h(Bt). \tag{3.2.10}$$

And

$$\frac{h(t)}{t} \leq h(Dt) \tag{3.2.11}$$

for all $t > 0$, where D is chosen sufficiently large such that $m_{n+1} \leq D^n m_n$ for all n; the existence of a suitable D is ensured by property (3.2.4).

In the proof of theorem 3.2.2 we will need a function N defined as

$$N(r) := \min \{n \in \mathbb{N} : h(r) = m_n r^n\}. \tag{3.2.12}$$

In other words, $N(r)$ denotes the smallest n with $\frac{m_n}{m_{n+1}} \leq r$. It is easy to see that N is decreasing.

Before proving the existence of almost holomorphic extensions we need an auxiliary result. Lemma 3.2.1 and the subsequent theorem 3.2.2 are taken from [5, p. 41].

Lemma 3.2.1. *Let $f \in C_{gl}\{M_n\}(\mathbb{R})$, and for $\xi \in \mathbb{R}$, $z \in \mathbb{C}$ and $n \in \mathbb{N}$ define*

$$T^n_\xi(f, z) := \sum_{k=0}^{n} f^{(k)}(\xi) \frac{(z-\xi)^k}{k!}.$$

Then, for $\xi_1, \xi_2 \in \mathbb{R}$, we have

$$\left|T^n_{\xi_1}(f, z) - T^n_{\xi_2}(f, z)\right| \leq ||f||_{\mathbb{R},B} B^{n+1} m_{n+1} (|\xi_1 - \xi_2| + |z - \xi_1|)^{n+1} \tag{3.2.13}$$

for any $B > 0$ with $||f||_{\mathbb{R},B} < \infty$.

Proof. Assume w.l.o.g. $\xi_1 \leq \xi_2$. We use Taylor's formula to rewrite $T^n_{\xi_2}(f, z)$ as follows

$$\begin{aligned}
T^n_{\xi_2}(f, z) &= \sum_{k=0}^{n} f^{(k)}(\xi_2) \frac{(z-\xi_2)^k}{k!} \\
&= \sum_{k=0}^{n} \left(\left(\sum_{j=k}^{n} f^{(j)}(\xi_1) \frac{(\xi_2-\xi_1)^{j-k}}{(j-k)!} \right) + f^{(n+1)}(\xi(k)) \frac{(\xi_2-\xi_1)^{n+1-k}}{(n+1-k)!} \right) \frac{(z-\xi_2)^k}{k!} \\
&= \left(\sum_{j=0}^{n} f^{(j)}(\xi_1) \sum_{k=0}^{j} \frac{(\xi_2-\xi_1)^{j-k}}{(j-k)!} \frac{(z-\xi_2)^k}{k!} \right) + \sum_{k=0}^{n} f^{(n+1)}(\xi(k)) \frac{(\xi_2-\xi_1)^{n+1-k}}{(n+1-k)!} \frac{(z-\xi_2)^k}{k!} \\
&= \underbrace{\sum_{j=0}^{n} f^{(j)}(\xi_1) \frac{(z-\xi_1)^j}{j!}}_{=T^n_{\xi_1}(f,z)} + \sum_{k=0}^{n} f^{(n+1)}(\xi(k)) \frac{(\xi_2-\xi_1)^{n+1-k}}{(n+1-k)!} \frac{(z-\xi_2)^k}{k!}
\end{aligned}$$

where $\xi(k) \in (\xi_1, \xi_2)$. We used the binomial theorem to derive the last line.

Observe that there exists $B > 0$ such that $|f^{(n+1)}(\xi(k))| \leq ||f||_{\mathbb{R},B} B^{n+1} M_{n+1} < \infty$ for all k. Therefore

$$\begin{aligned} \left|T^n_{\xi_1}(f,z) - T^n_{\xi_2}(f,z)\right| &= \left|\sum_{k=0}^{n} f^{(n+1)}(\xi(k)) \frac{(\xi_2-\xi_1)^{n+1-k}}{(n+1-k)!} \frac{(z-\xi_2)^k}{k!}\right| \\ &\leq ||f||_{\mathbb{R},B} B^{n+1} M_{n+1} \sum_{k=0}^{n} \frac{|\xi_2-\xi_1|^{n+1-k}}{(n+1-k)!} \frac{|z-\xi_2|^k}{k!} \\ &= ||f||_{\mathbb{R},B} B^{n+1} \underbrace{\frac{M_{n+1}}{(n+1)!}}_{=m_{n+1}} (|\xi_2-\xi_1| + |z-\xi_2|)^{n+1}. \end{aligned}$$

Again we used the binomial theorem to derive the last line. □

Theorem 3.2.2. *Let $f \in C_{gl}\{M_n\}(\mathbb{R})$, then there exists $F \in C^\infty(\mathbb{C})$ with $F|_{\mathbb{R}} = f$ and constants $B, C > 0$ such that for all $z = x + iy \in \mathbb{C}$*

$$\left|\overline{\partial} F(z)\right| \leq Ch(B|y|).$$

In addition F can be chosen to be bounded and to have globally bounded partial derivatives of first order.

Proof. Assume w.l.o.g. $||f||_{\mathbb{R},1} < \infty$; this is no restriction because if $||f||_{\mathbb{R},B} < \infty$, then, for $g(x) := f(B^{-1}x)$ we have $||g||_{\mathbb{R},1} = ||f||_{\mathbb{R},B} < \infty$. If G is an extension of g, then $F(z) := G(Bz)$ is the required extension of F.

For $z = x + iy \in \mathbb{C} \backslash \mathbb{R}$ define

$$G(z) := T_x^{N(2|y|)}(f, z),$$

where N is defined as in (3.2.12). Observe that as all notions involved in the definition of G are measurable, G is measurable.

In addition G is locally bounded: For $\zeta = \xi + i\eta \in B(z, \frac{|y|}{2})$ we have $\frac{|y|}{2} \leq |\eta| \leq \frac{3|y|}{2}$, thus $N(|y|) \geq N(2|\eta|)$, and hence

$$|G(\zeta)| \leq \max_{n \leq N(|y|)} \sup_{\zeta \in B(z,|y|/2)} |T^n_\xi(f, \zeta)| < \infty.$$

Let ψ be a smooth, non-negative and radially symmetric function defined on $\mathbb{C}$ with $\text{supp}(\psi) \subseteq \mathbb{D}$ and

$$\int_{\mathbb{C}} \psi(z) dx \wedge dy = 1.$$

We claim that the extension defined by

$$F(z) := \begin{cases} f(z) & , z \in \mathbb{R}, \\ \frac{4}{y^2} \int_{\mathbb{C}} \psi\left(2\frac{\zeta - z}{y}\right) G(\zeta) d\xi \wedge d\eta & , z \in \mathbb{C} \backslash \mathbb{R}. \end{cases}$$

is as required.

Observe that the integrand in the above definition of $F(z)$ is constant zero for $\zeta \notin B(z, \frac{|y|}{2})$. Since the integrand is smooth in z for all ζ and since G is locally bounded, it follows that F is smooth on $\mathbb{C}\backslash\mathbb{R}$. In addition

$$\begin{aligned}
\frac{4}{y^2}\int_{\mathbb{C}} \psi\left(2\frac{\zeta - z}{y}\right)\zeta^k d\xi \wedge d\eta &= \int_{\mathbb{C}} \psi(\zeta)\left(\frac{y}{2}\zeta + z\right)^k d\xi \wedge d\eta \\
&= \int_{B(0,1)} \psi(\zeta)\left(\frac{y}{2}\zeta + z\right)^k d\xi \wedge d\eta \\
&= \int_0^1 \psi(r) r \int_0^{2\pi} (\frac{y}{2} r e^{i\phi} + z)^k d\phi dr \\
&= \int_0^1 \psi(r) r \frac{1}{i} \int_{\partial B(z, \frac{y}{2} r)} \frac{\zeta^k}{\zeta - z} d\zeta dr \\
&= \int_0^1 \psi(r) r 2\pi z^k dr \\
&= z^k \int_{\mathbb{C}} \psi(\zeta) d\xi \wedge d\eta = z^k.
\end{aligned}$$

By linearity of the integral it thus follows for an arbitrary polynomial p

$$\frac{4}{y^2}\int_{\mathbb{C}} \psi\left(2\frac{\zeta - z}{y}\right) p(\zeta) d\xi \wedge d\eta = p(z).$$

Now set $p(z) = T^n_{x_0}(f, z)$ for an arbitrary $x_0 \in \mathbb{R}$ which implies

$$F(z) = T^n_{x_0}(f, z) + \frac{4}{y^2}\int_{\mathbb{C}} \psi\left(2\frac{\zeta - z}{y}\right)(G(\zeta) - T^n_{x_0}(f, \zeta)) d\xi \wedge d\eta, \tag{3.2.14}$$

and applying $\overline{\partial}$ to the above equality yields

$$\overline{\partial} F(z) = \underbrace{\overline{\partial}(T^n_{x_0}(f, z))}_{=0} + \int_{\mathbb{C}} \overline{\partial}\left(\frac{4}{y^2}\psi\left(2\frac{\zeta - z}{y}\right)\right)(G(\zeta) - T^n_{x_0}(f, \zeta)) d\xi \wedge d\eta.$$

Now let $x_0 = x$, which leads to

$$\overline{\partial} F(z) = \int_{\mathbb{C}} \overline{\partial}\left(\frac{4}{y^2}\psi\left(2\frac{\zeta - z}{y}\right)\right)(G(\zeta) - T^n_x(f, \zeta)) d\xi \wedge d\eta; \tag{3.2.15}$$

observe that $T^n_x(f, \zeta)$ is not in the scope of the $\overline{\partial}$-operator.

Computing the partial derivatives with respective to x and y separately and adding their respective absolute values immediately implies that

$$\left|\overline{\partial}\left(\frac{4}{y^2}\psi\left(2\frac{\zeta - z}{y}\right)\right)\right| \leq \frac{A}{|y|^3}$$

for some positive A. Therefore we get by using the representation (3.2.15)

$$\begin{aligned} |\overline{\partial} F(z)| &\leq \frac{A}{|y|^3} \int_{B(z,\frac{|y|}{2})} |G(\zeta) - T_x^n(f,\zeta)| d\xi \wedge d\eta \\ &\leq \frac{\tilde{C}}{|y|} \sup_{\zeta \in B(z,\frac{|y|}{2})} |G(\zeta) - T_x^n(f,\zeta)| \end{aligned} \tag{3.2.16}$$

which holds for arbitrary $n \in \mathbb{N}$.

For $\zeta = \xi + i\eta \in B(z, \frac{|y|}{2})$, we have $|\xi - x| \leq \frac{|y|}{2}$ and $|\eta| \leq \frac{3}{2}|y|$, so by applying lemma 3.2.1 we get for all n

$$\begin{aligned} \left|T_\xi^n(f,\zeta) - T_x^n(f,\zeta)\right| &\leq ||f||_{\mathbb{R},1} m_{n+1} (|\xi - x| + |\zeta - \xi|)^{n+1} \\ &\leq ||f||_{\mathbb{R},1} m_{n+1} (2|y|)^{n+1}. \end{aligned} \tag{3.2.17}$$

Now let $n = N(4|y|) - 1$. Then we have for $\zeta \in B(z, \frac{|y|}{2})$

$$|G(\zeta) - T_x^n(f,\zeta)| \leq |T_\xi^{N(2|\eta|)}(f,\zeta) - T_\xi^n(f,\zeta)| + |T_\xi^n(f,\zeta) - T_x^n(f,\zeta)|.$$

We estimate the summands separately. By (3.2.17) and the definition of n (see also (3.2.12))

$$|T_\xi^n(f,\zeta) - T_x^n(f,\zeta)| \leq ||f||_{\mathbb{R},1} m_{n+1} (4|y|)^{n+1} \leq ||f||_{\mathbb{R},1} h(4|y|). \tag{3.2.18}$$

Recall that N is decreasing, the sequence $(r^n m_n)_n$ is decreasing for $n \leq N(r)$, and $2|\eta| \leq 3|y| \leq 4|y|$. Thus we get for the other summand

$$\begin{aligned} |T_\xi^{N(2|\eta|)}(f,\zeta) - T_\xi^n(f,\zeta)| &\leq \sum_{k=n+1}^{N(2|\eta|)} |f^{(k)}(\xi)| \frac{|\eta|^k}{k!} \\ &\leq ||f||_{\mathbb{R},1} \sum_{k=n+1}^{N(2|\eta|)} m_k |\eta|^k \\ &= ||f||_{\mathbb{R},1} \sum_{k=n+1}^{N(2|\eta|)} m_k (2|\eta|)^k \frac{1}{2^k} \\ &\leq ||f||_{\mathbb{R},1} m_{n+1} (2|\eta|)^{n+1} \sum_{k=1}^{\infty} \frac{1}{2^k} \\ &= ||f||_{\mathbb{R},1} m_{n+1} (2|\eta|)^{n+1} \\ &\leq ||f||_{\mathbb{R},1} m_{n+1} (4|y|)^{n+1} = ||f||_{\mathbb{R},1} h(4|y|). \end{aligned} \tag{3.2.19}$$

Together with (3.2.18) this shows that for $\zeta \in B(z, \frac{|y|}{2})$,

$$|G(\zeta) - T_x^n(f,\zeta)| \leq 2||f||_{\mathbb{R},1} h(4|y|)$$

for $n = N(4|y|) - 1$.

Thus we may continue to estimate (3.2.16) and get

$$|\overline{\partial} F(z)| \leq \frac{2\tilde{C}}{|y|} ||f||_{\mathbb{R},1} h(4|y|).$$

As by (3.2.11) there is a constant D such that $\frac{h(t)}{t} \leq h(Dt)$, there exist positive constants B, C such that

$$|\overline{\partial} F(z)| \leq C h(B|y|). \tag{3.2.20}$$

It remains to show that F is smooth on $\mathbb{R}$. A similar argument shows that $|F(z) - T_x^n(f,z)| = o(|y|^n)$ as $|y| \to 0$ for all n: For given n we get for sufficiently small t that $N(2t) \geq n+1$. By (3.2.14), there is a constant $K > 0$ such that

$$\begin{aligned} |F(z) - T_x^n(f,z)| &\leq K \sup_{\zeta \in B(z, \frac{|y|}{2})} |G(\zeta) - T_x^n(f,z)| \\ &\leq K \sup_{\zeta \in B(z, \frac{|y|}{2})} \left(|T_\xi^n(f,\zeta) - T_x^n(f,\zeta)| + |T_\xi^{N(2|\eta|)}(f,\zeta) - T_\xi^n(f,\zeta)| \right) \\ &\leq K ||f||_{\mathbb{R},1} m_{n+1} \left((2|y|)^{n+1} + (4|y|)^{n+1} \right) \leq K ||f||_{\mathbb{R},1} m_{n+1} (6|y|)^{n+1} \end{aligned} \tag{3.2.21}$$

for sufficiently small $|y|$. We used (3.2.17) and (3.2.19) to derive the third inequality. This implies that F is actually n-times differentiable at points $z \in \mathbb{R}$.

Now we modify the given extension F to get a new extension which has the additional boundedness properties. If we set $n = 0$ in (3.2.21), we get $|F(z) - f(x)| \leq K ||f||_{\mathbb{R},1} m_1(6|y|)$ for $z = x + iy \in \mathbb{C}$ with sufficiently small imaginary part. This shows that for z in some small open strip $S \supseteq \mathbb{R}$ with height l

$$|F(z)| \leq K ||f||_{\mathbb{R},1} m_1(6|y|) + ||f||_{\mathbb{R},1} M_0. \tag{3.2.22}$$

Let $\rho \in C^\infty(\mathbb{R})$ be non-negative, compactly supported in $(-l, l)$ and constant 1 on a small neighbourhood around 0. Then we define a new extension by $\tilde{F}(z) := F(z)\rho(y)$. It can be easily verified that $\tilde{F}$ is again an extension of f with property (3.2.20) (with different constants $\tilde{B}, \tilde{C}$). (3.2.22) immediately implies that $\tilde{F}$ is bounded.

Global boundedness of the partial derivatives of $\tilde{F}$ follows by a similar version of (3.2.16) for $n = 1$ where $\overline{\partial}$ is replaced by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, and a subsequent analogous argumentation. □

Remark 3.2.3. *It is also possible to prove a local form of the above theorem (see [5, Theorem 2, p. 41]):*

Let $f \in C\{M_n\}(U)$, for some open $U \subseteq \mathbb{R}$, and h the associated weight function. Then for all compact sets $K \subseteq U$ there exists a continuously differentiable, compactly supported function F_K defined on $\mathbb{C}$ with $F_K|_K = f|_K$ and positive constants B_K, C_K such that

$$\left|\overline{\partial} F_K(z)\right| \leq C_K h(B_K \rho(z, K))$$

for all $z \in \mathbb{C}$, where $\rho(z,K) := \inf\{|z - w| : w \in K\}$.

The next theorem gives the converse direction. It is a special case of [5, Theorem 4, p. 44].

Theorem 3.2.4. *Let $F \in C^1(\mathbb{C})$ be bounded and suppose there are positive constants B, C such that*

$$|\overline{\partial} F(z)| \leq Ch(B|y|)$$

for all $z \in \mathbb{C}$. Then $f := F|_{\mathbb{R}} \in C_{gl}\{M_n\}(\mathbb{R})$.

Proof. Let $m \in \mathbb{Z}$. Then for $|x - m| < 1$ we have the following representation (see theorem 1.0.11)

$$f(x) = \underbrace{\frac{1}{2\pi i} \int_{\partial B(m,2)} \frac{F(\zeta)}{\zeta - x} d\zeta}_{=:h(x)} + \underbrace{\frac{1}{2\pi i} \int_{B(m,2)} \frac{\overline{\partial} F(\zeta)}{\zeta - x} d\zeta \wedge d\overline{\zeta}}_{=:g(x)}.$$

It is clear that the first summand is smooth on $(m-1, m+1)$ and we get for all $n \in \mathbb{N}$ and $x \in (m-1, m+1)$

$$h^{(n)}(x) = \frac{n!}{2\pi i} \int_{\partial B(m,2)} \frac{F(\zeta)}{(\zeta - x)^{n+1}} d\zeta.$$

For g we proceed inductively. Suppose we have already shown for some $n \in \mathbb{N}$ and all $x \in (m-1, m+1)$

$$g^{(n)}(x) = \frac{n!}{2\pi i} \int_{B(m,2)} \frac{\overline{\partial} F(\zeta)}{(\zeta - x)^{n+1}} d\zeta \wedge d\overline{\zeta}.$$

Then take some fixed $x_0 \in (m-1, m+1)$ and an arbitrary sequence $(x_k)_k$ (with $|x_k - m| < 1$ for all k) converging to x_0. Then we get

$$\frac{g^{(n)}(x_k) - g^{(n)}(x_0)}{x_k - x_0} = \frac{n!}{2\pi i} \int_{B(m,2)} \underbrace{\frac{1}{x_k - x_0} \overline{\partial} F(\zeta) \left(\frac{1}{(\zeta - x_k)^{n+1}} - \frac{1}{(\zeta - x_0)^{n+1}} \right)}_{=:\tau_k(\zeta)} d\zeta \wedge d\overline{\zeta}$$

and observe that for all $\zeta \in \mathbb{C}$ we have

$$\begin{aligned}
\tau_k(\zeta) &= \frac{1}{x_k - x_0} \overline{\partial} F(\zeta) \left(\frac{\sum_{j=0}^{n} \binom{n+1}{j} \zeta^j (-1)^{n+1-j} (x_0^{n+1-j} - x_k^{n+1-j})}{(\zeta - x_k)^{n+1} (\zeta - x_0)^{n+1}} \right) \\
&= \frac{1}{x_k - x_0} \overline{\partial} F(\zeta) \left(\frac{\sum_{j=0}^{n} \binom{n+1}{j} \zeta^j (-1)^{n-j} (x_k - x_0) p_{n-j}(x_0, x_k)}{(\zeta - x_k)^{n+1} (\zeta - x_0)^{n+1}} \right) \\
&= \overline{\partial} F(\zeta) \left(\frac{\sum_{j=0}^{n} \binom{n+1}{j} \zeta^j (-1)^{n-j} p_{n-j}(x_0, x_k)}{(\zeta - x_k)^{n+1} (\zeta - x_0)^{n+1}} \right)
\end{aligned}$$

where we have set $p_j(x_0, x_k) := \sum_{l=0}^{j} x_0^l x_k^{j-l}$. Using the assumption $\overline{\partial} F(z) \leq Ch(B|y|)$ and that $\frac{1}{|\xi+i\eta-x_k|^{n+1}|\xi+i\eta-x_0|^{n+1}} \leq \frac{1}{|\eta|^{2(n+1)}}$, we thus can derive for $\zeta = \xi + i\eta \in B(m,2)$

$$\begin{aligned}|\tau_k(\zeta)| &\leq \frac{Ch(B|\eta|)}{|\eta|^{2(n+1)}} \left| \sum_{j=0}^{n} \binom{n+1}{j} \zeta^j (-1)^{n-j} p_{n-j}(x_0, x_k) \right| \\ &\leq CB^{2(n+1)} m_{2(n+1)} \underbrace{\left| \sum_{j=0}^{n} \binom{n+1}{j} \zeta^j (-1)^{n-j} p_{n-j}(x_0, x_k) \right|}_{\leq D<\infty \text{ uniformly for all } \zeta\in B(m,2),\ k\in\mathbb{N}}\end{aligned}$$

where we used (3.2.8). Thus we can bound all τ_k by a uniform constant on $B(m,2)$ (the constant may depend on m, but not on k). Observe in addition that $\tau_k(\zeta)$ converges pointwise to $\overline{\partial} F(\zeta) \frac{n+1}{(\zeta - x_0)^{n+2}}$. Thus we can apply the dominated convergence theorem and derive

$$g^{(n+1)}(x_0) = \frac{(n+1)!}{2\pi i} \int_{B(m,2)} \frac{\overline{\partial} F(\zeta)}{(\zeta - x_0)^{n+2}} d\zeta \wedge d\overline{\zeta}.$$

By induction we may conclude for all $n \in \mathbb{N}$ and $x \in (m-1, m+1)$

$$f^{(n)}(x) = \frac{n!}{2\pi i} \int_{\partial B(m,2)} \frac{F(\zeta)}{(\zeta - x)^{n+1}} d\zeta + \frac{n!}{2\pi i} \int_{B(m,2)} \frac{\overline{\partial} F(\zeta)}{(\zeta - x)^{n+1}} d\zeta \wedge d\overline{\zeta}.$$

Let $K := \sup_{z\in\mathbb{C}} |F(z)|$ which is less than infinity by boundedness of F. We get for $|x - m| < 1$

$$\begin{aligned}|f^{(n)}(x)| &\leq \left| \frac{n!}{2\pi i} \int_{\partial B(m,2)} \frac{F(\zeta)}{(\zeta - x)^{n+1}} d\zeta \right| + \left| \frac{n!}{2\pi i} \int_{B(m,2)} \frac{\overline{\partial} F(\zeta)}{(\zeta - x)^{n+1}} d\zeta \wedge d\overline{\zeta} \right| \\ &\leq \frac{n!}{2\pi} K 4\pi + \frac{n!}{\pi} \int_{B(m,2)} \frac{Ch(B|\eta|)}{|\eta|^n |\zeta - x|} d\xi \wedge d\eta \\ &\leq \frac{n!}{2\pi} K 4\pi + \frac{n!}{\pi} CB^n m_n \int_{B(m,2)} \frac{1}{|\zeta - x|} d\xi \wedge d\eta \\ &\leq \frac{n!}{2\pi} K 4\pi + \frac{n!}{\pi} CB^n m_n \underbrace{\int_{B(x,3)} \frac{1}{|\zeta - x|} d\xi \wedge d\eta}_{=6\pi;\text{ see } (1.0.12)} \\ &\leq \underbrace{(2K + 6C)}_{=:\tilde{C}} B^n M_n\end{aligned}$$

where we used (3.2.8) to derive the third inequality. Observe that the constants $B, \tilde{C}$ are independent of m. As every x lies in some $B(m,1)$ we thus have global estimates for $|f^{(n)}(x)|$ which finally shows that $f \in C_{gl}\{M_n\}(\mathbb{R})$. □

Remark 3.2.5. *A similar proof can be used to prove the converse to remark 3.2.3: Let $U \subseteq \mathbb{R}$ be open and let f be a function on U. Suppose for all compact sets $K \subseteq U$ there is a continuously differentiable extension F_K of $f|_K$ to $\mathbb{C}$ and constants $B_K, C_K > 0$ depending on K such that*

$$\overline{\partial} F_K(z) \leq C_K h(B_K \rho(z, K))$$

for all $z \in \mathbb{C}$. An analogous argumentation as in the last proof then shows that $f \in C\{M_n\}(U)$. Now the appearing constants cannot be chosen uniformly. But this is not necessary as we only want to prove the local result.

