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„Competition vs. Collusion with asymmetric costs in Cournot Duopoly"

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Abstract

The focus of this thesis is to examine the ways in which companies could collude in quantity under asymmetric costs and investigates whether collusion is advantageous for all companies involved. In particular, with which method can a collusive outcome in a duopoly be achieved based on an asymmetric Cournot duopoly. With asymmetric costs it is more difficult to sustain collusion in comparison to symmetric costs. Nevertheless, this is an important topic, since it reflects reality much better than symmetric costs do. General knowledge conveys that under asymmetric costs and in the absence of side payments, the maximization of the profit of the whole industry is unlikely. This thesis is primarily based on the paper *"Competitive Advantage and Collusive Optima"* by Schmalensee (1987) [17]. The main emphasis is on the characterization of four distinct collusive procedures, from which a respective optimal solution is found by means of methods taken from the axiomatic bargaining theory. Results show that the collusive outcome for the low-cost firm might be under-performing. Finally, Schmalensee's research (1987) [17] is expanded in different ways.

Keywords. asymmetric costs, bargaining theory, cartel, collusion, Cournot, duopoly

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List of abbreviations

EG Equal Gain

FOC First order condition

K-S Kalai-Smorodinsky

MD Market Division

MS Market Sharing

PR Proportional Reduction

SOC Second order condition

SP Side Payments

C The superscript C characterizes either the quantity or the profit in Cournot equilibrium

M The superscript M characterizes either the quantity, the profit or the price as monopolist

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1. Introduction

This thesis deals with collusion under asymmetric costs to a great extent in a duopoly. In particular, the emphasis is on collusion in quantity. Thus, the focus mainly lies on the paper *"Competitive Advantage and Collusive Optima"*, written by Richard Schmalensee (1987) [17], which deals with this issue. In his article, Schmalensee introduced four different technologies for collusion.

The *main research question* is therefore, how should firms now collude in such situations with asymmetric costs that are based on an asymmetric Cournot duopoly? And is it profitable for all companies?

What does that mean in particular? Four different ways are described, how the companies can possibly collude and accordingly four methods to solve them optimally, established by Schmalensee (1987) [17]. The collusive outcome is then compared to that of Cournot competition, which holds as an important benchmark in this study. A higher profit is expected from collusion. Hence, the objective is to show that more profit can be obtained from collusion than from Cournot Competition.

This thesis should be a simplification for other researchers who are interested in this topic, especially in these four particular collusion technologies introduced by Schmalensee (1987) [17]. For this reason, I describe his model and the four collusion technologies in detail to make them easier to understand, since Schmalensee (1987) [17] did not always go into too much detail with his explanations and formulated some aspects in an abstract manner. While there is nothing wrong with his paper, I provided accurate computations and formulas to expand upon his explanations so that these are easy to follow. Including such computations also means that no additional calculations on the part of the reader are required. In addition, his results are partly verified. Although this thesis deals with theoretical rather than existing companies, it questions whether existing companies may apply this theoretical approach in reality.

In my research I do not take into account the stability of the derived collusive solutions and neglect the fact that the companies may have an incentive to deviate from the collusive outcome, as Schmalensee (1987) [17] did. In particular, the firms could cheat on the agreements in not holding on to their assigned quota or part of the market.

1.1. Motivation

A lot of literature concerning collusion in oligopolies deals with symmetric costs, since it is generally common knowledge that asymmetric costs prevent the companies from collusion, see Fischer and Normann (2017) [5]. Nevertheless, it is a topic of importance and deserves more attention, as asymmetric costs play an significant role in reality. In the actual world, two different companies would rarely have identical costs, namely fixed costs, production costs or the same technology, just to mention a few potential differences. Hence, when assuming asymmetric costs in contrast to symmetric costs, the reality is better reflected and it is more practical. Still, the theory about the case with symmetric costs is substantial.

If firms want to collude under asymmetric costs, it would be straight forward to maximize the outcome of the whole industry, which is commonly known as the purpose of a cartel. But solving this maximization task with asymmetric costs leads to an algebraic problem, which will be explained in section 2.2. However, pursuing this purpose stands to reason under the condition that some types of compensation payments are allowed, as Schmalensee (1987) [17], Bain (1948) [1] and some others have emphasized. Furthermore, since compensation payments can be prohibited in some countries and as a consequence of diseconomies of scale, it is misplaced in many cases for firms to use side payments or for them to merge (cf. Stigler 1964, page 45) [20]. Though there are also other studies which claim that collusion can sustain asymmetric costs. The approach of these studies, is on the one hand “balanced temptation” by Collie (2004) [3] and on the other hand the Cournot duopoly over an infinite horizon by Escrihuela-Villar (2012) [4].

1.2. Structure of the thesis

Initially, I will shortly introduce the Cournot model and then explain what the exact problem is that comes up when maximizing the joint profit of the whole industry. In Part I, to which section 3 – 6 belong, I explicitly describe the approach of Schmalensee (1987) [17], dealing with the aforementioned problem. In the third section, the simple model of Schmalensee (1987) [17] is illustrated. Then follows the core part of the thesis: the characterization of the four different collusion technologies, originally presented from Schmalensee (1987) [17]. These are described in detail in section 4 and are compared at the end of the section. In section 5, four collusive solution methods from the axiomatic bargaining theory, which are chosen from Schmalensee (1987) [17], are presented to find the optimal solution for the collusion technologies. The results of Schmalensee (1987) [17] are introduced at the end of Part I in section 6.

In Part II, I expand on Schmalensee's research (1987) [17]. In fact, I generalize his model by modifying first the demand function, the cost function and following this, I then expand the duopoly to a triopoly.

Finally, I shortly summarize the main issues and make a conclusion on the key results. Additional calculations can be found in the appendix.

2. The Problem

Before having a look at the actual problem of colluding firms, Cournot competition will shortly be introduced. In the following sections, this model will hold as a standard for comparison, especially if the firms are not colluding, and to examine the performance of the collusion technologies. A higher profit will be expected from collusion than from competition.

2.1. Cournot Competition with asymmetric costs

I now consider a simple model of Cournot competition with two or more firms in the market, which have different costs. Here there is assumed that two or more firms are competing in quantity in a market, in which a homogeneous product is sold. Below, the variable $1+N$ will describe the number of firms operating in the market. Note that N is a natural number. Why the number of firms is $1+N$ and not just N, will be explained below. I try to make it more general in using an arbitrary natural number of firms without loosing the focus.

The Cournot model is a static game with complete information of all players, who are just the firms. That means the equilibrium derived from Cournot competition is a Nash equilibrium in a normal form game.

Here the same demand function is chosen as from Schmalensee (1987) [17]. This demand function will always be used in Part I. That must not be true for the extensions in Part II. Thus, Schmalensee (1987) [17] assumed a linear inverse demand function, which looks like this:

$$P(Q) = \max_{0 \leq Q < \infty} \{1 - Q,\ 0\}. \tag{2.1}$$

In the equation above P characterizes the market price and $Q = q_i + Q_{-i}$ the total quantity, where q_i indicates the individual quantity of firm i and Q_{-i} is the sum of the quantities of all firms except firm i. In other words Q_{-i} indicates which quantities the other companies except firm i are choosing. Of course the quantity, which is the number of units the firms are producing, cannot take a negative value. Therefore, $q_i \in [0, \infty)$ for $i = 1, \ldots, 1+N$. As written in the headline it is assumed to have asymmetric cost functions with constant unit costs:

$$C(q_i) = c_i \cdot q_i, \quad \text{for } i = 1, \ldots, 1+N \quad c_1 < c_j \quad \text{for } j = 2, \ldots, 1+N. \tag{2.2}$$

Without loss of generality it is assumed that firm 1 has lower costs than any other firm. Each firm i has its individual cost parameter c_i. The profit function Π_i of firm i is determined by revenue reduced by costs, while the revenue is the price multiplied by the quantity.

$$\Pi_i(Q) = P(Q) \cdot q_i - C(q_i) = (1 - Q - c_i) \cdot q_i \quad \text{for } i = 1, \dots, 1+N \tag{2.3}$$

By applying the demand and cost function from above the profit function (2.3) is obtained. The goal is now to maximize the profit of each firm i, respectively:

$$\max_{0 \leq q_i < \infty} \Pi_i(Q) = \max_{0 \leq q_i < \infty} (1 - Q - c_i) \cdot q_i \quad \text{for } i = 1, \dots, 1+N. \tag{2.4}$$

Differentiating the profit of firm i with respect to their quantity q_i results in the following first order conditions (FOC):

$$\frac{\partial \Pi_i(Q)}{\partial q_i} = 1 - 2q_i - Q_{-i} - c_i \overset{!}{=} 0 \quad \text{for } i = 1, \dots, 1+N. \tag{2.5}$$

The equation above is set to zero to determine the extremum. Since the profit function is a quadratic function, which is a strictly concave function, the extreme value will be a global maximum.

