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„Ultradifferentiable CR Manifolds“

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Preface

The primary focus of this thesis is the study of certain results on the regularity of CR mappings, which have been traditionally referred to as reflection principles. The epynom of these kind of statements is the classical Schwarz reflection principle, which in fact may be viewed as a regularity result: Any real valued continuous function on the real line that extends holomorphically to one side is actually real analytic. Note that $\mathbb{R} \subseteq \mathbb{C}$ is a totally real submanifold and hence all continuous real valued function can be considered as CR mappings on $\mathbb{R}$.

The Schwarz reflection principle can easily be generalized to mappings between totally real submanifolds of $\mathbb{C}^n$. However it was a surprise when in the second half of the last century an increasing number of reflection principles for CR mappings between more general CR submanifolds were proven, beginning with the epochal theorem of Fefferman [**34**] on the smooth extension of biholomorphisms of bounded strictly pseudoconvex domains in $\mathbb{C}^n$. Among the important results on the boundary regularity that were shown after the theorem of Fefferman we would like to mention the reflection principle of Nirenberg-Webster-Yang [**60**] and the reflection principle for CR diffeomorphisms on essential finite real analytic hypersurfaces of Baouendi-Jacobowitz-Treves [**6**] to name only a few.

Most of these theorems are of a similar form, which can be summarized as follows. We consider a CR mapping H between two CR submanifolds M and M' with some a-priori regularity that extends holomorphically into a wedge with edge M. If the mapping and/or the manifolds satisfy certain nondegeneracy conditions then it is proven that H is actually of optimal regularity, that is smooth if M and M' are smooth, or real-analytic if the manifolds are analytic. The nondegeneracy assumptions mentioned are heavily tailored towards the methods applied in the various different proofs.

In particular, it is worth noting that in most instances the conditions in the smooth setting differ sharply from those used in the analytic category. One of the rare cases, where under the identical assumptions it could be shown that H is smooth if the manifolds are smooth and analytic if M and M' are both analytic manifolds, have been the results of Bernhard Lamel [**52, 53**]. He proved that every finitely nondegenerate CR mapping between two generic submanifolds that extends holomorphically is smooth and even analytic if both manifolds are real-analytic.

Recently Berhanu-Xiao [**10**] were able to strengthen this result in the smooth case by relaxing partially its assumptions. They require only the target manifold to be an embedded CR manifold, the source manifold could be only an abstract CR manifold. The finitely nondegenerate condition on the mapping remains unchanged but the holomorphic extension obviously makes no sense in this situation. It is replaced in the theorem of Berhanu-Xiao with the assumptation that the fibers of the wavefront set of H do not include opposite directions.

This microlocal assumption is automatically satisfied in the embedded setting if extension to a wedge is assumed since Baouendi-Chang-Treves [**4**] showed that for CR distributions on CR submanifolds of $\mathbb{C}^N$ the holomorphic extension into wedges is in fact a microlocal condition, which they used to define the hypoanalytic wavefront set of CR distributions. It coincides with the analytic wavefront set if the manifold is analytic. If the manifold is only smooth then the hypoanalytic wavefront set includes the smooth wavefront set.

Since the results of Lamel and Berhanu-Xiao suggest that finite nondegeneracy preserves regularity quite well, the following question arises naturally. Given a subsheaf $\mathcal{A}$ of the sheaf of smooth functions we may ask that if in the formulation of the theorem of Lamel the manifolds are assumed to be of class $\mathcal{A}$, does it follow that the CR mapping has to be of class $\mathcal{A}$ as well?

Of course we have to assume that $\mathcal{A}$ satisfies certain properties. First of all, in order for the conjecture above to make sense, $\mathcal{A}$ must be closed under composition and the implicit function theorem must hold in the category of mappings of class $\mathcal{A}$. Furthermore if we try to modify the existing proofs in the smooth category then we need some version of $\mathcal{A}$-wavefront set or more precisely a definition of $\mathcal{A}$-microlocal regularity. We should note at this point that in both Lamel's proof and that of Berhanu-Xiao the characterization of the smooth wavefront set by almost-analytic extensions was heavily used as both relied on an almost-analytic version of the implicit function theorem.

Several different kinds of ultradifferentiable classes of smooth functions have been used in various areas of mathematics, one of the most prominent cases being the famous Gevrey classes. These classes are often defined by putting growth conditions either on the derivatives or the Fourier transform of its elements.

One of the most explored families of ultradifferentiable classes, that also includes the Gevrey classes, is the category of Denjoy-Carleman classes. The elements of a Denjoy-Carleman class satisfy generalized Cauchy estimates of the form

$$\left|\partial^\alpha f(x)\right| \leq C h^{|\alpha|} m_{|\alpha|} |\alpha|!$$

on compact sets, where C and h are constants indepedent of α and $\mathcal{M} = (m_j)_j$ is a sequence of positive real numbers, the socalled weight sequence associated to the Denjoy-Carleman class. Such classes of smooth functions were first investigated by Borel and Hadamard, but were named after Denjoy and Carleman when they characterized quasianalyticity of those classes using its weight sequence.

There is a rich literature concerning the Denjoy-Carleman classes and their properties. Obviously conditions on the weight sequence translate to stability conditions of the associated class. For example, if $\mathcal{M}$ is a regular weight sequence in the sense of [**29**], then it is well known that the Denjoy-Carleman class is closed under composition, solving ordinary differential equations and the implicit function theorem holds in the class, c.f. e.g. [**67**]. Hence it makes sense in this situation to consider manifolds of Denjoy-Carleman type.

There have been also several attempts to define wavefront sets with respect to Denjoy-Carleman classes, see e.g. [**51**] and [**24**]. But the most widereaching definition of an ultradifferentiable wavefront set both with respect to conditions imposed on the weight sequence and scope of achieved results was given by Hörmander [**42**]. However his definition is a little bit too general for the purposes of this thesis. Due to his relative weak conditions on the weight sequences Hörmander was only able to define the ultradifferentiable wavefront set $\mathrm{WF}_{\mathcal{M}}\, u$ of distributions u on real-analytic manifolds but not distributions defined on ultradifferentiable manifolds.

However Dyn'kin proved that for regular weight sequences locally each Denjoy-Carleman function has an almost-analytic extension, whose dbar-derivative satisfies near $\operatorname{Im} z = 0$ a certain exponential decrease in terms of the weight sequence. In this thesis we use this result and several statements of Hörmander [**45**] to prove that the Denjoy-Carleman wavefront set can be characterized by $\mathcal{M}$-almost-analytic extensions. Using this characterization it is possible to modify Hörmander's proof to show that in this situation the ultradifferentiable wavefront set for distributions on Denjoy-Carleman manifolds can be well defined.

One of the fundamental results on the wavefront set is the elliptic regularity theorem which states that for all partial differential operators P with smooth coefficients we have that $\mathrm{WF}\, u \subseteq \mathrm{WF}\, Pu \cup \operatorname{Char} P$ for all distributions. Similarly Hörmander proved that $\mathrm{WF}_{\mathcal{M}}\, u \subseteq \mathrm{WF}_{\mathcal{M}}\, u \cup \operatorname{Char} P$ holds for operators with real-analytic coefficients. However, recently several authors [**3**], [**65**] have used the pattern of Hörmander's proof to show this inclusion for ultradifferentiable wavefront sets and operators with ultradifferentiable coefficients for variously defined ultradifferentiable classes.

Arguing similarly we prove that, if $\mathcal{M}$ is a regular weight sequence, then $\operatorname{WF}_{\mathcal{M}} u \subseteq \operatorname{WF}_{\mathcal{M}} Pu \cup \operatorname{Char} P$ for operators P with coefficients in the Denjoy-Carleman class associated to $\mathcal{M}$. In fact, we show this inclusion for vector-valued distributions and square matrices of operators with ultradifferentiable coefficients.

With this results on hand and an $\mathcal{M}$-almost analytic version of the almost-analytic implicit function theorem of Lamel we can now prove ultradifferentiable versions of the reflection principles of Lamel and Berhanu-Xiao for Denjoy-Carleman classes given by regular weight sequences.

More precisely this thesis is structured as follows. In chapter 1 we develop the theory of Denjoy-Carleman classes that is necessary for our purposes. In particular, the basic definitions and a summary of known results for classes given by regular weight sequences are given in section 1.1. Furthermore, after presenting the aforementioned result of Dyn'kin we prove here the $\mathcal{M}$-almost analytic version of the almost-analytic implicit function theorem mentioned above. In section 1.2 we note that by the results cited in the previous section it is possible to consider the category of manifolds of Denjoy-Carleman type if the weight sequence is regular. We observe also that this allows us to give an ultradifferentiable version of Sussmann's Theorem and to generalize the Theorem of Nagano for vector fields with coefficients in quasianalytic Denjoy-Carleman classes. The last section of chapter 1 contains proofs of generalizations of the basic smooth division theorems given in [**35**] to the category of Denjoy-Carleman classes and a brief discussion on the algebraic structure of quasianalytic classes.

In the first section of chapter 2 the basic theory of the ultradifferentiable wavefront set as presented in [**45**] is reviewed. We start section 2.2 with a result on the wavefront set of boundary values of $\mathcal{M}$-almost analytic functions with parameter. This generalized form is later on needed in the proof of the ultradifferentiable reflection principle. Here, however the statement without parameter together with results of Hörmander and the theorem of Dyn'kin leads to the characterization of the ultradifferentiable wavefront set by $\mathcal{M}$-almost analytic extensions, which in turn is crucial to show that the wavefront set can be invariantly defined on manifolds of Denjoy-Carleman type. In section 2.3 a generalized version of the famous theorem of Bony [**18**] on the characterizations of the analytic wavefront set is presented. In particular, we characterize the wavefront set with respect to regular Denjoy-Carleman classes by the generalized FBI transform introduced by Berhanu-Hounie. A similar result was recently given by Hoepfner-Medrado [**39**]. We shall note that in contrast to their result we allow here also quasianalytic classes. Section 2.4 is dedicated to the proof of the ultradifferentiable elliptic regularity theorem mentioned above, which in turn is used in section 2.5 together with a result of Hörmander [**41**] to prove a quasianalytic version of the Uniqueness Theorem of Holmgren [**40**], see also [**41**]. This enables us to show generalizations of statements of Bony [**16, 17**], Sjöstrand [**75**] and, applying the quasianalytic Nagano theorem, Zachmanoglou [**82, 83**].

In chapter 3 CR manifolds of Denjoy-Carleman type are considered at last. In section 3.1 basic definitions and first results are given, whereas the proofs of the ultradifferentiable versions of the reflection principles of Lamel and Berhanu-Xiao are presented in section 3.2. The last two sections are devoted to present essentially the generalization of [**35**] concerning the smoothness of infinitesimal CR automorphisms to regular Denjoy-Carleman classes. We end by examining smooth infinitesimal CR automorphisms on formally holomorphic nondegenerate quasianalytic CR submanifolds.

I would like to thank my supervisor Bernhard Lamel for his support and advice during the long journey that has led to this thesis. I would also like to express my gratitude to Armin Rainer and Gerhard Schindl, who introduced me to the theory of Denjoy-Carleman classes and its intricacies. Finally I wish to thank Michael Reiter.

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Preliminaries

We will summarize some basic notions and definitions that are going to be used throughout the thesis.

We will use the standard (subspace) topology on $\Omega \subseteq \mathbb{R}^n$. In particular we denote the system of neighbourhoods of a point $p \in \Omega$ by $\mathcal{U}(p) = \mathcal{U}_\Omega(p)$. Occasionally we are going to write $K \subset\subset \Omega$ to denote a compact subset K of Ω. If U is an open set then $U \subset\subset \Omega$ means that U is a relatively compact subset of Ω.

The standard scalar product in $\mathbb{R}^n$ will be written as

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j y_j.$$

Sometimes we will also use the convention $x \cdot y = \langle x, y \rangle$. A subset $\Gamma \in \mathbb{R}^n$ is a cone iff for all $\lambda > 0$ and $x \in \Gamma$ it holds that also $\lambda x \in \Gamma$. The set of positive integers is denoted by $\mathbb{N}$ whereas $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An element $\alpha \in \mathbb{N}_0^n$ is said to be a multi-index. The length of a multi-index α is defined as

$$|\alpha| = \sum_{j=1}^{n} \alpha_j.$$

Similarly the Euclidean norm in $\mathbb{R}^n$ is denoted by

$$|x| = \sqrt{\sum_{j=1}^{n} |x_j|^2}$$

for $x \in \mathbb{R}^n$.

If R is a ring, E a module over R and $f_1 \dots, f_d \in E$ then we denote the submodule of E that is generated by $f_1, \dots, f_d$ by

$$\operatorname{span}_R\{f_1, \dots, f_d\}.$$

If $\Omega \subseteq \mathbb{R}^n$ is open then we say that a function f defined on Ω is an element of $\mathcal{C}^1(\Omega)$ iff all partial derivatives

$$\frac{\partial f}{\partial x_j}(x), \qquad j = 1, \dots, n,$$

exist and define continuous functions on Ω. The spaces $\mathcal{C}^k(\Omega)$, $k \in \mathbb{N}$, are defined analogously, whereas $\mathcal{C}(\Omega) = \mathcal{C}^0(\Omega)$ is the space of continuous functions on Ω. Accordingly we write $\mathcal{E}(\Omega) = \mathcal{C}^\infty(\Omega) = \bigcap_{k=0}^\infty \mathcal{C}^k(\Omega)$ for the space of smooth functions. Note that usually all functions are considered to be complex-valued, if not stated otherwise. We may write

$$\partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}, \qquad j = 1, \dots, n,$$

and, if $\alpha \in \mathbb{N}_0^n$ is a multi-index, $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$. We shall also rarely use the following notation: Let $v \in \mathbb{R}^n$ then

$$\partial_v f = \sum_{j=1}^{n} v_j \partial_j f$$

is the directional derivative of f in direction v.

We write $\mathcal{C}^k(\Omega, E)$ for the k-times differentiable mappings, $k \in N_0 \cup \{\infty\}$, from Ω into a vector space E. If $k = \infty$ then we use also the notation $\mathcal{E}(\Omega, E)$. The Jacobi matrix, or Jacobian, of a map $F = (F_1, \dots, F_m) \in \mathcal{C}^1(\Omega, \mathbb{K}^m)$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, at $p \in \Omega$ is the matrix

$$\begin{pmatrix} \partial_1 F_1(p) & \dots & \partial_n F_1(p) \\ \vdots & & \vdots \\ \partial_1 F_m(p) & \dots & \partial_n F_m(p) \end{pmatrix}.$$

If $K \subseteq \Omega$ is compact then $\mathcal{E}(K)$ is the space consisting of those continuous functions on K that can be extended to smooth functions defined in some neighbourhood of K in Ω.

The space of test functions, that is smooth functions with compact support, i.e. functions $f \in \mathcal{E}(\Omega)$ such that

$$\operatorname{supp} f = \big\{p \in \Omega \mid \nexists U \in \mathcal{U}_\Omega(p)\colon f|_U \equiv 0\big\}$$

is compact, is denoted by $\mathcal{D}(\Omega)$. If $\mathcal{D}(\Omega)$ and $\mathcal{E}(\Omega)$ are equipped with their usual local convex topologies then the dual spaces $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$ are the usual spaces of distributions and distributions with compact support, respectively, on Ω. The duality bracket on $\mathcal{D}'$ is denoted by $\langle u, \varphi\rangle = u(\varphi)$ where $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. A linear form u on $D(\Omega)$ is an element of $\mathcal{D}'(\Omega)$ if and only if for each compact subset $K \subset\subset \Omega$ there are constants $C > 0$ and $k \in \mathbb{N}_0$ such that for all $\varphi \in \mathcal{D}(K) = \{\psi \in \mathcal{D}(\Omega \mid \operatorname{supp}\psi \subseteq K\}$

$$\langle u, \varphi\rangle \leq C \sum_{|\alpha|\leq k} \sup_{x\in K} \big|\partial^\alpha \varphi(x)\big|.$$

We say that the distribution u is of finite order iff the constant k does not depend on K. If k_0 is the smallest number such that the above estimate holds then u is a distribution of order k_0. The space of distributions of order k on Ω is denoted by $\mathcal{D}'^{,k}(\Omega)$. Any distribution with compact support is of finite order and we set $\mathcal{E}'^{,k} = \mathcal{D}'^{,k} \cap \mathcal{E}'$. For more details see e.g. [**45**], [**46**] or [**27**].

If $\Omega \subseteq \mathbb{C}^n$ is open with coordinates $Z = (Z_1, \dots, Z_n)$, $x = \operatorname{Re} Z$, $y = \operatorname{Im} Z$ and $f \in \mathcal{C}^1(\Omega)$ then we set

$$\frac{\partial f}{\partial Z_j} = \frac{1}{2}\bigg(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\bigg)$$
$$\frac{\partial f}{\partial \bar{Z}_j} = \frac{1}{2}\bigg(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\bigg).$$

Since a function $f \in \mathcal{C}^1(\Omega)$ is holomorphic if and only if $\bar{\partial}_j f = \frac{\partial f}{\partial \bar{Z}_j} = 0$ for all $j = 1, \dots, n$, we write frequently $g(p, \bar{p})$ for the value of an arbitrary function $g \in \mathcal{C}^1(\Omega)$ at the point $p \in \Omega$ in order to indicate that generally $\bar{\partial}_j g \neq 0$.

We recall that a paracompact, Hausdorff topological space M is an abstract smooth manifold of dimension n iff there is an atlas $\mathcal{A} = \{(V_\alpha, \varphi_\alpha)\}$ of charts φ_α, i.e. homeomorphisms $\varphi_\alpha : V_\alpha \to \mathbb{R}^n$ such that $M = \bigcup_\alpha V_\alpha$ is the union of the open subsets $V_\alpha \subset M$ and two arbitrary charts $\varphi_\alpha : V_\alpha \to \mathbb{R}^n$ and $\varphi_\beta : V_\beta \to \mathbb{R}^n$ in $\mathcal{A}$ are compatible, ithat means $\varphi_\alpha \circ \varphi_\beta^{-1} \in \mathcal{E}$ wherever the composition is defined.

If $\varphi : V \to \mathbb{R}^n$ is a chart then $\varphi^{-1} : U = \varphi(V) \to M$ is called a (local) parametrization of M and $(x_1, \dots, x_n) := \varphi^{-1}(q)$ are local coordinates on U

A map $F : M \to N$ between two manifolds is $\mathcal{C}^k$, $k \in \mathbb{N}_0 \cup \{\infty\}$, iff $\psi \circ F \circ \varphi^{-1}$ for any choice of charts φ of M and ψ of N. In particular, a function $f : M \to \mathbb{C}$ is $\mathcal{C}^k$ if and only if $\varphi^* f = f \circ \varphi$ is $\mathcal{C}^k$ for any local parametrization (U, φ) of M.

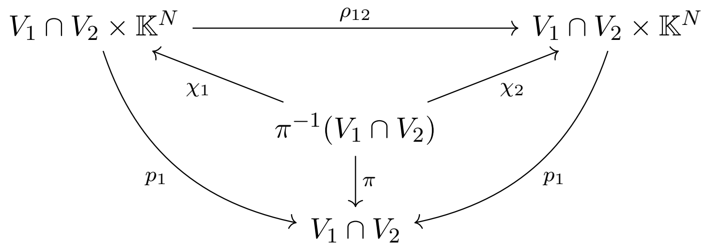
$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & \mathbb{C} \\ {\scriptstyle\varphi}\uparrow & \nearrow_{\varphi^* f} & \\ U & & \end{array}$$

We are going to identify occasionally a chart neighbourhood V with the open subset $U = \varphi(V) \subseteq \mathbb{R}^n$. We refer e.g. to [**25**] for a detailed account of the theory of manifolds.

When $\mathbb{K}$ denotes either the field $\mathbb{R}$ or $\mathbb{C}$, then a manifold E is said to be a ($\mathbb{K}$-)vector bundle over M of fiber dimension N, if the following holds: There is a smooth surjective map $\pi : E \to M$ such that $E_p = E|_p := \pi^{-1}(p)$ is an N-dimensional vector space over $\mathbb{K}$, the socalled fiber of E at p, for each $p \in M$. Furthermore for each $p \in M$ there is an open neighbourhood $V \subseteq M$ and a diffeomorphism χ such that the following diagrams commutes

$$\begin{array}{ccc} \pi^{-1}(V) & \xrightarrow{\chi} & V \times \mathbb{K}^N \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_1} \\ V & \xrightarrow{\text{id}} & V \end{array}$$

and such that the mapping $\chi|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \{q\} \times \mathbb{K}^N \cong \mathbb{K}^N$ is a linear isomorphism for each $q \in V$. The diffeomorphism χ is called a local trivialization of E. Local trivializations satisfy the following compatibility condition. Let χ_1 and χ_2 be local trivializations of a vector bundle E on the subsets V_1 and V_2 of M, then



commutes, where $\rho_{12} = \chi_2 \circ \chi_1^{-1}$ is linear in the last component. More precisely, we can consider ρ_{12} as a smooth mapping

$$\rho : V_1 \cap V_2 \to GL(N, \mathbb{K})$$

into the Lie group of invertible $N \times N$-matrices with entries in $\mathbb{K}$. The map ρ_{12} is called a transition function of E. If χ_3 is a local trivialization of E on a further open subset V_3 of M and $\rho_{23} = \chi_3 \circ \chi_2^{-1}$, $\rho_{31} = \chi_1 \circ \chi_3^{-1}$ the corresponding transition functions then the socalled cocyle condition is satisfied on $V_1 \cap V_2 \cap V_3$, namely

$$\rho_{12}(x) \cdot \rho_{23}(x) \cdot \rho_{31}(x) = \text{Id}$$

for $x \in V_1 \cap V_2 \cap V_3$. Note that it possible to reconstruct the bundle E from the transition functions defined on a covering of M.

A map f between two vector bundles E and F over the manifold M is a vector bundle homomorphism iff f is smooth and linear in the fiber, i.e.

$$f|_{E_p} : E_p \longrightarrow F_{\pi \circ f(p)}$$

is linear for all $p \in M$. If f is additionally a diffeomorphism and invertible in each fiber then it is called a vector bundle isomorphism.

If $U \subseteq M$ is an open subset then we write $E|_U = E(U)$ for the vector bundle $\pi^{-1}(U)$ over U.

If E is some vector bundle on M then a section of E is a mapping $X\colon M \to E$ that satisfies $\pi \circ X = \text{id}$. Note that we have not required X to be smooth. The space of sections of E is denoted by $\Gamma(M, E)$, whereas $\mathcal{E}(M, E)$ is the space of smooth sections. We define similarly $\mathcal{C}^k(M, E)$, $k \in \mathbb{N}_0$.

A local basis of $\mathcal{E}(M, E)$ on $U \subseteq M$ is given by smooth sections $f_j \in \mathcal{E}(U, E|_U) = \mathcal{E}(U, E)$, $j = 1, \ldots, N$, that are linearly independent at any point of U, such that any $X \in \mathcal{E}(M, E)$ can be written locally as

$$X|_U = \sum_{j=1}^{N} X_j f_j$$

with coefficients $X_j \in \mathcal{E}(U)$.

If $\pi : E \to M$ is a vector bundle then $\pi' : F \to M$ is a vector subbundle of E iff $F \subseteq E$ and $\pi' = \pi|_F$. The dual bundle E^* of a bundle E is defined by setting

$$E^* = \bigsqcup_{p \in M} (E_p)^*.$$

If ψ is a local trivialization on U then the dual map ψ^* is defined by $\psi^*(p, .) = (\psi(p, .))^*$ and $\varphi = (\psi^*)^{-1}$ is a local trivialization of E^*. Note also that if ρ is a transition function of E then $({}^\tau\rho)^{-1}$ is a transition function of E^*.

If $F \subseteq E$ is a subbundle, we can define a subbundle $F^\perp \subseteq E^*$ by

$$F_p^\perp := \left\{\sigma \in E_p^* \mid \sigma(v) = 0 \quad \forall v \in F_p\right\}.$$

Other constructions from linear algebra that transfer easily to the category of vector bundles include the tensor product. If E and F are two $\mathbb{K}$-vector bundles then the tensor product $E \otimes F = E \otimes_{\mathbb{K}} F$ is defined fiberwise by $(E \otimes F)_p = E_p \otimes F_p$. Note that $E \otimes F$ satisfies the following universal property. Let G be another $\mathbb{K}-$vector bundle and $\varphi : E \times F \to G$ a bilinear vector bundle morphism. Then there is a unique linear vector bundle morphism $\tilde{\varphi} : E \otimes F \to G$ such that the diagram

$$\begin{array}{ccc} E \times F & \xrightarrow{\otimes} & E \otimes F \\ & \underset{\varphi}{\searrow} & \downarrow \tilde{\varphi} \\ & & G \end{array}$$

commutes, where $\otimes$ is the morphism that maps $(e_p, f_p) \in E_p \times F_p$ to its tensor product $e_p \otimes f_p$. In particular, if E is a real vector bundle over M and if we denote the trivial complex bundle $M \times \mathbb{C}$ in a slight abuse of notation as $\mathbb{C}$ then the tensor product $\mathbb{C} \otimes_{\mathbb{R}} E$ is a complex vector bundle.

Another construction, that we need to mention is the exterior power $\bigwedge^k E$ of a vector bundle E. It satisfies the following universal property. If F is another vector bundle and $\psi : \prod^k E \to F$ is an anti-symmetric k-multilinear morphism then there exists a unique vector bundle homomorphism $\hat{\psi} : \bigwedge^k E \to F$ such that

$$\begin{array}{ccc} E \times \cdots \times E & \xrightarrow{\wedge} & \bigwedge^k E \\ & \underset{\psi}{\searrow} & \downarrow \hat{\psi} \\ & & G \end{array}$$

commutes. Here $\wedge$ is the following multilinear morphism. If $(v_p^1, \ldots, v_p^k) \in \prod_{j=1}^k E_p$ then

$$v_p^1 \wedge \cdots \wedge v_p^k = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) v_p^{\sigma(1)} \otimes \cdots \otimes v^{\sigma(k)_p}$$

where S_k is the symmetric group of degree k. For more details on the algebraic background of these constructions, see e.g. [**54**]. Note in particular that the fiber dimension of $\bigwedge^k E$ equals $\binom{N}{k}$. We set $\bigwedge^0 E = M \times \mathbb{K}$.

The basic examples of vector bundles are the tangent bundle $TM = \bigsqcup T_pM$, where T_pM is the usual tangent space at $p \in M$, of a manifold M and its dual the cotangent bundle T^*M. We denote the tangent map (or push-forward) of a $\mathcal{C}^1$-mapping $F : M \to N$ at the point p by

$$(F_*)_p : T_pM \to T_{F(p)}N$$

and the dual map to $F_*(p) = (F_*)_p$ is the cotangent map of F

$$F_p^* : T^*_{F(p)}N \to T_pM.$$

Thus, if φ is a chart of M on $U \subseteq M$, a local trivialization of TM on U is given by

$$\varphi_* : \pi^{-1}(U) = \bigsqcup_{p \in U} T_p M \longrightarrow \varphi(U) \times \mathbb{R}^n \cong U \times \mathbb{R}^n$$
$$(p, v_p) \longmapsto (\varphi(p), \varphi_*(p) v_p).$$

The transition function ρ of TM associated to two charts φ and ψ of M, i.e. associated to the local trivializations φ_* and ψ_*, is just the Jacobi matrix of $\psi \circ \varphi^{-1}$. Hence, if $\varphi^*(p) = (\varphi_*(p))^*$, then

$$\varphi^* : \pi^{-1}(U) = \bigsqcup_{p \in U} T_p^* M \longrightarrow \varphi(U) \times \mathbb{R}^n \cong U \times \mathbb{R}^n$$
$$(p, \xi_p) \longmapsto (\varphi(p), \varphi^*(p) \xi_p).$$

and the transition function ρ is the transpose of the Jacobi matrix of $\psi \circ \varphi^{-1}$. The smooth sections of TM and T^*M are called the vector fields of M and the 1-forms of M, respectively. The Lie bracket $[X, Y]$ of two vector fields X and Y is the vector field given by

$$[X, Y]f = X(Yf) - Y(Xf) \qquad f \in \mathcal{E}(M).$$

The set of vector fields $\mathfrak{X}(M) = \mathcal{E}(M, TM)$ thus is a Lie algebra, i.e. an algebra with the Lie bracket as multiplication.

An integral curve of $X \in \mathcal{C}^1(M, TM)$ is a curve $\gamma : \mathbb{R} \supseteq I \to M$ that satisfies the equation

$$\frac{d\gamma(t)}{dt} = X \circ \gamma(t).$$

If $p \in M$ and $X \in \mathcal{C}^1(M, TM)$ then there is always an integral curve γ_X^p of X such that the domain of definition $(\delta_p, \varepsilon_p) \subseteq \mathbb{R}$ of γ is maximal. The (local) flow $H = H_X$ of X is defined as the map

$$H : \mathbb{R} \times M \supseteq \big\{(\tau, p) \mid p \in M,\ \tau \in (\delta_p, \varepsilon_p)\big\} \longrightarrow M$$

that is defined by $H^\tau(p) = H(\tau, p) = \gamma_X^p(\tau)$.

A mapping $F : M \to N$ is said to be an immersion iff the tangent map $F_* : T_p M \to T_{F(p)} N$ is injective for all $p \in M$. If $M' \subseteq M$ is a subset of a manifold M and M' is itself a manifold such that the inclusion $\iota : M' \to M$ is an immersion then M' is called an immersed submanifold of M. If ι additionally is an homeomorphism on the image then we say that M' is an (regular) submanifold of M.

Let $\mathcal{L} \subseteq \mathfrak{X}(M)$ a Lie subalgebra of vector fields on M. We say that an immersed submanifold M' of M is an integral manifold of $\mathcal{L}$ iff

$$\iota_*\big(T_p M'\big) = \mathcal{L}(p) = \big\{X(p) \mid X \in \mathcal{L}\big\}$$

for all $p \in M'$. An integral manifold M' of $\mathcal{L}$ is called maximal if for any integral manifold M'' with $M' \subseteq M''$ it follows that $M' = M''$.

In general, the differential forms of degree k on M are the smooth sections of $\bigwedge^k(T^*M)$, i.e. the elements of $\mathcal{A}^k(M) := \mathcal{E}(M, \bigwedge^k(T^*M))$. If $\alpha \in \mathcal{A}^k(M)$ is a k-form and $\beta \in \mathcal{A}^\ell(M)$ then the exterior product $\alpha \wedge \beta \in \mathcal{A}^{k+\ell}(M)$ is defined by

$$(\alpha \wedge \beta)_p = \alpha_p \wedge \beta_p.$$

If $F : M \to N$ is a smooth map then the pullback of a k-form $\omega \in \mathcal{A}^k(N)$ by F is the k-form $F^*\omega \in \mathcal{A}^k(M)$ that is pointwise defined by

$$F^*\omega_p\big(X_p^1, \ldots, X_p^k\big) = \omega\big(F_* X_p^1, \ldots, F_* X_p^k\big)$$

where $X^1, \ldots, X^n \in \mathfrak{X}(M)$. Obviously the definition makes also sense for F only a $\mathcal{C}^1$-mapping and a k-form ω of class $\mathcal{C}^1$, i.e. $\omega \in \mathcal{C}^1(N, \bigwedge^k T^*N)$. That leads to $F^*\omega \in \mathcal{C}^1(M, \bigwedge^k T^*M)$.

If (U,φ) is a local chart of M with coordinate functions $\varphi(p) = (x_1(p),\dots,x_n(p)$ then a local basis of vector fields on U, i.e. a set of elements $V_1,\dots,V_N \in \mathcal{E}(U,TM)$ such that the vector fields V_j are linearly indepedent on U, is given by

$$V_j = \varphi_*^{-1}\left(\frac{\partial}{\partial x_j}\right) \qquad j = 1,\dots,n.$$

We may identify the coordinates on U and $\varphi(U)$ and write $V_j = \partial_{x_j}$. Similarly a local basis of 1-forms on U is given by dx_j, $j = 1,\dots,n$. Then $dx_1 \wedge \cdots \wedge dx_n$ is a local basis of $\mathcal{A}^n = \mathcal{E}(M, \bigwedge^n T^*M)$. More generally, the k-forms of the form $dx_{j_1} \wedge \cdots \wedge dx_{j_k}$, where $1 \le j_1 < j_2 < \cdots < j_k \le n$, constitute a local basis of $\mathcal{A}^k(M)$.

The exterior derivative of a k-form ω that is locally given by

$$\omega = \sum_{1\le j_1<\cdots<j_k\le n} f_{j_1\dots j_k} dx_{j_1} \wedge \cdots \wedge dx_{j_k}$$

is defined by

$$d\omega = \sum_{1\le j_1<\cdots<j_k\le n} df_{j_1\dots j_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}$$

where $df_{j_1\dots j_k} = \sum_{j=1}^n \partial_j f_{j_1\dots j_k} dx_j$ is the usual exterior derivative of the function $f_{j_1\dots j_k}$. It can be shown that the extorier derivative $d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$ is well defined and satisfies $d \circ d = 0$.

The Lie derivative of an k-form $\omega \in \mathcal{A}^k(M)$ with respect to a vector field $X \in \mathfrak{X}(M)$ is the k-form given by

$$\mathcal{L}_X \omega = \frac{d}{d\tau}\Big|_{\tau=0} \left(H^\tau\right)^* \omega.$$

where H^τ is the flow of X, c.f. [**38**].

A function $f : M \to \mathbb{C}$ is said to be locally integrable, iff for any parametrization $\varphi : U \to M$ the composition $f \circ \varphi$ is locally integrable on U.

A complex density on a (real) vector space V of dimension N is a mapping $d : \bigwedge^N V^* \backslash \{0\} \to \mathbb{C}$ such that for all $\lambda \in \mathbb{R}\backslash\{0\}$ and all $w \in \bigwedge^N V^* \backslash \{0\}$ we have

$$d(\lambda w) = |\lambda| \cdot d(w).$$

Since $\bigwedge^N V^*$ is 1-dimensional a density is completely determined by its value on one element of $\bigwedge^N V^* \backslash \{0\}$. Hence the space of densities $\mathsf{vol}(V)$ is a complex vector space of dimension 1.

If M is a manifold then the complex density bundle $\mathsf{vol}(M)$ is defined fiberwise by $\mathsf{vol}(M)_p = \mathsf{vol}(T_pM)$. For more details, see e.g. [**74**] or [**37**]. The complex density bundle is a complex line bundle, i.e. its complex fiber dimension is 1. If (U,φ) is a local chart and $\varphi(p) = (x_1(p),\dots,x_n(p))$ for $p \in U$ and consider the section $|dx_1 \wedge \cdots \wedge dx_n|$ of $\mathsf{vol}M$ that is defined by $|dx_1 \wedge \cdots \wedge dx_n|_p((dx_1 \wedge \cdots \wedge dx_n)_p) = 1$ for all $p \in U$. then $|dx_1 \wedge \cdots \wedge dx_n|$ generates $\mathcal{C}(M, \mathsf{vol}(M))$.

One important feature of the complex density bundle is that it is possible to integrate continuous sections of $\mathsf{vol}(M)$. More precisely, let φ be a chart of M on $U \subseteq M$, $K \subset\subset U$ a compact set and $d \in \mathcal{C}(M, \mathsf{vol}(M))$ a density with support in K. Then d is of the form

$$d = \tilde{d} |dx_1 \wedge \cdots \wedge dx_n|$$

where $\tilde{d} \in \mathcal{C}(U)$ with $\operatorname{supp} \tilde{d} \subseteq K$ and we define

$$\int_K d := \int_{\varphi(K)} \tilde{d}\big(\varphi^{-1}(x)\big)\, dx.$$

It can be shown to be well-defined, c.f. [**74**]. If one uses partitions of unity then the integral over more general sections of $\mathsf{vol}(M)$ can be defined in the usual way.

If $\mathsf{vol}(M)$ is the complex density bundle we define

$$\mathcal{D}(M,\mathsf{vol}(M)) := \big\{\psi \in \mathcal{E}(M,\mathsf{vol}(M))\colon\ \operatorname{supp}\psi \subset\subset M\big\}$$

as the space of compactly supported sections of $\mathsf{vol}(M)$ equipped with the usual topology. Its strong dual $\mathcal{D}'(M)$ is the space of distributions on M, for more details see e.g. [**23**] or [**37**]. Note that a function $f : M \to \mathbb{C}$ is locally integrable if and only if

$$\int_M |f\tau| < \infty$$

for all $\tau \in \mathcal{D}(M,\mathsf{vol}(M))$. Therefore any locally integrable function f can be considered as a distribution on M in the usual way.

If E is a vector bundle on M then we consider similarly

$$\mathcal{D}(M,E\otimes\mathsf{vol}(M)) = \big\{\omega \in \mathcal{E}(M,E\otimes\mathsf{vol}(M))\colon\ \operatorname{supp}\omega \subset\subset M\big\}$$

the space of compactly supported smooth sections of $E\otimes\mathsf{vol}(M)$.

The strong dual of $\mathcal{D}(M,E\otimes\mathsf{vol}(M))$ is the space of distributions (or generalized sections) on M with values in E^*

$$\mathcal{D}'(M,E^*) = \big(\mathcal{D}(M,E\otimes\mathsf{vol}(M))\big)'$$

If $\omega^1,\dots,\omega^\nu$ is a local basis of $\mathcal{E}(U,E|_U)$, $U \subseteq M$ open, and $\omega_j = (\omega^j)^*$, $j = 1,\dots,\nu$, the dual basis then a distribution $\mathfrak{Y} \in \mathcal{D}'(M,E^*)$ is locally of the form

$$\mathfrak{Y}|_U = \sum_{j=1}^{\nu} u_j\omega_j \tag{A}$$

where $u_j \in \mathcal{D}'(U)$. We also say that a section $\mathfrak{F} \in \Gamma(M,E^*)$ is locally integrable iff

$$\int_M |\mathfrak{F}(\tau)| < \infty$$

for all $\tau \in \mathcal{D}(M,E\otimes\mathsf{vol}(M))$.

We note that, beside the usual duality bracket for $\mathfrak{Y} \in \mathcal{D}'(M,E^*)$ and $\omega \in \mathcal{D}(M,E)$ by $\langle\mathfrak{Y},\omega\rangle$, there is another bracket

$$\{\,.\,,\,.\,\} : \mathcal{D}'(M,E^*) \times \mathcal{E}(M,E) \longrightarrow \mathcal{D}'(M),$$

which is defined locally as follows: On $U \subseteq M$ open as above we have the local representation (A) for $\mathfrak{Y}$ and we can write $\omega|_U = \sum_j f_j\omega^j$ with $f_j \in \mathcal{E}(U)$. We define

$$\{\mathfrak{Y},\omega\}|_U := \sum_j^{\nu} f_ju_j \in \mathcal{D}'(U).$$

We may write $\mathfrak{Y}(\omega) = \omega(\mathfrak{Y}) = \{\mathfrak{Y},\omega\}$.

Contents

CHAPTER 1

Denjoy-Carleman functions

1.1. Introduction

Troughout this and the next chapter Ω is going to denote an open subset of $\mathbb{R}^n$. A *weight sequence* is a sequence of positive real numbers $(M_j)_{j\in\mathbb{N}_0}$ with the following properties

$$\begin{aligned} &M_0 = 1 \\ &M_j^2 \le M_{j-1}M_{j+1} \qquad j \in \mathbb{N}. \end{aligned}$$

DEFINITION 1.1.1. Let $\mathcal{M} = (M_j)_j$ be a weight sequence. We say that a function $f \in \mathcal{E}(\Omega)$ is *ultradifferentiable of class* $\{\mathcal{M}\}$ iff for every compact set $K \subset\subset \Omega$ there exist constants C and h such that for all multi-indices $\alpha \in \mathbb{N}_0^n$

$$\left|D^\alpha f(x)\right| \le Ch^{|\alpha|}M_{|\alpha|} \qquad x \in K. \tag{1.1.1}$$

We denote the space of ultradifferentiable functions of class $\{M\}$ on Ω as $\mathcal{E}_{\mathcal{M}}(\Omega)$. It is always a subalgebra of $\mathcal{E}(\Omega)$ ([**48**]).

EXAMPLE 1.1.2. For any $s \ge 0$ consider the sequence $\mathcal{M}^s = ((k!)^{s+1})_k$. The space of ultradifferentiable functions associated to $\mathcal{M}^s$ is the well known space of Gevrey functions $\mathcal{G}^{s+1} = \mathcal{E}_{\mathcal{M}^s}$ of order $s+1$, c.f. e.g. [**68**]. If $s = 0$ then $\mathcal{G}^1 = \mathcal{E}_{\mathcal{M}^0} = \mathcal{O}$ is the space of real-analytic functions.

REMARK 1.1.3. It is easy to see that $\mathcal{E}_{\mathcal{M}}(\Omega)$ is an infinite-dimensional vector space, since it contains all polynomials. In fact $\mathcal{E}_{\mathcal{M}}(\Omega)$ is a complete locally convex space, see e.g. [**48**]. The topology on $\mathcal{E}_{\mathcal{M}}(\Omega)$ is defined as follows. If $K \subset\subset \Omega$ is a compact set such that $K = \overline{K^\circ}$ then we define for $f \in \mathcal{E}(K)$

$$\|f\|_K^h := \sup_{\substack{x\in K\\ \alpha\in\mathbb{N}_0^n}} \left|\frac{D^\alpha f(x)}{h^{|\alpha|}M_{|\alpha|}}\right|$$

and set

$$\mathcal{E}_{\mathcal{M}}^h(K) := \left\{f \in \mathcal{E}(K) \mid \|f\|_K^h < \infty\right\}.$$

It is easy to see that $\mathcal{E}_{\mathcal{M}}^h(K)$ is a Banach space. Moreover, $\mathcal{E}_{\mathcal{M}}^h(K) \subsetneq \mathcal{E}_{\mathcal{M}}^k(K)$ for $h < k$ and the inclusion mapping $\iota_h^k : \mathcal{E}_{\mathcal{M}}^h(K) \to \mathcal{E}_{\mathcal{M}}^k(K)$ is compact. Hence the space

$$\mathcal{E}_{\mathcal{M}}(K) := \left\{f \in \mathcal{E}(K) \mid \exists h > 0\colon \|f\|_K^h < \infty\right\} = \varinjlim_h \mathcal{E}_{\mathcal{M}}^h(K)$$

is a (LB)-space. We can now write

$$\mathcal{E}_{\mathcal{M}}(\Omega) = \varprojlim_K \mathcal{E}_{\mathcal{M}}(K)$$

as a projective limit. For more details on the topological structure of $\mathcal{E}_{\mathcal{M}}(\Omega)$ see [**48**].

We call $\mathcal{E}_{\mathcal{M}}(\Omega)$ also the *Denjoy-Carleman class on* Ω *associated to the weight sequence* $\mathcal{M}$. If $\mathcal{M}$ and $\mathcal{N}$ are two weight sequences then

$$\mathcal{M} \preccurlyeq \mathcal{N} :\Longleftrightarrow \sup_{k\in\mathbb{N}_0}\left(\frac{M_k}{N_k}\right)^{\frac{1}{k}} < \infty$$

defines a reflexive and transitive relation on the space of weight sequences. Furthermore it induces an equivalence relation by setting

$$\mathcal{M} \approx \mathcal{N} :\Longleftrightarrow \mathcal{M} \preccurlyeq \mathcal{N} \text{ and } \mathcal{N} \preccurlyeq \mathcal{M}.$$

It holds that $\mathcal{E}_\mathcal{M} \subseteq \mathcal{E}_\mathcal{N}$ if and only if $\mathcal{M} \preccurlyeq \mathcal{N}$ and $\mathcal{E}_\mathcal{M} = \mathcal{E}_\mathcal{N}$ if and only if $\mathcal{M} \approx \mathcal{N}$, see [**56**], c.f. also [**66**] and [**78**]. For example, if $r < s$ then $\mathcal{G}^{r+1} \subsetneq \mathcal{G}^{s+1}$.

