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Abstract

In the main part of this thesis we study the utility maximization problem from terminal wealth in a financial market with transaction costs. The main concern is the existence of a so-called shadow price, i.e., a least favorable frictionless market extension which lies within the bid-ask spread of the original market with transaction costs, such that trading in this fictitious market leads to the same maximal expected utility and optimal strategy. If the shadow price exists, the behavior of an economic agent in the market with transaction costs can be explained by passing to a suitable frictionless shadow market. Using duality methods, we show the existence of shadow prices in different settings.

First, we consider the problem with utility functions defined only on the positive real line. In a financial market driven by a continuous price process and with proportional transaction costs, we show the existence of shadow price processes, if the price process satisfies the condition (*NUPBR*) of “no unbounded profit with bounded risk”. We may furthermore prove that shadow prices exist, if the price process satisfies the weaker condition (*TWC*) of “two way crossing”. Examples and counterexamples are given. Special emphasis is put on financial models based on the fractional Brownian motion.

We also consider the case when the price process is a general càdlàg process and the agent receives an exogenous endowment. Under no-short-selling constraints, we are able to prove the existence of the primal optimizer and shadow price processes.

If we consider the utility maximization problem with utility functions defined on the whole real line, the picture changes. The existence of strict consistent price systems with “finite entropy” guarantees the existence of shadow prices, even in the case with bounded random endowment. If we only require in the definition that the shadow market yields the same optimal utility, without considering the optimal strategy, then such a shadow price in the weaker sense could always be constructed from the dual optimizer.

In the last part, we study the dual problem of utility maximization in incomplete frictionless markets with bounded random endowment and show that in the Brownian framework the countably additive part of the dual optimizer obtained in [23] can be represented by the terminal value of a supermartingale deflator which is defined in [70]. Furthermore, we show that this supermartingale deflator is a local martingale.

Zusammenfassung

Der Hauptteil dieser Dissertation beschäftigt sich mit dem Nutzenmaximierungsproblem in einem Finanzmarkt mit Transaktionskosten. Das Grundproblem ist die Frage, ob ein sogenannter Schattenpreis existiert, welcher innerhalb der Geld-Brief-Spanne des ursprünglichen Markts mit Transaktionskosten liegt und zum gleichen maximalen Erwartungsnutzen und zur gleichen optimalen Handelsstrategie führt. Falls der Schattenpreis existiert, kann das Verhalten des Investors auf dem Markt mit Transaktionskosten mithilfe eines geeigneten Schattenmarkts ohne Transaktionskosten beschrieben werden. Mithilfe von Methoden der Dualitätstheorie zeigen wir die Existenz von Schattenpreisen unter verschiedenen Rahmenbedingungen.

Wir betrachten zunächst das Nutzenmaximierungsproblem mit Nutzenfunktionen, die nur auf der positiven reellen Halbachse definiert sind. Für einen Finanzmarkt mit einem stetigen Preisprozess und proportionalen Transaktionskosten zeigen wir die Existenz eines Schattenpreisprozesses, wenn der Preisprozess die Bedingung “No unbounded profit with bounded risk” (*NUPBR*) erfüllt. Später zeigen wir, dass die Existenz von Schattenpreisen auch unter einer schwächeren Bedingung gilt, nämlich wenn der Preisprozess die Bedingung “Two way crossing” (*TWC*) erfüllt. Beispiele und Gegenbeispiele werden angegeben, vor allem Finanzmodelle, die auf der fraktionalen Brownschen Bewegung basieren.

Wir betrachten auch den Fall, in dem der Preisprozess *càdlàg* ist und der Investor eine zufällige finanzielle Ausstattung bekommt. Unter den Leerverkaufsbeschränkungen können wir die Existenz des Optimierers und der Schattenpreise beweisen.

Wenn wir die Nutzenfunktionen, die auf der ganzen reellen Achse definiert sind, in Betracht ziehen, stellt sich die Situation anders dar. Die Existenz des Schattenpreises folgt dann aus der Existenz eines sogenannten Strict Consistent Price Systems mit “endlicher Entropie”, auch im Fall mit beschränkter zufälliger finanzieller Ausstattung. Verlangen wir in seiner Definition nur, dass der Schattenpreis zum gleichen maximalen Erwartungsnutzen führt, ohne die optimale Handelsstrategie zu berücksichtigen, dann können solche Schattenpreise in diesem Rahmen immer aus den dualen Optimierern konstruiert werden.

Im letzten Teil der Dissertation untersuchen wir das duale Problem des Nutzenmaximierungsproblems auf unvollständigen Märkten mit beschränkten zufälligen finanziellen Ausstattungen. Wir beweisen, dass der endlich additive Teil des dualen Optimierers dem Endwert eines Supermartingaldeflators entspricht, der überdies ein lokales Martingal ist, wenn die Filtrierung von einer Brownschen Bewegung erzeugt ist.

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Chapter 1

Introduction

1.1 Background and motivation

One of the main topics of mathematical finance is the valuation of options and other derivatives.

In a frictionless market, valuation theory is based on the notion of replication in a complete market (or superreplication in an incomplete one). In this setting, the asset price processes should be semimartingales and the condition (*NFLVR*) of “no free lunch with vanishing risk”, which is equivalent to the existence of an equivalent local martingale measure, should be satisfied. The superreplication price of a contingent claim is the supremum of its expected value under all equivalent local martingale measures [33]. If there exists a unique equivalent local martingale measure, the market is complete, and the price of a contingent claim is uniquely determined.

The presence of transaction costs changes everything. In this case one cannot deduce any nontrivial information from the concepts of replication or superreplication. It was proved in [99] that, under transaction costs, the bounds for option prices obtained from superreplication arguments are only the trivial bounds. This result was extended from the Black-Scholes case to a much larger class of models, where the asset processes share the so-called conditional full support property, in [50] for the one-dimensional case and [9] for the multidimensional case.

While the concepts of replication and superreplication do not make sense economically in the presence of transaction costs, the theory of utility indifference pricing does. In this setting, we need to rigorously solve the portfolio optimization problem. This is why we should consider the utility maximization problem with transaction costs.

Utility maximization itself is also a classical problem in mathematical finance. Here, an economic agent invests in a financial market so as to maximize the expected utility of her terminal wealth.

In the framework of a continuous-time model, the problem was studied for the first time by Merton in two seminal papers [80, 81], employing the methods of stochastic optimal control. This approach requires stock prices to be governed by Markovian dynamics.

To avoid this strong assumption, a different approach, called the “duality method” or “martingale method”, has been developed since the 1990s. This approach is based on duality characterizations of portfolios provided by the set of “martingale measures”.

The main idea is to solve a dual variational problem and then to find the solution of the original problem by convex duality.

For the case of a complete financial market, where the set of martingale measures is a singleton, the dual method was developed by Pliska [90], Cox and Huang [20, 21] and Karatzas, Lehoczky and Shreve [64]. The case of incomplete financial models was studied in a discrete-time, finite probability space model by He and Pearson [52] and in a continuous-time diffusion model by He and Pearson [53] and by Karatzas, Lehoczky, Shreve and Xu [65].

The first study of the case of general utility within the framework of a general incomplete semimartingale model of a financial market was done by Kramkov and Schachermayer [70]. The authors considered an agent endowed with a deterministic initial wealth and a utility function supporting positive wealth. They established an abstract duality theorem for the primal and dual problems.

For a utility function supporting both positive and negative wealth, Schachermayer [94] established duality results for a locally bounded semimartingale stock price process. Biagini and Frittelli [7] generalized the result of [94] by removing the local boundedness assumption.

Cvitanić, Schachermayer and Wang [23] generalized [70] by allowing for an additional bounded random endowment. They defined the dual problem on the enlarged domain of finitely additive measures. Owen [86] treated the utility maximization problem with a utility function supporting negative wealth and a bounded random endowment in a financial market driven by a locally bounded semimartingale. Hugonnier and Kramkov [57] considered an unbounded random endowment for a general utility function defined on the positive real line. The case of a general utility function defined on the whole real line and unbounded random endowments with a locally bounded semimartingale model was treated by Owen and Žitković [87]. Recently, Biagini, Frittelli and Grasselli [8] relaxed the boundedness assumption on the random endowment and on the stock price process, and generalized the results of [86] and [87] by using an Orlicz space technique.

Utility maximization under transaction costs is essentially as old as its frictionless counterpart, dating back to Magill and Constantinides [77] and Constantinides [19]. From heuristic arguments, they concluded that it is optimal to keep existing holdings in all assets in a no-trade region and tradings should merely take place at its boundaries. Later on, Davis and Norman [31] considered this problem as a stochastic control problem and gave a rigorous proof for the heuristic derivation of [77]. Shreve and Soner extended the analysis by using the theory of viscosity solutions. See also [102, 38]. They all used the dynamic programming approach to treat optimization problems with Markovian state processes.

The first paper to use the duality method in the setting of proportional transaction costs was [22]. In that paper, Cvitanić and Karatzas modeled a bond and a stock as Itô processes and assumed constant proportional transaction costs. At the end of trading, the agent was assumed to liquidate her portfolio to the bond. In this setting, they proved the existence of a solution to the problem of utility maximization under the assumption that a dual minimization problem admits a solution. The existence of a solution to the dual problem was subsequently proved by Cvitanić and Wang [24]. Bouchard [10] considered the utility maximization problem with a utility function defined on the whole real line and

a bounded random endowment and provided a static duality result. If the utility function is defined on the positive half line, the duality theorem was proved by Czichowsky, Muhle-Karbe and Schachermayer [25] in finite discrete time. Recently, Czichowsky and Schachermayer established general duality results for the utility maximization problem in [27] for utility functions defined on the positive half line, and in [28] for utility functions defined on the whole real line under the assumption that the underlying price process is locally bounded.

In a frictionless market, we usually assume that there is a single consumption asset, which is used as a numéraire. Mathematically it does not make any difference whether or not the agent liquidates her holding in stock. However, this does matter in the transaction costs setting. Therefore, it is quite natural to allow the agent to have access to several consumption assets. Kabanov [59] introduced a much more general formulation of a transaction costs model for a currency market based on the concept of solvency cone. In this setting, the utility maximization problem with a multivariate utility function was afterwards studied by Deelstra, Pham and Touzi [32], Campi and Owen [14] without random endowment, and Benedetti and Owen [3] with bounded random endowment. They provided static duality results in different ways.

A crucial question in the theory of portfolio optimization with proportional transaction costs is whether or not there exists a so-called shadow price, i.e., a least favorable frictionless market extension, that leads to the same optimal strategy and utility. If the answer is affirmative, the behavior of a given economic agent can be explained by passing to a suitable frictionless shadow market. It is then possible to reduce the problem to a corresponding problem in the well-studied frictionless case and the shadow price corresponds to the dual optimizer [25, 27]. Starting from [61], the concept of shadow prices has been successfully applied to utility maximization problems in various concrete models, see [43, 44, 55].

Kallsen and Muhle-Karbe [62] showed that shadow prices always exist for utility maximization problems in finite probability spaces. In an Itô process setting, Cvitanić and Karatzas [22] proved that a shadow price exists and corresponds to the solution of a suitable dual problem, if the latter is attained in a set of martingales. Loewenstein [75] showed that shadow prices exist for continuous bid-ask price processes whenever short positions are ruled out. This result was generalized by Benedetti, Campi, Muhle-Karbe and Kallsen [4] to Kabanov's general multi-currency market models.

However, several counterexamples have been constructed showing that shadow prices may generally fail to exist for the utility maximization problem under transaction costs with a utility function defined on the positive half line without further assumptions, see [93, 25, 4, 27, 30].

In the general càdlàg framework, Czichowsky and Schachermayer [27] showed that the dual optimizer, which is not necessarily a local martingale, can be interpreted as shadow price in a generalized sense defined via a “sandwiched” process consisting of a predictable and an optional strong supermartingale and pertains to all strategies, which remain solvent under transaction costs.

If we consider utility functions defined on the whole real line, the picture changes. Recently, Czichowsky and Schachermayer [28] affirmed that the existence of a strictly consistent price system with “finite entropy” guarantees the existence of shadow prices

in the classical sense.

1.2 Main Results

This thesis is based on two published papers [30, 46], two submitted preprints [47, 74] and one working paper [26], which were jointly written with coauthors.

In Chapter 2, we introduce the financial market with proportional transaction costs and recall some basic results, which are used in the subsequent chapters. In Chapter 3, we summarize the main results of [27], where the authors provide the general duality theory for utility maximization problem under proportional transaction costs and introduce the notion of shadow price processes both in the classical as well as in the “sandwiched” sense. Czichowsky and Schachermayer generalized a result on the existence of a shadow price in the classical sense, which was proved in Itô process models by Cvitanić and Karatzas [22]: in the setting of general càdlàg processes, if the solution to the dual problem is attained by a local martingale (\hat{Z}^0, \hat{Z}^1) , then $\hat{S} := \frac{\hat{Z}^1}{\hat{Z}^0}$ is a shadow price process in the classical sense.

Chapter 4 is based on two papers. The first one is [30] “Shadow prices for continuous processes”, which is joint work with Christoph Czichowsky and Walter Schachermayer. In this paper we investigate the problem of utility maximization in a financial market with a continuous price process and proportional transaction costs. We show that the theory simplifies considerably if we restrict ourselves to continuous processes and obtain sharper results than in the general càdlàg setting on the existence of a shadow price in the classical sense. We state that, if the price process S is continuous and satisfies the condition $(NUPBR)$ of “no unbounded profit with bounded risk”, then the liquidation value process with respect to the optimal trading strategy is strictly positive. This ensures that the dual optimizer is induced by a local martingale (\hat{Z}^0, \hat{Z}^1) , hence a shadow price process defined as $\hat{S} = \frac{\hat{Z}^1}{\hat{Z}^0}$ exists. By a counterexample, we show that it is not possible to replace the assumption $(NUPBR)$ by the assumption of the existence of a consistent price system (CPS^μ) for each level $\mu \in (0, 1)$, which at first glance might seem to be the natural condition in the context of transaction costs. Through another counterexample, we show that, although the price process is continuous, shadow price processes are not necessarily continuous.

As the price process S has to be a semimartingale under the condition $(NUPBR)$, we could not apply this result to price processes based on fractional Brownian motion $B^H = (B_t^H)_{0 \leq t \leq T}$ such as the fractional Black-Scholes model

$$S_t = \exp(\mu t + \sigma B_t^H), \quad 0 \leq t \leq T,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $H \in (0, 1) \setminus \{\frac{1}{2}\}$ denotes the Hurst parameter of the fractional Brownian motion B^H . The second paper in Chapter 4 is [26] “Shadow prices, fractional Brownian motion, and portfolio optimization”, which is joint work with Christoph Czichowsky, Walter Schachermayer and Rémi Peyre. In this paper, we derive the existence of a shadow price process under the weaker condition (TWC) of “two way crossing”, which does not require S to be a semimartingale. Recently, Peyre [89] proved that the fractional Brownian motion does have the property (TWC) . By estimating the fluctuations of the

fractional Brownian motion, we show the existence of a shadow price for price processes based on the fractional Brownian motion.

Chapter 5 is based on joint work with Yiqing Lin [74], “Utility maximization problem with random endowment and transaction costs: when wealth becomes negative”. In this paper, we generalize the result of [28] and provide the duality theory for the utility maximization problem under transaction with a bounded random endowment, where the utility function is defined on the whole real line and the underlying price process is locally bounded. To achieve this, we first have an intermediate duality result for the problem on the positive half line, which could be proved by following the argument in [23]. This intermediate duality result is similar to [3], however, it is more straightforward and adapted to the numéraire-based setting, which is necessary for the subsequent approximation. For the problem on the whole real line, we first construct auxiliary primal and dual functions by proper truncation in order to come back to the case of the intermediate result. Then, we exhibit similar procedures as in [86] to complete the proof by approximating both optimizers and expected value functions. In the presence of a bounded random endowment, we show that, similarly as in [28], the existence of a strictly consistent price system satisfying the “finite generalized entropy” condition guarantees the existence of a shadow price. This is based on the fact that the dual optimizer is associated with a λ -consistent price system. If we generalize the shadow price definition, i.e., if we only require that the shadow price market yields the same optimal utility, then such kind of shadow price could be always constructed from the dual optimizer.

In Chapter 6, we consider a utility maximization problem with proportional transaction costs and random endowment under no-short-selling constraints. This work is based on [47] “On the existence of shadow prices for optimal investment with random endowment”, which is a joint project with Lingqi Gu and Yiqing Lin. First, we are inspired by the argument in [95, Section 3.3] to prove the existence of constrained primal solutions. Then, assuming the agent has a positive random endowment, we follow the lines of [75, 4] to construct a shadow price directly from the primal solution. In addition, we discuss the existence of shadow prices when the constraints are violated and the random endowment is allowed to be negative. We provide an example in the Black-Scholes framework with a constructive random endowment. In this example, shadow prices exist (but are not unique) and can be explicitly defined.

In Chapter 7, we focus our attention on the study of the dual problem of the expected utility maximization in incomplete frictionless markets with a bounded random endowment e_T . This is based on a collaboration with Lingqi Gu and Yiqing Lin [46] “On the dual problem of utility maximization in incomplete markets”. In order to solve the utility maximization problem with bounded random endowment, the authors of [23] employ the duality between L^∞ and its topological dual space $(L^\infty)^*$ and solve a dual minimization problem over the subset \mathcal{D} of $(L^\infty)^*$, which can be regarded as the weak-star closure of the set $\mathcal{M}_e(S)$ of equivalent local martingale measures. It is stated in [23] that a dual optimizer \widehat{Q} can be found in \mathcal{D} , which is unique up to the singular part, and moreover the primal optimizer can be formulated in terms of \widehat{Q}^r . In this chapter, we study the regular part of the dual optimizer and establish the following result: if the underlying filtration is Brownian, then the regular part \widehat{Q}^r of the dual optimizer \widehat{Q} can be attained by an equivalent local martingale deflator. When $e_T = 0$, Karatzas and Žitković [67] observed that, for Itô process models, the dual optimizer can be attained by an equivalent local

martingale deflator. Subsequently, Larsen and Žitković [73] generalized this result to all continuous semimartingale models. The present work generalizes the result in [73] to the case that e_T is a bounded random variable, which increases the complexity of the dynamics.

Chapter 2

Financial Market under Transaction Costs

2.1 Definitions and Notations

2.1.1 Market and Trading Strategies

We consider a financial market consisting of one riskless asset and one risky asset. The riskless asset has constant price one and can be traded without transaction costs. The price of the risky asset is given by a strictly positive adapted càdlàg stochastic process $S = (S_t)_{0 \leq t \leq T}$ on some underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ with fixed finite time horizon $T \in (0, \infty)$ satisfying the usual assumptions of right-continuity and completeness. In addition, we assume that \mathcal{F}_0 is trivial.

Trading in the risky asset incurs proportional transaction costs of size $\lambda \in (0, 1)$. This means that one has to pay a higher ask price S_t when buying risky shares but only receives a lower bid price $(1 - \lambda)S_t$ when selling them.

Remark 2.1.1. We assume without loss of generality that we pay transaction costs only when we sell risky shares, and pay nothing when we buy them. Indeed, if we set $\bar{S} = \frac{2-\lambda}{2}S$ and $\bar{\lambda} = \frac{\lambda}{2-\lambda}$, we obtain that $[(1 - \lambda)S, S]$ coincides with $[(1 - \bar{\lambda})\bar{S}, (1 + \bar{\lambda})\bar{S}]$. Conversely, any bid-ask spread $[(1 - \bar{\lambda})\bar{S}, (1 + \bar{\lambda})\bar{S}]$ with $\bar{\lambda} \in (0, 1)$ equals $[(1 - \lambda)S, S]$ for $S = (1 + \bar{\lambda})\bar{S}$ and $\lambda = \frac{2\bar{\lambda}}{1 + \bar{\lambda}}$.

We model **trading strategies** by \mathbb{R}^2 -valued, predictable processes $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ of finite variation, where φ_t^0 and φ_t^1 denote the holdings in units of the riskless and the risky asset, respectively, after rebalancing the portfolio at time t .

Remark 2.1.2. For any process $\varphi = (\varphi_t)_{0 \leq t \leq T}$ of finite variation, we denote by

$$\varphi_t = \varphi_0 + \varphi_t^\uparrow - \varphi_t^\downarrow, \quad 0 \leq t \leq T,$$

its Jordan-Hahn decomposition into two nondecreasing processes φ^\uparrow and φ^\downarrow both null at time zero. The total variation $\text{Var}_t(\varphi)$ of φ on $(0, T]$ is then given by

$$\text{Var}_t(\varphi) = \varphi_t^\uparrow + \varphi_t^\downarrow, \quad 0 \leq t \leq T.$$

We note that any process φ of finite variation is $\text{l\`a}d\text{l\`a}g$, and denote by φ^c its continuous part

$$\varphi_t^c := \varphi_t - \sum_{s < t} \Delta_+ \varphi_s - \sum_{s \leq t} \Delta \varphi_s,$$

where $\Delta_+ \varphi_s := \varphi_{s+} - \varphi_s$ and $\Delta \varphi_s := \varphi_s - \varphi_{s-}$ are its right and left jumps, respectively.

Definition 2.1.3. A strategy $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ is called **self-financing under transaction costs** λ , if

$$\int_s^t d\varphi_u^{0,\uparrow} \leq \int_s^t (1 - \lambda) S_u d\varphi_u^{1,\downarrow}, \quad \int_s^t d\varphi_u^{0,\downarrow} \geq \int_s^t S_u d\varphi_u^{1,\uparrow}, \quad (2.1.1)$$

for all $0 \leq s \leq t \leq T$, where the integrals are defined via

$$\begin{aligned} \int_s^t S_u d\varphi_u^{1,\uparrow} &:= \int_s^t S_u d\varphi_u^{1,\uparrow,c} + \sum_{s < u \leq t} S_{u-} \Delta \varphi_u^{1,\uparrow} + \sum_{s \leq u < t} S_u \Delta_+ \varphi_u^{1,\uparrow}, \\ \int_s^t S_u d\varphi_u^{1,\downarrow} &:= \int_s^t S_u d\varphi_u^{1,\downarrow,c} + \sum_{s < u \leq t} S_{u-} \Delta \varphi_u^{1,\downarrow} + \sum_{s \leq u < t} S_u \Delta_+ \varphi_u^{1,\downarrow}. \end{aligned}$$

The self-financing condition (2.1.1) states that purchases and sales of the risky asset are accounted for in the riskless position:

$$\begin{aligned} \int_s^t d\varphi_u^{0,\uparrow,c} &\leq \int_s^t (1 - \lambda) S_u d\varphi_u^{1,\downarrow,c}, & \int_s^t d\varphi_u^{0,\downarrow,c} &\geq \int_s^t S_u d\varphi_u^{1,\uparrow,c}, \\ \Delta \varphi_t^{0,\uparrow} &\leq (1 - \lambda) S_{t-} \Delta \varphi_t^{1,\downarrow}, & \Delta \varphi_t^{0,\downarrow} &\geq S_{t-} \Delta \varphi_t^{1,\uparrow}, \\ \Delta_+ \varphi_t^{0,\uparrow} &\leq (1 - \lambda) S_t \Delta_+ \varphi_t^{1,\downarrow}, & \Delta_+ \varphi_t^{0,\downarrow} &\geq S_t \Delta_+ \varphi_t^{1,\uparrow}, \end{aligned} \quad (2.1.2)$$

for $0 \leq s < t \leq T$.

Definition 2.1.4. We define the **liquidation value** at time t by

$$V_t^{\text{liq}}(\varphi) := \varphi_t^0 + (\varphi_t^1)^+ (1 - \lambda) S_t - (\varphi_t^1)^- S_t.$$

Remark 2.1.5. It follows by integration by parts that

$$V_t^{\text{liq}}(\varphi) = \varphi_0^0 + \varphi_0^1 S_0 + \int_0^t \varphi_u^1 dS_u - \lambda \int_0^t S_u d\varphi_u^{1,\downarrow} - \lambda S_t (\varphi_t^1)^+,$$

which means that the liquidation value $V_t^{\text{liq}}(\varphi)$ is given by the initial value of the position $\varphi_0^0 + \varphi_0^1 S_0$ plus the gains from trading $\int_0^t \varphi_u^1 dS_u$ minus the transaction costs for rebalancing the portfolio $\lambda \int_0^t S_u d\varphi_u^{1,\downarrow}$ minus the costs $\lambda S_t (\varphi_t^1)^+$ for liquidating the position at time t .

Definition 2.1.6. A self-financing trading strategy φ is called **admissible**, if there exists $M > 0$ such that we have

$$V_\tau^{\text{liq}}(\varphi) \geq -M, \quad \text{a.s.}$$

for every $[0, T]$ -valued stopping time τ .

Remark 2.1.7. The admissibility condition above is in the numéraire-based sense, which means that an agent can cover the trading strategy φ by holding M units of bond. There is another definition of the admissibility, namely in the numéraire-free sense, which means that an agent can cover the trading strategy φ by holding M units of bond as well as M units of stock.

2.1.2 Consistent Price System

Definition 2.1.8. Fix a price process $S = (S_t)_{0 \leq t \leq T}$ and transaction costs $0 < \lambda < 1$ as above. A λ -consistent price system is a pair (\tilde{S}, \mathbf{Q}) such that

1. \mathbf{Q} is a probability measure equivalent to \mathbf{P} ,
2. $\tilde{S} = (\tilde{S}_t)_{0 \leq t \leq T}$ takes its values in the bid-ask spread $[(1 - \lambda)S, S]$,
3. \tilde{S} is a local martingale under \mathbf{Q} .

The condition (EMM) of the “existence of an equivalent local martingale measure” in the frictionless setting corresponds to the following notion.

Definition 2.1.9. For $0 < \lambda < 1$, we say that a price process $S = (S_t)_{0 \leq t \leq T}$ satisfies (CPS $^\lambda$), if there exists a consistent price system.

We have also another way to define the consistent price system.

Definition 2.1.10. Fix $0 < \lambda < 1$ and $S = (S_t)_{0 \leq t \leq T}$ as above. A λ -consistent price system is a two-dimensional strictly positive process $Z = (Z_t^0, Z_t^1)_{0 \leq t \leq T}$ with $Z_0^0 = 1$, that consists of a martingale Z^0 and a local martingale Z^1 under \mathbf{P} such that

$$\tilde{S}_t := \frac{Z_t^1}{Z_t^0} \in [(1 - \lambda)S_t, S_t], \quad a.s. \quad (2.1.3)$$

for $0 \leq t \leq T$.

We denote by $\mathcal{Z}_e^\lambda(S)$ the set of λ -consistent price systems.

We call a process **absolutely continuous λ -consistent price system**, if we replace the strict positivity in the above definition by nonnegativity (where we consider (2.1.3) to be satisfied if $\frac{Z_t^1}{Z_t^0} = 0$).

By $\mathcal{Z}_a^\lambda(S)$ we denote the set of absolutely λ -consistent price systems.

Remark 2.1.11. In the above definition, Z^0 defines a density process of an equivalent local martingale measure $\mathbf{Q} \sim \mathbf{P}$ for a price process \tilde{S} evolving in the bid-ask spread $[(1 - \lambda)S, S]$, and $Z^1 = Z^0 \tilde{S}$.

Definition 2.1.12. Fix $0 < \lambda < 1$ and a price process $S = (S_t)_{0 \leq t \leq T}$ as above. We call a two-dimensional strictly positive process $Z = (Z_t^0, Z_t^1)_{0 \leq t \leq T}$ **local λ -consistent price system** for S , if there exists a localizing sequence of $[0, T]$ -valued stopping times $(\tau_n)_{n \in \mathbb{N}}$ increasing to T with

$$\lim_{n \rightarrow \infty} \mathbf{P}[\tau_n < T] = 0,$$

such that each stopped process $Z^{\tau_n} = (Z_{t \wedge \tau_n}^0, Z_{t \wedge \tau_n}^1)_{0 \leq t \leq T}$ defines a λ -consistent price system for the stopped process S^{τ_n} .

We denote by $\mathcal{Z}_{loc,e}^\lambda(S)$ the set of local λ -consistent price systems for S .

We say that a property (P) of a stochastic process $S = (S_t)_{0 \leq t \leq T}$ holds locally, if there exists a localizing sequence of $[0, T]$ -valued stopping times $(\tau_n)_{n \in \mathbb{N}}$ increasing to T with $\lim_{n \rightarrow \infty} \mathbf{P}[\tau_n < T] = 0$, such that each stopped process $S^{\tau_n} = (S_{t \wedge \tau_n})_{0 \leq t \leq T}$ has property (P).

Lemma 2.1.13. *Fix a strictly positive adapted càdlàg process $S = (S_t)_{0 \leq t \leq T}$ and transaction costs $0 < \lambda < 1$. If S admits a local λ -consistent price system, then S satisfies locally (CPS^λ) .*

Proof. By Definition 2.1.12, there exists a localizing sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that each stopped process S^{τ_n} admits a λ -consistent price system Z^{τ_n} , which already shows that S satisfies locally (CPS^λ) . \square

Proposition 2.1.14. *Fix a strictly positive adapted càdlàg process $S = (S_t)_{0 \leq t \leq T}$ and transaction costs $0 < \lambda < 1$. The following assertions are equivalent*

- (i) *S satisfies locally (CPS^μ) for all $0 < \mu < \lambda$, i.e., there exists a localizing sequence $(\tau_n)_{n \in \mathbb{N}}$, such that each stopped process S^{τ_n} satisfies (CPS^μ) for all $0 < \mu < \lambda$.*
- (ii) *For all $0 < \mu < \lambda$, S satisfies locally (CPS^μ) , i.e., for each $\mu \in (0, \lambda)$, there exists a localizing sequence $(\tau_n)_{n \in \mathbb{N}}$ (which may depend on μ), such that each stopped process S^{τ_n} satisfies (CPS^μ) .*
- (iii) *For all $0 < \mu < \lambda$, we have $\mathcal{Z}_{loc,e}^\mu(S) \neq \emptyset$.*

Proof. (i) \Rightarrow (iii) : Fix $\mu \in (0, \lambda)$. We have to show the existence of a strictly positive process $Z = (Z^0, Z^1)$ and a localizing sequence $(\tau_n)_{n \in \mathbb{N}}$, such that each Z^{τ_n} defines a μ -consistent price system for S^{τ_n} .

Since $\mu > 0$, there exists an $\varepsilon_0 \in (0, \mu)$ satisfying $(1 - \mu) = (1 - \varepsilon_0)^3$. Let $\varepsilon_n := 1 - (1 - \varepsilon_0)^{2^{-n}}$ so that

$$\prod_{n=1}^{\infty} (1 - \varepsilon_n)^2 = \prod_{n=1}^{\infty} (1 - \varepsilon_0)^{2^{-n}} = (1 - \varepsilon_0)^{\sum_{n=1}^{\infty} \frac{1}{2^n}} = 1 - \varepsilon_0.$$

The assertion (i) implies the existence of a localizing sequence $(\tau_n)_{n \in \mathbb{N}_0}$, such that S^{τ_n} satisfies (CPS^{ε_n}) for each $n \in \mathbb{N}_0$. It also follows that $(1 - \varepsilon_n)S^{\tau_n}$ satisfies (CPS^{ε_n}) for each $n \in \mathbb{N}_0$. Let $Z^{(n)} = (Z_t^{(n),0}, Z_t^{(n),1})_{0 \leq t \leq T}$ denote the ε_n -consistent price system for the stopped process $(1 - \varepsilon_n)S^{\tau_n}$, for each $n \in \mathbb{N}_0$. To concatenate these objects, define the following processes: $(\bar{Z}_t^{(0),0}, \bar{Z}_t^{(0),1}) := (Z_t^{(0),0}, Z_t^{(0),1})$ for $0 \leq t \leq \tau_0$ and

$$\bar{Z}_t^{(n+1),0} := \begin{cases} \bar{Z}_t^{(n),0} & \text{for } 0 \leq t \leq \tau_n, \\ Z_t^{(n+1),0} \frac{\bar{Z}_{\tau_n}^{(n),0}}{Z_{\tau_n}^{(n+1),0}} & \text{for } \tau_n \leq t \leq \tau_{n+1}, \end{cases}$$

and

$$\bar{Z}_t^{(n+1),1} := \begin{cases} \bar{Z}_t^{(n),1} & \text{for } 0 \leq t \leq \tau_n, \\ Z_t^{(n+1),1} \frac{\bar{Z}_{\tau_n}^{(n),1}}{Z_{\tau_n}^{(n+1),1}} & \text{for } \tau_n \leq t \leq \tau_{n+1}, \end{cases}$$

for $n \geq 1$. Now define $(Z_t^0, Z_t^1) := (\bar{Z}_t^{(n),0}, \bar{Z}_t^{(n),1})$ for $0 \leq t \leq \tau_n$, which is in $\mathcal{Z}_{loc,e}^\mu(S)$. Since $\{\tau_n = T\} \nearrow \Omega$ almost surely as n goes to infinity, we may define (Z_T^0, Z_T^1) by

$$(Z_T^0, Z_T^1) := \lim_{n \rightarrow \infty} (\bar{Z}_{\tau_n}^{(n),0}, \bar{Z}_{\tau_n}^{(n),1}).$$

Indeed, clearly, Z^0 and Z^1 are strictly positive local martingale under \mathbf{P} and $Z_0^0 = 1$. We now show that $\frac{Z^1}{Z^0}$ takes its values in $[(1 - \mu)S, S]$. Using induction we may show that, for each $n \in \mathbb{N}$,

$$\frac{\overline{Z}_t^{(n),1}}{\overline{Z}_t^{(n),0}} \in \left[(1 - \varepsilon_0)^2 \prod_{k=1}^n (1 - \varepsilon_k) S_t, \frac{(1 - \varepsilon_0)}{\prod_{k=1}^n (1 - \varepsilon_k)} S_t \right],$$

for $0 \leq t \leq \tau_n$. (For $n = 0$ it is clear, as $Z^{(0)}$ is an ε_0 -consistent price system for $(1 - \varepsilon_0)S^{\tau_0}$. Assume this holds for n . For $0 \leq t \leq \tau_n$, it satisfies by the assumption hypothesis

$$\frac{\overline{Z}_t^{(n+1),1}}{\overline{Z}_t^{(n+1),0}} = \frac{\overline{Z}_t^{(n),1}}{\overline{Z}_t^{(n),0}} \in \left[(1 - \varepsilon_0)^2 \prod_{k=1}^n (1 - \varepsilon_k) S_t, \frac{(1 - \varepsilon_0)}{\prod_{k=1}^n (1 - \varepsilon_k)} S_t \right],$$

and for $\tau_n \leq t \leq \tau_{n+1}$

$$\frac{\overline{Z}_t^{(n+1),1}}{\overline{Z}_t^{(n+1),0}} = \frac{Z_t^{(n+1),1}}{Z_t^{(n+1),0}} \frac{\overline{Z}_{\tau_n}^{(n),1}}{\overline{Z}_{\tau_n}^{(n),0}} \frac{1}{\frac{Z_{\tau_n}^{(n+1),1}}{Z_{\tau_n}^{(n+1),0}}} \in \left[(1 - \varepsilon_0)^2 \prod_{k=1}^{n+1} (1 - \varepsilon_k) S_t, \frac{(1 - \varepsilon_0)}{\prod_{k=1}^{n+1} (1 - \varepsilon_k)} S_t \right],$$

since $\frac{\overline{Z}_t^{(n+1),1}}{\overline{Z}_t^{(n+1),0}} \in [(1 - \varepsilon_{n+1})^2 S_t, (1 - \varepsilon_{n+1}) S_t]$ for $0 \leq t \leq \tau_{n+1}$.) Hence for each $n \in \mathbb{N}_0$ we have that

$$\frac{Z_t^1}{Z_t^0} \in \left[(1 - \varepsilon_0)^2 \prod_{k=1}^n (1 - \varepsilon_k) S_t, \frac{(1 - \varepsilon_0)}{\prod_{k=1}^n (1 - \varepsilon_k)} S_t \right] \subseteq [(1 - \mu)S_t, S_t],$$

for $0 \leq t \leq \tau_n$.

(iii) \Rightarrow (ii) : It follows immediately by Lemma 2.1.13.

(ii) \Rightarrow (i) : Let $\varepsilon_k := \frac{1}{k}$. By (ii), for each $k \in \mathbb{N}$, the process S satisfies locally (CPS^{ε_k}) , i.e., we may find a localizing sequence of stopping times $(\rho_n^k)_{n \in \mathbb{N}}$ increasing to T almost surely as $n \rightarrow \infty$ with $\lim_{n \rightarrow \infty} \mathbf{P}[\rho_n^k < T] = 0$, such that the stopped process $S^{\rho_n^k}$ satisfies an ε_k -consistent price system for all $n \in \mathbb{N}$.

Find an increasing sequence of integers $(n_{m,k})_{k \in \mathbb{N}}$, such that

$$\mathbf{P}[\rho_{n_{m,k}}^k < T] < \frac{1}{2^{m+k}}.$$

Letting $\tau_m := \bigwedge_{k=1}^{\infty} \rho_{n_{m,k}}^k$ we obtain that

$$\mathbf{P}[\tau_m < T] = \mathbf{P} \left[\bigcup_{k \in \mathbb{N}} \{\rho_{n_{m,k}}^k < T\} \right] \leq \sum_{k \in \mathbb{N}} \mathbf{P}[\rho_{n_{m,k}}^k < T] < \sum_{k \in \mathbb{N}} \frac{1}{2^{m+k}} = \frac{1}{2^m}.$$

By Borel-Cantelli's Lemma, $(\tau_m)_{m \in \mathbb{N}}$ is a localizing sequence increasing to T with

$$\lim_{m \rightarrow \infty} \mathbf{P}[\tau_m < T] = 0,$$

uniformly for each $k \in \mathbb{N}$.

Therefore we obtain a localizing sequence $(\tau_m)_{m \in \mathbb{N}}$, such that S^{τ_m} satisfies (CPS^{ε_k}) for each $k \in \mathbb{N}$. For each $0 < \mu < \lambda$ there exists an ε_k -consistent price system, with $\varepsilon_k \leq \mu$, which is also a μ -consistent price system. \square

A crucial feature of the proposition above is that all equivalent statements contain the quantifier “for all μ ”.

Remark 2.1.15. There is a subtle difference between the frictionless and the transaction cost case. In the frictionless case the set $\mathcal{M}_e(S)$ of equivalent local martingale measures for the process S has the following concatenation property: let $\mathbf{Q}^1, \mathbf{Q}^2 \in \mathcal{M}_e(S)$ and associate the density processes $Z_t^1 = \mathbf{E}\left[\frac{d\mathbf{Q}^1}{d\mathbf{P}} \mid \mathcal{F}_t\right]$ and $Z_t^2 = \mathbf{E}\left[\frac{d\mathbf{Q}^2}{d\mathbf{P}} \mid \mathcal{F}_t\right]$. For a stopping time τ we define the concatenated process

$$Z_t := \begin{cases} Z_t^1, & \text{for } 0 \leq t \leq \tau, \\ Z_\tau^1 \frac{Z_t^2}{Z_\tau^2}, & \text{for } \tau \leq t \leq T. \end{cases} \quad (2.1.4)$$

Then, $\frac{d\mathbf{Q}}{d\mathbf{P}} = Z_T$ defines again an equivalent local martingale measure for S .

For $\lambda > 0$ the sets $\mathcal{Z}_{loc,e}^\lambda(S)$ and $\mathcal{Z}_e^\lambda(S)$ do not have this property any more. However, as in the proof of Proposition 2.1.14 above, under the local version of the condition “(CPS $^\mu$) for all μ ”, we may use the similar technique of concatenation as in the frictionless setting.

2.2 Fundamental Theorem of Asset Pricing

2.2.1 FTAP for Continuous Processes

Definition 2.2.1. *The process S admits **arbitrage** with λ -transaction costs, if there is a self-financing trading strategy φ starting at $(\varphi_0^0, \varphi_0^1) = (0, 0)$, which is admissible such that*

$$V_T^{\text{liq}}(\varphi) \geq 0, \text{ a.s.} \quad \text{and} \quad \mathbf{P}\left[V_T^{\text{liq}}(\varphi) > 0\right] > 0$$

Theorem 2.2.2. *Let $S = (S_t)_{0 \leq t \leq T}$ be an adapted strictly positive continuous price process. The following assertions are equivalent:*

1. *For each $0 < \mu < 1$ there exists an μ -consistent price system.*
2. *For each $0 < \mu < 1$ there is no arbitrage for μ -transaction costs.*

Proof. See [51, Theorem 4]. □

2.2.2 Local Version of FTAP for Continuous Processes

In this subsection, we give a local version of the fundamental theorem of asset pricing in the context of transaction costs. We shall use the subsequent variants of the concept of no arbitrage.

Definition 2.2.3. *Let $S = (S_t)_{0 \leq t \leq T}$ be a strictly positive, continuous process. We say that S allows for an **obvious arbitrage**, if there are $\alpha > 0$ and $[0, T] \cup \{\infty\}$ -valued stopping times $\sigma \leq \tau$ with $\mathbf{P}[\sigma < \infty] = \mathbf{P}[\tau < \infty] > 0$ such that either*

$$(a) \quad S_\tau \geq (1 + \alpha)S_\sigma, \quad \text{a.s. on } \{\sigma < \infty\},$$

or

$$(b) S_\tau \leq \frac{1}{1+\alpha} S_\sigma, \quad \text{a.s. on } \{\sigma < \infty\}.$$

In the case of (b) we also assume that $(S_t)_{\sigma \leq t \leq \tau}$ is uniformly bounded.

We say that S allows for an **obvious immediate arbitrage**, if, in addition, we have either

$$(a) S_t \geq S_\sigma, \quad \text{for } \sigma \leq t \leq \tau, \text{ a.s. on } \{\sigma < \infty\},$$

or

$$(b) S_t \leq S_\sigma, \quad \text{for } \sigma \leq t \leq \tau, \text{ a.s. on } \{\sigma < \infty\}.$$

We say that S satisfies the condition *(NOA)* (respectively, *(NOIA)*) of no obvious arbitrage (respectively, no obvious immediate arbitrage), if no such opportunity exists.

It is indeed rather obvious how to make an arbitrage if *(NOA)* fails, provided the transaction costs $0 < \lambda < 1$ are smaller than α . Assuming, e.g., condition (a), one goes long in the asset S at time σ and closes the position at time τ . In case of an obvious immediate arbitrage one is in addition assured that during such an operation the stock price will never fall under the initial value S_σ . In particular this gives an unbounded profit with bounded risk under transaction costs λ .

In the case of condition (b) one does a similar operation by going short in the asset S . The boundedness condition in the case (b) of *(NOA)* makes sure that this strategy is admissible.

We now formulate a local version of the fundamental theorem of asset pricing in the setting of transaction costs.

Theorem 2.2.4. *Let $S = (S_t)_{0 \leq t \leq T}$ be a strictly positive, continuous process. The following assertions are equivalent.*

- (i) *Locally, there is no obvious immediate arbitrage (NOIA).*
- (ii) *Locally, there is no obvious arbitrage (NOA).*
- (iii) *Locally, for each $0 < \mu < 1$, the condition (CPS $^\mu$) of existence of a μ -consistent price system holds true.*

Proof. See [51, Theorem 1] and [98, Theorem 5.11]. □

2.3 Superreplication Theorem

2.3.1 Theorem on Admissibility

Proposition 2.3.1. *Fix a strictly positive adapted càdlàg process $S = (S_t)_{0 \leq t \leq T}$ and transaction costs $0 < \lambda < 1$. Let $\varphi = (\varphi^0, \varphi^1)$ be an admissible λ -self-financing trading strategy. Suppose that (\tilde{S}, \mathbf{Q}) is a λ -consistent price system.*

Then the process $(\tilde{V}_t)_{0 \leq t \leq T}$ defined by

$$\tilde{V}_t := \varphi_t^0 + \varphi_t^1 \tilde{S}_t, \quad 0 \leq t \leq T,$$

satisfies $\tilde{V} \geq V^{\text{liq}}(\varphi)$ almost surely, and is an optional strong supermartingale under \mathbf{Q} .

Proof. See [96, Proposition 2]. □

Theorem 2.3.2. *Fix a strictly positive adapted càdlàg process $S = (S_t)_{0 \leq t \leq T}$, transaction costs $0 < \lambda < 1$. Suppose that S satisfies (CPS^μ) for each $0 < \mu < \lambda$. Let $\varphi = (\varphi^0, \varphi^1)$ be an admissible λ -self-financing trading strategy and suppose that there is a positive constant $M > 0$ such that $V_T^{\text{liq}}(\varphi) \geq -M$ almost surely.*

Then we have that $V_\tau^{\text{liq}}(\varphi) \geq -M$ almost surely, for every stopping time $0 \leq \tau \leq T$.

Proof. See [96, Theorem 1]. □

The assumption (CPS^μ) , for each $0 < \mu < \lambda$, cannot be dropped in Proposition 2.3.2 as shown by the counterexample presented in [96, Lemma 1].

In the market with transaction costs, we have the a priori assumption that the strategies φ have finite variation. Under the assumption that $(CPS^{\lambda'})$ holds true for some $0 < \lambda' < \lambda$, the convex hull of the set of variations $\text{Var}_T(\varphi)$ is bounded in probability.

Lemma 2.3.3. *Fix a strictly positive adapted càdlàg process $S = (S_t)_{0 \leq t \leq T}$ and transaction costs $0 < \lambda < 1$. Suppose that $(CPS^{\lambda'})$ is satisfied for some $0 < \lambda' < \lambda$.*

Then the convex hull of the random variables $\varphi_T^{0,\uparrow}$ (also $\varphi_T^{0,\downarrow}$, $\varphi_T^{1,\uparrow}$ and $\varphi_T^{1,\downarrow}$) remains bounded in $L^0(\mathbf{P})$, when φ runs through the self-financing M -admissible trading strategies.

More precisely: the set

$$\text{conv}(\varphi_T^{0,\uparrow} : \varphi \text{ is self-financing } M\text{-admissible})$$

is bounded in $L^0(\mathbf{P})$.

Proof. See [97, Lemma 3.1, Remark 3.2]. □

2.3.2 Superreplication Theorem

The following superreplication theorem describes the set of contingent claims allowing an agent in the market with proportional transaction costs λ to superreplicate with a given initial endowment by following some admissible λ -self-financing trading strategy.

Theorem 2.3.4 (Superreplication). *Fix a strictly positive adapted càdlàg process $S = (S_t)_{0 \leq t \leq T}$, transaction costs $0 < \lambda < 1$, and a contingent claim which pays g many units of bond at time T .*

Assume that the random variable g is uniformly bounded from below, and the process S satisfies (CPS^μ) for each $0 < \mu < \lambda$.

For a number $x \in \mathbb{R}$, the following assertions are equivalent:

(i) *There is an admissible λ -self-financing portfolio $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ such that*

$$\varphi_0 = (x, 0) \quad \text{and} \quad \varphi_T = (g, 0).$$

(ii) *For every λ -consistent price system (\tilde{S}, \mathbf{Q}) , we have $\mathbf{E}_{\mathbf{Q}}[g] \leq x$.*

Proof. See [97, Theorem 1.4, Section 5]. □

Chapter 3

Duality Theory for Utility Maximization under Transaction Costs

In this chapter, we review the general duality theory for utility maximization problem with general càdlàg price processes in the presence of proportional transaction costs, obtained in [27].

3.1 Formulation of the Problem

Let $U : (0, \infty) \rightarrow \mathbb{R}$ be a standard utility function defined on the positive real line, i.e., a strictly increasing, strictly concave, continuously differentiable function satisfying the Inada conditions:

$$U'(0) := \lim_{x \rightarrow 0} U'(x) = \infty \quad \text{and} \quad U'(\infty) := \lim_{x \rightarrow \infty} U'(x) = 0.$$

Define $U(x) = -\infty$ whenever $x \leq 0$. We assume that the utility function U satisfies the reasonable asymptotic elasticity, i.e.,

$$AE(U) := \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

This condition was firstly given by Kramkov and Schachermayer [70]. We may find financial interpretations and more results about it in [70, 71] as well as Appendix B.1.2.

We consider an agent who maximizes the expected utility of her terminal wealth. Fix an $x > 0$. The maximization problem is to find the optimal trading strategy $\hat{\varphi} = (\hat{\varphi}^0, \hat{\varphi}^1)$ to

$$\mathbf{E}[U(V_T^{\text{liq}}(\varphi))] \rightarrow \max!, \quad \varphi \in \mathcal{A}_0^\lambda(x), \quad (3.1.1)$$

where $\mathcal{A}_0^\lambda(x)$ denote the set of all λ -self-financing, 0-admissible trading strategies under transaction costs λ , starting with initial endowment $(\varphi_0^0, \varphi_0^1) = (x, 0)$. We denote the value function by

$$u(x) := \sup_{\varphi \in \mathcal{A}_0^\lambda(x)} \mathbf{E}[U(V_T^{\text{liq}}(\varphi))].$$

The problem (3.1.1) can also be formulated as the problem for random variables

$$\mathbf{E}[U(g)] \rightarrow \max!, \quad g \in \mathcal{C}_0^\lambda(x), \quad (3.1.2)$$

where

$$\mathcal{C}_0^\lambda(x) := \left\{ V_T^{\text{liq}}(\varphi) \mid \varphi \in \mathcal{A}_0^\lambda(x) \right\} \subseteq L_+^0(\mathbf{P})$$

denotes the set of all attainable payoffs under transaction costs. Define $\mathcal{C}_0^\lambda := \mathcal{C}_0^\lambda(1)$.

We can always assume without loss of generality that the price process cannot jump at the terminal time T , while the investor can still liquidate her position in the risky asset. Indeed, we may enlarge the time interval $[0, T]$ to $[0, T + 1]$, the underlying filtration as well as the price process do not change, i.e., for every $t \in [T, T + 1]$ we have $\mathcal{F}_t = \mathcal{F}_T$ and $S_t = S_T$. Between T and $T + 1$, an agent is allowed to make a final self-financing change in her portfolio, according to the terms fixed by the market at time T . (See [15, Remark 4.2].) This implies that we can assume without loss of generality that $\varphi_T^1 = 0$ and therefore

$$\mathcal{C}_0^\lambda(x) = \left\{ \varphi_T^0 \mid \varphi \in \mathcal{A}_0^\lambda(x) \right\} \subseteq L_+^0(\mathbf{P}).$$

3.2 Duality Theory

Let $Z = (Z^0, Z^1) \in \mathcal{Z}_{loc,e}^\lambda(S)$ be any local λ -consistent price system. By definition of $\mathcal{Z}_{loc,e}^\lambda(S)$, we may find a localizing sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times, such that the stopped process Z^{τ_n} defines a λ -consistent price system, for each n . Let $\varphi \in \mathcal{A}_0^\lambda(x)$ be arbitrary. By Proposition 2.3.1, the process $\varphi_{t \wedge \tau_n}^0 Z_{t \wedge \tau_n}^0 + \varphi_{t \wedge \tau_n}^1 Z_{t \wedge \tau_n}^1$ is an optional strong supermartingale. As a consequence, $Z^0 \varphi^0 + Z^1 \varphi^1$ is a local optional strong supermartingale. It follows from $Z^0 \varphi^0 + Z^1 \varphi^1 \geq Z^0 V^{\text{liq}}(\varphi) \geq 0$ that $Z^0 \varphi^0 + Z^1 \varphi^1$ is an optional strong supermartingale. In particular, $\mathbf{E}[Z_T^0 g] \leq x$ for each $g \in \mathcal{C}_0^\lambda(x)$ and each $Z = (Z^0, Z^1) \in \mathcal{Z}_{loc,e}^\lambda(S)$.

Let us introduce the convex conjugate function $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ of U

$$V(y) := \sup_{x > 0} \{U(x) - xy\}, \quad y > 0.$$

Note that $V(y)$ is strictly decreasing, strictly convex and continuously differentiable and satisfies

$$V(0) = U(\infty), \quad V(\infty) = U(0).$$

By Fenchel's inequality, we obtain that

$$u(x) = \sup_{g \in \mathcal{C}_0^\lambda(x)} \mathbf{E}[U(g)] \leq \sup_{g \in \mathcal{C}_0^\lambda(x)} \mathbf{E}[V(yZ_T^0) + yZ_T^0 g] \leq \mathbf{E}[V(yZ_T^0)] + xy,$$

for all $Z = (Z^0, Z^1) \in \mathcal{Z}_{loc,e}^\lambda(S)$ and $y > 0$. Therefore, we consider

$$\mathbf{E}[V(yZ_T^0)] \rightarrow \min!, \quad Z = (Z^0, Z^1) \in \mathcal{Z}_{loc,e}^\lambda(S), \quad (3.2.1)$$

as our dual problem. Again, the problem (3.2.1) can be also alternatively formulated as a problem over a set of random variables

$$\mathbf{E}[V(h)] \rightarrow \min!, \quad h \in D^\lambda(y), \quad (3.2.2)$$

where

$$D^\lambda(y) := \left\{ yZ_T^0 \mid Z = (Z^0, Z^1) \in \mathcal{Z}_{loc,e}^\lambda(S) \right\} = yD^\lambda$$

for $y > 0$ and $D^\lambda := D^\lambda(1)$.

Similarly to the frictionless case [70], the solution \widehat{h} to (3.2.2) is in general only attained as a \mathbf{P} -a.s. limit

$$\widehat{h} = y \lim_{n \rightarrow \infty} Z_T^{n,0} \quad (3.2.3)$$

of a minimizing sequence $(Z^n)_{n \in \mathbb{N}} \in \mathcal{Z}_{loc,e}^\lambda(S)$. To ensure the existence of an optimizer, we have to enlarge the sets $\mathcal{Z}_{loc,e}^\lambda(S)$ and $D^\lambda(y)$, and work with relaxed versions of the dual problems (3.2.1) and (3.2.2).

On the level of random variables, we consider

$$\mathbf{E}[V(h)] \rightarrow \min!, \quad h \in \overline{\text{sol}(D^\lambda(y))}^{\mathbf{P}}, \quad (3.2.4)$$

where $\overline{\text{sol}(D^\lambda(y))}^{\mathbf{P}}$ is the closed (in probability) convex solid hull of $D^\lambda(y)$ in $L_+^0(\mathbf{P})$ for $y > 0$. By Theorem A.1.4 (Bipolar Theorem), we obtain that $\overline{\text{sol}(D^\lambda(y))}^{\mathbf{P}} = (D^\lambda(y))^{\circ\circ}$ and it is easy to show that

$$\overline{\text{sol}(D^\lambda(y))}^{\mathbf{P}} = \left\{ yh \in L_+^0(\mathbf{P}) \mid \exists (Z^n)_{n \in \mathbb{N}} \subseteq \mathcal{Z}_{loc,e}^\lambda(S) \text{ such that } h \leq \lim_{n \rightarrow \infty} Z_T^{n,0} \right\}.$$

As sets $\mathcal{C}_0^\lambda(x)$ and $\overline{\text{sol}(D^\lambda(y))}^{\mathbf{P}}$ are polar to each other in $L_+^0(\mathbf{P})$ by Lemma 3.2.2, the abstract versions of the main results of [70], Theorems 3.1 and Theorem 3.2, carry over verbatim to the present setting under transaction costs. This gives static duality results in the sense that they provide duality relations between the solutions to the problems (3.1.2) and (3.2.4), which are problems for random variables rather than stochastic processes.

In order to establish dynamic duality results, similarly as in the frictionless duality [70], we consider supermartingale deflators as dual variables. These are nonnegative (not necessarily càdlàg) supermartingales $Y = (Y^0, Y^1) \geq 0$ such that $\widetilde{S} := \frac{Y^1}{Y^0}$ is valued in the bid-ask spread $[(1 - \lambda)S, S]$ and that $Y^0\varphi^0 + Y^1\varphi^1 = Y^0(\varphi^0 + \varphi^1\widetilde{S})$ is a supermartingale for all $\varphi \in \mathcal{A}_0^\lambda(1)$.

In the frictionless case [70], the solution to the dual problem for a semimartingale price process $\widetilde{S} = (\widetilde{S}_t)_{0 \leq t \leq T}$ is attained in the set of one-dimensional càdlàg supermartingale deflators. The reason for this is that, in the frictionless setting, the value process $\varphi^0 + \varphi^1\widetilde{S} = x + \varphi^1 \cdot \widetilde{S}$ is right-continuous (as a stochastic integral is càdlàg). Hence, the optimal supermartingale deflator to the dual problem can be obtained as the càdlàg Fatou limit of a minimizing sequence of equivalent local martingale or supermartingale deflators; see [70, Lemma 4.1 and Lemma 4.2] and A.1.5.

However, for càdlàg price processes $S = (S_t)_{0 \leq t \leq T}$ under transactions costs λ , we have to use predictable finite variation strategies $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ that can have left and right jumps to model trading strategies. This is unavoidable in order to obtain that the set $\mathcal{C}_0^\lambda(x)$ of attainable payoffs under transaction costs is closed in $L_+^0(\mathbf{P})$, see [15, Theorem 3.5] or [97, Theorem 3.4]. As we have to optimize simultaneously over Y^0 and Y^1 to obtain the optimal supermartingale deflator, we need a different limit than the Fatou

limit in $Y^0\varphi^0 + Y^1\varphi^1$ to remain in the class of supermartingale deflators. This limit also needs to ensure the convergence of a minimizing sequence $Z^n = (Z_t^{n,0}, Z_t^{n,1})_{0 \leq t \leq T}$ of local λ -consistent price systems at the jumps of the trading strategies. It turns out that the convergence in probability at all finite stopping times is the right topology to work with. We note that the limit of nonnegative local martingales with respect to this convergence is an optional strong supermartingale.

Definition 3.2.1. *We call a nonnegative process $Y = (Y^0, Y^1)$ an **optional strong supermartingale deflator** starting at y for some $y > 0$, if*

- (i) $Y_0^0 = y$,
- (ii) $\frac{Y^1}{Y^0} \in [(1 - \lambda)S, S]$,
- (iii) $Y^0\varphi^0 + Y^1\varphi^1$ is a nonnegative optional strong supermartingale for all $\varphi \in \mathcal{A}_0^\lambda(1)$.

As dual variables we consider the set of optional strong supermartingale deflators starting at y , denoted by $\mathcal{B}^\lambda(y)$, and accordingly,

$$\mathcal{D}^\lambda(y) := \{Y_T^0 \mid (Y^0, Y^1) \in \mathcal{B}^\lambda(y)\}.$$

Hence, the dual problem is now

$$\mathbf{E}[V(h)] \rightarrow \min!, \quad h \in \mathcal{D}^\lambda(y), \quad (3.2.5)$$

and the dual value function is defined by

$$v(y) := \inf_{h \in \mathcal{D}^\lambda(y)} \mathbf{E}[V(h)].$$

The following lemma was shown in [27, Lemma A.1], which establishes the polar relation of \mathcal{C}_0^λ and \mathcal{D}^λ .

Lemma 3.2.2. *Suppose that S satisfies locally (CPS $^\mu$) for all $\mu \in (0, \lambda)$. Then:*

- (1) \mathcal{C}_0^λ and \mathcal{D}^λ are convex, solid and closed in the topology of convergence in measure.
- (2) $g \in \mathcal{C}_0^\lambda$, if and only if $\mathbf{E}[gh] \leq 1$, for all $h \in \mathcal{D}^\lambda$, and $h \in \mathcal{D}^\lambda$, if and only if $\mathbf{E}[gh] \leq 1$, for all $g \in \mathcal{C}_0^\lambda$.
- (3) \mathcal{C}^λ is a bounded subset of $L_+^0(\mathbf{P})$ and contains the constant function $\mathbf{1}$.
- (4) The closed, convex, solid hull of \mathcal{D}^λ in $L_+^0(\mathbf{P})$ is given by \mathcal{D}^λ , i.e.,

$$(\mathcal{D}^\lambda)^{\circ\circ} = \overline{\text{sol}(\mathcal{D}^\lambda)}^{\mathbf{P}} = \mathcal{D}^\lambda.$$

- (5) \mathcal{D}^λ is closed under countable convex combinations.

- (6) For any $g \in \mathcal{C}_0^\lambda$, we have

$$\sup_{h \in \mathcal{D}^\lambda} \mathbf{E}[gh] = \sup_{h \in \mathcal{D}^\lambda} \mathbf{E}[gh].$$

Proof. (1). The convexity of \mathcal{C}_0^λ and \mathcal{D}^λ is clear.

For $g \in \mathcal{C}_0^\lambda$ and $0 \leq h \leq g$ we can use the same trading strategy for g and then throw away money to get h . Therefore $h \in \mathcal{C}_0^\lambda$, and the solidity of \mathcal{C}_0^λ follows.

As regards the solidity of \mathcal{D}^λ : let $Y_T^0 \in \mathcal{D}^\lambda$ with $Y \in \mathcal{B}^\lambda$ and $h \in L_+^0(\mathbf{P})$ satisfying $0 \leq h \leq Y_T^0$. We may define $Z \in \mathcal{B}^\lambda$ by

$$(Z_t^0, Z_t^1) := \begin{cases} (Y_t^0, Y_t^1), & 0 \leq t < T, \\ (h, Y_T^1 \frac{h}{Y_T^0}), & t = T. \end{cases}$$

Indeed, we have $Z_0^0 = Y_0^0 = 1$,

$$\frac{Z^1}{Z^0} = \frac{Y^1}{Y^0} \in [(1-\lambda)S, S],$$

and for all stopping times σ and τ with $0 \leq \sigma \leq \tau \leq T$, we have by $0 \leq h \leq Y_T^0$ that

$$\mathbf{E}[Z_\tau^0 \varphi_\tau^0 + Z_\tau^1 \varphi_\tau^1 | \mathcal{F}_\sigma] \leq \mathbf{E}[Y_\tau^0 \varphi_\tau^0 + Y_\tau^1 \varphi_\tau^1 | \mathcal{F}_\sigma] \leq Y_\sigma^0 \varphi_\sigma^0 + Y_\sigma^1 \varphi_\sigma^1 = Z_\sigma^0 \varphi_\sigma^0 + Z_\sigma^1 \varphi_\sigma^1,$$

for all $\varphi \in \mathcal{A}_0^\lambda(1)$.

To prove the closedness of \mathcal{C}_0^λ , let $(\varphi^n)_{n \in \mathbb{N}} \subseteq \mathcal{A}_0^\lambda(1)$ be such that $g^n := V_T^{\text{liq}}(\varphi^n)$ converges to some $g \in L_+^0(\mathbf{P})$ in probability. Since S satisfies locally (CPS^μ) for all $\mu \in (0, \lambda)$, by Proposition 2.1.14, there exist a $Z \in \mathcal{Z}_{loc,e}^{\lambda'}(S)$ for some $\lambda' \in (0, \lambda)$ and a localizing sequence $(\tau_m)_{m \in \mathbb{N}}$ such that each Z^{τ_m} defines a λ' -consistent price system for S^{τ_m} . By Lemma 2.3.3 ([97, Remark 3.2]) we have that the convex combination of the variation of φ^n on $\llbracket 0, \tau_m \rrbracket$ of φ^n

$$A_m := \text{conv}\{\text{Var}_{\tau_m}(\varphi^n) \mid n \in \mathbb{N}\}$$

is bounded in probability for each $m \in \mathbb{N}$, which is equivalent to the boundedness of $A := \text{conv}\{\text{Var}_T(\varphi^n) \mid n \in \mathbb{N}\}$ in probability.

Indeed, fix $\varepsilon > 0$ and let $m, N(\varepsilon, m) \in \mathbb{N}$ be such that $\mathbf{P}[\tau_m < T] < \frac{\varepsilon}{2}$ and

$$\sup_{g \in A_m} \mathbf{P}[g > N(\varepsilon, m)] < \frac{\varepsilon}{2},$$

by the $L^0(\mathbf{P})$ -boundedness of A_m . Since $\text{Var}_{\tau_m}(\varphi^n) = \text{Var}_T(\varphi^n)$ on $\{\tau_m = T\}$ for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \sup_{g \in A} \mathbf{P}[g > N(\varepsilon, m)] &= \sup_{g \in A} \{\mathbf{P}[g > N(\varepsilon, m), \tau_m = T] + \mathbf{P}[g > N(\varepsilon, m), \tau_m < T]\} \\ &\leq \sup_{g \in A_m} \{\mathbf{P}[g > N(\varepsilon, m)] + \mathbf{P}[\tau_m < T]\} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, by Theorem A.1.13 ([15, Proposition 3.4]) and Remark A.1.14, there exist a sequence of convex combinations $\tilde{\varphi}^n \in \text{conv}(\varphi^n, \varphi^{n+1}, \dots)$ and a predictable finite variation process φ such that

$$\mathbf{P}[\tilde{\varphi}_t^n \rightarrow \varphi_t, \forall t \in [0, T]] = 1,$$

which already implies that $\varphi \in \mathcal{A}_0^\lambda(1)$.

The closedness of \mathcal{D}^λ follows by combining similar arguments as in [70, Lemma 4.1] with a new version of Komlós' lemma for nonnegative optional strong supermartingales in [29]. To that end, let $(h^n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{D}^λ converging to some h in probability. Then there exists a sequence $((Y^{n,0}, Y^{n,1}))_{n \in \mathbb{N}} \subseteq \mathcal{B}^\lambda(1)$ such that $Y_T^{n,0} = h^n$ for each $n \in \mathbb{N}$. Since $Y^{n,0}$ and $Y^{n,1}$ are nonnegative optional strong supermartingales, there exist by Theorem A.1.16 ([29, Theorem 2.7]) a sequence

$$(\tilde{Y}^{n,0}, \tilde{Y}^{n,1}) \in \text{conv}((Y^{n,0}, Y^{n,1}), (Y^{n+1,0}, Y^{n+1,1}), \dots)$$

and optional strong supermartingales \tilde{Y}^0 and \tilde{Y}^1 such that

$$(\tilde{Y}_\tau^{n,0}, \tilde{Y}_\tau^{n,1}) \xrightarrow{\mathbf{P}} (\tilde{Y}_\tau^0, \tilde{Y}_\tau^1), \quad (3.2.6)$$

for all $[0, T]$ -valued stopping times τ . This convergence in probability is then sufficient to deduce that $\tilde{Y}_0^0 = 1$, $\tilde{Y}_T^0 = h$, and that $\tilde{Y}^0 \varphi^0 + \tilde{Y}^1 \varphi^1$ is a nonnegative optional strong supermartingale for all $(\varphi^0, \varphi^1) \in \mathcal{A}_0^\lambda(1)$. To see the latter, observe that, for all stopping times σ and τ such that $0 \leq \sigma \leq \tau \leq T$, we have that

$$\begin{aligned} \tilde{Y}_\sigma^0 \varphi_\sigma^0 + \tilde{Y}_\sigma^1 \varphi_\sigma^1 &= \liminf_{n \rightarrow \infty} (\tilde{Y}_\sigma^{n,0} \varphi_\sigma^0 + \tilde{Y}_\sigma^{n,1} \varphi_\sigma^1) \geq \liminf_{n \rightarrow \infty} \mathbf{E}[\tilde{Y}_\tau^{n,0} \varphi_\tau^0 + \tilde{Y}_\tau^{n,1} \varphi_\tau^1 | \mathcal{F}_\sigma] \\ &\geq \mathbf{E} \left[\liminf_{n \rightarrow \infty} (\tilde{Y}_\tau^{n,0} \varphi_\tau^0 + \tilde{Y}_\tau^{n,1} \varphi_\tau^1) \middle| \mathcal{F}_\sigma \right] = \mathbf{E}[\tilde{Y}_\tau^0 \varphi_\tau^0 + \tilde{Y}_\tau^1 \varphi_\tau^1 | \mathcal{F}_\sigma] \end{aligned}$$

by Fatou's lemma for conditional expectations.

To conclude that $(\tilde{Y}^0, \tilde{Y}^1) \in \mathcal{B}^\lambda(1)$ and hence that $h \in \mathcal{D}^\lambda$, it remains to show that $(\tilde{Y}^0, \tilde{Y}^1)$ is \mathbb{R}_+^2 -valued and $\tilde{S} := \frac{\tilde{Y}^1}{\tilde{Y}^0}$ is valued in $[(1-\lambda)S, S]$. Indeed, as $\tilde{S}_\tau^n \in [(1-\lambda)S_\tau, S_\tau]$, this implies that also \tilde{S}_τ is valued in $[(1-\lambda)S_\tau, S_\tau]$. By Theorem A.1.8 (section theorem), the assertion follows. The assertion that $(\tilde{Y}^0, \tilde{Y}^1)$ is \mathbb{R}_+^2 -valued follows by the same arguments.

(2). The first assertion, $\mathcal{C}_0^\lambda = (\mathcal{D}^\lambda)^\circ$, follows by Lemma 3.2.3 below, which is the local version of the superreplication theorem under transaction costs.

The second assertion follows from the fact that $(\mathcal{C}_0^\lambda)^\circ = (\mathcal{D}^\lambda)^{\circ\circ} = \mathcal{D}^\lambda$, since by (1) the set \mathcal{D}^λ is convex solid and closed in the topology of convergence in measure.

(3). Assume \mathcal{C}_0^λ fails to be bounded in $L^0(\mathbf{P})$. Then there exists $\alpha > 0$ such that, for all $M > 0$, we may find a $\varphi_T^0 \in \mathcal{C}_0^\lambda$ such that $\mathbf{P}[\varphi_T^0 \geq M] \geq \alpha$. Fix an $Z \in \mathcal{Z}_{loc, \varepsilon}^\lambda(S)$. The strict positivity of Z implies that

$$\beta := \inf \{ \mathbf{E}[Z_T^0 \mathbf{1}_A] : \mathbf{P}[A] \geq \alpha \} > 0.$$

(Indeed, For an $\alpha > 0$, we may find an $\varepsilon > 0$ such that $\alpha - \varepsilon > 0$. Since $Z_T^0 > 0$ a.s., there exists $N \in \mathbb{N}$ such that $\mathbf{P}[Z_T^0 < \frac{1}{N}] < \alpha - \varepsilon$, for all $n \geq N$. The worse case is that $\{Z_T^0 < \frac{1}{N}\} \subseteq A$. Hence,

$$\mathbf{E}[Z_T^0 \mathbf{1}_A] = \mathbf{E}[Z_T^0 \mathbf{1}_A \mathbf{1}_{\{Z_T^0 < \frac{1}{N}\}}] + \mathbf{E}[Z_T^0 \mathbf{1}_A \mathbf{1}_{\{Z_T^0 \geq \frac{1}{N}\}}] \geq \frac{1}{N} \mathbf{P}[A \cap \{Z_T^0 \geq \frac{1}{N}\}] \geq \frac{\varepsilon}{N} > 0,$$

and the assertion follows.) Let $M > \frac{1}{\beta}$. We arrive at a contradiction to the supermartingale property

$$1 = \mathbf{E}[Z_0^0 \varphi_0^0] \geq \mathbf{E}[Z_T^0 \varphi_T^0] \geq \mathbf{E}[Z_T^0 \varphi_T^0 \mathbf{1}_{\{\varphi_T^0 \geq M\}}] \geq M \mathbf{E}[Z_T^0 \mathbf{1}_{\{\varphi_T^0 \geq M\}}] \geq \beta M > 1.$$

The fact that \mathcal{C}_0^λ contains the constant function $\mathbf{1}$ follows by definition.

(4). It follows from Lemma 3.2.3 that $\mathcal{C}_0^\lambda = (D^\lambda)^\circ$, therefore

$$(\mathcal{C}_0^\lambda)^\circ = (D^\lambda)^{\circ\circ} = \overline{\text{sol}(D^\lambda)}^{\mathbf{P}}.$$

Since $(\mathcal{C}_0^\lambda)^\circ = \mathcal{D}^\lambda$ by (2), we obtain $(D^\lambda)^{\circ\circ} = \overline{\text{sol}(D^\lambda)}^{\mathbf{P}} = \mathcal{D}^\lambda$.

(5). Given $(Z^{n,0}, Z^{n,1})_{n \in \mathbb{N}} \subseteq \mathcal{Z}_{loc,e}^\lambda(S)$ and $(\alpha_n)_{n \in \mathbb{N}}$ positive numbers such that $\sum_{n=1}^\infty \alpha_n = 1$, we have that $\sum_{n=1}^\infty \alpha_n Z^{n,0}$ is a nonnegative local martingale starting at 1, $\sum_{n=1}^\infty \alpha_n Z^{n,1}$ is a local martingale and

$$\frac{\sum_{n=1}^\infty \alpha_n Z^{n,1}}{\sum_{n=1}^\infty \alpha_n Z^{n,0}} \in [(1-\lambda)S, S],$$

which already gives (5).