Now q_i is expressed for every i in each equation from (2.5) and the so-called reaction function is obtained, which shows the best 'reaction' of firm i in choosing its quantity against which quantity the other firms are selecting.

$$q_i = \begin{cases} \frac{1}{2} \cdot (1 - Q_{-i} - c_i) & \text{for} \quad Q_{-i} \leq 1 - c_i \\ 0 & \text{otherwise} \end{cases} \quad \text{for } i = 1, \dots, 1+N \tag{2.6}$$

This distinction of cases is made, because if the other firms increase their quantities, firm i will have more costs to produce than it can earn. In that case it will stay out of the market and produce zero goods.

From here first the quantity of each firm is derived by solving the system of equations (2.6) of the reaction functions for each i and then the profit by setting the calculated quantities into the profit function (2.3) for each i. Before this is done, it will be anticipated a little bit from the next section 3 to make it more suitable to the model of Schmalensee (1987) [17] in the next section. In this model there are two types of firms. I will apply this here for simplification. Hence there is firm 1 and one or more firms 2, the latter are basically identical. That means all firms 2 have the same costs c_2 and the same quantity q_2. Let the number of firms 2 be N. Therefore, the total quantity is $Q = q_1 + Nq_2$. Since all firms 2 are identical, one can apply the symmetric argument, which means $q_2 = \cdots = q_{1+N}$. By means of that argument, the following quantities below are obtained for the general case for $1+N$ firms. In the following the quantity and profit in Cournot equilibrium will be marked with a superscript C.

$$q_1^C = \frac{1}{2+N} \cdot [1-(1+N)\cdot c_1 + Nc_2] \tag{2.7}$$

$$q_2^C = \frac{1}{2+N} \cdot [1-2c_2+c_1] \tag{2.8}$$

In addition, the quantity for all N high-cost firms is given by Nq_2^C. Hence the corresponding profit functions are obtained by inserting the quantities above in the profit function from equation (2.3). Thus, the profit functions of firm 1 and one firm 2, especially for N firms 2 $N\Pi_2^C = N \cdot (q_2^C)^2$, in Cournot equilibrium are given by:

$$\Pi_1^C = (q_1^C)^2 = \frac{1}{(2+N)^2} \cdot [1-(1+N)\cdot c_1 + Nc_2]^2 \tag{2.9}$$

$$\Pi_2^C = (q_2^C)^2 = \frac{1}{(2+N)^2} \cdot [1-2c_2+c_1]^2. \tag{2.10}$$

The subsequent equations focus on the two firm case. This is needed later on for some calculations.

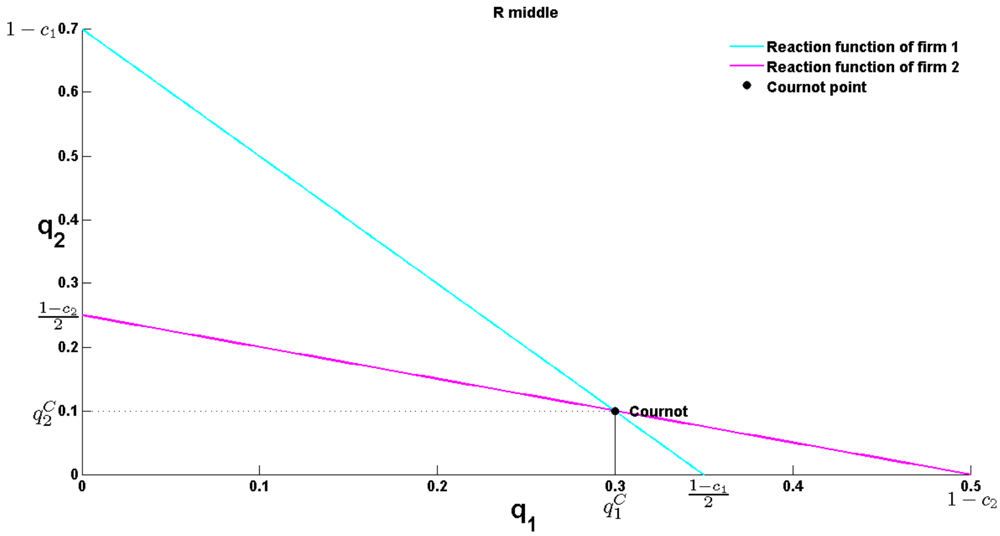
$$q_i^C = \frac{1-2c_i+c_{-i}}{3} \tag{2.11}$$

$$\Pi_i^C = \frac{(1-2c_i+c_{-i})^2}{9} \tag{2.12}$$

The quantities in Nash equilibrium are characterized by the point (q_1^C, q_2^C), as the profits in Nash equilibrium by the point (Π_1^C, Π_2^C). Both equations above (2.11) and (2.12) and accordingly (2.7) – (2.10) hold as a benchmark for the following study about collusion.

The figure 2.1 below is plotted in MATLAB and shows the reaction function of only two firms from equation (2.6). To be more precise the graph shows the reaction functions of one single firm 1 competing against one single firm 2. The intersection is the Nash equilibrium for both firms, where both firms choose their optimal quantity, respectively called Cournot point. In this point neither of the firms want to deviate from their produced quantity, since they cannot increase their profit by changing the quantity. The exact values for the intersection point in the figure below are determined from equation (2.11) with the costs denoted in the notes.

Figure 2.1: Reaction function of two firms in Cournot competition



Notes: Figure 2.1 is plotted in MATLAB with the following values $c_1 = 0.3$ and $c_2 = 0.5$. This corresponds to the cost difference parameter R_{middle}, which will be introduced in the next section 3.

2.2. The Problem of Collusion

Starting from now the focus lays on how firms can or should collude. The easiest way how firms can collude and to use the whole potential of the market is to maximize joint profits. To receive these joint profit, the profits of each firm in the market are summed up. It is shown not only for the two firm case, but directly for the general case with $1+N$ firms operating in the market. Since I want to show that this works analogous for an arbitrary natural number of firms, namely N. There are $1+N$ firms, because there is one single firm competing against N other identical firms. That means the same problem is faced, if there are an arbitrary number of firms competing against one single firm with low costs. But also if there are three different firms competing in the market. Here I mean different in terms of various costs. The proof can be seen in the appendix A.4. This also works analogous for the cost function introduced in the next section 3.

Hence, the joint profit is given by $\Pi = \Pi_1 + N \cdot \Pi_2$. That means the following formula is obtained with the profit function (2.3) from above:

$$\Pi(Q) = (1 - q_1 - Nq_2 - c_1) \cdot q_1 + (1 - q_1 - Nq_2 - c_2) \cdot N \cdot q_2, \quad \text{where} \quad c_1 \neq c_2. \tag{2.13}$$

Without loss of generality, it is assumed that $c_1 < c_2$. It is important to remark that asymmetric costs are supposed, because a contradiction will be induced at the end of the argumentation. However, here the joint profit Π will be maximized similar to section 2.1. In contrast to the previous case now the joint profit will be differentiated and not only the firm's own profit separately with respect to the quantity of each firm. Then, it is set to zero. Therefore, the FOC's are given by:

$$\frac{\partial \Pi(Q)}{\partial q_1} = 1 - 2q_1 - 2Nq_2 - c_1 \stackrel{!}{=} 0 \tag{2.14}$$

$$\frac{\partial \Pi(Q)}{\partial q_2} = 1 - 2q_1 - 2Nq_2 - c_2 \stackrel{!}{=} 0. \tag{2.15}$$

If one is solving these equations for q_1 and q_2, these two reaction functions are obtained:

$$q_1 = \begin{cases} \frac{1}{2} \cdot (1 - 2Nq_2 - c_1) & \text{for} \quad Nq_2 \leq \frac{1-c_1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{2.16}$$

$$q_2 = \begin{cases} \frac{1}{2N} \cdot (1 - 2q_1 - c_2) & \text{for} \quad q_1 \leq \frac{1-c_2}{2} \\ 0 & \text{otherwise} \end{cases}. \tag{2.17}$$

Trying to solve the two previous equations by means of setting q_2 from equation (2.17) into q_1 from equation (2.16) leads to the cancellation of the quantities q_1 and q_2. This results in:

$$\Longleftrightarrow c_1 = c_2 \quad \lightning. \tag{2.18}$$

This means there is no solution to the system of equations (2.16) – (2.17) for the quantities q_1 and q_2 with the assumption of asymmetric costs. The equation above says that firm 1 and all N firms 2 have the same costs. But this is a contradiction to the assumption in the beginning of having asymmetric costs. In other words, the problem cannot be solved algebraically in this way. What works in the symmetric cost case, ends up with an algebraic problem in the asymmetric cost case. In the following section, Schmalensee's approach to solve this issue will be discussed.

Part I.
Schmalensee's approach

In the following part of my thesis, I work on Schmalensees paper (1987) [17]. I describe it in more detail such that the topic is more comprehensible. Furthermore, I will provide precise calculations to make it easier to follow without making own computations. That means this part leans mainly on his paper and also his structure. In addition, whenever I use the phrasing "Schmalensee's model", I mean the model from Schmalensee (1987) [17].

3. Framework of Schmalensee

Richard Schmalensee (1987) [17] tried to solve the problem of the previous subsection 2.2 in another way using the following simple model. He assumed two types of companies competing in quantity in a market with a homogeneous product. The first type is a single *low-cost firm*, which will also be called *firm 1*. The second type consists of N *high-cost firms*, which are basically identical and will be named *firms of type 2*. Remember that N is a natural number. In total, there are $1+N$ firms operating in the market.