The weight function $\omega_\mathcal{M}$ (c.f. [**56**] and[**48**]) associated to the weight sequence $\mathcal{M}$ is defined by

$$\begin{aligned} \omega_\mathcal{M}(t) &:= \sup_{j\in\mathbb{N}_0} \log \frac{t^j}{M_j} \qquad t > 0, \\ \omega_\mathcal{M}(0) &:= 0. \end{aligned}$$

It follows that $\omega_\mathcal{M}$ is a continuous increasing function on $[0,\infty)$, vanishes on the interval $[0,1]$ and $\omega_\mathcal{M} \circ \exp$ is convex. In particular $\omega_\mathcal{M}(t)$ increases faster than $\log t^p$ for any $p > 0$ as t tends to infinity [**48, 56**]. It is possible to extract the weight sequence from the weight function ([**56**], [**48**]), i.e.

$$M_k = \sup_t \frac{t^k}{e^{\omega_\mathcal{M}(t)}}.$$

If f and g are two continuous functions defined on $[0,\infty)$ then we set $f \sim g$ iff $f(t) = O(g(t))$ and $g(t) = O(f(t))$ for $t \to \infty$. It can be shown that the weight function ω_s for the Gevrey space $\mathcal{G}^{s+1}$ satisfies

$$\omega_s(t) \sim t^{\frac{1}{s+1}}.$$

Sometimes the classes $\mathcal{E}_\mathcal{M}$ are defined using the sequence $m_k = \frac{M_k}{k!}$ instead of $(M_k)_k$ and (1.1.1) is replaced by

$$\left|D^\alpha f(x)\right| \leq Ch^{|\alpha|}|\alpha|!m_{|\alpha|}.$$

Infrequently the sequences $\mu_k = \frac{M_{k+1}}{M_k}$ or $L_k = M_k^{\frac{1}{k}}$ are also used, with an accordingly modified version of (1.1.1), c.f. also Remark 2.1.3. The main reason for the different ways of defining the Denjoy-Carleman classes is the following. In order to show that these classes satisfy certain properties, like the inverse function theorem, one has to put certain conditions on the defining data of the spaces, i.e. the weight sequence, c.f. [**67**]. Often these conditions are easier to write down in terms of these other sequences instead of using $(M_j)_j$. In the following our point of view is that the sequences $(M_k)_k$, $(m_k)_k$, $(\mu_k)_k$ and $(L_k)_k$ are all associated to the weight sequence $\mathcal{M}$. We are going to use especially the two sequences $(m_j)_j$ and $(M_j)_j$ indiscriminately.

We may note that sometimes ultradifferentiable functions associated to the weight sequence $\mathcal{M}$ are defined as smooth functions satisfying (1.1.1) for all $h > 0$ on each compact K, see e.g. [**32**]. One says then that f is ultradifferentiable of class $(\mathcal{M})$ and the corresponding space is the Beurling class associated to $\mathcal{M}$. On the other hand $\mathcal{E}_\mathcal{M}$ is then usually called the Romieu class associated to $\mathcal{M}$, c.f. [**48**] and [**67**].

From now on we shall put certain conditions on the weight sequences under consideration.

Definition 1.1.4. We say that a weight sequence $\mathcal{M}$ is *regular* iff it satisfies the following conditions:

$$m_0 = m_1 = 1 \tag{M1}$$

$$\sup_k \sqrt[k]{\frac{m_{k+1}}{m_k}} < \infty \tag{M2}$$

$$m_k^2 \leq m_{k-1}m_{k+1} \qquad k \in \mathbb{N} \tag{M3}$$

$$\lim_{k\to\infty} \sqrt[k]{m_k} = \infty \tag{M4}$$

The last condition just means that the space $\mathcal{O}$ of all real-analytic functions is strictly contained in $\mathcal{E}_\mathcal{M}$ whereas the first is an useful normalization condition that will help simplify certain computations. It is obvious that if we replace in (M1) the number 1 with some other positive real number we would not change the resulting space $\mathcal{E}_\mathcal{M}$.

If $\mathcal{M}$ is a regular weight sequence, then it is well known that the associated Denjoy-Carleman class satisfies certain stability properties, c.f. e.g. [**12, 67**]. For example $\mathcal{E}_\mathcal{M}$ is *closed under differentiation*, i.e. if $f \in \mathcal{E}_\mathcal{M}(\Omega)$ then $D^\alpha f \in \mathcal{E}_\mathcal{M}(\Omega)$ for all $\alpha \in \mathbb{N}_0^n$.

Remark 1.1.5. The fact that $\mathcal{E}_\mathcal{M}(\Omega)$ is closed under differentiation implies immediately another stability condition, namely *closedness under division by a coordinate* ([**12**]):

Suppose that $f \in \mathcal{E}_\mathcal{M}(\Omega)$ and $f(x_1, \dots, x_{j-1}, a, x_{j+1}, \dots, x_n) = 0$ for some fixed $a \in \mathbb{R}$ and all x_k, $k \neq j$, with the property that $(x_1, \dots, x_{j-1}, a, x_{j+1}, \dots, x_n) \in \Omega$. Then we apply the Fundamental Theorem of Calculus to the function

$$f_j : t \longmapsto f(x_1, \dots, x_{j-1}, t(x_j - a) + a, x_{j+1}, \dots, x_n)$$

and obtain

$$f(x) = \int_0^1 \frac{\partial f_j}{\partial t}(t)\, dt = (x_j - a) \int_0^1 \frac{\partial f}{\partial x_j}(x_1, \dots, x_{j-1}, t(x_j - a) + a, x_{j+1}, \dots, x_n)\, dt = (x_j - a) g(x).$$

It is easy to see that $g \in \mathcal{E}_\mathcal{M}(\Omega)$ using $\frac{\partial f}{\partial x_j} \in \mathcal{E}_\mathcal{M}(\Omega)$.

For the proof of the properties above only (M2) was used. If we apply also (M3) then it is possible to show that $\mathcal{E}_\mathcal{M}(\Omega)$ is *inverse closed*, i.e. if $f \in \mathcal{E}_\mathcal{M}(\Omega)$ does not vanish at any point of Ω then

$$\frac{1}{f} \in \mathcal{E}_\mathcal{M}(\Omega),$$

c.f. [**67**] and the remarks therein.

In fact, if $\mathcal{M}$ is a regular weight sequence then the associated Denjoy-Carleman class satisfies also the following stability properties.

Theorem 1.1.6. *Let $\mathcal{M}$ be a regular weight sequence and $\Omega_1 \subseteq \mathbb{R}^m$ and $\Omega_2 \subseteq \mathbb{R}^n$ open sets. Then the following holds:*

(1) *The class $\mathcal{E}_\mathcal{M}$ is* closed under composition *(Romieu [**70**] see also [**12**]) i.e. let $F : \Omega_1 \to \Omega_2$ be a $\mathcal{E}_\mathcal{M}$-mapping, that is each component F_j of F is ultradifferentiable of class $\{\mathcal{M}\}$ in Ω_1, and $g \in \mathcal{E}_\mathcal{M}(\Omega_2)$. Then also $g \circ F \in \mathcal{E}_\mathcal{M}(\Omega_1)$.*
(2) *The inverse function theorem holds in the Denjoy-Carleman class $\mathcal{E}_\mathcal{M}$ (Komatsu [**49**]): Let $F : \Omega_1 \to \Omega_2$ be a $\mathcal{E}_\mathcal{M}$-mapping and $p_0 \in \Omega_1$ such that the Jacobian $F'(p_0)$ is invertible. Then there exist neighbourhoods U of p_0 in Ω_1 and V of $q_0 = F(x_0)$ in Ω_2 and a $\mathcal{E}_\mathcal{M}$-mapping $G : V \to U$ such that $G(q_0) = p_0$ and $F \circ G = \mathrm{id}_V$.*
(3) *The implicit function theorem is valid in $\mathcal{E}_\mathcal{M}$ ([**49**]): Let $F : \mathbb{R}^{n+d} \supseteq \Omega \to \mathbb{R}^d$ be a $\mathcal{E}_\mathcal{M}$-mapping and $(x_0, y_0) \in \Omega$ such that $F(x_0, y_0) = 0$ and $\frac{\partial F}{\partial y}(x_0, y_0)$ is invertible. Then there exist open sets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^d$ with $(x_0, y_0) \in U \times V \subseteq \Omega$ and a $\mathcal{E}_\mathcal{M}$-mapping $G : U \to V$ such that $G(x_0) = y_0$ and $F(x, G(x)) = 0$ for all $x \in V$.*

Furthermore it is true that $\mathcal{E}_\mathcal{M}(\Omega)$ is *closed under solving ODEs*.

Theorem 1.1.7 (Yamanaka [**81**] see also [**50**]). *Let $\mathcal{M}$ be a regular weight sequence, $0 \in I \subseteq \mathbb{R}$ an open interval, $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^d$ open and $F \in \mathcal{E}_\mathcal{M}(I \times U \times V)$.*

Then the initial value problem

$$\begin{aligned} x'(t) &= F(t, x(t), \lambda) \qquad && t \in I,\ \lambda \in V \\ x(0) &= x_0 && x_0 \in U \end{aligned}$$

has locally a unique solution x that is ultradifferentiable near 0.

More precisely, there is an open set $\Omega \subseteq I \times U \times V$ that contains the point $(0, x_0, \lambda)$ and an $\mathcal{E}_{\mathcal{M}}$-mapping $x = x(t, y, \lambda) : \Omega \to U$ such that the function $t \mapsto x(t, y_0, \lambda_0)$ is the solution of the initial value problem

$$
\begin{aligned}
x'(t) &= F(t, x(t), \lambda_0) \\
x(0) &= y_0.
\end{aligned}
$$

For any regular weight sequence $\mathcal{M}$ we can define the associated weight by

$$
h_{\mathcal{M}}(t) = \inf_k t^k m_k \quad \text{if } t > 0 \quad \& \quad h_{\mathcal{M}}(0) = 0. \tag{1.1.2}
$$

Similarly to above we have that

$$
m_k = \sup_t \frac{h_{\mathcal{M}}(t)}{t^k}
$$

In order to describe the connection between the weight and the weight function associated to a regular weight sequence we set

$$
\begin{aligned}
\tilde{\omega}_{\mathcal{M}}(t) &:= \sup_{j \in \mathbb{N}_0} \log \frac{t^j}{m_j} \\
\tilde{h}_{\mathcal{M}}(t) &= \inf_k t^k M_k
\end{aligned}
$$

for $t > 0$ and $\tilde{\omega}_{\mathcal{M}}(0) = \tilde{h}_{\mathcal{M}}(0) = 0$.

Lemma 1.1.8. *If $\mathcal{M}$ is a regular weight sequence then*

$$
\begin{aligned}
h_{\mathcal{M}}(t) &= e^{-\tilde{\omega}_{\mathcal{M}}\left(\frac{1}{t}\right)} \\
\tilde{h}_{\mathcal{M}}(t) &= e^{-\omega_{\mathcal{M}}\left(\frac{1}{t}\right)}
\end{aligned} \tag{1.1.3}
$$

Proof. We prove only the equality for $h_{\mathcal{M}}$. Of course, the verification of the other equation is completely analogous. If $t > 0$ is chosen arbitrarily we have by the monotonicity of the exponential function that

$$
\exp\left(\tilde{\omega}_{\mathcal{M}}\left(\frac{1}{t}\right)\right) = \exp\left(\sup_k \log \frac{1}{m_k t^k}\right) = \sup_k \frac{1}{m_k t^k} = \frac{1}{\inf_k m_k t^k} = \frac{1}{h_{\mathcal{M}}(t)}.
$$

□

We obtain that $h_{\mathcal{M}}$ is continuous with values in $[0, 1]$, equals 1 on $[1, \infty)$ and goes more rapidly to 0 than t^p for any $p > 0$ for $t \to 0$. Albeit the weight function is the prevalant concept, the weight was used e.g. by Dyn'kin [**28, 29**] and Thilliez [**77**].

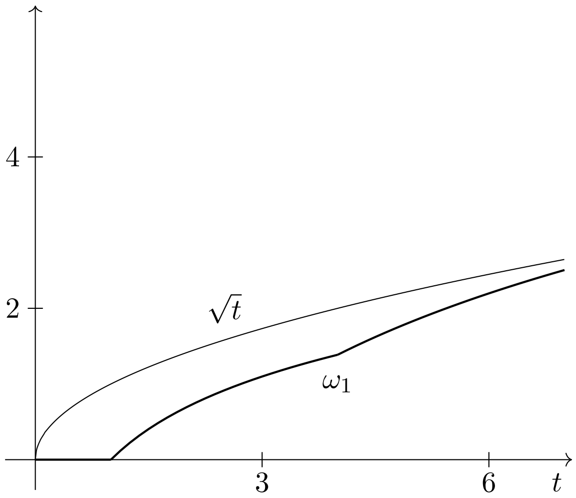
Example 1.1.9. If $\mathcal{M} = \mathcal{M}^s$ is the Gevrey sequence of order s then we know already that the associated weight function satisfies $\omega_s(t) \sim t^{\frac{1}{1+s}}$. Hence (1.1.3) shows for $s > 0$ that if we set

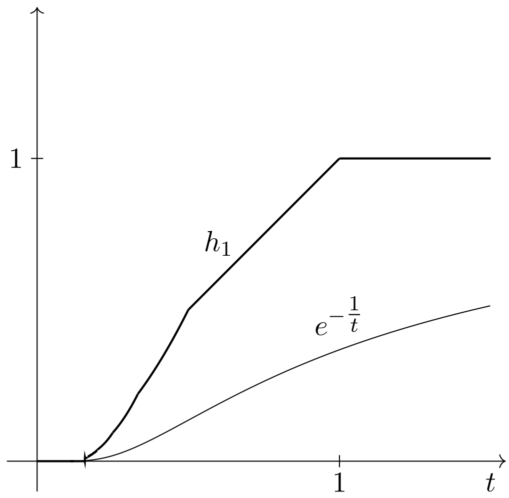
$$
f_s(t) = e^{-\frac{1}{t^s}}
$$

then there are constants C_1, C_2, Q_1 and $Q_2 > 0$ such that

$$
C_1 f_s(Q_1 t) \leq h_s(t) \leq C_2 f_s(Q_2 t)
$$

for $t > 0$.





$s = 1$

It is well known (see e.g. [**57**], [**58**] or [**79**]) that a function f is smooth on Ω if and only if there is an almost-analytic extension F of f, i.e. there exists a smooth function F on some open set $\tilde{\Omega} \subseteq \mathbb{C}^n$ with $\tilde{\Omega} \cap \mathbb{R}^n = \Omega$ such that

$$\bar{\partial}_j F = \frac{\partial}{\partial \bar{z}_j} F = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) F$$

is flat on Ω and $F|_\Omega = f$. The idea is now that if f is ultradifferentiable then one should find an extension F of f such that the regularity of f is translated in a certain uniform decrease of $\tilde{\partial}_j F$ near Ω (c.f. [**30**]). Such extensions were constructed e.g. by [**63**] and [**2**] under relative restrictive conditions on the weight sequence. The most general result in this regard though was given by Dyn'kin [**28, 29**].

THEOREM 1.1.10. *Let $\mathcal{M}$ be a regular weight sequence, $K \subset\subset \mathbb{R}^n$ a compact set with $K = \overline{K^\circ}$. Then $f \in \mathcal{E}_{\mathcal{M}}(K)$ if and only if there exists a test function $F \in \mathcal{D}(\mathbb{C}^n)$ with $F|_K = f$ and if there are constants $C, Q > 0$ such that*

$$\bar{\partial}_j F(z) \leq C h_{\mathcal{M}}(Q d_K(z)) \tag{1.1.4}$$

where $1 \leq j \leq n$ and d_K is the distance function with respect to K on $\mathbb{C}^n \backslash K$.

We shall note that Dyn'kin used the function $h_1(t) = \inf m_k t^{k-1}$ instead of the weight $h_{\mathcal{M}}$. It is easy to see that $h_1(t) = h_{\mathcal{M}}(t)/t$. Since $h_{\mathcal{M}}$ is rapidly decreasing for $t \to 0$ we can interchange these two functions in the formulation of Theorem 1.1.10. In fact, Dyn'kin's proof gives immediately the following result.

COROLLARY 1.1.11. *Let $\mathcal{M}$ be a regular weight sequence, $p \in \Omega$ and $f \in \mathcal{D}'(\Omega)$. If f is ultradifferentiable of class $\{\mathcal{M}\}$ near p, i.e. there exists a compact neighbourhood K of p such that $f|_K \in \mathcal{E}_{\mathcal{M}}(K)$, then there are an open neighbourhood $W \subseteq \Omega$, a constant $\rho > 0$ and a function $F \in \mathcal{E}(W + iB(0,\rho))$ such that $F|_W = f|_W$ and*

$$\left|\bar{\partial}_j F(x+iy)\right| \leq C h_{\mathcal{M}}(Q|y|) \tag{1.1.5}$$

for some positive constants C, Q and all $1 \leq j \leq n$ and $x + iy \in W + iB(0,\rho)$.

The following theorem is the $\mathcal{M}$-almost analytic version of the "almost-holomorphic" implicit function theorem of Lamel [**53**].

THEOREM 1.1.12. *Let $\mathcal{M}$ be a regular weight sequence, $U \subseteq \mathbb{C}^N$ a neighbourhood of the origin, $A \in \mathbb{C}^p$ and $F : U \times \mathbb{C}^p \to \mathbb{C}^N$ of class $\{\mathcal{M}\}$ on U and polynomial in the last variable with $F(0, A) = 0$ and $F_Z(0, A)$ invertible. Then there exists a neighbourhood $U' \times V'$ of $(0, A)$ and a smooth mapping $\phi = (\phi_1, \dots, \phi_N) : U' \times V' \to \mathbb{C}^N$ with $\phi(0, A) = 0$ with the property* *that if* $F(Z,\bar{Z},W)=0$ *for some* $(Z,W)\in U'\times V'$ *then* $Z=\phi(Z,\bar{Z},W)$. *Furthermore, there are constants* $C,\gamma>0$ *such that*

$$\left|\frac{\partial \phi_j}{\partial Z_k}(Z,\bar{Z},W)\right| \le Ch_{\mathcal{M}}\big(\gamma|\phi(Z,\bar{Z},W)-Z|\big) \tag{1.1.6}$$

for all $1\le j,k\le N$ *and* ϕ *is holomorphic in* W.

Proof. We write $F(Z,\bar{Z},W)=F(x,y,W)$, where $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ are the underlying real coordinates of $\mathbb{C}^N$, i.e. $Z_j=x_j+iy_j$ for $1\le j\le N$. Let $U_0\subseteq\mathbb{R}^N$ be a neighbourhood of 0 such that $\overline{U_0\times U_0}\subseteq U$. Using Theorem 1.1.10 we find a smooth mapping

$$\tilde{F}=U_0\times\mathbb{R}^N\times U_0\times\mathbb{R}^N\times\mathbb{C}^p\longrightarrow\mathbb{C}^N$$

such that $\tilde{F}(x,x',y,y',W)\big|_{x'=y'=0}=F(x,y,W)$ and if we write $\xi_k=x_k+ix'_k$, $\eta_k=y_k+iy'_k$ for $k=1,\dots,N$ and set $\zeta=(\xi,\eta)$, then for each compact subset $K\subset\subset\mathbb{C}^p$ there are constants $C,\gamma>0$ such that

$$\left|\frac{\partial \tilde{F}_j}{\partial\bar{\xi}_k}(\zeta,\bar{\zeta},W)\right|\le Ch_{\mathcal{M}}(\gamma|\operatorname{Im}\zeta|) \tag{1.1.7a}$$

$$\left|\frac{\partial \tilde{F}_j}{\partial\bar{\eta}_k}(\zeta,\bar{\zeta},W)\right|\le Ch_{\mathcal{M}}(\gamma|\operatorname{Im}\zeta|) \tag{1.1.7b}$$

for $(\zeta,W)\in(U_0+i\mathbb{R}^N)^2\times K$ and $1\le j,k\le N$. Note also that $\tilde{F}$ is still polynomial in the variable W.

We introduce new variables $\chi=(\chi_1,\dots,\chi_N)\in\mathbb{C}^N$ by

$$\xi_k=\frac{Z_k+\chi_k}{2}\qquad\qquad \eta_k=\frac{Z_k-\chi_k}{2i}\qquad\qquad 1\le k\le N$$

and note that

$$x_k=\frac{Z_k+\chi_k}{2}\bigg|_{\chi_k=\bar{Z}_k}\qquad\qquad y_k=\frac{Z_k-\chi_k}{2i}\bigg|_{\chi_k=\bar{Z}_k}.$$

We also set $G(Z,\bar{Z},\chi,\bar{\chi},W)=\tilde{F}(\xi,\bar{\xi},\eta,\bar{\eta},W)$. The function G is therefore smooth in the first $2N$ variables in some neighbourhood of the origin and polynomial in the last p variables. Due to the definition of G we have

$$\frac{\partial G}{\partial\bar{Z}}=\frac{1}{2}\frac{\partial\tilde{F}}{\partial\bar{\xi}}+\frac{1}{2i}\frac{\partial\tilde{F}}{\partial\bar{\eta}}$$

$$\frac{\partial G}{\partial\bar{\chi}}=\frac{1}{2}\frac{\partial\tilde{F}}{\partial\bar{\xi}}-\frac{1}{2i}\frac{\partial\tilde{F}}{\partial\bar{\eta}}.$$

We are going to compute the real Jacobian of G at the point $(0,A)$. We obtain

$$\frac{\partial G}{\partial Z}(0,A)=\frac{\partial F}{\partial Z}(0,A)$$

and

$$\frac{\partial G}{\partial\bar{Z}}(0,A)=\frac{1}{2}\left(\frac{\partial\tilde{F}}{\partial\bar{\xi}}(0,A)-i\frac{\partial\tilde{F}}{\partial\bar{\eta}}(0,A)\right)=0$$

and thus

$$\det\begin{pmatrix}\frac{\partial G}{\partial Z} & \frac{\partial G}{\partial\bar{Z}}\\ \frac{\partial\bar{G}}{\partial Z} & \frac{\partial\bar{G}}{\partial\bar{Z}}\end{pmatrix}(0,A)=\left|\det\frac{\partial F}{\partial Z}(0,A)\right|^2\neq 0$$

by assumption. Hence, by the smooth implicit function theorem, there is a smooth mapping ψ defined in some open neighbourhood of $(0,A)$, valued in $\mathbb{C}^N$ and holomorphic in the variable W such that $Z=\psi(\chi,\bar{\chi},W)$ solves the equation $G(Z,\bar{Z},\chi,\bar{\chi},W)=0$ uniquely. Since $G(Z,\bar{Z},\bar{Z},Z,W)=F(Z,\bar{Z},W)$, this fact implies that if $F(Z,\bar{Z},W)=0$ then $Z=\psi(\bar{Z},Z,W)$. We set $\phi(Z,\bar{Z},W)=\psi(\bar{Z},Z,W)$ and claim that φ satisfies (1.1.6).

In fact, if we differentiate the implicit equation $G(\psi(\chi,\bar\chi,W),\overline{\psi(\chi,\bar\chi,W)},\chi,\bar\chi,W)=0$ then we obtain

$$\begin{aligned} G_Z\psi_{\bar\chi}+G_{\bar Z}\bar\psi_{\bar\chi}+G_{\bar\chi}&=0\\ \bar G_{\bar Z}\bar\psi_{\bar\chi}+\bar G_Z\psi_{\bar\chi}+\bar G_{\bar\chi}&=0.\end{aligned}$$

If we multiply the last line with $G_{\bar Z}\bar G_{\bar Z}^{-1}$ and substract the result from the first line then

$$\bigl(G_Z-G_{\bar Z}\bar G_{\bar Z}^{-1}\bar G_Z\bigr)\psi_{\bar\chi}=G_{\bar Z}\bar G_{\bar Z}^{-1}\bar G_{\bar\chi}-G_{\bar\chi}.$$

Hence we have in a small neighbourhood of $(0,A)$ that

$$\phi_Z(Z,\bar Z,W)=\psi_{\bar\chi}(\bar Z,Z,W)=\left(\frac{G_{\bar Z}\bar G_{\bar Z}^{-1}\bar G_{\bar\chi}-G_{\bar\chi}}{G_Z-G_{\bar Z}\bar G_{\bar Z}^{-1}\bar G_Z}\right)\Bigl(\psi(\bar Z,Z,W),\overline{\psi(\bar Z,Z,W)},\bar Z,Z,W\Bigr).$$

This formula shows that any function $\partial_{Z_k}\varphi_j$ is a sum of products each of which contains a factor of the form $G_{\bar Z_\ell}$ or $G_{\bar\chi_\ell}$ for some ℓ. Note also that by definition $\operatorname{Im}\xi=\frac{1}{2}(\operatorname{Im}Z+\operatorname{Im}\chi)$ and $\operatorname{Im}\eta=-\frac{1}{2}(\operatorname{Re}Z-\operatorname{Re}\chi)$.

Hence (1.1.7) implies on some compact neighbourhood of $(0,A)$, where $\det G_Z^{-1}$ is bounded,

$$\begin{aligned}\bigl|\phi_Z(Z,\bar Z,W)\bigr|&\le Ch_{\mathcal M}\left(\frac{1}{2}\gamma\bigl(|\operatorname{Im}\phi(Z,\bar Z,W)-\operatorname{Im}Z|^2+|\operatorname{Re}Z-\operatorname{Re}\phi(Z,\bar Z,W)|^2\bigr)^{\frac{1}{2}}\right)\\ &=Ch_{\mathcal M}\bigl(\gamma|\phi(Z,\bar Z,W)-Z|\bigr)\end{aligned}$$

for some positive constants C and γ. □

One of the main questions in the study of ultradifferentiable functions is if the class under consideration behaves more like the ring of real-analytic functions or the ring of smooth functions. E.g., does the class contain flat functions, that means nonzero elements whose Taylor series at some point vanishes? That leads to following definition.

Definition 1.1.13. Let $E\subseteq\mathcal E(\Omega)$ be a subalgebra. We say that E is quasianalytic iff for $f\in E$ the fact that $D^\alpha f(p)=0$ for some $p\in\Omega$ and all $\alpha\in\mathbb N_0^n$ implies that $f\equiv 0$ in the connected component of Ω that contains p.

In the case of Denjoy-Carleman classes quasianalyticity is characterized by the following theorem.

Theorem 1.1.14 (Denjoy[**26**]-Carleman[**22, 21**]). *The space $\mathcal E_{\mathcal M}(\Omega)$ is quasianalytic if and only if*

$$\sum_{k=1}^{\infty}\frac{M_{k-1}}{M_k}=\infty. \tag{1.1.8}$$

We say that a weight sequence is quasianalytic iff it satisfies (1.1.8) and non-quasianalytic otherwise.

Example 1.1.15. Let $\sigma>0$ be a parameter. We define a family $\mathcal N^\sigma$ of weight sequences by

$$N_k^\sigma=k!\bigl(\log(k+e)\bigr)^{\sigma k}.$$

The weight sequence $\mathcal N^\sigma$ is quasianalytic if and only if $0<\sigma\le 1$ [**78**].

Remark 1.1.16. Obviously $\mathcal D_{\mathcal M}(\Omega)=\mathcal D(\Omega)\cap\mathcal E_{\mathcal M}(\Omega)$ is nontrivial if and only if $\mathcal E_{\mathcal M}(\Omega)$ is non-quasianalytic [**71**]. It is well known that the sequences $\mathcal M^s$ are non-quasianalytic if and only if $s>0$. In fact there is a non-quasianalytic regular weight sequence $\tilde{\mathcal M}$ such that $\tilde{\mathcal M}\precnapprox\mathcal M^s$ for all $s>0$ [**66**, p.125]. Hence

$$\mathcal O\subsetneq\mathcal E_{\tilde{\mathcal M}}\subsetneq\bigcap_{s>0}\mathcal G^{s+1}.$$

1.2. Ultradifferentiable manifolds

From now on, unless explicitly stated otherwise, $\mathcal{M}$ will always be assumed to be a regular weight sequence. Using Theorem 1.1.6 we are able to define

Definition 1.2.1. Let M be a smooth manifold and $\mathcal{M}$ a weight sequence. We say that M is an ultradifferentiable manifold of class $\{\mathcal{M}\}$ iff there is an atlas $\mathcal{A}$ of M that consists of charts such that

$$\varphi' \circ \varphi^{-1} \in \mathcal{E}_{\mathcal{M}}$$

for all $\varphi, \varphi' \in \mathcal{A}$.

If $M \subseteq \mathbb{R}^N$ is an ultradifferentiable submanifold of class $\{\mathcal{M}\}$ then the following characterization is proven exactly as the analogous result in the smooth setting.

Proposition 1.2.2. *Let $M \subset \mathbb{R}^N$ be a smooth manifold of dimension n and $p \in \mathcal{M}$ and $\mathcal{M}$ be a weight sequence. The following statements are equivalent:*

(1) *The manifold M is ultradifferentiable of class $\{\mathcal{M}\}$ near p.*
(2) *There are an open neighbourhood $U \subseteq \mathbb{R}^N$ of p and an $\mathcal{E}_{\mathcal{M}}$-mapping $\rho : U \to \mathbb{R}^{N-n}$ such that $d\rho \neq 0$ on W and*

$$\rho^{-1}(0) = M \cap U.$$

A mapping $F\colon M \to N$ between two manifolds of class $\{\mathcal{M}\}$ is ultradifferentiable of class $\{\mathcal{M}\}$ iff $\psi \circ F \circ \varphi^{-1} \in \mathcal{E}_{\mathcal{M}}$ for any charts φ and ψ of M and N, respectively. We can now consider the category of ultradifferentiable manifolds of class $\{\mathcal{M}\}$. In particular, it is possible to consider the usual constructions like vector bundles, vector fields or differential forms.

Definition 1.2.3. Let M be an ultradifferentiable manifold of class $\{\mathcal{M}\}$. We say that a smooth vector bundle $\pi : E \to M$ is an ultradifferentiable vector bundle of class $\{M\}$ iff for any point $p \in M$ there is a neighbourhood U of p and a local trivialization χ of class $\{\mathcal{M}\}$ on U.

Remark 1.2.4. Let E be an ultradifferentiable vector bundle of class $\{\mathcal{M}\}$. Then E can also be considered as a smooth vector bundle or as a vector bundle of class $\{\mathcal{N}\}$ for any weight sequence $\mathcal{N} \succcurlyeq \mathcal{M}$. We observe in particular that a local basis of $\mathcal{E}_{\mathcal{M}}(M,E)$ is also a local basis of $\mathcal{E}_{\mathcal{N}}(M,E)$ and $\mathcal{E}(M,E)$, respectively.

We denote by $\mathfrak{X}_{\mathcal{M}}(M) = \mathcal{E}_{\mathcal{M}}(M,TM)$ the Lie algebra of ultradifferentiable vector fields on M. Note that, if $\mathcal{M}$ is a regular weight sequence, an integral curve of an ultradifferentiable vector field of class $\{\mathcal{M}\}$ is an $\mathcal{E}_{\mathcal{M}}$-curve by Theorem 1.1.7.

The next result is an ultradifferentiable version of Sussmann's Theorem [**76**].

Theorem 1.2.5. *Let $p_0 \in \Omega$ and a collection $\mathfrak{D}$ of ultradifferentiable vector fields of class $\{\mathcal{M}\}$. Then there exists an ultradifferentiable submanifold W of Ω through p_0 such that all vector fields in $\mathfrak{D}$ are tangent to W at all points of W and such that the following holds:*

(1) *The germ of W at p_0 is unique, i.e. if W' is an ultradifferentiable submanifold of Ω containing p_0 and to which all vector fields of $\mathfrak{D}$ are tangent at every point of W' then there is a neighbourhood $V \subseteq \Omega$ of p_0 such that $W \cap V \subseteq W' \cap V$.*
(2) *For every open set $U \subseteq \Omega$ containing p_0 there exists $J \in \mathbb{N}$ and open neighbourhoods $V_1 \subseteq V_2 \subseteq U$ of p_0 such that every point $p \in W \cap V_1$ can be reached from p_0 by a polygonal path of J integral curves of vector fields in $\mathfrak{D}$ contained in $W \cap V_2$.*

The proof of Theorem 1.2.5 is essentially the same as in the smooth setting, c.f. e.g. [**8**], due to Theorem 1.1.7.

The (unique) germ of the manifold W will be denoted as the *local Sussmann orbit* of p_0 relative to $\mathfrak{D}$. The local Sussman orbit does not depend on Ω.

We are going to close this section with a proof of a quasianalytic version of Nagano's theorem [**59**]. We follow mainly the presentation given in [**8**].

Theorem 1.2.6. *Let U be an open neighbourhood of $p_0 \in \mathbb{R}^n$ and $\mathcal{M}$ a quasianalytic regular weight sequence. Furthermore let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{X}_{\mathcal{M}}(U)$ that is also an $\mathcal{E}_{\mathcal{M}}$-module, i.e. if $X \in \mathfrak{g}$ and $f \in \mathcal{E}_{\mathcal{M}}(U)$ then $fX \in \mathfrak{g}$.*

Then there exists an ultradifferentiable submanifold W of class $\{\mathcal{M}\}$ in U, such that

$$T_pW = \mathfrak{g}(p) \qquad \forall p \in W. \tag{1.2.1}$$

Moreover, the germ of W at p_0 is uniquely defined by this property.

Proof. We choose coordinates $x = (x_1, \dots, x_n)$ vanishing at p_0 and vector fields $X_1, \dots, X_r$ in $\mathfrak{g}$,

$$X_j = \sum_{k=1}^{n} a_{jk}(x) \frac{\partial}{\partial x_k} \qquad a_{jk} \in \mathcal{E}_{\mathcal{M}}(U),\ j = 1, \dots, r$$

such that $X_1(0), \dots, X_r(0)$ form a basis of $\mathfrak{g}(0)$ and

$$X_j(0) = \frac{\partial}{\partial x_j}\Big|_0 \qquad j = 1, \dots, r.$$

Hence

$$\det\bigl(a_{jk}(x)\bigr)_{1 \le j,k \le r} \neq 0$$

for x in some neighbourhood of 0. Since the conclusion of the theorem is local, we shall assume that this neighbourhood is U. Thus, after an $\mathcal{E}_{\mathcal{M}}(U)$-linear transformation on the vector fields $X_1, \dots, X_r$, we may write

$$X_j = \frac{\partial}{\partial x_j} + \sum_{k=r+1}^{n} b_{jk}(x) \frac{\partial}{\partial x_k} \qquad j = 1, \dots, r$$

with $b_{jk}(0) = 0$. Let $\mathcal{Y}$ be the vector space over $\mathbb{R}$ spanned by the vector fields $X_1, \dots, X_r$ above and denote by $\mathfrak{g}_0$ the set of vector fields in $\mathfrak{g}$ which are of the form

$$\sum_{k=r+1}^{n} c_k(x) \frac{\partial}{\partial x_k}.$$

Note that $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$ and a $\mathcal{E}_{\mathcal{M}}(U)$-module. Moreover all elements in $\mathfrak{g}_0$ vanish at the origin. We put

$$\mathcal{Z} := \mathcal{Y} + \mathfrak{g}_0$$

and deduce

$$[Z_1, Z_2] \in \mathfrak{g}_0 \subset \mathcal{Z} \qquad \forall Z_1, Z_2 \in \mathcal{Z}.$$

Hence $\mathcal{Z}$ is a Lie subalgebra of $\mathfrak{g}$, that is proper if $r > 0$ and we have $\mathcal{Z}(x) = \mathfrak{g}(x)$ for all $x \in U$. In order to finish the proof we need a lemma:

Lemma 1.2.7. *Let V be a neighbourhood of 0 in $\mathbb{R}^n$ and $\mathcal{A}$ a Lie subalgebra of $\mathfrak{X}_{\mathcal{M}}(V)$ with the property that all commutators of vector fields in $\mathcal{A}$ vanish at 0. If $X \in \mathcal{A}$ vanishes at the origin then it vanishes on any integral curve $t \mapsto \exp_0 tY$ for $Y \in \mathcal{A}$.*

Proof. Let $X, Y \in \mathcal{A}$ as above and assume $Y(0) \neq 0$ (otherwise, there is nothing to prove). We write

$$X = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j}, \quad Y = \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j}.$$

If $(\operatorname{ad} Y)(X) = [Y, X]$ then it is easy to conclude that

$$(\operatorname{ad} Y)^k = \sum_{j=1}^{n} \bigl(Y^k a_j\bigr) \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \sum_{p=1}^{q_j} \bigl(S_{pj} b_j\bigr) \frac{\partial}{\partial x_j}$$

where $S_{pj} = V_1 V_2 \dots V_{\ell_{pj}}$ is a string of length $\ell_{pj} \leq k$ with $V_i \in \mathcal{A}$ such that at least one V_i vanishes at 0. Indeed, for $k = 1$ the commutator

$$[Y, X] = \sum_{j=1}^{n} (Y a_j) \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} (X b_j) \frac{\partial}{\partial x_j}$$

is of the desired form. If we suppose that we have for $k = k_0 \geq 1$ a representation of $(\operatorname{ad} Y)^{k_0}(X)$ as above, then

$$\begin{aligned}
(\operatorname{ad} Y)^{k_0+1} X &= \big[Y, (\operatorname{ad} Y)^{k_0} X\big] \\
&= \sum_{j=1}^{n} Y \Big(Y^k a_j + \sum_{p=1}^{q_j} S_{pj} b_j \Big) \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \big((\operatorname{ad} Y)^{k_0} X \big) b_j \frac{\partial}{\partial x_j} \\
&= \sum_{j=1}^{n} Y^{k+1} a_j \frac{\partial}{\partial x_j} - \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{p=1}^{q_j} \big(Y S_{pj} - (\operatorname{ad} Y)^{k_0} X \big) b_j
\end{aligned}$$

is also of the form as wished since $(\operatorname{ad} Y)^{k_0} X = [Y, (\operatorname{ad} Y)^{k_0 - 1} X]$ vanishes as a commutator of two vector fields in $\mathcal{A}$. Now let $S = V_1 V_2 \dots V_j$ be a string of length j with $V_i \in \mathcal{A}$ and at least one of the V_i vanishes at 0. Then all coefficients of the operator S vanish. This is obvious if $V_1(0) = 0$. If $V_2(0) = 0$ then we use the fact that

$$V_1 V_2 V_3 \dots V_j = V_2 V_1 V_3 \dots V_j + [V_1, V_2] V_3 \dots V_j.$$

By the assumption on $\mathcal{A}$ we have that $[V_1, V_2](0) = 0$ and hence the right-hand side of the equation above vanishes at 0. The general statement follows in a straight-forward manner by induction.

For $k \geq 1$ we have that $(\operatorname{ad} Y)^k (X)(0) = 0$ and thus by the arguments above we conclude $Y^k a_j(0) = 0$ for all $j = 1, \dots, n$. Now, let $\gamma(t) = \exp_0(tY)$ be the integral curve of Y through the origin and put $\tilde{a}_j = a_j \circ \gamma$. Then

$$\frac{d^k \tilde{a}_j}{dt^k} = Y^k a$$

and we conclude that the curve $\tilde{a}_j$ is flat at the origin. Since the class $\mathcal{E}_{\mathcal{M}}$ is quasianalytic it follows that a_j vanishes on the complete curve γ. □

We continue with the proof of Theorem 1.2.6. By Lemma 1.2.7 we conclude that for any $X \in \mathfrak{g}_0$ and $Y \in \mathcal{Y}$, X vanishes on the integral curve $t \mapsto \exp_0 tY$.

We define the manifold $W \subset U$ by the following parametrization

$$\mathbb{R}^r \ni (t_1, \dots, t_r) \longmapsto \Phi(t_1, \dots, t_r) := \exp_0 \left(\sum_{j=1}^{r} t_j X_j \right) \in U$$

for $(t_1, \dots, t_r)$ in a sufficiently small neighbourhood V of 0 in $\mathbb{R}^r$, such that the rank of Φ is r in V. Thus the parametrization defines a manifold in a neighbourhood of 0 in U. Lemma 1.2.7 implies that $\mathfrak{g}_0(x) = 0$ for all $x \in W$ and hence $\mathfrak{g}(x) = \mathcal{Z}(x) = \mathcal{Y}$ for $x \in W$. In order to prove (1.2.1) it suffices then to show, due to dimensionality, that $\mathcal{Y} \subseteq T_x W$ for all $x \in W$. For this, we choose $p \in W$ and $X \in \mathcal{Y}$. We want to show that $X(p) \in T_p W$. Since $p \in W$, there exists $(t_1^0, \dots, t_r^0) \in V$ such that

$$p = \exp_0 \left(\sum_{j=1}^{r} t_j^0 X_j \right).$$

In other words, p is the point with time one on the integral curve of the vector field $Y = \sum_j t_j^0 X_j$ from 0. Consider the mapping

$$f(s, t) := \exp_0\big(t(sX + Y)\big).$$

It is defined on $R = \{(s,t) \in \mathbb{R}^2 \mid |s| < \varepsilon,\ t \in (-\delta, 1+\delta)\}$, where $\delta, \varepsilon > 0$ are chosen suitably, and maps R into W. We claim that for any $t \in (-\delta, 1+\delta)$ we have

$$\frac{\partial f}{\partial s}(0,t) = tX\big((f(0,t)\big) = tX\big(f(0,t)\big) \tag{1.2.2}$$

We regard f and all other vector fields like, e.g., $X \circ f$ as vector-valued functions $R \to \mathbb{R}^n$. We first differentiate $f(s,t)$ with respect to t

$$\frac{\partial f}{\partial t} = sX(f(s,t)) + Y(f(s,t)) \tag{1.2.3}$$

and hence

$$\frac{\partial^2 f}{\partial s \partial t}(0,t) = X(f(0,t)) + \sum_{j=1}^{n} \frac{\partial Y}{\partial x_j}\big(f(0,t)\big)\frac{\partial f_j}{\partial s}(0,t).$$

Note that

$$\frac{\partial f}{\partial s}(0,0) = 0.$$

We conclude that the function

$$u:\ t \longmapsto \frac{\partial f}{\partial s}(0,t)$$

satisfies the following system of ordinary differential equations

$$\frac{\partial u}{\partial t}(t) = X\big(f(0,t)\big) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\big(f(0,t)\big)u_j(t), \quad u(0) = 0. \tag{1.2.4}$$

The claim, i.e. (1.2.2), will be proven, in view of the uniqueness of solutions of ordinary differential equations, if we show that the function $\tilde{u}(t) = tX(f(0,t))$ also solves (1.2.4). Obviously $\tilde{u}(0) = 0$. Furthermore

$$\frac{d}{dt}\big(tX(f(0,t))\big) = X(f(0,t)) + t\sum_{j=1}^{n} \frac{\partial X}{\partial x_j}\big(f(0,t)\big)\frac{\partial f_j}{\partial t}(0,t)$$

and using (1.2.3) we obtain

$$\begin{aligned}\frac{d}{dt}\big(tX(f(0,t)) &= X(f(0,t)) + t\sum_{j=1}^{n} \frac{\partial X}{\partial x_j}\big(f(0,t)\big)Y_j(f(0,t)) \\ &= X(f(0,t)) + t[Y,X](f(0,t)) + \sum_{j=1}^{n} \frac{\partial Y}{\partial x_j}(f(0,t))\big(tX_j(f(0,t))\big).\end{aligned}$$

Lemma 1.2.7 gives that $[Y,X](f(0,t)) = 0$ for all t and hence it follows that $\tilde{u}$ satisfies (1.2.4).

Since f maps R into W, the vector $\frac{\partial f}{\partial s}(s,t)$ is in the tangent space $T_{f(s,t)}W$. In particular, (1.2.2) implies that $X(p)$ is in T_pW and since both $p \in W$ and $X \in \mathcal{Y}$ were chosen arbitrarily we have $\mathcal{Y}(x) \subseteq T_xW$ for all $x \in W$ which proves (1.2.1) as indicated above.

It remains to prove the uniqueness. Suppose that W' is another manifold of class $\{\mathcal{M}\}$ through 0 satisfying (1.2.1). Necessarily $\dim W' = \dim \mathfrak{g}(0) = \dim W$. Thus it suffices to show that there is an open neighbourhood U_1 of the origin in U such that

$$W \cap U_1 \subseteq W' \cap U_1$$

Let $\hat{V}$ be a convex neighbourhood of 0 in $V \subseteq \mathbb{R}^r$ and define $\hat{W} = \Phi(\hat{V}) \subseteq W$. We choose an open neighbourhood U_1 of 0 such that $W \cap U_1 = \hat{W}$. We can choose $\hat{V}$ and U_1 so small that $W' \cap U_1$ is closed in U_1. Let $p_1 \in \hat{W}$. By definition, there exists a vector field $Y \in \mathfrak{g}$ such that the integral curve $\gamma(t) = \exp_0(tY)$ goes through p_1 at time 0. Since $\hat{V}$ is convex we have that $\gamma(t) \in \hat{W} \subset U_1$ for $t \in [0,1]$. Furthermore, since

$$Y(p) \in T_pW' \tag{1.2.5}$$

for all $p \in W'$ by assumption we infer that $\gamma(t) \in W' \cap U_1$ if t is small enough. The proof is finished if we can show that $p_1 = \gamma(1) \in W' \cap U_1$. Let $E := \{t_0 \in [0,1] \mid \gamma(t) \in W' \cap U_1 \ \ \forall t \in [0,t_0]\} \subseteq [0,1]$. By (1.2.5) E is open, but E is also closed since $W' \cap U_1$ is closed in V and $\gamma([0,1])$ is contained in V. Thus $E = [0,1]$ and therefore $W \cap U_1 = W' \cap U_1$. □

We call the uniquely defined germ $\gamma_{p_0}(\mathfrak{g})$ of the manifold constructed in Theorem 1.2.6 the local Nagano leaf of $\mathfrak{g}$ at p_0. From now on all Lie algebras of ultradifferentiable vector fields that are considered are assumed to be also $\mathcal{E}_{\mathcal{M}}$-modules. As in the analytic category, c.f. [**8**], we have the following result.