(6). Suppose that there exists a $\hat{g} \in \mathcal{C}_0^\lambda$ such that there is a $\hat{h} \in \mathcal{D}^\lambda \setminus D^\lambda$ such that

$$\mathbf{E}[\hat{g}h] < \mathbf{E}[\hat{g}\hat{h}]$$

for all $h \in D^\lambda$. Therefore we may find an $0 < \alpha$ such that

$$\mathbf{E}[\hat{g}h] \leq \alpha < \mathbf{E}[\hat{g}\hat{h}].$$

for all $h \in D^\lambda$, which implies that $\frac{\hat{g}}{\alpha} \in \mathcal{C}_0^\lambda$ by Lemma 3.2.3 below. On the other hand, we obtain

$$\mathbf{E}\left[\frac{\hat{g}}{\alpha}\hat{h}\right] > 1,$$

which contradicts the fact that $\hat{h} \in \mathcal{D}^\lambda = (\mathcal{C}_0^\lambda)^\circ$. \square

Lemma 3.2.3. ([27, Lemma A.2]) *Suppose that S satisfies locally (CPS $^\mu$) for all $\mu \in (0, \lambda)$. Let $g \in L_+^0(\mathbf{P})$.*

Then we have that $g \in \mathcal{C}_0^\lambda$ if and only if $\mathbf{E}[gZ_T^0] \leq 1$ for all $Z \in \mathcal{Z}_{loc,e}^\lambda(S)$.

Proof. The ‘‘only if’’ part follows from the fact that, for all $\varphi \in \mathcal{A}_0^\lambda(1)$ and $Z \in \mathcal{Z}_{loc,e}^\lambda(S)$, the process $Z^0\varphi^0 + Z^1\varphi^1$ is a nonnegative local optional strong supermartingale by Proposition 2.3.1 ([96, Proposition 2]) and hence a true optional strong supermartingale.

For the ‘‘if’’ part, let $(\tau_m)_{m \in \mathbb{N}}$ be a localizing sequence of stopping times for some $Z \in \mathcal{Z}_{loc,e}^\lambda(S)$ such that Z^{τ_m} is a λ' -consistent price system for S^{τ_m} for some $\lambda' \in (0, \lambda)$. Then,

$$g_m := g\mathbf{1}_{\{\tau_m=T\}} \in \mathcal{C}_{0,m}^\lambda := \left\{ V_{\tau_m}^{\text{liq}}(\varphi) \mid \varphi \in \mathcal{A}_0^\lambda(1) \right\},$$

and $g \in \mathcal{C}_0^\lambda$ if and only if $g_m \in \mathcal{C}_{0,m}^\lambda$ for each $m \in \mathbb{N}$, as $\mathcal{C}_{0,m}^\lambda \subseteq \mathcal{C}_0^\lambda$, $g_m \xrightarrow{\mathbf{P}} g$ and \mathcal{C}_0^λ is closed with respect to the topology of convergence in probability.

Assume now for a proof by contradiction that there exists some $m' \in \mathbb{N}$ such that $g_{m'} \notin \mathcal{C}_{0,m'}^\lambda$. As $S^{\tau_{m'}}$ satisfies the assumptions of the Superreplication Theorem 2.3.4 ([97, Theorem 1.4]), there exists a λ' -consistent price system $\bar{Z} = (\bar{Z}^0, \bar{Z}^1)$ for $S^{\tau_{m'}}$ such that

$$\mathbf{E}\left[g_{m'}\bar{Z}_{\tau_{m'}}^0\right] > 1.$$

By the assumption that S admits a local μ -consistent price system for any $\mu \in (0, \lambda)$ we can extend \bar{Z} to a local λ -consistent price system $\tilde{Z} = (\tilde{Z}^0, \tilde{Z}^1)$ by setting

$$\tilde{Z}_t^0 := \begin{cases} \bar{Z}_t^0 & \text{for } 0 \leq t < \tau_{m'}, \\ \check{Z}_t^0 \frac{\bar{Z}_{\tau_{m'}}^0}{\check{Z}_{\tau_{m'}}^0} & \text{for } \tau_{m'} \leq t \leq T, \end{cases}$$

and

$$\tilde{Z}_t^1 := \begin{cases} (1 - \mu') \bar{Z}_t^1 & \text{for } 0 \leq t < \tau_{m'}, \\ (1 - \mu') \check{Z}_t^1 \frac{\bar{Z}_{\tau_{m'}}^1}{\check{Z}_{\tau_{m'}}^1} & \text{for } \tau_{m'} \leq t \leq T, \end{cases}$$

for some local μ' -consistent price system $\check{Z} = (\check{Z}^0, \check{Z}^1)$ with $0 < \mu' < \frac{\lambda - \lambda'}{2}$. Indeed, clearly, \tilde{Z}^0 and \tilde{Z}^1 are strictly positive local martingale under \mathbf{P} and $\tilde{Z}_0^0 = 1$. To show that $\frac{\tilde{Z}_t^1}{\tilde{Z}_t^0}$ takes its values in $[(1 - \lambda)S, S]$ note that, for $0 \leq t < \tau_{m'}$, the quotient $\frac{\tilde{Z}_t^1}{\tilde{Z}_t^0}$ lies in $[(1 - \lambda')(1 - \mu')S_t, (1 - \mu')S_t]$. For $\tau_{m'} \leq t \leq T$ we still obtain that

$$\frac{\tilde{Z}_t^1}{\tilde{Z}_t^0} = (1 - \mu') \frac{\check{Z}_t^1}{\check{Z}_t^0} \frac{\bar{Z}_{\tau_{m'}}^1}{\bar{Z}_{\tau_{m'}}^0} \frac{1}{\frac{\check{Z}_{\tau_{m'}}^1}{\check{Z}_{\tau_{m'}}^0}} \in [(1 - \mu')^2(1 - \lambda')S_t, S_t],$$

which is contained in $[(1 - \lambda)S_t, S_t]$ as $0 < \mu' < \frac{\lambda - \lambda'}{2}$. Since

$$\mathbf{E}[g\tilde{Z}_T^0] \geq \mathbf{E}\left[g_{m'} \bar{Z}_{\tau_{m'}}^0\right] > 1,$$

we obtain a contradiction to the assumption that $\mathbf{E}[gZ_T^0] \leq 1$ for all $Z \in \mathcal{Z}_{loc,e}^\lambda(S)$. \square

We recall the duality theorem stated in [27, Theorem 3.2].

Theorem 3.2.4. *Let S be an adapted strictly positive càdlàg process. Suppose that*

- *S admits locally a μ -consistent price system for all $\mu \in (0, \lambda)$.*
- *The asymptotic elasticity of U is strictly less than one, i.e.,*

$$AE(U) := \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

- *The maximal expected utility is finite, i.e., $u(x) < \infty$, for some $x \in (0, \infty)$.*

Then

(1) *We have the following properties for the value functions:*

- *$u(x) < \infty$ for all $x > 0$. u is strictly concave, strictly increasing and continuously differentiable on $(0, \infty)$, and satisfy the Inada conditions*

$$u'(0) = \infty \quad \text{and} \quad u'(\infty) = 0.$$

- $v(y) < \infty$ for all $y > 0$. v is strictly convex, strictly decreasing and continuously differentiable on $(0, \infty)$, and satisfy the Inada conditions

$$v'(0) = -\infty \quad \text{and} \quad v'(\infty) = 0.$$

- The primal value function u and the dual value function v are conjugate, i.e.,

$$u(x) = \inf_{y>0} \{v(y) + xy\}, \quad v(y) = \sup_{x>0} \{u(x) - xy\}.$$

- (2) For all $x, y > 0$, the solutions $\widehat{g}(x) \in \mathcal{C}_0^\lambda(x)$ and $\widehat{h}(y) \in \mathcal{D}^\lambda(y)$ to the primal problem (3.1.2) and the dual problem (3.2.5), respectively, exist, are unique, and there are $(\widehat{\varphi}^0(x), \widehat{\varphi}^1(x)) \in \mathcal{A}_0^\lambda(x)$ and $(\widehat{Y}^0(y), \widehat{Y}^1(y)) \in \mathcal{B}^\lambda(y)$ such that

$$V_T^{\text{liq}}(\widehat{\varphi}(x)) = \widehat{g}(x) \quad \text{and} \quad \widehat{Y}_T^0(y) = \widehat{h}(y). \quad (3.2.7)$$

- (3) For all $x > 0$, let $\widehat{y}(x) = u'(x) > 0$, which is the unique solution to

$$v(y) + xy \rightarrow \min!, \quad y > 0,$$

and is equivalent to $x = -v'(\widehat{y}(x))$. Then, $\widehat{g}(x)$ and $\widehat{h}(\widehat{y}(x))$ are given by the first order conditions

$$\widehat{h}(\widehat{y}(x)) = U'(\widehat{g}(x)) \quad \text{and} \quad \widehat{g}(x) = -V'(\widehat{h}(\widehat{y}(x))), \quad (3.2.8)$$

and we have that

$$\mathbf{E}[\widehat{g}(x)\widehat{h}(\widehat{y}(x))] = x\widehat{y}(x).$$

In particular, the process $\widehat{Y}^0(\widehat{y}(x))\widehat{\varphi}^0(x) + \widehat{Y}^1(\widehat{y}(x))\widehat{\varphi}^1(x)$ is a càdlàg uniformly integrable martingale for all $(\widehat{\varphi}^0(x), \widehat{\varphi}^1(x)) \in \mathcal{A}_0^\lambda(x)$ and $(\widehat{Y}^0(\widehat{y}(x)), \widehat{Y}^1(\widehat{y}(x))) \in \mathcal{B}^\lambda(\widehat{y}(x))$ satisfying (3.2.7) with $y = \widehat{y}(x)$.

- (4) We have that

$$v(y) = \inf_{(Z^0, Z^1) \in \mathcal{Z}_{loc, e}^\lambda(S)} \mathbf{E}[V(yZ_T^0)]. \quad (3.2.9)$$

Proof. The proof follows immediately from Theorem B.1.20 ([70, Theorem 3.2 and Proposition 3.2]). The process $\widehat{Y}^0(\widehat{y}(x))\widehat{\varphi}^0(x) + \widehat{Y}^1(\widehat{y}(x))\widehat{\varphi}^1(x)$ is a martingale, as it is an optional strong supermartingale with constant expectation. \square

3.3 Shadow Price

Let \widetilde{S} be any fictitious price process, that takes values in the bid-ask spread $[(1 - \lambda)S, S]$ of the original market S and can be traded in a frictionless way. As purchases and sales can be carried out at potentially more favorable prices, any attainable payoff in the market with transaction costs can be dominated by a payoff in the frictionless market \widetilde{S} . As a consequence, we obtain

$$u(x) = \sup_{\varphi \in \mathcal{A}_0^\lambda(x)} \mathbf{E}[U(V_T^{\text{liq}}(\varphi))] \leq \sup_{\varphi \in \mathcal{X}(x; \widetilde{S})} \mathbf{E}[U(x + (\varphi^1 \cdot \widetilde{S})_T)] =: u(x; \widetilde{S}). \quad (3.3.1)$$

Here $\mathcal{X}(x; \tilde{S})$ denotes the set of all self-financing and admissible trading strategies $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ for the price process $\tilde{S} = (\tilde{S}_t)_{0 \leq t \leq T}$ in the frictionless setting, i.e., that $\varphi^1 = (\varphi_t^1)_{0 \leq t \leq T}$ is an \tilde{S} -integrable predictable process such that $x + (\varphi^1 \cdot \tilde{S})_t \geq 0$ for all $t \in [0, T]$ and $\varphi^0 = (\varphi_t^0)_{0 \leq t \leq T}$ is defined via

$$\varphi_t^0 = x + (\varphi^1 \cdot \tilde{S})_t - \varphi_t^1 \tilde{S}_t, \quad t \in [0, T].$$

Note that $\mathcal{A}_0^\lambda(x) \subseteq \mathcal{X}(x; \tilde{S})$.

The natural question here is whether we can find a least unfavorable frictionless market evolving in the bid-ask spread with the same maximal expected utility as the original market with transaction costs. This leads to the following definition.

Definition 3.3.1. *In the above setting, a semimartingale $\tilde{S} = (\tilde{S}_t)_{0 \leq t \leq T}$ is called a **shadow price process** for the optimization problem (3.1.1), if*

(i) $\tilde{S} = (\tilde{S}_t)_{0 \leq t \leq T}$ takes values in the bid-ask spread $[(1 - \lambda)S, S]$.

(ii) The solution $\tilde{\varphi} = (\tilde{\varphi}^0, \tilde{\varphi}^1)$ to the corresponding frictionless utility maximization problem

$$\mathbf{E}[U(x + (\varphi^1 \cdot \tilde{S})_T)] \rightarrow \max!, \quad (\varphi^0, \varphi^1) \in \mathcal{X}(x; \tilde{S}), \quad (3.3.2)$$

exists and coincides with the solution $\hat{\varphi} = (\hat{\varphi}^0, \hat{\varphi}^1)$ to (3.1.1) under transaction costs.

Note that a shadow price $\tilde{S} = (\tilde{S}_t)_{0 \leq t \leq T}$ depends on the process S , the agent's utility function, and on her initial endowment.

If a shadow price \tilde{S} exists, then an optimal strategy $\tilde{\varphi} = (\tilde{\varphi}^0, \tilde{\varphi}^1)$ for the frictionless utility maximization problem (3.3.2) can also be realized in the market with transaction costs in the sense spelled out in (3.3.3) below. As the expected utility for \tilde{S} without transaction costs is by (3.3.1) a priori higher than that of any other strategy under transaction costs, it is – a fortiori – also an optimal strategy under transaction costs. The existence of a shadow price \tilde{S} implies in particular that the optimal strategy $\hat{\varphi} = (\hat{\varphi}^0, \hat{\varphi}^1)$ under transaction costs only trades, if \tilde{S} is at the bid or ask price, i.e.,

$$\{d\hat{\varphi}^1 > 0\} \subseteq \{\tilde{S} = S\} \quad \text{and} \quad \{d\hat{\varphi}^1 < 0\} \subseteq \{\tilde{S} = (1 - \lambda)S\}$$

in the sense that

$$\begin{aligned} \{d\hat{\varphi}^{1,c} > 0\} &\subseteq \{\tilde{S} = S\}, & \{d\hat{\varphi}^{1,c} < 0\} &\subseteq \{\tilde{S} = (1 - \lambda)S\}, \\ \{\Delta\hat{\varphi}^1 > 0\} &\subseteq \{\tilde{S}_- = S_-\}, & \{\Delta\hat{\varphi}^1 < 0\} &\subseteq \{\tilde{S}_- = (1 - \lambda)S_-\}, \\ \{\Delta_+\hat{\varphi}^1 > 0\} &\subseteq \{\tilde{S} = S\}, & \{\Delta_+\hat{\varphi}^1 < 0\} &\subseteq \{\tilde{S} = (1 - \lambda)S\}. \end{aligned} \quad (3.3.3)$$

The precise mathematical meaning of (3.3.3) is given by

$$\begin{aligned} \int_0^T \mathbf{1}_{\{\tilde{S} \neq S\}}(u) d\hat{\varphi}_u^{1,\uparrow} &= \int_0^T \mathbf{1}_{\{\tilde{S} \neq S\}}(u) d\hat{\varphi}_u^{1,\uparrow,c} + \sum_{0 < u \leq T} \mathbf{1}_{\{\tilde{S}_- \neq S_-\}}(u) \Delta\hat{\varphi}_u^{1,\uparrow} \\ &+ \sum_{0 \leq u < T} \mathbf{1}_{\{\tilde{S} \neq S\}}(u) \Delta_+\hat{\varphi}_u^{1,\uparrow} = 0, \end{aligned}$$

and

$$\begin{aligned} \int_0^T \mathbf{1}_{\{\widehat{S} \neq (1-\lambda)S\}}(u) d\widehat{\varphi}_u^{1,\downarrow} &= \int_0^T \mathbf{1}_{\{\widehat{S} \neq (1-\lambda)S\}}(u) d\widehat{\varphi}_u^{1,\downarrow,c} + \sum_{0 < u < T} \mathbf{1}_{\{\widehat{S}_- \neq (1-\lambda)S_-\}}(u) \Delta \widehat{\varphi}_u^{1,\downarrow} \\ &+ \sum_{0 \leq u < T} \mathbf{1}_{\{\widehat{S} \neq (1-\lambda)S\}}(u) \Delta_+ \widehat{\varphi}_u^{1,\downarrow} = 0. \end{aligned}$$

In general, shadow prices fail to exit in the sense of Definition 3.3.1 (see the counterexamples in [4, 25, 27]). The reason for this is that, similarly to the frictionless case [70], the solution \widehat{h} to (3.2.2) is in general only attained as a \mathbf{P} -a.s. limit

$$\widehat{h} = y \lim_{n \rightarrow \infty} Z_T^{n,0}$$

of a minimizing sequence $(Z^n)_{n=1}^\infty$ of local λ -consistent price systems.

In [27, Example 4.1] we also see that, it may happen that the dual optimizer $\widehat{Y} = (\widehat{Y}^0, \widehat{Y}^1)$ as well as its ratio \widehat{S} do not have càdlàg trajectories and therefore fail to be semimartingales. Though we are not in the standard setting of stochastic integration, we can still define the stochastic integral $\widehat{\varphi}^1 \cdot \widehat{S}$ of a predictable finite variation process $\widehat{\varphi}^1 = (\widehat{\varphi}_t^1)_{0 \leq t \leq T}$ with respect to the làdlàg process $\widehat{S} = (\widehat{S}_t)_{0 \leq t \leq T}$ by integration by parts; see (A.1.1). This yields

$$(\widehat{\varphi}^1 \cdot \widehat{S})_t = \int_0^t \widehat{\varphi}_u^{1,c} d\widehat{S}_u + \sum_{0 < u \leq t} \Delta \widehat{\varphi}_u^1 (\widehat{S}_t - \widehat{S}_{u-}) + \sum_{0 \leq u < t} \Delta_+ \widehat{\varphi}_u^1 (\widehat{S}_t - \widehat{S}_u). \quad (3.3.4)$$

The integral (3.3.4) can still be interpreted as the gains from trading of the self-financing trading strategy $\widehat{\varphi}^1 = (\widehat{\varphi}_t^1)_{0 \leq t \leq T}$ without transaction for the price process $\widehat{S} = (\widehat{S}_t)_{0 \leq t \leq T}$.

It turns out that the left jumps $\Delta \widehat{\varphi}_u^1$ of the optimizer $\widehat{\varphi}^1$ need special care. The crux here is that, as shown in (3.3.4), the trades $\Delta \widehat{\varphi}_u^1$ are not carried out at the price \widehat{S}_u but rather at its left limit \widehat{S}_{u-} . We need to consider a pair of processes $Y^p = (Y_t^{p,0}, Y_t^{p,1})_{0 \leq t \leq T}$ and $Y = (Y_t^0, Y_t^1)_{0 \leq t \leq T}$, that correspond to the limit of the left limits $Z_-^n = (Z_-^{n,0}, Z_-^{n,1})$ and the limit of the approximating consistent price systems $Z^n = (Z^{n,0}, Z^{n,1})$ themselves retrospectively. As shown in [27, Example 4.2], the process Y^p and Y_- do not need to coincide so that we have that “limit of left limits \neq left limits of limits”.

Like the left limits $Z_-^n = (Z_-^{n,0}, Z_-^{n,1})$, their limit $Y^p = (Y^{p,0}, Y^{p,1})$ is a predictable strong supermartingale.

In the context of Theorem 3.2.4 above, we call $\mathcal{Y} = (Y^p, Y) = ((Y^{p,0}, Y^{p,1}), (Y^0, Y^1))$ a **sandwiched strong supermartingale deflator**, if

- $Y = (Y^0, Y^1) \in \mathcal{B}^\lambda(y)$,
- $(Y^{p,0}, Y^0)$ and $(Y^{p,1}, Y^1)$ are sandwiched strong supermartingales,
- the process \widetilde{S}^p lies in the bid-ask spread, i.e.,

$$\widetilde{S}_t^p := \frac{Y_t^{p,1}}{Y_t^{p,0}} \in [(1-\lambda)S_{t-}, S_{t-}], \quad t \in [0, T].$$

The definitions above allow us to obtain the following extension of Theorem 3.2.4. Roughly speaking, it states that the hypotheses of Theorem 3.2.4 suffice to yield a shadow price in a more general “sandwiched sense”.

Theorem 3.3.2. *Under the assumptions of Theorem 3.2.4, let $(Z^n)_{n \in \mathbb{N}}$ be a minimizing sequence of local λ -consistent price systems for the dual problem (3.2.9), i.e.,*

$$\mathbf{E}[V(\hat{y}(x)Z_T^{n,0})] \searrow v(\hat{y}(x)), \quad \text{as } n \rightarrow \infty.$$

Then there exist convex combinations $\tilde{Z}^n \in \text{conv}(Z^n, Z^{n+1}, \dots)$ and a sandwiched strong supermartingale deflator $\hat{\mathcal{Y}} = (\hat{Y}^p, \hat{Y})$ such that

$$\hat{y}(x)(\tilde{Z}_{\tau-}^{n,0}, \tilde{Z}_{\tau-}^{n,1}) \xrightarrow{\mathbf{P}} (\hat{Y}_{\tau}^{p,0}, \hat{Y}_{\tau}^{p,1}) \quad \text{and} \quad \hat{y}(x)(\tilde{Z}_{\tau}^{n,0}, \tilde{Z}_{\tau}^{n,1}) \xrightarrow{\mathbf{P}} (\hat{Y}_{\tau}^0, \hat{Y}_{\tau}^1),$$

for all $[0, T]$ -valued stopping times τ and we have, for any primal optimizer $\hat{\varphi} = (\hat{\varphi}^0, \hat{\varphi}^1)$, that

$$\hat{Y}^0 \hat{\varphi}^0 + \hat{Y}^1 \hat{\varphi}^1 = \hat{Y}^0 (x + \hat{\varphi}^1 \cdot \hat{\mathcal{S}}), \quad (3.3.5)$$

where

$$\hat{\mathcal{S}} = (\hat{S}^p, \hat{S}) = \left(\frac{\hat{Y}^{p,1}}{\hat{Y}^{p,0}}, \frac{\hat{Y}^1}{\hat{Y}^0} \right)$$

and

$$x + (\hat{\varphi}^1 \cdot \hat{\mathcal{S}})_t := x + \int_0^t \hat{\varphi}_u^{1,c} d\hat{S}_u + \sum_{0 < u \leq t} \Delta \hat{\varphi}_u^1 (\hat{S}_t - \hat{S}_u^p) + \sum_{0 \leq u < t} \Delta_+ \hat{\varphi}_u^1 (\hat{S}_t - \hat{S}_u). \quad (3.3.6)$$

This implies (after choosing a suitable version of $\hat{\varphi}^1(x)$) that

$$\begin{aligned} \{d\hat{\varphi}^{1,c} > 0\} &\subseteq \{\hat{S} = S\}, & \{d\hat{\varphi}^{1,c} < 0\} &\subseteq \{\hat{S} = (1 - \lambda)S\}, \\ \{\Delta \hat{\varphi}^1 > 0\} &\subseteq \{\hat{S}^p = S_-\}, & \{\Delta \hat{\varphi}^1 < 0\} &\subseteq \{\hat{S}^p = (1 - \lambda)S_-\}, \\ \{\Delta_+ \hat{\varphi}^1 > 0\} &\subseteq \{\hat{S} = S\}, & \{\Delta_+ \hat{\varphi}^1 < 0\} &\subseteq \{\hat{S} = (1 - \lambda)S\}. \end{aligned} \quad (3.3.7)$$

Proof. See [27, Theorem 3.5]. □

For any sandwiched supermartingale deflator $\mathcal{Y} = (Y^p, Y)$, with the associated price process

$$\tilde{\mathcal{S}} = (\tilde{S}^p, \tilde{S}) = \left(\frac{Y^{1,p}}{Y^{0,p}}, \frac{Y^1}{Y^0} \right),$$

and any trading strategy $\varphi \in \mathcal{A}_0^\lambda(x)$, we have for the liquidation value $V^{\text{liq}}(\varphi)$ that

$$\begin{aligned} V_t^{\text{liq}}(\varphi) &\leq x + (\varphi^1 \cdot \tilde{\mathcal{S}})_t := x + \int_0^t \varphi_u^{1,c} d\tilde{S}_u + \sum_{0 < u \leq t} \Delta \varphi_u^1 (\tilde{S}_t - \tilde{S}_u^p) \\ &\quad + \sum_{0 \leq u < t} \Delta_+ \varphi_u^1 (\tilde{S}_t - \tilde{S}_u). \end{aligned} \quad (3.3.8)$$

Indeed, it is obvious that a self-financing trading in the frictionless way for a price process $\tilde{\mathcal{S}} = (\tilde{S}^p, \tilde{S})$ taking values in the bid-ask spread is at least as favorable as trading for S with transaction costs. The relations (3.3.5) and (3.3.7) illustrate that the agent only trades, when $\hat{\mathcal{S}} = (\hat{S}^p, \hat{S})$ matches the bid or ask price.

Let us give some comments on the class of trading strategies competing against $\hat{\varphi}^1$.

- (i) The process φ^1 should be predictable and of finite variation so that the stochastic integral (3.3.6) is well-defined.
- (ii) Associate to the process φ^1 the process φ^0 by

$$\varphi_t^0 := x + (\varphi^1 \cdot \widehat{\mathcal{S}})_t - \varphi_t^1 \widehat{\mathcal{S}}_t, \quad 0 \leq t \leq T. \quad (3.3.9)$$

One may check that φ^0 is a predictable finite variation process and also satisfies

$$\varphi_{t-}^0 = x + (\varphi^1 \cdot \widehat{\mathcal{S}})_{t-} - \varphi_{t-}^1 \widehat{\mathcal{S}}_{t-}^p.$$

Therefore, the process $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ models the holdings in bond and stock induced by the process φ^1 considered as trading strategy without transaction costs on $\widehat{\mathcal{S}}$.

- (iii) The natural admissibility condition in the frictionless setting is $\varphi_t^0 + \varphi_t^1 \widehat{\mathcal{S}}_t \geq 0$, for all $0 \leq t \leq T$, which is used in Definition 3.3.1. However, it is shown by counterexamples in [4, 25, 27] that this notion is too wide in order to obtain a positive result in the present general context. Instead, we define the admissibility in terms of the original process S under transaction costs λ , i.e.,

$$V_t^{\text{liq}}(\varphi) := \varphi_t^0 + (\varphi_t^1)^+(1 - \lambda)S_t - (\varphi_t^1)^- S_t \geq 0. \quad (3.3.10)$$

Theorem 3.3.3. *Under the assumptions of Theorem 3.3.2, let φ be a predictable process of finite variation, which is self-financing for $\widehat{\mathcal{S}}$ without transaction costs, i.e., satisfies (3.3.9) and is admissible in the sense of (3.3.10).*

Then the process

$$\widehat{Y}_t^0 \varphi_t^0 + \widehat{Y}_t^1 \varphi_t^1 = \widehat{Y}_t^0 (x + (\varphi^1 \cdot \widehat{\mathcal{S}})_t), \quad 0 \leq t \leq T, \quad (3.3.11)$$

is a nonnegative optional strong supermartingale and

$$\begin{aligned} \mathbf{E}[U(x + (\varphi^1 \cdot \widehat{\mathcal{S}})_T)] &\leq \mathbf{E}[U(x + (\widehat{\varphi}^1 \cdot \widehat{\mathcal{S}})_T)] \\ &= \mathbf{E}[U(\widehat{\varphi}_T^0 + \widehat{\varphi}_T^1 \widehat{\mathcal{S}}_T)] = \mathbf{E}[U(V_T^{\text{liq}}(\widehat{\varphi}))]. \end{aligned} \quad (3.3.12)$$

Proof. See [27, Theorem 3.6]. □

Theorem 3.3.3 states that the sandwiched strong supermartingale deflator $\widehat{\mathcal{S}} = (\widehat{\mathcal{S}}^p, \widehat{\mathcal{S}})$ may be viewed as a frictionless shadow price, if we use a more liberal concept than Definition 3.3.1.

However, if the solution $\widehat{Z} = (\widehat{Z}^0, \widehat{Z}^1) \in \mathcal{Z}_{loc,e}^\lambda(S)$ to problem (3.2.1) exists, i.e., if there is no “loss of mass”, the ratio $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ is a shadow price in the sense Definition 3.3.1.

Theorem 3.3.4. (Shadow Price Theorem, [27, Proposition 3.7]). *If there is a minimizer $(\widehat{Y}^0, \widehat{Y}^1) \in \mathcal{B}^\lambda(\widehat{y}(x))$ of the dual problem (3.2.5) which is a local martingale, then $\widehat{S} := \frac{\widehat{Y}^1}{\widehat{Y}^0}$ is a shadow price in the sense of Definition 3.3.1.*

Proof. Suppose that $(\widehat{Y}^0, \widehat{Y}^1) \in \mathcal{B}^\lambda(\widehat{y}(x))$ is a local martingale and hence càdlàg. Then the process $(\widehat{Y}^{p,0}, \widehat{Y}^{p,1})$ coincides with $(\widehat{Y}_-^0, \widehat{Y}_-^1)$ (see Remark A.1.21) and the integral $x + \widehat{\varphi}^1 \cdot \widehat{S}$ reduces to the usual stochastic integral $x + \widehat{\varphi}^1 \cdot \widehat{S}$ with $\widehat{S} := \frac{\widehat{Y}^1}{\widehat{Y}^0}$.

Moreover, by Theorem 3.3.2, we have that

$$\widehat{Y}^0 \varphi^0 + \widehat{Y}^1 \varphi^1 = \widehat{Y}^0(x + \varphi^1 \cdot \widehat{S})$$

which is a nonnegative local martingale and hence a supermartingale for all $(\varphi^0, \varphi^1) \in \mathcal{X}(x; \widehat{S})$, which implies that \widehat{Y}^0 is an equivalent local martingale deflator for \widehat{S} without transaction costs starting at $\widehat{Y}_0^0 = \widehat{y}(x)$ and hence $\widehat{Y}^0 \in \mathcal{Y}(\widehat{y}(x); \widehat{S})$. Recall that

$$\mathcal{Y}(y; \widehat{S}) := \left\{ (Y_t)_{0 \leq t \leq T} \geq 0 \left| \begin{array}{l} Y_0 = y \text{ and } Y(\varphi^0 + \varphi^1 \widehat{S}) = Y(1 + \varphi^1 \cdot \widehat{S}) \text{ is} \\ \text{a càdlàg supermartingale for all } \varphi \in \mathcal{X}(1; \widehat{S}) \end{array} \right. \right\}.$$

Let $u(x; \widehat{S})$ and $v(y; \widehat{S})$ denote the value functions of the primal and dual problems, respectively, in the frictionless market \widehat{S} .

As $\widehat{Y}_T^0 = U'(V_T^{\text{liq}}(\widehat{\varphi}))$ and $\widehat{Y}^0 \widehat{\varphi}^0 + \widehat{Y}^1 \widehat{\varphi}^1 = \widehat{Y}^0(x + \widehat{\varphi}^1 \cdot \widehat{S})$ is a martingale by Theorem 3.2.4(3), we obtain by the duality for the frictionless utility maximization problem, i.e., [70, Theorem 2.2], that $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in \mathcal{X}(x; \widehat{S})$ and $\widehat{Y}^0 \in \mathcal{Y}(\widehat{y}(x); \widehat{S})$ are the solutions to the frictionless primal and dual problem for \widehat{S} , if $\widehat{y}(x; \widehat{S}) = \widehat{y}(x)$.

To see the latter, we observe that $u(x) = v(\widehat{y}(x)) + x\widehat{y}(x)$ by Theorem 3.2.4 and therefore by (3.3.1)

$$\begin{aligned} v(\widehat{y}(x)) + x\widehat{y}(x) &= u(x) \leq u(x; \widehat{S}) \leq v(\widehat{y}(x); \widehat{S}) + x\widehat{y}(x) \\ &\leq \mathbf{E}[V(\widehat{Y}_T^0)] + x\widehat{y}(x) = v(\widehat{y}(x)) + x\widehat{y}(x), \end{aligned}$$

since $v(\widehat{y}(x)) = \mathbf{E}[V(\widehat{Y}_T^0)]$ and $\widehat{Y}^0 \in \mathcal{Y}(\widehat{y}(x); \widehat{S})$. Therefore, we obtain $\widehat{y}(x; \widehat{S}) = \widehat{y}(x)$ and $u(x) = u(x; \widehat{S})$. \square

Conversely, it was proved in [27, Proposition 3.8] that if a shadow price in the classical sense exists, it is necessarily derived from a dual minimizer.

Note that by [63, Proposition 4.19] the existence of an optimal strategy to the frictionless utility maximization problem (3.3.2) for \widehat{S} essentially implies that \widehat{S} satisfies (NUPBR).

Chapter 4

Shadow Prices for Continuous Processes

In Chapter 3, we analyzed the duality theory in full generality, i.e., in the framework of càdlàg processes S . In this chapter, we show that the theory simplifies considerably, if we restrict ourselves to continuous processes S . More importantly, we obtain sharper results than in the general càdlàg setting.

4.1 Simplification of the Duality Theory

There is a pleasant simplification as compared to the general setting. As the price process does not jump, it does not matter, if one is trading immediately before, or just at a given time, while in the case of a càdlàg process S it does make a difference whether the jumps of φ are on the left or on the right side. If φ satisfies the self-financing condition (2.1.1), then its left-continuous version φ^l as well as its right-continuous version φ^r also satisfies (2.1.1). It turns out that the convenient choice is to impose that the process φ is right-continuous, and therefore càdlàg. Indeed, in this case φ is a semimartingale so that the Riemann-Stieltjes integrals in (2.1.1) may also be interpreted as Itô integrals and we are in the customary realm of stochastic analysis. Therefore, we define trading strategies as \mathbb{R}^2 -valued càdlàg adapted finite variation processes $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$. But occasionally it will be convenient also to consider the left-continuous version φ^l , which has the advantage of being predictable. We shall indicate if we pass to the left-continuous version φ^l . Again by the continuity of S , trading strategies can be assumed to be optional.

As we deal with the right-continuous processes φ , we have the usual notational problem of a jump at time zero. This is done by distinguishing between the value $\varphi_{0-} = (x, 0)$ above and $\varphi_0 = (\varphi_0^0, \varphi_0^1)$. In accordance with (2.1.1) we must have

$$\varphi_0^0 - \varphi_{0-}^0 \leq -S_0(\varphi_0^1 - \varphi_{0-}^1)^+ + (1 - \lambda)S_0(\varphi_0^1 - \varphi_{0-}^1)^-$$

i.e.,

$$\varphi_0^0 \leq x - S_0(\varphi_0^1)^+ + (1 - \lambda)S_0(\varphi_0^1)^-.$$

Now we adapt the general duality theorem to its special case.

Let $\mathcal{A}_0^\lambda(x)$ be the set of 0-admissible \mathbb{R}^2 -valued adapted càdlàg finite variation processes φ , starting with initial endowment $(\varphi_{0-}^0, \varphi_{0-}^1) = (x, 0)$ and satisfying the self-financing condition (2.1.1). Denote by $\mathcal{C}_0^\lambda(x)$ the convex subset in L_+^0

$$\mathcal{C}_0^\lambda(x) := \{V_T^{\text{liq}}(\varphi) \mid \varphi \in \mathcal{A}_0^\lambda(x)\}. \quad (4.1.1)$$

For given initial endowment $x > 0$, the agent wants to maximize expected utility at terminal time T , i.e.,

$$\mathbf{E}[U(g)] \rightarrow \max!, \quad g \in \mathcal{C}_0^\lambda(x). \quad (4.1.2)$$

Contrary to Chapter 3, where we were forced to consider optional strong supermartingales, in the present setting of continuous S , we may remain in the usual realm of càdlàg supermartingales. Indeed, as the trading strategies is right-continuous, we are allowed to pass the supermartingale property onto to the Fatou limit as in the frictionless case. Summing up, the dual optimizer is then attained as Fatou limit under transaction costs as well, if the price process S is continuous.

Definition 4.1.1. *For $y > 0$, we call a nonnegative process $Y = (Y^0, Y^1)$ an **supermartingale deflator** starting at y , if*

- (i) $Y_0^0 = y$,
- (ii) $\frac{Y^1}{Y^0} \in [(1 - \lambda)S, S]$,
- (iii) $Y^0\varphi^0 + Y^1\varphi^1$ is a nonnegative supermartingale for all $\varphi \in \mathcal{A}_0^\lambda(1)$.

We denote by $\mathcal{B}^\lambda(y)$ the set of supermartingale deflators with $Y_0^0 = y$ and by $\mathcal{D}^\lambda(y)$ the set of random variables $h \in L_+^0(\Omega, \mathcal{F}, \mathbf{P})$ such that there is a supermartingale deflator $(Y_t^0, Y_t^1)_{0 \leq t \leq T} \in \mathcal{B}^\lambda(y)$, whose first coordinate has terminal value $Y_T^0 = h$. We denote by \mathcal{B}^λ and \mathcal{D}^λ the sets $\mathcal{B}^\lambda(1)$ and $\mathcal{D}^\lambda(1)$, respectively.

Recall that $D^\lambda := \{Z_T^0 \mid Z \in \mathcal{Z}_{loc, e}^\lambda(S)\}$. Now we can state the polar relation between \mathcal{C}_0^λ and \mathcal{D}^λ .

Proposition 4.1.2. ([30, Proposition 2.9]). *Fix the continuous process $S = (S_t)_{0 \leq t \leq T}$ and transaction costs $0 < \lambda < 1$. Suppose that S satisfies (CPS $^\mu$) locally for all $0 < \mu < \lambda$. We then have:*

- (i) *The sets \mathcal{C}_0^λ and \mathcal{D}^λ are solid, convex subsets of $L_+^0(\mathbf{P})$ which are closed with respect to convergence in probability.*
- (ii) *For $g \in L_+^0$, we have that $g \in \mathcal{C}_0^\lambda$ iff we have $\mathbf{E}[gh] \leq 1$, for all $h \in \mathcal{D}^\lambda$. For $h \in L_+^0$, we have that $h \in \mathcal{D}^\lambda$ iff we have $\mathbf{E}[gh] \leq 1$, for all $g \in \mathcal{C}_0^\lambda$.*
- (iii) *The set \mathcal{C}_0^λ is bounded in L^0 and contains the constant function $\mathbf{1}$.*
- (iv) *The set \mathcal{D}^λ equals the closed (in probability) convex solid hull of D^λ in $L_+^0(\mathbf{P})$.*

Proof. The proof is analogous to Lemma 3.2.2. However, we should pay some attention, since $\varphi \in \mathcal{A}_0^\lambda(1)$ and $Y \in \mathcal{B}^\lambda$ are assumed to be càdlàg.

Let $(g^n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{C}_0^λ converging in probability to some $g \in L_+^0$ and associate to each g^n the càdlàg trading strategy $\varphi^n \in \mathcal{A}_0^\lambda(1)$. Following the proof of Lemma 3.2.2

(1), after passing to convex combinations, we may find a finite variation process φ such that

$$\mathbf{P}[\varphi_t^n \rightarrow \varphi_t, \forall t \in [0, T]] = 1.$$

As we restrict ourselves to continuous process S , we may pass to the right-continuous version of φ , denoted again by φ , which is in $\mathcal{A}_0^\lambda(1)$ and $\varphi_T^0 = g$.

For the closedness of \mathcal{D}^λ , we use here the classical Fatou convergence for supermartingales as [70, Lemma 4.1]. Take $(h^n)_{n \in \mathbb{N}}$ in \mathcal{D}^λ converging to some h in probability. Our goal is to prove that $h \in \mathcal{D}^\lambda$. For each $n \in \mathbb{N}$, associate to h^n the càdlàg positive supermartingale $(Y^{n,0}, Y^{n,1}) \in \mathcal{B}^\lambda$ with $Y_T^{n,0} = h^n$. We are going to construct a càdlàg nonnegative supermartingale $(Y^0, Y^1) \in \mathcal{B}^\lambda$ such that $Y_T^0 = h$. By the positivity, we may use the Komlós type Theorem A.1.2 to find convex combinations $(\tilde{h}_n)_{n \in \mathbb{N}}$ and $(\tilde{Y}^{n,1}, \tilde{Y}^{n,1})_{n \in \mathbb{N}}$. More precisely a diagonal argument gives this \mathbf{P} -a.s. converging simultaneously for $(\tilde{h}_n)_{n \in \mathbb{N}}$ and $(\tilde{Y}_t^{n,1}, \tilde{Y}_t^{n,1})_{n \in \mathbb{N}}$ for all t from a countable subset $\mathcal{T} \subseteq [0, T]$ which is dense in $[0, T]$ and contains 0 and T , i.e., we always take the same convex combinations for all $(\tilde{Y}_t^{n,1}, \tilde{Y}_t^{n,1})_{n \in \mathbb{N}}$, such that

$$(\tilde{Y}_t^{n,1}, \tilde{Y}_t^{n,1}) \rightarrow (Y_t^{\infty,1}, Y_t^{\infty,1}), \quad a.s. \text{ for all } t \in \mathcal{T}.$$

Hence, $Y_T^{n,0} = h^n$ implies that $\tilde{Y}_T^{n,0} = \tilde{h}^n$ and so $Y_T^{\infty,0} = h^\infty$. As $h^n \rightarrow h$ in probability and $\tilde{h}^n \rightarrow h^\infty$ almost surely, we obtain that $h^\infty = h$ almost surely, so $h = Y_T^{\infty,0}$ almost surely. Therefore, we only needs to show that $Y_T^{\infty,0} = Y_T^0$ for some $(Y^0, Y^1) \in \mathcal{B}^\lambda$.

By the convexity of \mathcal{B}^λ , we have that, for each $(\varphi^0, \varphi^1) \in \mathcal{A}_0^\lambda(1)$, the process $\varphi^0 \tilde{Y}^{n,0} + \varphi^1 \tilde{Y}^{n,0}$ is a nonnegative supermartingale. Therefore, by Fatou's lemma and the supermartingale property, we obtain that

$$\begin{aligned} \mathbf{E}[\varphi_t^0 Y_t^{\infty,0} + \varphi_t^1 Y_t^{\infty,1} | \mathcal{F}_s] &\leq \mathbf{E}\left[\lim_{n \rightarrow \infty} (\varphi_t^0 \tilde{Y}_t^{n,0} + \varphi_t^1 \tilde{Y}_t^{n,1}) \middle| \mathcal{F}_s\right] \\ &\leq \liminf_{n \rightarrow \infty} \mathbf{E}\left[\varphi_t^0 \tilde{Y}_t^{n,0} + \varphi_t^1 \tilde{Y}_t^{n,1} \middle| \mathcal{F}_s\right] \\ &\leq \liminf_{n \rightarrow \infty} (\varphi_s^0 \tilde{Y}_s^{n,0} + \varphi_s^1 \tilde{Y}_s^{n,1}) \\ &= \varphi_s^0 Y_s^{\infty,0} + \varphi_s^1 Y_s^{\infty,1}, \end{aligned}$$

for each $s \leq t$ with $s, t \in \mathcal{T}$, and so the process $(\varphi_t^0 Y_t^{\infty,0} + \varphi_t^1 Y_t^{\infty,1})_{t \in \mathcal{T}}$ is a supermartingale over \mathcal{T} for each $(\varphi^0, \varphi^1) \in \mathcal{A}_0^\lambda(1)$. In particular, $(Y_t^{\infty,0}, Y_t^{\infty,1})_{t \in \mathcal{T}}$ is a nonnegative supermartingale by taking simply $(\varphi^0, \varphi^1) = (1, 0)$ and $(\varphi^0, \varphi^1) = (0, 1)$, respectively. Now, we pass to a càdlàg process by

$$(Y_t^0, Y_t^1) := \begin{cases} \lim_{s \searrow t, s \in \mathcal{T}} (Y_s^{\infty,0}, Y_s^{\infty,1}), & 0 \leq t < T, \\ (Y_T^{\infty,0}, Y_T^{\infty,1}), & t = T. \end{cases}$$

Clearly, $Y_T^0 = h$, and $Y^0(1 - \lambda)S \leq Y^1 \leq Y^0 S$ as S is continuous. Using Fatou and backward martingales we can prove that $(\varphi^0 Y^0 + \varphi^1 Y^1)$ is a supermartingale for each (φ^0, φ^1) . Indeed, let $0 \leq s \leq t < T$. As (φ^0, φ^1) is càdlàg, we have

$$\begin{aligned} \mathbf{E}[\varphi_t^0 Y_t^0 + \varphi_t^1 Y_t^1 | \mathcal{F}_s] &= \mathbf{E}\left[\lim_{u \searrow t, u \in \mathcal{T}} (\varphi_u^0 Y_u^{\infty,0} + \varphi_u^1 Y_u^{\infty,1}) \middle| \mathcal{F}_s\right] \\ &\leq \liminf_{u \searrow t, u \in \mathcal{T}} \mathbf{E}\left[\varphi_u^0 Y_u^{\infty,0} + \varphi_u^1 Y_u^{\infty,1} \middle| \mathcal{F}_s\right]. \end{aligned}$$

Let $(s_n)_{n \in \mathbb{N}_0}$ be a sequence in \mathcal{T} such that $s_0 = u$ and $s_n \searrow s$. By the continuity of the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$, we have $\mathcal{F}_s = \bigcap_{n \in \mathbb{N}_0} \mathcal{F}_{s_n}$. For $-m \in \mathbb{N}_0$, define

$$\mathcal{G}_m := \bigcap_{n=0}^{-m} \mathcal{F}_{s_n} \quad \text{and} \quad M_m := \mathbf{E}[\varphi_u^0 Y_u^{\infty,0} + \varphi_u^1 Y_u^{\infty,1} | \mathcal{G}_m].$$

Clearly, $\mathcal{G}_0 = \mathcal{F}_{s_0} = \mathcal{F}_u$ and $\mathcal{G}_{-\infty} = \mathcal{F}_s$, and $(M_m)_{m \leq 0}$ is a martingale. By Backwards Convergence Theorem (see, e.g., [91, Theorem I.10]), we obtain that

$$\lim_{m \rightarrow -\infty} M_m = \mathbf{E} \left[M_0 \middle| \bigcap_{m=-\infty}^0 \mathcal{G}_m \right].$$

Hence,

$$\begin{aligned} \mathbf{E} [\varphi_t^0 Y_t^0 + \varphi_t^1 Y_t^1 | \mathcal{F}_s] &\leq \liminf_{u \searrow t, u \in \mathcal{T}} \mathbf{E} \left[\mathbf{E} [\varphi_u^0 Y_u^{\infty,0} + \varphi_u^1 Y_u^{\infty,1} | \mathcal{F}_u] \middle| \mathcal{F}_s \right] \\ &= \liminf_{u \searrow t, u \in \mathcal{T}} \mathbf{E} \left[M_0 \middle| \bigcap_{m=-\infty}^0 \mathcal{G}_m \right] = \liminf_{u \searrow t, u \in \mathcal{T}} \lim_{n \rightarrow \infty} M_{-n} \\ &= \lim_{n \rightarrow \infty} \mathbf{E} [\varphi_{s_n}^0 Y_{s_n}^{\infty,0} + \varphi_{s_n}^1 Y_{s_n}^{\infty,1} | \mathcal{F}_{s_n}] \\ &\leq \lim_{n \rightarrow \infty} \varphi_{s_n}^0 Y_{s_n}^{\infty,0} + \varphi_{s_n}^1 Y_{s_n}^{\infty,1} \\ &= \varphi_s^0 Y_s^0 + \varphi_s^1 Y_s^1. \end{aligned}$$

Therefore, $(Y^0, Y^1) \in \mathcal{B}^\lambda$ with $Y_T^0 = h$, so $h \in \mathcal{D}^\lambda$. \square

Define now the dual minimization problem

$$\mathbf{E}[V(h)] \rightarrow \min!, \quad h \in \mathcal{D}^\lambda(y). \quad (4.1.3)$$

We now can conclude from Proposition 4.1.2 that the duality results of portfolio optimization, as obtained in [70, Theorem 3.1 and 3.2] (Theorem B.1.20), carry over verbatim to the present setting as these results only need the validity of Proposition 4.1.2 as input. We recall the essence of these results.

Theorem 4.1.3 (Duality Theorem). *In addition to the hypotheses of Proposition 4.1.2 suppose that there is a utility function $U : (0, \infty) \rightarrow \mathbb{R}$ satisfying $AE(U) < 1$. Define the primal and dual value functions as*

$$u(x) := \sup_{g \in \mathcal{C}_0^\lambda(x)} \mathbf{E}[U(g)], \quad v(y) := \inf_{h \in \mathcal{D}^\lambda(y)} \mathbf{E}[V(h)],$$

and suppose that $u(x) < \infty$, for some $x > 0$.

Then, the following statements hold true.

- (i) *The functions $u(x)$ and $v(y)$ are finitely valued, for all $x, y > 0$, and mutually conjugate*

$$v(y) = \sup_{x > 0} \{u(x) - xy\}, \quad u(x) = \inf_{y > 0} \{v(y) + xy\}.$$

The functions u and v are continuously differentiable and strictly concave (respectively, convex) and satisfy

$$u'(0) = -v'(0) = \infty, \quad u'(\infty) = v'(\infty) = 0.$$

(ii) For all $x, y > 0$, the solutions $\widehat{g}(x) \in \mathcal{C}_0^\lambda(x)$ in (4.1.2) and $\widehat{h}(y) \in \mathcal{D}^\lambda(y)$ in (4.1.3) exist, are unique and take their values a.s. in $(0, \infty)$. There are $(\widehat{\varphi}^0(x), \widehat{\varphi}^1(x)) \in \mathcal{A}_0^\lambda(x)$ and $(\widehat{Y}^0(y), \widehat{Y}^1(y)) \in \mathcal{B}^\lambda(y)$ such that

$$V_T^{\text{liq}}(\widehat{\varphi}(x)) = \widehat{g}(x) \quad \text{and} \quad \widehat{Y}_T^0(y) = \widehat{h}(y).$$

(iii) If $x > 0$ and $y > 0$ are related by $u'(x) = y$, or equivalently $x = -v'(y)$, then $\widehat{g}(x)$ and $\widehat{h}(y)$ are related by the first order conditions

$$\widehat{h}(y) = U'(\widehat{g}(x)) \quad \text{and} \quad \widehat{g}(x) = -V'(\widehat{h}(y)), \quad (4.1.4)$$

and we have that

$$\mathbf{E}[\widehat{g}(x)\widehat{h}(y)] = xy. \quad (4.1.5)$$

In particular, the process $\widehat{\varphi}_t^0(x)\widehat{Y}_t^0(y) + \widehat{\varphi}_t^1(x)\widehat{Y}_t^1(y)$ is a uniformly integrable \mathbf{P} -martingale.

(iv) We have that

$$v(y) = \inf_{(Z^0, Z^1) \in \mathcal{Z}_{loc, e}^\lambda(S)} \mathbf{E}[V(yZ_T^0)].$$

Remark 4.1.4. The Duality Theorem 4.1.3 asserts the existence of a strictly positive dual optimizer $\widehat{h}(y) \in \mathcal{D}^\lambda(y)$, which implies that there is an equivalent supermartingale deflator $\widehat{Y}(y) = (\widehat{Y}_t^0(y), \widehat{Y}_t^1(y))_{0 \leq t \leq T} \in \mathcal{B}^\lambda(y)$ such that $\widehat{h}(y) = \widehat{Y}_T^0$. We are interested in the question whether the supermartingale $\widehat{Y}(y)$ can be chosen to be a local martingale. We say “can be chosen” for the following reason: the dual optimizer $\widehat{Y}(y)$ is not necessarily unique, especially the the second coordinate $\widehat{Y}^1(y)$ (see [98, Remark 6.9] for a counterexample).

The phenomenon that the dual optimizer may be induced by a supermartingale only, rather than by a local martingale, is well-known in the frictionless theory (see [70, Example 5.1 and 5.1']). This phenomenon is related to the singularity of the utility function U at the left boundary of its domain, where we have

$$U'(0) := \lim_{x \searrow 0} U'(x) = \infty.$$

If one passes to utility functions U which take finite values on the entire real line, e.g., $U(x) = -e^{-x}$, the present “supermartingale phenomenon” does not occur any more (compare [94]).

4.2 Existence of Shadow Prices under (NUPBR)

In the present context of portfolio optimization under transaction costs, the question of the local martingale property of the dual optimizer $\widehat{Y}(y)$ is of crucial relevance in view of the subsequent *Shadow Price Theorem*. It states that, if the dual optimizer is induced by a local martingale, there is a shadow price. This theorem essentially goes back to the work of Cvitanić and Karatzas [22]. While these authors did not explicitly crystallize the

notation of a shadow price, subsequently Loewenstein [75] explicitly formulated the relation between a financial market under transaction costs and a corresponding frictionless market. Later this has been termed “shadow price process” (compare also [62, 4] as well as [61, 44, 43, 17, 55] for constructions in the Black-Scholes model).

We start by giving a precise meaning to this notion.

Definition 4.2.1. *In the above setting, a semimartingale $\tilde{S} = (\tilde{S}_t)_{0 \leq t \leq T}$ is called a **shadow price process** for the optimization problem (4.1.2), if*

- (i) \tilde{S} takes its values in the bid-ask spread $[(1 - \lambda)S, S]$.
- (ii) The optimizer to the corresponding frictionless utility maximization problem

$$\mathbf{E}[U(\tilde{g})] \rightarrow \max!, \quad \tilde{g} \in \mathcal{C}(x; \tilde{S}), \quad (4.2.1)$$

exists and coincides with the solution $\hat{g}(x) \in \mathcal{C}_0^\lambda(x)$ for the optimization problem (4.1.2) under transaction costs. In (4.2.1) the set $\mathcal{C}(x; \tilde{S})$ consists of all nonnegative random variables, which are attainable by starting with initial endowment x and then trading in an admissible way the stock price process \tilde{S} in a frictionless way, as defined in [70].

- (iii) The optimal trading strategy \hat{H} (in the sense of predictable, \tilde{S} -integrable process for the frictionless market \tilde{S} , as in [70]) is equal to the left-continuous version of the finite variation process $\hat{\varphi}^1(x)$ of the optimizer $(\hat{\varphi}_t^0(x), \hat{\varphi}_t^1(x))_{0 \leq t \leq T}$ of the optimization problem (4.1.2).

The essence of the above definition is that the value function $u(x; \tilde{S})$ of the optimization problem for the frictionless market \tilde{S} is equal to the value function $u(x)$ of the optimization problem for S under transaction costs, i.e.,

$$u(x; \tilde{S}) := \sup_{\tilde{g} \in \mathcal{C}(x; \tilde{S})} \mathbf{E}[U(\tilde{g})] = \sup_{g \in \mathcal{C}_0^\lambda(x)} \mathbf{E}[U(g)] =: u(x), \quad (4.2.2)$$

although the set $\mathcal{C}(x; \tilde{S})$ contains the set $\mathcal{C}_0^\lambda(x)$ defined in (4.1.1).

Theorem 4.2.2 (Shadow Price Theorem). *Under the hypothesis of Theorem 4.1.3 fix $x > 0$ and $y > 0$ such that $u'(x) = y$. Assume that the dual optimizer $\hat{h}(y)$ equals $\hat{Z}_T^0(y)$, where $\hat{Z}(y) \in \mathcal{B}^\lambda(y)$ is a local martingale under \mathbf{P} .*

Then the strictly positive semimartingale $\hat{S} := \frac{\hat{Z}^1(y)}{\hat{Z}^0(y)}$ is a shadow price process (in the sense of Definition 4.2.1) for the optimization problem (4.1.2).

Proof. By hypothesis, there is a local martingale $(\hat{Z}^0(y), \hat{Z}^1(y))$ such that $\hat{h}(y) = \hat{Z}_T^0(y)$. The process $\hat{S} = \frac{\hat{Z}^1(y)}{\hat{Z}^0(y)}$ then satisfies condition (i) of the above definition, i.e., $\hat{S} \in [(1 - \lambda)S, S]$.

To verify (ii) first observe the economically rather obvious relation

$$\mathcal{C}(x; \hat{S}) \supseteq \mathcal{C}_0^\lambda(x). \quad (4.2.3)$$

Indeed, every frictionless trade on the process \widehat{S} (which takes values in the bid-ask spread $[(1-\lambda)S, S]$) is at least as favorable as a trade on S under transaction costs λ (where the agent always gets the less favorable choice between $(1-\lambda)S$ and S). Therefore, it is intuitively obvious that any claim, which can be attained by trading in S under transaction costs λ , can also be attained by trading in \widehat{S} in a frictionless way.

More formally, let $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ be an admissible trading strategy for the process S , which is self-financing under transaction costs λ . The value process

$$\widehat{V}_t := \varphi_t^0 + \varphi_t^1 \widehat{S}_t \quad (4.2.4)$$

gives the value of this portfolio if we evaluate the position φ_t^1 in stocks using the price \widehat{S}_t . As the process φ^1 has finite variation and \widehat{S} is a semimartingale, we obtain by [58, Proposition I.4.49]

$$d\widehat{V}_t = \varphi_{t-}^1 d\widehat{S}_t + \left(d\varphi_t^0 + \widehat{S}_t d\varphi_t^1 \right). \quad (4.2.5)$$

The increment in the bracket is nonpositive in view of the inequalities (2.1.1) and the fact that \widehat{S} takes values in the bid-ask spread $[(1-\lambda)S, S]$. On the other hand, consider the left-continuous version of the process $(\varphi_t^1)_{0 \leq t \leq T}$ as a predictable integrand for the semimartingale \widehat{S} . For the corresponding value process $X_t := x + (\varphi_-^1 \cdot \widehat{S})_t$ we obtain that $\widehat{V}_0 = X_0 = x$ and

$$dX_t = \varphi_{t-}^1 d\widehat{S}_t. \quad (4.2.6)$$

Comparing (4.2.5) and (4.2.6) we obtain that $(X_t - \widehat{V}_t)_{0 \leq t \leq T}$ is a nondecreasing process so that $X_T \geq \widehat{V}_T$ almost surely. Hence we obtain for the liquidation value

$$V_T^{\text{liq}}(\varphi) \leq \widehat{V}_T \leq X_T, \quad a.s., \quad (4.2.7)$$

which shows the inclusion (4.2.3) and therefore the inequality \geq in (4.2.2).

We now use the duality results to show the reverse inequality. By the definition of \widehat{S} and the assumption that $(\widehat{Z}^0, \widehat{Z}^1)$ is a local martingale, we have that $\widehat{Z}^0 \widehat{S}$ is a local martingale under measure \mathbf{P} , so that \widehat{Z}^0 may be viewed as a strict martingale density process for the semimartingale \widehat{S} , i.e., $\mathcal{Z}_e(\widehat{S}) \neq \emptyset$. By Proposition B.1.21 we have that the sets $\mathcal{C}(1; \widehat{S})$ and $\mathcal{D}(1; \widehat{S})$ satisfy the assumptions of [70, Theorem 3.2] (Theorem B.1.20), where $\mathcal{D}(1; \widehat{S})$ is the set of nonnegative random variables g , which can be dominated by the terminal value Y_T of a supermartingale deflator $(Y_t)_{0 \leq t \leq T}$ in the frictionless setting.

Now make the crucial observation that we have the following inclusion:

$$\mathcal{D}(y; \widehat{S}) \subseteq \mathcal{D}^\lambda(y), \quad (4.2.8)$$

Indeed, by Proposition B.1.21 and Proposition 4.1.2 we have the polar relations

$$\begin{aligned} \mathcal{D}(y; \widehat{S}) &= \left\{ \widetilde{h} \in L_+^0 : \mathbf{E}[\widetilde{g}\widetilde{h}] \leq xy, \text{ for all } \widetilde{g} \in \mathcal{C}(x; \widehat{S}) \right\}, \\ \mathcal{D}^\lambda(y) &= \left\{ h \in L_+^0 : \mathbf{E}[gh] \leq xy, \text{ for all } g \in \mathcal{C}_0^\lambda(x) \right\}. \end{aligned}$$

We have seen in (4.2.3) that the reverse inclusion holds true for the primal sets, which proves (4.2.8).

Next observe that $\widehat{h}(y)$, which is the dual optimizer in the (larger) set $\mathcal{D}^\lambda(y)$, actually lies already in the (smaller) set $\mathcal{D}(y; \widehat{S})$. Indeed, $\frac{\widehat{h}(y)}{y}$ is the terminal value of a martingale density process for \widehat{S} , hence a terminal value of a supermartingale deflator in the frictionless setting, i.e.,

$$\widehat{h}(y) \in \mathcal{D}(y; \widehat{S}). \quad (4.2.9)$$

Looking at the frictionless dual problem

$$\mathbf{E}[V(\widetilde{h})] \rightarrow \min! \quad \widetilde{h} \in \mathcal{D}(y; \widehat{S}), \quad (4.2.10)$$

and the corresponding dual problem under transaction costs

$$\mathbf{E}[V(h)] \rightarrow \min! \quad h \in \mathcal{D}^\lambda(y), \quad (4.2.11)$$

we deduce from (4.2.8) and (4.2.9) that $\widehat{h}(y)$ a fortiori is the optimizer for (4.2.10) in $\mathcal{D}(y; \widehat{S})$ with respect to the frictionless setting. In particular, the dual value functions $v(y)$ and $v(y; \widehat{S})$ coincide.

Turning to the primal side, for $x = -v'(y)$ we note that the value function $u(x; \widehat{S})$ is finite, since for any $\widetilde{g} \in \mathcal{C}(x; \widehat{S})$ we deduce from (4.1.4) in Theorem 4.1.3 and Fenchel's inequalities that

$$\begin{aligned} \mathbf{E}[U(\widetilde{g})] &\leq \mathbf{E}[U(\widehat{g}(x)) + U'(\widehat{g}(x))(\widetilde{g} - \widehat{g}(x))] \\ &= \mathbf{E}[U(\widehat{g}(x)) - \widehat{g}(x)\widehat{h}(y)] + \mathbf{E}[\widetilde{g}\widehat{h}(y)] \\ &\leq v(y) + xy < \infty. \end{aligned}$$

Hence, we are in the setting of [70, Theorem 3.2] (Theorem B.1.20).

We need to show that the Lagrange multiplier $\widehat{y}(x; \widehat{S})$ coincides with $y = u'(x)$. Indeed, by the Fenchel's inequality and the fact that $u(x; \widehat{S}) \geq u(x)$, we obtain

$$u(x) = v(y) + xy = v(y; \widehat{S}) + xy \geq v(\widehat{y}(x; \widehat{S}); \widehat{S}) + x\widehat{y}(x; \widehat{S}) = u(x; \widehat{S}) \geq u(x),$$

which follows that $u'(x; \widehat{S}) = \widehat{y}(x; \widehat{S}) = y = u'(x)$

From Theorem 4.1.3 and [70, Theorem 3.2] (Theorem B.1.20) the primal and dual optimizers are related by the first order condition

$$\widehat{g}(x) = -V'(\widehat{h}(y)), \quad \text{a.s.} \quad (4.2.12)$$

which shows that the primal optimizer $\widehat{g}(x) \in \mathcal{C}_0^\lambda(x)$ also is the optimizer to the problem (4.2.1) in the larger set $\mathcal{C}(x; \widehat{S})$.

Hence, the condition (ii) in the above definition is satisfied.

Now we show the condition (iii) in the above definition. By Theorem 4.1.3 we have that

$$(\widehat{M}_t)_{0 \leq t \leq T} := (\widehat{\varphi}_t^0(x)\widehat{Z}_t^0(y) + \widehat{\varphi}_t^1(x)\widehat{Z}_t^1(y))_{0 \leq t \leq T}$$

is a martingale. Applying [58, Theorem I.4.49] we get

$$d\widehat{M}_t = \widehat{\varphi}_{t-}^0(x)d\widehat{Z}_t^0(y) + \widehat{\varphi}_{t-}^1(x)d\widehat{Z}_t^1(y) + \widehat{Z}_t^0(y)\left(d\widehat{\varphi}_t^0(x) + \widehat{S}_t d\widehat{\varphi}_t^1(x)\right).$$

Note again that by (2.1.1) the term in the bracket is a nonpositive increment.

The martingale property of \widehat{M} implies that

$$\int_0^t \widehat{Z}_u^0(y) \left(d\widehat{\varphi}_u^0(x) + \widehat{S}_u d\widehat{\varphi}_u^1(x) \right) = 0, \quad a.s. \quad 0 \leq t \leq T.$$

Since $\widehat{Z}_t^0(y)$ is strictly positive, we obtain that

$$\int_0^t d\widehat{\varphi}_u^0(x) + \int_0^t \widehat{S}_u d\widehat{\varphi}_u^1(x) = 0, \quad a.s. \quad (4.2.13)$$

for all $0 \leq t \leq T$.

We now denote by $(\widehat{H}_t)_{0 \leq t \leq T}$ the left-continuous version of $(\widehat{\varphi}_t^1(x))_{0 \leq t \leq T}$ as a predictable integrand for \widehat{S} . By (4.2.7) $(\widehat{H}_t)_{0 \leq t \leq T}$ is admissible in the frictionless setting. For the process $\widehat{\varphi}^0(x) + \widehat{\varphi}^1(x)\widehat{S}$, we obtain by [58, Theorem I.4.49] and (4.2.13) that

$$\begin{aligned} \widehat{\varphi}_t^0(x) + \widehat{\varphi}_t^1(x)\widehat{S}_t &= x + \int_0^t d\widehat{\varphi}_u^0(x) + \int_0^t \widehat{S}_u d\widehat{\varphi}_u^1(x) + \int_0^t \widehat{\varphi}_{u-}^1(x) d\widehat{S}_u \\ &= x + \int_0^t \widehat{\varphi}_{u-}^1(x) d\widehat{S}_u, \end{aligned}$$

so that

$$\widehat{g}(x) = \widehat{\varphi}_T^0(x) = x + (\widehat{\varphi}_-^1(x) \cdot \widehat{S})_T = x + (\widehat{H} \cdot \widehat{S})_T.$$

Since the primal optimizer to (4.2.1) coincides with the one to (4.1.2), $(\widehat{H}_t)_{0 \leq t \leq T}$ is the optimal strategy to (4.2.1). \square

Remark 4.2.3. Let \widehat{S} be the shadow price process as above and define the optional sets in $\Omega \times [0, T]$

$$A^{buy} = \left\{ \widehat{S}_t = S_t \right\} \quad \text{and} \quad A^{sell} = \left\{ \widehat{S}_t = (1 - \lambda)S_t \right\}.$$

It deduces from (2.1.1) and (4.2.13) that

$$\begin{aligned} \int_0^T d\widehat{\varphi}_u^0(x) &\leq \int_0^T (1 - \lambda)S_u d\widehat{\varphi}_u^{1,\downarrow}(x) - \int_0^T S_u d\widehat{\varphi}_u^{1,\uparrow}(x) \\ &\leq \int_0^T \widehat{S}_u d\widehat{\varphi}_u^{1,\downarrow}(x) - \int_0^T \widehat{S}_u d\widehat{\varphi}_u^{1,\uparrow}(x) = \int_0^T d\widehat{\varphi}_u^0(x), \end{aligned}$$

which implies that

$$\int_0^T (\widehat{S}_u - (1 - \lambda)S_u) d\widehat{\varphi}_u^{1,\downarrow}(x) + \int_0^T (S_u - \widehat{S}_u) d\widehat{\varphi}_u^{1,\uparrow}(x) = 0, \quad a.s.$$

Therefore, the optimizer $\widehat{\varphi} = (\widehat{\varphi}^0, \widehat{\varphi}^1)$ of the optimization problem (4.1.2) for S under transaction costs λ satisfies

$$\begin{aligned} \{d\widehat{\varphi}_t^1(x) < 0\} &\subseteq \left\{ \widehat{S}_t = (1 - \lambda)S_t \right\}, \\ \{d\widehat{\varphi}_t^1(x) > 0\} &\subseteq \left\{ \widehat{S}_t = S_t \right\}, \end{aligned}$$

for all $0 \leq t \leq T$, i.e., the measures associated to the increasing process $\widehat{\varphi}^{1,\uparrow}$ (respectively, $\widehat{\varphi}^{1,\downarrow}$) are supported by A^{buy} (respectively, A^{sell}). This crucial feature has been originally shown by Cvitanić and Karatzas [22] in an Itô process setting. In the present form it is a special case of [27, Theorem 3.5] (Theorem 3.3.2).

Proposition 4.2.4. *Fix $0 < \lambda < 1$ and let S be a continuous semimartingale satisfying (NUPBR). Under the assumption of Theorem 4.1.3, the liquidation value process associated to the optimizer $\widehat{\varphi} = (\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)_{0 \leq t \leq T}$*

$$\widehat{V}_t^{\text{liq}} := \widehat{\varphi}_t^0 + (1 - \lambda)(\widehat{\varphi}_t^1)^+ S_t - (\widehat{\varphi}_t^1)^- S_t$$

is strictly positive almost surely for each $0 \leq t \leq T$, i.e.,

$$\inf_{0 \leq t \leq T} \widehat{V}_t^{\text{liq}} > 0, \quad a.s.$$

Proof. As shown by Choulli and Stricker [18, Théorème 2.9] (Theorem B.1.6, compare also [63, 69, 100, 101]), the condition (NUPBR) implies the existence of a *strict martingale density* for the continuous semimartingale S , i.e., a $(0, \infty)$ -valued local martingale Z such that ZS is a local martingale. Note that $(\widehat{V}_t^{\text{liq}})_{0 \leq t \leq T}$ is a semimartingale as we assumed φ to be optional and càdlàg, which makes the application of Itô's lemma legitimate. Applying Itô's lemma to the semimartingale $Z\widehat{V}^{\text{liq}}$ and recalling that φ has finite variation, we get

$$\begin{aligned} d(Z_t \widehat{V}_t^{\text{liq}}) &= Z_t [d\widehat{\varphi}_t^0 + (1 - \lambda)S_t d(\widehat{\varphi}_t^1)^+ - S_t d(\widehat{\varphi}_t^1)^-] \\ &\quad + \widehat{\varphi}_{t-}^0 dZ_t + [(1 - \lambda)(\widehat{\varphi}_{t-}^1)^+ - (\widehat{\varphi}_{t-}^1)^-] d(Z_t S_t). \end{aligned}$$

By (2.1.1) the increment in the first bracket is nonpositive. The two terms dZ_t and $d(Z_t S_t)$ are the increments of a local martingale. Therefore the process $Z\widehat{V}^{\text{liq}}$ is a local supermartingale under \mathbf{P} . As $Z\widehat{V}^{\text{liq}} \geq 0$, it is, in fact, a supermartingale.

Since Z_T is strictly positive and the terminal value $\widehat{V}_T^{\text{liq}}$ is strictly positive almost surely by Theorem 4.1.3, we have that the trajectories of $Z\widehat{V}^{\text{liq}}$ are strictly positive almost surely, by the supermartingale property of $Z\widehat{V}^{\text{liq}}$. This implies that the process \widehat{V}^{liq} is strictly positive almost surely. \square

Proposition 4.2.5. *Fix $0 < \lambda < 1$. Under the assumptions of Theorem 4.1.3 (where we do not impose the assumption (NUPBR)), suppose that the liquidation value process $\widehat{V}^{\text{liq}}(\varphi)$ associated to the optimizer $\widehat{\varphi} = (\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)_{0 \leq t \leq T}$ is strictly positive almost surely for each $0 \leq t \leq T$.*

Then the dual optimizer $\widehat{h}(y)$ is induced by a local martingale $\widehat{Z} = (\widehat{Z}_t^0, \widehat{Z}_t^1)_{0 \leq t \leq T}$.

Proof. Fix $y > 0$ and assume without loss of generality that $y = 1$. We have to show that there is a local λ -consistent price system $\widehat{Z} = (\widehat{Z}_t^0, \widehat{Z}_t^1)_{0 \leq t \leq T}$ with $\widehat{Z}_0^0 = 1$ and $\widehat{Z}_T^0 = \widehat{h}$, where \widehat{h} is the dual optimizer in Theorem 4.1.3 for $y = 1$.

By Proposition 4.1.2 (iv), we know that there is a sequence $(Z^n)_{n \in \mathbb{N}}$ of local λ -consistent price systems such that

$$\lim_{n \rightarrow \infty} Z_T^{n,0} \geq \widehat{h}, \quad a.s.$$

By the optimality of \widehat{h} we must have equality above. Using Lemma 4.2.6 we may assume, by passing to convex combinations, that the sequence $(Z^n)_{n \in \mathbb{N}}$ converges to a supermartingale, denoted by \widehat{Z} , in the sense of (4.2.19).

By passing to a localizing sequence of stopping times, we may assume that all processes Z^n are uniformly integrable martingales, that S is bounded from above and bounded away from zero, and that the process $\widehat{\varphi}$ is bounded.

To show that the supermartingale \widehat{Z} is a local martingale, consider its Doob-Meyer decomposition

$$d\widehat{Z}_t^0 = d\widehat{M}_t^0 - d\widehat{A}_t^0, \quad (4.2.14)$$

$$d\widehat{Z}_t^1 = d\widehat{M}_t^1 - d\widehat{A}_t^1, \quad (4.2.15)$$

where the predictable processes \widehat{A}^0 and \widehat{A}^1 are nondecreasing. We have to show that \widehat{A}^0 and \widehat{A}^1 vanish. By stopping once more, we may assume that these two processes are bounded and that \widehat{M}^0 and \widehat{M}^1 are true martingales.

