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# MASTERARBEIT

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## Old and new results on ordinal definability

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# Contents

# Introduction

The notion of *ordinal definability* was introduced by Kurt Gödel in [**2**], where he conjectured that the collection $OD$ of ordinal definable sets would lead to a model of set theory, and that the axiom of choice would be true in this model.

Following Gödel's remarks, John Myhill and Dana Scott investigated the collection $HOD$ of hereditarily ordinal definable sets, and confirmed that $HOD$ is indeed a model of ZFC. This yielded a somewhat easier consistency proof for the axiom of choice than the one by means of Gödels constructibility theory.

Unlike $L$, the inner model $HOD$ is very sensitive to the surrounding universe $V$, which makes it difficult to give a general analysis. Quoting Kenneth Kunen from his classic [**17**], p.162:

> Some mathematicians might find the definitions of $OD$ and $HOD$ somewhat fishy because of their extremely non-constructive nature.

and later

> [...] the non-constructive nature of $OD$ makes it very difficult to deal with.

Kunen's use of the term *non-constructiveness* refers to the fact that $OD$ is about definability in the *whole universe* $V$. This is very much unlike the case of $L$, where one only has to consider definability in small set-size structures $L_\alpha$. In light of Tarski's theorem on the undefinability of truth, it is quite a surprise that $OD$ is indeed a definable class.

The question whether $HOD$ has any first-order properties besides AC was settled in the negative by Stansiław Roguski, who showed in 1976 that *every* model of ZFC arises as the $HOD$ of a class-forcing extension of the universe. (This can be paraphrased by saying that $HOD$ has no internal structure.)

The question about the relation between $HOD$ and $V$ remains a meaningful one. A first result in this direction is due to Petr Vopěnka, who showed in 1972 that every set of ordinals is contained in a generic extension of $HOD$. In 2012 Sy-David Friedman generalized this result to show that indeed the whole universe $V$ is a (class-)forcing extension of $HOD$.

One may also ask whether the existence of large cardinals is reflected in $HOD$. Here, the results have been exclusively negative for all but the smallest large cardinals. It has also been shown that the cardinal arithmetic may differ substantially between $HOD$ and $V$.

*About this thesis*:

This thesis is divided into five chapters.

In Chapter 1, we develop the basic theory of ordinal definability. Most of the results here are old and well-known. We have included a section on Paris models and pointwise definability, two (less-known) concepts related to ordinal definability.

In Chapter 2, we use forcing to prove the aforementioned result by Roguski, and subsequently prove Vopěnka's Theorem.

In Chapter 3, we introduce Friedman's *Stable Core* and use it to generalize Vopěnka's Theorem.

In Chapter 4, we prove that measurability is not necessarily witnessed in $HOD$, and quote some related results.

Chapter 5 contains Background material. The choice about what to include here has been somewhat random and unbalanced. If the reader feels like something important has been left out, she can most certainly find it in the classic introductory books such as Kunen's [**16**] and Jech's [**15**].

CHAPTER 1

# The basic theory of $HOD$

## 1. Ordinal definability

Let $\mathfrak{M} = (M, \in_M)$ be a model of ZF set theory. Following the usual abuse of notation, we will identify $\mathfrak{M}$ with its underlying set $M$.
A set $x \in \mathfrak{M}$ is called *definable* in $\mathfrak{M}$ if

$$\mathfrak{M} \models \varphi(x) \quad \text{and} \quad \mathfrak{M} \models \exists! y \varphi(y)$$

for some first-order formula $\varphi \in \mathcal{L}_{\mathrm{ZF}}$, the language of set theory[1]. A class $A \subseteq \mathfrak{M}$ is called *definable* in $\mathfrak{M}$ if for all $x \in \mathfrak{M}$

$$x \in A \leftrightarrow \mathfrak{M} \models \varphi(x)$$

for some first-order formula $\varphi$. If $A$ happens to be a set in $\mathfrak{M}$, these two notions coincide by means of the Axiom of Extensionality.
Denote the collection of $\mathfrak{M}$-definable sets by $\mathrm{Df}(\mathfrak{M})$. It is well known that $\mathrm{Df}(\mathfrak{M})$ is in general not definable in $\mathfrak{M}$: For example, consider the case where $ON^{\mathfrak{M}}$ is uncountable. Then not all $\mathfrak{M}$-ordinals can be definable, because there are only countably many formulas. Now if $\mathrm{Df}(\mathfrak{M})$ was definable, the least element $\alpha$ of $ON^{\mathfrak{M}} \setminus \mathrm{Df}(\mathfrak{M})$ could be defined in $\mathfrak{M}$ by the formula

$$\alpha \notin \mathrm{Df}(\mathfrak{M}) \wedge \forall \beta < \alpha (\beta \in \mathrm{Df}(\mathfrak{M}))$$

contradicting $\alpha \notin \mathrm{Df}(\mathfrak{M})$.
To give a more general argument, it is well-known that first-order logic cannot capture infinite cardinalities. But $\mathrm{Df}(\mathfrak{M})$ clearly has to be countable (as viewed externally), and so it cannot be completely described by a first-order formula.

In this section, we show that there is an $\mathfrak{M}$-definable class

$$OD(\mathfrak{M}) \supseteq \mathrm{Df}(\mathfrak{M})$$

which can be regarded as a canonical definable approximation to $\mathrm{Df}(\mathfrak{M})$. The elements of $OD(\mathfrak{M})$ will themelves be characterized by some generalized definability property.
The following lemma shows how large such a class must be.

[1]By using the notation $\varphi(x)$, we mean to indicate that $\varphi$ has at most one free variable $x$. If $\varphi$ is to contain parameters, this will be noted in context.

LEMMA 1.1. *Let $N \subseteq \mathfrak{M}$ be an $\mathfrak{M}$-definable class containing all $\mathfrak{M}$-definable sets, and let $A$ be an $\mathfrak{M}$-definable class with a $\mathfrak{M}$-definable well-order. Then $A \subseteq N$.*

PROOF. If $A \setminus N \neq \emptyset$, let $x \in A \setminus N$ be minimal with respect to the well-order of $A$. Then $x$ is $\mathfrak{M}$-definable, and so $x \in N$. Contradiction! □

In particular, if such an $N$ exists, it must contain the class of all $\mathfrak{M}$-ordinals (and therefore fails to be countable in general).
This proof was of course just a generalization of the argument in the first paragraph.
In light of Lemma 1.1, a natural candidate for our class $OD(\mathfrak{M})$ could therefore be the union of all $\mathfrak{M}$-definable classes which have $\mathfrak{M}$-definable well-orders. This indeed works, but we do not want to use it as the official definition of $OD(\mathfrak{M})$ since it is too blatantly second order. So let us first make one further remark.
Let $A$ be an $\mathfrak{M}$-definable class, and assume there is an $\mathfrak{M}$-definable well-order on $A$ as in Lemma 1.1. We may additionally assume[2] that the well-order on $A$ is set-like. We then can define a rank function $\mathrm{rk}_A : A \to ON^{\mathfrak{M}}$ in $\mathfrak{M}$ by recursion on $A$. Let $\varphi_A(x, y)$ be the formula

$$(x \in A) \wedge (\mathrm{rk}_A(x) = y)$$

Now if $a \in A$ and $\mathrm{rk}_A(a) = \alpha$, then

$$\mathfrak{M} \models \varphi_A(a, \alpha) \quad \text{and} \quad \mathfrak{M} \models \exists! z \varphi_A(z, \alpha)$$

And hence every set in $A$ can be uniformly defined from only one ordinal parameter, namely its rank in the well-order of $A$.

This motivates the following definition:

DEFINITION 1.2. A set $x \in \mathfrak{M}$ is called *ordinal definable in* $\mathfrak{M}$ if there is a formula $\varphi(x, y)$ and $\alpha \in ON^{\mathfrak{M}}$ such that

$$\mathfrak{M} \models \varphi(x, \alpha) \quad \text{and} \quad \mathfrak{M} \models \exists! z \varphi(z, \alpha)$$

Let $OD(\mathfrak{M})$ denote the class of all sets which are ordinal definable in $\mathfrak{M}$. Of course, $OD(\mathfrak{M}) \supseteq \mathrm{Df}(\mathfrak{M})$ (just discard the ordinal parameter).

Let us make two remarks on this definition.

REMARK 1.3. $OD(\mathfrak{M})$ contains all $\mathfrak{M}$-definable classes with $\mathfrak{M}$-definable well-orders; this follows from the previous discussion. So for example, $ON^{\mathfrak{M}} \subseteq OD(\mathfrak{M})$. This can of course also be checked directly

[2]If $\preceq$ is *any* well-order on $A$, then

$$x \preceq' y \leftrightarrow (\mathrm{rk}(x) < \mathrm{rk}(y) \vee (\mathrm{rk}(x) = \mathrm{rk}(y) \wedge x \preceq y))$$

is a set-like well-order on $A$. Hence a class has a definable well-order iff it has a set-like definable well-order.

from the definition of ordinal definability: Each ordinal $\alpha$ is defined by the formula $x = \alpha$, using $\alpha$ as a parameter.

REMARK 1.4. The restriction to only one ordinal parameter is not essential, for any finite number of parameters can be coded into one using a definable pairing function.

REMARK 1.5. For any $\alpha, \beta$ the following sets (as calculated in $\mathfrak{M}$) are ordinal definable:

$$\alpha \cap \beta, \alpha^\beta, \mathcal{P}(\alpha), V_\alpha, H(|\alpha|), \ldots$$

It is now left to show that $OD(\mathfrak{M})$ is itself definable in $\mathfrak{M}$. This will follow from the Reflection Theorem in $\mathfrak{M}$.
First, let us denote by $\mathrm{Df}(V_\beta)^{\mathfrak{M}}$ and $OD(V_\beta)^{\mathfrak{M}}$ the definable resp. ordinal definable elements of $V_\beta^{\mathfrak{M}}$ *as defined within* $\mathfrak{M}$, i.e. using the $\mathfrak{M}$-definable satisfaction relation for set structures in $M$. For example,

$$x \in \mathrm{Df}(V_\beta)^{\mathfrak{M}} \leftrightarrow \mathfrak{M} \models \exists \varphi(x) \in \mathcal{L}_{\mathrm{ZF}}(x \text{ is unique such that } V_\beta \models \varphi(x))$$

PROPOSITION 1.6. $OD(\mathfrak{M}) = \bigcup_{\beta \in ON^{\mathfrak{M}}} OD(V_\beta)^{\mathfrak{M}} = \bigcup_{\beta \in ON^{\mathfrak{M}}} \mathrm{Df}(V_\beta)^{\mathfrak{M}}$.
*In particular, the class $OD(\mathfrak{M})$ is $\mathfrak{M}$-definable.*

PROOF. We first show the equality on the left side.
Let $x \in OD(\mathfrak{M})$. Then $x$ is the unique solution in $\mathfrak{M}$ to some formula $\varphi(x, \alpha)$ with $\alpha \in ON^{\mathfrak{M}}$. By the Reflection Theorem (applied in $\mathfrak{M}$), there is a $\beta > \alpha$ such that

$$\mathfrak{M} \models x \text{ is the unique solution to } \varphi(x, \alpha) \text{ in } V_\beta$$

So $x \in OD(V_\beta)^{\mathfrak{M}}$.
Conversely assume now that $x \in OD(V_\beta)^{\mathfrak{M}}$ for some $\beta \in ON^{\mathfrak{M}}$. Pick $\varphi \in (\mathcal{L}_{\mathrm{ZF}})^{\mathfrak{M}}$, $\alpha \in ON^{\mathfrak{M}}$ such that $\mathfrak{M}$ thinks that $\varphi$ defines $x$ in $V_\beta$ from the ordinal parameter $\alpha < \beta$. Assume that $\varphi = \varphi_n$ where $(\varphi_n)_{n \in \omega^{\mathfrak{M}}}$ is an $\mathfrak{M}$-definable enumeration of $(\mathcal{L}_{\mathrm{ZF}})^{\mathfrak{M}}$. Then the formula

$$V_\beta \models \varphi_n(x, \alpha)$$

defines $x$ in $\mathfrak{M}$ from ordinal parameters $\alpha, \beta$ and $n$, and so $x \in OD(\mathfrak{M})$.
One can prove the equality of $OD(\mathfrak{M})$ and $\bigcup_{\beta \in ON^{\mathfrak{M}}} \mathrm{Df}(V_\beta)^{\mathfrak{M}}$ in the same way, using the *Extended Reflection principle* in $\mathfrak{M}$, which is proved as Lemma 5.1 in the Background material. It says that $\beta$ can always be chosen in such a way that the ordinal parameter $\alpha$ becomes definable - and therefore eliminable - in $V_\beta$.
For the "In particular…" part, one now only has to note that the definition of $OD(V_\beta)^{\mathfrak{M}}$ is uniform in $\beta$, and thus

$$x \in OD(\mathfrak{M}) \leftrightarrow \mathfrak{M} \models \exists \beta (x \in OD(V_\beta))$$

is a definition of $OD(\mathfrak{M})$ inside the model $\mathfrak{M}$. $\square$

Remark 1.7. Although this proof is not difficult, there is one subtle point to it. Namely, to argue that $\bigcup_{\beta\in ON^{\mathfrak{M}}} OD(V_\beta)^{\mathfrak{M}}$ is contained in $OD(\mathfrak{M})$, one can *not* proceed as follows: "Pick $\varphi \in M, \alpha \in ON^{\mathfrak{M}}$ such that $\mathfrak{M}$ thinks that $\varphi$ defines $x$ in $V_\beta$ from the ordinal parameter $\alpha < \beta$. Then the formula

$$V_\beta \models \varphi(x, \alpha)$$

defines $x$ in $\mathfrak{M}$ from ordinal parameters $\alpha$ and $\beta$, and so $x \in OD(\mathfrak{M})$." The problem is that if $\mathfrak{M}$ is an $\omega$-nonstandard model, then the formula $\varphi \in M$ might have nonstandard length, and so one cannot make sense of $V_{v_0} \models \varphi(x, v_1)$ as a formula in the meta-theory. We avoided this by coding $\varphi$ into a (possibly nonstandard) number $n \in \omega^{\mathfrak{M}}$ and using this as an additional ordinal parameter.
(Recall also that in the definition of ordinal definability, the ordinal parameters are ordinals "in the sense of $\mathfrak{M}$".)
Now if one is only interested in standard models, all these distinctions become void, and the theory becomes somewhat neater. It is however still of interest that the concept of ordinal definability can be developed in nonstandard models as well.

Remark 1.8. One may replace the $V_\alpha$'s in the above lemma by the stages of any ordinal-indexed hierarchy which has the Reflection property with respect to $V$. For example, the same proof yields

$$OD(\mathfrak{M}) = \bigcup_{\kappa\in Card^{\mathfrak{M}}} \mathrm{Df}(H(\kappa))^{\mathfrak{M}}$$

where $H(\kappa)$ denotes the collection of all sets $x$ such that $|\,\mathrm{trcl}(x)| < \kappa$.

Now that we have seen that $OD(\mathfrak{M})$ has a first-order definition which does not depend on the model $\mathfrak{M}$, we will use $OD$ as a class term in the way one is used to from set-theoretic practice. In particular, we write $OD^{\mathfrak{M}}$ for the interpretation of $OD$ inside $\mathfrak{M}$, and this yields exactly $OD(\mathfrak{M})$.

We now show that $OD^{\mathfrak{M}}$ itself has an $\mathfrak{M}$-definable well-order. The proof is basically the same as for the inner model $L$.

Lemma 1.9. *There is an $\mathfrak{M}$-definable surjective function $F$ from $ON^{\mathfrak{M}}$ to $OD^{\mathfrak{M}}$.*

Proof. Using the representation

$$OD^{\mathfrak{M}} = \bigcup_{\beta\in ON^{\mathfrak{M}}} \mathrm{Df}(V_\beta)$$

each ordinal definable set in $\mathfrak{M}$ is identified by the stage $V_\beta$ where it is defined and the defining formula. So the function

$$G : ON^{\mathfrak{M}} \times \omega^{\mathfrak{M}} \to OD^{\mathfrak{M}}$$

$$G(\beta, n) = \begin{cases} x & \text{if } \mathfrak{M} \models (x \text{ is the unique solution to } \varphi_n \text{ in } V_\beta) \\ \emptyset & \text{else} \end{cases}$$

- where $\{\varphi_n \mid n \in \omega\}$ is some fixed $\mathfrak{M}$-definable enumeration of $(\mathcal{L}_{\mathrm{ZF}})^{\mathfrak{M}}$ - is well-defined and surjective. Next, it is not hard to find a definable surjection $H : ON^{\mathfrak{M}} \to ON^{\mathfrak{M}} \times \omega^{\mathfrak{M}}$. Then $F = G \circ H$ is as desired. □

COROLLARY 1.10. *There is an $\mathfrak{M}$-definable well-order $<_{OD}$ of the class $OD^{\mathfrak{M}}$.*

PROOF. Let $F$ be the function from Lemma 1.9. For $x, y \in OD^{\mathfrak{M}}$ set $x <_{OD} y$ iff

$$\min\{\alpha \in ON^{\mathfrak{M}} \mid F(\alpha) = x\} < \min\{\alpha \in Ord^{\mathfrak{M}} \mid F(\alpha) = y\}$$

This is a well-order on $OD^{\mathfrak{M}}$. □

COROLLARY 1.11. *$OD^{\mathfrak{M}}$ is the smallest $\mathfrak{M}$-definable class which contains all $\mathfrak{M}$-definable sets.*

PROOF. By Lemma 1.1 and Corollary 1.10. □

$V = OD$ denotes the statement that every set is definable from ordinal parameters. Equivalently and maybe more intuitive, $V = OD$ holds iff every set is definable in an initial segment of the universe (see Lemma 1.6).

LEMMA 1.12. *$\mathfrak{M} \models (V = OD)$ iff $\mathfrak{M}$ has an $\mathfrak{M}$-definable well-order.*

PROOF. If $\mathfrak{M} \models (V = OD)$ then $\mathfrak{M}$ has a definable well-order by Corollary 1.10. Conversely, if $\mathfrak{M}$ has an $\mathfrak{M}$-definable well-order then $\mathfrak{M} \subseteq OD^{\mathfrak{M}}$ and so $\mathfrak{M} \models (V = OD)$. □

Clearly, if a model has a definable well-order then it satisfies the Axiom of Choice, and so:

COROLLARY 1.13. $\mathrm{ZF} \vdash (V = OD \to \mathrm{AC})$

From what we have proved so far, $V = OD$ is easily seen to be consistent:

LEMMA 1.14. $\mathrm{ZF} \vdash L \subseteq OD$

PROOF. $L$ is definably well-orderable in any model $\mathfrak{M}$ of set theory, and thus $\mathfrak{M} \models L \subseteq OD$ by Lemma 1.1. □

COROLLARY 1.15. $\mathrm{Con(ZF)} \to \mathrm{Con}(\mathrm{ZF}{+}V = OD)$

PROOF. $V = L$ implies $V = OD$ and is consistent relative to ZF. □

Corollary 1.16. Con(ZF) $\to$ Con(ZFC)

A proof for the relative consistency of AC with ZF which does not build on results about the $L$ hierarchy will be given later.

We will see that also the theory $ZF+V \neq OD$ is consistent relative to ZF. In fact, one may find models for all of the theories

$$\mathrm{ZF} + L = OD \neq V$$
$$\mathrm{ZF} + L \neq OD = V$$
$$\mathrm{ZF} + L \neq OD \neq V$$

## 2. Some model-theoretic results

In this section, we use model-theoretic methods to prove some results about $\mathrm{Df}(\mathfrak{M})$ and $OD^{\mathfrak{M}}$, now seen as structures for the language $\mathcal{L}_{\mathrm{ZF}}$ with the $\in$-relation inherited from $\mathfrak{M}$.

Definition 1.17.

(1) A substructure $\mathfrak{N} \subseteq \mathfrak{M}$ has $\mathfrak{M}$-*definable witnesses* if $\mathrm{Df}(\mathfrak{M}) \subseteq \mathfrak{N}$ and whenever $\mathfrak{N} \models \exists x \varphi(x, a)$ for some formula $\varphi$ and some parameter $a \in \mathrm{Df}(\mathfrak{M})$, there is an $x_0 \in \mathrm{Df}(\mathfrak{M})$ such that $\mathfrak{N} \models \varphi(x_0, a)$.

(2) $\mathfrak{M}$ has *definable witnesses* if the above holds for $\mathfrak{N} = \mathfrak{M}$. Equivalently, $\mathfrak{M}$ has definable witnesses if whenever $\mathfrak{M} \models \exists x \varphi(x)$ for some formula $\varphi$, there is an $x_0 \in \mathrm{Df}(\mathfrak{M})$ such that $\mathfrak{M} \models \varphi(x_0)$.

Lemma 1.18. *Assume that* $\mathfrak{N}$ *has* $\mathfrak{M}$*-definable witnesses. Then* $\mathrm{Df}(\mathfrak{M}) \preceq \mathfrak{N}$.

Proof. By the Tarski-Vaught criterion. □

Clearly, if $\mathrm{Df}(\mathfrak{M}) \subseteq \mathfrak{N}$ and $\mathfrak{N}$ has an $\mathfrak{M}$-definable well-order, then $\mathfrak{N}$ has $\mathfrak{M}$-definable witnesses: One can always pick the least witness with respect to that well-order. In particular:

Lemma 1.19. $\mathrm{Df}(\mathfrak{M}) \preceq OD^{\mathfrak{M}}$

We can now give a nice model-theoretic characterization of the axiom $V = OD$, which tells us that - loosely speaking - models of $V = OD$ are completely determined by their definable elements.

Lemma 1.20. $\mathfrak{M} \models (V = OD)$ *iff* $\mathrm{Df}(\mathfrak{M}) \preceq \mathfrak{M}$

Proof. If $\mathfrak{M}$ is a model of $V = OD$, then $\mathrm{Df}(\mathfrak{M}) \preceq \mathfrak{M}$ by Lemma 1.19.
Conversely, assume $\mathrm{Df}(\mathfrak{M}) \preceq \mathfrak{M}$. Now if $x \in \mathrm{Df}(\mathfrak{M})$, then $x$ is the unique solution to some formula $\varphi$ in $\mathfrak{M}$. But by elementarity, $x$ is also the unique solution to $\varphi$ in $\mathrm{Df}(\mathfrak{M})$. Hence every element of $\mathrm{Df}(\mathfrak{M})$ is parameter-free definable in $\mathrm{Df}(\mathfrak{M})$, and so in particular $\mathrm{Df}(\mathfrak{M}) \models V = OD$. But then by elementarity $\mathfrak{M} \models V = OD$.