COROLLARY 1.2.8. *Let $\mathcal{M}$ be quasianalytic and $\mathfrak{D} \subseteq \mathfrak{X}_{\mathcal{M}}(\Omega)$ a collection of ultradifferentiable vector fields. If $\mathfrak{g} = \mathfrak{g}_{\mathfrak{D}}$ is the Lie algebra generated by $\mathfrak{D}$ and $p_0 \in \Omega$ then the local Sussman orbit of p_0, relative to $\mathfrak{D}$, coincides with the local Nagano leaf of $\mathfrak{g}$.*

PROOF. Let W_N be a representative of the local Nagano leaf of $\mathfrak{g}$ at p_0 and W_S a representative of the local Sussman orbit of p_0, relative to $\mathfrak{D}$. By Theorem 1.2.5 (1) there exists an open neighbourhood V of p_0 such that $W_S \cap V \subseteq W_N \cap V$. On the other hand $\mathfrak{g}(p) = T_pW_N$ for all $p \in W_N$ and $\mathfrak{g}(p) \subseteq T_pW_S$ at every $p \in W_S$, hence $\mathfrak{g}(p) = T_pW_S$ for $p \in W_S \cap V$. The uniqueness part of Theorem 1.2.6 gives the equality of the local Nagano leaf and the local Sussman orbit. □

Following [**59**], c.f. also [**8**], we can also give a global version of Theorem 1.2.6.

THEOREM 1.2.9. *Let $\mathcal{M}$ be a quasianalytic regular weight sequence. If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{X}_{\mathcal{M}}(\Omega)$ then $\mathfrak{g}$ admits a foliation of Ω, that is a partition of Ω by maximal integral manifolds.*

PROOF. For $x \in \Omega$ set $\mathfrak{M}_x$ to be the set of all embedded connected submanifolds $W \subseteq \Omega$ such that (1.2.1) holds in some neighbourhood of x. We need a Lemma in order to proceed.

LEMMA 1.2.10. *Let $W \subseteq \Omega$ be an immersed connected $\mathcal{E}_{\mathcal{M}}$-manifold such that*

$$\iota_* T_w W = \mathfrak{g}(\iota w) \quad \forall w \in W' \tag{1.2.6}$$

where ι is the embedding of W into Ω and W' is an open subset of W. Then (1.2.6) *holds for all points in W.*

PROOF. Suppose that $W' \neq W$ otherwise there would be nothing to prove. W.l.o.g. assume that W' is the maximal open set such that (1.2.6) holds. Let $w_0 \in \partial W' \subseteq W$ and choose a local basis of the ultradifferentiable vector fields $\xi_1, \dots, \xi_k$ tangent to W near w_0. If we choose a small enough neighbourhood W_0 of w_0 then due to ι being an immersion there is similar to the smooth case (c.f. [**25**, Corollary 2.4.10]) an ultradifferentiable local diffeomorphism $\psi : \mathbb{R}^n \supseteq U_0 \to \Omega$ near $\iota(w_0)$ such that U_0 is open and connected, $\varphi(0) = \iota(w_0)$ and

$$\varphi = \iota|_{U_0}^{-1} \circ \psi : \ U_0 \cap \mathbb{R}^k \longrightarrow W_0$$

is a well-defined ultradifferentiable diffeomorphism. If U_0 is small enough, then after a coordinate change we may write

$$\eta_j = \varphi_*^{-1} \xi_j = \frac{\partial}{\partial x_j} \qquad j = 1, \dots, k,$$

on $U_0 \cap \mathbb{R}^k$. On the other hand let $X_1, \dots, X_m$ be a local basis of $\mathfrak{g}$ near $\iota(w_0)$ and thus

$$Y_\nu = \psi_*^{-1} X_\nu = \sum_{\ell=1}^{n} a_{\ell,\nu} \frac{\partial}{\partial x_\ell} \qquad \nu = 1, \dots, m$$

where $a_{\ell,\nu} \in \mathcal{E}_{\mathcal{M}}(U_0)$. We observe that by assumption we have that on $U' := \varphi^{-1}(W_0 \cap W')$

$$Y_\nu|_{U'} \in \operatorname{span}_{\mathcal{E}_{\mathcal{M}}}(\eta_1, \dots, \eta_k) \qquad \nu = 1, \dots, m.$$

However that means $b_{\ell,\nu} = (a_{\ell,\nu})|_{\{0\} \times \mathbb{R}^{n-k}}$ is zero on U' for $\ell = k+1, \dots, n$. Thence the functions $b_{\ell,\nu}$, $\ell = k+1, \dots, n$ have to vanish on $\varphi^{-1}(W_0)$. That is a contradiction to the assumption that W' is maximal relative to the property (1.2.6). □

We continue the proof of Theorem 1.2.9 and define the global Nagano leaf through x as the manifold

$$\Gamma_x(\mathfrak{g}) = \bigcup_{W \in \mathfrak{M}_x} W$$

together with the final topology induced by the embeddings $W \to \Gamma_x(\mathfrak{g})$. Then $\Gamma_x(\mathfrak{g})$ is an immersed connected ultradifferentiable manifold of class $\mathcal{M}$ and by Lemma 1.2.10 at any point $y \in \Gamma_x(\mathfrak{g})$ the global Nagano leaf $\Gamma_x(\mathfrak{g})$ contains the local Nagano leaf $\gamma_y(\mathfrak{g})$ through y. That shows $\Gamma_y(\mathfrak{g}) = \Gamma_x(\mathfrak{g})$. Hence the global Nagano leafs define a foliation of Ω. □

1.3. Division Theorems

In this section we want to transfer the results pertaining the division of smooth functions in [**35**, section 4] to the category of ultradifferentiable functions of class $\{\mathcal{M}\}$. This is possible because these classes are closed under division by a coordinate, c.f. Remark 1.1.5.

Lemma 1.3.1. *Let λ be an ultradifferentiable function of class $\{\mathcal{M}\}$ defined near $0 \in \mathbb{R}$ that is non-flat at the origin, i.e. there is a positive integer $k \in \mathbb{N}$ such that $\lambda^{(j)}(0) = 0$ for all integers $0 \leq j \leq k-1$ and $\lambda^{(k)}(0) \neq 0$. Further assume that there is a locally integrable function u defined near 0 such that the product $f = \lambda u$ is of class $\{\mathcal{M}\}$ in some neighbourhood of the origin.*

Then u is ultradifferentiable of class $\{\mathcal{M}\}$ near the origin.

Proof. First, we note that the zero of λ at 0 is isolated. Therefore we restrict ourselves to an open interval I that contains the origin and such that 0 is the only zero of λ on I. Iterating the argument given in Remark 1.1.5 we see that there is a function $\tilde{\lambda}$ of class $\{\mathcal{M}\}$ defined near 0 such that $\tilde{\lambda}(0) \neq 0$ and

$$\lambda(x) = x^k \tilde{\lambda}(x).$$

In order to proceed we want a similar decomposition of f. But, since we are not able to say anything apriori about the values of the derivatives of f at the origin, we can only find an ultradifferentiable function f_1 such that

$$f(x) = x f_1(x)$$

in a neighbourhood of 0. If $k > 1$ then we would have that

$$u(x) = x^{1-k} \frac{f_1(x)}{\tilde{\lambda}(x)}$$

in a punctured neighbourhood of 0. Hence, if $f_1(0) \neq 0$ then $u \sim x^{1-k}$ for $x \to 0$. This is a contradiction to the assumption that u is locally integrable. Therefore $f_1(0) = 0$ and there has to be a function f_2 of class $\{\mathcal{M}\}$ such that $f(x) = x^2 f_2(x)$ near 0. Repeating this argument if necessary, we obtain that there is a function f_k ultradifferentiable of class $\{\mathcal{M}\}$ defined near the origin such that

$$f(x) = x^k f_k(x).$$

It follows that

$$u(x) = \frac{f_k(x)}{\tilde{\lambda}(x)}$$

in some neighbourhood of 0. □

Proposition 1.3.2. *Let $p_0 \in \mathbb{R}^n$ and λ an ultradifferentiable function of class $\{\mathcal{M}\}$ defined in a neighbourhood of p_0 and $\lambda(p_0) = 0$. Suppose that $\lambda^{-1}(0)$ is a hypersurface of class $\{\mathcal{M}\}$ near p_0 and that there are $v \in \mathbb{R}^n$ and $k \in \mathbb{N}$ such that $\partial_v^j(p) = 0$ for $0 \leq j < k$ and $\partial_v^k(p) \neq 0$ for all $p \in \lambda^{-1}(0) \cap U$ where U is a neighbourhood of p_0.*

If u is a locally integrable function defined near the origin in $\mathbb{R}^n$ such that $\lambda \cdot u = f$ is ultradifferentiable of class $\{\mathcal{M}\}$ near p_0 then u has also to be of class $\{\mathcal{M}\}$ in some neighbourhood of p_0.

Proof. We can choose ultradifferentiable coordinates $(x_1,\dots,x_{n-1},x_n)=(x',x_n)$ in a neighbourhood V of p_0 in $\mathbb{R}^n$ such that $p_0=0$, $\lambda^{-1}(0)\cap V=\{(x',x_n)\in V\mid x_n=0\}$ and

$$\frac{\partial^j\lambda}{\partial x_n^j}(0)=0,\qquad 0\le j<k,$$
$$\frac{\partial^k\lambda}{\partial x_n^k}(0)\neq 0.$$

Similarly to above, using Remark 1.1.5 we conclude, if we shrink V, that there is $\tilde\lambda\in\mathcal{E}_{\mathcal{M}}(V)$ with the following properties: $\tilde\lambda(x)\neq 0$ and $\lambda(x)=x_n^k\tilde\lambda(x)$ for all points $x\in V$. There is also a Denjoy-Carleman function f_1 on V such that $f(x',x_n)=x_nf_1(x',x_n)$. We want to show, as in the 1-dimensional case, that $f_1(x',0)=0$ for $(x',0)\in V$ if $k>1$: Suppose that there exists some $y\in\mathbb{R}^{n-1}$ with $(y,0)\in V$ and $f_1(y,0)\neq 0$. Then there is a neighbourhood W of $(y,0)$ such that $f_1(x)\neq 0$ and also $\tilde\lambda(x)\neq 0$ for $x\in W$. W.l.o.g. the open set W is of the form $W=W'\times I\subset\mathbb{R}^{n-1}\times\mathbb{R}$ and set

$$F(x_n):=\int\limits_{W'}\left|\frac{f_1}{\tilde\lambda}(x)\right|dx$$

for $x_n\in I$. We conclude that

$$\int\limits_W|u(x)|\,dx=\int\limits_I|x_n|^{1-k}F(x_n)\,dx=\infty$$

and hence u cannot be locally integrable near $(y,0)$ which contradicts our assumption. Therefore we obtain by iteration a function $\tilde f$ of class $\{\mathcal{M}\}$ defined near the origin in $\mathbb{R}^n$ such that $f(x',x_n)=x_n^k\tilde f(x',x_n)$. Hence $u=\tilde f/\tilde\lambda$ is also of class $\{\mathcal{M}\}$ in a neighbourhood of 0. □

Corollary 1.3.3. *Let $U\subseteq\mathbb{R}^n$ a neighbourhood of 0, $\lambda\in\mathcal{E}_{\mathcal{M}}(U)$ and suppose that λ is of the form $\lambda(x)=x^\alpha\tilde\lambda(x)$ where $\alpha\in\mathbb{N}_0^n$ and $\tilde\lambda\in\mathcal{E}_{\mathcal{M}}(U)$ with $\tilde\lambda(0)\neq 0$.*

If u is a locally integrable function near 0 with the property that the product $f:=\lambda\cdot u$ is of class $\{\mathcal{M}\}$ near the origin, then u is also ultradifferentiable near 0.

Proof. Note first that, if $\alpha=\alpha_je_j$ then the statement is just Proposition 1.3.2. In the general case we argue as follows: Set $\tilde f=f/\tilde\lambda$ and

$$u_k(x)=\prod_{j=k+1}^n x_j^{\alpha_j}u(x).$$

The function $\tilde f$ is of class $\{\mathcal{M}\}$ whereas the functions u_k are locally integrable near 0. Furthermore we define $u_{n+1}=u$ and obtain

$$x_1^{\alpha_1}u_1(x)=\tilde f(x)$$
$$x_{k+1}^{\alpha_{k+1}}u_{k+1}(x)=u_k(x)\qquad 1\le k\le n.$$

Hence repeated application of Proposition 1.3.2 finishes the proof. □

In the literature the focus regarding questions of divisibility of functions seems to be more on the problem if it is possible to show that functions that are formally divisible, i.e. their Taylor series are divisible, are actually divisible. Indeed, the Weierstrass division theorem for example implies that two real-analytic functions that are formally divisible are also divisible as functions.

However, the equivalent of the Weierstrass division theorem does not hold for general quasianalytic Denjoy-Carleman classes [**1**],[**62**], c.f. also [**33**]. In general the algebraic structure of quasianalytic Denjoy-Carleman classes is far more complicated than that of the space of real-analytic functions, c.f. the survey of Thilliez [**78**].

Despite this there are some positive results known for quasianalytic regular classes, e.g. Bierstone and Milman [**12**] showed that certain desingularization theorems hold in these classes whereas Rolin, Speissegger and Wilkie [**69**] proved that quasianalytic regular Denjoy-Carleman classes define o-minimal structures. Both of these approaches can be used to prove division theorems. Especially the following result was shown by Nowak [**61**].

THEOREM 1.3.4. *Let $p \in \mathbb{R}^n$, $\mathcal{M}$ quasianalytic and $f, g \in \mathcal{E}_{\mathcal{M}}$ are defined near p with power series expansions $\hat{f}$ and $\hat{g}$ at p. If $\hat{f} \in \hat{g} \cdot \mathbb{C}[[x]]$ then $f \in g \cdot \mathcal{E}_{\mathcal{M}}$ near p.*

CHAPTER 2

Geometric microlocal analysis in the ultradifferentiable category

2.1. Introduction

In 1971 Hörmander [**41**] proved the following local characterization of $\mathcal{E}_{\mathcal{M}}$ via the Fourier transform:

Proposition 2.1.1. *Let $u \in \mathcal{D}'(\Omega)$ and $p_0 \in \Omega$. Then u is ultradifferentiable of class $\{\mathcal{M}\}$ near p_0 if and only if there are an open neighbourhood V of p_0, a bounded sequence $(u_N)_N \subseteq \mathcal{E}'(U)$ such that $u|_V = (u_N)|_V$ and some constant $Q > 0$ so that*

$$\sup_{\substack{\xi \in \mathbb{R}^n \\ N \in \mathbb{N}_0}} \frac{|\xi|^N |\hat{u}_N(\xi)|}{Q^N M_N} < \infty.$$

Subsequently he used this fact to define analogously to the smooth category:

Definition 2.1.2. Let $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$. We say that u is *microlocal ultradifferentiable of class* $\{\mathcal{M}\}$ at (x_0, ξ_0) iff there is a bounded sequence $(u_N)_N \subseteq \mathcal{E}'(\Omega)$ such that $u_N|_V \equiv u|_V$, where $V \in \mathcal{U}(x_0)$ and a conic neighbourhood Γ of ξ_0 such that for some constant $Q > 0$

$$\sup_{\substack{\xi \in \Gamma \\ N \in \mathbb{N}_0}} \frac{|\xi|^N |\hat{u}_N|}{Q^N M_N} < \infty. \tag{2.1.1}$$

The ultradifferentiable wavefront set $\operatorname{WF}_{\mathcal{M}} u$ is then defined as

$$\operatorname{WF}_{\mathcal{M}} u := \bigl\{(x, \xi) \in T^*\Omega \setminus \{0\} \mid u \text{ is not microlocal of class } \{\mathcal{M}\} \text{ at } (x, \xi)\bigr\}.$$

Remark 2.1.3. We need to point out that Hörmander in [**41**] defined $\operatorname{WF}_{\mathcal{M}}$ for weight sequences that satisfy weaker conditions then those we imposed in Definition 1.1.4. He required, as we have done, (M2) and that $\mathcal{O} \subseteq \mathcal{E}_{\mathcal{M}}$, but (M3) is replaced by the monotonic growth of the sequence

$$L_N = (M_N)^{\frac{1}{N}}. \tag{2.1.2}$$

This condition still implies that $\mathcal{E}_{\mathcal{M}}$ is an algebra but gives only that $\mathcal{E}_{\mathcal{M}}$ is closed under composition with analytic mappings.

More precisely, in terms of the sequence $(L_N)_N$ the conditions that Hörmander imposed take the following form. First, $N \leq L_N$ and $L_N \leq CL_{N+1}$ for all N and a constant $C > 0$ independent of N. Furthermore as mentioned before the sequence $(L_N)_N$ is also assumed to be increasing.

Note that his classes might not even be defined by weight sequences in the sense of section 1.1. Hence Hörmander in [**45**] was able to define $\operatorname{WF}_{\mathcal{M}} u$ for distributions u on real analytic manifolds but not on arbitrary ultradifferentiable manifolds of class $\{\mathcal{M}\}$; note that the implicit function theorem may not hold in an arbitrary ultradifferentiable class defined by weight sequences obeying his conditions. Similarly he proved that

$$\operatorname{WF}_{\mathcal{M}} u \subseteq \operatorname{WF}_{\mathcal{M}} Pu \cup \operatorname{Char} P$$

for linear partial differential operators P with analytic coefficients but not for operators whose coefficients might be only of class $\{\mathcal{M}\}$.

As mentioned before it is possible to modify the arguments of Hörmander in the case of regular weight sequences to show that the above inclusion holds for partial differential operators with ultradifferentiable coefficients. Similarly we are able to define $\mathrm{WF}_{\mathcal{M}} u$ for distributions defined on manifolds of class $\{\mathcal{M}\}$, in this instance using Dyn'kin's almost-analytic extension of ultradifferentiable functions.

However, since regular weight sequences also fulfill the conditions of Hörmander we can use all of his results on $\mathrm{WF}_{\mathcal{M}}$. Indeed, in terms of L_N, we have that (M4) implies that $k \leq \gamma L_k$ for all $k \in \mathbb{N}_0$ and a constant $\gamma > 0$ independent of k by Sterling's formula whereas (M2) is equivalent to the existence of a constant $A > 0$ such that $L_k \leq A L_{k-1}$. We note that the last estimate implies $L_N \leq A^N$ for $N \in \mathbb{N}_0$ since $L_1 = 1$. On the other hand, it is well-known that if $(M_N)_N$ satisfies (M3) then $(L_N)_N$ is an increasing fsequence, see [**56**].

The following result by Hörmander shows that we may choose the distributions u_N in Definition 2.1.2 in a special manner.

Proposition 2.1.4 ([**45**] Lemma 8.4.4.). *Let $u \in \mathcal{D}'(\Omega)$ and let $K \subset \Omega$ be compact, $F \subset \mathbb{R}^n$ a closed cone such that $\mathrm{WF}_{\mathcal{M}} u \cap (K \times F) = \emptyset$. If $\chi_N \in \mathcal{D}(K)$ and for all α*

$$\left|D^{\alpha+\beta}\chi_N\right| \leq C_\alpha h_\alpha^{|\beta|} M_N^{\frac{|\beta|}{N}} \qquad |\beta| \leq N$$

for some constants C_α, $h_\alpha > 0$.

Then it follows that $\chi_N u$ is bounded in $\mathcal{E}'^S$ if u is of order S in a neighbourhood of K, and further

$$|\widehat{\chi_N u}(\xi)| \leq C \frac{Q^N M_N}{|\xi|^N} \qquad N \in \mathbb{N},\ \xi \in F$$

for some constants $C, Q > 0$.

We summarize the basic properties of $\mathrm{WF}_{\mathcal{M}}$ according to [**45**].

Theorem 2.1.5 ([**45**] Theorem 8.4.5-8.4.7). *Let $u \in \mathcal{D}'(\Omega)$ and $\mathcal{M}, \mathcal{N}$ weight sequences. Then we have*

1. $\mathrm{WF}_{\mathcal{M}} u$ *is a closed conic subset of* $\Omega \times \mathbb{R}^n \backslash \{0\}$.
2. *The projection of* $\mathrm{WF}_{\mathcal{M}} u$ *in* Ω *is*
$$\pi_1\big(\mathrm{WF}_{\mathcal{M}} u\big) = \operatorname{sing\,supp}_{\mathcal{M}} u = \overline{\big\{x \in \Omega \mid \nexists V \in \mathcal{U}(x) :\ u|_V \in \mathcal{E}_{\mathcal{M}}(U)\big\}}$$
3. $\mathrm{WF}\, u \subseteq \mathrm{WF}_{\mathcal{M}} u \subseteq \mathrm{WF}_{\mathcal{N}} u$ *if* $\mathcal{M} \preccurlyeq \mathcal{N}$.
4. *If $P = \sum p_\alpha D^\alpha$ is a partial differential operator with ultradifferentiable coefficents of class $\{\mathcal{M}\}$ then* $\mathrm{WF}_{\mathcal{M}} Pu \subseteq \mathrm{WF}_{\mathcal{M}} u$.

Additionally we note $\mathrm{WF}_{\mathcal{M}} u$ satisfies the following *microlocal reflection property*:

$$(x,\xi) \notin \mathrm{WF}_{\mathcal{M}} u \Longleftrightarrow (x,-\xi) \notin \mathrm{WF}_{\mathcal{M}} \bar{u}. \tag{2.1.3}$$

In particular, if u is a real-valued distribution, i.e. $\bar{u} = u$, then $\mathrm{WF}_{\mathcal{M}} u|_x := \{\xi \in \mathbb{R}^n \mid (x,\xi) \in \mathrm{WF}_{\mathcal{M}} u\}$ is symmetric at the origin.

Example 2.1.6. It is easy to see that $\mathrm{WF}_{\mathcal{M}} \delta_p = \{p\} \times \mathbb{R}^n \backslash \{0\}$ for any regular weight sequence $\mathcal{M}$.

Remark 2.1.7. The complicated form of Definition 2.1.2 compared with the definition of the smooth wavefront set stems from the fact that quasianalytic weight sequences are allowed. Thus in general there may not be any nontrivial test functions of class $\{\mathcal{M}\}$. However if $\mathcal{D}_{\mathcal{M}} \neq \{0\}$ then we can choose in Definition 2.1.2 the constant sequence $u_N = \varphi u$ for some $\varphi \in \mathcal{D}_{\mathcal{M}}(\Omega)$ with $\varphi(x_0) = 1$ and (2.1.1) is equivalent to

$$\exists C, Q > 0 \quad \left|\widehat{\varphi u}(\xi)\right| \leq C \inf_N Q^N M_N |\xi|^{-N} \qquad \forall \xi \in \Gamma$$

thus 1.1.3 implies

$$\left|\widehat{\varphi u}(\xi)\right| \le C\tilde{h}_{\mathcal{M}}\left(\frac{Q}{|\xi|}\right) \le C\exp\left(-\omega_{\mathcal{M}}\left(\frac{|\xi|}{Q}\right)\right).$$

We conclude that (c.f. e.g. [**68**] in the case of Gevrey-classes) that for non-quasianalytic weight sequences $\mathcal{M}$ (2.1.1) is equivalent to

$$\exists Q > 0 \quad \sup_{\xi\in\Gamma} e^{\omega_{\mathcal{M}}(Q|\xi|)}\left|\widehat{\varphi u}(\xi)\right| < \infty.$$

Proposition 2.1.1 is then only a restatement to the well-known fact that for non-quasianalytic weight sequences we have that $\varphi \in \mathcal{D}_{\mathcal{M}}$ if and only if $\hat{\varphi} \le Ce^{-\omega_{\mathcal{M}}(Q|\xi|)}$ for some constants C, Q. Therefore it is possible to define ultradifferentiable classes using appropriately defined weight functions instead of weight sequences, see e.g. in a somehow generalized setting [**13**]. However, this approach leads only to non-quasianalytic spaces. This restriction was removed by [**19**] who reformulated the defining estimates of these classes to allow also quasianalytic classes. A wavefront set relative to these classes was introduced in [**3**], c.f. section 2.4. The complicated connection between the classes defined by weight sequences and those given by weight functions was investigated in [**15**]. Recently a new approach to define spaces of ultradifferentiable functions was introduced in [**66**], which encompasses the classes given by weight sequences and weight functions, see also [**67**].

2.2. Invariance of the wavefront set under ultradifferentiable mappings

Our aim in this section is to develop, using the almost-analytic extension of functions in $\mathcal{E}_{\mathcal{M}}$ given by Dyn'kin, a geometric description of $\mathrm{WF}_{\mathcal{M}}$ similarly to the one that was presented in [**55**, section 4] for the smooth wavefront set.

We need to fix some notations: If $\Gamma \subseteq \mathbb{R}^d$ is a cone and $r > 0$ then

$$\Gamma_r := \big\{y \in \Gamma \mid\ |y| < r\big\}.$$

If $\Gamma' \subseteq \Gamma$ is also a cone we write $\Gamma' \subset\subset \Gamma$ iff $(\Gamma' \cap S^{d-1}) \subset\subset (\Gamma \cap S^{d-1})$.

Similarly to [**55**, section 2.1] (c.f. also [**53**, section 2]) in the smooth category we say that, if $\mathcal{M}$ is a weight sequence, a function $F \in \mathcal{E}(\Omega \times U \times \Gamma_r)$, $U \subseteq \mathbb{R}^d$ open, is *$\mathcal{M}$-almost analytic* in the variables $(x, y) \in U \times \Gamma_r$ with parameter $x' \in \Omega$ iff for all $K \subset\subset \Omega$, $L \subset\subset U$ and cones $\Gamma' \subset\subset \Gamma$ there are constants $C, Q > 0$ such that for some r' we have

$$\left|\frac{\partial F}{\partial \bar{z}_j}(x', x, y)\right| \le Ch_{\mathcal{M}}(Q|y|) \qquad (x', x, y) \in K \times L \times \Gamma'_{r'},\ j = 1, \dots, d \tag{2.2.1}$$

where $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ and $h_{\mathcal{M}}$ is the weight associated to the regular weight sequence $\mathcal{M}$ as defined by (1.1.2).

We may also say generally that a function $g \in \mathcal{C}(\Omega \times U \times \Gamma_r)$ is of *slow growth* in $y \in \Gamma_r$ if for all $K \subset\subset \Omega$, $L \subset\subset U$ and $\Gamma' \subset\subset \Gamma$ there are constants $c, k > 0$ such that

$$|g(x', x, y)| \le c|y|^{-k} \qquad (x', x, y) \in K \times L \times \Gamma'_r. \tag{2.2.2}$$

The next theorem is a generalization of [**45**, Theorem 4.4.8].

THEOREM 2.2.1. *Let $F \in \mathcal{E}(\Omega\times U\times\Gamma_r)$ be $\mathcal{M}$-almost analytic in the variables $(x, y) \in U\times\Gamma_r$ and of slow growth in the variable $y \in \Gamma_r$. Then the distributional limit u of the sequence $u_\varepsilon = F(\,.\,, .\,, \varepsilon) \in \mathcal{E}(\Omega \times U)$ exists. We say that $u = b_\Gamma(F) \in \mathcal{D}'(\Omega \times U)$ is the boundary value of F. Furthermore, we have*

$$\mathrm{WF}_{\mathcal{M}}\, u \subseteq \big(\Omega \times U\big) \times \big(\mathbb{R}^n \times \Gamma^\circ\big)$$

where $\Gamma^\circ = \{\eta \in \mathbb{R}^d \mid \langle y, \eta\rangle \ge 0\ \ \forall y \in \Gamma\}$ is the dual cone of Γ in $\mathbb{R}^d$.

PROOF. Let $\varphi \in \mathcal{D}(\Omega \times U)$ and $Y_0 \in \Gamma_\delta$. Then there are $K \subset\subset \Omega$, $L \subset\subset U$ such that $\operatorname{supp}\varphi \subseteq K \times L$ and constants $c, k > 0$ exists such that (2.2.2) holds. We set

$$\Phi_\kappa(x', x, y) = \sum_{|\alpha| \le \kappa} \partial_x^\alpha \varphi(x', x) \frac{(iy)^\alpha}{\alpha!}$$

for $\kappa \ge k$. Obviously $F \cdot \Phi_\kappa$ can be extended to a smooth function on $\mathbb{R}^n \times \mathbb{R}^d \times \Gamma_\delta$ that vanishes outside $K \times L \times \Gamma_\delta$. We consider the function

$$u_\varepsilon : \mathbb{R}^2 \ni (\sigma, \tau) \longmapsto F(x', \tilde{x} + \sigma Y_0, \varepsilon + \tau Y_0)\Phi_\kappa(x', \sigma Y_0, \tau Y_0)$$

where $x' \in \mathbb{R}^n$, $\tilde{x} \in Y_0^\perp = \{z \in \mathbb{R}^d \mid \langle z, Y_0 \rangle = 0\}$. If $a < b$ are chosen such that $\varphi(x', \tilde{x} + \sigma Y_0) = 0$ for all $x' \in \mathbb{R}^n$, $\tilde{x} \in Y_0^\perp$ and $\sigma \le a$ or $\sigma \ge b$ then $u_\varepsilon(\sigma, \tau) = 0$ for all $\tau \in [0, 1]$. If $R = [a, b] \times [0, 1]$ then Stokes' Theorem states that

$$\int_{\partial R} u_\varepsilon \, d\zeta = \int_R \frac{\partial u_\varepsilon}{\partial \bar{\zeta}} \, d\bar{\zeta} \wedge d\zeta \tag{2.2.3}$$

where we have set $\zeta = \sigma + i\tau$.

A simple computation gives

$$\begin{aligned} 2i \frac{\partial}{\partial \bar{\zeta}} \big(\Phi_\kappa(x', \tilde{x} + \sigma Y_0, \tau Y_0)\big) &= \sum_{|\alpha| \le \kappa} \sum_{j=1}^{d} \partial_x^{\alpha + e_j} \varphi(x', \tilde{x} + \sigma Y_0) \tau^{|\alpha|} \frac{(iY_0)^{\alpha + e_j}}{\alpha!} \\ &\qquad - \sum_{|\alpha| \le \kappa} \partial_x^\alpha \varphi(x', \tilde{x} + \sigma Y_0) |\alpha| \tau^{|\alpha| - 1} \frac{(iY_0)^\alpha}{\alpha!} \\ &= \sum_{1 \le |\alpha| \le \kappa + 1} \partial_x^\alpha \varphi(x', \tilde{x} + \sigma Y_0) \tau^{|\alpha| - 1} \frac{(iY_0)^\alpha}{\alpha!} \sum_{j=1}^{d} \alpha_j \\ &\qquad - \sum_{1 \le |\alpha| \le \kappa} \partial_x^\alpha \varphi(x', \tilde{x} + \sigma Y_0) |\alpha| \tau^{|\alpha| - 1} \frac{(iY_0)^\alpha}{\alpha!} \\ &= (\kappa + 1) \tau^\kappa \sum_{|\alpha| = \kappa + 1} \partial_x^\alpha \varphi(x', \tilde{x} + \sigma Y_0) \frac{(iY_0)^\alpha}{\alpha!}. \end{aligned}$$

Hence formula (2.2.3) means in detail that

$$\begin{aligned} \int_a^b F(x', \sigma Y_0, \varepsilon) \varphi(x', \sigma Y_0) \, d\sigma &= \int_a^b F(x', \sigma Y_0, \varepsilon + Y_0) \Phi_\kappa(x', \sigma Y_0, Y_0) \, d\sigma \\ &\quad + 2i \int_a^b \int_0^1 \langle \bar{\partial} F(x', \sigma Y_0, \varepsilon + \tau Y_0), Y_0 \rangle \Phi_\kappa(x', \sigma Y_0, \tau Y_0) \, d\tau d\sigma \\ &\quad + (\kappa + 1) \int_a^b \int_0^1 F(x', \sigma Y_0, \varepsilon + \tau Y_0) \tau^\kappa \sum_{|\alpha| = \kappa + 1} \frac{\partial_x^\alpha \varphi}{\beta!} \, d\tau d\sigma \end{aligned}$$

and thus integrating over $\Omega \times Y_0^\perp$ yields

$$\begin{aligned}\int\limits_{\Omega\times U} F(x',x,\varepsilon)\varphi(x',x)\,d\lambda(x',x) &= \int\limits_{\Omega\times U} F(x',x,\varepsilon+Y_0)\Phi_\kappa(x',x,Y_0)\,d\lambda(x',x)\\ &+ 2i\int\limits_{\Omega\times U}\int\limits_0^1 \langle\bar\partial F(x',x,\varepsilon+\tau Y_0),Y_0\rangle\Phi_\kappa(x',x,\tau Y_0)\,d\tau d\lambda(x',x)\\ &+(\kappa+1)\int\limits_{\Omega\times U}\int\limits_0^1 F(x',x,\varepsilon+\tau Y_0)\tau^\kappa\sum_{|\alpha|=\kappa+1}\partial_x^\alpha\varphi(x',x)\frac{(iY_0)^\alpha}{\alpha!}d\lambda(x',x).\end{aligned} \tag{2.2.4}$$

Since by assumption $|\tau^\kappa F(x',x,\varepsilon+\tau Y_0)|\le c$ for some constant c and $\bar\partial_j F$ decreases rapidly for $\Gamma_r \ni y \to 0$ (c.f. the remarks after Lemma 1.1.8) the bounded convergence theorem implies that the right-hand side converges for $\varepsilon \to 0$. Hence we define

$$\begin{aligned}\langle u,\varphi\rangle := &\int\limits_{\Omega\times U} F(x',x,Y_0)\Phi_\kappa(x',x,Y_0)\,d\lambda(x',x)\\ &+2i\int\limits_{\Omega\times U}\int\limits_0^1\langle\bar\partial F(x',x,\tau Y_0),Y_0\rangle\Phi_\kappa(x',x,\tau Y_0)\,d\tau d\lambda(x',x)\\ &+(\kappa+1)\int\limits_{\Omega\times U}\int\limits_0^1 F(x',x,\tau Y_0)\tau^\kappa\sum_{|\alpha|=\kappa+1}\partial_x^\alpha\varphi(x',x)\frac{(iY_0)^\alpha}{\alpha!}\,d\tau d\lambda(x',x).\end{aligned} \tag{2.2.5}$$

Since there is a constant $\tilde C$ only depending on F and $K\times L$ such that

$$|\langle u,\varphi\rangle| \le \tilde C \sup_{(x',x)\in K\times L}\left(\sum_{|\beta|\le\kappa+1}\left|\partial_x^\beta\varphi(x',x)\right|\right)$$

we deduce that the linear form u on $\mathcal{D}(\Omega\times U)$ given by (2.2.5) is a distribution.

Now, let $p_0\in\Omega\times U$ and $\omega_2\times V_2 \subset\subset \omega_1\times V_1\subset\subset\Omega\times U$ two open neighbourhoods of p_0. Using [**45**, Theorem 1.4.2] we can choose a sequence $(\varphi_\kappa)_\kappa\subset\mathcal{D}(\omega_1\times V_1)$ such that $\varphi_\kappa|_{\omega_2\times V_2}\equiv 1$ and for all $\gamma\in\mathbb{N}_0^{n+d}$ we have that

$$\left|D^{\gamma+\beta}\varphi_\kappa\right| \le \left(C_\gamma(\kappa+1)\right)^{|\beta|} \qquad |\beta|\le\kappa+1 \tag{2.2.6}$$

for a constant $C_\gamma\ge 1$ independent of κ. As before we set for each κ

$$\Phi_\kappa(x',x,y) = \sum_{|\alpha|\le\kappa}\partial_x^\alpha\varphi_\kappa(x',x)\frac{(iy)^\alpha}{\alpha!}.$$

We aim to estimate $\widehat{\varphi_\kappa u}$. In order to do so let $(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^d$ and notice that (2.2.5) implies for $\kappa\geq k$

$$\begin{aligned}\widehat{\varphi_\kappa u}(\xi,\eta) &= \Big\langle u, e^{-i\langle\,.\,,(\xi,\eta)\rangle}\varphi_\kappa\Big\rangle\\ &= \int\limits_{\Omega\times U} F(x',x,Y_0)e^{-i(x'\xi+(x+iY_0)\eta)}\Phi_\kappa(x',x,Y_0)\,d\lambda(x',x)\\ &\quad + 2i\int\limits_{\Omega\times U}\int\limits_0^1 \big\langle\bar\partial F(x',x,\tau Y_0),Y_0\big\rangle e^{-i(x'\xi+(x+i\tau Y_0)\eta)}\Phi_\kappa(x',x,\tau Y_0)\,d\tau d\lambda(x',x)\\ &\quad + (\kappa+1)\int\limits_{\Omega\times U}\int\limits_0^1 F(x',x,\tau Y_0)e^{-i(x'\xi+(x+i\tau Y_0)\eta)}\tau^\kappa\sum_{|\alpha|=\kappa+1}\partial_x^\alpha\varphi(x',x)\frac{(iY_0)^\alpha}{\alpha!}\,d\tau d\lambda(x',x)\end{aligned}$$

for some fixed, but arbitrary $Y_0\in\Gamma_r$ (note that k depends on u, $\omega_1\times V_1$ and Y_0). Condition (2.2.6) gives the following estimate for $0\leq\mu\leq\kappa+1$

$$\left|\sum_{|\alpha|=\mu}\partial_x^\alpha\varphi_\kappa(x',x)\frac{(iY)^\alpha}{\alpha!}\right|\leq C_0^\mu(\kappa+1)^\mu\sum_{|\alpha|=\mu}\frac{|Y^\alpha|}{\alpha!}=C_0^\mu(\kappa+1)^\mu\frac{|Y|_1^\mu}{\mu!}$$

where $|Y|_1=\sum_j|Y_j|$ for $Y=(Y_1,\dots,Y_d)\in\mathbb{R}^d$. Hence we have

$$\begin{aligned}\big|\Phi_\kappa(x',x,\tau Y_0)\big| &\leq C_1^{\kappa+1}\\ \left|(\kappa+1)\sum_{|\alpha|=\kappa+1}\partial_x^\alpha\varphi_\kappa(x',x)\frac{(iY_0)^\alpha}{\alpha!}\right| &\leq C_1^{\kappa+1}\end{aligned}$$

for $C_1=2e^{C_0|Y_0|_1}$ and $\tau\in[0,1]$. We obtain

$$\begin{aligned}|\widehat{\varphi_\kappa u}(\xi,\eta)| &\leq C_1^{\kappa+1}e^{\eta Y_0}+2C_1^{\kappa+1}C\int\limits_0^1 h_{\mathcal M}(Q\tau|Y_0|)e^{\tau\eta Y_0}\,d\tau+C_1^{\kappa+1}\int\limits_0^1\tau^{\kappa-k}e^{\tau\eta Y_0}\,d\tau\\ &\leq C_2Q_1^\kappa\Bigg(e^{\eta Y_0}+m_{\kappa-k}\int\limits_0^1\tau^{\kappa-k}e^{\eta Y_0}\Bigg)=C_2Q_1^\kappa\Big(e^{\eta Y_0}+m_\kappa(\kappa-k)!(-Y_0\eta)^{k-\kappa-1}\Big)\end{aligned}$$

for some constants C_2,Q_1 and $Y_0\eta<0$. If we set $\tilde Y_0=(0,Y_0)\in\mathbb{R}^n\times\mathbb{R}^d$ then obviously

$$\big\langle\tilde Y_0,(\xi,\eta)\big\rangle=\langle Y_0,\eta\rangle.$$

Therefore we have for $\kappa\geq k$ and $\zeta=(\xi,\eta)$ that

$$\big|\widehat{\varphi_\kappa u}(\zeta)\big|=C_3Q_1^\kappa\Big(e^{\tilde Y_0\zeta}+m_{\kappa-k}(\kappa-k)!\big(-\tilde Y_0\zeta\big)^{k-\kappa-1}\Big)$$

and $\tilde Y_0\zeta<0$.

Now for any $\zeta_0\in\mathbb{R}^{n+d}$ with $\langle\tilde Y_0,\zeta_0\rangle<0$ we can choose an open cone $V\subseteq\mathbb{R}^{n+d}$ such that $\zeta_0\in V$ and for some constant $c>0$ we have $\langle\tilde Y_0,\zeta\rangle<-c|\zeta|$ if $\zeta\in V$. Furthermore we set $u_\kappa=\varphi_{k+\kappa-1}u$. Clearly the sequence $(u_\kappa)_\kappa$ is bounded in $\mathcal{E}'(\Omega\times U)$ and $u_\kappa|_{\omega_2\times V_2}\equiv u|_{\omega_2\times V_2}$. Also using the inequality $e^{-c|\zeta|}\leq\kappa!(c|\zeta|)^{-\kappa}$ we conclude

$$\big|\hat u_\kappa(\zeta)\big|=C_3Q_1^\kappa\Big(\kappa!(c|\zeta|)^{-\kappa}+m_{\kappa-1}(\kappa-1)!\big(c|\zeta|\big)^{-\kappa}\Big)\leq C_3Q_2^\kappa m_\kappa\kappa!|\zeta|^{-\kappa}\qquad\zeta\in V.$$

Hence $(p_0,\zeta_0)\notin\operatorname{WF}_{\mathcal M}u$ and therefore

$$\operatorname{WF}_{\mathcal M}u\subseteq\big(\Omega\times U\big)\times\big(\mathbb{R}^n\times\Gamma^\circ\big)\setminus\{(0,0)\}$$

□

It is clear that the proof requires only $F \in \mathcal{C}^1$. From now the constants used in the proofs will be generic, i.e. they may change from line to line.

REMARK 2.2.2. If $F \in \mathcal{E}(\Omega \times U \times V)$ is $\mathcal{M}$-almost analytic with respect to the variables $(x, y) \in U \times V$ we will often write $F(x', x+iy)$ or $F(x', z, \bar{z})$ and consider F as a smooth function on $\Omega \times (U + iV)$. If $\Omega = \emptyset$ then we just say that F is $\mathcal{M}$-almost analytic.

EXAMPLE 2.2.3. Consider the holomorphic function $F(z) = \frac{1}{z}$ on $\mathbb{C}\backslash\{0\}$. It is well known that the boundary values of F onto the real line from above and beneath, commonly denoted by

$$\frac{1}{x + i0} = b_+F = \lim_{y\to 0+} \frac{1}{x + iy}$$
$$\frac{1}{x - i0} = b_+F = \lim_{y\to 0+} \frac{1}{x - iy}$$

satisfy the jump relations (c.f. e.g. [**27**]), in particular

$$2i\delta = \frac{1}{x - i0} - \frac{1}{x + i0}.$$

We have that both $\frac{1}{x+i0}$ and $\frac{1}{x-i0}$ are real-analytic outside the origin. Hence the application of Theorem 2.2.1 together with the jump relations imply that

$$\mathrm{WF}_{\mathcal{M}}\left(\frac{1}{x \pm i0}\right) = \{0\} \times \mathbb{R}_{\pm}.$$

There is a partial converse to the last theorem.

THEOREM 2.2.4. *Let $\Gamma \subset \mathbb{R}^n$ be an open convex cone and $u \in \mathcal{D}'(\Omega)$ with $\mathrm{WF}_{\mathcal{M}}\, u \in \Omega \times \Gamma^{\circ}$. If $V \subset\subset \Omega$ and Γ' is an open convex cone with $\overline{\Gamma}' \subseteq \Gamma \cup \{0\}$ then there is an $\mathcal{M}$-almost analytic function F on $V + i\Gamma'_r$ of slow growth for some $r > 0$ such that $u|_V = b_{\Gamma'}(F)$*

PROOF. By [**45**, Theorem 8.4.15] we have that u can be written on a bounded neighbourhood U of V as a sum of a function $f \in \mathcal{E}_{\mathcal{M}}(U)$ and the boundary value of a holomorphic function of slow growth on $U + i\Gamma'_r$ for some r. To obtain the assertion use Corollary 1.1.11 to extend f almost-analytically on V. □

In order to proceed we need a further refinement of a result of Hörmander.