We start by showing that \widehat{A}^0 and \widehat{A}^1 are aligned in the following way:

$$(1 - \lambda)S_t d\widehat{A}_t^0 \leq d\widehat{A}_t^1 \leq S_t d\widehat{A}_t^0, \quad (4.2.16)$$

which is the differential notation for the integral inequality

$$\int_0^T (1 - \lambda)S_t \mathbf{1}_D d\widehat{A}_t^0 \leq \int_0^T \mathbf{1}_D d\widehat{A}_t^1 \leq \int_0^T S_t \mathbf{1}_D d\widehat{A}_t^0, \quad (4.2.17)$$

which we require to hold true for every optional subset $D \subseteq \Omega \times [0, T]$. Turning to the differential notation again, inequality (4.2.16) may be intuitively interpreted that $\frac{d\widehat{A}_t^1}{d\widehat{A}_t^0}$ takes values in the bid-ask spread $[(1 - \lambda)S_t, S_t]$. The proof of the claim (4.2.17) is formalized in the subsequent Lemma 4.2.7 below.

The process $\widehat{V}_t = \widehat{\varphi}_t^0 \widehat{Z}_t^0 + \widehat{\varphi}_t^1 \widehat{Z}_t^1$ is a uniformly integrable martingale by Theorem 4.1.3. By Itô's lemma and using the fact that $\widehat{\varphi}$ is of finite variation, we have

$$d\widehat{V}_t = \widehat{\varphi}_{t-}^0 (d\widehat{M}_t^0 - d\widehat{A}_t^0) + \widehat{\varphi}_{t-}^1 (d\widehat{M}_t^1 - d\widehat{A}_t^1) + \widehat{Z}_t^0 d\widehat{\varphi}_t^0 + \widehat{Z}_t^1 d\widehat{\varphi}_t^1.$$

Hence, we may write the process \widehat{V}_t as the sum of three integrals

$$\begin{aligned} \widehat{V}_t = x &+ \int_0^t \left(\widehat{Z}_u^0 d\widehat{\varphi}_u^0 + \widehat{Z}_u^1 d\widehat{\varphi}_u^1 \right) + \int_0^t \left(\widehat{\varphi}_{u-}^0 d\widehat{M}_u^0 + \widehat{\varphi}_{u-}^1 d\widehat{M}_u^1 \right) \\ &- \int_0^t \left(\widehat{\varphi}_{u-}^0 d\widehat{A}_u^0 + \widehat{\varphi}_{u-}^1 d\widehat{A}_u^1 \right). \end{aligned}$$

The first integral defines a nonincreasing process by the self-financing condition (2.1.1) and the fact that $\frac{\widehat{Z}_u^1}{\widehat{Z}_u^0}$ takes values in $[(1 - \lambda)S_u, S_u]$. The second integral defines a local martingale.

As regards the third term, we claim that

$$\int_0^t \left(\widehat{\varphi}_{u-}^0 d\widehat{A}_u^0 + \widehat{\varphi}_{u-}^1 d\widehat{A}_u^1 \right) \quad (4.2.18)$$

defines a nondecreasing process. As \widehat{V} is a martingale, this will imply that the process (4.2.18) vanishes.

We deduce from (4.2.17) that

$$\begin{aligned}
\int_0^t \left(\widehat{\varphi}_{u-}^0 d\widehat{A}_u^0 + \widehat{\varphi}_{u-}^1 d\widehat{A}_u^1 \right) &= \int_0^t \left(\widehat{\varphi}_{u-}^0 d\widehat{A}_u^0 + \widehat{\varphi}_{u-}^1 d\widehat{A}_u^1 \right) \mathbf{1}_{\{\widehat{\varphi}_{u-}^1 \leq 0\}} \\
&\quad + \int_0^t \left(\widehat{\varphi}_{u-}^0 d\widehat{A}_u^0 + \widehat{\varphi}_{u-}^1 d\widehat{A}_u^1 \right) \mathbf{1}_{\{\widehat{\varphi}_{u-}^1 > 0\}} \\
&\geq \int_0^t \left(\widehat{\varphi}_{u-}^0 - \widehat{\varphi}_{u-}^1 S_u \right) \mathbf{1}_{\{\widehat{\varphi}_{u-}^1 \leq 0\}} d\widehat{A}_u^0 \\
&\quad + \int_0^t \left(\widehat{\varphi}_{u-}^0 + \widehat{\varphi}_{u-}^1 (1 - \lambda) S_u \right) \mathbf{1}_{\{\widehat{\varphi}_{u-}^1 > 0\}} d\widehat{A}_u^0 \\
&= \int_0^t V_{u-}^{\text{liq}}(\widehat{\varphi}) d\widehat{A}_u^0.
\end{aligned}$$

As we have assumed that the liquidation value $V^{\text{liq}}(\widehat{\varphi})$ almost surely satisfies that

$$\inf_{0 \leq t \leq T} V_t^{\text{liq}}(\widehat{\varphi}) > 0, \quad a.s.,$$

and the process \widehat{A}^0 is nondecreasing, the vanishing of the process in (4.2.18) implies that \widehat{A}^0 vanishes. By (4.2.16) the processes \widehat{A}^0 and \widehat{A}^1 vanish simultaneously.

Summing up, modulo the (still missing) proof of (4.2.17), we deduce from the fact that \widehat{V} is a martingale that \widehat{A}^0 and \widehat{A}^1 vanish. Therefore, \widehat{Z}^0 and \widehat{Z}^1 are local martingales. \square

In the above proof of Proposition 4.2.5 we have used the following consequence of the Fatou-limit construction of Föllmer and Kramkov [41, Lemma 5.2]. (Compare also [29, Proposition 2.3] for a more refined result.)

Lemma 4.2.6. *Let $(Z^n)_{n \in \mathbb{N}}$ be a sequence of $[0, \infty)$ -valued (càdlàg) supermartingales $Z^n = (Z_t^n)_{0 \leq t \leq T}$, all starting at $Z_0^n = 1$. There exists a sequence of forward convex combinations, still denoted by $(Z^n)_{n \in \mathbb{N}}$, a limiting (càdlàg) supermartingale Z as well as a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times such that, for every stopping time $0 \leq \tau \leq T$ with $\mathbf{P}[\tau = \tau_n] = 0$, for each $n \in \mathbb{N}$, we have*

$$Z_\tau = \mathbf{P} - \lim_{n \rightarrow \infty} Z_\tau^n, \quad (4.2.19)$$

the convergence holding true in probability.

Proof. In [29, Theorem 2.7] it is shown that there exists a (làdlàg) optional strong supermartingale $\overline{Z} = (\overline{Z}_t)_{0 \leq t \leq T}$ such that, after passing to forward convex combinations of $(Z^n)_{n \in \mathbb{N}}$, we have

$$\overline{Z}_\tau = \mathbf{P} - \lim_{n \rightarrow \infty} Z_\tau^n, \quad (4.2.20)$$

for all stopping times $0 \leq \tau \leq T$. We shall see that the càdlàg version of \overline{Z} then is our desired supermartingale Z . We note in passing that Z is the Fatou-limit of $(Z^n)_{n \in \mathbb{N}}$ as constructed by Föllmer and Kramkov in [41].

Indeed, we may find a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times exhausting all the jumps of \overline{Z} . Therefore for a stopping time τ avoiding all the τ_n , we have $Z_\tau = \overline{Z}_\tau$ so that in this case (4.2.20) implies (4.2.19). \square

Lemma 4.2.7. *In the setting of Proposition 4.2.5, let $\widehat{A}^0, \widehat{A}^1$ be the bounded, predictable processes in (4.2.14) and (4.2.15), and let $0 \leq \sigma \leq T$ be a stopping time. For $\varepsilon > 0$, define*

$$\tau_\varepsilon := \inf \left\{ t \geq \sigma : \frac{S_t}{S_\sigma} = 1 + \varepsilon \text{ or } 1 - \varepsilon \right\}. \quad (4.2.21)$$

Then

$$\begin{aligned} (1 - \varepsilon)(1 - \lambda)S_\sigma \mathbf{E} \left[\widehat{A}_{\tau_\varepsilon}^0 - \widehat{A}_\sigma^0 \middle| \mathcal{F}_\sigma \right] &\leq \mathbf{E} \left[\widehat{A}_{\tau_\varepsilon}^1 - \widehat{A}_\sigma^1 \middle| \mathcal{F}_\sigma \right] \\ &\leq (1 + \varepsilon)S_\sigma \mathbf{E} \left[\widehat{A}_{\tau_\varepsilon}^0 - \widehat{A}_\sigma^0 \middle| \mathcal{F}_\sigma \right]. \end{aligned} \quad (4.2.22)$$

Before aboarding the proof we remark that it is routine to deduce (4.2.17) from the preceding lemma.

Proof. The processes \widehat{A}^0 and \widehat{A}^1 are càdlàg, being defined as the differences of two càdlàg processes. Hence, we have

$$\mathbf{E} \left[\widehat{A}_{\tau_\varepsilon}^1 - \widehat{A}_\sigma^1 \middle| \mathcal{F}_\sigma \right] = \lim_{\delta \searrow 0} \mathbf{E} \left[\widehat{A}_{\tau_\varepsilon + \delta}^1 - \widehat{A}_{\tau_\delta}^1 \middle| \mathcal{F}_\sigma \right].$$

Fix the sequence $(Z^n)_{n \in \mathbb{N}}$ of local martingales as above. It follows from (4.2.19) that we have for all but countably many $\delta > 0$, that $(Z_{\tau_\delta}^n)_{n \in \mathbb{N}}$ converges to $\widehat{Z}_{\tau_\delta}$ in probability. The bottom line is that it will suffice to prove (4.2.22) under the additional assumption that $(Z_\sigma^n)_{n \in \mathbb{N}}$ and $(Z_{\tau_\varepsilon}^n)_{n \in \mathbb{N}}$ converge to \widehat{Z}_σ and $\widehat{Z}_{\tau_\varepsilon}$ in probability and – after passing once more to a subsequence – almost surely.

To simplify notation we drop the subscript ε from τ_ε . We then have almost surely that

$$\lim_{n \rightarrow \infty} (Z_\tau^{n,0} - Z_\sigma^{n,0}) = \left(\widehat{Z}_\tau^0 - \widehat{Z}_\sigma^0 \right) = \left(\widehat{M}_\tau^0 - \widehat{M}_\sigma^0 \right) - \left(\widehat{A}_\tau^0 - \widehat{A}_\sigma^0 \right), \quad (4.2.23)$$

and

$$\lim_{n \rightarrow \infty} (Z_\tau^{n,1} - Z_\sigma^{n,1}) = \left(\widehat{Z}_\tau^1 - \widehat{Z}_\sigma^1 \right) = \left(\widehat{M}_\tau^1 - \widehat{M}_\sigma^1 \right) - \left(\widehat{A}_\tau^1 - \widehat{A}_\sigma^1 \right). \quad (4.2.24)$$

We also have that

$$\lim_{C \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{E} \left[(Z_\tau^{n,0} - Z_\sigma^{n,0}) \mathbf{1}_{\{Z_\tau^{n,0} - Z_\sigma^{n,0} \geq C\}} \middle| \mathcal{F}_\sigma \right] = \mathbf{E} \left[\widehat{A}_\tau^0 - \widehat{A}_\sigma^0 \middle| \mathcal{F}_\sigma \right], \quad (4.2.25)$$

holds true a.s., and similarly

$$\lim_{C \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{E} \left[(Z_\tau^{n,1} - Z_\sigma^{n,1}) \mathbf{1}_{\{Z_\tau^{n,1} - Z_\sigma^{n,1} \geq C\}} \middle| \mathcal{F}_\sigma \right] = \mathbf{E} \left[\widehat{A}_\tau^1 - \widehat{A}_\sigma^1 \middle| \mathcal{F}_\sigma \right]. \quad (4.2.26)$$

Indeed, we have for fixed $C > 0$

$$\begin{aligned} 0 &= \mathbf{E} \left[Z_\tau^{n,0} - Z_\sigma^{n,0} \middle| \mathcal{F}_\sigma \right] \\ &= \mathbf{E} \left[(Z_\tau^{n,0} - Z_\sigma^{n,0}) \mathbf{1}_{\{Z_\tau^{n,0} - Z_\sigma^{n,0} \geq C\}} \middle| \mathcal{F}_\sigma \right] + \mathbf{E} \left[(Z_\tau^{n,0} - Z_\sigma^{n,0}) \mathbf{1}_{\{Z_\tau^{n,0} - Z_\sigma^{n,0} < C\}} \middle| \mathcal{F}_\sigma \right]. \end{aligned}$$

Note that

$$\begin{aligned} \lim_{C \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{E} \left[(Z_\tau^{n,0} - Z_\sigma^{n,0}) \mathbf{1}_{\{Z_\tau^{n,0} - Z_\sigma^{n,0} < C\}} \middle| \mathcal{F}_\sigma \right] &= \mathbf{E} \left[\widehat{Z}_\tau^0 - \widehat{Z}_\sigma^0 \middle| \mathcal{F}_\sigma \right] \\ &= -\mathbf{E} \left[\widehat{A}_\tau^0 - \widehat{A}_\sigma^0 \middle| \mathcal{F}_\sigma \right], \end{aligned}$$

where the last equality follows from (4.2.23). We thus have shown (4.2.25), and (4.2.26) follows analogously.

We even obtain from (4.2.25) and (4.2.26) that

$$\lim_{C \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{E} \left[Z_\tau^{n,0} \mathbf{1}_{\{Z_\tau^{n,0} \geq C\}} \middle| \mathcal{F}_\sigma \right] = \mathbf{E} \left[\widehat{A}_\tau^0 - \widehat{A}_\sigma^0 \middle| \mathcal{F}_\sigma \right] \quad (4.2.27)$$

and

$$\lim_{C \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{E} \left[Z_\tau^{n,1} \mathbf{1}_{\{Z_\tau^{n,1} \geq C\}} \middle| \mathcal{F}_\sigma \right] = \mathbf{E} \left[\widehat{A}_\tau^1 - \widehat{A}_\sigma^1 \middle| \mathcal{F}_\sigma \right] \quad (4.2.28)$$

Indeed, the sequence $(Z_\sigma^{n,0})_{n \in \mathbb{N}}$ converges a.s. to \widehat{Z}_σ^0 so that by Egoroff's Theorem it converges uniformly on sets of measure bigger than $1 - \delta$. As we condition on \mathcal{F}_σ in (4.2.25), we may suppose without loss of generality that $(Z_\sigma^{n,0})_{n \in \mathbb{N}}$ converges uniformly to \widehat{Z}_σ^0 . Therefore, the terms involving $Z_\sigma^{n,0}$ in (4.2.25) disappear in the limit $C \rightarrow \infty$.

Finally, observe that

$$\frac{Z_\tau^{n,1}}{Z_\tau^{n,0}} \in [(1 - \lambda)S_\tau, S_\tau] \subseteq [(1 - \varepsilon)(1 - \lambda)S_\sigma, (1 + \varepsilon)S_\sigma].$$

Conditioning again on \mathcal{F}_σ , this implies on the one hand

$$\lim_{C \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{E} \left[Z_\tau^{n,1} \mathbf{1}_{\{Z_\tau^{n,0} \geq C\}} \middle| \mathcal{F}_\sigma \right] = \mathbf{E} \left[\widehat{A}_\tau^1 - \widehat{A}_\sigma^1 \middle| \mathcal{F}_\sigma \right],$$

and on the other hand

$$\begin{aligned} \frac{\mathbf{E} \left[\widehat{A}_\tau^1 - \widehat{A}_\sigma^1 \middle| \mathcal{F}_\sigma \right]}{\mathbf{E} \left[\widehat{A}_\tau^0 - \widehat{A}_\sigma^0 \middle| \mathcal{F}_\sigma \right]} &= \lim_{C \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\mathbf{E} \left[Z_\tau^{n,1} \mathbf{1}_{\{Z_\tau^{n,0} \geq C\}} \middle| \mathcal{F}_\sigma \right]}{\mathbf{E} \left[Z_\tau^{n,0} \mathbf{1}_{\{Z_\tau^{n,0} \geq C\}} \middle| \mathcal{F}_\sigma \right]} \\ &\in [(1 - \varepsilon)(1 - \lambda)S_\sigma, (1 + \varepsilon)S_\sigma], \end{aligned}$$

which is assertion (4.2.22). □

Remark 4.2.8. We note that we do not need the strict positivity of $V^{\text{liq}}(\widehat{\varphi})$ in the above lemma.

Summing up, we have that

Theorem 4.2.9. *Fix the level $0 < \lambda < 1$ of transaction costs and assume that the assumptions of Theorem 4.1.3 plus the assumption of (NUPBR) are satisfied. To resume: $S = (S_t)_{0 \leq t \leq T}$ is a continuous, strictly positive semimartingale satisfying the condition (NUPBR) of “no unbounded profit with bounded risk”, and $U : (0, \infty) \rightarrow \mathbb{R}$ is a utility function satisfying the condition of reasonable asymptotic elasticity. We also suppose that the value function $u(x)$ is finite, for some $x > 0$.*

Then, for each $y > 0$, the dual optimizer $\widehat{h}(y)$ in Theorem 4.1.3 is induced by a local martingale $\widehat{Z} = (\widehat{Z}_t^0, \widehat{Z}_t^1)_{0 \leq t \leq T}$. Hence, by Theorem 4.2.2 the process $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ is a shadow price.

Following the proof of Proposition 4.2.5 we may obtain the following properties of primal and dual optimizers, similarly as in Theorem 3.3.2.

Proposition 4.2.10. *Under the assumptions of Theorem 4.1.3, let $(Z^n)_{n \in \mathbb{N}}$ be a minimizing sequence of local λ -consistent price systems for the dual problem, i.e.,*

$$\mathbf{E}[V(yZ_T^{n,0})] \searrow v(y), \quad \text{as } n \rightarrow \infty,$$

where $y = u'(x)$.

Then, there exists a sequence of convex combinations, still denoted by $(Z^n)_{n \in \mathbb{N}}$, a limiting càdlàg supermartingale \hat{Y} as well as a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times such that $\hat{Y}_T = \hat{h}$, and we have

$$\hat{Y}_\tau = \mathbf{P} - \lim_{n \rightarrow \infty} yZ_\tau^n,$$

for every stopping time $0 \leq \tau \leq T$ with $\mathbf{P}[\tau = \tau_n] = 0$ for each $n \in \mathbb{N}$, and

$$\begin{aligned} \{d\hat{\varphi}_t^1 > 0\} &\subseteq \{\hat{S}_t = S_t\}, \\ \{d\hat{\varphi}_t^1 < 0\} &\subseteq \{\hat{S}_t = (1 - \lambda)S_t\}, \end{aligned} \tag{4.2.29}$$

for any primal optimizer $\hat{\varphi} = (\hat{\varphi}^0, \hat{\varphi}^1)$, where $\hat{S} = \frac{\hat{Y}^1}{\hat{Y}^0}$. This implies that

$$\hat{Y}^0 \hat{\varphi}^0 + \hat{Y}^1 \hat{\varphi}^1 = \hat{Y}^0 (x + \hat{\varphi}_-^1 \cdot \hat{S}).$$

Proof. By the first step of Proposition 4.2.5, we may find a limiting supermartingale deflator $(\hat{Y}^0, \hat{Y}^1) \in \mathcal{B}^\lambda(y)$ with $\hat{Y}_T^0 = \hat{h}$ and $\hat{V}_t = \hat{\varphi}_t^0 \hat{Y}_t^0 + \hat{\varphi}_t^1 \hat{Y}_t^1$ is a uniformly integrable martingale for each primal optimizer $\hat{\varphi} = (\hat{\varphi}^0, \hat{\varphi}^1)$ by Theorem 4.1.3.

By Itô's lemma (see [58, Proposition I.4.49]) and using the fact that $\hat{\varphi}$ is of finite variation, we obtain

$$\begin{aligned} \hat{V}_t = xy + \int_0^t \left(\hat{Y}_u^0 d\hat{\varphi}_u^0 + \hat{Y}_u^1 d\hat{\varphi}_u^1 \right) + \int_0^t \left(\hat{\varphi}_{u-}^0 d\hat{M}_u^0 + \hat{\varphi}_{u-}^1 d\hat{M}_u^1 \right) \\ - \int_0^t \left(\hat{\varphi}_{u-}^0 d\hat{A}_u^0 + \hat{\varphi}_{u-}^1 d\hat{A}_u^1 \right), \end{aligned}$$

where $\hat{Y}_t^0 = \hat{Y}_0^0 + \hat{M}_t^0 - \hat{A}_t^0$ and $\hat{Y}_t^1 = \hat{Y}_0^1 + \hat{M}_t^1 - \hat{A}_t^1$ denote the Doob-Meyer decomposition.

The first integral defines a nonincreasing process by the λ -self-financing condition (2.1.1) and the fact that $\frac{\hat{Z}_u^1}{\hat{Z}_u^0}$ takes values in $[(1 - \lambda)S_u, S_u]$. The second integral defines a local martingale. By (4.2.16) we may deduce that the third integral defines a nondecreasing process.

As \hat{V} is a martingale, this imply that the first and the third integrals vanish,

$$0 = \int_0^t \left(\hat{Y}_u^0 d\hat{\varphi}_u^0 + \hat{Y}_u^1 d\hat{\varphi}_u^1 \right) = \int_0^t \hat{Y}_u^0 \left(d\hat{\varphi}_u^0 + \hat{S}_u d\hat{\varphi}_u^1 \right).$$

Since \hat{Y}^0 is strictly positive, by following the proof of Theorem 4.2.2, we see that the assertion (4.2.29) holds true.

Using Itô's lemma again it follows that

$$\hat{V}_t = \hat{Y}_t^0 (\hat{\varphi}_t^0 + \hat{\varphi}_t^1 \hat{S}_t) = \hat{Y}_t^0 \left(x + (\hat{\varphi}_-^1 \cdot \hat{S})_t + R_t \right)$$

where R_t is a nonpositive nonincreasing process given by

$$\begin{aligned} R_t &:= \int_0^t d\widehat{\varphi}_u^0 + \int_0^t \widehat{S}_u d\widehat{\varphi}_u^1 \\ &= \int_0^t -S_u d\widehat{\varphi}_u^{1,\uparrow} + \int_0^t (1-\lambda)S_u d\widehat{\varphi}_u^{1,\downarrow} + \int_0^t \widehat{S}_u d\widehat{\varphi}_u^{1,\uparrow} - \int_0^t \widehat{S}_u d\widehat{\varphi}_u^{1,\downarrow} \\ &= \int_0^t (\widehat{S}_u - S_u) d\widehat{\varphi}_u^{1,\uparrow} + \int_0^t ((1-\lambda)S_u - \widehat{S}_u) d\widehat{\varphi}_u^{1,\downarrow}. \end{aligned}$$

It follows from (4.2.29) that $R_t = 0$ almost surely. \square

4.3 Two Counterexamples

The assumption of S satisfying $(NUPBR)$, which is the local version of the customary assumption $(NFLVR)$, is quite natural in the present context. Nevertheless one might be tempted (as the present authors originally have been) to conjecture that this assumption could be replaced by a weaker assumption as used in Proposition 4.1.2, i.e., that for every $0 < \mu < \lambda$ there exists a μ -consistent price system, at least locally. Unfortunately, this idea was wishful thinking and such hopes turned out to be futile. In this section we show that the assumption of $(NUPBR)$ in Theorem 4.2.9 cannot be replaced by the assumption of the local existence of μ -consistent price systems, for all $0 < \mu < 1$.

Proposition 4.3.1. *There is a continuous, strictly positive semimartingale $S = (S_t)_{0 \leq t \leq T}$ with the following properties.*

- (i) *S satisfies the stickiness property introduced by Guasoni in [49]. Hence, for every $0 < \mu < 1$, there is a μ -consistent price system.*
- (ii) *For fixed $0 < \lambda < 1$ and $U(x) = \log(x)$, the value function $u(x)$ is finite so that by Theorem 4.1.3 there is a dual optimizer $\widehat{Y} = (\widehat{Y}_t^0, \widehat{Y}_t^1) \in \mathcal{B}^\lambda$.*
- (iii) *The optimizer \widehat{Y} fails to be a local martingale.*

In fact, there is no shadow price in the sense of Definition 4.2.1, i.e., no semimartingale $(\widetilde{S}_t)_{0 \leq t \leq T}$ such that \widetilde{S} takes its values in the bid-ask spread $[(1-\lambda)S, S]$ and such that equality (4.2.2) holds true.

Remark 4.3.2. The construction in the proof will yield a nondecreasing process S which will imply in a rather spectacular way that S does not satisfy $(NUPBR)$.

We start by outlining the proof in an informal way, banning the technicalities into the Appendix. First note that, for logarithmic utility $U(x) = \log(x)$, the normalized dual optimizer $\frac{\widehat{Y}(y)}{y}$ does not depend on $y > 0$; we therefore dropped the dual variable $y > 0$ in (ii) and (iii) above.

Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion on some underlying probability space $(\Omega, \mathcal{F}, \mathbf{P})$, starting at $B_0 = 0$, and let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the \mathbf{P} -augmented filtration generated by B . For $w \geq 0$, define the Brownian motion W^w with drift, starting at $W_0^w = w$, by

$$W_t^w := w + B_t - t, \quad t \geq 0.$$

Define the stopping time

$$\tau^w := \inf\{t > 0 \mid W_t^w \leq 0\}$$

and observe that the law of τ^w is inverse Gaussian with mean w and variance w^2 (see e.g. [92, I.9]).

For fixed $w > 0$, the stock price process $S = S^w$ is defined by

$$S_t^w := e^{t \wedge \tau^w}, \quad t \geq 0. \quad (4.3.1)$$

Let us comment on this peculiar definition of a stock price process S : the price can only *move upwards*, as it equals the exponential function up to time τ^w ; from this moment on S remains constant (but never goes down).

It is notationally convenient to let t range in the time interval $[0, \infty]$. To transform the construction into our usual setting of bounded time intervals $[0, T]$, note that τ^w is a.s. finite so that the deterministic time change $u = \arctan(t)$ defines a process $\bar{S}_u^w = S_{\arctan(t)}^w$ which can be continuously extended to all $u \in [0, \frac{\pi}{2}]$. We prefer not to do this notational change and to let $T = \infty$ be the terminal horizon of the process $S = (S_t)_{0 \leq t \leq \infty}$ and of our optimization problem.

Fix transaction costs $\lambda \in (0, 1)$, the utility function $U(x) = \log(x)$, and initial endowment $x = 1$. We consider the portfolio optimization problem (4.1.2), i.e.,

$$\mathbf{E}[\log(g)] \rightarrow \max!, \quad g \in \mathcal{C}_0^{\lambda, w}. \quad (4.3.2)$$

The superscript w pertains to the initial value W_0^w of the process W^w and will be dropped if there is no danger of confusion.

We shall verify below that S admits a μ -consistent price system, for all $0 < \mu < 1$, and that the value (4.3.2) of the optimization problem is finite.

Let us discuss on an intuitive level what the optimal strategy for the log-utility optimizing agent should look like. Obviously she will never want to go short on a stock S which only can go up. Rather, she wants to invest substantially into this bonanza. For an agent without transaction costs, there is no upper bound for such an investment as there is no downside risk. Hence S allows for an “unbounded profit with bounded risk” and the utility optimization problem degenerates in this case, i.e., $u(x) \equiv \infty$.

More interesting is the situation when the agent is confronted with transaction costs $0 < \lambda < 1$. Starting from initial endowment $x = 1$, i.e., $(\varphi_{0-}^0, \varphi_{0-}^1) = (1, 0)$, there is an upper bound for her investment into the stock at time $t = 0$, namely $\frac{1}{\lambda}$ many stocks.

This is the maximal amount of holdings in stock which yields a nonnegative liquidation value $V_0^{\text{liq}}(\varphi)$. Indeed, in this case $(\varphi_0^0, \varphi_0^1) = (1 - \frac{1}{\lambda}, \frac{1}{\lambda})$ implies that $V_0^{\text{liq}}(\varphi) = 1 - \frac{1}{\lambda} + (1 - \lambda)\frac{1}{\lambda} = 0$.

This gives rise to the following notation.

Definition 4.3.3. Let $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq \infty}$ be a self-financing trading strategy for S such that $\varphi_t^0 + \varphi_t^1 S_t > 0$. The leverage process is defined as

$$L_t(\varphi) = \frac{\varphi_t^1 S_t}{\varphi_t^0 + \varphi_t^1 S_t}, \quad t \geq 0.$$

The process $L_t(\varphi)$ may be interpreted as the ratio of the value of the position in stock to the total value of the portfolio if we do not consider transaction costs. We obtain from the above discussion that the process $L_t(\varphi)$ is bounded by $\frac{1}{\lambda}$ if φ is admissible, i.e., if

$$V_t^{\text{liq}}(\varphi) = \varphi_t^0 + (1 - \lambda)\varphi_t^1 S_t \geq 0,$$

for $t \geq 0$.

What is the optimal leverage which the log-utility maximizer chooses, say at time $t = 0$? The answer depends on the initial value w of the process W^w . If w is very small, it is intuitively rather obvious that the optimal strategy $\hat{\varphi}$ only uses leverage $L_0(\hat{\varphi}) = 0$ at time $t = 0$, i.e., it is optimal to keep all the money in bond. Indeed, in this case τ^w takes small values with high probability. If the economic agent decides to buy stock at time $t = 0$, then — due to transaction costs — she will face a loss with high probability, as she has to liquidate the stock before it has substantially risen in value. For sufficiently small w these losses will outweigh the gains which can be achieved when τ^w takes large values. Hence for w sufficiently small, say $0 < w \leq \underline{w}$, we expect that the best strategy is not to buy any stock at time $t = 0$.

Now we let the initial value w range above this lower threshold \underline{w} . As w increases it again is rather intuitive from an economic point of view that the agent will dare to take an increasingly higher leverage at time $t = 0$. Indeed, the stopping times τ^w are increasing in w so the prospects for a substantial rise of the stock price become better as w increases.

The crucial feature of the example is that we will show that there is a finite upper threshold $\bar{w} > 0$ such that, for $w \geq \bar{w}$, the optimal strategy $\hat{\varphi}$ at time $t = 0$ takes *maximal leverage*, i.e., $L_0(\hat{\varphi}) = \frac{1}{\lambda}$. In fact, the optimal strategy $\hat{\varphi}$ will then satisfy $L_t(\hat{\varphi}) = \frac{1}{\lambda}$ and therefore $V_t^{\text{liq}}(\hat{\varphi}) = 0$ as long as W_t^w remains above the threshold \bar{w} .

Lemma 4.3.4. *Using the above notation there is $\bar{w} > 0$ such that, for $w \geq \bar{w}$, the optimizer $\hat{\varphi}^w$ of the optimization problem (4.3.2) satisfies*

$$L_0(\hat{\varphi}^w) = \frac{1}{\lambda}.$$

More precisely, fix $w = \bar{w} + 1$, and define $\sigma := \inf\{t > 0 \mid W_t^w \leq \bar{w}\}$. Then

$$L_t(\hat{\varphi}^w) = \frac{1}{\lambda}, \tag{4.3.3}$$

for $0 \leq t \leq \sigma$.

For $0 \leq t \leq \sigma$ we then may explicitly calculate the primal optimizer

$$\hat{\varphi}_t^0 = \left(1 - \frac{1}{\lambda}\right) \exp\left(\frac{1}{\lambda}t\right), \quad \hat{\varphi}_t^1 = \frac{1}{\lambda} \exp\left(\left(\frac{1}{\lambda} - 1\right)t\right), \tag{4.3.4}$$

and the dual optimizer

$$\hat{Y}_t^0 = \exp\left(-\frac{1}{\lambda}t\right), \quad \hat{Y}_t^1 = \exp\left(\left(1 - \frac{1}{\lambda}\right)t\right), \tag{4.3.5}$$

so that

$$\hat{S}_t := \frac{\hat{Y}_t^1}{\hat{Y}_t^0} = S_t, \tag{4.3.6}$$

for $0 \leq t \leq \sigma$.

Admitting this lemma, we can quickly show Proposition 4.3.1. The crucial assertion is that there is no shadow price \widehat{S} .

Proof of Proposition 4.3.1. Using the above notation fix $w = \bar{w} + 1$ and drop this superscript to simplify notation.

(i) We claim that the process S has the **stickiness property** as defined by Guasoni [49, Definition 2.2]: this property states that, for any $\varepsilon > 0$ and any stopping time σ with $\mathbf{P}[\sigma < \infty] > 0$, we have, conditionally on $\{\sigma < \infty\}$, that the set of paths $(S_t)_{t \geq \sigma}$, which do not leave the price corridor $[\frac{1}{1+\varepsilon}S_\sigma, (1+\varepsilon)S_\sigma]$, has strictly positive measure. By [49, Corollary 2.1] the stickiness property implies no arbitrage under transaction costs μ for each $\mu \in (0, 1)$. Together with Theorem 2.2.2 we have that, for the continuous process S , the stickiness property of Y implies that S verifies (CPS^μ) for all $0 < \mu < 1$.

To show the stickiness property simply observe that, for each $\delta > 0$ and each stopping time σ such that $\mathbf{P}[\sigma < \tau] > 0$, we have

$$\mathbf{P}[\sigma < \tau, |\tau - \sigma| < \delta] > 0.$$

Indeed, given σ such that $\mathbf{P}[\sigma < \tau] > 0$, i.e., W has not yet reached zero at time σ , $(W_t)_{t \geq \sigma}$ will hit zero with positive probability before more than δ units of time elapse.

(ii) For fixed $0 < \lambda < 1$ and $\varphi \in \mathcal{A}_0^\lambda(x)$, we observe that $V_\tau^{\text{liq}}(\varphi) \leq x \exp(\frac{\tau}{\lambda})$. As τ has expectation $\mathbf{E}[\tau] = w$, we obtain that

$$u(x) \leq \mathbf{E} \left[\log \left(x \exp \left(\frac{\tau}{\lambda} \right) \right) \right] = \log(x) + \frac{1}{\lambda} \mathbf{E}[\tau] = \log(x) + \frac{w}{\lambda} < \infty.$$

Hence, by Theorem 4.1.3 there is a dual optimizer $\widehat{Y} \in \mathcal{B}^\lambda$.

(iii) Lemma 4.3.4 provides very explicitly the form of the primal and dual optimizer $\widehat{\varphi}$ and \widehat{Y} , respectively, for $0 \leq t \leq \sigma$. In particular, \widehat{Y} is a supermartingale, which fails to be a local martingale.

We now turn to the final assertion of Proposition 4.3.1. We know from Theorem 3.3.3 that the process $\widehat{S}_t := \frac{\widehat{Y}_t^1}{\widehat{Y}_t^0}$ is a shadow price process in a generalized sense. By (4.3.6) we have $\widehat{S}_t = S_t$, for $0 \leq t \leq \sigma$.

Let us recall this generalized sense of a shadow price as stated in Theorem 3.3.3: for every competing finite variation, self-financing trading strategy $\varphi \in \mathcal{A}_0^\lambda(x)$ such that the liquidation value remains nonnegative, i.e.,

$$V_t^{\text{liq}}(\varphi) = \varphi_t^0 + (\varphi_t^1)^+(1-\lambda)S_t - (\varphi_t^1)^-S_t \geq 0, \quad (4.3.7)$$

for all $0 \leq t \leq T$, we have

$$\mathbf{E} \left[U \left(x + (\varphi^1 \cdot \widehat{S})_T \right) \right] \leq \mathbf{E} \left[U \left(V_T^{\text{liq}}(\widehat{\varphi}) \right) \right] = u(x). \quad (4.3.8)$$

Here

$$(\varphi^1 \cdot \widehat{S})_t = \int_0^t \varphi_u^1 d\widehat{S}_u, \quad 0 \leq t \leq T,$$

denotes the stochastic integral with respect to the semimartingale \widehat{S} .

This generalized shadow price property does hold true for the above process $\widehat{S} := \frac{\widehat{Y}^1}{\widehat{Y}^0}$ by Theorem 3.3.3. In fact, as everything is very explicit in the present example, at least for $0 \leq t \leq \sigma$, this also can easily be verified directly.

But presently we are considering the shadow price property in the more classical sense of Definition 4.2.1, where we allow φ^1 in (4.3.8) to range over all predictable \widehat{S} -integrable processes which are admissible only in the sense

$$x + (\varphi^1 \cdot \widehat{S})_t \geq 0, \quad (4.3.9)$$

for all $0 \leq t \leq T$. This condition is much weaker than (4.3.7).

Clearly \widehat{S} is the only candidate for a shadow price process in the sense of Definition 4.2.1. But as \widehat{S} only moves upwards, for $0 \leq t \leq \sigma$, it can certainly not satisfy this property. Indeed, the left-hand side of (4.2.2) must be infinity:

$$\sup_{\tilde{g} \in \mathcal{C}(x; \widehat{S})} \mathbf{E}[U(\tilde{g})] = \infty. \quad (4.3.10)$$

For example, it suffices to consider the integrands $\varphi^1 = C \mathbf{1}_{]0, \tau]}$ to obtain $(\varphi^1 \cdot \widehat{S})_T = C(S_\sigma - S_0) = C(e^\sigma - 1)$. Sending C to infinity we obtain (4.3.10).

This shows that there cannot be a shadow price process \widetilde{S} as in Definition 4.2.1. \square

Remark 4.3.5. The construction of Proposition 4.3.1 uses the Brownian filtration $\mathbb{F}^B = (\mathcal{F}_t^B)_{t \geq 0}$, while the natural (\mathbf{P} -augmented) filtration $\mathbb{F}^S = (\mathcal{F}_t^S)_{t \geq 0}$ generated by the price process $S = (S_t)_{t \geq 0}$ is much smaller. Fortunately, this discrepancy of the filtrations may be avoided. Let us define

$$G_t := \int_0^t W_u^w du, \quad t \geq 0,$$

where $(W_u^w)_{u \geq 0}$ is defined above. Note that the process G generates the Brownian filtration \mathbb{F}^B . Also note that the process G is increasing up to time τ . We may replace the process $S_t = e^{t \wedge \tau}$ in the construction of Proposition 4.3.1 by the process

$$\overline{S}_t := \exp((t + G_t) \wedge (\tau + G_\tau)).$$

The reader may verify that \overline{S} also has the properties claimed in Proposition 4.3.1 and has the additional feature to generate the Brownian filtration up to time τ .

Let us now prove Lemma 4.3.4.

Consider the price process $(S_t^w)_{t \geq 0}$ as in (4.3.1). Fix proportional transaction costs $\lambda \in (0, 1)$ as well as real numbers φ_0^0 and φ_0^1 . We consider the problem

$$\mathbf{E}[\log(V_{\tau^w}^{\text{liq}}(\varphi^0, \varphi^1))] \rightarrow \max!, \quad (\varphi^0, \varphi^1) \in \mathcal{A}_0^{\lambda, w}(\varphi_0^0, \varphi_0^1), \quad (4.3.11)$$

where $\mathcal{A}_0^{\lambda, w}(\varphi_0^0, \varphi_0^1)$ denotes the set of all self-financing and admissible trading strategies (φ^0, φ^1) under transaction costs λ starting with initial endowment $(\varphi_0^0, \varphi_0^1)$. If we do not need the dependence on w explicitly, we drop the superscript w in the sequel to lighten the notation and simply write W , τ , S and $\mathcal{A}_0^\lambda(\varphi_0^0, \varphi_0^1)$.

Proposition 4.3.6. Fix $w \geq 0$. For all $(\varphi_0^0, \varphi_0^1)$ with $V_0^{\text{liq}}(\varphi_0^0, \varphi_0^1) > 0$, there exists an optimal strategy $\widehat{\varphi} = (\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)_{0 \leq t < \infty}$ to problem (4.3.11) and we have that

$$\begin{aligned} u(\varphi_0^0, \varphi_0^1) &:= \sup_{(\varphi^0, \varphi^1) \in \mathcal{A}_0^\lambda(\varphi_0^0, \varphi_0^1)} \mathbf{E}[\log(V_\tau^{\text{liq}}(\varphi^0, \varphi^1))] \\ &= \inf_{y > 0} \left\{ \inf_{(Z^0, Z^1) \in \mathcal{Z}_e^\lambda(S)} \left\{ \mathbf{E}[-\log(yZ_T^0) - 1] + y\mathbf{E}[Z_0^0\varphi_0^0 + Z_0^1\varphi_0^1] \right\} \right\}. \end{aligned}$$

Proof. Since $U(x) = \log(x)$ has reasonable asymptotic elasticity, $S = (S_t)_{0 \leq t < \infty}$ satisfies the condition (CPS^μ) for all $\mu \in (0, 1)$ by Proposition 4.3.1.(i), the assertions follow from the general static duality results for utility maximization under transaction costs as soon as we have shown that $u(\varphi_0^0, \varphi_0^1) < \infty$; compare [32] and Section 3.2 in [11].

For the latter, we observe that

$$V_\tau^{\text{liq}}(\varphi^0, \varphi^1) \leq (\varphi_0^0 + \varphi_0^1) \exp\left(\frac{1}{\lambda}\tau\right)$$

and τ has an inverse Gaussian distribution with mean $\mathbf{E}[\tau] = w$, which implies

$$u(\varphi_0^0, \varphi_0^1) \leq \log(\varphi_0^0 + \varphi_0^1) + \frac{1}{\lambda}\mathbf{E}[\tau] = \log(\varphi_0^0 + \varphi_0^1) + \frac{1}{\lambda}w < \infty,$$

hence the proof is completed. \square

In order to show Lemma 4.3.4, we define the value function $v(l, w)$ on $[0, \frac{1}{\lambda}] \times [0, \infty)$ by

$$v(l, w) := \sup_{(\varphi^0, \varphi^1) \in \mathcal{A}_0^{\lambda, w}(1-l, l)} \mathbf{E}[\log(V_{\tau^w}^{\text{liq}}(\varphi^0, \varphi^1))],$$

where $(\varphi^0, \varphi^1) \in \mathcal{A}_0^{\lambda, w}(1-l, l)$ ranges through all admissible trading strategies starting at $(\varphi_{0-}^0, \varphi_{0-}^1) = (1-l, l)$. We shall see that, for fixed w , the function $v(l, w)$ is *decreasing* in l : indeed, one may always move at time $t = 0$ to a higher degree of leverage; but not vice versa, in view of the transaction costs λ .

Lemma 4.3.7. For fixed $0 < \lambda < 1$. The value function $v : [0, \frac{1}{\lambda}] \times [0, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$ has the following properties:

- (1) $v(l, w)$ is concave and nonincreasing in l for all $w \in [0, \infty)$ and $v(l, 0) = \log(1 - \lambda)$.
- (2) $v(l, w)$ is nondecreasing in w for all $l \in [0, \frac{1}{\lambda}]$.
- (3) v is jointly continuous and $v(l, w) = -\infty$ if and only if $(l, w) = (\frac{1}{\lambda}, 0)$.¹
- (4) v satisfies the dynamic programming principle, i.e.,

$$v(l, w) = \sup_{(\varphi^0, \varphi^1) \in \mathcal{A}_0^{\lambda, w}(1-l, l)} \mathbf{E}[\log(\varphi_{\tau \wedge \sigma}^0 + \varphi_{\tau \wedge \sigma}^1 S_{\tau \wedge \sigma}^w) + v(L_{\tau \wedge \sigma}(\varphi), W_{\tau \wedge \sigma}^w)]$$

for all stopping times σ .

¹With continuity at $-\infty$ defined in the usual way.

(5) There exists a nondecreasing, càdlàg function $\ell : [0, \infty) \rightarrow [0, \frac{1}{\lambda}]$ given by

$$\ell(w) := \max \{l \in [0, \frac{1}{\lambda}] \mid v(l, w) = v(0, w)\} \quad (4.3.12)$$

such that

$$(i) \ v(l, w) = \max_{k \in [0, \frac{1}{\lambda}]} v(k, w) \text{ for all } l \in [0, \ell(w)].$$

$$(ii) \ v(l, w) \text{ is strictly concave and strictly decreasing in } l \text{ on } (\ell(w), \frac{1}{\lambda}].$$

Proof. (1) As

$$\mathcal{A}_0^\lambda \left(1 - (\mu l_1 + (1 - \mu)l_2), (\mu l_1 + (1 - \mu)l_2)\right) \subseteq \mu \mathcal{A}_0^\lambda(1 - l_1, l_1) + (1 - \mu) \mathcal{A}_0^\lambda(1 - l_2, l_2)$$

for all $l_1, l_2 \in [0, \frac{1}{\lambda}]$ and $\mu \in [0, 1]$, the concavity of $v(l, w)$ in l follows immediately from that of $\log(x)$ and $V_\tau(\varphi^0, \varphi^1)$, as $\log(x)$ is nondecreasing.

If $l_1 < l_2$, the investor with initial endowment $(\varphi_0^0, \varphi_0^1) = (1 - l_1, l_1)$ can immediately buy $(l_2 - l_1)$ units of stock at time $t = 0$ for the price $S_0 = 1$ to get $(\varphi_0^0, \varphi_0^1) = (1 - l_2, l_2)$. This implies that $\mathcal{A}_0^\lambda(1 - l_1, l_1) \supseteq \mathcal{A}_0^\lambda(1 - l_2, l_2)$ and therefore $v(l_1, w) \geq v(l_2, w)$.

The assertion that $v(l, 0) = \log(1 - \lambda l)$ follows immediately from $S^0 \equiv 1$.

(2) As $\tau^{w_1} < \tau^{w_2}$ for all $0 \leq w_1 < w_2$ and hence $S_t^{w_1} \leq S_t^{w_2}$ for all $t \geq 0$, it is clear that $v(l, w_1) \leq v(l, w_2)$.

(3) The continuity of the function $v(\cdot, w) : [0, \frac{1}{\lambda}] \rightarrow \mathbb{R} \cup \{-\infty\}$ for fixed $w \geq 0$ on $(0, \frac{1}{\lambda})$ follows immediately from the fact that any finitely valued concave function is on the relative interior of its effective domain continuous. At $l = 0$, it follows from the fact that $v(\cdot, w)$ is concave and nonincreasing.

The argument for the continuity at $l = \frac{1}{\lambda}$ is slightly more involved. To that end, let $\lambda_n \in (0, 1)$ such that $\lambda_n \nearrow \lambda$ and consider for any $n \in \mathbb{N}$ the optimization problem

$$\mathbf{E}[\log(V_{\tau^w}^{\lambda_n, w}(\varphi^0, \varphi^1))] \rightarrow \max!, \quad (\varphi^0, \varphi^1) \in \mathcal{A}_0^{\lambda_n, w}(1 - l, l), \quad (4.3.13)$$

where

$$V_{\tau^w}^{\lambda_n, w}(\varphi^0, \varphi^1) := \varphi_{\tau^w}^0 + (\varphi_{\tau^w}^1)^+(1 - \lambda_n)S_{\tau^w}^w - (\varphi_{\tau^w}^1)^-S_{\tau^w}^w$$

denotes the terminal liquidation value with transaction costs λ_n and $\mathcal{A}_0^{\lambda_n, w}(\varphi_0^0, \varphi_0^1)$ the set of all self-financing and admissible trading strategies (φ^0, φ^1) under transaction costs λ_n starting with initial endowment $(\varphi_0^0, \varphi_0^1)$. By Proposition 4.3.6, the solution $\widehat{\varphi}^n(l, w) = (\widehat{\varphi}^{n,0}(l, w), \widehat{\varphi}^{n,1}(l, w))$ to (4.3.13) exists for all $(l, w) \in [0, \frac{1}{\lambda_n}] \times [0, \infty) \setminus \{(\frac{1}{\lambda_n}, 0)\}$ and $n \in \mathbb{N}$. So we can define the functions $v^n : [0, \frac{1}{\lambda_n}] \times [0, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$ for $n \in \mathbb{N}$ by

$$v^n(l, w) := \sup_{(\varphi^0, \varphi^1) \in \mathcal{A}_0^{\lambda_n, w}(1-l, l)} \mathbf{E}[\log(V_{\tau^w}^{\lambda_n, w}(\varphi^0, \varphi^1))],$$

that can by Proposition 4.3.6 be represented as

$$v^n(l, w) = \inf_{y > 0} \inf_{(Z^0, Z^1) \in \mathcal{Z}_e^{\lambda_n}(S)} \{ \mathbf{E}[-\log(yZ_T^0) - 1] + y(1 - l + l\mathbf{E}[Z_0^1]) \}. \quad (4.3.14)$$

As $\mathcal{Z}_e^{\lambda_n}(S) \subseteq \mathcal{Z}_e^{\lambda_{n+1}}(S)$ and $\bigcup_{n=1}^{\infty} \mathcal{Z}_e^{\lambda_n}(S)$ is $L^1(\mathbb{R}^2)$ -dense in $\mathcal{Z}_e^\lambda(S)$ and closed under countable convex combinations by martingale convergence, we have by (4.3.14) and in [70, Proposition 3.2] that

$$v^n(l, w) \searrow v(l, w) \quad (4.3.15)$$

for all $(l, w) \in [0, \frac{1}{\lambda}] \times [0, \infty)$. To see that (4.3.15) also holds for $(l, w) = (\frac{1}{\lambda}, 0)$, choose $(Z^{n,0}, Z^{n,1}) \equiv (1, 1 - \lambda_n) \in \mathcal{Z}_e^{\lambda_n}(S)$. Then

$$v^n\left(\frac{1}{\lambda}, 0\right) \leq \inf_{y>0} \left\{ -\log(y) - 1 + y \left(\frac{\lambda - \lambda_n}{\lambda} \right) \right\} \leq -\log\left(\frac{\lambda}{\lambda - \lambda_n}\right) \rightarrow -\infty,$$

as n goes to infinity. Hence, we have for each $w \in [0, \infty)$ a sequence of continuous, nonincreasing functions $v^n(\cdot, w) : [0, \frac{1}{\lambda}] \rightarrow \mathbb{R}$ that converges pointwise to the function $v(\cdot, w) : [0, \frac{1}{\lambda}] \rightarrow \mathbb{R} \cup \{-\infty\}$ from above and this already implies that $v(\cdot, w)$ is continuous at $\frac{1}{\lambda}$.

Indeed, let $l_m \in (0, \frac{1}{\lambda})$ such that $l_m \nearrow \frac{1}{\lambda}$ and choose, for $\varepsilon > 0$ and $w > 0$, some $n \in \mathbb{N}$ such that

$$0 \leq v^n\left(\frac{1}{\lambda}, w\right) - v\left(\frac{1}{\lambda}, w\right) \leq \varepsilon$$

and then $m(\varepsilon) \in \mathbb{N}$ such that

$$0 \leq v^n(l_m, w) - v^n\left(\frac{1}{\lambda}, w\right) \leq \varepsilon$$

for all $m \geq m(\varepsilon)$. Since $v^n(l_m, w) \geq v(l_m, w)$, we have that

$$0 \leq v(l_m, w) - v\left(\frac{1}{\lambda}, w\right) \leq v^n(l_m, w) - v^n\left(\frac{1}{\lambda}, w\right) + v^n\left(\frac{1}{\lambda}, w\right) - v\left(\frac{1}{\lambda}, w\right) \leq 2\varepsilon$$

for all $m \geq m(\varepsilon)$, which proves the continuity at $l = \frac{1}{\lambda}$ for $w > 0$. For $w = 0$ and $N \in \mathbb{N}$, choose $n \in \mathbb{N}$ such that $v^n(\frac{1}{\lambda}, 0) \leq -N$ and then $m(N) \in \mathbb{N}$ such that $0 \leq v^n(l_m, w) - v^n(\frac{1}{\lambda}, w) \leq 1$ for all $m \geq m(N)$. Using the same arguments as above, we then obtain that $v(l_m, w) \leq -N + 1$ for all $m \geq m(N)$, which implies that

$$\lim_{m \rightarrow \infty} v(l_m, 0) = -\infty$$

and therefore the continuity of $v(\cdot, 0)$ at $l = \frac{1}{\lambda}$.

For the proof of the continuity of $v(l, w)$ in w , we observe that $v(l, w)$ is continuous in l for each fixed $w \in [0, \infty)$ and nondecreasing and hence Borel-measurable in w for each fixed $l \in [0, \frac{1}{\lambda}]$. Therefore, $v(l, w)$ is a Carathéodory function (see [1, Definition 4.50]) and hence jointly Borel-measurable by [1, Lemma 4.51]. Combining the first part of the proof of [12, Theorem 3.5] with [12, Remark 5.2] this implies that

$$\begin{aligned} v(l, w) \leq & \sup_{(\varphi^0, \varphi^1) \in \mathcal{A}_0^{\lambda, w}(1-l, l)} \mathbf{E} \left[\log \left(\varphi_{\tau^w \wedge \sigma}^0 + \varphi_{\tau^w \wedge \sigma}^1 S_{\tau^w \wedge \sigma}^w \right) \right. \\ & \left. + v \left(\frac{\varphi_{\tau^w \wedge \sigma}^1 S_{\tau^w \wedge \sigma}^w}{\varphi_{\tau^w \wedge \sigma}^0 + \varphi_{\tau^w \wedge \sigma}^1 S_{\tau^w \wedge \sigma}^w}, W_{\tau^w \wedge \sigma}^w \right) \right] \end{aligned} \quad (4.3.16)$$

for all stopping times σ , where we use the joint measurability of $v(l, w)$ to replace the upper semicontinuous envelope of the value function V^* by the value function V itself (both in the notation of [12]).

For $0 \leq w_1 < w_2$, we then have by (4.3.16) that

$$\begin{aligned} 0 &\leq v(l, w_2) - v(l, w_1) \\ &\leq \sup_{(\varphi^0, \varphi^1) \in \mathcal{A}_0^{\lambda, w_2}(1-l, l)} \mathbf{E} \left[\log(\varphi_\sigma^0 + \varphi_\sigma^1 e^\sigma) + v(L_\sigma(\varphi), w_1) \right] - v(l, w_1) \\ &\leq \mathbf{E} \left[\frac{\sigma}{\lambda} + v\left(\frac{le^\sigma}{1+l(e^\sigma-1)}, w_1\right) \right] - v(l, w_1) \end{aligned}$$

with $\sigma := \inf\{t > 0 \mid W_t^{w_2} = w_1\}$, where we used that $L(\widehat{\varphi}(l, w_2)) \leq \frac{1}{\lambda}$ and $v(l, w)$ is nonincreasing in l . As σ has an inverse Gaussian distribution with mean $\mathbf{E}[\sigma] = (w_2 - w_1)$ and variance $\text{Var}[\sigma] = (w_2 - w_1)^2$, we can make $v(l, w_2) - v(l, w_1)$ arbitrary small by choosing w_2 sufficiently close to w_1 using the continuity of $v(\cdot, w_1)$, which proves the continuity of $v(l, w)$ in w from above.

To prove the continuity of $v(l, w)$ in w from below, consider the stopping time $\rho := \inf\{t > 0 \mid W_t^{w_1} = w_2\}$. Then

$$\begin{aligned} 0 &\leq v(l, w_2) - v(l, w_1) \\ &\leq v(l, w_2) - \mathbf{E} \left[\left\{ \log(1 + l(e^\rho - 1)) + v\left(\frac{le^\rho}{1+l(e^\rho-1)}, w_2\right) \right\} \mathbf{1}_{\{\rho \leq \varepsilon\}} \right. \\ &\quad \left. + \left\{ \log\left(\frac{1}{\lambda} - 1\right) + \log(\tau \wedge 1) \right\} \mathbf{1}_{\{\rho > \varepsilon\}} \right] \end{aligned} \quad (4.3.17)$$

for all $\varepsilon > 0$ again by (4.3.16), as

$$\log(V_\tau^{\text{liq}}(\varphi^0, \varphi^1)) \geq \log\left(\frac{1}{\lambda} - 1\right) + \log(\tau \wedge 1)$$

for $(\varphi^0, \varphi^1) \equiv (1 - \frac{1}{\lambda}, \frac{1}{\lambda})$. Now, since

$$\mathbf{P}[\rho > \varepsilon] \leq \mathbf{P} \left[\sup_{0 \leq u \leq \varepsilon} B_u < w_2 - w_1 + \varepsilon \right] = \mathbf{P} \left[|Z| < \frac{w_2 - w_1 + \varepsilon}{\sqrt{\varepsilon}} \right]$$

by the reflection principle for some normally distributed random variable $Z \sim N(0, 1)$, we can make the right-hand side of (4.3.17) arbitrarily small by choosing $\varepsilon = w_2 - w_1$ and w_1 sufficiently close to w_2 using the continuity of $v(\cdot, w_2)$.

Having the continuity of $v(l, w)$ in l and w separately, the joint continuity follows from the fact that $v(l, w)$ is nonincreasing in l for fixed w and nondecreasing in w for fixed l . Indeed, fix $(l, w) \in (0, \frac{1}{\lambda}) \times [0, \infty)$ and $\varepsilon > 0$ and let $0 \leq l_1 < l < l_2 \leq \frac{1}{\lambda}$ be such that

$$|v(l', w) - v(l, w)| < \varepsilon$$

for all $l' \in [l_1, l_2]$. Now choose $w_1 \leq w$ and $w_2 > w$ such that

$$0 \leq v(l_2, w) - v(l_2, w_1) < \varepsilon \quad \text{and} \quad 0 \leq v(l_1, w_2) - v(l_1, w) < \varepsilon.$$

Then

$$v(l', w') - v(l, w) \leq v(l_1, w_2) - v(l, w) < 2\varepsilon$$

and

$$v(l, w) - v(l', w') \leq v(l, w) - v(l_2, w_1) < 2\varepsilon$$

for all $(l', w') \in [l_1, l_2] \times [w_1, w_2]$, which gives the joint continuity. If $l = 0$, the joint continuity follows by simply choosing $l_1 = 0$ in the above and, if $l = \frac{1}{\lambda}$ and $w > 0$,

by setting $l_2 = \frac{1}{\lambda}$. To prove the joint continuity for $(l, w) = (\frac{1}{\lambda}, 0)$, observe that there exists for any $N \in \mathbb{N}$ some $w_1 > 0$ such that $v(\frac{1}{\lambda}, w_1) \leq -N$ and $l_1 < \frac{1}{\lambda}$ such that $v(l_1, w_1) - v(\frac{1}{\lambda}, w_1) \leq 1$. Then $v(l', w') \leq -N + 1$ for all $(l', w') \in [l_1, \frac{1}{\lambda}] \times [0, w_1]$ and hence $v(l, w)$ is also jointly continuous at $(l, w) = (\frac{1}{\lambda}, 0)$.

(4) As the value function $v(l, w)$ is jointly continuous, it coincides with its lower semicontinuous and upper semicontinuous envelope. Therefore, the dynamic programming principle follows from the weak dynamic programming principle in [12, Theorem 3.5] using [12, Remark 5.2] and observing that the set of controls does not depend on the current time.

(5) Because $v(l, w)$ is continuous and nonincreasing in l , the set $\{k \in [0, \frac{1}{\lambda}] \mid v(k, w) = v(0, w)\}$ is a compact interval and so we can define $\ell(w)$ for all $w \geq 0$ via (4.3.12).

By the joint continuity of $v(l, w)$, we obtain that the function $\ell : [0, \infty) \rightarrow [0, \frac{1}{\lambda}]$ is upper semicontinuous and hence càdlàg, as it is also nondecreasing.

Indeed, suppose by way of contradiction that there exists a sequence (w_n) in $[0, \frac{1}{\lambda}]$ such that $w_n \rightarrow w$ and $\lim_{n \rightarrow \infty} \ell(w_n) =: k > \ell(w)$ along a subsequence again indexed by n . Then

$$\lim_{n \rightarrow \infty} v(\ell(w_n), w_n) = v(k, w) < v(\ell(w), w)$$

by the joint continuity of v and the definition of $\ell(w)$. But this yields a contradiction, as we also have

$$\lim_{n \rightarrow \infty} v(\ell(w_n), w_n) = \lim_{n \rightarrow \infty} v(0, w_n) = v(0, w) = v(\ell(w), w)$$

again using the definition of $\ell(w)$ and the joint continuity of v .

To see that $\ell(w)$ is also nondecreasing, denote the optimal strategy to problem (3.3.11) for $(\varphi_0^0, \varphi_0^1) = (1 - l, l)$ and $W_0 = w$ by $\widehat{\varphi}(l, w) = (\widehat{\varphi}^0(l, w), \widehat{\varphi}^1(l, w))$ and consider $0 \leq w_1 < w_2$. Then $\widehat{\varphi}(\ell(w_2), w_2)$ satisfies $L_t(\widehat{\varphi}(\ell(w_2), w_2)) \geq \ell(w_1)$ for all $t \leq \sigma := \inf\{t > 0 \mid W_t^{w_2} = w_1\}$, as we could otherwise construct a better strategy for the investor trading at S^{w_2} and starting with $(\varphi_0^0, \varphi_0^1) = (1 - \ell(w_2), \ell(w_2))$. For this, we observe that

$$dL_t(\varphi) = L_t(\varphi)(1 - L_t(\varphi))\mathbf{1}_{[0, \tau]}dt + \frac{L_t(\varphi)}{\varphi_t^1}d\varphi_t^{1, \uparrow} - \frac{L_t(\varphi)(1 - \lambda L_t(\varphi))}{\varphi_t^1}d\varphi_t^{1, \downarrow},$$

which implies that we can always trade in such a way to keep the leverage $L_t(\varphi) \equiv \ell(w_1)$. For $\ell(w_1) > 1$, we buy stocks at the rate

$$d\varphi_t^{1, \uparrow} = -\varphi_t^1(1 - L_t(\varphi))\mathbf{1}_{[0, \tau]}dt$$

and for $\ell(w_1) < 1$ we sell at

$$-d\varphi_t^{1, \downarrow} = -\varphi_t^1 \frac{(1 - L_t(\varphi))}{(1 - \lambda L_t(\varphi))} \mathbf{1}_{[0, \tau]}dt.$$

This gives

$$d \log(\varphi_t^0 + \varphi_t^1 S_t) = \ell(w_1) \mathbf{1}_{[0, \tau]} dt$$

and

$$d \log(\varphi_t^0 + \varphi_t^1 S_t) = \ell(w_1) \frac{1 - \lambda}{1 - \lambda \ell(w_1)} \mathbf{1}_{[0, \tau]} dt,$$

respectively. As $\frac{1-\lambda}{1-\lambda\ell(w_1)} > 1$ for $\ell(w_1) < 1$, we obtain by part (4) that the strategy $\varphi = (\varphi^0, \varphi^1) \in \mathcal{A}_0^{\lambda, w_2}(1 - \ell(w_2), \ell(w_2))$ that keeps $L_t(\varphi) = L_t(\widehat{\varphi}(\ell(w_2), w_2)) \vee \ell(w_1)$ for all $t \leq \sigma$ and then continues with $\widehat{\varphi}(\ell(w_1), w_1)$, if $L_t(\widehat{\varphi}(\ell(w_2), w_2)) \leq \ell(w_1)$, or $\widehat{\varphi}(\ell(w_2), w_2)$, if $L_t(\widehat{\varphi}(\ell(w_2), w_2)) > \ell(w_1)$, yields a higher expected utility, i.e.,

$$\mathbf{E}[\log(V_{\tau w_2}^{\text{liq}}(\varphi^0, \varphi^1))] > \mathbf{E}[\log(V_{\tau w_2}^{\text{liq}}(\widehat{\varphi}^0(\ell(w_2), w_2), \widehat{\varphi}^1(\ell(w_2), w_2)))].$$

As $v(l, w) = v(\ell(w), w)$ for $l \in [0, \ell(w)]$ and $v(l, w) < v(\ell(w), w)$ for $l \in (\ell(w), \frac{1}{\lambda}]$, it follows from the concavity of $v(l, w)$ in l that $v(l, w)$ is strictly decreasing in l on $(\ell(w), \frac{1}{\lambda}]$. This implies that

$$g(l_1, w) := V_{\tau}^{\text{liq}}(\widehat{\varphi}^0(l_1, w), \widehat{\varphi}^1(l_1, w)) \neq V_{\tau}^{\text{liq}}(\widehat{\varphi}^0(l_2, w), \widehat{\varphi}^1(l_2, w)) =: g(l_2, w)$$

for $\ell(w) < l_1 < l_2 \leq \frac{1}{\lambda}$ and hence the strict concavity of $v(l, w)$ in l on $(\ell(w), \frac{1}{\lambda}]$, as

$$\begin{aligned} \mu v(l_1, w) + (1 - \mu)v(l_2, w) &= \mu \mathbf{E}[\log(g(l_1, w))] + (1 - \mu) \mathbf{E}[\log(g(l_2, w))] \\ &< \mathbf{E}[\log(\mu g(l_1, w) + (1 - \mu)g(l_2, w))] \\ &\leq v(\mu l_1 + (1 - \mu)l_2, w) \end{aligned}$$

for all $\mu \in (0, 1)$ by Jensen's inequality. \square

Lemma 4.3.8. *Let $\ell : [0, \infty) \rightarrow [0, \frac{1}{\lambda}]$ be an increasing function (no left- or right-continuity is assumed). Recall that the optimizer $\widehat{\varphi} = (\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)_{t \geq 0}$ is right-continuous and that we have to distinguish between $\widehat{\varphi}_{0-}$ and $\widehat{\varphi}_0$.*

If

$$\mathbf{P}\left[\inf_{0 \leq t < \tau} (L_t(\widehat{\varphi}) - \ell(W_t)) < 0\right] > 0, \quad (4.3.18)$$

then there are stopping times $0 \leq \sigma_1 \leq \sigma_2$ and $\alpha > 0$, such that $\mathbf{P}[\sigma_1 < \sigma_2 \leq \tau] > 0$ and $L_t(\widehat{\varphi}) < \ell(W_t) - \alpha$ on $]\sigma_1, \sigma_2]$.

Proof. Assuming (4.3.18), there is $\varepsilon > 0$ such that $\sigma := \inf\{t > 0 \mid L_t(\widehat{\varphi}) < \ell(W_t) - \varepsilon\}$ satisfies $\mathbf{P}[\sigma < \tau] > 0$. To see that σ is a stopping time, we observe that it is the first hitting time of the progressively measurable set $\{(\omega, t) \mid L_t(\widehat{\varphi})(\omega) < \ell(W_t(\omega)) - \varepsilon\}$. By the càdlàg property of $\widehat{\varphi}$ we have

$$L_{\sigma}(\widehat{\varphi}) \leq \lim_{w \searrow W_{\sigma}} \ell(w) - \varepsilon$$

on $\{\sigma < \tau\}$. Now we distinguish two cases.

Case 1: Let $A := \{\omega \mid \ell \text{ has a continuity point at } W_{\sigma}\}$ and

$$\mathbf{P}[A, \sigma < \tau] > 0. \quad (4.3.19)$$

Define $\sigma_1 := \sigma \mathbf{1}_A + \infty \mathbf{1}_{A^c}$ and the Borel-measurable function $\delta(w)$ by

$$\delta(w) := \frac{\sup\{|w' - w| \mid \ell(w') \geq \ell(w) - \frac{\varepsilon}{3}\}}{2}$$

so that $\delta(W_\sigma) > 0$ on $A \cap \{\sigma < \tau\}$ and $\ell(w') > \ell(w) - \frac{\varepsilon}{3}$ for every $w' \geq w - \delta(w)$. As regards the process $L_t(\widehat{\varphi})$ let

$$\varrho := \inf \{t > \sigma \mid L_t(\widehat{\varphi}) > L_\sigma(\widehat{\varphi}) + \frac{\varepsilon}{3}\}.$$

We cannot deduce that $L_\varrho(\widehat{\varphi}) \leq L_\sigma(\widehat{\varphi}) + \frac{\varepsilon}{3}$, as $L_t(\widehat{\varphi})$ may have an upward jump at time ϱ . To remedy this difficulty, we may use the fact that the stopping time ϱ is *predictable*, as every stopping time in a Brownian filtration is predictable (see e.g. [84, Example 4.12]). We therefore may find an increasing sequence $(\varrho_n)_{n=1}^\infty$ of announcing stopping times, i.e., $\varrho_n < \varrho$ and $\lim_{n \rightarrow \infty} \varrho_n = \varrho$, almost surely. As $\varrho > \sigma_1$ on A we may find n such that $\mathbf{P}[\{\varrho_n > \sigma_1\} \cap A] > 0$. For this n , we may define

$$\sigma_2 := \inf\{t > \sigma_1 \mid W_t \leq W_\sigma - \delta(W_\sigma)\} \wedge \varrho_n \wedge \tau$$

on $A \cap \{\varrho_n > \sigma_1\}$ and $+\infty$ elsewhere. Then $\sigma_1 < \sigma_2$ on A and σ_1, σ_2 and $\alpha = \frac{\varepsilon}{3}$ satisfy the assertion of the lemma.

Case 2: If (4.3.19) fails, there must be one point $\tilde{w} \in (0, \infty)$ with $\lim_{w \nearrow \tilde{w}} \ell(w) < \lim_{w \searrow \tilde{w}} \ell(w)$ such that $\mathbf{P}[W_\sigma = \tilde{w}] > 0$. For each real number $w > \tilde{w}$, we define the stopping time σ^w by

$$\sigma^w := \inf\{t > \sigma \mid W_t = w\}.$$

We may find $w > \tilde{w}$ which is a continuity point of ℓ and sufficiently close to \tilde{w} such that $\mathbf{P}[\sigma^w < \tau] > 0$. We may then proceed as in Case 1 by letting $\sigma_1 := \sigma^w$, which completes the proof. \square

Proposition 4.3.9. *The optimal strategy $\widehat{\varphi} = (\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)_{t \geq 0}$ is determined by the nondecreasing function $\ell : [0, \infty) \rightarrow [0, \frac{1}{\lambda}]$ in (4.3.12) in the following way:*

(i) $(\widehat{\varphi}_t^1)_{0 \leq t < \tau}$ is nondecreasing while $(\widehat{\varphi}_t^0)_{0 \leq t < \tau}$ is nonincreasing and satisfies

$$d\widehat{\varphi}_t^0 = -S_t d\widehat{\varphi}_t^1 = -e^t d\widehat{\varphi}_t^1, \quad 0 \leq t < \tau.$$

(ii) $(\widehat{\varphi}_t^1)_{0 \leq t < \tau}$ is the smallest nondecreasing process such that

$$L_t(\widehat{\varphi}) = \frac{\widehat{\varphi}_t^1 e^t}{1 + \int_0^t \widehat{\varphi}_u^1 e^u du} \geq \ell(W_t), \quad 0 \leq t < \tau. \quad (4.3.20)$$

Proof. (i) This follows immediately from the following fact: As S is strictly increasing on $\llbracket 0, \tau \rrbracket$, any strategy selling stock shares before time τ sells them at a lower price and hence has a smaller liquidation value at time τ as the strategy not selling stock shares before time τ .

Here is the formal argument. Let $(\varphi^0, \varphi^1) \in \mathcal{A}_0^\lambda(\varphi_0^0, \varphi_0^1)$ and $\varphi^1 = \varphi_0^1 + \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$ the Jordan-Hahn decomposition of φ^1 into two nondecreasing processes $\varphi^{1,\uparrow}$ and $\varphi^{1,\downarrow}$ starting at 0. Define a strategy $(\tilde{\varphi}^0, \tilde{\varphi}^1) \in \mathcal{A}_0^\lambda(\varphi_0^0, \varphi_0^1)$ by

$$\tilde{\varphi}^1 = \varphi_0^1 + \varphi^{1,\uparrow} \quad \text{and} \quad \tilde{\varphi}^0 = \varphi_0^0 - \int S_u d\varphi_u^{1,\uparrow}.$$

Then,

$$\begin{aligned}
V_\tau^{\text{liq}}(\varphi^0, \varphi^1) &= \varphi_0^0 + \int_0^\tau (1-\lambda) S_u d\varphi_u^{1,\downarrow} - \int_0^\tau S_u d\varphi_u^{1,\uparrow} + (\varphi_\tau^1)^+(1-\lambda)S_\tau - (\varphi_\tau^1)^- S_\tau \\
&\leq \varphi_0^0 - \int_0^\tau S_u d\varphi_u^{1,\uparrow} + (\varphi_0^1 + \varphi_\tau^{1,\uparrow})^+(1-\lambda)S_\tau - (\varphi_0^1 + \varphi_\tau^{1,\uparrow})^- S_\tau \\
&= V_\tau^{\text{liq}}(\tilde{\varphi}^0, \tilde{\varphi}^1),
\end{aligned}$$

since $\varphi_\tau^1 \leq \tilde{\varphi}_\tau^1 = \varphi_\tau^1 + \varphi_\tau^{1,\downarrow}$ and S is nondecreasing and therefore

$$\int_0^\tau (1-\lambda) S_u d\varphi_u^{1,\downarrow} + (\varphi_\tau^1)^+(1-\lambda)S_\tau - (\varphi_\tau^1)^- S_\tau \leq (\tilde{\varphi}_\tau^1)^+(1-\lambda)S_\tau$$

for $\tilde{\varphi}_\tau^1 \geq 0$ and

$$\int_0^\tau (1-\lambda) S_u d\varphi_u^{1,\downarrow} - (\varphi_\tau^1)^- S_\tau \leq -(\tilde{\varphi}_\tau^1)^- S_\tau$$

for $\tilde{\varphi}_\tau^1 < 0$.