(A quicker but less informative proof is this: If $\mathfrak{M} \not\models V = OD$, then $M \models \exists x(x \notin OD)$, but this statement has no definable witness. Hence $\mathrm{Df}(\mathfrak{M}) \not\preceq \mathfrak{M}$ by the Tarski-Vaught-Test.) □

While having a definable well-order gives one a *uniform* selector of definable witnesses to all properties $\varphi$, having *definable witnesses* seemingly only yields one selector for each property $\varphi$. Over ZF however, the concepts turn out to be equivalent:

COROLLARY 1.21. *If $\mathfrak{M}$ has definable witnesses, then $\mathfrak{M}$ has a definable well-order.*

PROOF. If $\mathfrak{M}$ has definable witnesses, then $\mathrm{Df}(\mathfrak{M}) \preceq \mathfrak{M}$ by Lemma 1.18, and so $\mathfrak{M} \models V = OD$ by the previous lemma. But this means exactly that $\mathfrak{M}$ has a definable well-order. □

Going back to the proof of Lemma 1.20, we have seen that if $\mathfrak{M} \models V = OD$, the structure $\mathrm{Df}(\mathfrak{M})$ has the curious property that all of its elements are definable in $\mathrm{Df}(\mathfrak{M})$. Let us call such a structure *pointwise definable.* Of course, any pointwise definable model of a countable theory has itself to be countable, and so pointwise definability cannot be first-order expressible.

The following example of a pointwise definable model is due to Paul Cohen.

LEMMA 1.22. *Let $\alpha$ be minimal such that $L_\alpha \models$ ZF (if such an $\alpha$ exists). Then $L_\alpha$ is pointwise definable.*

PROOF. One easily sees that $\alpha$ is a limit, and so $L_\alpha \models V = L$. In particular, $L_\alpha \models V = OD$, and so $\mathrm{Df}(L_\alpha) \preceq L_\alpha$ by Lemma 1.20. By condensation, $\mathrm{Df}(L_\alpha)$ is therefore isomorphic to some $L_\beta$, where $\beta \leq \alpha$. Now since this $L_\beta$ satisfies ZF and $\alpha$ was chosen minimal, $\beta$ cannot be strictly smaller than $\alpha$. Hence $\alpha = \beta$, and it follows that $\mathrm{Df}(L_\alpha)$ must have been equal to $L_\alpha$. □

LEMMA 1.23. *The following are equivalent for any $\mathfrak{M} = (M, \in_{\mathfrak{M}})$:*

(1) *$\mathfrak{M}$ is a pointwise definable model of* ZF
(2) *$\mathfrak{M}$ is a* prime model *of* ZF + $V = OD$*, i.e. $\mathfrak{M}$ is a model of* ZF + $V = OD$ *which elementary embeds into any structure $\mathfrak{N} = (N, \in_{\mathfrak{N}})$ having the same first-order theory as $\mathfrak{M}$*
(3) *$\mathfrak{M} \cong \mathrm{Df}(\mathfrak{N})$ for some $\mathfrak{N} = (N, \in_{\mathfrak{N}}) \models V = OD$*

PROOF. (1)→(2): If $\mathfrak{M}$ is pointwise definable, then clearly $\mathfrak{M} \models V = OD$. Now if $\mathfrak{N}$ and $\mathfrak{M}$ share the same first-order theory, then in particular $\mathfrak{N} \models V = OD$ and so $\mathrm{Df}(\mathfrak{N})$ is a pointwise definable model of the same theory as $\mathfrak{N}$. But then $\mathrm{Df}(\mathfrak{M}) = \mathfrak{M}$ and $\mathrm{Df}(\mathfrak{N})$ are pointwise definable models of the same theory, and therefore easily seen to be isomorphic. So $\mathfrak{M}$ elementarily embeds into $\mathfrak{N}$.

(2)→(3): Since $\mathfrak{M} \models V = OD$, $\mathrm{Df}(\mathfrak{M})$ has the same first order theory as $\mathfrak{M}$ and so $\mathfrak{M}$ embeds into $\mathrm{Df}(\mathfrak{M})$. It follows that $\mathfrak{M}$ is pointwise definable (because $\mathrm{Df}(\mathfrak{M})$ is), and so $\mathfrak{M} = \mathrm{Df}(\mathfrak{M})$.
(3)→(1): This follows from the proof of Lemma 1.20. □

The equivalence of (1) and (3) tells us that $V = OD$ is exactly the first order content of pointwise definability: Any pointwise definable model satisfies $V = OD$, and any first-order property consistent with $V = OD$ can be enjoyed by a pointwise definable model.

**2.1. Paris models.** It is easy to see that in a model $\mathfrak{M} \models \mathrm{ZF}$, the inclusion $\mathrm{Df}(\mathfrak{M}) \subseteq OD^{\mathfrak{M}}$ is proper if and only if there is some ordinal in $M$ which is not definable without parameters. The extra strength of ordinal definability compared to parameter-free definability then lies exactly in the admission of these non-definable ordinals as parameters. Conversely, we call $\mathfrak{M}$ a *Paris model* if all of its ordinals are definable in $\mathfrak{M}$. Any pointwise definable model is obviously a Paris model, but the class of Paris models is richer. This is shown in the following Proposition:

PROPOSITION 1.24 (Enayat, [**12**]). *Every consistent extension $T$ of ZF has a Paris model.*

PROOF. Let $T$ be a consistent extension of ZF. For any formula $\varphi(x)$ in the language $\mathcal{L}_{\mathrm{ZF}}$ of set theory, consider the formula

$$\bar{\varphi}(x) \equiv \varphi(x) \to \exists y(\varphi(y) \wedge y \neq x)$$

which says that $x$ is not defined by the formula $\varphi$. Now consider the 1-type

$$p(x) = \{x \in ON\} \cup \{\bar{\varphi}(x) \mid \varphi \in \mathcal{L}_{\mathrm{ZF}}\}$$

Any realization of $p(x)$ in a model of ZF is a non-definable ordinal. Consequently, a model of ZF is Paris if and only if $p(x)$ is omitted in this model. By the omitting types theorem, we are thus done if we can show that the type $p(x)$ is non-isolated over ZF.
Assume towards a contradiction that some formula $\psi(x)$ isolates $p$, i.e. $\psi(x)$ is a satisfiable formula such that every witness to $\psi(x)$ is an undefinable ordinal. Now consider the formula

$$\psi_0(x) \equiv x \text{ is the least witness to } \psi(x)$$

Then $\psi_0(x)$ uniquely defines an ordinal $\alpha$. This definable ordinal $\alpha$ is a witness to $\psi(x)$, contradicting the choice of $\psi(x)$. □

In particular, it follows that there are Paris models of $\mathrm{ZF}+V \neq OD$, which therefore cannot be pointwise definable.
Just like a pointwise definable model has to be countable, a Paris model can contain only countably many ordinals[3]. It follows that being Paris

[3]If $\mathfrak{M}$ satisfies AC, $|\mathfrak{M}| = |ON^{\mathfrak{M}}|$, and so $\mathfrak{M}$ itself will be countable if $\mathfrak{M}$ is Paris.

is not first-order expressible. Note in this context that the above Proposition also shows that being Paris is first-order conservative over ZF. There is no guarantee that the models constructed in Proposition 1.24 are well-founded, and indeed they cannot be if, for example, $T$ contains the sentence $\neg\,\mathrm{Con}(\mathrm{ZF})$. However, we have the following proposition:

PROPOSITION 1.25. *Let $T$ be a consistent completion of ZF, and assume that $T$ has a well-founded model. Then any Paris model of $T$ is well-founded.*

PROOF. Assume to the contrary that the completion $T$ has an ill-founded Paris model $\mathfrak{M}$. Then there is an infinite descending sequence

$$\alpha_0 \ni \alpha_1 \ni \alpha_2 \ni \dots$$

of ordinals in $\mathfrak{M}$. Since $\mathfrak{M}$ is Paris, each $\alpha_n$ has a defining formula $\varphi_n$. So $\mathfrak{M}$ is a model of the sentence

$$\begin{aligned} &\exists! x \exists! y (\varphi_n(x) \wedge \varphi_{n+1}(y)) \\ \wedge \quad &\forall x \forall y (\varphi_n(x) \wedge \varphi_{n+1}(y) \rightarrow y \in x) \end{aligned}$$

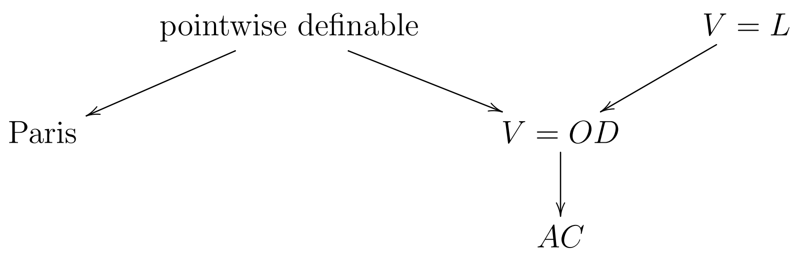
for every $n \in \omega$, and so the collection of these sentences is contained in $T$ (since $T$ is complete). But then clearly every model $\mathfrak{N}$ of $T$ must be ill-founded, since the witnesses to the $\varphi_n$'s form an infinite descending chain in $\mathfrak{N}$. □

Returning to older questions, we now see:

LEMMA 1.26. *Let $\mathfrak{M}$ be a model of ZF. Then* $\mathrm{Df}(\mathfrak{M})$ *is definable in* $\mathfrak{M}$ *iff* $\mathfrak{M}$ *is Paris.*

PROOF. If $\mathfrak{M}$ is Paris, then $\mathrm{Df}(\mathfrak{M}) = OD^{\mathfrak{M}}$, and so $\mathrm{Df}(\mathfrak{M})$ is definable. If $\mathfrak{M}$ fails to be Paris, i.e. if $\mathfrak{M}$ has undefinable ordinals, then $\mathrm{Df}(\mathfrak{M})$ cannot be definable because of the paradox of the least undefinable ordinal. □

To conclude this section, the diagram below shows the implications between the various concepts we have discussed. We will show later that none of the arrows is reversible (assuming Con(ZF)).



## 3. The inner model $HOD$

In this section, we work inside inside some fixed background universe $V \models \mathrm{ZF}$.
We have seen that there is a definable class $OD$ containing exactly those sets in $V$ which are definable from ordinal parameters. If $V \neq OD$, the first-order analysis of $OD$ is difficult because $OD$ then fails be transitive. This is because all the $V_\alpha, \alpha \in ON$ are ordinal definable, and thus $OD$ cannot be transitive unless $V = OD$.
To give a more concrete example for the trouble arising with the non-transitivity of $OD$, assume that not all reals in $V$ are ordinal definable (a situation which is easily seen to be consistent by later results). Then both $\mathbb{R}$ and $\mathbb{R} \cap OD$ are (ordinal) definable (the latter so because $OD$ is a definable class), but they differ only on sets which are not ordinal definable. So in particular, the Axiom of Extensionality can fail in $OD$.
One overcomes these difficulties by considering only ordinal definable sets having the property that also their members and their members' members etc. are ordinal definable. This is captured in the definition below. Here, $\mathrm{trcl}(x)$ denotes the *transitive closure* of $x$, i.e. the smallest transitive set containing all elements of $x$.

DEFINITION 1.27. $HOD = \{x \in OD \mid \mathrm{trcl}(x) \subseteq OD\}$

$HOD$ is the class of *hereditarily ordinal definable* sets. It is definable, transitive and contains all the ordinals.
Let us quickly note that

$$V = HOD \leftrightarrow V = OD$$

since if $V = OD$, then $OD$ is transitive and so $OD = HOD$. But the whole point of the introduction of $HOD$ is that it has nice first-order properties even in the case that $V \neq OD$.

The following easy result will be useful to check that certain sets are in $HOD$.

LEMMA 1.28. *$(x \in HOD \leftrightarrow x \in OD \wedge x \subseteq HOD)$.*

PROOF. By $\in$-induction. □

For sets of ordinals $x$, this becomes $(x \in HOD \leftrightarrow x \in OD)$. This will be used often.
Some of the properties of $OD$ discussed in the previous section easily translate to $HOD$:

COROLLARY 1.29. *There is a definable well-order $<_{HOD}$ of the class $HOD$.*

PROOF. Let $F : ON \to OD$ be the surjection defined in Lemma 1.9, and set

$$F'(\alpha) = \begin{cases} F(\alpha) & \text{if } F(\alpha) \in HOD \\ \emptyset & \text{else} \end{cases}$$

Then $F' : OD \to HOD$ is a surjection, and one can define a well-order $<_{HOD}$ on $HOD$ using $F'$ just as we did for $OD$ in Corollary 1.10. □

PROPOSITION 1.30. *All axioms of* ZF *are true in* $HOD$.

PROOF. $HOD$ satisfies Extensionality because it is a transitive class.
If $x, y \in HOD$ then clearly both $\{x, y\}$ and $\bigcup x$ are ordinal definable. But $\{x, y\} \subseteq HOD$ and $\bigcup x \subseteq HOD$ by assumption, so both are in fact hereditarily ordinal definable by Lemma 1.28. Hence Pairing and Union are true in $HOD$.
For Powerset, let $x \in HOD$ and let $y = \mathcal{P}(x) \cap HOD$. Then $y$ is ordinal definable and all of its elements lie in $HOD$. So $y \in HOD$, and $y = \mathcal{P}(x)^{HOD}$ by absoluteness.
For Replacement, assume that $u, v \in HOD$ and $\varphi(x, y, z)$ is a formula such that

$$\forall x \in u \exists! y \varphi(x, y, v)$$

is true in $HOD$. In other words,

$$\forall x \in u \exists! y (y \in HOD \wedge \varphi(x, y, v)^{HOD})$$

is true in $V$, and so by Replacement and Comprehension in $V$ there is a set $w$ such that

$$y \in w \leftrightarrow \exists x \in u (y \in HOD \wedge \varphi(x, y, v)^{HOD})$$

This $w$ is definable by the formula above using parameters $u, v \in HOD$, and so $w \in OD$, and clearly also $w \subseteq HOD$. So $w \in HOD$, and

$$\forall x \in u \exists y \in w \varphi(x, y, v)$$

holds in $HOD$. □

The proof that all ZF axioms are true in $HOD$ becomes almost a triviality if one uses the following fact:

FACT 1.31. *A transitive class $M$ is an inner model of* ZF *iff $M$ is closed under Gödel operations and $M$ is* almost universal*, i.e. for every set $x \subseteq M$ there is an $y \in M$ such that $x \subseteq y$.*

Now Gödel operations are absolutely definable functions, and so $OD$ is obviously closed under them. In fact, one can check the defining clauses of the Gödel operations to see that they map $HOD$-sets to $HOD$.
To check that $HOD$ is almost universal, let $x \subseteq HOD$ and pick an ordinal $\alpha$ such that $x \subseteq V_\alpha$. Then $V_\alpha \cap HOD$ is an element of $HOD$ containing $x$ as a subset.

PROPOSITION 1.32. AC *is true in* $HOD$.

PROOF. Let $x \in HOD$. Then $<_{HOD} \cap (x \times x)$ is hereditarily ordinal definable, since $<_{HOD}$ is a definable class and $x \in HOD$. So $<_{HOD} \cap (x \times x)$ is a well-order of $x$ in $HOD$. $\square$

We therefore arrive (again) at:

PROPOSITION 1.33. Con(ZF) $\rightarrow$ Con(ZFC)

PROOF. By the general observation that inner models can be used to yield relative consistency results. Namely, if ZFC $\vdash \perp$ then ZF $\vdash \perp^{HOD}$ by Proposition 1.32, but ZF $\vdash (\perp^{HOD} \leftrightarrow \perp)$. So ZF $\vdash \perp$. $\square$

As promised before, this result needs neither results on $L$ nor the consistency of $V = OD$.

REMARK 1.34. We have seen that $HOD \models$ AC, and so it is completely determined by its sets of ordinals. Now these are exactly the ordinal definable sets of ordinals (since for sets $x$ of ordinals, being ordinal definable and being hereditarily ordinal definable is the same). In this sense, $HOD$ is canonically related to the class $OD$: It is the unique inner model of ZF whose sets of ordinals are exactly the ordinal definable sets of ordinals.

The ease with which the various set existence axioms are proved in $HOD$ is based on the fact that the concept of ordinal definability refers to definability in $V$, and not to definability in (some initial segment of) $HOD$, as it is the case with the $L$ hierarchy. So there is never a need to relativize complex logical expressions to $HOD$.
It is by this reference to $V$-definability that $HOD$ (and likewise, $OD$) strongly depends on the surrounding universe $V$. So if $V, W$ are two models of set theory, there is no a priori reason to believe that $HOD^V = HOD^W$, even if $V \subseteq W$ and both have the same ordinals. In particular,

FACT 1.35. *It is consistent (relative to* ZF*) that* $HOD^{HOD} \subsetneq HOD$.

A proof will be given later. In the situation described in Fact 1.35, $HOD$ fails to satisfy the statement $V = OD$. The intuition here is that that although $HOD$ consists of ordinal definable sets, it does not necessarily see why they are ordinal definable. So moving from $V$ to $HOD$, some sets might lose their definability properties.[4]

[4]One can also arrange that $HOD^{HOD^{HOD}} \subsetneq HOD^{HOD} \subsetneq HOD$, and in fact it is possible that there is even an Ord-length sequence of nested $HOD$'s (taking intersections at limit points) which never stabilizes. Moreover, every model of ZFC can be obtained by first moving to a generic extension, and then applying the $HOD$ operator transfinitely many times. These and other funny results have been proven by Jech, McAloon and Zadrozny in the 70's and early 80's.

Definition 1.36. Let $M$ be an inner model[5] of ZF. The ZF-*theory of* $M$, denoted by $\mathrm{Th}_{\mathrm{ZF}}(M)$, is the set of all $\varphi \in \mathcal{L}_{\mathrm{ZF}}$ such that $\mathrm{ZF} \vdash \varphi^M$.

Speaking model-theoretically, $\mathrm{Th}_{\mathrm{ZF}}(M)$ consists of all sentences which are true in "the $M$ of" every model of ZF. For example, Proposition 1.32 says that $\mathrm{AC} \in \mathrm{Th}_{\mathrm{ZF}}(HOD)$.
Again, a distinction to the inner model $L$ arises:
$\mathrm{Th}_{\mathrm{ZF}}(L)$ is really just (the deductive closure of) the theory $ZF+V = L$. This is because $V = L$ is true in $L$ by the absoluteness of the $L$-hierarchy, and so consequences of $V = L$ hold in the $L$ of every model. In other words, some model $\mathfrak{M}$ of set theory is "the $L$ of some universe" if and only if $\mathfrak{M} \models V = L$.
With $HOD$, things are different. Although every $\varphi \in \mathrm{Th}_{\mathrm{ZF}}(HOD)$ certainly holds in all models of $\mathrm{ZF} + V = HOD$, saying that some structure $\mathfrak{M}$ models $\mathrm{ZF} + V = HOD$ is in general *stronger* than saying that $\mathfrak{M}$ arises as the $HOD$ of some model of ZF (see Fact 1.35). So there are really two different concepts to consider.
In section 3.2, we will see that in fact *every* model of ZFC (note the AC!) arises as the $HOD$ of some model of ZF, and so $\mathrm{Th}_{\mathrm{ZF}}(HOD)$ is simply the deductive closure of ZFC.
Furthermore, we present a partial result on the implications of $V = HOD$ in section 3.1.

We conclude this section with three more characterisations of $HOD$.

Proposition 1.37. *$HOD$ is the largest transitive class (and therefore the largest inner model) which has a definable well-order.*

Proof. Let $M \subseteq V$ be a transitive class with a definable well-order. Then $M \subseteq OD$. By transitivity of $M$, $M \subseteq HOD$. □

Proposition 1.38. *$HOD = L[A]$ for a $V$-definable class $A \subseteq ON$*

Proof. Let $\{x_\alpha \mid \alpha \in ON\}$ be the definable enumeration of $HOD$ induced by its well-order. Let $\Gamma$ be the canonical pairing function of ordinals and set $A = \{\Gamma(\alpha, \beta) \mid \alpha \in x_\beta\}$.
Since $A$ is definable, $L[A] \subseteq HOD$. Conversely, let $x \in HOD$. We may assume that $x$ is a set of ordinals, since $HOD \models AC$ and so every set is coded into a set of ordinals. Pick $\beta \in ON$ such that $x = x_\beta$. Using Replacement, we can find a limit ordinal $\gamma > \beta$ such that $L_\gamma$ is closed under $\Gamma$ and for all $\alpha \in ON$, $\alpha \in x_\beta \to \alpha < \gamma$.
Then $x = \{\xi < \gamma \mid \Gamma(\xi, \beta) \in A \cap \gamma\}$ is definable in $(L_\gamma, A \cap \gamma)$, and therefore $x \in L[A]$. □

[5]We treat inner models as syntactical objects. So $M$ is really a formula such that the class defined by $M$ is transitive, contains all the ordinals and satisfies the axioms of ZF.

PROPOSITION 1.39. *$V = OD$ iff $V = L[A]$ for some $V$-definable class $A \subseteq ON$*

PROOF. If $V = OD$, then $V = L[A]$ for a definable $A \subseteq ON$ by the previous proposition. Assume conversely that $V = L[A]$ for a definable $A$. Then we can define a well-order of $V$ in the same way as one does for $L$, and so $V = OD$ follows. □

The following curious characterisation of $HOD$ is taken from the original paper [**1**] on ordinal definability. By $L^1$ we denote the version of the constructible hierarchy where first-order definability is replaced by second-order definability, i.e. $L^1 = \bigcup_{\alpha \in ON} L^1_\alpha$ where

$$\begin{aligned} L^1_0 &= \emptyset \\ L^1_{\alpha+1} &= \{x \subseteq L^1_\alpha \mid x \text{ is second-order definable over } L^1_\alpha\} \\ L^1_\gamma &= \bigcup_{\alpha<\gamma} L^1_\alpha \text{ for limit } \gamma \end{aligned}$$

In the definition of $L^1_{\alpha+1}$, the second-order quantifiers range over subsets of $L^1_\alpha$ *in* $V$. So there is no reason to expect that the $L^1$ hierarchy is absolute. As in the definition of $L$, we allow the defining formulas to contain parameters from $L^1_\alpha$.

PROPOSITION 1.40 (Myhill-Scott). *$AC \rightarrow L^1 = HOD$*

PROOF. The proof for $L^1 \subseteq HOD$ is similar to the proof that $L \subseteq HOD$.
Conversely, let $A \in OD$ be a set of ordinals, and assume that $A$ is definable in some stage $V_\alpha$ from a formula $\varphi_A(x)$, i.e.

$$x \in A \leftrightarrow (V_\alpha, \in) \models \varphi_A(x)$$

We may assume[6] that $\alpha$ is a limit. Using AC, let $\kappa = |V_\alpha|$. We claim that $A$ is second-order definable in any $L^1_\theta$ which contains $\kappa$ as an element. First note that any bijection $F : V_\alpha \rightarrow \kappa$ induces a relation $E_0$ on $\kappa$ such that $(\kappa, E_0) \cong (V_\alpha, \in)$. We now want to define $A$ in $L^1_\theta$ by saying that

$$\xi \in A \leftrightarrow \varphi_A \text{ holds in } (\kappa, E_0) \text{ for the element corresponding to } \xi$$

However, this uses a second-order parameter $E_0$. So instead we will use the formula

(*)
$$\begin{aligned} \xi \in A \leftrightarrow \exists E\, ((\kappa, E) \cong (V_\alpha, \in) \wedge \\ \varphi_A \text{ holds in } (\kappa, E) \text{ for the element corresponding to } \xi) \end{aligned}$$

The rest of the proof is devoted to showing that (*) can be transformed to a second-order statement over $L^1_\theta$.