LEMMA 2.2.5. *Let $\Gamma_j \subseteq \mathbb{R}^n\backslash\{0\}$, $j = 1, \dots, N$, be closed cones such that $\bigcup_j \Gamma_j = \mathbb{R}^n\backslash\{0\}$ and $V \subset\subset \Omega$. Any $u \in \mathcal{D}'(\Omega)$ can be written on V as a linear combination $u|_V = \sum_j u_j$ of distributions $u_j \in \mathcal{D}'(V)$ that satisfy*

$$\mathrm{WF}_{\mathcal{M}}\, u_j \subseteq \mathrm{WF}_{\mathcal{M}}\, u \cap \left(V \times \Gamma_j\right)$$

PROOF. Set $v = \varphi u$ where $\varphi \in \mathcal{D}(\Omega)$ such that $\varphi \equiv 1$ on V. [**45**, Corollary 8.4.13] gives the existence of $v_j \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\mathrm{WF}_{\mathcal{M}}\, v_j \subseteq \mathrm{WF}_{\mathcal{M}}\, v \cap \left(\mathbb{R}^n \times \Gamma_j\right).$$

Set $u_j = (v_j)|_U$. □

Together with the above Lemma Theorem 2.2.4 implies

COROLLARY 2.2.6. *Let $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n\backslash\{0\}$. Then $(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$ if and only if there are a neighbourhood U of x_0, open convex cones $\Gamma_1, \dots, \Gamma_N$ with the properties $\xi_0\Gamma_j < 0$, $j = 1, \dots N$ and $\Gamma_j \cap \Gamma_k = \emptyset$ for $j \neq k$, and $\mathcal{M}$-almost analytic functions h_j on $U + i\Gamma_{r_j}$, $r_j > 0$, of slow growth such that*

$$u|_U = \sum_{j=1}^{N} b_{\Gamma_j}(h_j)$$

Proof. W.l.o.g. assume that $\mathrm{WF}_{\mathcal{M}}\, u \neq \emptyset$. If $(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$ one can find closed cones $V_1, \dots, V_N$ with nonempty interior and $V_j \cap V_k$ has measure zero for $j \neq k$ such that ξ_0 is contained in the interior of V_1 and $V_1 \cap \mathrm{WF}_{\mathcal{M}}\, u = \emptyset$ whereas $\xi_0 \notin V_j$ are acute cones and $\mathrm{WF}_{\mathcal{M}}\, u \cap V_j \neq \emptyset$ for $j = 2, \dots, N$. By Lemma 2.2.5 we can write u on an open neighbourhood U of x_0 as a sum $u = u_1 + \sum_{j=2}^{N} u_j$ with u_1 being an ultradifferentiable function defined on U and $u_j \in \mathcal{D}'(U)$ such that $\mathrm{WF}_{\mathcal{M}}\, u_j \subseteq \mathrm{WF}_{\mathcal{M}}\, u \cap V_j$, $j = 2, \dots, N$. The cones $V_2, \dots, V_N$ are the dual cones of open convex cones $\Gamma_2, \dots, \Gamma_N$, i.e. $\Gamma_j^\circ = V_j$. We can choose cones $\Gamma_j' \subset\subset \Gamma_j$ and using Theorem 2.2.4 we find $\mathcal{M}$-almost analytic functions h_j on $U + i\Gamma_{j,r}'$ of slow growth such that $u_j = b_{\Gamma_j'}(h_j)$. It remains to note that $\xi_0 y < 0$ for all $y \in \Gamma_j'$, $j = 2, \dots, N$. □

Let $\Omega_1 \subseteq \mathbb{R}^m$ and $\Omega_2 \subseteq \mathbb{R}^n$ be open. If $F : \Omega_1 \to \Omega_2$ is a $\mathcal{E}_{\mathcal{M}}$-mapping then we denote as in [**45**, page 263] the set of normals by

$$N_F = \big\{ (F(x), \eta) \in \Omega_2 \times \mathbb{R}^n :\ DF(x)\eta = 0 \big\}.$$

where DF denotes the transpose of the Jacobian of F. The following is a generalization of [**45**, Theorem 8.5.1]

Theorem 2.2.7. *For any $u \in \mathcal{D}'(\Omega_2)$ with $N_F \cap \mathrm{WF}_{\mathcal{M}}\, u = \emptyset$ we obtain that the pull-back $F^* u \in \mathcal{D}'(\Omega_1)$ is well defined and*

$$\mathrm{WF}_{\mathcal{M}}\big(F^* u\big) \subseteq F^*\big(\mathrm{WF}_{\mathcal{M}}\, u\big). \tag{2.2.7}$$

Proof. The first part of the statement is [**45**, Theorem 8.2.4]. For the proof of the second part of the theorem assume first that there is an open convex cone Γ such that u is the boundary value of an $\mathcal{M}$-almost analytic function Φ on $\Omega_2 + i\Gamma_r$ of slow growth. Hence $\mathrm{WF}_{\mathcal{M}}\, u \subseteq \Omega_2 \times \Gamma^\circ$. If $x_0 \in \Omega_1$ and $DF(x_0)\eta \neq 0$ for $\eta \in \Gamma^\circ \backslash \{0\}$ then $DF(x_0)\Gamma^\circ$ is a closed convex cone. We claim that

$$\mathrm{WF}_{\mathcal{M}}(F^* u)|_{x_0} \subseteq \big\{ (x_0, DF(x_0)\eta) :\ \eta \in \Gamma^\circ \backslash \{0\} \big\}.$$

We adapt as usual the argument of [**45**]. We can write (see [**45**, page 296])

$$DF(x_0)\Gamma^\circ = \big\{ \xi \in \mathbb{R}^n \mid \langle h, \xi \rangle \geq 0,\ F'(x_0))h \in \Gamma \big\}.$$

If $\tilde{F}$ denotes an $\mathcal{M}$-almost analytic extension of F onto $X_0 + i\mathbb{R}^n$, $X_0 \in \mathcal{U}(x_0)$ relatively compact in Ω_1, which exists due to Theorem 1.1.10, then Taylor's formula implies that

$$\operatorname{Im} \tilde{F}(x + i\varepsilon h) \in \Gamma \qquad x \in X_0$$

for $F'(x_0)h \in \Gamma$ if X_0 and $\varepsilon > 0$ are small.

Recalling (2.2.4) we see that the map

$$\mathbb{R}_{\geq 0} \times \big(\Gamma \cup \{0\}\big) \ni (\varepsilon, y) \longmapsto \tilde{\Phi}(\varepsilon, y) := \Phi\big(\tilde{F}(\,.\, + i\varepsilon h) + iy\big) \in \mathcal{D}'(X_0)$$

is continuous. If $\varepsilon \to 0$ then $\tilde{\Phi}(\varepsilon, y) \to \tilde{\Phi}(0, y) = \Phi(\tilde{F}(\,.\, + 0i) + iy)$ in $\mathcal{D}'$ and if now $y \to 0$ we have by definition $\tilde{\Phi}(0, y) \to F^* u$. On the other hand if first $y \to 0$ then $\tilde{\Phi}(\varepsilon, y) \to \tilde{\Phi}(\varepsilon, 0) = \Phi(\tilde{F}(\,.\, + i\varepsilon h))$. Hence by continuity

$$F^* u = \lim_{\varepsilon \to 0} \Phi\big(\tilde{F}(\,.\, + i\varepsilon h)\big)$$

in $\mathcal{D}'(X_0)$ and by the proof of Theorem 2.2.1

$$\mathrm{WF}_{\mathcal{M}}|_{x_0} \subseteq \{(x_0, \xi) \mid \langle h, \xi \rangle \geq 0\}.$$

The claim follows.

Now suppose that $(F(x_0), \eta_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$. By Corollary 2.2.6 we can write a general distribution u on some neighbourhood U_0 of $F(x_0)$ as $\sum_{j=1}^{N} u_j$ where the distributions u_j, $j = 1, \dots, N$, are the boundary values of some $\mathcal{M}$-almost analytic functions Φ_j on $U_0 + i\Gamma_j$, where the Γ_j are some open convex cones such that $\eta_0 \Gamma_j < 0$ for all $j = 1, \dots, N$. By assumption $DF(x)\eta \neq 0$ when $(F(x), \eta) \in \mathrm{WF}_{\mathcal{M}}\, u$ for $x \in F^{-1}(U_0)$. Hence we can assume that $DF(x)\eta \neq 0$ for $\eta \in \Gamma_j^\circ$ for all $j = 1, \dots, N$ and $x \in F^{-1}(U_0)$ since in the proof of Corollary 2.2.6 the cones Γ_j,

$j = 1, \dots, N$, can be chosen such that the set $\Gamma_j^\circ \backslash \mathrm{WF}_\mathcal{M}\, u$ has small measure. By the arguments above we have for a small neighbourhood V of x_0 that

$$F^* u|_V = \sum_{j=1}^{N} F^* u_j|_V$$

and $\mathrm{WF}_\mathcal{M}(F^* u_j)|_{x_0} \subseteq \{(x_0, DF(x_0)\eta) \mid \eta \in \Gamma_j^\circ \backslash \{0\}\}$ for all j. However, since $\eta_0 \Gamma_j < 0$ it follows that $(x_0, DF(x_0)\eta_0) \notin \mathrm{WF}_\mathcal{M}(F^* u_j)$ and therefore $(x_0, DF(x_0)\eta_0) \notin \mathrm{WF}_\mathcal{M}(F^* u)$. □

Remark 2.2.8. If F is an $\mathcal{E}_\mathcal{M}$-diffeomorphism we obtain from Theorem 2.2.7 that

$$\mathrm{WF}_\mathcal{M}\big(F^* u\big) = F^*\big(\mathrm{WF}_\mathcal{M}\, u\big).$$

Hence if M is an $\mathcal{E}_\mathcal{M}$-manifold and $u \in \mathcal{D}'(M)$ we can define $\mathrm{WF}_\mathcal{M}\, u$ invariantly as a subset of $T^*M \backslash \{0\}$. More precisely, there is a subset K_u of T^*M such that the diagram

$$\begin{array}{ccc} & K_u & \\ \swarrow & & \searrow \\ T^*\varphi(U \cap V) \supseteq \mathrm{WF}_\mathcal{M}\, v_1 & \xrightarrow{\quad \rho^* \quad} & \mathrm{WF}_\mathcal{M}\, v_2 \subseteq T^*\psi(U \cap V) \end{array}$$

commutes for any two charts φ and ψ of M on $U \subseteq M$ and $V \subseteq M$, respectively. We have set $\rho = \psi \circ \varphi^{-1}$, $v_1 = \varphi^* u \in \mathcal{D}'(\varphi(U \cap V))$ and $v_2 = \psi^* u \in \mathcal{D}'(\psi(U \cap V))$. It follows that $K_u \subseteq T^*M \backslash \{0\}$ has to be closed and fiberwise conic. We set $\mathrm{WF}_\mathcal{M}\, u := K_u$.

Analogously we define the wavefront set of a distribution $u \in \mathcal{D}'(M, E)$ with values in an ultradifferentiable vector bundle locally by setting

$$\mathrm{WF}_\mathcal{M}\, u|_V = \bigcup_{j=1}^{\nu} u_j$$

where $V \subseteq M$ is an open subset such that there is a local basis $\omega^1, \dots, \omega^\nu$ of $\mathcal{E}_\mathcal{M}(V, E)$ and $u_j \in \mathcal{D}'(V)$ are distributions on V such that

$$u|_V = \sum_{j=1}^{\nu} u_j \omega^j.$$

We close this section by observing that Theorem 2.2.7 allows us to strengthen a uniqueness result of Boman [**14**]:

Theorem 2.2.9. *Let $\mathcal{M}$ be a quasianalytic weight sequence and $S \subseteq \mathbb{R}^n$ an $\mathcal{E}_\mathcal{M}$-submanifold. If u is a distribution defined on a neighbourhood of S such that*

$$\mathrm{WF}_\mathcal{M}\, u \cap N^* S = \emptyset$$

and

$$\partial^\alpha u|_S = 0 \qquad \forall \alpha \in \mathbb{N}_0^n,$$

then u vanishes on some neighbourhood of S.

Indeed, locally S is diffeomorphic to

$$S' = \big\{(x', x'') \in \mathbb{R}^{m+d} \mid x'' = 0\big\} \subseteq \mathbb{R}^n$$

and the assumptions of the theorem translate to the corresponding conditions for the pullback $w = F^* u$ where $F : \mathbb{R}^n \to \mathbb{R}^n$ is the local $\mathcal{E}_\mathcal{M}$-diffeomorphism that maps S' to S. Then the proof of Theorem 1 in [**14**] gives $w = 0$ in a neighbourhood of S'.

2.3. A generalized version of Bony's Theorem

We have seen that for a distribution u the wavefront set $\mathrm{WF}_{\mathcal{M}}\, u$ can be described either using the Fourier transform or by its $\mathcal{M}$-almost analytic extensions. The similar fact is true for the analytic wavefront set using holomorphic extensions. The latter was the original approach of Sato [**72**]. However, [**20**] used the classical FBI-Transform to describe the set of microlocal analytic singularities. It was Bony [**18**] who proved that all three methods describe actually the same set. In the ultradifferentiable setting [**24**], see also [**47**], used the FBI transform to define an ultradifferentiable singular spectrum for Fourier hyperfunctions. However, they did not mention how this singular spectrum in the case of distributions may be related to $\mathrm{WF}_{\mathcal{M}}$ as defined by Hörmander. Our next aim is to show an ultradifferentiable version of Bony's theorem. We will work in the generalized setting of Berhanu and Hoepfner[**9**]. We shall note that recently Hoepfner and Medrado [**39**] also proved a characterization of the ultradifferentiable wavefront set by this generalized FBI transform for a certain class of non-quasianalytic weight sequences.

Let p be a real, homogeneous, positive, elliptic polynomial of degree $2k$, $k \in \mathbb{N}$, on $\mathbb{R}^n$, i.e.

$$p(x) = \sum_{\alpha=2k} a_\alpha x^\alpha \qquad a_\alpha \in \mathbb{R},$$

and there are constants $c, C > 0$ such that

$$c|x|^{2k} \leq p(x) \leq C|x|^{2k} \qquad x \in \mathbb{R}^n.$$

Let $c_p^{-1} = \int e^{-p(x)} dx$. As in [**9**, section 4] we consider the generalized FBI transform with generating function e^{-p} of a distribution of compact support $u \in \mathcal{E}'(\mathbb{R}^n)$, i.e.

$$\mathfrak{F}u(t,\xi) = c_p \Big\langle u(x), e^{i\xi(t-x)} e^{-|\xi| p(t-x)} \Big\rangle.$$

The inversion formula is

$$u = \lim_{\varepsilon \to \infty} \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\xi(x-t)} e^{-\varepsilon|\xi|^2} \mathfrak{F}u(t,\xi) |\xi|^{\frac{n}{2k}}\, dt d\xi \tag{2.3.1}$$

where of course the distributional limit is meant.

Theorem 2.3.1. *Let $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$. Then $(x_0,\xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$ if and only if there is a test function $\psi \in \mathcal{D}(\Omega)$ with $\psi|_U \equiv 1$ for some neighbourhood U of x_0 such that*

$$\sup_{(t,\xi) \in V \times \Gamma} e^{\omega_{\mathcal{M}}(\gamma|\xi|)} \big|\mathfrak{F}(\psi u)(t,\xi)\big| < \infty \tag{2.3.2}$$

for some conic neighbourhood $V \times \Gamma$ of (x_0,ξ_0) and some constant $\gamma > 0$.

Proof. First, assume that $(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$. By Corollary 2.2.6 we know that for some neighbourhood U of x_0

$$u|_U = \sum_{j=1}^N b_{\Gamma^j}(F_j)$$

where F_j are $\mathcal{M}$-almost analytic on $U \times \Gamma^j_{r_j}$ for cones Γ^j with $\xi_0 \Gamma^j < 0$. Hence it suffices to prove the necessity of (2.3.2) for $u = b_\Gamma(F)$ being the boundary value of an $\mathcal{M}$-almost analytic function on $U \times \Gamma_d$ where Γ is a cone with the property that $\xi_0\Gamma < 0$. W.l.o.g. $x_0 = 0$ and let $r > 0$ such that $B_{2r}(0) \subset\subset U$ and $\psi \in \mathcal{D}(B_{2r}(0))$ such that $\psi|_{B_r(0)} \equiv 1$. Furthermore we choose $v \in \Gamma_d$ and set

$$Q(t,\xi,x) = i\xi(t-x) - |\xi| p(t-x).$$

Then

$$\mathfrak{F}(\psi u)(t,\xi) = \lim_{\tau \to 0+} \int\limits_{B_{2r}(0)} e^{Q(t,\xi,x+i\tau v)} \psi(x) F(x+i\tau v)\, dx.$$

As in the proof of Theorem 4.2 in [**9**] we put $z = x + iy$, $\psi(z) = \psi(x)$ and

$$D_\tau := \{x + i\sigma v \in \mathbb{C}^n \mid x \in B_{2r} = B_{2r}(0),\ \tau \le \sigma \le \lambda\}$$

for some $\lambda > 0$ to be determined later and consider the n-form

$$e^{Q(t,\xi,z)}\psi(z)F(z)\,dz_1 \wedge \cdots \wedge dz_n.$$

Since $\psi \in \mathcal{D}(B_{2r}(0))$ Stokes' theorem implies

$$\begin{aligned}\int\limits_{B_{2r}} e^{Q(t,\xi,x+i\tau v)}\psi(x)F(x+i\tau v)\,dx &= \int\limits_{B_{2r}} e^{Q(t,\xi,x+i\lambda v)}\psi(x)F(x+i\lambda v)\,dx \\ &\quad + \sum_{j=1}^{n}\int\limits_{D_\tau} e^{Q(t,\xi,z)}\frac{\partial}{\partial \bar z_j}\big(\psi(z)F(z)\big)\,d\bar z_j \wedge dz_1 \wedge \cdots \wedge dz_n \\ &= \int\limits_{B_{2r}} e^{Q(t,\xi,x+i\lambda v)}\psi(x)F(x+i\lambda v)\,dx \\ &\quad + \sum_{j=1}^{n}\int\limits_{B_{2r}}\int\limits_{\tau}^{\lambda} e^{Q(t,\xi,x+i\sigma v)}\frac{\partial \psi}{\partial \bar z_j}(x+i\sigma v)F(x+i\sigma v)\,d\sigma dx \\ &\quad + \sum_{j=1}^{n}\int\limits_{B_{2r}}\int\limits_{\tau}^{\lambda} e^{Q(t,\xi,x+i\sigma v)}\psi(x+i\sigma v)\frac{\partial F}{\partial \bar z_j}(x+i\sigma v)\,d\sigma dx.\end{aligned} \tag{2.3.3}$$

We need to estimate the integrals on the right-hand side of (2.3.3). Since $\xi_0 \cdot v < 0$ there is an open cone Γ_1 containing ξ_0 such that $\xi \cdot v \le -c_0|\xi||v|$ for all $\xi \in \Gamma_1$ and some constant $c_0 > 0$. We note that for $\xi \in \Gamma_1$ and t in some bounded neighbourhood of the origin we have

$$\begin{aligned}\operatorname{Re} Q(t,\xi,x+i\lambda v) &= \lambda(\xi v) - |\xi|\operatorname{Re} p(t-x-i\lambda v) \\ &= \lambda(\xi v) - |\xi|\big(\operatorname{Re} p(t-x) + O(\lambda^2)|v|^2\big) \\ &\le \lambda(\xi v) - c|\xi|\big(|t-x|^{2k} + O(\lambda^2)|v|^2\big) \\ &\le -c_0\lambda|v||\xi| + O\big(\lambda^2\big)|\xi|.\end{aligned}$$

Hence for λ small enough

$$\operatorname{Re} Q(t,\xi,x+i\lambda v) \le -\frac{c_0}{2}\lambda|v||\xi| \tag{2.3.4}$$

where $\xi \in \Gamma_1$, $x \in B_{2r}$ and t is in a bounded neighbourhood V of 0. We conclude that

$$\left|\int\limits_{B_{2r}} e^{Q(t,\xi,x+i\lambda v)}\psi(x)F(x+i\lambda v)\,dx\right| \le C_1 e^{-\gamma_1|\xi|}$$

for some constants $\gamma_1, C_1 > 0$ and $(t,\xi) \in V \times \Gamma_1$. We note that (M4) implies that $\omega_{\mathcal{M}}(t) = O(t)$ for $t \to \infty$, c.f. e.g. [**48**] or [**15**], thence

$$\left|\int\limits_{B_{2r}} e^{Q(t,\xi,x+i\lambda v)}\psi(x)F(x+i\lambda v)\,dx\right| \le C_1 e^{-\omega_{\mathcal{M}}(\gamma_1|\xi|)}$$

for $(t,\xi) \in V \times \Gamma_1$.

On the other hand we can also estimate

$$\begin{aligned}\operatorname{Re} Q(t,\xi,x+i\sigma v) &\le \sigma(\xi v) - c|t-x|^{2k}|\xi| + O\big(\lambda^2\big)|\xi| \\ &\le -c|t-x|^{2k}|\xi| + O\big(\lambda^2\big)|\xi|\end{aligned}$$

since $\xi v < 0$ for all $\xi \in \Gamma_1$. If $x \in \operatorname{supp}(\partial\psi/\partial\bar{z}_j)$ then $|x| \geq r$. Therefore if $|t| \leq r/2$ and λ small enough we obtain that there is a constant $\gamma_2 > 0$ such that

$$\operatorname{Re} Q(t,\xi,x+i\sigma v) \leq -\gamma_2|\xi|$$

for all $\xi \in \Gamma_1$. Hence

$$\left| \sum_{j=1}^{n} \int_{B_{2r}} \int_{\tau}^{\lambda} e^{Q(t,\xi,x+i\sigma v)} \frac{\partial\psi}{\partial\bar{z}_j}(x+i\sigma v)F(x+i\sigma v)\, d\sigma dx \right| \leq C_2 e^{-\gamma_2|\xi|} \leq C_2 e^{-\omega_{\mathcal{M}}(\gamma_2|\xi|)}$$

for $\xi \in \Gamma_1$, $|t| \leq r/2$ and all $0 < \tau < \lambda$.

In order to estimate the third integral in (2.3.3) we remark that by (2.3.4) we have for a generic constant $C_3 > 0$ and all $k \in \mathbb{N}_0$ that

$$\begin{aligned}\left| \sum_{j=1}^{n} \int_{B_{2r}} \int_{\tau}^{\lambda} e^{Q(t,\xi,x+i\sigma v)} \psi(x) \frac{\partial F}{\partial\bar{z}_j}(x+i\sigma v)\, d\sigma dx \right| &\leq C_3 \int_0^\infty e^{-c_0\sigma|v||\xi|} h_{\mathcal{M}}(Q\sigma|v|)\, d\sigma \\ &\leq C_3 \int_0^\infty e^{-c_0\sigma|v||\xi|} Q^k\sigma^k|v|^k m_k\, d\sigma \\ &= C_3 Q^k m_k c_0^{-k}|\xi|^{-k}k! \\ &= C_3 Q_1^k M_k|\xi|^{-k}.\end{aligned}$$

Hence by Lemma 1.1.8

$$\begin{aligned}\left| \sum_{j=1}^{n} \int_{B_{2r}} \int_{\tau}^{\lambda} e^{Q(t,\xi,x+i\sigma v)} \psi(x) \frac{\partial F}{\partial\bar{z}_j}(x+i\sigma v)\, d\sigma dx \right| &\leq C_3 \tilde{h}_{\mathcal{M}}\big(Q_1|\xi|^{-1}\big) \\ &\leq C_3 e^{-\omega_{\mathcal{M}}(Q_2|\xi|)}.\end{aligned}$$

In view of (2.3.3) we have shown that for $\xi \in \Gamma_1$ and t in a small enough neighbourhood of 0 there are constants $C, Q > 0$ such that

$$\left| \int_{B_{2r}} e^{Q(t,\xi,x+i\tau v)} \psi(x) F(x+i\tau v)\, dx \right| \leq C e^{-\omega_{\mathcal{M}}(Q|\xi|)}.$$

Note that in the estimate the constants C and Q depend on λ but not on $\tau < \lambda$. Thus (2.3.2) is proven.

On the other hand, assume that (2.3.2) holds for a point (x_0,ξ_0), i.e. that there is a neighbourhood V of x_0, an open cone $\Gamma \subseteq \mathbb{R}^n$ containing ξ_0 and constants $C, \gamma > 0$ such that

$$\big|\mathfrak{F}(\psi u)(x,\xi)\big| \leq C e^{-\omega_{\mathcal{M}}(\gamma|\xi|)} \qquad x \in V,\ \xi \in \Gamma \tag{2.3.5}$$

for some test function $\psi \in \mathcal{D}(\Omega)$ that is 1 near x_0. We may assume that $x_0 = 0$. We have to prove that $(0,\xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$ or, equivalently, $(0,\xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, v$ where $v = \psi u$. We invoke the inversion formula (2.3.1) for the FBI transform

$$v = \lim_{\varepsilon\to\infty} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\xi(x-t)} e^{-\varepsilon|\xi|^2} \mathfrak{F}v(t,\xi)|\xi|^{\frac{n}{2k}}\, dt d\xi$$

and split the occuring integral into 4 parts

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\xi(x-t)} e^{-\varepsilon|\xi|^2} \mathfrak{F}v(t,\xi)|\xi|^{\frac{n}{2k}}\, dt d\xi = I_1^\varepsilon(x) + I_2^\varepsilon(x) + I_3^\varepsilon(x) + I_4^\varepsilon(x) \tag{2.3.6}$$

where

$$I_1^\varepsilon(x) = \int\limits_{\mathbb{R}^n} \int\limits_{|t|\leq a} e^{i\xi(x-t)} e^{-\varepsilon|\xi|^2} \mathfrak{F}v(t,\xi)|\xi|^{\frac{n}{2k}}\, dtd\xi$$

$$I_2^\varepsilon(x) = \int\limits_{|\xi|\leq B} \int\limits_{a\leq|t|\leq A} e^{i\xi(x-t)} e^{-\varepsilon|\xi|^2} \mathfrak{F}v(t,\xi)|\xi|^{\frac{n}{2k}}\, dtd\xi$$

$$I_3^\varepsilon(x) = \int\limits_{\mathbb{R}^n} \int\limits_{|t|\geq A} e^{i\xi(x-t)} e^{-\varepsilon|\xi|^2} \mathfrak{F}v(t,\xi)|\xi|^{\frac{n}{2k}}\, dtd\xi$$

$$I_4^\varepsilon(x) = \int\limits_{|\xi|\geq B} \int\limits_{a\leq|t|\leq A} e^{i\xi(x-t)} e^{-\varepsilon|\xi|^2} \mathfrak{F}v(t,\xi)|\xi|^{\frac{n}{2k}}\, dtd\xi$$

for certain constants a, A and B to be determined. We modify the approach in [**11, 9**] and analogously to the analytic case we are going to show that the last three integrals converge for ε tending to 0 to holomorphic functions that are defined near the origin in $\mathbb{C}^n$ without using (2.3.5). Our assumption that (2.3.5) holds will allow us to prove that I_1^ε converge to a distributions that can be written as the sum of boundary values of certain $\mathcal{M}$-almost analytic functions.

We begin with the easiest case. We see immediately that for any choice of these constants the function I_2^ε extends to an entire function on $\mathbb{C}^n$ and if $\varepsilon \to 0$ these functions converge uniformly on compact subsets to the entire function

$$I_2(z) = \int\limits_{|\xi|\leq B} \int\limits_{a\leq|t|\leq A} e^{i\xi(z-t)} \mathfrak{F}v(t,\xi)|\xi|^{\frac{n}{2k}}\, dtd\xi.$$

If we choose $A \geq 4$ large enough for

$$\operatorname{supp}(v) \subseteq \left\{ y \in \mathbb{R}^n \;\middle|\; |y| \leq \frac{A}{4} \right\} \tag{2.3.7}$$

to hold, then we have similar to before that for $|t| \geq A$

$$\begin{aligned}
\operatorname{Re} Q(t,\xi,x) &= -p(t-x)|\xi| \\
&\leq -c|t-x|^{2k}|\xi| \\
&\leq -c|\xi|\big(|t| - A/4\big)^{2k} \\
&= -c|\xi|\left(|t|^2 - \frac{1}{2}|t|A + \frac{A^2}{2^4}\right)^k \\
&\leq -c|\xi|\left(\frac{1}{2}|t|^2 + \frac{A^2}{2^4}\right)^k \\
&\leq -c|\xi|\sum_{j=1}^k \binom{k}{j}\frac{|t|^{2j}}{2^j}\frac{A^{2(k-j)}}{2^{4(k-j)}} \\
&\leq -c|\xi|\left(\frac{|t|^{2k}}{2^k} + \frac{A^{2k}}{2^{4k}}\right) \\
&\leq -c|\xi|\left(\frac{|t|}{2} + \frac{A}{4}\right).
\end{aligned}$$

Hence

$$|\mathfrak{F}v(t,\xi)| \leq Ce^{-\tilde{c}|\xi|\left(|t|+\frac{A}{2}\right)}$$

for some generic constants C and $\tilde{c}$ independent from ξ and thus we conclude that

$$\begin{aligned}\left|\int\limits_{|t|\geq A} e^{i\xi t}\mathfrak{F}v(t,\xi)\,dt\right| &\leq Ce^{-\tilde{c}\frac{A}{2}|\xi|}\int\limits_A^\infty \rho^{n-1}e^{-\tilde{c}|\xi|\rho}\,d\rho \\ &= Ce^{-\tilde{c}\frac{A}{2}|\xi|}\left(\frac{A^{n-1}e^{-\tilde{c}|\xi|A}}{\tilde{c}|\xi|}+\frac{n-1}{\tilde{c}|\xi|}\int\limits_A^\infty \rho^{n-2}e^{-\tilde{c}|\xi|}\,d\rho\right) \\ &\leq Ce^{-\tilde{c}A|\xi|}\end{aligned}$$

when $|\xi|\geq 1$ and the constants do not depend on ξ. But this means

$$\left|e^{i\xi(x+iy)-\varepsilon|\xi|^2}|\xi|^{\frac{n}{2k}}\int\limits_{|t|\geq A} e^{-i\xi t}\mathfrak{F}v(t,\xi)\,dt\right| \leq C|\xi|^{\frac{n}{2k}}e^{(-c_1+|y|)|\xi|}$$

and hence

$$I_3(z)=\int\limits_{\mathbb{R}^n}\int\limits_{|t|\geq A} e^{i\xi(z-t)}\mathfrak{F}v(t,\xi)|\xi|^{\frac{n}{2k}}\,dtd\xi$$

constitutes a holomorphic function near the origin of $\mathbb{C}^n$. Therefore we obverse that the entire functions I_3^ε converge uniformly in some neighbourhood of 0 to I_3 for $\varepsilon\to 0$.

In order to examine I_4^ε we write

$$I_4^\varepsilon(x)=\iint\limits_{\substack{|\xi|\geq B\\ a\leq|t|\leq A}}|\xi|^{\frac{n}{2k}}\left\langle v(y),e^{i(x-y)\xi-|\xi|p(t-y)-\varepsilon|\xi|^2}\right\rangle_y d\xi dt.$$

Since $v\in\mathcal{E}'(\Omega)$ there has to be a sequence $v_j\in\mathcal{D}(\Omega)$ such that $v_j\to v$ in $\mathcal{E}'$ and without loss of generality $\operatorname{supp} v_j\subseteq K=\left\{y\in\mathbb{R}^n \mid |y|\leq\frac{A}{2}\right\}$. Then

$$I_4^\varepsilon(x)=\lim_{j\to\infty}\iint\limits_{\substack{|\xi|\geq B\\ a\leq|t|\leq A}}|\xi|^{\frac{n}{2k}}\int\limits_K v_j(y)e^{i(x-y)\xi-|\xi|p(t-y)-\varepsilon|\xi|^2}\,dyd\xi dt$$

By the Theorem of Fubini and the exponential decrease in the variable ξ we deduce

$$\iint\limits_{\substack{|\xi|\geq B\\ a\leq|t|\leq A}}|\xi|^{\frac{n}{2k}}\int\limits_K v_j(y)e^{i(x-y)\xi-|\xi|p(t-y)-\varepsilon|\xi|^2}\,dyd\xi dt=\iint\limits_{\substack{a\leq|t|\leq A\\ K}} v_j(y)\int\limits_{|\xi|\geq B}|\xi|^{\frac{n}{2k}}e^{i(x-y)\xi-|\xi|p(t-y)-\varepsilon|\xi|^2}\,d\xi dydt$$

and thus

$$I_4^\varepsilon(x)=\langle v,G^\varepsilon(x,y)\rangle_y$$

where

$$G^\varepsilon(x,y):=\int\limits_{a\leq|t|\leq A}\int\limits_{|\xi|\geq B}|\xi|^{\frac{n}{2k}}e^{i(x-y)\xi-|\xi|p(t-y)-\varepsilon|\xi|^2}\,d\xi dt.$$

Note that G^ε and therefore also I_4^ε extend to entire functions for all $\varepsilon>0$.

We recall from **[11]** that the function $g(\xi)=\log|\xi|$ has a holomorphic extension into the region $W=\{\zeta\in\mathbb{C}^n \mid |\operatorname{Re}\zeta|>|\operatorname{Im}\zeta|\}$ which we denote by $\log\langle\zeta\rangle$, where

$$\log\langle\zeta\rangle=\frac{1}{2}\log\sum_{j=1}^n\zeta_j^2=\log\left(\sum_{j=1}^n\zeta_j^2\right)^{\frac{1}{2}}$$

for a suitable branch of the complex logarithm. Of course, the function $g_1(\zeta) = \langle\zeta\rangle^{\frac{1}{2k}}$ and $g_2(\zeta) = \langle\zeta\rangle$ are also holomorphic on W. We consider the exact form

$$F^\varepsilon(\zeta; x, y, t) = g_1(\zeta)^n e^{i(x-y)\zeta - g_2(\zeta)p(t-y) - \varepsilon g_2(\zeta)^2}\, d\zeta_1 \wedge \cdots \wedge d\zeta_n$$

on W and the n-cycle

$$\Gamma_R = \Gamma_R^1 \cup \Gamma^2 \cup \Gamma_R^3 \cup \Gamma_R^4$$

consisting of the regions

$$\begin{aligned}
\Gamma_R^1 &= \{\zeta \in \mathbb{C}^n \mid \operatorname{Im}\zeta = 0,\, B \le |\operatorname{Re}\zeta| \le R\} \\
\Gamma^2 &= \{\zeta \in \mathbb{C}^n \mid |\operatorname{Re}\zeta| = B,\, \operatorname{Im}\zeta = \sigma s(x-y),\ 0 \le \sigma \le 1\} \\
\Gamma_R^3 &= \{\zeta \in \mathbb{C}^n \mid \zeta = \xi + is|\xi|(x-y),\ \xi \in \mathbb{R}^n,\ B \le |\xi| \le R\} \\
\Gamma_R^4 &= \{\zeta \in \mathbb{C}^n \mid |\operatorname{Re}\zeta| = R,\, \operatorname{Im}\zeta = \sigma s(x-y),\ 0 \le \sigma \le 1\}
\end{aligned}$$

where s is a parameter that is later specified and $R > B$. Stokes' Theorem tells us that for x, y and t fixed we have

$$\int_{\Gamma_R} F^\varepsilon(\zeta; x, y, t) = 0.$$

If $R \to \infty$ we observe that $\int_{\Gamma_R^4} F^\varepsilon(\zeta; x, y, t) \to 0$ uniformly for x, y and t varying in compact subsets. As a result we obtain that

$$\begin{aligned}
G^\varepsilon(x, y) = &\int_{a\le|t|\le A} \int_{\Gamma^3} g_1(\zeta)^{\frac{n}{2k}} e^{i(x-y)\zeta - g_2(\zeta)p(t-y) - \varepsilon g_2(\zeta)^2}\, d\zeta dt \\
&- \int_{a\le|t|\le A} \int_{\Gamma^2} g_1(\zeta)^{\frac{n}{2k}} e^{i(x-y)\zeta - g_2(\zeta)p(t-y) - \varepsilon g_2(\zeta)^2}\, d\zeta dt
\end{aligned} \tag{2.3.8}$$

where $\Gamma^3 = \{\zeta \in \mathbb{C}^n \mid \zeta = \xi + is|\xi|(x-y),\ \xi \in \mathbb{R}^n,\ B \le |\xi|\}$.

Since Γ_2 is compact we conclude that the second integral on the right-hand side constitutes an entire function that converges to an entire function for ε tending to 0.

On the other hand let us consider

$$P_\varepsilon(z, y, t, \xi) := i(z-y)\xi - s|z-y|^2|\xi| - g_2(\zeta(\xi))p(t-y)\varepsilon g_2(\zeta(\xi))^2$$

with $\zeta(\xi) := \xi + is|\xi|(\operatorname{Re} z - y)$. We need to estimate $\operatorname{Re} P_\varepsilon$. If we assume that $|z| \le \delta$ for δ small, $|y| \le \frac{A}{2}$ (recall (2.3.7)) and $s = s(\delta, A)$ small enough then

$$s^2|z-y|^2 \le \frac{1}{2}.$$

We conclude for $|z| \le \delta$ and $|y| \le \frac{A}{2}$ that

$$\begin{aligned}
\operatorname{Re} P_\varepsilon(z, y, t, \xi) &\le |\operatorname{Im} z||\xi| - \big(s|z-y|^2 + p(t-y)\big)|\xi| - \varepsilon|\xi|^2\big(1 - s^2|z-y|^2\big) \\
&\le \delta|\xi| - \big(s|z-y|^2 + p(t-y)\big)|\xi| - \frac{\varepsilon}{2}|\xi|^2 \\
&\le \delta|\xi| + \min\big(-s|z-y|^2, -p(t-y)\big)|\xi| - \frac{\varepsilon}{2}|\xi|^2.
\end{aligned}$$

If $|y| \le \frac{a}{2}$ then

$$\min\big(-s|z-y|^2, -p(t-y)\big) \le -c|t-y|^{2k} \le -c\Big(\frac{a}{2}\Big)^{2k}.$$

On the other hand, if $\frac{a}{2} \le |y| \le \frac{A}{2}$ and $\delta \le \frac{a}{2}$ then

$$\min\big(-s|z-y|^2, -p(t-y)\big) \le -s\Big(\frac{a}{4}\Big)^2.$$

So if we choose $\delta > 0$ small enough and let $z \in B_\delta(0) \subseteq \mathbb{C}^n$, $|y| \le \frac{A}{2}$ and $|t| \ge a$ then

$$\operatorname{Re} P_\varepsilon(z, y, t, \xi) \le -c'|\xi|$$

for some constant $c' > 0$. It follows that the first integral in (2.3.8) extends to an entire function with respect to the variable x and converges uniformly for z in a small neighbourhood of the origin and $|y| \le \frac{A}{4}$ to

$$\int\limits_{a\le|t|\le A}\int\limits_{\Gamma_3} g_1(\zeta)^{\frac{n}{2k}} e^{i(x-y)\zeta - g_2(\zeta)p(t-y)}\, d\zeta dt.$$

This fact implies the uniform convergence of $I_4^\varepsilon(z) = \langle v, G^\varepsilon(z, \,.\,)\rangle$ to the holomorphic function $I_4(z) = \langle v, G(z, \,.\,)\rangle$ as long as z is in a small neighbourhood of 0 in $\mathbb{C}^n$.

It remains to look at I_1^ε. Suppose that a is small enough such that $B_a(0) \subseteq V$. Let $\mathcal{C}_j$, $1 \le j \le N$ be open, acute cones such that

$$\mathbb{R}^n = \bigcup_{j=1}^{N} \overline{\mathcal{C}}_j$$

and the intersection $\overline{\mathcal{C}}_j \cap \overline{\mathcal{C}}_k$ has measure zero for $j \ne k$. Furthermore, let $\xi_0 \in \mathcal{C}_1$, $\mathcal{C}_1 \subseteq \Gamma$ and $\xi_0 \notin \overline{\mathcal{C}}_j$ for $j \ne 1$. In particular that means that (2.3.5) holds on $B_a(0) \times \mathcal{C}_1$, i.e.

$$\left|\mathfrak{F}(\psi u)(x,\xi)\right| \le Ce^{-\omega_{\mathcal{M}}(\gamma|\xi|)} \qquad x \in B_a(0),\ \xi \in \mathcal{C}_1 \tag{2.3.9}$$

Furthermore for $j = 2, \dots, N$ we can choose open cones Γ_j with the property that $\xi_0\Gamma_j < 0$ and there is some positive constant c_j such that

$$\langle v, \xi\rangle \ge c_j|v|\cdot|\xi| \qquad \forall v \in \Gamma_j,\ \forall \xi \in \mathcal{C}_j. \tag{2.3.10}$$

We set

$$f_j^\varepsilon(x+iy) = \int\limits_{\mathcal{C}_j}\int\limits_{B_a(0)} e^{i\xi(x+iy-t)-\varepsilon|\xi|^2}\mathfrak{F}v(t,\xi)|\xi|^{\frac{n}{2k}}\, dt d\xi$$

for $j \in \{2, \dots, N\}$. Note that each f_j^ε is entire if $\varepsilon > 0$ and for ε tending to 0 the functions f_j^ε converge uniformly on compact subsets of the wedge $\mathbb{R}^m + i\Gamma_j$ to

$$f_j(x+iy) = \int\limits_{\mathcal{C}_j}\int\limits_{B_a(0)} e^{i\xi(x+iy-t)}\mathfrak{F}v(t,\xi)|\xi|^{\frac{n}{2k}}\, dt d\xi$$

which are also holomorphic on $\mathbb{R}^m \times i\Gamma_j$ due to (2.3.10).

Similarly we define

$$f_1^\varepsilon(x) = \int\limits_{\mathcal{C}_1}\int\limits_{B_a(0)} e^{i\xi(x-t)-\varepsilon|\xi|^2}\mathfrak{F}v(t,\xi)|\xi|^{\frac{n}{2k}}\, dt d\xi$$

and

$$f_1(x) = \int\limits_{\mathcal{C}_1}\int\limits_{B_a(0)} e^{i\xi(x-t)}\mathfrak{F}v(t,\xi)|\xi|^{\frac{n}{2k}}\, dt d\xi.$$

The functions f_1^ε, $\varepsilon > 0$, extend to entire functions whereas f_1 is smooth due to (2.3.9) since $e^{-\omega_{\mathcal{M}}}$ is rapidly decreasing (c.f. the remark after the proof of Lemma 1.1.3). This decrease also shows that f_1^ε converges uniformly to f_1 in a neighbourhood of 0 since

$$\begin{aligned}\left|f_1(x) - f_1^\varepsilon(x)\right| &\le \int\limits_{\mathcal{C}_1}\int\limits_{B_a(0)} \left|\mathfrak{F}v(t,\xi)\right||\xi|^{\frac{n}{2k}}\left|1 - e^{-\varepsilon|\xi|^2}\right| dt d\xi \\ &\le C\int\limits_{\mathcal{C}_1} |\xi|^{\frac{n}{2k}} e^{-\omega_{\mathcal{M}}(\gamma|\xi|)}\left|1 - e^{-\varepsilon|\xi|^2}\right| d\xi\end{aligned}$$

and the last integral converges to 0 by the monotone convergence theorem.

In fact $f_1 \in \mathcal{E}_{\mathcal{M}}$ because

$$\begin{aligned}
\left|D^\alpha f_1(x)\right| &\leq \int_{\mathcal{C}_1} |\xi|^{\frac{n}{2k}} \left|\xi^\alpha \mathfrak{F} v(t,\xi)\right| dt d\xi \\
&\leq C \int_{\mathcal{C}_1} |\xi|^{\frac{n}{2k}+|\alpha|} e^{-\omega_{\mathcal{M}}(\gamma|\xi|)}\, d\xi \\
&\leq C \int_{\mathcal{C}_1} |\xi|^{\frac{n}{2k}-2n} |\xi|^{2n+|\alpha|} \tilde{h}_{\mathcal{M}}\left(\frac{1}{\gamma|\xi|}\right) d\xi \\
&\leq C\gamma^{-2n+|\alpha|} M_{2n+|\alpha|} \int_{\mathcal{C}_1} |\xi|^{\frac{n}{2k}-2n}\, d\xi \\
&\leq C\gamma^{|\alpha|} M_{|\alpha|},
\end{aligned}$$

where in the last step (M2) is used.

So we have showed that on an open neighbourhood U of the origin and some open cones Γ_j, $j = 2, \dots, N$ that satisfy $\xi_0 \Gamma_j < 0$ we can write

$$v|_U = v_0 + \sum_{j=2}^{N} b_{\Gamma_j} f_j$$

with $v_0 \in \mathcal{E}_{\mathcal{M}}(U)$ and f_j holomorphic on $U + i\Gamma_j$ for $j = 2, \dots, N$. Hence $(0, \xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, v$. $\square$

We summarize our results regarding the description of $\mathrm{WF}_{\mathcal{M}}\, u$ in order to obtain the generalized Bony's Theorem alluded in the beginning of this section (c.f. [**39**]).