Schmalensee (1987) [17] used the wording "leader" and "follower" to characterize the two types of firms, which is maybe a bit misleading, because there is no time difference in moving. The firms are just moving once at the same time. That is why I do not use this wording in my thesis. That means a static game with complete information is considered.

The inverse demand function from equation (2.1) is as well employed by Schmalensee (1987) [17]. This simple market demand function and the following cost function is selected to simplify the calculations, particularly since one of the collusion technologies involves more complicated computations. The subsequent cost function is chosen, which is valid for both types of firms:

$$C(q_i) = (1-\theta_i)\cdot q_i, \quad \text{for} \quad i=1,2 \quad \text{and} \quad \theta_i \in [0,1]. \tag{3.1}$$

That means there are constant unit costs with a cost parameter θ_i, which can take values between zero and one. The variable q_1 characterizes the quantity of firm 1, while q_2 indicates the quantity of a single high-cost firm. Accordingly, the quantity of all N firms of type 2 is $N\cdot q_2$. As explained in section 2.1 the quantities cannot be negative, consequently $q_i \in [0,\infty)$ for $i=1,2$. The cost parameters must fulfill the subsequent inequality. The reason will be explained below.

$$\theta_2 < \theta_1 < 2 \cdot \theta_2 \tag{3.2}$$

The first inequality just ensures that firm 1 has lower costs than any firm of type 2, since firm 1 is the low-cost firm. However, the second inequality makes sure that the market price of firm 1 as a monopolist, which is $1 - \frac{\theta_1}{2}$, is higher than the costs of each firm of type 2, which are $1 - \theta_2$. Expressed in formula this means $1 - \theta_2 = c_2 < P_1^M = 1 - q_1^M$. In this formula the superscript M denotes the price, respectively the quantity, of firm 1 as a monopolist. In the appendix A.3, the determination of firm 1 and N firms of type 2 as a monopolist is shown. The market price of firm 1 as a monopolist has to be higher, because if firm 1 drops its price, firms of type 2 will have more costs to produce the product than they would be able to gain. In this case, it will stay out of the market and produce nothing.

With this demand (2.1) and cost function (3.1), the following profit function is obtained

$$\Pi_i(Q) = (\theta_i - Q) \cdot q_i \quad \text{for } i = 1,2. \tag{3.3}$$

One must not forget that the determination of the total quantity is as follows $Q = q_1 + N \cdot q_2$, since there are N high-cost firms. Hence $N \cdot q_2$ is the quantity of all firms of type 2. Moreover, the corresponding joint industry profit is given by:

$$\Pi = \Pi_1 + N \cdot \Pi_2. \tag{3.4}$$

The variable Π characterizes the profit of the whole industry. Thus it is the sum of the profits of each operating firm in the market. Furthermore, Π_1 is the profit of the low-cost firm, whereas Π_2 denotes the profit of a single high-cost firm. The total profit of firms of type 2 is characterized by $N \cdot \Pi_2$.

Schmalensee (1987) [17] introduced the parameter R, which measures the relative difference in costs, while the natural number N describes the number of high-cost firms. In addition, R can signify how profitable the market is. With the following equation the market can be characterized with only these two parameters, namely R and N.

$$R = \frac{\theta_1}{\theta_2} \tag{3.5}$$

From the previous equation (3.5), equation (3.2) and the range of values of θ_i, it follows that $1 < R < 2$. Small values of R, that means values near 1, signify that the firms have nearly the same costs, while high values of R, near 2, indicate a proportional big difference in costs. Schmalensee (1987) [17] used three different values of R to measure most suitably the effect of relative changes in costs, respectively changes in R. This is done in choosing the values relatively well distributed over the interval of R. Therefore, there is a value near 1, which I will name R_{low} and it takes the value 1.143. Secondly, a value in the middle of the interval of R is selected, namely $R_{\text{middle}} = 1.4$. Thirdly, Schmalensee (1987) [17] introduced a value near 2, which is characterized by $R_{\text{high}} = 1.727$. The calculations of this values can be found in the appendix A.1. Especially the values are summed up in table A.1. Note that in the symmetric cost case $\theta_1 = \theta_2$ and therefore $R = 1$.

Another parameter S is introduced, since it has a stronger correlation to the measured variables and thus is easier to determine. S measures the market share of the low-cost firm in Cournot equilibrium. The market share of firm 1 in Cournot equilibrium is determined as follows:

$$S = \frac{q_1^C}{Q^C} = \frac{q_1^C}{q_1^C + N \cdot q_2^C} \tag{3.6}$$

$$= \frac{(1+N)\cdot\theta_1 - N\theta_2}{\theta_1 + N\theta_2}. \tag{3.7}$$

In the second equation, the values of the Cournot quantities from equation (3.16) and (3.17), which are determined in section 3.1, are set in the formula of S. Note that $S + S_2 = 1$, where $S_2 = 1 - S$ is the market share of a all N firms of type 2. With small transformations R can now also be expressed with these two parameters S and N:

$$R = \frac{N\cdot(1+S)}{N+1-S}. \tag{3.8}$$

The market can now be characterized instead of the parameters R and N with the above introduced parameter S and N. More details on the derivation of the parameter S is to be found in the appendix A.2. This equation (3.8) and the range of values of R imply that S lies in the interval $S \in \left[\frac{1}{N+1}, 1\right]$, see Schmalensee (1985) [16]. With easy transformation of inequalities the interval above for S can be obtained, see in the appendix A.2 in the second part. If one of the parameters either S or N is kept constant and the other one is raised to its limit, then the cost benefit of the low-cost firm rises. Of course the rivalry between the different types of firms increases with increasing values of N.

Note that Schmalensee (1987) [17] assumed that "if collusion does not occur, the market is in Cournot equilibrium" (Schmalensee 1987, page 354) [17] and so do I. Therefore, the Cournot equilibria for Schmalensee's model are determined.

3.1. Cournot competition with Schmalensee's model

To have the exact Cournot values for Schmalensee's model, I now shortly calculate the Cournot quantities and profits. In section 2.1, the model of Cournot competition is introduced. There I used the general cost function (2.2). In contrast, Schmalensee chose this cost function (3.1). I utilize a more general cost function, such that I can adapt a different cost function more easily in Part II, where I extend Schmalensee's model. I chose the same demand function (2.1) in section 2.1 as Schmalensee (1987) [17], such that there is no need for adaption. This demand function will always be used in Part I, while in Part II some modifications can be made.

Since only the cost function is changed, one does not have to repeat the whole calculation. Accordingly, I will determine the Cournot quantity and the Cournot profit for Schmalensee's model in just setting in the already calculated Cournot values from equations (2.7) – (2.10). That means the subsequent transformation has to be done:

$$c_i = 1 - \theta_i \quad \text{for } i = 1, 2. \tag{3.9}$$

As previously described, firm 1's quantity is characterized by q_1^C and consequently q_2^C is the quantity of a single high-cost firm. To obtain the Cournot quantity of all N firms of type 2 just multiply q_2^C with N. Now one sets the transformation (3.9) in the equation (2.7). Thus the Cournot quantity of firm 1 according to Schmalensee's model is the following:

$$q_1^C = \frac{1}{2+N} \cdot [1 - (1+N)c_1 - Nc_2] \tag{3.10}$$

$$= \frac{1}{2+N} \cdot [1 - (1+N) \cdot (1-\theta_1) + N \cdot (1-\theta_2)] \tag{3.11}$$

$$= \frac{1}{2+N} \cdot [(1+N)\theta_1 - N\theta_2]. \tag{3.12}$$

In addition, firms of type 2's Cournot quantity is analogous transformed by setting in the equation (2.8).

$$q_2^C = \frac{1}{2+N} \cdot (1 - 2c_2 + c_1) \tag{3.13}$$

$$= \frac{1}{2+N} \cdot (1 - 2 \cdot (1-\theta_2) + 1 - \theta_1) \tag{3.14}$$

$$= \frac{1}{2+N} \cdot (2\theta_2 - \theta_1) \tag{3.15}$$

To sum up, the Cournot quantities for Schmalensee's model of both types of firms are:

$$q_1^C = \frac{1}{2+N} \cdot [(1+N)\theta_1 - N\theta_2] \tag{3.16}$$

$$q_2^C = \frac{1}{2+N} \cdot (2\theta_2 - \theta_1) \tag{3.17}$$

$$N \cdot q_2^C = \frac{N}{2+N} \cdot (2\theta_2 - \theta_1)\,. \tag{3.18}$$

The previous characterization of the Cournot profit functions is kept, namely Π_1^C for the Cournot profit of the low-cost firm and accordingly Π_2^C for a single high-cost firm. Hence for all N firms of type 2 the term $N\Pi_2^C$ is used. To determine the Cournot profit of firm 1, respectively of firms of type 2, one directly uses that $\Pi_i^C = (q_i^C)^2$ for $i = 1,2$ from equation (2.9), respectively from equation (2.10). That means one only has to square the Cournot quantites from equation (3.16), analogue from equation (3.17).

$$\Pi_1^C = (q_1^C)^2 = \frac{1}{(2+N)^2} \cdot [(1+N)\theta_1 - N\theta_2]^2 \tag{3.19}$$

$$\Pi_2^C = (q_2^C)^2 = \frac{1}{(2+N)^2} \cdot [2\theta_2 - \theta_1]^2 \tag{3.20}$$

$$N \cdot \Pi_2^C = N \cdot (q_2^C)^2 = \frac{N}{(2+N)^2} \cdot [2\theta_2 - \theta_1]^2 \tag{3.21}$$

Be careful in Part I always to these Cournot values will be referred. That must not hold for my extensions in Part II.