Combining theorems 3.2.2 and 3.2.4 (remarks 3.2.3 and 3.2.5) we get a characterization of $C_{gl}\{M_n\}(\mathbb{R})$ ($C\{M_n\}(U)$):

Corollary 3.2.6. *Let f be a function defined on $\mathbb{R}$. Then $f \in C_{gl}\{M_n\}(\mathbb{R})$ if and only if there exists a bounded continuously differentiable extension F of f to $\mathbb{C}$ and constants $B, C > 0$ such that for all $z \in \mathbb{C}$*

$$|\overline{\partial} F(z)| \leq C h(B|y|).$$

Let $U \subseteq \mathbb{R}$ be open and let f be a function defined on U. Then $f \in C\{M_n\}(U)$ if and only if for all compact $K \subseteq U$ there exists a continuously differentiable extension F_K of $f|_K$ to $\mathbb{C}$ and constants $B_K, C_K > 0$ such that for all $z \in \mathbb{C}$

$$|\overline{\partial} F_K(z)| \leq C_K h(B_K \rho(z, K)).$$

Now this characterization can be used to give a short proof of the stability under composition of regular DC classes. The proof reduces to an application of the chain rule of the d-bar operator (see lemma 1.0.1).

Theorem 3.2.7. *Let U, V be open subsets of $\mathbb{R}$ and (M_n) a regular weight sequence. Assume $f \in C_{gl}\{M_n\}(U)$ (or $C\{M_n\}(U)$) and $g \in C_{gl}\{M_n\}(V)$ (or $C\{M_n\}(V)$) with $g : V \to U$. Then it follows*

$$f \circ g \in C_{gl}\{M_n\}(V) \text{ (or } C\{M_n\}(V)).$$

Proof. The standard proof (see e.g. [1, Theorem 4.7, p. 11-12]) uses the "Faà di Bruno formula" (see e.g. [1, Proposition 4.3, p. 9]), which gives a formula for the n-th derivative of a composition of functions. In the following we use the techniques developed in this chapter to derive an alternative proof (for global DC classes with $U = V = \mathbb{R}$).

Let $f, g \in C_{\mathrm{gl}}\{M_n\}(\mathbb{R})$. Let F, G be extensions of f, g as in theorem 3.2.2. We take them to be bounded and to have globally bounded partial derivatives. If we can show $|\overline{\partial}(F \circ G)(z)| \leq C h(B|y|)$, this implies the desired result by theorem 3.2.4.

By global boundedness of $\left|\frac{\partial}{\partial y} G(z)\right|$, there is a constant D such that for all $z = x + iy \in \mathbb{C}$

$$|G(x + iy) - G(x)| \leq D|y|,$$

and as $|G(x+iy) - G(x)| \geq |\Im G(x+iy)|$ (use that $G(x) \in \mathbb{R}$), we get

$$|\Im G(x+iy)| \leq D|y|. \tag{3.2.23}$$

Using lemma 1.0.1, we get

$$\begin{aligned}
|\overline{\partial}(F \circ G)(z)| &\leq |\partial F(G(z))||\overline{\partial} G(z)| + |\overline{\partial} F(G(z))||\overline{\partial G(z)}| \\
&\leq K_2 C_1 h(B_1|y|) + K_1 C_2 h(B_2|\Im G(z)|) \\
&\leq K_2 C_1 h(B_1|y|) + K_1 C_2 h(B_2 D|y|) \\
&\leq Ch(B|y|),
\end{aligned}$$

where K_1 is a global bound for ∂G (which is possible since the partial derivatives are globally bounded) and B_1, C_1 are chosen such that $|\overline{\partial} G(z)| \leq C_1 h(B_1|y|)$; the constants B_2, C_2, K_2 are chosen analogously for F. We used (3.2.23) to derive the third inequality and chose B, C sufficiently large to get the last inequality. This completes the proof. □

4 An application of 3.2. A stronger Denjoy Carleman theorem

In this chapter we present a proof of the famous Denjoy Carleman theorem based on asymptotically holomorphic extensions. We will use the characterization developed in section 3.2. Using some additional assumptions, we will actually show more; namely in the case of a non-quasianalytic weight sequence (i.e. $(M_n)_n$ fulfils $\neg$(ii) from theorem 4.0.1) we will give a precise estimate from below and from above of the maximal possible growth of a function in a neighbourhood of a point where it vanishes of infinite order (see theorem 4.3.1). We will mainly follow [5] and [6].

In order to formulate the Denjoy-Carleman theorem we need to define the notion of *quasianalyticity* of a class of smooth functions and of a regular weight sequence:

Definition A class of smooth functions defined on some open set $U \subseteq \mathbb{R}$ is called *quasianalytic* if for each element f in this class $f^{(n)}(x_0) = 0$ for some $x_0 \in U$ and all $n \in \mathbb{N}$ implies $f \equiv 0$ on the connected component of x_0.

A regular weight sequence $(M_n)_n$ is called *quasianalytic* if $\sum_{n=1}^{\infty} M_n^{-1/n} = \infty$.

Observe that the class of analytic functions is clearly quasianalytic. The goal of the Denjoy-Carleman theorem is now to describe quasianalyticity for DC classes in terms of a growth condition for the defining weight sequence; i.e. quasianalyticity of the defining weight sequence is equivalent to quasinalayticity of the DC class.

Before formulating the theorem, let us introduce another weight function ϕ for regular DC classes (we have already defined the associated weight function h, see section 3.2): Given the associated weight function h of a regular DC class, we define $\phi(t) := \log\left(\log\left(\frac{1}{h(t)}\right)\right)$ for $0 < t \leq t_0$ with $t_0 := h^{-1}(\min\{\frac{1}{2e}, \frac{m_0}{2}\})$ and $\phi(t) = \phi(t_0)$ for $t_0 \leq t \leq 1$. Note that h is strictly increasing and continuous on $(0, \frac{m_0}{m_1}]$, thus bijective onto its range there; and for $r \geq \frac{m_0}{m_1}$, we have $h(r) = m_0$. By construction ϕ is positive on $(0, 1]$, decreasing and converging to ∞ near 0.

Theorem 4.0.1 (Denjoy-Carleman theorem).
Let $C\{M_n\}(U)$ be a regular DC class (i.e. properties (3.2.3), (3.2.4) *and* (3.2.5) *are fulfilled). Then the following are equivalent:*

(i) $C\{M_n\}(U)$ *is quasianalytic,*

(ii) $(M_n)_n$ *is quasianalytic (i.e.* $\sum_{n=1}^{\infty} M_n^{-1/n} = \infty$*),*

(iii) $\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} = \infty$,

(iv) $\int_0^1 \phi(t)dt = \infty$.

Remark 4.0.2. *The equivalence of (i), (ii) and (iii) is part of the classical Denjoy-Carleman theorem (see e.g. [13, 19.11 Theorem, p. 380]). That quasianalyticity can be characterized by (iv) is stated for example in [6, Theorem 1, p. 55].*

The equivalence of (ii), (iii) and (iv) can be verified by a direct computation. We do not carry out the computations here; for a proof(idea) we refer to [2, Lemma 5, p. 64]. From now on, we take for granted that properties (ii), (iii) and (iv) are equivalent, thus quasianalyticity for regular weight sequences from now on can be characterized by any of the properties (ii), (iii) and (iv).

The following sections of this chapter are concerned with the equivalence of (i) and (iv). We will not prove this equivalence in full generality; i.e. (4.1.1) *will be assumed in general for our classes and for the proof of (i)⇒(iv) we will assume in addition* (4.3.40)*. These additional assumptions will make it possible to give an alternative proof based on asymptotically holomorphic extensions and even to derive quantitative results in the proof of ¬(iv)⇒ ¬(i); see section 4.3.*

4.1 Preparation

In what follows $C\{M_n\}$ denotes a regular DC class and h the associated weight function. The equivalence (i)⇔(iv) from theorem 4.0.1 will be proved using the additional assumption

$$t|\phi'(t)| \to \infty, \text{ as } t \to 0. \tag{4.1.1}$$

Observe that the notion $t|\phi'(t)|$ makes sense for all points except $t = \frac{m_n}{m_{n+1}}$; differentiation is not be available at such points. We assume from now on that (4.1.1) is satisfied.

Due to (4.1.1) an application of the mean value theorem implies that for given $C, D > 0$, there is $T(C, D) > 0$ such that for $0 < t < T$ we have

$$|\phi(t) - \phi(Dt)| \geq C. \tag{4.1.2}$$

Before we prove some preparatory results, we observe that condition (4.1.1) is actually restrictive: For non-quasianalytic weight sequences an example is given by the Gevrey-classes; they are defined by the weight sequence $M_n := (n!)^\alpha$, where $\alpha > 1$; see [5, Example, p. 41].

But even in the quasianalytic case, condition (4.1.1) is restrictive. More specifically: We construct a regular quasianalytic weight sequence (where quasianalyticity means for a weight sequence to fulfil one of the equivalent conditions (ii), (iii) or (iv) from theorem 4.0.1; see remark 4.0.2) with $\liminf_{t\to 0} t|\phi'(t)| = 0$. In view of lemma 4.1.2 (that is proved without using (4.1.1)) this is somewhat surprising. The subsequent example was suggested by Gerhard Schindl.

Before constructing a suitable weight sequence, we observe: Given any regular weight sequence $(M_n)_n$ and denoting $\mu_n := \frac{m_n}{m_{n-1}}$ (where $m_n = \frac{M_n}{n!}$), an application of (3.2.6) shows for $t \in \left(\frac{1}{\mu_{k+1}}, \frac{1}{\mu_k}\right)$

$$t|\phi'(t)| = \frac{k}{-k\log(t) - \log(m_k)}.$$

Thus for given $\varepsilon > 0$, the existence of some $t \in \left(\frac{1}{\mu_{k+1}}, \frac{1}{\mu_k}\right)$ with

$$t|\phi'(t)| \leq \varepsilon$$

is equivalent to the existence of some $t \in \left(\frac{1}{\mu_{k+1}}, \frac{1}{\mu_k}\right)$ with

$$t \leq \exp\left(-\frac{1}{\varepsilon}\right)\frac{1}{m_k^{1/k}}.$$

This now implies for given $\varepsilon > 0$

$$\inf_{t \in \left(\frac{1}{\mu_{k+1}}, \frac{1}{\mu_k}\right)} t|\phi'(t)| \leq \varepsilon \Leftrightarrow \exp\left(\frac{1}{\varepsilon}\right) m_k^{1/k} < \mu_{k+1}. \tag{4.1.3}$$

Let $(N_k)_k$ be a given regular quasianalytic weight sequence with $N_0 = 1$. We denote $n_k := \frac{N_k}{k!}$ and $\nu_k := \frac{n_k}{n_{k-1}}$, we set $\nu_0 = 1$. An example for such a sequence is given by $\nu_k := \log(k+e)$. We now use the given weight sequence $(N_k)_k$, that may fulfil (4.1.1), to construct a new regular quasianalytic weight sequence $(M_k)_k$ with $\liminf_{t \to 0} t|\phi'(t)| = 0$.

First we construct an increasing sequence $(a_k)_k$ iteratively as follows: Set $a_0 := 0, a_1 := 1$. Suppose a_l is already defined for $l \leq k$ such that $a_l < a_{l+1}$. Then let $b_k > a_k + 2$ be chosen such that

$$\frac{\nu_{b_k}}{e^k \nu_{a_k}} > 2. \tag{4.1.4}$$

As $\nu_l \to \infty$ with $l \to \infty$ (which follows from regularity conditions (3.2.3) and (3.2.5)), the definition of b_k makes sense. Let $c_k \geq b_k$ be defined sufficiently large to get

$$\frac{n_{b_k}}{\left(\prod_{p=a_k+1}^{b_k} e^k n_{a_k}^{1/a_k}\right)\left(\prod_{l=2}^{k}\prod_{r=a_{l-1}+1}^{a_l} e^{l-1} n_{a_{l-1}}^{1/a_{l-1}}\right)\nu_1} 2^{c_k - b_k} > 1. \tag{4.1.5}$$

Let $d_k > a_k + 2$ be defined such that

$$\sum_{l=a_{k+1}}^{d_k} \frac{1}{l n_l^{1/l}} \geq e^k. \tag{4.1.6}$$

The definition of d_k makes sense as $(N_k)_k$ was assumed to be quasianalytic (use Stirling's formula together with property (ii) from theorem 4.0.1). Now we set

$$a_{k+1} := \max\{c_k, d_k\}.$$

Then $(a_k)_k$ is by definition an increasing sequence with $a_{k+1} > a_k + 2$.

Finally set $\mu_0 := 1$, $\mu_1 := \nu_1$ and for $k \geq 1$ and $a_k + 1 \leq l \leq a_{k+1}$

$$\mu_l := e^k n_{a_k}^{1/a_k}.$$

Set $m_n := \prod_{l=0}^{n} \mu_l$ and $M_n := n! m_n$ for all $n \in \mathbb{N}$. The regularity conditions (3.2.3), (3.2.4) and (3.2.5) follow directly from the assumed regularity of the sequence $(N_k)_k$.

We observe

$$\begin{aligned} \frac{n_{a_{k+1}}}{m_{a_{k+1}}} &= \frac{n_{b_k}}{m_{b_k}} \frac{\nu_{b_k+1}}{\mu_{b_k+1}} \cdots \frac{\nu_{a_{k+1}}}{\mu_{a_{k+1}}} \\ &\geq \frac{n_{b_k}}{m_{b_k}} \prod_{l=b_k+1}^{a_{k+1}} \frac{\nu_{b_k}}{e^k n_{a_k}^{1/a_k}} \\ &\geq \frac{n_{b_k}}{m_{b_k}} \prod_{l=b_k+1}^{a_{k+1}} \frac{\nu_{b_k}}{e^k \nu_{a_k}} \\ &> \frac{n_{b_k}}{m_{b_k}} 2^{a_{k+1}-b_k} > 1 \end{aligned} \tag{4.1.7}$$

where the first line holds by the definitions of m_l and μ_l (resp. n_l and ν_l). For the second line we used the strong logarithmic convexity of $(N_k)_k$ (i.e. the sequence $(\nu_l)_l$ is increasing) and the definition of μ_l for $a_k + 1 \leq l \leq a_{k+1}$. The third line follows since $n_l^{1/l} \leq \nu_l$ for all $l \in \mathbb{N}$. For the last line we used (4.1.4) (for the second to the last inequality) and (4.1.5) (for the last inequality; observe that the denominator of the quotient in (4.1.5) is exactly m_{b_k}).

This now means that $n_{a_k} > m_{a_k}$ for all $k \in \mathbb{N}$. Therefore

$$\mu_{a_k+1} = e^k n_{a_k}^{1/a_k} > e^k m_{a_k}^{1/a_k}.$$

Applying (4.1.3) we get

$$\inf_{t \in \left(\frac{1}{\mu_{a_k+1}}, \frac{1}{\mu_{a_k}}\right)} t|\phi'(t)| \leq \frac{1}{k}.$$

Therefore $\liminf_{t\to 0} t|\phi'(t)| = 0$ when ϕ is defined by the weight sequence $(M_k)_k$.

It is still left to show qusianalyticity of $(M_k)_k$: To this end we verify condition (iii) from theorem 4.0.1:

$$\begin{aligned} \sum_{l=1}^{\infty} \frac{1}{l\mu_l} &= \frac{1}{\mu_1} + \sum_{k=1}^{\infty} \sum_{l=a_k+1}^{a_{k+1}} \frac{1}{l\mu_l} \\ &= \frac{1}{\mu_1} + \sum_{k=1}^{\infty} \sum_{l=a_k+1}^{a_{k+1}} \frac{1}{l e^k n_{a_k}^{1/a_k}} \\ &\geq \frac{1}{\mu_1} + \sum_{k=1}^{\infty} \frac{1}{e^k} \sum_{l=a_k+1}^{a_{k+1}} \frac{1}{l n_l^{1/l}} \\ &\geq \frac{1}{\mu_1} + \sum_{k=1}^{\infty} \frac{1}{e^k} e^k = \infty \end{aligned}$$

where we used (4.1.6) to derive the last line. This finishes the counterexample for the quasianalytic case.

Remark 4.1.1. *An example of a class which satisfies* (4.1.1) *is given by the weight sequence* $M_n := n!(\log n)^{\alpha n}$*, where* $\alpha > 0$*; see [6, Remarks (3), p. 56]. As* $\phi(t) = \frac{1}{t^{1/\alpha}}$*,* (4.1.1) *is satisfied. This class is quasianalytic for* $\alpha \leq 1$ *and non-quasianalytic for* $\alpha > 1$*.*

The next lemma shows that condition (iv) from theorem 4.0.1 can be formulated differently.

Lemma 4.1.2.

$$\int_0^1 \phi(t)dt = \infty \Leftrightarrow \int_0^1 t|\phi'(t)|dt = \infty.$$

Proof. Since ϕ is decreasing, we have $|\phi'(t)| = -\phi'(t)$. Using integration by parts, we get for $0 < r < 1$

$$\begin{aligned}\int_r^1 \phi(t)dt &= \phi(1) - r\phi(r) - \int_r^1 t\phi'(t)dt \\ &= \phi(1) - r\phi(r) + \int_r^1 t|\phi'(t)|dt.\end{aligned} \tag{4.1.8}$$

As $-r\phi(r) \leq 0$ for all r, we get

$$\int_r^1 \phi(t)dt \leq \phi(1) + \int_r^1 t|\phi'(t)|dt.$$

Therefore $\int_0^1 \phi(t)dt = \infty$ immediately implies $\int_0^1 t|\phi'(t)|dt = \infty$.

To prove the other direction, assume $\int_0^1 t|\phi'(t)|dt = \infty$ and $\int_0^1 \phi(t)dt < \infty$ and derive a contradiction: Under these assumptions, we get by (4.1.8) that

$$\phi(1) - r\phi(r) + \int_r^1 t|\phi'(t)|dt$$

is decreasing in r (because the left-hand side of (4.1.8) is decreasing in r) and converging to some limit $C < \infty$ as $r \to 0$. Since $\int_0^1 t|\phi'(t)|dt = \infty$, there is some T such that for $r \leq T$

$$\int_r^1 t|\phi'(t)|dt \geq C + 1 - \phi(1).$$

As $\phi(1) - r\phi(r) + \int_r^1 t|\phi'(t)|dt \leq C$ for all r by (4.1.8), we thus get for $r \leq T$

$$\phi(1) - r\phi(r) + C + 1 - \phi(1) \leq C$$

which is equivalent to $r\phi(r) \geq 1$ for $r \leq T$. But then $\phi(r) \geq \frac{1}{r}$ for small r and thus ϕ is not integrable near 0 which contradicts the assumption $\int_0^1 \phi(t)dt < \infty$. Thus the proof is completed.

□

The idea for the following lemma is taken from `http://functions.wolfram.com/ElementaryFunctions/ArcTan/20/02/0002/`.

Lemma 4.1.3. *For $x \in \mathbb{R}$ and $n \in \mathbb{N}$*

$$|\arctan^{(n)}(x)| \leq n!.$$

Therefore $\arctan \in C_{gl}\{M_n\}(\mathbb{R})$ *for an arbitrary regular weight sequence.*

Proof. Observe that $\arctan'(x) = \frac{1}{1+x^2}$. A simple computation shows

$$\frac{1}{1+x^2} = \frac{-i}{2}\left(\frac{1}{x-i} - \frac{1}{x+i}\right).$$

Therefore we have for the n-th derivative of arctan (by differentiating the right-hand side of the above equality $(n-1)$ times)

$$\arctan^{(n)}(x) = \frac{i(-1)^n(n-1)!}{2}\left(\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right).$$

As $\frac{1}{|x \pm i|^n} \leq 1$ for $x \in \mathbb{R}$, we get $|\arctan^{(n)}(x)| \leq (n-1)! < n!$. $\square$

Lemma 4.1.4. *Let $a, b > 0$. Set $S_{2b} := \{z \in \mathbb{C} : 0 \leq y \leq 2b\}$. Let $\psi : S_{2b} \to \mathbb{D}$ be defined as $\psi(z) := \frac{e^{\frac{\pi}{2b}z} - i}{e^{\frac{\pi}{2b}z} + i}$ (see section 2.5) and let $\tau : \mathbb{R} \to [0,1]$ be defined as*

$$\tau(x) := \omega(\psi(x+ib), \psi([-a,a]), \mathbb{D}).$$

Then τ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

If $a \geq b$, we have the following estimate for all x

$$\tau(x) \geq C_0 e^{-\frac{\pi}{2b}|x|}.$$

For $|x| \geq 2a (\geq 2b)$ we have in addition

$$\tau(x) \geq e^{-C\frac{|x|-a}{b}}.$$

Here C_0, C are positive constants independent of a and b.

