MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

The surface symmetric Einstein-Vlasov System with positive cosmological constant in the massless case

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master MSc

Wien, 2017 / Vienna, 2017

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 066 876
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Physik UG2002
Betreut von / Supervisor:	Dr. David Miro Fajman

Preface

The dominant interaction in cosmology is the gravitational interaction. It is a very rare event that stars or even bigger structures like galaxies come sufficiently close to each other that they collide and thereby interact non-gravitationally. Even if they do so, the effect can be considered very small. This gives rise to investigate cosmological models where non-gravitational interactions are neglected. In these models the matter is described by a time-dependent distribution function on phase-space that evolves accordingly to the so-called Vlasov equation, which is then coupled to the gravitational field equations. The Vlasov equation basically states that all the particles contained in the model stay in free fall for all times. In the theory of general relativity these paths of free fall are described by the Einstein equations making the system a so-called Einstein-Vlasov system.

In order to be able to make statements about such a system one often needs to simplify it by introducing symmetries and add restrictions to the matter model. Our investigation will focus on matter distributions that carry spherical, plane or hyperbolic symmetry. Because these form two-dimensional orbits of symmetry they will be referred to as surface symmetries. The formalism for the different type of symmetries are closely related which often makes it possible to make statements to the whole class of surface symmetric systems rather then handle every type separately.

In this work we will study the Einstein-Vlasov model with massless matter as an initial value problem. We will prescribe initial data at some point in time and then investigate the development of the system towards the past and future direction. We will prove that for a class of initial data the problem has a unique solution starting from an initial point in the past and is complete towards the future. Furthermore, we will investigate the cosmology for late times and show that the matter dilutes in a specific way in the expanding direction. A particular issue that arises in the massless system is that the momentum support of the distribution function must exclude the point of zero momentum. We show that the support in our models always remains away from this point.

Acknowledgements:

I want to thank Dr. David Fajman for his help and patience during the course of my thesis and for his lectures and seminars on Partial Differential equations and Sobolev Spaces.
Furthermore, I want to thank the people of Austria who funded almost all of my education and gave me the opportunity to study under the guidance of brilliant people like Prof. Dr. Markus Arndt, Prof. Dr. Harald Rindler or Prof. Dr. Piotr T. Chruściel at the University of Vienna. I hope that I can pay my dues for the upcoming generations.
But most of all, I want to thank Neptun Yousefi for her support during some hard episodes of my studies and life. She shone light on my path when I could only see darkness. I love you.

Contents

Chapter 1

Introduction

In general relativity the gravitational field is described in terms of the geometry of manifolds. These manifolds are n-dimensional Lorentzian manifolds often referred to as space-times, that is that they carry a non-degenerate however indefinite metric tensor g with signature $(n-1,1)$. The use of such a metric allows one to classify three types of vectors depending on their sign of their metric square length

$$g_{\mu\nu}v^{\mu}v^{\nu}.$$

Spacelike timelike and nullvectors have positive, negative and zero squares respectively. In this work we will be particularly interested in the latter kind of vectors, as they may represent the 4-momenta of massless matter like light. In general relativity the main focus lies on the metric tensor $g_{\mu\nu}$. The left hand side of the Einstein equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 4\pi T_{\mu\nu}$$

is only dependent on the metric tensor and its derivatives. On the other side, there is the energy-momentum-tensor $T_{\mu\nu}$ for the specific problem describing the type of matter at hand. Solving this equation can be a highly complicated task and explicit solutions are only known for setups that carry a high degree of symmetry like spherical symmetry. One vacuum solution being spherically symmetric is known to be the famous Schwarzschild-metric [13]

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

with r being the radius, m the mass of the central object and $d\Omega$ the solid angle element. There are two issues with this metric that will be of some concern in this work. Firstly the metric becomes undefined for the radius $r = 2m$ the famous Schwarzschild-radius. This singularity is however a so called coordinate singularity that can be dealt with by using other coordinates like the Eddington-Finkelstein coordinates although at the cost of a non-diagonal metric. Notice that inside the Schwarzschild-radius the coefficient to dt^2 gets positive, whereas the coefficient to dr^2 gets negative. So, in order to get a total negative square of velocity $v^\nu v^\mu g_{\mu\nu}$, which physical objects should have, the dominating component can no longer be the time component v^t but the radial component v^r of the velocity vector. So, moving forward in 'time' this object has to predominantly decrease or increase its radius, which is an indication that the metric describes a black or white hole respectively. But there is another singularity at $r = 0$ that is qualitatively different. In fact one can show that this singularity arises due to a 'real' divergence of the Riemann curvature at the center by evaluating the closely related Kretschmann scalar meaning that it cannot be overcome by changing to a different set of coordinates.

In this work, we will on the one hand ensure that we only consider scenarios in which no coordinate singularities arise, one the other hand we will show that we encounter a curvature singularity by evaluating the Kretschmann scalar when approaching $t = 0$. For a closer and more rigorous look on all that see [13].

1.1 Einstein-Vlasov System

For a deeper understanding of this section we refer to [13] and [17] In the description of these models we use a distribution function f that defines the matter density for a specific point in phase space. We are not dealing with negative mass densities so the distribution function has to be non-negative. We consider the matter that has no (non-gravitational) interactions between the particles which can be thought of as the stars or whole galaxies. As these particles are in permanent free fall they entirely follow geodesics and that gives rise to the so-called an Einstein-Vlasov system. The more common approach is to consider massive particles which have possible momenta on a hypersurface P of the tangent bundle TM called the mass shell. We will however consider massless particles with the corresponding momenta lying on the forward light-cone. One particle with 4-momentum p^μ then defines a curve on the light-cone via the geodesic equation and the collection of all particles define a geodesic flow field X on P. The Vlasov equation follows from the condition that the distribution function should be constant along the flow, i.e. that $Xf = 0$.

We will choose coordinates x^μ such that the hypersurfaces with x^0 being constant are spacelike. That means that we can interpret x^0 as the time component t and we will call a vector future-pointing iff its time component v^0 is positive. Together with the momenta p^μ we can express the Vlasov equation by:

$$\frac{\partial f}{\partial t} + \frac{p^a}{p^0}\frac{\partial f}{\partial x^a} - \Gamma^a_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^a} = 0.$$

Here $\Gamma^\alpha_{\mu\nu}$ denotes the Christoffel symbol and a summation over Latin indices is to be summed over the spacial indices $(1, 2, 3)$ only. Note that p^0 here can be expressed by the spacial momentum coordinates p^a via $g_{\alpha\beta} p^\alpha p^\beta = 0$ in the massless case. The Vlasov equation can now be solved using the so-called method of characteristics. That is to use the fact, that the equation transports values along lines (characteristics) in phase-space. These characteristics can now be computed via the system:

$$\frac{dX^a}{ds} = P^a$$

$$\frac{dP^a}{ds} = -\Gamma^a_{\alpha\beta} P^\beta P^\alpha.$$

This can be a valuable simplification as we now only dealing with ODEs and we will use this extensively in course of this work.

These equations resemble the geodesic equation, as we have modelled the system in a way that it transports the particles along geodesics through phase-space.

Finally, in order to have a complete Einstein-Vlasov system we need to define the energy-momentum tensor $T_{\mu\nu}$. In the coordinates (x^μ, p^a) the explicit form is:

$$T_{\alpha\beta} = -\int f p_\alpha p_\beta \frac{\sqrt{g}}{p_0} dp^1 dp^2 dp^3.$$

A simple computation now shows that this tensor is divergence-free and satisfies the dominant energy condition $T_{\alpha\beta} V^\alpha W^\beta \geq 0$ for all future-pointing timelike vectors V^α and W^β.

In this work, we will use the fact that space-time in general relativity can be interpreted as a Cauchy problem. One can prescribe data on a spacelike hypersurface S, that is the induced metric g_{ab}, the second fundamental form k_{ab} and appropriate matter data on S by f. These tensors together with the Riemann curvature tensor have to obey the so-called constraint equations [13]

$$R - k_{ab}k^{ab} + (k^a_a)^2 = 16\pi\rho$$

and

$$\nabla_a k^a_b - \nabla_b (k^a_a) = 8\pi j_b$$

with ρ and j_b given by

$$\rho = \int f_0 \frac{(p^a) p^a p_a}{\sqrt{p^a p_a}} \sqrt{g} dp^1 dp^2 dp^3$$

and

$$j_a = \int f_0 (p^a) p_a \sqrt{g} dp^1 dp^2 dp^3.$$

g here is the induced metric on the hypersurface and R the Ricci curvature scalar of the induced metric.

As one can see there have to be other conditions to ensure that these integrals converge. One can impose fall-off conditions to the distribution functions, the simplest one is to require the distribution function to have compact support. We will choose to do so in this work.

1.2 Overview

We will show that there is a class of compactly supported distribution functions such that the Cauchy problem of the Einstein-Vlasov system in surface symmetry with a positive cosmological constant and massless matter has a unique solution outgoing from initial data prescribed on a spacelike hypersurface. In order to show that we will have to prove local existence in a neighborhood of this hypersurface and investigate certain continuation criteria in the past and future direction, partly directly related to the metric components. Furthermore, we will show that this solution cannot be extended beyond a certain point in the past direction because of a divergence of the Kretschmann scalar. In the future direction however, the solutions can be extended infinitely. We will also show that this cosmology is future geodesically complete meaning that all the geodesics can be continued for all times towards the future. Furthermore we will show that the universe approaches a de-Sitter type cosmology at late times. The matter will get dustlike meaning that all the energy density ρ dominates the other matter terms at late times.

A technical issue that arises with massless matter is that the lightcone unlike the mass-shell in the massive case is not differentiable in the limit towards vanishing momentum. We will for the local existence result require the distribution function to stay away from the tip of the lightcone and thereby ensure differentiability for a small time interval. For the global existence result towards the future we have to investigate the behaviour of the geodesics approaching the tip to ensure geodesic completeness and thereby avoiding breakdown of the system at the same time.

The ideas and techniques for much of this work were described in [4], [7], [8], and [11].

Chapter 2

The Einstein-Vlasov System in Surface Symmetry

In a surface symmetric Einstein model the Topology is $\mathbb{R}^+ \times \Sigma$ with a Riemannian submanifold Σ and the metric is assumed to have the form

$$ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + t^2(d\theta^2 + sin_k^2\theta d\varphi^2) \quad \text{with}$$

$$\sin_k \theta := \begin{cases} \sin\theta & \text{for } k = 1 \text{ spherical symmetry} \\ 1 & \text{for } k = 0 \text{ plane symmetry} \\ \sinh\theta & \text{for } k = -1 \text{ hyperbolic symmetry.} \end{cases} \tag{2.1}$$

$t > 0$ denotes the timelike coordinate, $r \in [0, 1]$ and $\mu(t, r)$ and $\lambda(t, r)$ are 1-periodic in r. φ ranges in the domain $[0, 2\pi]$ and θ from $[0, \pi[$, $[0, 2\pi]$ or $[0, \infty[$ in the case of $k = 1$, $k = 0$ or $k = -1$ respectively.
This topology together with the metric should be seen as the definition of the symmetry. The orbits of the symmetry action are spheres, flat tori and hyperbolic spaces respectively. We first calculate the Christoffel symbols. As discussed in [11] the non-zero components read:

$$\Gamma^0_{00} = \dot{\mu}, \quad \Gamma^0_{01} = \mu', \quad \Gamma^0_{11} = e^2\lambda - \mu\dot{\lambda},$$

$$\Gamma^1_{00} = e^2\mu - \lambda\mu', \quad \Gamma^1_{01} = \dot{\lambda}, \quad \Gamma^1_{11} = \lambda',$$

$$\Gamma^0_{22} = e^{-2\mu}t, \quad \Gamma^0_{33} = \sin^2_k\theta e^{-2\mu}t,$$

$$\Gamma^2_{02} = \Gamma^3_{03}\frac{1}{t}, \quad \Gamma^2_{33} = -k^2 \sin_k\theta \cos_k\theta, \quad \Gamma^3_{23} = k^2 \cot_k\theta$$

with

$$\cos_k \theta := \begin{cases} \cos\theta & \text{for } k = 1 \\ 1 & \text{for } k = 0 \\ \cosh\theta & \text{for } k = -1 \end{cases} \tag{2.2}$$

and $\cot_k \theta = \cos_k \theta \sin_k \theta$.

Then the components of the Riemann curvature tensor can be expressed in the form

$$R^d_{abc} = \left(e^{-2\lambda}\left(\mu'' + \mu'(\mu' - \lambda')\right) - e^{-2\mu}\left(\ddot{\lambda} + \dot{\lambda}\left(\dot{\lambda} - \dot{\mu}\right)\right)\right)\left(g_{bc}\delta^b_a - g_{ac}\delta^d_b\right),$$

$$R^D_{ABC} = \frac{1}{t^2}\left(e^{-2\mu} + k\right)\left(g_{AC}\delta^D_B - g_{BC}\delta^D_A\right),$$

$$R^d_{AbC} = \frac{1}{t} g_{AC} g^{de} \Gamma^0_{be}.$$

The lower case Latin indices take the value 0 and 1 and the upper case ones 2 and 3. One can derive the missing components by the standard symmetry relations of the tensor.

The non-zero components of the Einstein tensor $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ are

$$G_{00} = \frac{1}{t^2} e^{2\mu}\left(e^{-2\mu}\left(2t\dot{\lambda} + 1\right) + k\right),$$

$$G_{01} = \frac{2}{t}\mu',$$

$$G_{11} = \frac{1}{t^2} e^{2\lambda}\left(e^{-2\mu}\left(2t\dot{\mu} - 1\right) - k\right),$$

$$G_{22} = t^2 e^{-2\lambda}\left(\mu'' + \mu'\left(\mu' - \lambda'\right)\right) - t^2 e^{-2\mu}\left(\ddot{\lambda} + \left(\dot{\lambda} - \dot{\mu}\right)\left(\dot{\lambda} + \frac{1}{t}\right)\right) \quad \text{and}$$

$$G_{33} = \sin_k^2 \theta G_{22}.$$

The matter model is set up as follows: we choose the distribution function $f(t, r, w, F)$ to be a function of

$$t, r, w := e^{\lambda} p^1 \quad \text{and} \quad F := t^4 (p^2)^2 + t^4 \sin_k^2 \theta (p^3)^2.$$

In the massless case

$$p^0 = e^{-\mu}\sqrt{w^2 + F/t^2}$$

and we define

$$\langle p \rangle := \sqrt{w^2 + F/t^2}$$

Analogous to the massive case handled in [7] the Einstein-Vlasov system in the massless case is

$$\partial_t f + \frac{e^{\mu-\lambda} w}{\sqrt{w^2 + F/t^2}} \partial_r f - \left(\dot{\lambda} w + e^{\mu-\lambda}\mu'\sqrt{w^2 + F/t^2}\right)\partial_w f = 0 \tag{2.3}$$

$$e^{-2\mu}(2t\dot{\lambda}+1)+k-\Lambda t^2=8\pi t^2\rho \tag{2.4}$$

$$e^{-2\mu}(2t\dot{\mu}-1)-k+\Lambda t^2=8\pi t^2 p \tag{2.5}$$

$$\mu'=-4\pi t e^{\lambda+\mu} j \tag{2.6}$$

$$e^{-2\lambda}\left(\mu''+\mu'(\mu'-\lambda')\right)-e^{-2\mu}\left(\ddot{\lambda}+\left(\dot{\lambda}-\dot{\mu}\right)\left(\dot{\lambda}+\frac{1}{t}\right)\right)+\Lambda=4\pi q \tag{2.7}$$

where

$$\rho(t,r):=\frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\langle p\rangle f(t,r,w,F)dFdw=e^{-2\mu}T_{00}(t,r) \tag{2.8}$$

$$p(t,r):=\frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{w^2}{\langle p\rangle}f(t,r,w,F)dFdw=e^{-2\lambda}T_{11}(t,r) \tag{2.9}$$

$$j(t,r):=\frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}wf(t,r,w,F)dFdw=-e^{\lambda+\mu}T_{01}(t,r) \tag{2.10}$$

$$q(t,r):=\frac{\pi}{t^4}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{F}{\langle p\rangle}f(t,r,w,F)dFdw=\frac{2}{t^2}T_{22}(t,r)=\frac{2}{t^2 sin_k^2\theta}T_{33} \tag{2.11}$$

$$\langle p\rangle:=\sqrt{w^2+F/t^2} \tag{2.12}$$

In the case of mass less matter the energy-momentum-tensor should be trace free and a simple computation shows that it is indeed.

$$\begin{aligned}
g^{\mu\nu}T_{\mu\nu}&=g^{00}T_{00}+g^{11}T_{11}+g^{22}T_{22}+g^{33}T_{33}=\\
&=-e^{-2\mu}e^{2\mu}\frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\langle p\rangle f(t,r,w,F)dFdw+\\
&+e^{-2\lambda}e^{2\lambda}\frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{w^2}{\langle p\rangle}f(t,r,w,F)dFdw+\\
&+\frac{1}{t^2}\frac{t^2}{2}\frac{\pi}{t^4}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{F}{\langle p\rangle}f(t,r,w,F)dFdw+\\
&+\frac{1}{t^2 sin_k^2\theta}\frac{t^2 sin_k^2\theta}{2}\frac{\pi}{t^4}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{F}{\langle p\rangle}f(t,r,w,F)dFdw+=
\end{aligned}$$

$$\begin{aligned}
&=-\frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\langle p\rangle f(t,r,w,F)dFdw+\\
&+\frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{w^2}{\langle p\rangle}f(t,r,w,F)dFdw+\\
&+\frac{\pi}{2t^4}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{F}{\langle p\rangle}f(t,r,w,F)dFdw+\\
&+\frac{\pi}{2t^4}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{F}{\langle p\rangle}f(t,r,w,F)dFdw+=
\end{aligned}$$

$$= \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \left(-\langle p \rangle + \frac{w^2}{\langle p \rangle} + \frac{1}{2t^2} \frac{F}{\langle p \rangle} + \frac{1}{2t^2} \frac{F}{\langle p \rangle} \right) f(t, r, w, F) dF dw =$$

$$= \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{1}{\langle p \rangle} \underbrace{\left(-w^2 - \frac{F}{t^2} + w^2 + \frac{F}{2t^2} + \frac{F}{2t^2} \right)}_{=0} f(t, r, w, F) dF dw = 0$$

The initial data at time set to $t = 1$ will be denoted as

$f(t_0, r, w, F) =: \tilde{f}(t_0, r, w, F) =: f_0(r, w, F), \lambda(t_0, r) = \tilde{\lambda}(r) =: \lambda_0, \mu(t_0, r) =: \tilde{\mu}(r) =: \mu_0$

Initially $(t_0, r, 0, 0)$ is not in the support of f. As long as this holds p and q are defined.

Chapter 3

The Main Theorem

We will here state the main theorem that is about to be proved in this work. The functions involved require a minimal amount of regularity in all following statements.

First, we will define the notion of a regular solution. The functions have to be differentiable to some degree in order for the equations (2.3)-(2.7) to make sense. Furthermore, we require every function to be 1-periodic in r.

Definition 3.1. *Let $I \subset]0, \infty[$ be an interval.*

1. *The non-negative function $f(s, r, w, F)$ is called* regular *iff $C^1(I \times \mathbb{R}^2 \times [0, \infty]) \ni f(s, r, w, F) = f(s, r+1, w, F)$ everywhere and there is a set $B_\xi(s, r)$ being a 2-half-Ball centered at $(0, 0)$ in the w, F half-plane with a radius $\xi(s, r) > 0$ such that $B_\xi(s, r) \cap supp f(1, r, ., .) = \emptyset$ for every $r \in \mathbb{R}$.*

2. *The functions $\bar{\mu}(s, r), \rho(s, r), q(s, r), j(s, r)$ and $p(s, r)$ are* regular *iff they are element of $C^1(I \times \mathbb{R})$and 1-periodic in r.*

3. *The functions $\lambda(s, r)$ and $\mu(s, r)$ are* regular *iff $\dot{\lambda}, \dot{\mu} \in C^1(I \times \mathbb{R})$ respectively and 1-periodic in r.*

4. *The set of functions $(\tilde{f}(r, w, F), \tilde{\mu}(r), \tilde{\lambda}(r))$ defined as $(f(1, r, w, F), \mu(1, r), \lambda(1, r))$ will be referred to as initial data. Every function of this set is called regular iff the corresponding functions are regular at $s = 1$.*

We are now in the position to state the main theorem of this work.

Theorem 3.1. *Consider an Einstein-Vlasov system with spherical, plane or hyperbolic symmetry and positive cosmological constant for massless matter with prescribed regular initial data, if we assume that* $\Lambda > 3$ *in the spherical case and*

$$\tilde{\mu}(r) < ln\left(\left(\Lambda/3 - k\right)^{-1/2}\right), \forall r \in \mathbb{R},$$

generally, then there exists a unique, left maximal, regular solution (f, λ, μ) *of the Einstein-Vlasov system* (2.3)-(2.7) *on a time interval* $]T, \infty]$ *with* $T \in [0, 1[$ *which is future geodesically complete.*

Chapter 4

Local Existence

The main concern of this chapter will be to show that we have local-in-time unique solutions for the massless Einstein-Vlasov system with surface symmetry and $\Lambda > 0$ in the spherical, hyperbolic and flat case. However, to achieve this we first have to study the nature of the system.