[6]See Remark 1.8.

In the following, we use uppercase letters for second order variables throughout.

We first show that for a second-order variable $E$, the statement

$$(\kappa, E) \cong (V_\alpha, \in)$$

is definable by a second-order formula $\exists R\, \Gamma(E, R, \kappa, \alpha)$. The additional second order variable $R$ will describe a rank function on $(\kappa, E)$. All first-order quantifiers in $\Gamma(E, R)$ (we suppress the parameters $\alpha, \kappa$ from now on) will be bounded to $\kappa$, $E$ or $R$, and so basic absoluteness arguments will show that $\Gamma(E, R)$ works inside $L^1_\theta$ as expected. The formula $\Gamma(E, R)$ is given as

$E$ is an extensional binary relation on $\kappa$ with minimal element 0

$\wedge$ $R$ is a function from $\kappa$ to $\alpha$

$\wedge$ $\Phi(R, E)$

where $\Phi(R, E)$ is some yet to be determined formula which says that $R$ is a rank function for $(\kappa, E)$ making this structure isomorphic to $(V_\alpha, \in)$. To define $\Phi(R, E)$, recall that non-zero ranks are always successor ordinals. For any $x \in \kappa$, denote by $pred_E(x)$ the definable class $\{y \in \kappa \mid E(y, x)\}$. We now imitate the recursive definition of the $V_\alpha$ hierarchy for the structure $(\kappa, E)$:

$$\begin{aligned}\Phi(R, E) \equiv\ & \forall x \in \kappa\, R(x) = (\bigcup\{R(y) \mid E(y, x)\}) + 1 \\ & \wedge\quad \forall \beta < \alpha \forall Y \subseteq \kappa (\forall y \in Y (R(y) \leq \beta) \rightarrow \exists x \in \kappa (Y = pred_E(x)))\end{aligned}$$

$\Phi(R, E)$ says that every node in $(\kappa, E)$ has as predecessors elements of lower $R$-rank, and for every collection $Y$ of elements of bounded $R$-rank there is a node having exactly the elements of $Y$ as predecessors. The essential point of the proof is that second-order logic allows as to quantify over *all* subsets $Y \subseteq \kappa$ (and not only over those in $L^1_\theta$), thereby guaranteeing that the structure $(\kappa, E)$ is as rich as $(V_\alpha, \in)$.

So assume $\Gamma(E_0, R_0)$ holds for some $E_0, R_0 \subseteq L^1_\theta$ and let $\pi$ denote the Mostowski collapse on $E_0$. Clearly $\pi : (\kappa, E_0) \cong (V_\alpha, \in)$. Moreover $\pi$ preserves ranks: $R_0(x) = \xi$ iff $\mathrm{rk}(\pi(x)) = \xi$ (where rk denotes the $\in$-rank). In particular, if $(E_1, R_1)$ is another pair such that $\Gamma(E_1, R_1)$ then there is an isomorphism $\iota : (\kappa, E_1) \cong (\kappa, E_2)$ and for all $x \in \kappa$, $R_0(x) = R_1(\iota(x))$.

Let $ord(x)$ be any first-order formula defining the ordinals in $(V_\alpha, \in)$, and define a relation $=_E$ on $\kappa \times \alpha$ via

$$x =_E \xi :\leftrightarrow R(x) = \xi \wedge (\kappa, E) \models ord(x)$$

$=_E$ is $(L^1_\theta, E, R)$-definable end expresses that the element $x$ in the structure $(\kappa, E)$ corresponds to the ordinal $\xi$ in $(V_\alpha, \in)$: If $\Gamma(E_0, R_0)$ and $L^1_\theta \models (x =_E \xi)$, then $R_0(x) = \mathrm{rk}(\pi(x)) = \xi$ and $V_\alpha \models ord(\pi(x))$, and

thus $\pi(x) = \xi$.
Furthermore, for $\xi \in \alpha$ and any formula $\varphi(x)$ we define

$$(\kappa, E) \models_{\text{ord}} \varphi(\xi) :\leftrightarrow \exists x \in \kappa(x =_E \xi \wedge (\kappa, E) \models \varphi(x))$$

$(\kappa, E) \models_{\text{ord}} \varphi(\xi)$ says that $\varphi$ holds in $(\kappa, E)$ for the element corresponding to the ordinal $\xi$.
The relation $\models_{\text{ord}}$ is independent of the choice of $E$: That is, if $\Gamma(E_0, R_0)$ and $\Gamma(E_1, R_1)$, then $(\kappa, E_0) \models_{\text{ord}} \varphi(\xi)$ iff $(\kappa, E_1) \models_{\text{ord}} \varphi(\xi)$. To see this, assume that $(\kappa, E_0) \models_{\text{ord}} \varphi(\xi)$ and pick an $x \in \kappa$ such that $R_0(x) = \xi$ and $(\kappa, E_0) \models \mathit{ord}(x) \wedge \varphi(x)$. Let $\iota : (\kappa, E_0) \cong (\kappa, E_1)$ be an isomorphism. Then $R_1(\iota(x)) = \xi$ and $(\kappa, E_1) \models \mathit{ord}(\iota(x)) \wedge \varphi(\iota(x))$, and so $(\kappa, E_1) \models_{\text{ord}} \varphi(\xi)$.
Finally, we can define $A$ in $L^1_\theta$ by saying

$$\xi \in A \leftrightarrow \exists E \exists R(\Gamma(E, R) \wedge (\kappa, E) \models_{\text{ord}} \varphi_A(\xi))$$

This completes the proof.

□

Remark 1.41. Without $AC$, $HOD$ can be strictly larger than $L^1$. This is shown in [**11**].

## 4. Forcing and ordinal definability

We now investigate the relation between ordinal definability in the ground model and in some generic extension[7].
First, an easy observation shows us that being (not) ordinal definable is not an intrinsic property of a set.

Lemma 1.42. *Let $x \in V$ be a set. Then there is a generic extension $V[G] \supseteq V$ such that $x$ is ordinal definable in $V[G]$.*

Proof. It suffices to show this when $x$ is a set of ordinals. We may further assume that the GCH holds up to a sufficiently large cardinal, since this can be forced.
Pick $x \subseteq \lambda$, and let $(\kappa_\alpha)_{\alpha<\lambda}$ be an increasing enumeration of $\lambda$-many infinite regular cardinals. For each $\alpha < \lambda$, let $C_\alpha$ be the usual partial order which forces $2^{\kappa_\alpha} = \kappa_\alpha^{++}$. We now let $P$ be the Easton-support product of $(P_\alpha)_{\alpha<\lambda}$, where $P_\alpha = C_\alpha$ if $\alpha \in x$ and $P_\alpha = \{\emptyset\}$ otherwise.
Let $G$ be $P$-generic over $V$. By Easton's theorem, $V[G]$ has the same cofinalities and cardinals as $V$, and furthermore for each $\alpha < \lambda$,

$$(2^{\kappa_\alpha})^{V[G]} = \begin{cases} \kappa_\alpha^{++} & \text{if } \alpha \in x \\ \kappa_\alpha^{+} & \text{if } \alpha \notin x \end{cases}$$

[7]Unless noted otherwise, *forcing* and *generic* always mean *set-forcing* and *set-generic*.

This obviously makes $x$ definable from the sequence $(\kappa_\alpha)_{\alpha<\lambda}$ in $V[G]$. Now one just has to chose $(\kappa_\alpha)_{\alpha<\lambda}$ sufficiently definable; for example, one can take

$$(\kappa_\alpha)_{\alpha<\lambda} = \text{ the first } \lambda\text{-many infinite regular cardinals}$$

which is definable from $\lambda$ (both in $V$ and $V[G]$, by cofinality preservation) and therefore makes $x$ ordinal definable in the generic extension. $\square$

Remark 1.43. We say that in the situation of the above proof, $x$ *gets coded into the continuum pattern.* This technique will be used for proving various results, and we will omit the details from now on.
If one wants to make certain sets ordinal definable without messing up the continuum pattern, different techniques have to be used. One of them is the $\diamond^*$-coding described in [**14**].

Remark 1.44. If one can arrange $(\kappa_\alpha)_{\alpha<\lambda}$ to be definable without parameters in $V[G]$, then so will be $x$. For example, $x$ is a real number, one can code $x$ into the continuum pattern from $\aleph_0$ up to the definable cardinal $\aleph_\omega$, and this will make $x$ definable without parameters in $V[G]$.

Remark 1.45. By a slightly different method, we can make an arbitrary set $x \subseteq \kappa$ of ordinals definable in $V[G]$. Assume that the characteristic function of $x$ has the course of values

$$a_0 a_1 \dots a_\xi a_{\xi+1} \dots$$

where $a_\xi \in \{0, 1\}$ for all $\xi < \kappa$. Then one can code the sequence

$$a_0 a_0 a_1 a_1 \dots a_\xi a_\xi a_{\xi+1} a_{\xi+1} \dots 01$$

into the GCH pattern starting at $\aleph_0$. Then the place where the coding of the $a_\xi$'s ends can be defined in $V[G]$ as the smallest even ordinal $\alpha$ such that the $GCH$ holds at $\aleph_\alpha$ but not at $\aleph_{\alpha+1}$. So the sequence

$$a_0 a_0 a_1 a_1 \dots a_\xi a_\xi a_{\xi+1} a_{\xi+1} \dots 01$$

is definable in $V[G]$ without parameters, and it follows that $x$ is definable in $V[G]$ without parameters.

Remark 1.46. Clearly, we can modify this method further to make any finite list

$$x_0, x_1, \dots, x_n \in V$$

of sets definable in a generic extension $V[G]$. It is not clear how to extend this into the transfinite, and indeed there is a natural limit to this coding: We clearly cannot make uncountably many (as viewed externally) $x \in V$ definable in $V[G]$. If $V$ itself is countable, it *is* possible to make all $x \in V$ definable in a generic extension $V[G]$; see Proposition 2.12.
Turning to the case of ordinal definability, we will prove that a "generic iteration" of the coding method does produce an extension $V[G]$ in which all sets are ordinal definable; this is Theorem 2.2.

Our second main technique for proving independence results about ordinal definability uses the concept of *weak homogenity.*

Definition 1.47. A partial order $P$ is called *weakly homogeneous* if for any $p, q \in P$ there is an automorphism $i$ of $P$ such that $i(p)$ is compatible with $q$.

The following result about weakly homogeneous forcings is well-known.

Lemma 1.48. *Let $P$ be a weakly homogeneous forcing, and let $G$ be $V$-generic over $P$. Then for all statements $\varphi$ of the forcing language which contain only check names, $\exists p \in P(p \Vdash \varphi)$ iff $1 \Vdash \varphi$.*

Proof. Assume $p \Vdash \varphi$ for some $p \in P$, and let $q$ be any other condition. Pick an automorphism $i$ of $P$ such that $i(p)$ is compatible with $q$. Let $r \leq i(p), q$. Since $\varphi$ contains only check names, $i(p) \Vdash \varphi$, and so $r \Vdash \varphi$. Since $q$ was arbitrary, it follows that the collection $\{s \in P \mid s \Vdash \varphi\}$ is dense, and so $1 \Vdash \varphi$. □

This means in particular that the first-order theory of a generic extension by $P$ is independent of the choice of the generic and definable in $V$.

Proposition 1.49. *Let $P$ be an ordinal definable and weakly homogenous forcing and let $G$ be a $V$-generic filter over $P$. Then if $x \in OD^{V[G]}$ and $x \subseteq V$, it follows that $x \in OD^V$. In particular, $HOD^{V[G]} \subseteq HOD^V$.*

Proof. Let $x \subseteq V$ be ordinal definable in $V[G]$. Choose a formula $\varphi(v, w)$ and ordinals $\alpha, \xi$ such that $x = \{y \in (V_\alpha)^V \mid V[G] \models \varphi(y, \xi)\}$. Let $\dot{x}$ be a name for $x$.
For all $y \in (V_\alpha)^V$ we have

$$y \in x \Leftrightarrow V[G] \models \varphi(y, \xi) \Leftrightarrow \exists p \in G(p \Vdash \varphi(\check{y}, \check{\xi})) \Leftrightarrow 1 \Vdash \varphi(\check{y}, \check{\xi})$$

where the last equivalence is by Lemma 1.48, since $\varphi(\check{y}, \check{\xi})$ contains only ground model parameters. Therefore

$$x = \{y \in V_\alpha \mid 1 \Vdash \varphi(\check{y}, \check{\xi})\}$$

Note that the forcing relation $\Vdash$ for the formula $\varphi$ is ordinal definable in $V$ since $P$ is. Thus it follows from this representation that $x$ is ordinal definable in $V$.
$HOD^{V[G]}$ has consequently no new sets of ordinals, and since $HOD^{V[G]} \models$ AC, this suffices to show that $HOD^{V[G]} \subseteq HOD^V$. □

Lemma 1.50. *Let $P$ be an ordinal definable and weakly homogenous forcing and let $G$ be a $V$-generic filter over $P$. Then $HOD^{V[G]}$ is a $V$-definable class. In particular, $HOD^{V[G]}$ does not depend on $G$.*

Proof.

$$x \in HOD^{V[G]} \leftrightarrow x \in V \wedge 1 \Vdash_P \check{x} \in HOD$$

□

Remark 1.51. Note that the forcing $P$ from Lemma 1.42 which makes $x$ definable is a product of weakly homogeneous forcings, and therefore itself weakly homogeneous. This is consistent with Lemma 1.49, since $P$ is defined from $x$, and so usually $P \notin OD$. In fact, $P \in OD \leftrightarrow x \in OD$.

Remark 1.52. It follows immediately that one cannot force $V = OD$ by an ordinal definable weakly homogeneous forcing (unless $V = OD$ already holds in the ground model).

It is now easy to show from Proposition 1.49 that $V \neq HOD$ is consistent:

Proposition 1.53. $\mathrm{Con}(\mathrm{ZF}) \to \mathrm{Con}(\mathrm{ZF} + V \neq HOD)$

Proof. Let $P$ be any nontrivial definable forcing which is weakly homogeneous, for example Cohen forcing. Then if $G$ is $V$-generic over $P$,

$$HOD^{V[G]} \subseteq HOD^V \subseteq V \subsetneq V[G]$$

and therefore $V[G] \models (V \neq HOD)$. □

Remark 1.54. Assume $V = L$, and let $P$ be an ordinal definable weakly homogeneous forcing. Then $HOD^{V[G]} = HOD^V$, since

$$L = L^{V[G]} \subseteq HOD^{V[G]} \subseteq HOD^V = L$$

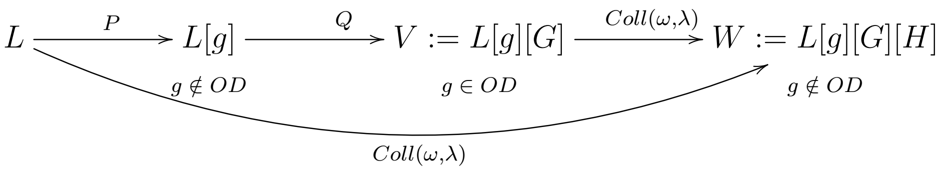
We now combine the continuum coding technique and weak homogenity to give an example where the $HOD$ of a generic extension is *properly* contained in the $HOD$ of the ground model.

Lemma 1.55. *There is a model $V$ and a generic extension $W \supseteq V$ such that $HOD^W \subsetneq HOD^V$.*

Proof. Let $P$ be Cohen forcing and $g$ a real which is $P$-generic over $L$. Of course, $g \notin L$. Now in $L[g]$, let $Q$ be the Easton poset which codes $g$ into the continuum pattern below $\aleph_\omega$. Let $G$ be $Q$-generic over $L[g]$. Then $g \in HOD^{L[g][G]}$ . Finally, let $H$ be generic for $\mathrm{Coll}(\omega, \lambda)$ where $\lambda$ is some cardinal larger than $|P * Q|$. We claim that $g \notin HOD^{L[g][G][H]}$. To see this, note that $P * Q * \mathrm{Coll}(\omega, \lambda) \cong \mathrm{Coll}(\omega, \lambda)$ by a well-known result of Levy, since the former is a forcing of size $\lambda$ which collapses $\lambda$. Hence there is a $\mathrm{Coll}(\omega, \lambda)$-generic filter $J$ which does the same as $g * G * H$. But $\mathrm{Coll}(\omega, \lambda)$ is weakly homogeneous, and so

$$L = HOD^{L[J]} = HOD^{L[g][G][H]}$$

Hence $g \notin HOD^{L[g][G][H]}$, but $g \in HOD^{L[g][G]}$.




□

We may isolate an idea of the previous proof to get the following:

Lemma 1.56. *Assume $x \in HOD$ and $x$ remains hereditarily ordinal definable in any extension by forcings of the form $Coll(\omega, \alpha)$. Then $x$ remains hereditarily ordinal definable in any set generic extension.*

Proof. Assume $x$ is as in the assumption, and let $V[G]$ be a generic extension of $V$ by the forcing poset $P$. Pick an $\alpha > |P|$. Then $P * Coll(\omega, \alpha) \cong Coll(\omega, \alpha)$. Choose a $V[G]$-generic $H \subseteq Coll(\omega, \alpha)$ and a $V$-generic $H' \subseteq Coll(\omega, \alpha)$ such that $V[G][H] = V[H']$. By assumption, $x \in HOD^{V[H']} = HOD^{V[G][H]}$. Now by the weak homogenity of $Coll(\omega, \alpha)$, $x \in HOD^{V[G]}$. □

In [**8**], it is asked which $x$ remain hereditarily ordinal definable in *every* set-generic extension. The collection of all such elements is there called $gHOD$, the *generic HOD*. $gHOD$ is a definable class, since

$$x \in gHOD \leftrightarrow \forall P \forall p \in P(p \Vdash_P \check{x} \in HOD)$$

Proposition 1.57 (Fuchs-Haminks-Reitz). *$gHOD$ is an inner model of ZFC.*

Proof. Denote by $H^\alpha$ the $HOD$ of $V[G]$, where $G$ is any $V$-generic filter on $Coll(\omega, \alpha)$. This is well defined because of Lemma 1.50. Furthermore, the above equivalence shows that $H^\alpha$ is uniformly definable in the parameter $\alpha$. If $\alpha < \beta$, then $Coll(\omega, \alpha) \times \mathrm{Coll}(\omega, \beta) \cong \mathrm{Coll}(\omega, \beta)$, and so $H^\beta = (H^\beta)^{V[G]}$ whenever $G$ is $V$-generic over $Coll(\omega, \alpha)$. Since $Coll(\omega, \beta)$ is weakly homogeneous, it follows that $H^\beta \subseteq H^\alpha$. So $(H^\alpha)_{\alpha \in ON}$ is a descending sequence of models of ZFC.
By Lemma 1.56, $gHOD = \bigcap\limits_\alpha H^\alpha$. $gHOD$ is therefore transitive and contains all the ordinals.
We now show that $gHOD$ is almost universal. So let $x \subseteq gHOD$ and pick an $\xi$ such that $x \subseteq V_\xi$. We are done if we can show that $V_\xi \cap gHOD \in gHOD$. Since $V_\xi$ is a set, we can use Replacement to find an $\alpha_0$ such that $V_\xi \cap gHOD = V_\xi \cap H^{\alpha_0}$. Now if $\alpha \geq \alpha_0$, then $V_\xi \cap H^{\alpha_0} = V_\xi \cap H^\alpha \in H^\alpha$. If $\alpha < \alpha_0$, then $V_\xi \cap H^{\alpha_0} \subseteq V_\xi \cap H^\alpha \in H^\alpha$ by the monotonicity of the sequence $(H^\alpha)_\alpha$. This together shows that $V_\xi \cap H^{\alpha_0} \cap gHOD$.
We have thus shown that $gHOD \models$ ZF. For AC, Let $x \in gHOD$ and let $U$ be the collection of all well-orders on $x$. Since $U$ is a

set, we can again use Replacement to find an $\alpha \in ON$ such that $U \cap gHOD = U \cap H^\alpha$. So $U \cap gHOD \neq \emptyset$, since $H^\alpha \models$ ZFC. $\square$

CHAPTER 2

# The advanced theory

## 1. $V = OD$ by forcing

We have already seen that $V = OD$ holds in $L$ and in all models of the form $L[A]$ where $A$ is a definable class of ordinals. In this section, we show how one can start with an arbitrary model $V$ and get an extension $W \supseteq V$ satisfying $V = OD$ by the method of forcing. There is some flexibility in this approach which allows us to code parts of $V$ into the extension $W$. For example, if $V$ contains a measurable cardinal $\kappa$ then $W$ can be chosen such that $\kappa$ remains measurable in $W$. This yields the relative consistency of "$V = OD + \exists\kappa$ measurable".
Lemma 1.42 showed how one makes a single set ordinal definable. One imaginable way of forcing $V = OD$ is to use this method in an iterated forcing construction, using a bookkepping function which will eventually list all sets. Of course, the iteration will add new sets, and the bookkepping would have to take care of these new sets as well.
It turns out that such a rather complicated bookkeeping is not needed: One can basically take any $ON$-length iteration of a forcing which potentially codes sets (say, into the continuum pattern), and then use a genericity argument to show that indeed all sets get coded. This is the approach we will follow here.

DEFINITION 2.1. Let $(P, \leq_P), (Q, \leq Q)$ be forcing posets. The *direct sum* $(P \oplus Q, \leq_{P\oplus Q})$ of $P$ and $Q$ is the forcing given by the following data:

- The underlying set of $P \oplus Q$ is the disjoint union of $P$ and $Q$ together with a new element 1
- $\leq_{P\oplus Q} = \leq_P \cup \leq_Q \cup \{(x, 1) \mid x \in P \cup Q\}$

In other words, elements from the $P$-part and from the $Q$-part of $P \oplus Q$ are incompatible, and within $P$ and $Q$ everything stays as before. If $G \subseteq P \oplus Q$ is a generic filter, then either $G \cap P \neq \emptyset$ or $G \cap Q \neq \emptyset$. In the first case, $G^* := G \setminus \{1\} \subseteq P$ and $G^*$ is $P$-generic. In the second case, $G^* \subseteq Q$ and $G^*$ is $Q$-generic.
The direct sum of two forcings is usually not weakly homogeneous. For example, work in $L$ and consider the definable forcing $P \oplus Q$ where $P = \{1_P\}$ is trivial, and $Q = \mathrm{Add}(\omega, \aleph_2)$. Let $q \in Q$ be arbitrary. Then $q \Vdash_{P\oplus Q} 2^{\aleph_0} = \aleph_2$, while $1_P \Vdash_{P\oplus Q} 2^{\aleph_0} = \aleph_1$. So the first-order theory of $L[G]$ depends on the choice of $G$, and thus $P \oplus Q$ cannot be weakly homogeneous by Lemma 1.48.