Proof. A simple calculation shows

$$\psi(x+ib) = \frac{e^{\frac{\pi}{2b}x} - 1}{e^{\frac{\pi}{2b}x} + 1}. \tag{4.1.9}$$

Thus ψ maps points of the form $z = x + ib$ to the interval $(-1,1) \subseteq \mathbb{R}$. In addition, we have

$$\psi([-a,a]) = \left[\frac{e^{-\frac{\pi}{2b}a} - i}{e^{-\frac{\pi}{2b}a} + i}, \frac{e^{\frac{\pi}{2b}a} - i}{e^{\frac{\pi}{2b}a} + i}\right]_{\partial\mathbb{D}},$$

where we used the notation from section 2.2 for arcs in $\partial\mathbb{D}$. It can be easily seen that $\psi([-a,a])$ is a subarc of the lower hemisphere of $\partial\mathbb{D}$ for all a, b (here it is not necessary that $a \geq b$). In addition all those arcs are symmetric with respect to the imaginary axis; i.e. $x + iy \in \psi([-a,a])$ implies $-x + iy \in \psi([-a,a])$. If we write $\psi([-a,a]) = [e^{i\phi_1(a,b)}, e^{i\phi_2(a,b)}]_{\partial\mathbb{D}}$, we thus know that we can choose $\phi_j(a,b)$ for $j = 1,2$ such that

$$-\pi < \phi_1(a,b) < -\frac{\pi}{2} < \phi_2(a,b) < 0,$$

and $\frac{1}{2}(\phi_1(a,b) + \phi_2(a,b)) = -\frac{\pi}{2}$ (due to symmetry of the arcs $\psi([-a,a])$).

By the definition of the harmonic measure for $\mathbb{D}$, we have for $t \in (-1,1)$

$$\begin{aligned}
\omega(t,\psi([-a,a]),\mathbb{D}) &= \frac{1}{2\pi i}\int_{[e^{i\phi_1(a,b)},e^{\phi_2(a,b)}]_{\partial\mathbb{D}}} \frac{1-t^2}{\zeta|\zeta - t|^2} d\zeta \\
&= (1-t^2)\frac{1}{2\pi i}\int_{\phi_1(a,b)}^{\phi_2(a,b)} \frac{1}{e^{is}|e^{is}-t|^2} ie^{is} ds \\
&= (1-t^2)\frac{1}{2\pi}\int_{\phi_1(a,b)}^{\phi_2(a,b)} \frac{1}{(\cos(s)-t)^2 + \sin(s)^2} ds \\
&= (1-t^2)\frac{1}{2\pi}\int_{\phi_1(a,b)}^{\phi_2(a,b)} \frac{1}{1-2\cos(s)t+t^2} ds \\
&\geq (1-|t|)(1+|t|)\frac{1}{2\pi}\int_{\phi_1(a,b)}^{\phi_2(a,b)} \frac{1}{1+2|t|+|t|^2} ds \\
&= \frac{1-|t|}{1+|t|}\frac{\phi_2(a,b)-\phi_1(a,b)}{2\pi}.
\end{aligned} \tag{4.1.10}$$

Using the geometric description of the harmonic measure for $\mathbb{D}$ (see section 2.2) and the symmetry of $\psi([-a,a])$, it follows that the function $t \mapsto \omega(t,\psi([-a,a]),\mathbb{D})$ is increasing for $t \in (-1,0)$ and decreasing for $t \in (0,1)$. Since the function $s \mapsto \frac{s-1}{s+1}$ is increasing on $(0,\infty)$, we may conclude that $x \mapsto \psi(x+ib)$ is increasing as composition of increasing functions (use the representation (4.1.9)). As $x \mapsto \psi(x+ib)$ maps $(-\infty,0)$ to $(-1,0)$ and $(0,\infty)$ to $(0,1)$, this implies the monotonicity properties of τ.

If we assume $a \geq b$, we immediately get $\phi_2(a,b) - \phi_1(a,b) \geq \phi_2(b,b) - \phi_1(b,b) = \phi_2(1,1) - \phi_1(1,1) =: \tilde{C_0} > 0$, and thus a uniform bound from below. Therefore we can continue to estimate (4.1.10) and get

$$\omega(t,\psi([-a,a]),\mathbb{D}) \geq \frac{1-|t|}{1+|t|}\frac{\tilde{C_0}}{2\pi} \geq (1-|t|)\underbrace{\frac{\tilde{C_0}}{4\pi}}_{=:C_0}. \tag{4.1.11}$$

Using (4.1.9) and (4.1.11), it thus follows for $a \geq b$ and $x \geq 0$

$$\begin{aligned}
\tau(x) &\geq C_0\left(1 - \frac{e^{\frac{\pi}{2b}x}-1}{e^{\frac{\pi}{2b}x}+1}\right) \\
&\geq C_0\left(1 - \frac{e^{\frac{\pi}{2b}x}-1}{e^{\frac{\pi}{2b}x}}\right) \\
&= C_0 e^{-\frac{\pi}{2b}|x|},
\end{aligned}$$

and for $x < 0$ we have

$$\begin{aligned}\tau(x) &\geq C_0\left(1 - \frac{1-e^{\frac{\pi}{2b}x}}{e^{\frac{\pi}{2b}x}+1}\right)\\ &\geq C_0\left(1-(1-e^{\frac{\pi}{2b}x})\right)\\ &= C_0 e^{-\frac{\pi}{2b}|x|}.\end{aligned}$$

Now choose $z = x + ib$ with $|x| \geq 2a (\geq 2b)$; let C be a constant such that $e^{-(C-\pi)} \leq C_0$. Then

$$\begin{aligned}C_0 e^{-\frac{\pi}{2b}|x|} &\geq e^{-(C-\pi)}e^{-\frac{\pi}{2b}|x|} \geq e^{-(C-\pi)\frac{a}{b}}e^{-\frac{\pi}{2b}|x|}\\ &= e^{-(2C-\pi)\frac{a}{b}+C\frac{a}{b}-\frac{\pi}{2b}|x|} = e^{-(C-\frac{\pi}{2})\frac{2a}{b}+C\frac{a}{b}-\frac{\pi}{2b}|x|}\\ &\geq e^{-(C-\frac{\pi}{2})\frac{|x|}{b}+C\frac{a}{b}-\frac{\pi}{2b}|x|} = e^{-C\frac{|x|}{b}+C\frac{a}{b}}\\ &= e^{-C\frac{|x|-a}{b}}.\end{aligned}$$

□

With these preparations we can prove an important tool needed in the proofs later on. It makes it possible to "spread" certain estimates to larger intervals. It is taken from [6, Theorem 2 (Spreading Lemma), p. 57].

Lemma 4.1.5. *Let h be the associated weight function of a regular DC class $C\{M_n\}$. Then there exists $A > 0$ (depending on the DC class) such that the following holds: Let $F \in C^1(\mathbb{C})$ with*

$$|F(z)| \leq \frac{1}{2|z+i|^2}, \tag{4.1.12}$$

$$|\overline{\partial}F(z)| \leq \frac{h(y)}{|z+i|^2}, \tag{4.1.13}$$

for $z \in \mathbb{H}$, and

$$|F(z)| \leq h(4y_0) \tag{4.1.14}$$

for $z \in [-a, a] + iy_0 =: I(a, y_0)$, where $0 < y_0 < A$ and $a > y_0$ are fixed. Then F satisfies

$$|F(z)| \leq h(8y_0) \tag{4.1.15}$$

for $z \in I(\tilde{a}, 2y_0)$ where $\tilde{a} := a + C_1 \int_{4y_0}^{8y_0} t|\phi'(t)|dt$ for some positive constant C_1 (independent of the DC class).

Remark 4.1.6. *In the assumptions (4.1.12) and (4.1.13) the term $\frac{1}{|z+i|^2}$ can be replaced with some function integrable on strips $\{z \in \mathbb{C} : 0 < \Im(z) < h\}$ and vanishing at infinity. As the theorem is taken from [6], we restrict to the formulation given there.*

Proof of lemma 4.1.5.
The constant A will be specified throughout the proof; conditions on the smallness of y_0 needed to justify certain arguments are subsummed in A. That means A will become smaller in every step and y_0 is always assumed to be smaller than the current A. To begin with, let $A := 1$.

As $\overline{\partial}F$ is absolutely integrable on $S_{y_0} := \{z \in \mathbb{C} : 0 < y < 3y_0\}$ by (4.1.13), it follows by lemma 1.0.13 that

$$F_1(z) := F(z) - \frac{1}{2\pi i}\int_{S_{y_0}} \frac{\overline{\partial}F(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}$$

is analytic on S_{y_0} and continuous up to the boundary.

As we want to apply the two constants theorem 2.5.2 to F_1 we also need to check that $F_1 \in C_0(S_{y_0})$. As $F \in C_0(S_{y_0})$ by (4.1.12) it suffices to show $F - F_1 \in C_0(S_{y_0})$: If $|z| \geq R > 1$, then

$$\begin{aligned}
&|F_1(z) - F(z)| \leq \frac{1}{\pi}\int_{S_{y_0}} \frac{|\overline{\partial}F(\zeta)|}{|\zeta - z|} d\xi \wedge d\eta \leq \frac{h(3y_0)}{\pi}\int_{S_{y_0}} \frac{1}{|\zeta + i|^2}\frac{1}{|\zeta - z|} d\xi \wedge d\eta \\
&= \frac{h(3y_0)}{\pi}\left(\int_{S_{y_0}\cap B(0,R/2)} \frac{1}{|\zeta + i|^2}\frac{1}{|\zeta - z|} d\xi \wedge d\eta + \int_{S_{y_0}\setminus B(0,R/2)} \frac{1}{|\zeta + i|^2}\frac{1}{|\zeta - z|} d\xi \wedge d\eta\right) \\
&\leq \frac{h(3y_0)}{\pi}\left(\frac{2}{R}\int_{S_{y_0}\cap B(0,R/2)} \frac{1}{|\zeta + i|^2} d\xi \wedge d\eta + \frac{4}{R^2}\int_{(S_{y_0}\setminus B(0,R/2))\cap B(z,1)} \frac{1}{|\zeta - z|} d\xi \wedge d\eta \right.\\
&\qquad\left. + \int_{S_{y_0}\setminus(B(0,R/2)\cup B(z,1))} \frac{1}{|\zeta + i|^2} d\xi \wedge d\eta\right) \\
&\leq \frac{h(3y_0)}{\pi}\left(\frac{2}{R}\int_{S_{y_0}} \frac{1}{|\zeta + i|^2} d\xi \wedge d\eta + \frac{8\pi}{R^2} + \underbrace{\int_{S_{y_0}\setminus B(0,R/2)} \frac{1}{|\xi|^2} d\xi \wedge d\eta}_{\leq \frac{K}{R}}\right) \\
&\leq \frac{\tilde{K}}{R}
\end{aligned}$$

for some constant $\tilde{K}$. Thus $F_1 - F \in C_0(S_{y_0})$ and therefore also F_1. In addition for arbitrary $z \in S_{y_0}$ we have the following estimate

$$\begin{aligned}
|F_1(z) - F(z)| &\leq \frac{1}{\pi}\int_{S_{y_0}} \frac{|\overline{\partial} F(\zeta)|}{|\zeta - z|} d\xi \wedge d\eta \\
&= \frac{1}{\pi}\int_{S_{y_0}\cap B(z,1/4)} \frac{|\overline{\partial} F(\zeta)|}{|\zeta - z|} d\xi \wedge d\eta + \frac{1}{\pi}\int_{S_{y_0}\setminus B(z,1/4)} \frac{|\overline{\partial} F(\zeta)|}{|\zeta - z|} d\xi \wedge d\eta \\
&\leq h(3y_0)\left(\frac{1}{2} + \frac{1}{\pi}\int_{S_{y_0}\setminus B(z,1/4)} \frac{1}{|\zeta + i|^2}\frac{1}{|\zeta - z|} d\xi \wedge d\eta\right) \\
&\leq h(3y_0)\left(\frac{1}{2} + \frac{4}{\pi}\int_{S_{y_0}} \underbrace{\frac{1}{|\xi + i|^2}}_{=\frac{1}{\xi^2+1}} d\xi \wedge d\eta\right) \\
&\leq h(3y_0)\left(\frac{1}{2} + \frac{12 y_0}{\pi}\underbrace{\int_{-\infty}^{\infty} \frac{1}{\xi^2 + 1} d\xi}_{=\pi}\right) \leq h(3y_0)\underbrace{\left(\frac{1}{2} + 12A\right)}_{=:C}.
\end{aligned}$$

where we used that $\frac{1}{|z+i|^2} \leq 1$ for $z \in S_{y_0}$ and lemma 1.0.12 to get the third line. For A smaller than $\frac{1}{24}$, we get $C \leq 1$. So assume that from now on.

Using these estimates together with (4.1.12) we get for $z \in S_{y_0}$

$$|F_1(z)| \leq |F(z)| + |F_1(z) - F(z)| \leq \frac{1}{2|z+i|^2} + h(3y_0)$$

and as $h(t)$ tends to 0 as t does, for sufficiently small t we get $h(3t) \leq \frac{1}{2}$. So suppose in addition to the above requirements on A, that it is chosen small enough to fulfil $h(3t) \leq \frac{1}{2}$ for $t < A$. Thus we have for $y_0 < A$ and $z \in S_{y_0}$

$$|F_1(z)| \leq 1.$$

Let $\tilde{S}_{y_0} := \{z \in \mathbb{C} : y_0 < \Im(z) < 3y_0\}$. As $\tilde{S}_{y_0} \subseteq S_{y_0}$, F_1 is analytic on $\tilde{S}_{y_0}$ and the above global estimates for $z \in S_{y_0}$ hold for $\tilde{S}_{y_0}$ as well. Thus we have for $z \in \tilde{S}_{y_0}$

$$|F_1(z)| \leq 1. \tag{4.1.16}$$

And by assumption (4.1.14) we get for $z \in I(a, y_0) \subseteq \partial\tilde{S}_{y_0}$

$$|F_1(z)| \leq |F(z)| + |F_1(z) - F(z)| \leq h(4y_0) + h(3y_0) \leq 2h(4y_0). \tag{4.1.17}$$

Therefore we may now apply the two constants theorem 2.5.2 to F_1 (with $\Omega = \tilde{S}_{y_0}$, $m = 2h(4y_0)$, $M = 1$ and $\phi(z) = \frac{e^{\pi\frac{z-y_0}{2y_0}} - i}{e^{\pi\frac{z-y_0}{2y_0}} + i}$ using the terminology from lemma 2.5.2; see section 2.5). To avoid confusion with $\phi = \log\log\frac{1}{h}$, we denote the biholomorphic map from lemma 2.5.2 by $\psi(z) := \frac{e^{\pi\frac{z-y_0}{2y_0}}-i}{e^{\pi\frac{z-y_0}{2y_0}}+i}$. An application of the two constants theorem now yields for $z \in \tilde{S}_{y_0}$

$$|F_1(z)| \leq (2h(4y_0))^{\omega(\psi(z),\psi(I(a,y_0)),\mathbb{D})} 1^{1-\omega(\psi(z),\psi(I(a,y_0)),\mathbb{D})}. \tag{4.1.18}$$

To justify the following considerations, assume A is small enough to get $h(8t) < \frac{1}{8}$ for $t < A$. Since $h(4t) \leq h(8t)$, this implies $2h(4t) \leq 1$ for $t \leq A$. Now for $y_0 < A$

$$\begin{aligned}
(2h(4y_0))^{2\exp(\phi(8y_0)-\phi(4y_0))} &= (2h(4y_0))^{2\frac{\log(h(8y_0))}{\log(h(4y_0))}} \\
&= 2^{2\frac{\log(h(8y_0))}{\log(h(4y_0))}} h(8y_0)^2 \\
&\leq 2^2 h(8y_0)^2 \\
&\leq \frac{1}{2}h(8y_0).
\end{aligned}$$

So if for some $z \in \tilde{S}_{y_0}$ we have $\omega(\psi(z),\psi(I(a,y_0)),\mathbb{D}) \geq 2\exp(\phi(8y_0)-\phi(4y_0))$, it follows by (4.1.18) that

$$|F_1(z)| \leq (2h(4y_0))^{\omega(\psi(z),\psi(I(a,y_0)),\mathbb{D})} \leq \frac{1}{2}h(8y_0).$$

And therefore

$$\begin{aligned}
|F(z)| &\leq |F_1(z)| + |F(z) - F_1(z)| \\
&\leq \frac{1}{2}h(8y_0) + h(3y_0).
\end{aligned}$$

Take A in addition sufficiently small to get $h(4t) \leq \frac{1}{2}h(8t)$ for $t \leq A$; see (3.2.9). Then it follows for $y_0 < A$

$$\frac{1}{2}h(8y_0) + h(3y_0) \leq \frac{1}{2}h(8y_0) + \frac{1}{2}h(8y_0) \leq h(8y_0).$$

Thus we have shown that for $z \in \tilde{S}_{y_0}$ with $\omega(\psi(z),\psi(I(a,y_0)),\mathbb{D}) \geq 2\exp(\phi(8y_0) - \phi(4y_0))$ and $y_0 < A$

$$|F(z)| \leq h(8y_0). \tag{4.1.19}$$

<u>Claim 1:</u> There exists an absolute constant $C_2 > 0$ such that for $z = x + i2y_0$, with $|x| \leq a + C_2 y_0(\phi(4y_0) - \phi(8y_0) - \log(2))$, we have

$$\omega(\psi(z),\psi(I(a,y_0)),\mathbb{D}) \geq 2\exp(\phi(8y_0) - \phi(4y_0))$$

and therefore

$$|F(z)| \leq h(8y_0).$$

Since $y_0 < a$ by assumption, we know by lemma 4.1.4 that $\omega(\psi(z), \psi(I(a, y_0)), \mathbb{D}) \geq \exp\left(-C\frac{|x|-a}{y_0}\right)$ for $z = x + i2y_0$ with $|x| \geq 2a$, where C is an absolute constant. Now observe

$$\exp\left(-C\frac{|x|-a}{y_0}\right) \geq 2\exp(\phi(8y_0) - \phi(4y_0))$$

is equivalent to

$$|x| \leq a + \underbrace{\frac{1}{C}}_{=:C_2} y_0(\phi(4y_0) - \phi(8y_0) - \log(2)).$$

This means that we can estimate the harmonic measure from below by $2\exp(\phi(8y_0) - \phi(4y_0))$ on $2a \leq |x| \leq a + C_2 y_0(\phi(4y_0) - \phi(8y_0) - \log(2))$. But since the harmonic measure can only be larger than this lower bound on $[-2a, 2a]$ by its monotonicity properties (see lemma 4.1.4), the estimate from below actually holds for $|x| \leq a + C_2 y_0(\phi(4y_0) - \phi(8y_0) - \log(2))$. This finishes the proof of claim 1.

Claim 2: There exists an absolute constant $C_1 > 0$ such that for sufficiently small y_0

$$C_1 \int_{4y_0}^{8y_0} t|\phi'(t)|dt \leq C_2 y_0(\phi(4y_0) - \phi(8y_0) - \log(2)). \tag{4.1.20}$$

Note that ϕ is decreasing and therefore $|\phi'| = -\phi'$. Thus the mean value theorem yields the existence of $\tau \in [4y_0, 8y_0]$ such that

$$\begin{aligned}\int_{4y_0}^{8y_0} t|\phi'(t)|dt &= \tau \int_{4y_0}^{8y_0} -\phi'(t)dt \\ &= \tau(\phi(4y_0) - \phi(8y_0)).\end{aligned}$$

As $\tau \leq 8y_0$ it follows

$$\int_{4y_0}^{8y_0} t|\phi'(t)|dt \leq 8y_0(\phi(4y_0) - \phi(8y_0)). \tag{4.1.21}$$

By (4.1.1) we can find $\delta > 0$ such that $t|\phi'(t)| \geq 4\log(2)$ for $0 < t < \delta$. Assume from now on that $A \leq \frac{\delta}{8}$. Then we get for $y_0 < A$

$$\int_{4y_0}^{8y_0} t|\phi'(t)|dt \geq 4\log(2)4y_0. \tag{4.1.22}$$

Therefore

$$\begin{aligned}\frac{C_2}{16}\int_{4y_0}^{8y_0} t|\phi'(t)|dt &= \frac{C_2}{8}\int_{4y_0}^{8y_0} t|\phi'(t)|dt - \frac{C_2}{16}\int_{4y_0}^{8y_0} t|\phi'(t)|dt \\ &\leq C_2 y_0(\phi(4y_0) - \phi(8y_0)) - C_2 y_0 \log(2).\end{aligned}$$

Thus let $C_1 := \frac{C_2}{16}$ (independent of the DC class, as $C_2 = \frac{1}{C}$ comes from lemma 4.1.4), finishing the proof of claim 2. Together with claim 1, this finishes the proof.

□

4.2 A qualitative result, proof of theorem 4.0.1 (iv)⇒(i)

The subsequent proof is taken from [6, Theorem 1(i), p. 55]. We assume (iv) from theorem 4.0.1 and, as mentioned before, that (4.1.1) is satisfied.

First let us show that it is possible to assume w.l.o.g. that $f \in C_{\text{gl}}\{M_n\}(\mathbb{R})$ with estimates of the form $|f^{(n)}(x)| \leq M_n$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$: By assumption $f \in C\{M_n\}(U)$ for some open $U \subseteq \mathbb{R}$. For a given f with $f^{(n)}(x_0) = 0$ for some $x_0 \in U$ and all $n \in \mathbb{N}$, we have to show that $f \equiv 0$ on the connected component of x_0 in U. So there is no loss of generality in assuming that U is an interval and $x_0 = 0$ (translate the argument of the given function f). For $[-\delta, \delta] \subseteq U$ we get $f \in C_{\text{gl}}\{M_n\}(-\delta, \delta)$. Define $g := \frac{2\delta}{\pi} \arctan$, then by lemma 4.1.3 and theorem 3.2.7 we get $f \circ g \in C_{\text{gl}}\{M_n\}(\mathbb{R})$ and a simple computation shows that $(f \circ g)^{(n)}(0) = 0$ for all n. So if the theorem is already proved for classes of the form $C_{\text{gl}}\{M_n\}(\mathbb{R})$, we may conclude $f \circ g \equiv 0$ which implies that f vanishes on $(-\delta, \delta)$. Thus it follows immediately that f vanishes on U.

For $f \in C_{\text{gl}}\{M_n\}(\mathbb{R})$ there are by definition positive constants $A, B > 0$ such that $|f^{(n)}(x)| \leq AB^n M_n$ for all n uniformly in x. Define $g(x) := \frac{1}{A} f(\frac{1}{B}x)$, then $|g^{(n)}(x)| \leq M_n$. So if we can conclude that g vanishes identically, the same holds for f.

So assume from now on

$$f^{(n)}(0) = 0, \tag{4.2.23}$$

$$|f^{(n)}(x)| \leq M_n \tag{4.2.24}$$

for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. It is our goal to show that $f \equiv 0$ on $\mathbb{R}$.

Due to theorem 3.2.2, there exists a bounded extension $F_1 \in C^1(\mathbb{C})$ of f and positive constants B_1, C_1 such that $|\overline{\partial} F_1(z)| \leq C_1 h(B_1|y|)$ for all $z \in \mathbb{C}$. Let b be defined on $\mathbb{R}$ such that $b(x) = 1$ for $|x| \leq \frac{1}{4}$ and $b(x) = 0$ for $|x| \geq \frac{1}{2}$ and set

$$F_2(z) = b(y) F_1(z)$$

for $z \in \mathbb{C}$. It is clear that F_2 is again an extension of our given f, $\text{supp}(F_2) \subseteq \{z \in \mathbb{C} : |y| \leq \frac{1}{2}\}$ and there are again positive constants B_2, C_2 such that $|\overline{\partial} F_2(z)| \leq C_2 h(B_2|y|)$; note that $F_1(z)$ and $F_2(z)$ coincide for $|\Im(z)| < \frac{1}{4}$. Finally we set

$$F(z) = \frac{F_2(z)}{(z+i)^2}$$

for all $z \in \mathbb{C}$ which extends $x \mapsto g(x) := \frac{f(x)}{(x+i)^2}$. A simple calculation shows that $g^{(n)}(0) = 0$ for all n and that we have

$$|F(z)| \leq \frac{D}{|z+i|^2}, \tag{4.2.25}$$

$$|\overline{\partial} F(z)| \leq \frac{C h(B|y|)}{|z+i|^2}, \tag{4.2.26}$$

where $B = B_2, C = C_2$ and D is the maximum of $|F_2|$. F is by construction C^1 on $\mathbb{C}$ and $\mathrm{supp}(F) \subseteq \{z \in \mathbb{C} : |y| \leq \frac{1}{2}\}$.