After defining the regularity conditions for all the functions contained in the system, we will study the characteristic system concerning its regularity and solvability. We will show that the Einstein-Vlasov system as defined above is overdefined, a property we will later use in the local existence theorem.

4.1 Preliminary Results

The theorem proved of this section is stating that the Einstein-Vlasov system as described above is overdefined. We will exploit this fact by setting up a modified system that is easier to handle when we construct an iterative scheme to establish local existence. We will then not have the burden to check if (2.6) and (2.7) are fulfilled in every step but can rather conclude that they have to be fulfilled in the limit of the iterative system.

Theorem 4.1. *Let* $(f, \lambda, \mu, \bar{\mu})$ *be a regular solution of the modified system*

$$\partial_t f + \frac{e^{\mu-\lambda} w}{\sqrt{w^2 + F/t^2}} \partial_r f - \left(\dot{\lambda} w + e^{\mu-\lambda} \bar{\mu} \sqrt{w^2 + F/t^2} \right) \partial_w f = 0 \tag{4.1}$$

and

$$\bar{\mu} = -4\pi t e^{\lambda+\mu} j \tag{4.2}$$

together with (2.4), (2.5),(2.8), (2.9),(2.10) (2.11) *on some time interval* $I \subset]0, \infty[$ *with* $1 \in I$, *and let the initial data satisfy* (2.6) *at* $t = 1$. *Then* (f, λ, μ) *solves the Einstein-Vlasov system* (2.3)-(2.11).

We will now setup the characteristic system by defining the 2-component function $G(s,r,w,F)$ and show that it allows one to compute the evolution of all the geodesics locally unless you start at the (unphysical) point of zero momentum where the momentum variables w and F both vanish. This evolution then allows us to define the distribution function f at times different then $t=1$ by tracking where the geodesics transport the matter.

Lemma 4.1. *Let λ, μ and μ' be regular on $I \times \mathbb{R}$, and define*

$$G(s,r,w,F) := \left(\frac{e^{\mu-\lambda} w}{\sqrt{w^2 + \frac{F}{s^2}}}, -\dot{\lambda} w - e^{\mu-\lambda} \mu' \sqrt{w^2 + \frac{F}{s^2}} \right)$$

for $(s,r,w,F) \in I \times \mathbb{R}^2 \times [0,\infty[\setminus I \times \mathbb{R} \times \{0\} \times \{0\}$. Then the following holds

1.
$$G \in C^1(I \times \mathbb{R}^2 \times [0,\infty[\setminus I \times \mathbb{R} \times B_\zeta)$$
with $\zeta > 0$ and G is 1-periodic in r.

2. *$\forall t \in I$ and $(r,w,F) \in \mathbb{R}^2 \times [0,\infty[\setminus \mathbb{R} \times B_\zeta$ the characteristic system*
$$\frac{d}{ds}(R,W) = G(s,R,W,F)$$
has a unique solution
$$T \ni s \mapsto (R,W)(s,t,r,w,F)$$
with $(R,W)(t,t,r,w,F) = (r,w)$ on some time interval $T \subset I$ and $t \in T$ called the characteristic of (t,r,w,F). Moreover, $(R,W) \in C^1(I^2 \times \mathbb{R}^2 \times [0,\infty[\setminus I^2 \times \mathbb{R} \times \{0\} \times \{0\})$, for $s,t \in T$ the mapping
$$(r,w,F) \mapsto \big(R(s,t,r,w,F), W(s,t,r,w,F), F\big)$$
is a C^1-diffeomorphism on $\mathbb{R}^2 \times [0,\infty[\setminus \mathbb{R} \times \{0\} \times \{0\}$ with the inverse
$$(r,w,F) \mapsto \big(R(t,s,r,w,F), W(t,s,r,w,F), F\big)$$
and
$$(R,W)(s,t,r+1,w,F) = (R,W)(s,t,r,w,F) + (1,0)$$
for $(r,w,F) \in \mathbb{R}^2 \times [0,\infty[\setminus \mathbb{R} \times \{0\} \times \{0\}$.

3. *Let $1 \in I$. For a non-negative function $\bar{f}(r,w,F) \in C^1(\mathbb{R}^2 \times [0,\infty[\setminus \mathbb{R} \times \{0\} \times \{0\})$ 1-periodic in r and compact support with respect to w and F then there exists an interval $T \ni 1$ such that the equation*
$$f(t,r,w,F) := \tilde{f}((R,W)(1,t,r,w,F),F) \quad \forall t \in T \quad and \quad \forall (r,w,F) \in I \times \mathbb{R}^2 \times [0,\infty[\setminus I \times \mathbb{R} \times \{0\} \times$$
and
$$f(t,r,w,F) := 0$$
for $f(t,r,0,0)$ with $(t,r) \in I \times \mathbb{R}$ defines a unique, regular solution to (2.3) with $\tilde{f} = f(1)$

4. *If* $f(t,r,w,F)$ *is regular and satisfies* (2.3) *then*

$$\partial_t\left(e^{\lambda}\int_{-\infty}^{\infty}\int_0^{\infty} f(t,r,w,F)dFdw\right)+\partial_r\left(e^{\mu}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{w}{\sqrt{w^2+F/t^2}}f(t,r,w,F)dFdw\right)=0 \quad (4.3)$$

and the quantity

$$\int_0^1\int_{-\infty}^{\infty}\int_0^{\infty} e^{\lambda(t,r)}f(t,r,w,F)dFdwdr \quad (4.4)$$

is conserved.

Proof. 1. The periodicity of G follows from the fact that every function contained in the differential equation is a 1-periodic C^1-function in r.

2. We can establish a Lipschitz constant for every point in the domain of G so existence and uniqueness follows by Picard-Lindelöv.

3. Follows by the theory of characteristics.

4. From the Vlasov equation (2.3) we get by multiplying with e^{λ}

$$e^{\lambda}\partial_t f+\frac{e^{\mu}w}{\sqrt{w^2+F/t^2}}\partial_r f-\left(\dot{\lambda}we^{\lambda}+e^{\mu}\mu'\sqrt{w^2+F/t^2}\right)\partial_w f=0$$

Integrating with respect to F and w yields

$$\int_{-\infty}^{\infty}\int_0^{\infty} e^{\lambda}\partial_t f dFdw+\int_{-\infty}^{\infty}\int_0^{\infty}\frac{e^{\mu}w}{\sqrt{w^2+F/t^2}}\partial_r f dFdw-$$

$$-\int_{-\infty}^{\infty}\int_0^{\infty}\underbrace{\dot{\lambda}e^{\lambda}}_{=\partial_t\left(e^{\lambda}\right)} w\partial_w f dFdw-\int_{-\infty}^{\infty}\int_0^{\infty}\underbrace{e^{\mu}\mu_r}_{=\partial_r\left(e^{\mu}\right)}\sqrt{w^2+F/t^2}\partial_w f dFdw=0$$

Iterating the second line by parts with respect to w gives

$$\int_{-\infty}^{\infty}\int_0^{\infty} e^{\lambda}\partial_t f dFdw+\int_{-\infty}^{\infty}\int_0^{\infty}\frac{e^{\mu}w}{\sqrt{w^2+F/t^2}}\partial_r f dFdw-$$

$$-\int_{-\infty}^{\infty}\int_0^{\infty}\partial_t\left(e^{\lambda}\right)fdFdw+\int_{-\infty}^{\infty}\int_0^{\infty}\partial_r\left(e^{\mu}\right)\frac{w}{\sqrt{w^2+F/t^2}}fdFdw$$

$$=\int_{-\infty}^{\infty}\int_0^{\infty}\underbrace{\left(e^{\lambda}\partial_t f+\partial_t\left(e^{\lambda}\right)f\right)}_{=\partial_t\left(e^{\lambda}f\right)}dFdw+$$

$$+\int_{-\infty}^{\infty}\int_0^{\infty}\underbrace{\left(e^{\mu}\partial_r f+\left(e^{\mu}\right)f\right)}_{=\partial_r\left(e^{\mu}f\right)}\frac{w}{\sqrt{w^2+F/t^2}}dFdw=0$$

Pulling out the derivatives now yields the first statement. For the conservation law we integrate with respect to r

$$\partial_t \left(\int_0^1 \int_{-\infty}^{\infty} \int_0^{\infty} e^{\lambda} f dr dF dw \right) + \left(e^{\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w}{\sqrt{w^2 + F/t^2}} f dF dw \right) \Bigg|_0^1 = 0$$

Note that the second term vanishes because of 1-periodicity in r of μ and f.

□

To control the second order derivatives of the characteristics we look at the geodesic deviation equation or Jacobi equation as it describes how neighboring geodesics change relative to each other cf. [13] in 3.3.18. The coefficients used in the following equations are now derived directly via the Riemann tensor rather then the Christoffel symbols as the Riemann curvature tensor appears in the geodesic deviation equation.

Lemma 4.2. *Let $I \subset]0, \infty[$ be an interval, let $\lambda, \mu, \tilde{\mu} : I \times \mathbb{R} \mapsto \mathbb{R}$ be regular, and define $(R, W)(s) := (R, W)(s, t, r, w, F)$as above in Lemma 4.1 with $(s, t, r, w, F) \in I^2 \times \mathbb{R}^2 \times [0, \infty[\backslash I^2 \times \mathbb{R} \times 0 \times 0$ For $\partial \in \{\partial_r, \partial_w\}$ define*

$$\xi(s) := e^{\lambda(s,R) - \mu(s,R)} \partial R(s, t, r, w, F)$$

$$\eta(s) := \partial W(s, t, r, w, F) + \left(\sqrt{w^2 + F/s^2} e^{\lambda - \mu} \right) \Big|_{(s,(R,W)(s,t,r,w,F))} \partial R(s, t, r, w, F).$$

Then

$$\dot{\xi}(s) = a_1(s, R(s), W(s), F)\xi(s) + a_2(s, R(s), W(s), F)\eta(s),$$
$$\dot{\eta}(s) = (a_3 + a_5)(s, R(s), W(s), F)\xi(s) + a_4(s, R(s), W(s), F)\eta(s),$$

with

$$a_1(s, r, w, F) := \frac{w^2}{w^2 + F/s^2} \dot{\lambda} - \dot{\mu},$$

$$a_2(s, r, w, F) := \frac{F/s^2}{(w^2 + F/s^2)^{3/2}},$$

$$a_3(s, r, w, F) := -\frac{1}{s} \sqrt{w^2 + F/s^2} \left(\dot{\lambda} - \dot{\mu} + \frac{F/s^2}{w^2 + F/s^2} \dot{\lambda} \right)$$

$$a_4(s, r, w, F) := -\frac{w}{\sqrt{w^2 + F/s^2}} \left(e^{\mu - \lambda} \tilde{\mu} + \frac{w}{\sqrt{w^2 + F/s^2}} \dot{\lambda} \right)$$

$$a_5(s, r, w, F) := -\sqrt{w^2 + F/s^2} e^{2\mu} \left(e^{-2\lambda} \left(\tilde{\mu}' + \tilde{\mu} \left(\mu' - \lambda' \right) \right) - e^{-2\mu} \left(\ddot{\lambda} + \left(\dot{\lambda} + 1/s \right) \left(\dot{\lambda} - \dot{\mu} \right) \right) \right)$$

If $\mu \in C^2(I \times \mathbb{R})$ *and if we let* $\tilde{\mu} = \mu'$ *then*

$$a_3(s, r, w, F) := -\frac{1}{s}\sqrt{w^2 + F/s^2}\left(\dot{\lambda} - \dot{\mu} + \frac{F/s^2}{w^2 + F/s^2}\dot{\lambda}\right) -$$

$$-e^{2\mu(s,r)} H \sqrt{w^2 + F/s^2},$$

with

$$H := e^{-2\lambda}\left(\mu'' + \mu'\left(\mu' - \lambda'\right)\right) - e^{-2\mu}\left(\ddot{\lambda} + \left(\dot{\lambda} + 1/s\right)\left(\dot{\lambda} - \dot{\mu}\right)\right)$$

and $a_5 = 0$.

Proof. We prove the statement by directly computing the derivatives.

The right hand side of G is continuously differentiable and the functions ξ and η exist and are continuously differentiable with respect to s. Keep in mind that

$$\langle p \rangle := \sqrt{w^2 + F/t^2}$$

and therefore

$$\partial \langle p \rangle = -\frac{w}{\langle p \rangle}\partial W$$

for example.

$$\dot{\xi} = \frac{d}{ds}\left(e^{\lambda - \mu}\partial R\right) =$$

$$= e^{\lambda - \mu}\left[\left(\dot{\lambda} - \dot{\mu}\right)\partial R + \left(\lambda' - \mu'\right)\dot{R}\partial R + \frac{d}{ds}\partial R\right] =$$

Now using that

$$\frac{d}{ds}\partial R = \partial \dot{R} = \partial\left(\frac{e^{\mu - \lambda} w}{\langle p \rangle}\right)$$

we get

$$\dot{\xi} = \left(\dot{\lambda} - \dot{\mu}\right)\left(e^{\lambda - \mu}\right)\partial R + \left(\lambda' - \mu'\right)e^{\lambda - \mu}\dot{R}\partial R + e^{\lambda - \mu}\partial\frac{dR}{ds} =$$

$$= \left(\dot{\lambda} - \dot{\mu}\right)e^{\lambda - \mu}\partial R + \left(\lambda' - \mu'\right)e^{\lambda - \mu}\frac{e^{\mu - \lambda} w}{\langle p \rangle}\partial R +$$

$$+ e^{\lambda - \mu}\left(\left(\mu' - \lambda'\right)\frac{e^{\mu - \lambda} w}{\langle p \rangle}\partial R + \frac{e^{\mu - \lambda}}{\langle p \rangle}\partial W - \frac{e^{\mu - \lambda} w^2}{\langle p \rangle^3}\partial W\right) =$$

$$\left(\dot{\lambda} - \dot{\mu}\right)\xi + \frac{F/s^2}{\langle p \rangle^3}\left(\eta - \langle p \rangle e^{\lambda - \mu}\dot{\lambda}\partial R\right) =$$

$$\frac{w^2}{\langle p \rangle}\left(\dot{\lambda} - \dot{\mu}\right)\xi + \frac{F/s^2}{\langle p \rangle^3}\eta.$$

Remember that

$$\eta(s) := \partial W(s,t,r,w,F) + \left(\sqrt{w^2+F/s^2}e^{\lambda-\mu}\right)\Big|_{(s,(R,W)(s,t,r,w,F))} \partial R(s,t,r,w,F).$$

and

$$\dot{W}(s) = -\dot{\lambda}W - e^{\mu-\lambda}\tilde{\mu}\sqrt{W^2+\frac{F}{s^2}}$$

$$\dot{\eta} = \partial\dot{W} + \frac{d}{ds}\left[\left(\sqrt{w^2+F/s^2}e^{\lambda-\mu}\right)\Big|_{(s,(R,W)(s,t,r,w,F))} \partial R(s,t,r,w,F)\right]$$

$$= -\dot{\lambda}'w\partial R - \dot{\lambda}\partial W \text{-} \left(\mu'-\lambda'\right)e^{\mu-\lambda}\tilde{\mu}\langle p\rangle\partial R -$$

$$-e^{\mu-\lambda}\tilde{\mu}'\langle p\rangle\partial R - \frac{w}{\langle p\rangle}e^{\mu-\lambda}\tilde{\mu}\partial W + \frac{1}{\langle p\rangle}\left(-\frac{F}{s^3}\right)e^{\lambda-\mu}\dot{\lambda}\partial R +$$

$$+\langle p\rangle e^{\lambda-\mu}\dot{\lambda}\left(-\dot{\lambda}w - e^{\mu-\lambda}\tilde{\mu}\langle p\rangle\right)\partial R + \langle p\rangle\left(\dot{\lambda}-\dot{\mu}\right)\dot{\lambda}e^{\lambda-\mu}\frac{e^{\mu-\lambda}}{\langle p\rangle}\partial R +$$

$$+\langle p\rangle e^{\lambda-\mu}\ddot{\lambda}\partial R + \langle p\rangle e^{\lambda-\mu}\dot{\lambda}'\frac{e^{\mu-\lambda}w}{\langle p\rangle}\partial R +$$

$$+\langle p\rangle e^{\lambda-\mu}\dot{\lambda}\left(\left(\mu'-\lambda'\right)\frac{e^{\mu-\lambda}w}{\langle p\rangle}\partial R + \frac{e^{\mu-\lambda}}{\langle p\rangle}\partial W - \frac{e^{\mu-\lambda}w^2}{\langle p\rangle^3}\partial W\right) =$$

$$= e^{2\left(\mu-\lambda\right)}\langle p\rangle\left(\left(\mu'-\lambda'\right)\tilde{\mu} + \tilde{\mu}'\right)\xi - \frac{F}{s^3}\frac{\dot{\lambda}}{\langle p\rangle}\xi -$$

$$-e^{\mu-\lambda}\dot{\lambda}\tilde{\mu}w\xi - \frac{w^2}{\langle p\rangle}\dot{\lambda}^2\xi + \langle p\rangle\left(\ddot{\lambda} + \left(\dot{\lambda}-\dot{\mu}\right)\dot{\lambda}\right)\xi -$$

$$-\frac{e^{\mu-\lambda}w}{\langle p\rangle}\left(\tilde{\mu} + \frac{\dot{\lambda}e^{\lambda-\mu}w}{\langle p\rangle}\right)\left(\eta - \langle p\rangle\dot{\lambda}\xi\right) =$$

$$= -\langle p\rangle e^{2\mu}\left(e^{-2\lambda}\left(\tilde{\mu}' + \tilde{\mu}\left(\mu'-\lambda'\right)\right) - e^{2\mu}\left(\ddot{\lambda} + \left(\dot{\lambda}+1/s\right)\left(\dot{\lambda}-\dot{\mu}\right)\right)\right) -$$

$$-\frac{1}{s}\langle p\rangle\left(\dot{\lambda} - \dot{\mu} + \frac{F/s^2}{\langle p\rangle}\dot{\lambda}\right)\xi - \frac{w}{\langle p\rangle}\left(e^{\mu-\lambda}\tilde{\mu} + \frac{w}{\langle p\rangle}\dot{\lambda}\right)\eta.$$

□

We will now investigate the time development of $\mu(t,r)$ and $\lambda(t,r)$.

Lemma 4.3. *Let $\rho, p : I \times \mathbb{R} \mapsto \mathbb{R}$ be regular, $I \in]0, \infty[$ an interval containing 1 and assume that*

$$\frac{e^{-2\tilde{\mu}} + k}{t} - k - \frac{8\pi}{t}\int_1^t s^2 p(s,r)ds + \frac{\Lambda}{3t}\left(t^3 - 1\right) > 0 \quad for \quad t \in I$$

Then the equations (2.5) *and* (2.4) *have a unique, regular solution on $I \times \mathbb{R}$ The solution of is given by*

$$e^{-2\mu} = \frac{e^{-2\tilde{\mu}} + k}{t} - k - \frac{8\pi}{t}\int_1^t s^2 p ds + \frac{\Lambda}{3t}\left(t^3 - 1\right) \tag{4.5}$$

$$\dot{\lambda}(t,r) = 4\pi t e^{2\mu}\rho(t,r) - \frac{1 + \left(k - \Lambda t^2\right)e^{2\mu}}{2t} \tag{4.6}$$

$$\lambda(t,r) = \tilde{\lambda}(r) + \int_1^t \dot{\lambda}(s,r)ds \tag{4.7}$$

Proof. (2.5) reads

$$e^{-2\mu}(2t\dot{\mu} - 1) - k + \Lambda t^2 = 8\pi t^2 p$$

Note that

$$e^{-2\mu}(2t\dot{\mu} - 1) = -\frac{d}{dt}\left(te^{-2\mu}\right).$$

So,

$$-\frac{d}{dt}\left(te^{-2\mu}\right) = 8\pi t^2 p + k - \Lambda t^2$$

The statement now follows by integrating the above equation from 1 to t with respect to t.

$$se^{-2\mu(s,r)}\Big|_{s=1}^{s=t} = -\int_1^t \left(8\pi s^2 p(s,r) + k - \Lambda s^2\right) ds \Leftrightarrow$$

$$te^{-2\mu} = e^{-2\tilde{\mu}} - kt + \Lambda t^3/3 + k - \Lambda/3 - 8\pi \int_1^t s^2 p ds \Leftrightarrow$$

$$e^{-2\mu} = \frac{e^{-2\tilde{\mu}} + k}{t} - k - 8\pi \int_1^t s^2 p ds + \frac{\Lambda}{3t}\left(t^3 - 1\right)$$

The statement regarding (2.4) is obvious. □

Here one can clearly see that in the case of negative Λ equation (4.5) breaks down for some sufficient $t > 1$ independent of the type of symmetry. That is the main reason why we consider the case with non-negative cosmological constant.