(ii) That $(\tilde{\varphi}_t^1)_{0 \leq t < \tau}$ is a nondecreasing process such that $L_t(\tilde{\varphi}) \geq \ell(W_t)$ for $0 \leq t < \tau$ follows immediately from part (i) above and by combining Lemmas 4.3.7 and 4.3.8. Indeed, suppose that

$$\mathbf{P} \left[\inf_{0 \leq t < \tau} (L_t(\tilde{\varphi}) - \ell(W_t)) < 0 \right] > 0.$$

Then there exist two stopping times σ_1 and σ_2 and $\alpha > 0$ such that $\mathbf{P}[\sigma_1 < \sigma_2 \leq \tau] > 0$ and $L_t(\tilde{\varphi}) < \ell(W_t) - \alpha$ on $]\sigma_1, \sigma_2]$ by Lemma 4.3.8. Therefore, we can define a strategy $\tilde{\varphi}$ such that $\tilde{\varphi} = \tilde{\varphi}$ on $[[0, \sigma_1]]$ and $L_t(\tilde{\varphi}) = L_t(\tilde{\varphi}) + \alpha$ on $[[\sigma_1, \sigma_2]]$. Then,

$$\begin{aligned}
&\mathbf{E} \left[\log(\tilde{\varphi}_{\sigma_2}^0 + \tilde{\varphi}_{\sigma_2}^1 S_{\sigma_2}) + v(L_{\sigma_2}(\tilde{\varphi}), W_{\sigma_2}) \right] \\
&= \mathbf{E} \left[\int_0^{\sigma_2} L_t(\tilde{\varphi}) dt + \alpha(\sigma_2 - \sigma_1) + v(L_{\sigma_2}(\tilde{\varphi}), W_{\sigma_2}) \right] \\
&= v(l, w) + \alpha \mathbf{E}[\sigma_2 - \sigma_1] > v(l, w)
\end{aligned}$$

by part (4) of Lemma 4.3.7, since $L_{\sigma_2}(\tilde{\varphi}) \leq L_{\sigma_2}(\tilde{\varphi}) \leq \ell(W_{\sigma_2})$ and $v(\cdot, W_{\sigma_2})$ is constant on $[0, \ell(W_{\sigma_2})]$. But this contradicts the optimality of $\tilde{\varphi}$ by part (4) of Lemma 4.3.7.

To see that $\Delta L(\tilde{\varphi}) = \ell(W) - L_-(\tilde{\varphi})$, assume by way of contradiction that there exists a stopping time σ such that $\mathbf{P}[A] > 0$ for $A := \{\Delta L_{\sigma \wedge \tau}(\tilde{\varphi}) > \ell(W_{\sigma \wedge \tau}) - L_{\sigma \wedge \tau-}(\tilde{\varphi}) \geq 0\}$. Then we have

$$\begin{aligned}
v(L_{\sigma \wedge \tau}(\tilde{\varphi}), W_{\sigma \wedge \tau}) &= v(L_{\sigma \wedge \tau-}(\tilde{\varphi}) + \Delta L_{\sigma \wedge \tau}(\tilde{\varphi}), W_{\sigma \wedge \tau}) \\
&< v(L_{\sigma \wedge \tau-}(\tilde{\varphi}) + (\ell(W_{\sigma \wedge \tau}) - L_{\sigma \wedge \tau-}(\tilde{\varphi})), W_{\sigma \wedge \tau}) \\
&= v(\ell(W_{\sigma \wedge \tau}), W_{\sigma \wedge \tau})
\end{aligned}$$

on A , as $v(l, w)$ is strictly decreasing on $(\ell(w), \frac{1}{\lambda}]$. But this contradicts the optimality of $\tilde{\varphi}$ by part 4) of Lemma 4.3.7. Indeed, the strategy $(\tilde{\varphi}^0, \tilde{\varphi}^1) \in \mathcal{A}_0^\lambda(\varphi_0^0, \varphi_0^1)$ given by

$$d\tilde{\varphi}_t^1 = \mathbf{1}_{[0, \sigma \wedge \tau]} d\tilde{\varphi}_t^1 + \mathbf{1}_{[\sigma \wedge \tau]} \left(\frac{\ell(W_{\sigma \wedge \tau})(\tilde{\varphi}_{\sigma \wedge \tau-}^0 + \tilde{\varphi}_{\sigma \wedge \tau-}^1 e^{\sigma \wedge \tau})}{e^{\sigma \wedge \tau}} - \tilde{\varphi}_{\sigma \wedge \tau-}^1 \right)$$

and $d\tilde{\varphi}^0 = -Sd\tilde{\varphi}^1$ satisfies $L_t(\tilde{\varphi}) = L_t(\hat{\varphi})$ on $\llbracket 0, \sigma \wedge \tau \llbracket$ and $L_{\sigma \wedge \tau}(\tilde{\varphi}) = \ell(W_{\sigma \wedge \tau})$ and therefore yields

$$\begin{aligned} v(l, w) &= \mathbf{E} \left[\log \left(\tilde{\varphi}_{\sigma \wedge \tau}^0 + \tilde{\varphi}_{\sigma \wedge \tau}^1 S_{\sigma \wedge \tau} \right) + v \left(L_{\sigma \wedge \tau}(\hat{\varphi}), W_{\sigma \wedge \tau} \right) \right] \\ &< \mathbf{E} \left[\log \left(\tilde{\varphi}_{\sigma \wedge \tau}^0 + \tilde{\varphi}_{\sigma \wedge \tau}^1 S_{\sigma \wedge \tau} \right) + v \left(L_{\sigma \wedge \tau}(\tilde{\varphi}), W_{\sigma \wedge \tau} \right) \right], \end{aligned}$$

where we used that $\tilde{\varphi}_{\sigma \wedge \tau}^0 + \tilde{\varphi}_{\sigma \wedge \tau}^1 S_{\sigma \wedge \tau} = \hat{\varphi}_{\sigma \wedge \tau}^0 + \hat{\varphi}_{\sigma \wedge \tau}^1 S_{\sigma \wedge \tau}$. Since $L_t(\hat{\varphi}) \geq \ell(W_t)$ for all $0 \leq t < \tau$, this proves $\Delta L(\hat{\varphi}) = \ell(W) - L_-(\hat{\varphi})$.

Let $\hat{\varphi} \in \mathcal{A}_0^\lambda(\varphi_0^0, \varphi_0^1)$ be the solution and $\tilde{\varphi} \in \mathcal{A}_0^\lambda(\varphi_0^0, \varphi_0^1)$ be the strategy such that $(\tilde{\varphi}_t^1)_{0 \leq t < \tau}$ is the smallest nondecreasing process with $L_t(\tilde{\varphi}) \geq \ell(W_t)$ for all $0 \leq t < \tau$. Define a nonnegative predictable process $(\tilde{\psi}_t)_{0 \leq t < \tau}$ of finite variation by $\tilde{\psi}_t := L_t(\hat{\varphi}) - L_t(\tilde{\varphi})$ and suppose by way of contradiction that

$$\mathbf{P} \left[\sup_{0 \leq t < \tau} \tilde{\psi}_t > \varepsilon \right] > 0 \quad (4.3.21)$$

for some $\varepsilon > 0$ or, equivalently, that $\mathbf{P}[\tau_\varepsilon < \tau] > 0$ for the stopping time

$$\tau_\varepsilon := \inf \{ t > 0 \mid \tilde{\psi}_t > \varepsilon \} \wedge \tau.$$

Next observe that

$$\Delta L_t(\tilde{\varphi}) \geq \ell(W_t) - L_{t-}(\hat{\varphi}) = \Delta L_t(\hat{\varphi})$$

for all $0 \leq t < \tau$, since $L_t(\hat{\varphi}) \geq L_t(\tilde{\varphi}) \geq \ell(W_t)$ for all $0 \leq t < \tau$ and $L(\tilde{\varphi})$ and $L(\hat{\varphi})$ also only jump upwards. This implies that $\tilde{\psi}_t^\uparrow$ is continuous, where $\tilde{\psi}_t = \tilde{\psi}_t^\uparrow - \tilde{\psi}_t^\downarrow$ denotes the Jordan-Hahn decomposition of $\tilde{\psi}_t$, and therefore that $L_{\tau_\varepsilon}(\hat{\varphi}) = L_{\tau_\varepsilon}(\tilde{\varphi}) + \varepsilon$.

Now consider the trading strategy $\varphi \in \mathcal{A}_0^\lambda(\varphi_0^0, \varphi_0^1)$ such that $\varphi^1 = \hat{\varphi}^1$ on $\llbracket 0, \tau_\varepsilon \llbracket$ and buys the minimal amount to keep $L_t(\varphi) \geq \ell(W_t)$ on $\llbracket \tau_\varepsilon, \tau \llbracket$ and $d\varphi^0 = Sd\varphi^1$. Define, similarly as above, a nonnegative predictable process $(\psi_t)_{0 \leq t < \tau}$ of finite variation by $\psi_t := L_t(\hat{\varphi}) - L_t(\varphi)$ and the stopping times

$$\tau_{\varepsilon, h} := \inf \{ t > \tau_\varepsilon \mid \psi_t > h \} \wedge \tau, \quad h > 0,$$

that satisfy $L_{\tau_{\varepsilon, h}}(\hat{\varphi}) = L_{\tau_{\varepsilon, h}}(\varphi) + h$ on $\{\tau_{\varepsilon, h} < \tau\}$ and $\tau_{\varepsilon, h} \searrow \tau_\varepsilon$ for $h \searrow 0$ on $\{\tau_\varepsilon < \tau\}$, since ψ^\uparrow is again continuous. Then we have by the optimality of $\hat{\varphi}$ and by the part (4) of Lemma 4.3.7 that

$$\frac{\mathbf{E} \left[\int_{\tau_\varepsilon}^{\tau_{\varepsilon, h}} (L_s(\hat{\varphi}) - L_s(\varphi)) ds + v(L_{\tau_{\varepsilon, h}}(\hat{\varphi}), W_{\tau_{\varepsilon, h}}) - v(L_{\tau_{\varepsilon, h}}(\varphi), W_{\tau_{\varepsilon, h}}) \middle| \mathcal{F}_{\tau_\varepsilon} \right]}{h} \geq 0 \quad (4.3.22)$$

on $\{\tau_\varepsilon < \tau\}$ for all $h > 0$. On the other side, we have

$$\lim_{h \searrow 0} \frac{\mathbf{E} \left[\int_{\tau_\varepsilon}^{\tau_{\varepsilon, h}} (L_s(\hat{\varphi}) - L_s(\varphi)) ds \middle| \mathcal{F}_{\tau_\varepsilon} \right]}{h} \leq \lim_{h \searrow 0} \mathbf{E}[(\tau_{\varepsilon, h} - \tau_\varepsilon) \middle| \mathcal{F}_{\tau_\varepsilon}] = 0$$

on $\{\tau_\varepsilon < \tau\}$ by Lebesgue's dominated convergence theorem and

$$\frac{\mathbf{E} \left[v(L_{\tau_{\varepsilon, h}}(\hat{\varphi}), W_{\tau_{\varepsilon, h}}) - v(L_{\tau_{\varepsilon, h}}(\varphi), W_{\tau_{\varepsilon, h}}) \middle| \mathcal{F}_{\tau_\varepsilon} \right]}{h} \leq \mathbf{E} \left[v'_-(L_{\tau_{\varepsilon, h}}(\varphi), W_{\tau_{\varepsilon, h}}) \middle| \mathcal{F}_{\tau_\varepsilon} \right]$$

on $\{\tau_\varepsilon < \tau\}$, since $L_{\tau_{\varepsilon,h}}(\widehat{\varphi}) - L_{\tau_{\varepsilon,h}}(\varphi) = h$ on $\{\tau_{\varepsilon,h} < \tau\}$. As

$$v'_-(l, w) := \inf_{h>0} \frac{v(l, w) - v(l - h, w)}{h}$$

is as the infimum of continuous functions upper semicontinuous and

$$L_{\tau_\varepsilon}(\varphi) = L_{\tau_\varepsilon}(\widetilde{\varphi}) + \varepsilon \geq \ell(W_{\tau_\varepsilon}) + \varepsilon$$

on $\{\tau_\varepsilon < \tau\}$, we obtain by Fatou's lemma that

$$\lim_{h \searrow 0} \mathbf{E} \left[v'_-(L_{\tau_{\varepsilon,h}}(\varphi), W_{\tau_{\varepsilon,h}}) \middle| \mathcal{F}_{\tau_\varepsilon} \right] \leq v'_-(L_{\tau_\varepsilon}(\varphi), W_{\tau_\varepsilon}) \leq v'_-(\ell(W_{\tau_\varepsilon}) + \varepsilon, W_{\tau_\varepsilon}) < 0$$

on $\{\tau_\varepsilon < \tau\}$, which is a contradiction to (4.3.22) and hence (4.3.21). \square

The following result is the crucial property of the function ℓ .

Lemma 4.3.10. *There is \bar{w} such that $\ell(w) = \frac{1}{\lambda}$ for all $w \geq \bar{w}$.*

Proof. Suppose to the contrary that $\ell(w) < \frac{1}{\lambda}$ for all $w \geq 0$. It is straightforward to check that $\lim_{w \rightarrow \infty} \ell(w) = \frac{1}{\lambda}$.

The basic idea is now to construct a strategy $\bar{\varphi}$ that yields, for sufficiently large $W_0 = w$, a higher expected utility than the optimal strategy $\widehat{\varphi}$ and hence a contradiction proving the lemma.

For this, we define the strategy $\bar{\varphi}$ in the following way: We start with $(\bar{\varphi}_0^0, \bar{\varphi}_0^1) = (1 - \frac{1}{\lambda}, \frac{1}{\lambda})$, i.e., with maximal leverage $L_0(\bar{\varphi}) = \frac{1}{\lambda}$, continue to leave $(\bar{\varphi}_t^0, \bar{\varphi}_t^1)$ constant until the stopping time

$$\varrho := \inf\{t > 0 \mid L_t(\bar{\varphi}) = L_t(\widehat{\varphi})\}$$

and trade such that $L_t(\bar{\varphi}) = L_t(\widehat{\varphi})$ after time ϱ . Note that the strategy $\widehat{\varphi}$ only trades at time $t < \tau$, if $L_t(\widehat{\varphi}) = \ell(W_t)$, by part (ii) of Proposition 4.3.9 and $L_{t_1}(\bar{\varphi}) > L_{t_1}(\widehat{\varphi})$, if $L_{t_0}(\bar{\varphi}) > L_{t_0}(\widehat{\varphi})$ and $\widehat{\varphi}$ does not trade between t_1 and t_0 for $0 \leq t_0 \leq t_1 < \tau$, which follows by a direct computation. Combing both we obtain that $L_t(\bar{\varphi}) > L_t(\widehat{\varphi}) \geq \ell(W_t)$ for $0 \leq t < \varrho$ and $L_\varrho(\bar{\varphi}) = L_\varrho(\widehat{\varphi}) = \ell(W_\varrho)$. Using the decreasing function

$$f(t) := \frac{\frac{1}{\lambda} e^t}{1 - \frac{1}{\lambda} + \frac{1}{\lambda} e^t}$$

starting at $f(0) = \frac{1}{\lambda}$ and satisfying $f(t) = L_t(\bar{\varphi})$ for $0 \leq t \leq \varrho$ and the ‘‘obstacle function’’

$$b(t) := \ell^{-1}(f(t))$$

then allows us to rephrase the definition of ϱ as $\varrho = \inf\{t > 0 \mid W_t = b(t)\}$. Here $\ell^{-1}(\cdot)$ denotes the right-continuous generalized inverse.

As $b : (0, \infty) \rightarrow (0, \infty)$ is nonincreasing and satisfies $\lim_{t \searrow 0} b(t) = \infty$, we obtain a sequence $(a_n)_{n=1}^\infty$ of nonpositive numbers with $\sum_{n=1}^\infty a_n = \infty$ by setting $a_n := b(2^{-n}) - b(2^{-n+1})$. Hence we may find, for any $\varepsilon > 0$, a number n such that

$$\varepsilon a_n > 2^{-n/4}, \tag{4.3.23}$$

as $\varepsilon \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \varepsilon a_n \leq \sum_{n=1}^{\infty} 2^{-n/4} < \infty$ would lead to a contradiction otherwise. Now we estimate

$$\mathbf{P}[\varrho > 2^{-n+1} \mid \varrho > 2^{-n}]$$

with $W_0 = w_n = \frac{a_n}{2} + b(2^{-n+1})$ which becomes small, if $\frac{2^{-n/2}}{a_n}$ becomes small. By (4.3.23) we have

$$\frac{2^{-n/2}}{a_n} < \varepsilon 2^{-n/4},$$

so that by elementary estimates on the Gaussian distribution, we have that

$$\mathbf{P}[\varrho > 2^{-n+1} \mid \varrho > 2^{-n}] < \delta 2^{-2n}, \quad (4.3.24)$$

for a pregiven $\delta > 0$. To see this, observe that

$$\mathbf{P}[\varrho > 2^{-n+1} \mid \varrho > 2^{-n}] = \frac{\mathbf{P}[\varrho > 2^{-n+1}]}{\mathbf{P}[\varrho > 2^{-n}]} \leq \frac{\mathbf{P}[W_{2^{-n+1}} \leq b(2^{-n+1})]}{\mathbf{P}\left[\sup_{0 \leq u \leq 2^{-n}} W_u < b(2^{-n})\right]}, \quad (4.3.25)$$

where we can estimate the probabilities on the right-hand side separately.

As

$$\mathbf{P}\left[\sup_{0 \leq u \leq 2^{-n}} W_u < b(2^{-n})\right] \geq \mathbf{P}\left[\sup_{0 \leq u \leq 2^{-n}} B_u < b(2^{-n}) - w_n\right],$$

we obtain by the reflection principle that

$$\begin{aligned} \mathbf{P}\left[\sup_{0 \leq u \leq 2^{-n}} W_u < b(2^{-n})\right] &\geq 1 - \mathbf{P}\left[\sup_{0 \leq u \leq 2^{-n}} B_u \geq b(2^{-n}) - w_n\right] \\ &= 1 - 2\mathbf{P}\left[B_{2^{-n}} \geq \frac{a_n}{2}\right] = 1 - \mathbf{P}\left[|Z| \geq \frac{1}{2} \frac{a_n}{2^{-n/2}}\right] \end{aligned}$$

for a standard normal distributed random variable $Z \sim N(0, 1)$ and therefore

$$\mathbf{P}\left[\sup_{0 \leq u \leq 2^{-n}} W_u < b(2^{-n})\right] \geq 1 - \left(2 \frac{2^{-n/2}}{a_n}\right)^2 > 1 - (2\varepsilon 2^{-n/4})^2 \quad (4.3.26)$$

by applying Chebyscheff's inequality with $\mathbf{E}[Z^2] = 1$.

For $\varepsilon > 0$ sufficiently small such that $a_n^3 \varepsilon^4 \leq \frac{1}{8}$, we have

$$-\frac{a_n}{2} + 2(\varepsilon a_n)^4 \leq -\frac{a_n}{4}.$$

Hence for the second probability we obtain

$$\begin{aligned} \mathbf{P}[W_{2^{-n+1}} \leq b(2^{-n+1})] &= \mathbf{P}[B_{2^{-n+1}} \leq b(2^{-n+1}) - w_n + 2^{-n+1}] \\ &\leq \mathbf{P}[B_{2^{-n+1}} \leq b(2^{-n+1}) - w_n + 2(\varepsilon a_n)^4] \\ &= \mathbf{P}\left[\sqrt{2^{-n+1}} Z \leq -\frac{a_n}{2} + 2(\varepsilon a_n)^4\right] \\ &\leq \mathbf{P}\left[Z \leq -\frac{a_n}{4\sqrt{2}2^{-n/2}}\right] = \frac{1}{2} \mathbf{P}\left[|Z| \geq \frac{a_n}{4\sqrt{2}2^{-n/2}}\right] \end{aligned}$$

with a standard normal distributed random variable $Z \sim N(0, 1)$. Then, applying again Chebyscheff's inequality this time with $\mathbf{E}[Z^8] = 105$ gives

$$\begin{aligned} \mathbf{P} [W_{2^{-n+1}} \leq b(2^{-n+1})] &\leq \frac{1}{2} \cdot 105(4\sqrt{2})^8 \left(\frac{2^{-n/2}}{a_n}\right)^8 \\ &\leq \frac{1}{2} \cdot \left(105(4\sqrt{2})^8 \varepsilon^8\right) 2^{-2n} =: \frac{1}{2}\delta 2^{-2n}. \end{aligned} \quad (4.3.27)$$

Plugging (4.3.26) and (4.3.27) into (4.3.25) then yields (4.3.24) after choosing ε small enough such that

$$\mathbf{P} \left[\sup_{0 \leq u \leq 2^{-n}} W_u < b(2^{-n}) \right] \geq \frac{1}{2}.$$

On the set $\{\varrho < \infty\}$ we can estimate the positive effect of the strategy $\bar{\varphi}$ on the value function by

$$\begin{aligned} &\mathbf{E} \left[\left(\log (V_\tau^{\text{liq}}(\bar{\varphi}^0, \bar{\varphi}^1)) - \log (V_\tau^{\text{liq}}(\hat{\varphi}^0, \hat{\varphi}^1)) \right) \mathbf{1}_{\{\varrho < \infty\}} \right] \\ &\geq \mathbf{E} \left[\int_0^\varrho (L_s(\bar{\varphi}) - L_s(\hat{\varphi})) ds \mathbf{1}_{\{\varrho < \infty\}} \right] \\ &\geq \mathbf{E} \left[\int_0^{2^{-n}} (L_s(\bar{\varphi}) - L_s(\hat{\varphi})) ds \mathbf{1}_{\{2^{-n} < \varrho \leq 2^{-n+1}\}} \right]. \end{aligned}$$

Using that

$$\max_{0 \leq u \leq 2^{-n}} L_u(\hat{\varphi}) = \max_{0 \leq u \leq 2^{-n}} \ell(W_u) \leq \ell(b(2^{-n})) = f(2^{-n}) = L_{2^{-n}}(\bar{\varphi})$$

on $\{\sup_{0 \leq u \leq 2^{-n}} W_u < b(2^{-n})\}$ and that

$$\mathbf{P}[2^{-n} < \varrho \leq 2^{-n+1}] = \mathbf{P}[\varrho > 2^{-n}] \cdot (1 - \mathbf{P}[\varrho > 2^{-n+1} \mid \varrho > 2^{-n}]) \geq \frac{1}{2} \mathbf{P}[\varrho > 2^{-n}]$$

and

$$\mathbf{P} \left[\sup_{0 \leq u \leq 2^{-n}} W_u < b(2^{-n}) \mid 2^{-n} < \varrho \leq 2^{-n+1} \right] \geq \frac{1}{2}$$

by (4.3.26) for sufficiently large n , we get

$$\begin{aligned} &\mathbf{E} \left[\left(\log (V_\tau^{\text{liq}}(\bar{\varphi}^0, \bar{\varphi}^1)) - \log (V_\tau^{\text{liq}}(\hat{\varphi}^0, \hat{\varphi}^1)) \right) \mathbf{1}_{\{\varrho < \infty\}} \right] \\ &\geq \int_0^{2^{-n}} (f(s) - f(2^{-n})) ds \cdot \frac{1}{4} \mathbf{P}[\varrho > 2^{-n}]. \end{aligned}$$

As

$$f(s) - f(2^{-n}) \geq \min_{u \in [0, 2^{-n}]} (-f'(u))(2^{-n} - s)$$

for $s \in [0, 2^{-n}]$ and $f'(u) = f(u)(1 - f(u))$ satisfies

$$-f'(u) \geq \frac{1}{2} f(0)(f(0) - 1) = \frac{1}{2} \frac{1}{\lambda} \left(\frac{1}{\lambda} - 1\right)$$

for all $u \in [0, 2^{-n}]$ by continuity of f for sufficiently large n , we obtain that

$$\mathbf{E} \left[\left(\log (V_\tau^{\text{liq}}(\bar{\varphi}^0, \bar{\varphi}^1)) - \log (V_\tau^{\text{liq}}(\hat{\varphi}^0, \hat{\varphi}^1)) \right) \mathbf{1}_{\{\varrho < \infty\}} \right] \geq c_1 2^{-2n} \mathbf{P}[\varrho > 2^{-n}] \quad (4.3.28)$$

for sufficiently large n with $c_1 := \frac{1}{16} \frac{1}{\lambda} \left(\frac{1}{\lambda} - 1 \right) > 0$.

For the estimate of the negative effect of the strategy $\bar{\varphi}$ on the set $\{\varrho = \infty\}$, we observe that, if $V_\tau^{\text{liq}}(\hat{\varphi}^0, \hat{\varphi}^1) \geq 1$, then

$$1 \leq V_\tau^{\text{liq}}(\hat{\varphi}^0, \hat{\varphi}^1) \leq 1 + \hat{\varphi}_\tau^1 ((1 - \lambda) S_\tau - 1) \leq 1 + \bar{\varphi}_0^1 ((1 - \lambda) S_\tau - 1) = V_\tau^{\text{liq}}(\bar{\varphi}^0, \bar{\varphi}^1),$$

since $\hat{\varphi}_t^1 \leq \bar{\varphi}_0^1 = \frac{1}{\lambda}$ for all $0 \leq t < \varrho$, and therefore

$$\log (V_\tau^{\text{liq}}(\bar{\varphi}^0, \bar{\varphi}^1)) - \log (V_\tau^{\text{liq}}(\hat{\varphi}^0, \hat{\varphi}^1)) \geq 0$$

on $\{\varrho = \infty, V_\tau^{\text{liq}}(\hat{\varphi}^0, \hat{\varphi}^1) \geq 1\}$. Hence, it is sufficient to consider $\{\varrho = \infty, V_\tau^{\text{liq}}(\hat{\varphi}^0, \hat{\varphi}^1) < 1\}$, where we can estimate the negative effect of $\bar{\varphi}$ as follows:

$$\begin{aligned} & \log (V_\tau^{\text{liq}}(\bar{\varphi}^0, \bar{\varphi}^1)) - \log (V_\tau^{\text{liq}}(\hat{\varphi}^0, \hat{\varphi}^1)) \\ & \geq \log (V_\tau^{\text{liq}}(\bar{\varphi}^0, \bar{\varphi}^1)) = \log \left(\left(\frac{1}{\lambda} - 1 \right) (e^\tau - 1) \right) \\ & \geq \log \left(\left(\frac{1}{\lambda} - 1 \right) \tau \right) \geq \log \left(\left(\frac{1}{\lambda} - 1 \right) \sigma \wedge 1 \right), \end{aligned}$$

where $\sigma := \inf\{t > 0 \mid W_t^1 \leq 0\} \leq \tau$ for $W_0^1 = 1$. As

$$\begin{aligned} 0 & \geq \mathbf{E} \left[\log \left(\left(\frac{1}{\lambda} - 1 \right) \sigma \wedge 1 \right) \right] \\ & = \int_0^{\frac{\lambda}{1-\lambda}} \log \left(\frac{1-\lambda}{\lambda} z \right) \left(\frac{1}{2\pi z^3} \right)^{\frac{1}{2}} \exp \left(-\frac{(z-1)^2}{2z} \right) dz \\ & =: -c_2 > -\infty, \end{aligned}$$

we obtain for the negative effect that

$$\mathbf{E} \left[\left(\log (V_\tau^{\text{liq}}(\bar{\varphi}^0, \bar{\varphi}^1)) - \log (V_\tau^{\text{liq}}(\hat{\varphi}^0, \hat{\varphi}^1)) \right) \mathbf{1}_{\{\varrho = \infty\}} \right] \geq -c_2 \mathbf{P}[\varrho = \infty]. \quad (4.3.29)$$

Combining (4.3.28) and (4.3.29) then gives

$$\mathbf{E} \left[\log (V_\tau^{\text{liq}}(\bar{\varphi}^0, \bar{\varphi}^1)) - \log (V_\tau^{\text{liq}}(\hat{\varphi}^0, \hat{\varphi}^1)) \right] \geq c_1 2^{-2n} \mathbf{P}[\varrho > 2^{-n}] - c_2 \mathbf{P}[\varrho = \infty]$$

and finally

$$\mathbf{E} \left[\log (V_\tau^{\text{liq}}(\bar{\varphi}^0, \bar{\varphi}^1)) - \log (V_\tau^{\text{liq}}(\hat{\varphi}^0, \hat{\varphi}^1)) \right] \geq (c_1 - c_2 \delta) 2^{-2n} \mathbf{P}[\varrho > 2^{-n}] > 0$$

by (4.3.24), as δ can be chosen arbitrarily small. \square

We finish this section by considering a variant of the example constructed in Proposition 4.3.1. The predictable stopping time τ used in the above example will now be replaced by a totally inaccessible stopping time.

The main feature of this modified example is to show that, for a continuous process S , it may happen that $\widehat{S} := \frac{\widehat{Y}^1}{\widehat{Y}^0}$ is a shadow price in the sense of Definition 4.2.1, but fails to be continuous.

Consider the first jump time τ^α of a Poisson process $(N_t)_{t \geq 0}$ with parameter $\alpha > 0$. It is exponentially distributed with parameter $\alpha > 0$, so that $\mathbf{E}[\tau^\alpha] = \alpha^{-1}$. The stock price process $S = S^\alpha$ is defined by

$$S_t^\alpha := e^{t \wedge \tau^\alpha}.$$

Similarly, as in the previous example, the price moves upwards up to time τ^α , and then remains constant. As information available to the investor we use the \mathbf{P} -augmented filtration $\mathbb{F}^{S^\alpha} = (\mathcal{F}_t^{S^\alpha})_{t \geq 0}$ generated by the price process $S^\alpha = (S_t^\alpha)_{t \geq 0}$.

For fixed transaction costs $\lambda \in (0, 1)$ such that $\lambda < \alpha^{-1}$, and initial endowment $x > 0$, we consider the portfolio optimization problem (4.1.2) with logarithmic utility function, i.e.,

$$\mathbf{E}[\log(g)] \rightarrow \max!, \quad g \in \mathcal{C}_0^\lambda(x). \quad (4.3.30)$$

Proposition 4.3.11. *The process S^α has the following properties.*

- (i) *The price process S^α satisfies the condition (CPS $^\mu$) for all $\mu \in (0, 1)$, but does not satisfy the condition (NUPBR).*
- (ii) *The value function $u(x)$ is finite, for $x > 0$.*
- (iii) *The dual optimizer $\widehat{Y} \in \mathcal{B}^\lambda$ is induced by a martingale \widehat{Z} and therefore Theorem 4.2.2 implies that $\widehat{S} = \frac{\widehat{Z}^1}{\widehat{Z}^0}$ is a shadow price in the sense of Definition 4.2.1.*
- (iv) *The shadow price \widehat{S} fails to be continuous. In fact it has a jump at time $t = \tau^\alpha$.*

Again, we start by arguing heuristically to derive candidates for primal and dual optimizer. Then we shall verify, using the duality theorem, that they are actually optimizers to the primal and dual problem, respectively.

Since S^α can never move downwards, it is rather intuitive that the agent will never go short on this (see Proposition 4.3.9 for a formal argument), hence the leverage process is always positive, i.e.,

$$L_t(\varphi) = \frac{\varphi_t^1 S_t}{\varphi^0 + \varphi_t^1 S_t} \geq 0.$$

By the memorylessness of the exponential distribution and the properties of $U(x) = \log(x)$, the optimal leverage should remain constant on the stochastic time interval $\llbracket 0, \tau^\alpha \rrbracket$. Under transaction costs $\lambda > 0$, the upper bound for the leverage $L_t(\varphi)$ is $\frac{1}{\lambda}$ as above, which is the maximal proportion of holdings in stock to the total wealth such that the liquidation value is nonnegative.

Fix $\ell \in [0, \frac{1}{\lambda}]$. Starting with initial endowment $(\varphi_{0-}^0, \varphi_{0-}^1) = (x, 0)$, we buy ℓx shares, i.e., $(\varphi_0^0, \varphi_0^1) = ((1 - \ell)x, \ell x)$.

Now we are looking for an optimal leverage $\widehat{\ell}$. From the λ -self-financing condition, we have

$$-d\varphi_t^0 = e^t d\varphi_t^1,$$

for $0 \leq t \leq \tau^\alpha$, where the agent will never sell stocks. As the leverage $L_t(\varphi)$ remains constant on $\llbracket 0, \tau^\alpha \rrbracket$, we obtain

$$(1 - \ell)\varphi_t^1 e^t = \ell\varphi_t^0,$$

for $0 \leq t \leq \tau^\alpha$. Solving the equations above with initial conditions from initial investment

$$(\varphi_0^0, \varphi_0^1) = ((1 - \ell)x, \ell x),$$

we obtain the explicit form of a λ -self-financing trading strategy with constant leverage ℓ and initial investment $((1 - \ell)x, \ell x)$

$$(\varphi_t^0, \varphi_t^1) = ((1 - \ell)x e^{\ell t}, \ell x e^{(\ell - 1)t}),$$

on $\llbracket 0, \tau^\alpha \rrbracket$, which yields that the liquidation value at time τ^α is

$$V_{\tau^\alpha}^{\text{liq}}(\varphi^0, \varphi^1) = (1 - \ell\lambda)x e^{\ell\tau^\alpha}.$$

Hence, the expected utility is

$$\begin{aligned} f(\ell) &:= \mathbf{E} \left[\log (V_{\tau^\alpha}^{\text{liq}}(\varphi^0, \varphi^1)) \right] = \mathbf{E} \left[\log ((1 - \ell\lambda)x e^{\ell\tau^\alpha}) \right] \\ &= \log(x) + \log(1 - \ell\lambda) + \frac{\ell}{\alpha}. \end{aligned}$$

Maximizing over $\ell \in [0, \frac{1}{\lambda}]$, we get the optimal leverage

$$\hat{\ell} = \frac{1 - \alpha\lambda}{\lambda} \vee 0.$$

Therefore, the educated guess for the optimal strategy is

$$\begin{aligned} (\hat{\varphi}_t^0, \hat{\varphi}_t^1) &= \left((1 - \hat{\ell})x e^{\hat{\ell}t}, \hat{\ell}x e^{(\hat{\ell} - 1)t} \right) \\ &= \left(\frac{\lambda - 1 + \alpha\lambda}{\lambda} x \exp\left(\frac{1 - \alpha\lambda}{\lambda}t\right), \frac{1 - \alpha\lambda}{\lambda} x \exp\left(\frac{1 - \alpha\lambda - \lambda}{\lambda}t\right) \right), \end{aligned}$$

for $0 \leq t \leq \tau^\alpha$. At τ^α the portfolio may be liquidated so that $(\hat{\varphi}_{\tau^\alpha}^0, \hat{\varphi}_{\tau^\alpha}^1) = (V_{\tau^\alpha}^{\text{liq}}(\hat{\varphi}), 0)$ for $t > \tau^\alpha$. This yields as candidate for the value function $u(x)$

$$\bar{u}(x) = \log(x) + \log(1 - \hat{\ell}\lambda) + \frac{\hat{\ell}}{\alpha} = \log(x) + \log(\alpha\lambda) + \frac{1 - \alpha\lambda}{\alpha\lambda},$$

which satisfies $\bar{u}(x) \leq u(x)$.

Remark 4.3.12. We see that $\hat{\ell}$ converges to $\frac{1}{\lambda}$, as α goes to 0. For a small $\alpha > 0$, τ^α takes big values with high probability. It is rather intuitive that in this case the agent will dare to take higher leverage. For a big α , τ^α takes small values with high probability. In this case, if the agent would decide to buy some stocks, she will face a loss with high probability, as she has to liquidate the stock before it has substantially risen in value.

Let us continue our heuristic search for the dual optimizer \hat{Z} and the shadow price $\hat{S} = \frac{\hat{Z}^1}{\hat{Z}^0}$.

As for a Poisson process $(N_t)_{t \geq 0}$ and $u < 1$ the process

$$\exp(\log(1 - u)N_t + u\alpha t), \quad t \geq 0$$

is a martingale, we use the following ansatz to look for the dual optimizer, where $u, v < 1$ are still free variables.

Set

$$\begin{aligned} Z_{\tau^\alpha \wedge t}^0 &:= \exp(\log(1-u)N_{\tau^\alpha \wedge t} + u\alpha(\tau^\alpha \wedge t)), \\ Z_{\tau^\alpha \wedge t}^1 &:= \exp(\log(1-v)N_{\tau^\alpha \wedge t} + v\alpha(\tau^\alpha \wedge t)), \\ \tilde{S}_t &:= \frac{Z_{\tau^\alpha \wedge t}^1}{Z_{\tau^\alpha \wedge t}^0} = \exp\left(N_{\tau^\alpha \wedge t} \log\left(\frac{1-v}{1-u}\right)\right) \exp((v-u)\alpha(\tau^\alpha \wedge t)). \end{aligned}$$

By the definition of τ^α , we have

$$\tilde{S}_t = \begin{cases} \exp((v-u)\alpha t), & \text{if } 0 \leq t < \tau^\alpha, \\ \frac{1-v}{1-u} \exp((v-u)\alpha\tau^\alpha), & \text{if } t \geq \tau^\alpha. \end{cases}$$

In order to be a candidate for a shadow price, \tilde{S} should satisfy

$$\tilde{S}_t = \begin{cases} S_t, & \text{if } 0 \leq t < \tau^\alpha, \\ (1-\lambda)S_{\tau^\alpha}, & \text{if } t \geq \tau^\alpha, \end{cases}$$

therefore the parameters u and v should solve the following equations

$$v - u = \frac{1}{\alpha}, \quad \frac{1-v}{1-u} = 1 - \lambda.$$

Solving the equations above, we obtain $u = 1 - \frac{1}{\alpha\lambda}$ and $v = 1 + \frac{1}{\alpha} - \frac{1}{\alpha\lambda}$ so that

$$\begin{aligned} \hat{Z}_t^0 &:= \left(\frac{1}{\alpha\lambda}\right)^{N_{\tau^\alpha \wedge t}} \exp\left(\left(\alpha - \frac{1}{\lambda}\right)(\tau^\alpha \wedge t)\right), \\ \hat{Z}_t^1 &:= \left(\frac{1}{\alpha\lambda} - \frac{1}{\alpha}\right)^{N_{\tau^\alpha \wedge t}} \exp\left(\left(1 + \alpha - \frac{1}{\lambda}\right)(\tau^\alpha \wedge t)\right), \\ \hat{S}_t &:= \frac{\hat{Z}_t^1}{\hat{Z}_t^0} = (1-\lambda)^{N_{\tau^\alpha \wedge t}} e^{\tau^\alpha \wedge t}. \end{aligned}$$

This finishes our heuristic considerations. We shall now apply duality theory to verify the above guesses.

Proof of Proposition 4.3.11. (i) Consider the process $Y_t := \log(S_t^\alpha) = t \wedge \tau^\alpha$. For all ε , $T > 0$ and all stopping times σ such that $\mathbf{P}[\sigma < \tau^\alpha] > 0$, we have that

$$\mathbf{P} \left[\sup_{t \in \llbracket \sigma, \tau^\alpha \rrbracket} |Y_t - Y_\sigma| < \varepsilon, \sigma < \tau^\alpha \right] = \mathbf{P}[\tau^\alpha - \sigma < \varepsilon | \sigma < \tau^\alpha] \mathbf{P}[\sigma < \tau^\alpha] > 0,$$

since τ^α is exponential distributed. Therefore the process $\log(S_t^\alpha)$ is sticky, see [49, Definition 2.2], which implies by [49, Corollary 2.1] the absence of arbitrage of S^α under ε -propositional transaction costs, for all $\varepsilon > 0$. This is equivalent to (CPS^μ) for all $\mu > 0$, by [51, Theorem 2].

Since $\mathbf{P}[\tau^\alpha > 0] > 0$ and trajectories of S^α are strictly increasing on $\llbracket 0, \tau^\alpha \rrbracket$, the simple buy and hold produces an obvious immediate arbitrage in the frictionless market, which shows the violation of $(NFLVR)$.

We now show that the process S^α allows for an unbounded profit with bounded risk, i.e., there exists an $\gamma > 0$, such that for all $C > 0$, there exists predictable S^α -integrable strategy H such that

$$(H \cdot S^\alpha)_t \geq -1 \quad a.s. \quad \forall t \in [0, 1] \quad \text{and} \quad \mathbf{P}[(H \cdot S^\alpha)_T \geq C] \geq \gamma.$$

Indeed: We consider the horizon $T = \infty$. This is only done for notational convenience, since τ^α is almost surely finite, so that almost all paths are eventually constant. Choose one $\gamma \in (0, 1)$. Let $C > 0$ arbitrary. Choose one $M \geq \frac{C\gamma^{1/\alpha}}{1-\gamma^{1/\alpha}}$. Let us consider a simple buy and hold strategy $H_t := M\mathbf{1}_{[0, \tau^\alpha]}$, which give

$$(H \cdot S)_t = M(e^{\tau^\alpha \wedge t} - 1) > 0 > -1,$$

and

$$\begin{aligned} \mathbf{P}[(H \cdot S)_T \geq C] &= \mathbf{P}[M(e^{\tau^\alpha} - 1) \geq C] = \mathbf{P}\left[\tau^\alpha \geq \log\left(\frac{M+C}{M}\right)\right] \\ &= \exp\left(-\alpha \log\left(\frac{M+C}{M}\right)\right) = \left(\frac{M}{M+C}\right)^\alpha \geq \gamma. \end{aligned}$$

Hence, the above assertion follows.

(ii) For fixed $0 < \lambda < 1$ and $\varphi \in \mathcal{A}_0^\lambda(x)$, we observe that $V_\tau^{\text{liq}}(\varphi) \leq x \exp\left(\frac{\tau^\alpha}{\lambda}\right)$. As τ^α has expectation $\mathbf{E}[\tau^\alpha] = \alpha^{-1}$, we obtain that

$$u(x) \leq \mathbf{E}\left[\log\left(x \exp\left(\frac{\tau^\alpha}{\lambda}\right)\right)\right] = \log(x) + \frac{1}{\lambda} \mathbf{E}[\tau^\alpha] = \log(x) + \frac{1}{\alpha\lambda} < \infty.$$

Hence, by Theorem 4.1.3 there is a dual optimizer $\hat{Y} \in \mathcal{B}$.

As regards (iii) and (iv) note that $(\hat{Z}_t^0, \hat{Z}_t^1)_{t \geq 0}$ is \mathbf{P} -martingale. Since $(\hat{Z}_t^0, \hat{Z}_t^1)_{t \geq 0}$ is strictly positive and satisfies

$$(1 - \lambda)S_t \leq \frac{\hat{Z}_t^1}{\hat{Z}_t^0} \leq S_t,$$

for all $t \geq 0$, it defines a λ -consistent price system.

For $\hat{y} := \bar{u}'(x) = \frac{1}{x}$, we have

$$\begin{aligned} v(\hat{y}) &\leq \mathbf{E}\left[-\log\left(\hat{y}\hat{Z}_{\tau^\alpha}^0\right) - 1\right] = -\mathbf{E}\left[\log\left(\frac{1}{x} \frac{1}{\alpha\lambda} e^{(\alpha-\frac{1}{\lambda})\tau^\alpha}\right)\right] - 1 \\ &= \log(x) + \log(\alpha\lambda) + \frac{\alpha\lambda-1}{\alpha\lambda} - 1 = \bar{u}(x) - x\hat{y} \\ &\leq u(x) - x\hat{y}. \end{aligned}$$

Combining this inequality with the trivial Fenchel inequality $v(\hat{y}) \geq u(x) - x\hat{y}$, we obtain

$$u(x) - x\hat{y} = \bar{u}(x) - x\hat{y} = v(\hat{y}),$$

in particular $u(x) = \bar{u}(x)$. From Theorem 4.1.3, $(\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{t \geq 0}$ is indeed an optimal strategy of the problem defined in (4.3.30), and $(\hat{Z}_t^0, \hat{Z}_t^1)_{t \geq 0}$ is a dual optimizer, which is a \mathbf{P} -martingale. According to Theorem 4.2.2, it follows that \hat{S} is a shadow price. \square

4.4 Shadow Prices for Fractional Brownian Motion

We continue the analysis pertaining to the existence of a shadow price process for portfolio optimization under proportional transaction costs. That is, a price process such that the solution to the frictionless utility maximization problem for this price process gives the same optimal strategy and utility as the original problem under transaction costs. We established in Theorem 4.2.9 a positive answer for a continuous price process $S = (S_t)_{0 \leq t \leq T}$ satisfying the condition (*NUPBR*) of no unbounded profit with bounded risk. The assumption of (*NUPBR*) implies that S has to be a semimartingale. Therefore, our result does not yet apply to price processes based on fractional Brownian motion $B^H = (B_t^H)_{0 \leq t \leq T}$ such as the fractional Black-Scholes model

$$S_t = \exp(\mu t + \sigma B_t^H), \quad 0 \leq t \leq T, \quad (4.4.1)$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $H \in (0, 1) \setminus \{\frac{1}{2}\}$ denotes the Hurst parameter of the fractional Brownian motion B^H . In this section, we combine a recent result of Rémi Peyre with a slight strengthening of our existence result in Theorem 4.2.9 to fill this gap.

In order to formulate the main result, we still need the following notation, which was introduced by Christian Bender [2] as a generalization of continuous martingales. Rémi Peyre [89] showed that fractional Brownian motion enjoys this property.

Definition 4.4.1. *Let $X = (X_t)_{0 \leq t \leq T}$ be a real-valued continuous stochastic process and $\sigma \leq T$ a stopping time. Define*

$$\begin{aligned} \sigma_+ &:= \inf\{t > \sigma \mid X_t - X_\sigma > 0\} \wedge T, \\ \sigma_- &:= \inf\{t > \sigma \mid X_t - X_\sigma < 0\} \wedge T. \end{aligned}$$

*Then, we say that X satisfies the condition (*TWC*) of “two way crossing”, if $\sigma_+ = \sigma_-$ a.s.*

(*TWC*) is a condition on the fine structure of the paths, which means whenever the process moves from X_σ , it will cross the level X_σ infinitely often in time intervals of length ε for each $\varepsilon > 0$.

In the subsequent theorem, we observe that it is sufficient for the conclusion of Theorem 4.2.9 to assume that the price process satisfies the condition (*TWC*) instead of assuming (*NUPBR*).

Theorem 4.4.2. *Let $U : (0, \infty) \rightarrow \mathbb{R}$ be a strictly concave, increasing, smooth utility function, satisfying the Inada conditions,*

$$U'(0) = \infty, \quad U'(\infty) = 0,$$

*and the condition (*RAE*) of reasonable asymptotic elasticity*

$$AE(U) := \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1. \quad (4.4.2)$$

*Fix transaction costs $\lambda \in (0, 1)$ and a continuous process $S = (S_t)_{0 \leq t \leq T}$ satisfying (*TWC*). Suppose that*

$$u(x) := \sup_{\varphi \in \mathcal{A}_0^\lambda(x)} \mathbf{E}[U(V_T^{\text{liq}}(\varphi))] < \infty \quad (4.4.3)$$

for some $x > 0$, where $\mathcal{A}_0^\lambda(x)$ denotes the set of all λ -self-financing and admissible trading strategies $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ under transaction costs starting with $\varphi_0 = (x, 0)$ and $V_T^{\text{liq}}(\varphi)$ their liquidation value at time T .

Then, the conclusion of Theorem 4.2.9 holds true. In particular, there exists an optimal trading strategy $\widehat{\varphi} = (\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)_{0 \leq t \leq T}$ and a shadow price $\widehat{S} = (\widehat{S}_t)_{0 \leq t \leq T}$.

Proof. We observe that, for continuous price processes $S = (S_t)_{0 \leq t \leq T}$, the condition (TWC) of two way crossing implies the no obvious immediate arbitrage condition (NOIA) locally. It follows by Theorem 2.2.4 that S satisfies locally the condition (CPS $^\mu$) of existence of a μ -consistent price system for each $\mu \in (0, 1)$. Therefore, the assumptions of Theorem 4.1.3 are satisfied and there exists an optimal trading strategy $\widehat{\varphi} = (\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)_{0 \leq t \leq T}$ that attains the supremum in (4.4.3) as well as a supermartingale deflator $\widehat{Z} = (\widehat{Z}_t^0, \widehat{Z}_t^1)_{0 \leq t \leq T}$ that solves the dual problem.

To obtain the existence of a shadow price $\widehat{S} = (\widehat{S}_t)_{0 \leq t \leq T}$ for problem (4.4.3), it is by Theorem 4.2.2 sufficient to show that the dual optimizer $\widehat{Z} = (\widehat{Z}_t^0, \widehat{Z}_t^1)_{0 \leq t \leq T}$ is a local martingale. By Proposition 4.2.5, this follows as soon as we have that the liquidation value

$$V_t^{\text{liq}}(\widehat{\varphi}) := \widehat{\varphi}_t^0 + (\widehat{\varphi}_t^1)^+(1 - \lambda)S_t - (\widehat{\varphi}_t^1)^-S_t$$

is strictly positive almost surely for all $t \in [0, T]$, i.e.,

$$\inf_{0 \leq t \leq T} V_t^{\text{liq}}(\widehat{\varphi}) > 0, \quad a.s. \quad (4.4.4)$$

To show (4.4.4), we argue by contradiction. Define

$$\sigma_\varepsilon := \inf \left\{ t \in [0, T] \mid V_t^{\text{liq}}(\widehat{\varphi}) \leq \varepsilon \right\}, \quad (4.4.5)$$

and let $\sigma := \lim_{\varepsilon \searrow 0} \sigma_\varepsilon$. We have to show that $\sigma = \infty$, almost surely. Suppose that $\mathbf{P}[\sigma < \infty] > 0$ and let us work towards a contradiction.

First observe that $V_\sigma^{\text{liq}}(\widehat{\varphi}) = 0$ on $\{\sigma < \infty\}$. Indeed, as $(V_t^{\text{liq}}(\widehat{\varphi}))_{0 \leq t \leq T}$ is càdlàg, we have that

$$0 \leq V_\sigma^{\text{liq}}(\widehat{\varphi}) \leq \lim_{\varepsilon \searrow 0} V_{\sigma_\varepsilon}^{\text{liq}}(\widehat{\varphi}) \leq 0$$

on the set $\{\sigma < \infty\}$.

So suppose that $V_\sigma^{\text{liq}}(\widehat{\varphi}) = 0$ on the set $\{\sigma < \infty\}$ with $\mathbf{P}[\sigma < \infty] > 0$. We may and do assume that S “moves immediately after σ ”, i.e., $\sigma = \inf\{t > \sigma \mid S_t \neq S_\sigma\}$. Indeed, we may replace σ on $\{\sigma < \infty\}$ by the stopping time $\sigma_+ = \sigma_-$, which satisfies $\sigma_+ < T$ on $\{\sigma < \infty\}$ almost surely as $V_T^{\text{liq}}(\widehat{\varphi}) > 0$ almost surely.

We shall repeatedly use the fact established in Theorem 4.1.3 that the process

$$\widehat{V} = (\widehat{\varphi}_t^0 \widehat{Z}_t^0 + \widehat{\varphi}_t^1 \widehat{Z}_t^1)_{0 \leq t \leq T}$$

is a uniformly integrable \mathbf{P} -martingale satisfying $\widehat{V}_T > 0$ almost surely.

Firstly, this implies that $\widehat{\varphi}_\sigma^1 \neq 0$ a.s. on $\{\sigma < \infty\}$. Indeed, otherwise $\widehat{Z}_\sigma^0 V_\sigma^{\text{liq}}(\widehat{\varphi}) = \widehat{V}_\sigma = 0$ on $\{\sigma < \infty\}$. As \widehat{V} is a uniformly integrable martingale with strictly positive terminal value $\widehat{V}_T > 0$, we arrive at the desired contradiction.

We consider here only the case that $\widehat{\varphi}_\sigma^1 > 0$ on $\{\sigma < \infty\}$ almost surely. The case $\widehat{\varphi}_\sigma^1 < 0$ with strictly positive probability on $\{\sigma < \infty\}$ can be dealt with in an analogous

way. Next, we show that we cannot have $\widehat{S}_\sigma = (1 - \lambda)S_\sigma$ with strictly positive probability on $\{\sigma < \infty\}$. Indeed, this again would imply that $\widehat{Z}_\sigma^0 V_\sigma^{\text{liq}}(\widehat{\varphi}) = \widehat{V}_\sigma = 0$ on this set which yields a contradiction as in the previous paragraph.

Hence we assume that $\widehat{S}_\sigma > (1 - \lambda)S_\sigma$ on $\{\sigma < \infty\}$. This implies by Proposition 4.2.10 that the utility-optimizing agent defined by $\widehat{\varphi}$ cannot sell stock at time σ as well as for some time after σ , as S is continuous and \widehat{S} càdlàg. Note, however, that the optimizing agent may very well buy stock. But we shall see that this is not to her advantage.

Define the stopping time ϱ_n as the first time after σ when one of the following events happens

- (i) $\widehat{S}_t - (1 - \lambda)S_t < \frac{1}{2}(\widehat{S}_\sigma - (1 - \lambda)S_\sigma)$ or
- (ii) $S_t < S_\sigma - \frac{1}{n}$.

By the hypothesis of *(TWC)* of two way crossing, we conclude that, a.s. on $\{\sigma < \infty\}$, we have that ϱ_n decreases to σ and that we have $S_{\varrho_n} = S_\sigma - \frac{1}{n}$, for n large enough. Choose n large enough such that $S_{\varrho_n} = S_\sigma - \frac{1}{n}$ on a subset of $\{\sigma < \infty\}$ of positive measure. Then $V_{\varrho_n}^{\text{liq}}(\widehat{\varphi})$ is strictly negative on this set which will give the desired contradiction. Indeed, the assumption $\widehat{\varphi}_\sigma^1 > 0$ implies that the agent suffers a strict loss from this position as $S_{\varrho_n} < S_\sigma$. The condition (i) makes sure that the agent cannot have sold stock between σ and ϱ_n . The agent may have bought additional stock during the interval $[\sigma, \varrho_n]$. However, this cannot result in a positive effect either as the subsequent calculation, which holds true on $\{S_{\varrho_n} = S_\sigma - \frac{1}{n}\}$, reveals

$$\begin{aligned} V_{\varrho_n}^{\text{liq}}(\widehat{\varphi}) &= \widehat{\varphi}_{\varrho_n}^0 + (1 - \lambda)\widehat{\varphi}_{\varrho_n}^1 S_{\varrho_n} \\ &\leq \widehat{\varphi}_\sigma^0 - \int_\sigma^{\varrho_n} S_u d\widehat{\varphi}_u^{1,\uparrow} + (1 - \lambda) \left(\widehat{\varphi}_\sigma^1 + \int_\sigma^{\varrho_n} d\widehat{\varphi}_u^{1,\uparrow} \right) S_{\varrho_n} \\ &= V_\sigma^{\text{liq}}(\widehat{\varphi}) + \widehat{\varphi}_\sigma^1 (1 - \lambda) \underbrace{(S_{\varrho_n} - S_\sigma)}_{=-\frac{1}{n}} - \int_\sigma^{\varrho_n} \underbrace{(S_u - (1 - \lambda)S_{\varrho_n})}_{\geq S_u - S_{\varrho_n} \geq 0} d\widehat{\varphi}_u^{1,\uparrow} < 0. \end{aligned}$$

This contradiction finishes the proof of the theorem. \square

The significance of the condition *(TWC)* in the above result is that it holds for the fractional Black-Scholes model (4.4.1) and does not require that S is a semimartingale. It allows us to conclude the existence of a shadow price process for the fractional Black-Scholes model and utility functions that are bounded from above, like power utility $U(x) = \frac{x^\alpha}{\alpha}$ with risk aversion parameter $\alpha < 0$. It remains as an open questions whether or not the indirect utility (4.4.3) is finite in the fractional Black-Scholes model for utility functions $U : (0, \infty) \rightarrow \mathbb{R}$ that are not bounded from above like logarithmic utility $U(x) = \log(x)$ or power utility $U(x) = \frac{x^\alpha}{\alpha}$ with risk aversion parameter $\alpha \in (0, 1)$. By estimating the number of fluctuations of the fractional Brownian motion, we may show the validity of (4.4.3).

We need the following notation. Fix $\delta > 0$ and define inductively the stopping times $(\tau_j)_{j \in \mathbb{N}_0}$ by $\tau_0 = 0$ and

$$\tau_j := \inf \{t > \tau_{j-1} \mid |B_t^H - B_{\tau_{j-1}}^H| \geq \delta\}.$$

We define the number of δ -fluctuations up to time T as the random variable

$$F_T^\delta := \sup\{j \geq 0 \mid \tau_j \leq T\}.$$

Lemma 4.4.3. *Fix $H \in (0, 1)$, the model (4.4.1) with $\mu \geq 0$ and $\sigma > 0$, as well as $\lambda > 0$, $T > 0$ and $\delta > 0$ such that $(1 - \lambda)e^{2\sigma\delta + \mu T} < 1$.*

There exists a constant $K = K(\delta, \lambda, \mu, \sigma)$, depending only on δ, λ, μ and σ such that, for each $g \in \mathcal{C}_0^\lambda(x)$ we have $g \leq xK^n$ on $\{F_T^\delta \leq n\}$. In particular, the value function $u(x) < \infty$ for each utility function $U : (0, \infty) \rightarrow \mathbb{R}$.

Proof. For an admissible λ -self-financing trading strategy φ define the optimistic value process $(V_t^{\text{opt}}(\varphi))_{0 \leq t \leq T}$ by

$$V_t^{\text{opt}}(\varphi) := \varphi_t^0 + (\varphi_t^1)^+ S_t - (\varphi_t^1)^-(1 - \lambda)S_t.$$

We note that we interchange the roles of S and $(1 - \lambda)S$ in the definition of the liquidation value $V_t^{\text{liq}}(\varphi)$ of φ . It is clear that $V_t^{\text{opt}}(\varphi) \geq V_t^{\text{liq}}(\varphi)$ for each admissible λ -self-financing trading strategy φ .

Fix a trajectory $(B_t^H(\omega))_{0 \leq t \leq T}$ and $j \in \mathbb{N}_0$ such that $\tau_j(\omega) < T$. We claim that there exists a constant $K = K(\delta, \lambda)$ such that we have

$$V_t^{\text{opt}}(\varphi(\omega)) \leq KV_{\tau_j}^{\text{opt}}(\varphi(\omega)),$$

for each $\tau_j(\omega) \leq t \leq \tau_{j+1}(\omega) \wedge T$.

We would like to do some rough estimates. Hence, we assume the agent knows the entire trajectory $(S_t(\omega))_{0 \leq t \leq T}$. As

$$S_t(\omega) = S_{\tau_j}(\omega) \exp\left(\sigma(B_t^H(\omega) - B_{\tau_j}^H(\omega)) + \mu(t - \tau_j(\omega))\right),$$

we see that $S_t(\omega)$ is in the interval $[e^{-\sigma\delta}S_{\tau_j}(\omega), e^{\sigma\delta + \mu T}S_{\tau_j}(\omega)]$.

Let us assume that the $B_t^H = B_{\tau_j}^H + \delta$. The agent, who is maximizing $V_t^{\text{opt}}(\varphi(\omega))$, wants to exploit the up-movement by investing into the stock S as much as possible. However, she cannot make $\varphi_u^1 \geq 0$ arbitrarily large, as she is restricted by the admissibility condition $V_t^{\text{liq}}(\varphi(\omega)) = \varphi_u^0(\omega) + \varphi_u^1(\omega)(1 - \lambda)S_u(\omega) \geq 0$, for every $\tau_j(\omega) \leq u \leq t$. As $S_u(\omega) \leq e^{\sigma\delta + \mu T}S_{\tau_j}(\omega)$ for $\tau_j(\omega) \leq u \leq t$, we have

$$\varphi_u^0(\omega) + \varphi_u^1(\omega)(1 - \lambda)e^{\sigma\delta + \mu T}S_{\tau_j}(\omega) \geq 0. \quad (4.4.6)$$

As any value $\varphi_{\tau_j}(\omega) = (\varphi_{\tau_j}^0(\omega), \varphi_{\tau_j}^1(\omega))$ with $V_{\tau_j}^{\text{opt}}(\varphi(\omega)) = V$ may be reached from $(V, 0)$ by either buying stock at time $\tau_j(\omega)$ at price $S_{\tau_j}(\omega)$ or selling it at price $(1 - \lambda)S_{\tau_j}(\omega)$, we may assume without loss of generality that $\varphi_{\tau_j}(\omega) = (V, 0)$.

The best strategy in our situation is to wait until the moment $\tau_j(\omega) \leq \bar{t} \leq t$ when $S_{\bar{t}}(\omega)$ is minimal in the interval $[\tau_j(\omega), t]$, then to buy at time \bar{t} as much stock as is allowed by (4.4.6), and then to keep the positions constant until time t . Assuming the most favorable case $S_{\bar{t}}(\omega) = e^{-\sigma\delta}S_{\tau_j}(\omega)$, we obtain $\varphi_u = (V, 0)$ for $\tau_j(\omega) \leq u < \bar{t}$ and

$$\varphi_u = \left(V - \frac{V}{1 - (1 - \lambda)e^{2\sigma\delta + \mu T}}, \frac{V}{1 - (1 - \lambda)e^{2\sigma\delta + \mu T}} \frac{1}{e^{-\sigma\delta}S_{\tau_j}(\omega)} \right)$$

for $\bar{t} \leq u \leq t$. Therefore,

$$\begin{aligned} V_t^{\text{opt}}(\varphi(\omega)) &\leq V \left(1 - \frac{1}{1 - (1 - \lambda)e^{2\sigma\delta + \mu T}} + \frac{e^{2\sigma\delta + \mu T}}{1 - (1 - \lambda)e^{2\sigma\delta + \mu T}} \right) \\ &= V \left(1 + \frac{e^{2\sigma\delta + \mu T} - 1}{1 - (1 - \lambda)e^{2\sigma\delta + \mu T}} \right). \end{aligned}$$

Due to the assumption $(1 - \lambda)e^{2\sigma\delta + \mu T} < 1$ the term in the bracket is a finite positive constant K , depending on λ, δ, μ and σ .

We have assumed a maximal up-movement. As regards the case of a maximal down-movement as well as any intermediate case, the argument goes in an analogous way and we yield the same estimate. Therefore, $g(\omega) \leq xK^{F_T^\delta(\omega)}$ for each $g \in \mathcal{C}_0^\lambda(x)$.

As any concave function $U(x)$ is dominated by an affine function $C + kx$, by Corollary A.1.25 we obtain that

$$\mathbf{E}[U(g)] \leq C + kx\mathbf{E}[K^{F_T^\delta}] = C + kx\mathbf{E}[\exp(\log(K)F_T^\delta)] < \infty,$$

for every $g \in \mathcal{C}_0^\lambda(x)$. □

By the same token, we may prove the same assertion for the case $\mu \leq 0$.

Thus, we may formulate the existence of a shadow price process for the fractional Black-Scholes model and utility functions satisfying the condition (RAE).

Proposition 4.4.4. *Let $U : (0, \infty) \rightarrow \mathbb{R}$ be a strictly concave, increasing, smooth utility function, satisfying the Inada conditions and (RAE). Fix transaction costs $\lambda \in (0, 1)$ and the fractional Black-Scholes model (4.4.1).*

Then the conclusion of Theorem 4.2.9 holds true. In particular, there exists an optimal trading strategy $\hat{\varphi} = (\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{0 \leq t \leq T}$ and a shadow price $\hat{S} = (\hat{S}_t)_{0 \leq t \leq T}$.

Proof of Proposition 4.4.4. The fractional Black-Scholes model satisfies (TWC) by the law of iterated logarithm for fractional Brownian motion at stopping times in [89, Theorem 34]. Together with Lemma 4.4.3 we have shown that the assumptions of Theorem 4.4.2 are satisfied. □

As in [28], where we established in Theorem 4.1 the existence of a shadow price for the fractional Black-Scholes model and utility functions $U : \mathbb{R} \rightarrow \mathbb{R}$ that are bounded from above, the proposition above allows us to obtain the optimal trading strategy to portfolio optimization problem under transaction costs for a non-semimartingale price process S by passing to a frictionless problem for a semimartingale price process \hat{S} . To the frictionless problem for \hat{S} , we can apply all known results from frictionless markets to derive the optimal trading strategy. In this fashion, we obtain similarly as in [28, Section 5] that \hat{S} is an Itô process that touches S and $(1 - \lambda)S$ without reflection, whenever the optimal trading strategy buys or sells risky assets.

Chapter 5

Utility Maximization with Random Endowment and Transaction Costs

In this chapter we study the utility maximization problem on the terminal wealth with proportional transaction costs and a random endowment.

As in Chapter 2, we consider a model of a financial market which consists of two assets, one bond and one stock. The price of the bond B is constant and normalized to $B \equiv 1$ and the stock price process $S = (S_t)_{0 \leq t \leq T}$ is strictly positive and càdlàg, based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ satisfying the usual hypotheses of right continuity and saturatedness, where \mathcal{F}_0 is assumed to be trivial.

We also assume here that the agent is endowed with initial wealth $x \in \mathbb{R}$ and receives an exogenous endowment e_T at time T , which is \mathcal{F}_T -measurable and satisfies $\rho := \|e_T\|_\infty < \infty$.

Under the assumption of the existence of λ -consistent price systems, to make the duality approach possible, we enlarge the dual domain L^1 to its topological bidual $(L^\infty)^* = ba$, the space of finitely additive measures. Then, we show that the results remain valid as they were pointed out in [23]. We also consider the utility functions $U : \mathbb{R} \rightarrow \mathbb{R}$, which are finitely valued for all $x \in \mathbb{R}$. Using the similar approximation technique as in [94] and [86], we provide duality results and the existence of shadow prices in this setting.

5.1 Utility Maximization Problem on the Positive Real Line

In this section, we suppose the agent's preferences over terminal wealth are modeled by a standard utility function $U : (0, \infty) \rightarrow \mathbb{R}$, which is strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions:

$$U'(0) := \lim_{x \rightarrow 0} U'(x) = \infty \quad \text{and} \quad U'(\infty) := \lim_{x \rightarrow \infty} U'(x) = 0,$$

and has reasonable asymptotic elasticity, i.e.,

$$AE(U) := \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

Without loss of generality, we may assume $U(\infty) > 0$ to simplify the analysis. Define also $U(x) = -\infty$ whenever $x \leq 0$.

Then we restrict our attention to the terminal liquidation wealth, for $x > 0$, the primal problem is to maximize the expected utility function from terminal wealth

$$\mathbf{E}[U(x + \varphi_T^0 + e_T)] \rightarrow \max!, \quad (\varphi^0, \varphi^1) \in \mathcal{A}_{adm}^\lambda(0), \quad (5.1.1)$$

where we denote by $\mathcal{A}_{adm}^\lambda(x)$, for $x \in \mathbb{R}$, the set of all λ -self-financing and admissible trading strategies starting from initial endowment $(\varphi_0^0, \varphi_0^1) = (x, 0)$.

Define

$$\mathcal{C}^\lambda(x) := \left\{ V_T^{\text{liq}}(\varphi) \mid \varphi = (\varphi^0, \varphi^1) \in \mathcal{A}_{adm}^\lambda(x) \right\}.$$

Again, we assume without loss of generality that $\varphi_T^1 = 0$ and therefore

$$\mathcal{C}^\lambda(x) = \left\{ \varphi_T^0 \mid \varphi = (\varphi^0, \varphi^1) \in \mathcal{A}_{adm}^\lambda(x) \right\}.$$

We denote $\mathcal{C}^\lambda := \mathcal{C}^\lambda(0)$, the set of random variables g in $L^0(\Omega, \mathcal{F}, \mathbf{P})$ such that there is an admissible λ -self-financing trading strategy $(\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$, with initial value $(0, 0)$ and terminal value $(g, 0)$.

Then the problem (5.1.1) can be rewritten as

$$\mathbf{E}[U(x + g + e_T)] \rightarrow \max!, \quad g \in \tilde{\mathcal{C}}^\lambda, \quad (5.1.2)$$

where the set $\tilde{\mathcal{C}}^\lambda$ consists of those elements of \mathcal{C}^λ for which the above expectation is well defined.

Finally, in order to exclude trivial case, we have the following assumption:

Assumption 5.1.1. *The value function*

$$u(x) := \sup_{g \in \tilde{\mathcal{C}}^\lambda} \mathbf{E}[U(x + g + e_T)]$$

is finitely valued for some $x > \rho$.

The concavity of $u(x)$ and Assumption 5.1.1 imply that $u(x) < \infty$ for all $x \in \mathbb{R}$.

Let $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ be the convex conjugate function of $U(x)$ defined by

$$V(y) := \sup_{x > 0} \{U(x) - xy\}, \quad y > 0.$$

We also define $I : (0, \infty) \rightarrow (0, \infty)$ the inverse function of U' on $(0, \infty)$, which is strictly decreasing, and satisfies $I(0) = \infty$, $I(\infty) = 0$ and $I = -V'$.

Throughout this chapter we have the following assumption:

Assumption 5.1.2. *S satisfies (CPS $^\mu$) for all $\mu \in (0, 1)$.*

For a treatment of the problem at hand, the usual dual space

$$\mathcal{M}_a^\lambda := \left\{ Z_T^0 \in L^1(\mathbf{P}) \mid (Z^0, Z^1) \in \mathcal{Z}_a^\lambda(S) \right\},$$

which is a subset of L^1 , is too small. As in [23], we extend the usual domain to $(L^\infty)^* = ba$, the dual space of L^∞ and define the following subset of ba , which is equipped with the weak-star topology $\sigma(ba, L^\infty)$,

$$\mathcal{D}^\lambda := \{Q \in ba \mid \|Q\| = 1 \text{ and } \langle Q, g \rangle \leq 0 \text{ for all } g \in \mathcal{C}^\lambda \cap L^\infty\},$$

and $\mathcal{D}^{\lambda,r} := \mathcal{D}^\lambda \cap L^1$, where r stands for regular.

Remark 5.1.3. We note that the set \mathcal{D}^λ is convex and also $\sigma(ba, L^\infty)$ -compact by Alaoglu's theorem. Since $-L_+^\infty \subseteq \mathcal{C}^\lambda$, we see that $\mathcal{D}^\lambda \subseteq ba_+$, and hence $\mathcal{D}^{\lambda,r} \subseteq L_+^1$. By Theorem 2.3.4, each $g \in \mathcal{C}^\lambda$ satisfies $\mathbf{E}[Z_T^0 g] \leq 0$, for every λ -consistent price system (Z^0, Z^1) , therefore $\mathcal{M}_a^\lambda \subseteq \mathcal{D}^{\lambda,r} \subseteq \mathcal{D}^\lambda$. As a consequence, the Assumption 5.1.2 implies that sets \mathcal{D}^λ and $\mathcal{D}^{\lambda,r}$ are nonempty.

Lemma 5.1.4. *The set \mathcal{D}^λ is the $\sigma(ba, L^\infty)$ -closure of \mathcal{M}_a^λ .*

Proof. It is clear that $\mathcal{M}_a^\lambda \subseteq \mathcal{D}^\lambda$ and \mathcal{D}^λ is $\sigma(ba, L^\infty)$ -closed, hence

$$\overline{\mathcal{M}_a^\lambda}^{\sigma(ba, L^\infty)} \subseteq \mathcal{D}^\lambda.$$

Assume now that there exists an element $\tilde{Q} \in \mathcal{D}^\lambda$ satisfying $\tilde{Q} \notin \overline{\mathcal{M}_a^\lambda}^{\sigma(ba, L^\infty)}$. As \mathcal{M}_a^λ is a convex set, the $\sigma(ba, L^\infty)$ -closure $\overline{\mathcal{M}_a^\lambda}^{\sigma(ba, L^\infty)}$ is also convex. By the Hahn-Banach theorem, there exists $f \in L^\infty = (ba, \sigma(ba, L^\infty))^*$, such that $\langle \tilde{Q}, f \rangle > \alpha$ and

$$\langle Q, f \rangle \leq \alpha, \quad \forall Q \in \overline{\mathcal{M}_a^\lambda}^{\sigma(ba, L^\infty)},$$

for some $\alpha \in \mathbb{R}$. In particular, $\mathbf{E}[Z_T^0 f] \leq \alpha$ for all $Z_T^0 \in \mathcal{M}_a^\lambda$, which follows by Theorem 2.3.4 that $f \in \mathcal{C}^\lambda(\alpha)$, therefore $f - \alpha \in \mathcal{C}^\lambda$. By the definition of \mathcal{D}^λ , we obtain that

$$\langle \tilde{Q}, f - \alpha \rangle = \langle \tilde{Q}, f \rangle - \alpha > 0,$$

which contradicts the fact that $\langle \tilde{Q}, f \rangle > \alpha$. \square

Lemma 5.1.5. *Let $g \in L^\infty$. Then $g \in \mathcal{C}^\lambda$ if and only if $\langle Q, g \rangle \leq 0$ for all $Q \in \mathcal{D}^{\lambda,r}$.*

Proof. The necessity follows directly from the definition of \mathcal{D}^λ . The sufficiency follows from the superreplication theorem (Theorem 2.3.4), since $\mathcal{M}_e^\lambda \subseteq \mathcal{D}^{\lambda,r} \subseteq \mathcal{D}^\lambda$. \square

Now we define the dual optimization problem by

$$v(y) := \inf_{Q \in \mathcal{D}^\lambda} J(y, Q) := \inf_{Q \in \mathcal{D}^\lambda} \left\{ \mathbf{E} \left[V \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] + y \langle Q, e_T \rangle \right\}. \quad (5.1.3)$$

In the following theorem, we see that even by adding transaction costs, the results are similar as in [23]. Now we state the main result:

Theorem 5.1.6. *Under Assumptions 5.1.2, 5.1.1, we have*

1. $u(x) < \infty$ for all $x \in \mathbb{R}$ and $v(y) < \infty$ for all $y > 0$.