As $\int_{\partial B(0,r)} \frac{F(\zeta)}{\zeta - z} d\zeta \to 0$ for $r \to \infty$ and as $\overline{\partial} F$ is absolutely integrable on $\mathbb{C}$ because of (4.2.26) and $\mathrm{supp}(F) \subseteq \{z \in \mathbb{C} : |y| \leq \frac{1}{2}\}$, we get

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\overline{\partial} F(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}. \tag{4.2.27}$$

By differentiation under the integral sign in x-direction at 0, and (4.2.23), we get for all $n \geq 1$

$$\int_{\mathbb{C}} \frac{\overline{\partial} F(\zeta)}{\zeta^n} d\zeta \wedge d\overline{\zeta} = 0. \tag{4.2.28}$$

Observe that the integral in (4.2.28) is absolutely convergent as

$$\frac{|\overline{\partial} F(z)|}{|z|^n} \leq \frac{Ch(B|z|)}{|z|^n |z+i|^2} \leq \frac{CB^n m_n}{|z+i|^2} \tag{4.2.29}$$

where the first inequality holds due to (4.2.26) and the second due to (3.2.8). By multiplying the equation (4.2.28) with z^{n-1} for $n \geq 1$, we immediately get

$$\int_{\mathbb{C}} \frac{\overline{\partial} F(\zeta) z^{n-1}}{\zeta^n} d\zeta \wedge d\overline{\zeta} = 0. \tag{4.2.30}$$

Suppose we already know the following representation for some $n \in \mathbb{N}$

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \overline{\partial} F(\zeta) \frac{z^n}{(\zeta - z)\zeta^n} d\zeta \wedge d\overline{\zeta}. \tag{4.2.31}$$

Then subtracting (4.2.30) from this representation leads to

$$\begin{aligned} F(z) &= \frac{1}{2\pi i} \int_{\mathbb{C}} \overline{\partial} F(\zeta) \left(\frac{z^n}{(\zeta - z)\zeta^n} - \frac{z^n}{\zeta^{n+1}} \right) d\zeta \wedge d\overline{\zeta} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \overline{\partial} F(\zeta) \left(\frac{z^{n+1}}{(\zeta - z)\zeta^{n+1}} \right) d\zeta \wedge d\overline{\zeta}. \end{aligned}$$

Together with (4.2.27) which is just (4.2.31) for $n = 0$ we now can conclude by induction that the representation (4.2.31) holds for all $n \in \mathbb{N}$. We thus have for arbitrary n

$$\begin{aligned} |F(z)| &\leq \frac{1}{\pi} \int_{\mathbb{C}} |\overline{\partial} F(\zeta)| \frac{|z|^n}{|\zeta - z||\zeta|^n} d\xi \wedge d\eta \\ &\leq \frac{|z|^n}{\pi} C B^n m_n \underbrace{\int_{\{\zeta : |\Im(\zeta)| \leq 1/2\}} \frac{1}{|\zeta - z||\zeta + i|^2} d\xi \wedge d\eta}_{\leq \tilde{C} \text{ for all } z \in \mathbb{C}} \\ &\leq \frac{C\tilde{C}}{\pi} (B|z|)^n m_n \end{aligned}$$

where we applied (4.2.29) to derive the second line. As the above holds for all n, we get $|F(z)| \leq \gamma h(\beta|z|)$ with $\beta := B$ and $\gamma := \frac{C\tilde{C}}{\pi}$. Now let $E \geq \max\{B, C, 2D, \beta, \gamma, 2\}$. Then a straightforward computation shows that the function $\tilde{F}(z) := \frac{1}{E}F(z/E)$ has the following properties

$$|\tilde{F}(z)| \leq \frac{1}{2}\frac{1}{|z/E+i|^2}, \tag{4.2.32}$$

$$|\tilde{F}(z)| \leq h(|z|), \tag{4.2.33}$$

$$|\overline{\partial}\tilde{F}(z)| \leq \frac{h(|y|)}{|z/E+i|^2}, \tag{4.2.34}$$

and $\tilde{F}$ extends the function $\frac{1}{E}\frac{f(x/E)}{(x/E+i)^2}$. If we can show that $\tilde{F}|_{\mathbb{R}} \equiv 0$, this implies $f \equiv 0$ on $\mathbb{R}$.

As the proof of the spreading lemma 4.1.5 also works with conditions (4.2.32) and (4.2.34) instead of (4.1.12) and (4.1.13) (cf. remark 4.1.6), we may apply lemma 4.1.5 to $\tilde{F}$.

Claim: Let A, C_1 be defined by lemma 4.1.5. Let $B > 0$ be chosen such that $t|\phi'(t)| \geq \frac{1}{C_1}$ for $0 < t < B$ (possible due to (4.1.1)) and set $\tilde{A} := \min\{A, B\}$. Suppose that for $z_0 = x_0 + iy_0$ with $0 < y_0 < \tilde{A}$, there exists $n \in \mathbb{N}$ with $C_1\int_{4y_0/2^n}^{4y_0} t|\phi'(t)|dt > x_0$. Then we have

$$|\tilde{F}(z_0)| \leq h(4y_0).$$

The claim is proved by an iterated application of lemma 4.1.5: We set $a_n := 2\frac{y_0}{2^n}$. Due to (4.2.33), we have $|\tilde{F}(x+i\frac{y_0}{2^n})| \leq h(\frac{4y_0}{2^n})$ for $|x| \leq a_n$. Since $a_n \geq \frac{y_0}{2^n}$, lemma 4.1.5 implies for $z = x + i\frac{y_0}{2^{n-1}}$ with $|x| \leq a_n + C_1\int_{4\frac{y_0}{2^n}}^{4\frac{y_0}{2^{n-1}}} t|\phi'(t)|dt =: a_{n-1}$ that

$$|\tilde{F}(z)| \leq h(4\frac{y_0}{2^{n-1}}).$$

We observe that $a_{n-1} \geq a_n = \frac{y_0}{2^{n-1}}$. Suppose, we have already shown for some k with $0 < k < n$ that $|\tilde{F}(x+i\frac{y_0}{2^{n-k}})| \leq h(4\frac{y_0}{2^{n-k}})$ for $|x| \leq a_{n-k}$ with $a_{n-k} \geq \frac{y_0}{2^{n-k}}$. Then we can apply lemma 4.1.5 and get for $z = x+i\frac{y_0}{2^{n-k-1}}$ with $|x| \leq a_{n-k}+C_1\int_{4\frac{y_0}{2^{n-k}}}^{4\frac{y_0}{2^{n-k-1}}} t|\phi'(t)|dt =: a_{n-k-1}$ that

$$|\tilde{F}(z)| \leq h(4\frac{y_0}{2^{n-k-1}}).$$

By the definition of B, we have $a_{n-k-1} \geq 4\frac{y_0}{2^{n-k-1}} - 4\frac{y_0}{2^{n-k}} \geq \frac{y_0}{2^{n-k-1}}$. Thus it follows inductively that we can apply lemma 4.1.5 n times. Since $a_0 = 2\frac{y_0}{2^n} + C_1\int_{4\frac{y_0}{2^n}}^{4y_0} t|\phi'(t)|dt$, this finishes the proof of the claim.

Now take $z_0 = x_0+iy_0$ fixed with small imaginary part as in the claim. As by (iv) and lemma 4.1.2 it follows $\int_0^1 t|\phi'(t)| = \infty$, there exists $n \in \mathbb{N}$ such that $C_1\int_{4y_0/2^n}^{4y_0} t|\phi'(t)| > x_0$. And thus $\tilde{F}(x_0 + iy_0) \leq h(4y_0)$.

Since $h(y)$ tends to 0 as $y \to 0$, we therefore have $\tilde{F}(x) = 0$ for all $x \in \mathbb{R}$. As mentioned above this implies that f has to vanish identically on $\mathbb{R}$.

□

4.3 A quantitative result, proof of theorem 4.0.1 (i)⇒(iv)

As always $C\{M_n\}$ is a regular DC class and h is its associated weight function. We will show ¬(iv)⇒ ¬(i) by proving a stronger implication, namely theorem 4.3.1. Let us introduce two functions:

$$P(x) := \sup\left\{|f(x)| : f^{(n)}(0) = 0, ||f^{(n)}||_\infty \leq M_n \ \forall n \in \mathbb{N}\right\}, \quad x \geq 0. \tag{4.3.35}$$

If we assume ¬(iv), it is possible to define $\theta(x)$ for $0 \leq x \leq \int_0^1 t|\phi'(t)|dt$ by requiring

$$\int_0^{\theta(x)} t|\phi'(t)|dt = x. \tag{4.3.36}$$

Observe that θ is clearly an increasing continuous function with $\theta(0) = 0$. Applying assumption (4.1.1), we get:

For all $\varepsilon > 0$ there exists $T > 0$ such that for $0 < x, x_0 < T$

$$|\theta(x) - \theta(x_0)| \leq \varepsilon|x - x_0|. \tag{4.3.37}$$

To see (4.3.37) we investigate the difference quotient of θ. Let x_0 be fixed and $x > x_0$, then

$$\begin{aligned}\frac{\theta(x) - \theta(x_0)}{x - x_0} &= \frac{\theta(x) - \theta(x_0)}{\int_{\theta(x_0)}^{\theta(x)} t|\phi'(t)|dt} \\ &= \frac{\theta(x) - \theta(x_0)}{\theta(\xi(x)) \int_{\theta(x_0)}^{\theta(x)} |\phi'(t)|dt} \\ &= \frac{\theta(x) - \theta(x_0)}{\theta(\xi(x))(\phi(\theta(x_0)) - \phi(\theta(x)))}\end{aligned}$$

where the existence of some suitable $\xi(x) \in (x_0, x)$ is ensured by the mean value theorem. By taking the limit for $x \to x_0$ in the above chain of equalities, we get for points where ϕ is differentiable (i.e. $x_0 \neq \frac{m_n}{m_{n+1}}$ for any n, see section 3.2)

$$\theta'(x_0) = -\frac{1}{\theta(x_0)\phi'(\theta(x_0))} = \frac{1}{\theta(x_0)|\phi'(\theta(x_0))|}. \tag{4.3.38}$$

By (4.1.1), the right-hand side tends to 0 as $x_0 \to 0$, thus an application of the mean value theorem yields (4.3.37). The following theorem is taken from [6, Theorem 1(ii), p. 56].

Theorem 4.3.1. *Assume ¬(iv). Then there exist* $0 < Q_1, Q_2$ *such that for small* x

$$P(x) \leq h(Q_1\theta(Q_2 x)). \tag{4.3.39}$$

If, in addition, there exists $k > 0$ *such that for* x *sufficiently small*

$$2\phi(\theta(x)) \leq \phi(k\theta(x)), \tag{4.3.40}$$

then there exists $0 < q$ *such that for small* x

$$h(q\theta(x)) \leq P(x). \tag{4.3.41}$$

Remark 4.3.2. (4.3.41) *implies non-quasianalyticity of the respective DC class and thus* $\neg$*(i). The example from remark 4.1.1 also provides an example for a DC class which satisfies the additional assumption* (4.3.40)*.*

Before proving the upper estimate of theorem 4.3.1 we need an auxiliary result. The following lemma and its proof is inspired by [6, 1.4 Lemma, p. 52] (where it is proved for different function classes).

Lemma 4.3.3. *Let* h *be the weight function of a regular DC class and* $\varepsilon, B, C > 0$ *and* τ *an integrable function on* $\mathbb{C}$*. Then we have the following:*

Let $F \in C^1(\mathbb{C}) \cap C_0(\mathbb{C})$ *such that* $\overline{\partial} F$ *is absolutely integrable on* $\mathbb{C}$ *with*

$$|\overline{\partial} F(z)| \leq Ch(B|y|)\tau(z) \text{ for all } z = x + iy \in \mathbb{C}, \tag{4.3.42}$$

and for some $z_0 = x_0 + iy_0 \in \mathbb{C}$

$$|F(z)| \leq Ch(B|y_0|) \text{ for } z \in B(z_0, \varepsilon|y_0|) =: B_0. \tag{4.3.43}$$

Then for all $A > 0$ *there exists* K *(independent of* z_0 *and* F*) such that for* $z \in B(z_0, A|y_0|)$

$$|F(z)| \leq Kh(K|y_0|). \tag{4.3.44}$$

Proof. For $z \in \mathbb{C}$, set

$$F_1(z) := F(z) - \frac{1}{2\pi i} \int_{B_0} \frac{\overline{\partial} F(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}.$$

Due to lemma 1.0.13, F_1 is analytic on B_0. It is clear by the definition of F_1 that $|F_1(z) - F(z)| \leq Kh(K|y_0|)$ everywhere for some constant K independent of F and z_0. So it suffices to prove for given $A > 0$ that $|F_1(z)| \leq Kh(K|y_0|)$ for $z \in B(z_0, A|y_0|)$, where K does not depend on F and z_0. Since $F \in C_0(\mathbb{C})$ it follows, by applying theorem 1.0.11 to F, that

$$F_1(z) = \frac{1}{2\pi i} \int_{\mathbb{C} \setminus B_0} \frac{\overline{\partial} F(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}. \tag{4.3.45}$$

Using that $F_1(z) = \frac{1}{2\pi i} \int_{\partial B_0} \frac{F(\zeta)}{\zeta - z} d\zeta$ for $z \in B_0$, implies for arbitrary $k \in \mathbb{N}$

$$F_1^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\partial B_0} \frac{F(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta.$$

Using (4.3.43) therefore implies

$$|F_1^{(k)}(z_0)| \leq Ch(B|y_0|) \frac{k!}{(\varepsilon|y_0|)^k}. \tag{4.3.46}$$

Differentiating (4.3.45) gives another representation of $F_1^{(k)}(z_0)$

$$F_1^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\mathbb{C} \setminus B_0} \frac{\overline{\partial} F(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta \wedge d\overline{\zeta}. \tag{4.3.47}$$

A simple calculation shows for arbitrary n

$$\left(\sum_{k=0}^{n-1}\frac{(z-z_0)^k}{(\zeta-z_0)^{k+1}}\right)+\frac{(z-z_0)^n}{(\zeta-z_0)^n(\zeta-z)}=\frac{1}{\zeta-z}. \tag{4.3.48}$$

Using (4.3.47) and (4.3.48), we get by (4.3.45) the following representation for F_1 for arbitrary $n\in\mathbb{N}$

$$F_1(z)=\sum_{k=0}^{n-1}F_1^{(k)}(z_0)\frac{(z-z_0)^k}{k!}+\frac{1}{2\pi i}\int_{\mathbb{C}\setminus B_0}\frac{\overline{\partial}F(\zeta)(z-z_0)^n}{(\zeta-z_0)^n(\zeta-z)}d\zeta\wedge d\overline{\zeta}. \tag{4.3.49}$$

Take $\tilde{C}$ (depending only on ε) such that $\frac{1}{|\zeta-z_0|}\leq\tilde{C}\frac{1}{|\eta|}$ for all $\zeta\in\mathbb{C}\setminus B_0$: If $\zeta=\xi+i\eta$ is chosen with $|\eta|\leq(1+\varepsilon)|y_0|$ we have by the definition of B_0

$$\frac{|\eta|}{|\zeta-z_0|}\leq\frac{(1+\varepsilon)|y_0|}{\varepsilon|y_0|}=\frac{(1+\varepsilon)}{\varepsilon}.$$

If $|\eta|>(1+\varepsilon)|y_0|$ we have $|\eta|=t|y_0|$ for some $t>(1+\varepsilon)$. Thus

$$\frac{|\eta|}{|\zeta-z_0|}\leq\frac{|\eta|}{|\eta-y_0|}\leq\frac{|\eta|}{||\eta|-|y_0||}=\frac{t|y_0|}{(t-1)|y_0|}=\frac{t}{t-1}$$

and $\frac{t}{t-1}\leq\frac{1+\varepsilon}{\varepsilon}$ for $t>(1+\varepsilon)$. Thus $\tilde{C}=\frac{1+\varepsilon}{\varepsilon}$ is a suitable choice.

We investigate the summands of (4.3.49) separately. First take some $A>0$, assume w.l.o.g. $A\geq 2\varepsilon$. By (4.3.46) it follows for $z\in B(z_0,A|y_0|)$

$$\begin{aligned}\left|\sum_{k=0}^{n-1}F_1^{(k)}(z_0)\frac{(z-z_0)^k}{k!}\right| &\leq Ch(B|y_0|)\sum_{k=0}^{n-1}\frac{(A|y_0|)^k}{(\varepsilon|y_0|)^k}\\ &=Ch(B|y_0|)\sum_{k=0}^{n-1}\left(\frac{A}{\varepsilon}\right)^k\\ &\leq C(B|y_0|)^n m_n\left(\frac{A}{\varepsilon}\right)^n\\ &=C\left(\frac{AB|y_0|}{\varepsilon}\right)^n m_n.\end{aligned}$$

In addition we get for the other summand

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\mathbb{C}\setminus B_0}\frac{\overline{\partial}F(\zeta)(z-z_0)^n}{(\zeta-z_0)^n(\zeta-z)}d\zeta\wedge d\overline{\zeta}\right| &\leq\frac{(A|y_0|)^n}{\pi}\int_{\mathbb{C}\setminus B_0}\frac{\tilde{C}^nCh(B|\eta|)\tau(\zeta)}{|\eta|^n|\zeta-z|}d\xi\wedge d\eta\\ &\leq\frac{C(AB\tilde{C}|y_0|)^n m_n}{\pi}\underbrace{\int_{\mathbb{C}\setminus B_0}\frac{\tau(\zeta)}{|\zeta-z|}d\xi\wedge d\eta}_{=D<\infty}\end{aligned}$$

where D only depends on ε. Now take $K := 2\max\{AB\tilde{C}, \frac{AB}{\varepsilon}, C, \frac{CD}{\pi}\}$ and observe that K does not depend on z_0 and F. Then we get for all n and $z \in B(z_0, A|y_0|)$

$$|F_1(z)| \leq K(K|y_0|)^n m_n.$$

By taking $n = N(K|y_0|)$ we thus get the desired result. □

Proof of $P(x) \leq h(Q_1\theta(Q_2 x))$.
Take some f as in the definition of P, i.e. $f^{(n)}(0) = 0$ and $|f^{(n)}(x)| \leq M_n$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ (this means $||f||_{\mathbb{R},1} \leq 1$). As in section 4.2, we define an extension $\tilde{F}$ of $\frac{1}{E}\frac{f(x/E)}{(x/E+i)^2}$ with properties (4.2.32), (4.2.33) and (4.2.34). Observe that the constants in the proof of theorem 3.2.2 only depend on $||f||_{\mathbb{R},1}$; in fact they are increasing in terms of $||f||_{\mathbb{R},1}$. Thus the same constant E can be chosen for all extensions of functions with $||f||_{\mathbb{R},1} \leq 1$.

Applying the claim from the proof of (iv)⇒(i) (section 4.2), we get for $z = x+iy$ with $y \geq \frac{\theta(x/C_1)}{4}$ that $|\tilde{F}(z)| \leq h(4y)$. Set $z_0 := x_0 + i\theta(x_0/C_1)$ and let $B_0 := B(z_0, \frac{\theta(x_0/C_1)}{2})$. Observe that $B_0 \subseteq G := \left\{z = x + iy : y \geq \frac{\theta(x/C_1)}{4}\right\}$ for sufficiently small x_0: Take $z = x + iy \in B_0$, we have by the definition of B_0 that $y \geq \frac{\theta(x_0/C_1)}{2}$. If we can show for $|x - x_0| \leq \frac{\theta(x_0/C_1)}{2}$

$$\frac{\theta(x/C_1)}{4} \leq \frac{\theta(x_0/C_1)}{2} \tag{4.3.50}$$

this implies $B_0 \subseteq G$. By (4.3.37), we have for sufficiently small x, x_0 (with $\varepsilon := C_1$)

$$|\theta(x/C_1) - \theta(x_0/C_1)| \leq |x - x_0|.$$

Thus we have for $|x - x_0| \leq \frac{\theta(x_0/C_1)}{2}$

$$\frac{1}{4}\theta(x/C_1) \leq \frac{1}{4}\left(\theta(x_0/C_1) + \frac{\theta(x_0/C_1)}{2}\right) \leq \frac{\theta(x_0/C_1)}{2},$$

and this is exactly (4.3.50).

Thus we may conclude $|\tilde{F}(z)| \leq h(4y)$ for $z \in B_0$ and by the definition of B_0 therefore in addition $|\tilde{F}(z)| \leq h(6\theta(x_0/C_1))$ for $z \in B_0$.

Observe that $\tilde{F}$ has all the necessary properties to apply lemma 4.3.3 with $\varepsilon = \frac{1}{2}$. Now take $A = 3$ (as in lemma 4.3.3) and we get a constant $\tilde{Q}$ independent of x_0 such that

$$|\tilde{F}(z)| \leq h(\tilde{Q}\theta(x_0/C_1))$$

for $z \in B(x_0 + i\theta(x_0/C_1), \frac{3}{2}\theta(x_0/C_1))$ and as $x_0 \in B(x_0 + i\theta(x_0/C_1), \frac{3}{2}\theta(x_0/C_1))$ we get especially

$$\left|\frac{1}{E}\frac{f(x_0/E)}{(x_0/E + i)^2}\right| = |\tilde{F}(x_0)| \leq h(\tilde{Q}\theta(x_0/C_1)). \tag{4.3.51}$$

As $E|x_0/E + i|^2 \leq 2E$ for sufficiently small x_0 and as $2Eh(\tilde{Q}\theta(x_0)) \leq h(2E\tilde{Q}\theta(x_0/C_1))$ for small x_0, we may set $Q_1 := 2E\tilde{Q}$ and $Q_2 := \frac{E}{C_1}$ and the proof is finished. □

For the lower estimate (4.3.41), we need a result on biholomorphic functions defined on symmetric domains.

Lemma 4.3.4. *Let $\Omega_j \subseteq \mathbb{C}$ for $j = 1, 2$ be two simply connected domains symmetric with respect to the real line; i.e. $z \in \Omega_j \Rightarrow \overline{z} \in \Omega_j$. If $h : \Omega_1 \to \Omega_2$ is biholomorphic, then also*

$$g(z) := \overline{h(\overline{z})}$$

is biholomorphic between Ω_1 and Ω_2.