The aim of the following statement is to show that the Einstein-Vlasov system (2.3)-(2.9) is a over defined system. We will later use this fact to leave us a bit of flexibility when dealing with the iterative scheme we will set up to prove the local existence.

Lemma 4.4. *Let* (f, λ, μ) *be a regular solution of* (2.3), (2.4), (2.5), (2.8) *and* (2.9) *on some interval* $I \subset]0, \infty[, 1 \in I$ *and let the initial data satisfy* (2.6) *and* $\tilde{\mu} \in C^2(\mathbb{R})$. *Then* (2.6) *and* (2.7) *hold* $\forall t \in T$ *and* $\mu \in C^2(I \times \mathbb{R})$.

Proof. (4.5) reads

$$e^{-2\mu(t,r)} = \frac{e^{-2\tilde{\mu}} + k}{t} - k - 8\pi \int_1^t s^2 p(s,r)ds + \frac{\Lambda}{3t}\left(t^3 - 1\right).$$

Taking its derivative with respect to r yields

$$-2t\mu' e^{-2\mu} = -2\tilde{\mu}'(r)e^{-2\tilde{\mu}} - 8\pi \int_1^t s^2 p'(s,r)ds$$

and making use of the Vlasov equation (2.3)

$$\partial_t f + \frac{e^{\mu-\lambda} w}{\sqrt{w^2 + F/t^2}}\partial_r f - \left(\dot{\lambda} w + e^{\mu-\lambda}\mu' \sqrt{w^2 + F/t^2}\right)\partial_w f = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{w}{\sqrt{w^2 + F/s^2}}\partial_r f = -e^{\lambda-\mu}\partial_t f + \left(e^{\lambda-\mu}\dot{\lambda} w + \mu' \sqrt{w^2 + F/t^2}\right)\partial_w f$$

we see that

$$\int_1^t p'(s,r)s^2 ds \overset{(2.9)}{=} \pi \int_1^t \int_{-\infty}^{\infty}\int_0^{\infty} \frac{w^2}{\sqrt{w^2 + F/s^2}}\partial_r f(s,r,w,F)dFdwds \overset{(2.3)}{=}$$

$$\overset{(2.3)}{=} \pi \int_1^t \int_{-\infty}^{\infty}\int_0^{\infty} w\left(-e^{\lambda-\mu}\partial_t f + \left(e^{\lambda-\mu}\dot{\lambda} w + \mu' \sqrt{w^2 + F/t^2}\right)\partial_w f\right)dFdwds =$$

$$= \pi \int_1^t \int_{-\infty}^{\infty}\int_0^{\infty} \left(-e^{\lambda-\mu} w\partial_t f + \left(e^{\lambda-\mu}\dot{\lambda} w^2 + \mu' w\sqrt{w^2 + F/t^2}\right)\partial_w f\right)dFdwds \overset{p.i}{=}$$

$$= \int_1^t \left(\dot{\lambda} - \dot{\mu}\right)e^{\lambda-\mu} \underbrace{\pi \int_{-\infty}^{\infty}\int_0^{\infty} wfdFdw}_{\overset{2.10}{=} js^2} ds - e^{\lambda-\mu} \underbrace{\pi \int_{-\infty}^{\infty}\int_0^{\infty} wfdFds}_{\overset{2.10}{=} Js^2}\Bigg|_1^t +$$

$$- \int_1^t 2e^{\lambda-\mu}\dot{\lambda} \underbrace{\pi \int_{-\infty}^{\infty}\int_0^{\infty} wdFdw}_{\overset{2.10}{=} js^2} ds -$$

$$- \int_1^t \mu' \underbrace{\pi \int_{-\infty}^{\infty}\int_0^{\infty} \sqrt{w^2 + F/t^2}}_{\overset{2.8}{=} \rho s^2} - \pi \int_1^t \mu' \underbrace{\pi \int_{-\infty}^{\infty}\int_0^{\infty} \frac{w^2}{\sqrt{w^2 + F/s^2}} fdFdw}_{\overset{(2.9)}{=} ps^2} ds =$$

$$= \int_1^t \left[\left(\cancel{\dot{\lambda}} - \dot{\mu}\right)e^{\lambda-\mu} j - \cancel{2}e^{\lambda-\mu}\dot{\lambda} j - \mu'(\rho + p)\right]s^2 ds - e^{\lambda-\mu} js^2\Big|_1^t =$$

$$= \int_1^t \left[-\left(\dot{\lambda} + \dot{\mu}\right)e^{\lambda-\mu} j - \mu'(\rho + p)\right]s^2 ds - e^{\lambda-\mu} js^2\Big|_1^t.$$

Thereby the above equation for μ' gets

$$-2t\mu' e^{-2\mu} = -2\tilde{\mu}'(r)e^{-2\tilde{\mu}}+8\pi\left(\int_1^t \left[-\left(\dot{\lambda}+\dot{\mu}\right)e^{\lambda-\mu}j - \mu'\left(\rho+p\right)\right]s^2 ds - e^{\lambda-\mu}js^2\Big|_1^t\right) \Leftrightarrow$$

$$\Leftrightarrow t\mu' e^{-2\mu} = \tilde{\mu}'(r)e^{-2\tilde{\mu}}-4\pi\left(\int_1^t \left[-\left(\dot{\lambda}+\dot{\mu}\right)e^{\lambda-\mu}j - \mu'\left(\rho+p\right)\right]s^2 ds - e^{\lambda-\mu}js^2\Big|_1^t\right).$$

Adding (2.5) and (2.4) we get

$$\dot{\lambda}+\dot{\mu} = 4\pi t e^{2\mu}(\rho+p). \tag{4.8}$$

Using this yields

$$t\mu' e^{-2\mu} = \tilde{\mu}'(r)e^{-2\tilde{\mu}}+4\pi\left(\int_1^t \left[-4\pi t e^{2\mu}(\rho+p)e^{\lambda-\mu}j - \mu'\left(\rho+p\right)\right]s^2 ds - e^{\lambda-\mu}js^2\Big|_1^t\right) \Leftrightarrow$$

$$\Leftrightarrow t\mu' e^{-2\mu} = \tilde{\mu}'(r)e^{-2\tilde{\mu}}-4\pi\left(\int_1^t \left[\left(4\pi t e^{\lambda+\mu}j+\mu'\right)\left(\rho+p\right)\right]s^2 ds + e^{\lambda-\mu}js^2\Big|_1^t\right) \Leftrightarrow$$

$$\Leftrightarrow t\mu' e^{-2\mu} = \tilde{\mu}'(r)e^{-2\tilde{\mu}}-4\pi\left(\int_1^t \left[\left(4\pi t e^{\lambda+\mu}j+\mu'\right)\left(\rho+p\right)\right]s^2 ds + e^{\lambda-\mu}jt^2 - e^{\tilde{\lambda}-\tilde{\mu}}j(1,r)\right) \Leftrightarrow$$

$$\Leftrightarrow te^{-2\mu}\left(\mu' + 4\pi t e^{\lambda+\mu}\right) = e^{-2\tilde{\mu}}\left(\tilde{\mu}' + 4\pi e^{\tilde{\lambda}+\tilde{\mu}}j\right)+4\pi\int_1^t\left(4\pi t e^{\lambda+\mu}j+\mu'\right)(\rho+p)s^2 ds.$$

Now because we assume the equation 2.6 to hold for the initial time

$$\tilde{\mu}' = -4\pi t e^{\tilde{\lambda}+\tilde{\mu}}j(1,r),$$

the first term on the right hand side vanishes and we are left with

$$te^{-2\mu}\left(\mu' + 4\pi t e^{\lambda+\mu}\right) = 4\pi\int_1^t\left(4\pi t e^{\lambda+\mu}j+\mu'\right)(\rho+p)s^2 ds$$

and therefore

$$\mu' + 4\pi t e^{\lambda+\mu}j = 0 \quad \forall t \in I.$$

In addition we get that μ' is continuously differentiable. Taking the derivative with respect to r yields

$$\mu'' = -4\pi t e^{\lambda+\mu}\left(\left(\lambda'+\mu'\right)j + j'\right) = \left(\lambda'+\mu'\right)\mu' - 4\pi t e^{\lambda+\mu}j'.$$

Now from (2.3) we get that

$$w\partial_r f = -\sqrt{w^2+F/s^2}e^{\lambda-\mu}\partial_t f + \left(\dot{\lambda}w\sqrt{w^2+F/s^2}e^{\lambda-\mu} + \mu'\left(w^2+F/s^2\right)\right)\partial_w f$$

$$j' = \frac{\pi}{t^2}\int_0^\infty\int_{-\infty}^\infty w\partial_r f dF dw \overset{(2.3)}{=}$$

$$= \frac{\pi}{t^2}\int_0^\infty\int_{-\infty}^\infty\left(-\sqrt{w^2+F/s^2}e^{\lambda-\mu}\partial_t f+\left(\dot\lambda w\sqrt{w^2+F/s^2}e^{\lambda-\mu}+\mu'\left(w^2+F/s^2\right)\right)\partial_w f\right)dFdw \overset{p.i.}{=}$$

$$= -\frac{\pi e^{\lambda-\mu}}{t^2}\int_0^\infty\int_{-\infty}^\infty\sqrt{w^2+F/s^2}\partial_t f dFdw+$$

$$+\frac{\dot\lambda}{t^2}\underbrace{\pi\int_0^\infty\int_{-\infty}^\infty\sqrt{w^2+F/s^2}fdFdw}_{=\rho}e^{\lambda-\mu}+\frac{\dot\lambda}{t^2}\underbrace{\pi\int_0^\infty\int_{-\infty}^\infty\frac{w^2}{\sqrt{w^2+F/s^2}}fdFdw}_{=p}e^{\lambda-\mu}+$$

$$+\frac{2\mu'}{t^2}\underbrace{\pi\int_0^\infty\int_{-\infty}^\infty wfdFdw}_{=j}=$$

$$= -\frac{\pi e^{\lambda-\mu}}{t^2}\int_0^\infty\int_{-\infty}^\infty\sqrt{w^2+F/s^2}\partial_t f dFdw+\dot\lambda e^{\lambda-\mu}(\rho+p)+2\mu' j.$$

So

$$\mu''=\left(\lambda'+\mu'\right)\mu'+4\pi te^{\lambda+\mu}\left(\frac{\pi e^{\lambda-\mu}}{t^2}\int_0^\infty\int_{-\infty}^\infty\sqrt{w^2+F/s^2}\partial_t f dFdw+\dot\lambda e^{\lambda-\mu}(\rho+p)+2\mu' j\right)=$$

$$=\left(\lambda'+\mu'\right)\mu'+\frac{4\pi^2e^{2\lambda}}{t}\int_0^\infty\int_{-\infty}^\infty\sqrt{w^2+F/s^2}\partial_t f dFdw+4\pi t\dot\lambda e^{2\lambda}(\rho+p)+8\pi te^{\lambda+\mu}\mu' j \tag{4.9}$$

Since

$$\dot\rho \overset{2.8}{=} -2\underbrace{\frac{\pi}{t^3}\int_0^\infty\int_{-\infty}^\infty\sqrt{w^2+F/s^2}fdFdw}_{=\rho/t}+$$

$$-\underbrace{\frac{\pi}{t^2}\int_0^\infty\int_{-\infty}^\infty\frac{F/t^3}{\sqrt{w^2+F/s^2}}fdFdw}_{=q/t}+$$

$$+\frac{\pi}{t^2}\int_0^\infty\int_{-\infty}^\infty\sqrt{w^2+F/s^2}\partial_t f dFdw=$$

$$=-\frac{2\rho}{t}-\frac{q}{t}+\frac{\pi}{t^2}\int_0^\infty\int_{-\infty}^\infty\sqrt{w^2+F/s^2}\partial_t f dFdw$$

computing $\ddot\lambda$ yields

$$\ddot\lambda \overset{4.6}{=} 4\pi e^{2\mu}\rho+8\pi t\dot\mu e^{2\mu}\rho+4\pi te^{2\mu}\dot\rho+\frac{1+\left(k-\cancel{\Lambda t^2}\right)e^{2\mu}}{2t^2}+\frac{\cancel{2}\Lambda e^{2\mu}}{2}-\frac{\left(k-\Lambda t^2\right)e^{2\mu}2\dot\mu}{2t}=$$

$$=\cancel{4\pi e^{2\mu}\rho}+\underline{8\pi t\dot\mu e^{2\mu}\rho}+4\pi te^{2\mu}\left(-\frac{\cancel{2}\rho}{t}-\frac{q}{t}+\frac{\pi}{t^2}\int_0^\infty\int_{-\infty}^\infty\sqrt{w^2+F/s^2}\partial_t f dFdw\right)+$$

$$+\frac{1+ke^{2\mu}}{2t^2}+\frac{\Lambda e^{2\mu}}{2}-\underline{\frac{\left(k-\Lambda t^2\right)e^{2\mu}2\dot\mu}{2t}}=$$

$$= -4\pi e^{2\mu}(\rho+q) + 2\dot\mu \underbrace{\left(4\pi e^{2\mu} - \frac{\left(k-\Lambda t^2\right)e^{2\mu}}{2t}\right)}_{=\dot\lambda+\frac{1}{2t}} + \frac{4\pi^2 e^{2\mu}}{t}\int_0^\infty\int_{-\infty}^\infty \sqrt{w^2+F/s^2}\,\partial_t f dF dw +$$

$$+\frac{1+ke^{2\mu}}{2t^2} + \frac{\Lambda e^{2\mu}}{2} =$$

$$= -4\pi e^{2\mu}(\rho+q) + 2\dot\lambda\dot\mu + \frac{\dot\mu}{t} + \frac{4\pi^2 e^{2\mu}}{t}\int_0^\infty\int_{-\infty}^\infty \sqrt{w^2+F/s^2}\,\partial_t f dF dw + \frac{1+ke^{2\mu}}{2t^2} + \frac{\Lambda e^{2\mu}}{2} \tag{4.10}$$

Now using these identities in (2.7) yields

$$e^{-2\lambda}\left(\mu'' + \mu'\left(\mu'-\lambda'\right)\right) - e^{-2\mu}\left(\ddot\lambda + \left(\dot\lambda - \dot\mu\right)\left(\dot\lambda + \frac{1}{t}\right)\right) + \Lambda =$$

$$\overset{(4.9),(4.10)}{=} e^{-2\lambda}\left(\cancel{\frac{4\pi^2 e^{2\lambda}}{t}\int_0^\infty\int_{-\infty}^\infty \sqrt{w^2+F/s^2}\,\partial_t f dF dw} + 4\pi t\dot\lambda e^{2\lambda}(\rho+p) + 8\pi t e^{\lambda+\mu}\mu' j\right) +$$

$$\underbrace{e^{-2\lambda}\left(\cancel{\lambda'}+\mu'\right)\mu' + e^{-2\lambda}\left(\mu'(\mu'-\cancel{\lambda'})\right)}_{=2e^{-2\lambda}\mu'^2} -$$

$$-e^{-2\mu}\left(-4\pi e^{2\mu}(\rho+q) + 2\dot\lambda\dot\mu + \frac{\dot\mu}{t} + \cancel{\frac{4\pi^2 e^{2\mu}}{t}\int_0^\infty\int_{-\infty}^\infty \sqrt{w^2+F/s^2}\,\partial_t f dF dw} + \right.$$

$$\left. + \frac{1+ke^{2\mu}}{2t^2} + \frac{\Lambda e^{2\mu}}{2}\right) -$$

$$-e^{-2\mu}\left(\left(\dot\lambda - \dot\mu\right)\left(\dot\lambda + \frac{1}{t}\right)\right) + \Lambda =$$

$$\underbrace{4\pi t\dot\lambda(\rho+p)}_{=e^{-2\mu}\left(\dot\lambda+\cancel{\dot\mu}\right)\dot\lambda} + \underbrace{8\pi t e^{\mu-\lambda}\mu' j}_{=\cancel{-2e^{-2\lambda}\mu'^2}} + \cancel{2e^{-2\lambda}\mu'^2} +$$

$$+4\pi(\rho+q) - \cancel{2e^{-2\mu}\dot\lambda\dot\mu} - \cancel{\frac{e^{-2\mu}\dot\mu}{t}} - \frac{e^{-2\mu}+k}{2t^2} - \frac{\Lambda}{2} -$$

$$-\cancel{e^{-2\mu}\dot\lambda^2} + \cancel{e^{-2\mu}\dot\lambda\dot\mu} - e^{-2\mu}\frac{\dot\lambda}{t} + \cancel{e^{-2\mu}\frac{\dot\mu}{t}} + \Lambda =$$

$$4\pi(\rho+q) - \frac{e^{-2\mu}+k}{2t^2} - e^{-2\mu}\frac{\dot\lambda}{t} + \frac{\Lambda}{2} \overset{(2.4)}{=}$$

$$= 4\pi q$$

□

We will now show that Theorem 4.1 indeed holds.

Proof. If we have a regular solution of the auxiliary system $(f, \lambda, \mu, \bar{\mu})$ then by taking the derivative of (4.5) we obtain

$$t\mu'(t,r)e^{-2\mu} = \tilde{\mu}(r)e^{-2\tilde{\mu}} + 4\pi \int_1^t p'(s,r)s^2 ds$$

and

$$\int_1^t p'(s,r)s^2 ds = \int_1^t \left[-\left(\dot{\lambda} + \dot{\mu}\right) e^{\lambda-\mu} j - \bar{\mu}\,(\rho + p) \right] s^2 ds - e^{\lambda-\mu} j s^2 \Big|_1^t$$

we obtain

$$t\mu'(t,r)e^{-2\mu} = \tilde{\mu}(r)e^{-2\tilde{\mu}} + 4\pi \left(\int_1^t \left[-\left(\dot{\lambda} + \dot{\mu}\right) e^{\lambda-\mu} j - \bar{\mu}\,(\rho + p) \right] s^2 ds - e^{\lambda-\mu} j s^2 \Big|_1^t \right) =$$

$$\overset{(4.8)}{=} \tilde{\mu}(r)e^{-2\tilde{\mu}} + 4\pi \left(\int_1^t \underbrace{\left[-4\pi t e^{2\mu}(\rho + p)e^{\lambda-\mu} j - \bar{\mu}\,(\rho + p) \right]}_{=0} s^2 ds - e^{\lambda-\mu} j s^2 \Big|_1^t \right) =$$

$$= -4\pi t^2 e^{\lambda-\mu} j + \underbrace{\tilde{\mu}(r)e^{-2\tilde{\mu}} + 4\pi e^{\tilde{\lambda}-\tilde{\mu}} j(1)}_{=0} \Leftrightarrow$$

$$\Leftrightarrow \mu' = -4\pi t e^{\lambda+\mu} \quad \forall t \in I$$

Thus (f, λ, μ) is a solution of the subsystem (2.3),(2.5),(2.4)2.8,(2.9) and by the previous theorem (2.7) is thereby fulfilled automatically. □

4.2 Local Existence in the Past Direction

We want to prove the local existence and uniqueness of solutions of the system. The steps are based on the work of Rein [11] who proved the following statement in Theorem 4.2 in the massive case with vanishing cosmological constant. Because we would encounter difficulties in the characteristic system at the point of zero momentum, we will exclude this area from the process by requiring the distribution function to vanish in an area containing this point initially, that is to say that we can put a uniform lower positive bound on the particles absolute momenta.

We now state the main local existence theorem for the past direction.

Theorem 4.2. *Let* $\tilde{f} \in C^1(\mathbb{R}^2 \times [0, \infty[)$ *with* $\tilde{f}(r + 1, w, F) = \tilde{f}(r, w, F)$ $\forall (r, w, F)$, $\tilde{f} > 0$ *and*

$$w_0 := \sup_{(r,w,F)\in supp \tilde{f}} |w| < \infty$$

$$F_0 := \sup_{(r,w,F)\in supp \tilde{f}} F < \infty$$

Furthermore we require $\tilde{f}(r,.,.)$ to be compactly supported and disjoint to a 2-half-ball $B_\zeta(0,0)$ with radius ζ centred at the origin in the (w,F) half plane. Let $\tilde{\lambda} \in C^1(\mathbb{R}), \tilde{\mu} \in C^2(\mathbb{R})$ with $\tilde{\lambda}(r) = \tilde{\lambda}(r+1)$, $\tilde{\mu}(r) = \tilde{\mu}(r+1)$ $\forall r \in \mathbb{R}$ and

$$\tilde{\mu}(r) = -4\pi e^{\tilde{\lambda}+\tilde{\mu}} j(r).$$

In addition we assume that $\Lambda > 3$ in the case of $k = 1$ and that

$$\tilde{\mu}(r) < ln\left(\left(\Lambda/3 - k\right)^{-1/2}\right), \forall r \in \mathbb{R},$$

generally, then there exists a unique, left maximal, regular solution (f, λ, μ) of the Einstein-Vlasov system (2.3)-(2.7) with the initial data $(f, \lambda, \mu)(1) =: (\tilde{f}, \tilde{\lambda}, \tilde{\mu})$ on a time interval $]T, 1]$ with $T \in [0, 1[$. If

$$\sup_{(t,r,w,F)\in suppf} |w| < \infty$$

then $T = 0$.