The figure 2.1 from section 2.1 is also valid for the two equations (3.16) – (3.17) above. The values $c_1 = 0.3$ and $c_2 = 0.5$ are transformed according to equation (3.9) into $\theta_i = 1 - c_i$ for $i = 1,2$. The transformed values for the parameter θ are the following $\theta_1 = 0.7$ and $\theta_2 = 0.5$. To prevent confusion, the parameter θ_1 has to be larger than θ_2, since the cost function is $1 - \theta_i$ for $i = 1,2$. Therefore, firm 1 would have costs of 0.3 and types of firms 2 0.5 in this case.

4. Collusion Technologies

Schmalensee (1987) [17] introduced four general technologies for collusion. In other words, four different methods how firms can collude, which try to solve the problem from section 2.2 in another way. These will be presented in the following subsections. That means in sections 4.1 – 4.4 Schmalensee's collusion technologies will be illustrated in more detail and therefore lean mainly on his paper, Schmalensee (1987) [17]. The first technology allows compensation payments, while under the other three technologies such payments are forbidden. At the end of this section, all technologies are compared to each other and with the symmetric and the Cournot case.

4.1. Side Payments

Sometimes side payments are forbidden by law or because of other restrictions, which Schmalensee (1987) [17] did not mention in detail. He took it more for granted that compensations payments are used in terms of building a cartel, which can be prohibited by competition laws. In most states building a cartel is not allowed. But in this technology the firms are allowed to make side payments or to merge. In other words, they are permitted to build a cartel. Hence, they will behave as they would have a monopoly. This is a good way to maximize total industry profits and to use the whole potential of the market. Below, the quantity and profit for this technology will be characterized with the superscript *SP*. Consequently, the low-cost firm will do all the production and firms of type 2 will produce nothing which means the quantity q_2^{SP} is zero. That is just because firm 1 can produce cheaper. Thus, just firm 1's profit is maximized as it has a monopoly.

$$\max_{0 \leq q_1 < \infty} \Pi(q_1) = \max_{0 \leq q_1 < \infty} (\theta_1 - q_1) \cdot q_1 \tag{4.1}$$

The detailed calculation of the case of a monopoly can be seen in the appendix A.3. In the following, the superscript M denotes the corresponding quantity or profit of a monopolist. In this technology, it is the same to maximize the total profit or firm 1's profit, since here it holds $\Pi = \Pi_1^M$. Solving the maximization task analog to section 2.1, the total collusive quantity is identical to the monopoly quantity of firm 1.

$$Q^{SP} = q_1^M = \frac{\theta_1}{2} = q_1^{SP} \tag{4.2}$$

Therefore, there is an explicit solution at the point:

$$\left(q_1^{SP},\, Nq_2^{SP}\right) = \left(\frac{\theta_1}{2},\, 0\right). \tag{4.3}$$

The equation above characterizes the optimal quantity for the Side Payments technology (SP). To derive the profit, Q^{SP} from equation (4.2) is set in equation (3.3) for $i = 1$. Accordingly, the total collusive profit equals the monopoly profit of the low-cost firm:

$$\Pi_1^M = \Pi^{SP} = \Pi_1^{SP} + N\Pi_2^{SP} = \frac{\theta_1^2}{4}. \tag{4.4}$$

This implies that the solution of the joint profit is explicit. Afterwards, this gain has to be divided between the firms. It follows that there is no explicit solution for the individual firm, since they are behaving as a cartel, instead there is a profit possibility frontier:

$$\Pi_2^{SP} = \frac{1}{N}\left(\frac{\theta_1^2}{4} - \Pi_1^{SP}\right). \tag{4.5}$$

Theoretically, the profit can be split freely. But the optimal solution for each type of firm should be found. In section 5, four solution methods are introduced, from which three of them can be applied to this collusion technology. Why will be explained later on in section 5.4. For every of the three individual solution methods there exists an optimal outcome which will be determined from the explicit profit possibility frontier (4.5).

Schmalensee (1987) [17] stated that this collusion technology should be used more to compare the other technologies with and to examine what profits can be possible with collusion. But it may not necessarily be employed for all cases as an alternative, which is close to reality.

4.2. Market Sharing

If it is not allowed to pay compensation payments or to merge, the profits of each company can be determined "only from its own production and sales" (Schmalensee 1987, page 354) [17]. That means each firm has to produce a certain number of products at a fixed price. The following three technologies deal with these particular 'output quotas' which are assigned to each type of firm. The companies have to keep this rate for the quantity. In other words, the cartel allocates particular 'output quotas' to each firm. Therefore, each firm determine its profit with respect to its rate in quantity and as well to that of the other firm. Furthermore, each firm holds the other firm's profit fixed at their output quota and maximizes its own profit with the information which quota is assigned to itself and the other type of firm. Then the explicit reaction function is calculated from the restricted profit function, limited in terms of quantity, and not the other way around.

In this specific technology, the profit function (3.3) is transformed into functions of total quantity Q to determine the profit of each type of firm with respect to their assigned output quotas. Hence from equation (3.3) it follows:

$$q_i = \frac{\Pi_i}{\theta_i - Q} \quad \text{for } i = 1,2. \tag{4.6}$$

Now the total quantity can be described by:

$$Q = q_1 + N \cdot q_2 = \frac{\Pi_1}{\theta_1 - Q} + N \cdot \frac{\Pi_2}{\theta_2 - Q}. \tag{4.7}$$

From here, the profit of all N high-cost firms is expressed as a function of the profit of firm 1 and the other way around. This is done by setting Q in the profit function (3.3), but transforming the part for the quantity q_i from equation (4.7) for $i = 1,2$ into:

$$q_1 = Q - N \cdot \frac{\Pi_2}{\theta_2 - Q} \quad \text{and analogue} \tag{4.8}$$

$$N \cdot q_2 = Q - \frac{\Pi_1}{\theta_1 - Q}. \tag{4.9}$$

With these transformations the subsequent equations are obtained:

$$\Pi_1(N\Pi_2) = \left[\theta_1 - \left(Q - \frac{N\Pi_2}{\theta_2 - Q} + \frac{N\Pi_2}{\theta_2 - Q}\right)\right] \cdot \left(Q - \frac{N\Pi_2}{\theta_2 - Q}\right) \tag{4.10}$$

$$N\Pi_2(\Pi_1) = \left[\theta_2 - \left(\frac{\Pi_1}{\theta_1 - Q} + Q - \frac{\Pi_1}{\theta_1 - Q}\right)\right] \cdot \left(Q - \frac{\Pi_1}{\theta_1 - Q}\right). \tag{4.11}$$

Converting the equations above, yields the following expressions for the profits Π_1 and $N\Pi_2$:

$$\max_{0 \le Q < \infty} N\Pi_2(\Pi_1) = \max_{0 \le Q < \infty} \left(Q - \frac{\Pi_1}{\theta_1 - Q}\right) \cdot (\theta_2 - Q) \tag{4.12}$$

$$\max_{0 \le Q < \infty} \Pi_1(\Pi_2) = \max_{0 \le Q < \infty} \left(Q - \frac{N\Pi_2}{\theta_2 - Q}\right) \cdot (\theta_1 - Q). \tag{4.13}$$

In the next step, both equations (4.12) and (4.13) are maximized with respect to the total quantity Q and then set to zero. Therefore, the FOC's read as follows:

$$\frac{\partial N\Pi_2(\Pi_1)}{\partial Q} = \theta_2 - 2Q + \frac{\Pi_1 \cdot (\theta_1 - \theta_2)}{(\theta_1 - Q)^2} \stackrel{!}{=} 0 \tag{4.14}$$

$$\frac{\partial \Pi_1(N\Pi_2)}{\partial Q} = \theta_1 - 2Q + \frac{N\Pi_2 \cdot (\theta_2 - \theta_1)}{(\theta_2 - Q)^2} \stackrel{!}{=} 0. \tag{4.15}$$

From the two first order conditions above, Π_1 will be expressed from equation (4.14) and analogue $N\Pi_2$ from equation (4.15), such that the explicit profits for the Market Sharing technology (MS) are given by:

$$\Pi_1^{MS} = \frac{(\theta_1 - Q)^2 \cdot (2Q - \theta_2)}{\theta_1 - \theta_2} \tag{4.16}$$

$$N\Pi_2^{MS} = \frac{(\theta_2 - Q)^2 \cdot (\theta_1 - 2Q)}{\theta_1 - \theta_2}. \tag{4.17}$$

To be sure that these are maximum values, the second derivations from equations (4.14) and (4.15) are computed in the appendix A.5. The calculations come to the result that the second order conditions (SOC)'s are fulfilled and the equations (4.16) and (4.17) are indeed maximized.