If $\Omega_j \cap \mathbb{R} \neq \emptyset$ for $j = 1, 2$ and $h(x_0), h'(x_0) \in \mathbb{R}$ for some $x_0 \in \Omega_1 \cap \mathbb{R}$, then $h|_{\Omega_1 \cap \mathbb{R}}$ is a homeomorhpism of $\Omega_1 \cap \mathbb{R}$ and $\Omega_2 \cap \mathbb{R}$.

Proof. Since $z \mapsto \overline{z}$ is bijective on Ω_j for $j = 1, 2$ due to symmetry, we get that g is bijective between Ω_1 and Ω_2 as it is the composition of bijective functions. Analyticity follows since

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} \overline{\left(\frac{h(\overline{z}) - h(\overline{z_0})}{\overline{z} - \overline{z_0}} \right)} = \overline{h'(\overline{z_0})}.$$

If $h(x_0), h'(x_0) \in \mathbb{R}$ for some $x_0 \in \Omega_1 \cap \mathbb{R}$, we have by the above considerations $h(x_0) = g(x_0)$ and $h'(x_0) = g'(x_0) \in \mathbb{R}$.

Now take $\phi_1 : \mathbb{D} \to \Omega_1$ biholomorphic with $\phi_1(0) = x_0$ and $\phi_1'(0) > 0$ and $\phi_2 : \Omega_2 \to \mathbb{D}$ biholomorphic with $\phi_2(h(x_0)) = 0$ and $\phi_2'(h(x_0)) > 0$. Observe that $\phi_2 \circ h \circ \phi_1(0) = 0 = \phi_2 \circ g \circ \phi_1(0)$ and $(\phi_2 \circ h \circ \phi_1)'(0) = (\phi_2 \circ g \circ \phi_1)'(0) \in \mathbb{R}$. Thus an application of the uniqueness result in the Riemann mapping theorem 1.0.15 yields that $\phi_2 \circ h \circ \phi_1 = \phi_2 \circ g \circ \phi_1$ and therefore $h = g$.

In other words $h(z) = \overline{h(\overline{z})}$ and thus $h(x) \in \Omega_2 \cap \mathbb{R}$ for $x \in \Omega_1 \cap \mathbb{R}$. If for some $z = x + iy$ with $y \neq 0$ we would have $h(z) \in \mathbb{R}$, this would imply that $h(z) = h(\overline{z})$ contradicting injectivity of h. Thus $h|_{\Omega_1 \cap \mathbb{R}} : \Omega_1 \cap \mathbb{R} \to \Omega_2 \cap \mathbb{R}$ is bijective and as restriction of a biholomorphic function also a homeomorphism. □

In addition we need a result dealing with the asymptotic behaviour of biholomorphic functions mapping certain domains, so-called $L-$strips, to strips of fixed height. The definitions and results are taken from [14].

A domain $\Omega \subseteq \mathbb{C}$ is called *$L-$strip* if there are differentiable functions $\phi_+, \phi_- : (x_0, \infty) \to \mathbb{R}$ with $\phi_+ > \phi_-$ and a Jordan arc $\gamma \subseteq \{z : \Re(z) \leq x_0\}$ such that $\partial\Omega = \phi_+((x_0, \infty)) \cup \phi_-((x_0, \infty)) \cup \gamma$ and $\{z = x + iy \in \mathbb{C} : x_0 \leq x < \infty, \phi_-(x) < y < \phi_+(x)\} \subseteq \Omega$; in addition it is assumed that

$$\lim_{t \to \infty} \phi_+'(t) = \lim_{t \to \infty} \phi_-'(t) = \tan(\gamma)$$

for some $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Such a domain is called L-strip with boundary inclination γ at infinity.

Theorem 4.3.5. *Let Ω be an $L-$strip with boundary inclination $\gamma = 0$ at infinity and $H : \Omega \to \{z \in \mathbb{C} : |\Im(z)| < \frac{\pi}{2}\}$ a biholomorphic function with $H(x+iy) \to \infty$ as $x \to \infty$ uniformly in y. Then for $z = x + iy$*

$$\Re(H(z)) \backsimeq \pi \int_{x_0}^{x} \frac{1}{\phi_+(t) - \phi_-(t)} dt$$

as $x \to \infty$ uniformly in y.

Proof. See [14, Theorem X, p. 315]. □

We write $f(t) \backsimeq g(t)$ as $t \to c$ if $\frac{f(t)}{g(t)} \to 1$ as $t \to c$.

In the proof of the lower estimate we need theorem 4.3.5 in a slightly different setting.

Corollary 4.3.6. *Let $\Omega \subseteq \mathbb{C}$ be a domain such that*

$$\Omega = \{z = x + iy \in \mathbb{C} : 0 < x < x_0, |y| < \tau(x)\}$$

where τ is an increasing continuous function on $[0, x_0)$, differentiable on $(0, x_0)$ with $\tau(0) = 0$ and

$$\lim_{x \to 0} \tau'(x) = 0.$$

Let $G : \Omega \to \{z = x + iy \in \mathbb{C} : |y| < \frac{\pi}{2}\}$ be a biholomorphic function with

$$\lim_{x \to 0} G(x + iy) = +\infty$$

uniformly in y. Then

$$\Re(G(x + iy)) \backsimeq \frac{\pi}{2} \int_{x}^{x_0} \frac{1}{\tau(t)} dt.$$

as $x \to 0$ uniformly in y.

Proof. To apply theorem 4.3.5 we need to transform the given domain Ω to an $L-$strip. Define

$$\tilde{\Omega} := \left\{ h(z) := \log\left(\frac{1}{z}\right) : z \in \Omega \right\}.$$

First, we observe that $\tilde{\Omega}$ is indeed an $L-$strip: Let $r(t) := \sqrt{t^2 + \tau(t)^2}$ and $\rho(t) := \arctan\left(\frac{\tau(t)}{t}\right)$. It is clear from the definition of r that it is strictly increasing and $r(t) > t$. Thus we get the existence of $r^{-1} : (0, r(x_0)) \to (0, x_0)$ and (as $r(x_0) > x_0$) r^{-1} is defined on $(0, x_0)$. In addition

$$r'(t) \to 1 \text{ as } t \to 0 \tag{4.3.52}$$

which follows since $\tau(0) = 0$ and $\tau'(t) \to 0$ as $t \to 0$. Moreover,

$$t\rho'(t) \to 0 \text{ as } t \to 0 \tag{4.3.53}$$

which holds as $t\rho'(t) = \left(\frac{1}{1+\frac{\tau(t)^2}{t^2}}\right) \frac{\tau'(t)t-\tau(t)}{t}$ and the first factor converges to 1 and the second factor to 0 (since τ' tends to 0!).

Now we describe a subdomain of Ω in polar coordinates as follows

$$\begin{aligned}\Omega &\supseteq \left\{z = r(t)e^{i\alpha} \in \mathbb{C} : 0 < t < r^{-1}(x_0), |\alpha| < \rho(t)\right\} \\ &= \left\{z = te^{i\alpha} \in \mathbb{C} : 0 < t < x_0, |\alpha| < \rho(r^{-1}(t))\right\} =: \Omega'.\end{aligned}$$

Therefore

$$\begin{aligned}h(\Omega') &= \left\{\log\left(\frac{1}{t}\right) - i\alpha : 0 < t < x_0, |\alpha| < \rho(r^{-1}(t))\right\} \\ &= \left\{z = x + iy : x > \log\left(\frac{1}{x_0}\right), |y| < \underbrace{\left(\rho \circ r^{-1} \circ \exp\right)(-x)}_{=:\beta(x)}\right\}.\end{aligned}$$

To see that $\tilde{\Omega}$ is an $L-$strip, we have to differentiate the boundary curve:

$$\beta'(x) = -\rho'(r^{-1}(e^{-x}))\left(r^{-1}\right)'(e^{-x})e^{-x}.$$

Since $r'(t) \to 1$ as $t \to 0$ due to (4.3.52), the same holds for r^{-1} and thus

$$\beta'(x) \backsimeq -\rho'(r^{-1}(e^{-x}))e^{-x} = -\rho'(r^{-1}(e^{-x}))r^{-1}(e^{-x})\frac{e^{-x}}{r^{-1}(e^{-x})}$$

as $x \to \infty$. Now we use (4.3.53) which implies that $\beta'(x)$ tends to 0 at infinity. Therefore $\tilde{\Omega}$ is an $L-$strip with boundary inclination $\gamma = 0$.

Before proving the desired result, we need an auxiliary result: Given two positive functions f, g defined on some intervals $(0, t_0)$ and $(0, t_1)$ respectively, both converging to ∞ near 0 with divergent integral near 0 and $f \backsimeq g$ at 0, then

$$\int_t^{t_0} f(s)ds \backsimeq \int_t^{t_1} g(s)ds \tag{4.3.54}$$

as $t \to 0$.

This can be seen as follows: Let $\varepsilon > 0$ be fixed. Then there is $t_2 < t_0, t_1$ such that $f(s) \leq (1+\varepsilon)g(s)$ for $s < t_2$. Therefore we get for $t < t_2$

$$\begin{aligned}\frac{\int_t^{t_0} f(s)ds}{\int_t^{t_1} g(s)ds} &\leq \frac{(1+\varepsilon)\int_t^{t_2} g(s)ds + \int_{t_2}^{t_0} f(s)ds}{\int_t^{t_2} g(s)ds + \int_{t_2}^{t_1} g(s)ds} \\ &= (1+\varepsilon)\frac{\int_t^{t_2} g(s)ds}{\int_t^{t_2} g(s)ds + \int_{t_2}^{t_1} g(s)ds} + \frac{\int_{t_2}^{t_0} f(s)ds}{\int_t^{t_2} g(s)ds + \int_{t_2}^{t_1} g(s)ds}.\end{aligned}$$

Since $t \mapsto \int_t^{t_2} g(s)ds$ tends to ∞ as $t \to 0$, it is clear that the first summand tends to $(1+\varepsilon)$ and the second summand to 0 as $t \to 0$. The other estimate follows analogously.

Set

$$\begin{aligned}\tilde{G} : \tilde{\Omega} &\to \left\{z = x + iy : |y| < \frac{\pi}{2}\right\}\\ z &\mapsto G(e^{-z})\end{aligned}$$

which is biholomorphic on $\tilde{\Omega} \cap \{z \in \mathbb{C} : |\Im(z)| < \pi\}$. Since $\tilde{G}(z) \to \infty$ as $z \to \infty$, we can apply theorem 4.3.5 and get for $z = te^{i\alpha} \in \Omega'$

$$\begin{aligned}\Re(G(z)) &= \Re\left(\tilde{G}\left(\log\left(\frac{1}{t}\right) - i\alpha\right)\right)\\ &\backsimeq \pi \int_{\log\left(\frac{1}{x_0}\right)}^{\log\left(\frac{1}{t}\right)} \frac{1}{2\rho \circ r^{-1}(e^{-s})} ds\\ &= -\frac{\pi}{2} \int_{x_0}^{t} \frac{1}{(\rho \circ r^{-1}(s))s} ds\\ &= \frac{\pi}{2} \int_{t}^{x_0} \frac{1}{(\rho \circ r^{-1}(s))s} ds\\ &\backsimeq \frac{\pi}{2} \int_{t}^{x_0} \frac{1}{\tau(s)} ds.\end{aligned}$$

The last line follows, since $\rho(r^{-1}(s)) = \arctan\left(\frac{\tau(r^{-1}(s))}{r^{-1}(s)}\right) \backsimeq \frac{\tau(r^{-1}(s))}{r^{-1}(s)}$ as $s \to 0$ (since $\arctan'(0) = 1$), and as $\tau(r^{-1}(s)) \backsimeq \tau(s)$ as $s \to 0$ (due to (4.3.55)).

Now observe that for $z = x + iy \in \Omega$ we have $z = te^{i\alpha}$ for some $t \in [x, r(x))$. And since for given $\varepsilon > 0$ we get for sufficiently small s

$$\begin{aligned}1 \leq \frac{\tau(r(s))}{\tau(s)} &\leq \frac{\tau(s + \tau(s))}{\tau(s)}\\ &\leq \frac{\tau(s) + \varepsilon\tau(s)}{\tau(s)} = 1 + \varepsilon.\end{aligned} \tag{4.3.55}$$

Combining this with $r'(t) \to 1$ as $t \to 0$, we get $\tau(r(s))r'(s) \backsimeq \tau(s)$ and therefore

$$\int_{t}^{x_0} \frac{1}{\tau(s)} ds \backsimeq \int_{r(t)}^{x_0} \frac{1}{\tau(s)} ds$$

as $t \to 0$. Finally we thus get

$$\Re(G(x + iy)) \backsimeq \frac{\pi}{2} \int_{x}^{x_0} \frac{1}{\tau(s)} ds$$

as $x \to 0$ uniformly in y. □

Now, we have all the necessary tools to prove the lower estimate from theorem 4.3.1.

Proof of $h(q\theta(x)) \leq P(x)$.
Let $\varepsilon < \frac{10}{132}$. By (4.3.37) we can find a $T_0 > 0$ such that $|\theta(x) - \theta(y)| \leq \varepsilon|x-y|$ for $0 < x, y < T_0$. Define

$$\Omega := \left\{ z = x + iy : 0 < x < T_0, |y| < \frac{5}{4}\theta(x) \right\},$$
$$S := \left\{ z = x + iy : |y| < \frac{\pi}{2} \right\}.$$

Both domains are simply connected and symmetric with respect to the real axis. By the Riemann mapping theorem 1.0.15, we get the existence of a biholomorphic function $\tilde{G} : \Omega \to S$ with $\tilde{G}(\frac{T_0}{2}) = 0$ and $\tilde{G}'(\frac{T_0}{2}) < 0$. An application of lemma 4.3.4 yields that

$$\tilde{G}(z) = \overline{\tilde{G}(\overline{z})}$$

and $\tilde{G}|_{(0,T_0)}$ is a homeomorphism from $(0, T_0)$ to $\mathbb{R}$. As $\tilde{G}'(\frac{T_0}{2}) < 0$, we get that $\tilde{G}|_{(0,T_0)}$ is decreasing, and thus

$$\lim_{x\to 0} \tilde{G}(x) = +\infty, \tag{4.3.56}$$
$$\lim_{x\to T_0} \tilde{G}(x) = -\infty. \tag{4.3.57}$$

Using the continuity of $\tilde{G}$ we also get that $\tilde{G}(x + iy) \to +\infty$ as $x \to 0$ uniformly in y. Now set

$$G(z) := \frac{3}{\pi}\tilde{G}(z).$$

Then G is a biholomorphic map from Ω to the strip $\{z = x + iy : |y| < \frac{3}{2}\}$.

Due to (4.3.38) we have $\theta'(t) \to 0$ as $t \to 0$, thus we can apply corollary 4.3.6 and get for $x \to 0$

$$\Re G(x+iy) = \frac{3}{\pi}\Re\tilde{G}(x+iy) \backsimeq \frac{3}{\pi}\frac{\pi}{2}\int_x^{T_0} \frac{1}{\frac{5}{4}\theta(s)}ds = \frac{12}{10}\int_x^{T_0} \frac{1}{\theta(s)}ds. \tag{4.3.58}$$

Applying (4.3.38), we get

$$(\phi \circ \theta)'(x) = \phi'(\theta(x))\theta'(x) = -\frac{1}{\theta(x)}.$$

In other words $-(\phi \circ \theta)$ is an anti-derivative of $\frac{1}{\theta}$. We continue the asymptotic representation (4.3.58) of $\Re G(x + iy)$ and get

$$\Re G(x+iy) \backsimeq \frac{12}{10}(\phi(\theta(x)) - \phi(\theta(T_0))) \backsimeq \frac{12}{10}\phi(\theta(x)). \tag{4.3.59}$$

Now set for $z \in \Omega$

$$F(z) := \exp(-\exp(G(z))).$$

F is by definition an analytic function on Ω. Choose $T_1 > 0$ sufficiently small to get $h(\theta(T_1)) \leq 1$ and for $x \leq T_1$:

$$\Re(G(x+iy)) \geq \frac{11}{10}\phi(\theta(x)),$$
$$\frac{1}{10}\phi(\theta(x)) \geq -\log\left(\cos\left(\frac{3}{2}\right)\right) =: -\log(\alpha).$$

Then it follows for $x = \Re(z) \leq T_1$

$$\begin{aligned} |F(z)| &= \exp(-\exp(\Re G(x+iy))\cos(\Im G(x+iy))) \\ &\leq \exp\left(-\exp\left(\frac{11}{10}\phi(\theta(x)\right)\alpha\right) \\ &\leq h(\theta(x)) \leq h(\theta(T_1)) \leq 1. \end{aligned} \tag{4.3.60}$$

An application of the additional assumption (4.3.40) implies for small x the existence of a small $1 > q > 0$ with

$$|F(x)| = \exp(-\exp(G(x))) \geq \exp(-\exp(2\phi(\theta(x)))) \geq h(q\theta(x)). \tag{4.3.61}$$

Define for $z \in \tilde{\Omega} := \Omega \cap (T_1 - \Omega)$

$$f(z) := F(z)F(T_1 - z).$$

f is analytic on $\tilde{\Omega}$. As $|F| \leq 1$, we get immediately by (4.3.60)

$$|f(z)| \leq |F(z)| \leq h(\theta(x)). \tag{4.3.62}$$

Since $|F(T_1 - x)| \geq c > 0$ (assume w.l.o.g. $c < 1$) as $x \to 0$ we get by (3.2.9) and (4.3.61)

$$|f(x)| \geq c|F(x)| \geq ch(q\theta(x)) \geq h(cq\theta(x)). \tag{4.3.63}$$

Set for $x_0 < \frac{T_1}{2}$, $B_0 := B\left(x_0, \frac{11}{10}\theta(x_0)\right)$. Then we claim

$$B_0 \subseteq \tilde{\Omega}.$$

To prove this inclusion it is enough to show

$$\frac{12}{10}\theta\left(x_0 - \frac{11}{10}\theta(x_0)\right) \geq \frac{11}{10}\theta(x_0).$$

This follows easily from (4.3.37): Observe

$$\begin{aligned} & \theta(x_0) \leq \theta\left(x_0 - \frac{11}{10}\theta(x_0)\right) + \varepsilon\frac{11}{10}\theta(x_0) \\ \Leftrightarrow \quad & \frac{12}{10}\theta\left(x_0 - \frac{11}{10}\theta(x_0)\right) \geq \underbrace{\frac{12}{10}\theta(x_0) - \varepsilon\frac{132}{100}\theta(x_0)}_{\geq\frac{11}{10}\theta(x_0);\text{ by the choice of }\varepsilon}. \end{aligned}$$

We also get $\theta\left(x_0 + \frac{11}{10}\theta(x_0)\right) \leq \frac{11}{10}\theta(x_0)$ by the choice of ε. Therefore by (4.3.62) for $z = x + iy \in B_0$

$$|f(x+iy)| \leq h(\theta(x)) \leq h\left(\theta\left(x_0 + \frac{11}{10}\theta(x_0)\right)\right) \leq h\left(\frac{11}{10}\theta(x_0)\right).$$

So f is analytic on B_0 and bounded by $h\left(\frac{11}{10}\theta(x_0)\right)$ there. Therefore we may conclude by Cauchy's estimates (see e.g. [13, 10.26 Theorem, p. 213])

$$|f^{(k)}(x_0)| \leq h\left(\frac{11}{10}\theta(x_0)\right) k! \left(\frac{11}{10}\theta(x_0)\right)^{-k} \leq M_k,$$

where we applied (3.2.8) to derive the second inequality. By symmetry we get $|f^{(k)}(x_0)| \leq M_k$ also for $x_0 \geq \frac{T_1}{2}$. In addition we have

$$\left|f^{(k)}(x_0)\right| \leq h\left(\frac{11}{10}\theta(x_0)\right) k! \left(\frac{11}{10}\theta(x_0)\right)^{-k} \leq M_{k+1}\frac{11}{10}\theta(x_0)$$

which tends to 0 as $x_0 \to 0$. Thus f is flat at 0. Again by symmetry the same argument holds for T_1. This implies that we can extend f off $[0, T_1]$ by $f \equiv 0$.

Thus we have found a function $f \in C^\infty(\mathbb{R})$ with

$$f^{(n)}(0) = 0 \text{ for all } n \in \mathbb{N},$$
$$|f^{(n)}(x)| \leq M_n \text{ for all } n \in \mathbb{N} \text{ and all } x \in \mathbb{R},$$
$$|f(x)| \geq h(cq\theta(x)) \text{ for small } x > 0.$$

This completes the proof. □

5 Two results of Borichev and Volberg on asymptotically holomorphic functions

In this chapter we present a Phragmén-Lindelöf theorem and a uniqueness theorem for asymptotically holomorphic functions. The uniqueness result may be understood as a Denjoy-Carleman theorem at infinity. In sections 5.1 and 5.2 we focus on understanding the proofs given by Borichev and Volberg. They use very involved constructions based upon an application of the two constants theorem (see theorems 2.4.2 and 2.5.2) in both proofs. In section 5.3 we will finally illustrate how the theorems can be used to solve a very interesting problem. The theorems as well as their application are taken from [3].

Before treating the mentioned theorems, we need two auxiliary results. One of which is the ordinary Phragmén-Lindelöf theorem (see [9, III C, p. 25]).

Theorem 5.0.1. *Let f be analytic in a sector S of opening 2γ with $\gamma < \pi$ (i.e. $S = (T \circ R)(\tilde{S}) := (T \circ R)(\{z \in \mathbb{C} : -\gamma < \arg(z) < \gamma\})$, where T is a translation and R a rotation). Suppose*

$$|f(z)| \leq Ce^{A|z|^\alpha} \quad \forall z \in S$$

for some $\alpha < \pi/(2\gamma)$ and $\limsup_{z\to\zeta} |f(z)| \leq m < \infty$ for all $\zeta \in \partial S$. Then

$$|f(z)| \leq m \quad \forall z \in S.$$

Proof. We can assume w.l.o.g. that f is defined on a sector $S = \{z \in \mathbb{C} : -\gamma < \arg(z) < \gamma\}$; if not, replace f by $f \circ (T \circ R)$, then the new f still fulfils all assumptions and is defined on a sector $\{z \in \mathbb{C} : -\gamma < \arg(z) < \gamma\}$.

Pick β such that $\alpha < \beta < \pi/(2\gamma)$ and $\varepsilon > 0$. Define

$$v_\varepsilon(z) := \log |f(z)| - \varepsilon \Re(z^\beta).$$

We claim that v_ε is subharmonic on S: Due to lemma 1.0.3 $\log |f(z)|$ is subharmonic. As the domain of definition is a sector, there exists a holomorphic logarithm on S. Therefore $z^\beta = e^{\beta \log(z)}$ can be holomorphically defined on S, so $\Re(z^\beta)$ is harmonic (as real part of a holomorphic function). This shows the claim.