Proof. Step 1 (construct sequences launched by initial data)

First, we construct a sequences of functions that satisfy the equations of our system.

Let $\lambda_0(t,r) := \tilde{\lambda}(r), \mu_0(t,r) := \tilde{\mu}(r), \mu_0'(t,r) := \tilde{\mu}'(r)$ for $t \in]0,1]$. If λ_{n-1}, μ_{n-1} and $\bar{\mu}_{n-1}$ are defined and regular on $]0,1] \times \mathbb{R}$ then we define $G_{n-1}(t,r,w,F)$ to be a function going from $]T,1] \times \mathbb{R} \times U$ with $U := \mathbb{R} \times [0,\infty[\backslash B_{\zeta/2}(0,0)$ onto its image contained in $\mathbb{R}^2$

$$G_{n-1}(t,r,w,F) := \left(\frac{e^{\mu_{n-1}-\lambda_{n-1}} w}{\sqrt{w^2 + \frac{F}{t^2}}}, -\dot{\lambda}_{n-1} w - e^{\mu_{n-1}-\lambda_{n-1}} \bar{\mu}_{n-1} \sqrt{w^2 + \frac{F}{t^2}} \right) \tag{4.11}$$

and and denote by $(R_n, W_n)(s,t,r,w,F)$ the solution of the characteristic system

$$\frac{d}{ds}(R,W) = G_{n-1}(s,R,W,F) \tag{4.12}$$

with the initial data

$$(R_n, W_n)(t,t,r,w,F) = (r,w) \tag{4.13}$$

outside of a region $B_{\sigma_n}(t)$ for which $|W_n| > \sigma_n(t)$ and $F = 0$. with a function $\sigma_n(t)$ chosen such that the support of $f_n(t)$ is disjoint to $B_{\rho_n}(t)$.

We now define

$$f_n(t,r,w,F) := \tilde{f}((R_n, W_n)(1,t,r,w,F), F) \tag{4.14}$$

in the region Thereby f_n is the solution of the Vlasov equation containing μ_{n-1} and λ_{n-1} instead of μ and λ respectively with $f_n(1,r,w,F) = \tilde{f}(r,w,F)$. The quantities ρ_n, p_n, j_n, q_n are defined by the usual momentum space integrals with f replaced by f_n.

We now define the sequence μ_n by

$$e^{-2\mu_n(t,r)} := \frac{e^{-2\tilde{\mu}} + k}{t} - k - \frac{8\pi}{t}\int_1^t s^2 p_n(s,r)ds + \frac{\Lambda}{3t}\left(t^3 - 1\right) \tag{4.15}$$

or differently stated

$$e^{-2\mu_n(t,r)} = \frac{e^{-2\tilde{\mu}} + k - \Lambda/3}{t} - k - \frac{8\pi}{t}\int_1^t s^2 p_n(s,r)ds + \frac{\Lambda t^2}{3}$$

In the case of $t \leq 1$ the last two terms are non-negative so

$$e^{-2\mu_n(t,r)} \geq \frac{e^{-2\tilde{\mu}} + k - \Lambda/3}{t} - k.$$

To ensure that the right hand of this equation is positive in the past direction, we need the assumption that

$$\tilde{\mu}(r) < ln\left(\left(\Lambda/3 - k\right)^{-1/2}\right)$$

everywhere. Note that for the case of negative Λ (4.15) would imply that there is some maximal time for which the right hand side gets negative and the system breaks down in these coordinates. Finally, we need to define

$$\dot{\lambda}_n(t,r) := 4\pi t e^{2\mu_n}\rho_n(t,r) - \frac{1 + \left(k - \Lambda t^2\right)e^{2\mu_n}}{2t} \tag{4.16}$$

$$\lambda_n(t,r) := \tilde{\lambda}(r) + \int_1^t \dot{\lambda}_n(s,r)ds, \tag{4.17}$$

and

$$\bar{\mu}_n(t,r) := -4\pi t e^{\mu_n + \lambda_n} j_n(t,r). \tag{4.18}$$

G_{n-1} is linear bounded with respect to w and therefore R_n and W_n exist on the time intervall $]0,1]$.

First, we need to establish uniform bounds of the momenta in the support of f. We define

$$P_n(t) := \sup\left\{|w|\,\middle|(r,w,f) \in supp f_n(t)\right\} \tag{4.19}$$

and have to show that $P_{n+1}(t)$ is bounded as well.

In the support of f we have

$$\underbrace{\sqrt{w^2 + F/t^2}}_{=:\langle p\rangle} \leq \sqrt{P_n(t)^2 + F_0/t^2} \leq P_n(t) + \sqrt{F_0}/t \overset{t<1}{\leq} \frac{1+F_0}{t}\left(1 + P_n(t)\right)$$

and

$$\frac{w^2}{\sqrt{w^2+F/t^2}} \leq \frac{P(t)^2}{\sqrt{P(t)^2+F/t^2}} \leq \frac{P(t)^2}{\sqrt{P(t)^2}} = P(t)$$

We now use these estimates to bound ρ_n, p_n and j_n from above.

$$\rho_n := \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\sqrt{w^2+F/t^2}f_n(r,t,w,F)dFdw \leq$$

$$\leq \frac{\pi||f||}{t^2}\int_{-P_n(t)}^{P_n(t)}\int_0^{F_0}\sqrt{w^2+F/t^2}dFdw \leq$$

$$\leq \frac{\pi||f||}{t^2}\int_{-P_n(t)}^{P_n(t)}\int_0^{F_0}\frac{1+F_0}{t}\big(1+P_n(t)\big)dFdw \leq$$

$$\leq \frac{2\pi||f||P_n(t)}{t^2}\int_0^{F_0}\frac{1+F_0}{t}\big(1+P_n(t)\big)dFdw$$

and thus

$$||\rho_n(t)|| \leq c_1\frac{(1+F_0)^2}{t^3}||f||\big(1+P_n(t)\big)^2$$

For $||p_n(t)||$ the situation is

$$p_n(t,r) := \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{w^2}{\langle p\rangle}f_n(t,r,w,F)dFdw =$$

$$= \frac{\pi}{t^2}\int_{-P_n(t)}^{P_n(t)}\int_0^{F_0}\frac{w^2}{\langle p\rangle}f_n(t,r,w,F)dFdw \leq$$

Thereby

$$||p_n(t)|| \leq \frac{\pi}{t^2}\int_{-P_n(t)}^{P_n(t)}\int_0^{F_0}P_n(t)||f||dFdw =$$

$$= \frac{2\pi}{t^2}P_n(t)^2F_0||f||$$

Similarly

$$j_n(t,r) := \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}wf_n(t,r,w,F)dFdw \Rightarrow$$

$$\Rightarrow ||j_n(t,r)|| \leq \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}P_n(t)||f||dFdw \leq$$

$$\leq \frac{\pi}{t^2}\int_{-P_n(t)}^{P_n(t)}\int_0^{F_0}P_n(t)||f||dFdw =$$

$$= \frac{2\pi}{t^2}P_n(t)^2F_0||f||$$

$||.||$ denotes the L^∞ norm. Note that $||f_n|| = ||\tilde{f}|| = ||f||$,

From (4.15) we know that

$$e^{-2\mu_n(t,r)} \geq \frac{c_4}{t}$$

with

$$c_4(\tilde{\mu}, \Lambda) := e^{-2\tilde{\mu}} + k - \frac{\Lambda}{3}$$

By (4.16) and (4.18) and the above estimates on ρ_n and j_n we get

$$\left| e^{\mu_n - \lambda_n} \bar{\mu}_n(s,r) \right| \leq 4\pi t e^{\mu_n - \lambda_n} e^{\mu_n + \lambda_n} \left| j_n(t,r) \right| =$$

$$= 4\pi s e^{2\mu_n} |j_n(s,r)| \leq 8\pi^2 \frac{F_0}{c_4} ||f|| P_n(s)^2$$

and

$$\left| \dot{\lambda}_n(s,r) \right| \leq 4\pi s e^{2\mu_n} \rho_n(s,r) + \frac{1 + \left(1 + \Lambda s^2\right) e^{2\mu_n}}{2s} \leq$$

$$\leq \frac{8\pi^2}{c_4} \left(1 + F_0\right)^2 ||f|| \frac{\left(1 + P_n(s)\right)^2}{s} + \frac{1}{2s} + \frac{1}{2c_4} + \frac{s^2 \Lambda}{2c_4}$$

By the definition given in (4.12)

$$\left| \dot{W}_{n+1}(s) \right| = -\dot{\lambda}_n W_{n+1} - e^{\mu_n - \lambda_n} \bar{\mu}_n \sqrt{W_{n+1}^2 + \frac{F}{s^2}}$$

Now, with the above estimates we get

$$\left| \dot{W}_{n+1}(s) \right| \leq \left(\frac{8\pi^2}{c_4} \left(1 + F_0\right)^2 ||f|| \frac{\left(1 + P_n(s)\right)^2}{s} + \frac{1}{2s} + \frac{1}{2c_4} + \frac{s^3 \Lambda}{2c_4} \right) \left| W_{n+1}(s) \right| +$$

$$+ 8\pi^2 \frac{F_0}{c_4} ||f|| P_n(s)^2 \frac{1 + F_0}{s} \left(1 + \left| W_{n+1}(s) \right| \right) =$$

$$= \frac{8\pi^2 ||f|| \left(1 + F_0\right) \left(1 + P_n(s)\right)}{c_4 s} \underbrace{\left(\left(1 + F_0\right) \left(1 + P_n(s)\right) \left| W_{n+1} \right| + F_0 P_n(s) \left(1 + \left| W_{n+1}(s) \right| \right) \right)}_{\leq 2\left(1+F_0\right)\left(1+P_n(s)\right)\left(1+\left|W_{n+1}\right|\right)} +$$

$$+ \left(\frac{1}{2s} + \frac{1}{2sc_4} + \frac{s\Lambda}{2c_4} \right) \left| W_{n+1}(s) \right| \leq$$

$$\leq \frac{c_2}{s} \left(1 + P_n(s)\right)^2 \left(1 + \left| W_{n+1}(s) \right| \right) + \left(\frac{1}{2s} + \frac{1}{2c_4} + \frac{s^2 \Lambda}{c_4} \right) \left| W_{n+1}(s) \right|$$

with

$$c_2(\tilde{f}, F_0, \tilde{\mu}) := \frac{16\pi^2}{c_4} (1 + F_0)^2 ||f||.$$

This implies that

$$P_{n+1}(t) \leq w_0 + c_2 \int_t^1 \left(\frac{1}{s}(1+P_n(s))^2(1+P_{n+1}(s)) + \left(\frac{1}{2s} + \frac{1}{2c_4} + \frac{s^2\Lambda}{c_4} \right) P_{n+1}(s) \right) ds.$$

Let z_1 be the left maximal solution of the equation

$$z_1(t) = w_0 + c_2 \int_t^1 \frac{1}{s}(1+z_1(s))^3 + \left(\frac{1}{2s} + \frac{1}{2c_4} + \frac{s^2\Lambda}{c_4} \right) z_1(s) ds,$$

which exists on some interval $]T_1, 1]$ with $T_1 \in [0, 1[$. By induction

$$P_n(t) \leq z_1(t), \forall n \quad \text{and} \quad t \in]T, 1].$$

Proof. To begin with we see that

$$P_0(t) = w_0 \leq z_1(t).$$

If we now assume that

$$P_n(t) \leq z_1(t),$$

we need to show that $P_{n+1}(t)$ still is. We know that P_{n+1} satisfies this integral inequality

$$P_{n+1} \leq w_0 + c_2 \int_t^1 \frac{1}{s}(1+z_1(s))^2 \left(1+P_{n+1}\right) + \left(\frac{1}{2s} + \frac{1}{2c_4} + \frac{s^2\Lambda}{c_4} \right) P_{n+1}(s) ds$$

and is therefore certainly bounded by the solution of the integral equality

$$z_1(t) = w_0 + c_2 \int_t^1 \frac{1}{s}(1+z_1(s))^3 + \left(\frac{1}{2s} + \frac{1}{2c_4} + \frac{s^2\Lambda}{c_4} \right) z_1(s) ds.$$

□

Step 2:

In this step, we establish bounds on certain derivatives of the iterates. We need a uniform bound on the Lipschitz-constant of G_n of the characteristic system in order to prove convergence in the next step. Differentiating (4.15) and (4.16) with respect to r one obtains

$$\mu_n'(t,r) = \frac{e^{2\mu_n}}{t}\left(\tilde{\mu}' e^{-2\tilde{\mu}} + 4\pi \int_1^t s^2 p_n'(s,r)ds\right),$$

$$\dot{\lambda}_n'(t,r) = e^{2\mu_n}\left(8\pi t \mu_n'(t,r)\rho_n(t,r) + 4\pi t \rho_n'(t,r) - \frac{k+\Lambda t^2}{t}\mu_n'(t,r)\right),$$

$$\lambda_n'(t,r) = \tilde{\lambda}'(r) + \int_1^t \dot{\lambda}_n'(s,r)ds.$$

Now, C_1 denotes a continuous function on $]T,1]$ that only contains powers of z_1.

By Step 1

$$||\rho_n'(t)||, ||p_n'(t)||, ||j_n'(t)|| \le C_1(t)||\partial_r f_n(t)||.$$

For example

$$||\rho_n'(t)|| \overset{\text{Step 1}}{\le} ||\partial_r\left(\frac{2\pi}{t^2}P_n(t)^2 F_0||f||\right)|| \le \frac{2\pi}{t^2}z_1(t)^2 F_0||\partial_r f_n|| = C_1(t)||\partial_r f_n||.$$

We define

$$D_n(t) := sup\left\{||\partial_r f_n(s)|| : t \le s \le 1\right\}.$$

The above estimates show that

$$||\mu_n'(t)||, ||\lambda_n'(t)||, ||\dot{\lambda}_n'(t)|| \le C_1(t)\left(c_5 + D_n(t)\right),$$

where

$$c_5 := ||e^{-2\tilde{\mu}}\tilde{\mu}'|| + ||\tilde{\lambda}'|| + 1.$$

From (2.6) it follows that

$$e^{\mu_n - \lambda_n}\mu_n' = -4\pi t e^{2\mu_n} j_n,$$

and

$$\left|\left(e^{\mu_n-\lambda_n}\right)'(t,r)\right| \le C_1(t)\left(c_3 + D_n(t)\right).$$

Now we can make an estimate for the derivatives of G_n with respect to r and w:

$$\partial_r G_n(t,r,w,F) = \left((\mu_n - \lambda_n)' \, e^{\mu_n - \lambda_n} \frac{w}{\sqrt{w^2 + F/t^2}}, -(e^{\mu_n - \lambda_n} \mu_n')' \sqrt{w^2 + F/t^2} - \dot{\lambda}_n' w \right)$$

$$\partial_w G_n(t,r,w,F) = \left(e^{\mu_n - \lambda_n} \frac{F/t^2}{\left(w^2 + F/t^2\right)^{3/2}}, -e^{\mu_n - \lambda_n} \mu_n' \frac{w}{\sqrt{w^2 + F/t^2}} - \dot{\lambda}_n \right),$$

and thus

$$\left|\partial_r G_n(t,r,w,F)\right| \leq C_1(t)\left(c_3 + D_n(t)\right)$$
$$\left|\partial_w G_n(t,r,w,F)\right| \leq C_1(t)$$

for $t \in]T_1, 1], r \in \mathbb{R}, F \in [0, F_0]$ and $|w| \leq z_1(t)$. By differentiating the characteristic system we obtain

$$\left| \frac{d}{ds} \partial_r \left(R_{n+1}, W_{n+1}\right)(s,t,r,w,F) \right| \leq C_1(s)(c_3 + D_n(s) \left|\partial_r (R_{n+1}, W_{n+1})(s,t,r,w,T)\right|.$$

Thereby for $(r,w,F) \in suppf_{n+1}(t) \cup suppf_n(t)$ we get the estimate,

$$\left|\partial_r (R_{n+1}, W(n+1))(1,t,r,w,f)\right| \leq e^{\int_t^1 C_1(s)(c_5 + D_n(s))ds}.$$

By definition of D_n this implies that

$$D_{n+1}(t) \leq ||\partial_{(r,w)} \tilde{f}|| e^{\int_t^1 C_1(s)(c_5 + D_n(s))ds}$$

Let z_2 be the left maximal solution of

$$z_2(t) = ||\partial_{(r,w)} \tilde{f}|| e^{\int_t^1 C_1(s)(c_5 + z_2(s))ds}$$

on an interval $]T_2, 1] \subset]T_1, 1]$. Then, by similar arguments like in the previous step

$$D_n(t) \leq z_2(t), \forall t \in]T_2, 1].$$

Now all the quantities estimated against D_n above can be bounded in terms of z_2 on $]T_2, 1]$.

Step 3 (iterates converge uniformly):

Let $[\delta, 1] \subset]T_2, 1]$. We will show that on such an interval the iterates converge uniformly. Define

$$\alpha_n(t) := sup\left\{||f_{n+1}(\tau) - f_n(\tau)|| \tau \in [t,1]\right\},$$

and let C denote a constant, which may depend on the functions z_1 and z_2 introduced before. Then

$$||\rho_{n+1}(t) - \rho_n(t)||, ||p_{n+1}(t) - p_n(t)||, ||j_{n+1}(t) - j_n(t)|| \leq C\alpha_n(t)$$

and thus

$$||\lambda_{n+1}(t) - \lambda_n(t)||, ||\dot{\lambda}_{n+1}(t) - \dot{\lambda}_n(t)||, ||\mu_{n+1}(t) - \mu_n(t)||, ||\mu_{n+1}'(t) - \mu_n'(t)|| \leq C\alpha_n(s).$$

Therefore,

$$\left|G_{n+1} - G_n\right|(s, r, w, F) \le C\alpha_n(s)$$

and by Step 2

$$\left|\partial_{(r,w)} G_n(s, r, w, F)\right| \le C$$

$\forall s \in [\delta, 1]$, $n \in \mathbb{N}$, and $(r, w, F) \in \mathbb{R}^2 \times [0, F_0]$ with $|w| \le z_x(s)$. For $(r, w, F) \in supp f_n(t) \cup supp f_{n+1}(t)$ we get

$$\left|\frac{d}{ds}(R, W)_{n+1} - \frac{d}{ds}(R, W)_n\right|(s, t, r, w, F) \le C\left|(R, W)_{n+1} - (R, W)_n\right|(s, t, r, w, F) + C\alpha_{n-1}(s).$$

Now, by integration with respect to time and then applying Grönwall's inequality we obtain

$$\left|(R, W)_{n+1} - (R, W)_n\right|(1, t, r, w, F) \le C\int_t^1 \alpha_{n-1}(s)ds$$

By definition of f_n this implies that

$$\alpha_n(t) \le C\int_t^1 \alpha_{n-1}(s)ds, \quad \forall n \ge 1.$$

By induction we get

$$a_n(t) \le C\frac{C^n(1-t)^n}{n!} \le \frac{C^{n+1}}{n!}.$$

We give a small proof of this statement:

Proof. First, we note that because of the boundedness of f_1 and f_2 the statement for α_1 is trivially true for some large enough C. Secondly, if the statement is true for some α_n $n \ge 1$ then

$$\alpha_{n+1} \le C\int_t^1 \alpha_n(s)ds \le C\int_t^1 \frac{C^{n+1}(1-s)^n}{n!}ds \le \frac{C^{n+2}\,(1-t)^{n+1}}{(n+1)!} \le \frac{C^{n+2}}{(n+1)!}$$

on the interval $[\delta, 1]$. □

This shows that f_n and thereby all other quantities converge uniformly. However, we still have to prove that these functions still obey the same regularity conditions.