Theorem 2.3.2. *Let $u \in \mathcal{D}'(\Omega)$. For $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$ the following statements are equivalent:*

1. *$(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$*
2. *There are $U \in \mathcal{U}(x_0)$, open convex cones $\Gamma^j \subseteq \mathbb{R}^n$ with $\xi_0 \Gamma^j < 0$ and $\mathcal{M}$-almost analytic functions F_j of slow growth in $U \times \Gamma^j_{\rho_j}$, $\rho_j > 0$ and $j = 1, \dots, N$ for some $N \in \mathbb{N}$ such that*
$$u|_U = \sum_{j=1}^{N} b_{\Gamma^j} F_j.$$
3. *There are $\varphi \in \mathcal{D}(\Omega)$ with $\varphi \equiv 1$ near x_0, $V \in \mathcal{U}(x_0)$ and an open cone Γ containing ξ_0 such that (2.3.2) holds.*

We can also give a local version of Theorem 2.3.2.

Corollary 2.3.3. *Let $u \in \mathcal{D}'(\Omega)$ and $p \in \Omega$. Then the following is equivalent:*

1. *The distribution u is of class $\mathcal{E}_{\mathcal{M}}$ near p.*
2. *There is a bounded sequence $(u_N)_N \subseteq \mathcal{E}'(\Omega)$ and an open neighbourhood $V \subseteq \Omega$ of p such that $u_N|_V = u|_V$ for all $N \in \mathbb{N}_0$ and (2.1.1) holds for $\Gamma = \mathbb{R}^n$ and some constant $Q > 0$.*
3. *There exists an open neighbourhood $W \subseteq \Omega$ of p, $r > 0$ and a smooth function F on $W + iB(0,r)$ such that $F|_W = u|_W$ and (1.1.5) holds for some constants $C, Q > 0$.*
4. *There is a testfunction $\psi \in \mathcal{D}(\Omega)$ such that $\varphi_{|U} \equiv 1$ for some neighbourhood U of p and constants $C, \gamma > 0$ such that*
$$\sup_{(t,\xi) \in V \times \mathbb{R}^n} e^{\omega_{\mathcal{M}}(\gamma|\xi|)} \left|\mathfrak{F}(\psi u)(t,\xi)\right| < \infty$$
for some $V \in \mathcal{U}(p)$.

Proof. The equivalence of (1) and (2) is just Proposition 2.1.1, whereas Corollary 1.1.11 shows that (1) implies (3). For the fact that (4) implies (1) we note that by Theorem 2.3.1 we have that for all $\xi \in \mathbb{R}^n \setminus \{0\}$ $(p, \xi) \notin \mathrm{WF}_{\mathcal{M}} u$. Therefore u has to be ultradifferentiable of class $\{\mathcal{M}\}$ near p. Now we show that (4) follows from (3): Suppose that $u \in \mathcal{E}_{\mathcal{M}}(V)$ on a neighbourhood of p and let $F \in \mathcal{E}(W + i\mathbb{R}^n)$ be an $\mathcal{M}$-almost analytic extension of u on a relatively compact neighbourhood $W \subset\subset V$ of p. We choose $\varphi \in \mathcal{D}(W)$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ near p. We consider the map

$$\theta : \ y \longmapsto \theta(y) = y - is\varphi(y)\frac{\xi}{|\xi|}.$$

for some $1 > s > 0$ to be determined.

Finally let $\psi \in \mathcal{D}(V)$ such that $\psi \equiv 1$ on W. As in the proof of Theorem 2.3.1 we set $\psi(z) = \psi(x)$ for $z = x + iy \in \mathbb{C}^n$. We put $v = \psi F$ and consider the n-form

$$e^{Q(t,\xi,z)} v(z)\, dz_1 \wedge \cdots \wedge dz_n$$

on

$$D_s = \left\{ x + i\sigma\varphi(x)\frac{\xi}{|\xi|} \in \mathbb{C}^n \;\middle|\; 0 < \sigma < s,\ x \in \operatorname{supp} v \right\}.$$

Stokes' Theorem gives us

$$\begin{aligned} \mathfrak{F}v(t,\xi) = c_p \int\limits_{\theta(\mathbb{R}^n)} & e^{Q(t,\xi,z)} v(z,\bar{z})\, dz_1 \wedge \cdots \wedge dz_n \\ &+ c_p \sum_{j=1}^{n} \int\limits_{D_s} e^{Q(t,\xi,z)} \frac{\partial v}{\partial \bar{z}_j}(z,\bar{z})\, d\bar{z}_j \wedge dz_1 \wedge \cdots \wedge dz_n. \end{aligned}$$

The second integral above is estimated in the same way as the last integral in (2.3.3). On the other hand the first integral on the right-hand side equals

$$G(t,\xi) = c_p \int\limits_{\mathbb{R}^n} e^{Q(t,\xi,\theta(y))} v(\theta(y)) \det \theta'(y)\, dy$$

We note that

$$\operatorname{Re} Q(t,\xi,\theta(y)) \leq -s\varphi(y)|\xi|\big(1 + O(s\varphi(y)\big) - c_0|t-y|^{2k}$$

and hence

$$\begin{aligned} |G(t,\xi)| &\leq C \int\limits_{B_\delta(p)} e^{\operatorname{Re} Q(t,\xi,\theta(y))}\, dy + C \int\limits_{\substack{\mathbb{R}^n \setminus B_\delta(p) \\ y \in \operatorname{supp}(v\circ\theta)}} e^{\operatorname{Re} Q(t,\xi,\theta(y))}\, dy \\ &= I_1(t,\xi) + I_2(t,\xi), \end{aligned}$$

where $B_\delta(p) \subseteq \{x \in \mathbb{R}^n \mid \varphi(x) = 1\}$, can be estimated as follows, c.f. [**11**]. Set $s = \delta/4$. We obtain

$$I_1(t,\xi) \leq Ce^{-c|\xi|}$$

for all $\xi \in \mathbb{R}^n$ if t is in some bounded neighbourhood of p. Furthermore

$$I_2(t,x) \leq C \int\limits_{\substack{\mathbb{R}^n \setminus B_r(p) \\ y \in \operatorname{supp}(u\circ\theta)}} e^{-|\xi||t-y|^{2k}}\, dy \leq Ce^{-\left(\frac{\delta}{2}\right)^{2k}|\xi|}$$

for all ξ and $|t-p| \leq \frac{\delta}{2}$.

Hence we have showed that there are constants $c, C > 0$ such that

$$|\mathfrak{F}u(t,\xi)| \leq Ce^{-\omega_{\mathcal{M}}(c|\xi|)}$$

for all $\xi \in \mathbb{R}^n$ and t in a bounded neighbourhood of p. □

2.4. Elliptic regularity

As mentioned in the introduction Albanese, Jornet and Oliaro [**3**] used the pattern of Hörmander's proof [**45**, Theorem 8.6.1] to prove elliptic regularity for operators with coefficients that are all in the same ultradifferentiable class defined by a weight function, c.f. Remark 2.1.7. Similarly Hörmander's methods were applied in [**64**] and [**65**] for certain classes that are defined by more degenerate sequences. It is easy to see that the approach of Albanese, Jornet and Oliaro can be used to show elliptic regularity for operators with $\mathcal{E}_{\mathcal{M}}$-coefficients as long as $\mathcal{M}$ is a regular weight sequence. However, they considered only scalar operators. We show here that Hörmander's proof can be modified in a way to investigate the regularity of solutions of a determined system of linear partial differential equations

$$\begin{aligned} P_{11}u_1 + \dots + P_{1\nu}u_\nu &= f_1 \\ \vdots \qquad\qquad & \vdots \\ P_{\nu 1}u_1 + \dots + P_{\nu\nu}u_\nu &= f_\nu \end{aligned}$$

where $P_{j,k}$, $1 \leq j,k \leq \nu$, is a partial differential operator with $\mathcal{E}_{\mathcal{M}}$-coefficients. Since we have showed in section 2.2 that $\mathrm{WF}_{\mathcal{M}}\, u$ is well defined for distributions u on $\mathcal{E}_{\mathcal{M}}$-manifolds, we can work in the following setting (see [**45**, chapter 6] and [**23**]).

Let M be an ultradifferentiable manifold of class $\{\mathcal{M}\}$ and E and F two vector bundles of class $\{\mathcal{M}\}$ on M with the same fiber dimension ν. An ultradifferentiable partial differential operator $P:\ \mathcal{E}_{\mathcal{M}}(M,E) \to \mathcal{E}_{\mathcal{M}}(M,F)$ of class $\{\mathcal{M}\}$ is given locally by

$$Pu = \begin{pmatrix} P_{11} & \cdots & P_{1\nu} \\ \vdots & \ddots & \vdots \\ P_{\nu 1} & \cdots & P_{\nu\nu} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_\nu \end{pmatrix} \tag{2.4.1}$$

where the P_{jk} are linear partial differential operators with ultradifferentiable coefficients defined in suitable chart neighbourhoods. If

$$Q(x,D) = \sum_{|\alpha|\leq m} q_\alpha(x) D^\alpha$$

is a differential operator of order $\leq m$ on some open set $\Omega \subseteq \mathbb{R}^n$ then the principal symbol q is defined to be

$$q(x,\xi) = \sum_{|\alpha|=m} q_\alpha D^\alpha.$$

Hence the order of P is of order $\leq m$ iff no operator P_{jk} on any chart neighbourhood is of order higher than m and P is of order m if the operator is not of order $\leq m-1$. The principal symbol p of P is an ultradifferentiable mapping defined on $T^*\Omega$ with values in the space of fiber-linear maps from E to F that is homogenous of degree m in the fibers of $T^*\Omega$. It is given locally by

$$p(x,\xi) = \begin{pmatrix} p_{11}(x,\xi) & \dots & p_{1\nu}(x,\xi) \\ \vdots & \ddots & \vdots \\ p_{\nu 1}(x,\xi) & \dots & p_{\nu\nu}(x,\xi) \end{pmatrix} \tag{2.4.2}$$

where p_{jk} is the principal symbol of the operator P_{jk}. See [**23**] for more details. We say that P is non-characteristic at a point $(x,\xi) \in T^*M\backslash\{0\}$ if $p(x,\xi)$ is an invertible linear mapping. We define the set of all characteristic points

$$\operatorname{Char} P = \{(x,\xi) \in T^*M\backslash\{0\} : P \text{ is characteristic at } (x,\xi)\}.$$

THEOREM 2.4.1. *Let M be a $\mathcal{E}_{\mathcal{M}}$-manifold and E,F two ultradifferentiable vector bundles on M of the same fiber dimension. If $P(x,D)$ is a differential operator between E and F with $\mathcal{E}_{\mathcal{M}}$-coefficients and p its principal symbol, then*

$$\mathrm{WF}_{\mathcal{M}}\, u \subseteq \mathrm{WF}_{\mathcal{M}}(Pu) \cup \operatorname{Char} P \qquad u \in \mathcal{D}'(M,E). \tag{2.4.3}$$

PROOF. We write $f = Pu$. Since the problem is local we work on some chart neighbourhood Ω such that in suitable trivializations of E and F we may write $u = (u_1, \dots, u_\nu) \in \mathcal{D}'(\Omega, \mathbb{C}^\nu)$, $f = (f_1, \dots, f_\nu) \in \mathcal{D}'(\Omega, \mathbb{C}^\nu)$ and P and its principal symbol p are of the form (2.4.1) and (2.4.2), respectively. In particular, P is of order m on Ω.

We have to prove that if $(x_0, \xi_0) \notin \mathrm{WF}_\mathcal{M} f \cup \mathrm{Char}\, P$ then $(x_0, \xi_0) \notin \mathrm{WF}_\mathcal{M} u$. Assuming this we find that there has to be a compact neighbourhood K of x_0 and a closed conic neighbourhood V of ξ_0 in $\mathbb{R}^n \setminus \{0\}$ satisfying

$$\det p(x,\xi) \neq 0 \qquad (x,\xi) \in K \times V \tag{2.4.4}$$

$$(K \times V) \cap \mathrm{WF}_\mathcal{M}(Pu)_j = \emptyset \qquad j = 1, \dots, \nu. \tag{2.4.5}$$

We consider the formal adjoint $Q = P^t$ of P with respect of the pairing

$$\langle f, g\rangle = \sum_{\tau=1}^{\nu} \int f_\tau(x) g_\tau(x)\, dx \qquad f, g \in \mathcal{D}(\Omega, \mathbb{C}^\nu).$$

If $P = (P_{jk})_{jk}$ then $Q = (Q_{jk})_{jk} = (P^t_{kj})_{jk}$ where P^t_{jk} denotes the formal adjoint of the scalar operator $P_{jk}(x, D) = \sum p^\alpha_{jk}(x) D^\alpha$, i.e. for $v \in \mathcal{E}(\Omega)$

$$P^t_{jk}(x,D)v = \sum_{|\alpha|\le m} (-D)^\alpha \Big(p^\alpha_{jk}(x) v(x)\Big).$$

Let $(\lambda_N)_N \subseteq \mathcal{D}(K)$ be a sequence of test functions satisfying $\lambda_N|_U \equiv 1$ on a fixed neighbourhood U of x_0 for all N and for all $\alpha \in \mathbb{N}_0^n$ there are constants $C_\alpha, h_\alpha > 0$ such that

$$\left| D^{\alpha+\beta} \lambda_N \right| \le C_\alpha (h_\alpha N)^{|\beta|}, \qquad |\beta| \le N. \tag{2.4.6}$$

If $u = (u^1, \dots, u^\nu) \in \mathcal{D}'(\Omega, \mathbb{C}^\nu)$, then we have that the sequence $u^\tau_N = \lambda_{2N} u^\tau$ is bounded in $\mathcal{E}'$ and each of these distributions is equal to u^τ in U for all τ. Hence we have to prove that $(u^\tau_N)_N$ satisfies (2.1.1), i.e.

$$\sup_{\substack{\xi\in V\\ N\in\mathbb{N}_0}} \frac{|\xi|^N \left|\hat{u}^\tau_N\right|}{Q^N M_N} < \infty$$

for a constant $Q > 0$ independent of N.

In order to do so, set $\Lambda^\tau_N = \lambda_N e_\tau \in \mathcal{D}'(\Omega, \mathbb{C}^\nu)$ and observe

$$\hat{u}^\tau_N(\xi) = \left\langle u^\tau, e^{-i\langle\,\cdot\,,\xi\rangle} \lambda_{2N}\right\rangle = \left\langle u, e^{-i\langle\,\cdot\,,\xi\rangle} \Lambda^\tau_{2N}\right\rangle.$$

Following the argument of Hörmander [**45**, Theorem 8.6.1] we want to solve the equation $Qg^\tau = e^{-ix\xi}\Lambda^\tau_{2N}$. We make the ansatz

$$g^\tau = e^{-ix\xi} B(x,\xi) w^\tau$$

where $B(x,\xi)$ is the inverse matrix of the transpose of $p(x,\xi)$, which exists if $(x,\xi) \in K \times V$ and is homogeneous of degree $-m$ in ξ; note that the principal symbol of $Q = P^t$ is $B^{-1}(x, -\xi)$. Using this we conclude that w has to satisfy

$$w^\tau - Rw^\tau = \Lambda^\tau_{2N}. \tag{2.4.7}$$

Here $R = R_1 + \dots + R_m$ with $R_j|\xi|^j$ being (matrix) differential operators of order $\le j$ with coefficients in $\mathcal{E}_\mathcal{M}$ that are homogeneous of degree 0 in ξ if $x \in K$ and $\xi \in V$.

A formal solution of (2.4.7) would be

$$w^\tau = \sum_{k=0}^{\infty} R^k \Lambda^\tau_{2N}.$$

However, this sum may not converge and even if it would converge, in the estimates we want to obtain we are not allowed to consider derivatives of arbitrary high order. Hence we set

$$w^\tau_N := \sum_{j_1+\dots+j_k \le N-m} R_{j_1} \cdots R_{j_k} \Lambda^\tau_{2N}$$

and compute

$$w_N^\tau - R w_N^\tau = \Lambda_{2N}^\tau - \sum_{\sum_{s=1}^k j_s > N-m \geq \sum_{s=2}^k j_s} R_{j_1} \dots R_{j_k} \Lambda_{2N}^\tau = \Lambda_{2N}^\tau - \rho_N^\tau .$$

Equivalently, we have

$$Q\big(e^{-ix\xi} B(x,\xi) w_N^\tau\big) = e^{-ix\xi}\big(\Lambda_{2N}^\tau(x) - \rho_N^\tau(x,\xi)\big).$$

We obtain now

$$\begin{aligned} \hat{u}_N^\tau(\xi) &= \big\langle u, e^{-i\langle\,.\,,\xi\rangle} \Lambda_{2N}^\tau \big\rangle \\ &= \big\langle u, Q\big(e^{-i\langle\,.\,,\xi\rangle} B(\,.\,,\xi) w_N^\tau\big)\big\rangle + \langle u, e^{-i\langle\,.\,,\xi\rangle} \rho_N^\tau(\,.\,,\xi)\rangle \\ &= \langle f, e^{-i\langle\,.\,,\xi\rangle} B(\,.\,,\xi) w_N^\tau\rangle + \langle u, e^{-i\langle\,.\,,\xi\rangle} \rho_N^\tau(\,.\,,\xi)\rangle \end{aligned} \tag{2.4.8}$$

and continue by estimating the right-hand side of (2.4.8). For this purpose we need the following Lemma.

Lemma 2.4.2. *There exists constants C and h depending only on R and the constants appearing in* (2.4.6) *such that, if $j = j_1 + \cdots + j_k$ and $j + |\beta| \leq 2N$, we have*

$$\Big| D^\beta \big(R_{j_1} \dots R_{j_k} \Lambda_{2N}^\tau\big)_\sigma \Big| \leq C h^N M_{2N}^{\frac{j+|\beta|}{2N}} |\xi|^{-j} \qquad \xi \in V, \sigma = 1, \dots, \nu. \tag{2.4.9}$$

Proof. Since both sides of (2.4.9) are homogeneous of degree $-j$ in $\xi \in V$ it suffices to prove the lemma for $|\xi| = 1$. Moreover we can write

$$\big(R_{j_1} \cdots R_{j_k} \Lambda_{2N}^\tau\big)_\sigma = \tilde{R}_\sigma^\tau \lambda_{2N} \qquad \sigma = 1, \dots, \nu$$

with $\tilde{R}_\sigma^\tau$ being a certain linear combination of products of components of the operators R_{j_s}. Especially the coefficients of $\tilde{R}_\sigma^\tau$ are all of class $\{\mathcal{M}\}$ on a common neighbourhood of K and since there are only finitely many of them we may assume that they all can be considered as elements of $\mathcal{E}_{\mathcal{M}}^q(K)$ for some $q > 0$. Recall also from Remark 2.1.3 that $\sqrt[N]{M_N} \to \infty$ and that there has to be a constant $\delta > 0$ such that $N \leq \delta \sqrt[N]{M_N}$. Hence (2.4.6) implies that for all $\alpha \in \mathbb{N}_0^n$ there are constants $C_\alpha > 0$ and $h_\alpha > 0$ such that

$$\big|D^{\alpha+\beta} \lambda_N\big| \leq C_\alpha h_\alpha^{|\beta|} M_N^{\frac{|\beta|}{N}} \tag{2.4.10}$$

for $|\beta| \leq N$. Therefore the proof of the lemma is a consequence of the following result. □

Lemma 2.4.3. *Let $K \subseteq \Omega$ be compact, $(\lambda_N)_N \subset \mathcal{D}(K)$ a sequence satisfying* (2.4.10) *and $a_1, \dots, a_{j-1} \in \mathcal{E}_{\mathcal{M}}^q(K)$. Then there are constants $C, h > 0$ independent of N such that for $j \leq N$ we have*

$$\big|D_{i_1}(a_1 D_{i_2}(a_2 \dots D_{i_{j-1}}(a_{j-1} D_{i_j} \lambda_N) \dots))\big| \leq C h^j M_N^{\frac{j}{N}}. \tag{2.4.11}$$

Proof. We begin by noting that (M3) implies that $m_j m_{k-j} \leq m_k$ for all $j \leq k \in \mathbb{N}$ c.f. [**56**]. Obviously the expression $D_{i_1} a_1 D_{i_2} a_2 \dots D_{i_{j-1}} a_{j-1} D_{i_j} \lambda_N$ is a sum of terms of the form $(D^{\alpha_1} a_1) \cdots (D^{\alpha_{j-1}} a_{j-1}) D^{\alpha_j} \lambda_N$ where $|\alpha_1| + \cdots + |\alpha_j| = j$.

We set $h \geq \max(q, h_0)$. If there are $C_{k_1,\dots,k_j}$ terms with $|\alpha_1| = k_1, \dots, |\alpha_j| = k_j$ then we have the following estimate on K

$$\begin{aligned} \big|D_{i_1} a_1 D_{i_2} a_2 \dots D_{i_{j-1}} a_{j-1} D_{i_j} \lambda_N\big| &\leq C \sum q^{j-k_j} C_{k_1,\dots,k_j} m_{k_1} \cdots m_{k_{j-1}} k_1! \cdots k_{j-1}! h_0^{k_j} M_N^{\frac{k_j}{N}} \\ &\leq C h^j \sum m_{j-k_j} C_{k_1,\dots,k_j} k_1! \cdots k_{j-1}! M_N^{\frac{k_j}{N}} \\ &\leq C h^j \sum C_{k_1,\dots,k_j} \frac{k_1! \cdots k_{j-1}!}{(j-k_j)!} M_{j-k_j} M_N^{\frac{k_j}{N}}. \end{aligned}$$

Now observe that since $\sqrt[N]{M_N}$ is increasing we have

$$M_{j-k_j} M_N^{\frac{k_j}{N}} = M_{j-k_j}^{\frac{j-k_j}{j-k_j}} M_N^{\frac{k_j}{N}} \le M_N^{\frac{j-k_j}{N}} M_N^{\frac{k_j}{N}} = M_N^{\frac{j}{N}}.$$

As noted in [**3**] it is possible to estimate

$$\frac{k_1! \cdots k_{j-1}!}{(j-k_j)!} = \frac{k_1! \cdots k_{j-1}! k_j! j!}{(j-k_j)! k_j! j!} \le 2^j \frac{k_1! \cdots k_j!}{j!},$$

and also (c.f. [**45**, p. 308])

$$\sum C_{k_1,\dots,k_j} k_1! \cdots k_j! = (2j-1)!!.$$

Since $\frac{(2j-1)!!}{j!2^j} \le 1$ we obtain

$$\begin{aligned}\left|D_{i_1} a_1 D_{i_2} a_2 \dots D_{i_{j-1}} a_{j-1} D_{i_j} \lambda_N\right| &\le C(4h)^j \frac{(2k-1)!!}{j!2^j} M_N^{\frac{j}{N}} \\ &\le C(4h)^j M_N^{\frac{j}{N}}.\end{aligned}$$

□

In order to estimate $\hat{u}_N^\tau$, we note that due to the boundedness of the sequence $(u_N^\tau)_N \subseteq \mathcal{E}'$ the Banach-Steinhaus theorem implies that there are constants κ and c such that

$$|\hat{u}_N^\tau| \le c\big(1+|\xi|\big)^\kappa$$

for all N and therefore if $|\xi| \le N$

$$|\xi|^N |\hat{u}_N^\tau| \le cN^N \big(1+N\big)^\kappa \le c\delta^N \tilde{C}^N M_N \tag{2.4.12}$$

for some constant $\tilde{C}$. Hence it suffices to estimate the terms on the right-hand side of (2.4.8) for $\xi \in V$, $|\xi| > N$. We begin with the second term.

As in the scalar case there are constants μ and $C > 0$ that only depend on u and K such that for all $\psi \in \mathcal{D}(\Omega, \mathbb{C}^\nu)$ with $\operatorname{supp} \psi \subseteq K$

$$\left|\langle u, \psi\rangle\right| \le C \sum_{|\alpha| \le \mu} \sup_K \left|D^\alpha \psi\right|.$$

Note that $\operatorname{supp}_x \rho_N^\tau(\,.\,, \xi) \subseteq K$ for all $\xi \in V$ and $N \in \mathbb{N}$. Thence

$$\begin{aligned}\left|\langle u, e^{-i\langle\,.\,,\xi\rangle} \rho_N^\tau(\,.\,,\xi)\rangle\right| &\le C \sum_{|\alpha|\le\mu} \sum_{\beta\le\alpha} |\xi|^{|\alpha|-|\beta|} \sup_{x\in K} \left|D_x^\beta \rho_N^\tau(x,\xi)\right| \\ &\le C \sum_{|\alpha|\le\mu} |\xi|^{\mu-|\alpha|} \sup_{x\in K} \left|D_x^\alpha \rho_N^\tau(x,\xi)\right|\end{aligned}$$

for $\xi \in V$, $|\xi| \ge 1$ and $N \in \mathbb{N}$. There are at most 2^N terms of the form $R_{j_1} \dots R_{j_k} \Lambda_{2N}^\tau$ in ρ_N^τ and each term can be estimated by (2.4.9) setting $j > N - m$ and hence

$$\left|D_x^\alpha \rho_N^\tau(x,\xi)\right| \le Ch^N 2^N |\xi|^{m-N} M_N^{\frac{N+|\alpha|}{N}}$$

for $x \in K$ and $\xi \in V$, $|\xi| > 1$. Therefore

$$\left|\langle u, e^{-i\langle\,.\,,\xi\rangle} \rho_N^\tau(\,.\,,\xi)\rangle\right| \le Ch^N 2^{N+\mu} |\xi|^{\mu+m-N} M_N^{\frac{N+\mu}{N}} \le Ch^N |\xi|^{\mu+m-N} M_N. \tag{2.4.13}$$

The first term in (2.4.8) is more difficult to estimate. Recall from Remark 2.1.3 that by assumption $\sqrt[N]{M_N}$ is increasing and that there is a constant δ such that $N \le \delta \sqrt[N]{M_N}$. Lemma

2.4.2 gives

$$\begin{aligned}
\left|D^\beta w_N^\tau(x,\xi)\right| &\le Ch^N M_{2N}^{\frac{N-m+|\beta|}{2N}}|\xi|^{m-N}\\
&\le Ch^N M_{2N}^{\frac{N-m+|\beta|}{2N}}N^{m-N}\\
&\le Ch^N M_{2N}^{\frac{N-m+|\beta|}{2N}}\delta^{m-N}M_N^{\frac{m-N}{N}}\\
&\le Ch^N M_{2N}^{\frac{N-m+|\beta|}{2N}}M_{2N}^{\frac{m-N}{2N}}\\
&\le Ch^N M_{2N}^{\frac{|\beta|}{2N}}
\end{aligned}$$

for $N > m$, $|\beta| \le N$ and $\xi \in V$, $|\xi| > N$. Recall that for $N \le m$ we have set $w_N^\tau = \Lambda_{2N}^\tau = \lambda_{2N}^\tau e_\tau$. Hence by the above and (2.4.10) it follows that

$$\left|D^\beta w_N^\tau(x,\xi)\right| \le Ch^N M_{2N}^{\frac{|\beta|}{2N}} \tag{2.4.14}$$

for all $N \in \mathbb{N}$, $|\beta| \le N$ and $\xi \in V$, $|\xi| > N$.

On the other hand, since the components of $B(x,\xi)$ are ultradifferentiable of class $\{\mathcal{M}\}$ and homogeneous in $\xi \in V$ of degree $-m$ we note that it is possible to show similarly to above, using an analogue to Lemma 2.4.2, the following estimate for N.

$$\left|D_x^\beta\big(w_N^\tau(x,\xi)|\xi|^m B(x,\xi)\big)\right| \le Ch^N M_{2N}^{\frac{|\beta|}{2N}} \qquad |\beta| \le N,\ \xi \in V,\ |\xi| > N. \tag{2.4.15}$$

In order to finish the proof of Theorem 2.4.1 we need an additional lemma.

Lemma 2.4.4. *Let $f \in \mathcal{D}'(\Omega)$, K be a compact subset of Ω and $V \subset \mathbb{R}^n\backslash\{0\}$ a closed cone such that*

$$\mathrm{WF}_{\mathcal{M}} f \cap (K \times V) = \emptyset.$$

Furthermore let $w_N \in \mathcal{E}(\Omega \times V)$ such that $\operatorname{supp} w_N \subseteq K \times V$ *and* (2.4.14) *holds.*

If μ denotes the order of f in a neighbourhood of K then

$$\left|\widehat{w_N f}(\xi)\right| = \left|\langle w_N(\,.\,,\xi)f, e^{-i\langle\,.\,,\xi\rangle}\rangle\right| \le Ch^N|\xi|^{\mu+n-N}M_{N-\mu-n}, \tag{2.4.16}$$

for $N > \mu + n$ and $\xi \in \Gamma$, $|\xi| > N$.

Proof. By Proposition 2.1.4 we can find a sequence $(f_N)_N$ that is bounded in $\mathcal{E}'^{,\mu}$ and equal to f in some neighbourhood of K and

$$\left|\hat{f}_N(\eta)\right| \le C\frac{Q^N M_N}{|\eta|^N} \qquad \eta \in W \tag{2.4.17}$$

where W is a conic neighbourhood of Γ. Then $w_N f = w_N f_{N'}$ for $N' = N - \mu - n$.

If we denote the partial Fourier transform of $w_N(x,\xi)$ by

$$\hat{w}_N(\eta,\xi) = \int_\Omega e^{-ix\eta}w_N(x,\xi)\,dx$$

then obviously (2.4.14) is equivalent to

$$\left|\eta^\beta \hat{w}_N(\eta,\xi)\right| \le Ch^N M_{2N}^{\frac{|\beta|}{2N}}$$

for $|\beta| \le N$, $\xi \in V$, $|\xi| > N$ and $\eta \in \mathbb{R}^n$. Since $|\eta| \le \sqrt{n}\max|\eta_j|$ we conclude that

$$|\eta|^\ell|\hat{w}_N(\eta,\xi)| \le Ch^N M_{2N}^{\frac{\ell}{2N}}$$

for $\ell \le N$, $\eta \in \mathbb{R}^n$ and $\xi \in V$, $|\xi| > N$. Hence we obtain

$$\begin{aligned}\left(|\eta| + M_{2N}^{\frac{1}{2N}}\right)^N \big|\hat{w}_N(\eta,\xi)\big| &= \sum_{k=0}^{N} \binom{N}{k} M_{2N}^{\frac{k}{2N}} |\eta|^{N-k} |\hat{w}_N(\eta,\xi)\big| \\ &\le Ch^N \sum_{k=0}^{N} \binom{N}{k} M_{2N}^{\frac{k}{2N}} M_N^{\frac{N-k}{2N}} \\ &\le Ch^N M_{2N}^{\frac{N}{2N}}\end{aligned} \tag{2.4.18}$$

if $\eta \in \mathbb{R}^n$, $\xi \in V$ and $|\xi| > N$. Like [**45**] and [**3**] we consider

$$\begin{aligned}\widehat{w_N f}(\xi) &= \frac{1}{(2\pi)^n} \int \hat{w}_N(\eta,\xi) \hat{f}_{N'}(\xi-\eta)\, d\eta \\ &= \frac{1}{(2\pi)^n} \int\limits_{|\eta|<c|\xi|} \hat{w}_N(\eta,\xi) \hat{f}_{N'}(\xi-\eta)\, d\eta + \frac{1}{(2\pi)^n} \int\limits_{|\eta|>c|\xi|} \hat{w}_N(\eta,\xi) \hat{f}_{N'}(\xi-\eta)\, d\eta\end{aligned}$$

for some $0 < c < 1$. The boundedness of the sequence $(f_N)_N$ in $\mathcal{E}'^{,\nu}$ implies as before that

$$\big|\hat{f}_N(\xi)\big| \le C\big(1+|\xi|\big)^\mu.$$

Hence we conclude that

$$(2\pi)^n \Big|\widehat{w_N f}(\xi)\Big| \le \big\|\hat{w}_N(\,.\,,\xi)\big\|_{L^1} \sup_{|\xi-\eta|<c|\xi|} \big|\hat{f}_{N'}(\eta)\big| + C \int\limits_{|\eta|>c|\xi|} \big|\hat{w}_N(\eta,\xi)\big| \big(1+c^{-1}\big)^\mu \big(1+|\eta|\big)^\mu\, d\eta$$

since $|\eta| \ge c|\xi|$ gives $|\xi+\eta| \le (1-c^{-1})|\eta|$.

On the other hand there is a constant $0 < c < 1$ such that $\eta \in W$ when $\xi \in V$ and $|\xi-\eta| \le c|\xi|$. Then $|\eta| \ge (1-c)|\xi|$ and we can replace the supremum above by $\sup_{\eta\in W} |\hat{f}_{N'}(\eta)|$. Furthermore by (2.4.18)

$$\begin{aligned}\big\|\hat{w}_N(\,.\,,\xi)\big\|_{L_1} &= \int\limits_{\mathbb{R}^n} \big|\hat{w}_N(\eta,\xi)\big|\, d\eta \\ &\le Ch^N M_{2N}^{\frac{N}{2N}} \int\limits_{\mathbb{R}^n} \Big(|\eta| + \sqrt[2N]{M_{2N}}\Big)^{-N} d\eta \\ &= Ch^N M_{2N}^{\frac{N}{2N}} \int\limits_0^\infty \Big(r + \sqrt[2N]{M_{2N}}\Big)^{-N} r^{n-1}\, dr \\ &\le Ch^N M_{2N}^{\frac{N}{2N}} \int\limits_0^\infty \Big(r + \sqrt[2N]{M_{2N}}\Big)^{-N'-1} dr \\ &\le Ch^N M_{2N}^{\frac{N}{2N}} \int\limits_{\sqrt[2N]{M_{2N}}}^\infty s^{-N'-1}\, ds \\ &\le Ch^N M_{2N}^{\frac{N}{2N}} \frac{M_{2N}^{-\frac{N'}{2N}}}{N'} \\ &\le Ch^N M_{2N}^{\frac{\mu+n}{2N}}\end{aligned}$$

if $N \geq \mu + n$. Note that (M2) implies that there is a constant δ such that $\sqrt[N]{M_N} \leq \delta^N$ for all $N \in \mathbb{N}$. Thence it follows for $\xi \in V$, $|\xi| > N$, that

$$\begin{aligned}
|\xi|^{N'}\left|\widehat{w_N f}(\xi)\right| &\leq C_1(1-c)^{-N'}\left\|\hat{w}_N(\,.\,,\xi)\right\|_{L^1} \sup_{\eta\in W}\left|\hat{f}_{N'}(\eta)\right||\eta|^{N'} \\
&\quad + C_2\left(1+c^{-1}\right)^{N'+\mu}\int(1+|\eta|)^{\mu}|\eta|^{N'}|\hat{w}_N(\eta,\xi)|\,d\eta \\
&\leq C_1 h^N M_{2N}^{\frac{n+\mu}{2N}} Q^{N'} M_{N'} + C_2\tilde{h}^{N'+\mu}M_{N'+\mu} \\
&\leq C_1 h^N \delta^{2N(n+\mu)} M_{N'} + C_2\tilde{h}^{N'}M_{N'} \\
&\leq C h^N M_{N'}
\end{aligned}$$

where we have also used (2.4.17). □

Due to (2.4.15) we can replace w_N in (2.4.16) with $(w_N^\tau|\xi|^m B)_\sigma$, $\sigma = 1,\dots,\nu$, and obtain

$$\left|\left\langle f, e^{-i\langle\,.\,,\xi\rangle}B(\,.\,,\xi)w_N^\tau\right\rangle\right| \leq Ch^N|\xi|^{\mu+n-N}M_{N-\mu-n} \tag{2.4.19}$$

for $\xi \in V$, $|\xi| > N$.

We consider now the sequence $(v_N^\tau)_N = (u^\tau_{N+m+n+\mu})_N$. If $\xi \in V$, $|\xi| \leq N$, then by (2.4.12)

$$|\xi|^N\left|\hat{v}_N^\tau\right| \leq Ch^N M_N.$$

On the other hand (2.4.8), (2.4.13) and (2.4.19) give

$$\begin{aligned}
|\xi|^N\left|\hat{v}_N^\tau(\xi)\right| &\leq C_1 h_1^N M_{N+m}|\xi|^{-m} + C_2 h_2^N M_{N+\mu+m+n}|\xi|^{-n} \\
&\leq C_1 h_1^N M_N N^{-m} + C_2 h_2^N M_N N^{-n} \\
&\leq Ch^N M_N
\end{aligned}$$

for $\xi \in V$, $|\xi| > N$.

Therefore we have shown for all $\tau = 1,\dots,\nu$ that the bounded sequence $(v_N^\tau)_N \subset \mathcal{E}'(\Omega)$ satisfies

$$\sup_{\substack{\xi\in V\\ N\in\mathbb{N}}} \frac{|\xi|^N\left|v_N^\tau(\xi)\right|}{Q^N M_N} < \infty$$

for some $Q > 0$. Obviously $u^\tau|_U \equiv (v_N^\tau)|_U$ and hence

$$(x_0,\xi_0) \notin \operatorname{WF}_{\mathcal{M}} u^\tau$$

for all $\tau = 1,\dots,\nu$. □

For elliptic operators, i.e. operators P with $\operatorname{Char} P = \emptyset$, the following holds obviously.

Corollary 2.4.5. *If P is elliptic and $u \in \mathcal{D}'$ then*

$$\operatorname{WF}_{\mathcal{M}} Pu = \operatorname{WF}_{\mathcal{M}} u$$

for all weight sequences $\mathcal{M}$.

2.5. Uniqueness Theorems

Hörmander [**41**] and Kawai (see [**73**]) independently noticed that results like Theorem 2.4.1 in the analytic category can be used to prove Holgrem's Uniqueness Theorem [**40**]. We show here that Theorem 2.4.1 can also be used to give a quasianalytic version of Holgrem's Uniqueness Theorem. We follow mainly the presentation in [**45**].

First recall [**44**, Theorem 6.1.]:

Proposition 2.5.1. *Let $I \subseteq \mathbb{R}$ be an interval and $x_0 \in \partial \operatorname{supp} u$ then $(x_0, \pm 1) \in \operatorname{WF}_{\mathcal{M}} u$ for any quasianalytic weight sequence $\mathcal{M}$.*

As Hörmander noted in [**44**] Proposition 2.5.1 immediately generalizes to a result in higher dimensions (c.f. [**45**][Theorem 8.5.6], see [**47**] for a similar result):

THEOREM 2.5.2. *Let $\mathcal{M}$ be a quasianalytic weight sequence, $u \in \mathcal{D}'(\Omega)$, $x_0 \in \operatorname{supp} u$ and $f : \Omega \to \mathbb{R}$ a function of class $\{\mathcal{M}\}$ with the following properties:*

$$df(x_0) \neq 0, \quad f(x) \leq f(x_0) \quad \text{if } x_0 \neq x \in \operatorname{supp} u$$

Then we have

$$(x_0, \pm df(x_0)) \in \mathrm{WF}_{\mathcal{M}}\, u.$$

PROOF. If we replace f by $f(x) - |x - x_0|^2$ we see that we may assume that $f(x) < f(x_0)$ for $x_0 \neq x \in \operatorname{supp} u$. Furthermore, since $df(x_0) \neq 0$ we can assume that $x_0 = 0$ and $f(x) = x_n$. Next we choose a neighbourhood U of 0 in $\mathbb{R}^{n-1}$ so that $U \times \{0\} \subset\subset \Omega$. By assumption $\operatorname{supp} u \cap (\bar{U} \times \{0\}) = \{0\}$. Hence there is an open interval $I \subset \mathbb{R}$ with $0 \in I$ such that

$$U \times I \subset\subset \Omega \quad \& \quad \operatorname{supp} u \cap \big(\partial U \times I\big) = \emptyset.$$

If A is an entire analytic function in the variables $x' = (x_1, \ldots, x_{n-1})$ then we consider the pushforward $U_A = A_* u$ (c.f. [**27**]). By [**45**, Theorem 8.5.4'] we have that

$$\mathrm{WF}_{\mathcal{M}}\big(U_A\big) \subseteq \big\{(x_n, \xi_n) \in I \times \mathbb{R}\backslash\{0\} \mid \exists x' \in U : \ (x', x_n, 0, \xi_n) \in \mathrm{WF}_{\mathcal{M}}\, u\big\}.$$

Note that (x', x_n) above must be close to 0 for x_n small.

Assume, e.g., that $(0, e_n) \notin \mathrm{WF}_{\mathcal{M}}\, u$, $e_n = (0, \ldots, 0, 1)$. Then I can be chosen so small that $(x, e_n) \notin \mathrm{WF}_{\mathcal{M}}\, u$ for $x \in U \times I$. We conclude that $(x_n, 1) \notin \mathrm{WF}_{\mathcal{M}}\, U_A$ if $x_n \in I$. Proposition 2.5.1 implies that $U_A = 0$ on I since $U_A = 0$ on $I \cap \{x_n > 0\}$. That means actually that

$$\Big\langle u|_{U \times I}, A \otimes \varphi \Big\rangle = 0$$

for all $\varphi \in \mathcal{D}(I)$. Since A was chosen arbitrarily from a dense subset of $\mathcal{E}(\mathbb{R}^{n-1})$ it follows that $u = 0$ on $U \times I$. □

In order to give Theorem 2.5.2 a more invariant form we need to recall some facts from [**45**].

DEFINITION 2.5.3. Let F be a closed subset of a $\mathcal{C}^2$ manifold X. The *exterior normal set* $N_e(F) \subseteq T^*X\backslash\{0\}$ is defined as the set of all points (x_0, ξ_0) such that $x_0 \in F$ and there exists a real valued function $f \in \mathcal{C}^2(X)$ with $df(x_0) = \xi_0 \neq 0$ and $f(x) \leq f(x_0)$ when $x \in F$.

In fact, following the remarks in [**45**, p. 300] we observe that it would be sufficient for f to be defined locally around x_0. Furthermore f could then also be chosen real-analytic in a chart neighbourhood near x_0. If g is $\mathcal{C}^1$ near a point $\tilde{x} \in F$ and $dg(\tilde{x}) = \tilde{\xi} \neq 0$ then $(\tilde{x}, \tilde{\xi}) \in \overline{N_e(F)} \subseteq T^*X\backslash\{0\}$. It is clear that if $(x_0, \xi_0) \in N_e(F)$ then $x_0 \in \partial F$. In fact, if $\pi : T^*\Omega \to \Omega$ is the canonical projection then $\pi(N_e(F))$ is dense in ∂F, see [**45**, Proposition 8.5.8.]. The *interior normal set* $N_i(F) \subseteq T^*X\backslash\{0\}$ consists of all points (x_0, ξ_0) with $(x_0, -\xi_0) \in N_e(F)$. The *normal set* of F is defined as $N(F) = N_e(F) \cup N_i(F) \subseteq T^*X\backslash\{0\}$.

In this notation Theorem 2.5.2 takes the following form.

THEOREM 2.5.4. *Let $\mathcal{M}$ be a quasianalytic weight sequence and $u \in \mathcal{D}'(\Omega)$. Then*

$$\overline{N(\operatorname{supp} u)} \subseteq \mathrm{WF}_{\mathcal{M}}\, u$$

Theorem 2.5.4 combined with Theorem 2.4.1 implies

THEOREM 2.5.5. *Let $\mathcal{M}$ be a quasianalytic weight sequence, P a partial differential operator with $\mathcal{E}_{\mathcal{M}}$-coefficients and $u \in \mathcal{D}'(\Omega)$ a solution of $Pu = 0$. Then*

$$\overline{N(\operatorname{supp} u)} \subseteq \operatorname{Char} P,$$

i.e., the principal symbol p_m of P must vanish on $N(\operatorname{supp} u)$.

In fact, we can now derive the *quasianalytic Holgrem Uniqueness Theorem*. We recall that a $\mathcal{C}^1$-hypersurface M is characteristic at a point x with respect to a partial differential operator P, iff for a defining function φ of M near x we have that $(x, d\varphi(x)) \in \operatorname{Char} P$.

Corollary 2.5.6. *Let $\mathcal{M}$ be quasianalytic and P a partial differential operator with $\mathcal{E}_{\mathcal{M}}$-coefficients. If X is a $\mathcal{C}^1$-hypersurface in Ω that is non-characteristic at x_0 and $u \in \mathcal{D}'(\Omega)$ a solution of $Pu = 0$ that vanishes on one side of X near x_0 then $u \equiv 0$ in a full neighbourhood of x_0.*

In fact, (c.f. Zachmanoglou [**82**]) it is possible to reformulate Corollary 2.5.6

Corollary 2.5.7. *Let $\mathcal{M}$ be quasianalytic and P a differential operator with coefficients in $\mathcal{E}_{\mathcal{M}}(\Omega)$. Furthermore let $F \in \mathcal{E}_{\mathcal{M}}(\mathbb{R}^n)$ be a real-valued function of the form*

$$F(x) = f\bigl(x'\bigr) - x_n, \qquad x' = (x_1, \dots, x_{n-1})$$

where $f \in \mathcal{E}_{\mathcal{M}}(\mathbb{R}^{n-1})$ and suppose that the level hypersurfaces of F are nowhere characteristic with respect to P in Ω. Set also $\Omega_c = \{x \in \Omega \mid F(x) < c\}$ for $c \in \mathbb{R}$. If $u \in \mathcal{D}'(\Omega)$ is a solution of $P(x,D)u = 0$ and there is $c \in \mathbb{R}$ such that $\Omega_c \cap \operatorname{supp} u$ is relatively compact in Ω, then $u = 0$ in Ω_c.