Now the explicit reaction functions will be determined from the FOC's (4.14) and (4.15) above. Therefore, the reaction function can be derived by means of the equations (4.16), (4.17), resulting from these FOC's, and the profit function (3.3). Recall that $Q = q_1 + Nq_2$. To determine the reaction function of q_1, first the profit function (3.3) is set in the explicit profit Π_1^{MS} from equation (4.16).

$$(\theta_1 - Q) \cdot q_1 = \frac{(\theta_1 - Q)^2 \cdot (2Q - \theta_2)}{\theta_1 - \theta_2} \tag{4.18}$$

In the second step, the equation of the total quantity $Q = q_1 + Nq_2$ is used and all variables are brought on one side.

$$(\theta_1 - q_1 - Nq_2) \cdot \left[\frac{(\theta_1 - q_1 - Nq_2) \cdot (2 \cdot (q_1 + Nq_2) - \theta_2)}{\theta_1 - \theta_2} - q_1 \right] = 0 \tag{4.19}$$

This leads to the cubic equation above, which means that there are three solutions for the quantity q_1. This three solutions are marked with the superscript (1) – (3). In this case it is not an exponent.

Hence, the first solution $q_1^{(1)}$ from equation (4.19) is the subsequent 'guessed zero' point, details see in the appendix A.6:

$$q_1^{(1)} = \theta_1 - Nq_2. \tag{4.20}$$

Now only the quadratic part of equation (4.19) in the square brackets has to be solved, which can easily be done with the small solution formula. The precise calculations of all zeros of the polynomial (4.19) can be found in the appendix A.6, which result in the subsequent roots:

$$q_1^{(2,3)} = \frac{\theta_1 + 2\theta_2}{4} \pm \frac{1}{4} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 8N \cdot q_2 \cdot (\theta_1 - \theta_2)} - Nq_2. \tag{4.21}$$

The last equation shows the second and third solution for the reaction function of q_1.

Then, the determination of the reaction function of q_2 follows. This works analogue to the derivation of the reaction function of q_1. Hence, the general profit function (3.3) is set in $N\Pi_2^{MS}$ from equation (4.17).

$$N \cdot (\theta_2 - Q) \cdot q_2 = \frac{(\theta_2 - Q)^2 \cdot (\theta_1 - 2Q)}{\theta_1 - \theta_2} \tag{4.22}$$

Additionally, the formula of $Q = q_1 + Nq_2$ is used, which leads to the following cubic equation:

$$(\theta_2 - q_1 - Nq_2) \cdot \left[\frac{(\theta_2 - q_1 + Nq_2) \cdot (\theta_1 - 2 \cdot (q_1 + Nq_2))}{\theta_1 - \theta_2} - Nq_2 \right] = 0. \tag{4.23}$$

As above, these three solutions will be characterized by the superscript (1) – (3). Again, this is not an exponent. Furthermore, the first solution $q_2^{(1)}$ is derived from equation (4.23), analogue to q_1:

$$q_2^{(1)} = \frac{1}{N} \cdot (\theta_2 - q_1). \tag{4.24}$$

In a next step, the part of the equation (4.23) in the square brackets, which is quadratic, should be solved with the small solution formula. In the appendix A.6, the detailed computation of this quadratic equation is shown. The result this calculation leads to the second and third solutions of the reaction function q_2, which are given by:

$$q_2^{(2,3)} = \frac{2\theta_1 + \theta_2}{4N} \pm \frac{1}{4N} \cdot \sqrt{(2\theta_1 - \theta_2)^2 - 8 \cdot q_1 \cdot (\theta_1 - \theta_2)} - \frac{q_1}{N}. \tag{4.25}$$

Finally, these are the explicit reaction functions for q_1 and q_2 for the MS technology. The explanation follows below.

$$q_1^{MS} = \frac{\theta_1 + 2\theta_2 - \sqrt{(\theta_1 - 2\theta_2)^2 + 8 \cdot N \cdot q_2 \cdot (\theta_1 - \theta_2)}}{4} - N \cdot q_2 \tag{4.26}$$

$$q_2^{MS} = \frac{2\theta_1 + \theta_2 - \sqrt{(2\theta_1 - \theta_2)^2 - 8 \cdot q_1 \cdot (\theta_1 - \theta_2)}}{4N} - \frac{q_1}{N} \tag{4.27}$$

The equations (4.19) and (4.23) demand three solutions for q_1 and q_2, but in this case there is only one for each q_i for $i = 1,2$ that make sense, namely the third solutions $q_1^{(3)}$ and $q_2^{(3)}$. The first solutions of the reaction functions $q_1^{(1)}$ and $q_2^{(1)}$ from equation (4.20) and (4.24) can be excluded because of a contradiction. Details and the whole calculation can be found in the appendix A.8. The proof for the elimination for $q_1^{(2)}$ and $q_2^{(2)}$ from equation (4.21) and (4.25) is discussed in the appendix A.7. It is an easy proof with a counterexample. In other words, it is proven that $q_1^{(2)}$ and $q_2^{(2)}$ are not convex in the relevant area. Note that the second solutions $q_1^{(2)}$ and $q_2^{(2)}$ are the solutions with the 'plus'.

To understand why the second solutions cannot be used, while the third solutions can, a small excursus to section 5 is made. In that section, bargaining solution methods are represented which are applied to the four collusion technologies. For these bargaining methods a convex "cooperative payoff region" is needed to receive a collusive outcome (cf. Binmore 1992, page 174) [2]. In other words, a concave function with a convex hull, respectively frontier is demanded (cf. Binmore 1992, page 174,193) [2]. Furthermore, whenever the expression "cooperative payoff region" is used, I refer to the relevant frontier, which is needed for the bargaining solutions, discussed in Binmore (1992) [2]. In contrast, Schmalensee (1987) [17] called the same collusive area the "profit possibility frontier". I will use both expressions as they describe the same thing. What does this mean to the second and third solutions of the reaction functions? First, note that the third solutions are the solutions with the 'minus' in front of the root. The third solution $q_1^{(3)}$ comes from equation (4.21) and $q_2^{(3)}$ from equation (4.25). On this subject, Schmalensee (1987) [17] referred to another paper by Bishop, stating that "Bishop (1960, p. 948) asserts the convexity of the contract curve in this case." (Bishop 1960, cited from Schmalensee 1987, page 356) [17]. 'Contract curve' in this context refers to the reaction function q_2 from equation (4.27), respectively the third solution $q_2^{(3)}$. I therefore accept that the third solutions $q_1^{(3)}$ and $q_2^{(3)}$ are valid.

In addition, it should not be forget to calculate the quantities for the MS technology. But if someone tries to determine the exact quantities, a problem is met. That is due to the fact that if the reaction function q_1^{MS} from equation (4.26), respectively q_2^{MS} from equation (4.27), is set in q_2^{MS}, respectively q_1^{MS}, an unsolvable equation is found, at least if exact solutions should be found. Details and the determination can be found in the appendix A.9. Since these equations are unsolvable in terms of exactness, this means there are no exact quantities for the MS technology. In this case, a numeric approach has to be applied.

4.3. Market Division

As the name says, in this technology the firms divide the market. That is a method to obtain certainly a convex cooperative payoff region. This can be a realistic possibility how firms could act in a cartel. For simplicity, Schmalensee (1987) [17] assumed that the customers are identical regarding all characteristics of importance. In addition, each potential purchaser is allotted to one firm in the market by the cartel. The low-cost firm gets assigned the particular section W^* of customers and accordingly firms of type 2 are allocated a fragment $1-W^*$.

Now the market is divided. It can be imagined that there are two different markets, but it should be kept in mind that actually this is not the case. Both types of firms are operating in the same market, although each in a particular isolated part of the market, which each firm gets assigned to. The firms cannot access the fraction of the market from the other type of firm. Thus, each type of firm is acting in its own section of the market as a monopoly. Therefore, it is obvious that all firms sell their product for their own monopoly price.

Furthermore, this segmentation of the market has to be taken into account and change the original demand function from equation (2.1). I name the inverse demand function for this technology for firm 1 $P_1(.)$, respectively for firms of type 2 $P_2(.)$. Hence, it can be said that there is one market for firm 1, where just the low-cost firm produces the quantity q_1. Consequently, the market demand function is $P_1(q_1)$. Analogue, there is another market for the N high-cost firms, in which only firms of type 2 are selling Nq_2 products. That means the inverse demand function changes to $P_2(N \cdot q_2)$.