Step 4: derivatives converge uniformly

In this step we need to show uniform convergence of the derivatives of the involved functions. We now make use of the system derived in Lemma 4.2 for every single iterate. So, we now denote the coefficients by $a_{n,i}$ with n being the number of the iterate and $i \in \{1, ..., 5\}$ shall mark the different coefficients. Because we are able to bound the distribution function and its derivatives with respect to w and r we have

$$\left|a_{n,i}(t,r,w,F)\right| + \left|\partial_{(r,w)} a_{n,i}(t,r,w,F)\right| \leq C \quad \forall n \in \mathbb{N} \text{ and } i \in \{1, \dots, 4\}$$

$u < |w| < U$, $t \in [\delta, 1]$ with $\delta \in]T_2, 1]$. The only derivatives we have not checked for boundedness are $\dot{\mu}_n$ and $\dot{\mu}_n'$ but by definiton of μ_n

$$\dot{\mu}_n = 4\pi t e^{\mu_n} p_n + \frac{1 + e^{2\mu_n}\left(k - t^2\Lambda\right)}{2t}$$

and

$$\dot{\mu}_n' = 2\mu_n'\left(\dot{\mu}_n - \frac{1}{2t}\right) + 4\pi t e^{2\mu_n} p_n'$$

We showed boundedness of these terms in Step 1 and 2. By step 3 we know that

$$a_{n,i}(t,r,w,F) - a_{m,i}(t,r,w,F) \mapsto 0 \quad \text{for} \quad n, m \to \infty \quad \text{and} \quad i \in 1, ..., 4$$

uniformly on $[\delta, 1] \times \mathbb{R} \times [-U, U] \times [0, F_0]$. We will now prove that $H_n - 4\pi q_n \to 0$ for $n \to \infty$ by repeating the proof of Lemma 4.4 but now considering the iterates of the corresponding expressions.

$$H_n := e^{-2\lambda_n}\left(\bar{\mu}_n' + \bar{\mu}_n\left(\mu_n' - \lambda_n'\right)\right) - e^{-2\mu_n}\left(\ddot{\lambda}_n + \left(\dot{\lambda}_n + 1/s\right)\left(\dot{\lambda}_n - \dot{\mu}_n\right)\right) + \Lambda$$

.

From (4.18) we get that

$$\bar{\mu}_n' = \left(\mu_n' + \lambda_n'\right)\bar{\mu}_n - 4\pi t e^{\mu_n + \lambda_n} j_n'$$

and via (4.16) we can derive

$$\ddot{\lambda} = 4\pi e^{2\mu_n}\rho_n + 8\pi t\dot{\mu}_n e^{2\mu_n}\rho_n + 4\pi t e^{2\mu_n}\dot{\rho}_n + \underbrace{\frac{1 + \left(k - \Lambda t^2\right)e^{2\mu_n}}{2t^2}}_{=4\pi e^{2\mu_n}\rho_n - \frac{\dot{\lambda}_n}{t}} + \Lambda e^{2\mu_n} - \frac{\left(k - \Lambda t^2\right)e^{2\mu_n} 2\dot{\mu}_n}{2t} =$$

$$= 4\pi t e^{2\mu_n}\left(\frac{2\rho_n}{t} + \dot{\rho}_n\right) + 2\dot{\mu}_n\dot{\lambda}_n + \frac{\dot{\mu}_n - \dot{\lambda}_n}{t} + \Lambda e^{2\mu_n}$$

By definition of ρ_n and by use of the Vlasov equation we get

$$\dot{\rho}_n = -\frac{2}{t}\rho_n - \frac{1}{t}q_n - \frac{\pi}{t^2}\int_0^\infty \int_{-\infty}^\infty \sqrt{w^2 + F/s^2}\,\partial_t f_n dw dF \stackrel{(2.3)}{=}$$

$$-\frac{2}{t}\rho_n - \frac{1}{t}q_n - \left(\underbrace{\frac{\pi}{t^2}\int_0^\infty\int_{-\infty}^\infty e^{\mu_{n-1}-\lambda_{n-1}} w\partial_r f_n dwdF}_{=e^{\mu_{n-1}-\lambda_{n-1}}j_n'} - \int_0^\infty\int_{-\infty}^\infty \sqrt{w^2+F/s^2}\dot{\lambda}_{n-1} w\partial_w f_n dwdF - \right.$$

$$\left. -\int_0^\infty\int_{-\infty}^\infty e^{\mu_{n-1}-\lambda_{n-1}}\bar{\mu}_{n-1}\left(w^2+F/s^2\right)\partial_w f_n dwdF \right) \overset{p.i.}{=}$$

$$\overset{p.i.}{=} -\frac{2}{t}\rho_n - \frac{1}{t}q_n + e^{\mu_{n-1}-\lambda_{n-1}}j_n' -$$

$$-\dot{\lambda}_{n-1}\left(\underbrace{\int_0^\infty\int_{-\infty}^\infty \frac{w^2}{\sqrt{w^2+F/s^2}} f_n dwdF}_{=p_n} + \underbrace{\int_0^\infty\int_{-\infty}^\infty \sqrt{w^2+F/s^2} f_n dwdF}_{=\rho_n} \right) -$$

$$-2e^{\mu_{n-1}-\lambda_{n-1}}\bar{\mu}_{n-1}\underbrace{\int_0^\infty\int_{-\infty}^\infty w f_n dwdF}_{=j_n} =$$

$$= -\frac{2}{t}\rho_n - \frac{1}{t}q_n - e^{\mu_{n-1}-\lambda_{n-1}}j_n' - 2\bar{\mu}_{n-1}e^{\mu_{n-1}-\lambda_{n-1}}j_n - \dot{\lambda}_{n-1}\left(\rho_n + p_n\right)$$

Now we can express H_n by

$$H_n := e^{-2\lambda_n}\left(\left(\mu_n' + \lambda_n'\right)\bar{\mu}_n - 4\pi t e^{\mu_n+\lambda_n}j_n' + \bar{\mu}_n\left(\mu_n' - \lambda_n'\right)\right) -$$

$$-e^{-2\mu_n}\left(4\pi t e^{2\mu_n}\left(\frac{2\rho_n}{t} + \dot{\rho_n}\right) + 2\dot{\mu}_n\dot{\lambda}_n + \frac{\dot{\mu}_n - \dot{\lambda}_n}{t} + \Lambda e^{2\mu_n} + \left(\dot{\lambda}_n + 1/t\right)\left(\dot{\lambda}_n - \dot{\mu}_n\right)\right) + \Lambda =$$

$$= e^{-2\lambda_n}\left(\left(\mu_n' + \cancel{\lambda_n'}\right)\bar{\mu}_n - 4\pi t e^{\mu_n+\lambda_n}j_n' + \bar{\mu}_n\left(\mu_n' - \cancel{\lambda_n'}\right)\right) -$$

$$-e^{-2\mu_n}\left(4\pi t e^{2\mu_n}\left(\cancel{\frac{2\rho_n}{t}} - \cancel{\frac{2\rho_n}{t}} - \frac{1}{t}q_n - e^{\mu_{n-1}-\lambda_{n-1}}j_n' - 2\bar{\mu}_{n-1}e^{\mu_{n-1}-\lambda_{n-1}}j_n - \dot{\lambda}_{n-1}\left(\rho_n + p_n\right)\right) + \right.$$

$$\left. + \cancel{2}\dot{\mu}_n\dot{\lambda}_n + \cancel{\frac{\dot{\mu}_n - \dot{\lambda}_n}{t}} + \cancel{\Lambda e^{2\mu_n}} + \left(\dot{\lambda}_n + \cancel{1/t}\right)\left(\dot{\lambda}_n - \cancel{\dot{\mu}_n}\right)\right) + \cancel{\Lambda} =$$

$$= 2e^{-2\lambda_n}\bar{\mu}_n\mu_n' + 4\pi t j_n'\left(e^{\mu_{n-1}-\lambda_{n-1}} - e^{\mu_n-\lambda_n}\right) + 4\pi q_n + \underbrace{8\pi t\bar{\mu}_{n-1}e^{\mu_{n-1}-\lambda_{n-1}}j_n}_{=-2\bar{\mu}_{n-1}e^{-2\lambda_n}\left(e^{\mu_n-\mu_{n-1}+\lambda_n-\lambda_{n-1}}\bar{\mu}_n\right)}$$

$$+e^{-2\mu_n}\left(\dot{\lambda}_{n-1}\left(\dot{\lambda}_n + \dot{\mu}_n\right) + \dot{\lambda}_n\left(\dot{\lambda}_n + \dot{\mu}_n\right)\right)$$

$$= 2e^{-2\lambda_n}\bar{\mu}_n\left(\mu_n' - e^{\mu_{n-1}-\mu_n+\lambda_n-\lambda_{n-1}}\bar{\mu}_{n-1}\right) +$$

$$+4\pi t j_n' \left(e^{\mu_{n-1}-\lambda_{n-1}} - e^{\mu_n-\lambda_n}\right) +$$

$$+e^{-2\mu_n} \left(\dot\lambda_n + \dot\mu_n\right) \left(\dot\lambda_{n-1} - \dot\lambda_n\right) + 4\pi q_n$$

Now as we have established convergence of λ_n, μ_n and $\dot\lambda_n$ we know that the second and third term will vanish. But we still need $\mu_n' - \bar\mu_n \to 0$ to conclude that $H_n \to 4\pi q_n$. Now taking the r derivative of μ_n yields

$$\mu_n'(t,r) = \frac{e^{2\mu_n}}{t} \left(\tilde\mu' e^{-2\tilde\mu} + 4\pi \int_1^t s^2 p_n'(s,r) ds\right). \tag{4.20}$$

From the definiton of p_n we get

$$p_n' = \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{\sqrt{w^2+F/t^2}} \partial_r f_n dF dw.$$

Again expressing $\partial_r f$ using the Vlasov equation we get

$$\frac{w^2}{\sqrt{w^2+\frac{F}{t^2}}} \partial_r f_n = -e^{\lambda_{n-1}-\mu_{n-1}} w \partial_t f_n + \left(e^{\lambda_{n-1}-\mu_{n-1}} \dot\lambda_{n-1} w^2 + \bar\mu_{n-1} w \sqrt{w^2+\frac{F}{t^2}}\right) \partial_w f_n.$$

Inserting that into (4.20) yields

$$\mu_n'(t,r) = \frac{e^{2\mu_n}}{t} \left(\tilde\mu' e^{-2\tilde\mu} - 4\pi \int_1^t s^2 \left(\frac{\pi}{s^2} \int_{-\infty}^{\infty} \int_0^{\infty} e^{\lambda_{n-1}-\mu_{n-1}} w \partial_s f_n dF dw\right) ds + \right)$$

$$+4\pi \frac{e^{2\mu_n}}{t} \int_1^t s^2 \left(\frac{\pi}{s^2} \int_{-\infty}^{\infty} \int_0^{\infty} \left(e^{\lambda_{n-1}-\mu_{n-1}} \dot\lambda_{n-1} w^2 + \bar\mu_{n-1} w \sqrt{w^2+\frac{F}{s^2}}\right) \partial_w f_n dF dw\right) ds \overset{p.i.}{=}$$

$$\overset{p.i.}{=} \frac{e^{2\mu_n}}{t}\left(\tilde{\mu}'e^{-2\tilde{\mu}} + 4\pi\int_1^t s^2\frac{\pi}{s^2}\int_{-\infty}^{\infty}\int_0^{\infty}\partial_s\left(e^{\lambda_{n-1}-\mu_{n-1}}w\right)f_n dF dw ds\right) +$$

$$-\frac{e^{2\mu_n}}{t}4\pi e^{\lambda_{n-1}-\mu_{n-1}}s^2\underbrace{\frac{\pi}{s^2}\int_{-\infty}^{\infty}\int_0^{\infty}wf_n dF dw\Bigg|_1^t}_{=j_n-\tilde{j}_n} +$$

$$-4\pi\frac{e^{2\mu_n}}{t}\int_1^t s^2\frac{\pi}{s^2}\int_{-\infty}^{\infty}\int_0^{\infty}\partial_w\left(e^{\lambda_{n-1}-\mu_{n-1}}\dot{\lambda}_{n-1}w^2 + \bar{\mu}_{n-1}w\sqrt{w^2+\frac{F}{s^2}}\right)f_n dF dw ds =$$

$$\frac{e^{2\mu_n}}{t}\left(\tilde{\mu}'e^{-2\tilde{\mu}} + 4\pi e^{\lambda_{n-1}-\mu_{n-1}}\tilde{j}_n - 4\pi t^2 e^{\lambda_{n-1}-\mu_{n-1}}j_n\right) +$$

$$+4\pi\frac{e^{2\mu_n}}{t}\int_1^t e^{\lambda_{n-1}-\mu_{n-1}}\left(\dot{\lambda}_{n-1}-\dot{\mu}_{n-1}\right)s^2\underbrace{\frac{\pi}{s^2}\int_{-\infty}^{\infty}\int_0^{\infty}wf_n dF dw}_{j_n(s,r)}\,ds -$$

$$-4\pi\frac{e^{2\mu_n}}{t}\int_1^t 2s^2 e^{\lambda_{n-1}-\mu_{n-1}}\dot{\lambda}_{n-1}\underbrace{\frac{\pi}{s^2}\int_{-\infty}^{\infty}\int_0^{\infty}wf_n dF dw}_{j_n}\,ds -$$

$$-4\pi\frac{e^{2\mu_n}}{t}\int_1^t s^2\bar{\mu}_{n-1}\underbrace{\frac{\pi}{s^2}\int_{-\infty}^{\infty}\int_0^{\infty}\left(\sqrt{w^2+\frac{F}{s^2}}+\frac{w^2}{\sqrt{w^2+F/s^2}}\right)f_n dF dw}_{=\rho_n+q_n=\left(\dot{\lambda}_n+\dot{\mu}_n\right)e^{-2\mu_n}/(4\pi s)}\,ds =$$

$$= \frac{e^{2\mu_n}}{t}\left(\tilde{\mu}'e^{-2\tilde{\mu}} + 4\pi e^{\lambda_{n-1}-\mu_{n-1}}\tilde{j}_n - 4\pi t^2 e^{\lambda_{n-1}-\mu_{n-1}}j_n\right) +$$

$$+4\pi\frac{e^{2\mu_n}}{t}\int_1^t s^2 e^{\lambda_{n-1}-\mu_{n-1}}\left(\dot{\lambda}_{n-1}-\dot{\mu}_{n-1}\right)j_n ds -$$

$$-\frac{e^{2\mu_n}}{t}\int_1^t s\left(2e^{\lambda_{n-1}-\mu_{n-1}}\dot{\lambda}_{n-1}4\pi s j_n + \bar{\mu}_{n-1}\left(\dot{\lambda}_n+\dot{\mu}_n\right)e^{-2\mu_n}\right)ds \overset{(4.18)}{=}$$

$$\frac{e^{2\mu_n}}{t}\left(\tilde{\mu}'e^{-2\tilde{\mu}} + 4\pi e^{\lambda_{n-1}-\mu_{n-1}}\tilde{j}_n + te^{\lambda_{n-1}-\lambda_n-\mu_{n-1}-\mu_n}\bar{\mu}_n\right) -$$

$$-\frac{e^{2\mu_n}}{t}\int_1^t s\left(e^{\lambda_{n-1}-\mu_{n-1}-\lambda_n-\mu_n}\left(\cancel{\dot{\lambda}_{n-1}}-\dot{\mu}_{n-1}\right)\bar{\mu}_n - \cancel{2}e^{\lambda_{n-1}-\lambda_n-\mu_{n-1}-\mu_n}\dot{\lambda}_{n-1}\bar{\mu}_n\right)ds =$$

$$\frac{e^{2\mu_n}}{t}\left(\tilde{\mu}'e^{-2\tilde{\mu}} + 4\pi e^{\lambda_{n-1}-\mu_{n-1}}\tilde{j}_n\right) + e^{\lambda_{n-1}-\lambda_n-\mu_{n-1}+\mu_n}\bar{\mu}_n -$$

$$+\frac{e^{2\mu_n}}{t}\int_1^t se^{-2\mu_n}\left(e^{\lambda_{n-1}-\mu_{n-1}-\lambda_n+\mu_n}\left(\dot{\lambda}_{n-1}+\dot{\mu}_{n-1}\right)-\bar{\mu}_{n-1}\left(\dot{\lambda}_n+\dot{\mu}_n\right)\right)ds.$$

Taking the limit $n \to \infty$ the integral vanishes and the first exponential goes to 1 and we are left with

$$\mu_n' = \frac{e^{2\mu_n}}{t}\left(\tilde{\mu}'e^{-2\tilde{\mu}} + 4\pi e^{\lambda_{n-1}-\mu_{n-1}}\tilde{j}_n\right) + \bar{\mu}_n.$$

Now, for the initial values $\bar{\mu}_n(1,r) = \tilde{\mu}'_n(r)$ so the first term vanishes and we have $\bar{\mu}_n = \mu'_n$ for all times and thereby $\mu'_n - \bar{\mu}_{n-1} \to 0$ as needed to conclude that $H_n \to 4\pi q_n$ and thereby the convergence of $a_{n,5}$. In Lemma 4.2 we have shown that $\dot{\xi}$ and $\dot{\eta}$ are linear combinations of ξ and η but it is now easy to see that the same relation holds for the corresponding $\dot{\xi}_n$ and $\dot{\eta}_n$. Thus, the convergence of $\partial_{(r,w)}(R_n, W_n)(1,t,r,w,F)$ is established. Therefore, the characteristics $(R,W)(1,t,r,w,f)$ are differentiable with respect to r and w and so is f. Now, this implies that the moments ρ, p, j, q all are continuously differentiable with respect to r. The coefficients of the characteristic system are now continuously differentiable with respect to r, w and F and thus the characteristics $(R,W)(1,t,r,w,f)$ themselves are differentiable with respect to F and t. Thereby, we have established the regularity of the system (f, λ, μ).

Step 5: (uniqueness) In Step 3 we have established estimates on the difference of two consecutive iterates of f. The same argumentation can be used to control the difference of two solutions f and g outgoing from the same initial data. We get

$$sup\left\{||f(\tau) - g(\tau)||\,\middle|\tau \in [t,1]\right\} \le C\int_t^1 sup\left\{||f(\tau) - g(\tau)||\,\middle|\tau \in [t,1]\right\} ds$$

for any compact subinterval of $]0,1]$ and thus $f = g$ there.

Step 6: (continuation criterion) We will prove that the left maximal solution (f, λ, μ) of the Einstein-Vlasov system (2.3)-(2.11) exist on $]T,1]$ $T = 0$ now assuming that

$$P^\star := sup\left\{|w|\,\middle|(r,w,F) \in suppf(t), t \in]T,1]\right\} < \infty$$

We will therefore assume that the $T > 0$ and prove that if you have initial data prescribed at any point $t_0 \in]T,1]$ one can always establish an existence interval $[t_0 - \delta, t_0]$ with a $\delta > 0$ independent of t_0. There then, would be the contradiction to the assumption that T is the lower bound of the maximal existence interval.

In the previous steps, we have shown that the solution exists at least to the intersection of left maximal existence intervals of z_1 and z_2

$$z_1(t) = w_0 + c_2\int_t^{t_0} \frac{1}{s}(1 + z_1(s))^3 + \left(\frac{1}{2s} + \frac{1}{2c_4} + \frac{s^2\Lambda}{c_4}\right) z_1(s)ds$$

$$z_2(t) = ||\partial_{(r,w)}f(t_0)||e^{\int_t^{t_0} C_1(s)(c_5+z_2(s))ds}$$

with

$$w_0 := \sup_{(r,w,F)\in suppf(t_0)} |w| < \infty$$

$$c_4(\mu(t_0),\Lambda) := \inf e^{-2\mu(t_0)} + \min(k,0) - \frac{\Lambda}{3}$$

$$c_5 := ||e^{-2\mu(t_0)}\mu'(t_0)|| + ||\lambda'(t_0)|| + 1$$

$$c_2(f, F_0, \mu) := \frac{16\pi^2}{c_4}(1+F_0)^2||f||$$

and the function C_1 depends only on powers of z_1 Now $W_0 \leq P^\star$, $||f(t_0)|| = ||\tilde{f}||$ F_0 does not change because F is a constant along characteristics. (4.5) reads

$$e^{-2\mu} = \frac{e^{-2\tilde{\mu}}+k}{t} - k \underbrace{- \frac{8\pi}{t}\int_1^{t_0} s^2 p ds + \frac{\Lambda}{3t}\left(t^3-1\right)}_{\geq 0}$$

the last two terms are negative so in the case of t_0 we have

$$e^{-2\mu(t_0)} + k \geq \frac{e^{-2\tilde{\mu}}+k}{t_0}$$

and thereby that

$$c_4(\mu(t_0), \Lambda) \geq c_4(\tilde{\mu}, \Lambda)$$

Thus there are positive constants $c_2^\star$ and $c_6^\star$ for $t_0 \in]T, 1]$ and say $s \in [T/2, 1]$ such that

$$c_2(f(t_0), F_0, \mu(t_0))/s \leq c_2^\star$$

and

$$c_6^\star \geq \left(\frac{1}{2s} + \frac{1}{2c_4} + \frac{s^2\Lambda}{c_4}\right)$$

We then define $z_1^\star(t)$ to be the left maximal solution of

$$z_1^\star(t) := P^\star + c_2^\star \int_t^{t_0} \left(1 + z_1^\star(s)\right)^3 + \left(\frac{1}{2s} + \frac{1}{2c_4} + \frac{s^2\Lambda}{c_4}\right) z_1^\star(s) ds$$

Now, the coefficients $a_1, ..., a_5$ are uniformly bounded along the characteristics in the support of f is the case that $\bar{\mu} = \mu'$. From Lemma 4.2 we get an upper bound on the derivatives of f

$$D^\star := sup \left\{ ||\partial_{(r,w)} f(t)|| \, | T < t < 1 \right\} < \infty$$

Now, the same argumentation that let us establish bounds on $\mu_n'(t,r)$, $\dot{\lambda}_n'(t,r)$ and $\lambda_n'(t,r)$ can now be used to obtain a uniform bound $c_5^\star$ that satisfies $c_5(\mu(t_0), \lambda(t_0)) \leq c_5^\star$ from

$$\mu'(t,r) = \frac{e^{2\mu}}{t}\left(\tilde{\mu}' e^{-2\tilde{\mu}} + 4\pi \int_1^t s^2 p'(s,r) ds\right)$$

$$\dot{\lambda}'(t,r) = e^{2\mu}\left(8\pi t \mu'(t,r)\rho(t,r) + 4\pi t \rho'(t,r) - \frac{k + \Lambda t^2}{t}\mu'(t,r)\right)$$

$$\lambda'(t,r) = \tilde{\lambda}'(r) + \int_1^t \dot{\lambda}'(s,r) ds$$

Then, we define

$$C_1^\star(s) := C_1(z_1^\star(s))$$

and $z_2^\star$ as the left maximal solution of

$$z_2^\star = D^\star e^{\int_t^{t_0} C_1^\star(s)\left(c_5^\star + z_2^\star(s)\right)^3 ds}$$

Now, we can find a $\delta < T/2$ independet of t_0 such that $z_1^\star$ and $z_2^\star$ exist on $[t_0 - \delta, t]$. on this interval we then have $z_1 \leq z_1^\star$ and $z_2 \leq z_2^\star$ and thereby extended the interval of existence for z_1 and z_2 in the past of T contradicting the assumption that T is maximally chosen. □

We have completed the proof of the local existence theorem in the past direction and will now consider the future direction.