Proof. We set for $c \in \mathbb{R}$

$$\omega_c = \bigl\{x \in \Omega \mid F(x) = c\bigr\}.$$

Note that each $c \in \mathbb{R}$ ω_c is not relatively compact in Ω. Therefore also Ω_c is not relatively compact in Ω for any c since $\partial\Omega_c = \omega_c$.

By assumption there is a $c \in \mathbb{R}$ such that $K = \operatorname{supp} u \cap \overline{\Omega}_c$ is compact in Ω. In particular, K is bounded in Ω. Hence there has to be $\tilde{c} < c$ such that

$$K \subseteq \bigl\{x \in \Omega \mid \tilde{c} \le F(x) \le c\bigr\}.$$

Let $c_1 < c$ be the greatest real number such that the inclusion above holds for $\tilde{c} = c_1$. Since K is compact there is a point $p \in \partial K$ such that $F(p) = c_1$. It follows that $p \in \partial \operatorname{supp} u \cap \omega_{c_1}$. Thus we can apply Corollary 2.5.6 because ω_{c_1} is nowhere characteristic for P. Hence u vanishes in a full neighbourhood of p. This contradicts the choice of c_1. We conclude that u has to vanish on Ω_c. □

Note that in [**43**] Hörmander used the analytic version of Corollary 2.5.7 to prove Holgrem's Uniqueness Theorem.

Remark 2.5.8. We have formulated our results for scalar operators on open sets of $\mathbb{R}^n$ but they remain of course valid on ultradifferentiable manifolds. Actually, all the conclusions in this section hold even for determined systems of operators and vector-valued distributions. Indeed, we have only to verify that Theorem 2.5.2 holds also for distributions with values in $\mathbb{C}^\nu$, but this is trivial: If $f(x) \le f(x_0)$ for $x \in \operatorname{supp} u$ then $f(x) \le f(x_0)$ for all $x \in \operatorname{supp} u_j$ and any $1 \le j \le n$, since $\operatorname{supp} u = \bigcup_{j=1}^{\nu} \operatorname{supp} u_j$. Hence Theorem 2.5.2 implies

$$(x_0, \pm df(x_0)) \in \bigcap_{j=1}^{\nu} \mathrm{WF}_{\mathcal{M}}\, u_j \subseteq \mathrm{WF}_{\mathcal{M}}\, u.$$

Following an idea of Bony ([**16, 17**]) it is possible to generalize the results above. For the formulation we need some additional notation. Consider a smooth real valued function p on $T^*\Omega$. The *Hamiltonian vector field* H_p of p is defined by

$$H_p = \sum_{j=1}^{n} \biggl(\frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial}{\partial \xi_j} \biggr).$$

An integral curve of H_p, i.e. a solution of the Hamilton-Jacobi equations

$$\frac{dx_j}{dt} = \frac{\partial p}{\partial \xi_j}(x,\xi),$$
$$\frac{d\xi_j}{dt} = -\frac{\partial p}{\partial x_j}(x,\xi),$$

$j = 1, \ldots, n$, is called a *bicharacteristic* if p vanishes on it. If q is another smooth real valued function on $T^*\Omega$ then the *Poisson bracket* is defined by $\{p, q\} := H_p(q)$ or in coordinates

$$\{p, q\} = \sum_{j=1}^{n} \left(\frac{\partial p}{\partial \xi_j} \frac{\partial q}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial q}{\partial \xi_j} \right).$$

See [**36**] or [**45**] for more details.

We continue by recalling a result of Sjöstrand [**75**] (see also [**45**]).

THEOREM 2.5.9. *Let F be a closed subset of Ω and suppose that $p \in \mathcal{E}(T^*\Omega \setminus \{0\})$ is real valued and vanishes on $N_e(F)$. If $(x_0, \xi_0) \in N_e(F)$ then the bicharacteristic $t \mapsto (x(t), \xi(t))$ with $(x(0), \xi(0)) = (x_0, \xi_0)$ stays for $|t|$ small in $N_e(F)$.*

The analogous statement is of course also true for $N_i(F)$ replacing $N_e(F)$. It follows

COROLLARY 2.5.10 (Bony). *Let F be a closed subset of Ω and set*

$$\mathcal{N}_F := \{ p \in \mathcal{E}(T^*\Omega \setminus \{0\}) \mid p \equiv 0 \text{ on } N(F) \}.$$

Then $\mathcal{N}_F$ is an ideal in $\mathcal{E}(T^*\Omega \setminus \{0\})$ that is closed under Poisson brackets.

We obtain the quasianalytic version of a result of Bony [**16, 17**].

THEOREM 2.5.11. *Let $\mathcal{M}$ be quasianalytic, P a differential operator with $\mathcal{E}_{\mathcal{M}}$-coefficients on Ω and Π the Poisson algebra that is generated by all functions $f \in \mathcal{E}(T^*\Omega \setminus \{0\})$ that vanish on* $\operatorname{Char} P$.

If $u \in \mathcal{D}'(\Omega)$ is a solution of the homogeneous equation $Pu = 0$ then all functions in Π have to vanish on $N(\operatorname{supp} u)$.

COROLLARY 2.5.12. *If the elements of Π have no common zeros and u vanishes in a neighbourhood of a point $p_0 \in \Omega$ then u must vanish in the connected component of Ω that contains p_0.*

We continue by taking a closer look at Theorem 2.5.9. Let $\pi : T^*\Omega \to \Omega$ be the canonical projection and $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$. If q is a smooth function on $T^*\Omega \setminus \{0\}$ that vanishes on $N(F)$, $F \subseteq \Omega$ closed, and $\lambda(t)$ the bicharacteristic through (x_0, ξ_0) then we conclude that the bicharacteristic curve $\gamma(t) = \pi \circ \lambda$ must stay in ∂F for small t in view of the remarks before Theorem 2.5.4.

Now suppose that Q is a real vector field on Ω and q its symbol. If we denote by γ the integral curve of Q through x_0 and by λ the bicharacteristic of q through (x_0, ξ_0) where (x_0, ξ_0) then it is trivial that $\gamma = \pi \circ \lambda$.

DEFINITION 2.5.13. We say that a partial differential operator P on Ω with $\mathcal{E}_{\mathcal{M}}$-coefficients is $\mathcal{M}$-admissible iff there are ultradifferentiable real-valued vector fields $Q_1, \ldots, Q_d$ with symbols $q_1, \ldots, q_d$ such that each q_j vanishes on $\operatorname{Char} P$.

Following the approach of Sjöstrand [**75**] we can generalize results of Zachmanoglou [**83**] (c.f. also [**17**]) to the quasianalytic setting.

PROPOSITION 2.5.14. *Let $\mathcal{M}$ be quasianalytic and P be an $\mathcal{M}$-admissible operator. If $\mathcal{L} = \mathcal{L}(Q_1, \ldots, Q_d)$ is the Lie algebra generated by the vector fields Q_j, $j = 1, \ldots, d$, $\varphi \in \mathcal{C}^1(\Omega, \mathbb{R})$ near a point $x_0 \in \Omega$ such that $(x_0, \varphi'(x_0)) \in \operatorname{Char} P$ and $u \in \mathcal{D}'(\Omega)$ a solution of $Pu = 0$ such that near x_0 we have $x_0 \in \operatorname{supp} u \subseteq \{\varphi \geq 0\}$. Then each $Q \in \mathcal{L}$ is tangent to $\{\varphi = 0\}$ at x_0 and the local Nagano leaf $\gamma_{x_0}(\mathcal{L})$ is contained in* $\operatorname{supp} u$.

PROOF. By assumption all $Q_1, \ldots, Q_d$ are tangent to $\{\varphi = 0\}$ at x_0 and hence also all $Q \in \mathcal{L}$. From the remarks before Definition 2.5.13 and Theorem 2.5.4 we see that all integral curves of the vector fields in $\mathcal{L}$ must be contained in $\partial \operatorname{supp} u$ for a small neighbourhood of x_0. Inspecting the construction of the representative of the local Nagano leaf in the proof of Theorem 1.2.6 we see that $\gamma_{x_0}(\mathcal{L}) \subseteq \operatorname{supp} u$ near x_0. □

In fact, we have the following global theorem (see for the analytic case [**83**], c.f. [**17**, Theorem 2.4.])

THEOREM 2.5.15. *Let* $\mathcal{M}$ *be quasianalytic and* P *an* $\mathcal{M}$*-admissable differential operator. If* $u \in \mathcal{D}'(\Omega)$ *is a solution of* $Pu = 0$ *and* $p_0 \notin \operatorname{supp} u$ *then every integral curve of the vector fields* $Q_1, \dots, Q_d$ *through* p_0 *stays in* $\Omega \setminus \operatorname{supp} u$.

PROOF. Let $\Gamma = \Gamma_{p_0}(\mathcal{L})$ be the global Nagano leaf of $\mathcal{L} = \mathcal{L}(Q_1, \dots, Q_d)$ through p_0 and suppose that $\partial \operatorname{supp} u \cap \Gamma \neq \emptyset$. Then there has to be a point $q_0 \in \Gamma \cap \partial \operatorname{supp} u$ such that for all neighbourhoods $V \subseteq \Omega$ of x_0 we have

$$\big(\Gamma \cap V\big) \cap \big(\Omega \setminus \operatorname{supp} u\big) \neq \emptyset.$$

Let V small enough such that $\Gamma \cap V$ is the representative of the local Nagano leaf of $\mathcal{L}$ at q_0 constructed in the proof of Theorem 1.2.6. Then

$$\Gamma \setminus \operatorname{supp} u \cap V \neq \emptyset.$$

Thence there is a vector field $X \in \mathcal{L}$ such that if $\gamma(t) = \exp tX$ is the integral curve of X through q_0 then $\gamma(0) = q_0$ and $\gamma(1) = q_1 \in V \setminus \operatorname{supp} u$. Possibly shrinking V and applying an ultradifferentiable coordinate change in V we may assume that $q_0 = 0$, $q_1 = (0, \dots, 0, 1)$ and

$$X = \frac{\partial}{\partial x_n}.$$

We note that in these new coordinates the assumption on P can be stated in the following way. Let $\xi \in \mathbb{R}^n$ with $\xi_n \neq 0$ then $p_m(x, \xi) \neq 0$ for all $x \in V$. There is also a neighbourhood $V_1 \subseteq V$ of q_1 such that u vanishes on V_1.

We adapt the proof of [**82**, Theorem 1]. Let $r > 0$ and $\delta > 0$ small enough so that

$$U = \big\{x \in \mathbb{R}^n \mid |x'| < r,\ -\delta < x_n < 1\big\}$$

is contained in V and

$$\big\{x \in \mathbb{R}^n \mid |x'| < r,\ x_n = 1\big\} \subseteq V_1.$$

We consider the real-analytic function

$$F(x) = (1 + \delta)\frac{|x'|^2}{r^2} - \delta - x_n.$$

The normals of the level hypersurfaces of F are always nonzero in the direction of the n-th unit vector. It follows that the level hypersurfaces are everywhere non-characteristic with respect to P in V. Set

$$U_1 = \left\{x \in U : \ F(x) < -\frac{\delta}{2}\right\}$$

and note that if $x \in U_1$ then $x_n > -\delta/2$. It is easy to see that $U_1 \cap \operatorname{supp} u$ is relatively compact in U. We conclude that $u = 0$ in U_1 by Corollary 2.5.7. That is a contradiction to the assumption $q_0 \in \partial \operatorname{supp} u$. □

EXAMPLE 2.5.16. If $Q_1, \dots, Q_d$ are real valued vector fields with $\mathcal{E}_{\mathcal{M}}$-coefficients, then the operators

$$P_0 = Q_1 + iQ_2$$

$$P_k = \sum_{j=1}^{d} Q_j^{2k} \qquad k \in \mathbb{N}$$

are $\mathcal{M}$-admissible.

CHAPTER 3

CR manifolds of Denjoy-Carleman type

In this chapter M is always going to denote an ultradifferentiable (sub-)manifold of class $\{\mathcal{M}\}$, where $\mathcal{M}$ is a regular weight sequence. Here though we may also allow to let $\mathcal{M} = \emptyset$ be the empty sequence, i.e. $\mathcal{E}_{\mathcal{M}} = \mathcal{E}$. In this particular case this chapter might not contain any new results, c.f. the references given in the individual sections for the results in the smooth case.

3.1. Introduction

In this section we rapidly recall the basic definitions of CR geometry, for more details see [**8**]. We begin with the embedded case. Let $M \subseteq \mathbb{C}^N$ be a real submanifold of $\mathbb{C}^N$, then $T_pM \subseteq T_p\mathbb{C}^N$ ($p \in M$) as real vector spaces, but $T_p\mathbb{C}^N = \mathbb{R}^{2N} \cong \mathbb{C}^N$ inherits a complex structure from $\mathbb{C}^N$. Hence there is a maximal complex subspace T_p^cM of $T_p\mathbb{C}^N$ such that $T_p^cM \subseteq T_pM \subseteq T_p\mathbb{C}^N$.

Definition 3.1.1. A submanifold $M \subseteq \mathbb{C}^N$ is said to be CR if the mapping

$$M \ni p \longmapsto \dim_{\mathbb{C}} T_p^c M$$

is constant. The CR dimension of M is then defined as $\dim_{CR} M := \dim_{\mathbb{C}} T_p^c M$.

Note that any real hypersurface $M \subseteq \mathbb{C}^N$ is CR. An arbitrary submanifold $M \subseteq \mathbb{C}^N$ of codimension d is said to be *generic* iff it can be realized as the intersection of d real hypersurfaces whose complex tangent spaces are in general position as complex vector spaces. The manifold M is said to be generic at a point $p \in M$ iff there is a neighbourhood U of p in $\mathbb{C}^N$ such that $M \cap U$ is generic. We recall that if $M \subseteq \mathbb{C}^N$ is a generic submanifold of CR dimension n and real codimension d then $n + d = N$.

It is easy to see that for a CR manifold M we can consider the complex tangent bundle $T^cM \subseteq TM$. However the complex tangent bundle, although being a vector bundle over $\mathbb{C}$, is realized as a subbundle of the real bundle TM. Often it is more convenient to take a different approach for the definition of CR manifolds. For this end consider the complexified tangent bundle $\mathbb{C}TM = \mathbb{C} \otimes TM$ of a manifold $M \subseteq \mathbb{C}^N$. Furthermore let $p \in M$ and set $\mathbb{C}T_p\mathbb{C}^N = T_p^{1,0}\mathbb{C}^N \oplus T_p^{0,1}\mathbb{C}^N$. If $z_j = x_j + iy_j$, $j = 1, \dots, N$ denote the coordinates of $\mathbb{C}^N$ then the spaces $T_p^{1,0}\mathbb{C}^N$ and $T_p^{0,1}\mathbb{C}^N$ are generated by $\partial/\partial z_j|_p$ and $\partial/\partial \bar{z}_j|_p$, $j = 1, \dots, N$, respectively. If we set $\mathcal{V}_p = \mathbb{C}T_pM \cap T_p^{0,1}\mathbb{C}^N$ then $\dim_{\mathbb{C}} \mathcal{V}_p = \dim_{\mathbb{C}} T_p^cM$. If M is a CR submanifold, then $\mathcal{V} = \bigsqcup_p \mathcal{V}_p$ is said to be the CR bundle associated to M. It is easy to see that $\mathcal{V}$ is involutive, i.e. $[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V}$, and $\mathcal{V} \cap \bar{\mathcal{V}} = \{0\}$. Using this it is possible to generalize the notion of CR manifold.

Definition 3.1.2. Let M be a manifold (not necessarily embedded) and $\mathcal{V} \subseteq \mathbb{C}TM$ a subbundle. We say that $(M, \mathcal{V})$ is an abstract CR manifold iff $\mathcal{V}$ is an involutive bundle and $\mathcal{V} \cap \bar{\mathcal{V}} = \{0\}$. The CR dimension of M is defined as $\dim_{CR} M = \dim \mathcal{V}$. If $\dim_{\mathbb{R}} M = m + n$ then the CR codimension is given by $d = m - n$.

If M is a CR manifold of class $\{\mathcal{M}\}$ then a CR vector field L is an ultradifferentiable section of $\mathcal{V}$, i.e. $L \in \mathcal{E}_{\mathcal{M}}(M, \mathcal{V})$. If $p \in M$ and $n = \dim_{CR} M$ then a local basis of CR vector fields near p consists of n CR vector fields $L_1, \dots, L_n$ defined near p that are linearly independent. We also set $L^\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n}$ for $\alpha \in \mathbb{N}_0^n$.

A CR function or CR distribution is a function or distribution on M that is annihilated by all CR vector fields. We refer to $T'M := \mathcal{V}^\perp$ as the holomorphic cotangent bundle. $T'M$ is a complex vector bundle on M with fiber dimension $N = n + d$. Its ultradifferentiable sections are called holomorphic forms. The real subbundle $T^0M \subseteq T'M$ that consists of the real dual vectors that vanish on $\mathcal{V} \oplus \bar{\mathcal{V}}$ is called the characteristic bundle of M and its sections of class $\{\mathcal{M}\}$ are the characteristic forms on M. Note that if L is a CR vector field, we have generally that $\operatorname{Char} L \subseteq T^0M$, hence we obtain for any CR distribution u that $\mathrm{WF}_{\mathcal{M}}\, u \subseteq T^0M$.

A $\mathcal{C}^1$-mapping H between two CR manifolds $(M, \mathcal{V})$ and $(M', \mathcal{V}')$ is CR iff for all $p \in M$ we have $H_*(\mathcal{V}_p) \subseteq \mathcal{V}'_{H(p)}$. Here H_* denotes the tangent map of H. If $M' \subseteq \mathbb{C}^{N'}$ is an embedded CR submanifold and $Z' = (Z'_1, \dots, Z'_{N'})$ some set of local holomorphic coordinates in $\mathbb{C}^{N'}$ then $H_j = Z'_j \circ H$, $1 \leq j \leq N'$ is a CR function on the CR manifold M for all $1 \leq j \leq N'$.

We continue with a first look at specific results about ultradifferentiable CR manifolds.

PROPOSITION 3.1.3. *Let $M \subseteq \mathbb{C}^N$ be a generic manifold of class $\{\mathcal{M}\}$ of codimension d and $p_0 \in M$. If n denotes the CR dimension of M then there are holomorphic coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ defined near p_0 that vanish at p_0 and a function $\varphi \in \mathcal{E}_{\mathcal{M}}(U \times V, \mathbb{R}^d)$ defined on a neighbourhood $U \times V$ of the origin in $\mathbb{R}^{2n} \times \mathbb{R}^d$ with $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$, such that near p_0 the manifold M is given by*

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w). \tag{3.1.1}$$

PROOF. We follow the proof of [**8**] for the result in the smooth category.

After an affine transformation we may assume that $p_0 = 0$. Let $\rho = (\rho_1, \dots, \rho_d)$ be a defining function for M near 0. The complex differentials $\partial\rho_1, \dots, \partial\rho_d$ are linearly independent over $\mathbb{C}$ near 0 since M is generic. For each $k \in \{1, \dots, d\}$ we write

$$\rho_k(Z, \bar{Z}) = \sum_{r=1}^{N} \Big(a_{kr} x_r + b_{kr} y_r \Big) + O(2)$$

where $O(2)$ denotes terms that vanish at least of quadratic order at 0. Since ρ_k is real-valued, the coefficients a_{kr} and b_{kr} have to be real numbers. We define a linear form ℓ_k on $\mathbb{C}^N$ by

$$\ell_k(Z) = \sum_{r=1}^{N} (b_{kr} + i a_{kr}) Z_r$$

and thus the above equation becomes

$$\rho_k(Z, \bar{Z}) = \operatorname{Im} \ell_k(Z) + O(2).$$

The linear forms ℓ_k, $k = 1, \dots, d$ are linearly indepedent over $\mathbb{C}$ since the differentials $\partial\rho_k$, $k = 1, \dots, d$, are $\mathbb{C}$-linearly indepedent. After renumbering the coordinates Z_j we can assume that

$$Z_1, \dots, Z_n, \ell_1, \dots, \ell_k$$

are linearly indepedent as linear forms over $\mathbb{C}$.

We define new holomorphic coordinates (z, w) near $(0, 0) \in \mathbb{C}^{n+d}$ by

$$\begin{aligned} z_j &= Z_j & & 1 \leq j \leq n \\ w_k &= \ell_k(Z) & & n+1 \leq k \leq N = n + d. \end{aligned}$$

In these new coordinates we have, if we set $\tilde{\rho}(z, \bar{z}, w, \bar{w}) = \rho(Z(z, w), \overline{Z(z, w)})$,

$$\tilde{\rho}(z, \bar{z}, w, \bar{w}) = \operatorname{Im} w + O(2) \tag{3.1.2}$$

and therefore we can locally near 0 solve the equation

$$\tilde{\rho}(z, \bar{z}, w, \bar{w}) = 0 \tag{3.1.3}$$

with respect to $t = \operatorname{Im} w$ according to Theorem 1.1.6. We obtain an ultradifferentiable solution φ of class $\{\mathcal{M}\}$ defined near $0 \in \mathbb{R}^{2n+d} = \mathbb{C}^n \times \mathbb{R}^d$ and valued in $\mathbb{R}^d$. The properties $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$ are easy consequences of (3.1.2) and (3.1.3). We also see that in view of (3.1.2) and

$$\tilde{\rho}(z, \bar{z}, s + i\varphi(z, \bar{z}, s), s - i\varphi(z, \bar{z}, s)) = 0$$

the function $\psi(z,\bar z,s,t)=t-\varphi(z,\bar z,s)$ is also a defining function for M near 0. This finishes the proof. □

Remark 3.1.4. We note that Proposition 3.1.3 can be used to give a special local basis of CR vector fields. Indeed, let $M\subseteq\mathbb{C}^N$ be a generic submanifold of codimension d that is given locally near a point $p_0\in M$ by a defining function $\rho=(\rho_1,\dots,\rho_d)$. If we use the coordinates $(z,w)\in\mathbb{C}^{n+d}$ from above then we can formally view ρ as a function on the variables $(z,\bar z,w,\bar w)$. Let ρ_z, $\rho_{\bar z}$, ρ_w and $\rho_{\bar w}$ the Jacobi matrices of ρ with respect to z, $\bar z$, w and $\bar w$ respectively. We can assume that ρ_w and $\rho_{\bar w}$ are invertible in a neighbourhood of p_0. According to [**8**, §1.6] a local basis of CR vector fields near p_0 is given by

$$(L)=\bigl(\partial_{\bar z}\bigr)-{}^\tau\rho_{\bar z}\,{}^\tau\rho_{\bar w}{}^{-1}\bigl(\partial_{\bar w}\bigr)$$

where we have used the following notation

$$(L)=\begin{pmatrix}L_1\\ \vdots\\ L_n\end{pmatrix},\qquad \bigl(\partial_{\bar z}\bigr)=\begin{pmatrix}\partial_{\bar z_1}\\ \vdots\\ \partial_{\bar z_n}\end{pmatrix},\qquad \bigl(\partial_{\bar w}\bigr)=\begin{pmatrix}\partial_{\bar w_1}\\ \vdots\\ \partial_{\bar w_d}\end{pmatrix}.$$

If we use the defining function $\rho=t-\varphi$ induced by (3.1.1) then this local basis is of the following form

$$\begin{aligned}L_j&=\frac{\partial}{\partial\bar z_j}-\sum_{\mu=1}^d 2b_\mu^j\frac{\partial}{\partial\bar w_\mu}\\ &=\frac{\partial}{\partial\bar z_j}-\sum_{\mu=1}^d b_\mu^j\frac{\partial}{\partial s_\mu}\end{aligned}$$

with

$$b_\mu^j=i\frac{\det B_\mu^j}{\det\Phi}.$$

Here we used

$$\Phi=\rho_{\bar w}=\begin{pmatrix}1+i(\varphi_1)_{s_1}&\cdots&i(\varphi_1)_{s_d}\\ \vdots&\ddots&\vdots\\ i(\varphi_d)_{s_1}&\cdots&1+i(\varphi_d)_{s_d}\end{pmatrix}$$

and B_μ^j is the following matrix. Let $\delta_{\mu\nu}$ be the Kronecker delta defined by $\delta_{\nu\nu}=1$ and $\delta_{\mu\nu}=0$ otherwise and set

$$(\varphi)_{s_\nu}=\begin{pmatrix}\delta_{1\nu}+i(\varphi_1)_{s_\nu}\\ \vdots\\ \delta_{d\nu}+i(\varphi_d)_{s_\nu}\end{pmatrix}\qquad\text{and}\qquad(\varphi)_{\bar z_j}=\begin{pmatrix}(\varphi_1)_{\bar z_j}\\ \vdots\\ (\varphi_d)_{\bar z_j}\end{pmatrix}.$$

Then

$$B_{j\mu}=\bigl((\varphi)_{s_1}\quad\cdots\quad(\varphi)_{s_{\mu-1}}\quad(\varphi)_{\bar z_j}\quad(\varphi)_{s_{\mu+1}}\quad\cdots\quad(\varphi)_{s_d}\bigr).$$

In particular, if $M\subseteq\mathbb{C}^{n+1}$ is a real hypersurface of class $\{\mathcal{M}\}$ locally given by the equation $\operatorname{Im}w=\varphi(z,\bar z,\operatorname{Re}w)$ where $\varphi\in\mathcal{E}_{\mathcal{M}}$ then the vector fields

$$L_j=\frac{\partial}{\partial\bar z_j}-2i\frac{\varphi_{\bar z_j}}{1+i\varphi_s}\frac{\partial}{\partial\bar w}\qquad j=1,\dots,n$$

form a local basis of the CR vector fields of M. When we use the local coordinates $(z,\bar z,s)$ of M induced by (3.1.1) then this basis takes the form

$$L_j=\frac{\partial}{\partial\bar z_j}-i\frac{\varphi_{\bar z_j}}{1+i\varphi_s}\frac{\partial}{\partial s}\qquad j=1,\dots,n.$$

We close the section with a first result on the structure of ultradifferentiable CR manifolds.

Definition 3.1.5. Let $M \subseteq \mathbb{C}^N$ a CR submanifold. The *CR orbit* Orb_p of $p \in M$ is the local Sussman orbit of p in M relative to the set of ultradifferentiable sections of T^cM.

Note that if $p_0 \in M$ then by construction $T^c_p\mathrm{Orb}_{p_0} = T^c_pM$ for all $p \in \mathrm{Orb}_{p_0}$ thence Orb_{p_0} is the germ of a CR submanifold of $\mathbb{C}^N$ of CR dimension n.

Definition 3.1.6. Let $M \subseteq \mathbb{C}^N$ a CR manifold and $p_0 \in M$.

(1) We say that M is minimal at p_0 iff there is no submanifold $S \subseteq M$ through p_0 such that $T^c_pM \subseteq T_pS$ for all $p \in S$ and $\dim_{\mathbb{R}} S < \dim_{\mathbb{R}} M$.
(2) The manifold M is said to be of finite type at p_0 iff there are vector fields $X_1, \dots, X_k \in \mathcal{E}_{\mathcal{M}}(M, T^cM)$ such that the Lie algebra generated by the $X_1, \dots, X_k$ evaluated at p_0 is isomorphic to $T_{p_0}M$.

It is well known that finite type implies minimality and that the two notions coincide for real-analytic CR manifolds, c.f. [**8**]. We are going to show that this fact holds also for quasianalytic CR submanifolds.

Theorem 3.1.7. *Let $\mathcal{M}$ be a quasianalytic weight sequence and $M \subseteq \mathbb{C}^N$ an ultradifferentiable CR manifold of class $\{\mathcal{M}\}$. The following statements are equivalent:*

(1) *M is minimal at p_0.*
(2) $\dim_{\mathbb{R}} \mathrm{Orb}_{p_0} = \dim_{\mathbb{R}} M$
(3) *M is of finite type at p_0.*

Proof. The equivalence of (1) and (2) holds even if $\mathcal{M}$ is non-quasianalytic. Following the arguments in [**8**, §4.1.] we see that, if we assume that M is nonminimal then $\dim_{\mathbb{R}} \mathrm{Orb}_{p_0} < \dim_{\mathbb{R}} M$. On the other hand if $\dim_{\mathbb{R}} \mathrm{Orb}_{p_0} < \dim_{\mathbb{R}} M$ then any representative W of Orb_{p_0} is by the remark below Definition 3.1.5 a submanifold of M and $T^c_pW = T^c_pM$ for all $p \in W$. It remains to prove that (2) implies (3).

By Corollary 1.2.8 we have that $\mathrm{Orb}_{p_0} = \gamma_{p_0}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra generated by the ultradifferentiable sections of T^cU with U being a sufficiently small neighbourhood of p_0 and $\gamma_{p_0}(\mathfrak{g})$ the local Nagano leaf of $\mathfrak{g}$ at p_0. Hence $\dim_{\mathbb{R}} \mathrm{Orb}_{p_0} = \dim_{\mathbb{R}} \gamma_{p_0}(\mathfrak{g}) = \dim_{\mathbb{R}} \mathfrak{g}(p_0)$.

On the other hand M is of finite type at p_0 if and only if $\dim_{\mathbb{R}} \mathfrak{g}(p_0) = \dim_{\mathbb{R}} M$. □

We shall note we could have shown the equivalence of (1) and (2) by citing the corresponding proof in the smooth category in [**8**, Theorem 4.1.3.]. Indeed, let $M \subseteq \mathbb{C}^N$ be an ultradifferentiable CR submanifold of class $\{\mathcal{M}\}$ and $p_0 \in M$. Then we can consider M also as an smooth CR manifold and define similar to [**8**] $\widetilde{\mathrm{Orb}}_{p_0}$ as the Sussman Orbit relative to the smooth sections of T^cM near p_0.

However, if $X_1, \dots, X_n$ is a local basis of $\mathcal{E}_{\mathcal{M}}(M, T^cM)$ near p_0 then we have that Orb_{p_0} is generated by $\mathfrak{D} = \{X_1, \dots, X_n\}$, c.f. Theorem 1.2.5. On the other hand, since the vector fields $X_1, \dots, X_n$ constitute also a local basis of $\mathcal{E}(M, T^cM)$ near p_0 we obtain also that $\widetilde{\mathrm{Orb}}_{p_0}$ is generated by $\mathfrak{D}$. It follows that $\mathrm{Orb}_{p_0} = \widetilde{\mathrm{Orb}}_{p_0}$ as germs of manifolds at p_0.

The next example is a straightforward generalization of [**8**, Example 1.5.16.].

Example 3.1.8. Let $\mathcal{M}$ be a non-quasianalytic weight sequence and $\psi \in \mathcal{E}_{\mathcal{M}}(\mathbb{R})$ a real valued function such that $\psi(y) = 0$ for $y \leq 0$ and $\psi(y) > 0$ for $y > 0$. We define a real hypersurface in $\mathbb{C}^2$ by

$$M = \bigl\{(z, w) \in \mathbb{C}^2 \mid \operatorname{Im} w = \varphi(\operatorname{Im} z)\bigr\}.$$

Then M is minimal at the origin but not of finite type at 0. Indeed, if M is non-minimal at 0 then according to [**8**, Theorem 1.5.15] there is a holomorphic hypersurface $S \subseteq M$ through the origin. Since $\partial/\partial z$ is tangent to S at 0 it follows that S is given near the origin by the defining equation $w = h(z)$ where h is a holomorphic function defined in some neighbourhood of $0 \in \mathbb{C}$ with $h(0) = 0$. We conclude that due to $S \subseteq M$ we necessarily have that

$$h(z) - \overline{h(z)} = 2i\psi(\operatorname{Re} z)$$

in some neighbourhood of 0. It follows that ψ has to be real-analytic near 0 which contradicts the definition of ψ.

Since ψ is flat at the origin, it follows that M cannot be of finite type at 0.

3.2. An ultradifferentiable reflection principle

The aim of this section is to prove generalizations of results of Bernhard Lamel and Berhanu-Xiao. Lamel proved that a finitely nondegenerate CR mapping that extends holomorphically to a wedge between two generic submanifolds is real analytic if the manifolds are real analytic ([**52**]) and smooth if the manifolds are both smooth ([**53**]). Our main result states that if the two CR manifolds are both ultradifferentiable of class $\{\mathcal{M}\}$ then H has to be ultradifferentiable of the *same* class $\{\mathcal{M}\}$. We begin with recalling the definition of finite nondegeneracy of a CR map.

Definition 3.2.1. Let M be an abstract CR manifold and $M' \subseteq \mathbb{C}^{N'}$ a generic submanifold. Furthermore let $\rho' = (\rho'_1, \ldots, \rho'_{d'})$ be a defining function of M' near a point $q_0 \in M'$, $L_1, \ldots, L_n$ a local basis of CR vector fields on M near $p_0 \in M$ and $H : M \to M'$ a $\mathcal{C}^m$-CR mapping with $H(p_0) = q_0$.

For $0 \leq k \leq m$ define an increasing sequence of subspaces $E_k(p_0) \subseteq \mathcal{C}^{N'}$ by

$$E_k(p_0) := \operatorname{span}_{\mathbb{C}} \left\{ L^\alpha \frac{\partial \rho'}{\partial Z'} \big(H(Z), \overline{H(Z)}\big)|_{Z=p_0} \ : \ 0 \leq |\alpha| \leq k, \, 1 \leq l \leq d' \right\}.$$

We say that H is k_0-nondegenerate at p_0 $(0 \leq k_0 \leq m)$ iff $E_{k_0-1}(p_0) \subsetneq E_{k_0}(p_0) = \mathbb{C}^{N'}$.

Furthermore if $\Gamma \subseteq \mathbb{R}^d$ is an open convex cone, $p_0 \in M$ and $U \subseteq \mathbb{C}^N$ an open neighbourhood of p_0 then a wedge $\mathcal{W}$ with edge M centered at p_0 is an open subset of the form $\mathcal{W} := \{Z \in U \mid \rho(Z, \bar{Z}) \in \Gamma\}$, where ρ is a local defining function of M.

Theorem 3.2.2. *Let $M \subseteq \mathbb{C}^N$ and $M' \subseteq \mathbb{C}^{N'}$ be two generic ultradifferentiable submanifolds of class $\{\mathcal{M}\}$, $p_0 \in M$, $p'_0 \in M'$ and $H : (M, p_0) \to (M', p'_0)$ a $\mathcal{C}^{k_0}$-CR mapping that is k_0-nondegenerate at p_0. Suppose furthermore that H extends continuously to a holomorphic map in a wedge $\mathcal{W}$ with edge M. Then H is ultradifferentiable of class $\{\mathcal{M}\}$ in a neighbourhood of p_0.*

Proof. Since the assertion of the theorem is local, we are going to work on a neighbourhood $\Omega \subseteq \mathbb{C}^N$ of p_0. If Ω is small enough then by Proposition 3.1.3 there are open neigbourhoods $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{R}^d$ of the origin and a function $\varphi \in \mathcal{E}_{\mathcal{M}}(U \times V, \mathbb{R}^d)$ with $\varphi(0,0) = 0$ and $\nabla\varphi(0,0) = 0$ such that

$$M \cap \Omega = \big\{(z,w) \in \Omega \mid \operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w)\big\}.$$

From now we denote $M \cap \Omega$ by M. If we choose U and V to be small enough we can consider the diffeomorphism

$$\begin{aligned} \Psi : \ U \times V &\longrightarrow \quad M \\ (z,s) &\longmapsto (z, s + i\varphi(z, \bar{z}, s)). \end{aligned}$$

If we shrink the neighbourhoods U, V a little bit (such that $\varphi \in \mathcal{E}_{\mathcal{M}}(\overline{U \times V}, \mathbb{R}^d)$) we can extend the mapping Ψ $\mathcal{M}$-almost analytically in the s-variables, i.e. there exists a smooth function $\tilde{\Psi} : U \times V \times \mathbb{R}^d \to \mathbb{C}^N$ such that $\tilde{\Psi}|_{U \times V \times \{0\}} = \Psi$ and for each component $\tilde{\Psi}_k$, $k = 1, \ldots, N$, of $\tilde{\Psi}$ we have

$$\left| \frac{\partial \tilde{\Psi}_k}{\partial \bar{w}'_j}(z, \bar{z}, s, t) \right| \leq C h_{\mathcal{M}}(\gamma |t|) \qquad j = 1, \ldots, d, \tag{3.2.1}$$

for some constants $C, \gamma > 0$. Here $w' = s + it \in V + i\mathbb{R}^d$. We see that there is some $r > 0$ such that $\tilde{\Psi}|_{U \times V \times B_r(0)}$ is a diffeomorphism.

By assumption $H = (H_1, \dots, H_{N'})$ extends continuously to a holomorphic mapping on a wedge $\mathcal{W}$ near 0. If we shrink $\mathcal{W}$ we may assume that ∂H_j, $j = 1, \dots, N'$, is bounded on $\mathcal{W}$. By definition

$$\mathcal{W} = \bigl\{Z \in \Omega_0 \mid \rho(Z, \bar{Z}) \in \tilde{\Gamma}\bigr\}$$

for a neighbourhood Ω_0 of the origin in $\mathbb{C}^N$ and an open acute cone $\tilde{\Gamma} \in \mathbb{R}^d$. If we shrink U, V, when necessary, and choose a suitable open and acute cone Γ, we achieve that

$$\tilde{\Psi}\Bigl(U \times V \times \Gamma_\delta\Bigr) \subset \mathcal{W}$$

for some $r \geq \delta > 0$. Note that $\tilde{\Psi}(U \times V \times \Gamma_\delta)$ is open in $\mathbb{C}^N$. For each $j = 1, \dots, N'$ set $h_j = H_j \circ \tilde{\Psi}$ and $u_j = H_j \circ \Psi$. Since

$$\frac{\partial h_j}{\partial \bar{w}'_k} = \sum_{\ell=1}^{N} \frac{\partial H_j}{\partial Z_\ell} \frac{\partial \tilde{\Psi}_\ell}{\partial \bar{w}'_k} \qquad j = 1, \dots, N',\ k = 1, \dots, d,$$

and ∂H_j is bounded, each function h_j is $\mathcal{M}$-almost analytic on $U \times V \times \Gamma_\delta$ due to (3.2.1) and extends $u_j \in \mathcal{C}^{k_0}(U \times V)$. Hence Theorem 2.2.1 implies

$$\mathrm{WF}_{\mathcal{M}}\, u_j \subseteq \bigl(U \times V\bigr) \times \bigl(\mathbb{R}^{2n} \times \Gamma^\circ\bigr) \backslash \{0\}. \tag{3.2.2}$$

If L_j, $j = 1, \dots, n$, is a basis of the CR vector fields on $M = M \cap \Omega$, then $\Lambda_j = \Psi^* L_j$ defines a CR structure on $U \times V$ and $\Lambda_j u_k = 0$ for $j = 1, \dots, n$ and $k = 1, \dots, N'$.

Let ρ' be a defining function of M' near $p'_0 = 0 \in \mathbb{C}^{N'}$. Then there are ultradifferentiable functions $\Phi_{\ell,\alpha}(Z', \bar{Z}', W)$ for $|\alpha| \leq k_0$, $\ell = 1, \dots, d'$, defined in a neighbourhood of $\{0\} \times \mathbb{C}^{K_0} \subseteq \mathbb{C}^{N'} \times \mathbb{C}^{K_0}$ and polynomial in the last $K_0 = N' \cdot |\{\alpha \in \mathbb{N}_0^n \mid |\alpha| \leq k_0\}|$ variables such that

$$\Lambda^\alpha\bigl(\rho'_\ell \circ u\bigr)(z, \bar{z}, s) = \Phi_{\ell,\alpha}\Bigl(u(z, \bar{z}, s), \bar{u}(z, \bar{z}, s), \bigl(\Lambda^\beta \bar{u}(z, \bar{z}, s)\bigr)_{|\beta| \leq k_0}\Bigr) = 0 \tag{3.2.3}$$

and

$$\Lambda^\alpha \rho'_{\ell,Z'}\bigl(u, \bar{u}\bigr)(0,0,0) = \Phi_{\ell,\alpha,Z'}\bigl(0, 0, (\Lambda^\beta \bar{u}(0,0,0))_{|\beta| \leq k_0}\bigr)$$

Since H is k_0-nondegenerate there are multi-indices $\alpha^1, \dots, \alpha^{N'}$ and $\ell^1, \dots, \ell^{N'} \in \{1, \dots, d'\}$ such that if we set

$$\Phi = \bigl(\Phi_{\ell^1,\alpha^1}, \dots, \Phi_{\ell^{N'},\alpha^{N'}}\bigr)$$

the matrix $\Phi_{Z'}$ is invertible. Hence by Theorem 1.1.12 there is a smooth function $\phi = (\phi_1, \dots, \phi_{N'})$ defined in a neighbourhood of $(0, (\Lambda^\beta \bar{u}(0,0,0))_{|\beta|})$ in $\mathbb{C}^{N'} \times \mathbb{C}^{K_0}$ such that, if we shrink $U \times V$ accordingly,

$$u_j(z, \bar{z}, s) = \phi_j\bigl(u(z, \bar{z}, s), \bar{u}(z, \bar{z}, s), (\Lambda^\beta \bar{u}(z, \bar{z}, s))_{|\beta| \leq k_0}\bigr) \qquad (z, s) \in U \times V,\ \ j = 1, \dots, N'$$

and (1.1.6) holds. If we further shrink $U \times V$ and δ and choose $\Gamma' \subset\subset \Gamma$ appropriately we see that

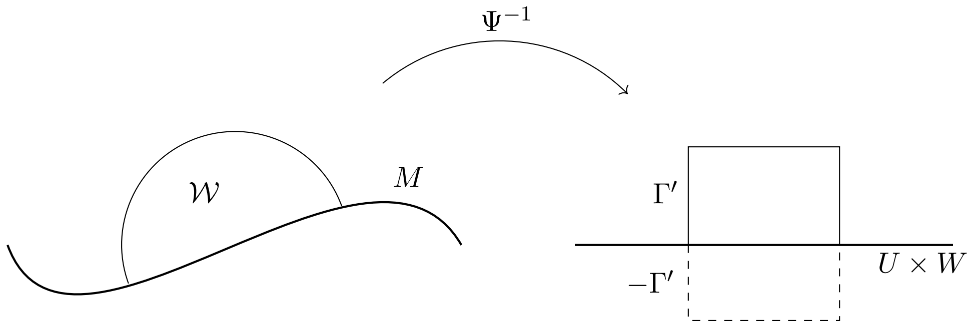
$$g_j(z, \bar{z}, s, t) = \phi_j\bigl(h(z, \bar{z}, s, -t), \bar{h}(z, \bar{z}, s, -t), (\tilde{h}_{\ell,\beta}(z, \bar{z}, s, t)_{\ell \in \{1, \dots, N'\}; |\beta| \leq k_0}\bigr) \tag{3.2.4}$$

is well defined for $t \in -\Gamma'_\delta$. Here $\tilde{h}_{j,\beta}$ is the $\mathcal{M}$-almost analytic extension of $\Lambda^\beta \bar{u}_j$ on $U \times V \times (-\Gamma'_\delta)$, which exists due to (3.2.2), (2.4.3), Proposition 2.1.5 and Theorem 2.2.4. It is also easy to see that $\bar{h}(z, \bar{z}, s, -t)$ is $\mathcal{M}$-almost analytic on $U \times V \times (-\Gamma'_\delta)$. We have that

$$\frac{\partial g_j}{\partial \bar{w}'_\ell} = \sum_{k=1}^{N'} \frac{\partial \phi_j}{\partial Z'_k} \frac{\partial h_k}{\partial w'_\ell} + \sum_{k=1}^{N'} \frac{\partial \phi_j}{\partial \bar{Z}'} \frac{\partial \bar{h}}{\partial w'_\ell} + \sum_{k=1}^{N'} \sum_{|\beta| \leq k_0} \frac{\partial \phi_j}{\partial W_{k,\beta}} \frac{\partial \tilde{h}_{k,\beta}}{\partial w'_\ell}$$

for $j = 1, \dots, N'$ and $\ell = 1, \dots, d$. Note that we can choose $U \times V$ and Γ'_δ so small that all functions appearing on the right-hand side are uniformly bounded. Hence, since $\partial_{w'_\ell} \bar{h} = \overline{\partial_{\bar{w}'_\ell} h}$, g_j is an $\mathcal{M}$-almost analytic extension on $U \times V \times (-\Gamma'_\delta)$ of u_j due to (1.1.6) and thus

$$\mathrm{WF}_{\mathcal{M}}\, u_j \subseteq \bigl(U \times V\bigr) \times \bigl(\mathbb{R}^n \times (\Gamma' \cup -\Gamma')^\circ\bigr) \backslash \{0\} = \bigl(U \times V\bigr) \times \bigl(\mathbb{R}^n \backslash \{0\} \times \{0\}\bigr).$$



On the other hand, since each u_j is CR we have that $\mathrm{WF}_{\mathcal{M}}\, u_j|_0 \subseteq \{0\} \times \mathbb{R}^d \setminus \{0\}$ and we deduce that in fact $\mathrm{WF}_{\mathcal{M}}\, u_j|_0 = \emptyset$ for all $j = 1, \dots, N'$. Hence the mapping H is ultradifferentiable of class $\{\mathcal{M}\}$ near p_0. □

If we recall the well-known result of Tumanov [**80**] which states that any CR function on a minimal CR submanifold M extends to a holomorphic function on a wedge with edge M, then we obtain the following corollary.