PROPOSITION 2.2. *Let $M$ be a countable transitive model of ZFC. For any cardinal $\eta$ in $M$, there is a class-generic extension $M[G] \supseteq M$ such that $H(\eta)^{M[G]} = H(\eta)^M$ and $M[G] \models (V = OD)$.*

PROOF. Let $\lambda = \eta^+$. We may assume that the GCH holds above $\lambda$ since this can be forced without changing $V_\lambda$. Now for each regular $\kappa > \lambda$, let $Q_\kappa$ denote the partial order which forces $2^\kappa = \kappa^{++}$, and let $T$ denote some trivial forcing. Our forcing $P$ is the Easton iteration of $Q_\kappa \oplus T$ for all regular $\kappa > \lambda$.
So let $G \subseteq P$ be $M$-generic. First, since $P$ is $\lambda^+$-closed, $V_\lambda^{M[G]} = V_\lambda$. Since $H(\eta)$ is definable from $\eta$ inside $V_\lambda$ (in any universe), it follows that $H(\eta)$ is preserved as well.
We have to show that each set in $M[G]$ is ordinal definable. Since $M \models$ AC, it suffices to show this for some set of ordinals $x \in M[G]$. Given such, pick a regular $\kappa \supseteq x$. Since $P^{\geq\kappa}$ is $\kappa^+$-closed, $x$ must have been added in the intermediate extension $M[G \restriction P^{<\kappa}]$. Now arguing in $M[G \restriction P^{<\kappa}]$, we claim that it is dense that $x$ gets coded into the continuum function above $\kappa$ by the tail forcing $P^{\geq\kappa}$. To see this, let $p \in P^{\geq\kappa}$ be arbitrary and let $\alpha = \sup(\mathrm{dom}(p))$. Now $p$ can be extended to a condition $\bar{p}$ with support in $\alpha + \kappa$ by setting $\bar{p} \restriction \alpha = p$ and $\bar{p}(\alpha+\xi) \in Q_{\alpha+\xi}$ arbitrary if $\xi \in x$ and $\bar{p}(\alpha+\xi) \in T$ if $\xi \notin x$. Then $\bar{p}$ forces that the continuum function codes $x$.
By genericity, $G$ picks such a condition $\bar{p}$, and so in the final model $M[G]$, $x$ can be read off from the continuum function starting at some ordinal $\gamma > \kappa$. Hence $x$ is definable from $\gamma$. □

COROLLARY 2.3. *If* $\mathrm{ZFC} + \exists\kappa$ measurable *is consistent, then so is* $\mathrm{ZFC} + V = OD + \exists\kappa$ measurable.

PROOF. Let $\kappa$ be measurable in $M$ and let $\lambda = (2^\kappa)^+$. Now apply the above theorem to get $M[G] \supseteq M$ satisfying $V = OD$ and $H(\lambda)^{M[G]} = H(\lambda)^M$. Because of the latter, $M[G]$ has no new $< \kappa$-sequences of subsets of $\kappa$, and so any $\kappa$-complete ultrafilter on $\kappa$ in $M$ remains $< \kappa$-complete in $M[G]$. □

The only thing we needed to know for this corollary was that the measurability of $\kappa$ is absolute for any sufficiently large $H(\lambda)$. So there is a more general phenomenon behind this, which we will discuss next. We start with an observation of Azriel Lévy.

LEMMA 2.4 (Lévy). *Assume AC. For every uncountable $\lambda$, $H(\lambda) \preceq_1 V$.*

PROOF. Assume $\exists x\varphi(x, a)$ holds for some $\Delta_0$-formula $\varphi$ and $a \in H(\lambda)$. By $\Delta_0$-absoluteness, it suffices to show that there is an $u \in H(\lambda)$ such that $\varphi(u, a)$ holds.
By the reflection principle, we can find a $V_\alpha \preceq_1 V$ such that $\mathrm{trcl}(\{a\}) \subseteq V_\alpha$ (choose $V_\alpha$ such that it reflects the formula defining $\Sigma_1$-truth). Now, using $AC$, construct an elementary submodel $M \preceq V_\alpha$ of size $< \lambda$ which

contains trcl($\{a\}$). This is possible because $\lambda > \omega$ and $|\,\mathrm{trcl}(\{a\})| < \lambda$ by assumption. It follows that $M \preceq_1 V$. Let $M'$ be the Mostowski collapse of $M$. $M'$ is transitive and of size $< \lambda$ and therefore $M' \in H(\lambda)$. Furthermore $a$ was not collapsed since trcl(($\{a\}$) $\subseteq M$. So $a \in M'$ and $M' \preceq_1 V$, hence there is an $u \in M' \subseteq H(\lambda)$ such that $\varphi(u, a)$ holds. □

LEMMA 2.5. *Assume AC and let* $\varphi \in \mathcal{L}_{\mathrm{ZF}}$. *Then* $\varphi$ *is* $\Sigma_2^{\mathrm{ZF}}$ *iff*

$$(*) \qquad \varphi \quad \leftrightarrow \quad \exists \lambda > \omega\, H(\lambda) \models \varphi$$

PROOF. Assume first that $\varphi \in \Sigma_2^{\mathrm{ZF}}$.
If $\varphi$ is true, then $\varphi$ holds in some $H(\lambda)$ by the Reflection Theorem. Conversely, assume that $H(\lambda) \models \varphi$ for some uncountable $\lambda$, and write $\varphi$ as $\exists x \psi(x)$ where $\psi \in \Delta_0^{\mathrm{ZF}}$. Then there is an $a \in H(\lambda)$ such that $H(\lambda) \models \psi(a)$. By Lemma 2.4, $\psi(a)$ is true in $V$, and therefore also $\varphi$. For the other direction, assume now that $\varphi$ is a formula satisfying (*). We argue that the right-hand side of the equivalence is $\Sigma_2^{\mathrm{ZF}}$. To see this, note that the relations $Card(y)$ and $x = H(y)$ are $\Pi_1$. The satisfaction relation $\models$ is $\Delta_0$ since all quantifiers are bounded to the domain of the structure in question. Thus $\varphi$ is equivalent to the $\Sigma_2$ statement

$$\exists x \exists y \exists \varphi (Card(y) \wedge y > \omega \wedge x = H(y) \wedge x \models \varphi).$$

□

The $\Sigma_2$ properties of set theory are therefore sometimes called *locally verifiable*. Proposition 2.2 shows that we can force $V = OD$ over models of ZFC while preserving arbitrary large $H(\lambda)$'s, and thus we can force $V = OD$ while preserving any particular locally verifiably property. This is summed up in the following Proposition:

PROPOSITION 2.6 (Roguski). $V = OD$ *is* $\Pi_2$*-conservative over ZFC. Equivalently, whenever* $\varphi \in \Sigma_2^{ZFC}$ *and* $\mathrm{Con}(\mathrm{ZFC} + \varphi)$, *then* $\mathrm{Con}(\mathrm{ZFC} + V = OD + \varphi)$.

PROOF. Assume that $\varphi \in \Sigma_2^{\mathrm{ZFC}}$ holds in $M \models$ ZFC. Pick an uncountable cardinal $\lambda$ in $M$ such that $H(\lambda)^M \models \varphi$. Now using Proposition 2.2, we can find a class-generic extension $M[G] \models V = OD$ which has the same $H(\lambda)$ (here we use the assumption that $M \models AC$). By the absoluteness of the satisfaction relation, $(H(\lambda) \models \varphi)^{M[G]}$, and so $M[G] \models \varphi$ by Lemma 2.5. □

COROLLARY 2.7. *The following statements are* $\Sigma_2^{\mathrm{ZFC}}$ *and therefore consistent with* $V = OD$*:*

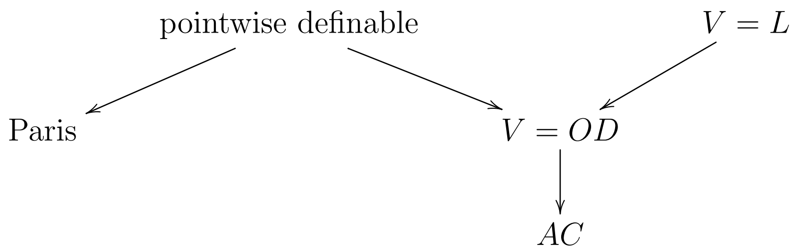
(1) *There is a measurable cardinal*
(2) $\neg GCH$
(3) $V \neq L$

Proof. The easiest way to show that the statements are $\Sigma_2$ is by showing that they are locally verifiable in the sense of Lemma 2.5. For example, if $V \neq L$, then there is a non-constructible set $x$ in some $H(\lambda)$. Then $x \notin L^{H(\lambda)} = L \cap H(\lambda)$, and so $H(\lambda) \models V \neq L$. □

Remark 2.8. The proposition is optimal in the sense that $V = OD$ is not $\Pi_2$-conservative over ZFC: the sentence $\varphi \equiv \exists x(x \subseteq \omega \wedge x \notin OD)$ is consistent with ZFC and is a $\Pi_2^{\mathrm{ZFC}}$ sentence.[1]

Remark 2.9. Similarly, $V = OD$ is not $\Pi_2$-conservative over ZF (without Choice), since AC is a $\Pi_2^{\mathrm{ZF}}$-statement implied by $V = OD$.

**1.1. Intermezzo.** Let us quickly recall the diagram of implications




We can now easily show that none of the arrows is invertible, assuming some mild consistency assumptions about ZF.

- *Paris but not pointwise definable*: Consider a Paris model of $V \neq OD$. Such a model exists by Lemma 1.24, but it cannot be pointwise definable.
- *$V = OD$ but not pointwise definable*: Consider any uncountable model of $V = OD$.
- *$AC$ but not $V = OD$*: Consider $L[g]$ where $g$ is Cohen generic over $L$. $L[g]$ satisfies $AC$, but $OD^{L[g]} = L$ by the weak homogenity of Cohen forcing.
- *$V = OD$ but not $V = L$*: Start with a model of $V \neq OD$, and consider the model $V[G]$ from Proposition 2.2.

## 2. Pointwise definability by forcing

We now use the method of the previous section to generically extend a given countable transitive model $M$ of ZFC to a pointwise definable model $M[G] \supseteq M$. Since pointwise definable models are countable, the countability of the ground model $M$ is essential for the proof. This

[1] $\varphi$ is equivalent to

$$\underbrace{\forall y\,(\underbrace{y = \mathcal{P}(\omega)}_{\Pi_1} \to \exists x \in y\,\underbrace{\forall\alpha\forall z(\underbrace{z = V_\alpha}_{\Pi_1} \to \underbrace{x \notin \mathrm{Df}(V_\alpha)}_{\Delta_1}))}_{\Pi_2})}_{\Pi_2}$$

is unlike in many forcing proofs, where the countability of the ground model only plays a minor technical role.
Let $A \subseteq M$. We say that $(M, A)$ is pointwise definable if every set in $M$ is definable in a formula in the language $\{\in, A\}$.

PROPOSITION 2.10. *Let $M$ be a countable transitive model of ZFC. There is a set $A \subseteq M$ such that $(M, A)$ is a pointwise definable and $(M, A) \models$ ZFC.*

PROOF. Note first that the statement is trivial if one does not require $(M, A) \models$ ZFC, since if $A$ is any well-order of $M$ in order type $\omega$, then every set in $M$ will be definable from its rank in $A$, which is a standard natural number and therefore definable. So clearly $(M, A)$ will be pointwise definable. But $(M, A)$ fails to satisfy Replacement, since $A$ gives rise to an unbounded map $\omega \to M$.
The $A$ we use will be class-generic over $M$ for a set-closed class forcing, and so $(M, A) \models$ ZFC will follow from the Forcing Theorem.
Since $M \models$ ZFC, it suffices to make all sets of ordinals definable in $(M, A)$. In fact, every set $x$ in $M$ can be decoded from a map $a_x \in 2^{<ON}$ which ends with two consecutive 1's and has value 0 at all odd places apart from that. We will restrict our attention to those $a$ for technical reasons.
Denote by $P$ the partial order $(2^{<ON}, \supseteq)^M$. Using the countability of $M$, let $\{x_n \mid n \in \omega\}$ and $\{D_n \mid n \in \omega\}$ be enumerations of $M$ and all parametrically definable dense subsets of $P$ in $M$ respectively. For each $n$, let $\varphi_n$ be a defining formula of $D_n$. By choosing the sequence $\{D_n \mid n \in \omega\}$ appropriately, we may assume that the parameters in $\varphi_n$ are among $x_0, \ldots, x_{n-1}$ ($\varphi_0$ is parameter-free, so for example $D_0$ could be $P$).
The desired class $A$ is the union of conditions $(p_n)_{n\in\omega}$, where the $p_n$ are defined inductively as follows:

(1) $p_0 = \emptyset$
(2) For even $n > 0$, $p_n$ is the minimal-length extension of $p_{n-1}$ to a condition in $D_n$ (if there are multiply such $p_n$, take the least with respect to the lexicographical ordering)
(3) For odd $n > 0$, $p_n$ is the concatenation of $p_{n-1}$ with the sequence $a_x$.

It follows from (2) that $A$ is $P$-generic over $M$ and so $(M, A) \models$ ZFC. We now show that $(M, A)$ is pointwise definable by giving definitions of $p_{2m}, p_{2m+1}, a_{x_m}$ and $x_m$ in the language $\{\in, A\}$ for every $m$ via (external) induction. Assume that such definitions have been found for all $n < m$. Then:

(1) If $m = 0$, then $p_0 = \emptyset$. Otherwise,
(2) $p_{2m}$ is the minimal-length and lexicographically least extension of $p_{m-1}$ to an initial segment of $A$ such that $M \models \varphi(p_m)$

(3) $p_{2m+1}$ is the length-minimal extension of $p_{2m-1}$ to an initial segment of $A$ which ends with two consecutive 1's
(4) $a_{x_m}$ is the tail segment of $p_{2m+1}$ given by removing the initial segment $p_{2m}$
(5) $x_m$ is the set coded by $a_{x_m}$

Note that in (2), all $M$-parameters in $\varphi_n$ are among $x_0, \ldots, x_{m-1}$ which are definable by induction hypothesis. Apart from that, it is clear that the clauses $(1) - (5)$ give rise to parameter-free definitions of $p_{2m}, p_{2m+1}, a_{x_m}$ and finally $x_m$ in $(M, A)$. □

REMARK 2.11. We remark that the constructed definition of $x_m$ in $(M, A)$ is by no means uniform in $m$, since the defining formulas of the various $D_n$'s are incorporated into the definition of $x_m$.

PROPOSITION 2.12. *Let $M$ be a countable transitive model of ZFC. There is a generic extension $M[G] \supseteq M$ such that $M[G]$ is pointwise definable.*

PROOF. Using the previous Proposition, we first find a class $A \subseteq M$ such that $(M, A) \models$ ZFC and $(M, A)$ is pointwise definable. Next, we use an $(M, A)$-definable forcing $Q$ which forces $V = OD$ and simultaneously makes $A$ definable. This can be achieved by modifying the construction in Proposition 2.2 such that any generic $H \subseteq Q$ uses one definable unbounded class of regular cardinals to force $V = OD$ and a disjoint definable unbounded class of regular cardinals to code $A$ (by using a class-version of Lemma 1.42). We omit the details here.
So in the resulting model $M[H]$, $V = OD$ holds and $A$ is a definable class. It follows that $M$ is also definable in $M[H]$, since it is the collection of all sets which are coded into $A$. Now since $(M, A)$ was pointwise definable, it follows that every set in $M$ is definable in $M[H]$. In particular, all ordinals are definable in $M[H]$ (i.e. $M[H]$ is a Paris model), and since $M[H] \models V = OD$, it follows that $M[H]$ is pointwise definable. □

## 3. The theory of *HOD*

We have seen that the axiom of choice holds in the *HOD* of any model of ZF. In this section we show that this is already the only example of a statement which provably holds in *HOD*.

PROPOSITION 2.13 (Roguski). *Let $M$ be a countable transitive model of ZFC. Then $M$ has a class-generic extension $M[G] \supseteq M$ such that $HOD^{M[G]} = M$.*

This can be paraphrased as saying that *HOD* has no internal structure, since *every* model of ZFC can arise as the *HOD* of some other model.

Corollary 2.14. *For any sentence* $\varphi \in \mathcal{L}_{\mathrm{ZF}}$*,* $\mathrm{ZF} \vdash \varphi^{HOD}$ *iff* $\mathrm{ZFC} \vdash \varphi$*, and so the theory of HOD is just (the deductive closure of) ZFC.*

Proof. If $\mathrm{ZFC} \vdash \varphi$ then certainly $\mathrm{ZF} \vdash \varphi^{HOD}$, since the $HOD$ of any model of ZF satisfies ZFC.
Assume conversely that $\mathrm{ZFC} \nvdash \varphi$, and let $M \models \mathrm{ZFC} + \neg\varphi$. By Proposition 2.13 there is a generic extension $N \supseteq M$ such that $M = HOD^N$. Then $N \models \mathrm{ZF}$ but $N \nvDash \varphi^{HOD}$, since $(\varphi^{HOD})^N \leftrightarrow \varphi^M$. Hence $\mathrm{ZF} \nvdash \varphi^{HOD}$. □

We first need some extensions and class versions of previous results. Let $(M, A)$ be a model of $\mathrm{ZF}(A)$. $OD^{(M,A)}$ is the class of all sets definable in $(M, A)$ from ordinal parameters. $HOD^{(M,A)}$ is defined in the obvious way.

Lemma 2.15.
1. $HOD^{(M,A)}$ *is* $(M, A)$*-definable*
2. *There is an* $(M, A)$*-definable class* $K \subseteq ON$ *such that* $HOD^{(M,A)} = L[K]$

Proof. By straighforward generalizations of the proofs of Lemmas 1.19 and 1.38. □

Lemma 2.16. *Let* $P \subseteq M$ *be a class forcing which preserves ZF and assume that the forcing relation for* $P$ *is* $(M, A)$*-definable. If* $P$ *is weakly homogeneous an* $G$ *is* $V$*-generic, then*

$$HOD^{(M[G],A,M)} = HOD^{(M,A)}$$

Proof. By generalizing the proof of Lemma 1.49. □

We are now set to prove the proposition.

Proof of Proposition 2.13. Let $M$ be a countable transitive model of ZFC. First, let $P$ be the set-closed forcing which adds a global well-order to $M$, and let $G$ be $P$-generic. Then

$$(M, G) \models \mathrm{ZFC}(G) + V = HOD^{(M,G)}$$

We now flatten the continuum function to do some coding in the next step. Let $Q \subseteq M$ be the class size partial order which forces $GCH$, and let $H$ be $Q$-generic over $(M, G)$. Then by the weak homogenity of $H$,

$$(M[H], G, M) \models \mathrm{ZFC}(G, M) + M = HOD^{(M[H],G,M)}$$

Note that we do not add $H$ as a predicate. Using $AC$ in $M[H]$ and Lemma 2.15, pick an $(M[H], G, M)$-definable class $A \subseteq ON$ such that $M = HOD^{(M[H],G,M)} = L[A]$. Finally, let $R$ be the $(M[H], G, M)$-definable forcing which codes the class $A$ into the continuum pattern. Let $I$ be $R$-generic over $(M[H], G, M)$, and let $N = M[H][I]$. Then

$N \models$ ZFC, and we claim that $HOD^N = M$.
First, $A$ can be read off the continuum function in $N$, and so

$$M = L[A] \subseteq HOD^N$$

Conversely,

$$HOD^N \subseteq HOD^{(N,G,M,M[H])}$$

since adding predicates makes more sets definable, and

$$HOD^{(N,G,M,M[H])} = HOD^{(M[H],G,M)}$$

by the weak homogenity of $R$, and $HOD^{(M[H],G,M)} = M$ by the previous discussion. So $HOD^N = M$ and the Proposition is proved. $\square$

## 4. Vŏpenka's Theorem

In this section, we give a proof that every set of ordinals is generic over $HOD$.
We start with some easy observations. Let $X \in OD$ be any set. $\mathcal{P}(X) \cap OD$, the collection of ordinal definable subsets of $X$, contains $\emptyset$ and $X$ and is closed under finite intersections and complements. In other words, it forms a subalgebra of $\mathcal{P}(X)$. Furthermore, $\mathcal{P}(X) \cap OD$ is *$OD$-complete*: if $A \subseteq \mathcal{P}(X) \cap OD$ is ordinal definable, then $\bigcap A \in \mathcal{P}(X) \cap OD$.
Let $Q(X)$ denote the set $\mathcal{P}(X) \cap OD \setminus \{\emptyset\}$, viewed as a forcing poset with the partial order $\leq = \subseteq$. This forcing has lots of atoms, namely all singletons $\{x\}$ where $x \in OD$, and so $G(x) := \{p \in Q \mid x \in p\}$ is an ordinal definable generic filter for each such $x$. If $x \notin OD$, we can still define the filter $G(x) := \{p \in Q \mid x \in p\}$ in $V$ as above. This is like a principal filter, only that the generating element is not in $Q$.

Lemma 2.17. *$G(x)$ intersects all $OD$-definable maximal antichains in $Q$.*

Proof. If $A \in OD$ is an antichain, i.e. $A$ consists of pairwise disjoint sets, then $\bigcap_{p \in A} X \setminus p$ is ordinal definable and disjoint from any set in $A$. So if $A$ is maximal, $\bigcap_{p \in A} X \setminus p$ cannot be a condition in $Q$, so it has to be the empty set. In other words, $\bigcup A = X$. It follows that $x$ is contained in some element of $A$, and thus $G(x)$ intersects $A$ by the definition of $G(x)$. $\square$

This is already the main ingredient of Vŏpenka's proof. We just need a transfer principle from $OD$ to $HOD$.