2. The primal value function is continuously differentiable on (x_0, ∞) and $u(x) = -\infty$ for all $x < x_0$, where $x_0 := -v'(\infty) = \sup_{Q \in \mathcal{D}^\lambda} \langle Q, -e_T \rangle$. The dual value function v is continuously differentiable on $(0, \infty)$.

3. The functions u and v are conjugate in the sense that

$$v(y) = \sup_{x > x_0} \{u(x) - xy\}, \quad y > 0, \quad (5.1.4)$$

$$u(x) = \inf_{y > 0} \{v(y) + xy\}, \quad x > x_0. \quad (5.1.5)$$

4. For all $y > 0$, there exists a solution $\widehat{Q}_y \in \mathcal{D}^\lambda$ to the dual problem, which is unique up to the singular part. For all $x > x_0$, $\widehat{g} := I\left(\widehat{y} \frac{d\widehat{Q}_y^r}{d\mathbf{P}}\right) - x - e_T$ is the solution to the primal problem, where $\widehat{y} = u'(x)$, which attains the infimum of $\{v(y) + xy\}$.

The rest of this section is devoted to the proof of the above main theorem. We split the proof in several lemmas and propositions, where we may see the use of the required assumptions for each step.

Lemma 5.1.7. For all $x \in \mathbb{R}$,

$$u(x) \leq \inf_{y > 0} \inf_{Q \in \mathcal{D}^\lambda} \{J(y, Q) + xy\} = \inf_{y > 0} \{v(y) + xy\}.$$

Proof. For the case $x + g + e_T \leq 0$ on a measurable set $A \in \mathcal{F}$ with $\mathbf{P}[A] > 0$, we get $u(x) = -\infty$, therefore the assertion satisfies trivially. We only have to consider the case $x + g + e_T > 0$ \mathbf{P} -a.s. As g is bounded from below by $-(x + \rho)$ and S satisfies (CPS^μ) for all $\mu \in (0, 1)$, it follows by [96, Theorem 1] that g can be attained by some $(x + \rho)$ -admissible, self-financing trading strategy.

From the definition of $V(y)$, positivity of $x + g + e_T$, and $\langle Q, g \rangle \leq 0$, it follows

$$\begin{aligned} \mathbf{E}[U(x + g + e_T)] &\leq \mathbf{E} \left[V \left(y \frac{dQ^r}{d\mathbf{P}} \right) + y \frac{dQ^r}{d\mathbf{P}}(x + g + e_T) \right] \\ &\leq \mathbf{E} \left[V \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] + y \langle Q, x + g + e_T \rangle \\ &\leq \mathbf{E} \left[V \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] + y \langle Q, e_T \rangle + xy \\ &= J(y, Q) + xy \end{aligned} \quad (5.1.6)$$

for all $y > 0$, $X \in \widetilde{\mathcal{C}}^\lambda$, $Q \in \mathcal{D}^\lambda$. Taking supremum and infimum at left-and right-hand side respectively, we obtain the assertion. \square

We now study the dual value function.

Lemma 5.1.8. The dual value function $v(y)$ is finitely valued for all $y > 0$.

Proof. By Jensen's inequality, the fact that V is decreasing and $\mathbf{E}\left[\frac{dQ^r}{d\mathbf{P}}\right] \leq 1$, we have

$$\begin{aligned} v(y) &= \inf_{Q \in \mathcal{D}^\lambda} \left\{ \mathbf{E} \left[V \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] + y \langle Q, e_T \rangle \right\} \\ &\geq \inf_{Q \in \mathcal{D}^\lambda} V \left(y \mathbf{E} \left[\frac{dQ^r}{d\mathbf{P}} \right] \right) - y\rho \\ &\geq V(y) - y\rho \\ &> -\infty \end{aligned} \tag{5.1.7}$$

for all $y > 0$.

To show $v(y) < \infty$, we recall the duality result without random endowment in the previous chapter. We denote by $\tilde{u}(x)$ and $\tilde{v}(y)$ be the primal and dual value function, respectively,

$$\begin{aligned} \tilde{u}(x) &:= \sup_{(\varphi^0, \varphi^1) \in \mathcal{A}_{adm}^\lambda(0)} \mathbf{E}[U(x + \varphi_T^0)] = \sup_{g \in \tilde{\mathcal{C}}^\lambda} \mathbf{E}[U(x + g)], \\ \tilde{v}(y) &:= \inf_{(Z^0, Z^1) \in \mathcal{Z}_e^\lambda(S)} \mathbf{E} \left[V(y Z_T^0) \right]. \end{aligned}$$

By Assumption 5.1.1, we obtain

$$\tilde{u}(x) \leq \sup_{g \in \tilde{\mathcal{C}}^\lambda} \mathbf{E}[U(x + g + \rho + e_T)] = u(x + \rho) < \infty, \tag{5.1.8}$$

for all $x > 0$. On the other hand, by Theorem 3.2.4

$$\tilde{v}(y) = \sup_{x > 0} \{ \tilde{u}(x) - xy \} = \tilde{u}(\hat{x}_y) - \hat{x}_y y < \infty,$$

for all $y > 0$. It follows from

$$\begin{aligned} v(y) &= \inf_{Q \in \mathcal{D}^\lambda} \left\{ \mathbf{E} \left[V \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] + y \langle Q, e_T \rangle \right\} \\ &\leq \inf_{(Z^0, Z^1) \in \mathcal{Z}_e^\lambda(S)} \mathbf{E} \left[V(y Z_T^0) \right] + y\rho \\ &= \tilde{v}(y) + y\rho, \end{aligned}$$

that $v(y) < \infty$, for all $y > 0$. □

Lemma 5.1.9. *For any $y > 0$, the infimum of the left-hand side of (5.1.3) is attained by some $\hat{Q}_y \in \mathcal{D}^\lambda$.*

Proof. Let $(Q_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}^\lambda$ be the minimizing sequence, i.e.,

$$v(y) = \lim_{n \rightarrow \infty} \left\{ \mathbf{E} \left[V \left(y \frac{dQ_n^r}{d\mathbf{P}} \right) \right] + y \langle Q_n, e_T \rangle \right\}.$$

Since \mathcal{D}^λ is convex and $\left(\frac{dQ_n^r}{d\mathbf{P}}\right)_{n \in \mathbb{N}}$ is L^1 -bounded, we can find a sequence $(\tilde{Q}_n)_{n \in \mathbb{N}}$ with $\tilde{Q}_n \in \text{conv}(Q_k; k \geq n)$ such that $\frac{d\tilde{Q}_n^r}{d\mathbf{P}}$ converges almost surely to some $f \geq 0$.

Clearly $\left| \langle \tilde{Q}_n, e_T \rangle \right| \leq \rho$. Then we can extract a subsequence of \tilde{Q}_n , which is still denoted by \tilde{Q}_n , such that $\langle \tilde{Q}_n, e_T \rangle$ converges.

Note that \mathcal{D}^λ is $\sigma(ba, L^\infty)$ -compact, thus the sequence $(\tilde{Q}_n)_{n \in \mathbb{N}}$ has a cluster point $\hat{Q}_y \in \mathcal{D}^\lambda$. From Proposition A.2.1 (4) we have

$$\frac{d\hat{Q}_y^r}{d\mathbf{P}} = f = \lim_{n \rightarrow \infty} \frac{d\tilde{Q}_n^r}{d\mathbf{P}}.$$

Similarly to [70, Lemma 3.2], we obtain the uniform integrability of $\left\{ V^- \left(y \frac{d\tilde{Q}_n^r}{d\mathbf{P}} \right) \right\}_{n \in \mathbb{N}}$. By Fatou's lemma, we have

$$\liminf_{n \rightarrow \infty} \mathbf{E} \left[V \left(y \frac{d\tilde{Q}_n^r}{d\mathbf{P}} \right) \right] \geq \mathbf{E} \left[V \left(y \frac{d\hat{Q}_y^r}{d\mathbf{P}} \right) \right].$$

Since $\langle \hat{Q}_y, e_T \rangle$ is a cluster point of $(\langle \tilde{Q}_n, e_T \rangle)_{n \in \mathbb{N}}$, which converges, we have

$$\langle \hat{Q}_y, e_T \rangle = \lim_{n \rightarrow \infty} \langle \tilde{Q}_n, e_T \rangle.$$

Hence,

$$\begin{aligned} J(y, \hat{Q}_y) &= \mathbf{E} \left[V \left(y \frac{d\hat{Q}_y^r}{d\mathbf{P}} \right) \right] + y \langle \hat{Q}_y, e_T \rangle \\ &\leq \liminf_{n \rightarrow \infty} \left\{ \mathbf{E} \left[V \left(y \frac{d\tilde{Q}_n^r}{d\mathbf{P}} \right) \right] + y \langle \tilde{Q}_n, e_T \rangle \right\} \\ &\leq \lim_{n \rightarrow \infty} \left\{ \mathbf{E} \left[V \left(y \frac{dQ_n^r}{d\mathbf{P}} \right) \right] + y \langle Q_n, e_T \rangle \right\} \\ &= v(y), \end{aligned}$$

which gives the optimality of $\hat{Q}_y \in \mathcal{D}^\lambda$. □

Lemma 5.1.10. *The solution of the dual problem might not be unique, but its countably additive part is unique.*

Proof. Assume that Q_1 and Q_2 are two minimizers such that $Q_1^r \neq Q_2^r$. Let $Q := \frac{1}{2}Q_1 + \frac{1}{2}Q_2 \in \mathcal{D}^\lambda$. By the strict convexity of V ,

$$\mathbf{E} \left[V \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] < \frac{1}{2} \mathbf{E} \left[V \left(y \frac{dQ_1^r}{d\mathbf{P}} \right) \right] + \frac{1}{2} \mathbf{E} \left[V \left(y \frac{dQ_2^r}{d\mathbf{P}} \right) \right],$$

hence,

$$J(y, Q) < \frac{1}{2} J(y, Q_1) + \frac{1}{2} J(y, Q_2) = J(y, \hat{Q}_y),$$

which is in contradiction to the optimality of \hat{Q}_y . □

Lemma 5.1.11. *The dual value function $v(\cdot)$ is strictly convex.*

Proof. It follows directly from the strict convexity of the function V . □

Proposition 5.1.12. For all $y > 0$, $\frac{d\widehat{Q}_y^r}{d\mathbf{P}} I\left((y - \varepsilon) \frac{d\widehat{Q}_y^r}{d\mathbf{P}}\right)$ is uniformly integrable for sufficiently small $\varepsilon > 0$.

To prove this proposition, we recall a result from [70] (see Lemma B.1.19).

Lemma 5.1.13. Under the assumption $AE(U) < 1$, there exist $y_0 > 0$ and $0 < \gamma < 1$ such that

$$yI(y) < \frac{\gamma}{1 - \gamma} V(y) \quad \text{and} \quad V(\beta y) < \beta^{-\frac{\gamma}{1 - \gamma}} V(y)$$

for all $0 < y < y_0$ and $0 < \beta < 1$.

Proof of Proposition 5.1.12. By Lemma 5.1.13, we can find a $y_0 > 0$, such that, for all $0 < y < y_0$ and sufficiently small $\varepsilon > 0$,

$$\begin{aligned} 0 &\leq \frac{d\widehat{Q}_y^r}{d\mathbf{P}} I\left((y - \varepsilon) \frac{d\widehat{Q}_y^r}{d\mathbf{P}}\right) \mathbf{1}_{\left\{y \frac{d\widehat{Q}_y^r}{d\mathbf{P}} < y_0\right\}} \\ &= \frac{1}{y - \varepsilon} \frac{y - \varepsilon}{y} y \frac{d\widehat{Q}_y^r}{d\mathbf{P}} I\left(\frac{y - \varepsilon}{y} y \frac{d\widehat{Q}_y^r}{d\mathbf{P}}\right) \mathbf{1}_{\left\{y \frac{d\widehat{Q}_y^r}{d\mathbf{P}} < y_0\right\}} \\ &\leq \frac{1}{y - \varepsilon} \frac{\gamma}{1 - \gamma} V\left(\frac{y - \varepsilon}{y} y \frac{d\widehat{Q}_y^r}{d\mathbf{P}}\right) \mathbf{1}_{\left\{y \frac{d\widehat{Q}_y^r}{d\mathbf{P}} < y_0\right\}} \\ &\leq \frac{\gamma C}{(y - \varepsilon)(1 - \gamma)} \left| V\left(y \frac{d\widehat{Q}_y^r}{d\mathbf{P}}\right) \right|, \end{aligned}$$

where $C = \left(\frac{y - \varepsilon}{y}\right)^{-\frac{\gamma}{1 - \gamma}}$. Since I is decreasing and positive,

$$0 \leq \frac{d\widehat{Q}_y^r}{d\mathbf{P}} I\left((y - \varepsilon) \frac{d\widehat{Q}_y^r}{d\mathbf{P}}\right) \mathbf{1}_{\left\{y \frac{d\widehat{Q}_y^r}{d\mathbf{P}} \geq y_0\right\}} \leq \frac{d\widehat{Q}_y^r}{d\mathbf{P}} I\left(\frac{y - \varepsilon}{y} y_0\right).$$

Therefore,

$$0 \leq \frac{d\widehat{Q}_y^r}{d\mathbf{P}} I\left((y - \varepsilon) \frac{d\widehat{Q}_y^r}{d\mathbf{P}}\right) \leq K \left| V\left(y \frac{d\widehat{Q}_y^r}{d\mathbf{P}}\right) \right| + \frac{d\widehat{Q}_y^r}{d\mathbf{P}} I\left(\frac{y_0}{2}\right),$$

for some constant $K > 0$. Since the right-hand side is an element in L^1 , we obtain the uniform integrability of $\frac{d\widehat{Q}_y^r}{d\mathbf{P}} I\left((y - \varepsilon) \frac{d\widehat{Q}_y^r}{d\mathbf{P}}\right)$ for sufficiently small $\varepsilon > 0$. \square

Lemma 5.1.14. The dual value function is continuously differentiable on $(0, \infty)$,

$$v'(y) = - \left\langle \widehat{Q}_y^r, I\left(y \frac{d\widehat{Q}_y^r}{d\mathbf{P}}\right) \right\rangle + \langle \widehat{Q}_y, e_T \rangle.$$

Proof. Let $y > 0$ be arbitrary. Define

$$f(z) := \mathbf{E} \left[V\left(z \frac{d\widehat{Q}_y^r}{d\mathbf{P}}\right) \right] + z \langle \widehat{Q}_y, e_T \rangle.$$

It is easy to see that $f(z)$ is convex, $f(\cdot) \geq v(\cdot)$ and $f(y) = v(y)$, which implies that $\Delta^- f(y) \leq \Delta^- v(y) \leq \Delta^+ v(y) \leq \Delta^+ f(y)$, where Δ^\pm describe the left and the right derivatives, respectively.

By the convexity of $V(\cdot)$ and the Fatou's lemma, it follows that

$$\begin{aligned} \Delta^+ f(y) &\leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbf{E} \left[V \left((y + \varepsilon) \frac{d\widehat{Q}_y^r}{d\mathbf{P}} \right) - V \left(y \frac{d\widehat{Q}_y^r}{d\mathbf{P}} \right) \right] + \langle \widehat{Q}_y, e_T \rangle \\ &\leq \limsup_{\varepsilon \rightarrow 0} \mathbf{E} \left[\frac{d\widehat{Q}_y^r}{d\mathbf{P}} V' \left((y + \varepsilon) \frac{d\widehat{Q}_y^r}{d\mathbf{P}} \right) \right] + \langle \widehat{Q}_y, e_T \rangle \\ &\leq \mathbf{E} \left[\frac{d\widehat{Q}_y^r}{d\mathbf{P}} V' \left(y \frac{d\widehat{Q}_y^r}{d\mathbf{P}} \right) \right] + \langle \widehat{Q}_y, e_T \rangle \\ &= - \left\langle \widehat{Q}_y, I \left(y \frac{d\widehat{Q}_y^r}{d\mathbf{P}} \right) \right\rangle + \langle \widehat{Q}_y, e_T \rangle. \end{aligned}$$

On the other side, by Proposition 5.1.12, we can apply Fatou's lemma again, and it follows that

$$\begin{aligned} \Delta^- f(y) &\geq \liminf_{\varepsilon \rightarrow 0} \mathbf{E} \left[- \frac{d\widehat{Q}_y^r}{d\mathbf{P}} I \left((y - \varepsilon) \frac{d\widehat{Q}_y^r}{d\mathbf{P}} \right) \right] + \langle \widehat{Q}_y, e_T \rangle \\ &\geq - \left\langle \widehat{Q}_y, I \left(y \frac{d\widehat{Q}_y^r}{d\mathbf{P}} \right) \right\rangle + \langle \widehat{Q}_y, e_T \rangle. \end{aligned}$$

Thus, $\Delta^- f(y) = \Delta^- v(y) = v'(y) = \Delta^+ v(y) = \Delta^+ f(y)$.

By strict convexity, $v(\cdot)$ is continuously differentiable. □

Lemma 5.1.15. *In particular,*

$$v'(0+) = -\infty, \quad v'(\infty) \in \left[\inf_{Q \in \mathcal{D}^\lambda} \langle Q, e_T \rangle, \sup_{Q \in \mathcal{D}^\lambda} \langle Q, e_T \rangle \right].$$

Proof. From (5.1.7), we have $v(0+) \geq V(0+)$. On the other hand, by the definition of $v(\cdot)$ and the decrease of $V(\cdot)$, we have that, for any $Q \in \mathcal{D}^\lambda$,

$$v(y) \leq \mathbf{E} \left[V \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] + y \langle Q, e_T \rangle \leq V(0+) + y\rho,$$

which implies $v(0+) \leq V(0+)$. Hence $v(0+) = V(0+) = U(\infty)$. We only need to consider the case that $U(\infty) < \infty$, indeed, if $U(\infty) = \infty$, we get $v(0+) = \infty$, and it follows trivially $v'(0+) = -\infty$.

By the convexity of v and V , (5.1.7), we have

$$\begin{aligned} v'(0+) &\leq \frac{v(y) - v(0+)}{y} \leq \frac{\mathbf{E} [V(y \frac{dQ^r}{d\mathbf{P}}) - V(0+)] + y\rho}{y} \\ &\leq -\mathbf{E} \left[\frac{dQ^r}{d\mathbf{P}} I \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] + \rho, \end{aligned}$$

for all $y > 0$ and $Q \in \mathcal{D}^\lambda$. Letting $y \rightarrow 0$, we obtain $v'(0+) = -\infty$ by monotone convergence theorem.

By the definition of $v(\cdot)$ and l'Hôpital's rule, we have

$$\begin{aligned} v'(\infty) &= \lim_{y \rightarrow \infty} \frac{v(y)}{y} = \lim_{y \rightarrow \infty} \frac{\inf_{Q \in \mathcal{D}^\lambda} \{ \mathbf{E} [V(y \frac{dQ^r}{d\mathbf{P}})] + y \langle Q, e_T \rangle \}}{y} \\ &\in \left[K + \inf_{Q \in \mathcal{D}^\lambda} \langle Q, e_T \rangle, K + \sup_{Q \in \mathcal{D}^\lambda} \langle Q, e_T \rangle \right], \end{aligned}$$

where

$$K = \lim_{y \rightarrow \infty} \frac{1}{y} \inf_{Q \in \mathcal{D}^\lambda} \mathbf{E} \left[V \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right].$$

Since $-V(\cdot)$ is increasing and $I(y) \rightarrow 0$ as $y \rightarrow \infty$, we have that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$-V(y) \leq C_\varepsilon + \varepsilon y,$$

for all $y > 0$. Hence

$$0 \leq -K = \lim_{y \rightarrow \infty} \frac{\sup_{Q \in \mathcal{D}^\lambda} \mathbf{E} [-V(y \frac{dQ^r}{d\mathbf{P}})]}{y} \leq \lim_{y \rightarrow \infty} \frac{C_\varepsilon + \varepsilon y}{y} = \varepsilon.$$

Consequently, $K = 0$ and the claim follows. \square

Now let us consider the next step, $\inf_{y>0} \{v(y) + xy\}$:

If $x < x_0 := -v'(\infty)$ we have $v'(y) + x < 0$ for all $y > 0$, hence $\inf_{y>0} \{v(y) + xy\} = -\infty$ and by Lemma 5.1.7 we have

$$u(x) \leq \inf_{y>0} \{v(y) + xy\} = -\infty.$$

In this case the optimization problem is trivial.

For each $x > x_0$, there exists a unique $\hat{y} > 0$, such that $v'(\hat{y}) + x = 0$, and \hat{y} attains the infimum of $\{v(y) + xy\}$. After having shown the existence of optimizer of the dual problem, we come back to the primal problem. For simplicity, denote $\hat{Q} := \hat{Q}_{\hat{y}}$. Let us consider

$$\hat{g} := I \left(\hat{y} \frac{d\hat{Q}^r}{d\mathbf{P}} \right) - x - e_T.$$

Since $I(\cdot)$ is positive, we have that $x + \hat{g} + e_T > 0$ \mathbf{P} -a.s. It follows from Lemma 5.1.14

$$\begin{aligned} -x = v'(\hat{y}) &= - \left\langle \hat{Q}^r, I \left(\hat{y} \frac{d\hat{Q}^r}{d\mathbf{P}} \right) \right\rangle + \langle \hat{Q}, e_T \rangle \\ &= - \left\langle \hat{Q}^r, x + \hat{g} + e_T \right\rangle + \langle \hat{Q}, e_T \rangle \\ &= - \left\langle \hat{Q}^r, x + \hat{g} \right\rangle + \langle \hat{Q}^s, e_T \rangle. \end{aligned} \tag{5.1.9}$$

The following lemmas will show that \hat{g} is an element in \mathcal{C}^λ .

Lemma 5.1.16.

$$\sup_{Q \in \mathcal{D}^\lambda} \{ \langle Q^r, x + \hat{g} \rangle - \langle Q^s, e_T \rangle \} = \langle \hat{Q}^r, x + \hat{g} \rangle - \langle \hat{Q}^s, e_T \rangle = x.$$

Proof. Given a $Q \in \mathcal{D}^\lambda$ which is a convex set, and an $\varepsilon \in (0, 1)$, define

$$Q_\varepsilon := (1 - \varepsilon)\hat{Q} + \varepsilon Q \in \mathcal{D}^\lambda.$$

It follows $Q_\varepsilon^r = (1 - \varepsilon)\hat{Q}^r + \varepsilon Q^r$. By the optimality of \hat{Q} and the convexity of $V(\cdot)$, we have

$$\begin{aligned} 0 &\geq \frac{1}{\varepsilon \hat{y}} \left\{ \mathbf{E} \left[V \left(\hat{y} \frac{d\hat{Q}^r}{d\mathbf{P}} \right) \right] + \hat{y} \langle \hat{Q}, e_T \rangle - \mathbf{E} \left[V \left(\hat{y} \frac{dQ_\varepsilon^r}{d\mathbf{P}} \right) \right] - \hat{y} \langle Q_\varepsilon, e_T \rangle \right\} \\ &= \frac{1}{\varepsilon \hat{y}} \mathbf{E} \left[V \left(\hat{y} \frac{d\hat{Q}^r}{d\mathbf{P}} \right) - V \left(\hat{y} \frac{dQ_\varepsilon^r}{d\mathbf{P}} \right) \right] + \langle \hat{Q}, e_T \rangle - \langle Q, e_T \rangle \\ &\geq \frac{1}{\varepsilon \hat{y}} \mathbf{E} \left[\hat{y} \left(\frac{d\hat{Q}^r}{d\mathbf{P}} - \frac{dQ_\varepsilon^r}{d\mathbf{P}} \right) V' \left(\hat{y} \frac{dQ_\varepsilon^r}{d\mathbf{P}} \right) \right] + \langle \hat{Q}, e_T \rangle - \langle Q, e_T \rangle \\ &= \mathbf{E} \left[\left(\frac{dQ^r}{d\mathbf{P}} - \frac{d\hat{Q}^r}{d\mathbf{P}} \right) I \left(\hat{y} \frac{dQ_\varepsilon^r}{d\mathbf{P}} \right) \right] + \langle \hat{Q}, e_T \rangle - \langle Q, e_T \rangle. \end{aligned}$$

We now claim that $\left(\left(\frac{dQ^r}{d\mathbf{P}} - \frac{d\hat{Q}^r}{d\mathbf{P}} \right) I \left(\hat{y} \frac{dQ_\varepsilon^r}{d\mathbf{P}} \right) \right)^-$ is uniformly integrable. Indeed,

$$\left(\left(\frac{dQ^r}{d\mathbf{P}} - \frac{d\hat{Q}^r}{d\mathbf{P}} \right) I \left(\hat{y} \frac{dQ_\varepsilon^r}{d\mathbf{P}} \right) \right)^- \leq \frac{d\hat{Q}^r}{d\mathbf{P}} I \left(\hat{y} \frac{dQ_\varepsilon^r}{d\mathbf{P}} \right) \leq \frac{d\hat{Q}^r}{d\mathbf{P}} I \left(\hat{y} (1 - \varepsilon) \frac{dQ_\varepsilon^r}{d\mathbf{P}} \right),$$

where the last term is uniformly integrable for sufficiently small ε by Lemma 5.1.12. Hence we can apply Fatou's lemma, and obtain

$$\begin{aligned} 0 &\geq \liminf_{\varepsilon \rightarrow 0} \mathbf{E} \left[\left(\frac{dQ^r}{d\mathbf{P}} - \frac{d\hat{Q}^r}{d\mathbf{P}} \right) I \left(\hat{y} \frac{dQ_\varepsilon^r}{d\mathbf{P}} \right) \right] + \langle \hat{Q}, e_T \rangle - \langle Q, e_T \rangle \\ &\geq \mathbf{E} \left[\left(\frac{dQ^r}{d\mathbf{P}} - \frac{d\hat{Q}^r}{d\mathbf{P}} \right) I \left(\hat{y} \frac{d\hat{Q}^r}{d\mathbf{P}} \right) \right] + \langle \hat{Q}, e_T \rangle - \langle Q, e_T \rangle \\ &= \langle Q^r, x + \hat{X} \rangle - \langle \hat{Q}^r, x + \hat{X} \rangle + \langle \hat{Q}^s, e_T \rangle - \langle Q^s, e_T \rangle, \end{aligned}$$

which implies our assertion. □

Lemma 5.1.17. *We have $\hat{g} \in \mathcal{C}^\lambda$.*

Proof. Firstly, we show that $\hat{g} \wedge n \in \mathcal{C}^\lambda$ for all $n \in \mathbb{N}$.

As \hat{g} is uniformly bounded from below, $\hat{g} \wedge n \in L^\infty$. For any $Q \in \mathcal{D}^{\lambda, r}$, we have $Q^r = Q$. It follows from Lemma 5.1.16 and $Q^s = 0$ that

$$\langle Q, x + \hat{g} \wedge n \rangle \leq \langle Q, x + \hat{g} \rangle \leq x + \langle Q^s, e_T \rangle = x.$$

Therefore

$$\langle Q, \hat{g} \wedge n \rangle \leq x - \langle Q, x \rangle = 0,$$

for all $Q \in \mathcal{D}^{\lambda,r}$ and $n \in \mathbb{N}$. By Lemma 5.1.5, $\widehat{g} \wedge n \in \mathcal{C}^\lambda$. As by [97, Theorem 3.4] \mathcal{C}^λ is closed with respect to the topology of convergence in measure, and $\widehat{g} \wedge n \rightarrow \widehat{g}$ almost surely, we then obtain that $\widehat{g} \in \mathcal{C}^\lambda$. \square

Proof of main theorem. As $\widehat{g} \in \mathcal{C}^\lambda$ bounded from below, we have that $\langle \widehat{Q}, \widehat{g} \rangle \leq 0$. By (5.1.9) and the positivity of $x + \widehat{g} + e_T$, we get

$$\begin{aligned} \langle \widehat{Q}, e_T \rangle + x &= \langle \widehat{Q}^r, x + \widehat{g} + e_T \rangle \leq \langle \widehat{Q}, x + \widehat{g} + e_T \rangle \\ &\leq \langle \widehat{Q}, e_T \rangle + \langle \widehat{Q}, x \rangle \leq \langle \widehat{Q}, e_T \rangle + x, \end{aligned}$$

which implies

$$\langle \widehat{Q}^s, x + \widehat{g} + e_T \rangle = 0, \quad \langle \widehat{Q}, \widehat{g} \rangle = 0, \quad \langle \widehat{Q}, x \rangle = x.$$

Together with

$$x + \widehat{g} + e_T = I \left(\widehat{y} \frac{d\widehat{Q}^r}{d\mathbf{P}} \right)$$

we obtain equalities instead of inequalities in (5.1.6), i.e.,

$$\mathbf{E}[U(x + \widehat{g} + e_T)] = \mathbf{E} \left[V \left(\widehat{y} \frac{d\widehat{Q}^r}{d\mathbf{P}} \right) \right] + \widehat{y} \langle \widehat{Q}, e_T \rangle + x\widehat{y}.$$

Hence, for $x > x_0$, we have

$$\begin{aligned} u(x) &\geq \mathbf{E}[U(x + \widehat{g} + e_T)] = \mathbf{E} \left[V \left(\widehat{y} \frac{d\widehat{Q}^r}{d\mathbf{P}} \right) \right] + \widehat{y} \langle \widehat{Q}, e_T \rangle + x\widehat{y} \\ &\geq v(\widehat{y}) + x\widehat{y} = u(x), \end{aligned}$$

which shows the optimality of $\widehat{g} \in \mathcal{C}^\lambda$ and (5.1.5). As u is differentiable, (5.1.4) follows from the convex duality theory.

By the positivity of $x + \widehat{g} + e_T$, we obtain that

$$u(x) = \mathbf{E}[U(x + \widehat{g} + e_T)] > -\infty,$$

for all $x > x_0$, which implies the existence of an $g \in \mathcal{C}^\lambda$ such that $x + g + e_T > 0$, \mathbf{P} -a.s., hence $\langle Q, x + g + e_T \rangle \geq 0$, and therefore

$$x \geq \langle Q, x \rangle \geq \langle Q, x \rangle + \langle Q, g \rangle \geq \langle Q, -e_T \rangle,$$

for all $Q \in \mathcal{D}^\lambda$, which follows that

$$x_0 \geq \sup_{Q \in \mathcal{D}^\lambda} \langle Q, -e_T \rangle.$$

By lemma 5.1.15, we have that

$$x_0 = \sup_{Q \in \mathcal{D}^\lambda} \langle Q, -e_T \rangle.$$

This completes the proof. \square

5.2 Optimal Investment When Wealth May Become Negative

In this section we consider the problem with a utility function $U : \mathbb{R} \rightarrow \mathbb{R}$, which is defined and finitely valued everywhere on the real line. We make the usual assumptions that U is continuously differentiable, strictly increasing, strictly concave and satisfies Inada conditions:

$$U'(-\infty) := \lim_{x \rightarrow -\infty} U'(x) = \infty \quad \text{and} \quad U'(\infty) := \lim_{x \rightarrow \infty} U'(x) = 0.$$

We also assume that the function U has reasonable asymptotic elasticity as defined in [94], i.e.,

$$AE_{-\infty} := \liminf_{x \rightarrow -\infty} \frac{xU'(x)}{U(x)} > 1 \quad \text{and} \quad AE_{+\infty} := \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

Economically, the marginal utility $U'(x)$ should be substantially smaller than the average utility $\frac{U(x)}{x}$, as $x \rightarrow \infty$, and substantially bigger as $x \rightarrow -\infty$.

Our aim this section is to study the optimization problem

$$\mathbf{E}[U(x + g + e_T)] \rightarrow \max!, \quad g \in \mathcal{C}^\lambda. \quad (5.2.1)$$

As before, we denote by u the value function

$$u(x) := \sup_{g \in \mathcal{C}^\lambda} \mathbf{E}[U(x + g + e_T)].$$

As pointed out in [28], see also [10] in the case with transaction costs and [94], [86] in the frictionless case, the optimum may not be attained by an admissible trading strategy, since U is defined on the whole real line. Hence we should define our optimization problem over the enlarged set \mathcal{C}_U^λ defined below:

$$\mathcal{C}_U^\lambda := \left\{ g \in L^0(\mathbf{P}; \mathbb{R} \cup \{\infty\}) \left| \begin{array}{l} \exists (g_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}^\lambda \text{ s.t. } U(x + g_n + e_T) \in L^1(\mathbf{P}) \\ \text{and } U(x + g_n + e_T) \xrightarrow{L^1(\mathbf{P})} U(x + g + e_T) \end{array} \right. \right\}.$$

Economically speaking, \mathcal{C}_U^λ describes all random variables g such that the utility $U(x + g + e_T)$ may be approximated by the utility $U(x + V_T^{\text{liq}}(\varphi) + e_T)$ with respect to the L^1 -norm, where φ ranges in the set of admissible λ -self-financing trading strategies. We now introduce the optimization problem:

$$\mathbf{E}[U(x + g + e_T)] \rightarrow \max!, \quad g \in \mathcal{C}_U^\lambda. \quad (5.2.2)$$

It follows from the definition of \mathcal{C}_U^λ that we have clearly the equality

$$u(x) = \sup_{g \in \mathcal{C}^\lambda} \mathbf{E}[U(x + g + e_T)] = \sup_{g \in \mathcal{C}_U^\lambda} \mathbf{E}[U(x + g + e_T)].$$

Note that $U(x + g_n + e_T) \xrightarrow{L^1(\mathbf{P})} U(x + g + e_T)$ implies that $g_n \rightarrow g$ with respect to convergence in probability, since U is strictly increasing.

In order to rule out trivial cases, we shall make the following assumption:

Assumption 5.2.1. *The value function satisfies $u(x) < U(\infty)$, for some $x \in \mathbb{R}$.*

Noting that a convex combination of admissible trading strategies is an admissible trading strategy, we deduce from the Assumption 5.2.1 that $u(x)$ is finitely valued for each $x \in \mathbb{R}$.

To formulate the dual problem to (5.2.1), we define the conjugate function $V(y)$ of $U(x)$ by

$$V(y) := \sup_{x \in \mathbb{R}} \{U(x) - xy\}, \quad y > 0,$$

which is a continuously differentiable, strictly convex function on $(0, \infty)$ satisfying

$$V(0) = U(\infty), \quad V(\infty) = \infty, \quad V'(0) = -\infty, \quad V'(\infty) = \infty.$$

We also have the following relations

$$U(x) = \inf_{y > 0} \{V(y) + xy\}, \quad x \in \mathbb{R},$$

and

$$V(y) = U(I(y)) - yI(y),$$

where I is the inverse function $(U')^{-1}$, which is equal to $-V'$.

Without loss of generality we assume that $U(0) > 0$ after possibly adding a constant to U . This implies the strict positivity of $V(y)$.

We recall a result from [94].

Corollary 5.2.2. *If $U : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $U(0) > 0$ and has reasonable asymptotic elasticity, and $[\lambda_0, \lambda_1]$ is a compact interval contained in $(0, \infty)$, we may find a constant $C > 0$ such that*

1. $V(\lambda y) \leq CV(y)$, for $y > 0$ and $\lambda_0 \leq \lambda \leq \lambda_1$.
2. $y|V'(y)| \leq CV(y)$, for $y > 0$.
3. For $\varepsilon > 0$ we may find $\delta > 0$, such that for all $(1 - \delta) < \lambda < (1 + \delta)$ we have

$$(1 - \varepsilon)V(y) < V(\lambda y) < (1 + \varepsilon)V(y), \quad \text{for } y > 0.$$

Proof. See [94, Corollary 4.2]. □

Our dual problem is then defined as

$$v(y) := \inf_{(Z^0, Z^1) \in \mathcal{Z}_a^\lambda(S)} \mathbf{E} [V(yZ_T^0) + yZ_T^0 e_T] = \inf_{Z_T^0 \in \mathcal{M}_a^\lambda} \mathbf{E} [V(yZ_T^0) + yZ_T^0 e_T]. \quad (5.2.3)$$

Remark 5.2.3. For all $g \in \mathcal{C}^\lambda$, $y > 0$ and $(Z^0, Z^1) \in \mathcal{Z}_a^\lambda(S)$, we have

$$\mathbf{E}[U(x + g + e_T)] \leq \mathbf{E} [V(yZ_T^0) + yZ_T^0(x + g + e_T)],$$

and therefore

$$u(x) \leq \inf_{y > 0} \{v(y) + xy\}.$$

Now, we state the main result:

Theorem 5.2.4. *Let S be a locally bounded strictly positive process. Under Assumptions 5.1.2 and 5.2.1, we have*

1. *The value functions u and v are finitely valued, strictly concave (respectively convex), continuously differentiable functions defined on \mathbb{R} (respectively \mathbb{R}_+); they are conjugate and satisfy*

$$u'(-\infty) = \infty, \quad u'(\infty) = 0, \quad v'(0) = -\infty, \quad v'(\infty) = \infty.$$

The value function u has reasonable asymptotic elasticity.

2. *For $y > 0$, the optimal solution $\widehat{Z}_T^0(y) \in \mathcal{M}_a^\lambda$ to the dual problem (5.2.3) exists and is unique. The map $y \mapsto \widehat{Z}_T^0(y)$ is continuous in the variation norm.*
3. *For $x \in \mathbb{R}$, the optimal solution $\widehat{g}(x)$ to the primal problem (5.2.1) exists, is unique and satisfies*

$$x + \widehat{g}(x) + e_T = I\left(\widehat{y}\widehat{Z}_T^0(\widehat{y})\right),$$

where $\widehat{y} = u'(x)$.

4. *We have the formulae*

$$\begin{aligned} v'(y) &= \mathbf{E}\left[\widehat{Z}_T^0(\widehat{y})\left(V'(\widehat{y}\widehat{Z}_T^0(\widehat{y})) + e_T\right)\right], \\ u'(x) &= \mathbf{E}[U'(x + \widehat{g}(x) + e_T)], \\ xu'(x) &= \mathbf{E}[(x + \widehat{g}(x))U'(x + \widehat{g}(x) + e_T)], \end{aligned}$$

where the usual rule $0 \cdot \infty = 0$ is applied, if the integrands are of this form.

We break the proof into several steps.

Following the approach used in [94, 10, 86], we approximate our optimization problem by a sequence of problems. It is always possible to construct an increasing sequence $(U_n)_{n \in \mathbb{N}}$, such that for all $n \in \mathbb{N}$

- $U_n = U$ on $[-n, \infty)$,
- $-\infty < U_n \leq U$ on $-(n+1), -n)$,
- $U_n = -\infty$ on $(-\infty, -(n+1)]$,
- U_n is increasing, strictly concave, continuously differentiable on $-(n+1), \infty)$, and satisfies

$$\lim_{x \rightarrow -(n+1)} U_n(x) = -\infty, \quad \lim_{x \rightarrow -(n+1)} U_n'(x) = \infty.$$

Define for $y \geq 0$,

$$V_n(y) := \sup_{x \in \mathbb{R}} \{U_n(x) - xy\} = U_n(I_n(y)) - yI_n(y),$$

where $I_n := (U_n')^{-1} = -V_n'$.

Remark 5.2.5.

1. $(U_n)_{n \in \mathbb{N}}$ is an increasing sequence of continuous function that converges to the continuous function $U_\infty = U$ and therefore the convergence holds uniformly on compact subsets.
2. $V_n \nearrow V$, $I_n \rightarrow I$ uniformly on compact subsets.
3. For all $n \in \mathbb{N}$,

$$\lim_{y \rightarrow 0} I_n(y) = \infty.$$

4. For all $n \in \mathbb{N}$, if $U_n(0) > 0$, we have that for all $y \geq 0$

$$V_n(y) = \sup_{x \in \mathbb{R}} (U_n(x) - xy) \geq U_n(0) > 0.$$

5. If U has reasonable asymptotic elasticity and $U(0) > 0$, we can choose U_n such that for all compact interval $[\lambda_0, \lambda_1]$ contained in $(0, \infty)$, we can find some $C > 0$, such that for all $n \in \mathbb{N}$, $y > 0$, $\lambda \in [\lambda_0, \lambda_1]$ we have

$$V_n(\lambda y) \leq C V_n(y), \quad y |I_n(y)| \leq C V_n(y).$$

For $\varepsilon > 0$ we may find $\delta > 0$, such that for all $(1 - \delta) < \lambda < (1 + \delta)$ we have

$$(1 - \varepsilon)V_n(y) < V(\lambda y) < (1 + \varepsilon)V_n(y), \quad \text{for } y > 0, \quad (5.2.4)$$

for all $n \in \mathbb{N}$.

Define $\tilde{U}_n(x) := U_n(x - (n + 1))$, which is a finitely valued for $x > 0$ and satisfies the Inada conditions at 0 and $+\infty$, and the condition of reasonable asymptotic elasticity at $+\infty$. For $x > x_0$, we consider the problem

$$\tilde{u}_n(x) := \sup_{g \in \mathcal{C}^\lambda} \mathbf{E} \left[\tilde{U}_n(x + g + e_T) \right], \quad (5.2.5)$$

which has a unique optimal solution $\tilde{g}_n(x) \in \mathcal{C}^\lambda$.

Hence, by a simple shift on the real line, we obtain that, for $x > x_0 - (n + 1)$,

$$\begin{aligned} \tilde{u}_n(x + n + 1) &= \mathbf{E} \left[\tilde{U}_n(x + n + 1 + \tilde{g}_n(x + n + 1) + e_T) \right] \\ &= \sup_{g \in \mathcal{C}^\lambda} \left[\tilde{U}_n(x + n + 1 + g + e_T) \right] = \sup_{g \in \mathcal{C}^\lambda} [U_n(x + g + e_T)] \end{aligned}$$

and $\hat{g}_n(x) := \tilde{g}_n(x + n + 1)$ is the solution to the following optimization problem

$$u_n(x) := \sup_{g \in \mathcal{C}^\lambda} [U_n(x + g + e_T)]. \quad (5.2.6)$$

Passing to the dual problem, fix $x > x_0 - (x + 1)$ and let $y = u'_n(x) = \tilde{u}'_n(x + n + 1)$. Let us consider the problem

$$\tilde{v}_n(y) := \inf_{Q \in \mathcal{D}^\lambda} \left\{ \mathbf{E} \left[\tilde{V}_n \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] + y \langle Q, e_T \rangle \right\}, \quad (5.2.7)$$

where \tilde{V}_n is the convex conjugate function to \tilde{U}_n . By Theorem 5.1.6, there exists a solution $\tilde{Q}_n(y) \in \mathcal{D}^\lambda$ to the problem (5.2.7) with

$$y \frac{d\tilde{Q}_n^r(y)}{d\mathbf{P}} = \tilde{U}'_n(x + n + 1 + \tilde{g}_n(x + n + 1) + e_T) = U'_n(x + \hat{g}_n(x) + e_T), \quad (5.2.8)$$

and $\tilde{u}_n(x)$, $\tilde{v}_n(y)$ are conjugate in the sense of (5.1.4) and (5.1.5). From Lemma 5.1.14 we have

$$\tilde{v}'_n(y) = \mathbf{E} \left[\frac{d\tilde{Q}_n^r(y)}{d\mathbf{P}} \tilde{V}'_n \left(y \frac{d\tilde{Q}_n^r(y)}{d\mathbf{P}} \right) \right] + \langle \tilde{Q}_n(y), e_T \rangle. \quad (5.2.9)$$

Let V_n be the convex conjugate of U_n and v_n be the conjugate of u_n in the sense of (5.1.4). We obtain the simple equalities

$$V_n(y) = \tilde{V}_n(y) + (n + 1)y, \quad v_n(y) = \tilde{v}_n(y) + (n + 1)y. \quad (5.2.10)$$

Therefore,

$$\begin{aligned} v_n(y) &= \inf_{Q \in \mathcal{D}^\lambda} \left\{ \mathbf{E} \left[\tilde{V}_n \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] + y \langle Q, e_T \rangle \right\} + (n + 1)y \\ &= \mathbf{E} \left[\tilde{V}_n \left(y \frac{d\tilde{Q}_n^r(y)}{d\mathbf{P}} \right) \right] + y \langle \tilde{Q}_n(y), e_T \rangle + (n + 1)y \\ &= \mathbf{E} \left[V_n \left(y \frac{d\tilde{Q}_n^r(y)}{d\mathbf{P}} \right) \right] + y \langle \tilde{Q}_n(y), e_T \rangle + (n + 1)y \left(1 - \mathbf{E} \left[\frac{d\tilde{Q}_n^r(y)}{d\mathbf{P}} \right] \right). \end{aligned} \quad (5.2.11)$$

It follows that the conjugate v_n to u_n is given by

$$v_n(y) = \inf_{Q \in \mathcal{D}^\lambda} \left\{ \mathbf{E} \left[V_n \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] + y \langle Q, e_T \rangle + (n + 1)y \left(1 - \mathbf{E} \left[\frac{dQ^r}{d\mathbf{P}} \right] \right) \right\}, \quad (5.2.12)$$

and this optimization problem has a solution $\hat{Q}_n(y) = \tilde{Q}_n(y)$ and the singular part $\hat{Q}_n^r(y)$ is unique. From (5.2.9) and (5.2.10) we obtain

$$\begin{aligned} v'_n(y) &= \mathbf{E} \left[\frac{d\hat{Q}_n^r(y)}{d\mathbf{P}} V'_n \left(y \frac{d\hat{Q}_n^r(y)}{d\mathbf{P}} \right) \right] + \langle \hat{Q}_n(y), e_T \rangle \\ &\quad + (n + 1) \left(1 - \mathbf{E} \left[\frac{d\hat{Q}_n^r(y)}{d\mathbf{P}} \right] \right), \end{aligned} \quad (5.2.13)$$

and from (5.2.8) we may deduce for $y = u'_n(x)$ that

$$x + \hat{g}_n(x) + e_T = -V'_n \left(y \frac{d\hat{Q}_n^r(y)}{d\mathbf{P}} \right). \quad (5.2.14)$$

We note that, for fixed $y > 0$, $v_n(y)$ is monotonic increasing in n and bounded above by $v(y)$, since by $V_n \leq V$ and $\mathcal{M}_a^\lambda \subseteq \mathcal{D}^\lambda$ we have

$$\begin{aligned} v_n(y) &= \inf_{Q \in \mathcal{D}^\lambda} \left\{ \mathbf{E} \left[V_n \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] + y \langle Q, e_T \rangle + (n+1)y \left(1 - \mathbf{E} \left[\frac{dQ^r}{d\mathbf{P}} \right] \right) \right\} \\ &\leq \inf_{Z_T^0 \in \mathcal{M}_a^\lambda} \mathbf{E} [V (yZ_T^0) + yZ_T^0 e_T] = v(y). \end{aligned} \quad (5.2.15)$$

Therefore we may define now the function

$$v_\infty(y) := \lim_{n \rightarrow \infty} v_n(y),$$

which turns out later to be the function v .

Lemma 5.2.6. *The function v_∞ is finitely valued and dominated by v .*

Proof. Since by Assumption 5.2.1 the value function $u(x)$ is finite for all $x \in \mathbb{R}$, the conjugate function u^* is finite for at least one $\bar{y} > 0$. Thus,

$$v_n(\bar{y}) = \sup_{x \in \mathbb{R}} \{u_n(x) - x\bar{y}\} \leq \sup_{x \in \mathbb{R}} \{u(x) - x\bar{y}\} = u^*(\bar{y}) < \infty.$$

Let $\widehat{Q}_n(\bar{y})$ denote the optimizer of the problem $v_n(\bar{y})$. By Remark 5.2.5. (5), we see that for each $y > 0$, there exists a constant $C = C(y/\bar{y})$ such that

$$\begin{aligned} v_n(y) &\leq \mathbf{E} \left[V_n \left(y \frac{d\widehat{Q}_n^r(\bar{y})}{d\mathbf{P}} \right) \right] + y \langle \widehat{Q}_n(\bar{y}), e_T \rangle + (n+1)y \left(1 - \mathbf{E} \left[\frac{d\widehat{Q}_n^r(\bar{y})}{d\mathbf{P}} \right] \right) \\ &\leq C \mathbf{E} \left[V_n \left(\bar{y} \frac{d\widehat{Q}_n^r(\bar{y})}{d\mathbf{P}} \right) \right] \\ &\quad + \frac{y}{\bar{y}} \left\{ \bar{y} \langle \widehat{Q}_n(\bar{y}), e_T \rangle + (n+1)\bar{y} \left(1 - \mathbf{E} \left[\frac{d\widehat{Q}_n^r(\bar{y})}{d\mathbf{P}} \right] \right) + \bar{y}\rho \right\} \\ &\leq \max\{C, y/\bar{y}\} (v_n(\bar{y}) + \bar{y}\rho) \\ &\leq \max\{C, y/\bar{y}\} (u^*(\bar{y}) + \bar{y}\rho) < \infty. \end{aligned}$$

The fact that v_∞ is dominated by v follows from (5.2.15). \square

Let now recall a convergence result of convex functions, which can be found in [86].

Lemma 5.2.7. *Suppose that $(v_n)_{n \in \mathbb{N}}$ is a sequence of convex (or concave) functions, which increases (or decreases) monotonically pointwise to a convex (respectively, concave) function v_∞ . Let $(y_n)_{n \in \mathbb{N}}$ be a sequence of real numbers tending to y in the domain of v_∞ . Then $v_n(y_n) \rightarrow v_\infty(y)$ as $n \rightarrow \infty$ and, provided the derivatives exist, $v'_n(y_n) \rightarrow v'_\infty(y)$ as $n \rightarrow \infty$.*

Proof. See [86, Lemma 2.3]. \square

Lemma 5.2.8. *Let $(y_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers tending to y . Denote by $\widehat{Q}_n(y_n) \in \mathcal{D}^\lambda$ the optimal solution to the optimization problem $v(y_n)$ as in (5.2.12).*

Then $\left(\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right)_{n \in \mathbb{N}}$ converges in the norm of $L^1(\mathbf{P})$ to a random variable $\widehat{Z}_T^0(y) \in L^1(\mathbf{P})$, which satisfies $\mathbf{E}[\widehat{Z}_T^0(y)] = 1$.

Proof. To do this, we shall show that this sequence is uniformly integrable and Cauchy in the topology of convergence in probability.

Suppose for a contradiction that the sequence $\left(\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}}\right)_{n \in \mathbb{N}}$ fails to be uniformly integrable, or equivalently that the sequence $\left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}}\right)_{n \in \mathbb{N}}$ fails to be uniformly integrable, that is, there exists an $\alpha > 0$ such that for each $C > 0$,

$$\limsup_{n \rightarrow \infty} \mathbf{E} \left[y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \mathbf{1}_{\left\{y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \geq C\right\}} \right] > \alpha.$$

It follows from the inequality

$$V_m(z) \geq U_m(-m) + mz$$

and the assumption $U_m(-m) > -\infty$ that

$$\lim_{z \rightarrow \infty} \frac{V_m(z)}{z} \geq m.$$

Fix $m \in \mathbb{N}$, find $C_m > 0$ such that $V_m(z) \geq (m-1)z$ for $z \geq C_m$, and find $n > m$ such that

$$\mathbf{E} \left[y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \mathbf{1}_{\left\{y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \geq C_m\right\}} \right] > \alpha.$$

Using the definition of v_n , we obtain

$$\begin{aligned} v_n(y_n) &= \mathbf{E} \left[V_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right] + y_n \langle \widehat{Q}_n(y_n), e_T \rangle + (n+1)y_n \left(1 - \mathbf{E} \left[\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right] \right) \\ &\geq \mathbf{E} \left[V_m \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \mathbf{1}_{\left\{y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \geq C_m\right\}} \right] - y_n \rho \\ &\geq (m-1) \mathbf{E} \left[y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \mathbf{1}_{\left\{y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \geq C_m\right\}} \right] - y_n \rho \\ &> (m-1)\alpha - y_n \rho, \end{aligned}$$

which contradicts the boundedness of $(v_n(y_n))_{n \in \mathbb{N}}$, showing the uniform integrability of $\left(\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}}\right)_{n \in \mathbb{N}}$.

To show that $\left(\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}}\right)_{n \in \mathbb{N}}$ is Cauchy with respect to the topology of convergence in probability, suppose to the contrary that there is an $\alpha > 0$ such that there are arbitrarily large n and m satisfying

$$\mathbf{P} \left[\left| \frac{dQ_n^r(y_n)}{d\mathbf{P}} - \frac{dQ_m^r(y_m)}{d\mathbf{P}} \right| > \alpha \right] > \alpha. \quad (5.2.16)$$

We claim that, for all $\alpha > 0$, there exists $N \in \mathbb{N}$ and a compact set

$$K \subseteq \{y \geq 0 \mid V_N(y) = V(y) < \infty\}$$

such that, for $n \geq N$,

$$\mathbf{P} \left[\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \notin K \right] < \frac{\alpha}{3}. \quad (5.2.17)$$

For suppose not, and suppose that $V(0) = \infty$. Choose $N \in \mathbb{N}$ such that

$$\min \{V(U'(N)), V(U'(-N))\} > \frac{3}{\alpha} \left(\sup_{n \in \mathbb{N}} v_n(y_n) + \sup_{n \in \mathbb{N}} y_n \rho \right).$$

Define $K_N := [U'(N), U'(-N)] \subseteq (0, \infty)$. Then there exists $n \geq N$ such that

$$\mathbf{P} \left[y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \notin K_N \right] \geq \frac{\alpha}{3}.$$

Therefore,

$$\begin{aligned} v_n(y_n) &\geq \mathbf{E} \left[V_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right] - y_n \rho \\ &\geq \mathbf{E} \left[V_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \mathbf{1}_{\left\{ y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \notin K_N \right\}} \right] - y_n \rho \\ &\geq \frac{\alpha}{3} \min \{V(U'(N)), V(U'(-N))\} - y_n \rho \\ &> \sup_{n \in \mathbb{N}} v_n(y_n) + \sup_{n \in \mathbb{N}} y_n \rho - y_n \rho \\ &> \sup_{n \in \mathbb{N}} v_n(y_n), \end{aligned}$$

which is a contradiction. In the case where $V(0) < \infty$, one can similarly find a compact set $K_N := [0, U'(-N)]$ for which (5.2.17) holds.

Note that the inequalities (5.2.16) and (5.2.17) imply that there are arbitrarily large n and m satisfying

$$\mathbf{P} \left[\left| \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} - \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \right| > \alpha \text{ and } \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}}, \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \in K \right] > \frac{\alpha}{3}. \quad (5.2.18)$$

By the strict convexity of V and the compactness of K , we may find an $\eta > 0$ such that, for $y_1, y_2 \in K$ with $|y_1 - y_2| > \alpha$, we have

$$V \left(\frac{y_1 + y_2}{2} \right) \leq \frac{V(y_1) + V(y_2)}{2} - \eta.$$

Choose $\varepsilon > 0$ small enough such that

$$\varepsilon \sup_{k \in \mathbb{N}} v_k(y_k) < \frac{\alpha \eta}{12} \quad \text{and} \quad \varepsilon \rho \sup_{k \in \mathbb{N}} y_k < \frac{\alpha \eta}{12}.$$

Using Remark 5.2.5 (5), there exists a constant δ such that for all λ such that $1 - \delta < \lambda < 1 + \delta$ and for all $n \in \mathbb{N}$ we have

$$(1 - \varepsilon)V_n(y) < V_n(\lambda y) < (1 + \varepsilon)V_n(y).$$

Choose N so that for all $n \geq m \geq N$,

$$1 - \delta \leq \frac{y_m}{y_n} \leq \min\{1 + \delta, 1 + \varepsilon\}, \quad (y_n - y_m)\rho < \frac{\alpha\eta}{12}, \quad v_n(y_n) - v_m(y_m) < \frac{\alpha\eta}{12}.$$

Finally, choose $n \geq m \geq N$ so that (5.2.18) holds.

It now follows that

$$\begin{aligned} v_m(y_m) &\leq \mathbf{E} \left[V_m \left(\frac{y_m}{2} \left(\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} + \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \right) \right) \right] + y_m \left\langle \frac{\widehat{Q}_n(y_n) + \widehat{Q}_m(y_m)}{2}, e_T \right\rangle \\ &\quad + (m+1)y_m \left(1 - \frac{1}{2} \mathbf{E} \left[\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} + \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \right] \right) \\ &\leq \frac{1}{2} \mathbf{E} \left[V_m \left(y_m \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right] + \frac{1}{2} \mathbf{E} \left[V_m \left(y_m \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \right) \right] \\ &\quad - \eta \mathbf{P} \left[\left| \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} - \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \right| > \alpha \text{ and } \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}}, \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \in K \right] \\ &\quad + \frac{1}{2} y_m \langle \widehat{Q}_n(y_n), e_T \rangle + \frac{1}{2} (m+1) y_m \left(1 - \mathbf{E} \left[\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right] \right) \\ &\quad + \frac{1}{2} y_m \langle \widehat{Q}_m(y_m), e_T \rangle + \frac{1}{2} (m+1) y_m \left(1 - \mathbf{E} \left[\frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \right] \right) \\ &\leq \frac{1}{2} \mathbf{E} \left[V_n \left(\frac{y_m}{y_n} y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right] + \frac{1}{2} (n+1) \frac{y_m}{y_n} y_n \left(1 - \mathbf{E} \left[\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right] \right) \\ &\quad + \frac{1}{2} \frac{y_m}{y_n} y_n \langle \widehat{Q}_n(y_n), e_T \rangle + \frac{1}{2} v_m(y_m) - \frac{\alpha\eta}{3} \\ &\leq \frac{1}{2} (1 + \varepsilon) \mathbf{E} \left[V_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right] + \frac{1}{2} (1 + \varepsilon) (n+1) y_n \left(1 - \mathbf{E} \left[\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right] \right) \\ &\quad + \frac{1}{2} (1 + \varepsilon) y_n \left(\langle \widehat{Q}_n(y_n), e_T \rangle + \rho \right) - \frac{1}{2} y_m \rho + \frac{1}{2} v_m(y_m) - \frac{\alpha\eta}{3} \\ &\leq \frac{1}{2} (1 + \varepsilon) v_n(y_n) + \frac{1}{2} v_m(y_m) + \frac{1}{2} ((1 + \varepsilon) y_n - y_m) \rho - \frac{\alpha\eta}{3} \\ &< v_m(y_m) + \frac{1}{2} (v_n(y_n) - v_m(y_m)) + \frac{1}{2} \varepsilon \sup_{k \in \mathbb{N}} v_k(y_k) \\ &\quad + \frac{1}{2} (y_n - y_m) \rho + \frac{1}{2} \varepsilon \sup_{k \in \mathbb{N}} y_k \rho - \frac{\alpha\eta}{3} \\ &= v_m(y_m) - \frac{\alpha\eta}{6}. \end{aligned}$$

This contradiction shows that $\left(\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right)_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to the topology of convergence in probability. Therefore, it converges in L^1 to a random variable, denoted by $\widehat{Z}_T^0(y)$, in particular,

$$\mathbf{E} \left[\widehat{Z}_T^0(y) \right] = \lim_{n \rightarrow \infty} \mathbf{E} \left[\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right].$$

We show now that $\mathbf{E}[\widehat{Z}_T^0(y)] = 1$.

Suppose to the contrary that $\mathbf{E}[\widehat{Z}_T^0(y)] < 1$, that is

$$\liminf_{n \rightarrow \infty} \mathbf{E} \left[\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right] < 1.$$

It follows by Lemma 5.2.6 that

$$\begin{aligned} v_\infty(y) &= \limsup_{n \rightarrow \infty} v_n(y_n) \\ &= \limsup_{n \rightarrow \infty} \left\{ \mathbf{E} \left[V_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right] + y_n \langle \widehat{Q}_n(y_n), e_T \rangle \right. \\ &\quad \left. + (n+1)y_n \left(1 - \mathbf{E} \left[\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right] \right) \right\} \\ &\geq \limsup_{n \rightarrow \infty} \left\{ 0 - y_n \rho + (n+1)y_n \left(1 - \mathbf{E} \left[\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right] \right) \right\} = \infty, \end{aligned}$$

which is a contradiction. \square

Remark 5.2.9. From the lemma above, we note that the singular parts of $\widehat{Q}_n(y_n)$ tend to $\widehat{Z}_T^0(y)$ in ba -norm as $n \rightarrow \infty$. By

$$\left\| \widehat{Q}_n(y_n) - \widehat{Z}_T^0(y) \right\|_{ba} \leq \left\| \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} - \widehat{Z}_T^0(y) \right\|_{L^1} + \left\| \widehat{Q}_n^s(y_n) \right\|_{ba} \rightarrow 0,$$

we obtain that

$$\widehat{Q}_n(y_n) \xrightarrow{ba} \widehat{Z}_T^0(y).$$

By the $\sigma(ba, L^\infty)$ -closedness of \mathcal{D}^λ we see that $\widehat{Z}_T^0(y) \in \mathcal{D}^\lambda$.

Lemma 5.2.10. *The random variable $\widehat{Z}_T^0(y)$ is an element of \mathcal{M}_a^λ .*

Proof. We are going to construct an absolutely continuous λ -consistent price system $(\widehat{Z}^0(y), \widehat{Z}^1(y)) \in \mathcal{Z}_a^\lambda(S)$, such that $\widehat{Z}_T^0(y)$ is the terminal value of its first component. To simplify notation, we drop y .

Since we have by Lemma 5.1.4 that \mathcal{D}^λ is the $\sigma(ba, L^\infty)$ -closure of the convex subset $\mathcal{M}_a^\lambda \subseteq L^1$, we may find by Proposition A.2.2 ([46, Proposition 5.1]) a sequence $(Z^{n,0}, Z^{n,1})_{n \in \mathbb{N}} \subseteq \mathcal{Z}_a^\lambda(S)$ such that

$$Z_T^{n,0} \rightarrow \widehat{Z}_T^0, \quad a.s.$$

As $\mathbf{E}[Z_T^{n,0}] = \mathbf{E}[\widehat{Z}_T^0] = 1$, it follows by Scheffé's lemma that

$$Z_T^{n,0} \xrightarrow{L^1} \widehat{Z}_T^0,$$

which implies that $(Z_T^{n,0})_{n \in \mathbb{N}}$ is uniformly integrable. Taking conditional expectation, we may identify \widehat{Z}_T^0 with a nonnegative martingale $\widehat{Z}^0 = (\widehat{Z}_t^0)_{0 \leq t \leq T}$.

It remains to show the existence of a local martingale $\widehat{Z}^1 = (\widehat{Z}_t^1)_{0 \leq t \leq T}$ such that $(\widehat{Z}^0, \widehat{Z}^1) \in \mathcal{Z}_a^\lambda(S)$.

By the local boundedness of S , there exists a sequence of $[0, T] \cup \{\infty\}$ -valued stopping times $(\tau_m)_{m \in \mathbb{N}}$, which is increasing and converges a.s. to ∞ , such that each stopped process S^{τ_m} is bounded. By the definition of \mathcal{Z}_a^λ we have, for each $n \in \mathbb{N}$, that

$$(1 - \lambda)Z_{\tau_m \wedge t}^{n,0} S_{\tau_m \wedge t} \leq Z_{\tau_m \wedge t}^{n,1} \leq Z_{\tau_m \wedge t}^{n,0} S_{\tau_m \wedge t}, \quad a.s. \quad (5.2.19)$$

As $(Z_{\tau_m \wedge t}^{n,0})_{0 \leq t \leq T}$ is a uniformly integrable martingale and S^{τ_m} is bounded, we have by (5.2.19) that $(Z_{\tau_m \wedge t}^{n,1})_{0 \leq t \leq T}$ is a uniformly integrable martingale for each $m \in \mathbb{N}$. Hence, for every $n \in \mathbb{N}$, $(\tau_m)_{m \in \mathbb{N}}$ is the localizing sequence for the local martingale $(Z_t^{n,1})_{0 \leq t \leq T}$.

It follows from the uniform integrability of $(Z_T^{n,0})_{n \in \mathbb{N}}$ and the optional sampling theorem that $(Z_{\tau_m \wedge T}^{n,0})_{n \in \mathbb{N}}$ is also uniformly integrable for each $m \in \mathbb{N}$. Therefore, by (5.2.19), we see that $(Z_{\tau_m \wedge T}^{n,1})_{n \in \mathbb{N}}$ is uniformly integrable for every $m \in \mathbb{N}$. By [36, Theorem I.20], the convex hull of $(Z_{\tau_m \wedge T}^{n,1})_{n \in \mathbb{N}}$ in L^1 is uniformly integrable for every $m \in \mathbb{N}$. Hence, we may extract a sequence of convex combinations $(\widetilde{Z}^{n,0}, \widetilde{Z}^{n,1}) \in \text{conv} \{(Z^{k,0}, Z^{k,1}); k \geq n\}$, such that, for each $m \in \mathbb{N}$, $\widetilde{Z}_T^{n,0} \rightarrow \widehat{Z}_T^0$ and $\widetilde{Z}_{\tau_m \wedge T}^{n,1} \rightarrow Y_m \in L^1$ a.s. and in L^1 -norm. Using $(Y_m)_{m \in \mathbb{N}}$ we may define an adapted process $\widehat{Z}^1 = (\widehat{Z}_t^1)_{0 \leq t \leq T}$ by

$$\widehat{Z}_t^1 := \mathbf{E}[Y_1 | \mathcal{F}_0] \mathbf{1}_{\{0\}}(t) + \sum_{k=1}^{\infty} \mathbf{E}[Y_k | \mathcal{F}_t] \mathbf{1}_{\llbracket \tau_{k-1}, \tau_k \rrbracket}(t), \quad 0 \leq t \leq T,$$

which is a local martingale. Indeed, let $0 \leq s < t \leq T$. On $\{\tau_{j-1} < \tau_m \wedge s \leq \tau_j\}$,

$$\begin{aligned} \mathbf{E} \left[\widehat{Z}_{\tau_m \wedge t}^1 \middle| \mathcal{F}_{\tau_m \wedge s} \right] &= \mathbf{E} \left[\sum_{k=1}^m \mathbf{E}[Y_k | \mathcal{F}_{\tau_m \wedge t}] \mathbf{1}_{\llbracket \tau_{k-1}, \tau_k \rrbracket}(\tau_m \wedge t) \middle| \mathcal{F}_{\tau_m \wedge s} \right] \\ &= \mathbf{E} \left[\sum_{k=1}^m \mathbf{E} \left[\lim_{n \rightarrow \infty} \widetilde{Z}_{\tau_k \wedge T}^{n,1} \middle| \mathcal{F}_{\tau_m \wedge t} \right] \mathbf{1}_{\llbracket \tau_{k-1}, \tau_k \rrbracket}(\tau_m \wedge t) \middle| \mathcal{F}_{\tau_m \wedge s} \right] \\ &= \lim_{n \rightarrow \infty} \mathbf{E} \left[\sum_{k=1}^m \mathbf{E} \left[\widetilde{Z}_{\tau_k \wedge T}^{n,1} \middle| \mathcal{F}_{\tau_m \wedge t} \right] \mathbf{1}_{\llbracket \tau_{k-1}, \tau_k \rrbracket}(\tau_m \wedge t) \middle| \mathcal{F}_{\tau_m \wedge s} \right] \\ &= \lim_{n \rightarrow \infty} \mathbf{E} \left[\sum_{k=1}^m \mathbf{E} \left[\mathbf{E} \left[\widetilde{Z}_{\tau_m \wedge T}^{n,1} \middle| \mathcal{F}_{\tau_k \wedge T} \right] \middle| \mathcal{F}_{\tau_m \wedge t} \right] \mathbf{1}_{\llbracket \tau_{k-1}, \tau_k \rrbracket}(\tau_m \wedge t) \middle| \mathcal{F}_{\tau_m \wedge s} \right] \\ &= \lim_{n \rightarrow \infty} \mathbf{E} \left[\mathbf{E} \left[\widetilde{Z}_{\tau_m \wedge T}^{n,1} \middle| \mathcal{F}_{\tau_m \wedge t} \right] \middle| \mathcal{F}_{\tau_m \wedge s} \right] \\ &= \lim_{n \rightarrow \infty} \mathbf{E} \left[\mathbf{E} \left[\mathbf{E} \left[\widetilde{Z}_{\tau_m \wedge T}^{n,1} \middle| \mathcal{F}_{\tau_m \wedge t} \right] \middle| \mathcal{F}_{\tau_j \wedge s} \right] \middle| \mathcal{F}_{\tau_m \wedge s} \right] \\ &= \lim_{n \rightarrow \infty} \mathbf{E} \left[\widetilde{Z}_{\tau_j \wedge T}^{n,1} \middle| \mathcal{F}_{\tau_m \wedge s} \right] \\ &= \mathbf{E} [Y_j | \mathcal{F}_{\tau_m \wedge s}] \\ &= \widehat{Z}_{\tau_m \wedge s}^1, \end{aligned}$$

due to the tower property and the L^1 -continuity of the conditional expectation. Analogously, we may show that $\mathbf{E}[\widehat{Z}_{\tau_m \wedge t}^1 | \mathcal{F}_0] = \widehat{Z}_0^1$. Therefore \widehat{Z}^1 is a local martingale.

It is easy to see that $(\widehat{Z}^0, \widehat{Z}^1)$ satisfies

$$(1 - \lambda)S_t \widehat{Z}_t^0 \leq \widehat{Z}_t^1 \leq S_t \widehat{Z}_t^0, \quad 0 \leq t \leq T,$$

which implies that $(\widehat{Z}^0, \widehat{Z}^1) \in \mathcal{Z}_a^\lambda(S)$. \square

Lemma 5.2.11. *The map $y \mapsto \widehat{Z}_T^0(y)$ is continuous in the $L^1(\mathbf{P})$ -norm.*

Proof. Take a sequence $(y_k)_{k \in \mathbb{N}}$ tending to $y > 0$. Using Lemma 5.2.8, we can find an increasing sequence $(n_k)_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} \left\| \widehat{Z}_T^0(y_k) - \frac{d\widehat{Q}_{n_k}^r(y_k)}{d\mathbf{P}} \right\|_{L^1(\mathbf{P})} = 0.$$

Using Lemma 5.2.8 once again, we see that

$$\lim_{k \rightarrow \infty} \left\| \frac{d\widehat{Q}_{n_k}^r(y_k)}{d\mathbf{P}} - \widehat{Z}_T^0(y) \right\|_{L^1(\mathbf{P})} = 0.$$

The continuity of the map $y \mapsto \widehat{Z}_T^0(y)$ in the $L^1(\mathbf{P})$ -norm follows from an application of the triangle inequality. \square

Lemma 5.2.12. *Let $(y_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers tending to $y > 0$. Then,*

$$\lim_{n \rightarrow \infty} v_n(y_n) = v(y) = \mathbf{E} \left[V \left(y \widehat{Z}_T^0(y) \right) + y \widehat{Z}_T^0(y) e_T \right] \quad (5.2.20)$$

and thus $\widehat{Z}_T^0(y) \in \mathcal{M}_a^\lambda$ is the unique minimizer of the dual problem (5.2.3). The dual value function v is strictly convex.

Proof. In the proof of Lemma 5.2.8 we showed that $\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}}$ converges in probability to $\widehat{Z}_T^0(y)$, hence

$$V_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \xrightarrow{\mathbf{P}} V \left(y \widehat{Z}_T^0(y) \right). \quad (5.2.21)$$

Using a version of Fatou's lemma for limits in probability, it follows by an application of Remark 5.2.9, Lemma 5.2.7 and Lemma 5.2.6 that

$$\begin{aligned} v(y) &\leq \mathbf{E} \left[V \left(y \widehat{Z}_T^0(y) \right) + y \widehat{Z}_T^0(y) e_T \right] \\ &\leq \liminf_{n \rightarrow \infty} \left\{ \mathbf{E} \left[V_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right] + y_n \left\langle \widehat{Q}_n(y_n), e_T \right\rangle \right\} \\ &\leq \liminf_{n \rightarrow \infty} v_n(y_n) = v_\infty(y) \leq v(y). \end{aligned} \quad (5.2.22)$$

We have equalities above and obtain in particular that $v_\infty(y) = v(y)$, and $\widehat{Z}_T^0(y) \in \mathcal{M}_a^\lambda$ is the minimizer of (5.2.3).