4.3 Local Existence in the Future Direction

We will now prove the existence for some time in the future, i.e. for $t \geq 1$. The argumentation is based on [11] in the case of massive particles and vanishing cosmological constant, or [8] in the case of positive cosmological constant.

Theorem 4.3. *There exists a unique, right maximal, regular solution to the system* (2.3)-(2.11) *with initial data* $(\tilde{f}, \tilde{\lambda}, \tilde{\mu})$ *on time interval* $[1, T[$ *with* $T > 1$. *If*

$$\sup \left\{ |w| \,\middle|\, (t, r, w, F) \in supp f \right\} < \infty,$$

and

$$\sup \left\{ e^{2\mu(t,r)} \,\middle|\, r \in [0,1[, t \in [1, T[\right\} < \infty$$

and there is no geodesic (r, w, F) *converging to* $(s, r, 0, 0)$ *when approaching some critical time* $T_c \in [1, T[$ *then* $T = \infty$

Proof. The proof is in most parts analogous to the one given for the past direction, so we will only point out the differences. For the future direction, we define the iterates the same way we defined them previously, only looking at (4.15) we see that its right hand side has to obey

$$\frac{\left(e^{-2\tilde{\mu}} + k\right)}{t} - k - \frac{8\pi}{t} \int_1^t s^2 p_n(s, r) ds + \frac{\Lambda}{3t}\left(t^3 - 1\right) > 0$$

and we will denote the supremum of the times for which the upper equation holds for all the previous iterates by T_n. Again we will define

$$P_n(t) := \sup \left\{ |w| : (r, w, f) \in supp f_n(t) \right\}$$

and now

$$Q_n(t) := \sup \left\{ e^{2\mu_n(s,r)} \,\middle|\, r \in [0,1], 1 \leq s \leq t \right\}$$

We establish bounds on the matter terms

$$||p_n(t)||, ||\rho_n(t)||, ||j_n(t)|| \leq c \left(1 + F_0\right)^2 ||\tilde{f}|| \left(1 + P_n(t)\right)^2 \frac{1}{t^2}$$

$$\left|e^{\mu_n-\lambda_n}\bar{\mu}_n(s,r)\right| \leq 4\pi t e^{\mu_n-\lambda_n}e^{\mu_n+\lambda_n}\left|j_n(t,r)\right| =$$

$$= 4\pi s e^{2\mu_n}|j_n(s,r)| \leq \frac{4\pi c}{s}(1+F_0)^2\,||\tilde{f}||\left(1+P_n(s)\right)^2 Q_n(s)$$

and

$$\left|\dot{\lambda}_n(s,r)\right| \leq 4\pi s e^{2\mu_n}\rho_n(s,r) + \frac{1+\left(1+\Lambda s^2\right)e^{2\mu_n}}{2s} \leq$$

$$\leq \frac{4\pi c}{s}(1+F_0)^2\,||\tilde{f}||\left(1+P_n(s)\right)^2 + \frac{1}{2} + \frac{1+s\Lambda}{2}Q_n(s)$$

So this time from

$$\left|\dot{W}_{n+1}(s)\right| = \left|-\dot{\lambda}_n W_{n+1} - e^{\mu_n-\lambda_n}\bar{\mu}_n\sqrt{W_{n+1}^2+\frac{F}{s^2}}\right|$$

we get the estimates

$$\left|\dot{W}_{n+1}(s)\right| \leq \frac{4\pi c}{s}(1+F_0)^2\,||\tilde{f}||\left(1+P_n(s)\right)^2\left|W_{n+1}(s)\right| + \frac{1}{2} + \frac{1+s\Lambda}{2}Q_n(s)\left|W_{n+1}(s)\right| +$$

$$+\frac{4\pi c}{s}(1+F_0)^2\,||\tilde{f}||\left(1+P_n(s)\right)^2 Q_n(s)\underbrace{\sqrt{W_{n+1}^2+\frac{F}{s^2}}}_{\leq\left(1+\left|W_{n+1}\right|\right)(1+F_0)} \leq$$

$$\leq \frac{4\pi c}{s}(1+F_0)^3\,||\tilde{f}||\left(1+P_n(s)\right)^2\left(1+\left|W_{n+1}(s)\right|\right)\left(1+Q_n(s)\right) + \frac{1}{2} + \frac{1+s\Lambda}{2}Q_n(s)\left|W_{n+1}(s)\right| \leq$$

$$\leq \frac{4\pi c}{s}(1+F_0)^3\,||\tilde{f}||\left(1+P_n(s)\right)^2\left(1+P_{n+1}(s)\right)\left(1+Q_n(s)\right) + \frac{1}{2} + \frac{1+s\Lambda}{2}Q_n(s)P_{n+1}(s),$$

which brings us to the integral inequality

$$P_{n+1}(s) \leq w_0 + 4\pi c\,(1+F_0)^3\,||\tilde{f}||\int_1^t\left[\frac{1}{s}\left(1+P_n(s)\right)^2\left(1+P_{n+1}(s)\right)\left(1+Q_n(s)\right) + \right.$$

$$\left. + \frac{1}{2} + \frac{1+s\Lambda}{2}Q_n(s)P_{n+1}(s)\right]ds$$

From (4.15) by differentiating by s we get

$$\dot{\mu} = 8\pi t e^{2\mu}p - \frac{1+\left(k-\Lambda t^2\right)e^{2\mu}}{2t}$$

multiplying by $e^{2\mu}$ yields

$$\frac{d}{dt}\left(e^{2\mu}\right) = 8\pi t e^{4\mu}p - \frac{e^{2\mu}+\left(k-\Lambda t^2\right)e^{4\mu}}{2t}$$

now integrating we get

$$e^{2\mu} = e^{2\tilde{\mu}} + \int_1^t\left[8\pi t e^{4\mu}p - \frac{e^{2\mu}+\left(k-\Lambda t^2\right)e^{4\mu}}{2t}\right]ds$$

and thereby the estimate

$$Q_n(s) \leq ||e^{2\tilde{\mu}}|| + c\left(1+F_0\right)^2 ||\tilde{f}|| \int_1^t \left[\left(1+P_n(s)\right)^2 \frac{1}{s^2}\left(1+Q_n(s)\right)^2 + \Lambda t Q_n(s)\right] ds$$

These integral equations let us now set up an system, which define two functions $z_1(s)$ and $z_2(s)$ that will serve us as estimates against $P_n(s)$ and $Q_n(s)$.

$$z_1(t) = w_0 + 4\pi c\left(1+F_0\right)^3 ||\tilde{f}|| \int_1^t \left[\frac{1}{s}\left(1+z_1(s)\right)^3\left(1+z_2(s)\right) + \frac{1}{2} + \frac{1+s\Lambda}{2} z_2(s) z_1(s)\right] ds$$

$$z_2(t) = ||e^{2\tilde{\mu}}|| + c\left(1+F_0\right)^2 ||\tilde{f}|| \int_1^t \left[\frac{1}{s^2}\left(1+z_1(s)\right)^2\left(1+z_2(s)\right)^2 + \Lambda s z_2(s)\right] ds$$

Now, by construction we have established bounds on $P_n(s)$ and $Q_n(s)$ and can state that the right maximal existence interval $[1, T[$ of the system (z_1, z_2) is exceeded by T_n. □

Chapter 5

Behaviour towards Zero Momentum

In the case of massless matter we are interested in the behaviour of the distribution function when getting close to the point of zero momentum. If the distribution function $f(t, r, w, F)$ would converge to non-zero values approaching the tip of the mass-shell this would have unphysical consequences as well as it would cause mathematical problems.

F is a conserved quantity along the characteristics so only geodesics with $F = 0$ can converge to $|v| = 0$. We want to show that for these geodesics one can always find an interval $(w_+, w_-) \ni w$ and 0 such that $(t, r, w, 0)$ is not contained in the support of $f(t, r, w, 0)$ for all t and r.

Theorem 5.1. *We consider the Einstein-Vlasov system* (2.3)-(2.11) *on an maximal interval* $(t_a, t_b) \ni 1$. *For every characteristic* $(r, w, F)(t)$ *with* $w(t) \neq 0$ *of the Einstein-Vlasov system* (2.3)-(2.11) *there is no time* $t_c \in]0, \infty[$ *for which*

$$\lim_{t \mapsto t_c} (r, w, F)(t) = (r, 0, 0)$$

under the assumption that

$$\sup \left\{ e^{2\mu(t,r)} \,\middle|\, r \in \mathbb{R}, t \in (t_a, t_b) \right\} < \infty.$$

Furthermore if $F = 0$ *there exist* C^1*-functions* $I(t)$ *and* $O(t)$ *and positive real numbers* K_I *and* K_0 *such that*

$$|w(t)| \leq K_O e^{O(t)}$$

and

$$|w(t)| \geq K_I e^{I(t)}$$

on (t_a, t_b) *for all characteristics.*

As F is constant along characteristics we only need to check if the ones with $F = 0$ will have a vanishing w component at some time. From the Vlasov equation (2.3) and the equations (2.4) − (2.6) we get for a characteristic with $F = 0$ the time evolution of w reduces to

$$\dot{w} = \frac{4\pi^2 e^{2\mu}}{s} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\tilde{w}|w| - w\sqrt{\tilde{w}^2 + \frac{\tilde{F}}{s^2}} \right) f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu}.$$

For $w \geq 0$

$$\dot{w} = \frac{4\pi^2 we^{2\mu}}{s} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\tilde{w} - \sqrt{\tilde{w}^2 + \frac{\tilde{F}}{s^2}} \right) f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu} =$$

$$= \frac{4\pi^2 we^{2\mu}}{s} \int_{P_-(s)}^{P_+(s)} \int_0^{F_0} \left(\tilde{w} - \sqrt{\tilde{w}^2 + \frac{\tilde{F}}{s^2}} \right) f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu} =$$

$P_\pm(s)$ denotes the upper or the lower bound on the value w in the support of f for a given time s respectively.

For the integrand we get estimates

$$0 \geq \left(\tilde{w} - \sqrt{\tilde{w}^2 + \frac{\tilde{F}}{s^2}} \right) f \geq \left(\tilde{w} - |\tilde{w}| - \frac{\sqrt{\tilde{F}}}{s} \right) f \geq \left(\tilde{w} - |\tilde{w}| - \frac{\sqrt{\tilde{F}}}{s} \right) ||f_0||_\infty$$

Therefore, we can establish bounds for the integral

$$0 \geq \int_{-\infty}^{\infty} \int_0^{\infty} \left(\tilde{w} - \sqrt{\tilde{w}^2 + \frac{\tilde{F}}{s^2}} \right) f d\tilde{F} d\tilde{w} \geq \int_{-\infty}^{\infty} \int_0^{\infty} \left(\tilde{w} - |\tilde{w}| - \frac{\sqrt{\tilde{F}}}{s} \right) f d\tilde{F} d\tilde{w} =$$

$$= \int_{-\infty}^{0} \int_0^{\infty} \left(2\tilde{w} - \frac{\sqrt{\tilde{F}}}{s} \right) f d\tilde{F} d\tilde{w} + \int_0^{\infty} \int_0^{\infty} \left(-\frac{\sqrt{\tilde{F}}}{s} \right) f d\tilde{F} \tilde{w} \geq$$

$$\geq ||f_0||_\infty \left(\int_{\min(P_-,0)}^{0} \int_0^{F_0} \left(2\tilde{w} - \frac{\sqrt{\tilde{F}}}{s} \right) d\tilde{F} d\tilde{w} + \int_0^{\max(P_+,0)} \int_0^{F_0} \left(-\frac{\sqrt{\tilde{F}}}{s} \right) d\tilde{F} d\tilde{w} \right) =$$

$$= ||f_0||_\infty \left(-\tilde{P}_-^2 F_0 + \frac{2\sqrt{F_0}^3 \tilde{P}_-}{3s} - \frac{2\sqrt{F_0}^3 \tilde{P}_+}{3s} \right)$$

We can conclude that

$$\underbrace{\frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu}}_{:= o_+(s) w} \geq \dot{w}$$

and

$$\dot{w} \geq \underbrace{\frac{4\pi^2 we^{2\mu}}{s}||f_0||_\infty \left(-\tilde{P}_-^2 F_0 + \frac{2\sqrt{F_0}^3\tilde{P}_-}{3s} - \frac{2\sqrt{F_0}^3\tilde{P}_+}{3s}\right) + \frac{1+ke^{2\mu}}{2s}w - \frac{\Lambda}{2}swe^{2\mu}}_{:=i_+(s)w}$$

The second inequality is now of special interest as it bounds the velocity of w from below when approaching zero. But the other inequality tells us that in the case of $F = 0$ $\tilde{P}_+$ will never diverge as $\dot{w}$ is bounded form above. In the case of $w < 0$ we have

$$\dot{w} = \frac{4\pi^2 we^{2\mu}}{s}\int_{-\infty}^{\infty}\int_0^{\infty}\left(-\tilde{w} - \sqrt{\tilde{w}^2 + \frac{\tilde{F}}{s^2}}\right) f d\tilde{F} d\tilde{w} + \frac{1+ke^{2\mu}}{2s}w - \frac{\Lambda}{2}swe^{2\mu}$$

analogous to the positive case we can sandwich the integral

$$0 \geq \int_{-\infty}^{\infty}\int_0^{\infty}\left(-\tilde{w} - \sqrt{\tilde{w}^2 + \frac{\tilde{F}}{s^2}}\right) f d\tilde{F} d\tilde{w} \geq \int_{-\infty}^{\infty}\int_0^{\infty}\left(-\tilde{w} - |\tilde{w}| - \frac{\sqrt{\tilde{F}}}{s}\right) f d\tilde{F} d\tilde{w} =$$

$$= \int_0^{\infty}\int_0^{\infty}\left(-2\tilde{w} - \frac{\sqrt{\tilde{F}}}{s}\right) f d\tilde{F} d\tilde{w} + \int_{-\infty}^{0}\int_0^{\infty}\left(-\frac{\sqrt{\tilde{F}}}{s}\right) f d\tilde{F}\tilde{w} \geq$$

$$\geq ||f_0||_\infty \left(\int_0^{\max(P_+,0)}\int_0^{F_0}\left(-2\tilde{w} - \frac{\sqrt{\tilde{F}}}{s}\right) d\tilde{F} d\tilde{w} + \int_{\min(P_-,0)}^{0}\int_0^{F_0}\left(-\frac{\sqrt{\tilde{F}}}{s}\right) d\tilde{F} d\tilde{w}\right) =$$

$$= ||f_0||_\infty \left(-\tilde{P}_+^2 F_0 + \frac{2\sqrt{F_0}^3\tilde{P}_-}{3s} - \frac{2\sqrt{F_0}^3\tilde{P}_+}{3s}\right)$$

and thereby we can also sandwich $\dot{w}$

$$i_-(s)w \geq \dot{w} \geq o_-(s)w$$

with

$$o_-(s) = \frac{1+ke^{2\mu}}{2s} - \frac{\Lambda}{2}se^{2\mu}$$

$$i_-(s) := \frac{4\pi^2 e^{2\mu}}{s}||f_0||_\infty \left(-\tilde{P}_+^2 F_0 + \frac{2\sqrt{F_0}^3\tilde{P}_-}{3s} - \frac{2\sqrt{F_0}^3\tilde{P}_+}{3s}\right) + \frac{1+ke^{2\mu}}{2s} - \frac{\Lambda}{2}se^{2\mu}$$

Here the $i_-(s)w \geq \dot{w}$ is the inequality of interest for the behaviour towards zero momentum. Here again the other inequality is of importance too, because it states that $\tilde{P}_-$ cannot diverge towards $-\infty$ in finite time.

We see that the coefficients to the outer bounds $o_+(s) =: o(s)$ and $o_-(s)$ are equal. So, we get inequalities for the outer bonds on $\dot{w}$

$$o(s)w \geq \dot{w} o(s)w \leq \dot{w} \tag{5.1}$$

Since $\dot{w}$ exists for all times greater than t_0 so does its antiderivative w and P_+ and P_-. So, all the functions contained in the sandwich are continuous on every interval in the future of t. Therefore, we can conclude by the fundamental theorem of calculus that there is an antiderivative $O(s)$ to the coefficient $o(s)$ of w. Now, by solving the differential equation we get outer bounds for w

$$|w(t)| \leq \left|w(t_0)\right| K e^{O(t)}$$

with

$$K := e^{O(t_0)}.$$

The coefficients of the inner bounds $i_\pm(s)$ for the positive and the negative case respectively on $\dot{w}$ are

$$i_\pm(s) := \left[\frac{4\pi^2 e^{2\mu}}{s} ||f_0||_\infty \left(-\tilde{P}_\pm^2 F_0 + \frac{2\sqrt{F_0}^3 \tilde{P}_-}{3s} - \frac{2\sqrt{F_0}^3 \tilde{P}_+}{3s} \right) + \frac{1 + ke^{2\mu}}{2s} - \frac{\Lambda}{2} s e^{2\mu} \right]$$

We now know that $\tilde{P}_\pm(t)$ are bounded so the whole expression stays bounded for all times in the future.

Chapter 6

Future Global Existence

We have shown that there are local solutions to this Einstein-Vlasov system. Now we will investigate the obvious question, if there are global solutions to this system in either the past or future direction.
We will show that global existence can be proven in the future direction. Under which circumstances if at all, one can achieve existence towards $t = 0$ remains an open question. The main theorem of the chapter reads:

Theorem 6.1. *For initial data and conditions as in Theorem 4.2 the solution of the Einstein-Vlasov system for the massless case* (2.3)-(2.11) *with positive cosmological constant and spherical, plane or hyperbolic symmetry, written in areal coordinates, exists for all* $t \in [1, \infty[$ *where* t *denotes the area radius of the surfaces of symmetry of the induced spacetime.*

In this segment, we will investigate under which circumstances the requirement for global existence of Theorem 4.3 is fulfilled, that is to investigate under which conditions a bound on μ and w for all times can be established. To achieve this goal, we will establish a series of bounds asymptotes for various objects of interest. The approach follows the methods of Tchapnda and Rendall [7] and Rein [11].

Lemma 6.1. *Consider a right maximal regular solution* (f, λ, μ) *of the Einstein-Vlasov system* (2.3)-(2.11) *under the conditions of Lemma 4.2 on an existence interval* $[1, T[$ *and if*

$$\sup \left\{ e^{2\mu(t,r)} \,\middle|\, r \in \mathbb{R}, t \in [1, T[\right\} < \infty$$

then $T = \infty$.