Now we observe that $\Re(z^\beta) = |z|^\beta \cos(\beta\phi) \geq |z|^\beta \cos(\beta\gamma)$ for $z \in S$ where ϕ is the argument of z. Since $0 < \beta\gamma < \pi/2$, we thus have $\Re(z^\beta) > 0$ for $z \in S$. This immediately implies

$$v_\varepsilon(z) \leq \log |f(z)|$$

and therefore

$$\limsup_{z\to\zeta} v_\varepsilon(z) \leq \log m \quad \forall \zeta \in \partial S.$$

By assumption we have $\log|f(z)| \leq \log C + A|z|^\alpha$. This yields the following inequality for v_ε

$$v_\varepsilon(z) = \log|f(z)| - \varepsilon\Re(z^\beta) \leq \underbrace{\log C + A|z|^\alpha - \varepsilon\cos(\beta\gamma)|z|^\beta}_{\to -\infty \text{ as } |z|\to\infty, \text{ since } \alpha<\beta}.$$

Therefore there exists some $R > 0$, such that $v_\varepsilon(z) \leq \log m$ for all $z \in S\backslash B(0,R)$. Let now $z_0 \in S$ be fixed. Since making R larger only makes it easier to fulfil the above inequality, it is possible to take the above $R > |z_0|$. This now gives

$$\limsup_{z\to\zeta} v_\varepsilon(z) \leq \log m$$

for all $\zeta \in \partial(S \cap B(0,R))$. Thus applying lemma 1.0.3 to this domain gives $v_\varepsilon(z_0) \leq \log m$ which is equivalent to $\log|f(z_0)| \leq \log m + \varepsilon\Re(z_0^\beta)$. Now let $\varepsilon \to 0$, which implies the statement of the theorem. □

The next theorem is a particular case of the two constants theorem 2.5.2 needed in the proof of theorem 5.1.1. It can be proved directly without an application of the more general two constants theorem. It is taken from [11, IX Lemma (Hadamard's three lines theorem), p. 33].

Theorem 5.0.2. *Let f be analytic on a vertical strip $S := \{z \in \mathbb{C} : a < \Re(z) < b\}$ and bounded on $\overline{S}$. Then the function $M : [a,b] \to \mathbb{R}$ defined by $M(x) = \sup_{\Re(z)=x} |f(z)|$ is logarithmically convex, i.e.*

$$M((1-t)a + tb) \leq M(a)^{1-t}M(b)^t \quad \textit{for} \quad 0 \leq t \leq 1.$$

Proof. Let $F(z) := f(z)M(a)^{\frac{z-b}{b-a}}M(b)^{\frac{a-z}{b-a}}$. Then F is analytic on the strip and as f is bounded, also F is bounded on $\overline{S}$. As $|F(z)| \leq 1$ for $\Re(z) \in \{a,b\}$, it follows by the maximum principle for bounded analytic functions (1.0.10) that $|F|$ is bounded by 1 on the whole strip which implies

$$|f(z)| \leq M(a)^{\frac{b-\Re(z)}{b-a}}M(b)^{\frac{\Re(z)-a}{b-a}}.$$

This completes the proof. □

5.1 Phragmén-Lindelöf theorem for asymptotically holomorphic functions

Theorem 5.1.1. *[3, Theorem 6.1, p. 358] Let f be continuously differentiable and bounded on $\mathbb{C}_+(=\{z \in \mathbb{C} : \Re(z) > 0\})$. Let ω be a positive decreasing function on $\mathbb{R}_+ \cup \{0\}$, where $\mathbb{R}_+ = \mathbb{C}_+ \cap \mathbb{R}$, with the following properties:*

$$\log\left(\frac{1}{\omega(x)}\right) \text{ is convex,}$$
$$\omega(x) = o(e^{-nx}) \quad x \to \infty, \quad \forall n \in \mathbb{N}.$$

If

$$|\overline{\partial} f(z)| < \omega(\Re(z)),$$
$$\left|f|_{\mathbb{R}_+}(x)\right| = O(\exp(-nx)), \quad x \to \infty, \quad \forall n \in \mathbb{N},$$

then there exists some $L > 0$, such that

$$|f(x)| < \omega(Lx)$$

for sufficiently large $x \in \mathbb{R}$.

Proof.
Claim 1: To simplify the following arguments, we observe that it is actually possible to assume without loss of generality that

$$|\overline{\partial} f(z)| < \frac{\omega(\Re(z))}{(1+|z|)^4}, \tag{5.1.1}$$

$$f(z) = \frac{1}{2\pi i}\int_{\zeta \in \mathbb{C}_+} \frac{\overline{\partial} f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}. \tag{5.1.2}$$

To see this, set

$$h_1(z) := \frac{f(z)}{(1+z)^6}.$$

Now observe, that there is some $R > 0$, such that

$$|1+z|^6 > (1+|z|)^4 \quad \text{for} \quad |z| > R.$$

As in addition $\frac{|1+z|^6}{(1+|z|)^4}$ is continuous and non-zero on $\overline{B(0,R) \cap \mathbb{C}_+}$, it is bounded from below by some $m \in (0,1)$. Therefore $|1+z|^6 > m(1+|z|)^4$ on $\mathbb{C}_+$, which implies $\frac{m}{|1+z|^6} \leq \frac{1}{(1+|z|)^4}$.

Let $\phi : \mathbb{R} \to [0,1]$ be a smooth function such that

$$\phi(t) = \begin{cases} 0 & , \quad t \leq \frac{1}{2} \\ 1 & , \quad t \geq 1 \end{cases}$$

Set $h_2 : \mathbb{C} \to \mathbb{C}$

$$h_2(z) = \begin{cases} 0 & , \quad \Re(z) \leq 0 \\ \phi(\Re(z))h_1(z) & , \quad \Re(z) > 0 \end{cases}$$

Then h_2 is now globally defined on $\mathbb{C}$ and coincides with h_1 for $\Re(z) \geq 1$. Therefore $\overline{\partial} h_2(z) = \overline{\partial} h_1(z)$ for $\Re(z) > 1$. Finally, we set

$$h_3(z) := \min\left\{ m, \frac{m\omega(1)}{\max_{0\leq\Re(z)\leq 1} |\overline{\partial}(\phi(\Re(z))f(z))|} \right\} h_2(z).$$

Now we can show that h_3 already fulfils (5.1.1) and (5.1.2): For $\Re(z) > 1$ we have

$$\begin{aligned} |\overline{\partial} h_3(z)| &\leq m \left| \overline{\partial} \left(\frac{f(z)}{(1+z)^6} \right) \right| \\ &= m \left| \frac{\overline{\partial} f(z)}{(1+z)^6} \right| \leq \frac{\omega(\Re(z))}{(1+|z|)^4}, \end{aligned}$$

and for $0 < \Re(z) \leq 1$ we have

$$\begin{aligned} |\overline{\partial} h_3(z)| &\leq \frac{m\omega(1)}{\max_{0\leq\Re(z)\leq 1} |\overline{\partial}(\phi(\Re(z))f(z))|} \left|\overline{\partial}(\phi(\Re(z))f(z))\right| \frac{1}{|1+z|^6} \\ &\leq \frac{\omega(1)}{(1+|z|)^4} \leq \frac{\omega(\Re(z))}{(1+|z|)^4}, \end{aligned}$$

where the last inequality holds as ω is decreasing. This shows, that (5.1.1) holds for h_3.

To see (5.1.2), we observe that

$$\begin{aligned} |h_3(z)| &\leq m|h_2(z)| \leq m \frac{|\phi(\Re(z))||f(z)|}{|1+z|^6} \\ &\leq m \frac{|f(z)|}{|1+z|^6} \leq \frac{|f(z)|}{(1+|z|)^4} \leq \frac{||f||_\infty}{(1+|z|)^4}. \end{aligned}$$

Since $h_3 \in C^1(\mathbb{C})$, it follows for $z \in B(0,k)$ and $n \geq k$ (see theorem 1.0.11)

$$h_3(z) = \frac{1}{2\pi i} \int_{\partial B(0,n)} \frac{h_3(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{B(0,n)} \frac{\overline{\partial} h_3(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}.$$

Now,

$$\begin{aligned} \left| \int_{\partial B(0,n)} \frac{h_3(\zeta)}{\zeta - z} d\zeta \right| &\leq 2\pi n \max_{\zeta\in\partial B(0,n)} \frac{|h_3(\zeta)|}{|\zeta - z|} \\ &\leq 2\pi n \frac{||f||_\infty}{(1+n)^4} \frac{1}{n-k}. \end{aligned}$$

The latter tends to 0 as $n \to \infty$, which now means that

$$h_3(z) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{B(0,n)} \frac{\overline{\partial} h_3(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta} = \frac{1}{2\pi i} \int_{\mathbb{C}_+} \frac{\overline{\partial} h_3(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}$$

where the second equality holds due to the dominated convergence theorem.

Now (5.1.1) and (5.1.2) hold for h_3 and for $\Re(z) > 1$, $h_3(z) = M\frac{f(z)}{(1+z)^6}$, with some small constant M.

Suppose, the theorem can be proved for h_3, then $M\frac{|f(x)|}{|1+x|^6} \le \omega(Kx)$ for some small K and large enough x. But the convexity of $\log(\frac{1}{\omega})$ just means $\omega(tx + (1-t)y) \ge \omega(x)^t \omega(y)^{1-t}$ for all $t \in [0,1]$, which shows

$$\frac{\omega(tx)}{\omega(x)} \ge \frac{\omega(x)^t \omega(0)^{1-t}}{\omega(x)} = c\omega(x)^{t-1} \ge \tilde{c}e^{(1-t)x}$$

where the existence of some $\tilde{c} > 0$ in the last inequality follows from the assumption that $\omega(x) = o(e^{-x})$ for $x \to \infty$. And the right hand side grows faster in x than any polynomial, which means that it is eventually larger than $M|1+x|^6$. Therefore

$$|f(x)| \le M|1+x|^6 \omega(Kx) \le \omega(\frac{K}{2}x)$$

for sufficiently large x. This now shows claim 1. So let us assume from now on that (5.1.1) and (5.1.2) hold for the given f.

The next goal is to prove

Claim 2:

$$|f(z)| = O(\exp(-n\Re(z))) \quad \forall n \in \mathbb{N} \text{ uniformly in } \Im(z). \tag{5.1.3}$$

To this end, we define $F_t(z) := f(z)e^{tz}$ for $t > 0$. F_t is bounded on the imaginary axis by assumption. In addition it is also bounded on the positive reals, because

$$|F_t(x)| = |f(x)e^{tx}| \le |f(x)e^{\lceil t \rceil x}| \overset{x\to\infty}{\to} 0.$$

Next, we define for $z \in \mathbb{C}_+$

$$a_{F_t}(z) := F_t(z) - \frac{1}{2\pi i} \int_{\mathbb{C}_+} \frac{\overline{\partial} F_t(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}. \tag{5.1.4}$$

Since

$$|\overline{\partial} F_t(z)| = \left|\overline{\partial} f(z)\right| |e^{tz}| \le \frac{1}{(1+|z|)^4} \underbrace{\omega(\Re(z))e^{t\Re(z)}}_{\le M_t < \infty}$$

and the right-hand side of the above inequality is bounded and absolutely integrable on $\mathbb{C}_+$, it is possible to apply Lemma 1.0.13 to derive analyticity of the function a_{F_t}.

Due to the above considerations, a_{F_t} is bounded on the real and imaginary axis and

$$|a_{F_t}(z)| \leq Me^{t|z|}$$

for some large constant M. So by applying the Phragmén-Lindelöf theorem 5.0.1 to the first quadrant, we get that a_{F_t} is bounded there. The same argument holds for the fourth quadrant. So a_{F_t} is bounded on $\mathbb{C}_+$.

Since $\frac{1}{2\pi i}\int_{\mathbb{C}_+} \frac{\overline{\partial} F_t(\zeta)}{z-\zeta} d\zeta \wedge d\overline{\zeta}$ is uniformly bounded (in z) on $\mathbb{C}_+$ (see the proof of 1.0.13), it follows that F_t is bounded on $\mathbb{C}_+$ as well which means that

$$|f(z)e^{tz}| \leq C_t < \infty \quad \forall z \in \mathbb{C}_+.$$

Therefore it follows that

$$|f(z)| = O(e^{-n\Re(z)}) \text{ for every } n \text{ uniformly in } \Im(z)$$

which is the desired result of claim 2.

Next, we define for each $t \in \mathbb{R}_+$ and $z \in \Omega_t := \{z \in \mathbb{C} : \Re(z) > t\}$

$$f_t(z) := \frac{1}{2\pi i} \int_{0<\Re(\zeta)<t} \frac{\overline{\partial} f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}. \tag{5.1.5}$$

Clearly for each fixed $z \in \Omega_t$, the integrand of (5.1.5) is absolutely integrable on $\{\zeta \in \mathbb{C} : 0 < \Re(\zeta) < t\}$ by (5.1.1). In addition the integrand is also analytic on Ω_t for each fixed $\zeta \in \{\zeta \in \mathbb{C} : 0 < \Re(\zeta) < t\}$.

In addition, given a compact disc $\overline{B(z_0, r)} \subseteq \Omega_t$, we get by the fact that there must be an $\varepsilon > 0$ such that $\overline{B(z_0, r+\varepsilon)} \subseteq \Omega_t$, the following

$$\left|\frac{\overline{\partial} f(\zeta)}{\zeta - z}\right| \overset{(5.1.2)}{\leq} \frac{\omega(0)}{(1+|\zeta|)^4 |\zeta - z|} \leq \frac{\omega(0)}{\varepsilon(1+|\zeta|)^4} \tag{5.1.6}$$

where the above inequalities hold for all ζ with $0 < \Re(\zeta) < t$ and all $z \in \overline{B(z_0, r)}$. As the right-hand side of (5.1.6) is absolutely integrable on $\{\zeta \in \mathbb{C} : 0 < \Re(\zeta) < t\}$, it is possible to apply Lemma 1.0.14 and derive analyticity of f_t on Ω_t. By (5.1.2) we also get the following estimate for $z \in \Omega_t$

$$\begin{aligned} |f(z) - f_t(z)| &\leq \frac{1}{\pi} \int_{\Omega_t} \left|\frac{\overline{\partial} f(\zeta)}{\zeta - z}\right| d\xi \wedge d\eta \\ &\leq \frac{\omega(t)}{\pi} \underbrace{\int_{\mathbb{C}} \frac{1}{(1+|\zeta|)^4 |\zeta - z|} d\xi \wedge d\eta}_{\leq \tilde{K} \text{ independent of } z} \\ &\leq K\omega(t). \end{aligned} \tag{5.1.7}$$

<u>Claim 3:</u> Let $A := \{n \in \mathbb{N} : \sup_{\Re(z) \geq e^n} |f(z)| < \omega(e^{n-2})\}$ and $B := \mathbb{N} \backslash A$. Then A is infinite.

Define

$$a(x) := -e^{-x} \log \left(\sup_{\Re(z) \geq e^x} |f(z)| \right). \tag{5.1.8}$$

Let us first check that $a(n) \to \infty$: By the definition of $a(n)$ this is clearly equivalent to

$$\forall R > 0 \; \exists N(R) \in \mathbb{N} : \forall n \geq N \qquad \sup_{\Re(z) \geq e^n} |f(z)| \leq e^{-Re^n}.$$

And as $|f(z)| = O(e^{-n\Re(z)})$ for all $n \in \mathbb{N}$ by (5.1.3), there exists some N such that $|f(z)| \leq e^{-R\Re(z)}$ for all z with $\Re(z) \geq e^N$. Therefore for all z with $\Re(z) \geq e^n \geq e^N$ we have

$$|f(z)| \leq e^{-R\Re(z)} \leq e^{-Re^n}.$$

So $\sup_{\Re(z) \geq e^n} |f(z)| \leq e^{-Re^n}$ and therefore $a(n) \to \infty$.

Now let $n \in B$ (i.e. $\sup_{\Re(z) \geq e^n} |f(z)| \geq \omega(e^{n-2})$). The function $f_{e^{n-2}}$ is bounded and analytic on $\Omega_{e^{n-2}}$. Boundedness follows from the boundedness of f and (5.1.7), analyticity was shown for general f_t. In addition the following estimates hold

$$\sup_{\Re(z) = e^{n-2}} |f_{e^{n-2}}(z)| \leq (K+1) e^{-a(n-2)e^{n-2}}, \tag{5.1.9}$$

$$\sup_{\Re(z) = e^{n}} |f_{e^{n-2}}(z)| \leq (K+1) e^{-a(n)e^{n}}. \tag{5.1.10}$$

(5.1.9) holds due to

$$\begin{aligned}
\sup_{\Re(z) = e^{n-2}} |f_{e^{n-2}}(z)| &\leq \sup_{\Re(z) = e^{n-2}} |f_{e^{n-2}}(z) - f(z)| + \sup_{\Re(z) = e^{n-2}} |f(z)| \\
&\overset{(5.1.7)}{\leq} K\omega(e^{n-2}) + \sup_{\Re(z) = e^{n-2}} |f(z)| \\
&\underset{n \in B}{\leq} K \sup_{\Re(z) \geq e^n} |f(z)| + \sup_{\Re(z) = e^{n-2}} |f(z)| \\
&\leq (K+1) \sup_{\Re(z) \geq e^{n-2}} |f(z)| \\
&= (K+1) e^{-a(n-2)e^{n-2}}.
\end{aligned}$$

(5.1.10) holds due to an analogous argument.

Due to the three lines theorem 5.0.2 (for $x \in [e^{n-2}, e^n]$ and $t \in [0,1]$ such that $x = (1-t)e^{n-2} + te^n$), it follows that

$$\sup_{\Re(z) = x} |f_{e^{n-2}}(z)| \leq (K+1) e^{\left(-(1-t)a(n-2)e^{n-2} - ta(n)e^n\right)}. \tag{5.1.11}$$

Since $f_{e^{n-2}}$ is bounded and analytic in the the half plane $\{z \in \mathbb{C} : \Re(z) \geq x\}$ for $x \geq e^{n-2}$, the respective maximum has to be attended at the boundary $\{z : \Re(z) = x\}$ by the maximum principle for bounded analytic functions (1.0.10). Therefore

$$\sup_{\Re(z) \geq x} |f_{e^{n-2}}(z)| \leq (K+1) e^{\left(-(1-t)a(n-2)e^{n-2} - ta(n)e^n\right)}. \tag{5.1.12}$$

For $e^{n-1} = \frac{e}{e+1}e^{n-2} + \frac{1}{e+1}e^n$ this means

$$\sup_{\Re(z)\geq e^{n-1}} |f_{e^{n-2}}(z)| \leq (K+1)e^{\left(-\frac{e}{e+1}a(n-2)e^{n-2}-\frac{1}{e+1}a(n)e^n\right)}. \tag{5.1.13}$$

Claim 3a: For $n \in B$ there exist absolute constants K_1, K_2, such that

$$a(n-1) \geq -K_1e^{-n} + \frac{a(n-2)+ea(n)}{1+e}, \tag{5.1.14}$$

$$a(n) - a(n-1) \leq K_2e^{-n} + \frac{1}{e}(a(n-1)-a(n-2)). \tag{5.1.15}$$

In order to show (5.1.14), we observe that

$$\begin{aligned}\sup_{\Re(z)\geq e^{n-1}} |f(z)| &\leq \sup_{\Re(z)\geq e^{n-1}} |f(z)-f_{e^{n-2}}(z)| + \sup_{\Re(z)\geq e^{n-1}} |f_{e^{n-2}}(z)| \\ &\overset{(5.1.7),(5.1.13)}{\leq} K\omega(e^{n-2}) + (K+1)e^{\left(-\frac{e}{e+1}a(n-2)e^{n-2}-\frac{1}{e+1}a(n)e^n\right)} \\ &\leq 2(K+1)e^{\left(-\frac{e}{e+1}a(n-2)e^{n-2}-\frac{1}{e+1}a(n)e^n\right)}\end{aligned} \tag{5.1.16}$$

where the last inequality holds, because $n \in B$ and therefore

$$\begin{aligned}e^{-a(n)e^n} &= \sup_{\Re(z)\geq e^n} |f(z)| \geq \omega(e^{n-2}) \\ e^{-a(n-2)e^{n-2}} &= \sup_{\Re(z)\geq e^{n-2}} |f(z)| \geq \sup_{\Re(z)\geq e^n} |f(z)| \geq \omega(e^{n-2}).\end{aligned}$$

This implies

$$\omega(e^{n-2}) \leq e^{\left(-\frac{e}{e+1}a(n-2)e^{n-2}-\frac{1}{e+1}a(n)e^n\right)}.$$

By applying $-e^{-(n-1)}\log$ to the chain of inequalities (5.1.16), we get

$$\begin{aligned}a(n-1) &= -e^{-(n-1)}\log\left(\sup_{\Re(z)\geq e^{n-1}} |f(z)|\right) \\ &\geq -e^{-(n-1)}\log\left(2(K+1)e^{\left(-\frac{e}{e+1}a(n-2)e^{n-2}-\frac{1}{e+1}a(n)e^n\right)}\right) \\ &= \underbrace{-e\log(2(K+1))}_{=:K_1}e^{-n} + \frac{a(n-2)}{e+1} + \frac{ea(n)}{e+1}.\end{aligned}$$

This is (5.1.14). (5.1.15) is then a simple manipulation of (5.1.14):

$$\begin{aligned}& -a(n-1) \leq K_1e^{-n} - \frac{a(n-2)}{1+e} - \frac{ea(n)}{1+e} \\ \Leftrightarrow\quad & \frac{e}{1+e}a(n) - \frac{e}{1+e}a(n-1) \leq K_1e^{-n} + \frac{1}{1+e}a(n-1) - \frac{1}{1+e}a(n-2) \\ \Leftrightarrow\quad & a(n)-a(n-1) \leq \underbrace{K_1\frac{1+e}{e}}_{=:K_2}e^{-n} + \frac{1}{e}(a(n-1)-a(n-2)).\end{aligned}$$

This is now used to show claim 3: Suppose A is finite, then beginning from some index n_0, all n are in B. Then (5.1.15) holds for all $n \geq n_0$. Set $b_n := a(n) - a(n-1)$, we can write $a(n) = b_n + \cdots + b_{n_0+1} + a(n_0)$. (5.1.15) now reads $b_n \leq K_2 e^{-n} + \frac{1}{e} b_{n-1}$. But even if equality holds, the solution of this recursion (with arbitrary initial value) is summable: W.l.o.g. let $n_0 = 0$ and observe

$$\sum_{k=0}^{n} b_k = \left(K_2 \sum_{k=1}^{n} k e^{-k} \right) + \left(\sum_{k=0}^{n} e^{-k} \right) b_0 \leq K_2 \frac{2 - 1/e}{(1-1/e)^2} + \frac{b_0}{1 - 1/e} < \infty$$

for all $n \in \mathbb{N}$.

But this means, that $a(n)$ has to be bounded in this case, which is a contradiction to $a(n) \to \infty$. Therefore, there are arbitrarily large integers in A and claim 3 holds.