Lemma 2.18. *Let $\mathcal{A}$ be an ordinal definable first-order structure in a finite language, and assume that all elements of the universe of $\mathcal{A}$ are ordinal definable. Then there is a $\mathcal{B} \in HOD$ such that $\mathcal{A} \cong \mathcal{B}$.*

PROOF. Note first that it follows from the assumptions that all relations and functions of $\mathcal{A}$ are ordinal definable. Now let $F : OD \to ON$ be a definable injection. We define the universe of $\mathcal{B}$ as $B := F''(A)$, where $A$ is the universe of $\mathcal{A}$. Then $B$ is an ordinal definable set of ordinals, so $B \in HOD$. Now do likewise for functions and relations in $\mathcal{A}$. Then $F : \mathcal{A} \cong \mathcal{B}$. $\square$

THEOREM 2.19 (Vŏpenka). *Every set of ordinals is generic over HOD.*

PROOF. Let $x$ be a set of ordinals and pick some $x \subseteq \kappa$. Let $Q$ be the forcing $\mathcal{P}(\mathcal{P}(\kappa)) \cap OD \setminus \emptyset$ ordered by $\subseteq$. Note that $Q \in OD$ (but usually $Q \notin HOD$). Now applying the above lemma, let $Q'$ be an isomorphic forcing in $HOD$. Working in $V$, let $G_x = \{p \in Q \mid x \in p\}$, and let $G'_x := F''G_x$ be the corresponding filter in $Q'$. Every maximal antichain $A' \subseteq Q'$ in $HOD$ corresponds to a maximal antichain $A \subseteq Q$ in $OD$ which is met by $G_x$ by the discussion above, and so $G'_x$ meets all maximal antichains in $HOD$. Furthermore,

$$\alpha \in x \leftrightarrow \{u \subseteq \kappa \mid \alpha \in u\} \in G_x \leftrightarrow F''(\{u \subseteq \kappa \mid \alpha \in u\}) \in G'_x$$

so that $x$ and $G'_x$ are mutually definable (note that the collection $\{u \subseteq \kappa \mid \alpha \in u\}$ is ordinal definable). Thus $HOD[G'_x] = HOD[x]$. $\square$

COROLLARY 2.20. *Assume $V = L[x]$ for some set $x$ of ordinals. Then $V$ is a set-generic extension of HOD.*

CHAPTER 3

# The stable core

In this section, we show that the whole universe is generic over $HOD$ for a forcing which is definable in $V$. More generally, we give a sufficient condition for an inner model to be only "one class forcing away" from $V$.
To this end, we define the *stability predicate* $S \subseteq V$ and show that for any definable inner model $M$, $V$ is generic over the extended model $(M[S], S)$. The smallest such model $(L[S], S)$ is called the *stable core* of $V$. It is consistent that $L[S] \subsetneq HOD$.
Since $S$ is definable, $HOD[S] = HOD$. We can therefore conclude that $V$ is class-generic over $HOD$ for a forcing definable in $V$.

All material in this section is taken from [**6**].
We always assume that $AC$ holds in $V$.

## 1. Proof outline

Vŏpenka's theorem can be rephrased in the following way: If $V = L[A]$ for some set of ordinals $A$, then $V$ is generic over $HOD$. The proof proceeded by finding a partial order in $HOD$ such that the set $A$ itself (up to some definable bijection) was a generic for this poset.
Turning to the class case, we have seen (12.1.1) that $V$ can always be written as $V = L[F]$ where $F$ is (the characteristic function of) a $V$-generic *class* of ordinals. Our aim is now to devise a forcing $P$ such that $F$ is $M$-generic over $P$ for any inner model $M \subseteq V$ in which $P$ is definable.
Since we have restricted ourselves to conditions which are sets, we cannot directly use the same forcing as in the proof of Vopenka's theorem. Instead we take a more syntactical approach, where a condition is a statement in an infinitary logic (defined in $M$) describing how an imagined function $\dot{F} : ON \to 2$ "could behave" on set-many values (it will be one of our tasks to determine what this "could behave" should mean).
Let $P \subseteq M$ denote this informally described forcing. Working in $V$, we identify $F$ with the class

$$G_F = \{\varphi \in P \mid \varphi \text{ is true when } \dot{F} \text{ is replaced by } F\}$$

which is a filter on $P$. Of course our dream is that $G_F$ is $M$-generic on $P$. So consider an $M$-definable antichain $A \subseteq P$. Let $\bar{A}$ be the

conjunction of $\{\neg\varphi \mid \varphi \in A\}$. If $G_F$ does not intersect $A$, then $F$ makes $\bar{A}$ true. This should qualify $\bar{A}$ as a statement about $\dot{F}$ which "could be possible" from $M$'s point of view, because we have found the witness $F$ in the outer model $V \supseteq M$. So $\bar{A}$ is a condition in $P$ and hence $A$ was not maximal.
The problem with this reasoning is that if $A$ is a proper class, then $\bar{A}$ contains class-many informations about $\dot{F}$ and therefore is too big to be a condition. We can resolve this by using a reflection argument. Namely, assume the antichain $A$ is $\Sigma_n$-definable in $V$ and pick an $\alpha$ such that

$$(V_\alpha, F \cap V_\alpha) \preceq_n (V, F) \tag{*}$$

Such an $\alpha$ exists because $(V, F)$ models Replacement. Then $F$ intersects $A$ iff $F$ intersects $A \cap V_\alpha$, which is a set. We will use this fact to refine our forcing $P$ to a forcing which has set-size antichains, and for which the genericity argument sketched above works.
Since this refined $P$ is defined using (*), it seems to be only definable in inner models which have access to $F$. This would of course be useless, since $F$ already codes all of $V$. However, we can work with much less information about $V$:
The key idea is that we can always choose the generic class $F$ in such a way that for a large $V$-definable class $C$ of $\alpha$, $H(\alpha) \preceq_n V$ already implies $(H(\alpha), F \restriction \alpha) \preceq_n (V, F)$. We call such an $F$ *stability-preserving.* Now to define $P$, our inner model only has to to have access to $V$'s stability relation on the class $C$.

## 2. The stability predicate

Let us start with some definitions.
For the reflection argument we are aiming at it suffices that $F$ is stability-preserving on an unbounded class of ordinals. For technical reasons, we restrict ourselves to the class of cardinals $\beta$ such that $H(\alpha) \in H(\beta)$ for all $\alpha < \beta$.
For $\alpha, \beta \in C$ we say that $\alpha$ is *$n$-Stable in $\beta$* if $\alpha < \beta$ are limit points of $C$ and $(H(\alpha), C \cap \alpha) \preceq_n (H(\beta), C \cap \beta)$. We say that $\alpha$ is *$n$-stable in ON* if $(H(\alpha), C \cap \alpha) \preceq_n (V, C)$.
We call $\alpha$ *$n$-Admissible* if $(H(\alpha), C \cap \alpha)$ is a model of $\Sigma_n$-Replacement. If $\alpha$ is $n$-Stable in some $\beta$, it follows that $\alpha$ is $n$-Admissible[1].
For our purpose, the $n$-Admissibility of $\alpha$ is used a an indicator that

[1]Assume $f : a \to H(\alpha)$ is $\Sigma_n$-definable over $(H(\alpha), C \cap \alpha)$, and $\alpha$ is $n$-Stable in $\beta$. By elementarity, the relation $f(x) = y$ is absolute between $(H(\alpha), C \cap \alpha)$ and $(H(\beta), C \cap \beta)$. Thus $(H(\beta), C \cap \beta) \models \forall x \in a \exists! y \in H(\alpha)(f(x) = y)$, and so in particular $(H(\beta), C \cap \beta) \models \exists D \forall x \in a \exists! y \in D(f(x) = y)$. The latter statement is $\Sigma_n$, and so it holds in $(H(\alpha), C \cap \alpha)$ as well by elementarity. But this means that $f$ is bounded in $H(\alpha)$.

an $n$-Stability relation holds between $\alpha$ and some larger $\beta$.
Finally we define the *Stability predicate*:

$$S = \{(\alpha, \beta, n) \mid \alpha \text{ is } n\text{-Stable in } \beta \text{ and } \beta \text{ is } n\text{-Admissible}\}$$

## 3. Forcing a stability-preserving predicate

We now show how one can find a generic $F : C \to 2$ which codes the universe (i.e. $V = L[F]$) and which is stability-preserving, meaning that whenever

$$(H(\alpha), C \cap \alpha) \preceq_n (H(\beta), C \cap \beta)$$

it follows that

$$(*) \qquad (H(\alpha), C \cap \alpha, F \upharpoonright \alpha) \preceq_n (H(\beta), C \cap \beta, F \upharpoonright \beta)$$

$F$ will be generic over a forcing $P$ which refines the tree forcing $(2^{<ON}, \supseteq)$.

The key idea is to chose $P$ in such a way that the initial segments $F \upharpoonright \alpha$ of $F$ have to be sufficiently generic for $H(\alpha)$. Namely every $\Sigma_n$ statement $\varphi$ about $(H(\alpha), C \cap \alpha, F \upharpoonright \alpha)$ should be decided by a proper initial segment $F \upharpoonright \alpha' \subseteq F \upharpoonright \alpha$. It will then be possible to speak about the predicate $F \upharpoonright \alpha$ inside $H(\alpha)$ by means of the forcing relation $\Vdash$ and a name $\dot{f}$ only, and so the predicate $F \upharpoonright \alpha$ becomes eliminable.

The forcing $P$ is defined as $\bigcup_{\alpha \in C} P(\alpha)$, where the sets $P(\alpha)$ are defined by induction on $\alpha \in C$. Every $P(\alpha)$ consists of functions from $C \cap \alpha$ to 2.
If $\alpha$ is an successor of some $\beta \in C$, we simply let $P(\alpha)$ be all possible extensions of maps from $P(\beta)$, i.e. $P(\alpha)$ consists of all $f : C \cap \alpha \to 2$ such that $f \upharpoonright \alpha \in P(\beta)$.
So let $\alpha$ be a limit, and assume that $P(\beta)$ has been defined for all $\beta < \alpha$. We let $P(< \alpha)$ be the union of all the $P(\beta)$'s, $\beta < \alpha$, and view this as a class forcing in $H(\alpha)$, where the order is inclusion.
We assume that $P(< \alpha)$ is *extendible*, meaning that for each $p \in P(\gamma)$ and $\gamma < \beta < \alpha$ (all in $C$), $p$ can be extended to a condition in $P(\beta)$. We postpone a proof until later.
Under this assumption, $P(< \alpha)$ adds a generic function from $\alpha \cap C$ to 2 whose canonical name we denote by $\dot{f}_\alpha$.
We say that $f : \alpha \cap C \to 2$ is $n$-generic for $P(< \alpha)$ if for any forcing statement $\varphi \in \Sigma_n(H(\alpha), C \cap \alpha, \dot{f}_\alpha)$, there is a $\beta < \alpha$ such that $f \upharpoonright \beta$ decides $\varphi$ in $P(< \alpha)$.
We now define $P(\alpha)$ to be the collection of all $f : \alpha \cap C \to 2$ such that

(1) $f \upharpoonright \beta \in P(\beta)$ for all $\beta \in C \cap \alpha$
(2) $f$ is $n$-generic for $P(< \alpha)$ for all $n$ such that $\alpha$ is $n$-Admissible

Again, extendibility of $P(< \alpha)$ will tell us that there are indeed $f$ satisfying (2).

LEMMA 3.1. *For any* $\varphi \in \Sigma_n(H(\alpha), C \cap \alpha, \dot{f}_\alpha)$*, the forcing relation* $f \Vdash \varphi$ *derived from* $P(<\alpha)$ *is* $\Sigma_n$*-definable over* $(H(\alpha), C \cap \alpha)$*.*

PROOF. Note that the forcing $P(<\alpha)$ does not add sets to $H(\alpha)$, and so any forcing statement $f \Vdash \varphi$ is essentially just a statement about the graph of $\dot{f} : C \cap \alpha \to 2$, where $\dot{f}$ extends $f$. If for example $\varphi$ is $\Pi_1$, then $f \vDash \varphi$ if for all $g \supset f$ in $P(<\alpha)$ and all transitive sets $M$ with $ord(M) = \mathrm{dom}(f)$, $(M, C \cap M, g) \models \varphi$. This relation is $\Pi_1$.
The full result follows by induction on $n$. □

Now let $F \subseteq P = \bigcup_{\alpha \in ON} P(\alpha)$ be a generic class.

LEMMA 3.2. *Let* $\alpha, \beta \in C$ *such that* $\alpha$ *is* $n$*-stable in* $\beta$*, and assume that* $\beta$ *is* $n$*-Admissible. Then* $(H(\alpha), F \restriction \alpha) \preceq_n (H(\beta), F \restriction \beta)$*.*

PROOF. First note that both $\alpha$ and $\beta$ are $n$-Admissible. Assume $(H(\alpha), F \restriction \alpha) \models \varphi$ for some $\varphi \in \Sigma_n(H(\alpha), \dot{f})$. By construction, $F \restriction \alpha$ is then an $n$-generic class for the forcing $P(<\alpha)$, and so by the Forcing Theorem for $P(<\alpha)$ there exists an $\alpha_0 < \alpha$ such that

$$H(\alpha) \models (F \restriction \alpha_0 \Vdash \varphi)$$

This statement is $\Sigma_n$ over $H(\alpha)$ by Lemma 3.1, and so by elementarity it follows that

$$H(\beta) \models (F \restriction \alpha_0 \Vdash \varphi)$$

which in turn implies that $(H(\beta), F \restriction \beta) \models \varphi$ since $F \restriction \beta$ is an $n$-generic class for $P(<\beta)$ which contains the condition $F \restriction \alpha_0$. □

PROPOSITION 3.3 (Extendibility). *Let* $\alpha < \beta$ *be cardinals in* $C$*. Then every condition in* $P(\alpha)$ *can be extended to a condition in* $P(\beta)$*.*

PROOF. By induction on $\beta$. So let $\alpha < \beta$ and $p \in P(\alpha)$. If $\beta$ is a successor of some $\gamma \in C$, we can just extend $p$ to a condition $q$ in $P(\gamma)$ by the induction hypothesis and then arbitrarily extend $q$ to a condition in $P(\beta)$ (since in the recursive definition of $P(\beta)$ for successors, no restrictions are made). So let us assume that $\beta$ is a limit.
The general strategy for producing an extension in $P(\beta)$ is this: We first pick an cofinal increasing sequence $(\beta_i)$ of elements of $(C \cap \beta)$ such that $\beta_0 > \alpha$, and then we successively extend $p$ to conditions $q_i$ in $P(\beta_i)$. Finally we set $q = \bigcup_i q_i$.
If we have found an extension of $p$ in $P(\beta_i)$, the existence of a further extension in $P(\beta_{i+1})$ is guaranteed by the induction hypothesis since $\beta_{i+1} < \beta$. Now assume that $i$ is a limit and we have already found extensions $p \geq q_0 \geq q_1 \geq \ldots \geq q_j \geq q_{j+1} \geq \ldots$ for all $j < i$. We want to set $q_i = \bigcup_{j<i} q_j$. However by the definition of $P(\beta_i)$, this $q_i$ is only a condition if it is $n$-generic for $P(<\beta_i)$ for every $n$ such that $\beta_i$ is $n$-Admissible. The rest of the proof shows that this situation can

always be arranged by choosing the sequence $(\beta_i)$ carefully.

Case 1: $\beta$ *is not* 1-*Admissible*
The failure of 1-Admissibility implies the existence of an unbounded increasing sequence $(\beta_i)_{i<\delta}$ in $C\cap\beta$ which is $\Delta_1$-definable[2] in $(H(\beta), C\cap\beta)$, where $\delta < \beta$. We may assume that $\delta$ and all parameters in the definition of $(\beta_i)_{i<\delta}$ are contained in $H(\beta_0)$ (otherwise, switch to an appropriate tail segment of $(\beta_i)_{i<\delta}$). Whenever $\beta_j$ is a limit of the sequence, it follows by $\Delta_1$-absoluteness that the restriction of $(\beta_i)_{i<\delta}$ to values below $\beta_j$ is $\Delta_1$-definable in $H(\beta_j, C\cap\beta_j)$, and so this subsequence witnesses the failure of 1-Admissibility for $(H(\beta_j), C\cap\beta_j)$.
It then follows from the definition of $P$ that for each limit $\beta_j < \beta$ and for $\beta$ itself, $P(\beta_j)$ and $P(\beta)$ are simply the set-union of the various $P(\beta_i)$ below. We can therefore extend $p$ successively to conditions $q_0, q_1, \ldots, q_i, \ldots$ where $q_i \in P(\beta_i)$, taking unions at limit points. $q = \bigcup_{i<\delta} q_i$ is then the desired extension of $p$ in $P(\beta)$.
Case 2: *For some* $0 < n < \omega$, $\beta$ *is* $n$-*Admissible, but not* $(n+1)$-*Admissible*
Let us first assume that in addition, there are cofinally many $\xi < \beta$ such that $\xi$ is $n$-Stable in $\beta$. Then a similar reasoning as in Case 1 yields: There is an increasing unbounded sequence $(\beta_i)_{i<\delta}$ in $C\cap\beta$, now consisting of $n$-Stables in $\beta$, which is $\Delta_{n+1}$-definable over $(H(\beta), C\cap\beta)$. For each limit $\beta_j$, it follows from the $\Delta_{n+1}$-definability of $(\beta_i)_{i<\delta}$ combined with the $n$-Stability of $\beta_j$ in $\beta$ that the restriction of $(\beta_i)_{i<\delta}$ to values below $\beta_j$ is $\Delta_{n+1}$-definable in $H(\beta_j, C\cap\beta_j)$, and so $\beta_j$ is not $(n+1)$-Admissible. Now extend $p$ successively to a sequence of $q_i$'s as in Case 1. At limit points $\beta_j$, $q_j$ is $n$-generic for $P(<\beta_j)$ since $\beta_j$ is a limit of $n$-Stables. So $q_j$ is indeed a condition in $P(\beta_j)$.
Assume now that $\beta$ is not a limit of $n$-Stables. Then $\beta$ must have cofinality $\omega$: Otherwise, we could use the $n$-Admissibility of $H(\beta)$ to close substructures of $(H(\beta), C\cap\beta)$ under $\Sigma_n$-Skolem functions, producing cofinally many $n$-Stables. It suffices to prove that $p$ can be extended to decide any given collection of $\Pi_n(H(\beta), C\cap\beta, \dot{f})$ sentences of size less than $\beta$. Once we have done that, we can extend $p$ in $\omega$ steps to a condition in $P(\beta)$ which is $n$-generic. Let $(\varphi_i)_{i<\delta}$, $\delta<\beta$ be an enumeration of such a collection of $\Pi_n$-sentences, where $\delta < \beta$. Furthermore, let $D$ be the club of all $\gamma < \beta$ such that $\gamma$ is a limit of $(n-1)$-Stables in $\beta$ and large enough so that $H(\gamma)$ contains $p$ and the enumeration $(\varphi_i)_{i<\delta}$. We now define by induction sequences $(\beta_i)$ of ordinals below $\beta$ and $(q_i)$ conditions in $P(\beta_i)$. $\beta_0$ is the least element of $D$, and $q_0$ is an extension of $p$ in $P(\beta_0)$. Now given $\beta_i$ and $q_i$, we let $\beta_{i+1}$ be the least element of $D$ above $\beta_i$ such that $D$ contains an extension of $q_i$ which

[2]Note that for functions, $\Sigma_{n+1}$ and $\Delta_{n+1}$ are the same: If the relation $f(x) = y$ is $\Sigma_{n+1}$, then so is $f(x) \neq y$, since this is equivalent to $\exists z(f(x) = z \wedge z \neq y)$

decides $\varphi_i$. Let $q_{i+1} \leq q_i$ be such an extension. Finally for limit $i$, we let $\beta_i = \bigcup_{j<i} \beta_j$ and $q_i = \bigcup_{j<i} q_j$. $q_j$ is indeed a condition in $P(\beta_j)$, since $\beta_j$ fails to be $n$-Admissible and $q_j$ is a limit of $(n-1)$-Stables.
<u>Case 3:</u> *$\beta$ is $n$-Admissible for every $n$*
This means that $(H(\beta), C \cap \beta)$ satisfies full Replacement. It is then easy to construct a cofinal sequence $(\beta_i)$ of elements of $C \cap \beta$ such that every limit $\beta_i$ is the limit of $n$-Stables for every $n \in \omega$. We can then extend $p$ successively to conditions $q_i \in P(\beta_i)$, taking unions at limit steps. This works since if $\beta_i$ is a limit, then for every $n$, $\beta_i$ is $n$-generic because it is a limit of $n$-Stables.

□

## 4. Truth in outer models

In this section we show that a model $M$ can reason about truth in its outer models $N \supseteq M$, at least for statements of some fixed bounded complexity. We are interested in the case where the outer model is of the form $L[F]$ for some class function $F : ON \to 2$. Thus statements about $N$ are essential statements about the function $F$.
We start by defining an infinitary language $\mathcal{L}$ in $M$ which describes a function $\dot{F} : ON \to 2$. The atomic sentences of $\mathcal{L}$ are

$$\dot{F}(\alpha) = 0$$
$$\dot{F}(\alpha) = 1$$

where $\dot{F}$ is a fixed symbol and $\alpha \in ON^M$, and inductively setting

$$\bigvee \Phi \in \mathcal{L}$$
$$\bigwedge \Phi \in \mathcal{L}$$

whenever $\Phi \in M$ is a set of $\mathcal{L}$-formulas. We may think of $\varphi \in \mathcal{L}$ as being coded as a tree of finite height and set-size width. We make the harmless technical assumption that if all ordinals mentioned in $\varphi \in \mathcal{L}$ are bounded below some $\alpha$, then $\varphi \in H(\alpha)$. Intuitively, each $\varphi \in \mathcal{L}$ contains set-many information about the function $\dot{F}$.
Let $o(\varphi)$ denote the set of ordinals occuring in $\varphi$, and let $2^{<ON}$ be the set of all functions from some ordinal to 2. If $f \in 2^{<ON}$ and $\operatorname{dom} f \supseteq o(\varphi)$, one can define $f \models \varphi$ in the natural way, that is $\varphi$ holds when $\dot{F}$ is replaced by $f$. For definiteness, let $f \models \varphi$ be true when $\operatorname{dom} f \not\supseteq o(\varphi)$. For $\varphi, \psi \in \mathcal{L}$ we also consider $\neg\varphi$ and $\varphi \to \psi$ to be part of $\mathcal{L}$, using the obvious semantics.

The relation $f \models \varphi$ is $\Delta_0(f, \varphi)$ (infinite junctions over $\Phi$ become quantifiers bounded by $\Phi$), so it can be evaluated in every transitive model containing $f, \varphi$. Keeping this in mind, we say that $\varphi$ is *valid*, written

$$\vdash \varphi$$

if for all set-generic extensions $N \supseteq M$ and all $f \in N$, $f \models \varphi$.

Lemma 3.4. *The relation $\vdash$ is $M$-definable.*

Proof. This follows from the definability of the forcing relation: $\varphi$ is valid if and only if

$$\forall P \forall p \in P(p \Vdash_P \forall f(f \models \varphi))$$

Note that on the right side of $\Vdash$ there is really only one formula in which $P$ and $\varphi$ act as parameters. □

Lemma 3.5. *Let $\varphi \in \mathcal{L}$ and let $N \subseteq$ be any outer model of $M$. Then $\varphi$ is valid in $N$ if and only if $\varphi$ is valid in $M$.*

Proof. Assume that $\varphi$ is not valid in $N$. So there is a generic extension $W \supseteq N$ such that $W \models \exists f(f \models \neg\varphi))$. By further forcing if necessary, we may assume that $\varphi$ is countable in $W$. Then $\exists f(f \models \neg\varphi))$ is $\Sigma_1(H(\omega_1), \varphi)$ in $W$, and so it holds in all inner models of $W$ which contain a real coding $\varphi$ by Shoenfield's absoluteness theorem. In particular, since $\varphi$ is countable in $M[G]$, we have that $M[G] \models \exists f(f \models \neg\varphi)$, and so $\varphi$ is not valid in $M$.
Conversely, assume that there is a counterexample to $\varphi$ in a set-generic extension of $M$. By further forcing if necessary, we may assume that this extension is of the form $M[G]$ where $G$ is Levy collapse generic over $M$. Pick a condition $p$ in the Levy collapse which forces $\exists f(f \models \neg\varphi)$. Now if $H$ is Levy collapse generic over $N$ and contains $p$, $N[H] \models \exists f(f \models \neg\varphi)$. So $\varphi$ is not valid in $N$. □

Lemma 3.6. *Let $\alpha \in ON$ and $\Phi \subseteq H(\alpha)$ be a set of $\mathcal{L}$-formulas. If $\Phi$ is $\Sigma_n$-definable over $H(\alpha)$ and $f : \alpha \to 2$, then the notion $f \models \bigwedge \Phi$ is $\Sigma_n$ over $(H(\alpha), f)$.*

If $T$ is an $\mathcal{L}$-theory (which may be a proper class) and $\varphi \in \mathcal{L}$, we say that $T$ *implies* $\varphi$ if for some set $T_0 \subseteq T$ the sentence $\bigwedge T_0 \to \varphi$ is valid. $\varphi$ is *consistent with* $T$ if $T$ does not imply $\neg\varphi$.