Proof. By Theorem 4.3 we have to show is that under the assumption that we have a bound on $e^{2\mu}$ we can establish a bound on

$$\sup \left\{ |w| \,\middle|\, (t, r, w, F) \in suppf \right\}.$$

We define

$$P_+(t) := \max\left\{0, \sup\left\{w \,\middle|\, (t,r,w,F) \in suppf\right\}\right\}$$

$$P_-(t) := \min\left\{0, \inf\left\{w \,\middle|\, (t,r,w,F) \in suppf\right\}\right\}$$

for some time $t \in [1, T[$.

$$\dot{w} = \frac{4\pi^2 e^{2\mu}}{s} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\tilde{w}\sqrt{w^2 + \frac{F}{s^2}} - w\sqrt{\tilde{w}^2 + \frac{\tilde{F}}{s^2}} \right) f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu} =$$

in the case of positive w and because f is regular in particular $f = 0$ if w and F vanish we have

$$= \frac{4\pi^2 e^{2\mu}}{s} \int_{-\infty}^{\infty} \int_0^{F_0} \frac{\tilde{w}^2 \left(\cancel{w^2} + \frac{F}{s^2} \right) - w^2 \left(\cancel{\tilde{w}^2} + \frac{\tilde{F}}{s^2} \right)}{\tilde{w}\sqrt{w^2 + \frac{F}{s^2}} + w\sqrt{\tilde{w}^2 + \frac{\tilde{F}}{s^2}}} f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu} \leq$$

$$\leq \frac{4\pi^2 e^{2\mu}}{s} \int_0^{P_+(s)} \int_0^{F_0} \frac{\tilde{w}^2 F}{\tilde{w}|w|} f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu} =$$

$$= \frac{4\pi^2 e^{2\mu}}{s} \int_0^{P_+(s)} \int_0^{F_0} \frac{\tilde{w} F}{w} f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu} \leq$$

$$\leq \frac{\pi^2 e^{2\mu} ||f|| F_0^2 P_+^2(s)}{sw} + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu} =$$

$$=: \frac{\kappa_1 P_+^2(s)}{sw} + \left(\frac{\kappa_2}{s} + \kappa_3 s \right) w.$$

The coefficients κ_i with $i = 1, 2, 3$ are bounded functions of time by assumption. So we can state that

$$w(t) \leq w_0 + \int_1^t \left[\frac{\kappa_1 P_+^2(s)}{sw} + \left(\frac{\kappa_2}{s} + \kappa_3 s \right) w \right] ds.$$

Now, because this argumentation is true for all characteristics of the system, we can certainly find a characteristic that is some $\delta > 0$ arbitrarily small away from $P_+(s)$ so

$$w(t) \geq P_+(s) - \delta$$

on a non-vanishing time interval $[1, t_1[$. Combining some characteristics that have this property on the time axis such that the time intervals $[t_i, t_{i+1}]$ fulfill

$$\bigcup_i [t_i, t_{i+1}] = [1, T[$$

Because they are sufficiently close to $P_+(s)$ we can state that

$$P_+(t) \leq w_0 + \int_1^t \left[\frac{\kappa_1 P_+(s)}{s} + \left(\frac{\kappa_2}{s} + \kappa_3 s \right) P_+(s) \right] ds + K(t, \delta)$$

with some continuous real function $K(t, \delta) > 0$ and sufficiently big to compensate for the small error due to δ. In the negative case, this computation yields a lower bound on $P_-(t)$ in an analogous manner so the proof is complete. □

We will now work towards the global existence theorem by showing that $e^{2\mu}$ can be bounded for all times.

Lemma 6.2. *Let (f, λ, μ) be right maximal regular solutions of the Einstein-Vlasov system (2.3)-(2.9) with initial conditions $\tilde{f}(r, w, F) \geq 0 \in C^1(\mathbb{R}^2 \times [t_0, \infty[)$, $\tilde{\mu} \in C^2(\mathbb{R})$ and $\tilde{\lambda} \in C^1(\mathbb{R})$, f, λ, μ entirely 1-periodic in r*

$$e^{2\mu} = \left[\frac{t_0\left(e^{2\tilde{\mu}} + k\right)}{t} - k - \frac{8\pi}{t}\int_{t_0}^{t} s^2 p(s,r)ds + \frac{\Lambda}{3t}\left(t^3 - t_0^3\right)\right]^{-1} \geq \frac{t}{C - kt + \frac{\Lambda}{3}t^3}, t \in [t_0, T[\tag{6.1}$$

With a constant

$$C = t_0\left(e^{2\tilde{\mu}} + k\right). \tag{6.2}$$

Proof. By integration of

$$e^{-2\mu}(2t\dot{\mu} - 1) - k + \Lambda t^2 = 8\pi t^2 p \Rightarrow$$

we obtain

$$\Rightarrow \int_{t_0}^{t}\left[\underbrace{e^{-2\mu}(2s\dot{\mu} - 1)}_{=\frac{d}{ds}\left(-se^{-2\mu}\right)} - k + \Lambda s^2\right] ds = \int_{t_0}^{t} 8\pi s^2 pds \Leftrightarrow$$

$$\Leftrightarrow t_0 e^{-2\tilde{\mu}} - te^{-2\mu} - k(t - t_0) + \frac{\Lambda}{3}(t^3 - t_0^3) = \int_{t_0}^{t} 8\pi s^2 pds \Leftrightarrow$$

$$\Leftrightarrow t_0(e^{-2\tilde{\mu}} + k) - te^{-2\mu} - kt + \frac{\Lambda}{3}(t^3 - t_0^3) = \int_{t_0}^{t} 8\pi s^2 pds \Leftrightarrow$$

$$\Leftrightarrow -te^{-2\mu} = \int_{t_0}^{t} 8\pi s^2 pds - \frac{\Lambda}{3}(t^3 - t_0^3) + kt - t_0(e^{-2\tilde{\mu}} + k) \Leftrightarrow$$

$$\Leftrightarrow e^{-2\mu} = -\frac{1}{t}\int_{t_0}^{t} 8\pi s^2 pds + \frac{\Lambda}{3t}(t^3 - t_0^3) - k + \frac{t_0(e^{-2\tilde{\mu}} + k)}{t} \Leftrightarrow$$

$$\Leftrightarrow e^{2\mu} = \left[\underbrace{-\frac{1}{t}\int_{t_0}^{t} 8\pi s^2 pds}_{\geq 0} + \frac{\Lambda}{3t}(t^3 - t_0^3) - k + \frac{t_0(e^{-2\tilde{\mu}} + k)}{t}\right]^{-1} \geq \frac{t}{C - kt + \frac{\Lambda t^3}{3}}$$

with a positive constant $C := t_0(e^{-2\tilde{\mu}} + k)$. □

C does not depend on Λ.

Lemma 6.3. *All the functions having the same regularity as in Lemma 6.2, then*

$$\int_0^1 e^{\mu+\lambda}\rho(t,r)dr \leq Ct^{-3}, t \in [t_0, T[\tag{6.3}$$

holds.

Proof. We compute the time derivative

$$\frac{d}{dt}\int_0^1 e^{\mu+\lambda}\rho(t,r)dr =$$

$$= \int_0^1 \left[e^{\mu+\lambda}\left(\dot{\mu}+\dot{\lambda}\right)\rho(t,r) + e^{\mu+\lambda}\dot{\rho}(t,r)\right]dr. \tag{6.4}$$

The time derivative of $\langle p\rangle$ reads

$$\dot{\langle p\rangle} = \frac{d}{dt}\sqrt{w^2+F/t^2} = \frac{\frac{-F}{t^3}}{\langle p\rangle} = -\frac{1}{t^3}\frac{F}{\langle p\rangle}.$$

With this we can compute the time derivative of ρ

$$\dot{\rho}(t,r) = \frac{d}{dt}\left(\frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\langle p\rangle f(t,r,w,F)dFdw\right) =$$

$$= -\frac{1}{t}\underbrace{\frac{2\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\langle p\rangle fdFdw}_{=2\rho} - \frac{1}{t}\underbrace{\frac{\pi}{t^4}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{F}{\langle p\rangle}fdFdw}_{=q} + \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\langle p\rangle\partial_t fdFdw. \tag{6.5}$$

For the last term appearing in the integral one can make use of the Vlasov equation (2.3)

$$\partial_t f = -\frac{e^{\mu-\lambda}w}{\sqrt{w^2+F/t^2}}\partial_r f + \left(\dot{\lambda}w + e^{\mu-\lambda}\mu'\sqrt{w^2+F/t^2}\right)\partial_w f$$

$$\frac{\pi}{t^2}\int_0^1 e^{\mu+\lambda}\int_{-\infty}^{\infty}\int_0^{\infty}\langle p\rangle\partial_t fdFdwdr \overset{(2.3)}{=}$$

$$\overset{(2.3)}{=} \frac{\pi}{t^2}\int_0^1 e^{\mu+\lambda}\int_{-\infty}^{\infty}\int_0^{\infty}\langle p\rangle\left(-\frac{e^{\mu-\lambda}w}{\sqrt{w^2+F/t^2}}\partial_r f + \left(\dot{\lambda}w + e^{\mu-\lambda}\mu'\sqrt{w^2+F/t^2}\right)\partial_w f\right)dFdwdr =$$

$$= \frac{\pi}{t^2}\int_0^1 e^{\mu+\lambda}\int_{-\infty}^{\infty}\int_0^{\infty}\left(-e^{\mu-\lambda}w\partial_r f + \left(\dot{\lambda}w\langle p\rangle + e^{\mu-\lambda}\mu'\langle p\rangle^2\right)\partial_w f\right)dFdwdr =$$

$$= \frac{\pi}{t^2}\int_0^1\int_{-\infty}^{\infty}\int_0^{\infty}\left(-e^{2\mu}w\partial_r f + \left(e^{\mu+\lambda}\dot{\lambda}w\langle p\rangle + e^{2\mu}\mu'\langle p\rangle^2\right)\partial_w f\right)dFdwdr \overset{p.i}{=}$$

$$\overset{p.i}{=} \frac{\pi}{t^2}\int_0^1\int_{-\infty}^{\infty}\int_0^{\infty}\left(\partial_r\left(e^{2\mu}w\right)f - \partial_w\left(e^{\mu+\lambda}\dot{\lambda}w\langle p\rangle + e^{2\mu}\mu'\langle p\rangle^2\right)f\right)dFdwdr =$$

$$= \frac{\pi}{t^2}\int_0^1\int_{-\infty}^{\infty}\int_0^{\infty}\left(\cancel{e^{2\mu}2\mu' w} - e^{\mu+\lambda}\dot{\lambda}\left(\langle p\rangle + \frac{w^2}{\langle p\rangle}\right) - \cancel{e^{2\mu}\mu' 2w}\right)fdFdwdr =$$

$$= \int_0^1 \left(e^{\mu+\lambda} \dot{\lambda} \left(\rho + p\right) \right) f dr.$$

From equation (2.4) and (2.5) respectively we get

$$e^{-2\mu}(2t\dot{\lambda} + 1) + k - \Lambda t^2 = 8\pi t^2 \rho \Leftrightarrow$$

$$\Leftrightarrow \dot{\lambda} = 4\pi t \rho e^{2\mu} - \frac{ke^{2\mu}}{2t} + \frac{\Lambda e^{2\mu}}{2t} - \frac{1}{2t}$$

and

$$e^{-2\mu}(2t\dot{\mu} - 1) - k + \Lambda t^2 = 8\pi t^2 p \Leftrightarrow$$

$$\Leftrightarrow \dot{\mu} = \frac{1}{2t} \left(8\pi t^2 p e^{2\mu} + k e^{2\mu} - \Lambda e^{2\mu} + 1 \right)$$

$$\Rightarrow \dot{\lambda} + \dot{\mu} = \frac{1}{2t} 8\pi t^2 e^{2\mu} \left(p + \rho\right) = \frac{1}{t} 4\pi t^2 e^{2\mu} \left(p + \rho\right). \tag{6.6}$$

Using these results in the integral above

$$\int_0^1 e^{\mu+\lambda} \dot{\lambda} \left(\rho + p\right) f dr \overset{(6.6)}{=} -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \frac{\rho + p}{2} \left(-8\pi t^2 \rho e^{2\mu} + 1 + k e^{2\mu} - \Lambda e^{2\mu} \right) f dr.$$

Together with 2ρ and q in (6.5) we get the expression:

$$-\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[2\rho + q + \frac{\rho + p}{2} \left(-8\pi t^2 \rho e^{2\mu} + 1 + k e^{2\mu} - \Lambda e^{2\mu} \right) \right] dr,$$

and with (6.4):

$$-\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[-\left(\dot{\mu} + \dot{\lambda} \right) t \rho(t,r) + 2\rho + q + \frac{\rho + p}{2} \left(-8\pi t^2 \rho e^{2\mu} + 1 + k e^{2\mu} - \Lambda e^{2\mu} \right) \right] dr =$$

$$= -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[-4\pi t^2 e^{2\mu} \left(p + \rho\right) \rho(t,r) + 2\rho + q + \frac{\rho + p}{2} \left(-8\pi t^2 \rho e^{2\mu} + 1 + k e^{2\mu} - \Lambda e^{2\mu} \right) \right] dr =$$

$$= -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[\cancel{-4\pi t^2 e^{2\mu} \left(p + \rho\right) \rho(t,r)} + 2\rho + q + \frac{\rho + p}{2} \left(\cancel{-8\pi t^2 \rho e^{2\mu}} + 1 + k e^{2\mu} - \Lambda e^{2\mu} \right) \right] dr =$$

$$= -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[2\rho + q + \frac{\rho + p}{2} \left(1 + k e^{2\mu} - \Lambda t^2 e^{2\mu} \right) \right] dr.$$

So we have

$$\frac{d}{dt} \int_0^1 e^{\mu+\lambda} \rho(t,r) dr = -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[2\rho + q + \frac{\rho + p}{2} \left(1 + k e^{2\mu} - \Lambda e^{2\mu} \right) \right] dr. \tag{6.7}$$

Using (6.1) yields

$$1 + k e^{2\mu} - \Lambda e^{2\mu} \overset{(6.1)}{\leq} 1 + \frac{kt - \Lambda t^3}{C - kt + \frac{\Lambda t^3}{3}} = \frac{C - \frac{2}{3}\Lambda t^3}{C - kt + \frac{1}{3}\Lambda t^3}. \tag{6.8}$$

We destinguish cases:

Case 1: If $t \geq \sqrt[3]{\frac{3C}{2\Lambda}}$ the expression on the right is negative and therefore the left hand side of the inequality is too. Together with the fact that $p, q \geq 0$ we get from (6.7) that

$$\frac{d}{dt}\int_0^1 e^{\mu+\lambda}\rho(t,r)dr \leq -\frac{1}{t}\int_0^1 e^{\mu+\lambda}\left[2\rho + \frac{\rho}{2}\left(1 + ke^{2\mu} - \Lambda e^{2\mu}\right)\right]dr. \qquad (6.9)$$

Defining $C'(\Lambda) := \frac{3}{\Lambda}(3C - 2k)$ we have the estimate

$$1 + ke^{2\mu} - \Lambda e^{2\mu} \leq 1 + \frac{kt - \Lambda t^3}{C - kt + \frac{\Lambda t^3}{3}} \leq C'(\Lambda)t^{-2} - 2,$$

and combining this with (6.9) yields

$$\frac{d}{dt}\int_0^1 e^{\mu+\lambda}\rho(t,r)dr \leq -\frac{3}{t}\int_0^1 e^{\mu+\lambda}\rho dr + \frac{C'(\Lambda)}{2t^3}\int_0^1 e^{\mu+\lambda}\rho dr. \qquad (6.10)$$

When multiplied with t^3 this equation gives

$$\frac{d}{dt}\left[t^3\int_0^1 e^{\mu+\lambda}\rho(t,r)dr\right] \leq \frac{C'(\Lambda)}{2}t^{-3}\left[t^3\int_0^1 e^{\mu+\lambda}\rho(t,r)dr\right] \Rightarrow$$

$$\Rightarrow \int_{T_0}^t \frac{d}{ds}\left[s^3\int_0^1 e^{\mu+\lambda}\rho(s,r)dr\right]ds \leq \int_{T_0}^t \frac{C's^{-3}}{2}\left[s^3\int_0^1 e^{\mu+\lambda}\rho(s,r)dr\right]ds \Leftrightarrow$$

$$\Leftrightarrow t^3\int_0^1 e^{\mu+\lambda}\rho(t,r)dr \leq C'' + \int_{T_0}^t \frac{C'(\Lambda)}{2}s^{-3}\left[s^3\int_0^1 e^{\mu+\lambda}\rho(s,r)dr\right]ds$$

with $C'' := T_0^3\int_0^1 e^{\mu+\lambda}\rho(T_0,r)ds$. Now, the Grönvall's inequality implies that

$$t^3\int_0^1 e^{\mu+\lambda}\rho(t,r)dr \leq Ke^{\int_0^1 e^{\mu+\lambda}\rho(s,r)dr} \Leftrightarrow$$

$$\Leftrightarrow \int_0^1 e^{\mu+\lambda}\rho(t,r)dr \leq Kt^{-3}.$$

Case 2: If $t < \sqrt[3]{\frac{3C}{2\Lambda}}$ from (6.7) we get that, since $q \geq 0$, $\rho \geq p$ and $k - \Lambda t^2 \leq 0$

$$\underbrace{\frac{d}{dt}\int_0^1 e^{\mu+\lambda}\rho(t,r)dr}_{} \overset{(6.7)}{\leq}$$

$$\stackrel{(6.7)}{\leq} -\frac{2}{t}\int_0^1 e^{\mu+\lambda}\rho(t,r)dr+\frac{1}{t}\int_0^1 e^{\mu+\lambda}\frac{\rho+p}{2}dr+\frac{1}{t}\int_0^1 e^{\mu+\lambda}\frac{\left(k-\Lambda t^2\right)e^{2\mu}}{2}(p+\rho)\,dr \leq$$

$$\leq -\underbrace{\frac{1}{t}\int_0^1 e^{\mu+\lambda}\rho dr}.$$

Defining

$$E(t) := \int_0^1 e^{\mu+\lambda}\rho dr$$

the statement reads

$$\frac{d}{dt}E(t) \leq -\frac{1}{t}E(t).$$

By Grönvall's lemma we obtain

$$E(t) \leq E(t_0)e^{\int_{t_0}^t \frac{1}{s}ds} \Rightarrow$$

$$\Rightarrow E(t) \leq E(t_0)\frac{t_0}{t}$$

So,

$$\int_0^1 e^{\mu+\lambda}\rho(t,r)dr \leq Ct^{-1} \leq$$

$$\leq \left(Ct^2+C\right)t^{-3} \leq \left[C\sqrt[3]{\frac{3C}{2\Lambda}}^2+C\right]t^{-3} \leq Ct^{-3}.$$

Thereby, (6.3) holds for $t < \sqrt[3]{\frac{3C}{2\Lambda}}$ as well. □

Lemma 6.4. *With the same conditions as in Lemma 6.2 and* $\mu' = -4\pi t e^{\lambda+\mu} j$ *we find*

$$\left|\mu(t,r) - \int_0^1 \mu(t,\sigma)d\sigma\right| \leq Kt^{-2} \tag{6.11}$$

with $K := 4\pi C$.

Proof. Using (6.3) and $\mu' = -4\pi t e^{\lambda+\mu} j$

$$\left|\mu(t,r) - \int_0^1 \mu(t,\sigma)d\sigma\right| = \left|\int_0^1\int_\sigma^r \mu'(t,\tau)d\tau d\sigma\right| \leq \int_0^1\int_0^1 \left|\mu'(t,\tau)\right| d\tau d\sigma \leq$$

$$4\pi t\int_0^1 e^{\lambda+\mu}\left|j(t,\tau)\right|d\tau \leq 4\pi t\int_0^1 e^{\mu+\lambda}\rho(t,\tau)d\tau \leq Kt^{-2}.$$

□

Lemma 6.5. *Under the circumstances of Lemma 6.4 above*

$$e^{\mu(t,r)-\lambda(t,r)} \leq Ct^{-2}. \tag{6.12}$$

Proof. For $t \in [t_0, T[$ and $r \in [0,1]$.

$$\frac{\partial}{\partial t} e^{\mu-\lambda} = e^{\mu-\lambda}\left[4\pi t e^{2\mu}(p-\rho) + \frac{1+ke^{2\mu}}{t} - \Lambda t e^{2\mu}\right] \overset{\rho<p}{\leq} e^{\mu-\lambda}\left[\frac{1+ke^{2\mu}}{t} - \Lambda t e^{2\mu}\right] \overset{(6.1)}{\leq}$$

$$\overset{(6.1)}{\leq} \left[\frac{1}{t} + \frac{k-\Lambda t^2}{C - kt + \frac{\Lambda t^3}{3}}\right] e^{\mu-\lambda}.$$

$$\frac{\partial}{\partial t} e^{\mu-\lambda} \leq \left[\frac{1}{t} + \frac{k-\Lambda t^2}{\underbrace{C - kt + \frac{\Lambda t^3}{3}}_{:=\kappa}}\right] e^{\mu-\lambda} = \left[\frac{1}{t} - \frac{\frac{\partial \kappa}{\partial t}}{\kappa}\right] e^{\mu-\lambda} \Leftrightarrow$$

$$\frac{\frac{\partial}{\partial t} e^{\mu-\lambda}}{e^{\mu-\lambda}} \leq \frac{1}{t} + \frac{k-\Lambda t^2}{C - kt + \frac{\Lambda t^3}{3}} = \frac{1}{t} - \frac{\frac{\partial \kappa}{\partial t}}{\kappa} \Rightarrow$$

$$\Rightarrow \ln\left(e^{\mu-\lambda}\right)\Big|_{T_0}^{t} \leq \ln t - \ln \kappa(t) - \ln T_0 + \ln \kappa(T_0) \Leftrightarrow$$

$$\Leftrightarrow \mu - \lambda \leq K' + \ln t - \ln \kappa \Leftrightarrow \quad \text{with} \quad K' := ln(K'(T_0)) - ln(T_0)$$

$$\Leftrightarrow e^{\mu-\lambda} \leq K' \frac{t}{\kappa} = K' \frac{t}{C - kt + \frac{\Lambda t^3}{3}} \leq K'' t^{-2}$$

with

$$K'' := K'C.$$

□

Lemma 6.6. *With the properties of Lemma 6.4*

$$\mu(t,r) \overset{(6.11)}{\leq} Kt^{-2} + \int_0^1 \mu(t,r)dr \leq C_1 + Kt^{-2} + \frac{1}{2}\ln\frac{t}{C - kt + \frac{\Lambda}{3}t^3} \leq K'''\left(1 + t^{-2} + \ln t^{-2}\right) \tag{6.13}$$

For some sufficient $K''' \in \mathbb{R}$.