Corollary 3.2.3. *Let $M \subseteq \mathbb{C}^N$ and $M' \subseteq \mathbb{C}^{N'}$ generic submanifolds of class $\{\mathcal{M}\}$, $p_0 \in M$, $p_0' \in M'$, M minimal at p_0 and $H : (M, p_0) \to (M', p_0')$ a $\mathcal{C}^{k_0}$-CR mapping that is k_0-nondegenerate at p_0. Then H is ultradifferentiable of class $\{\mathcal{M}\}$ in some neighbourhood of p_0.*

A CR manifold M is said to be k_0-nondegenerate, as introduced in [**5**], iff $\mathrm{id} : M \to M$ is k_0-nondegenerate. For a discussion of this nondegeneracy condition see [**8**] or [**31**]. We note here only the fact that any CR diffeomorphism between two k_0-nondegenerate CR manifolds is k_0-nondegenerate. This leads to the following.

Corollary 3.2.4. *Let $M \subseteq \mathbb{C}^N$ and $M' \subseteq \mathbb{C}^{N'}$ generic submanifolds of class $\{\mathcal{M}\}$ that are k_0-nondegenerate at $p_0 \in M$ and $p_0' \in M'$, respectively. Furthermore assume that M is minimal at p_0 and let $H : M \to M'$ a CR diffeomorphism that is $\mathcal{C}^{k_0}$ near p_0 and satisfies $H(p_0) = p_0'$. Then H has to be ultradifferentiable of class $\{\mathcal{M}\}$ near p_0.*

Recently Berhanu-Xiao [**10**] showed that it is possible to slightly weaken the prerequisites of the smooth reflection principle of Lamel. In particular, the source manifold M can be chosen to be an abstract CR manifold. Using the methods developed previously we can also generalize this result to the ultradifferentiable category.

Theorem 3.2.5. *Let $(M, \mathcal{V})$ be an abstract CR manifold and $M' \subseteq \mathbb{C}^{N'}$ be a generic submanifold, both of class $\{\mathcal{M}\}$. Furthermore let $p_0 \in M$, $H : M \to M'$ a $\mathcal{C}^{k_0}$-CR mapping that is k_0-nondegenerate at p_0 and there is a closed acute cone $\Gamma \subseteq \mathbb{R}^d$ such that $\mathrm{WF}_{\mathcal{M}}\, H|_{p_0} \subseteq \{0\} \times \Gamma$. Then H is ultradifferentiable of class $\{\mathcal{M}\}$ near p_0.*

Proof. Since the assertation is local we will work on a small chart neighbourhood $\Omega = U \times V \times W \subseteq \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ of M of $p_0 = 0$. Here n denotes the CR-dimension of M whereas d is the CR-codimension of M. We use coordinates (x, y, s) on Ω and write $z = x + iy$. In these coordinates a local basis of the CR vector fields of M is given by

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=1}^{n} a_{jk} \frac{\partial}{\partial z_k} + \sum_{\alpha=1}^{d} b_{j\alpha} \frac{\partial}{\partial s_\alpha} \qquad j = 1, \dots, n.$$

From the assumptions we conclude that if Ω is small enough that there is an open, convex cone $\Gamma_1 \subseteq \mathbb{R}^N \backslash \{0\}$ such that

$$\mathrm{WF}_{\mathcal{M}} H = \bigcup_{j=1}^{N'} \mathrm{WF}_{\mathcal{M}} H_j \subseteq \Omega \times \Gamma_1^\circ \tag{3.2.5}$$

due to the closedness of $\mathrm{WF}_{\mathcal{M}} H$ in $T^*M\backslash\{0\}$. If we further shrink Ω (resp. U, V and W) and choose an open convex cone $\Gamma_2 \subseteq \mathbb{R}^N\backslash\{0\}$ such that $\overline{\Gamma}_2 \subseteq \Gamma_1 \cup \{0\}$ we have by Theorem 2.2.4 that there is an $\mathcal{M}$-almost extension $\tilde{F}$ with slow growth of H onto $\Omega \times \Gamma_2$. If we now choose an open convex cone $\Gamma_3 \subseteq \mathbb{R}^d\backslash\{0\}$ with $\{0\} \times \Gamma_3 \subseteq \Gamma_2$ we infer that

$$F := \tilde{F}|_{\Omega\times(\{0\}\times\Gamma_3)}$$

is an $\mathcal{M}$-almost analytic function on $U \times V \times W \times \Gamma_3$ with values in $\mathbb{C}^{N'}$ and

$$\lim_{\Gamma_3 \ni t \to 0} F(\,.\,,\,.\,,\,.\,,t) = H$$

in the sense of distributions.

Let $\rho' = (\rho'_1, \dots, \rho'_{N'})$ be an ultradifferentiable defining function of M' near $p'_0 = H(p_0)$. As before in the proof of Theorem 3.2.2 we conclude that there are ultradifferentiable functions $\Phi_{\ell,\alpha}(Z', \bar{Z}', W)$ for $|\alpha| \le k_0$, $\ell = 1, \dots, d'$, defined in a neighbourhood of $\{0\} \times \mathbb{C}^{K_0} \subset \mathbb{C}^{N'} \times \mathbb{C}^{K_0}$ and polynomial in the last $K_0 = N'|\{\alpha \in \mathbb{N}_0^{n'} \mid |\alpha| \le k_0\}|$ variables. From now on we can follow the proof of Theorem 3.2.2 verbatim. □

3.3. Infinitesimal CR automorphisms and multipliers

In this and the next section we show how the results in [**35**] concerning the smoothness of infinitesimal CR automorphisms transfer to the ultradifferentiable setting. We begin with recalling the basic definitions. Here $(M, \mathcal{V})$ is always an ultradifferentiable abstract CR manifold of class $\{\mathcal{M}\}$.

Definition 3.3.1. Let $U \subseteq M$ an open subset and $X : U \to TM$ a vector field of class $\mathcal{C}^1$. We say that X is an infinitesimal CR automorphism iff its flow H^τ, defined for small τ, has the property, that there is $\varepsilon > 0$ such that H^τ is a CR mapping provided that $|\tau| \le \varepsilon$.

We need for the proofs of the regularity results a more suitable characterization of infinitesimal CR automorphisms. We call a section $\mathfrak{Y} \in \Gamma(M, (T'M)^*)$ a holomorphic vector field on M.

Apparently every vector field $X \in \Gamma(M, TM)$ gives rise to a holomorphic vector field by first extending X to $\mathbb{C}TM$ and then restricting the extension to T^*M. For a partial converse, we recall from [**35**] the following purely algebraic result.

Lemma 3.3.2. *Let $\mathfrak{Y} \in \Gamma(M, (T'M)^*)$. Then there exists a unique vector field $X \in \Gamma(M, TM)$ such that $\mathfrak{Y}$ is induced by X if and only if $\mathfrak{Y}(\tau) = \overline{\mathfrak{Y}(\tau)}$ for all characteristic forms τ.*

Indeed, since $(\mathbb{C}TM)^* = \mathcal{V}^\perp + \overline{\mathcal{V}}^\perp$ and $\mathbb{C}T^0M = (\mathcal{V} \oplus \overline{\mathcal{V}})^\perp$, we can decompose any form $\omega = \alpha + \bar{\beta}$ with α, β holomorphic forms in a nonunique manner. Thus $\mathfrak{Y}$ gives rise to a real vector field X via

$$X(\omega) = \frac{1}{2}\Big(\alpha(\mathfrak{Y}) + \overline{\beta(\mathfrak{Y})}\Big)$$

which is well defined provided that $\mathfrak{Y}(\bar{\tau}) = \overline{\mathfrak{Y}(\tau)}$ for all $\tau \in \Gamma(M, \mathbb{C}T^0M)$ or equivalently, that $\mathfrak{Y}(\tau) = \overline{\mathfrak{Y}(\tau)}$ for all $\tau \in \Gamma(M, T^0M)$, both of which are equivalent to the definition of X above being independent of the decomposition $\omega = \alpha + \bar{\beta}$. From now on we shall not distinguish between X being a real vector field or a holomorphic vector field.

We recall the well-known identity, see e.g. [**38**],

$$\mathcal{L}_X\alpha(Y) = d\alpha(X, Y) + Y\alpha(X) = X\alpha(Y) - \alpha([X, Y]),$$

which holds for arbitrary complex vector fields X, Y and complex forms α on smooth manifolds.

We conclude that accordingly the Lie derivative

$$\mathcal{L}_L\omega(\,.\,) = d\omega(L,\,.\,)$$

of a holomorphic form ω with respect to a CR vector field L is again a holomorphic form. It is now possible to make the following definition. We shall say that a holomorphic vector field $\mathfrak{Y} \in \Gamma(M, (T'M)^*)$ is CR iff

$$L\omega(\mathfrak{Y}) = d\omega(L, \mathfrak{Y})$$

for every CR vector field L and holomorphic form ω. In particular a real vector field X is CR if and only if

$$\omega([L, X]) = 0$$

for all CR vector fields L and holomorphic forms ω. We recall from [**35**] the following fact.

PROPOSITION 3.3.3. *If X is an infinitesimal CR automorphism on M, then X considered as a holomorphic vector field, i.e. $X \in \mathcal{C}^1(M, (T'M)^*)$ is CR.*

PROOF. Let H^τ denote the flow of X. By definition, H^τ satisfies the following differential equation

$$\frac{dH^\tau}{d\tau}(p) = X \circ H_\tau(p).$$

We note that $H^0 = \mathrm{id}_M$ is trivially a CR map, but by assumption we know that if τ is small then

$$\omega\big((H^\tau)_* L\big) = 0$$

for any CR vector field L and any holomorphic form ω, i.e. $\omega(L) = 0$.

We begin with the following general claim: For any triple (Y, B, α), where

$$Y = \sum_{j=1}^m Y_j \frac{\partial}{\partial x_j} \qquad Y_j \in \mathbb{R}$$

$$B = \sum_{j=1}^m B_j \frac{\partial}{\partial x_j}$$

$$\alpha = \sum_{j=1}^m \alpha^j dx^j$$

are defined near 0 and $\alpha(B) = 0$, we have, if K^τ is the flow of Y,

$$\frac{d}{d\tau}\big((K^\tau)^*\alpha(B)\big)\big|_{\tau=0} = \alpha\big([B, Y]\big)$$

near the origin. For the convenience of the reader, we shall include the computation below.

Recalling the fact

$$(K^\tau)^*\alpha\big(B\big)(p) = \alpha\big((K^\tau)_* B\big)(K^\tau(p)) = \sum_{j=1}^m \sum_{k=1}^m \big(\alpha^k \circ K^\tau\big)(p) B_j(p) \frac{\partial K_k^\tau}{\partial x_j}(p)$$

we can compute

$$\begin{aligned}
\frac{d}{d\tau}\big((K^\tau)^*\alpha(B)\big)(p) &= \sum_{j=1}^m \sum_{k=1}^m \frac{d}{d\tau}\left(\big(\alpha^k \circ K^\tau\big)(p) \frac{\partial K_k^\tau}{\partial x_j}(p) B_j(p)\right) \\
&= \sum_{j=1}^m \sum_{k=1}^m \sum_{\ell=1}^m \left(\frac{\partial \alpha^k}{\partial y_\ell} \circ K^\tau\right)(p) \big(Y_\ell \circ K^\tau\big)(p) \frac{\partial K_k^\tau}{\partial x_j}(p) B_j(p) \\
&\quad + \sum_{j=1}^m \sum_{k=1}^m \sum_{\ell=1}^m \big(\alpha^k \circ K^\tau\big)(p) \left(\frac{\partial Y_k}{\partial y_\ell} \circ K^\tau\right)(p) \frac{\partial K_\ell^\tau}{\partial x_j}(p) B_j(p).
\end{aligned}$$

This leads immediately to

$$\begin{aligned}\frac{d}{d\tau}\big((K^\tau)^*\alpha(B)\big)\big|_{\tau=0} &= \sum_{k=1}^m\sum_{\ell=1}^m\left(\frac{\partial\alpha^k}{\partial x_\ell}Y_\ell B_k + \alpha^k\frac{\partial Y_k}{\partial x_\ell}B_\ell\right)\\ &= \sum_{k=1}^m\sum_{\ell=1}^m\left(-\alpha^k Y_\ell\frac{\partial B_k}{\partial x_\ell} + \alpha^k\frac{\partial Y_k}{\partial x_\ell}B_\ell\right)\\ &= \alpha\big([B,Y]\big).\end{aligned}$$

Now we set $Y = X$, $B = L$ and $\alpha = \omega$ as above. Then we have

$$0 = \frac{d}{d\tau}\big(H_\tau^*\omega(L)\big)\big|_{\tau=0} = \omega\big([L,X]\big)$$

and hence X is CR. □

We are now able to generalize the notion of infinitesimal CR automorphism. To this end consider the space $\mathcal{D}'(M,(T'M)^*)$ of distributions with values in $(T'M)^*$.

Definition 3.3.4. An infinitesimal CR diffeomorphism with distributional coefficients on M is a generalized holomorphic vector field $\mathfrak{Y} \in \mathcal{D}'(M,(T'M)^*)$ that satisfies

$$L\omega(\mathfrak{Y}) = (\mathcal{L}_L\omega)(\mathfrak{Y}) \tag{3.3.1}$$

for every CR vector field L and holomorphic form ω and

$$\mathfrak{Y}(\tau) = \overline{\mathfrak{Y}(\tau)} \tag{3.3.2}$$

for all characteristic forms τ.

Note that (3.3.1) is in fact a CR equation for $\mathfrak{Y}$. If $U \subseteq M$ is an open subset of M then we say that $\mathfrak{Y} \in \mathcal{D}'(M,(T'M)^*)$ is an infinitesimal CR automorphism on U iff (3.3.1) and (3.3.2) hold for all local sections $L \in \mathcal{E}_{\mathcal{M}}(U,\mathcal{V}|_U)$ and $\theta \in \mathcal{E}_{\mathcal{M}}(U,T^0M|_U)$, respectively. Let the subset $U \subset M$ is small enough such that there is a local basis $L_1,\dots,L_n$ of CR vector fields and also a local basis $\{\omega^1,\dots,\omega^N\}$ of the space of holomorphic forms. We recall that locally a distribution $\mathfrak{Y} \in \mathcal{D}'(M,(T'M)^*)$ is of the form

$$\mathfrak{Y}|_U = \sum_{j=1}^N X_j\omega_j \tag{3.3.3}$$

with $X_j \in \mathcal{D}'(U)$. We introduce also the following operators on U

$$\mathbf{L}_j = L_j\cdot\mathbf{Id}_N = \begin{pmatrix} L_j & & 0\\ & \ddots & \\ 0 & & L_j\end{pmatrix}$$

and note that since $d\omega^k(L_j,\,.\,)$ is again a holomorphic form we have

$$d\omega^k(L_j,\,.\,) = \sum_{\ell=1}^N B^j_{k,\ell}\omega^\ell$$

with $B^k_{j,\ell} \in \mathcal{E}_{\mathcal{M}}(U)$. We observe that $\mathfrak{Y}$ is CR on U if and only if

$$L_jX_k = L_j\big(\omega^k(\mathfrak{Y})\big) = d\omega^k\big(L_j,\mathfrak{Y})\big) = \sum_{\ell=1}^N B^j_{k,\ell}X_\ell$$

for all $1 \le j \le n$ and $0 \le k \le N$. We set

$$B_j = \begin{pmatrix} B^1_{j,1} & \dots & B^1_{j,N}\\ \vdots & & \vdots\\ B^N_{j,1} & \dots & B^N_{j,N}\end{pmatrix}.$$

Furthermore, using its local representation (3.3.3), we can identify $\mathfrak{Y}$ with the vector $X = (X_1, \dots, X_N)$. Hence (3.3.1) turns into

$$\mathbf{L}_j X = B_j \cdot X$$

or

$$P_j X = 0$$

respectively, where

$$P_j = \mathbf{L}_j - B_j$$

In particular we infer from above and Theorem 2.4.1 that

$$\mathrm{WF}_{\mathcal{M}} \mathfrak{Y} \subseteq T^0 M. \tag{3.3.4}$$

For the formulation of the main regularity results we need one more definition. To begin we introduce for the ultradifferentiable CR manifold M the following sequence of spaces of sections.

$$E_k = \left\langle \mathcal{L}_{K_1} \dots \mathcal{L}_{K_j} \theta \colon j \le k,\ \ K_q \in \mathcal{E}_{\mathcal{M}}(M, \mathcal{V}),\, \theta \in \mathcal{E}_{\mathcal{M}}(M, T^0 M) \right\rangle.$$

We note that $E_0 = \mathcal{E}_{\mathcal{M}}(M, T^0 M)$, and $E_j \subseteq \mathcal{E}_{\mathcal{M}}(M, T'M)$ for all $j \in \mathbb{N}_0$, and set $E = \bigcup_{j \in \mathbb{N}_0} E_j$.

We associate to the increasing chain E_k the increasing sequence of ideals $\mathcal{S}^k \subset \mathcal{E}_{\mathcal{M}}(M, \mathbb{C})$, where

$$\mathcal{S}^k = \bigwedge\nolimits^N E_k = \left\{ \det \begin{pmatrix} V^1(\mathfrak{Y}_1) & \dots & V^1(\mathfrak{Y}_N) \\ \vdots & & \vdots \\ V^N(\mathfrak{Y}_1) & \dots & V^N(\mathfrak{Y}_N) \end{pmatrix} : V^j \in E_k,\, \mathfrak{Y}_j \in \mathcal{E}_{\mathcal{M}}(M, (T'M)^*) \right\}.$$

We set $\mathcal{S} = \mathcal{S}(M) = \bigcup_{k \in \mathbb{N}_0} \mathcal{S}^k$ and call it the space of multipliers of M. In fact each $\mathcal{S}^k$ and thus also $\mathcal{S}$ can be considered actually as ideal sheaves, if we define $E^k(U)$ and $\mathcal{S}^k(U)$ accordingly.

Note that locally one can find smaller sets of generators: Let $U \subset M$ be open, and assume that $L_1, \dots, L_n$ is a local basis for $\Gamma(U, \mathcal{V})$, that $\theta^1, \dots, \theta^d$ is a local basis for $\Gamma(U, T^0 M)$, and that $\omega^1, \dots, \omega^N$ is a local basis of $T'M$. We write $\mathcal{L}_j = \mathcal{L}_{L_j}$ for $j = 1, \dots, n$ and $\mathcal{L}^\alpha = \mathcal{L}_1^{\alpha_1} \dots \mathcal{L}_n^{\alpha_n}$ for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. We note that, since $\mathcal{V}$ is formally integrable, the $\mathcal{L}^\alpha$, where $|\alpha| = k$, generate *all* k-th order homogeneous differential operators in the $\mathcal{L}_j$, and we thus have

$$E_k\big|_U = \left\langle \mathcal{L}^\alpha \theta^\mu \ :\ \ 1 \le \mu \le d,\ \ |\alpha| \le k \right\rangle.$$

We can expand

$$\mathcal{L}^\alpha \theta^\mu = \sum_{\ell=1}^N A_\ell^{\alpha,\mu} \omega^\ell \tag{3.3.5}$$

and for any choice $\underline{\alpha} = (\alpha^1, \dots, \alpha^N)$ of multi-indices $\alpha^1, \dots, \alpha^N \in \mathbb{N}^n$ and $r = (r_1, \dots, r_N) \in \{1, \dots, d\}^N$ we define the functions

$$D(\underline{\alpha}, r) = \det \begin{pmatrix} A_1^{\alpha^1, r_1} & \dots & A_N^{\alpha^1, r_1} \\ \vdots & & \vdots \\ A_1^{\alpha^N, r_N} & \dots & A_N^{\alpha^N, r_N} \end{pmatrix}. \tag{3.3.6}$$

With this notation, we have

$$\mathcal{S}^k\big|_U = \left\langle D(\underline{\alpha}, r) \colon |\alpha^j| \le k \right\rangle;$$

we shall denote the stalk of $\mathcal{S}^k$ at p by $\mathcal{S}_p^k$.

The space of multipliers of a CR manifold M clearly encodes the nondegeneracy properties of M. We close this section by taking a closer look at the connection of $\mathcal{S}$ with finite nondegeneracy. We recall from [**8**] the definition of finite nondegeneracy for abstract CR manifolds.

DEFINITION 3.3.5. Let M be an abstract CR manifold and

$$E_k(p) = \big\langle \mathcal{L}_{K_1} \dots \mathcal{L}_{K_j}\theta(p)\colon j \le k,\ K_q \in \mathcal{E}(M,\mathcal{V}),\, \theta \in \mathcal{E}(M,T^0M)\big\rangle. \tag{3.3.7}$$

for $p \in M$ and $k \in \mathbb{N}$. Then M is k_0-nondegenerate at $p_0 \in M$ iff $E_{k_0-1} \subsetneq E_{k_0} = T'_{p_0}M$. We say that M is finite nondegenerate iff M is finite nondegenerate at every point.

REMARK 3.3.6. This definition is in fact local, since by [**8**, Proposition 11.1.10.] if $L_1,\dots,L_n$ is a local basis of CR vector fields and $\theta^1,\dots\theta^d$ is a local basis of characteristic forms near p_0 then M is k_0-nondegenerate if and only if

$$T'_{p_0}M = \operatorname{span}_{\mathbb{C}}\big\{\mathcal{L}^\alpha\theta^\mu(p_0)\colon |\alpha| \le k_0,\ \mu \in \{1,\dots,d\}\big\}.$$

Hence we may replace M with any open neighbourhood $U \subseteq M$ of p_0 in (3.3.7). Thus we observe that a CR submanifold M is k_0-nondegenerate at $p_0 \in M$ if and only if $\mathcal{S}^{k_0}_{p_0} = (\mathcal{E}_{\mathcal{M}})_{p_0}$.

More precisely, let $U \subseteq M$ be an open subset and $q \in U$. Then M is k_0-nondegenerate at q if and only if there is a multiplier $f \in \mathcal{S}^{k_0}(U)$ that does not vanish at q, i.e. $f(q) \neq 0$.

Indeed, if $f(q) \neq 0$ then obviously $E_{k_0}(q) = T'_qM$. On the other hand, if $g(q) = 0$ for all multipliers $g \in \mathcal{S}^{k_0}(U)$ then necessarily $E_{k_0}(q) \neq T'_qM$.

3.4. Regularity of infinitesimal CR automorphisms

DEFINITION 3.4.1. Let $(M,\mathcal{V})$ be an ultradifferentiable abstract CR manifold of class $\{\mathcal{M}\}$, and $\mathfrak{Y}$ an infinitesimal CR diffeomorphism with distributional coefficients of M, see section 3.3.

We say that $\mathfrak{Y}$ extends microlocally to a wedge with edge M iff there exists a set $\Gamma \subseteq T^0M$ such that for each $p \in M$, the fiber $\Gamma_p \subseteq T^0_pM\setminus\{0\}$ is a closed, convex cone, and

$$\mathrm{WF}_{\mathcal{M}}(\omega(\mathfrak{Y})) \subseteq \Gamma$$

for every holomorphic form $\omega \in \mathcal{E}_{\mathcal{M}}(M,T'M)$.

Note that the condition $\Gamma \subseteq T^0M$ is not as strict as it seems, because $\mathrm{WF}_{\mathcal{M}}(\omega(\mathfrak{Y})) \subseteq T^0M$ by (3.3.4).

THEOREM 3.4.2. *Let $(M,\mathcal{V})$ be an ultradifferentiable abstract CR structure of class $\{\mathcal{M}\}$, and $\mathfrak{Y}$ an infinitesimal CR diffeomorphism of M with distributional coefficients which extends microlocally to a wedge with edge M.*

Then, for any $\omega \in E$, the evaluation $\omega(\mathfrak{Y})$ is ultradifferentiable, and for any $\lambda \in \mathcal{S}$, the vector field $\lambda\mathfrak{Y}$ is also of class $\{\mathcal{M}\}$.

PROOF. Since the assertion is local we will work in a suitable small open set $U \subseteq M$ such that there are local bases $L_1,\dots,L_n$ of $\mathcal{E}_{\mathcal{M}}(U,\mathcal{V})$ and $\omega^1,\dots,\omega^N$ of $\mathcal{E}_{\mathcal{M}}(U,T'M)$, respectively. We recall that we can represent $\mathfrak{Y}$ on U by (3.3.3) or by $X = (X_1,\dots,X_N) \in \mathcal{D}'(U,\mathbb{C}^N)$. By assumption we know that there is a closed convex cone $\Gamma \subseteq T^0M\setminus\{0\}$ such that $\mathrm{WF}_{\mathcal{M}} X_j \subseteq \Gamma$ for each $j = 1,\dots,N$. If we set $W^+ = (\Gamma)^c \subseteq T^0M\setminus\{0\}$, then $\mathrm{WF}_{\mathcal{M}} X_j \cap W^+ = \emptyset$ for all $j = 1,\dots,N$. We may refer to this fact by saying that X_j *extends above*. On the other hand, if we analogously put $W^- = (-\Gamma)^c \subseteq T^0M\setminus\{0\}$ then $\mathrm{WF}_{\mathcal{M}} \bar{X}_j \cap W^- = \emptyset$ by (2.1.3); we say that $\bar{X}_j$ *extends below*.

Furthermore let $\{\theta^1,\dots,\theta^d\}$ be a generating set of $\mathcal{E}_{\mathcal{M}}(U,T^0M)$ and recall (3.3.5), i.e.

$$\mathcal{L}^\alpha\theta^\nu = \sum_{\ell=1}^N A^{\alpha,\nu}_\ell\omega^\ell$$

with $A^{\alpha,\nu}_\ell \in \mathcal{E}_{\mathcal{M}}(U)$ for $\alpha \in \mathbb{N}^n_0$ and $\nu = 1,\dots,d$. In particular, (3.3.2), i.e. $\theta(\mathfrak{Y}) = \overline{\theta(\mathfrak{Y})}$, turns into

$$\sum_{\ell=1}^N A^{0,\nu}_\ell X_\ell = \sum_{\ell=1}^N \bar{A}^{0,\nu}_\ell \bar{X}_\ell$$

and applying $\mathcal{L}^\alpha$ to (3.3.2) yields

$$\sum_{\ell=1}^N A_\ell^{\alpha,\nu} X_\ell = \sum_{\ell=1}^N \sum_{|\alpha|\leq|\alpha|} C_\ell^{\beta,\nu} L^\beta \bar{X}_\ell,$$

where $\mathbb{C}_\ell^{\beta,\nu} \in \mathcal{E}_\mathcal{M}(U)$. Note that in both equations above the left hand side extends above, while the right hand side extends below.

Now choose any N-tuple $\underline{\alpha} = (\alpha^1, \dots, \alpha^N) \in \mathbb{N}_0^{Nn}$ of multi-indices with $|\alpha| \leq k$ for all $j = 1, \dots, N$ and $r = (r_1, \dots, r^N) \in \{1, \dots, d\}^N$. Then we have

$$\begin{pmatrix} A_1^{\alpha^1,r_1} & \dots & A_N^{\alpha^1,r_1} \\ \vdots & \ddots & \vdots \\ A_1^{\alpha^N,r_N} & \dots & A_N^{\alpha^N,r_N} \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} = \begin{pmatrix} \sum C_\beta^{\alpha^1,\ell} L^\beta \bar{X}_\ell \\ \vdots \\ \sum C_\beta^{\alpha^N,\ell} L^\beta \bar{X}_\ell \end{pmatrix}.$$

If we multiply the equation with the classic adjoint of the matrix

$$\begin{pmatrix} A_1^{\alpha^1,r_1} & \dots & A_N^{\alpha^1,r_1} \\ \vdots & \ddots & \vdots \\ A_1^{\alpha^N,r_N} & \dots & A_N^{\alpha^N,r_N} \end{pmatrix}$$

then we obtain

$$D(\underline{\alpha}, r) X_j = \sum_{\substack{|\beta|\leq k \\ \ell=1,\dots,N}} D_{\beta,j}^{\underline{\alpha},r} L^\beta \bar{X}_j$$

for each $j = 1, \dots, N$ where the $D_{\beta,j}^{\underline{\alpha},r}$ are ultradifferentiable functions on U. It follows that the right hand side of this equation extends below, whereas the left hand side obviously extends above. Hence $\operatorname{WF}_\mathcal{M} D(\underline{\alpha}, r) X = \emptyset$. We conclude that $\lambda X \in \mathcal{E}_\mathcal{M}(U)$ for any $\lambda \in \mathcal{S}^k(U)$ since $\mathcal{S}^k(U)$ is generated by the functions $D(\underline{\alpha}, r)$. □

The next statement is an obvious corollary of Theorem 3.4.2.

Corollary 3.4.3. *Let $(M, \mathcal{V})$ be an ultradifferentiable finitely nondegenerate abstract CR structure and X an infinitesimal CR diffeomorphism of M with distributional coefficients which extends microlocally to a wedge with edge M. Then X is ultradifferentiable of class $\{\mathcal{M}\}$.*

However, the condition that M is actually finitely nondegenerate is far too restrictive. We shall say that $(M, \mathcal{V})$ is CR-regular if for every $p \in M$ there exists a multiplier $\lambda \in \mathcal{S}$ with the property that near p, the zero set of λ is a finite intersection of real hypersurfaces in M, and such that λ does not vanish to infinite order at p. Thence we can apply Proposition 1.3.2 or Corollary 1.3.3, respectively.

Theorem 3.4.4. *Let $(M, \mathcal{V})$ be an abstract CR structure, $p \in M$, and assume that M is CR-regular near p. Then any locally integrable infinitesimal CR diffeomorphism X of M which extends microlocally to a wedge with edge M is of class $\{\mathcal{M}\}$ near p.*

Without boundedness conditions on X this theorem is actually in some sense optimal as we are going to see later on.

In general it might be difficult to determine if a certain CR manifold is CR-regular. In the forthcoming we want to present some instances of CR-regular manifolds. But first we take a closer look at the Lie derivatives of characteristic forms.

Suppose that M is a CR manifold and near a point $p_0 \in M$ there are local coordinates (x, y, s) of M such that the vector fields

$$L_j = \frac{\partial}{\partial \bar{z}_j} - \sum_{\tau=1}^d b_\tau^j \frac{\partial}{\partial s_\tau}, \quad j = 1, \dots, n,\ z_j = x_j + y_j, \tag{3.4.1}$$

where $b_\tau^j \in \mathcal{E}_{\mathcal{M}}$, are a local basis of CR vector fields near p_0. In this setting (c.f. Remark 3.1.4) the characteristic bundle is spanned by the forms

$$\theta^\tau = ds_\tau + \sum_{j=1}^n b_\tau^j \, d\bar{z}_j + \sum_{j=1}^n \bar{b}_\tau^j \, dz_j, \quad \tau = 1, \dots, d.$$

Furthermore, the forms θ^τ, $\tau = 1, \dots, d$, and $\omega^j = dz_j$, $j = 1, \dots, n$, constitute a local basis of holomorphic forms on M near p_0. We also define the functions

$$\lambda_\mu^{j,k} := L_k \bar{b}_\mu^j - \bar{L}_j b_\mu^k$$

for $j, k = 1, \dots, n$ and $\mu = 1, \dots, d$.

Consider a general holomorphic form

$$\eta = \sum_{\mu=1}^d \sigma_\mu \theta^\mu + \sum_{j=1}^n \rho_j \omega^j.$$

The Lie derivative of η with respect to the CR vector field L_k is

$$\mathcal{L}_k \eta = d\eta(L_k, \,.\,) = \sum_{\mu=1}^d \left(L_k \sigma_\mu - \sum_{\nu=1}^d \sigma_\nu \big(b_\nu^k\big)_{s_\mu} \right) \theta^\mu + \sum_{j=1}^n \left(L_k \rho_j + \sum_{\mu=1}^d \sigma_\mu \lambda_\mu^{j,k} \right) \omega^j. \tag{3.4.2}$$

Let $\alpha \in \mathbb{N}_0^n$ a multi-index of length $|\alpha| = m$. We introduce the finite sequence $m_j := \sum_{\ell \le j} \alpha_\ell$, $j = 1, \dots, n$, and set $m_0 := 0$ and associate to α the function $p_\alpha : \{0, 1, \dots, m\} \to \{0, 1, \dots, n\}$ which is defined by

$$p_\alpha(\ell) = j \qquad \text{if } \ \ell \in (m_{j-1}, m_j]$$

for $\ell = 1, \dots, m$ and $p_\alpha(0) = 0$. We also associate the following sequences of multi-indices to α

$$\alpha(\ell) := \sum_{q \le \ell} e_{p_\alpha(q)} \qquad\qquad \ell = 0, 1, \dots, m,$$
$$\hat{\alpha}(\ell) := \sum_{q > \ell} e_{p(q)},$$

where e_j is the j-th standard unit vector in $\mathbb{R}^n$.

With this notation and (3.4.2) we can now state what the Lie derivative of the characteristic form θ^μ ($\mu = 1, \dots, d$) is:

$$\mathcal{L}^\alpha \theta^\mu = \sum_{\tau=1}^d T_\tau^{\alpha,\mu} \theta^\tau + \sum_{j=1}^n A_j^{\alpha,\mu} \omega^j \tag{3.4.3}$$

The functions $T_\tau^{\alpha,\mu}$ and $A_j^{\alpha,\mu}$ are defined iteratively by

$$T_\tau^{0,\mu} = \delta_{\mu\tau},$$
$$T_\tau^{\alpha,\mu} = L_{p_\alpha(1)} T_\tau^{\hat{\alpha}(1),\mu} - \sum_{\nu=1}^d \big(b_\nu^{p(1)}\big)_{s_\tau} T_\nu^{\hat{\alpha}(1),\mu} \tag{3.4.4a}$$

and

$$A_j^{\alpha,\mu} = \sum_{k=1}^m \sum_{\nu=1}^d L^{\alpha(k-1)} \Big(T_\nu^{\alpha-\alpha(k),\mu} \lambda_\nu^{j,p_\alpha(k)} \Big). \tag{3.4.4b}$$

We are now able to give the first example of a CR regular submanifold of $\mathbb{C}^N$.

Definition 3.4.5. We say that a real hypersurface $M \subset \mathbb{C}^N$ is weakly nondegenerate at p_0 iff there exist coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ near p_0 and numbers $k, m \in \mathbb{N}$ such that $p_0 = 0$ in these coordinates and near p_0 M is given by an equation of the form

$$\operatorname{Im} w = (\operatorname{Re} w)^m \varphi(z, \bar{z}, \operatorname{Re} w),$$

where

$$\frac{\partial^{|\alpha|}\varphi}{\partial z^\alpha}(0,0,0) = \frac{\partial^{|\alpha|}\varphi}{\partial \bar{z}^\alpha}(0,0,0) = 0, \quad |\alpha| \le k,$$

and

$$\operatorname{span}_{\mathbb{C}}\{\varphi_{z\bar{z}^\alpha}(0,0,0) \colon |\alpha| \le k\} = \mathbb{C}^n.$$

If k_0 is the smallest k for which the preceding condition holds, we say that M is weakly k_0-nondegenerate at p_0.

PROPOSITION 3.4.6. *Let $M \subseteq \mathbb{C}^N$ be an ultradifferentiable real hypersurface, $p_0 \in M$, and assume that M is weakly k_0-nondegenerate at p_0. Then M is CR regular near p_0. In particular, any locally integrable infinitesimal CR diffeomorphism of M which extends microlocally to a wedge with edge M near p_0 is ultradifferentiable near p_0.*

PROOF. In order to show that M is CR regular we are going to construct a multiplier $\lambda \in \mathcal{S}$ of the form

$$\lambda(z,\bar{z},s) = s^\ell \psi(z,\bar{z},s)$$

in suitable local coordinates and with $\psi \in \mathcal{E}_{\mathcal{M}}$ not vanishing at $s=0$ and $\ell \in \mathbb{N}$.

Recall that by assumption there are coordinates $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ such that $p_0 = 0$ and M is given locally by

$$\operatorname{Im} w = (\operatorname{Re} w)^m \varphi(z,\bar{z},\operatorname{Re} w)$$

where $m \in \mathbb{N}$ and φ is an ultradifferentiable real-valued function defined near 0 with the property that $\varphi_{z^\alpha}(0) = \varphi_{\bar{z}^\alpha}(0) = 0$ for $|\alpha| \le k_0$ and

$$\operatorname{span}_{\mathbb{C}}\{\varphi_{z\bar{z}^\alpha}(0,0,0) \colon 0 < |\alpha| \le k_0\} = \mathbb{C}^n.$$

In these coordinates a local basis of the CR vector fields on M is given by

$$L_j = \frac{\partial}{\partial \bar{z}_j} - b^j \frac{\partial}{\partial s}, \qquad 1 \le j \le n,$$

with

$$b^j = i\frac{s^m \varphi_{\bar{z}_j}}{1 + i(s^m\varphi)_s},$$

whereas the characteristic bundle is spanned near the origin by

$$\theta = ds + \sum_{j=1}^n b^j \, d\bar{z}_j + \sum_{j=1}^n b^j \, dz_j$$

and θ together with the forms $\omega^j = dz_j$ constitute a local basis of $T'M$ near the origin.

We observe that for $1 \le j, \ell \le n$

$$\begin{aligned}
\lambda^j_\ell &:= L_j \bar{b}^\ell - \bar{L}_\ell b^j \\
&= s^m \Bigg(\frac{i\varphi_{\bar{z}_j z_\ell}(1 + i(s^m\varphi)_s) + \varphi_{z_\ell}(s^m\varphi_{\bar{z}_j})_s}{(1+i(s^m\varphi)_s)^2} \\
&\qquad + \frac{\varphi_{\bar{z}_j}\big((s^m\varphi_{z_\ell})_s(1+i(s^m\varphi)_s) - is^m\varphi_{z_\ell}(s^m\varphi)_{ss}\big)}{(1+i(s^m\varphi)_s)^3} \\
&\qquad + \frac{i\varphi_{\bar{z}_j z_\ell}(1 + i(s^m\varphi)_s) + \varphi_{\bar{z}_j}(s^m\varphi_{z_\ell})_s}{(1+i(s^m\varphi)_s)^2} \\
&\qquad - \frac{\varphi_{z_\ell}\big((s^m\varphi_{\bar{z}_j})_s(1+i(s^m\varphi)_s) - s^m\varphi_{\bar{z}_j}(s^m\varphi)_{ss}\big)}{(1+i(s^m\varphi)_s)^3} \Bigg) \\
&= s^m \chi^j_\ell
\end{aligned}$$

and $\chi^j_\ell(0) = 2i\varphi_{\bar{z}_j z_\ell}(0)$ by the assumptions on φ.

In this setting (3.4.3) takes the form

$$\mathcal{L}^\alpha\theta = T^\alpha\theta + \sum_{j=1}^{n} A_j^\alpha \omega^j$$

and (3.4.4) implies that

$$T^\alpha = L_{p(1)}T^{\hat{\alpha}(1)} - \left(b^{p(1)}\right)_s T^{\hat{\alpha}(1)}, \quad T^0 = 1$$
$$A_j^\alpha = \sum_{k=1}^{|\alpha|} = L^{\alpha(k-1)}\left(T^{\alpha\hat{(k)}}\lambda^j_{p(k)}\right).$$

If we use the two simple facts for smooth functions f, g, namely $(s^q f)_s = s^{q-1}f + s^q f_s$ for $q \in \mathbb{N}$ we see that $T^\beta = s^{m-1}G^\beta$ for $|\beta| \geq 1$. Hence, if $m \geq 2$ we have

$$A_\ell^\alpha(z,\bar{z},s) = s^m \frac{2i\varphi_{\bar{z}^\alpha z_\ell}(z,\bar{z},s)}{1+(s^m\varphi(z,\bar{z},s))_s^2} + s^{2m-1}R_\ell^\alpha(z,\bar{z},s) = s^m B_\ell^\alpha(z,\bar{z},s).$$

On the other hand we obtain for $m = 1$ the following representation

$$A_\ell^\alpha(z,\bar{z},s) = s\frac{2i\varphi_{\bar{z}^\alpha z_\ell}(z,\bar{z},s)}{1+(\varphi(z,\bar{z},s)+s\varphi_s(z,\bar{z},s))^2} + sS_\ell^\alpha(z,\bar{z},s) + s^2R_\ell^\alpha(z,\bar{z},s) = sB_\ell^\alpha(z,\bar{z},s),$$

where S_ℓ^α is a sum of products of rational functions with respect to φ and its derivatives. Each of these summands contains at least one factor of the form $\varphi_{\bar{z}^\beta}$ or φ_{z^β} with $|\beta| \leq |\alpha| \leq k_0$ and therefore $S_\ell^\alpha(0) = 0$.

By assumption there have to be multi-indices $\alpha^1, \ldots, \alpha^n \neq 0$ of length shorter than k_0 such that

$$\{\varphi_{z\bar{z}^{\alpha^1}}(0), \ldots, \varphi_{z\bar{z}^{\alpha^n}}(0)\}$$

is a basis for $\mathbb{C}^n$. Now we choose $\underline{\alpha} = (0, \alpha^1, \ldots, \alpha^n)$ and calculate according to (3.3.6) the multiplier $D(\underline{\alpha}) = D(\underline{\alpha}, 1)$ (note that $d = 1$):

$$\begin{aligned} D(\underline{\alpha}) &= \det\begin{pmatrix} 1 & 0 & \ldots & 0 \\ A_\theta^{\alpha^1} & A_1^{\alpha^1} & \ldots & A_n^{\alpha^1} \\ \vdots & \vdots & \ddots & \vdots \\ A_\theta^{\alpha^n} & A_1^{\alpha^n} & \ldots & A_n^{\alpha^n} \end{pmatrix} \\ &= s^{n\cdot m}\det\begin{pmatrix} 1 & 0 & \ldots & 0 \\ A_\theta^{\alpha^1} & B_1^{\alpha^1} & \ldots & B_n^{\alpha^1} \\ \vdots & \vdots & \ddots & \vdots \\ A_\theta^{\alpha^n} & B_1^{\alpha^n} & \ldots & B_n^{\alpha^n} \end{pmatrix} \\ &= s^{n\cdot m}Q(\underline{\alpha}) \end{aligned}$$

where

$$Q(\underline{\alpha}) = \det\begin{pmatrix} 1 & 0 & \ldots & 0 \\ A_\theta^{\alpha^1} & B_1^{\alpha^1} & \ldots & B_n^{\alpha^1} \\ \vdots & \vdots & \ddots & \vdots \\ A_\theta^{\alpha^n} & B_1^{\alpha^n} & \ldots & B_n^{\alpha^n} \end{pmatrix} = \det\begin{pmatrix} B_1^{\alpha^1} & \ldots & B_n^{\alpha^1} \\ \vdots & \ddots & \vdots \\ B_1^{\alpha^n} & \ldots & B_n^{\alpha^n} \end{pmatrix},$$

hence

$$Q(\underline{\alpha})(0) = (2i)^n \det\begin{pmatrix} \varphi_{z\bar{z}^{\alpha^1}}(0) \\ \vdots \\ \varphi_{z\bar{z}^{\alpha^n}}(0) \end{pmatrix} \neq 0.$$

We conclude that M is CR-regular. □

Obviously, a similar approach as in the hypersurface case above can be used to find manifolds of higher codimension that are CR-regular.

Definition 3.4.7. We say that a CR manifold $M \subseteq \mathbb{C}^N$ of codimension d is weakly nondegenerate at $p_0 \in M$ (in the first codimension) iff there are local coordinates $(z,w) \in \mathbb{C}^{n+d}$ near p_0 such that M is given by the equations

$$\operatorname{Im} w_\mu = (\operatorname{Re} w)^{\gamma^\mu} \varphi_\mu(z,\bar z, \operatorname{Re} w), \quad \mu = 1,\dots,d,$$

with $\gamma^1 < \gamma^\nu$, $\nu = 2,\dots,d$, and $|\gamma^1| \geq 2$. Furthermore the function φ_1 satisfies for some k

$$\operatorname{span}_{\mathbb{C}}\bigl\{ \bigl(\varphi_1\bigr)_{z\bar z^\alpha}(0,0,0) :\ |\alpha| \leq k \bigr\} = \mathbb{C}^n.$$

If k_0 is the smallest integer k for which the above condition holds, we say that M is weakly k_0-nondegenerate at p_0.