The strict convexity of v and the uniqueness of $\widehat{Z}_T^0(y)$ follow from the strict convexity of V , convexity of the set \mathcal{M}_a^λ and formula (5.2.20). \square

Lemma 5.2.13. *We have*

$$V_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \xrightarrow{L^1(\mathbf{P})} V \left(y \widehat{Z}_T^0(y) \right).$$

Proof. Note that $\left(V_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right)_{n \in \mathbb{N}}$ is a sequence of nonnegative random variables in $L^1(\mathbf{P})$. It follows from (5.2.22) that $V(y \widehat{Z}_T^0(y)) \in L^1(\mathbf{P})$ and

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[V_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right] = \mathbf{E} \left[V \left(y \widehat{Z}_T^0(y) \right) \right]. \quad (5.2.23)$$

Using Scheffé's lemma, the assertion follows from (5.2.21) and (5.2.23). \square

Remark 5.2.14. We note that it follows from (5.2.22) that

$$\lim_{n \rightarrow \infty} (n+1) \left(1 - \mathbf{E} \left[\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right] \right) = 0. \quad (5.2.24)$$

Lemma 5.2.15. *The map $y \mapsto V \left(y \widehat{Z}_T^0(y) \right)$ is continuous in the $L^1(\mathbf{P})$ -norm.*

Proof. In the same way as the proof of Lemma 5.2.11, the assertion follows from Lemma 5.2.13. \square

Lemma 5.2.16. *For $(y_n)_{n \in \mathbb{N}}$ tending to $y > 0$, we have*

$$y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} V_n' \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \xrightarrow{L^1(\mathbf{P})} y \widehat{Z}_T^0(y) V' \left(y \widehat{Z}_T^0(y) \right). \quad (5.2.25)$$

Proof. By Corollary 5.2.2. (2), there is a constant C such that

$$y |V_n'(y)| \leq C V_n(y), \quad \text{for } y \geq 0,$$

uniformly in $n \in \mathbb{N}$, where, in the case $y = 0$, we adopt the rule $0 \cdot \infty = 0$. Hence the sequence of random variables $\left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} V_n' \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right)_{n \in \mathbb{N}}$ is dominated in absolute value by the $L^1(\mathbf{P})$ -convergent sequence $\left(C V_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right)_{n \in \mathbb{N}}$ and is therefore uniformly integrable.

By Lemma 5.2.8, Remark 5.2.5 and the continuity of the map $y \mapsto y V'(y)$ for $y > 0$ (for $y \geq 0$ in the case $V(0) = U(\infty) < \infty$), we have that $\left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} V_n' \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right)_{n \in \mathbb{N}}$ converges in probability to $y \widehat{Z}_T^0(y) V' \left(y \widehat{Z}_T^0(y) \right)$, and therefore converges in the $L^1(\mathbf{P})$ -norm. \square

Lemma 5.2.17. *The map $y \mapsto \widehat{Z}_T^0(y) V' \left(y \widehat{Z}_T^0(y) \right)$ is continuous in the $L^1(\mathbf{P})$ -norm. The function v is continuously differentiable and*

$$\lim_{n \rightarrow \infty} v_n'(y_n) = v'(y) = \mathbf{E} \left[\widehat{Z}_T^0(y) \left(V' \left(y \widehat{Z}_T^0(y) \right) + e_T \right) \right]. \quad (5.2.26)$$

Proof. The first assertion follows from Lemma 5.2.16 by the same argument as the proof of Lemma 5.2.11.

To prove the formula (5.2.26), we observe firstly that the term on the right hand side is a continuous function of $y > 0$ by the first assertion. The convexity of v (by Lemma 5.2.12) implies the existence of the derivative $v'(y)$ for all but countably many y 's. Therefore, it suffices to show (5.2.26) whenever the derivative $v'(y)$ exists. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence tending to $y > 0$ such that $v'(y)$ exists. Using formula (5.2.22), Lemma 5.2.7, formula (5.2.13), Lemma 5.2.16, Remark 5.2.9 and formula (5.2.24) we see that

$$\begin{aligned}
v'(y) &= v'_\infty(y) = \lim_{n \rightarrow \infty} v'_n(y_n) \\
&= \lim_{n \rightarrow \infty} \left\{ \mathbf{E} \left[\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right] + \langle \widehat{Q}_n(y_n), e_T \rangle \right. \\
&\quad \left. + (n+1) \left(1 - \mathbf{E} \left[\frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right] \right) \right\} \\
&= \mathbf{E} \left[\widehat{Z}_T^0(y) V' \left(y \widehat{Z}_T^0(y) \right) + \widehat{Z}_T^0(y) e_T \right].
\end{aligned} \tag{5.2.27}$$

□

Lemma 5.2.18. *For $(y_n)_{n \in \mathbb{N}}$ tending to $y > 0$, we have*

$$y \widehat{Z}_T^0(y) V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \xrightarrow{L^1(\mathbf{P})} y \widehat{Z}_T^0(y) V' \left(y \widehat{Z}_T^0(y) \right). \tag{5.2.28}$$

Proof. It is clear that

$$y \widehat{Z}_T^0(y) V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \xrightarrow{\mathbf{P}} y \widehat{Z}_T^0(y) V' \left(y \widehat{Z}_T^0(y) \right).$$

We have to prove the uniform integrability of the positive parts

$$y \widehat{Z}_T^0(y) V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right)^+ = y \widehat{Z}_T^0(y) V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \mathbf{1}_{\left\{ V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \geq 0 \right\}},$$

for $n \in \mathbb{N}$.

By distinguishing pointwise the cases $y \widehat{Z}_T^0(y) \geq y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}}$ and $y \widehat{Z}_T^0(y) < y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}}$, we have

$$\begin{aligned}
&y \widehat{Z}_T^0(y) V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \mathbf{1}_{\left\{ V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \geq 0 \right\}} \\
&\leq \max \left\{ y \widehat{Z}_T^0(y) V' \left(y \widehat{Z}_T^0(y) \right) \mathbf{1}_{\left\{ V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \geq 0 \right\}}, \right. \\
&\quad \left. y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \mathbf{1}_{\left\{ V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \geq 0 \right\}} \right\}.
\end{aligned} \tag{5.2.29}$$

As by Lemma 5.2.16 the family of functions on the right hand side of (5.2.29) is uniformly integrable, we obtain the uniform integrability of the positive parts of the sequence.

Let $x_n := -v'_n(y_n)$. By Lemma 5.2.7, we have $\lim_{n \rightarrow \infty} x_n = -v'(y)$.

As $\widehat{g}_n(x) \in \mathcal{C}^\lambda$ and $\widehat{Z}_T^0(y) \in \mathcal{M}_a^\lambda \subseteq \mathcal{D}^\lambda$, we deduce from equation (5.2.14) that

$$\mathbf{E} \left[\widehat{Z}_T^0(y) \left(x_n + V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) + e_T \right) \right] = -\mathbf{E} \left[\widehat{Z}_T^0(y) \widehat{g}_n(x) \right] \geq 0. \quad (5.2.30)$$

Therefore, using the formula (5.2.26), we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mathbf{E} \left[y \widehat{Z}_T^0(y) V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right] &\geq \lim_{n \rightarrow \infty} \left(-x_n y - \mathbf{E} \left[y \widehat{Z}_T^0(y) e_T \right] \right) \\ &= v'(y) y - \mathbf{E} \left[y \widehat{Z}_T^0(y) e_T \right] \\ &= \mathbf{E} \left[y \widehat{Z}_T^0(y) V' \left(y \widehat{Z}_T^0(y) \right) \right]. \end{aligned} \quad (5.2.31)$$

As the positive parts of $\left(y \widehat{Z}_T^0(y) V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right)_{n \in \mathbb{N}}$ is uniformly integrable, we deduce from Fatou's lemma that

$$\mathbf{E} \left[y \widehat{Z}_T^0(y) V' \left(y \widehat{Z}_T^0(y) \right) \right] \geq \limsup_{n \rightarrow \infty} \mathbf{E} \left[y \widehat{Z}_T^0(y) V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right].$$

Together with (5.2.31), we obtain that

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[y \widehat{Z}_T^0(y) V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right] = \mathbf{E} \left[y \widehat{Z}_T^0(y) V' \left(y \widehat{Z}_T^0(y) \right) \right].$$

Using Scheffé's lemma, the result now follows from the convergence in probability. \square

Lemma 5.2.19. *Let $\widehat{g}_n(x) \in \mathcal{C}^\lambda$ be the optimal solution to the primal problem (5.2.6). Define $\widehat{g}(x) := - \left(x + V' \left(y \widehat{Z}_T^0(y) \right) + e_T \right)$, where $y = -v'(x)$.*

Then

$$U(x + \widehat{g}_n(x) + e_T) \xrightarrow{L^1(\mathbf{P})} U(x + \widehat{g}(x) + e_T). \quad (5.2.32)$$

We have $\widehat{g}(x) \in \mathcal{C}^\lambda$, and

$$\lim_{n \rightarrow \infty} u_n(x) = u(x) = \mathbf{E} [U(x + \widehat{g}(x) + e_T)],$$

therefore $\widehat{g}(x)$ is the unique maximizer to the optimal problem (5.2.2).

Proof. Fix $x \in \mathbb{R}$ and let $y_n := u'_n(x)$. We observe that the concave functions u_n increases to a function, which is denoted by u_∞ , and which bounded from above by u and conjugate to $v = v_\infty$. As v is strictly convex, u_∞ is continuously differentiable, and using Lemma 5.2.7, we see that

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} u'_n(x) = u'_\infty(x) = -v'(x) = y.$$

By Remark 5.2.5 (ii) and Lemma 5.2.8 we have

$$V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \xrightarrow{\mathbf{P}} V' \left(y \widehat{Z}_T^0(y) \right),$$

and hence

$$U(x + \widehat{g}_n(x) + e_T) \xrightarrow{\mathbf{P}} U(x + \widehat{g}(x) + e_T). \quad (5.2.33)$$

Next we show that

$$\lim_{n \rightarrow \infty} \left\| U(x + \widehat{g}_n(x) + e_T) - U_n(x + \widehat{g}_n(x) + e_T) \right\|_{L^1(\mathbf{P})} = 0. \quad (5.2.34)$$

Indeed, since $U_{n+1}(x)$ coincides with $U(x)$ for $x \geq -(n+1)$ and $x + \widehat{g}_n(x) + e_T \geq -(n+1)$ we have

$$\begin{aligned} & \mathbf{E} [U(x + \widehat{g}_n(x) + e_T)] - \mathbf{E} [U_n(x + \widehat{g}_n(x) + e_T)] \\ &= \mathbf{E} [U_{n+1}(x + \widehat{g}_n(x) + e_T)] - \mathbf{E} [U_n(x + \widehat{g}_n(x) + e_T)] \\ &\leq u_{n+1}(x) - u_n(x). \end{aligned}$$

As the increasing bounded sequence $(u_n(x))_{n \in \mathbb{N}}$ is convergent, the right-hand side of this last inequality tends to zero, .

From equation (5.2.14), Lemma 5.2.13 and Lemma 5.2.16 we see that

$$\begin{aligned} & \left\| U_n(x + \widehat{g}_n(x) + e_T) - U_m(x + \widehat{g}_m(x) + e_T) \right\|_{L^1(\mathbf{P})} \\ &= \left\| U_n \left(-V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right) - U_m \left(-V'_m \left(y_m \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \right) \right) \right\|_{L^1(\mathbf{P})} \\ &\leq \left\| U_n \left(-V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right) + y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) \right. \\ &\quad \left. - U_m \left(-V'_m \left(y_m \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \right) \right) - y_m \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} V'_m \left(y_m \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \right) \right\|_{L^1(\mathbf{P})} \\ &\quad + \left\| y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) - y_m \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} V'_m \left(y_m \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \right) \right\|_{L^1(\mathbf{P})} \\ &= \left\| V_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) - V_m \left(y_m \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \right) \right\|_{L^1(\mathbf{P})} \\ &\quad + \left\| y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} V'_n \left(y_n \frac{d\widehat{Q}_n^r(y_n)}{d\mathbf{P}} \right) - y_m \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} V'_m \left(y_m \frac{d\widehat{Q}_m^r(y_m)}{d\mathbf{P}} \right) \right\|_{L^1(\mathbf{P})} \\ &\rightarrow 0, \end{aligned}$$

as $m, n \rightarrow \infty$. As a consequence,

$$\begin{aligned} & \left\| U(x + \widehat{g}_n(x) + e_T) - U(x + \widehat{g}_m(x) + e_T) \right\|_{L^1(\mathbf{P})} \\ &\leq \left\| U_n(x + \widehat{g}_n(x) + e_T) - U_m(x + \widehat{g}_m(x) + e_T) \right\|_{L^1(\mathbf{P})} \\ &\quad + \left\| U(x + \widehat{g}_m(x) + e_T) - U_m(x + \widehat{g}_m(x) + e_T) \right\|_{L^1(\mathbf{P})} \\ &\quad + \left\| U(x + \widehat{g}_n(x) + e_T) - U_n(x + \widehat{g}_n(x) + e_T) \right\|_{L^1(\mathbf{P})} \\ &\rightarrow 0, \end{aligned}$$

as $m, n \rightarrow \infty$. Together with (5.2.33), this implies (5.2.32).

From the definition of \mathcal{C}_U^λ , we have that the random variable $\widehat{g}(x)$ belongs to \mathcal{C}_U^λ .

We obtain that by an application of the triangle inequality that

$$u_\infty(x) = \lim_{n \rightarrow \infty} u_n(x) = \lim_{n \rightarrow \infty} \mathbf{E} [U_n(x + \widehat{g}_n(x) + e_T)] = \mathbf{E} [U(x + \widehat{g}(x) + e_T)].$$

It is clear that, for all $g \in \mathcal{C}_U^\lambda$, all $y > 0$ and all $\widehat{Z}_T^0 \in \mathcal{M}_a^\lambda$, we have

$$\mathbf{E} [U(x + g + e_T)] \leq \mathbf{E} [V(y\widehat{Z}_T^0) + y\widehat{Z}_T^0 e_T] + xy.$$

For $\widehat{g}(x)$, $x = -v'(y)$ and $\widehat{Z}_T^0(y)$ satisfying $x + \widehat{g}(x) + e_T = -V'(y\widehat{Z}_T^0(y))$, it follows from Fenchel's equality and equation (5.2.26)

$$\begin{aligned} \mathbf{E} [U(x + \widehat{g}(x) + e_T)] &= \mathbf{E} [V(y\widehat{Z}_T^0(y)) + y\widehat{Z}_T^0(y)(x + \widehat{g}(x) + e_T)] \\ &= \mathbf{E} [V(y\widehat{Z}_T^0(y)) - y\widehat{Z}_T^0(y)V'(y\widehat{Z}_T^0(y))] \\ &= \mathbf{E} [V(y\widehat{Z}_T^0(y))] - yv'(y) + y\mathbf{E} [\widehat{Z}_T^0(y)e_T] \\ &= \mathbf{E} [V(y\widehat{Z}_T^0(y)) + y\widehat{Z}_T^0(y)e_T] + xy, \end{aligned}$$

therefore,

$$u(x) = \mathbf{E} [U(x + \widehat{g}(x) + e_T)] = u_\infty(x),$$

and $\widehat{g}(x)$ is the unique optimizer of the primal problem. \square

Lemma 5.2.20. *We have the following formulae*

$$u'(x) = \mathbf{E} [U'(x + \widehat{g}(x) + e_T)], \quad (5.2.35)$$

$$xu'(x) = \mathbf{E} [(x + \widehat{g}(x))U'(x + \widehat{g}(x) + e_T)]. \quad (5.2.36)$$

Proof. To show (5.2.35), we note that

$$u'(x) = y = \mathbf{E} [y\widehat{Z}_T^0(y)] = \mathbf{E} [U'(-V'(y\widehat{Z}_T^0(y)))] = \mathbf{E} [U'(x + \widehat{g}(x) + e_T)].$$

The formula (5.2.36) is a reformulation of equation (5.2.26), indeed,

$$\begin{aligned} xu'(x) = -v'(y)y &= \mathbf{E} [y\widehat{Z}_T^0(y) (-V'(y\widehat{Z}_T^0(y)) - e_T)] \\ &= \mathbf{E} [(x + \widehat{g}(x))U'(x + \widehat{g}(x) + e_T)]. \end{aligned}$$

which follows from the relation between the primal and dual optimizers. \square

Following the same way as in the proof of [86, Corollary 3.2 (v)] we obtain the following properties of the value functions u and v .

Lemma 5.2.21. *The function u is conjugate to v , has reasonable asymptotic elasticity and satisfies the Inada conditions*

$$u'(-\infty) = \infty, \quad u'(\infty) = 0.$$

Proof. It follows from the fact that u_n and v_n are conjugate, and that u_n and v_n converge monotonically to u and v , respectively.

Using Corollary 5.2.2, we have that, for each $\lambda > 0$, there is a constant $C > 0$ such that

$$v(\lambda y) = \mathbf{E} \left[V \left(\lambda y \frac{d\widehat{Q}^r(\lambda y)}{d\mathbf{P}} \right) \right] \leq \mathbf{E} \left[V \left(\lambda y \frac{d\widehat{Q}^r(y)}{d\mathbf{P}} \right) \right] \leq C \mathbf{E} \left[V \left(y \frac{d\widehat{Q}^r(y)}{d\mathbf{P}} \right) \right] = Cv(y).$$

It follows from [94, Proposition 4.1] and [70, Corollary 6.1], that u has reasonable asymptotic elasticity.

Suppose for a contradiction that $u'(-\infty) = \alpha < \infty$. Then, for all $\varepsilon > 0$, there exists an x_0 such that for all $x \leq x_0$ we have $\alpha - \varepsilon \leq u'(x) \leq \alpha$. It follows that, for $x \leq x_0$,

$$AE_{-\infty}(u) \leq \liminf_{x \rightarrow -\infty} \frac{xu'(x)}{u(x)} \leq \liminf_{x \rightarrow -\infty} \frac{\alpha x}{u(x_0) + (\alpha - \varepsilon)(x - x_0)} = \frac{\alpha}{\alpha - \varepsilon}.$$

Since this holds true for all $\varepsilon > 0$, we have $AE_{-\infty}(u) = 1$, which is a contradiction.

Suppose for a contradiction that $u'(\infty) = \alpha > 0$. Then, for all $\varepsilon > 0$, there exists an x_0 such that for all $x \geq x_0$ we have $\alpha \leq u'(x) \leq \alpha + \varepsilon$. It follows that, for $x \geq x_0$,

$$AE_{+\infty}(u) \geq \limsup_{x \rightarrow \infty} \frac{xu'(x)}{u(x)} \geq \limsup_{x \rightarrow \infty} \frac{\alpha x}{u(x_0) + (\alpha + \varepsilon)(x - x_0)} = \frac{\alpha}{\alpha + \varepsilon}.$$

Since this holds true for all $\varepsilon > 0$, we have $AE_{+\infty}(u) = 1$, which is a contradiction. \square

We see that $g \in \mathcal{C}_U^\lambda$ may attain the value $+\infty$. Especially in the case when $U(\infty) < \infty$, this is natural. We now consider the question, whether there exists a self-financing trading strategy $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ under transaction costs λ , that attains the solution $\widehat{g}(x)$ to (5.2.1), i.e., $V_T^{\text{liq}}(\widehat{\varphi}) = \widehat{g}(x)$, and hence \widehat{g} is almost surely finite.

As in [28] we define the set $\mathcal{A}_U^\lambda(x)$ of all predictable finite variation processes (φ^0, φ^1) , starting at $(\varphi_0^0, \varphi_0^1) = (x, 0)$, satisfying the λ -self-financing condition (2.1.1) and such that there exists a sequence $(\varphi^{n,0}, \varphi^{n,1})_{n \in \mathbb{N}} \subseteq \mathcal{A}_{adm}^\lambda(x)$ verifying that $U(x + V_T^{\text{liq}}(\varphi^n) + e_T) \in L^1(\mathbf{P})$,

$$U(x + V_T^{\text{liq}}(\varphi^n) + e_T) \xrightarrow{L^1(\mathbf{P})} U(x + V_T^{\text{liq}}(\widehat{\varphi}) + e_T)$$

and

$$\mathbf{P} [(\widetilde{\varphi}_t^{n,0}, \widetilde{\varphi}_t^{n,1}) \rightarrow (\widehat{\varphi}_t^0, \widehat{\varphi}_t^1), \forall t \in [0, T]] = 1.$$

We simply write \mathcal{A}_U^λ for $\mathcal{A}_U^\lambda(0)$.

The following proposition shows that the existence of a strictly consistent price system with finite V -expectation guarantees the existence of trading strategies attaining the primal optimizer and the strict positivity of dual optimizers. It is a generalization of [10, Lemma 25] and [28, Proposition 3.2] to our setting and its proof follows by similar arguments.

Proposition 5.2.22. *Under the assumptions of Theorem 5.2.4, suppose further that, for some $\lambda' \in (0, \lambda)$, there exists a λ' -consistent price system $(\overline{Z}^0, \overline{Z}^1) \in \mathcal{Z}_e^{\lambda'}(S)$, such that*

$$\mathbf{E}[V(\overline{y}\overline{Z}_T^0)] < \infty,$$

for some $\bar{y} > 0$.

Then the solution to the primal problem (5.2.2) is attainable, i.e., there exists a $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in \mathcal{A}_U^\lambda$ such that $V_T^{\text{liq}}(\widehat{\varphi}) = \widehat{g}(x)$, and the dual optimizer $(\widehat{Z}^0, \widehat{Z}^0)$ is in $\mathcal{Z}_e^\lambda(S)$, i.e., a λ -consistent price system.

Proof. By Theorem 5.2.4, there exists a sequence $((\varphi^{n,1}, \varphi^{n,1}))_{n \in \mathbb{N}} \subseteq \mathcal{A}_{adm}^\lambda$ such that

$$U(x + V_T^{\text{liq}}(\varphi^n) + e_T) \xrightarrow{L^1(\mathbf{P})} U(x + \widehat{g}(x) + e_T). \quad (5.2.37)$$

Then, for $\bar{S} := \frac{\bar{Z}^1}{\bar{Z}^0}$, the process $(\bar{Z}_t^0(x + \varphi_t^{n,0} + \varphi_t^{n,1}\bar{S}_t + A_t^n))_{0 \leq t \leq T}$ is a supermartingale for each $n \in \mathbb{N}$, where

$$A_t^n := (\lambda - \lambda') \int_0^t S_u d\varphi_u^{n,1,\downarrow}.$$

Indeed, by integration by parts and the λ -self-financing condition (2.1.1), we obtain

$$\begin{aligned} x + \varphi_t^{n,0} + \varphi_t^{n,1}\bar{S}_t + A_t^n &= x + \varphi_t^{n,0} + \int_0^t \varphi_u^{n,1} d\bar{S}_u + \int_0^t \bar{S}_u d\varphi_u^{n,1} + A_t^n \\ &\leq x + \int_0^t \varphi_u^{n,1} d\bar{S}_u - \int_0^t S_u d\varphi_u^{n,1,\uparrow} + \int_0^t (1 - \lambda) S_u d\varphi_u^{n,1,\downarrow} + \int_0^t \bar{S}_u d\varphi_u^{n,1} + A_t^n \\ &= x + \int_0^t \varphi_u^{n,1} d\bar{S}_u - \int_0^t (S_u - \bar{S}_u) d\varphi_u^{n,1,\uparrow} - \int_0^t (\bar{S}_u - (1 - \lambda') S_u) d\varphi_u^{n,1,\downarrow} \\ &=: x + (\varphi^{n,1} \cdot \bar{S})_t - B_t^n \end{aligned}$$

Since $(1 - \lambda)S_u \leq \bar{S}_u \leq S_u$, the process $(B_t^n)_{0 \leq t \leq T}$ is increasing. It follows by Bayes' rule that \bar{S} is a local martingale under the measure $\bar{\mathbf{Q}} \sim \mathbf{P}$ defined by $\frac{d\bar{\mathbf{Q}}}{d\mathbf{P}} := \bar{Z}_T^0$. As $\varphi^{n,1}$ is of finite variation and hence locally bounded, the stochastic integral $\varphi^{n,1} \cdot \bar{S}$ is a local martingale under $\bar{\mathbf{Q}}$. Therefore, $x + (\varphi^{n,1} \cdot \bar{S})_t - B_t^n$ is a local supermartingale under $\bar{\mathbf{Q}}$. Using Bayes' rule once again, we obtain that

$$(\bar{Z}_t^0(x + \varphi_t^{n,0} + \varphi_t^{n,1}\bar{S}_t + A_t^n))_{0 \leq t \leq T} = (\bar{Z}_t^0(x + (\varphi^{n,1} \cdot \bar{S})_t - B_t^n))_{0 \leq t \leq T}$$

is a local supermartingale under \mathbf{P} . Since $(\varphi^{n,0}, \varphi^{n,1}) \in \mathcal{A}_{adm}^\lambda$, we have

$$\bar{Z}_t^0(x + \varphi_t^{n,0} + \varphi_t^{n,1}\bar{S}_t + A_t^n) \geq \bar{Z}_t^0 V_t^{\text{liq}}(\varphi^n) \geq -M^n \bar{Z}_t^0,$$

for some $M^n \geq 0$. As \bar{Z}^0 is a true martingale, the process $(\bar{Z}_t^0(x + \varphi_t^{n,0} + \varphi_t^{n,1}\bar{S}_t + A_t^n))_{0 \leq t \leq T}$ is a true supermartingale under \mathbf{P} , which implies in particular that

$$\mathbf{E}[\bar{Z}_T^0(x + \varphi_T^{n,0} + A_T^n)] \leq x,$$

and

$$\mathbf{E}[\bar{Z}_T^0(x + \varphi_T^{n,0} + A_T^n + e_T)] \leq x + \rho, \quad (5.2.38)$$

for all $n \in \mathbb{N}$.

By Fenchel's inequality and the monotonicity of U we can estimate

$$\bar{y} \bar{Z}_T^0(x + V_T^{\text{liq}}(\varphi^n) + A_T^n + e_T) \geq U(x + V_T^{\text{liq}}(\varphi^n) + e_T) - V(\bar{y} \bar{Z}_T^0).$$

By the assumption we have that $V(\bar{y}\bar{Z}_T^0) \in L^1(\mathbf{P})$, and it follows by (5.2.37), that $\left(\bar{y}\bar{Z}_T^0(x + V_T^{\text{liq}}(\varphi^n) + A_T^n + e_T)^-\right)_{0 \leq t \leq T}$ is uniformly integrable. Together with (5.2.38) we obtain the sequence $\left(\bar{y}\bar{Z}_T^0(x + V_T^{\text{liq}}(\varphi^n) + A_T^n + e_T)\right)_{0 \leq t \leq T}$ is $L^1(\mathbf{P})$ -bounded.

It follows from $\bar{Z}_T^0 > 0$ and $V_T^{\text{liq}}(\varphi^n) \xrightarrow{\mathbf{P}} \hat{g}(x)$, that $\text{conv}\{A_T^n; n \in \mathbb{N}\}$ is bounded in $L^0(\mathbf{P})$. Since \bar{S} is a nonnegative local martingale under $\bar{\mathbf{Q}}$, hence also a nonnegative supermartingale under $\bar{\mathbf{Q}}$, we see that

$$\inf_{0 \leq u \leq T} S_u \geq \inf_{0 \leq u \leq T} \bar{S}_u > 0$$

by [37, Theorem VI-17]. This implies that $\text{conv}\{\text{Var}_T(\varphi^{n,1}); n \in \mathbb{N}\}$ is bounded in $L^0(\mathbf{P})$, therefore the same for $\text{conv}\{\text{Var}_T(\varphi^{n,0}); n \in \mathbb{N}\}$. By [15, Proposition 3.4] there exists a sequence

$$(\tilde{\varphi}^{n,0}, \tilde{\varphi}^{n,1}) \in \text{conv}\{(\varphi^{k,0}, \varphi^{k,1}); k \geq n\}$$

of convex combinations and a predictable process $(\hat{\varphi}^0, \hat{\varphi}^1)$ of finite variation such that

$$\mathbf{P}[(\tilde{\varphi}_t^{n,0}, \tilde{\varphi}_t^{n,1}) \rightarrow (\hat{\varphi}_t^0, \hat{\varphi}_t^1), \forall t \in [0, T]] = 1.$$

It implies that $(\hat{\varphi}^0, \hat{\varphi}^1)$ is a λ -self-financing trading strategy such that $V_T^{\text{liq}}(\hat{\varphi}) = \hat{g}(x)$, therefore $(\hat{\varphi}^0, \hat{\varphi}^1) \in \mathcal{A}_U^\lambda$.

Since $\hat{g}(x) = V_T^{\text{liq}}(\hat{\varphi}) < \infty$, we have that

$$\hat{y}\hat{Z}_T^0(\hat{y}) = U'(x + \hat{g}(x) + e_T) > 0$$

by the Inada condition. Hence, $(\hat{Z}^0, \hat{Z}^1) \in \mathcal{Z}_e^\lambda(S)$. \square

5.3 Shadow Price

In this section, we study the existence of shadow prices.

For utility maximization problems under proportional transaction costs, it has been observed that the original market with transaction costs can sometimes be replaced by a frictionless shadow market, that yields the same optimal strategy and utility. We adapt the definition of shadow price in Chapter 3 to our setting with random endowment.

Definition 5.3.1. A semimartingale $\tilde{S} = (\tilde{S}_t)_{0 \leq t \leq T}$ is called a shadow price, if

(i) \tilde{S} takes values in the bid-ask spread $[(1 - \lambda)S, S]$.

(ii) The solution $\hat{H} = (\hat{H}_t)_{0 \leq t \leq T}$ to the frictionless utility maximization problem

$$u(x; \tilde{S}) := \sup_{H \in \mathcal{A}_U(\tilde{S})} \mathbf{E}[U(x + (H \cdot \tilde{S})_T + e_T)], \quad (5.3.1)$$

exists in the sense of [86], where $\mathcal{A}_U(\tilde{S})$ denotes the set of all \tilde{S} -integrable predictable processes H , such that there exists a sequence $(H^n)_{n \in \mathbb{N}}$ of admissible

self-financing trading strategies without transaction costs such that $U(x + (H^n \cdot \tilde{S})_T + e_T) \in L^1(\mathbf{P})$ and

$$U(x + (H^n \cdot \tilde{S})_T + e_T) \xrightarrow{L^1(\mathbf{P})} U(x + (H \cdot \tilde{S})_T + e_T).$$

(iii) The optimal trading strategy \hat{H} to the frictionless problem (5.3.1) coincides with the holdings in stocks $\hat{\varphi}^1$ to the utility maximization problem (5.2.2) under transaction costs such that $(\hat{H} \cdot \tilde{S})_T = \hat{g}(x) = V_T^{\text{liq}}(\hat{\varphi})$.

The basic idea is that, a shadow price \tilde{S} (if this exists) allows us to obtain the optimal trading strategy for the utility maximization problem (5.2.2) under transaction costs by solving the frictionless utility maximization problem (5.3.1). As the expected utility for \tilde{S} without transaction costs is a priori higher than that of any other strategy under transaction costs, the shadow price is a least favorable frictionless market lying in the bid-ask spread and leading to the same optimal strategy and utility.

Note that the existence of a shadow price implies that the optimal strategy \hat{H} to the frictionless problem (5.3.1) is of finite variation and that both optimal strategies \hat{H} and $\hat{\varphi}^1$, that coincide $\hat{H} = \hat{\varphi}^1$, only trade, if \hat{S} is at the bid or ask price, i.e.,

$$\{d\hat{\varphi}^1 > 0\} \subseteq \{\hat{S} = S\} \quad \text{and} \quad \{d\hat{\varphi}^1 < 0\} \subseteq \{\hat{S} = (1 - \lambda)S\}$$

in the sense that

$$\begin{aligned} \{d\hat{\varphi}^{1,c} > 0\} &\subseteq \{\tilde{S} = S\}, & \{d\hat{\varphi}^{1,c} < 0\} &\subseteq \{\tilde{S} = (1 - \lambda)S\}, \\ \{\Delta\hat{\varphi}^1 > 0\} &\subseteq \{\tilde{S}_- = S_-\}, & \{\Delta\hat{\varphi}^1 < 0\} &\subseteq \{\tilde{S}_- = (1 - \lambda)S_-\}, \\ \{\Delta_+\hat{\varphi}^1 > 0\} &\subseteq \{\tilde{S} = S\}, & \{\Delta_+\hat{\varphi}^1 < 0\} &\subseteq \{\tilde{S} = (1 - \lambda)S\}. \end{aligned} \quad (5.3.2)$$

Following the similar arguments of [28, Proposition 3.3], we show that the attainability of the primal optimizer by trading strategies implies the existence of a shadow price, which can be obtained as the quotient of a dual optimal process.

Proposition 5.3.2. *Under the assumptions of Theorem 5.2.4, suppose that the solution $\hat{g}(x)$ to the primal problem (5.2.2) is attainable.*

Then, the solution (\hat{Z}^0, \hat{Z}^1) to the dual problem (5.2.3) is a λ -consistent price system, and $\hat{S} := \frac{\hat{Z}^1}{\hat{Z}^0}$ is a shadow price to problem (5.2.2) in the sense of Definition 5.3.1.

Proof. The first assertion has been already shown in the proof of Proposition 5.2.22.

By the assumption there exists a sequence of admissible λ -self-financing trading strategies $(\varphi^{n,0}, \varphi^{n,1})_{n \in \mathbb{N}}$ such that

$$\mathbf{P} [(\varphi_t^{n,0}, \varphi_t^{n,1}) \rightarrow (\hat{\varphi}_t^0, \hat{\varphi}_t^1), \forall t \in [0, T]] = 1, \quad (5.3.3)$$

and

$$U(x + \varphi_T^{n,0} + e_T) \xrightarrow{L^1(\mathbf{P})} U(x + \hat{g}(x) + e_T). \quad (5.3.4)$$

Following along the same arguments as in the proof of Proposition 5.2.22 after replacing (\bar{Z}^0, \bar{Z}^1) by $(\widehat{Z}^0, \widehat{Z}^1)$ and setting $\lambda' = \lambda$ that $(\widehat{Z}_t^0(x + \varphi_t^{n,0}) + \widehat{Z}_t^1\varphi_t^{n,1})_{0 \leq t \leq T}$ is a supermartingale under \mathbf{P} , therefore

$$\mathbf{E} \left[\widehat{Z}_T^0(x + \varphi_T^{n,0} + e_T) \right] \leq x + \rho.$$

As $V(\widehat{y}\widehat{Z}_T^0) \in L^1(\mathbf{P})$, it follows from (5.3.4) and the Fenchel's inequality

$$\widehat{y}\widehat{Z}_T^0(x + \varphi_T^{n,0} + e_T) \geq U(x + \varphi_T^{n,0} + e_T) - V(\widehat{y}\widehat{Z}_T^0),$$

that $(\widehat{y}\widehat{Z}_T^0(x + \varphi_T^{n,0} + e_T)^-)_{0 \leq t \leq T}$ is uniformly integrable. Since $e_T \in L^\infty(\mathbf{P})$, we see that, for each $n \in \mathbb{N}$, the process $\left((\widehat{Z}_t^0(x + \varphi_t^{n,0}) + \widehat{Z}_t^1\varphi_t^{n,1})^- \right)_{0 \leq t \leq T}$ is a nonnegative submartingale and hence of class (D) so that $\left((\widehat{Z}_\tau^0(x + \varphi_\tau^{n,0}) + \widehat{Z}_\tau^1\varphi_\tau^{n,1})^- \right)_{n \in \mathbb{N}}$ is uniformly integrable for every $[0, T]$ -valued stopping time τ . Therefore we may use Fatou's lemma to show that $(\widehat{Z}_t^0(x + \widehat{\varphi}_t^0) + \widehat{Z}_t^1\widehat{\varphi}_t^1)_{0 \leq t \leq T}$ is a supermartingale under \mathbf{P} . By Theorem 5.2.4 (4) we have that

$$x = \mathbf{E} \left[\widehat{Z}_T^0(x + \widehat{\varphi}_T^0) \right],$$

hence $(\widehat{Z}_t^0(x + \widehat{\varphi}_t^0) + \widehat{Z}_t^1\widehat{\varphi}_t^1)_{0 \leq t \leq T}$ is a martingale under \mathbf{P} .

By integration by parts we obtain that

$$\begin{aligned} \widehat{Z}_t^0(x + \widehat{\varphi}_t^0) + \widehat{Z}_t^1\widehat{\varphi}_t^1 &= \widehat{Z}_t^0 \left(x + \widehat{\varphi}_t^0 + \widehat{\varphi}_t^1 \widehat{S}_t \right) \\ &= \widehat{Z}_t^0 \left(x + \int_0^t d\widehat{\varphi}_u^0 + (\widehat{\varphi}^1 \cdot \widehat{S})_t + \int_0^t \widehat{S}_u d\widehat{\varphi}_u^1 \right) \\ &= \widehat{Z}_t^0 \left(x + (\widehat{\varphi}^1 \cdot \widehat{S})_t - \int_0^t (S_u - \widehat{S}_u) d\widehat{\varphi}_u^{1,\uparrow} - \int_0^t (\widehat{S}_u - (1 - \lambda)S_u) d\widehat{\varphi}_u^{1,\downarrow} \right) \\ &=: \widehat{Z}_t^0 \left(x + (\widehat{\varphi}^1 \cdot \widehat{S})_t - A_t \right) = \widehat{Z}_t^0 \left(x + (\widehat{\varphi}^1 \cdot \widehat{S})_t \right) - \widehat{Z}_t^0 A_t. \end{aligned}$$

Again as in the proof of Proposition 5.2.22 by Bayes' rule $\widehat{Z}^0(x + \widehat{\varphi}^1 \cdot \widehat{S})$ is a local martingale and $\widehat{Z}^0 A$ is an increasing process. This implies that $A \equiv 0$ and therefore

$$\{d\widehat{\varphi}^1 > 0\} \subseteq \{\widehat{S} = S\} \quad \text{and} \quad \{d\widehat{\varphi}^1 < 0\} \subseteq \{\widehat{S} = (1 - \lambda)S\}$$

in the sense of (5.3.2).

It is clear that,

$$u(x; \widehat{S}) \leq \mathbf{E} [V(yZ_T) + yZ_T e_T] + xy,$$

for $y > 0$ and $Z_T \in \mathcal{Z}_a(\widehat{S})$. As $(\widehat{Z}^0, \widehat{Z}^1) \in \mathcal{Z}_e^\lambda(S)$, we obtain that \widehat{Z}^0 is the density process of an equivalent local martingale measure for the frictionless process \widehat{S} , therefore

$$\mathbf{E} \left[V(\widehat{y}\widehat{Z}_T^0) + \widehat{y}\widehat{Z}_T^0 e_T \right] + x\widehat{y} = u(x) \leq u(x; \widehat{S}) \leq \mathbf{E} \left[V(\widehat{y}\widehat{Z}_T^0) + \widehat{y}\widehat{Z}_T^0 e_T \right] + x\widehat{y}.$$

It follows from the frictionless duality theorem [86, Theorem 1.1] that $x + (\widehat{\varphi}^1 \cdot \widehat{S})_T + e_T = x + \widehat{g}(x) + e_T = -V'(\widehat{y}\widehat{Z}_T^0)$ is the optimal terminal wealth to the frictionless utility maximization problem (5.3.1) for \widehat{S} . Since $\widehat{\varphi} \cdot \widehat{S}$ is a martingale under $\widehat{\mathbf{Q}}$ defined by $\frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}} := \widehat{Z}_T^0$, we obtain that $\widehat{\varphi}^1$ has to be the optimal strategy and in $\mathcal{A}_U(x; \widehat{S})$ by [86, Theorem 1.1.(v)].

This implies that \widehat{S} is a shadow price in the sense of Definition 5.3.1 for the utility maximization problem (5.2.2) under transaction costs. \square

We observe that the solution to the primal problem (5.2.2) is not necessarily attainable, i.e., there may not exist an optimal λ -self-financing trading strategy $(\widehat{\varphi}^0, \widehat{\varphi}^1)$, such that $\widehat{\varphi}_T^0 = \widehat{g}(x)$. However, the solution $(\widehat{Z}^0, \widehat{Z}^1)$ to the dual problem is always a local martingale (an absolutely continuous consistent price system). We may define the following generalized shadow price, which only leads to the same optimal utility as the one under transaction costs.

Definition 5.3.3. *In the above setting, a semimartingale $\widetilde{S} = (\widetilde{S}_t)_{0 \leq t \leq T}$ is called a generalized shadow price for the optimization problem (5.2.1), if*

(i). \widetilde{S} takes values in the bid-ask spread $[(1 - \lambda)S, S]$.

(ii). The solution $\widetilde{g} \in \mathcal{C}_U(\widetilde{S})$ to the corresponding frictionless utility maximization problem

$$u(x; \widetilde{S}) := \sup_{g \in \mathcal{C}_U(\widetilde{S})} \mathbf{E}[U(x + g + e_T)] \quad (5.3.5)$$

exists and coincides with the solution $\widehat{g} \in \mathcal{C}_U^\lambda$ to (5.2.1) under transaction costs, where

$$\mathcal{C}_U(\widetilde{S}) := \left\{ g \in L^0(\mathbb{R} \cup \{\infty\}) \left| \begin{array}{l} \exists g_n \in \mathcal{C}(\widetilde{S}) \text{ s.t. } U(x + g_n + e_T) \in L^1(\mathbf{P}) \\ \text{and } U(x + g_n + e_T) \xrightarrow{L^1(\mathbf{P})} U(x + g + e_T) \end{array} \right. \right\},$$

and

$$\mathcal{C}(\widetilde{S}) := \{g \in L^0 \mid g \leq (H \cdot S)_T \text{ for some admissible portfolio } H\}.$$

Remark 5.3.4. In the duality theorem of the utility maximization problem with utility functions defined on the positive real line, the existence of an optimal trading strategy $\widehat{\varphi} \in \mathcal{A}_{adm}^\lambda$ follows directly from the existence of the dual optimizer $\widehat{g} \in \mathcal{C}^\lambda$. Therefore, it is quite natural to require in the classical definition of shadow price that the optimal trading strategy in the frictionless shadow market is also an optimal one in the original market with transaction costs. For the problem with utility functions defined on the whole real line, we have in general no chance to find the optimal strategy. This is our motivation to define the generalized shadow price in such a way.

Theorem 5.3.5. *The process $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ is a generalized shadow price by the definition above, where $(\widehat{Z}^0, \widehat{Z}^1) \in \mathcal{Z}_a^\lambda(S)$ is the solution to the dual problem (5.2.3).*

Remark 5.3.6. We consider (2.1.3) to be satisfied if $\frac{\widehat{Z}^1}{\widehat{Z}^0} = \frac{0}{0}$.

Proof. From the definition of $\mathcal{C}_U(\widehat{S})$ and $\mathcal{C}(\widehat{S})$, we know

$$u(x; \widehat{S}) = \sup_{g \in \mathcal{C}(\widehat{S})} \mathbf{E}[U(x + g + e_T)].$$

Since $\mathcal{C}^\lambda \subseteq \mathcal{C}(\widehat{S})$, then

$$u(x) = \sup_{g \in \mathcal{C}^\lambda} \mathbf{E}[U(x + g + e_T)] \leq \sup_{g \in \mathcal{C}(\widehat{S})} \mathbf{E}[U(x + g + e_T)] = u(x; \widehat{S}). \quad (5.3.6)$$

Moreover,

$$\begin{aligned} \mathcal{D}(\widehat{S}) &:= \{Q \in ba \mid \|Q\| = 1 \text{ and } \langle Q, g \rangle \leq 0 \text{ for all } g \in \mathcal{C}(\widehat{S}) \cap L^\infty\} \\ &\subseteq \{Q \in ba \mid \|Q\| = 1 \text{ and } \langle Q, g \rangle \leq 0 \text{ for all } g \in \mathcal{C}^\lambda \cap L^\infty\} = \mathcal{D}^\lambda. \end{aligned} \quad (5.3.7)$$

Let $\widehat{y} := u'(x)$. Now consider the following value function

$$v(\widehat{y}; \widehat{S}) := \inf_{\mathbf{Q} \in \mathcal{M}_a(\widehat{S})} \mathbf{E} \left[V \left(\widehat{y} \frac{d\mathbf{Q}}{d\mathbf{P}} \right) + \widehat{y} \frac{d\mathbf{Q}}{d\mathbf{P}} e_T \right].$$

By [46, Corollary A.2], the formulation of the function $v(\cdot)$ is equivalent to

$$v(\widehat{y}; \widehat{S}) = \inf_{Q \in \mathcal{D}(\widehat{S})} \left\{ \mathbf{E} \left[V \left(\widehat{y} \frac{dQ^r}{d\mathbf{P}} \right) \right] + \widehat{y} \langle Q, e_T \rangle \right\}. \quad (5.3.8)$$

Then, we deduce from (5.3.7),

$$v(\widehat{y}; \widehat{S}) \geq \inf_{Q \in \mathcal{D}^\lambda} \left\{ \mathbf{E} \left[V \left(\widehat{y} \frac{dQ^r}{d\mathbf{P}} \right) \right] + \widehat{y} \langle Q, e_T \rangle \right\} = v(\widehat{y}). \quad (5.3.9)$$

As $(\widehat{Z}^0, \widehat{Z}^1) \in \mathcal{Z}_a^\lambda(S)$, we have that the measure $\widehat{\mathbf{Q}}$, defined by $\frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}} = \widehat{Z}_T^0$, is an absolutely continuous martingale measure for \widehat{S} , i.e., $\widehat{\mathbf{Q}} \in \mathcal{M}_a(\widehat{S})$. Hence, we deduce that \widehat{Z}_T^0 a fortiori is the optimizer for $v(\widehat{y}; \widehat{S})$. In particular, $v(\widehat{y}) = v(\widehat{y}; \widehat{S})$. It follows from Theorem 5.2.4, Fenchel's inequality and (5.3.6) that

$$u(x) = v(\widehat{y}) + x\widehat{y} = v(\widehat{y}; \widehat{S}) + x\widehat{y} \geq \inf_{y > 0} \left\{ v(y; \widehat{S}) + xy \right\} \geq u(x; \widehat{S}) \geq u(x), \quad (5.3.10)$$

therefore the primal value functions coincide. In the frictionless market, we have a posteriori $u(x; \widehat{S}) < U(\infty)$. By the uniqueness of the primal solution and $\mathcal{C}^\lambda \subseteq \mathcal{C}(\widehat{S})$, the primal optimizer to (5.3.5) exists, is unique and coincides with the one to the optimization problem (5.2.1). \square

Remark 5.3.7. In the theorem above, it is not clear whether equivalent martingale measures for the shadow market \widehat{S} exist or not, except for the case where $(\widehat{Z}^0, \widehat{Z}^1)$ is strictly positive. We stress that the following inequity in (5.3.10) still holds true under the assumption $\mathcal{M}_a(\widehat{S}) \neq \emptyset$:

$$u(x; \widehat{S}) \leq \inf_{y > 0} \left\{ v(y; \widehat{S}) + xy \right\}.$$

Indeed, this follows from Fenchel's inequality and the easy part of the superreplication theorem in the frictionless setting, which could be deduced under the weaker assumption $\mathcal{M}_a(\widehat{S}) \neq \emptyset$. Furthermore, we observe that there is no duality gap, i.e.,

$$u(x; \widehat{S}) = v(\widehat{y}; \widehat{S}) + x\widehat{y} = \inf_{y>0} \left\{ v(y; \widehat{S}) + xy \right\},$$

and there exist at least a primal optimizer (which may not be attained by trading strategies) and a dual one in the shadow market, which coincide with the ones in the original market with transaction costs.

Remark 5.3.8. The fact that $\widehat{Z}_T^0 \in \mathcal{M}_a^\lambda$ (or \mathcal{M}_e^λ) is the unique solution to the dual problem (5.2.3) does not mean the uniqueness of the couple $(\widehat{Z}^1, \widehat{Z}^1) \in \mathcal{Z}_a^\lambda(S)$ (or $\mathcal{Z}_e^\lambda(S)$). In another word, the shadow price process need not be unique.

Conversely, the following result shows that, if a (generalized) shadow price \widehat{S} exists as above and satisfies $\mathcal{M}_e(\widehat{S}) \neq \emptyset$, it is necessarily derived from a dual minimizer. (Compare [28, Proposition 3.8].)

Proposition 5.3.9. *If a (generalized) shadow price \widehat{S} exists as above and satisfies $\mathcal{M}_e(\widehat{S}) \neq \emptyset$, then there exists a \mathbf{P} -martingale \widehat{Z}^0 , such that $(\widehat{Z}^0, \widehat{Z}^0\widehat{S}) \in \mathcal{Z}_a^\lambda(S)$ is a solution to the dual problem (5.2.3).*

Proof. Choose $\mathbf{Q} \in \mathcal{M}_a(\widehat{S})$ and denote by Z its density process. It obvious that $(Z^0, Z^1) := (Z, Z\widehat{S}) \in \mathcal{Z}_a^\lambda(\widehat{S})$. Moreover, from $\mathcal{M}_a(\widehat{S}) \subseteq \mathcal{M}_a^\lambda$ and [94, Theorem 2.2], we have

$$\begin{aligned} u(x) &= v(\widehat{y}(x)) + x\widehat{y}(x) \leq v(\widehat{y}(x; \widehat{S})) + x\widehat{y}(x; \widehat{S}) \\ &\leq v(\widehat{y}(x; \widehat{S}); \widehat{S}) + x\widehat{y}(x; \widehat{S}) = u(x; \widehat{S}) = u(x), \end{aligned}$$

which implies $\widehat{y}(x) = \widehat{y}(x; \widehat{S})$ and $v(\widehat{y}(x)) = v(\widehat{y}(x; \widehat{S}); \widehat{S})$, hence $(\widehat{Z}^0, \widehat{Z}^1) := (\widehat{Z}, \widehat{Z}\widehat{S}) \in \mathcal{Z}_a^\lambda$ is the solution to the frictional dual problem (5.2.3), where $\widehat{Z} \in \mathcal{M}_a(\widehat{S})$ is the solution to its frictionless counterpart for the shadow price process \widehat{S} . \square

Remark 5.3.10. The assumption $\mathcal{M}_e(\widehat{S}) \neq \emptyset$ ensures that we could apply the result of [94, Theorem 2.2] to the frictionless market with \widehat{S} , in particular, we could deduce the following equality

$$v(\widehat{y}(x; \widehat{S}); \widehat{S}) + x\widehat{y}(x; \widehat{S}) = u(x; \widehat{S}).$$

5.4 Application to exponential pricing

It is known from the so-called ‘‘face-lifting theorem’’ that, under transaction costs, the bounds for option prices obtained from superreplication arguments are only the trivial bounds. (See e.g., [50].) Therefore the concepts of superreplication do not make sense economically in the presence of transaction costs. However, the concept of a utility indifference price makes perfect economic sense in the presence of transaction costs. (See e.g., [56].)

We denote now the value function by $u^{e_T}(x)$ instead of $u(x)$ to emphasize the dependence on e_T and u^0 denotes the value function of utility maximization problem without random endowment. The utility indifference price is the solution $p(x)$ of

$$u^{e_T}(x - p(x)) = u^0(x).$$

Let us consider the exponential utility function

$$U(x) = -\exp(-\gamma x), \quad x \in \mathbb{R},$$

where $\gamma > 0$ stands for the absolute risk aversion parameter. In this case, using the duality result, we could obtain a dual formulation for the utility based price.

For the exponential utility function $U(x)$, we have

$$V(y) = \frac{y}{\gamma} \left(\log \left(\frac{y}{\gamma} \right) - 1 \right), \quad y > 0.$$

Lemma 5.4.1. *For the exponential utility function, we have that*

$$u^{e_T}(x) = \inf_{Z_T^0 \in \mathcal{M}_a^\lambda} U \left(\frac{1}{\gamma} \mathbf{E} [Z_T^0 \log (Z_T^0)] + \mathbf{E} [Z_T^0 e_T] + x \right),$$

for all $x \in \mathbb{R}$.

Proof. The proof of the above lemma follows from Theorem 5.2.4 and is similar to the one of [10, Proposition 11]. Fix $Z_T^0 \in \mathcal{M}_a^\lambda$. Then, we obtain that

$$\begin{aligned} \inf_{y>0} \{ \mathbf{E} [V(yZ_T^0) + yZ_T^0 e_T] + xy \} &= \inf_{y>0} \left\{ \mathbf{E} \left[\frac{yZ_T^0}{\gamma} \left(\log \left(\frac{yZ_T^0}{\gamma} \right) - 1 \right) + yZ_T^0 e_T \right] + xy \right\} \\ &= \inf_{y>0} \left\{ \frac{y}{\gamma} \left(\log \left(\frac{y}{\gamma} \right) - 1 \right) + y \left(\mathbf{E} \left[\frac{Z_T^0}{\gamma} \log (Z_T^0) + Z_T^0 e_T \right] + x \right) \right\} \\ &= U \left(\mathbf{E} \left[\frac{Z_T^0}{\gamma} \log (Z_T^0) + Z_T^0 e_T \right] + x \right). \end{aligned} \tag{5.4.1}$$

It follows by Theorem 5.2.4 that

$$\begin{aligned} u^{e_T}(x) &= \inf_{y>0} \inf_{Z_T^0 \in \mathcal{M}_a^\lambda} \{ \mathbf{E} [V(yZ_T^0) + yZ_T^0 e_T] + xy \} \\ &= \inf_{Z_T^0 \in \mathcal{M}_a^\lambda} U \left(\mathbf{E} \left[\frac{Z_T^0}{\gamma} \log (Z_T^0) + Z_T^0 e_T \right] + x \right), \end{aligned}$$

which finishes the proof. \square

Lemma 5.4.2. *For all $x \in \mathbb{R}$, the utility based price of e_T equals*

$$\begin{aligned} p(x) &= U^{-1}(u^{e_T}(x)) - U^{-1}(u^0(x)) \\ &= \inf_{Z_T^0 \in \mathcal{M}_a^\lambda} \mathbf{E} \left[\frac{Z_T^0}{\gamma} \log (Z_T^0) + Z_T^0 e_T + x \right] + \sup_{Z_T^0 \in \mathcal{M}_a^\lambda} \mathbf{E} \left[-\frac{Z_T^0}{\gamma} \log (Z_T^0) - x \right]. \end{aligned}$$

Proof. By the special property of the exponential function we have that

$$u^{e_T}(x+w) = e^{-\gamma w} u^{e_T}(x). \quad (5.4.2)$$

In particular, $u^{e_T}(x) = e^{-\gamma x} u^{e_T}(0)$, which follows that

$$\lim_{x \rightarrow -\infty} u^{e_T}(x) = -\infty, \quad \lim_{x \rightarrow \infty} u^{e_T}(x) = 0.$$

Since u^{e_T} is concave, continuous and strictly increasing, there exists a solution of the equation $u^{e_T}(x-p) = u^0(x)$, denoted by $p(x)$.

Again by (5.4.2) we have that

$$\exp(\gamma p(x)) u^{e_T}(x) = u^{e_T}(x-p(x)) = u^0(x).$$

The assertion follows by a simple computation and Lemma 5.4.1. \square

Corollary 5.4.3. *Under the assumptions for Theorem 5.2.4, the utility based pricing can be represented by the solution of dual problem on shadow markets, i.e.,*

$$\begin{aligned} p(x) &= \inf_{Z_T^0 \in \mathcal{M}_a^\lambda} \mathbf{E} \left[\frac{Z_T^0}{\gamma} \log \left(\frac{Z_T^0}{\gamma} \right) - \frac{Z_T^0}{\gamma} + Z_T^0 e_T \right] - \inf_{Z_T^0 \in \mathcal{M}_a^\lambda} \mathbf{E} \left[\frac{Z_T^0}{\gamma} \log \left(\frac{Z_T^0}{\gamma} \right) - \frac{Z_T^0}{\gamma} \right] \\ &= v(1; \widehat{S}(x; e_T)) - v(1; \widehat{S}(x)), \end{aligned}$$

where $\widehat{S}(x; e_T)$ is the generalized shadow price corresponding to the problem (5.2.1) with x and e_T , while $\widehat{S}(x)$ is the one corresponding to the (5.2.1) with x but without random endowment.

Remark 5.4.4. The choice of the generalized shadow price will not alter the above result.

Chapter 6

On the Existence of Shadow Prices for Optimal Investment with Random Endowment under No-Short-Selling Constraints

In this chapter, we consider a numéraire-based utility maximization problem under proportional transaction costs and random endowment. Assuming that the agent cannot short sell assets and is endowed with a strictly positive contingent claim, a primal optimizer of this utility maximization problem exists. Moreover, we observe that the original market with transaction costs can be replaced by a frictionless shadow market that yields the same optimality. On the other hand, we present an example to show that in some case when these constraints are relaxed, the existence of shadow prices is still warranted.

6.1 Formulation of the Problem

Again we consider a financial market consisting of two assets, one bond and one stock, where the price of the bond B is constant and normalized to $B \equiv 1$. The price process of the stock $S = (S_t)_{0 \leq t \leq T}$ is a strictly positive and càdlàg, which is based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ satisfying the usual hypotheses of right continuity and saturatedness, where \mathcal{F}_0 is assumed to be trivial. Here, T is a finite time horizon and we assume $\mathcal{F}_{T-} = \mathcal{F}_T$.

We introduce proportional transaction costs $0 < \lambda < 1$ for the trading of the stock, which models the width of the bid-ask spread $[(1 - \lambda)S, S]$.

In this chapter, we shall consider a utility maximization problem similar to the one in Benedetti et al. [4], where the agent is facing the no-short-selling constraint, which forces him to keep both the amount of bond and the number of stock shares positive. In other words, the agent is only allowed to trade with the admissible strategies defined as follows:

Definition 6.1.1. *Under transaction costs $\lambda \in (0, 1)$, a self-financing strategy $\varphi = (\varphi^0, \varphi^1)$ with $(\varphi_0^0, \varphi_0^1) = (x, 0)$ and $(\varphi_T^0, \varphi_T^1) = (\varphi_T^0, 0)$ is called admissible under the*

no-short-selling constraint, if for each $t \in [0, T]$ we have that

$$\varphi_t^0 \geq 0 \text{ and } \varphi_t^1 \geq 0, \quad a.s.$$

We denote by $\mathcal{A}_+^\lambda(x)$ the collection of all such strategies. Moreover, we define

$$\mathcal{C}_+^\lambda(x) := \left\{ g \in L_+^0 \mid g \leq \varphi_T^0, \text{ for some } (\varphi^0, \varphi^1) \in \mathcal{A}_+^\lambda(x) \right\}.$$

We also assume that as well as trading in this market, the agent is endowed with an exogenous random endowment \bar{e}_T at the terminal time T , which is represented by an \mathcal{F}_T -measurable random variable.

Assumption 6.1.2. *The endowment \bar{e}_T is a strictly positive and finite-valued random variable, which can be decomposed into a deterministic part and a random part, i.e., $\bar{e}_T = x + e_T$, where $x > 0$ and $e_T \geq 0$, a.s.*

Remark 6.1.3. Indeed, for the utility maximization problem, it has little matter when the agent receives the deterministic and riskless endowment x . Thus we may assume that x is the initial wealth of the agent and e_T is endowed at time T . We restrict our attention to the trading strategies starting from $(\varphi_0^0, \varphi_0^1) = (x, 0)$.

Let $U : (0, \infty) \rightarrow \mathbb{R}$ be a standard utility function defined on the positive real line satisfying the Inada conditions

$$U'(0) := \lim_{x \rightarrow 0} U'(x) = \infty \quad \text{and} \quad U'(\infty) := \lim_{x \rightarrow \infty} U'(x) = 0,$$

and the condition of reasonable asymptotic elasticity (RAE)

$$AE(U) := \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

Then, the problem for the agent is to maximize expected utility at terminal time T from his bond account derived from trading and the random endowment, i.e.,

$$u(x; e_T) := \sup_{g \in \mathcal{C}_+^\lambda(x)} \mathbf{E}[U(g + e_T)]. \quad (6.1.1)$$

Consistent price systems (CPSs) play an important role in the framework with transaction costs (compare, e.g., [60, 97]). In the present chapter, to establish the utility maximization problem under the no-short-selling constraint, we adopt an extended notion – λ -supermartingale-CPSs, similarly defined as in [4].

Definition 6.1.4. *Fix $\lambda > 0$ and the price process S . A λ -supermartingale-CPS is a couple of two positive processes $Z = (Z_t^0, Z_t^1)_{0 \leq t \leq T}$ consisting of two supermartingales Z^0 and Z^1 , such that*

$$S_t^Z := \frac{Z_t^1}{Z_t^0} \in [(1 - \lambda)S_t, S_t], \quad a.s., \quad (6.1.2)$$

for all $0 \leq t \leq T$.

The set of all λ -supermartingale-CPSs is denoted by $\mathcal{Z}_{sup}^\lambda$.

We introduce now the following assumption on the existence of a supermartingale-CPS, which is an analogue to the existence of an equivalent supermartingale density in the frictionless setting.

Assumption 6.1.5. *For some $0 < \lambda' < \lambda$, we have that $\mathcal{Z}_{sup}^{\lambda'} \neq \emptyset$.*

For $Z \in \mathcal{Z}_{sup}^{\lambda}$, define $S^Z := \frac{Z^1}{Z^0}$. By the definition, S^Z is a positive semimartingale taking values in $[(1 - \lambda)S, S]$. Then, we can construct a frictionless market consisting of one bond with zero interest rate and an underlying asset, whose price process is S^Z . Adapting the previous setting under transaction costs, we adopt the following notion of self-financing trading strategies.

Definition 6.1.6. *In the frictionless market associated with S^Z , an \mathbb{R}^2 -valued predictable process $\tilde{\varphi} := (\tilde{\varphi}_t^0, \tilde{\varphi}_t^1)$ starting from $(x, 0)$ is a self-financing trading strategy, if $\tilde{\varphi}^1$ is S^Z -integrable and*

$$\tilde{\varphi}_t^0 + \tilde{\varphi}_t^1 S_t^Z = x + \int_0^t \tilde{\varphi}_u^1 dS_u^Z, \quad 0 \leq t \leq T.$$

Here, $\tilde{\varphi}_t^0$ and $\tilde{\varphi}_t^1$ describe the amount of bond and the number of stock shares held at time $t \in [0, T]$.

We shall formulate a utility maximization problem for the frictionless model with S^Z . In accordance with (6.1.1), we always assume that neither asset can be shorted, so that the maximization problem is established over all admissible strategies defined as follows.

Definition 6.1.7. *Let $S^Z := \frac{Z^1}{Z^0}$, for some $Z \in \mathcal{Z}_{sup}^{\lambda}$. A self-financing strategy $\tilde{\varphi}$ is admissible under the no-short-selling constraint, if we have*

$$\tilde{\varphi}_t^0 \geq 0 \text{ and } \tilde{\varphi}_t^1 \geq 0, \quad \text{a.s.},$$

for all $0 \leq t \leq T$.

We denote by $\mathcal{A}_+^Z(x)$ the collection of all such admissible trading strategies under the no-short-selling constraint starting from $(x, 0)$. Moreover, we define

$$\mathcal{C}_+^Z(x) := \{ \tilde{g} \in L_+^0 \mid \tilde{g} \leq \tilde{\varphi}_T^0 + \tilde{\varphi}_T^1 S_T^Z, \text{ for some } (\tilde{\varphi}^0, \tilde{\varphi}^1) \in \mathcal{A}_+^Z(x) \}.$$

Lemma 6.1.8. *Any payoff in the original market S with transaction costs can be dominated by that in the potentially more favorable frictionless markets, which is within the bid-ask spread. Namely, fix $Z \in \mathcal{Z}_{sup}^{\lambda}$ and let $(\varphi^0, \varphi^1) \in \mathcal{A}_+^{\lambda}(x)$ be arbitrary, then there exists a $(\tilde{\varphi}^0, \tilde{\varphi}^1) \in \mathcal{A}_+^Z(x)$ such that*

$$\tilde{\varphi}_t^0 \geq \varphi_t^0 \text{ and } \tilde{\varphi}_t^1 \geq \varphi_t^1, \quad \text{a.s.}, \quad (6.1.3)$$

for all $0 \leq t \leq T$.

Proof. For any $(\varphi^0, \varphi^1) \in \mathcal{A}_+^{\lambda}(x)$, since (φ^0, φ^1) is a λ -self-financing trading strategy, we have that

$$\begin{aligned} \varphi_t^0 + \varphi_t^1 S_t^Z &= x + \int_0^t d\varphi_u^0 + \int_0^t \varphi_{u-}^1 dS_u^Z + \int_0^t S_u^Z d\varphi_u^1 \\ &= x + \int_0^t (d\varphi_u^0 + S_u^Z d\varphi_u^1) + \int_0^t \varphi_{u-}^1 dS_u^Z \leq x + \int_0^t \varphi_{u-}^1 dS_u^Z. \end{aligned}$$

Then, define a self-financing trading strategy in the frictionless market associated with S^Z by

$$\begin{cases} \tilde{\varphi}_t^0 := x + \int_0^t \varphi_{u-}^1 dS_u^Z - \varphi_t^1 S_t^Z \geq \varphi_t^0, \\ \tilde{\varphi}_t^1 := \varphi_t^1, \end{cases}$$

which satisfies (6.1.3). □

Obviously, $\mathcal{C}_+^\lambda(x) \subseteq \mathcal{C}_+^Z(x)$, for any $Z \in \mathcal{Z}_{sup}^\lambda$. Therefore, letting

$$u^Z(x; e_T) := \sup_{\tilde{g} \in \mathcal{C}_+^Z(x)} \mathbf{E}[U(\tilde{g} + e_T)],$$

it follows that

$$u(x; e_T) \leq \inf_{Z \in \mathcal{Z}_{sup}^\lambda} u^Z(x; e_T),$$

which means each frictionless market with S^Z affords better, at least not worse, investment opportunity than the original frictional market. An interesting question is whether there exists a least favorable $\tilde{Z} \in \mathcal{Z}_{sup}^\lambda$, such that the gap is closed, i.e., the inequality becomes equality. If so, the corresponding price process $S^{\tilde{Z}} := \frac{\tilde{Z}^1}{\tilde{Z}^0}$ is called shadow price. Below is the definition of the shadow price similar to [4, Definition 3.9].

Definition 6.1.9. *Fix the initial value x and the terminal random endowment e_T . We assume that short selling of either asset is not allowed. Then, the process $S^{\tilde{Z}}$ associated with some $\tilde{Z} := \tilde{Z}(x, e_T) \in \mathcal{Z}_{sup}^\lambda$ is called a shadow price process, if*

$$\sup_{g \in \mathcal{C}_+^\lambda(x)} \mathbf{E}[U(g + e_T)] = \sup_{\tilde{g} \in \mathcal{C}_+^{\tilde{Z}}(x)} \mathbf{E}[U(\tilde{g} + e_T)].$$

6.2 Solvability of the Primal Problem and Existence of Shadow Prices

In this section, we shall present our main result, that is, the solvability of (6.1.1) and the existence of shadow prices.

6.2.1 Main Theorems

The existence of shadow prices for the utility maximization problem with neither the no-short-selling constraint nor random endowment has been studied in [62, 25, 27, 30] by duality methods. By contrast, we shall solve (6.1.1) directly by following the line of [4].

First of all, we display a superreplication theorem as an analogue of [4, Lemma 4.1]:

Lemma 6.2.1. *For any $Z \in \mathcal{Z}_{sup}^\lambda$, the process $Z^0 \varphi^0 + Z^1 \varphi^1$ is a positive supermartingale, for any $(\varphi^0, \varphi^1) \in \mathcal{A}_+^\lambda(x)$.*

Proof. As (φ^0, φ^1) is of finite variation and (Z^0, Z^1) is a supermartingale, we obtain by [58, Proposition I.4.49] that

$$\begin{aligned} Z_t^0 \varphi_t^0 + Z_t^1 \varphi_t^1 &= (Z_0^0 \varphi_0^0 + Z_0^1 \varphi_0^1) + \int_0^t (\varphi_{u-}^0 dZ_u^0 + \varphi_{u-}^1 dZ_u^1) + \int_0^t (Z_u^0 d\varphi_u^0 + Z_u^1 d\varphi_u^1) \\ &= x + \int_0^t (\varphi_{u-}^0 dZ_u^0 + \varphi_{u-}^1 dZ_u^1) + \int_0^t (Z_u^0 d\varphi_u^0 + Z_u^1 d\varphi_u^1). \end{aligned}$$

The first integral defines a supermartingale due to the positivity of φ^0 and φ^1 . The second integral defines a decreasing process by the fact that (φ^0, φ^1) is λ -self-financing and that $\frac{Z_u^1}{Z_u^0}$ takes values in $[(1 - \lambda)S_u, S_u]$. Therefore, the process $Z^0 \varphi^0 + Z^1 \varphi^1$ is a positive supermartingale. \square

Remark 6.2.2. Comparing with [97, Theorem 1.4], we require less on the underlying asset price S for the superreplication theorem, since we are working with a smaller set of trading strategies.

Furthermore, we have some properties of the convex sets $\mathcal{A}_+^\lambda(x)$ and $\mathcal{C}_+^\lambda(x)$ as follows.

Lemma 6.2.3. *Under Assumption 6.1.5, the total variation $\text{Var}(\varphi^0)$ and $\text{Var}(\varphi^1)$ remain bounded in L^0 , when φ runs through $\mathcal{A}_+^\lambda(x)$.*

Proof. Write $\varphi^0 = \varphi^{0,\uparrow} - \varphi^{0,\downarrow}$ and $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$ as the canonical differences of increasing processes. Then, we could define a strategy $\tilde{\varphi} \in \mathcal{A}_+^{\lambda'}(x)$ by

$$\tilde{\varphi}_t := \left(\varphi_t^0 + \frac{\lambda - \lambda'}{1 - \lambda} \varphi_t^{0,\uparrow}, \varphi_t^1 \right), \quad 0 \leq t \leq T,$$

and prove by Lemma 6.2.1 that for $Z \in \mathcal{Z}_{sup}^{\lambda'}$,

$$\frac{\lambda - \lambda'}{1 - \lambda} \mathbf{E} \left[Z_T^0 \varphi_T^{0,\uparrow} \right] \leq \mathbf{E} [Z_T^0 \varphi_T^0 + Z_T^1 \varphi_T^1] + \frac{\lambda - \lambda'}{1 - \lambda} \mathbf{E} \left[Z_T^0 \varphi_T^{0,\uparrow} \right] \leq x.$$

The remainder of the proof is identical with the one of [97, Lemma 3.1]. \square

Then, we state the following lemma without proof and refer the reader to [97, Theorem 3.4].

Lemma 6.2.4. *Under Assumption 6.1.5, the set $\mathcal{C}_+^\lambda(x)$ is convex closed and bounded in L_+^0 .*

In what follows, we shall establish the existence and uniqueness result for the primal solution of (6.1.1). The spirit of the proof is revealed in [95] (compare also [48] and [4]). However, we observe that the positivity of $(f_n)_{n \in \mathbb{N}}$ in [95, Lemma 3.16] is not essentially needed for the proof of the existence. Thus, we reorganize a proof. Firstly, we introduce the following lemma.