Now we define a function

$$G(x) := x a(\log(x)).$$

Observe, that $G(x) = -\log(\sup_{\Re(z) \geq x} |f(z)|)$ is increasing. The goal is to show

<u>Claim 4:</u> There exists $L > 0$ such that for sufficiently large $x \in \mathbb{R}$

$$G(x) \geq \log(\frac{1}{\omega(Lx)}).$$

This implies the result of the theorem: Suppose we get some L such that the above inequality holds for large x. Then

$$\begin{aligned} & -\log(\sup_{\Re(z) \geq x} |f(z)|) \geq \log(\frac{1}{\omega(Lx)}) \\ \Leftrightarrow \quad & \frac{1}{\sup_{\Re(z) \geq x} |f(z)|} \geq \frac{1}{\omega(Lx)} \\ \Leftrightarrow \quad & \sup_{\Re(z) \geq x} |f(z)| \leq \omega(Lx). \end{aligned}$$

If $n \in A$, then we get by the definition of A that

$$G(e^n) = -\log(\sup_{\Re(z) \geq e^n} |f(z)|) \geq -\log(\omega(e^{n-2}))$$

and for $x \in [e^n, e^{n+1}]$, $L < e^{-3}$:

$$\log(\frac{1}{\omega(Lx)}) \leq \log(\frac{1}{\omega(e^{-3}x)}) \leq \log(\frac{1}{\omega(e^{n-2})}) \leq G(e^n) \leq G(x). \tag{5.1.17}$$

<u>Claim 4a:</u> There is an absolute constant K_3 such that for large enough $n \in A$

$$G(e^{n-1}) \geq -K_3 \log(\omega(e^{n-2})).$$

To see this, first observe that

$$\sup_{\Re(z)\geq e^{n-2}} |f_{e^{n-2}}(z)| \overset{(5.1.7)}{\leq} \underbrace{\sup_{\Re(z)\geq e^{n-2}} |f(z)| + K\omega(e^{n-2})}_{\leq 1 \text{ for large n by (5.1.3) and the definition of } \omega} .$$

Since $n \in A$, it follows by (5.1.7) that $\sup_{\Re(z)=e^n} |f_{e^{n-2}}(z)| \leq (K+1)\omega(e^{n-2})$. Therefore the three lines theorem (5.0.2) implies for sufficiently large $n \in A$

$$\sup_{\Re(z)\geq e^{n-1}} |f_{e^{n-2}}(z)| \leq \underbrace{(K+1)^{\frac{1}{1+e}}}_{=:P} \omega(e^{n-2})^{\frac{1}{1+e}}.$$

Using this, we get

$$\begin{aligned}
\sup_{\Re(z)\geq e^{n-1}} |f(z)| &\leq \sup_{\Re(z)\geq e^{n-1}} |f_{e^{n-2}}(z) - f(z)| + \sup_{\Re(z)\geq e^{n-1}} |f(z)| \\
&\leq (K+1)\omega(e^{n-2}) + P\omega(e^{n-2})^{\frac{1}{e+1}} \\
&\leq (K+1)\omega(e^{n-2})^{\frac{1}{e+1}} + P\omega(e^{n-2})^{\frac{1}{e+1}} \\
&\leq 2\max\{(K+1), P\}\omega(e^{n-2})^{\frac{1}{e+1}}.
\end{aligned}$$

Now, an application of $-\log$ to the above chain of inequalities gives

$$\begin{aligned}
G(e^{n-1}) &= -\log\left(\sup_{\Re(z)\geq e^{n-1}} |f(z)|\right) \\
&\geq -\log\left(2\max\{(K+1), P\}\omega(e^{n-2})^{\frac{1}{e+1}}\right) \\
&= -\log\left(2\max\{(K+1), P\}\right) - \frac{1}{1+e}\log\left(\omega(e^{n-2})\right) (= (+))
\end{aligned}$$

As $-\log(\omega(e^n)) \to +\infty$, for large enough n, we get

$$-\frac{1}{2(1+e)}\log(\omega(e^{n-2})) \geq -\log\left(2\max\{(K+1), P\}\right)$$

which now leads to

$$(+) \geq \frac{-1}{2(1+e)}\log(\omega(e^{n-2})).$$

This shows claim 4a.

By taking some smaller $K_4 < K_3$, we can achieve (by convexity) for large x that

$$\log(\frac{1}{\omega(K_4 x)}) \leq K_3 \log(\frac{1}{\omega(x)}).$$

This now shows that if L is in addition smaller than K_4e^{-2}, for $x \in [e^{n-1}, e^n]$

$$\log(\frac{1}{\omega(Lx)}) \leq \log(\frac{1}{\omega(K_4e^{n-2})}) \leq K_3\log(\frac{1}{\omega(e^{n-2})}) \leq G(e^{n-1}) \leq G(x). \qquad (5.1.18)$$

Claim 4b: For large enough integers $n_1, n_2 \in A$, such that $[n_1+1, n_2-1] \cap \mathbb{N} \subset B$, there are absolute constants K_5, K_6, such that for $x \in [e^{n_1}, e^{n_2-1}]$:

$$G(x) \geq K_5\frac{G(e^{n_1})(e^{n_2-1}-x)+G(e^{n_2-1})(x-e^{n_1})}{e^{n_2-1}-e^{n_1}} + K_6. \qquad (5.1.19)$$

We will first show that it is possible to find constants C_1 and C_2 such that

$$\pi(x) \geq \sigma(x) \geq C_1\frac{G(e^{n_1})(e^{n_2-1}-x)+G(e^{n_2-1})(x-e^{n_1})}{e^{n_2-1}-e^{n_1}} + C_2 \qquad (5.1.20)$$

where π is the polygon connecting the points of

$$\Pi := \{(e^n, G(e^n)) : n \in [n_1, n_2-1] \cap \mathbb{N}\}$$

and σ is it's convex minorant. As can be easily verified, σ is also a polygon connecting the points of some set $\Sigma := \{(e^{m_k}, G(e^{m_k})) : k = 1, \ldots, l\} \subseteq \Pi$, where $m_1 = n_1$ and $m_l = n_2 - 1$. Next observe that (5.1.14) is equivalent to

$$\frac{e^n a(n) - e^{n-1}a(n-1)}{e^n - e^{n-1}} \leq K_1 e^{-n} + \frac{e^{n-1}a(n-1) - e^{n-2}a(n-2)}{e^{n-1} - e^{n-2}}.$$

By the definition of G this means for $n \in B$

$$\frac{G(e^n) - G(e^{n-1})}{e^n - e^{n-1}} \leq K_1 e^{-n} + \frac{G(e^{n-1}) - G(e^{n-2})}{e^{n-1} - e^{n-2}}.$$

By denoting $d_m^n := \frac{G(e^n)-G(e^m)}{e^n-e^m}$ this means

$$d_{n-1}^n \leq \frac{K_1}{e^{n-1}} + d_{n-2}^{n-1} \qquad (5.1.21)$$

and therefore (note the difference of $m_{k\pm1}$ and $m_k \pm 1$!)

$$d_{m_k}^{m_{k+1}} \leq d_{m_k}^{m_k+1} \overset{(5.1.21)}{\leq} \frac{K_1}{e^{m_k}} + d_{m_k-1}^{m_k} \leq \frac{K_1}{e^{m_k}} + d_{m_{k-1}}^{m_k} \qquad (5.1.22)$$

where the first and last inequality hold due to the definition of σ as convex minorant. Because σ is a convex polygon, it immediately follows that $d_{m_1}^{m_2} < d_{m_2}^{m_3} < \cdots < d_{m_{l-1}}^{m_l}$ and $d_{m_1}^{m_2} \leq d_{m_1}^{m_l} \leq d_{m_{l-1}}^{m_l}$.

Now (5.1.20) takes the following form (observe that the first inequality $\pi(x) \geq \sigma(x)$ is trivially satisfied)

$$\sigma(x) \geq C_1 d_{n_1}^{n_2-1}x + C_1\frac{G(e^{n_1})e^{n_2-1} - G(e^{n_2-1})e^{n_1}}{e^{n_2-1}-e^{n_1}} + C_2. \qquad (5.1.23)$$

If $|\Sigma| = 2$ (i.e. σ is a line and therefore π a concave polygon) let $C_1 := 1$ and $C_2 := 0$ to get (5.1.23). So assume from now on $|\Sigma| \geq 3$.

If $\frac{e-1}{e} d_{m_1}^{m_l} \leq d_{m_1}^{m_2}$, $C_1 := \frac{e-1}{e}$ and $C_2 := 0$ are valid choices for the constants in (5.1.23). If not, let k_0 be the maximal integer such that

$$d_{m_{k_0-1}}^{m_{k_0}} \leq \frac{e-1}{e} d_{m_1}^{m_l}. \tag{5.1.24}$$

Note that $2 \leq k_0 \leq l-1$.

The next goal is to show that (5.1.23) holds with the choices $C_1 := \frac{e-1}{e}$ and $C_2 := -K_1$: First observe that in order to show that the above constants are good enough to get (5.1.23) it suffices to show

$$0 \leq \frac{e-1}{e}(e^{m_{k_0}} - e^{m_1}) d_{m_1}^{m_l} - \underbrace{(e^{m_{k_0}} - e^{m_1}) d_{m_1}^{m_{k_0}}}_{=G(e^{m_{k_0}})-G(e^{m_1})} \leq K_1. \tag{5.1.25}$$

To see that (5.1.25) already implies (5.1.23), observe that the above difference is exactly the difference of the value of the line L at $e^{m_{k_0}}$ and $G(e^{m_{k_0}})$, where L is a line with slope $\frac{e-1}{e} d_{m_1}^{m_l}$ passing through the point $(e^{m_1}, G(e^{m_1}))$. The left-hand side of (5.1.25) follows since $\frac{e-1}{e} d_{m_1}^{m_l} - d_{m_1}^{m_{k_0}} \geq d_{m_{k_0-1}}^{m_{k_0}} - d_{m_1}^{m_{k_0}} \geq 0$ due to convexity. By translating L by the value $-K_1$ the desired result (5.1.23) thus follows by the definition of k_0.

In order to derive (5.1.25) we need several observations.

$$d_{m_1}^{m_l} = \frac{e^{m_{k_0}} - e^{m_1}}{e^{m_l} - e^{m_1}} d_{m_1}^{m_{k_0}} + \sum_{k=k_0+1}^{l} \frac{e^{m_k} - e^{m_{k-1}}}{e^{m_l} - e^{m_1}} d_{m_{k-1}}^{m_k}.$$

By applying inequality (5.1.22) iteratively, we get for $k_0 < k \leq l$

$$\begin{aligned} d_{m_{k-1}}^{m_k} &\leq \frac{K_1}{e^{m_{k_0}}} \left(\sum_{i=k_0}^{l} \left(\frac{1}{e}\right)^i \right) + d_{m_{k_0-1}}^{m_{k_0}} \\ &\leq \frac{K_1}{e^{m_{k_0}}} \frac{e}{e-1} + d_{m_{k_0-1}}^{m_{k_0}}, \end{aligned}$$

and therefore

$$\begin{aligned} d_{m_1}^{m_l} &\leq \frac{e^{m_{k_0}} - e^{m_1}}{e^{m_l} - e^{m_1}} d_{m_{k_0-1}}^{m_{k_0}} + \sum_{k=k_0+1}^{l} \frac{e^{m_k} - e^{m_{k-1}}}{e^{m_l} - e^{m_1}} \left(\frac{K_1}{e^{m_{k_0}}} \frac{e}{e-1} + d_{m_{k_0-1}}^{m_{k_0}} \right) \\ &= \left(\frac{K_1}{e^{m_{k_0}}} \frac{e}{e-1} \sum_{k=k_0+1}^{l} \frac{e^{m_k} - e^{m_{k-1}}}{e^{m_l} - e^{m_1}} \right) + d_{m_{k_0-1}}^{m_{k_0}} \\ &\leq \frac{K_1}{e^{m_{k_0}}} \frac{e}{e-1} + d_{m_{k_0-1}}^{m_{k_0}}. \end{aligned} \tag{5.1.26}$$

Additionally we have

$$\begin{aligned}(e^{m_{k_0}} - e^{m_1})d_{m_1}^{m_{k_0}} &= G(e^{m_{k_0}}) - G(e^{m_1})\\ &\geq G(e^{m_{k_0}}) - G(e^{m_{k_0-1}})\\ &= (e^{m_{k_0}} - e^{m_{k_0-1}})d_{m_{k_0-1}}^{m_{k_0}}\end{aligned} \tag{5.1.27}$$

and

$$\begin{aligned}\frac{e-1}{e}(e^{m_{k_0}} - e^{m_1}) - (e^{m_{k_0}} - e^{m_{k_0-1}}) \leq &\quad \frac{e-1}{e}e^{m_{k_0}} - (e^{m_{k_0}} - e^{m_{k_0-1}})\\ = &\quad e^{m_{k_0}} - e^{m_{k_0}-1} - e^{m_{k_0}} + e^{m_{k_0-1}}\\ = &\quad e^{m_{k_0-1}} - e^{m_{k_0}-1} \leq 0.\end{aligned} \tag{5.1.28}$$

Now putting all these observations together we finally get

$$\begin{aligned}&\frac{e-1}{e}(e^{m_{k_0}} - e^{m_1})d_{m_1}^{m_l} - (e^{m_{k_0}} - e^{m_1})d_{m_1}^{m_{k_0}}\\ \overset{(5.1.27)}{\leq}\quad &\frac{e-1}{e}(e^{m_{k_0}} - e^{m_1})d_{m_1}^{m_l} - (e^{m_{k_0}} - e^{m_{k_0-1}})d_{m_{k_0-1}}^{m_{k_0}}\\ \overset{(5.1.26)}{\leq}\quad &\frac{K_1}{e^{m_{k_0}}}(e^{m_{k_0}} - e^{m_1}) + \left(\frac{e-1}{e}(e^{m_{k_0}} - e^{m_1}) - (e^{m_{k_0}} - e^{m_{k_0-1}})\right)d_{m_{k_0-1}}^{m_{k_0}}\\ \overset{(5.1.28)}{\leq}\quad &\frac{K_1}{e^{m_{k_0}}}(e^{m_{k_0}} - e^{m_1}) \quad \leq \quad K_1.\end{aligned}$$

This is exactly (5.1.25) which, as already shown, implies (5.1.23). We have thus found absolute constants such that (5.1.19) holds for $G = \pi$.

We still have to study the behaviour of G on intervals $[e^{n-1}, e^n]$ for $n \in B$ to finally give absolute constants K_5 and K_6. To this end, we observe that in analogy to (5.1.12) it holds that

$$\sup_{\Re(z)\geq x} |f_{e^{n-1}}(z)| \leq (K+1)e^{-(1-t)a(n-1)e^{n-1} - ta(n)e^n}$$

for $(1-t)e^{n-1} + te^n = x \in [e^{n-1}, e^n]$ and $t \in [0,1]$. With a justification as for (5.1.14) it is then possible to derive

$$G(x) \geq -K_1 e + (1-t)G(e^{n-1}) + tG(e^n) \tag{5.1.29}$$

with x and t as above. Now set

$$K_5 := \frac{e-1}{e}, \quad K_6 := -(1+e)K_1.$$

Then (5.1.29) together with (5.1.23) shows that these choices for K_5 and K_6 imply (5.1.19).

If n_1 is chosen large enough to get (for all $n_1 < n \in A$)

$$\frac{K_5}{2}G(e^n) \geq -K_6 + \log(1/\omega(0))$$

it follows by convexity that

$$\log\left(\frac{1}{\omega(\frac{K_5}{2}e^n)}\right) \leq K_5 G(e^n) + K_6.$$

A similar argument leads to the existence of a constant K_7 such that

$$\log\left(\frac{1}{\omega(K_7 e^{n_2-1})}\right) \leq K_5 G(e^{n_2-1}) + K_6.$$

Now taking $L < \min\{K_5/2, K_7\}$ shows that the values of $\log(\frac{1}{\omega(Lx)})$ on e^{n_1} and e^{n_2-1} are smaller than the respective values of

$$x \mapsto K_5 \frac{G(e^{n_1})(e^{n_2-1} - x) + G(e^{n_2-1})(x - e^{n_1})}{e^{n_2-1} - e^{n_1}} + K_6.$$

As $\log\left(\frac{1}{\omega(Lx)}\right)$ is assumed convex, it is smaller than the above affine linear function on the whole interval $[e^{n_1}, e^{n_2-1}]$.

Now we take $n_1 \in A$ large enough such that all above considerations hold and L smaller than the choices of the respective constants. Then it finally follows that

$$G(x) \geq \log\left(\frac{1}{\omega(Lx)}\right)$$

for $x \geq n_1$, which shows claim 4 and therefore the theorem. □

5.2 Uniqueness result

Theorem 5.2.1. *[3, Theorem 7.1, p. 362]*
Let ρ be a positive increasing function defined on $\mathbb{R}_+ \cup \{0\}$ with

$$\rho(0) = 0, \ \int_0^1 \log\log \frac{1}{\rho(x)} dx = \infty. \tag{5.2.30}$$

Let ϕ be a positive decreasing function defined on $\mathbb{R}_+ \cup \{0\}$ with

$$\int_0^\infty \phi(x) dx < \infty. \tag{5.2.31}$$

Let $S := \{z \in \mathbb{C}_+ : |\Im(z)| < 1\}$ and f a bounded continuously differentiable function defined on S with

$$\left|\overline{\partial} f(z)\right| < \rho(\phi(\Re(z))|\Im(z)|), \tag{5.2.32}$$

$$\left|f|_{\mathbb{R}_+}(x)\right| < \rho(\phi(x)) \quad x > 0. \tag{5.2.33}$$

Then

$$f(x) = 0$$

for sufficiently large $x \in \mathbb{R}_+$.

If $\rho = h$, where h is the corresponding weight function defined by some regular weight sequence (see section 3.2), then

$$f(x) = 0$$

for all $x \in \mathbb{R}_+$.

Proof.
Claim 1: It is possible to assume w.l.o.g. in addition that ϕ is continuous and ρ continuous, strictly increasing and $\lim_{x\to\infty} \rho(x) = \infty$.

To this end, it is enough to find continuous, decreasing $\tilde{\phi} \geq \phi$ and continuous, strictly increasing $\tilde{\rho} \geq \rho$ such that

$$\int_0^\infty \tilde{\phi}(x) dx < \infty, \quad \int_0^1 \log\log \frac{1}{\tilde{\rho}(x)} dx = \infty$$

because due to $\tilde{\phi} \geq \phi$, $\tilde{\rho} \geq \rho$ and monotony we have

$$\tilde{\rho}(\tilde{\phi}(x)) \geq \tilde{\rho}(\phi(x)) \geq \rho(\phi(x)).$$

Consider ρ: We define

$$M(x) := \log\log \frac{1}{\rho(x)}. \tag{5.2.34}$$

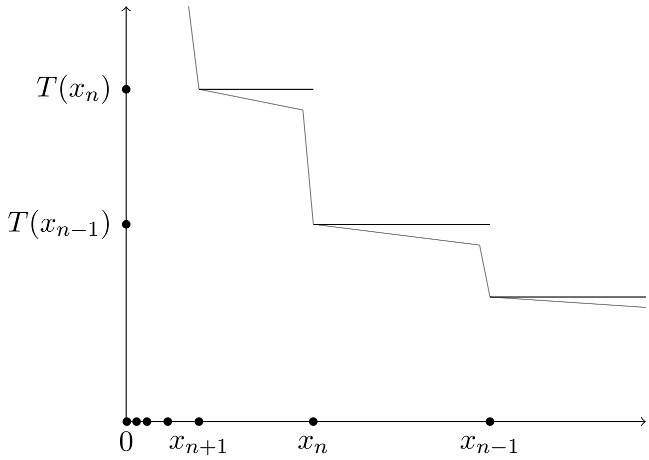
Then M is decreasing on $\mathbb{R}_+$ and $\lim_{x\to 0} M(x) = \infty$. If we find a continuous and strictly decreasing $\tilde{M} \le M$ with $\int_0^1 \tilde{M}(x)dx = \infty$, then $\exp(-\exp(\tilde{M}))$ is a suitable $\tilde{\rho}$.

Due to (5.2.30), there exists a decreasing function $T \le M$ defined on $(0, 1]$ and a strictly decreasing sequence $(x_n)_{n\in\mathbb{N}}$ converging to 0 such that $T|_{(x_{n+1},x_n]}$ is constant (i.e. T is a step function on each compact subinterval of $(0, 1]$), $T(x_{n+1}) > T(x_n)$ and $\int_t^1 T(x)dx \ge \int_t^1 M(x)dx - 1$ for all $t \in (0, 1]$.

Now define $\tilde{M}$ as $\tilde{M}(x_n) = T(x_{n-1})\ \forall n \in \mathbb{N}$ and $\tilde{M}$ continuous and strictly decreasing on $[x_{n+1}, x_n]$ such that

$$\int_{x_{n+1}}^{x_n} \tilde{M}(x)dx \ge \int_{x_{n+1}}^{x_n} T(x)dx - \frac{1}{2^n}$$

and $\tilde{M} \le M$ continuous on $[x_1, \infty)$ and tending to $-\infty$. Then $\tilde{M}$ fulfils the requirements above. $\tilde{M}$ may be imagined as follows (the grey graph represents a possible $\tilde{M}$):



A similar argument leads to the existence of a suitable $\tilde{\phi}$. This shows the claim.

Assume from now on that ρ and ϕ have the additional properties from claim 1. By the additional assumptions made for ρ it is now possible to define its inverse ρ^{-1} on all of $\mathbb{R}_+$, which is also continuous. Therefore it is possible to define

$$s(t) := \rho^{-1}\left(\sup_{x\ge t} |f(x)|\right)$$

which is continuous as it is the composition of continuous functions. As ρ is strictly increasing, s is decreasing (as the composition of an increasing and a decreasing function). Furthermore it holds that

$$s(t) = \rho^{-1}\left(\sup_{x\ge t} |f(x)|\right) < \rho^{-1}(\rho(\phi(t))) = \phi(t). \tag{5.2.35}$$

Assume from now on $s(t) > 0$ for all $t \in \mathbb{R}_+$.

If we can derive a contradiction this implies the desired result: The negation of $s(t) > 0$ for all $t > 0$ is $s(T_0) = 0$ for some $T_0 > 0$ (s is by definition non-negative). As s is decreasing by assumption, this implies that $s(t) = 0$ for $t \geq T_0$ which is the desired result for general ρ. If in addition $\rho = h$, remark 3.2.5 implies that $f|_{\mathbb{R}_+} \in C\{M_n\}(\mathbb{R}_+)$ and thus (5.2.30) implies with the application of theorem 4.0.1 that f is quasianalytic. So $f(t)$ being constant zero for large enough t implies that $f(t)$ is constant 0 for all $t > 0$.