Proposition 3.4.8. *Let $M \subseteq \mathbb{C}^N$ be a generic ultradifferentiable CR submanifold of codimension d, $p_0 \in M$, and assume that M is weakly nondegenerate at p_0. Then any locally integrable infinitesimal CR diffeomorphism of M which extends microlocally to a wedge with edge M near p_0 is ultradifferentiable near p_0.*

Proof. Similar to before we have to construct a multiplier $\lambda \in \mathcal{S}$ of the form $\lambda(z,\bar z,s) = s^\beta \psi(z,\bar z,s)$ where $\psi \in \mathcal{E}_\mathcal{M}$ and $\psi(0) \neq 0$. By assumption there are coordinates $(z,w) \in \mathbb{C}^{n+d}$ near $p_0 = 0$ such that M is given by

$$\operatorname{Im} w_\mu = (\operatorname{Re} w)^{\gamma^\mu} \varphi_\mu(z,\bar z, \operatorname{Re} w), \quad \mu = 1,\dots,d.$$

In particular note that $\alpha^1 \leq \alpha^\mu$ for $\mu = 2,\dots,d$.

We deduce from Remark 3.1.4 that the vector fields

$$L_j = \frac{\partial}{\partial \bar z_j} - \sum_{\mu=1}^{d} b_\mu^j \frac{\partial}{\partial s_\mu}$$

are a local basis of the CR vector fields near the origin. The coefficients b_μ^j are of the form

$$b_\mu^j = i\bigl(\det(\operatorname{Id}_d + i\Phi)\bigr)^{-1} \cdot \det B_\mu^j$$

where Φ denotes the Jacobi matrix of the map $(s^{\gamma^\mu}\varphi_\mu)_\mu$ with respect to the variables $s = (s_1 \dots, s_d)$ and

$$B_\mu^j = \begin{pmatrix} 1 + i(s^{\gamma^1}\varphi_1)_{s_1} & \dots & i(s^{\gamma^1}\varphi_1)_{s_{\mu-1}} & s^{\gamma^1}(\varphi_1)_{\bar z_j} & i(s^{\gamma^1}\varphi_1)_{s_{\mu+1}} & \dots & i(s^{\gamma^1}\varphi_1)_{s_d} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ i(s^{\gamma^\mu}\varphi_\mu)_{s_1} & \dots & i(s^{\gamma^\mu}\varphi_\mu)_{s_{\mu-1}} & s^{\gamma^\mu}(\varphi_\mu)_{\bar z_j} & i(s^{\gamma^\mu}\varphi_\mu)_{s_{\mu+1}} & \dots & i(s^{\gamma^\mu}\varphi_\mu)_{s_d} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ i(s^{\gamma^d}\varphi_d)_{s_1} & \dots & i(s^{\gamma^d}\varphi_d)_{s_{\mu-1}} & s^{\gamma^d}(\varphi_d)_{\bar z_j} & i(s^{\gamma^d}\varphi_d)_{s_{\mu+1}} & \dots & 1 + i(s^{\gamma^d}\varphi_d)_{s_d} \end{pmatrix}.$$

Hence for all $j = 1,\dots n$ and $\mu = 1,\dots,d$ we have

$$b_\mu^j = i s^{\gamma^1}\bigl(\det(\operatorname{Id}_d + i\Phi)\bigr)^{-1} \det C_\mu^j \tag{3.4.5}$$

with

$$C_\mu^j = \begin{pmatrix} 1 + i(s^{\gamma^1}\varphi_1)_{s_1} & \dots & i(s^{\gamma^1}\varphi_1)_{s_{\mu-1}} & (\varphi_1)_{\bar z_j} & i(s^{\gamma^1}\varphi_1)_{s_{\mu+1}} & \dots & i(s^{\gamma^1}\varphi_1)_{s_d} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ i(s^{\gamma^\mu}\varphi_\mu)_{s_1} & \dots & i(s^{\gamma^\mu}\varphi_\mu)_{s_{\mu-1}} & s^{\tilde\gamma^\mu}(\varphi_\mu)_{\bar z_j} & i(s^{\gamma^\mu}\varphi_\mu)_{s_{\mu+1}} & \dots & i(s^{\gamma^\mu}\varphi_\mu)_{s_d} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ i(s^{\gamma^d}\varphi_d)_{s_1} & \dots & i(s^{\gamma^d}\varphi_d)_{s_{\mu-1}} & s^{\tilde\gamma^d}(\varphi_d)_{\bar z_j} & i(s^{\gamma^d}\varphi_d)_{s_{\mu+1}} & \dots & 1 + i(s^{\gamma^d}\varphi_d)_{s_d} \end{pmatrix}$$

and $\tilde{\gamma}^\mu = \gamma^\mu - \gamma^1 > 0$. We observe that

$$\det C_1^j\big|_{s=0} = (\varphi_1)_{\bar{z}_j}(z,\bar{z},0) \tag{3.4.6a}$$

$$\det C_\mu^j = 0 \qquad \mu = 2,\dots,d, \tag{3.4.6b}$$

since $|\gamma^\mu| \geq |\gamma^1| \geq 2$.

Furthermore the forms

$$\theta^\mu = ds_\mu + \sum_{j=1}^n b_\mu^j d\bar{z}_j + \sum_{j=1}^n \bar{b}_\mu^j dz_j, \quad \mu = 1,\dots,d,$$

span the characteristic bundle near 0 and θ^μ, $\mu = 1,\dots,d$ and $\omega^j = dz_j$, $j = 1,\dots,n$, form a local basis of the holomorphic forms on M. From (3.4.3) we recall for $\alpha \in \mathbb{N}_0^n$ and $\mu = 1,\dots,d$ that

$$\mathcal{L}^\alpha \theta^\mu = \sum_{\tau=1}^d T_\tau^{\alpha,\mu}\theta^\tau + \sum_{j=1}^n A_j^{\alpha,\mu}\omega^j$$

and from (3.4.4)

$$T_\tau^{0,\mu} = \delta_{\mu\tau}$$

$$T_\tau^{\alpha,\mu} = L_{p_\alpha(1)} T_\tau^{\hat{\alpha}(1),\mu} - \sum_{\nu=1}^d \big(b_\nu^{p(1)}\big)_{s_\tau} T_\nu^{\hat{\alpha}(1),\mu}$$

$$A_j^{\alpha,\mu} = \sum_{k=1}^{|\alpha|}\sum_{\nu=1}^d L^{\alpha(k-1)}\Big(T_\nu^{\alpha-\alpha(k),\mu}\lambda_\nu^{j,p_\alpha(k)}\Big).$$

We recall that

$$\lambda_\nu^{j,k} = L_k \bar{b}_\nu^j - \bar{L}_j b_\nu^k$$
$$= \big(\bar{b}_\nu^j\big)_{\bar{z}_k} - \sum_{\mu=1}^d b_\mu^k \big(\bar{b}_\nu^j\big)_{s_\mu} - \big(b_\nu^k\big)_{z_j} + \sum_{\mu=1} \bar{b}_\mu^j \big(b_\nu^k\big)_{s_\mu}$$

and note that (3.4.5) and (3.4.6) imply that

$$\lambda_\nu^{j,k} = 2is^{\gamma^1} R_\nu^{j,k} \qquad \nu = 1,\dots,d,$$

where

$$R_1^{j,k}\Big|_{s=0} = \big(\varphi_1\big)_{\bar{z}_k z_j}\Big|_{s=0}$$

$$R_\nu^{j,k}\Big|_{s=0} = 0 \qquad \nu = 1,\dots,d.$$

It is easy to see that also $T_\tau^{\alpha,\mu}\big|_{s=0} = 0$ for $\alpha \neq 0$. We conclude that for all $\alpha \neq 0$, and $j = 1,\dots,n$

$$A_j^{\alpha,\mu} = 2is^{\gamma^1}\tilde{A}_j^{\alpha,\mu} \qquad \mu = 1,\dots,d$$

where

$$\tilde{A}_j^{\alpha,1}\Big|_{s=0} = \big(\varphi_1\big)_{\bar{z}^\alpha z_j}\Big|_{s=0}$$

$$\tilde{A}_j^{\alpha,\mu}\Big|_{s=0} = 0 \qquad \mu = 2,\dots,d.$$

By assumptation there are multi-indices $\alpha^1,\dots,\alpha^n \in \mathbb{N}_0^n$ of length at most k_0 such that the vectors

$$\big(\varphi_1\big)_{z\bar{z}^{\alpha^j}}(0), \qquad j = 1,\dots,n,$$

form a basis of $\mathbb{C}^n$.

We compute the multiplier $D(\overline{\alpha}, r)$ for $\underline{\alpha} = (0, \ldots, 0, \alpha^1, \ldots, \alpha^n)$ and $r = (1, \ldots, d, 1, \ldots, n)$. By (3.3.6) we have

$$\begin{aligned}
D(\underline{\alpha}, r) &= \det \begin{pmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & 1 & 0 & \ldots & 0 \\ T_1^{\alpha^1,1} & \ldots & T_d^{\alpha^1,1} & A_1^{\alpha^1,1} & \ldots & A_n^{\alpha^1,1} \\ \vdots & & \vdots & \vdots & & \vdots \\ T_1^{\alpha^n,1} & \ldots & T_d^{\alpha^n,1} & A_1^{\alpha^n,1} & \ldots & A_n^{\alpha^n,1} \end{pmatrix} \\
&= \det \begin{pmatrix} A_1^{\alpha^1,1} & \ldots & A_n^{\alpha^1,1} \\ \vdots & & \vdots \\ A_1^{\alpha^n,1} & \ldots & A_n^{\alpha^n,1} \end{pmatrix} \\
&= \det \begin{pmatrix} 2is^{\gamma^1} \tilde{A}_1^{\alpha^1,1} & \ldots & 2is^{\gamma^1} \tilde{A}_n^{\alpha^1,1} \\ \vdots & & \vdots \\ 2is^{\gamma^1} \tilde{A}_1^{\alpha^n,1} & \ldots & 2is^{\gamma^1} \tilde{A}_n^{\alpha^n,1} \end{pmatrix} \\
&= (2i)^n s^{n\gamma^1} \det \begin{pmatrix} \tilde{A}_1^{\alpha^1,1} & \ldots & \tilde{A}_n^{\alpha^1,1} \\ \vdots & & \vdots \\ \tilde{A}_1^{\alpha^n,1} & \ldots & \tilde{A}_n^{\alpha^n,1} \end{pmatrix} \\
&= (2i)^n s^{n\gamma^1} \Lambda(\underline{\alpha}, r).
\end{aligned}$$

We conclude

$$\Lambda(\underline{\alpha}, r)(0) = \det \begin{pmatrix} (\varphi_1)_{z\bar{z}^{\alpha^1}}(0) \\ \vdots \\ (\varphi_1)_{z\bar{z}^{\alpha^n}}(0) \end{pmatrix} \neq 0.$$

□

In the preceding results we required the involved manifolds to have a special form in order to simplify the necessary calculations, but of course there are many more CR regular manifolds. The next example gives a CR manifold that is not weakly nondegenerate at 0 in the sense of Definition 3.4.7 but is still CR regular.

Example 3.4.9. Let $M \subseteq \mathbb{C}^3$ the CR manifold given by

$$\begin{aligned}
\operatorname{Im} w_1 &= \operatorname{Re} w_1 \, |z|^2 \\
\operatorname{Im} w_2 &= \operatorname{Re} w_2 \, |z|^2.
\end{aligned}$$

The CR bundle $\mathcal{V}$ of M is spanned by

$$L = \frac{\partial}{\partial \bar{z}} - i \frac{s_1 z}{1 + i|z|^2} \frac{\partial}{\partial s_1} - i \frac{s_2 z}{1 + i|z|^2} \frac{\partial}{\partial s_2}.$$

Thus a basis of the characteristic form is given by

$$\begin{aligned}
\theta^1 &= ds_1 + i \frac{s_1 z}{1 + i|z|^2} d\bar{z} - i \frac{s_1 \bar{z}}{1 - i|z|^2} dz \\
\theta^2 &= ds_2 + i \frac{s_2 z}{1 + i|z|^2} d\bar{z} - i \frac{s_2 \bar{z}}{1 - i|z|^2} dz.
\end{aligned}$$

We know that θ^1, θ^2 and $\omega = dz$ is a basis of $T'M$. If $\alpha = e_1$ we recall from (3.4.3) that

$$\mathcal{L}^\alpha \theta^1 = T_1^{\alpha,1} \theta^1 + T_2^{\alpha,1} \theta^2 + A^{\alpha,1} \omega.$$

Using (3.4.4) we observe that

$$\begin{aligned}T_1^{\alpha,1} &= -i\frac{z}{1+i|z|^2}\\ T_2^{\alpha,1} &= 0\\ A^{\alpha,1} &= -2is_1\frac{1-|z|^4}{\left(1+|z|^4\right)^2}.\end{aligned}$$

Hence, if we set $\underline{\alpha} = (0,0,\alpha)$ and $r = (1,2,1)$ then the multiplier $D(\underline{\alpha}, r)$ of M given by (3.3.6) is

$$D(\underline{\alpha}, r) = \det\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -i\frac{z}{1+i|z|^2} & 0 & -2is_1\frac{1-|z|^4}{(1+|z|^4)^2}\end{pmatrix} = -2is_1\frac{1-|z|^4}{\left(1+|z|^4\right)^2}$$

and thus M is CR regular.

Next we are going to present an example that shows that the local integrability condition in Theorem 3.4.4, Proposition 3.4.6 and Proposition 3.4.8, respectively, is essential for the conclusions in these statements to hold. More precisely, we construct two different infinitesimal diffeomorphisms with distributional coefficents on a real hypersurface in $\mathbb{C}^2$ such that the two diffeomorphisms are not locally integrable. We also construct a multiplier such that the products of this multiplier with each diffeomorphism coincide and are ultradifferentiable. We further note that the coefficients of both diffeomorphisms are closely related to the non-extendable CR distribution for nonminimal CR submanifolds given by Baouendi and Rothschild [**7**].

Example 3.4.10. We begin with the calculation of the multiplier in a more general setting in order to simplify the computations. We will later on restrict ourselves to real hypersurfaces in $\mathbb{C}^2$. Let $(M, \mathcal{V})$ be a 3-dimensional abstract CR structure of hypersurface type that is generated in some coordinates by the vector field

$$L = \frac{\partial}{\partial \bar{z}} - s^m b(z,\bar{z})\frac{\partial}{\partial s}.$$

The characteristic bundle T^0M is spanned by

$$\theta = ds + s^m\bar{b}(z,\bar{z})dz + s^m b(z,\bar{z})d\bar{z}$$

and thus the forms $\omega = dz$ and θ form a basis of $T'M$. We obtain (c.f. (3.4.2))

$$d\theta(L, \,.\,) = -2is^m \operatorname{Im}\left(\frac{\partial b}{\partial z}\right)(z,\bar{z})\omega - ms^{m-1}b(z,\bar{z})\theta.$$

We calculate the simplest nontrivial multiplier: for $\alpha^1 = 0,\ \alpha^2 = 1$ and $r = (1,1)$ (note that $N = 2$ and $d = 1$) we have by (3.3.6)

$$\begin{aligned}D(\underline{\alpha}, r) &= \det\begin{pmatrix} 1 & 0\\ -ms^{m-1}b(z,\bar{z}) & -2is^m\operatorname{Im}\left(\frac{\partial b}{\partial z}\right)(z,\bar{z})\end{pmatrix}\\ &= -2is^m\operatorname{Im}\left(\frac{\partial b}{\partial z}\right)(z,\bar{z}).\end{aligned}$$

Now let $m = 1$, $b = i\frac{\psi_{\bar{z}}}{1+i\psi}$ for some ultradifferentiable real-valued function ψ defined in an open neighbourhood V of $0 \in \mathbb{C}$, i.e. M is an embedded real hypersurface of class $\{\mathcal{M}\}$ in $\mathbb{C}^2$ given near the origin by the defining function

$$\rho(z,\bar{z},w,\bar{w}) = \operatorname{Im} w - \operatorname{Re} w \cdot \psi(z,\bar{z}).$$

Then the multiplier $D(\underline{\alpha}, r)$ from above is of the form

$$D(\underline{\alpha}, r) = 2is\left(\frac{\psi_{z\bar{z}}}{|\Psi|^2} - 2\frac{\psi_z\psi_{\bar{z}}\psi}{|\Psi|^4}\right) = 2is\,G(z,\bar{z}),$$

where we have set $\Psi := 1+i\psi$. Note also that $\omega_1 = \omega = dz$ and $\omega_2 = dw = \Psi ds + is\psi_z dz + is\psi_{\bar z} d\bar z$ is an alternative basis for $T'M$ in this situation.

Since M is a real hypersurface in $\mathbb{C}^2$ we have the following decomposition of an open neighbourhood Ω of $0 \in \mathbb{C}^2$

$$\Omega = U_+ \cup M \cup U_-$$

with $U_+ = \{(z,w) \in \Omega\colon \rho(z,\bar z,\bar z,w,\bar w) > 0\}$ and $U_- = \{(z,w) \in \Omega\colon \rho(z,\bar z,w,\bar w) < 0\}$ being open subsets of Ω. We shall also assume that $\Omega \cap (\mathbb{C}\times\{0\}) = V \times \{0\}$.

If we consider the holomorphic function

$$F:\ (z,w) \longmapsto \frac{1}{w}$$

on $\mathbb{C}\times\mathbb{C}\backslash\{0\}$ then we see that F is of slow growth for $w \to 0$ on both U_+ and U_-. We write $u_+ = b_+F$ for the boundary value of $F|_{U_+}$ and $u_- = b_-F$ for the boundary value of $F|_{U_-}$, respectively. Note that by the Plemelj-Sokhotski jump relations (see, e.g., [**27**]) we have

$$u_0 = u_+ - u_- = -\frac{2\pi i}{\Psi}(1\otimes\delta).$$

Note also that u_0 is essentially (up to the factor $-2\pi i$) the non-extendable CR distribution from [**7**], c.f. also [**8**], for the hypersurface M.

We claim that $\mathrm{WF}_{\mathcal{M}}\, u_+ = \mathbb{R}_+\theta|_{V\times\{0\}}$ and $\mathrm{WF}_{\mathcal{M}}\, u_- = \mathbb{R}_-\theta|_{V\times\{0\}}$, respectively (c.f. Example 2.2.3): Note that u_+ and u_- are ultradifferentiable outside $V\times\{0\} \subset M$ and that $\mathrm{WF}_{\mathcal{M}}\, u_0 = (\mathbb{R}\backslash\{0\})\,\theta|_{V\times\{0\}}$. Furthermore we know that $\mathrm{WF}_{\mathcal{M}}\, u_+$ and $\mathrm{WF}_{\mathcal{M}}\, u_-$ must each be contained in $(\mathbb{R}\backslash\{0\})\theta$ since both are CR distributions. However, since u_+ extends holomorphically to U_+ it follows that $\mathrm{WF}_{\mathcal{M}}\, u_+ \cap \mathbb{R}_-\theta = \emptyset$ (c.f. the proof of Theorem 3.2.2) and by symmetry we have also $\mathrm{WF}_{\mathcal{M}}\, u_- \cap \mathbb{R}_+\theta = \emptyset$. Now let $p = (z,0) \in V\times\{0\}$ and suppose that, e.g., $\mathbb{R}_+\theta_p \cap \mathrm{WF}_{\mathcal{M}}\, u_+ = \emptyset$. Then we would have that $\mathbb{R}_+\theta_p \cap \mathrm{WF}_{\mathcal{M}}\, u_0 = \emptyset$ which is obviously a contradiction to above.

We consider the following vector fields with distributional coefficients

$$X_+ = u_+\frac{\partial}{\partial z}\Big|_M + \bar u_+\frac{\partial}{\partial\bar z}\Big|_M$$

and

$$X_- = u_-\frac{\partial}{\partial z}\Big|_M + \bar u_-\frac{\partial}{\partial\bar z}\Big|_M.$$

We claim that both vector fields constitute infinitesimal CR diffeomorphisms on M if

$$\frac{\partial\psi}{\partial x} = \psi\frac{\partial\psi}{\partial y}$$

where $z = x+iy$. We show this for X_+, the argument for X_- is completely analagous of course. First we see that X_+ is real since

$$X_+ = \operatorname{Re} u_+\frac{\partial}{\partial x}\Big|_M + \operatorname{Im} u_+\frac{\partial}{\partial y}\Big|_M.$$

Furthermore note that the regular distributions ($\nu > 0$)

$$u_\nu = \frac{1}{s\Psi + i\nu}$$

on M converge to u_+ in $\mathcal{D}'$ for $\nu \to 0$. We have

$$\begin{aligned}
X_+\rho &= -s\psi_x \operatorname{Re} u_+ - s\psi_y \operatorname{Im} u_+\\
&= \lim_{\nu\to 0}\big(-s\psi_x \operatorname{Re} u_\nu - s\psi_y \operatorname{Im} u_\nu\big)\\
&= \lim_{\nu\to 0}\left(\frac{-s^2(\psi_x - \psi\psi_y) + s\nu}{s^2 + (s\psi+\nu)^2}\right)\\
&= \lim_{\nu\to 0} s\nu|u_\nu|^2 = 0
\end{aligned}$$

with convergence in $\mathcal{D}'$. Hence $X_+ \in \mathcal{D}'(M, TM)$. We conclude further

$$L\big(\omega_1(X_+)\big) = Lu_+ = 0,$$
$$L\big(\omega_2(X_+)\big) = 0$$

and since $d\omega_j = 0$, $(j = 1, 2)$

$$d\omega_1(L, X_+) = 0,$$
$$d\omega_2(L, X_+) = 0.$$

Since $\omega_1(X_+) = \omega_1(X_-) = u_+$, $\omega_2(X_+) = \omega_2(X_+) = 0$ and $\omega_1(X_-) = u_-$ all the assumptions of Theorem 3.4.2 are satisfied for both X_+ and X_-.

Indeed

$$D(\underline{\alpha}, r)u_+ = D(\underline{\alpha}, r)u_- = 2i\frac{G(z, \bar{z})}{\Psi(z, \bar{z})} \in \mathcal{E}_{\mathcal{M}}(M)$$

hence $D(\underline{\alpha}, r)X_+ = D(\underline{\alpha}, r)X_- \in \mathcal{E}_{\mathcal{M}}$. Note also that $D(\underline{\alpha}, r)u_0 = 0$.

We close this section with a look into the case of quasianalytic manifolds. We begin with recalling the following definition from [**8**, § 11.7]. Let $M \subseteq \mathbb{C}^N$ be a CR submanifold with defining functions $\rho = (\rho_1, \dots, \rho_d)$ near $p_0 \in M$. A *formal holomorphic vector field* at p_0 is a vector field of the form

$$X = \sum_{j=1}^{N} a_j(Z)\frac{\partial}{\partial Z_j}$$

with the coefficients a_j being formal power series in $Z - p_0$ with complex coefficients. The formal vector field X is said to be tangent iff there exists a $d \times d$ matrix $c(Z, \bar{Z})$ consisting of formal power series in the variables $Z - p_0$ and $\bar{Z} - \bar{p}_0$ such that

$$X\rho(Z, \bar{Z}) \sim c(Z, \bar{Z})\rho(Z, \bar{Z}),$$

where $\sim$ denotes equality as formal power series in $Z - p_0$ and $\bar{Z} - \bar{p}_0$. Note that the existence of nontrivial holomorphic vector fields at p_0 tangent to M does not depend on the choice of holomorphic coordinates and defining equations near p_0.

Definition 3.4.11. A generic submanifold $M \subseteq \mathbb{C}^N$ is formally holomorphically nondegenerate at $p_0 \in M$ iff there is no nontrivial formal holomorphic vector field at p_0 that is tangent to M.

Remark 3.4.12. If M is formally holomorphically nondegenerate at p_0 then M is formally holomorphically nondegenerate at every point of some neighbourhood U of p_0. Furthermore if M is holomorphically nondegenerate on an open set $U \subseteq M$ then M is finitely nondegenerate on an open and dense subset $V \subseteq U$, c.f. [**8**, Theorem 11.7.5].

Theorem 3.4.13. *Let $\mathcal{M}$ be a quasianalytic regular weight sequence and $M \subseteq \mathbb{C}^N$ a generic submanifold of class $\{\mathcal{M}\}$ that is formally holomorphically nondegenerate.*

Every smooth CR diffeomorphism $\mathfrak{Y}$ that extends microlocally to a wedge with edge M is ultradifferentiable of class $\{\mathcal{M}\}$.

Proof. As usual we argue locally near a point p_0. After a choice of local bases of CR vector fields and holomorphic forms and selecting a generating set for the characteristic forms we can use the representation (3.3.3) near p_0. By Theorem 3.4.2 we know that for any multiplier λ the product $\Lambda_j = \lambda \cdot X_j$ is ultradifferentiable for $j = 1, \dots, N$. Since X_j is smooth by assumption we have that the equality holds also for the formal power series at p_0 of Λ_j, λ and X_j. Since M is formally holomorphically nondegenerate at p_0 there has to be a multiplier $\lambda \in \mathcal{S}$ with nontrivial formal power series at p_0. Indeed, if the power series of λ at p_0 equals 0 then λ itself has to vanish in a neighbourhood of p_0 by the quasianalyticity of $\mathcal{M}$. On the other hand in every neighbourhood of p_0 there is a point q at which M is finitely nondegenerate [**8**, Theorem 11.7.5]. Hence by Remark 3.3.6 there has to be a nontrivial multiplier λ' defined on some neighbourhood U of p_0.

We conclude that the formal power series of $\Lambda'_j = \lambda' X_j$ at p_0 is divisible by the Taylor series of λ' at p_0. Hence Theorem 1.3.4 gives that X_j is ultradifferentiable of class $\{\mathcal{M}\}$ near p_0. $\square$

Bibliography

[1] Francesca Acquistapace, Fabrizio Broglia, Michail Bronshtein, Andreea Nicoara, and Nahum Zobin. Failure of the Weierstrass preparation theorem in quasi-analytic Denjoy-Carleman rings. *Adv. Math.*, 258:397–413, 2014.

[2] Ziad Adwan and Gustavo Hoepfner. Approximate solutions and micro-regularity in the Denjoy-Carleman classes. *J. Differential Equations*, 249(9):2269–2286, 2010.

[3] A. A. Albanese, D. Jornet, and A. Oliaro. Quasianalytic wave front sets for solutions of linear partial differential operators. *Integral Equations Operator Theory*, 66(2):153–181, 2010.

[4] M. S. Baouendi, C. H. Chang, and F. Trèves. Microlocal hypo-analyticity and extension of CR functions. *J. Differential Geom.*, 18(3):331–391, 1983.

[5] M. S. Baouendi, Xiaojun Huang, and Linda Preiss Rothschild. Regularity of CR mappings between algebraic hypersurfaces. *Invent. Math.*, 125(1):13–36, 1996.

[6] M. S. Baouendi, H. Jacobowitz, and F. Trèves. On the analyticity of CR mappings. *Ann. of Math. (2)*, 122(2):365–400, 1985.

[7] M. S. Baouendi and Linda Preiss Rothschild. Cauchy-Riemann functions on manifolds of higher codimension in complex space. *Invent. Math.*, 101(1):45–56, 1990.

[8] M. Salah Baouendi, Peter Ebenfelt, and Linda Preiss Rothschild. *Real submanifolds in complex space and their mappings*, volume 47 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1999.

[9] S. Berhanu and J. Hounie. A class of FBI transforms. *Comm. Partial Differential Equations*, 37(1):38–57, 2012.

[10] S. Berhanu and Ming Xiao. On the C^∞ version of the reflection principle for mappings between CR manifolds. *Amer. J. Math.*, 137(5):1365–1400, 2015.

[11] Shiferaw Berhanu, Paulo D. Cordaro, and Jorge Hounie. *An introduction to involutive structures*, volume 6 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.

[12] Edward Bierstone and Pierre D. Milman. Resolution of singularities in Denjoy-Carleman classes. *Selecta Math. (N.S.)*, 10(1):1–28, 2004.

[13] Göran Björck. Linear partial differential operators and generalized distributions. *Ark. Mat.*, 6:351–407 (1966), 1966.

[14] Jan Boman. Microlocal quasianalyticity for distributions and ultradistributions. *Publ. Res. Inst. Math. Sci.*, 31(6):1079–1095, 1995.

[15] José Bonet, Reinhold Meise, and Sergej N. Melikhov. A comparison of two different ways to define classes of ultradifferentiable functions. *Bull. Belg. Math. Soc. Simon Stevin*, 14(3):425–444, 2007.

[16] J. M. Bony. Une extension du théorème de Holmgren sur l'unicité du problème de Cauchy. *C. R. Acad. Sci. Paris Sér. A-B*, 268:A1103–A1106, 1969.

[17] J. M. Bony. Extensions du théorème de Holmgren. In *Séminaire Goulaouic-Schwartz (1975/1976) Équations aux dérivées partielles et analyse fonctionnelle, Exp. No. 17*, page 13. Centre Math., École Polytech., Palaiseau, 1976.

[18] J. M. Bony. Équivalence des diverses notions de spectre singulier analytique. In *Séminaire Goulaouic-Schwartz (1976/1977), Équations aux dérivées partielles et analyse fonctionnelle, Exp. No. 3*, page 12. Centre Math., École Polytech., Palaiseau, 1977.

[19] R. W. Braun, R. Meise, and B. A. Taylor. Ultradifferentiable functions and Fourier analysis. *Results Math.*, 17(3-4):206–237, 1990.

[20] J. Bros and D. Iagolnitzer. Tuboïdes et structure analytique des distributions. II. Support essentiel et structure analytique des distributions. In *Séminaire Goulaouic-Lions-Schwartz 1974–1975: Équations aux dérivées partielles linéaires et non linéaires, Exp. No. 18*, page 34. Centre Math., École Polytech., Paris, 1975.

[21] T. Carleman. Sur les fonctions indéfiniment dérivables. *C. R. Acad. Sci., Paris*, 177:422–424, 1923.

[22] T. Carleman. Sur les fonctions quasi-analytiques. 5. Kongreß der Skandinav. Mathematiker in Helsingfors, 4.–7. Juli 1922. Helsingfors: Akadem. Buchh., 181-196 (1923)., 1923.

[23] Jacques Chazarain and Alain Piriou. *Introduction to the theory of linear partial differential equations*, volume 14 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam-New York, 1982. Translated from the French.

[24] Soon-Yeong Chung and Dohan Kim. A quasianalytic singular spectrum with respect to the Denjoy-Carleman class. *Nagoya Math. J.*, 148:137–149, 1997.

[25] Lawrence Conlon. *Differentiable manifolds*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, second edition, 2008.

[26] A. Denjoy. Sur les fonctions quasi-analytiques de variable réelle. *C. R. Acad. Sci., Paris*, 173:1329–1331, 1921.

[27] J. J. Duistermaat and J. A. C. Kolk. *Distributions*. Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2010. Theory and applications, Translated from the Dutch by J. P. van Braam Houckgeest.

[28] E. M. Dyn'kin. Pseudoanalytic continuation of smooth functions. Uniform scale. In *Mathematical programming and related questions (Proc. Seventh Winter School, Drogobych, 1974), Theory of functions and functional analysis (Russian)*, pages 40–73. Central Èkonom.-Mat. Inst. Akad. Nauk SSSR, Moscow, 1976.

[29] E. M. Dyn'kin. Pseudoanalytic extension of smooth functions. The uniform scale. *Transl., Ser. 2, Am. Math. Soc.*, 115:33–58, 1980.

[30] E. M. Dyn'kin. The pseudoanalytic extension. *J. Anal. Math.*, 60:45–70, 1993.

[31] Peter Ebenfelt. New invariant tensors in CR structures and a normal form for real hypersurfaces at a generic Levi degeneracy. *J. Differential Geom.*, 50(2):207–247, 1998.

[32] Leon Ehrenpreis. *Fourier analysis in several complex variables*. Pure and Applied Mathematics, Vol. XVII. Wiley-Interscience Publishers A Division of John Wiley & Sons, New York-London-Sydney, 1970.

[33] Abdelhafed Elkhadiri and Hassan Sfouli. Weierstrass division theorem in quasianalytic local rings. *Studia Math.*, 185(1):83–86, 2008.

[34] Charles Fefferman. The Bergman kernel and biholomorphic mappings of pseudoconvex domains. *Invent. Math.*, 26:1–65, 1974.

[35] Stefan Fürdös and Bernhard Lamel. Regularity of infinitesimal CR automorphisms. *Internat. J. Math.*, 27(14):1650112, 25pp, 2016.

[36] Alain Grigis and Johannes Sjöstrand. *Microlocal analysis for differential operators*, volume 196 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994. An introduction.

[37] Victor Guillemin and Shlomo Sternberg. *Geometric asymptotics*. American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14.

[38] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[39] G. Hoepfner and A. Medrado. The FBI transforms and their use in microlocal analysis. Preprint, 2017.

[40] E. Holmgren. Über Systeme von linearen partiellen Differentialgleichungen. *Öfversigt af Kongl. Vetenskaps-Akad. För.*, 58:91–103, 1901.

[41] Lars Hörmander. Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients. *Comm. Pure Appl. Math.*, 24:671–704, 1971.

[42] Lars Hörmander. A remark on Holmgren's uniqueness theorem. *J. Differential Geometry*, 6:129–134, 1971/72.

[43] Lars Hörmander. *Linear partial differential operators*. Springer Verlag, Berlin-New York, 1976.

[44] Lars Hörmander. Remarks on Holmgren's uniqueness theorem. *Ann. Inst. Fourier (Grenoble)*, 43(5):1223–1251, 1993.

[45] Lars Hörmander. *The analysis of linear partial differential operators. I*. Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[46] John Horváth. *Topological vector spaces and distributions. Vol. I*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.

[47] June Gi Kim, Soon-Yeong Chung, and Dohan Kim. Microlocal analysis in the Denjoy-Carleman class. *J. Korean Math. Soc.*, 38(3):561–575, 2001.

[48] Hikosaburo Komatsu. Ultradistributions. I. Structure theorems and a characterization. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 20:25–105, 1973.

[49] Hikosaburo Komatsu. The implicit function theorem for ultradifferentiable mappings. *Proc. Japan Acad. Ser. A Math. Sci.*, 55(3):69–72, 1979.

[50] Hikosaburo Komatsu. Ultradifferentiability of solutions of ordinary differential equations. *Proc. Japan Acad. Ser. A Math. Sci.*, 56(4):137–142, 1980.

[51] Hikosaburo Komatsu. Microlocal analysis in Gevrey classes and in complex domains. In *Microlocal analysis and applications (Montecatini Terme, 1989)*, volume 1495 of *Lecture Notes in Math.*, pages 161–236. Springer, Berlin, 1991.

[52] Bernhard Lamel. A reflection principle for real-analytic submanifolds of complex spaces. *J. Geom. Anal.*, 11(4):627–633, 2001.

[53] Bernhard Lamel. A C^{∞}-regularity theorem for nondegenerate CR mappings. *Monatsh. Math.*, 142(4):315–326, 2004.

[54] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[55] Otto Liess. Carleman regularization in the C^{∞}-category. *Ann. Univ. Ferrara Sez. VII (N.S.)*, 45(suppl.):213–240 (2000), 1999. Workshop on Partial Differential Equations (Ferrara, 1999).

[56] S. Mandelbrojt. *Séries adhérentes, régularisation des suites, applications*. Gauthier-Villars, Paris, 1952.

[57] J. N. Mather. On Nirenberg's proof of Malgrange's preparation theorem. In *Proceedings of Liverpool Singularities—Symposium, I (1969/70)*, pages 116–120. Lecture Notes in Mathematics, Vol. 192. Springer, Berlin, 1971.

[58] Anders Melin and Johannes Sjöstrand. Fourier integral operators with complex-valued phase functions. In *Fourier integral operators and partial differential equations (Colloq. Internat., Univ. Nice, Nice, 1974)*, pages 120–223. Lecture Notes in Math., Vol. 459. Springer, Berlin, 1975.

[59] Tadashi Nagano. Linear differential systems with singularities and an application to transitive Lie algebras. *J. Math. Soc. Japan*, 18:398–404, 1966.

[60] L. Nirenberg, S. Webster, and P. Yang. Local boundary regularity of holomorphic mappings. *Comm. Pure Appl. Math.*, 33(3):305–338, 1980.

[61] Krzysztof Jan Nowak. On division of quasianalytic function germs. *Internat. J. Math.*, 24(13):1350111, 5, 2013.

[62] Adam Parusiński and Jean-Philippe Rolin. A note on the Weierstrass preparation theorem in quasianalytic local rings. *Canad. Math. Bull.*, 57(3):614–620, 2014.

[63] Hans-Joachim Petzsche and Dietmar Vogt. Almost analytic extension of ultradifferentiable functions and the boundary values of holomorphic functions. *Math. Ann.*, 267(1):17–35, 1984.

[64] Stevan Pilipović, Nenad Teofanov, and Filip Tomić. Beyond Gevrey regularity. *J. Pseudo-Differ. Oper. Appl.*, 7(1):113–140, 2016.

[65] Stevan Pilipović, Nenad Teofanov, and Filip Tomić. Superposition and propagation of singularities for extended gevrey regularity. 07 2016.

[66] Armin Rainer and Gerhard Schindl. Composition in ultradifferentiable classes. *Studia Math.*, 224(2):97–131, 2014.

[67] Armin Rainer and Gerhard Schindl. Equivalence of stability properties for ultradifferentiable function classes. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 110(1):17–32, 2016.

[68] Luigi Rodino. *Linear partial differential operators in Gevrey spaces*. World Scientific Publishing Co., Inc., River Edge, NJ, 1993.

[69] J.-P. Rolin, P. Speissegger, and A. J. Wilkie. Quasianalytic Denjoy-Carleman classes and o-minimality. *J. Amer. Math. Soc.*, 16(4):751–777 (electronic), 2003.

[70] Charles Roumieu. Ultra-distributions définies sur R^n et sur certaines classes de variétés différentiables. *J. Analyse Math.*, 10:153–192, 1962/1963.

[71] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[72] M. Sato. Hyperfunctions and partial differential equations. In *Proc. Int. Conf. Funct. Anal. Rel. Topics, Tokyo 1969.*, pages 91–94. Tokyo University Press, 1970.

[73] Mikio Sato, Takahiro Kawai, and Masaki Kashiwara. Microfunctions and pseudo-differential equations. In *Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau)*, pages 265–529. Lecture Notes in Math., Vol. 287. Springer, Berlin, 1973.

[74] S. R. Simanca. *Pseudo-differential operators*, volume 236 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1990.

[75] Johannes Sjöstrand. Singularités analytiques microlocales. In *Astérisque, 95*, volume 95 of *Astérisque*, pages 1–166. Soc. Math. France, Paris, 1982.

[76] Héctor J. Sussmann. Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, 180:171–188, 1973.

[77] Vincent Thilliez. Division by flat ultradifferentiable functions and sectorial extensions. *Results Math.*, 44(1-2):169–188, 2003.

[78] Vincent Thilliez. On quasianalytic local rings. *Expo. Math.*, 26(1):1–23, 2008.

[79] François Trèves. *Introduction to pseudodifferential and Fourier integral operators. Vol. 2*. Plenum Press, New York-London, 1980. Fourier integral operators, The University Series in Mathematics.

[80] A. E. Tumanov. Extension of CR-functions into a wedge from a manifold of finite type. *Mat. Sb. (N.S.)*, 136(178)(1):128–139, 1988.

[81] Takesi Yamanaka. On ODEs in the ultradifferentiable class. *Nonlinear Anal.*, 17(7):599–611, 1991.

[82] E. C. Zachmanoglou. An application of Holmgren's theorem and convexity with respect to differential operators with flat characteristic cones. *Trans. Amer. Math. Soc.*, 140:109–115, 1969.

[83] E. C. Zachmanoglou. Foliations and the propagation of zeroes of solutions of partial differential equations. *Bull. Amer. Math. Soc.*, 78:835–839, 1972.

Abstract

The main topic of this thesis is the study of regularity of CR mappings between ultradifferentiable CR manifolds. Ultradifferentiable is understood in the sense of Denjoy-Carleman classes, i.e. subalgebras of smooth functions defined by weight sequences. We consider mainly Denjoy-Carleman classes that are defined by weight sequences, which are regular in the sense of Dyn'kin.

In particular, reflection principles of Lamel and Berhanu-Xiao for finitely nondegenerate CR mappings are generalized to the ultradifferentiable category. More precisely, any finitely nondegenerate CR mapping between two ultradifferentiable CR manifolds of the same Denjoy-Carleman class, that extends near a point holomorphically into a wedge, is ultradifferentiable near this point of the same regularity as the manifolds.

In order to prove the aforementioned result, a geometric theory of the ultradifferentiable wavefront set with respect to Denjoy-Carleman classes, that was initially defined by Hörmander, is developed for regular weight sequences. In particular, using a theorem of Dyn'kin on the characterizations of elements in regular Denjoy-Carleman class by almost-analytic extensions, a characterization of the ultradifferentiable wavefront set either by almost-analytic extensions into flat wedges or by the generalized FBI transform in the sense of Berhanu-Hounie is proven. This allows to show that the ultradifferentiable wavefront set can be invariantly defined on ultradifferentiable manifolds of the same Denjoy-Carleman class. Moreover an ultradifferentiable microlocal elliptic regularity theorem for vector-valued distributions and partial differential operators with ultradifferentiable coefficients is proven, what generalizes statements of Hörmander, Albanese-Jornet-Oliaro and others.

Besides the proof of the ultradifferentiable reflection principle, the statements mentioned above on the ultradifferentiable are used to generalize directly the results on the regularity of infinitesimal CR automorphisms on smooth abstract CR manifolds by Fürdös-Lamel to the ultradifferentiable setting. As a further straightforward application of the microlocal techniques quasianalytic generalizations of statements of Holmgren, Hörmander, Bony and Zachmanoglou about the uniqueness of solutions of homogeneous equations.

Zusammenfassung

Das Hauptthema dieser Arbeit ist die Untersuchung der Regularität von CR Abbildungen zwischen ultradifferenzierbaren CR Mannigfaltigkeiten. Ultradifferenzierbar ist hier im Sinne von Denjoy-Carleman Klassen gemeint, d.h. von Teilalgebren glatter Funktionen die durch Gewichtsfolgen definiert werden. Es werden hier hauptsächlich Denjoy-Carleman Klassen betrachtet, die (durch im Sinne von Dyn'kin reguläre) Gewichtsfolgen definiert sind.

Insbesondere werden Reflektionsprinzipe von Lamel und Berhanu-Xiao für endlich nichtdegenerierte CR Abbildungen in die ultradifferenzierbare Kategorie verallgemeinert. Genauer wird gezeigt, dass jede endlich nichtdegenerierte CR Abbildung zwischen zwei ultradifferenzierbaren CR Mannigfaltigkeiten von derselben Denjoy-Carleman Klasse, die nahe eines Punktes eine holomorphe Ausdehnung in einen Wedge besitzt, nahe dieses Punktes ultradifferenzierbar von der gleichen Regularität wie die Mannigfaltigkeiten ist.

Für den Beweis der obigen Aussage wird eine geometrische Theorie der ultradifferenzierbaren Wellenfrontmenge im Sinne von Denjoy-Carleman Klassen, welches ursprünglich von Hörmander definiert wurde, für reguläre Gewichtsfolgen entwickelt. Insbesonders wird ein Satz von Dyn'kin über die Charakterisierung von Elementen regulärer Denjoy-Carleman Klassen durch fast-analytische Ausdehnungen verwendet, um die Charakterisierung der ultradifferenzierbaren Wellenfrontmenge durch fast-analytische Ausdehnungen in flache Wedges bzw. durch die verallgemeinerte FBI Transformation im Sinne von Berhanu-Hounie zu zeigen. Dies erlaubt die invariante Definition der ultradifferenzierbare Wellenfrontmenge auf ultradifferenzierbare Mannigfaltigkeiten der selben Denjoy-Carleman Klasse zu geben. Weiters wird ein Satz über ultradifferenzierbare mikrolokale elliptische Regularität für vektorwertige Distributionen und Differentialoperatoren mit ultradifferenzierbaren Koeffizienten bewiesen, was Resultate von Hörmander, Albanese-Jornet-Oliaro und anderen verallgemeinert.

Weiters werden die oben genannten Resultate für die ultradifferenzierbare Wellenfrontmenge dazu verwendet die Aussagen von Fürdös-Lamel bezüglich der Regularität von infinitesimalen CR Automorphismen auf abstrakten CR Mannigfaltigkeiten in die ultradifferenzierbare Kategorie zuverallgemeinern.

Als weitere direkte Anwendung der mikrolokalen Techniken werden quasianalytische Verallgemeinerungen von Resultaten von Holmgren, Hörmander, Bony und Zachmanoglou über die Eindeutigkeit von Lösungen homogener Gleichungen gegeben.