We now extend the predicate $\models$ to proper classes. Let $N \models ZF^-$ and $F : N \to 2$ be a class function. Assume that $\Phi \subseteq N \cap \mathcal{L}$ is $\Sigma_n$-definable over $(N, F)$ and $(N, F)$ satisfies at least $\Sigma_n$-Replacement. Then we can define $(N, F) \models \Phi$ by

$$\forall \varphi \in \Phi(F \restriction_{ord(\varphi)} \models \varphi)$$

This notion is again $\Sigma_n$ over $(N, F)$.
There is an obvious forcing $P(\mathcal{L})$ definable from $\mathcal{L}$. First, we identify two sentences $\varphi, \psi$ if $\varphi \leftrightarrow \psi$ is valid. Now discard the equivalence class of $\perp$ (i.e. take any $\varphi$ such that $\neg\varphi$ is valid and discard the equivalence

class of $\varphi$). Then order the remaining equivalence classes by $[\varphi] \leq [\psi]$ iff $\vdash \varphi \rightarrow \psi$.

LEMMA 3.7. *Let $F : ON \rightarrow 2$ be a function which is definable in an outer model $N \supseteq M$ and let $G_F = \{\varphi \in P(\mathcal{L}) \mid F \models \varphi\}$. Then $G_F$ is a filter on $P(\mathcal{L})$.*

PROOF. This is straightforward. For example, assume $\varphi \in G_F$ and $\varphi \leq \psi$. Let $\alpha$ be the supremum of all ordinals occuring in $\varphi$ and $\psi$. By definition, $\varphi \rightarrow \psi$ is valid, so in particular it holds for the function $F \upharpoonright \alpha$ in the outer model $N$. Since $F \upharpoonright \alpha \models \varphi$ by assumption, it follows that $F \upharpoonright \alpha \models \psi$ and so $\psi \in G_F$. □

LEMMA 3.8. *Let $F : ON \rightarrow 2$ be a function and let*

$$G_F = \{\varphi \in P(\mathcal{L}) \mid F \models \varphi\}$$

*. Then $G_F$ is a filter on $P(\mathcal{L})$. Furthermore no set-size antichain $A \subseteq P(\mathcal{L})$ which is disjoint from $G_F$ is maximal.*

PROOF. It is easy to see that $G_F$ is a filter on $P(\mathcal{L})$. Now given a set-size antichain $A$, we may form $\bar{A} = \bigwedge\{\neg\varphi \mid \varphi \in A\}$. Since $G_F \cap A = \emptyset$ we know that $F \models \bar{A}$, so $\bar{A}$ is satisfiable in an outer model of $M$ and therefore belongs to $P(\mathcal{L})$. But $\bar{A}$ is incompatible with every element of $A$, so $A$ was not maximal. □

This does not tell us much because one easily sees that many antichains in $P(\mathcal{L})$ are proper classes.

## 5. Applying $V$'s reflection principle

We now want to make the reflection argument sketched in the proof outline available to our inner model $M \subseteq V = L[F]$. All the information needed for this is coded in $V$'s stability predicate $S$.
Choose an $r \in \omega$ such that both $M[S]$ and $S$ are $r$-definable in $V$.
We therefore work in the model $(M[S], S)$ and refine the forcing $P(\mathcal{L})$ from the last section.
Let $T$ be the $\mathcal{L}$-theory consisting of all sentences of the form

$$\bigwedge(\Phi \cap H(\alpha)^{M[S]}) \rightarrow \bigwedge(\Phi \cap H(\beta)^{M[S]})$$

where

(1) $\Phi$ is a set of $\mathcal{L}$-sentences
(2) For some $n \in \omega$, $\Phi$ is $\Sigma_n$-definable over $H(\alpha)^{M[S]}$
(3) $\alpha$ is $n+r$-Stable in $\beta$ and $\beta$ is $(n+r)$-Admissible (in $V$)

$T$ is $(M[S], S)$-definable precisely because the stability relations needed in (3) are coded into $S$.

LEMMA 3.9. *If $\Phi$ is $\Sigma_n$-definable over $H(\alpha)^{M[S]}$, then $\Phi$ is $\Sigma_{n+r}$-definable over $H(\alpha)^V$.*

PROOF. Recall that $(\beth_\alpha = \alpha)^V$. Since this is downwards absolute, $(\beth_\alpha = \alpha)^{M[S]}$ and so $H(\alpha)^{M[S]} = V_\alpha^{M[S]} = V_\alpha^V \cap M[S]$. Furthermore $V_\alpha^V \cap M[S] = (M[S])^{V_\alpha}$ is $\Sigma_r$-definable in $V_\alpha$, and it follows that $\Phi$ is $\Sigma_{n+r}$-definable over $V_\alpha = H(\alpha)$. □

COROLLARY 3.10. *$T$ is true in $(V, F)$.*

PROOF. This follows from Lemmas 3.6, 3.9 and the fact that $F$ is stability preserving: If $\bigwedge(\Phi \cap H(\alpha)^{M[S]}) \to \bigwedge(\Phi \cap H(\beta)^{M[S]})$ is a formula in $T$, then $(V_\alpha, F \restriction \alpha) \preceq_{n+r} (V_\beta, F \restriction \beta)$. □

Now let $Q$ consist of all sentences $\varphi \in P(\mathcal{L})$ which are consistent with $T$, meaning that there is no set $T_0 \subseteq T$ such that $\bigwedge T_0 \to \neg\varphi$ is valid. ONer $Q$ by $\varphi \leq \psi$ iff $\varphi \wedge \neg\psi$ is not consistent with $T$. It follows that $\varphi, \psi \in Q$ are incompatible iff $\varphi \wedge \psi$ is not consistent with $T$.
Now in $V$, we let $G = \{q \in Q \mid F \models q\}$. This is a filter on $Q$, which is proved in a straightforward manner using the fact that in $V$, $F \models T$.

LEMMA 3.11. *$G$ intersects all maximal antichains $A \subseteq Q$ which are sets.*

This is the same argument as in the proof of Vopenka's theorem. Let $A \in M[S]$ be an antichain and consider $\bar{A} = \{\neg\varphi \mid \varphi \in A\}$. Since $\bar{A}$ is a set we may form the conjunction $\bigwedge \bar{A}$. Assume that $G$ does not intersect $A$. Then $F \models \bigwedge \bar{A}$ by the definition of $G$ and hence $\bigwedge \bar{A}$ is consistent with $T$. Thus $\bigwedge \bar{A}$ is a condition in $Q$ which is incompatible with every element of $A$. So $A$ is not maximal.

LEMMA 3.12. *All $(M[S], S)$-definable antichains in $Q$ are sets.*

Let $A \subseteq Q$ be an $(M[S], S)$-definable antichain which might be a proper class in $M[S]$. Again, consider the class $\bar{A} = \{\neg\varphi \mid \varphi \in A\}$. Pick an $n \in \omega$ such that $\bar{A}$ is $\Sigma_n$-definable over $(M[S], S)$ and $\alpha \in ON$ which is $n$-Stable in $V$ and big enough for $H(\alpha)^{M[S]}$ to contain all parameters in the definition of $\bar{A}$. Then $\bar{A} \cap H(\alpha)^{M[S]}$ is $\Sigma_n$-definable over $H(\alpha)^{M[S]}$, using the same defining formula.
Whenever $\beta > \alpha$ is $n$-Stable in $V$ one has that $\alpha$ is $n$-Stable in $\beta$. Thus it is an axiom of $T$ that

$$\bigwedge(\bar{A} \cap H(\alpha)^{M[S]}) \to \bigwedge(\bar{A} \cap H(\beta)^{M[S]})$$

Since there are arbitrarily large such $\beta$, $T$ together with the sentence $\bigwedge(\bar{A} \cap H(\alpha)^{M[S]})$ implies every statement in $\bar{A}$. It follows that $A = A \cap H(\alpha)^{M[S]}$:
Assume otherwise that $\varphi \in A \setminus A \cap H(\alpha)^{M[S]}$. Since $A$ is an antichain, $\varphi$ implies $\bigwedge \bar{A} \cap H(\alpha)^{M[S]}$. $\neg\varphi \in \bar{A}$, so it is implied by $T + \bigwedge(\bar{A} \cap H(\alpha)^{M[S]})$. Hence $T + \varphi$ implies $\neg\varphi$, contradicting the fact that $\varphi$ is consistent with $T$.

Corollary 3.13. *$G$ is $Q$-generic over $(M[S], S)$ and $M[S][G] = V$. Hence, $V$ is a class-generic extension of $(M[S], S)$ by a forcing definable in $V$.*

Proof. The genericity of $G$ follows from Lemmas 3.11 and 3.12. Since $M[S]$ and $G$ are $V$-definable, $M[S][G] \subseteq V$. On the other hand, $V = L[F]$ and $F$ is definable from $G$, and so $V \subseteq M[S][G]$. $\square$

This finishes the proof of the main result.
To conclude this section, we sketch a proof that the Stable Core can be smaller than $HOD$.

Lemma 3.14. *It is consistent that $L[S] \subsetneq HOD$.*

Sketch of Proof. Using a variant of Jensen's technique for coding the universe into a real, one can find a class-generic extension $L[r]$ of $L$ where $r$ is real not set-generic over $L$ and $L[r]$ has the same stability predicate as $L$. It follows that inside the model $L[r]$, $L[S]$ equals $L$, and so $r$ is not set-generic over $L[S]^{L[r]}$. But $r$ *is* set-generic over $HOD^{L[r]}$ by Vŏpenka's Theorem 2.19, and so $(L[S] \neq HOD)^{L[r]}$. $\square$

CHAPTER 4

# Large cardinal witnessing

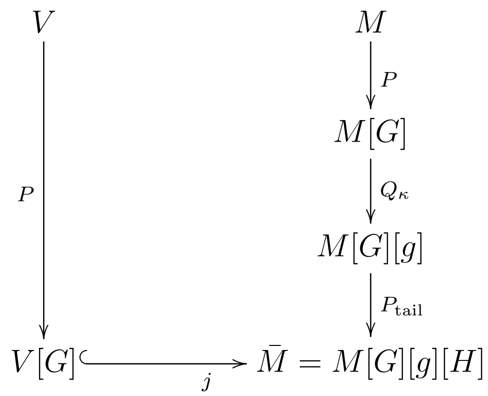
## 1. A measurable cardinal which is not measurable in HOD

In this section, we show that measurability is not witnessed in $HOD$, i.e. it is possible that for some measurable $\kappa$, $\kappa$ is not measurable in $HOD$.
The proof is a modification of a result due to Kunen, which we will prove first.

THEOREM 4.1 (Kunen). *Let $\kappa \in V$ be measurable. Then there is a forcing extension of $V$ in which $\kappa$ fails to be measurable, but becomes measurable again after forcing with $Add(\kappa, 1)$.*

PROOF. We may assume that the GCH holds in $V$, since this can be forced while preserving the measurability of $\kappa$. To fix some notation, let $P$ be the Easton-support iteration which adds a Cohen generic subset to every inaccessible cardinal below $\kappa$, and let $Q_\kappa = Add(\kappa, 1)$ be the forcing to add a Cohen generic subset to $\kappa$ itself.
Let $G$ be $P$-generic over $V$, and let $g \subseteq \kappa$ be $Q_\kappa$-generic over $V[G]$. We claim that (1) $\kappa$ is not measurable in $V[G]$ and (2) $\kappa$ is measurable in $V[G][g]$.
<u>Claim 1:</u> *$\kappa$ is not measurable in $V[G]$*
Assume to the contrary that $\kappa$ remains measurable in $V[G]$, and let $j : V[G] \preceq \bar{M}$ be the corresponding ultrapower embedding. The statement that $V[G]$ is a forcing extension of $V$ by $P$ is first-order definable in some parameter over $V[G]$ (see []), and so by elementarity we can conclude that $\bar{M} = M[j(G)]$ for some $\bar{M}$-definable inner model $M \subseteq \bar{M}$ where $j(G)$ is $j(P)$-generic over $M$. Also by elementarity, $j(P)$ is the forcing in $M$ which adds a Cohen subset to every $M$-inaccessible cardinal below $j(\kappa)$. Now for cardinals $\lambda \leq \kappa$, $M$-inaccessible means the same as inaccessible in $V$ because $V_\kappa^M = V_\kappa$. It follows that $j(P) = P * Q_\kappa * P_{\text{tail}}$ where the tail forcing $P_{\text{tail}}$ is $\kappa$-closed in $M$. Since each condition $p \in P$ has size $< \kappa$, $j(p) = p$ and therefore $j``G = G$. So $j(G)$ splits as $G * g * G_{\text{tail}}$ where $g \subseteq \kappa$ is $Q_\kappa$-generic over $M[G]$.
$\mathcal{P}(\kappa)^V \subseteq M$, since if $x \in \mathcal{P}(\kappa)^V$, then $j(x) \in \mathcal{P}(j(\kappa))^M$ by elementarity, and so $x = j(x) \cap \kappa \in M$. Using the fact that every set in $H(\kappa+)^V$ can be coded into a subset of $\kappa$, it follows that $H(\kappa+)^V \subseteq M$. Every $x \in P(\kappa)^{V[G]}$ has a (code of a) $P$-name $\dot{x}$ in $H(\kappa+)^V$. So every $x \in P(\kappa)^{V[G]}$ has a name in $M$ by the previous discussion, and

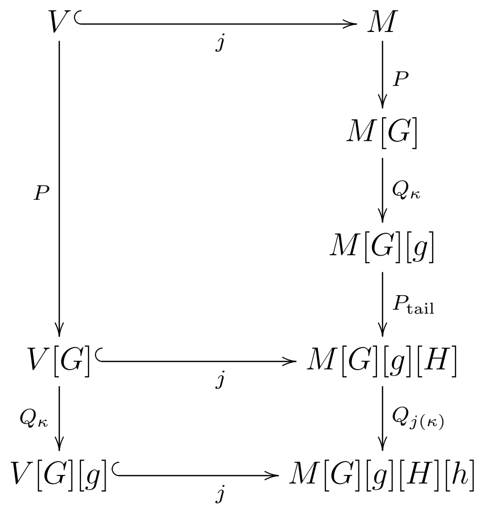
so $P(\kappa)^{V[G]} \subseteq M[G]$. In particular, $M[G]$ contains $g$, which is absurd since $g$ was generic over $M[G]$.




<u>Claim 2:</u> *$\kappa$ is measurable in $V[G][g]$*
Let $j : V \preceq M$ be the ultrapower embedding given by some $\kappa$-complete ultrafilter on $\kappa$. We show that $j$ can be lifted to $V[G][g]$ in a definable way.
By elementarity, $j(P)$ is the forcing in $M$ which adds a Cohen subset to every $M$-inaccessible cardinal below $j(\kappa)$. Now for cardinals $\lambda \leq \kappa$, $M$-inaccessible means the same as inaccessible in $V$ because $V_\kappa^M = V_\kappa$. It follows that $j(P) = P * Q_\kappa * P_{\text{tail}}$ where the tail forcing $P_{\text{tail}}$ is $\kappa$-closed in $M$. Since each condition $p \in P$ has size $< \kappa$, $j(p) = p$ and therefore $j"G = G$, which is also $P$-generic over $M$ since $M \subseteq V$.
We first construct a $V[G][g]$-definable lifting of $j$ to $V[G]$. For this we have to find a filter in $V[G][g]$ which is $j(P)$-generic over $M$ and extends $G$. By the product lemma, this amounts to finding a filter which is $Q_\kappa * P_{\text{tail}}$-generic over $M[G]$.
So let us work in $V[G][g]$. For the $Q_\kappa$-part of $Q_\kappa * P_{\text{tail}}$, we may just take $g$, which is generic over $V[G]$ and so also over $M[G]$. For the tail forcing $P_{\text{tail}}$ we make the following observations:
<u>Subclaim 2.1:</u> $|\mathcal{P}(P_{\text{tail}})^{M[G][g]}| = \kappa+$
$P_{\text{tail}}$ has size $j(\kappa)$, and so $|\mathcal{P}(P_{\text{tail}})^{M[G][g]}| = (2^{j(\kappa)})^{M[G][g]}$, which equals $(2^{j(\kappa)})^M$ since the forcing $P * Q_\kappa$ does not affect the cardinal arithmetic at $j(\kappa)$. Now $(2^{j(\kappa)})^M = |j(2^\kappa)|^V = |j(\kappa+)|^V$ by elementarity and the GCH in $V$. Furthermore $(|j(\kappa+)| = (\kappa+))^V$ by a basic property of the ultrapower embedding, and this remains true in the subsequent extension $V[G][g]$.
<u>Subclaim 2.2:</u> *$M[G][g]$ is $\leq \kappa$-closed*
$M$ is $\leq \kappa$-closed in $V$ by a basic property of the ultrapower embedding. By an argument similar to the one in the proof of (1), it then follows that $M[G][g]$ is $\leq \kappa$-closed in $V[G][g]$.
It follows from Claim 2.1 that we can list all dense subsets of $P_{\text{tail}}$ in $M[G][g]$ in a $\kappa+$-sequence. We now construct a descending sequence of conditions in $P_{\text{tail}}$ which hits every dense subset. The limit steps are

handled using Claim 2.2. The filter $H$ generated by this sequence is $P_{\mathrm{tail}}$-generic over $M[G][g]$.

We have thus shown that $j$ can be lifted to $j : V[G] \to M[G][g][H]$. To lift $j$ fully to $V[G][g]$, we have to find a $j(Q_\kappa) = Q_{j(\kappa)}$-generic filter over $M[G][g][H]$ which extends $g$. To do so, one repeats the arguments of Claim 1 and Claim 2 to show that one can construct in $V[G][g]$ descending sequences of conditions in $Q_{j(\kappa)}$ hitting all the dense sets in $M[G][g][H]$. $g$ can be viewed as a condition in $Q_{j(\kappa)}$; so take a sequence starting with $g$ which hits all dense sets in $M[G][g][H]$, and let $h$ be the filter generated by this sequence. Then $h \supset g$ and $h$ is $M[G][g][H]$-generic, and so we can lift $j$ to $j : V[G][g] \to M[G][g][H][h]$, proving that $\kappa$ is measurable in $V[G][g]$.




□

Proposition 4.2. *Let $\kappa \in V$ be measurable. Then there is a forcing extension of $V$ in which $\kappa$ is measurable, but not measurable in HOD.*

Proof. We modify the construction in Theorem 4.1 in the following way: After adjoining the $P$-generic filter $G$ to $V$, we do a further forcing $R$ which codes $(\mathcal{P}(\mathcal{P}(\kappa)))^{V[G]}$ into the continuum pattern sufficiently high above $\kappa$. Namely, $R$ should not add any new collections of subsets of $\kappa$. Let $I$ be $R$-generic over $V[G]$. Finally (as in Theorem 4.1), we add a Cohen generic subset $g \subseteq \kappa$. We will show that the resulting model $V[G][I][g]$ is as desired.

<u>Claim 1:</u> *$\kappa$ is not measurable in $W := HOD^{V[G][I][g]}$*
In the proof of Theorem 4.1, we showed that $\kappa$ is not measurable in $V[G]$. This carries over to $V[G][I]$ since $R$ preserves $(\mathcal{P}(\mathcal{P}(\kappa)))^{V[G]}$. So for the non-measurability of $\kappa$ in $W$, it suffices to show that $W$ and $V[G][I]$ have the same $\mathcal{P}(\mathcal{P}(\kappa))$.
By the weak homogenity of $Q_\kappa$, it follows that $W \subseteq V[G][I]$, and so

$(\mathcal{P}(\mathcal{P}(\kappa)))^W \subseteq (\mathcal{P}(\mathcal{P}(\kappa))^{V[G][I]}$. On the other hand, every set $x \in (\mathcal{P}(\mathcal{P}(\kappa))^{V[G][I]}$ is contained in $V[G]$ since $R$ does not add subsets to $\kappa$, and so $x$ can be read off the continuum function in $V[G][I]$. This remains so in the final model $V[G][I][g]$, and hence $x \in W$.

<u>Claim 2:</u> *$\kappa$ is measurable in $V[G][I][g]$*
The forcing $\mathrm{Add}(\kappa, 1)$ is the same in $V[G]$ and in $V[G][I]$, and so we can form the model $V[G][g]$. We have seen in Theorem 4.1 that $\kappa$ is measurable in $V[G][g]$. Now in $V[G]$, $R$ is $\leq \kappa$-closed and $\mathrm{Add}(\kappa, 1)$ is $\kappa$+-cc, and so by Easton's lemma $R$ remains $\leq \kappa$-distributive in $V[G][g]$. It follows that $R$ does not add subsets of $\kappa$ to $V[G][g]$, and so $\kappa$ remains measurable in $V[G][g][I] = V[G][I][g]$. □

## 2. Further results

The paper [**9**] contains many more constructions of models where large cardinals fail to be large in *HOD*.
For example, the following result strengthens Proposition 4.2:

PROPOSITION 4.3 ([**9**], p. 3f). *Assume that $\kappa$ is measurable in $V$. There is a forcing extension in which $\kappa$ is still measurable, but not weakly compact in HOD.*

And another variation on the same theme:

PROPOSITION 4.4 ([**9**], p. 4f). *Assume that $\kappa$ is supercompact in $V$. There is a forcing extension in which $\kappa$ is still supercompact, but not weakly compact in HOD.*

Using class forcing, one can get the following global result:

PROPOSITION 4.5 ([**9**], p.10f). *There is a class forcing extension of $V$ such that*

(1) *The supercompact cardinals of the extension are exactly the supercompact cardinals of V*
(2) *All supercompact cardinals fail to be weakly compact in the HOD of the extension*
(3) *There are no supercompact cardinals in the HOD of the extension.*

W. Hugh Woodin conjectures that there is a limit to these kinds of results, namely:

CONJECTURE 4.6 (Woodin). *If there is a supercompact cardinal, then there is a measurable cardinal in HOD.*

This conjecture is based on Woodin's *HOD dichotomy.* For the statement of this result, recall that a cardinal $\delta$ is called *extendible* if for every $\eta > \delta$ there is a $\theta > \eta$ and an elementary embedding $j : V_{\eta+1} \to V_{\theta+1}$ such that $crit(j) = \delta$ and $j(\delta) > \eta$.