As mentioned above, the market, respectively the customers, are divided into two fractions, where the first fragment is W^* and the second is $1-W^*$. These two sections are summing up to $1 = W^* + (1-W^*)$. Here 1 characterizes the 'whole market'. This implies that each firm cannot use the whole potential of the market, but only their fraction of the market. The total quantity Q is the sum of all quantities produced in the 'whole market', which corresponds to 100%, respectively 1. Firm 1, respectively firms of type 2, cannot access the 'whole market'. Actually, since the firms run their business in the same market and only can access their allocated part of the market and not the other, they are operating in their fraction only. Instead of the 'whole market' the low-cost firm is performing in a fraction W^* of 1, which is $\frac{1}{W^*}$. This works analogue for firms of type 2, which can only enter their fragment of the market, which is $\frac{1}{1-W^*}$. Thus, the inverse demand function of firm 1 in the MS technology is given by $P(\frac{q_1}{W^*})$ and firms of type 2's inverse demand function is $P(\frac{N \cdot q_2}{1-W^*})$.

$$P_1 = P\left(\frac{q_1}{W^*}\right) = 1 - \frac{q_1}{W^*} \tag{4.28}$$

$$P_2 = P\left(\frac{N \cdot q_2}{1-W^*}\right) = 1 - \frac{N \cdot q_2}{1-W^*} \tag{4.29}$$

Accordingly, the profit functions can be written as:

$$\Pi_1\left(\frac{q_1}{W^*}\right) = \left(\theta_1 - \frac{q_1}{W^*}\right) \cdot q_1 \tag{4.30}$$

$$\Pi_2\left(\frac{N \cdot q_2}{1-W^*}\right) = \left(\theta_2 - \frac{N \cdot q_2}{1-W^*}\right) \cdot q_2. \tag{4.31}$$

Taking a look on the equations (4.30) and (4.31), it is obvious that the profit of each firm only depend on the quantity of their own product and not on that of the other. Again, the aim is to maximize the profit. Therefore, maximization yields:

$$\max_{0 \leq q_1 < \infty} \Pi_1\left(\frac{q_1}{W^*}\right) = \left(\theta_1 - \frac{q_1}{W^*}\right) \cdot q_1 \tag{4.32}$$

$$\max_{0 \leq q_2 < \infty} \Pi_2\left(\frac{N \cdot q_2}{1-W^*}\right) = \left(\theta_2 - \frac{N \cdot q_2}{1-W^*}\right) \cdot q_2. \tag{4.33}$$

From the equations of the maximization above, it is evident to obtain the following FOC's:

$$\frac{\partial \Pi_1}{\partial q_1} = \theta_1 - \frac{2}{W^*} \cdot q_1 \stackrel{!}{=} 0 \tag{4.34}$$

$$\frac{\partial \Pi_2}{\partial q_2} = \theta_2 - \frac{2N}{1-W^*} \cdot q_2 \stackrel{!}{=} 0. \tag{4.35}$$

Immediately, the explicit values for the quantities are derived without the need of solving the reaction functions:

$$q_1 = \frac{W^*}{2} \cdot \theta_1 \tag{4.36}$$

$$q_2 = \frac{1-W^*}{2N} \cdot \theta_2, \quad \text{respectively} \quad Nq_2 = \frac{1-W^*}{2} \cdot \theta_2. \tag{4.37}$$

Now also the reaction function of firms of type 2 firms is determined, since it is needed for the representation of the graph. To get a connection between the both equations above (4.36) and (4.37), the fragment W^* is expressed from equation (4.36):

$$W^* = \frac{2}{\theta_1} \cdot q_1. \tag{4.38}$$

In setting the expression for W^* above in the first equation of (4.37), the explicit form of the reaction function looks like:

$$q_2^{MD} = \frac{\theta_2}{2N} \cdot \left(1 - \frac{2 \cdot q_1}{\theta_1}\right). \tag{4.39}$$

Therefore, the explicit profits can be determined from the quantities from equations (4.36) and (4.37).

$$\Pi_1 = \frac{W^*}{4} \cdot \theta_1^2 \tag{4.40}$$

$$\Pi_2 = \frac{1 - W^*}{4N} \cdot \theta_2^2, \quad \text{respectively} \quad N\Pi_2 = \frac{1 - W^*}{4} \cdot \theta_2^2 \tag{4.41}$$

Now the profit possibility frontier is determined to get a relationship between the profits of both types of firms. Hence, the fraction W^* is expressed analogous to the case of the quantity:

$$W^* = \frac{4}{\theta_1^2} \cdot \Pi_1. \tag{4.42}$$

Accordingly, the explicit profit possibility frontier is given by:

$$\Pi_2^{MD} = \frac{\theta_2^2}{4N} \cdot \left(1 - \frac{4 \cdot \Pi_1}{\theta_1^2}\right). \tag{4.43}$$

As mentioned above, the profit of each firm only depends on the quantity of their own product. Consequently, both types of firms request the price of a monopolist, which means that they are not charging the same price for the same product in this technology. In the following equation, P_1 characterizes the price of the product of firm 1 and P_2 is the price, which the high-cost firms are charging.

$$P_1 = 1 - \frac{q_1}{W^*} = 1 - \frac{\frac{W^*}{2} \cdot \theta_1}{W^*} = 1 - \frac{\theta_1}{2} \tag{4.44}$$

$$P_2 = 1 - \frac{Nq_2}{1 - W^*} = 1 - \frac{\frac{1-W^*}{2} \cdot \theta_2}{1 - W^*} = 1 - \frac{\theta_2}{2} \tag{4.45}$$

From equations (4.44) and (4.45), it can immediately be seen that firm 1 and firms of type 2 are not requesting the same price for the same kind of product. Recall that it holds $\theta_2 < \theta_1$ from the first inequality from (3.2) and $\theta_i \in [0,1]$. It follows that $1 - \frac{\theta_2}{2} > 1 - \frac{\theta_1}{2}$. Hence, the high-cost firms charge a higher price for the same product compared to the low-cost firm. The difference in price is simply traceable, because firms of type 2 have higher costs than firm 1.

Since there is a price difference for the same product, while the costs of the firms vary, a mechanism is needed that eliminates arbitrage. Schmalensee (1987) [17] did not discuss what kind of mechanism is needed to preclude arbitrage or any other details about the mechanism. He just noted that there should be applied some mechanism, which eliminates arbitrage at reasonable cost, because of the price difference (cf. Schmalensee 1987, page 356) [17].

4.4. Proportional Reduction

In comparison to the more complex Market Sharing technology, the Proportional Reduction (PR) is a more elementary method to impose particular quotas for the production volume. In other words, it is a simplification of the Market Sharing technology. Here market shares are preserved at their actual Cournot shares, see section 3 on page 10, and then the quantities of firm 1 as of firms of type 2 are decreased proportional. In fact, the shares are hold tight at competitive values. That means the market share for firm 1 in the PR technology is retained at the Cournot share of firm 1 which is $S = \frac{q_1^C}{q_1^C + Nq_2^C}$. For detailed values see equation (3.7). Accordingly, the share of the high-cost firms is preserved at their Cournot share which is $S_2 = \frac{Nq_2^C}{q_1^C + Nq_2^C}$. For this reason, the market share is hold tight at the ratio:

$$S = \frac{q_1^C}{Q^C} : \frac{Nq_2^C}{Q^C} = S_2 \quad \Rightarrow \quad q_1^C : Nq_2^C. \tag{4.46}$$

Obviously, this describes a linear function from the zero point to the Cournot quantity $\left(q_1^C,\, Nq_2^C\right)$. Therefore, all values are valid, for which the market shares have the same relation as the quantities in Cournot competition and which lie between the zero point and the Cournot quantity. As a consequence, this are all points, which fulfill $\left(\lambda \cdot q_1^C,\, \lambda \cdot Nq_2^C\right)$ for $\lambda \in [0,\ 1]$. Recall the quantities from Cournot competition from equation (3.12) and (3.15). Thus, the possible quantities in the PR technology can be written as:

$$\left(q_1^{PR},\, Nq_2^{PR}\right) = \left(\lambda \cdot q_1^C,\, \lambda \cdot Nq_2^C\right) \quad \text{for } \lambda \in [0,\ 1] \tag{4.47}$$

$$\left(q_1^{PR},\, Nq_2^{PR}\right) = \frac{1}{2+N} \cdot \Big(\lambda \cdot ((1+N)\,\theta_1 - N\theta_2),\, \lambda \cdot N\,(2\theta_2 - \theta_1)\Big). \tag{4.48}$$

Hence, the parameter λ can be seen as the factor, by which the Cournot quantities are reduced. It shows the actual percentage of reduction from the Cournot point. More precisely, λ can be iterated over the interval $[0,1]$ to find the optimal quantity, which maximizes the profit.

Shcherbakov and Wakamori (2015) [18] were also concerned with the PR technology, but they had another approach and used an inverse discount factor, which describes the percentage of the reduction. A small example will be presented, such that it is easier to understand. If their inverse factor would be $1.25 = \frac{1}{1-0.2}$, then this is a reduction by 20% in comparison to the Cournot case (cf. Shcherbakov and Wakamori 2015, page 6) [18]. That would mean the inverse of the inverse factor is 0.9, which is λ. Therefore, this would correspond to 80% of the Cournot quantity.