A crucial part of the proof is the following
Claim 2: There exists $A < \infty$ and $\tau > 2$ such that

$$0 \leq M(s(t)) - M\left(s\left(t - \frac{s(t)}{\phi(t-2)}\right)\right) \leq A, \quad \forall t \geq \tau. \tag{5.2.36}$$

To this end define for $t \geq 2$

$$P(t) := \left\{z \in \mathbb{C} : t - 2\frac{s(t)}{\phi(t-2)} \leq \Re(z) \leq t+1, |\Im(z)| \leq \frac{s(t)}{\phi(t-2)}\right\}.$$

Now define for $z \in P(t)$

$$f_{P(t)}(z) = f(z) - \frac{1}{2\pi i}\int_{\zeta \in P(t)} \frac{\overline{\partial} f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}.$$

Observe that since ϕ is decreasing and by (5.2.35) it follows that $\frac{s(t)}{\phi(t-2)} < 1$ and hence the definition of $f_{P(t)}$ makes sense (f is only defined on the strip S). By lemma 1.0.13 $f_{P(t)}$ is analytic on $P(t)^{\circ}$. And for $x \in [t, t+1]$ the following inequality holds

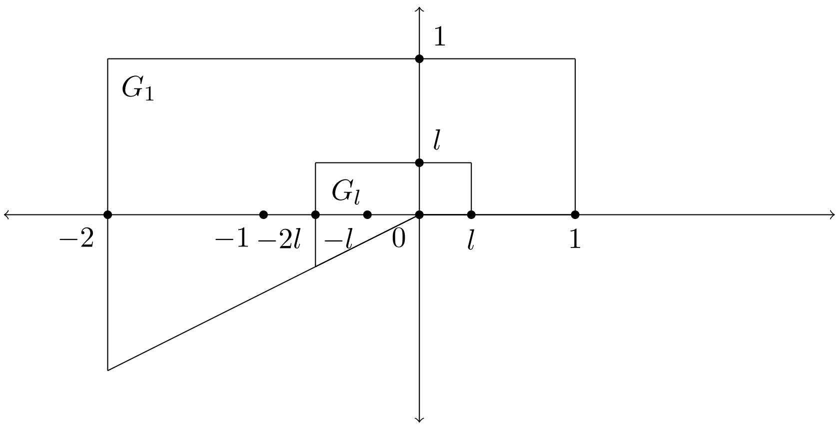
$$\begin{aligned}
|f_{P(t)}(x)| &\leq |f(x)| + \frac{1}{\pi}\int_{\zeta \in P(t)} \frac{|\overline{\partial} f(\zeta)|}{|\zeta - x|} d\xi \wedge d\eta \\
&\overset{(5.2.32)}{\leq} \sup_{y \geq t} |f(y)| + \frac{1}{\pi}\int_{\zeta \in P(t)} \frac{\rho(\phi(\Re(\zeta))|\Im(\zeta)|)}{|\zeta - x|} d\xi \wedge d\eta \\
&\leq \sup_{y \geq t} |f(y)| + \frac{1}{\pi}\int_{\zeta \in P(t)} \frac{\rho\left(\phi(t-2)\frac{s(t)}{\phi(t-2)}\right)}{|\zeta - x|} d\xi \wedge d\eta \\
&= \sup_{y \geq t} |f(y)| + \sup_{y \geq t} |f(y)| \underbrace{\frac{1}{\pi}\int_{\zeta \in P(t)} \frac{1}{|\zeta - x|} d\xi \wedge d\eta}_{\leq 8} \\
&\leq 9 \sup_{y \geq t} |f(y)| \\
&= 9 \exp\left(-\exp\left(M\left(s(t)\right)\right)\right)
\end{aligned}$$

where the third inequality holds due to the definition of $P(t)$ and the monotonicity properties of ρ and ϕ. That the integral in the fourth line is less than 8 holds due to the fact that $P(t) \subseteq B(x, 4)$ for any $x \in [t, t+1]$; apply lemma 1.0.12.

By the two constants theorem 2.4.2 it thus follows the existence of $A_1 > 0$ and $\tau_1 > 2$ such that for $t \geq \tau_1$

$$\left| f_{P(t)}\left(t - \frac{s(t)}{\phi(t-2)}\right)\right| \leq \exp\left(-\exp\left(M\left(s(t)\right) - A_1\right)\right). \tag{5.2.37}$$

This can be seen as follows: First define for $l \in (0,1]$ the set G_l as the interior of the closed polygon defined by connecting the points $0, l, l+il, -2l+il, -2l-il, 0$ in this order.



By the Riemann mapping theorem 1.0.15 it is possible to define a biholomorphic map $g_l : \mathbb{D} \to G_l$ with $g_l(0) = -l$ which extends continuously and bijectively to the boundary by the theorem of Carathéodory 2.3.1, and we get (naming the continuous extension from $\overline{\mathbb{D}}$ to $\overline{G_l}$ again g_l)

$$\omega(-l, [0,l], G_l) = \omega(0, g_l^{-1}([0,l]), \mathbb{D}).$$

Now observe that lg_1 is also a biholomorphic map from $\mathbb{D}$ to G_l with $(lg_1)(0) = -l$ and in addition we have $(lg_1)^{-1}([0,l]) = g_1^{-1}([0,1])$. Therefore we get

$$\omega(-l, [0,l], G_l) = \omega(0, g_1^{-1}([0,1]), \mathbb{D}).$$

As $g_1|_{S^1} : S^1 \to \partial G_1$ is a homeomorphism, $g_1^{-1}([0,1])$ has non-empty interior (as subset of S^1) and therefore $\omega(-l, [0,l], G_l) = \omega(0, g_1^{-1}([0,1]), \mathbb{D}) =: \alpha > 0$.

As $\left(t + G_{\frac{s(t)}{\phi(t-2)}}\right) \subseteq P(t)$, apply the two constants theorem 2.4.2 with $\Omega = t + G_{\frac{s(t)}{\phi(t-2)}}$, $z = t - \frac{s(t)}{\phi(t-2)}$, $E = [t, t + \frac{s(t)}{\phi(t-2)}]$, $m = 9\exp\left(-\exp\left(M\left(s(t)\right)\right)\right)$ and $M = C$ some constant uniformly bounding all $f_{P(t)}$ on their respective domains. To define C observe that $|\overline{\partial} f|$ is uniformly bounded on S by $\rho(\phi(0))$ due to (5.2.32). Thus we have for arbitrary $t > 2$ and $z \in P(t)$

$$\begin{aligned}\left|f_{P(t)}\left(z\right)\right| &\leq |f(z)| + \frac{1}{\pi}\int_{\zeta \in P(t)} \frac{|\overline{\partial} f(\zeta)|}{|\zeta - z|} d\xi \wedge d\eta \\ &\leq ||f||_\infty + 10\rho(\phi(0)) =: C\end{aligned}$$

where we used that $P(t) \subseteq B(z, 5)$ for any $z \in P(t)$ and applied lemma 1.0.12. Now using these choices for the parameters in 2.4.2 and assuming w.l.o.g. $C \geq 1$, yields

$$\begin{aligned}\left| f_{P(t)}\left(t - \frac{s(t)}{\phi(t-2)}\right)\right| &\leq \exp\left(-\exp\left(M\left(s(t)\right)\right)\right)^{\alpha} 9^{\alpha} C^{1-\alpha} \\ &\leq \underbrace{9C}_{=:K} \exp\left(-\exp\left(M\left(s(t)\right)\right)\right)^{\alpha}.\end{aligned}$$

As $\exp\left(-\exp\left(M\left(s(t)\right)\right)\right) = \sup_{x\geq t}|f(x)|$ tends to 0 as $t \to \infty$ by (5.2.33), there is some large τ_1 such that for $t \geq \tau_1$ we have $\exp\left(-\exp\left(M\left(s(t)\right)\right)\right)^{\frac{\alpha}{2}} \leq \frac{1}{K}$. Now set $A_1 := -\log(\alpha/2)$, then (5.2.37) holds with these choices for A_1 and τ_1. Now take some $\tau > \tau_1$ such that for $t \geq \tau$ we get

$$\exp\left(-\exp\left(M\left(s(t)\right) - A_1 + \log(1/2)\right)\right) \leq \frac{1}{9}. \tag{5.2.38}$$

Then for $t \geq \tau$ the following holds

$$\begin{aligned}&\left| f\left(t - \frac{s(t)}{\phi(t-2)}\right)\right| \\ &\leq \left| f_{P(t)}\left(t - \frac{s(t)}{\phi(t-2)}\right)\right| + \frac{1}{\pi}\int_{\zeta\in P(t)} \frac{|\overline{\partial} f(\zeta)|}{|\zeta - t - \frac{s(t)}{\phi(t-2)}|} d\xi \wedge d\eta \\ &\overset{(5.2.37)}{\leq} \exp\left(-\exp\left(M\left(s(t)\right) - A_1\right)\right) + \frac{1}{\pi}\int_{\zeta\in P(t)} \frac{|\overline{\partial} f(\zeta)|}{|\zeta - t - \frac{s(t)}{\phi(t-2)}|} d\xi \wedge d\eta \\ &\leq \exp\left(-\exp\left(M\left(s(t)\right) - A_1\right)\right) + 8\exp\left(-\exp\left(M\left(s(t)\right)\right)\right) \\ &\leq 9\exp\left(-\exp\left(M\left(s(t)\right) - A_1\right)\right) \\ &= 9\exp\left(-\exp\left(M\left(s(t)\right) - A_1 + \log(1/2)\right)\right)^2 \\ &\overset{(5.2.38)}{\leq} \exp\left(-\exp\left(M\left(s(t)\right) - A_1 + \log(1/2)\right)\right).\end{aligned}$$

Now set $A := A_1 - \log(1/2)$ and τ as above, applying $\log \circ -\log$ to the above chain of inequalities yields (5.2.36) and therefore the claim.

Now use claim 2 to show

Claim 3: There exists an increasing sequence of positive real numbers $(t_k)_{k=0}^{\infty}$ with $\lim_{k\to\infty} t_k = \infty$ and

$$t_{k+1} - \frac{s(t_{k+1})}{\phi(t_{k+1} - 2)} = t_k \quad \forall k \geq 0. \tag{5.2.39}$$

Take $t_0 := 2$ and set $r(t) := t - \frac{s(t)}{\phi(t-2)}$ for $t \geq 2$. As s and ϕ are continuous, the same holds for r. Now assume t_k is already defined for $k \leq n$. Recall the assumption $s(t) > 0$ for all t, so the same holds for t_n (i.e. $s(t_n) > 0$) and therefore $r(t_n) < t_n$. In addition $\frac{s(t)}{\phi(t-2)} < 1$ implies $r(t) \geq t-1$ which shows that $r(t) \to \infty$ as $t \to \infty$. Now the intermediate value theorem implies the existence of some $t_{n+1} > t_n$ with $r(t_{n+1}) = t_n$. The only thing left to prove is $\lim_{k\to\infty} t_k = \infty$. So assume not, i.e. there is some $t_\infty < \infty$ such that $\lim_{k\to\infty} t_k = t_\infty$, as this is the only alternative since t_k is increasing. But this would imply

$$t_\infty = \lim_{k\to\infty} t_k = \lim_{k\to\infty} r(t_{k+1}) = r(t_\infty)$$

where the last equality holds due to continuity of r. But this contradicts $r(t) < t\ \forall t$, which holds by the assumption $s(t) > 0\ \forall t$. Thus the proof of claim 3 is finished.

(5.2.36) and (5.2.39) imply for all $k \geq k_0$ (take k_0 such that $t_{k_0} \geq \tau$)

$$M(s(t_k)) - M(s(t_{k-1})) \leq A. \tag{5.2.40}$$

Assume w.l.o.g. $k_0 = 0$. Iterated application of (5.2.40) yields

$$M(s(t_k)) - M(s(t_0)) \leq kA. \tag{5.2.41}$$

As $M(t)$ is decreasing and tending to ∞ as t approaches 0, the divergence of the integral (5.2.30) happens arbitrarily close to 0, i.e.

$$\int_0^x M(t)dt = \infty \quad \forall x > 0. \tag{5.2.42}$$

By (5.2.35) s is smaller than a positive decreasing integrable function ϕ on $(0,\infty)$. As such a function ϕ clearly converges to 0 at infinity, the same holds for s. Therefore, after removing points t_k with $s(t_{k-1}) = s(t_k)$ from the sequence, $\{(s(t_k), s(t_{k-1})] : k \geq 0\}$ forms a partition of $(0, s(t_0)]$. Together with (5.2.42) this implies

$$\sum_{k=1}^{\infty} \int_{s(t_k)}^{s(t_{k-1})} M(t)dt = \infty. \tag{5.2.43}$$

Now by using the above considerations, we get

$$\begin{aligned}
\sum_{k=0}^{\infty} s(t_k) &= \sum_{k=1}^{\infty} k\,(s(t_{k-1}) - s(t_k)) \\
&\overset{(5.2.41)}{\geq} \sum_{k=1}^{\infty} \frac{M(s(t_k)) - M(s(t_0))}{A} (s(t_{k-1}) - s(t_k)) \\
&\geq \frac{1}{A} \sum_{k=1}^{\infty} \int_{s(t_k)}^{s(t_{k-1})} M(t)dt - \frac{1}{A} M(s(t_0))s(t_0) \overset{(5.2.43)}{=} \infty
\end{aligned}$$

where the last inequality holds as $M(t) \leq M(s(t_k))$ on the interval $(s(t_k), s(t_{k-1})]$. But also, by (5.2.39),

$$\sum_{k=0}^{\infty} s(t_k) = s(t_0) + \sum_{k=1}^{\infty} \phi(t_k - 2)(t_k - t_{k-1})$$
$$\leq s(t_0) + \sum_{k=1}^{\infty} \int_{t_{k-1}}^{t_k} \phi(t-2)dt \overset{(5.2.31)}{<} \infty$$

where the first inequality holds as $\phi(t-2) \geq \phi(t_k - 2)$ on the interval $[t_{k-1}, t_k]$.

Therefore the assumption that s is positive leads to a contradiction. Thus the theorem is proved. □

5.3 Application

Here we will sketch an application of theorems 5.1 and 5.2. Like the theorems, the application is taken from [3]. We will only outline the main steps of the proof as a detailed treatment would go beyond the scope of this thesis.

Borichev and Volberg use the technique of asymptotically holomorphic extensions to prove that the number of limit cycles lying in a compact subset of $\mathbb{R}^2$ defined by a system of ODEs of the form

$$\begin{aligned} \dot{x} &= \alpha(x, y) \\ \dot{y} &= \beta(x, y), \qquad (x, y) \in \mathbb{R}^2 \end{aligned} \tag{5.3.44}$$

is finite if α, β lie in a quasianalytic regular DC class $C_{gl}\{M_n\}$, provided the conditions (a), (b) and (c) below are satisfied.

Here we have functions α, β defined on $\mathbb{R}^2$, but the definition of DC classes is analogous to the one from section 3.2; the only difference being that we require that the absolute value of any partial derivative of order n (i.e. k-times differentiation with respect to x and $(n-k)$-times differentiation with respect to y for some $0 \leq k \leq n$) is locally (globally) bounded by $AB^n M_n$. The Denjoy-Carleman theorem also holds in the same form as in theorem 4.0.1, which can be seen by applying the one-dimensional result to a function of two variables composed with a curve parametrizing an affine line. Thus the assumption of quasianalyticity of $C_{gl}\{M_n\}$ is equivalent to

$$\int_0^1 \phi(t)dt = \infty \tag{5.3.45}$$

where $\phi(t) = \log\left(\log\left(\frac{1}{h(t)}\right)\right)$ as in section 3.2.

Apart from assuming (5.3.45) for the class $C_{gl}\{M_n\}$, the proof uses the following assumptions:

(a)
$$\lim_{x\to\infty} \frac{\phi(x)}{\log\left(\frac{1}{x}\right)} = \infty.$$

(b) For any compact set $K \subseteq \mathbb{C}$ there are only finitely many singular points $(x, y) \in K$. A point (x, y) is called singular if it is a common zero of α and β.

(c) For each singular point (x_0, y_0), the Jacobian matrix of the vector field $(x, y) \mapsto (\alpha(x, y), \beta(x, y))$ is invertible at (x_0, y_0). This means that all singular points on a polycycle (defined below) are saddle points.

Before illustrating a proof of the desired result, we need some definitions.

Definition

Cycle: A closed integral curve of the system (5.3.44) (i.e. a closed solution) is called cycle.

Polycycle: A closed union of integral curves connecting finitely many singular points of the system (5.3.44) is called a polycycle.

Limit cycle: A cycle is called limit cycle if it admits a neighbourhood without other cycles.

Monodromy transformation of a cycle: Given a cycle C and a transversal Γ; i.e. a one dimensional C^ω-manifold intersecting the cycle at some point (x_0, y_0) such that the linear approximation of C resp. Γ at (x_0, y_0) are linearly independent. Suppose there are no singular points in a neighbourhood of C. Then the monodromy transformation (with respect to the cycle C and the transversal Γ) maps a point $(x, y) \in \Gamma$ (close to (x_0, y_0)) to the point $m(x, y)$ on Γ that is reached in minimal positive time when going along the integral curve of (5.3.44) starting at (x, y). Using a suitable parametrization of Γ, m can be interpreted as a map from $(-1, 1)$ to $\mathbb{R}$.

Monodromy transformation of a polycycle: Is defined analogously as for cycles. Let C be a polycycle. The only difference is that transversals are replaced by semitransversals; i.e. one dimensional C^ω manifolds with boundary (point $(x_0, y_0) \in C$) whose (one-sided) linear approximation at (x_0, y_0) and the linear approximation of C at (x_0, y_0) are linearly independent. If the point (x_0, y_0) is a singular point of (5.3.44) the linear approximation of C has to be replaced with a one-sided linear approximation.

Observe that monodromy transformations of cycles always exist, whereas monodromy transformations of polycycles may not exist. Points lying (on some semitransveral Γ) arbitrarily close to a polycycle C may never return to Γ along an integral curve of (5.3.44).

In [3] Borichev and Volberg show

Theorem 5.3.1. *[3, Theorem 2.1, p. 348]*
Assume conditions (a), (b), (c). Let α, β from (5.3.44) *lie in some quasianalytic DC class. Then any polycycle of the system* (5.3.44) *has a neighbourhood without limit cycles.*

Due to Borichev and Volberg, [3, 2.2. Main theorem, p. 347], theorem 5.3.1 implies the desired result:

Theorem 5.3.2. *[3, Theorem 2.2, p. 348] Assume conditions (a), (b), (c). Let α, β from* (5.3.44) *lie in some quasianalytic DC class. Then for any compact subset $K \subseteq \mathbb{R}^2$, the number of limit cycles of* (5.3.44) *lying in K is finite.*

Proof sketch of theorem 5.3.1. Assume the contrary, i.e. there is a polycycle C of (5.3.44) with limit cycles lying arbitrarily close to the polycycle. Then there exists a sequence of limit cycles C_n converging to the polycycle. An application of [3, Theorem A, p. 347] yields the existence of a monodromy transformation $\delta_{C,\Gamma}$ of C (with some semitransversal Γ) and this monodromy transformation is of the form

$$\delta_{C,\Gamma}(t) = t + r(t)$$

where $r(t) = o(t^n)$ as $t \to 0$ for all $n \in \mathbb{N}$.

A quite involved construction then shows (see sections 4 and 5 in [3]) that it is possible to extend the mapping $t \mapsto \delta_{C,\Gamma}(e^{-t})$ to a continuously differentiable function $\zeta \mapsto g(\zeta)$ defined on $\Omega_\varepsilon := \{z = x + iy \in \mathbb{C}_+ : |y| < e^{\varepsilon x}\}$ for some ε such that for $f(\zeta) := g(\zeta) - e^\zeta$ (also defined on Ω_ε) it follows

$$\left|\overline{\partial} f(\zeta)\right| \leq \rho_0(e^{-\varepsilon\xi})$$

for $\zeta = \xi + i\eta \in \Omega_\varepsilon$. Moreover

$$|f|_{\mathbb{R}}(\xi)| = O(\exp(-n\xi)), \quad \xi \to \infty, \quad \forall n \in \mathbb{N}.$$

Here $\rho_0(t) = \gamma(h(\gamma t))^{\frac{1}{4}}$ (for some large γ). Since $\log\left(\frac{1}{\rho_0(e^{-\varepsilon\xi})}\right)$ is convex, an application of a corollary of theorem 5.1 (see [3, Corollary 6.2, p. 361]) implies the existence of a constant $0 < K < 1$ such that for large $\xi \in \mathbb{R}$

$$\left|\delta_{C,\Gamma}(e^{-\xi}) - e^{-\xi}\right| = |f|_{\mathbb{R}}(\xi)| \leq \rho_0(e^{-\varepsilon K\xi}). \tag{5.3.46}$$

In addition it is shown for $\zeta \in \{z = x + iy \in \mathbb{C} : |y| < 1\}$ that there exists some $D > 0$ with

$$|\overline{\partial} f(\zeta)| \leq \rho_0(D|\eta|e^{-\varepsilon\xi}) \leq \rho_0(D|\eta|e^{-\varepsilon K\xi}).$$

Now observe that the divergence of the integral near zero of $\log\left(\log\left(\frac{1}{h(t)}\right)\right)$ (which holds due to the assumed quasianalyticity) is equivalent to the divergence of the integral near zero of $\log\left(\log\left(\frac{1}{\rho_0(t)}\right)\right)$. In addition $\int_0^\infty e^{-\varepsilon Kt}dt < \infty$. Observe that (5.3.46) is condition (5.2.33) from theorem 5.2. Therefore theorem 5.2 is applicable in the current situation. Thus it follows that $\delta_{C,\Gamma}(e^{-\xi}) - e^{-\xi} = f(\xi) = 0$ for sufficiently large $\xi \in \mathbb{R}$.

This shows that the monodromy transformation of C is the identity. But this means that close to the given polycycle C all integral curves are cycles. And this implies for large enough n that arbitrarily close to C_n there is another cycle. Thus the given C_n are not limit cycles, contradicting the assumption.

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Bibliography

[1] E. Bierstone, P.D. Milman. Resolution of singularities in Denjoy-Carleman classes. *Selecta Mathematica, New Series*, vol. 10(1), 1-28, 2004.

[2] J. Boman. Uniqueness and non-uniqueness for microanalytic continuation of ultra-distributions. *Contemporary Mathematics*, vol. 251, 61-82, 2000.

[3] A.A. Borichev, A.L. Volberg. Finiteness of the set of limit cycles, and uniqueness theorems for asymptotically holomorphic functions. *St. Petersburg Mathematical Journal*, vol. 7(3), 343-368, 1996.

[4] J.B. Conway. *Functions in one complex variable I. Second edition.* Graduate texts in mathematics 11, Springer-Verlag, New York, 1978.

[5] E.M. Dynkin. Pseudoanalytic extension of smooth functions. The uniform scale. *American Mathematical Society Translations*, vol. 115(2), 33-58, 1980.

[6] E.M. Dynkin. The pseudoanalytic extension. *Journal d'analyse mathématique*, vol. 60, 45-70, 1993.

[7] J.B. Garnett, D.E. Marshall. *Harmonic measure.* New Mathematical Monographs, Cambridge University Press, Cambridge, 2005.

[8] L. Hörmander. *An introduction to complex analysis in several variables.* North-Holland Publishing Company, Amsterdam, 1973.

[9] P. Koosis. *The logarithmic integral I.* Cambridge Studies in Advanced Mathematics 12, Cambridge University Press, Cambridge, 1988.

[10] J.N. Mather. On Nierenberg's proof of Malgrange's preparation theorem. *Lecture Notes in Mathematics*, vol. 192, 116-120, 1971.

[11] M. Reed, B. Simon. *Methods of modern mathematical physics, volume 2: Fourier analysis, self-adjointness.* Academic Press, San Diego, 1975.

[12] R. Remmert, G. Schumacher. *Funktionentheorie 1. Fünfte, neu bearbeitete Auflage.* Springer-Verlag, Berlin Heidelberg, 2002.

[13] W. Rudin. *Real and complex analysis. Third edition.* McGraw-Hill Book Company, New York, 1987.

[14] S.E. Warschawski. On conformal mapping of infinite strips. *Transactions of the American Mathematical Society*, vol. 51, 280-335, 1942.