Proposition 4.7 ([**13**]). *Assume that there is an extendible cardinal $\delta$. Then exactly one of the following holds:*

1. *For every singular cardinal $\gamma > \delta$, $\gamma$ is singular in HOD and $(\gamma^+)^{HOD} = \gamma^+$.*
2. *Every regular cardinal above $\delta$ is measurable in HOD.*

In [**13**], it is conjectured that (2) fails.

CHAPTER 5

# Background material

## 1. The extended Reflection principle

For each formula $\varphi(x)$ with one free variable, let us denote by $\mathrm{ExRef}_\varphi(\alpha)$ the formula

$$\exists\beta(\alpha \in \mathrm{Df}(V_\beta) \wedge \forall x \in V_\beta(\varphi(x)^{V_\beta} \leftrightarrow \varphi(x)))$$

(where $\alpha$ is a free parameter). The *Extended Reflection principle for* $\varphi$, denoted by $\mathrm{ExRef}_\varphi$, is the sentence

$$\forall\alpha \mathrm{ExRef}_\varphi(\alpha)$$

LEMMA 5.1. *For all formulas $\varphi(x)$, $\mathrm{ExRef}_\varphi$ is provable in ZF.*

PROOF. Assume there is a $\varphi(x)$ for which $\mathrm{ExRef}_\varphi$ fails, and let $\alpha_0$ be the least witness to this failure.
Now consider the formula

$$\alpha \text{ is minimal such that } \neg\mathrm{ExRef}_\varphi(\alpha)$$

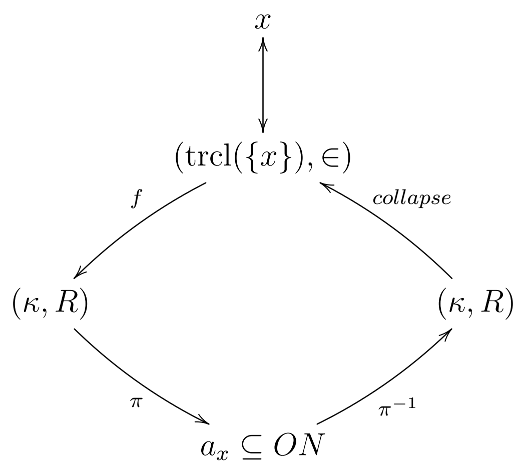
which we will denote by $\Phi(\alpha)$. $\alpha$ is treated as a free ordinal parameter again. By assumption, $\Phi(\alpha_0)$.
Now using the ordinary Reflection principle in ZF, pick $\beta$ such that $V_\beta$ reflects both $\Phi(\alpha)$ and $\varphi(x)$. Then $\alpha_0$ is the unique solution to $\Phi(\alpha)$ in $V_\beta$ by elementarity, and so $\alpha_0 \in \mathrm{Df}(V_\beta)$. But $V_\beta$ reflects $\varphi(x)$, contradicting the choice of $\alpha_0$. □

## 2. Coding sets into sets of ordinals

We give a proof that in the presence of the axiom of choice, every set can be coded into a set of ordinals. This often allows us to prove properties of the universe by proving them for sets of ordinals only.

LEMMA 5.2. *Assume $M$ is a transitive model of ZFC. Let $x$ be a set. Then there is a set of ordinals $a_x \in M$ which codes $x$ in the following way: whenever $N$ is a transitive model of ZF containing $a_x$, then $N$ already contains $x$.*

PROOF. Using the axiom of choice, let $\kappa = |\,\mathrm{trcl}(\{x\})|$ and pick a bijection $f : \mathrm{trcl}(\{x\}) \to \kappa$. Define a relation $R$ on $\kappa$ by setting $f(u)Rf(v) \leftrightarrow u \in v$ (so that $f : (\mathrm{trcl}(\{x\}), \in) \cong (\kappa, R)$). Finally let $\pi$ be a $\Delta_1$-definable pairing function of ordinals, and define $a_x := \pi''(R)$.




Now if $a_x \in N$ and $N$ is a transitive model of ZF, then $N$ can reconstruct $R$, since $\pi$ and its inverse are definable in $N$. The supremum of all ordinals occuring as a component in $R$ is exactly $\kappa$. Now $(\kappa, R)$ is a well-founded relation in $N$, since it was well-founded in $M$ and this is absolute. So $N$ can perform the Mostowski collapse on $(\kappa, R)$ to yield an isomorphic structure $(T, \in) \in N$ where $T$ is transitive. But now $(T, \in) \cong (\mathrm{trcl}(\{x\}), \in)$ and so $T = \mathrm{trcl}(\{x\})$. It follows that $x = \sup(T) \in N$. □

COROLLARY 5.3. *Let $M, N$ be two transitive models of ZFC. If $M$ and $N$ have the same sets of ordinals, then $M = N$.*

PROOF. Since the statement is symmetric in $M, N$, it suffices to show that $M \subseteq N$. So let $x \in M$. Since $M \models AC$, $x$ has an ordinal code $a_X$ which is contained in $N$ by assumption. But then $x \in N$ by the previous Lemma. □

## 3. Basic forcing facts

*Forcing* is a method to adjoin sets to a given countable transitive model $M$ of ZFC (called the *ground model*) while preserving the ZFC axioms. There has already been a notion of adjointment in the pre-forcing time of set theory, namely by means of *relative constructibility*. We briefly discuss a special case of this: If $L_\alpha \models$ ZF and $G \notin L_\alpha$ is a real, a structure $L_\alpha[G]$ can be defined by iterating definability up to $\alpha$ - as one does for $L_\alpha$ - but now with an additional predicate for the set $G$. It is then easy to see that $L_\alpha[G]$ is a transitive superset of $L_\alpha$ containing $G$ as an element. Furthermore, if $L_\alpha[G] \models$ ZF, then it is the smallest model of ZF with this property.
The bad news is that $L_\alpha[G]$ often fails to be a model of ZF. For example, if $L_\alpha$ is countable, $G$ can code an enumeration of $\alpha$ in order type $\omega$. Then $L_\alpha[G]$ contains an enumeration of its ordinal height, and so Replacement fails in $L_\alpha[G]$.

It was proved by Paul Cohen that if $G$ has a special property, called *genericity*, then all ZF axioms are preserved in $L_\alpha[G]$. The assumption that the ground model satisfies $V = L$ turned out to be unneccessary for the theory: Starting from any transitive model $M$ of ZF, one can construct a larger model $M[G] \models$ ZF by adjoining a generic set $G$. Apart from the ZF axioms, the first-order theory of $M[G]$ largely depends on the choice of the generic $G$.

We give a quick overview on the general theory. $M$ is always a transitive model of ZF.

Let $P \in M$ be a quasi-order with maximal element 1. A $P$-name is a set of tuples $(\pi, p)$ where $p \in P$ and $\pi$ is a $P$-name (this is of course a recursive definition). The class of $P$-names in $M$ is denoted by $M^P$. Given $\sigma \in M^P$ and any set $G \subseteq P$, we recursively define

$$\sigma[G] = \{\pi[G] \mid (\pi, p) \in \sigma \wedge p \in G\}$$

$\sigma[G]$ is called the *evaluation* of $\sigma$ by $G$. For every $x \in M$ we can define the *check name* $\check{x}$ recursively by

$$\check{x} = \{(\check{y}, 1) \mid y \in x\}.$$

Clearly, if $1 \in G$, then $\check{x}[G] = x$. Furthermore, we set

$$M[G] = \{\sigma[G] \mid \sigma \in M^P\}$$

If $1 \in G$, then $M \subseteq M[G]$, and $G \in M[G]$ since $G$ is the evaluation of the name $\{(\check{p}, p) \mid p \in P\}$. One may also check that $M[G]$ is transitive and closed under some basic set-theoretic operations like pairing. To get more structure, we have to put more restrictions on $G$.

First, call a set $D \subseteq P$ *dense* if for all $p \in P$, there is a $q \leq p$ such that $q \in D$. We now call a set $G$ *$P$-generic over $M$* if $G$ is a filter on $P$ and $G$ intersects all dense sets $D \in M$.

Lemma 5.4. *Assume that $M$ is countable. Then for every $p \in P$, there is a $P$-generic filter $G$ containing $p$. Furthermore, if $P$ is non-atomic*[1]*, then $G \notin M$.*

Proof. List all dense subsets of $P$ in $M$ as $D_1, D_2, D_3, \ldots$ Now define a sequence $(p_n)_{n\in\omega}$ by letting $p_0 = p$ and choosing $p_{n+1} \leq p_n$ such that $p_{n+1} \in D_{n+1}$. This is possible because $D_{n+1}$ is dense. Then the filter $G$ generated by the $p_n$'s is generic.

If $P$ is non-atomic, then $P \setminus G$ is a dense set not intersected by $G$, and so $P \setminus G$ cannot be contained in $M$ if $G$ is generic. It follows that $G$ cannot be contained in $M$ either. □

[1] i.e. $\forall p \in P \exists q, r \in P(q, r \leq p \wedge \neg\exists v \in P(v \leq q \wedge v \leq r))$

For $p \in P$, any formula $\varphi(x_0, \dots, x_n)$ and $\sigma_0, \dots, \sigma_n \in M^P$ we define the *forcing relation*

$$p \Vdash_P \varphi(\sigma_0, \dots, \sigma_n) :\Leftrightarrow \\ M[G] \models \varphi(\sigma_0[G], \dots, \sigma_n[G]) \text{ for all } P\text{-generic G}$$

The subscript $P$ is often dropped when clear from context. The essential results about forcing are the *Definability Theorem* and the *Forcing Theorem*.

THEOREM 5.5 (Definability Theorem). *For any formula $\varphi(x_0, \dots, x_n)$, the class*

$$\{(P, p, \sigma_0, \dots, \sigma_n) \mid p \in P,\ \sigma_1, \dots, \sigma_n \in M^P \text{ and } p \Vdash_P \varphi(\sigma_0, \dots, \sigma_n)\}$$

*is definable in $M$.*

PROOF. See [**16**, p. 251f]. □

THEOREM 5.6 (Forcing Theorem). *Assume that $G$ is $P$-generic over $M$. Then for any formula $\varphi(x_0, \dots, x_n)$ and any names $\sigma_0, \dots, \sigma_n \in M^P$*

$$M[G] \models \varphi(\sigma_0[G], \dots, \sigma_n[G]) \Leftrightarrow \exists p \in G(p \Vdash \varphi(\sigma_0, \dots, \sigma_n))$$

PROOF. See [**16**, p.257f]. □

Using both theorems, it is not too hard to prove that $M[G]$ satisfies ZF:

THEOREM 5.7. *Assume that $G$ is $P$-generic over $M$. Then $M[G] \models$ ZF. If $M$ satisfies AC, then so does $M[G]$.*

PROOF. See [**16**, p.252f]. (Very) roughly speaking, the proof proceeds as follows: To show that a certain set $x$ exists in $M[G]$, one uses the Definability Theorem to cook up a name $\sigma$ for it, and then uses the Forcing Theorem to show that indeed $\tau[G] = x$. □

## 4. Some forcings

Let $x, y$ be sets and $\kappa$ be a cardinal. $\mathrm{Fn}_\kappa(x, y)$ denotes the set of all partial functions of size $< \kappa$ from $x$ to $y$. We make this set into a partial order by setting

$$p \leq q :\leftrightarrow p \supseteq q$$

for $p, q \in \mathrm{Fn}_\kappa(x, y)$.
Any bijection $\pi : x \to x$ of the set $x$ induces an automorphism $\tilde{\pi}$ of $\mathrm{Fn}_\kappa(x, y)$ via $\tilde{\pi}(p) = p \circ \pi$. It follows that $\mathrm{Fn}_\kappa(x, y)$ is weakly homogeneous.

**4.1. Cohen forcing.** For regular $\kappa$ and some cardinal $\lambda$ we set

$$\mathrm{Add}(\kappa, \lambda) := \mathrm{Fn}_\kappa(\kappa \times \lambda, 2)$$

$\mathrm{Add}(\kappa, \lambda)$ is $\kappa$-closed and has the $(2^{<\kappa})^+$-cc by a $\Delta$-system argument. A standard density argument shows that

$$1 \Vdash_{\mathrm{Add}(\kappa,\lambda)} 2^\kappa \geq |\lambda|$$

**4.2. The Levy Collapse.** For infinite regular cardinals $\kappa < \lambda$ we set

$$\mathrm{Coll}(\kappa, \lambda) := \mathrm{Fn}_\kappa(\kappa, \lambda)$$

$\mathrm{Coll}(\kappa, \lambda)$ is $\kappa$-closed and has the $(\lambda^{<\kappa})^+$-cc by a $\Delta$-system argument. By a density argument,

$$1 \Vdash_{\mathrm{Coll}(\kappa,\lambda)} |\lambda| \leq \kappa$$

$\mathrm{Coll}(\omega, \kappa)$ has a nice uniqueness property by the following classic result:

PROPOSITION 5.8. *Let $P$ be a forcing of size $\kappa$ which collapses $\kappa$ to a countable ordinal. Then $P$ and* $\mathrm{Coll}(\omega, \kappa)$ *are forcing equivalent: There is a forcing $P'$ such that both $P$ and* $\mathrm{Coll}(\omega, \kappa)$ *can be densely embedded into $P'$.*

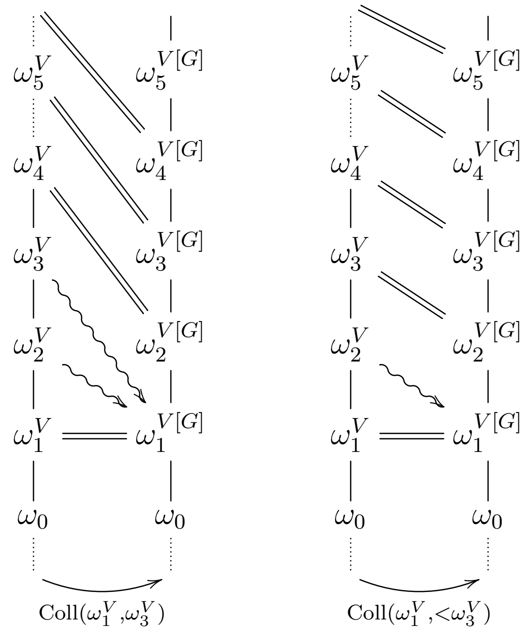
PROOF. See [**18**, p. 129]. □

We define $\mathrm{Coll}(\kappa, < \lambda)$ to be the $\kappa$-product of all $\mathrm{Coll}(\kappa, \alpha)$, where $\kappa < \alpha < \lambda$ is a cardinal. It follows that

$$\Vdash_{\mathrm{Coll}(\kappa,<\lambda)} \lambda = \kappa +$$

so that $\lambda$ is collapsed to the successor of $\kappa$.

If $\lambda^{<\kappa} = \lambda$, $\mathrm{Coll}(\kappa, \lambda)$ preserves all cardinals up to $\kappa$ (by $\kappa$-closure) and above $\lambda$ (by $\lambda^+$-cc), while $\lambda$ is collapsed to $\kappa$.




**4.3. Forcing the GCH.** We may use a product of collapses to force the GCH in a (class-)generic extension. For this, we take $P$ to be the Easton product of $\mathrm{Coll}(\omega_\alpha, < 2^{\omega_\alpha})$, which collapses $2^{\omega_\alpha}$ to $\omega_\alpha^+$ for all $\alpha \in ON$.

## 5. Easton forcing

An *Easton index function* is a function $E$ defined on a set of regular cardinals such that for all $\kappa, \lambda \in \mathrm{dom}(E)$

- $E(\kappa)$ is a cardinal
- $\kappa < \lambda \rightarrow E(\kappa) \leq E(\lambda)$
- $\mathrm{cof}(E(\kappa)) > \kappa$

Intuitively, $E$ is a possibility of how the continuum function could behave on $\mathrm{dom}(E)$.
In the following, fix some Easton index function $E$.
*Easton forcing* is the partial order which forces $2^\kappa = E(\kappa)$ for all $\kappa \in \mathrm{dom}(E)$, assuming that the ground model satisfies the GCH. It is defined in the following way:
First, consider the (full) product $R = \prod_{\kappa \in \mathrm{dom}(E)} \mathrm{Add}(\kappa, E(\kappa))$. For a condition $p \in R$, let $\mathrm{supp}(p) = \{\kappa \in \mathrm{dom}(E) \mid p(\kappa) \neq \emptyset\}$. We now let $P_E$ be the subposet of $R$ consisting of all conditions $p$ such that $\mathrm{supp}(p)$ is bounded below any regular cardinal $\kappa$ (not necessarily from $\mathrm{dom}(E)$). This condition is only then non-trivial when $\kappa$ is a limit, i.e. when $\kappa$ is an inaccessible cardinal.
For a regular cardinal cardinal $\lambda$, let $P_E^{\leq\lambda} = \{p \in P_E \mid \mathrm{supp}(p) \subseteq \lambda \cup \{\lambda\}\}$ and $P_E^{>\lambda} = \{p \in P_E \mid \mathrm{supp}(p) \cap \lambda = \emptyset\}$. Then $P_E = P_E^{\leq\lambda} \times P_E^{>\lambda}$.

LEMMA 5.9. *$P_E^{\leq\lambda}$ is $(\lambda+)$-cc and $P_E^{>\lambda}$ is $\leq \lambda$-closed.*

PROOF. See [**15**, p. 233]. □

The proof for the $(\lambda+)$-cc is where the bound on the supports is needed.

PROPOSITION 5.10. *Let $G \times H$ be $V$-generic over a product $P \times Q$, where $P$ is $\leq \kappa$-closed and $Q$ is $(\kappa+)$-cc. Then $P$ is $\leq \kappa$-distributive in $V[H]$.*

PROOF. See [**15**, p. 234]. □

COROLLARY 5.11. *$P_E$ preserves cardinals.*

PROOF. It suffices to show that $P_E$ preserves regular cardinals. Assume to the contrary that this fails, i.e. there is a regular cardinal $\kappa$ such that for some $P_E$-generic $G$, $V[G]$ has a cofinal map $f : \lambda \to \kappa$ where $\lambda < \kappa$ is regular in $V[G]$, and therefore also in $V$.
By factorizing $P_E$ into $P_E^{>\lambda} \times P_E^{\leq\lambda}$, we can write $V[G]$ as a two-step extension $V[G_0][G_1]$ where $G_0$ is $P_E^{>\lambda}$-generic over $V$ and $G_1$ is $P_E^{\leq\lambda}$-generic over $V[G_0]$. Now using Lemma 5.9 and Proposition 5.10, we conclude that $f$ must already exist in $V[G_0]$, and so $\kappa$ fails to be regular in $V[G_0]$. But this cannot be, since $\kappa$ was regular in $V$ and $P_E^{\leq\lambda}$ has the $(\lambda+)$-cc.

□

COROLLARY 5.12. *Let $G$ be $P_E$-generic. Then in $V[G]$, $2^\kappa = E(\kappa)$ for all $\kappa \in \mathrm{dom}(E)$.*

## 6. Automorphisms of partial orders

Let $P$ be a forcing poset. A map $i : P \to P$ is an *automorphism* of $P$ if is bijective, $i(1) = 1$ and $p \leq q \Leftrightarrow i(p) \leq i(q)$ holds for all $p, q \in P$.

Every automorphism $i$ of $P$ induces a bijection $i^*$ on the class of $P$-names by setting

$$i^*(\sigma) = \{(i^*(\tau), i(p)) \mid (\tau, p) \in \sigma\}$$

Note that $i^*(\check{x}) = \check{x}$ for all check names $\check{x}$. The relevance of automorphisms to the theory of forcing lies in the following Lemma, which can be proved by an induction on formulas.

LEMMA 5.13. *Let $i$ be an automorphism of $P$. Then for all formulas $\varphi(x_0, \ldots, x_n)$, $\sigma_0, \ldots \sigma_n \in M^P$ and all $p \in P$*

$$p \Vdash \varphi(\sigma_0, \ldots, \sigma_n) \quad \Leftrightarrow \quad i(p) \Vdash \varphi(i^*(\sigma_0), \ldots, i^*(\sigma_n))$$

PROOF. See [**16**, p. 270f]. □

## 7. The Lévy hierarchy

In the following, $\mathcal{L}$ is some fixed recursive extension of the language of set theory.
By $\Delta_0$ we denote the class of $\mathcal{L}$-formulas in which all quantifiers are bounded, i.e. of the form $\forall x \in t$ or $\exists x \in t$ for some $\mathcal{L}$-term $t$. Set $\Sigma_0 = \Pi_0 = \Delta_0$.
We inductively define the classes $\Sigma_n$ and $\Pi_n$ for $n > 0$.
A formula is $\Sigma_n$ if it is of the form

$$\exists x_0 \ldots \exists x_k \varphi$$

for some $k \in \omega$ and $\varphi \in \Pi_{n-1}$. A formula is $\Pi_n$ if it is of the form

$$\forall x_0 \ldots \forall x_k \varphi$$

for some $k \in \omega$ and $\varphi \in \Sigma_{n-1}$.

Let now $T$ be an $\mathcal{L}$-theory. By $\Sigma_n^T$ we denote the class of all $\mathcal{L}$ formulas which are $T$-provably equivalent to some $\Sigma_n$-formula, that is the class of all $\varphi$ for which there is a $\psi \in \Sigma_n$ such that $T \vdash (\varphi \leftrightarrow \psi)$. $\Pi_n$ is defined analogously.
Finally, we set $\Delta_n^T = \Sigma_n^T \cap \Pi_n^T$.
With this notation, it is easy to see that (for any $T$) every formula is contained in some $\Sigma_n^T$. Furthermore, the class $\Pi_n^T$ consists exactly of the negations of formulas in $\Sigma_n^T$ and vice versa. If $T$ contains some basic set theory including the Pairing axiom, then every $\Sigma_n^T$-formula has a representation of the form

$$\exists x_0 \forall x_1 \ldots Q x_{n-1} \psi$$

where $Q = \forall$ if $n$ is even and $Q = \exists$ otherwise, and $\pi$ is $\Delta_0$.
Given a model $\mathfrak{M}$ of (some fragment of) ZF, we say that $\varphi$ is $\Sigma_n$ over $\mathfrak{M}$ if there is is a $\Sigma_n$ formula $\bar{\varphi}$ such that $\mathfrak{M} \models \varphi \leftrightarrow \bar{\varphi}$. If $a \in \mathfrak{M}$, we say that $\varphi$ is $\Sigma_n(a)$ if in the language extended by a constant for $a$, $\varphi$ is $\Sigma_n$ over $\mathfrak{M}$.

## 8. Arithmetization and truth predicates

Every set-theoretic formula $\varphi$ has a natural definable representation within any model of ZF, which we will also denote by $\varphi$.
(The converse is not true: If $\varphi$ is a formula in the sense of some model $\mathfrak{M} \models$ ZF, it might not correspond to a formula in the metatheory. This happens exactly if $\mathfrak{M}$ contains non-standard natural numbers, and therefore formulas of non-standard length.)
A predicate $T$ is called a *truth predicate* if for all $\varphi \in \mathcal{L}_{\text{ZF}}$

$$\text{ZF} \vdash \varphi \leftrightarrow T(\varphi)$$

$T$ is called a $\Sigma_n$ *truth predicate for* $\Sigma_n$ if the above equivalence holds at least for all $\varphi \in \Sigma_n^{\mathrm{ZF}}$.