To get back to the point, remember the ratio of the market shares above and the following holds:

$$q_1^{PR} : N \cdot q_2^{PR} = q_1^C : N \cdot q_2^C. \tag{4.49}$$

Consequently, the explicit reaction functions are achieved in transforming the ratio from equation (4.49) and expressing q_1^{PR}:

$$q_1^{PR} = \frac{q_1^C}{q_2^C} \cdot q_2^{PR} = \frac{(1+N)\,\theta_1 - N\theta_2}{2\theta_2 - \theta_1} \cdot q_1^{PR}. \tag{4.50}$$

Similarly, the reaction function of the high-cost firms is determined.

$$q_2^{PR} = \frac{q_2^C}{q_1^C} \cdot q_1^{PR} = \frac{2\theta_2 - \theta_1}{(1+N)\,\theta_1 - N\theta_2} \cdot q_1^{PR} \tag{4.51}$$

Additionally, in the last steps of the two equations above the Cournot quantities are inserted in the reaction function. Note that for the plot of the reaction function the interval has to be changed from $[0,1]$ to $[0, q_1^C]$, because of the transformation above.

Accordingly, the profits are given by:

$$\Pi_1^{PR} = (\theta_1 - q_1^{PR} - N \cdot q_2^{PR}) \cdot q_1^{PR} \tag{4.52}$$

$$N\Pi_2^{PR} = (\theta_2 - q_1^{PR} - N \cdot q_2^{PR}) \cdot Nq_2^{PR}. \tag{4.53}$$

If the values for the Cournot quantities are set in the profit functions above, the following is obtained:

$$\Pi_1^{PR} = \left(\theta_1 - \frac{\lambda}{2+N}\left((1+N)\,\theta_1 - N\theta_2\right) - \frac{N\lambda}{2+N}\left(2\theta_2 - \theta_1\right)\right) \cdot \frac{\lambda}{2+N}\left((1+N)\,\theta_1 - N\theta_2\right) \quad (4.54)$$

$$= \frac{\lambda}{(2+N)^2} \cdot \left((1+N)\,\theta_1 - N\theta_2\right) \cdot \left(\theta_1\,(2+N) - \lambda\,(\theta_1 + N\theta_2)\right) \quad (4.55)$$

$$N\Pi_2^{PR} = \left(\theta_2 - \frac{\lambda}{2+N}\left((1+N)\,\theta_1 - N\theta_2\right) - \frac{N\lambda}{2+N}\left(2\theta_2 - \theta_1\right)\right) \cdot \frac{N\lambda}{2+N}\left(2\theta_2 - \theta_1\right) \quad (4.56)$$

$$= \frac{N\lambda}{(2+N)^2} \cdot \left(2\theta_2 - \theta_1\right) \cdot \left(\theta_2\,(2+N) - \lambda \cdot (\theta_1 + N\theta_2)\right). \quad (4.57)$$

4.5. Comparison of the collusion technologies

In this section, the differences and similarities between the four collusion technologies, which were introduced in the previous four sections 4.1 – 4.4, will be discussed on the basis of the subsequent figures. The differences in the graphs regarding to the different values of R are considered. These show the quantity and profit curves of all technologies, accordingly in the SP case a point. The subsequent figures in this section are plotted in MATLAB with 10.000 points in the relevant interval for three different values for R and $N = 1$. The values of R can be found in detail in table A.1 in the appendix A.1. These are the same three values for R, which Schmalensee (1987) [17] employed. Therefore, they are applied in this study, because the values are well spread over the interval of R, which is $[1,2]$. Recall $R = \frac{\theta_1}{\theta_2}$ indicates the measure of relative cost difference of both types of firms. The Cournot point $\left(q_1^C, q_2^C\right)$ used to compare the results with. It is the point $\left(q_1^C, q_2^C\right)$ and not $\left(q_1^C, N \cdot q_2^C\right)$, since firm 1 is plotted with one firm of type 2 in the following graphs in section 4.5.1. Sometimes, also the case of symmetric costs is taken for comparison.

4.5.1. Quantity curves of the collusion technologies

First, the collusion technologies are compared according to the quantity space and afterwards to the profit space. Below, the quantity curves of the four different technologies can be seen for different values of R. The three figures below show that the quantities are reduced in comparison to the Cournot case. This is done to achieve higher profits. The point of the Cournot quantity is labeled with a light blue dot in all three figures below 4.1 and 4.2a – 4.2b.

To be precise, all equations are specified, which are used to represent the curves in figures 4.1 and 4.2. In the Side Payments technology, the curve is an exact point from equation (4.3), which is labeled with a dark blue dot. In the MS technology, the reaction function of q_2 from equation (4.27) is plotted with a green marked line. Furthermore, the reaction curve for q_2 from equation (4.39) labeled as red line characterizes the Market Division (MD) technology. Lastly, the equation (4.51) of the reaction function of q_2 marked with a yellow line shows its graphic representation for the PR technology. For more comprehension, all used equations in the plots are listed below:

$$\left(q_1^{SP}, q_2^{SP}\right) = \left(\frac{\theta_1}{2}, 0\right) \tag{4.58}$$

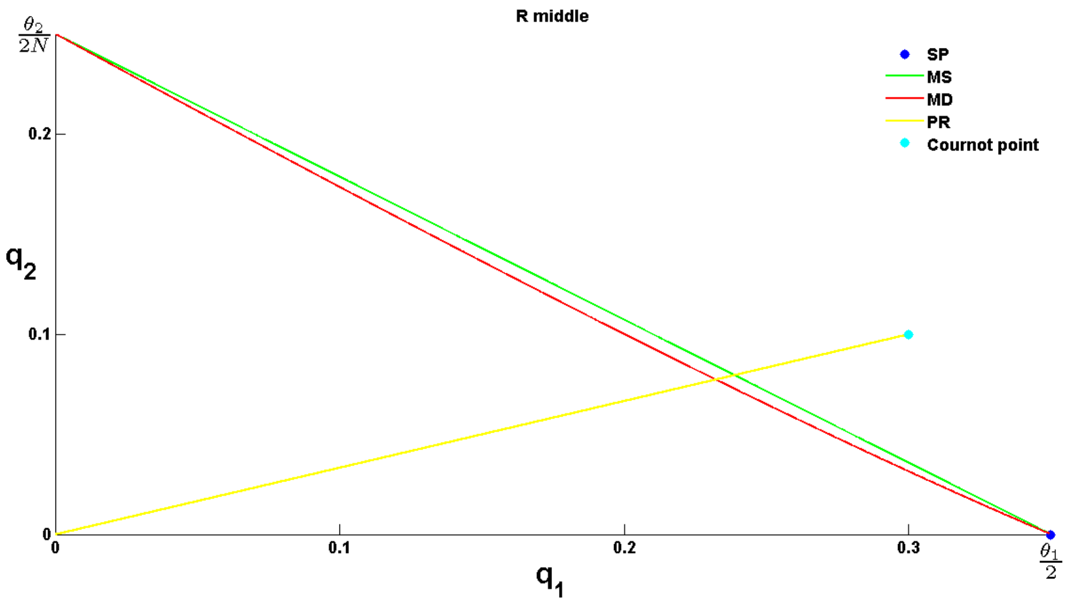
$$q_2^{MS} = \frac{2\theta_1 + \theta_2 - \sqrt{(2\theta_1 - \theta_2)^2 - 8 \cdot q_1 \cdot (\theta_1 - \theta_2)}}{4N} - \frac{q_1}{N} \tag{4.59}$$

$$q_2^{MD} = \frac{\theta_2}{2N} \cdot \left(1 - \frac{2 \cdot q_1}{\theta_1}\right) \tag{4.60}$$

$$q_2^{PR} = \frac{2\theta_2 - \theta_1}{(1+N)\,\theta_1 - N\theta_2} \cdot q_1^{PR}. \tag{4.61}$$

In the PR technology q_1^{PR} iterates over the interval $[0, q_1^C]$ with 10.000 points, while in the other two technologies, namely MS and MD, the interval $[0, q_1^{SP}]$ is used with the same number of points. That is because $q_1^{SP} = \frac{\theta_1}{2}$ is the maximum value for the quantities of all four technologies. Note that the SP technology is a point and therefore is plotted as a single point.

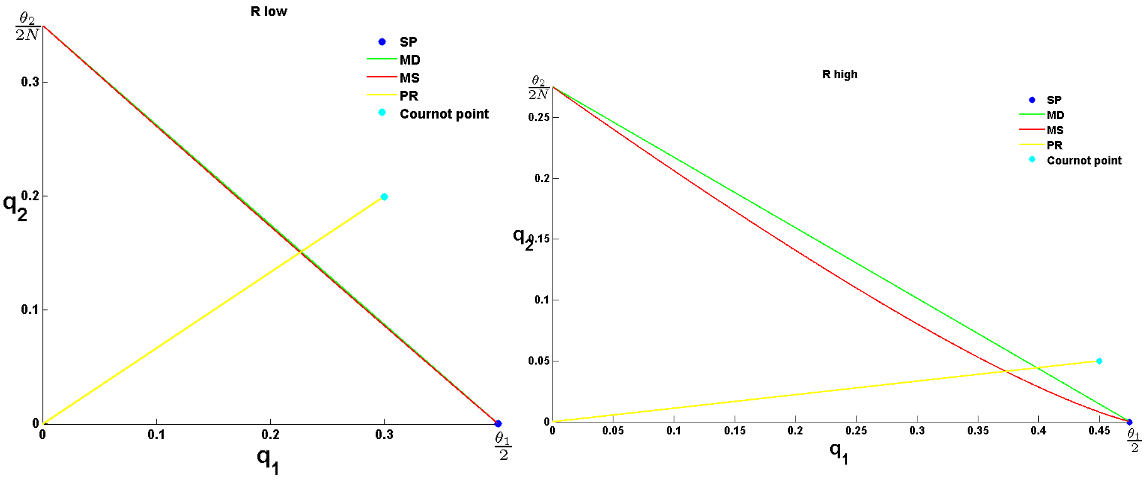
Figure 4.1: Quantity curves of all collusion technologies for R_{middle} and $N = 1$



Notes: Figure 4.1 is plotted in MATLAB for $N = 1$. The cost values for the parameter $R_{\text{middle}} = 1.4$ are the following: $\theta_1 = 0.7$ and $\theta_2 = 0.5$.