Lemma 6.2.5. *Suppose $\{f_n\}_{n \in \mathbb{N}}$ is a sequence in L^1 , $f_n \rightarrow f_0 \in L^0$, almost surely. Moreover, $\lim_{n \rightarrow \infty} \mathbf{E}[f_n]$ is finite and f_0^+ is integrable. We denote $\alpha := \lim_{n \rightarrow \infty} \mathbf{E}[f_n] - \mathbf{E}[f_0]$. In particular, if $\mathbf{E}[f_0] = -\infty$, we note $\alpha := \infty$. Then, we have*

(i) For any $M > 0$,

$$\limsup_{n \rightarrow \infty} \mathbf{E}[f_n \mathbf{1}_{\{f_n \geq M\}}] \geq \alpha. \quad (6.2.1)$$

(ii) For any $\alpha' < \alpha$ and $M > 0$, there exists a subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$ and a sequence of disjoint sets $(A_k)_{k \in \mathbb{N}}$, such that for each $k \in \mathbb{N}$, $f_{n_k} \geq M$ on A_k , and

$$\mathbf{E}[f_{n_k} \mathbf{1}_{A_k}] \geq \alpha'.$$

Proof. (i) We suppose, contrary to our claim (6.2.1), there exists $M > 0$, such that

$$\limsup_{n \rightarrow \infty} \mathbf{E}[f_n \mathbf{1}_{\{f_n \geq M\}}] =: \beta < \alpha, \quad (6.2.2)$$

which implies the boundedness of $\{\mathbf{E}[f_n \mathbf{1}_{\{f_n \geq M\}}]\}_{n \in \mathbb{N}}$. Suppose $\mathbf{E}[f_0] = -\infty$, then we have

$$\mathbf{E}[f_0 \mathbf{1}_{\{f_0 < M\}}] \leq \mathbf{E}[f_0] = -\infty.$$

Thus,

$$\limsup_{n \rightarrow \infty} \mathbf{E}[f_n \mathbf{1}_{\{f_n < M\}}] \leq \mathbf{E}[f_0 \mathbf{1}_{\{f_0 < M\}}] = -\infty. \quad (6.2.3)$$

From (6.2.3) and (6.2.2), we can conclude that $\lim_{n \rightarrow \infty} \{\mathbf{E}[f_n]\}_{n \in \mathbb{N}} = -\infty$, which contradicts to the assumption. Therefore, $\mathbf{E}[f_0] \in \mathbb{R}$. In this case, we have

$$\begin{aligned} \mathbf{E}[f_0 \mathbf{1}_{\{f_0 < M\}}] &\geq \limsup_{n \rightarrow \infty} \mathbf{E}[f_n \mathbf{1}_{\{f_n < M\}}] \\ &= \lim_{n \rightarrow \infty} \mathbf{E}[f_n] - \liminf_{n \rightarrow \infty} \mathbf{E}[f_n \mathbf{1}_{\{f_n \geq M\}}] \\ &\geq \mathbf{E}[f_0] + \alpha - \beta, \end{aligned} \quad (6.2.4)$$

where the equality is deduced from the convergence of $\{\mathbf{E}[f_n]\}_{n \in \mathbb{N}}$ and the boundedness of $\{\mathbf{E}[f_n \mathbf{1}_{\{f_n \geq M\}}]\}_{n \in \mathbb{N}}$. Obviously, (6.2.4) is a contradiction.

(ii) We first consider the case that $\alpha < \infty$. Fix α' , M and denote by $\varepsilon := \alpha - \alpha'$. We now construct inductively a subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$ as well as a sequence $\{\mathcal{A}^m\}_{m \in \mathbb{N}}$, where \mathcal{A}^m consists of m subsets of Ω , i.e.,

$$\mathcal{A}^m := \{A_1^m, A_2^m, \dots, A_m^m\}, \quad m \in \mathbb{N}.$$

Let $M_1 = M$. From (i), we could choose n_1 such that $\mathbf{E}[f_{n_1} \mathbf{1}_{\{f_{n_1} \geq M_1\}}] \geq \alpha$ and define $A_1^1 := \{f_{n_1} \geq M_1\}$. Suppose that $\{f_{n_k}\}_{k=1}^N$ and $\{\mathcal{A}^m\}_{m=1}^N$ are well-defined and we now define $f_{n_{N+1}}$ and \mathcal{A}^{N+1} . Note that for each $n \in \mathbb{N}$ and any $\widetilde{M} > 0$,

$$\mathbf{E}[\mathbf{1}_{\{f_n \geq \widetilde{M}\}}] = \mathbf{E}[\mathbf{1}_{\{f_n^+ \geq \widetilde{M}\}}] \leq \mathbf{E}[\mathbf{1}_{\{|f_n^+ - f_0^+| \geq \frac{\widetilde{M}}{2}\}}] + \mathbf{E}[\mathbf{1}_{\{f_0^+ \geq \frac{\widetilde{M}}{2}\}}].$$

Thus, by the integrability of f_0^+ , f_1 , f_2 , \dots , f_N and Markov's inequality, one can choose $M_{N+1} \geq M_N$ sufficiently large, such that for any $n \geq n_N$ and $k = 1, 2, \dots, N$,

$$\mathbf{E}[|f_n| \mathbf{1}_{\{f_{n_{N+1}} \geq M_{N+1}\}}] \leq \frac{\varepsilon}{2^{N+1}}.$$

Then, we fix $n_{N+1} \geq n_N$ satisfying $\mathbf{E}[f_{n_{N+1}} \mathbf{1}_{\{f_{n_{N+1}} \geq M_{N+1}\}}] \geq \alpha$ and define $A_{N+1}^{N+1} := \{f_{n_{N+1}} \geq M_{N+1}\}$ and $A_k^{N+1} := A_k^N \setminus A_{N+1}^{N+1}$. Note that the sequence $\{\mathcal{A}^m\}_{m \in \mathbb{N}}$ we defined above has the following properties:

- (a) For each k , $\{A_k^m\}_{m \geq k}$ is decreasing in m ;
- (b) For each m , $A_1^m, A_2^m, \dots, A_m^m$ are disjoint;
- (c) For each k and $m \geq k$,

$$\mathbf{E}[f_{n_k} \mathbf{1}_{A_k^m}] \geq \alpha - \sum_{i=k}^{m-1} \frac{\varepsilon}{2^i} > \alpha'.$$

Letting $A_k := \bigcap_{m \geq k} A_k^m$. It is easy to verify that the sequences $\{f_{n_k}\}_{k \in \mathbb{N}}$ and $\{A_k\}_{k \in \mathbb{N}}$ yield the desired result.

If $\alpha = \infty$, we could complete the proof by a similar argument. \square

Theorem 6.2.6. *Let Assumptions 6.1.5, 6.1.2 and 7.1.1 hold. Assume moreover that*

$$u(x; e_T) < \infty.$$

Then, the utility maximization problem (6.1.1) admits a unique solution $\hat{g} \in \mathcal{C}_+^\lambda(x)$.

Proof. The uniqueness is trivial due to the strict concavity of U . Thus, we only have to show the existence.

- (i) Since $u(x; e_T) < \infty$, we could pick a maximizing sequence for (7.1.1), i.e.,

$$u(x; e_T) = \lim_{n \rightarrow \infty} \mathbf{E}[U(g_n + e_T)].$$

By passing to a sequence of convex combinations $g'_n \in \text{conv}(g_n, g_{n+1}, \dots)$, still denoted by g_n , and applying the Komlós type theorem as well as Lemma 6.2.4, we may suppose that g_n converges a.s. to $\hat{g} \in \mathcal{C}^\lambda(x)$.

(ii) It is easy to verify that u is still a concave function in x and thus $u(x; e_T) < \infty$ implies $u(x+1; e_T) < \infty$. We claim that $\mathbf{E}[(U(\hat{g} + e_T))^+] < +\infty$. If not, we could deduce that $u(x+1; e_T) = \infty$, which is impossible. Therefore, $\mathbf{E}[U(\hat{g} + e_T)]$ exists.

(iii) We now prove that \hat{g} is the primal optimizer. If not, there exists an $\alpha \in (0, \infty]$ such that

$$\alpha = u(x; e_T) - \mathbf{E}[U(\hat{g} + e_T)].$$

For each n , denote by $f_n = U(g_n + e_T)$. Fixing $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$, such that for each $n \geq n_0$,

$$u(x; e_T) - \mathbf{E}[f_n] \leq \varepsilon. \tag{6.2.5}$$

Since $AE(U) < 1$, by Lemma 6.3 in [70], there exists some $\gamma > 1$, such that $U(\frac{x}{2}) > \frac{\gamma}{2}U(x)$, for all $x \geq x_0 > 0$. From Lemma 6.2.5, we can choose sufficiently large $M > 0$ and $m > n \geq n_0$, such that $U^{-1}(M) \geq 2x_0$,

$$\mathbf{E}[|f_n| \mathbf{1}_{\{f_m \geq M\}}] \leq \varepsilon \text{ and } \mathbf{E}[f_m \mathbf{1}_{\{f_m \geq M\}}] \geq \alpha - \varepsilon. \tag{6.2.6}$$

Then,

$$\begin{aligned} \mathbf{E} \left[U \left(\frac{g_n + g_m}{2} + e_T \right) \right] &= \mathbf{E} \left[U \left(\frac{g_n + g_m}{2} + e_T \right) \mathbf{1}_{\{f_m \geq M\}} \right] \\ &\quad + \mathbf{E} \left[U \left(\frac{g_n + g_m}{2} + e_T \right) \mathbf{1}_{\{f_m < M\}} \right]. \end{aligned}$$

Furthermore, due to the positivity of g_n , g_m and e_T , we have

$$\mathbf{E} \left[U \left(\frac{g_n + g_m}{2} + e_T \right) \mathbf{1}_{\{f_m \geq M\}} \right] \geq \frac{\gamma}{2} \mathbf{E} [U (g_n + g_m + 2e_T) \mathbf{1}_{\{f_m \geq M\}}] \geq \frac{\gamma}{2} \mathbf{E} [f_m \mathbf{1}_{\{f_m \geq M\}}]$$

and

$$\mathbf{E} \left[U \left(\frac{g_n + g_m}{2} + e_T \right) \mathbf{1}_{\{f_m < M\}} \right] \geq \frac{1}{2} \mathbf{E} [f_n \mathbf{1}_{\{f_m < M\}}] + \frac{1}{2} \mathbf{E} [f_m \mathbf{1}_{\{f_m < M\}}].$$

Therefore, we can deduce from (6.2.5) and (6.2.6),

$$\begin{aligned} \mathbf{E} \left[U \left(\frac{g_n + g_m}{2} + e_T \right) \right] &\geq \frac{1}{2} \mathbf{E} [f_n \mathbf{1}_{\{f_m < M\}}] + \frac{1}{2} \mathbf{E} [f_m] + \frac{\gamma - 1}{2} \mathbf{E} [f_m \mathbf{1}_{\{f_m \geq M\}}] \\ &\geq u(x; e_T) + \frac{(\gamma - 1)\alpha}{2} - \frac{\gamma + 2}{2} \varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we have

$$\mathbf{E} \left[U \left(\frac{g_n + g_m}{2} + e_T \right) \right] > u(x; e_T),$$

which is a contradiction to the maximality of $u(x; e_T)$. \square

Now we turn to consider the frictionless market associated with S^Z , for $Z \in \mathcal{Z}_{sup}^\lambda$. Similarly to Lemma 6.2.1, we have the supermartingale property of $Z^0 \tilde{\varphi}^0 + Z^1 \tilde{\varphi}^1$ for $(\tilde{\varphi}^0, \tilde{\varphi}^1) \in \mathcal{A}_+^Z(x)$. The subsequent lemma has been reviewed in [4, Lemma 4.1]. However, for the convenience of the reader, we prove it in the numéraire-based case.

Lemma 6.2.7. *Fix $Z \in \mathcal{Z}_{sup}^\lambda$. The process $Z^0 \tilde{\varphi}^0 + Z^1 \tilde{\varphi}^1$ is a positive supermartingale, for any $(\tilde{\varphi}^0, \tilde{\varphi}^1) \in \mathcal{A}_+^Z(x)$.*

Proof. Note that $Z^0 \tilde{\varphi}^0 + Z^1 \tilde{\varphi}^1 = Z^0(\tilde{\varphi}^0 + \tilde{\varphi}^1 S^Z) = Z^0(x + \tilde{\varphi}^1 \cdot S^Z)$ by the frictionless self-financing condition. Using Itô's formula and [45, Proposition A.1], we obtain that

$$\begin{aligned} Z_t^0 \tilde{\varphi}_t^0 + Z_t^1 \tilde{\varphi}_t^1 &= x Z_0^0 + ((\tilde{\varphi}_-^0 + \tilde{\varphi}_-^1 S_-^Z) \cdot Z^0)_t + (Z_-^0 \cdot (\tilde{\varphi}^1 \cdot S^Z))_t + [\tilde{\varphi}^1 \cdot S^Z, Z^0]_t \\ &= x Z_0^0 + ((\tilde{\varphi}_-^0 + \tilde{\varphi}_-^1 S_-^Z) \cdot Z^0)_t + (\tilde{\varphi}^1 \cdot (Z_-^0 \cdot S^Z))_t + (\tilde{\varphi}^1 \cdot [S^Z, Z^0])_t. \end{aligned}$$

It follows from the frictionless self-financing condition again and [58, I.4.36] that

$$\Delta(\tilde{\varphi}^0 + \tilde{\varphi}^1 S^Z) = \Delta(\tilde{\varphi}^1 \cdot S^Z) = \tilde{\varphi}^1 \Delta S^Z,$$

therefore, $\tilde{\varphi}_-^0 + \tilde{\varphi}_-^1 S_-^Z = \tilde{\varphi}^0 + \tilde{\varphi}^1 S_-^Z$. By [58, I.4.37, Definition I.4.45], we obtain

$$\begin{aligned} Z_t^0 \tilde{\varphi}_t^0 + Z_t^1 \tilde{\varphi}_t^1 &= x Z_0^0 + (\tilde{\varphi}^0 \cdot Z^0)_t + (\tilde{\varphi}^1 \cdot (S_-^Z \cdot Z^0 + Z_-^0 \cdot S^Z + [S^Z, Z^0]))_t \\ &= x Z_0^0 + (\tilde{\varphi}^0 \cdot Z^0)_t + (\tilde{\varphi}^1 \cdot (S^Z Z^0))_t \\ &= x Z_0^0 + (\tilde{\varphi}^0 \cdot Z^0)_t + (\tilde{\varphi}^1 \cdot Z^1)_t, \end{aligned}$$

which is a positive local supermartingale and hence a supermartingale. \square

Then, it is easy to deduce that for $\tilde{g} \in \mathcal{C}_+^Z(x)$,

$$\mathbf{E}[U(\tilde{g} + e_T)] \leq \mathbf{E}[V(Z_T^0) + Z_T^0(\tilde{g} + e_T)] \leq \mathbf{E}[V(Z_T^0)] + \mathbf{E}[Z_T^0 e_T] + Z_0^0 x, \quad (6.2.7)$$

for each $Z \in \mathcal{Z}_{sup}^\lambda$, where V is the conjugate of U . Therefore, to prove the existence of shadow prices, it suffices to have the following lemma, whose proof is postponed to the next subsection.

Lemma 6.2.8. *Let Assumptions 6.1.5 and 6.1.2 hold. There exists a $\hat{Z} \in \mathcal{Z}_{sup}^\lambda$, such that*

$$(i) \quad \hat{Z}_T^0 = U'(\hat{g} + e_T);$$

$$(ii) \quad \mathbf{E}[\hat{Z}_T^0 \hat{g}] = \hat{Z}_0^0 x.$$

Theorem 6.2.9. *The λ -supermartingale-CPS $\hat{Z} \in \mathcal{Z}_{sup}^\lambda$ satisfying Lemma 6.2.8 (i)-(ii) defines a shadow price $S^{\hat{Z}} := \frac{\hat{Z}_1^1}{\hat{Z}_0^0}$.*

Proof. Consider the frictionless market associated with $S^{\hat{Z}}$. By Lemma 6.2.8, we have

$$\begin{aligned} u^{\hat{Z}}(x; e_T) &\geq u(x; e_T) = \mathbf{E}[U(\hat{g} + e_T)] = \mathbf{E}[V(\hat{Z}_T^0) + \hat{Z}_T^0(\hat{g} + e_T)] \\ &= \mathbf{E}[V(\hat{Z}_T^0)] + \mathbf{E}[\hat{Z}_T^0 e_T] + \hat{Z}_0^0 x \geq u^{\hat{Z}}(x; e_T), \end{aligned}$$

where the last inequality follows from (6.2.7). The inequality above implies $u^{\hat{Z}}(x; e_T) = u(x; e_T)$, which proves that $S^{\hat{Z}}$ is a shadow price for the problem (6.1.1). \square

Remark 6.2.10. As has been indicated in [4], by the strict concavity of u , \hat{g} is the unique solution in $\mathcal{C}_+^{\hat{Z}}(x)$ for the frictionless problem $u^{\hat{Z}}(x; e_T)$. Moreover, the trading strategy $\hat{\varphi}$, that attains the maximality in the frictional market, does the same in the frictionless one associated with the shadow price $S^{\hat{Z}}$. Therefore, the optimal trading strategy $(\hat{\varphi}^0, \hat{\varphi}^1)$ for S under transaction costs λ satisfies

$$\begin{aligned} \{d\hat{\varphi}_t^1 > 0\} &\subseteq \{S_t^{\hat{Z}} = S_t\}, \\ \{d\hat{\varphi}_t^1 < 0\} &\subseteq \{S_t^{\hat{Z}} = (1 - \lambda)S_t\}, \end{aligned}$$

for all $0 \leq t \leq T$.

Remark 6.2.11. In our case, shadow prices are determined not only by the random endowment but also by its decomposition (see Assumption 6.1.2). The decomposition of the random endowment together with the no-short-selling constraints can be explained as the agent's trading rule created by her controller. Precisely, if the random endowment \tilde{e}_T that the agent will eventually receive is decomposed into $x + e_T$ by her controller, then it means that the agent is allowed to borrow at most x in the bond market for trading the stock. Thus, the different ways of decomposition mean the different limits of short selling in bond, which lead to different maximal utilities and also shadow prices.

6.2.2 Proof of Lemma 6.2.8

In this subsection, we prove Lemma 6.2.8. Generally speaking, we shall follow the line of the proof of [4, Proposition 4.2]. Thus, we only give the sketch in order to show how it develops in the numéraire-based context and how a positive random endowment works. The proof is divided in several stages.

Firstly, the following dynamic programming principle, similar to [4, Lemma 4.4], can be verified as a special case of [40, Theorem 1.17].

Proposition 6.2.12. *Define*

$$\mathcal{U}_s(\varphi_s^0, \varphi_s^1) := \operatorname{ess\,sup}_{(\psi^0, \psi^1) \in \mathcal{A}_{s,T}^\lambda(\varphi_s^0, \varphi_s^1)} \mathbf{E} [U(\psi_T^0 + e_T) | \mathcal{F}_s],$$

where $\mathcal{A}_{s,T}^\lambda(\varphi_s^0, \varphi_s^1)$ is the set of all admissible λ -self-financing trading strategies, which agree with $\varphi \in \mathcal{A}_+^\lambda(x)$ in $[0, s]$.

Then, the process $(\mathcal{U}_s(\widehat{\varphi}_s^0, \widehat{\varphi}_s^1))_{0 \leq t \leq T}$ is a martingale i.e.,

$$\mathcal{U}_s(\widehat{\varphi}_s^0, \widehat{\varphi}_s^1) = \mathbf{E} [\mathcal{U}_t(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1) | \mathcal{F}_s], \quad a.s.,$$

for all optimal trading strategies $\widehat{\varphi}$ attaining \widehat{g} .

Proof. In the numéraire-based context, this proposition can be directly proved. Without loss of generality, it suffices to verify the following claim

$$\mathbf{E} [U(\psi_T^0 + e_T) | \mathcal{F}_s] \leq \mathbf{E} [U(\widehat{\varphi}_T^0 + e_T) | \mathcal{F}_s], \quad (6.2.8)$$

for all $(\psi^0, \psi^1) \in \mathcal{A}_{s,T}^\lambda(\widehat{\varphi}_s^0, \widehat{\varphi}_s^1)$.

To obtain a contradiction, we suppose that (6.2.8) is not true, i.e., there exists a $(\psi^0, \psi^1) \in \mathcal{A}_{s,T}^\lambda(\widehat{\varphi}_s^0, \widehat{\varphi}_s^1)$ and a set $A \subseteq \Omega$ with $\mathbf{P}[A] > 0$ defined as

$$A := \{\mathbf{E}[U(\psi_T^0 + e_T) | \mathcal{F}_s] > \mathbf{E}[U(\widehat{\varphi}_T^0 + e_T) | \mathcal{F}_s]\} \in \mathcal{F}_s. \quad (6.2.9)$$

Then define

$$(\psi^0, \psi^1) \mathbf{1}_A + (\widehat{\varphi}^0, \widehat{\varphi}^1) \mathbf{1}_{A^c} =: (\eta^0, \eta^1) \in \mathcal{A}_{s,T}^\lambda(\widehat{\varphi}_s^0, \widehat{\varphi}_s^1).$$

We have that

$$\mathbf{E} [U(\eta_T^0 + e_T) | \mathcal{F}_s] > \mathbf{E} [U(\widehat{\varphi}_T^0 + e_T) | \mathcal{F}_s], \quad a.s.$$

on A , and

$$\mathbf{E} [U(\eta_T^0 + e_T) | \mathcal{F}_s] \geq \mathbf{E} [U(\widehat{\varphi}_T^0 + e_T) | \mathcal{F}_s], \quad a.s.,$$

which implies

$$\mathbf{E} [U(\eta_T^0 + e_T)] > \mathbf{E} [U(\widehat{\varphi}_T^0 + e_T)] = u(x; e_T).$$

This is in contradiction to the maximality of $\widehat{\varphi}$. Thus, by the definition of $\mathcal{U}_s(\varphi_s^0, \varphi_s^1)$ and (6.2.8), we obtain

$$\mathcal{U}_s(\widehat{\varphi}_s^0, \widehat{\varphi}_s^1) = \operatorname{ess\,sup}_{(\psi^0, \psi^1) \in \mathcal{A}_{s,T}^\lambda(\widehat{\varphi}_s^0, \widehat{\varphi}_s^1)} \mathbf{E} [U(\psi_T^0 + e_T) | \mathcal{F}_s] = \mathbf{E} [U(\widehat{\varphi}_T^0 + e_T) | \mathcal{F}_s].$$

Finally, the tower property of conditional expectations yields the desired result. \square

The next step goes in an exactly same way as in [4], i.e., we should first construct a pair $\widehat{Z} = (\widehat{Z}_t^0, \widehat{Z}_t^1)_{0 \leq t \leq T}$, then verify the shadow price can be defined by $\frac{\widehat{Z}_t^1}{\widehat{Z}_t^0}$. The additional positive e_T in the dynamic will not alter the following results.

Proposition 6.2.13. *The following processes are well-defined:*

$$\begin{cases} \widetilde{Z}_t^0 := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} (\mathcal{U}_t(\widehat{\varphi}_t^0 + \varepsilon, \widehat{\varphi}_t^1) - \mathcal{U}_t(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)), \\ \widetilde{Z}_t^1 := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} (\mathcal{U}_t(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1 + \varepsilon) - \mathcal{U}_t(\widehat{\varphi}_t^0, \widehat{\varphi}_t^1)), \end{cases} \quad (6.2.10)$$

for $0 \leq t < T$, and

$$\begin{cases} \widetilde{Z}_T^0 := U'(\widehat{\varphi}_T + e_T), \\ \widetilde{Z}_T^1 := U'(\widehat{\varphi}_T + e_T)(1 - \lambda)S_T. \end{cases} \quad (6.2.11)$$

Furthermore, define

$$\begin{cases} \widehat{Z}_t^i := \lim_{\substack{s \searrow t \\ s \in \mathbb{Q}}} \widetilde{Z}_s^i, & 0 \leq t < T; \\ \widehat{Z}_t^i := \widetilde{Z}_T^i, & t = T. \end{cases} \quad (6.2.12)$$

Then, the process \widehat{Z} is a càdlàg supermartingale and moreover, for all $0 \leq t \leq T$, we have

$$(1 - \lambda)S_t \leq \frac{\widehat{Z}_t^1}{\widehat{Z}_t^0} \leq S_t, \quad a.s. \quad (6.2.13)$$

Consequently, \widehat{Z} is a λ -supermartingale-CPS.

The proof of Proposition 6.2.13 will be splitted into several lemmata.

Lemma 6.2.14. *For each $0 \leq t \leq T$ and $i = 0, 1$, the random variables \widetilde{Z}_t^i is well-defined as the limit of an increasing sequence.*

Proof. Consider $\varepsilon_1, \varepsilon_2$ with $\varepsilon_1 > \varepsilon_2 > 0$,

$$\begin{aligned} \mathcal{U}_t(\widehat{\varphi}_t + \varepsilon_2 e_i) &= \mathcal{U}_t \left(\frac{\varepsilon_2}{\varepsilon_1} (\widehat{\varphi}_t + \varepsilon_1 e_i) + \left(1 - \frac{\varepsilon_2}{\varepsilon_1}\right) \widehat{\varphi}_t \right) \\ &\geq \frac{\varepsilon_2}{\varepsilon_1} \mathcal{U}_t(\widehat{\varphi}_t + \varepsilon_1 e_i) + \left(1 - \frac{\varepsilon_2}{\varepsilon_1}\right) \mathcal{U}_t(\widehat{\varphi}_t). \end{aligned}$$

The last inequality follows from the concavity of U and the definition of \mathcal{U}_t . It implies that

$$\frac{\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon_2 e_i) - \mathcal{U}_t(\widehat{\varphi}_t)}{\varepsilon_2} \geq \frac{\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon_1 e_i) - \mathcal{U}_t(\widehat{\varphi}_t)}{\varepsilon_1}.$$

Therefore,

$$\widetilde{Z}_t^i := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} (\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon e_i) - \mathcal{U}_t(\widehat{\varphi}_t))$$

is well-defined as the limit of an increasing sequence. \square

Lemma 6.2.15. *There exists a sequence of $(\psi^n)_{n \in \mathbb{N}} = (\psi^{n,0}, \psi^{n,1})_{n \in \mathbb{N}} \subseteq \mathcal{A}_{t,T}^\lambda(\widehat{\varphi}_t + \varepsilon e_i)$ such that*

$$\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon e_i) = \nearrow - \lim_{n \rightarrow \infty} \mathbf{E} [U(\psi_T^{n,0} + e_T) | \mathcal{F}_t].$$

Proof. It suffices to prove that the set $\{\mathbf{E}[U(\psi_T^0 + e_T) | \mathcal{F}_t] \mid (\psi^0, \psi^1) \in \mathcal{A}_{t,T}^\lambda(\widehat{\varphi}_t + \varepsilon e_i)\}$ is directed upwards. To show this, pick two processes $(\psi^0, \psi^1), (\eta^0, \eta^1) \in \mathcal{A}_{t,T}^\lambda(\widehat{\varphi}_t + \varepsilon e_i)$, and define the set

$$A := \{\mathbf{E}[U(\psi_T^0 + e_T) | \mathcal{F}_t] > \mathbf{E}[U(\eta_T^0 + e_T) | \mathcal{F}_t]\} \in \mathcal{F}_t$$

and the process

$$(\psi^0, \psi^1)\mathbf{1}_A + (\eta^0, \eta^1)\mathbf{1}_{A^c} =: (\xi^0, \xi^1) \in \mathcal{A}_{t,T}^\lambda(\widehat{\varphi}_t + \varepsilon e_i).$$

It is easy to see that

$$\mathbf{E}[U(\psi_T^0 + e_T) | \mathcal{F}_t] \vee \mathbf{E}[U(\eta_T^0 + e_T) | \mathcal{F}_t] \leq \mathbf{E}[U(\xi_T^0 + e_T) | \mathcal{F}_t].$$

The assertion follows by Theorem A.1.6. \square

Lemma 6.2.16. *The process $\widetilde{Z} = (\widetilde{Z}_t^0, \widetilde{Z}_t^1)_{0 \leq t \leq T}$ is a (not necessarily càdlàg) supermartingale.*

Proof. Clearly, we have

$$\mathcal{A}_{t,T}^\lambda(\widehat{\varphi}_t + \varepsilon e_i) \subseteq \mathcal{A}_{s,T}^\lambda(\widehat{\varphi}_s + \varepsilon e_i),$$

for $0 \leq s < t \leq T$. Then,

$$\begin{aligned} \mathcal{U}_s(\widehat{\varphi}_s + \varepsilon e_i) &= \operatorname{ess\,sup}_{(\psi^0, \psi^1) \in \mathcal{A}_{s,T}^\lambda(\widehat{\varphi}_s + \varepsilon e_i)} \mathbf{E}[U(\psi_T^0 + e_T) | \mathcal{F}_s] \\ &\geq \operatorname{ess\,sup}_{(\psi^0, \psi^1) \in \mathcal{A}_{t,T}^\lambda(\widehat{\varphi}_t + \varepsilon e_i)} \mathbf{E}[U(\psi_T^0 + e_T) | \mathcal{F}_s] \\ &\geq \mathbf{E}[U(\psi_T^{n,0} + e_T) | \mathcal{F}_s] = \mathbf{E}[\mathbf{E}[U(\psi_T^{n,0} + e_T) | \mathcal{F}_t] | \mathcal{F}_s]. \end{aligned}$$

So, by the monotone convergence theorem we have that

$$\begin{aligned} \mathcal{U}_s(\widehat{\varphi}_s + \varepsilon e_i) &\geq \lim_{n \rightarrow \infty} \mathbf{E}[\mathbf{E}[U(\psi_T^{n,0} + e_T) | \mathcal{F}_t] | \mathcal{F}_s] \\ &= \mathbf{E}\left[\lim_{n \rightarrow \infty} \mathbf{E}[U(\psi_T^{n,0} + e_T) | \mathcal{F}_t] \mid \mathcal{F}_s\right] = \mathbf{E}[\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon e_i) | \mathcal{F}_s], \end{aligned}$$

which shows

$$\frac{\mathcal{U}_s(\widehat{\varphi}_s + \varepsilon e_i) - \mathcal{U}_s(\widehat{\varphi}_s)}{\varepsilon} \geq \mathbf{E}\left[\frac{\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon e_i) - \mathcal{U}_t(\widehat{\varphi}_t)}{\varepsilon} \mid \mathcal{F}_s\right].$$

Therefore, we obtain that

$$\begin{aligned} \widetilde{Z}_s^i &:= \lim_{\varepsilon \searrow 0} \frac{\mathcal{U}_s(\widehat{\varphi}_s + \varepsilon e_i) - \mathcal{U}_s(\widehat{\varphi}_s)}{\varepsilon} \geq \lim_{\varepsilon \searrow 0} \mathbf{E}\left[\frac{\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon e_i) - \mathcal{U}_t(\widehat{\varphi}_t)}{\varepsilon} \mid \mathcal{F}_s\right] \\ &= \mathbf{E}\left[\lim_{\varepsilon \searrow 0} \frac{\mathcal{U}_t(\widehat{\varphi}_t + \varepsilon e_i) - \mathcal{U}_t(\widehat{\varphi}_t)}{\varepsilon} \mid \mathcal{F}_s\right] = \mathbf{E}[\widetilde{Z}_t^i | \mathcal{F}_s]. \end{aligned}$$

To this end, we should verify it at time $t = T$. For the first component \tilde{Z}^0 , it follows from

$$\mathcal{U}_s(\hat{\varphi}_s + \varepsilon e_1) = \mathcal{U}_s(\hat{\varphi}_s^0 + \varepsilon, \hat{\varphi}_s^1) \geq \mathbf{E} \left[U(\hat{\varphi}_T^0 + \varepsilon + e_T) \mid \mathcal{F}_s \right],$$

and the monotone convergence theorem that

$$\begin{aligned} \tilde{Z}_s^0 &:= \lim_{\varepsilon \searrow 0} \frac{\mathcal{U}_s(\hat{\varphi}_s + \varepsilon e_1) - \mathcal{U}_s(\hat{\varphi}_s)}{\varepsilon} \geq \lim_{\varepsilon \searrow 0} \mathbf{E} \left[\frac{U(\hat{\varphi}_T^0 + e_T + \varepsilon) - U(\hat{\varphi}_T^0 + e_T)}{\varepsilon} \mid \mathcal{F}_s \right] \\ &= \mathbf{E} \left[U'(\hat{\varphi}_T^0 + e_T) \mid \mathcal{F}_s \right] = \mathbf{E} \left[\tilde{Z}_T^0 \mid \mathcal{F}_s \right]. \end{aligned}$$

For the second component \tilde{Z}^1 , as

$$\mathcal{U}_s(\hat{\varphi}_s + \varepsilon e_2) = \mathcal{U}_s(\hat{\varphi}_s^0, \hat{\varphi}_s^1 + \varepsilon) \geq \mathbf{E} \left[U(\hat{\varphi}_T^0 + \varepsilon(1 - \lambda)S_T + e_T) \mid \mathcal{F}_s \right],$$

we obtain by the monotone convergence theorem that

$$\begin{aligned} \tilde{Z}_s^1 &:= \lim_{\varepsilon \searrow 0} \frac{\mathcal{U}_s(\hat{\varphi}_s + \varepsilon e_2) - \mathcal{U}_s(\hat{\varphi}_s)}{\varepsilon} \\ &\geq \lim_{\varepsilon \searrow 0} \mathbf{E} \left[\frac{U(\hat{\varphi}_T^0 + e_T + \varepsilon(1 - \lambda)S_T) - U(\hat{\varphi}_T^0 + e_T)}{\varepsilon} \mid \mathcal{F}_s \right] \\ &= \mathbf{E} \left[U'(\hat{\varphi}_T^0 + e_T)(1 - \lambda)S_T \mid \mathcal{F}_s \right] = \mathbf{E} \left[\tilde{Z}_T^1 \mid \mathcal{F}_s \right]. \end{aligned}$$

Recalling that $u(x; e_T)$ is finitely valued and concave on \mathbb{R}_+ , we have that

$$\tilde{y} := \tilde{Z}_0^0 = \lim_{\varepsilon \searrow 0} \frac{u(x + \varepsilon; e_T) - u(x; e_T)}{\varepsilon}$$

takes finite values for $x > 0$. The proof of the lemma is now complete. \square

Lemma 6.2.17. *The process $(\tilde{Z}^0, \tilde{Z}^1)$ satisfies*

$$(1 - \lambda)S_t \leq \frac{\tilde{Z}_t^1}{\tilde{Z}_t^0} \leq S_t, \quad a.s.$$

for all $0 \leq t \leq T$.

Proof. It is obviously at the terminal time T . We only consider the claim at time $t \in [0, T)$. Let $(k_l^n)_{l \geq 0}$ be a partition of $[0, \infty)$, with mesh size decreasing to 0 as n goes to infinity. For all $\varepsilon > 0$, on the set $B_l := \{k_l^n < S_t \leq k_{l+1}^n\}$, we observe that

$$\mathcal{U}_t(\hat{\varphi}_t^0 + \varepsilon, \hat{\varphi}_t^1) \geq \mathcal{U}_t \left(\hat{\varphi}_t^0, \hat{\varphi}_t^1 + \frac{\varepsilon}{S_t} \right) \geq \mathcal{U}_t \left(\hat{\varphi}_t^0, \hat{\varphi}_t^1 + \frac{\varepsilon}{k_{l+1}^n} \right).$$

Therefore,

$$\frac{\mathcal{U}_t(\hat{\varphi}_t^0 + \varepsilon, \hat{\varphi}_t^1) - \mathcal{U}_t(\hat{\varphi}_t^0, \hat{\varphi}_t^1)}{\varepsilon} \geq \frac{\mathcal{U}_t \left(\hat{\varphi}_t^0, \hat{\varphi}_t^1 + \frac{\varepsilon}{k_{l+1}^n} \right) - \mathcal{U}_t(\hat{\varphi}_t^0, \hat{\varphi}_t^1)}{\varepsilon}.$$

Using monotone convergence again, we have $\tilde{Z}_t^0 \geq \tilde{Z}_t^1 \frac{1}{k_{l+1}^n}$, and hence

$$\sum_{l \in \mathbb{N}} \mathbf{1}_{B_l} \tilde{Z}_t^0 \geq \sum_{l \in \mathbb{N}} \mathbf{1}_{B_l} \frac{1}{k_{l+1}^n} \tilde{Z}_t^1.$$

Letting $n \rightarrow \infty$, we obtain $S_t \geq \frac{\tilde{Z}_t^1}{\tilde{Z}_t^0}$ for each $0 \leq t \leq T$. Analogously, by the fact that

$$\mathcal{U}_t(\hat{\varphi}_t^0, \hat{\varphi}_t^1 + \varepsilon) \geq \mathcal{U}_t(\hat{\varphi}_t^0 + \varepsilon(1 - \lambda)S_t, \hat{\varphi}_t^1),$$

we obtain $\frac{\tilde{Z}_t^1}{\tilde{Z}_t^0} \geq (1 - \lambda)S_t$ for each $0 \leq t \leq T$, which completes the proof. \square

Lemma 6.2.18. *The process \hat{Z} is a well-defined càdlàg supermartingale satisfying*

$$(1 - \lambda)S_t \leq \frac{\hat{Z}_t^1}{\hat{Z}_t^0} \leq S_t, \quad (6.2.14)$$

for all $0 \leq t \leq T$.

Proof. The existence follows from [66, Proposition 1.3.14 (i),(iii)]. In particular, $\hat{Z}_0^0 \leq \tilde{Z}_0^0$. As the price process S is càdlàg, it is clear that (6.2.14) holds true. \square

Proof of Lemma 6.2.8. It remains to proof (2) in Lemma 6.2.8. Since $(\hat{Z}^0, \hat{Z}^1) \in \mathcal{Z}_{sup}^\lambda$, we have

$$\mathbf{E}[\hat{Z}_T^0 \hat{g}] \leq \hat{Z}_0^0 x. \quad (6.2.15)$$

It remains to show the reversed inequality. For $\alpha < 1$, we note that $u(\alpha x; e_T) \geq \mathbf{E}[U(\alpha \hat{g} + e_T)]$. By the convexity of u , we obtain

$$\tilde{Z}_0^0(x - \alpha x) \leq u(x; e_T) - u(\alpha x; e_T) \leq \mathbf{E}[U(\hat{g} + e_T)] - \mathbf{E}[U(\alpha \hat{g} + e_T)].$$

Therefore, it follows from the strict convexity and the continuous differentiability of U that

$$\tilde{Z}_0^0 x \leq \mathbf{E} \left[\frac{U(\hat{g} + e_T) - U(\alpha \hat{g} + e_T)}{1 - \alpha} \right] \leq \mathbf{E}[U'(\alpha \hat{g} + e_T) \hat{g}]. \quad (6.2.16)$$

Letting $\alpha \nearrow 1$, monotone convergence yields

$$\hat{Z}_0^0 x \leq \tilde{Z}_0^0 x \leq \mathbf{E}[U'(\hat{g} + e_T) \hat{g}] \leq \mathbf{E}[\hat{Z}_T^0 \hat{g}]. \quad (6.2.17)$$

We complete the proof by comparing (6.2.15) and (6.2.17). \square

Remark 6.2.19. We have seen here that the nonnegativity of the random endowment is important. This ensures the strict positivity of $\alpha \hat{g} + e_T$, for $0 < \alpha < 1$, such that $\mathbf{E}[U'(\alpha \hat{g} + e_T) \hat{g}]$ in (6.2.16) is well-defined.

Chapter 7

On the Dual Problem of Utility Maximization in Incomplete Markets

In this chapter, we study the dual problem of the expected utility maximization in incomplete markets with bounded random endowment. We start with the problem formulated in [23] and prove the following statement: in the Brownian framework, the countably additive part \widehat{Q}^r of the dual optimizer $\widehat{Q} \in ba = (L^\infty)^*$ obtained in [23] can be represented by the terminal value of a supermartingale deflator Y defined in [70], which is a local martingale.

7.1 Formulation of the Problem

In this section, we shall recall the formulation of the utility maximization problem in incomplete markets with random endowment and briefly introduce the results obtained in [23].

Consider the model of a financial market consisting of $d + 1$ assets: one bond and d stocks. Without loss of generality, we assume that the bond price is constant. The stock price process $S = (S^i)_{1 \leq i \leq d}$ is a strictly positive semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ satisfying the usual hypotheses of right continuity and saturatedness, where \mathcal{F}_0 is assumed to be trivial. Here, T is a finite time horizon.

Assume that the agent is endowed with initial wealth $x \in \mathbb{R}$ and her investment strategy is denoted by $H = (H^i)_{1 \leq i \leq d}$, which is a predictable S -integrable process specifying the number of shares of each stock held in her portfolio. We also assume that the agent receives an exogenous endowment e_T at time T , which is \mathcal{F}_T -measurable and satisfies $\rho := \|e_T\|_\infty < \infty$. Then, the total value of her portfolio at time T can be written into

$$W_T = x + (H \cdot S)_T + e_T,$$

where $(H \cdot S)_t = \int_0^t H_u dS_u$ denotes the stochastic integral with respect to S .

We call H an admissible strategy if the process $(H \cdot S)$ is uniformly bounded from below by a constant, and we denote by \mathcal{C}_0 the convex cone of \mathcal{F}_T -measurable random variables dominated by admissible stochastic integrals, i.e.,

$$\mathcal{C}_0 := \{g \in L^0(\mathcal{F}_T) \mid g \leq (H \cdot S)_T, \text{ for some admissible strategy } H\}.$$

Moreover, we define $\mathcal{C} := \mathcal{C}_0 \cap L^\infty$.

Suppose the agent's preferences over terminal wealth are modeled by a utility function $U : (0, \infty) \rightarrow \mathbb{R}$, which is strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions:

$$U'(0) := \lim_{x \rightarrow 0} U'(x) = \infty \quad \text{and} \quad U'(\infty) := \lim_{x \rightarrow \infty} U'(x) = 0.$$

Without loss of generality, we may assume $U(\infty) > 0$ and define $U(x) = -\infty$, if $x \leq 0$.

As usual, we assume the following assumptions, which ensure the existence of solutions of the primal and dual problems.

Assumption 7.1.1. *The utility function U satisfies the reasonable asymptotic elasticity, i.e.,*

$$AE(U) := \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

Then, the primal problem can be formulated in the following way:

$$u(x) = \sup_{g \in \mathcal{C}_0} \mathbf{E}[U(x + g + e_T)], \quad x \in \mathbb{R}. \quad (7.1.1)$$

We adopt the following assumption as in [23, 70], which ensures a (NFLVR) setting (see [33, 34]).

Assumption 7.1.2. *There exists at least one probability measure \mathbf{Q} equivalent to \mathbf{P} , such that for any H admissible, $(H \cdot S)$ is a local martingale under \mathbf{Q} . Namely, the set $\mathcal{M} := \mathcal{M}_e(S)$ of all equivalent local martingale measures is not empty.*

To establish the dual problem, we first define the dual domain, which is a nonempty subset of $ba_+ = (L^\infty)_+^*$, convex and compact with respect to the weak-star topology $\sigma(ba, L^\infty)$:

$$\mathcal{D} := \{Q \in ba_+ \mid \|Q\|_{ba} = 1 \text{ and } \langle Q, g \rangle \leq 0, \text{ for all } g \in \mathcal{C}\}. \quad (7.1.2)$$

Remark 7.1.3. We note that the space $(L^\infty)^*$ can be identified with the space of bounded additive measures denoted by ba . Each element in ba_+ admits a unique Yosida-Hewitt decomposition $Q = Q^r + Q^s$, where the regular part $Q^r \in L^1$ is countably additive and the singular part Q^s is purely finitely additive (see [103]).

Then, the dual problem can be formulated as

$$v(y) := \inf_{Q \in \mathcal{D}} \left\{ \mathbf{E} \left[V \left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] + y \langle Q, e_T \rangle \right\}, \quad y > 0, \quad (7.1.3)$$

where V is the conjugate of U .

Assumption 7.1.4. $|u(x)| < \infty$ holds for some $x > \rho$.

Now, we summarize the result obtained in [23] as the following theorem:

Theorem 7.1.5 (Theorem 3.1 and Lemma 4.4 in [23]). *Under Assumptions 7.1.2, 7.1.1, 7.1.4, we have*

- (1) The primal value function u is finitely valued and continuously differentiable on (x_0, ∞) , and $u(x) = -\infty$, for all $x < x_0$, where $x_0 := \sup_{Q \in \mathcal{D}} \langle Q, -e_T \rangle$.
- (2) The dual value function v is finitely valued and continuously differentiable on $(0, \infty)$.
- (3) The functions u and v are conjugate in the sense that

$$\begin{aligned} v(y) &= \sup_{x > x_0} \{u(x) - xy\}, \quad y > 0, \\ u(x) &= \inf_{y > 0} \{v(y) + xy\}, \quad x > x_0. \end{aligned}$$

- (4) For all $y > 0$, there exists a solution $\widehat{Q}_y \in \mathcal{D}$ to the dual problem, which is unique up to the singular part. For all $x > x_0$, $\widehat{g} := I\left(\widehat{y} \frac{d\widehat{Q}_y^r}{d\mathbf{P}}\right) - x - e_T$ is the solution to the primal problem, where $I = -V'$ and $\widehat{y} = u'(x)$, which attains the infimum of $\{v(y) + xy\}$. There is a unique admissible trading strategy \widehat{H} such that $\widehat{g} = (\widehat{H} \cdot S)_T$.
- (5) The following equality is verified for the solutions of the primal and dual problems:

$$\langle \widehat{Q}_y^r, x + (\widehat{H} \cdot S)_T + e_T \rangle = \langle \widehat{Q}_y, x + (\widehat{H} \cdot S)_T + e_T \rangle = x + \langle \widehat{Q}_y, e_T \rangle. \quad (7.1.4)$$

Remark 7.1.6.

- (i) Since the random variable $x + (\widehat{H} \cdot S)_T + e_T$ in Theorem 7.1.5 is uniformly bounded from below, then $\langle \widehat{Q}_y, x + (\widehat{H} \cdot S)_T + e_T \rangle$ is well-defined by

$$\langle \widehat{Q}_y, x + (\widehat{H} \cdot S)_T + e_T \rangle := \lim_{M \rightarrow \infty} \langle \widehat{Q}_y, (x + (\widehat{H} \cdot S)_T + e_T) \wedge M \rangle,$$

although it is not necessarily an element in L^∞ .

- (ii) From the construction of the primal solution, one can see that $\frac{d\widehat{Q}_y^r}{d\mathbf{P}} > 0$, \mathbf{P} -a.s., so that $\widehat{Q}_y^r \sim \mathbf{P}$.
- (iii) The equality of optimality (7.1.4) shows that the purely finitely additive part \widehat{Q}_y^s “concentrates” its mass on the sets,

$$\left\{ x + (\widehat{H} \cdot S)_T + e_T < \frac{1}{n} \right\}, \text{ for any } n \in \mathbb{N}.$$

7.2 Revisit the Dual Problem

In this section, we will present our main result, i.e., in the Brownian framework, the countably additive part \widehat{Q}^r of any dual optimizer $\widehat{Q} \in (L^\infty)^*$ obtained in [23] can be attained by the terminal value of a local martingale \widehat{Y} , which belongs to the set of all supermartingale deflators, defined by

$$\mathcal{Y}(1) := \{Y = (Y_t)_{0 \leq t \leq T} \mid Y_0 = 1, XY \text{ is a supermartingale for any } X \in \mathcal{X}(1)\},$$

where

$$\mathcal{X}(1) := \{1 + (H \cdot S) \mid 1 + (H \cdot S)_t \geq 0, \text{ for all } 0 \leq t \leq T\}.$$

We first observe that the dual optimizer for the problem (7.1.3) can be approximated by a sequence of equivalent local martingale measures.

Proposition 7.2.1. *Let Assumptions 7.1.2, 7.1.1, 7.1.4 hold. Let $\hat{y} := u'(x)$. If $\hat{Q}_{\hat{y}}$ is a dual optimizer (denoted by \hat{Q} for short) for the problem (7.1.3), then there exists a sequence $(\mathbf{Q}^n)_{n \in \mathbb{N}}$ of equivalent local martingale measures, such that*

$$\frac{d\mathbf{Q}^n}{d\mathbf{P}} \rightarrow \frac{d\hat{Q}}{d\mathbf{P}}, \text{ a.s.} \quad \text{and} \quad \langle \mathbf{Q}^n, e_T \rangle \rightarrow \langle \hat{Q}, e_T \rangle, \text{ as } n \rightarrow \infty. \quad (7.2.1)$$

Proof. First, we claim that \mathcal{D} is the weak-star closure of \mathcal{M} , which can be regarded as a subset of ba via the canonical embedding. For the convenience of the reader, we shall briefly prove this claim. Indeed, $\mathcal{D} \supseteq \overline{\mathcal{M}}^{\sigma((ba, L^\infty))}$ is trivial. To show the equality, we suppose, contrary to the claim, there exists a point $\bar{Q} \in \mathcal{D}$ but $\bar{Q} \notin \overline{\mathcal{M}}^{\sigma((ba, L^\infty))}$. From [23], \mathcal{D} is compact, thus by the Hahn-Banach separation theorem, one can find a function $f \in L^\infty$ and a constant α such that

$$\langle Q, f \rangle \leq \alpha, \text{ for all } Q \in \overline{\mathcal{M}}^{\sigma((ba, L^\infty))},$$

but

$$\langle \bar{Q}, f \rangle > \alpha. \quad (7.2.2)$$

Applying the superreplication theory, we conclude that $f - \alpha \in \mathcal{C}$ and thus, from the definition of \mathcal{D} , we have $\langle \bar{Q}, f \rangle \leq \alpha$, which is in contradiction to (7.2.2).

Then, it follows from Corollary A.2.3 that we can find a sequence $(\mathbf{Q}^n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$, such that (7.2.1) holds. \square

Remark 7.2.2. For any cluster point Q^* of the sequence $(\mathbf{Q}^n)_{n \in \mathbb{N}}$ in Proposition 7.2.1, Q^* is a dual optimizer for (7.1.3) by Proposition A.1 in [23].

Let $(\mathbf{Q}^n)_{n \in \mathbb{N}}$ be the sequence chosen in Proposition 7.2.1. Define for each $n \in \mathbb{N}$

$$Y_t^n := \mathbf{E} \left[\frac{d\mathbf{Q}^n}{d\mathbf{P}} \middle| \mathcal{F}_t \right],$$

which is the density process of \mathbf{Q}^n and therefore, a strictly positive martingale.

We recall the definition of optional strong supermartingales. These processes are introduced by Mertens [79] as a generalization of càdlàg supermartingales. We also refer to [37, Appendix I] for more properties of these processes.

Definition 7.2.3. *A real-valued stochastic process $Y = (Y_t)_{0 \leq t \leq T}$ is called an **optional strong supermartingale**, if*

1. Y is optional;
2. Y_τ is integrable for every $[0, T]$ -valued stopping time τ ;
3. For all stopping times σ and τ with $0 \leq \sigma \leq \tau \leq T$, we have

$$Y_\sigma \geq \mathbf{E}[Y_\tau | \mathcal{F}_\sigma].$$

By [29, Theorem 2.7], there exists a sequence $(\tilde{Y}^n)_{n \in \mathbb{N}}$ of convex combinations $\tilde{Y}^n \in \text{conv}(Y^n, Y^{n+1}, \dots)$, and a nonnegative optional strong supermartingale \hat{Y} (not necessarily càdlàg), such that for every $[0, T]$ -valued stopping time σ , we have

$$\tilde{Y}_\sigma^n \xrightarrow{\mathbf{P}} \hat{Y}_\sigma, \text{ as } n \rightarrow \infty. \quad (7.2.3)$$

Obviously,

$$\frac{d\tilde{\mathbf{Q}}^n}{d\mathbf{P}} = \tilde{Y}_T^n \longrightarrow \hat{Y}_T = \frac{d\hat{\mathbf{Q}}^r}{d\mathbf{P}}, \mathbf{P} - a.s., \text{ as } n \rightarrow \infty,$$

where $d\tilde{\mathbf{Q}}^n = \tilde{Y}_T^n d\mathbf{P}$. In the remainder of this chapter, our main goal is to show the claim that

$$\hat{Y} \text{ is a local martingale,} \quad (7.2.4)$$

under the following assumption:

Assumption 7.2.4. *The underlying filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ is generated by a Brownian motion.*

Once the claim (7.2.4) is verified, we know from the above assumption that \hat{Y} is continuous and thus a (càdlàg) supermartingale. By a similar argument as in the proof of [70, Lemma 4.1], namely, for any $X \in \mathcal{X}(1)$, applying [29, Theorem 2.7] again, one can see that $X\hat{Y}$ is still a supermartingale, which implies $\hat{Y} \in \mathcal{Y}(1)$.

Remark 7.2.5. One can also apply the well-known Fatou-convergence result (see [41, Lemma 5.2]) to construct \hat{Y}' as the Fatou limit of $(\tilde{Y}^n)_{n \in \mathbb{N}}$, whose terminal value is exactly the density $\frac{d\hat{\mathbf{Q}}^r}{d\mathbf{P}}$. Although the process \hat{Y}' constructed in this way is certainly càdlàg, yet (7.2.3) may fail. Therefore, the advantage of the result in [29] is that we could find a unified sequence which is not only the limit of \tilde{Y}^n at the terminal time T but also at any intermediate time. Particularly, one can pick a subsequence such that the convergence holds \mathbf{P} -a.s. at countably many times. Note that the difference between the two kinds of limit is only on the graph of countably many stopping times, see [30, Section A.1].

Remark 7.2.6. Under the above assumption, every local martingale has a continuous modification. In particular, from Assumption 7.1.2, the stock price process S in our setting is indeed continuous. It is not clear to us whether this assumption is really necessary for the following theorem or it could be weakened. We leave this as an open question.

Now, we are ready to state our main result. Its proof is postponed to the next section.

Theorem 7.2.7. *Under Assumptions 7.1.2, 7.1.1, 7.1.4, 7.2.4, the process \hat{Y} defined in (7.2.3) is a local martingale and thus, the regular part $\hat{\mathbf{Q}}^r$ of any dual optimizer obtained in [23] can be attained by a local martingale, which belongs to $\mathcal{Y}(1)$.*

7.3 Proof of Theorem 7.2.7

In this section, we shall prove Theorem 7.2.7. We break the proof into three main steps. In the sequel, each subsection stands for a step.

7.3.1 The Fictitious Optimal Wealth Process

In a first stage, we construct a fictitious optimal wealth process \widehat{W} , which attains the optimal terminal value $x + (\widehat{H} \cdot S)_T + e_T$. Then, we look for a sequence of stopping times, such that at each stopping time, the process \widehat{W} is bounded away from 0.

Define for every $n \in \mathbb{N}$ the process

$$\widetilde{W}_t^n := x + (\widehat{H} \cdot S)_t + \mathbf{E}_{\widetilde{\mathbf{Q}}^n}[e_T | \mathcal{F}_t], \quad 0 \leq t \leq T.$$

Since \mathcal{M} is closed with respect to convex combination, $\widetilde{\mathbf{Q}}^n$ is still an equivalent local martingale measures, so that \widetilde{W}^n is a $\widetilde{\mathbf{Q}}^n$ -supermartingale. It follows from the optimality of \widehat{H} that

$$\widetilde{W}_T^n = x + (\widehat{H} \cdot S)_T + e_T > 0, \quad \mathbf{P} - a.s.,$$

which holds also $\widetilde{\mathbf{Q}}^n$ -a.s., since \mathbf{P} and $\widetilde{\mathbf{Q}}^n$ are equivalent. By [37, Theorem VI-17], one can deduce that

$$\inf_{0 \leq t \leq T} \widetilde{W}_t^n > 0, \quad \widetilde{\mathbf{Q}}^n - a.s., \quad (7.3.1)$$

which holds also \mathbf{P} -a.s.

Consider the process

$$\widetilde{Y}_t^n \widetilde{W}_t^n = \widetilde{Y}_t^n (x + (\widehat{H} \cdot S)_t) + \mathbf{E}[\widetilde{Y}_T^n e_T | \mathcal{F}_t], \quad 0 \leq t \leq T,$$

which is obviously a strictly positive \mathbf{P} -supermartingale. Applying [29, Theorem 2.7] again, there exists a sequence of convex combinations of $(\widetilde{Y}^n)_{n \in \mathbb{N}}$, denoted still by $(\widetilde{Y}^n)_{n \in \mathbb{N}}$, and a nonnegative optional strong supermartingale \widehat{Z} , such that for every $[0, T]$ -valued stopping time σ we have

$$\widetilde{Y}_\sigma^n \widetilde{W}_\sigma^n \xrightarrow{\mathbf{P}} \widehat{Z}_\sigma, \quad \text{as } n \rightarrow \infty. \quad (7.3.2)$$

It is evident that (7.2.3) still holds for $(\widetilde{Y}^n)_{n \in \mathbb{N}}$ as well.

Proposition 7.3.1. *The process $\widehat{W} := \frac{\widehat{Z}}{\widehat{Y}}$ is well-defined.*

Proof. Since $\widehat{Y}_T = \frac{d\widehat{Q}^r}{d\mathbf{P}}$, as stated in Remark 7.1.6 (ii), we have that $\widehat{Y}_T > 0$, \mathbf{P} -a.s. Then, one can employ the same argument as (7.3.1) to deduce (see [37, Theorem VI-17, Appendix I Remark 5])

$$\inf_{0 \leq t \leq T} \widehat{Y}_t > 0, \quad \mathbf{P} - a.s., \quad (7.3.3)$$

which implies that \widehat{W} is well-defined. \square

Proposition 7.3.2. *The process \widehat{Z} is a martingale and from Assumption 7.2.4 it has a continuous modification.*

Proof. Note that

$$\widehat{W}_T = \widetilde{W}_T^n = x + (\widehat{H} \cdot S)_T + e_T,$$

for all $n \in \mathbb{N}$. We have, from (7.1.4) and (7.2.1),

$$\widehat{Z}_0 = \lim_{n \rightarrow \infty} \widetilde{Y}_0^n \widetilde{W}_0^n = x + \lim_{n \rightarrow \infty} \langle \widetilde{\mathbf{Q}}^n, e_T \rangle = x + \langle \widehat{Q}, e_T \rangle = \mathbf{E}[\widehat{Y}_T \widehat{W}_T] = \mathbf{E}[\widehat{Z}_T],$$

from which we conclude that the process \widehat{Z} is a martingale, since we have already known it is an optional strong supermartingale during the construction. \square

Proposition 7.3.3. *There exists a sequence of stopping times $(\tau_k)_{k \in \mathbb{N}}$, such that*

$$\widehat{W}_{t \wedge \tau_k} \geq \frac{1}{k},$$

for all $t \in [0, T]$ and $\mathbf{P}[\tau_k = T] \nearrow 1$, as $k \rightarrow \infty$.

Proof. Although \widehat{Y} is only an optional strong supermartingale, we can always apply the martingale inequality to show that

$$\sup_{0 \leq t \leq T} \widehat{Y}_t < \infty, \quad \mathbf{P} - a.s.$$

(see [37, Appendix I-3, page 395]). On the other hand, thanks to proposition 7.3.2, we could proceed with the same argument as (7.3.1) to obtain

$$\inf_{0 \leq t \leq T} \widehat{Z}_t > 0, \quad \mathbf{P} - a.s.$$

Clearly, we now have

$$\inf_{0 \leq t \leq T} \widehat{W}_t > 0, \quad \mathbf{P} - a.s.$$

Without loss of generality, we assume $\widehat{Z}_t = \widehat{Z}_T$, for $t \geq T$. Define, for $k \in \mathbb{N}$,

$$\sigma_k := \inf \left\{ t > 0 : \widehat{W}_t < \frac{1}{k} \right\}, \quad (7.3.4)$$

which goes to infinity. From Assumption 7.2.4, all the stopping times defined above are predictable. Therefore, for each k , we can choose a sequence $\sigma_{k,m} \rightarrow \sigma_k$ and $\sigma_{k,m} < \sigma_k$, whenever $\sigma_k > 0$. Define $\tau_k := \sigma_{k,m_k} \wedge T$, where

$$\mathbf{P} \left[|\sigma_{k,m_k} - \sigma_k| > \frac{1}{2^k} \right] < \frac{1}{2^k}.$$

The sequence $(\tau_k)_{k \in \mathbb{N}}$ yields the desired result. \square

7.3.2 The Fictitious Process for the Random Endowment

In the sequel, fix $k \in \mathbb{N}$ and denote by $\tau = \tau_k$. We shall first decompose \widehat{W} and obtain a fictitious process for the random endowment e_T . Then, we construct a dual optimizer Q^* and prove that the random variable e_τ is the conditional expectation of e_T under Q^* .

It follows from (7.2.3) and (7.3.2) that for every $[0, T]$ -valued stopping time τ ,

$$\widetilde{W}_\tau^n \xrightarrow{\mathbf{P}} \widehat{W}_\tau, \quad \text{as } n \rightarrow \infty. \quad (7.3.5)$$

Then, we rewrite the process \widehat{W} as

$$\widehat{W}_t = x + (\widehat{H} \cdot S)_t + e_t, \quad 0 \leq t \leq T,$$

where

$$e_t := \mathbf{P} - \lim_{n \rightarrow \infty} \mathbf{E}_{\widetilde{Q}^n} [e_T | \mathcal{F}_t] \quad (7.3.6)$$

with

$$\mathbf{E}_{\widetilde{Q}^n} [e_T | \mathcal{F}_t] = \frac{\mathbf{E}[\widetilde{Y}_T^n e_T | \mathcal{F}_t]}{\mathbf{E}[\widetilde{Y}_T^n | \mathcal{F}_t]}.$$

Remark 7.3.4. In [23], e_T is indeed associated with a cumulative process $e := (e_t)_{0 \leq t \leq T}$ with $e_0 = 0$, however, only the terminal value e_T influences the choice of the agent. Here, $e := (e_t)_{0 \leq t \leq T}$ is a fictitious value process with the terminal value e_T , which is constructed by (7.3.6) and should be differed from the one in [23].

With the stopping time τ , we see that $(\mathbf{E}[\tilde{Y}_T^n e_T | \mathcal{F}_\tau])_{n \in \mathbb{N}}$ and $(\mathbf{E}[\tilde{Y}_T^n | \mathcal{F}_\tau])_{n \in \mathbb{N}}$ are L^1 -bounded, then recalling Komlós' lemma, we can find a sequence $(\bar{Y}^n)_{n \in \mathbb{N}}$ of convex combinations $\bar{Y}^n \in \text{conv}(\tilde{Y}^n, \tilde{Y}^{n+1}, \dots)$ associated with $\bar{\mathbf{Q}}^n \in \text{conv}(\tilde{\mathbf{Q}}^n, \tilde{\mathbf{Q}}^{n+1}, \dots)$, such that \mathbf{P} -a.s., for some $g \in L^1(\Omega, \mathcal{F}_\tau, \mathbf{P})$,

$$\lim_{n \rightarrow \infty} \mathbf{E}[\bar{Y}_T^n e_T | \mathcal{F}_\tau] = g, \quad \lim_{n \rightarrow \infty} \mathbf{E}[\bar{Y}_T^n | \mathcal{F}_\tau] = \lim_{n \rightarrow \infty} \bar{Y}_\tau^n = \hat{Y}_\tau,$$

and

$$e_\tau = \lim_{n \rightarrow \infty} \mathbf{E}_{\bar{\mathbf{Q}}^n}[e_T | \mathcal{F}_\tau] = \frac{g}{\hat{Y}_\tau}.$$

Remark 7.3.5. The random variables $(\mathbf{E}_{\bar{\mathbf{Q}}^n}[e_T | \mathcal{F}_\tau])_{n \in \mathbb{N}}$ and e_τ are in $L^\infty(\Omega, \mathcal{F}_\tau, \mathbf{P})$, and

$$\mathbf{E}_{\bar{\mathbf{Q}}^n}[e_T | \mathcal{F}_\tau] \xrightarrow{\sigma(L^\infty, L^1)} e_\tau, \quad \text{as } n \rightarrow \infty.$$

Indeed, for each step function $\xi^m \in \mathcal{F}_\tau$, $m \in \mathbb{N}$, it follows from the bounded convergence theorem that

$$\langle \mathbf{E}_{\bar{\mathbf{Q}}^n}[e_T | \mathcal{F}_\tau], \xi^m \rangle \rightarrow \langle e_\tau, \xi^m \rangle, \quad \text{as } n \rightarrow \infty.$$

If $\xi^m \rightarrow \xi$ in $L^1(\Omega, \mathcal{F}_\tau, \mathbf{P})$, then from the uniform integrability of $(\mathbf{E}_{\bar{\mathbf{Q}}^n}[e_T | \mathcal{F}_\tau] \xi^m)_{m, n \in \mathbb{N}}$, we have

$$\langle \mathbf{E}_{\bar{\mathbf{Q}}^n}[e_T | \mathcal{F}_\tau], \xi \rangle \rightarrow \langle e_\tau, \xi \rangle, \quad \text{as } n \rightarrow \infty.$$

By Egorov's theorem, there exists an increasing sequence of sets $(\Omega_m)_{m \in \mathbb{N}}$, such that for each m , $\mathbf{P}[\Omega_m] > 1 - \frac{1}{2^m}$, and $(\bar{Y}_\tau^n)_{n \in \mathbb{N}}$ converges uniformly to \hat{Y}_τ on Ω_m . Observing that, for each $n \in \mathbb{N}$, \bar{Y}_τ^n is in $L^1_+(\Omega, \mathcal{F}_\tau, \mathbf{P})$, we know from Fatou's lemma that

$$\mathbf{E}_{\mathbf{P}}[|\hat{Y}_\tau|] \leq 1,$$

and thus

$$(\bar{Y}_\tau^n)_{n \in \mathbb{N}} \text{ is uniformly integrable on } \Omega_m. \quad (7.3.7)$$

Proposition 7.3.6. *The sequence $(\bar{\mathbf{Q}}^n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$ associated with $(\bar{Y}_\tau^n)_{n \in \mathbb{N}}$ admits a cluster point $Q^* \in \mathcal{D}$, such that*

(i) $\bar{Y}_T^n \rightarrow \frac{dQ^*}{d\mathbf{P}}$, \mathbf{P} -a.s.;

(ii) $\langle \hat{Q}, e_T \rangle = \langle Q^*, e_T \rangle$;

(iii) Q^* is a dual optimizer for the problem (7.1.3).

Proof. Note $(\bar{\mathbf{Q}}^n)_{n \in \mathbb{N}} \subseteq \mathcal{M} \subseteq \mathcal{D}$. Since \mathcal{D} is a weak-star compact subset of ba , the sequence $(\bar{\mathbf{Q}}^n)_{n \in \mathbb{N}}$ admits a cluster point $Q^* \in \mathcal{D}$. (i) follows from [23, Proposition A.1]; (ii) holds because of (7.2.1); (i) and (ii) imply (iii). \square

Immediately, we have the following corollary:

Corollary 7.3.7. *The finitely additive measure $Q^*|_{\mathcal{F}_\tau}$ is countably additive on Ω_m , for each $m \in \mathbb{N}$, where $Q^*|_{\mathcal{F}_\tau}$ denotes the restriction of Q^* on \mathcal{F}_τ . In other words, $(Q^*|_{\mathcal{F}_\tau})^s$ vanishes on Ω_m , i.e., for each $A \subseteq \Omega_m$, $A \in \mathcal{F}_\tau$, we have that $\langle (Q^*|_{\mathcal{F}_\tau})^s, \mathbf{1}_A \rangle = 0$.*

Proceeding as in the proof of [88, Corollary 5.2], we observe that $Q^*|_{\mathcal{F}_\tau}$ is also a cluster point of the sequence $(\overline{Q}^n|_{\mathcal{F}_\tau})_{n \in \mathbb{N}}$, and it follows again from [23, Proposition A.1.] that

$$\widehat{Y}_\tau = \lim_{n \rightarrow \infty} \mathbf{E}[\overline{Y}_T^n | \mathcal{F}_\tau] = \frac{d(Q^*|_{\mathcal{F}_\tau})^r}{d\mathbf{P}}, \quad \mathbf{P} - a.s. \quad (7.3.8)$$

Remark 7.3.8. We remark that the choice of the finitely additive measure Q^* depends on the stopping time τ , which may not be a Föllmer finitely additive measure for \widehat{Y} (see [88, Definition 2.6]).

The following corollary is straightforward from (7.3.3):

$$(Q^*|_{\mathcal{F}_\tau})^r \sim \mathbf{P}. \quad (7.3.9)$$

An argument similar to the one used in (7.3.8) shows that

Proposition 7.3.9.

$$g = \lim_{n \rightarrow \infty} \mathbf{E}[\overline{Y}_T^n e_T | \mathcal{F}_\tau] = \frac{d((Q^* e_T)|_{\mathcal{F}_\tau})^r}{d\mathbf{P}}, \quad \mathbf{P} - a.s.,$$

where $Q^* e_T$ denotes the linear operator $\langle Q^*, e_T \cdot \rangle \in ba$ and

$$\begin{aligned} ((Q^* e_T)|_{\mathcal{F}_\tau})^r &:= ((Q^* e_T^+)|_{\mathcal{F}_\tau})^r - ((Q^* e_T^-)|_{\mathcal{F}_\tau})^r; \\ ((Q^* e_T)|_{\mathcal{F}_\tau})^s &:= ((Q^* e_T^+)|_{\mathcal{F}_\tau})^s - ((Q^* e_T^-)|_{\mathcal{F}_\tau})^s. \end{aligned}$$

The lemma below is the core of the proof to Theorem 7.2.7:

Lemma 7.3.10. *The random variable e_τ is the conditional expectation of e_T under Q^* with respect to \mathcal{F}_τ . In particular,*

$$\langle Q^*|_{\mathcal{F}_\tau}, e_\tau \rangle = \langle Q^*, e_T \rangle.$$

Proof. It follows from the boundedness of e_T , there exists a unique random variable $\eta \in \mathcal{F}_\tau$ (see [76, Definition 7.1, Theorem 7.2]), such that for each $A \in \mathcal{F}_\tau$,

$$\langle Q^*|_{\mathcal{F}_\tau} \eta, \mathbf{1}_A \rangle = \langle Q^* e_T, \mathbf{1}_A \rangle.$$

Our aim now is to prove

$$\eta = e_\tau = \frac{d((Q^* e_T)|_{\mathcal{F}_\tau})^r}{\frac{d(Q^*|_{\mathcal{F}_\tau})^r}{d\mathbf{P}}}, \quad \mathbf{P} - a.s.$$

It is evident that Q^*e_T is a cluster point of the sequence $(\overline{Q}^n e_T)_{n \in \mathbb{N}}$, and similar to Corollary 7.3.7, we know that for each $m \in \mathbb{N}$, $((Q^*e_T)|_{\mathcal{F}_\tau})^s$ vanishes on Ω_m . Then, for any $A \subseteq \Omega_m$, $A \in \mathcal{F}_\tau$, we have

$$\begin{aligned} \left\langle (Q^*|_{\mathcal{F}_\tau})^r, \frac{d((Q^*e_T)|_{\mathcal{F}_\tau})^r}{\frac{d(Q^*|_{\mathcal{F}_\tau})^r}{d\mathbf{P}}} \mathbf{1}_A \right\rangle &= \mathbf{E} \left[\frac{d((Q^*e_T)|_{\mathcal{F}_\tau})^r}{d\mathbf{P}} \mathbf{1}_A \right] = \langle ((Q^*e_T)|_{\mathcal{F}_\tau})^r, \mathbf{1}_A \rangle \\ &= \langle (Q^*e_T)|_{\mathcal{F}_\tau}, \mathbf{1}_A \rangle = \langle Q^*e_T, \mathbf{1}_A \rangle = \langle Q^*|_{\mathcal{F}_\tau} \eta, \mathbf{1}_A \rangle \\ &= \langle (Q^*|_{\mathcal{F}_\tau})^r, \eta \mathbf{1}_A \rangle, \end{aligned}$$

which implies $\eta = e_\tau$, $(Q^*|_{\mathcal{F}_\tau})^r$ -a.s. on Ω_m . Thanks to (7.3.9), we obtain that $\eta = e_\tau$, \mathbf{P} -a.s. on Ω_m . Letting $m \rightarrow \infty$, we end the proof. \square

7.3.3 Proof of the Main Result

In this subsection, we show that $(\overline{Y}_\tau^n)_{n \in \mathbb{N}}$ is uniformly integrable so that $\widehat{Y}_{\cdot \wedge \tau}$ is a martingale. Then, substituting τ by τ_k , $k \in \mathbb{N}$, we can conclude that \widehat{Y} is a local martingale.

Let us first consider a dynamic version of (7.1.4):

Proposition 7.3.11. *It satisfies*

$$\langle Q^*|_{\mathcal{F}_\tau}, \widehat{W}_\tau \rangle = \langle Q^*|_{\mathcal{F}_\tau}, x + (\widehat{H} \cdot S)_\tau + e_\tau \rangle = x + \langle Q^*, e_T \rangle.$$

Proof. As $Q^* \in \mathcal{D}$, we have by definition that

$$\langle Q^*|_{\mathcal{F}_\tau}, (\widehat{H} \cdot S)_\tau \rangle = \langle Q^*, (\widehat{H} \cdot S)_\tau \rangle \leq 0.$$

Thus, from the positivity of $(Q^*|_{\mathcal{F}_\tau})^s$, the martingale property of $\widehat{Y}\widehat{W}$ together with Lemma 7.3.10, we obtain

$$\begin{aligned} x + \langle Q^*, e_T \rangle &= \langle (Q^*|_{\mathcal{F}_\tau})^r, x + (\widehat{H} \cdot S)_\tau + e_\tau \rangle \\ &\leq \langle Q^*|_{\mathcal{F}_\tau}, x + (\widehat{H} \cdot S)_\tau + e_\tau \rangle \leq x + \langle Q^*, e_T \rangle, \end{aligned}$$

which completes the proof. \square

Now we can deduce that $(\overline{Y}_\tau^n)_{n \in \mathbb{N}}$ is uniformly integrable.