LEMMA 5.14. *For each natural number n there is a definable truth predicate for* $\Sigma_n$*. In more detail, there is a formula* $SAT_n(w, \bar{x})$ *such that for each* $\varphi(\bar{x}) \in \Sigma_n$

$$\forall \bar{x}(\varphi(\bar{x}) \leftrightarrow SAT_n(\varphi, \bar{x}))$$

$SAT_n$ *is itself* $\Sigma_n$ *for every* $n > 0$*, and* $SAT_0$ *is* $\Delta_1$*.*

PROOF. We state the definition of $SAT_n(\bar{x}, z)$ by (meta-)induction on $n$. It will be clear that the formulas work as espected.
Note first that all syntactial notations like $x$ *is a formula*, $x$ *has n free variables* etc. can be written as $\Delta_0$ formulas using some reasonable encoding.
For $n = 0$, recall that $\Delta_0$-formulas are absolute for all transitive classes. Therefore we can set

$$SAT_0(\varphi, \bar{x}) \equiv \exists M(M = \mathrm{trcl}(\{\bar{x}\}) \wedge M \models \varphi(\bar{x}))$$

or equivalently,

$$SAT_0(\varphi, \bar{x}) \equiv \forall M(M = \mathrm{trcl}(\{\bar{x}\}) \rightarrow M \models \varphi(\bar{x}))$$

Clearly, $SAT_0$ is $\Delta_1$.
Assume now that for some $n \in \omega$, $SAT_n$ is a truth predicate for $\Sigma_n$ which is itself $\Sigma_n$ (or $\Delta_1$ if $n = 0$). First note that any $\varphi \in \Sigma_{n+1}$ is ZF-provably equivalent to a formula in prenex normal form where all blocks of quantifiers are contracted into a single quantifier. So we may assume that $\varphi \in \Sigma_{n+1}$ is of the form $\exists v \psi(v, x)$ where $\psi(v, x)$ is $\Pi_n$. Now using the equivalence $\varphi(x) \leftrightarrow \neg\forall v \neg\psi(v, x)$ and the fact that $\neg\psi(v, x)$ is $\Sigma_n$, we can set

$$SAT_{n+1}(\varphi, x) \equiv \neg\forall v SAT_n(\neg\psi, v, x)$$

which is $\Sigma_{n+1}$ as desired. □

COROLLARY 5.15. *For every* $n \in \omega$*, there is a club of* $\alpha$*'s such that* $V_\alpha \preceq_n V$*.*

PROOF. Apply the Reflection Theorem to $SAT_n$. □

In what follows, we tacitly assume Con(ZF).

LEMMA 5.16. *No* $\Sigma_n$ *truth predicate can be* $\Pi_n$*-definable.*

PROOF. Assume to the contrary that $T$ is a $\Pi_n$-definable truth predicate for $\Sigma_n$. ZF is strong enough to prove Gödel's fixed point lemma. So there is a sentence $\varphi$ satisfying

$$\varphi \leftrightarrow \neg T(\varphi)$$

Now if $T$ was $\Pi_n$, then $\varphi$ would be a $\Sigma_n$-sentence which could be evaluated using the predicate $T$. But this leads immediately to the contradiction

$$T(\varphi) \leftrightarrow \varphi \leftrightarrow \neg T(\varphi).$$

□

COROLLARY 5.17. *The Lévy hierarchy is proper:*

$$\Sigma_0^{\mathrm{ZF}} \subsetneq \Sigma_1^{\mathrm{ZF}} \subsetneq \Sigma_2^{\mathrm{ZF}} \subsetneq \dots$$

PROOF. For every $n \in \omega$, $SAT_{n+1}$ is $\Sigma_{n+1}$ but not $\Sigma_n$. □

COROLLARY 5.18 (Tarski). *There is no universal truth predicate. That is, there is no formula $SAT(w, \bar{x})$ such that for all $\varphi \in \mathcal{L}_{\mathrm{ZF}}$*

$$\forall \bar{x}(\varphi(\bar{x}) \leftrightarrow SAT(\varphi, \bar{x}))$$

PROOF. Otherwise, the Lévy hierarchy would collapse to the complexity of $SAT$, contradicting the previous corollary. □

## 9. Filters, ultrafilters and measurable cardinals

Let $\kappa$ be an uncountable regular cardinal.
A *filter* $U$ on $\kappa$ is a non-empty collection of subsets of $\kappa$ which is closed under taking supersets and under taking finite intersections. To avoid trivialities, one furthermore requires that $U$ contains no bounded subsets of $\kappa$.
$U$ is called *principal* if it is of the form $U = \{X \subseteq \kappa \mid A \subseteq X\}$ for some $A \subseteq \kappa$. Otherwise, $U$ is called *non-principal.*
$U$ is an *ultrafilter on* $\kappa$ if for all $X \subseteq \kappa$, either $X \in U$ or $\kappa \setminus X \in U$.
Principle ultrafilters are too simple to be of interest.
Let $\lambda$ be a cardinal. We say that $U$ is $\lambda$-*complete* if $U$ is closed under intersections of size $< \lambda$. *Countably complete* means the same as $\omega_1$-complete. So every filter $U$ is $\omega$-complete by definition, and in the natural situation that $U$ contains all tail intervals $[\alpha, \kappa)$ for $\alpha < \kappa$, it follows that $U$ can be at most $\kappa$-complete because $\bigcap_{\alpha<\kappa} [\alpha, \kappa) = \emptyset \notin U$.
$\kappa$ is called *measurable* if there is a non-principal, $\kappa$-complete ultrafilter on $\kappa$. This turns out to be a large cardinal notion:

PROPOSITION 5.19. *Every measurable cardinal is inaccessible.*

PROOF. See [**18**, p. 26]. □

## 10. Elementary embeddings and ultrapowers

**10.1. Elementary embeddings.** Let $\mathcal{M}, \mathcal{N}$ be structures for some first-order language $\mathcal{L}$.
A map $j : \mathcal{M} \to \mathcal{N}$ is called an $(\mathcal{L}-)$*elementary embedding* if for all $\mathcal{L}$-formulas $\varphi(v_0, \dots, v_n)$ and all $a_0, \dots, a_n \in \mathcal{M}$

$$\mathcal{M} \models \varphi(a_1, \dots, a_n) \quad \Leftrightarrow \quad \mathcal{N} \models \varphi(j(a_1), \dots, j(a_n))$$

$j : \mathcal{M} \to \mathcal{N}$ is called a $\Sigma_n$-elementary embedding if the above equivalence holds for all $\varphi \in \Sigma_n$.
In any case, it follows that $j$ is an injective $\mathcal{L}$-homomorphism.
We write $j : \mathcal{M} \preceq \mathcal{N}$ to denote that $j$ is an elementary embedding from $\mathcal{M}$ to $\mathcal{N}$. We write $\mathcal{M} \preceq \mathcal{N}$ if such a $j$ exists. Similarly, one defines $j : \mathcal{M} \preceq_n \mathcal{N}$ and $\mathcal{M} \preceq_n \mathcal{N}$.
We are mostly interested in the case where $\mathcal{M}, \mathcal{N}$ are transitive *class* models of ZF(C). In this case, the concept of elementary embeddability as stated above is not definable in the language of set theory. It is however possible to express the statement $\mathcal{M} \preceq_1 \mathcal{N}$, and we take $\mathcal{M} \preceq \mathcal{N}$ to mean exactly that in this context. This is justified by the observation that if $\mathcal{M} \preceq_1 \mathcal{N}$ and $\mathcal{M}, \mathcal{N} \models \mathrm{ZF}$, then in fact $\mathcal{M} \preceq_{\mathrm{n}} \mathcal{N}$ for every natural number $n$ in the meta-theory (see [**18**, p. 45f]).

**10.2. Ultrapowers.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $X$ a set and $U$ an ultrafilter on $X$. Then $\mathcal{M}^U$ is the $\mathcal{L}$-structure given by the following data:

(1) The universe of $\mathcal{M}^U$ is the set of all functions from $X$ to $\mathcal{M}$ modulo the equivalence relation
$$f \sim g \Leftrightarrow \{x \in X \mid f(x) = g(x)\} \in U$$
. As usual, let $[f]$ denote the equivalence class of $f : X \to \mathcal{M}$ under $\sim$.
(2) If $R$ is an unary relation symbol in $\mathcal{L}$, then
$$[f] \in R^{\mathcal{M}^U} \Leftrightarrow \{x \in X \mid f(x) \in R^{\mathcal{M}}\} \in U$$
. Similary for $n$-ary $R$ where $n > 0$.
(3) If $F$ is an unary function symbol in $\mathcal{L}$, then $F^{\mathcal{M}^U} : \mathcal{M}^U \to \mathcal{M}^U$ is the coordinate-wise application of $F^{\mathcal{M}}$. More explicitly, $F^{\mathcal{M}^U}([f]) = [(F^{\mathcal{M}}(f(x)))_{x \in X}]$.

Proposition 5.20 (Łos). *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $\kappa$ a cardinal and $U$ an ultrafilter on $\kappa$. Then for every $\mathcal{L}$-formula $\varphi(v_0, \ldots, v_n)$ and functions $f_0, \ldots, f_n : \kappa \to \mathcal{M}$*
$$\mathcal{M}^U \models \varphi([f_0], \ldots, [f_n]) \quad \Leftrightarrow \quad \{\alpha < \kappa \mid \mathcal{M} \models \varphi(f_0(\alpha), \ldots, f_n(\alpha)\} \in U$$

Under the above assumptions, one sees that the map $j : \mathcal{M} \to \mathcal{M}^U$ which sends an $m \in M$ to the equivalence class of the constant function $f : \kappa \to \{m\}$ is an elementary embedding, called the *canonical ultrapower embedding (given by U)*.

**10.3. Ultrapowers of V.** The ultrapower construction can be carried out inside a class model $V$ of set theory by a slight modification of the above construction. Let $\kappa \in V$ be a cardinal and $U \in V$ an ultrafilter over $\kappa$. Instead of working with the full eqivalence classes

$[f]$ (which are now class-size) one now picks a $V$-definable set of representatives for $[f]$ and defines $V^U$ to be the collection of all these sets of representatives. This yields a $V$-definable class model $V^U \subseteq V$ and a $V$-definable elementary embedding $j : V \preceq V^U$.
As usual, we want to work with transitive models.

LEMMA 5.21. *If $U$ is countably complete, then $V_U$ is well-founded and therefore isomorphic to a transitive model.*

PROOF. Assume there is an infinite descending chain

$$[f_0] \ni [f_1] \ni [f_2] \ni \dots$$

in $V^U$. This means that for each $i \in \omega$ there is a set $U_i \in U$ such that $f_i(\alpha) \ni f_{i+1}(\alpha)$ for all $\alpha \in U_i$. By countable completeness, pick an $\alpha \in \bigcap U_i \neq \emptyset$. Then $f_0(\alpha) \ni f_1(\alpha) \ni f_2(\alpha) \ni \dots$ is an infinite descending chain in $V$, which is absurd.
Of course $V^U$ is also extensional by elementarity. So $V^U$ is isomorphic to a transitive inner model $M \subseteq V$ by Mostowski's collapsing theorem. □

The situation is summed up by

$$j : V \preceq V^U \cong M \subseteq V$$

For $x \in V$, we usually identify $j(x)$ with its collapse in $M$.
Let $crit(j)$ be the least $\alpha$ such that $j(\alpha) \neq \alpha$, if such an $\alpha$ exists. By elementarity, $crit(j)$ is also an ordinal.

PROPOSITION 5.22. *Let $U$ be a $\kappa$-complete ultrafilter on $\kappa$ and let $j : V \preceq V^U \cong M$ be the corresponding ultrapower embedding. Then:*

(1) $crit(j) = \kappa$.
(2) $j(x) = x$ *for every* $x \in V_\kappa$.
(3) $2^\kappa \leq (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$.
(4) $M$ *is closed under taking* $\kappa$*-sequences.*

PROOF. See [**18**, p. 50]. □

PROPOSITION 5.23. *If $M$ is an inner model an $j : M \preceq V$, then $crit(j)$ is a measurable cardinal.*

PROOF. See [**18**, p. 49f]. □

## 11. Lifting elementary embeddings

Let $j : V \preceq M$ be an elementary embedding. Now assume we have a partial order $P \in V$ and a $V$-generic $G \subseteq P$. By elementarity, $j(P)$ is a partial order in $M$, and $j$"$G \subseteq j(P)$. What follows is the basic observation about lifting of elementary embeddings.

LEMMA 5.24. *Let $H \subseteq j(P)$ be generic over $M$. The following are equivalent:*

(1) *$j$"$G \subseteq H$*
(2) *There is an elementary embedding $j^+ : V[G] \preceq M[H]$ such that $j^+ \restriction_V = j$ and $j^+(G) = H$*

PROOF. For the backward direction, if $p \in G$ then $j^+(p) \in j^+(G)$ by elementarity, and $j^+(p) = p$, $j^+(G) = H$ by the assumptions.
Conversely, assume that $j$"$G \subseteq H$. For $x = \sigma^G \in V[G]$, we try to set $j^+(x) = j(\sigma)^H$. To see that this is well-defined, assume that $\sigma^G = \tau^G$ and pick $p \in G$ forcing this. By elementarity, $j(p) \Vdash j(\sigma) = j(\tau)$ in $M$. Now $j(p) \in j$"$G \subseteq H$ and so $j(\sigma)^H = j(\tau)^H$ in $M[H]$.
The elementarity of $j^+$ is proved similarly: If $V[G] \models \varphi(x)$ then pick $p \in G$ forcing $\varphi(\dot{x})$. By elementary $j(p) \Vdash \varphi(j(\dot{x}))$ in $M$ and so $M[H] \models \varphi(j(x))$ since $j(p) \in H$. If $V[G] \not\models \varphi(x)$, then $V[G] \models \neg\varphi(x)$ and one can use exactly the same argument as before to conclude that $M[H] \not\models \varphi(j(x))$.
If $x \in M$, then $x = \check{x}^G$ and so $j^+(x) = j(\check{x})^H$. But $j(\check{x}) = \check{j(x)}$ by absoluteness, and so $j^+(x) = j(\check{x})^H = \check{j(x)}^H = j(x)$.
Finally let $\dot{G}$ be the canonical $P$-name for $G$. Then $j(\dot{G}) = \dot{H}$ by elementarity and so $j^+(G) = H$ by the definition of $j^+$. □

It is important to note that $j^+$ does not need to be $V[G]$-definable, in fact $M[H]$ does not even need to be contained in $V[G]$.
Thus if $j$ is the ultrapower embedding induced by some measurable cardinal $\kappa \in V$, it does *not* follow in the above situation that $j^+$ witnesses the measurability of $\kappa$ in $V[G]$.

Let us discuss a special case where the forcing $P$ satisfies the following:
- $j(P) \cong P * Q$ for some partial order $Q \in V$
- $j$"$G = G$

For example, consider the case that $j$ is an ultrapower embedding and $crit(j) = \kappa$. Let $P$ be an iteration of forcings $P_\alpha$, $\alpha < \kappa$ such that $|P_\alpha| < \kappa$. By elementarity, $j(P)$ is an iteration of length $j(\kappa)$, and for $\alpha < \kappa$, $j(P_\alpha) = P_\alpha$ by the size restriction on $P_\alpha$. Thus $j(P)$ splits as $P * Q_{tail}$ for some tail iteration $Q_{tail}$.
If we require additionally that the supports of conditions in $P$ are bounded, i.e. of size $< \kappa$, then $j \restriction P = id$ and therefore $j$"$G = G$.
Let now $G$ be $P$-generic over $V$. Since $M \subseteq V$, $G$ is also $P$-generic over $M$. Furthermore let $K$ be $Q$-generic over $V[G]$. Again, $K$ is also $Q$-generic over $M[G]$. By the product lemma, $H = G * K$ is $j(P)$-generic over $V$ (and therefore over $M$) and thus Lemma 5.24 applies.

## 12. Class forcing

In *Class forcing*, one forces with a partial order $P$ which is a proper class in the ground model. There are some technical obstacles to make this work, and several distinctions to set forcing arise.

We will deal with structures of the form $(M, A_1, \ldots, A_n)$ where $M$ is a transitive model of some set theory and $A_i \subseteq M$ for each $i \leq n$. A class $U \subseteq M$ is called $(M, A_1, \ldots, A_n)$-definable if it is definable in $M$ from set parameters and the classes $A_1, \ldots, A_n$ (viewed as predicates). We say that $(M, A_1, \ldots, A_n) \models$ ZF if $M \models$ ZF and the Replacement scheme holds in $M$ for formulas mentioning the $A_i$'s as predicates.
Since each finite number of classes $A_1, \ldots A_n$ can be definably coded into a single class $A$, we will from now on restrict ourselves to the case that $n = 1$. So fix some ground model $(M, A) \models$ ZF.
A *class forcing* $P \subseteq M$ is a $(M, A)$-definable class quasi-order with maximal element 1. Given such, one defines the class $M^P \subseteq M$ of $P$-names as in the set forcing case (in particular, names are still sets). One does not have a name for the generic object, since this would have to be a proper class. However, for each $\alpha$ one can set

$$\dot{G}_\alpha = \{(\check{p}, p) \mid p \in P \cap V_\alpha\}$$

as an approximation.
Given any $G \subseteq P$, $M[G]$ is defined as in set forcing, and likewise we have to impose some structure on $G$ to achieve that $M[G]$ satisfies more than just the most elementary set theory. The right generalization here is this: We say that $G$ is *$P$-generic over* $(M, A)$ if $G$ intersects every dense $(M, A)$-definable subclass of $P$.
There is some flexibility in what one actually takes to be the generic extension by $G$. One may either look at the structure $M[G]$ only, or at the expanded structures $(M[G], G)$ or even $(M[G], M, A, G)$. Just for the moment, let $\mathfrak{M}$ denote one of these choices. The point is that we want to have $\mathfrak{M} \models$ ZF, which means exactly that $M[G] \models$ ZF and that the Replacement scheme holds for all $\mathfrak{M}$-definable classes.
The following is the class-version of Lemma 5.4:

Lemma 5.25. *Assume that $(M, A)$ is countable. Then for every $p \in P$, there is a $P$-generic filter $G$ over $(M, A)$ containing $p$. Furthermore, if $P$ is non-atomic, then $G$ is not definable in $(M, A)$.*

Of course, $G$ is definable in $(M[G], G)$ just by definition. The point is that *if* $(M[G], G) \models$ ZF - i.e. Replacement holds in $M[G]$ even for formulas mentioning $G$ as a predicate - we may work with $G$ in $M[G]$ as freely as with any $M[G]$-definable class. In this sense, one can say that the forcing $P$ *adds a class* $G$.
The bad news is that the analogues for the Definability and the Forcing Theorem can fail for proper class-size $P$, and $\mathfrak{M}$ may not satisfy ZF. To give an easy example, consider the class-size forcing $P$ consisting of finite functions from $\omega$ into the ordinals, ordered by reverse inclusion. Any $P$-generic filter gives rise to a cofinal map $G : \omega \to Ord$, and so Replacement fails in $M[G]$ relative to the predicate $G$.
Several sufficient conditions on a class forcing $P$ for the definability of

the forcing relation and for the preservation of the ZF(C) axioms are present in the literature. One of them is the notion of *tameness*, as developed by Sy Friedman in [**7**].

The blackbox assumption in this thesis is that all described class forcings are sufficiently well-behaved to make the argument at hand work. In particular, the Definability and Forcing Theorem holds for the three class forcings which are described in the next section.

**12.1. Some examples of class forcings.**

12.1.1. *Adding a generic class of ordinals.* In this forcing, conditions are functions $f : \alpha_f \to 2$, where $\alpha_f$ is some ordinal (different $f$'s may have different ordinal domains). The ordering is inclusion.
If $G$ is a generic for this forcing, then $F := \bigcup G$ is the characteristic function of a subclass of $ON$, as one can see by checking the usual density arguments. The forcing is $\kappa$-closed for every $\kappa$ (we also say that the forcing is *set-closed*). Hence if $V \models AC$, it follows that no sets are added: $V = V[F] = V[G]$.
By another density argument, any set-length sequence of zeros and ones in $V$ occurs at some place in $F$. It follows that any set of ordinals in $V$ can be read off from the class $F$, and so if we assume $V \models AC$, then $V = L[F]$.
Now by the Forcing Theorem, for a generic class $F$ any true statement in $(V, F)$ is forced by a condition in $G$, or equivalently, by an initial segment $F \upharpoonright \alpha$ of $F$. So deciding truth in $(V, F)$ is simply checking if $F \supseteq f$ for various $f : \alpha \to 2$.

12.1.2. *Easton Forcing.* This is like the forcing described in Section 5, only that the domain of the function $F$ is now the class of all regular cardinals.

12.1.3. *Forcing Global Choice.* Here conditions are functions
$f : \alpha_f \to V$, ordered by reverse inclusion. This forcing is set-closed. If $G$ is a generic, then $F := \bigcup G$ is a function from $ON$ onto $V$. One can thus read off a well-order of $V$ from the function $F$. In other words, the generic extension $(V[G], G)$ satisfies the axiom of Global Choice $GC$. Assuming $AC$, no sets are added, which shows that $GC$ is first-order conservative over ZFC.

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# Appendix

## CV (in German)

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## Abstract

This thesis in the area of set theory summarizes a couple of results on ordinal definability. A set is called ordinal definable if it can be described by a formula of set theory using ordinal parameters. This notion was first suggested by Kurt Gödel. Dana Scott, John Myhill and others began to study the related inner model $HOD$ of hereditarily ordinal definable sets and used it to prove (among other things) the relative consistency of the axiom of choice.

We give an introduction to the general theory and then prove some classic results by Myhill, Scott, Võpenka and Roguski, followed by more recent results of Friedman, Hamkins and others.

## Zusammenfassung (German Abstract)

Diese Masterarbeit aus dem Bereich der Mengenlehre fasst mehrere Ergebnisse über Ordinalzahl-Definierbarkeit zusammen. Hierbei heißt eine Menge ordinalzahl-definierbar, wenn sie durch eine Formel in der Sprache der Mengenlehre mit Ordinalzahlen als Parametern eindeutig beschrieben werden kann. Dieses Konzept wurde von Kurt Gödel erfunden. Dana Scott, John Myhill und andere untersuchten später das innere Modell $HOD$, welches gerade aus den erblich ordinalzahl-definierbaren Mengen besteht, und bewiesen damit (unter anderem) die relative Konsistenz des Auswahlaxioms.
Wir beginnen mit einer Einführung in die allgemeine Theorie und beweisen dann einige grundlegende Ergebnisse von Myhill, Scott, Võpenka und Roguski. Anschließend besprechen wir aktuelle Ergebnisse von Friedman, Hamkins und anderen Mathematikern.