Proof.

$$\int_0^1 \mu(t,r)dr = \int_0^1 \tilde{\mu}(r)dr + \int_{T_0}^t \int_0^1 \dot{\mu}(s,r)drds \overset{(2.5)}{=}$$

$$\overset{(2.5)}{=} C + \int_{T_0}^t \frac{1}{2s}\int_0^1 \left[e^{2\mu}\left(8\pi s^2 p + k - \Lambda s^2\right) + 1\right] drds =$$

$$= \int_0^1 \tilde{\mu}(r)dr + \frac{1}{2}\ln\left(\frac{t}{T_0}\right) + \int_{T_0}^t \frac{1}{2s}\int_0^1 e^{2\mu}8\pi s^2 p + e^{2\mu}\left(k - \Lambda s^2\right) drds =$$

$$= \int_0^1 \tilde{\mu}(r)dr + \underbrace{\frac{1}{2}\ln\left(\frac{t}{T_0}\right)}_{\leq 0} +$$

$$+ \int_{T_0}^t \frac{1}{2s}\int_0^1 \left[8\pi s^2 \underbrace{e^{\mu-\lambda}}_{\overset{(6.12)}{\leq} Cs^{-2}} \underbrace{e^{\mu+\lambda}p}_{\leq e^{\mu+\lambda}\rho \overset{(6.3)}{\to} Cs^{-3}} - \underbrace{e^{2\mu}}_{\overset{(6.1)}{\geq} \frac{s}{C-ks+\frac{\Lambda}{3}s^3}} \left(-k+\Lambda s^2\right)\right] drds \leq$$

$$\leq \int_0^1 \tilde{\mu}(r)dr + 4\pi \int_{T_0}^t s^{-4}ds - \frac{1}{2}\int_{T_0}^t \frac{-k+\Lambda s^2}{C-ks+\frac{\Lambda}{3}s^3} drds =$$

$$= \int_0^1 \tilde{\mu}(r)dr + 4\pi \int_{T_0}^t s^{-4}ds + \frac{1}{2}\ln \frac{s}{C-ks+\frac{\Lambda}{3}s^3}\Bigg|_{s=T_0}^{s=t} =$$

$$= \int_0^1 \tilde{\mu}(r)dr + 4\pi \int_{T_0}^t s^{-4}ds + \frac{1}{2}\ln \frac{t}{C-ks+\frac{\Lambda}{3}s^3}\Bigg|_{s=T_0}^{s=t} \leq$$

$$\leq C_1 + \frac{1}{2}\ln \frac{t}{C-ks+\frac{\Lambda}{3}s^3}$$

with

$$C_1 := \int_0^1 \tilde{\mu}(r)dr + CT_0^{-3} - \frac{1}{2}\ln \frac{T_0}{C-kT_0+\frac{\Lambda}{3}T_0^3}.$$

One can now use (6.11) to get an upper bound on μ. □

We have now established bounds for μ and w for all times. Furthermore we know that we will not encounter geodesics that approach a point with $(w, F) = (0, 0)$ in finite time. We can now prove theorem 6.1:

Proof. We have shown in Theorem 4.3 that we have future existence to infinity if we can bound w and μ for all times. Now, in Lemma 6.1 and Lemma 6.6 we showed that we can indeed find uniform bounds for these values. Furthermore, we know by Theorem 5.1 that we will not encounter regularity problems because no characteristic runs into the tip of the light-cone. Therefore all continuation criteria are met and we have $T = \infty$. □

Chapter 7

On Asymptotic Behaviour

We will elaborate on the evolution of the system for early and late times. The results are similar but not identical to the ones in the massive case with or without positive cosmological constant.

7.1 Future Asymptotic Behaviour

In this section, we are interested in the late time behaviour of the system. Many other cosmologies with positive cosmological constant relax to a de-Sitter cosmology at late times and we will show that this is also the case in this cosmology. Furthermore, we will show that the cosmology becomes more and more dustlike over time due to the relative dominance of the matter term ρ. Analogous statements in the massive case were studied in [11] and [7] with vanishing and positive cosmological constant respectively.

Theorem 7.1. *Consider solutions of the Einstein-Vlasov system* (f, λ, μ) *for initial data* $(\tilde{f}, \tilde{\lambda}, \tilde{\mu})$ *for times* $t \geq t_0$ *for some large time* t_0*. Let*

$$w_0 := \sup_{(r,w,F) \in supp \tilde{f}} |w|$$

$$F_0 := \sup_{(r,w,F) \in supp \tilde{f}} F$$

Furthermore, we define for $t > t_0$

$$P_+(t) := max(0, \sup_{supp f} w)$$

$$P_-(t) := min(0, \inf_{supp f} w)$$

Fix $\varepsilon \in]0, 1[$*. We claim that*

$$P_+(t) \leq w_0 \left(\frac{t}{t_0}\right)^{-1+\varepsilon} \quad and \quad P_-(t) \geq -w_0 \left(\frac{t}{t_0}\right)^{-1+\varepsilon}. \tag{7.1}$$

Proof. Assume that the estimate on P_+ were false for some $t \in \mathbb{R}$. Define

$$t_1 := \sup \left\{ t \geq t_0 | P_+(s) \leq w_0 \left(\frac{s}{t_0} \right)^{-1+\varepsilon}, t_0 \leq s \leq t \right\}$$

and $P_+(t_1) = w_0 \left(\frac{t_1}{t_0} \right)^{-1+\varepsilon} > 0$. By continuity we can then choose an $\alpha \in [0,1]$ so that there exists a time interval $[t_1, t_2]$ such that for some characteristic curve that satisfies

$$(1 - \alpha/2) P_+(t_2) \leq w(t_2) \leq P_+(t_2) \tag{7.2}$$

the estimate

$$0 < (1 - \alpha) P_+(t) \leq w(t) \leq P_+(t) \tag{7.3}$$

holds for all $t \in [t_1, t_2]$.

For such a characteristic curve on $s \in [t_1, t_2]$ we have

$$\dot{w} = \frac{4\pi^2 e^{2\mu}}{s} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\tilde{w} \sqrt{w^2 + \frac{F}{s^2}} - w \sqrt{\tilde{w}^2 + \frac{\tilde{F}}{s^2}} \right) f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu} \leq$$

$$\leq \frac{4\pi^2 e^{2\mu}}{s} \int_0^{P_+} \int_0^{F_0} \frac{\tilde{w}^2 \left(\cancel{w^2} + \frac{F}{s^2} \right) - w^2 \left(\cancel{\tilde{w}^2} + \frac{\tilde{F}}{s^2} \right)}{\tilde{w} \sqrt{w^2 + \frac{F}{s^2}} + w \sqrt{\tilde{w}^2 + \frac{\tilde{F}}{s^2}}} f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu} \leq$$

$$\leq \frac{4\pi^2 e^{2\mu}}{s} \int_0^{P_+} \int_0^{F_0} \frac{\tilde{w}^2 F}{\tilde{w}|w|} f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu} =$$

$$= \frac{4\pi^2 e^{2\mu}}{s} \int_0^{P_+} \int_0^{F_0} \frac{\tilde{w} F}{w} f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu} \leq$$

$$\leq \frac{4\pi^2 e^{2\mu}}{s} \frac{F_0^2 P_+^2(s)}{2w} ||f_0||_\infty + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu} \overset{(7.3)}{\leq}$$

$$\overset{(7.3)}{\leq} \frac{4\pi^2 e^{2\mu}}{s} \frac{F_0^2}{2(1-\alpha)^2} ||f_0||_\infty w + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} swe^{2\mu} \leq$$

$$\frac{1 + \left(C + k - \Lambda s^2 \right) e^{2\mu}}{2s} w$$

with a time constant $C(f_0, F_0, \alpha)$. Here we can assume that $s > \sqrt{\frac{C+k}{\Lambda}}$. Then the expression gets negative and using (6.1) we get

$$\dot{w} \leq \frac{1 + \frac{Cs + ks - \Lambda s^3}{C - ks + \frac{\Lambda}{3} s^3}}{2s} w \tag{7.4}$$

Now

$$3 + \frac{Cs + ks - \Lambda s^3}{C - ks + \frac{\Lambda}{3} s^3} = \frac{3C + Cs - 2ks}{C - ks + \frac{\Lambda}{3} s^3} \leq$$

$$\leq \left(\frac{9C}{\Lambda s} + \frac{3C}{\Lambda} - \frac{6k}{\Lambda} \right) s^{-2} \leq$$

$$\leq \underbrace{\frac{C-6k}{\Lambda}}_{=:C(\Lambda)} s^{-2}$$

We obtain the estimate

$$\frac{1+\frac{Cs+ks-\Lambda s^3}{C-ks+\frac{\Lambda}{3}s^3}}{2s} \leq C(\Lambda)s^{-2} - 2$$

Thus (7.4) implies that

$$\dot{w} \leq \left(-\frac{1}{s} + \frac{C(\Lambda)}{2s^3}\right) w \tag{7.5}$$

multiplying by $s^{1-\varepsilon}$ gives

$$\frac{d}{ds}\left(s^{1-\varepsilon} w\right) \leq s^{1-\varepsilon}\left(-\frac{\varepsilon}{s} + \frac{C(\Lambda)}{2s^3}\right) w \leq 0$$

for large s. Thus the function $s \mapsto s^{1-\varepsilon}$ is decreasing on $[t_1, t_2]$. This implies that

$$t^{1-\varepsilon} w(t) \leq t_1^{1-\varepsilon} \leq t_1^{1-\varepsilon} P_+(t_1) = \frac{w_0}{t}$$

by definition of t_1 and so

$$w(t) \leq w_0 \left(\frac{t}{t_1}\right)^{1-\varepsilon} \lightning \tag{7.6}$$

The argumentation for $w < 0$ is analogous and yields the bound for P_-. □

7.2 Future Geodesic Completeness

We want to investigate if this cosmology makes physical sense with respect to geodesic completeness, that is that all geodesics can be continued for all times. In the past direction, the geodesics come to an end approaching $t = 0$ so the future direction is of major concern in this case. Future geodesic completeness, then implies that no singularities of any kind appear for times greater then one. The methods are identical to the work done in [7]. We will exclude the tip of the light-cone in this step, by Theorem 5.1 we know that the geodesic of the tip has to stay there for all times.

Theorem 7.2. *Consider an Einstein-Vlasov system with plane or hyperbolic symmetry and positive cosmological constant with regularity conditions as described above. If the gradient of R is initially past-pointing, then there is a corresponding Cauchy development, which is future geodesically complete.*

For timelike geodesics we consider massive test-particles with

$$g_{\alpha\beta} p^\alpha p^\beta = -m^2$$

and

$$p^0 > 0$$

Now, since

$$dt = p^0 d\tau$$

with

$$p^0 = e^{-\mu}\sqrt{m^2 + w^2 + F/t^2}$$

by (6.1) we can estimate e^μ in the future

$$e^\mu \geq Ct^{-1}$$

and by Theorem 7.1 we can bound w with some constant K and F is constant along geodesics. So

$$d\tau = \frac{e^\mu}{\sqrt{m^2 + w^2 + F/t^2}} dt \geq \frac{Ct^{-1}}{\sqrt{m^2 + K^2 + F}} dt \quad \text{for} \quad t \geq 0.$$

Integrating with respect to time over the interval $[1, \infty[$ lets the right hand side diverge, so the proper time diverges with it to $+\infty$.

For null geodesics, we do not have a proper time at hand, however we can define an affine parameter that works analogous in this respect. For the massless case, we have to exclude geodesics that have vanishing momentum $(w, F) = (0, 0)$ from the argumentation. Note that by Theorem 5.1 we will not encounter characteristics that get vanishing momentum i.e. $(w, F) = (0, 0)$. The argumentation then is the same as in the timelike case.

7.3 Asymptotics of the Metric Tensor

Here we will investigate the development of the functions λ and μ to get and idea what shape the cosmology will have at late times. We will therefore work towards getting the generalized Kasner exponents in order to conclude that the cosmology will develop de-Sitter type properties. The result in [7] and [8] for the massive case holds with small changes in the our case so we will only state:

Theorem 7.3. *Consider a regular solution (f, μ, λ) of the Einstein-Vlasov system with spherical, plane or hyperbolic surface-symmetry with initial data as defined in Definition 3.1 then the following holds at late times:*

$$\dot{\lambda} = t^{-1}\left(1 + O(t^{-2})\right)$$

$$\lambda = \ln t\left[1 + O\left((\ln t)^{-1}\right)\right]$$

$$\dot{\mu} = -t^{-1}\left(1 + O(t^{-2})\right)$$

$$\lambda = -\ln t \left[1 + O\left((\ln t)^{-1}\right)\right]$$

$$\mu' = O(t^{-3+2\varepsilon})$$

with $\varepsilon \in]0, 2/3[$. *Furthermore,*

$$\lim_{t\to\infty} \frac{K_1^1(t,r)}{K(t,r)} = \lim_{t\to\infty} \frac{K_2^2(t,r)}{K(t,r)} = \lim_{t\to\infty} \frac{K_3^3(t,r)}{K(t,r)} = \frac{1}{3}$$

with $K(t,r) := K_i^i(t,r)$ *is the trace of the second fundamental form* K_{ij} *of the metric.*

7.4 Asymptotics of the Matter Terms

We will now discuss the development of the matter terms ρ, p, j and q and show that ρ dominates at late times making the cosmology increasingly dustlike. The proof of theorem the statements 6.1 and 6.2 in [8] can be proven in the same manner with the exception of the matter term p.

Theorem 7.4. *For any characteristic* (r, w, F) *and for any regular solution of the Einstein-Vlasov system with positive cosmological constant for the spherical, plane and hyperbolic case, the following properties hold at late times:*

$$\rho = O(t^{-3}); p = O(t^{-4}); j = O(t^{-4}); q = O(t^{-5})$$

and therefore

$$\frac{p}{\rho} = O(t^{-1}); \frac{j}{\rho} = O(t^{-1}); \frac{q}{\rho} = O(t^{-2}).$$

Proof. By (2.9) we have

$$p(t,r) = \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{F_0} \frac{w^2}{\sqrt{w^2 + F/t^2}} f(t,r,w,F) dF dw$$

So,

$$p(t,r) \geq \frac{\pi}{t^2} \int_{|w| \leq Ct^{-1}} \int_0^{F_0} w f(t,r,w,F) dF dw$$

and thus,

$$p \leq Ct^{-4}$$

for q and j similar arguments hold so the proof is complete. □

We have thereby shown that the cosmology evolves into a dustlike cosmology at late time due to the dominance of the energy-density term ρ compared to the matter terms $p(t,r), j(t,r)$ and $q(t,r)$ for large t.

7.5 The Nature of the Singularity at t=0

We will now try to understand what happens at $t = 0$. A priori it could be that it is just a singularity that appears because of a bad choice of coordinates in that time region. We will however prove that that is not the case by investigating the Kretschmann scalar for $t \to 0$. The proof is very much based on Theorem 5.1 in [11].

Theorem 7.5. *Let* (f, μ, λ) *be a regular solution of* (2.3)-(2.11) *for initial data presrcribed at* $t = 1$ *on the time-intervall* $]0, 1]$. *Then the Kretschmann scalar*

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \geq 6\left(\inf e^{-2\tilde{\mu}} + k\right)t^{-6}, \quad t \in]0,1], \quad r \in \mathbb{R}$$

Proof. We follow the argumentation of [11] which differs slightly because it lacks the cosmological constant. However, the Kretschmann scalar has the same appearance when derived from the metric although its time evolution may differ. As shown in [11] it reads:

$$\begin{aligned} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 4\left(e^{-2\lambda}\left(\mu'' + \mu'\left(\mu' - \lambda'\right)\right) - e^{-2\mu}\left(\ddot{\lambda} + \dot{\lambda}\left(\dot{\lambda} - \dot{\mu}\right)\right)\right)^2 + \\ + \frac{8}{t^2}\left(e^{-4\mu}\dot{\lambda}^2 + e^{-4\mu}\dot{\mu}^2 - 2e^{-2(\lambda+\mu)}\left(\mu'\right)^2\right) + \\ + \frac{4}{t^4}\left(e^{-2\mu} + k\right)^2 = \\ =: K_1 + K_2 + K_3 \end{aligned}$$

Now, the term in the first line K_1 is positive and can be dropped because we want to establish a lower bound on the scalar. For the term in the second line K_2 we use the equations (2.4), (2.5) and (2.6) to express the squares of

$$\begin{aligned} e^{-2\mu}\dot{\lambda} = 4\pi t\rho - \frac{k - \Lambda t^2 + e^{-2\mu}}{2t}, \\ e^{-2\mu}\dot{\mu} = 4\pi tp + \frac{k - \Lambda t^2 + e^{-2\mu}}{2t}, \quad \text{and} \\ e^{-\mu-\lambda}\mu' = -4\pi tj \end{aligned}$$

to get the term

$$\frac{8}{t^2}\left(16\pi^2 t^2\left(\rho^2 + p^2 - 2j^2\right) - 4\pi t\left(\rho - p\right)\frac{k - \Lambda t^2 + e^{-2\mu}}{t} + \frac{\left(k - \Lambda t^2 + e^{-2\mu}\right)^2}{2t^2}\right).$$

By using the Cauchy-Schwartz inequality we get

$$|j(t,r)| \leq \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\left(1 + w^2 + F/t^2\right)^{1/4}\sqrt{f}\frac{|w|}{\left(1 + w^2 + F/t^2\right)^{1/4}}\sqrt{f}dFdw \leq$$

$$\leq \sqrt{\rho(t,r)p(t,r)},$$

and thereby that

$$\rho^2 + p^2 - 2j^2 \geq \rho^2 + p^2 - 2\rho p = (\rho - p)^2 .$$

So, the second line can be estimated by

$$K_2 \geq \frac{8}{t^2}\left(4\pi t\,(\rho - p)^2 - 4\pi t\,(\rho - p)\,\frac{k - \Lambda t^2 + e^{-2\mu}}{t} + \frac{\left(k - \Lambda t^2 + e^{-2\mu}\right)^2}{2t^2}\right) =$$

$$= \frac{8}{t^2}\left(\left(4\pi t\,(\rho - p) - \frac{k - \Lambda t^2 + e^{-2\mu}}{2t}\right)^2 + \frac{1}{4}\frac{\left(k - \Lambda t^2 + e^{-2\mu}\right)^2}{t^2}\right).$$

Together with K_3 we can now state for the Kretschmann scalar

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \geq K_2 + K_3 \geq \frac{2}{t^2}\frac{\left(k - \Lambda t^2 + e^{-2\mu}\right)^2}{t^2} + \frac{4}{t^4}\left(e^{-2\mu} + k\right)^2 .$$

Now, for small t the cosmological constant becomes sufficiently small to neglect it. Then, by (4.5) we have that

$$e^{-2\mu} + k \geq \frac{e^{-2\tilde{\mu}+k}}{t}$$

and thereby we have proven the statement. □

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Zusammenfassung/Abstract

Abstract

It is shown that the cosmological surface symmetric Einstein-Vlasov system for massless matter has solutions in a neighborhood for a wide range of initial data. A minimal amount of regularity is required and it is shown that the solutions can be extended infinity towards the future. A proof of geodesic completeness is given and the behaviour of the matter close to the tip of the lightcone is investigated. The properties of the cosmology at late times are developed and that it tends towards a dustlike matter universe as time goes to infinity.

Zusammenfassung

Es wird gezeigt, dass ein kosmologisches, oberflächensymmetrisches Einstein-Vlasov System für masselose Materie für ein weites Spektrum an Anfangsdaten, Lösungen besitzt. Dafür müssen nur minimale Anforderungen an die Regularität gestellt werden, und es wird gezeigt, dass sich die Lösungen beliebig weit in die Zunkunft erstrecken. Es wird ein Beweis geführt, der die geodätische Zukunftsvollständigkeit zeigt, und das Verhalten der Materie nahe der Spizte des Lichtkegels untersucht. Weiters wird das Verhalten der Kosmologie für große Zeiten untersucht und gezeigt, dass es sich zunehmend staubartig verhält.