Proposition 7.3.12. *The sequence of random variables $(\overline{Y}_\tau^n)_{n \in \mathbb{N}} = (\mathbf{E}[\frac{d\overline{Q}^n}{d\mathbf{P}}|_{\mathcal{F}_\tau}])_{n \in \mathbb{N}}$ is uniformly integrable and*

$$\overline{Y}_\tau^n \xrightarrow{L^1} \widehat{Y}_\tau, \quad \text{as } n \rightarrow \infty.$$

Proof. From the proof the proposition above, we see that

$$\langle (Q^*|_{\mathcal{F}_\tau})^s, x + (\widehat{H} \cdot S)_\tau + e_\tau \rangle = 0,$$

on the other hand, according to Proposition 7.3.3,

$$\widehat{W}_\tau = x + (\widehat{H} \cdot S)_\tau + e_\tau > 0, \quad \mathbf{P} - a.s.$$

Thus, we derive that $(Q^*|_{\mathcal{F}_\tau})^s \equiv 0$. Recalling that $\|Q^*\|_{ba} = 1$, we have

$$\mathbf{E}[\widehat{Y}_\tau] = \mathbf{E}\left[\frac{d(Q^*|_{\mathcal{F}_\tau})^r}{d\mathbf{P}}\right] = 1.$$

We summarize as follows

$$\mathbf{E}[\overline{Y}_\tau^n] = 1 \text{ and } \overline{Y}_\tau^n \longrightarrow \widehat{Y}_\tau, \mathbf{P} - a.s., \text{ as } n \rightarrow \infty.$$

The desired result follows by Scheffé's lemma (compare with Lemma 7.4.1). \square

Proof of Theorem 7.2.7. For each τ_k defined in Proposition 7.3.3, it follows by Proposition 7.3.12 that

$$\overline{Y}_{\tau_k}^n \xrightarrow{L^1} \widehat{Y}_{\tau_k}, \text{ as } n \rightarrow \infty.$$

From the martingale property and (7.2.3), we also have for $t \in [0, T]$ that

$$\overline{Y}_{t \wedge \tau_k}^n \xrightarrow{L^1} \widehat{Y}_{t \wedge \tau_k}, \text{ as } n \rightarrow \infty,$$

which implies $\widehat{Y}_{\cdot \wedge \tau_k}$ is a martingale. By the definition of $(\tau_k)_{k \in \mathbb{N}}$, \widehat{Y} is a local martingale. As already discussed in the previous section, \widehat{Y} is consequently a supermartingale deflator, i.e., \widehat{Y} belongs to $\mathcal{Y}(1)$. \square

Remark 7.3.13. Indeed, for each k , the dual optimizer Q^* we constructed generates an equivalent local martingale measure on $\llbracket 0, \tau_k \rrbracket$ such that the pricing of the fictitious random endowment e_{τ_k} under this measure is exact $\langle \widehat{Q}, e_T \rangle$, in particular, $\langle \widehat{Q}, e_T \rangle$ is an arbitrage-free price for e_{τ_k} .

Remark 7.3.14. We would like to explain a little about the dynamics of $Q^*|_{\mathcal{F}_\cdot}$. Clearly, the underlying price process S and local martingale \widehat{Y} is continuous, so is the wealth process \widehat{W} . Consider a set $A \in \mathcal{F}_\tau$, where τ is a $[0, T]$ -valued stopping time, and suppose \widehat{W}_τ is strictly positive on A . For an infinitesimal δt such that $\widehat{W}_{\tau+\delta t}$ cannot suddenly jump to 0, we can show that if $(Q^*|_{\mathcal{F}_\tau})^s \equiv 0$ on A , then $(Q^*|_{\mathcal{F}_{\tau+\delta t}})^s \equiv 0$ on A . Otherwise,

$$\begin{aligned} \langle Q^*|_{\mathcal{F}_{\tau+\delta}}, \widehat{W}_{\tau+\delta} \mathbf{1}_A \rangle &= \langle (Q^*|_{\mathcal{F}_{\tau+\delta}})^r, \widehat{W}_{\tau+\delta} \mathbf{1}_A \rangle + \langle (Q^*|_{\mathcal{F}_{\tau+\delta}})^s, \widehat{W}_{\tau+\delta} \mathbf{1}_A \rangle \\ &> \langle (Q^*|_{\mathcal{F}_{\tau+\delta}})^r, \widehat{W}_{\tau+\delta} \mathbf{1}_A \rangle = \langle (Q^*|_{\mathcal{F}_\tau})^r, \widehat{W}_\tau \mathbf{1}_A \rangle = \langle Q^*|_{\mathcal{F}_\tau}, \widehat{W}_\tau \mathbf{1}_A \rangle. \end{aligned}$$

This implies $\langle Q^*, ((\widehat{H} \cdot S)_{\tau+\delta t} - (\widehat{H} \cdot S)_\tau) \mathbf{1}_A \rangle > 0$, which is a contradiction to the definition of \mathcal{D} .

7.4 Alternative Proof of a Result of Larsen and Žitković

In Section 3.2 of [73], the authors show that if the stock price process S is a continuous semimartingale in the problem without random endowment formulated in [70], then the dual optimizer is associated with a local martingale living in the set of supermartingale deflators. Here, we shall give an alternative proof for this assertion based on the dynamics

of the primal and dual solutions. We emphasize that no extra condition on the filtration \mathcal{F} is assumed in this subsection.

Before presenting the theorem, we first introduce the following lemma, which provides us an abstract structure.

Lemma 7.4.1. *Let $(Y^n)_{n \in \mathbb{N}} \subseteq L_+^1(\Omega, \mathcal{F}, \mathbf{P})$ and $(X^n)_{n \in \mathbb{N}} \subseteq L^0(\Omega, \mathcal{F}, \mathbf{P})$, where for each n , $X^n \geq a > 0$, \mathbf{P} -a.s. If there exists a pair of random variables $(Y, X) \in L^1(\Omega, \mathcal{F}, \mathbf{P}) \times L^0(\Omega, \mathcal{F}, \mathbf{P})$, such that*

$$Y^n \rightarrow Y, \quad X \leq \liminf_{n \rightarrow \infty} X^n, \quad \mathbf{P} - a.s.,$$

and

$$\mathbf{E}[YX] \geq \liminf_{n \rightarrow \infty} \mathbf{E}[Y^n X^n].$$

Then, $(Y^n)_{n \in \mathbb{N}}$ is uniformly integrable.

Proof. If not, by passing to a subsequence if necessary, there exists $\varepsilon > 0$, for each $n \in \mathbb{N}$, there exists A_n such that $\mathbf{P}[A_n] < \frac{1}{2^n}$ and

$$\mathbf{E}[Y^n \mathbf{1}_{A_n}] \geq \varepsilon.$$

Define

$$\eta^n := Y^n \mathbf{1}_{A_n}, \quad \xi^n := Y^n \mathbf{1}_{A_n^c}.$$

Then $\xi^n \rightarrow Y$, a.s., while by Fatou's lemma, we have

$$\mathbf{E}[YX] \leq \liminf_{n \rightarrow \infty} \mathbf{E}[\xi^n X^n] = \liminf_{n \rightarrow \infty} \mathbf{E}[Y^n X^n - \eta^n X^n] \leq \mathbf{E}[YX] - a\varepsilon,$$

which is a contradiction. □

In [70], the primal value function can be regarded as (7.1.1) with $e_T \equiv 0$. On the other hand, the dual domain is defined as the solid subset generated by all terminal values of supermartingale deflators, namely,

$$\mathcal{D} := \{h \in L_+^0(\Omega, \mathcal{F}_T, \mathbf{P}) \mid h \leq Y_T, \text{ for some } Y \in \mathcal{Y}(1)\}. \quad (7.4.1)$$

Then, the dual problem is formulated by

$$v(y) := \inf_{h \in \mathcal{D}} \mathbf{E}[V(yh)]. \quad (7.4.2)$$

It has been proved that for each $x > 0$, and $\hat{y} := u'(x)$, the value $v(y)$ is attained by a unique dual optimizer $\hat{h}_{\hat{y}} \in \mathcal{D}$, denoted by \hat{h} for short, and the primal solution $\hat{X}_T = x + (\hat{H} \cdot S)_T$ can be constructed in terms of \hat{h} . Moreover,

$$\mathbf{E}[\hat{h} \hat{X}_T] = x. \quad (7.4.3)$$

Instead of Assumption 7.2.4, we make the following assumption.

Assumption 7.4.2. *The stock price process S is continuous.*

Theorem 7.4.3. *Under Assumptions 7.1.2, 7.1.1, 7.1.4, 7.4.2, the dual optimizer of (7.4.2) obtained in [70] is associated with a supermartingale deflator, which is a local martingale.*¹

Proof. Since $(\widehat{H} \cdot S)$ is a supermartingale under each $\mathbf{Q} \in \mathcal{M}$, similarly to (7.3.1), we obtain

$$\inf_{0 \leq t \leq T} \widehat{X}_t > 0, \quad \mathbf{P} - a.s.$$

By the continuity of $\widehat{X} := x + (\widehat{H} \cdot S)$, one can thus define a sequence of stopping times as (7.3.4), such that $\widehat{X}_{\sigma_k \wedge T} \geq \frac{1}{k}$. Recalling [70, Proposition 3.2], for each $y > 0$, the value $v(y)$ of the dual problem can be approximated by choosing a minimizing sequence $(\mathbf{Q}^n)_{n \in \mathbb{N}}$ of equivalent local martingale measures, associated with the density process $(Y^n)_{n \in \mathbb{N}}$, such that $Y_T^n \rightarrow \widehat{h}$, \mathbf{P} -a.s. By [29, Theorem 2.7], we could find a sequence of convex combinations of $(Y^n)_{n \in \mathbb{N}}$, and an optional strong supermartingale \widehat{Y} , such that Y^n converges to \widehat{Y} in the sense of (7.2.3). In particular, after passing to a subsequence,

$$Y_{\sigma_k \wedge T}^n \rightarrow \widehat{Y}_{\sigma_k \wedge T} \quad \text{and} \quad Y_T^n \rightarrow \widehat{Y}_T = \widehat{h}, \quad \mathbf{P} - a.s.,$$

so that \widehat{Y}_T is the dual optimizer. Moreover, applying once again Theorem 2.7 in [29], we can show that $\widehat{Y}\widehat{X}$ is an optional strong supermartingale, then we can deduce that $\widehat{Y}\widehat{X}$ is a martingale from (7.4.3). Consequently, we arrive at

$$\begin{cases} \mathbf{E}[Y_{\sigma_k \wedge T}^n \widehat{X}_{\sigma_k \wedge T}] \leq x, \\ \mathbf{E}[\widehat{Y}_{\sigma_k \wedge T} \widehat{X}_{\sigma_k \wedge T}] = x. \end{cases} \quad (7.4.4)$$

It follows immediately by Lemma 7.4.1 that $(Y_{\sigma_k \wedge T}^n)_{n \in \mathbb{N}}$ is uniformly integrable so that $\widehat{Y}_{\cdot \wedge \sigma_k \wedge T}$ is a true martingale. We now can conclude that \widehat{Y} is a local martingale, and thus is càdlàg. For any $X \in \mathcal{X}(1)$, applying Theorem 2.7 in [29] again, one can see that $X\widehat{Y}$ is still a (càdlàg) supermartingale, which implies $\widehat{Y} \in \mathcal{Y}(1)$. \square

Remark 7.4.4. According to [63], the inspection of the proofs in [70] reveals that the usual assumption (*NFLVR*) can be replaced by a weaker one (*NUPBR*), which is equivalent to that $\mathcal{Y}(1) \neq \emptyset$ or the existence of equivalent local martingale densities (see e.g. [69]). In this case, to deduce that \widehat{Y} is a local martingale, we could proceed the same as above with a minimizing sequence of equivalent local martingale densities and a classical localization argument if necessary.

Remark 7.4.5. Compare with Section 4, the proof for the case of $e_T = 0$ is much simpler, and we only need the continuity of the stock price process rather than the assumption on the underlying filtration. We explain as follows. Firstly, the wealth process \widehat{X} in both lines of (7.4.4) is unified, in contrast, for the case with nontrivial e_T , our sequence of fictitious wealth processes depends on n , and it is not easy to find a sequence of stopping times that stop the fictitious wealth processes simultaneously to let all of them stay above 0. Secondly, by the continuity of $(\widehat{H} \cdot S)$, we can easily stop the process by some τ and let it be bounded away from 0, while in the other case, we lack the continuity of \widehat{W} .

¹Kramkov and Weston [72] also proved a similar result but using a different technique.

Appendix A

Mathematical Work Tools

A.1 Probability Theory and Stochastic Analysis

A.1.1 Some Compactness Theorems in Probability

We now consider the $L^0(\Omega, \mathcal{F}, \mathbf{P})$. We assume that \mathcal{F} is \mathbf{P} -complete, i.e., for $B \subseteq A \in \mathcal{F}$ with $\mathbf{P}[A] = 0$ we have $B \in \mathcal{F}$. A pleasant consequence is that: one can change any measurable function on a set of outer \mathbf{P} -measure 0, the resulting function is still measurable.

Proposition A.1.1. *We have the following properties*

(1) For $f, g \in L^0$, $d(f, g) := \mathbf{E}[1 \wedge |f - g|]$ is a metric on L^0 and satisfies

$$d(f, g) = \mathbf{P}[|f - g| > 1] + \mathbf{E}[|f - g| \mathbf{1}_{\{|f - g| \leq 1\}}] \geq \mathbf{P}[|f - g| > 1].$$

(2) We have the following equivalence: $d(f_n, f) \rightarrow 0 \iff f_n \xrightarrow{\mathbf{P}} f$.

(3) L^0 equipped with the topology of convergence in probability is a complete metric space.

(4) The definition of L^0 only depends on the nullsets: If $\mathbf{Q} \sim \mathbf{P}$, then $L^0(\mathbf{Q}) = L^0(\mathbf{P})$. The convergence in probability is the same under \mathbf{P} as under any $\mathbf{Q} \sim \mathbf{P}$.

(5) (L^0, d) is a topological vector space. However, L^0 is not locally convex: the topology cannot be generated by a family of seminorms. All general results on locally convex topological vector spaces cannot be directly applied.

A neighbourhood basis of 0 in L^0 is given by $B_\varepsilon := \{f \in L^0 : d(f, 0) < \varepsilon\}$.

A subset $A \subseteq L^0$ is **bounded in L^0** , if for all $\varepsilon > 0$ there exists $\lambda > 0$ such that

$$\lambda A = \{\lambda f \mid f \in A\} \subseteq B_\varepsilon,$$

or equivalently, if

$$\lim_{M \rightarrow \infty} \sup_{f \in A} \mathbf{P}[|f| \geq M] = 0.$$

A subset $A \subseteq L^0$ is not bounded in L^0 , iff there exist $\delta > 0$, $(\gamma_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ with $\gamma_n \rightarrow \infty$, and $(f_n)_{n \in \mathbb{N}} \subseteq A$ such that $\mathbf{P}[|f_n| \geq \gamma_n] \geq \delta$ for all $n \in \mathbb{N}$.

The following theorem is a Komlós type result in L^0 .

Theorem A.1.2. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of random variables in $L_+^0(\Omega, \mathcal{F}, \mathbf{P})$. Then, there exists a sequence $g_n \in \text{conv}(f_n, f_{n+1}, \dots)$, i.e.,

$$g_n = \sum_{k=n}^{N_n} \lambda_k f_k, \quad \text{where } \lambda_k \in [0, 1] \text{ with } \sum_{k=n}^{N_n} \lambda_k = 1,$$

such that the sequence $(g_n)_{n \in \mathbb{N}}$ converges a.s. to a random variable g valued in $[0, \infty]$.

Moreover, if $\text{conv}(f_n, n \geq 1)$ is bounded in L^0 , then g is finite a.s.

Moreover, if there exist $\alpha > 0$, $\delta > 0$ such that $\mathbf{P}[f_n > \alpha] > \delta$ for all $n \in \mathbb{N}$, then $\mathbf{P}[g > 0] > 0$.

Proof. See [35, Lemma 9.8.1]. □

A.1.2 Bipolar Theorem

We now present the nonlocally convex version of the Bipolar Theorem for $\langle L_+^0, L_+^0 \rangle$, proved by Brannath and Schachermayer [13].

We define the duality in $\langle L_+^0, L_+^0 \rangle$ given by $\langle f, g \rangle = \mathbf{E}[fg] \in [0, \infty]$. For a subset $C \subseteq L_+^0$ define the **polar** C° of C as

$$C^\circ = \{f \in L_+^0 : \mathbf{E}[fg] \leq 1, \forall g \in C\}.$$

A subset $C \subseteq L_+^0$ will be called **solid** if $g \in C$, $h \in L^0$ and $0 \leq h \leq g$ implies that $h \in C$. Note that the polar C° of C is closed with respect to the topology of convergence in probability, convex and solid.

Proposition A.1.3. Assume that the set $C \subseteq L_+^0$ is nonempty, closed in probability, convex and solid. Then $C = C^{\circ\circ}$.

Proof. See [95, Proposition A.1]. □

Theorem A.1.4 (Bipolar Theorem). Let $C \subseteq L_+^0$ be nonempty. Then the bipolar $C^{\circ\circ}$ is the closed convex solid hull of C in L_+^0 (closure in the topology of convergence in probability).

Proof. See [95, Theorem A.2]. □

A.1.3 Essential Supremum of a Family of Random Variables

Definition A.1.5. Let $(f_i)_{i \in I}$ be a family of \mathbb{R} -valued random variables. The **essential supremum** of this family, denoted by $\text{ess sup}_{i \in I} f_i$ is a random variable \widehat{f} such that

1. $f_i \leq \widehat{f}$ a.s., for all $i \in I$,
2. if g is a random variable satisfying $f_i \leq g$ a.s., for all $i \in I$, then $\widehat{f} \leq g$ a.s.

Theorem A.1.6. Let $(f_i)_{i \in I}$ be a family of \mathbb{R} -valued random variables. Then, $\widehat{f} = \text{ess sup}_{i \in I} f_i$ exists and is unique. Moreover, if the family $(f_i)_{i \in I}$ is directed upwards, i.e., for all i, j in I , there exists k in I such that $f_i \vee f_j \leq f_k$, then there exists an increasing sequence $(f_{i_n})_{n \in \mathbb{N}}$ in $(f_i)_{i \in I}$ satisfying

$$\widehat{f} = \lim_{n \rightarrow \infty} f_{i_n} \quad \text{a.s.}$$

Proof. See [83, Proposition V-1-1] or [42, Theorem A 32]. □

A.1.4 Section Theorem

Let us call $\pi : \Omega \times \mathbb{R}_+ \rightarrow \Omega$ the canonical projection.

Definition A.1.7. A set $A \in \bar{\Omega} := \Omega \times \mathbb{R}_+$ is called **P-evanescent**, if $\pi(A)$ has outer **P-measure** 0, i.e., $\pi(A)$ is a **P-nullset**. The process X is called **P-evanescent**, if $\{X \neq 0\}$ is **P-evanescent**. Write $X \leq Y$, if $\{X > Y\}$ is **P-evanescent**. Call X and Y **indistinguishable**, if $X - Y$ is **P-evanescent**.

Theorem A.1.8 (Section Theorem). Let $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ be two optional processes. Suppose that for every bounded stopping time τ , we have $X_\tau \leq Y_\tau$ a.s. Then, we have $X \leq Y$.

If X and Y are predictable, then it is enough to test it with predictable times.

Proof. See [54, Theorem 4.10]. □

A.1.5 Fatou Convergence and Föllmer-Kramkov Theorem

Definition A.1.9. Let $(Y^n)_{n \in \mathbb{N}}$ be a sequence of stochastic processes and \mathcal{T} be a dense subset of \mathbb{R}_+ . The sequence $(Y^n)_{n \in \mathbb{N}}$ is **Fatou convergent** on \mathcal{T} to a process Y , if $(Y^n)_{n \in \mathbb{N}}$ is uniformly bounded from below and

$$Y_t = \limsup_{s \searrow t} \limsup_{s \in \mathcal{T}} \limsup_{n \rightarrow \infty} Y_s^n = \liminf_{s \searrow t} \liminf_{s \in \mathcal{T}} \liminf_{n \rightarrow \infty} Y_s^n$$

almost surely for all $t \geq 0$. If $\mathcal{T} = \mathbb{R}_+$, then the sequence $(Y^n)_{n \in \mathbb{N}}$ is called simply **Fatou convergent**.

Lemma A.1.10. Let $(Y^n)_{n \in \mathbb{N}}$ be a sequence of supermartingales, $Y_0^n = 0$, $n \in \mathbb{N}$, which is uniformly bounded from below, and \mathcal{T} be a dense countable subset of \mathbb{R}_+ .

There is a sequence $\tilde{Y}^n \in \text{conv}(Y^n, Y^{n+1}, \dots)$, $n \in \mathbb{N}$, and a supermartingale Y , $Y_0 \leq 0$, such that $(\tilde{Y}^n)_{n \in \mathbb{N}}$ is Fatou convergent on \mathcal{T} to Y .

Proof. See [41, Lemma 5.2]. □

Proposition A.1.11. ([29, Proposition 2.3]). Let $(M^n)_{n \in \mathbb{N}}$ be a sequence of nonnegative martingales $M^n = (M_t^n)_{0 \leq t \leq T}$ starting at $M_0^n = 1$.

Then there exists a sequence $(\tilde{M}^n)_{n \in \mathbb{N}}$ of convex combinations

$$\tilde{M}^n \in \text{conv}(M^n, M^{n+1}, \dots)$$

and nonnegative random variables Z_q for $q \in \mathbb{Q} \cap [0, T] \cup \{T\}$ such that

(1) $\tilde{M}_q^n \xrightarrow{\mathbf{P}} Z_q$ for all $q \in \mathbb{Q} \cap [0, T] \cup \{T\}$.

(2) The process $\bar{Y} = (\bar{Y}_t)_{0 \leq t \leq T}$ given by

$$\bar{Y}_t := \begin{cases} \lim_{q \in \mathbb{Q} \cap [0, T], q \downarrow t} Z_q & 0 \leq t < T \\ Z_T & t = T \end{cases}$$

is a càdlàg supermartingale.

(3) The process $\bar{Y} = (\bar{Y}_t)_{0 \leq t \leq T}$ is the Fatou limit of the sequence of the sequence $(\widetilde{M}^n)_{n \in \mathbb{N}}$ along $\mathbb{Q} \cap [0, T] \cup \{T\}$, i.e., we have \mathbf{P} -almost surely that

$$\bar{Y}_t = \limsup_{q \in \mathbb{Q} \cap [0, T], q \downarrow t} \limsup_{n \rightarrow \infty} \widetilde{M}_q^n = \liminf_{q \in \mathbb{Q} \cap [0, T], q \uparrow t} \liminf_{n \rightarrow \infty} \widetilde{M}_q^n$$

for $0 \leq t < T$ and

$$\bar{Y}_T = \lim_{n \rightarrow \infty} \widetilde{M}_T^n.$$

Remark A.1.12. As we defined \bar{Y}_t as the limit for q strictly bigger than t , we do not have in general that $\bar{Y}_t = \lim_{n \rightarrow \infty} \widetilde{M}_t^n$ for $0 \leq t < T$, not even for $t \in \mathbb{Q} \cap [0, T]$. See counterexample in [29, Example 2.4].

A.1.6 A compactness result of processes of finite variation

For process of finite variation ψ we denote by $\psi_t = \psi_0 + \psi_t^\uparrow - \psi_t^\downarrow$ its Jordan-Hahn decomposition into two nondecreasing processes ψ^\uparrow and ψ^\downarrow both null at zero. The total variation $\text{Var}_t(\psi)$ of ψ on $[0, t]$ is then given by $\text{Var}_t(\psi) = \psi_t^\uparrow + \psi_t^\downarrow$.

Theorem A.1.13. *Let $(\psi^n)_{n \in \mathbb{N}}$ be a sequence of finite variation predictable processes, such that the corresponding sequence $(\text{Var}_T(\psi^n))_{n \in \mathbb{N}}$ is bounded in $L^1(\mathbf{Q})$ for some equivalent measure $\mathbf{Q} \sim \mathbf{P}$.*

Then there exists a sequence of convex combinations $\widetilde{\psi}^n \in \text{conv}(\psi^n, \psi^{n+1}, \dots)$ such that $(\widetilde{\psi}^n)_{n \in \mathbb{N}}$ converges for a.e. $\omega \in \Omega$ for every $t \in [0, T]$ to a finite variation predictable process $\widehat{\psi}$, i.e.,

$$\mathbf{P}[\widetilde{\psi}_t^n \rightarrow \widehat{\psi}_t, \forall t \in [0, T]] = 1.$$

Proof. See [15, Proposition 3.4]. □

Remark A.1.14. The assumption of $L^1(\mathbf{Q})$ -boundedness of $(\text{Var}_T(\psi^n))_{n \in \mathbb{N}}$ can be replaced by $L^0(\mathbf{P})$ -boundedness of its convex hull, i.e., the set $\text{conv}(\text{Var}_T(\psi^n), n \in \mathbb{N})$ is bounded in probability.

We note that any process of finite variation is $\text{l\`a}d\text{l\`a}g$.

For any $\text{l\`a}d\text{l\`a}g$ process X we denote by X^c its continuous part given by

$$X_t^c := X_t - \sum_{s < t} \Delta_+ X_s - \sum_{s \leq t} \Delta X_s,$$

where $\Delta_+ X_t := X_{t+} - X_t$ and $\Delta X_t := X_t - X_{t-}$ are its right and left jumps, respectively.

We can define for a finite variation process ψ and a $\text{l\`a}d\text{l\`a}g$ process X the integrals

$$\int_0^t X_u d\psi_u := \int_0^t X_u d\psi_u^c + \sum_{0 < u \leq t} X_{u-} \Delta \psi_u + \sum_{0 \leq u < t} X_u \Delta_+ \psi_u$$

and

$$\int_0^t \psi_u dX_u := \int_0^t \psi_u^c dX_u + \sum_{0 < u \leq t} \Delta \psi_u (X_t - X_{u-}) + \sum_{0 \leq u < t} \Delta_+ \psi_u (X_t - X_u) \quad (\text{A.1.1})$$

pathwise by using Riemann-Stieltjes integrals such that the integration by parts formula

$$\psi_t X_t = \psi_0 X_0 + \int_0^t \psi_u dX_u + \int_0^t X_u d\psi_u \quad (\text{A.1.2})$$

holds true. Note that, if X is a semimartingale and ψ is in addition predictable, the pathwise integral (A.1.1) coincides with the classical stochastic integral.

A.1.7 Strong Supermartingale

Definition A.1.15. A real-valued stochastic process $Y = (Y_t)_{0 \leq t \leq T}$ is called an **optional strong supermartingale**, if

- (1) Y is optional.
- (2) Y_τ is integrable for every $[0, T]$ -valued stopping time τ .
- (3) For all stopping times σ and τ with $0 \leq \sigma \leq \tau \leq T$ we have $\mathbf{E}[Y_\tau | \mathcal{F}_\sigma] \leq Y_\sigma$.

This notion has been introduced by Mertens [79] as a generalization of càdlàg supermartingales. Indeed, by the optional sampling theorem, each càdlàg supermartingale is an optional strong supermartingale, but not every optional strong supermartingale has a càdlàg modification. For example, every deterministic decreasing function $(Y_t)_{0 \leq t \leq T}$ is an optional strong supermartingale, but there is no reason why it should be càdlàg or càglàd. However, by [37, Appendix I, Theorem 4], every optional strong supermartingale is indistinguishable from a làdlàg process, and so we can assume without loss of generality that all optional strong supermartingales are làdlàg.

As the Doob-Meyer decomposition in the càdlàg case, every optional strong supermartingale admits a unique decomposition $Y = M - A$, called the **Mertens decomposition**, into a càdlàg local martingale M and a nondecreasing and hence làdlàg (but in general neither càdlàg nor càglàd) predictable process A starting at 0.

Theorem A.1.16. Let $(Y^n)_{n \in \mathbb{N}}$ be a sequence of nonnegative optional strong supermartingales $Y^n = (Y_t^n)_{0 \leq t \leq T}$ starting at $Y_0^n = 1$.

Then, there is a sequence $(\tilde{Y}^n)_{n \in \mathbb{N}}$ of convex combinations $\tilde{Y}^n \in \text{conv}(Y^n, Y^{n+1}, \dots)$ and a nonnegative optional strong supermartingale $Y = (Y_t)_{0 \leq t \leq T}$ such that, for every $[0, T]$ -valued stopping time τ , we have that $\tilde{Y}_\tau^n \xrightarrow{\mathbf{P}} Y_\tau$.

Proof. See [29, Theorem 2.7]. □

Remark A.1.17. At a single finite stopping time τ we may pass to a subsequence to obtain that \tilde{Y}_τ^n converges also \mathbf{P} -almost sure to Y_τ . By means of a counterexample ([29, Proposition 4.1]) this is not possible for all stopping times simultaneously, since the set of all stopping times is far from being countable.

Since an optional strong supermartingale X is làdlàg, we may define the stochastic integral $\varphi \cdot X$ with respect to X via (A.1.1) with a general predictable finite variation process φ . This integral depends not only on the values of the integrator X but also explicitly on that of its left limits X_- . As a consequence, in order to obtain a satisfactory

convergence result for the integrals $\varphi \cdot X^n$ to a limit $\varphi \cdot X$, we have to take special care of the left limits of the integrators.

As shown in [29, Theorem 2.9], the convergence $\tilde{X}_\tau^n \xrightarrow{\mathbf{P}} X_\tau$ for all $[0, T]$ -valued stopping times τ implies the convergence of the left limits, i.e., $\tilde{X}_{\sigma-}^n \xrightarrow{\mathbf{P}} X_{\sigma-}$ for all $[0, T]$ -valued totally inaccessible stopping times τ . However, it may fail to obtain the convergence of the left limits $\tilde{X}_{\sigma-}^n \xrightarrow{\mathbf{P}} X_{\sigma-}$ at accessible stopping times σ . Moreover, even if the left limits $\tilde{X}_{\sigma-}^n$ converge to some random variable Y in probability, it may happen that $Y \neq X_{\sigma-}$.

As a consequence, we need to consider two processes $X^{(0)}$ and $X^{(1)}$, which correspond to the limiting processes of the left limits \tilde{X}_-^n and the process \tilde{X}^n itself. We replace the time interval $I = [0, T]$ by the set $\tilde{I} = [0, T] \times \{0, 1\}$ with the lexicographic order and merge both processes $X^{(0)}$ and $X^{(1)}$ into one process

$$X_{\tilde{t}} := \begin{cases} X_t^{(0)}, & \text{for } \tilde{t} = (t, 0), \\ X_t^{(1)}, & \text{for } \tilde{t} = (t, 1), \end{cases}$$

for $\tilde{t} \in \tilde{I}$, which is by (A.1.3) below a supermartingale indexed by $\tilde{t} \in \tilde{I}$.

Definition A.1.18. *A real-valued stochastic process $X = (X_t)_{0 \leq t \leq T}$ is called a **predictable strong supermartingale**, if*

- (1) X is predictable.
- (2) X_τ is integrable for every $[0, T]$ -valued predictable stopping time τ .
- (3) For all predictable stopping times $0 \leq \sigma \leq \tau \leq T$, we have $\mathbf{E}[X_\tau | \mathcal{F}_{\sigma-}] \leq X_\sigma$.

We may extend Theorem A.1.16 to hold also for left limits.

Theorem A.1.19. *Let $(X^n)_{n \in \mathbb{N}}$ be a sequence of nonnegative optional strong supermartingales starting at $X_0^n = 1$.*

Then, there exist a sequence $(\tilde{X}^n)_{n \in \mathbb{N}}$ of convex combinations $\tilde{X}^n \in \text{conv}(X^k, k \geq n)$, a nonnegative optional strong supermartingale $X^{(1)}$ and a nonnegative predictable strong supermartingale $X^{(0)}$ such that

- (1) For all $[0, T]$ -valued stopping times τ , we have that

$$\tilde{X}_\tau^n \xrightarrow{\mathbf{P}} X_\tau^{(1)}, \quad \tilde{X}_{\tau-}^n \xrightarrow{\mathbf{P}} X_\tau^{(0)}.$$

- (2) For all $[0, T]$ -valued predictable stopping times τ , we have that

$$\mathbf{E}[X_\tau^{(1)} | \mathcal{F}_{\tau-}] \leq X_\tau^{(0)} \leq X_{\tau-}^{(1)}. \quad (\text{A.1.3})$$

- (3) For all predictable processes φ of finite variation, we have that

$$(\varphi \cdot \tilde{X}^n)_\tau \xrightarrow{\mathbf{P}} \int_0^\tau \varphi_u^c dX_u^{(1)} + \sum_{0 < u \leq \tau} \Delta \varphi_u (X_\tau^{(1)} - X_u^{(1)}) + \sum_{0 \leq u < \tau} \Delta_+ \varphi_u (X_\tau^{(1)} - X_u^{(1)})$$

for all $[0, T]$ -valued stopping times τ .

Proof. See [29, Theorem 2.11, Theorem 2.12]. \square

We combine predictable and optional strong supermartingales to obtain the following notion.

Definition A.1.20. A *sandwiched strong supermartingale* is a pair $\mathfrak{X} = (X^p, X)$ such that X^p (respectively, X) is a predictable (respectively, optional) strong supermartingale and such that

$$X_{\tau-} \geq X_{\tau}^p \geq \mathbf{E}[X_{\tau} | \mathcal{F}_{\tau-}], \quad (\text{A.1.4})$$

for all predictable stopping times τ .

Starting from an optional strong supermartingale $X = (X_t)_{0 \leq t \leq T}$, we may define the process $X_t^p := X_{t-}$ to obtain a “sandwiched” strong supermartingale $\mathfrak{X} = (X^p, X)$.

Remark A.1.21. If X is a local martingale, the choice is unique as we have equalities in (A.1.4). But in general, there may be strict inequalities: if $X_t = f(t)$ for a deterministic nonincreasing function f , we may choose $X_t^p = f^p(t)$, where $f^p(t)$ is any function sandwiched between $f(t-)$ and $f(t)$.

For a sandwiched strong supermartingale $\mathfrak{X} = (X^p, X)$ and a predictable process ψ of finite variation we may define a stochastic integral in “a sandwiched sense” by

$$(\psi \cdot \mathfrak{X})_t := \int_0^t \psi_u^c dX_u + \sum_{0 < u \leq t} \Delta \psi_u (X_t - X_u^p) + \sum_{0 \leq u < t} \Delta_+ \psi_u (X_t - X_u). \quad (\text{A.1.5})$$

We note that (A.1.5) differs from (A.1.1) only by replacing X_- by X^p . As we can extend every optional strong supermartingale X to a sandwiched strong supermartingale $\mathfrak{X} = (X^p, X)$ by $X^p := X_-$, the two formulas are consistent. Moreover, both integrals (A.1.5) and (A.1.1) are equal to the usual stochastic integral, if X is a local martingale.

We can also define for a sandwiched strong supermartingale \mathfrak{X} and a predictable process ψ of finite variation the following integral

$$\int_0^t \mathfrak{X}_u d\psi_u := \int_0^t X_u d\psi_u^c + \sum_{0 < u \leq t} X_u^p \Delta \psi_u + \sum_{0 \leq u < t} X_u \Delta_+ \psi_u, \quad (\text{A.1.6})$$

such that the integration by parts formula

$$\psi_t X_t = \psi_0 X_0 + (\psi \cdot \mathfrak{X})_t + \int_0^t \mathfrak{X}_u d\psi_u \quad (\text{A.1.7})$$

holds true.

A.1.8 Fractional Brownian motion

We recall here some results on the fractional Brownian motion, which is a generalization of Brownian motion. More results can be found in [85, 16, 6, 82].

Definition A.1.22. Let $H \in (0, 1)$ be a constant. A **fractional Brownian motion** $(B_t^H)_{t \geq 0}$ of Hurst index H is a continuous and centered Gaussian process with covariance function

$$\mathbf{E}[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

For $H = \frac{1}{2}$, the fractional Brownian motion is then a standard Brownian motion. By the definition a fractional Brownian motion has the following properties:

1. $B_0^H = 0$ and $\mathbf{E}[B_t^H] = 0$ for all $t \geq 0$.
2. B^H has homogeneous increments, i.e., $B_{t+s}^H - B_s^H$ has the same law of B_t^H for $s, t \geq 0$.
3. B^H is a Gaussian process and $\mathbf{E}[(B_t^H)^2] = t^{2H}$, $t \geq 0$.
4. The trajectories of B^H are continuous and Hölder continuous of order strictly less than H .
5. The sample paths of a fractional Brownian motion is not differentiable. For every $t \geq 0$, we have with probability 1 that

$$\limsup_{h \rightarrow 0} \left| \frac{B_{t+h}^H - B_t^H}{h} \right| = \infty.$$

Fractional Brownian motion fails to be a semimartingale (except for the classical Brownian case $H = \frac{1}{2}$), see, e.g., [16, Section 1.3].

For $H \in (0, 1)$, Mandelbrot and Van Ness [78] gave the following construction of fractional Brownian motion:

$$B_t^H = C_H \int_{\mathbb{R}} [\varphi_H(t-s) - \varphi_H(-s)] dW_s, \quad t \in \mathbb{R}$$

where $(W_s)_{s \in \mathbb{R}}$ is a two-sided Brownian motion,

$$\varphi_H(x) = \mathbf{1}_{\{x \geq 0\}} x^{H-\frac{1}{2}}, \quad x \in \mathbb{R},$$

and C_H is a normalizing constant.

Theorem A.1.23. For each $H \in (0, 1)$, fractional Brownian motion B^H has the condition of “two way crossing”.

Proof. See [89]. □

Fix $\delta > 0$ and define inductively the stopping times $(\tau_j)_{j \in \mathbb{N}_0}$ by $\tau_0 = 0$ and

$$\tau_j := \inf \left\{ t > \tau_{j-1} \mid |B_t^H - B_{\tau_{j-1}}^H| \geq \delta \right\}.$$

We define the **number of δ -fluctuations up to time T** as the random variable

$$F_T^\delta := \sup \{j \geq 0 \mid \tau_j \leq T\}.$$

We than have the following estimate.

Proposition A.1.24. *With the notation above, there exist finite positive constants $C = C(H)$, $C' = C'(H)$ only depending on H , such that*

$$\mathbf{P}[F_T^\delta \geq n] \leq C' \exp\left(-\frac{\delta^2 n^{1+(2H \wedge 1)}}{CT^{2H}}\right),$$

for $n \in \mathbb{N}$.

Proof. See [26, Proposition 5.1]. □

Corollary A.1.25. *The random variable F_T^δ does have exponential moments of all orders:*

$$\mathbf{E}[\exp(aF_T^\delta)] < \infty, \quad \text{for all } a < \infty.$$

Moreover, for $H \geq \frac{1}{2}$, there exists $a > 0$ such that

$$\mathbf{E}[\exp(a(F_T^\delta)^2)] < \infty.$$

Proof. See [26, Corollary 5.2]. □

A.2 Functional Analysis

A.2.1 Some Results on Finite Additive Measures

We state and prove some results on the space $(L^\infty)^*$, the dual space of L^∞ . A detailed discussion can be found in [39, 5, 103]. Denote by $(L^\infty)_+^*$ the set of all nonnegative elements in $(L^\infty)^*$.

The following proposition collects some properties of the space $(L^\infty)_+^*$; more information can be found in Appendix of [23] and references there.

Proposition A.2.1.

1. *The set $(L^\infty)_+^*$ can be identified as the set of all nonnegative finitely additive bounded set functions on \mathcal{F} , which vanish on the \mathbf{P} -null sets. This set is denoted by ba_+ .*
2. *Every $Q \in ba_+$ admits a unique decomposition in the form of*

$$Q = Q^r + Q^s, \quad Q^r \geq 0, \quad Q^s \geq 0,$$

where the regular part Q^r is the maximal countably additive measure on \mathcal{F} , that is dominated by Q , and the singular part Q^s is purely finitely additive and does not dominate any nontrivial countably additive measure.

3. *$Q \in ba_+$ is purely finitely additive, i.e., $Q^r = 0$, if and only if for every $\varepsilon > 0$, there exists a set $A_\varepsilon \in \mathcal{F}$ such that $\mathbf{P}[A_\varepsilon] > 1 - \varepsilon$ and $\langle Q, \mathbf{1}_{A_\varepsilon} \rangle = 0$.*
4. *Suppose $(Q_n)_{n \in \mathbb{N}} \subseteq ba_+$ is a sequence such that $\frac{dQ_n^r}{d\mathbf{P}} \rightarrow f$ almost surely for some $f \geq 0$. Then any weak-star cluster point Q of $(Q_n)_{n \in \mathbb{N}}$ satisfies $\frac{dQ^r}{d\mathbf{P}} = f$ almost surely.*

For any $Q \in ba_+$, we may define

$$\langle Q, X \rangle := \lim_{n \rightarrow \infty} \langle Q, X \wedge n \rangle \in [0, \infty],$$

for all $X \in L_+^0$. For $X \in L^0$, set $\langle Q, X \rangle = \langle Q, X^+ \rangle - \langle Q, X^- \rangle$ whenever this is well-defined.

Proposition A.2.2. *Suppose D is a convex subset of L_+^1 , which is also a subset of ba_+ via the canonical embedding. Denote by \mathcal{D} the weak-star closure of D in ba_+ . Then, for each $Q \in \mathcal{D}$, there exists a sequence $(Q_n)_{n \in \mathbb{N}} \subseteq D$, such that $Q_n \rightarrow Q^r$, \mathbf{P} -a.s., as $n \rightarrow \infty$.*

Proof. Fix $Q \in \mathcal{D}$. For each $n \in \mathbb{N}$, there exists a set $A_n \in \mathcal{F}$, such that $\mathbf{P}[A_n] > 1 - \frac{1}{2^n}$ and Q^s is null on A_n . By the definition of \mathcal{D} , we see that Q is a weak-star limit point of D and thus, $Q|_{A_n} \in ba_+$ is also a weak-star limit point of $D|_{A_n}$, where

$$D|_{A_n} := \{Q|_{A_n} \in L^1 \mid Q \in D\}.$$

From $Q|_{A_n} = Q^r|_{A_n} \in L^1$, we know that $Q^r|_{A_n}$ is a weak limit point of $D|_{A_n}$. Moreover, due to the fact that $D|_{A_n}$ is convex, $Q^r|_{A_n}$ belongs to the L^1 -closure of $D|_{A_n}$. Therefore, there exists an element $Q^n \in D$, such that

$$\|Q_n|_{A_n} - Q^r|_{A_n}\|_{L^1} < \frac{1}{2^n}.$$

Finally,

$$\|Q_n|_{A_n} - Q^r\|_{L^1} \leq \|Q_n|_{A_n} - Q^r|_{A_n}\|_{L^1} + \|Q^r|_{A_n} - Q^r\|_{L^1} \rightarrow 0,$$

which yields $Q_n \rightarrow Q^r$, \mathbf{P} -a.s. up to a subsequence. \square

Corollary A.2.3. *Suppose D is a convex subset of L_+^1 , which is also a subset of ba_+ via the canonical embedding. Denote by \mathcal{D} the weak-star closure of D in ba_+ . Then, for each $Q \in \mathcal{D}$, there exists a sequence $(Q_n)_{n \in \mathbb{N}} \subseteq D$, such that $Q_n \rightarrow Q^r$, \mathbf{P} -a.s., as $n \rightarrow \infty$, and for countably many $f_i \in L^\infty$, $i \in \mathbb{N}$,*

$$\langle Q_n, f_i \rangle \rightarrow \langle Q, f_i \rangle, \text{ as } n \rightarrow \infty.$$

Proof. For each n , define

$$D_n := \bigcap_{i=1}^n \left\{ \tilde{Q} \in D \mid |\langle \tilde{Q}, f_i \rangle - \langle Q, f_i \rangle| < \frac{1}{n} \right\}.$$

It is evident that Q belongs to the weak-star closure of D_n . Applying the proposition above, one can find a sequence $(Q_{n,m})_{m \in \mathbb{N}} \subseteq D_n$, such that $Q_{n,m} \rightarrow Q^r$, \mathbf{P} -a.s. By the diagonal argument, we complete the proof. \square

Remark A.2.4. We remark that the assumption that $D \subseteq L^1$ is crucial in the above proposition. For a general subset $D \subseteq ba$ and an element Q in its weak-star closure \mathcal{D} , one may not find a sequence $(Q_n)_{n \in \mathbb{N}}$ from D , such that $(Q_n)^r \rightarrow Q^r$. For instance, $\Omega = [0, 1]$, \mathcal{F} is the Lebesgue σ -algebra and $\mathbf{P} = \lambda$ is the Lebesgue measure. Define $D := \{Q \in ba_+ \mid \|Q\|_{ba} = 1, Q = Q^s\}$, then we find that statement of Proposition A.2.2 does not hold. Indeed, as $\{Q \in L_+^1 \mid \|Q\|_{ba} = 1\} \subseteq \{Q \in ba_+ \mid \|Q\|_{ba} = 1\} = \mathcal{D}$, we have that the Lebesgue measure λ is an element in \mathcal{D} . However, for each sequence $(Q_n)_{n \in \mathbb{N}}$, all the cluster points Q satisfy $Q^r = 0$. Therefore, we cannot find a sequence in D converging to the Lebesgue measure λ .

Appendix B

Utility Maximization Problem in Frictionless Markets

B.1 Utility Maximization on the positive real line under (NUPBR)

We consider a financial market consisting of two assets, one bond and one stock. We suppose that the price of the bond is constant, and denote by $S = (S)_{0 \leq t \leq T}$ the price process of the stock. The process S is assumed to be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ satisfying the usual hypotheses. Here T is a finite time horizon.

A self-financing portfolio is defined as a pair (x, H) , where the constant x is the initial value of the portfolio, and $H = (H_t)_{0 \leq t \leq T}$ is a predictable S -integrable process, where H_t specifies, how many units of asset S are held in the portfolio at time t . The value process $X = (X_t)_{0 \leq t \leq T}$ of such a portfolio is given by $X_t = x + (H \cdot S)_t$, $0 \leq t \leq T$.

B.1.1 NUPBR

Definition B.1.1. *We say that S allows for an unbounded profit with bounded risk, if there is $\alpha > 0$ such that, for every $C > 0$, there is an 1-admissible trading strategy H such that*

$$\mathbf{P}[(H \cdot S)_T \geq C] \geq \alpha.$$

Define

$$\mathcal{K}_1 := \{(H \cdot S)_T \mid H \text{ is 1-admissible}\}.$$

Definition B.1.2. *We say that S satisfies the condition (NUPBR) of **no unbounded profit with bounded risk**, if \mathcal{K}_1 is bounded in L^0 , i.e.,*

$$\lim_{M \rightarrow \infty} \sup_{X \in \mathcal{K}_1} \mathbf{P}[|X| > M] = 0.$$

Definition B.1.3. *A **strict martingale density** for S is a strictly positive local martingale $Z = (Z_t)_{0 \leq t \leq T}$ with $Z_0 = 1$ such that ZS is a local martingale.*

The set of strict martingale densities is denoted by $\mathcal{Z}_e(S)$.

Definition B.1.4. An *equivalent local martingale deflator* is a strictly positive process $Z = (Z_t)_{0 \leq t \leq T}$ with $Z_0 = 1$ such that $Z(1 + H \cdot S)$ is a local martingale for every 1-admissible trading strategy H .

By Itô's formula, we may see that these two definitions are essentially the same.

Theorem B.1.5. Suppose S is a strictly positive semimartingale. Then, the following two conditions are equivalent:

1. S satisfies (NUPBR).
2. There exists at least one strict martingale density.
3. There exists at least one equivalent local martingale deflator.

Proof. See [69, Theorem 2.1]. □

We present also results of special cases, which can be found in [18, Théorème 2.9].

Theorem B.1.6. Suppose S is a continuous semimartingale with $S_t = S_0 + M_t + A_t$. Then, the following three conditions are equivalent:

1. S satisfies (NUPBR).
2. There exists a strict martingale density for S .
3. S satisfies the **structure condition**, i.e., there exists an \mathbb{R} -valued predictable process such that

$$dA_t = h_t d\langle M \rangle_t \quad \text{and} \quad \int_0^T h_t^2 d\langle M \rangle_t < \infty \quad \text{a.s.}$$

Moreover, $Z := \mathcal{E}(-h \cdot M)$ is a strict martingale density for S .

Remark B.1.7. From the above theorems we see that (NUPBR) is the local version of the condition of no free lunch with vanishing risk (NFLVR). We also remark that (NUPBR) is a local property, i.e., if S satisfies (NUPBR) locally, it satisfies (NUPBR).

We denote by \mathcal{C} the set of positive contingent claims superreplicable at price 1

$$\mathcal{C} := \{g \in L_+^0 \mid g \leq 1 + (H \cdot S)_T \text{ for some 1-admissible strategy } H\}.$$

Theorem B.1.8 (Superreplication under (NUPBR)). Fix a strictly positive semimartingale S satisfying (NUPBR) and a random variable $g \in L_+^0$. Then,

$$g \in \mathcal{C} \iff \mathbf{E}[gZ_T] \leq 1, \quad \text{for each } Z \in \mathcal{Z}_e(S).$$

Proof. “ \Rightarrow ”: Let $Z \in \mathcal{Z}_e(S)$ be a strict martingale density. We may find a localizing sequence $(\tau_n)_{n \in \mathbb{N}}$ such that Z^{τ_n} defines a density process of an equivalent local martingale measure for S^{τ_n} . Therefore, by Superreplication Theorem in the frictionless setting, it follows that

$$X_t^{\tau_n} Z_t^{\tau_n} = (1 + (H \cdot S)_t^{\tau_n}) Z_t^{\tau_n} = (1 + (H \cdot S^{\tau_n})_t) Z_t^{\tau_n}$$

is a nonnegative supermartingale, hence $(X_t Z_t)_{0 \leq t \leq T}$ is a supermartingale, for each 1-admissible trading strategy H , which implies that

$$\mathbf{E}[g Z_T] \leq \mathbf{E}[(1 + (H \cdot S)_T) Z_T] \leq 1,$$

for each $g \in \mathcal{C}$ and $Z \in \mathcal{Z}_e(S)$.

“ \Leftarrow ”: Let $(\tau_m)_{m \in \mathbb{N}}$ be the localizing sequence for some $Z \in \mathcal{Z}_e$ such that Z^{τ_m} is the density process of an equivalent local martingale measure for S^{τ_m} . Define $g_m := g \mathbf{1}_{\{\tau_m = T\}}$ and

$$\mathcal{C}_m := \{g \in L_+^0 \mid g \leq 1 + (H \cdot S)_{\tau_m}, H \text{ is 1-admissible}\}.$$

It is obvious that $g_m \rightarrow g$ in probability. We claim that $g_m \in \mathcal{C}_m$ for all $m \in \mathbb{N}$. Indeed, assume that there exists $m' \in \mathbb{N}$ such that $g_{m'} \notin \mathcal{C}_{m'}$. We now have to work towards a contradiction. As $S^{\tau_{m'}}$ satisfies assumptions of Superreplication Theorem in the frictionless setting, we may find an equivalent local martingale measure $\bar{\mathbf{Q}}^{m'}$ for $S^{\tau_{m'}}$ such that $\mathbf{E}_{\bar{\mathbf{Q}}^{m'}}[g_{m'}] > 1$. Let us denote by $\bar{Z}^{\tau_{m'}}$ its density process. Pick another strict martingale density \hat{Z} and define

$$\tilde{Z}_t := \begin{cases} \bar{Z}_t, & \text{for } 0 \leq t \leq \tau_{m'}, \\ \hat{Z}_t \frac{\bar{Z}_{\tau_{m'}}}{\hat{Z}_{\tau_{m'}}}, & \text{for } \tau_{m'} \leq t \leq T. \end{cases}$$

Then, we obtain by the assumption that

$$1 \geq \mathbf{E}[g \tilde{Z}_T] \geq \mathbf{E}[g_{m'} \tilde{Z}_T] = \mathbf{E}[g_{m'} \tilde{Z}_{\tau_{m'}}] > 1,$$

which is a contradiction.

As $\mathcal{C}_m \subseteq \mathcal{C}$ for each $m \in \mathbb{N}$ and \mathcal{C} is closed with respect to the convergence in probability by [68, Theorem 2], it follows that $g \in \mathcal{C}$. \square

B.1.2 Utility Function, Reasonable Asymptotic Elasticity

In addition to the model S of a financial market, we now consider a function $U(x)$, modelling the utility of an agent's wealth x at the terminal T .

Definition B.1.9 (Utility function on \mathbb{R}_{++}). *A utility function $U : \mathbb{R}_{++} \rightarrow \mathbb{R} \cup \{\infty\}$ is a function with the following properties*

1. U is increasing on \mathbb{R}_{++} ,
2. U is continuous on $\{U > -\infty\}$,
3. U is differentiable and strictly concave on $\text{int}\{U > -\infty\}$,
4. U satisfies the Inada conditions i.e.,

$$U'(0) := \lim_{x \rightarrow 0} U'(x) = \infty, \quad U'(\infty) := \lim_{x \rightarrow \infty} U'(x) = 0.$$

The marginal utility tends to infinity, when the wealth x tends to the infimum of its allowed values of U ; and it tends to 0, when wealth tends to infinity.

Definition B.1.10. The conjugate function V to the utility function U is defined as

$$V(y) := \sup_{x>0} \{U(x) - xy\}, \quad y > 0.$$

Remark B.1.11. The function $V(y)$ is the Legendre transform of $-U(-x)$.

Lemma B.1.12 (Fenchel's inequality).

$$U(x) - V(y) \leq xy.$$

The inequality will be an equality, if and only if $x = \operatorname{argmax}\{U(x) - xy\}$ and $y = \operatorname{argmin}\{V(y) + xy\}$.

Proposition B.1.13. If U is a utility function, then V has the following properties

1. V is decreasing, strictly convex and continuously differentiable on $(0, \infty)$.
2. V satisfies

$$V'(0) = -\infty, \quad V'(\infty) = 0, \quad V(0) = U(\infty), \quad V(\infty) = 0.$$

3. V satisfies the following relation to U

$$U(x) = \inf_{y>0} \{V(y) + xy\}, \quad x > 0.$$

4. The derivative of U is the inverse function of the negative of the derivative of V , i.e., $I := (U')^{-1} = -V'$.
5. We have the formula $V(y) = U(I(y)) - yI(y)$.

Example B.1.14. Typical examples are

$$U(x) = \log(x), \quad x > 0 \quad \text{with} \quad V(y) = -\log(y) - 1,$$

and

$$U(x) = \frac{x^\alpha}{\alpha}, \quad \alpha \in (-\infty, 1) \setminus \{0\}, \quad x > 0 \quad \text{with} \quad V(y) = \frac{1-\alpha}{\alpha} y^{\alpha-1}.$$

Definition B.1.15. A utility function $U : \mathbb{R}_{++} \rightarrow \mathbb{R}$ satisfies the reasonable asymptotic elasticity (RAE), if

$$AE_{+\infty}(U) = \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

Remark B.1.16. The quantity $\frac{xU'(x)}{U(x)}$ is the elasticity of the function U at x . Intuitively, it is the percentage change in output for a percentage change in input, (if the quantities are all positive)

$$\frac{\frac{\Delta f(x)}{f(x)}}{\frac{\Delta x}{x}} = \frac{\Delta f(x)}{\Delta x} \frac{x}{f(x)} \doteq \frac{xf'(x)}{f(x)}.$$

The economic intuition behind decreasing marginal utility suggests that, for large x , the marginal utility $U'(x)$ should be substantially smaller than the average utility $\frac{U(x)}{x}$, as $x \rightarrow \infty$. The extreme case $AE_{+\infty}(U) = 1$ corresponds to the case when the marginal utility in the limit equals the average utility, as $x \rightarrow \infty$, which seems unreasonable.

Lemma B.1.17. *Let U be a utility function. Then,*

(1) $AE_{+\infty}(U)$ is well-defined: $AE_{+\infty}(U) \leq 1$.

(2) $AE_{+\infty}(U)$ depends on $U(\infty)$:

- $U(\infty) = \infty \implies AE_{+\infty}(U) \in [0, 1]$,
- $U(\infty) \in (0, \infty) \implies AE_{+\infty}(U) = 0$,
- $U(\infty) \in (-\infty, 0] \implies AE_{+\infty}(U) \in [-\infty, 0]$.

Proof. See [70, Lemma 6.1]. □

Example B.1.18. Typical examples (and counterexamples) of such utility functions are

- $U(x) = \log(x)$, for which $AE_{+\infty}(U) = 0$,
- $U(x) = \frac{x^\alpha}{\alpha}$, for which $AE_{+\infty}(U) = \alpha$,
- $U(x) = \frac{x}{\log(x)}$, for large enough, for which $AE_{+\infty}(U) = 1$.

Lemma B.1.19. *Let U be a utility function satisfying $U(\infty) > 0$. Then, the following assertions are equivalent:*

(1) $AE_{+\infty}(U) < 1$.

(2) There exist $x_0 > 0$ and $\gamma \in (0, 1)$ such that

$$xU'(x) < \gamma U(x), \quad \forall x \geq x_0.$$

(3) There exist $x_0 > 0$ and $\gamma \in (0, 1)$ such that

$$U(\lambda x) < \lambda^\gamma U(x), \quad \forall \lambda > 1, \forall x \geq x_0.$$

(4) There exist $y_0 > 0$ and $\gamma \in (0, 1)$ such that

$$V(\mu y) < \mu^{-\frac{\gamma}{1-\gamma}} V(y), \quad \forall 0 < \mu < 1, \forall 0 < y \leq y_0.$$

(5) There exist $y_0 > 0$ and $\gamma \in (0, 1)$ such that

$$-yV'(y) < \frac{\gamma}{1-\gamma} V(y), \quad \forall 0 < y \leq y_0.$$

The infimum of $\gamma > 0$, for which these hold true, equals the asymptotic elasticity $AE_{+\infty}(U)$.

Proof. See [70, Lemma 6.3] and [70, Corollary 6.1]. □

B.1.3 Abstract Version of the Theorems

Let \mathcal{C} and \mathcal{D} be two subsets of $L_+^0(\Omega, \mathcal{F}, \mathbf{P})$ and

$$\mathcal{C}(x) = x\mathcal{C} = \{xg : g \in \mathcal{C}\}, \text{ for } x > 0,$$

and

$$\mathcal{D}(y) = y\mathcal{D} = \{yh : h \in \mathcal{D}\}, \text{ for } y > 0.$$

The abstract versions of the optimization problems:

$$u(x) = \sup_{g \in \mathcal{C}(x)} \mathbf{E}[U(g)], \quad \text{and} \quad v(y) = \inf_{h \in \mathcal{D}(y)} \mathbf{E}[V(h)].$$

Theorem B.1.20 (Abstract version, $AE_{+\infty}(U) < 1$). *Assume that*

- *The sets \mathcal{C} and \mathcal{D} satisfy the following properties:*
 - (i) *\mathcal{C} , \mathcal{D} are subsets of $L_+^0(\Omega, \mathcal{F}, \mathbf{P})$, which are convex, solid and closed in the topology of convergence in measure.*
 - (ii)
$$\begin{aligned} g \in \mathcal{C} &\iff \mathbf{E}[gh] \leq 1 \text{ for all } h \in \mathcal{D}, \\ h \in \mathcal{D} &\iff \mathbf{E}[gh] \leq 1 \text{ for all } g \in \mathcal{C}. \end{aligned}$$
 - (iii) *\mathcal{C} is a bounded subset of $L^0(\Omega, \mathcal{F}, \mathbf{P})$ and contains the constant function $\mathbf{1}$.*
- *U is a utility function satisfying the reasonable asymptotic elasticity, i.e.,*

$$AE_{+\infty}(U) = \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

- *$u(x) := \sup_{g \in \mathcal{C}(x)} \mathbf{E}[U(g)] < \infty$ for some $x > 0$.*

Then:

- (i)
 - *$u(x) < \infty$ for all $x > 0$.*
 - *$v(y) < \infty$ for all $y > 0$.*
 - *The value function u continuously differentiable and strictly concave on $(0, \infty)$.*
 - *The value function v continuously differentiable and strictly convex on $(0, \infty)$.*
 - *The value functions u and v are conjugate:*

$$\begin{aligned} v(y) &= \sup_{x>0} \{u(x) - xy\}, \quad y > 0, \\ u(x) &= \inf_{y>0} \{v(y) + xy\}, \quad x > 0. \end{aligned}$$

- *The functions u' and $-v'$ are strictly decreasing and satisfy*

$$u'(0) = \infty, \quad u'(\infty) = 0, \quad v'(0) = -\infty, \quad v'(\infty) = 0.$$

- The asymptotic elasticity $AE_{+\infty}(u)$ of u is also less than 1, more precisely

$$AE_{+\infty}(u)_+ \leq AE_{+\infty}(U)_+ < 1.$$

- (ii) • The optimal solution $\widehat{g}(x) \in \mathcal{C}(x)$ to the primal problem exists and is unique.
- The optimal solution $\widehat{h}(y) \in \mathcal{D}(y)$ to the dual problem exists and is unique.
 - For $y = u'(x)$, we have the dual relation

$$\widehat{g}(x) = I(\widehat{h}(y)), \quad \widehat{h}(y) = U'(\widehat{g}(x)).$$

- Moreover:

$$\mathbf{E}[\widehat{g}(x)\widehat{h}(y)] = xy.$$

- (iii) We have the following relations between u' , v' and \widehat{g} , \widehat{h} respectively:

$$\begin{aligned} xu'(x) &= \mathbf{E}[\widehat{g}(x)U'(\widehat{g}(x))], \quad x > 0, \\ yv'(y) &= \mathbf{E}[\widehat{h}(y)V'(\widehat{h}(y))], \quad y > 0. \end{aligned}$$

- (iv) Let $\widetilde{\mathcal{D}}$ be a convex subset of \mathcal{D} , such that

- for any $g \in \mathcal{C}$: $\sup_{h \in \widetilde{\mathcal{D}}} \mathbf{E}[gh] = \sup_{h \in \mathcal{D}} \mathbf{E}[gh]$,
- the set $\widetilde{\mathcal{D}}$ is closed under countable convex combinations (i.e., for any sequence $(h^n)_{n \in \mathbb{N}}$ of elements of $\widetilde{\mathcal{D}}$ and any sequence of positive numbers $(a^n)_{n \in \mathbb{N}}$ such that $\sum_{n \in \mathbb{N}} a^n = 1$, the random variable $\sum_{n \in \mathbb{N}} a^n h^n$ belongs to $\widetilde{\mathcal{D}}$.)

then,

$$v(y) := \inf_{h \in \mathcal{D}} \mathbf{E}[V(yh)] = \inf_{h \in \widetilde{\mathcal{D}}} \mathbf{E}[V(yh)].$$

Proof. See [70, Theorem 3.2, Proposition 3.2 and Section 3]. □

B.1.4 Duality Result

We denote by $\mathcal{X}(x)$ the family of wealth processes with nonnegative capital at any instant, i.e., $X_t \geq 0$ for all $t \in [0, T]$, and with initial value equal to x , i.e.,

$$\mathcal{X}(x) := \{X \geq 0 \mid X_t = x + (H \cdot S)_t \geq 0, \text{ for all } t \in [0, T]\}.$$

We shall use the shorter notation \mathcal{X} for $\mathcal{X}(1)$. Clearly we have

$$\mathcal{X}(x) = x\mathcal{X} = \{x\mathcal{X} \mid X \in \mathcal{X}\}, \quad \text{for } x \geq 0.$$

For a given initial capital $x > 0$, the goal of the agent is to maximize the expected value of terminal utility. The value function of this problem is denoted by

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbf{E}[U(X_T)]. \tag{B.1.1}$$

We define the family $\mathcal{Y}(y)$ of nonnegative semimartingales Y with $Y_0 = y$ and such that, for any $X \in \mathcal{X}(1)$, the product XY is a supermartingale,

$$\mathcal{Y}(y) := \{Y \geq 0 \mid Y_0 = y \text{ and } XY \text{ is a supermartingale, for all } X \in \mathcal{X}(1)\}.$$

In particular, as $\mathcal{X}(1)$ contains the process $X \equiv 1$, any $Y \in \mathcal{Y}(y)$ is a supermartingale. Note that the set $\mathcal{Y}(1)$ contains the strict martingale density processes, i.e., $\mathcal{Z}_e(S) \subseteq \mathcal{Y}(1)$.

We define the value function of the dual problem by

$$v(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbf{E}[V(Y_T)].$$

We pass from the sets of processes $\mathcal{X}(x)$, $\mathcal{Y}(y)$ to the sets $\mathcal{C}(x)$, $\mathcal{D}(y)$ of random variables dominated by the final outcomes X_T , Y_T , respectively,

$$\begin{aligned} \mathcal{C}(x) &:= \{g \in L_+^0(\Omega, \mathcal{F}, \mathbf{P}) \mid 0 \leq g \leq X_T, \text{ for some } X \in \mathcal{X}(x)\}, \\ \mathcal{D}(y) &:= \{h \in L_+^0(\Omega, \mathcal{F}, \mathbf{P}) \mid 0 \leq h \leq Y_T, \text{ for some } Y \in \mathcal{Y}(y)\}. \end{aligned}$$

We write \mathcal{C} , \mathcal{D} , \mathcal{X} , \mathcal{Y} for $\mathcal{C}(1)$, $\mathcal{D}(1)$, $\mathcal{X}(1)$, $\mathcal{Y}(1)$ and observe that

$$\mathcal{C}(x) = x\mathcal{C} = \{xg \mid g \in \mathcal{C}\}, \text{ for } x > 0,$$

and the analogous relations for $\mathcal{D}(y)$, $\mathcal{X}(x)$ and $\mathcal{Y}(y)$. We denote by $\tilde{\mathcal{D}}$ the set of all terminal values of strict martingale density processes, i.e.,

$$\tilde{\mathcal{D}} := \{Z_T \mid Z \in \mathcal{Z}_e(S)\}.$$

Proposition B.1.21 (Duality relation). *Suppose $\mathcal{Z}_e(S) \neq \emptyset$, where S is a semimartingale. Then the sets \mathcal{C} , \mathcal{D} have the following properties:*

(i) \mathcal{C} , \mathcal{D} are subsets of $L_+^0(\Omega, \mathcal{F}, \mathbf{P})$, which are convex, solid and closed in the topology of convergence in measure.

(ii)

$$\begin{aligned} g \in \mathcal{C} &\iff \mathbf{E}[gh] \leq 1 \text{ for all } h \in \mathcal{D}, \\ h \in \mathcal{D} &\iff \mathbf{E}[gh] \leq 1 \text{ for all } g \in \mathcal{C}. \end{aligned}$$

(iii) \mathcal{C} is a bounded subset of $L^0(\Omega, \mathcal{F}, \mathbf{P})$ and contains the constant function $\mathbf{1}$.

Proof. (i). The convexity and solidity of \mathcal{C} and \mathcal{D} are rather obvious. The closedness of \mathcal{C} follows by [68, Theorem 2] and the proof of [70, Lemma 4.1] remains valid for the closedness of \mathcal{D} under the weaker assumption (*NUPBR*).

(ii). Here, we need to show the following polarities:

$$\mathcal{C} = \mathcal{D}^\circ \quad \text{and} \quad \mathcal{D} = \mathcal{C}^\circ.$$

From the definitions we obtain $\mathcal{C} \subseteq \mathcal{D}^\circ$ and $\mathcal{D} \subseteq \mathcal{C}^\circ$. It follows by Theorem B.1.8 that $\mathcal{C} = \tilde{\mathcal{D}}^\circ$. As $\tilde{\mathcal{D}} \subseteq \mathcal{D}$, we obtain $\mathcal{D}^\circ \subseteq \tilde{\mathcal{D}}^\circ = \mathcal{C}$, and therefore $\mathcal{C} = \mathcal{D}^\circ$. As the set \mathcal{D} is convex solid and closed in the topology of convergence in measure, we may apply Proposition A.1.3 to conclude that $\mathcal{C}^\circ = \mathcal{D}^{\circ\circ} = \mathcal{D}$.

(iii). The boundedness of \mathcal{C} follows from the definition of (*NUPBR*) and it is rather clear that \mathcal{C} contains the constant function $\mathbf{1}$. \square

Lemma B.1.22. *The set $\tilde{\mathcal{D}}$ is a convex subset of \mathcal{D} satisfying the following properties*

(i) for any $g \in \mathcal{C}$: $\sup_{h \in \tilde{\mathcal{D}}} \mathbf{E}[gh] = \sup_{h \in \mathcal{D}} \mathbf{E}[gh]$,

(ii) the set $\tilde{\mathcal{D}}$ is closed under countable convex combinations.

Proof. (i) follows from $\mathcal{D}^{\circ\circ} = \tilde{\mathcal{D}}^{\circ\circ}$ and the superreplication theorem under (NUPBR). It is clear that $\tilde{\mathcal{D}}$ is closed under countable convex combinations, which implies (ii). \square

Theorem B.1.23. *Assume that*

- S satisfies (NUPBR), equivalently $\mathcal{Z}_e(S) \neq \emptyset$.
- U is a utility function satisfying the reasonable asymptotic elasticity, i.e.,

$$AE_{+\infty}(U) = \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

- $u(x) := \sup_{X \in \mathcal{X}(x)} \mathbf{E}[U(X_T)] < \infty$ for some $x > 0$.

Then:

1.
 - $u(x) < \infty$ for all $x > 0$.
 - $v(y) < \infty$ for all $y > 0$.
 - The value function u continuously differentiable and strictly concave on $(0, \infty)$.
 - The value function v continuously differentiable and strictly convex on $(0, \infty)$.
 - The value functions u and v are conjugate:

$$v(y) = \sup_{x > 0} \{u(x) - xy\}, \quad y > 0,$$

$$u(x) = \inf_{y > 0} \{v(y) + xy\}, \quad x > 0.$$

- The functions u' and $-v'$ are strictly decreasing and satisfy

$$u'(0) = \infty, \quad u'(\infty) = 0, \quad v'(0) = -\infty, \quad v'(\infty) = 0.$$

- The asymptotic elasticity $AE_{+\infty}(u)$ of u is also less than 1, more precisely

$$AE_{+\infty}(u)_+ \leq AE_{+\infty}(U)_+ < 1.$$

2.
 - The optimal solution $\hat{X}(x) \in \mathcal{X}(x)$ to the primal problem exists and is unique.
 - The optimal solution $\hat{Y}(y) \in \mathcal{Y}(y)$ to the dual problem exists and is unique.
 - For $y = u'(x)$, we have the dual relation

$$\hat{X}_T(x) = I(\hat{Y}_T(y)), \quad \hat{Y}_T(y) = U'(\hat{X}_T(x)).$$

- The process $\hat{X}(x)\hat{Y}(y)$ is a uniformly integrable martingale on $[0, T]$.

3. We have the following relations between u' , v' and \widehat{X} , \widehat{Y} respectively:

$$\begin{aligned} xu'(x) &= \mathbf{E} \left[\widehat{X}_T(x) U'(\widehat{X}_T(x)) \right], \quad x > 0, \\ yv'(y) &= \mathbf{E} \left[\widehat{Y}_T(y) V'(\widehat{Y}_T(y)) \right], \quad y > 0. \end{aligned}$$

4. The dual value function v have the following representation

$$v(y) = \inf_{Z \in \mathcal{Z}_e(S)} \mathbf{E}[V(yZ_T)] = \inf_{h \in \widetilde{\mathcal{D}}} \mathbf{E}[V(yZ_T)].$$

Proof. See [70, Theorem 2.2 and Section 4]. □

Remark B.1.24. In general, the class $\mathcal{Y}(1)$ cannot be replaced by the small class \mathcal{M}^{loc} of equivalent local martingale deflators. However,

$$v(y) = \inf_{Y \in \mathcal{Y}(1)} \mathbf{E}[V(yY_T)] = \inf_{Z \in \mathcal{M}^{loc}} \mathbf{E}[V(yZ_T)].$$

Remark B.1.25. The dual optimizer $\widehat{Y}_T(1)$ may fail to be the density of a probability measure, i.e., $\mathbf{E}[\widehat{Y}_T(y)] < y$, for $y > 0$. In general

$$u'(x) \neq \mathbf{E}_{\mathbf{P}} \left[U'(\widehat{X}_T(x)) \right], \quad v'(y) \neq \mathbf{E}_{\widehat{\mathbf{Q}}} \left[V' \left(y \frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}} \right) \right].$$

The validity of $u'(x) = \mathbf{E}_{\mathbf{P}} [U'(\widehat{X}_T(x))]$ is tantamount to the validity of $y = \mathbf{E}_{\mathbf{P}} [\widehat{Y}_T(y)]$.

Remark B.1.26. The theorem states that under the assumption *(RAE)* on the utility function U , the duality theory works well in this context. Actually, the condition of *(RAE)* is minimal and cannot be relaxed in the sense that one can find counterexamples of continuous price processes S for which the value function $v(y)$ is not finite for all y and there does not exist a solution to the primal problem $u(x)$, whenever $AE_{+\infty}(U) = 1$. See [70, Example 5.2].

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