Figure 4.2: Quantity curves of all collusion technologies for R_{low} and R_{high} with $N = 1$

(a) R low (b) R high



Notes: Figures 4.2a and 4.2b are plotted in MATLAB for the parameter $N = 1$ and $R_{\text{low}} = \frac{8}{7}$ with the following values for the cost parameters $\theta_1 = 0.8$ and $\theta_2 = 0.7$, respectively for $R_{\text{high}} = 1.\overline{72}$ with $\theta_1 = 0.95$ and $\theta_2 = 0.55$.

The SP curve is only a point, as mentioned in section 4.1, while the curves of the MD and the PR technology is linear. The Market Sharing technology has a more complex quantity curve in comparison to the other technologies. It has a strictly convex curve as stated in section 4.2 on page 18. The PR curve is a linear curve between the origin and the Cournot point.

The Side Payments technology is the only technology of all four which can use the whole potential of the market. The other three fail to do so. But side payments are often forbidden by law or other limitations. Furthermore, it is not completely realistic that just one firm produces and the other firm is willing to shut its own production down or even exits the market. Hence, this technology is used more as a standard of comparison than as a real alternative.

The higher the value for R, that means the higher the difference in costs, the more the MD and MS curves differ. In addition, the Cournot quantity of firms of types 2 decreases. At low values of R, the quantity curve of the Market Sharing technology approximates the quantity curve of the Market Division technology.

4.5.2. Profit curves of the collusion technologies

Now the profit curves of all four technologies are discussed. Figures 4.3 and 4.4 show, that if the profit curves lie above the black dotted line in the figures, both vertical and horizontal, the technologies are producing more profit for both firms than in the Cournot case. This holds for all technologies. It does not have to be much more. The goal is to obtain more profit from collusion than from Cournot Competition. The solutions of all technologies determined by the solution methods, which are introduced in the next section 5, are all lying above this dashed lines. In the figures 4.3 and 4.4 this area above the dotted lines is called collusive profit area higher than Cournot in the legend.

Furthermore, the exact equations that describe the curves in the figures 4.3, 4.4a and 4.4b are illustrated. The following equation (4.5) specifies the profit possibility frontier for the Side Payments technology, which is labeled with a dark blue line in the three figures below. While in the MS technology the profit of both types of firms are represented by means of the equation (4.16) for firm 1 and equation (4.17) for the high-cost firms. It is marked with a red line in the following three graphs. The profit of Π_1^{MS} is plotted as x-coordinate, while Π_2^{MS} as y-coordinate. The green line in the three figures below is the curve for the profit possibility frontier of the Market Division technology. The equation (4.43) describes the profit curve for the MD technology. Finally the Proportional Reduction curve for the profit, which is labeled in yellow, is specified by these equations (4.52) for firm 1 and (4.53) for firms of type 2. For the PR technology the profit of the low-cost firm is plotted as x-coordinate and that of firms of type 2 as y-coordinate.

$$\Pi_2^{SP} = \frac{1}{N}\left(\frac{\theta_1^2}{4} - \Pi_1^{SP}\right) \tag{4.62}$$

$$\Pi_1^{MS} = \frac{(\theta_1 - Q)^2 \cdot (2Q - \theta_2)}{\theta_1 - \theta_2} \quad \text{and} \tag{4.63}$$

$$N\Pi_2^{MS} = \frac{(\theta_2 - Q)^2 \cdot (\theta_1 - 2Q)}{\theta_1 - \theta_2} \tag{4.64}$$

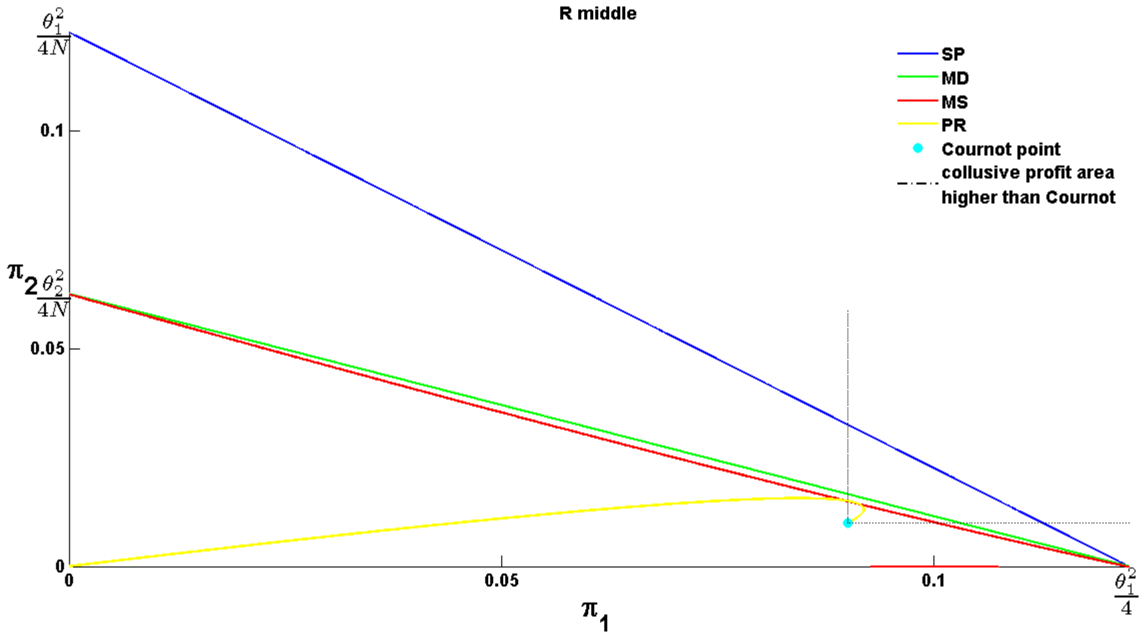
$$\Pi_2^{MD} = \frac{\theta_2^2}{4N} \cdot \left(1 - \frac{4 \cdot \Pi_1}{\theta_1^2}\right) \tag{4.65}$$

$$\Pi_1^{PR} = (\theta_1 - q_1^{PR} - N \cdot q_2^{PR}) \cdot q_1^{PR} \quad \text{and} \tag{4.66}$$

$$\Pi_2^{PR} = (\theta_2 - q_1^{PR} - N \cdot q_2^{PR}) \cdot N q_2^{PR}. \tag{4.67}$$

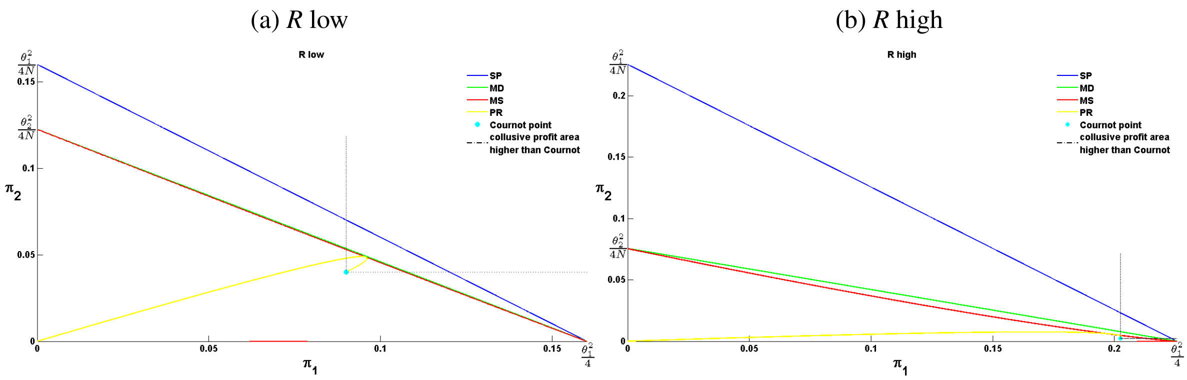
In the formulas of the profit of the MS technology the explicit reaction function for q_1^{MS} from equation (4.26) and for q_2^{MS} from equation (4.27) are used. Similarly, the reaction functions q_1^{PR} from equations (4.50) and q_2^{PR} from equation (4.51) are set in the corresponding expressions of the profit in the PR technology.

Figure 4.3: Profit curves of all collusion technologies for R_{middle}



Notes: Figure 4.3 is plotted in MATLAB for $N = 1$. The values for the parameter $R_{\text{middle}} = 1.4$ are the following: $\theta_1 = 0.7$ and $\theta_2 = 0.5$.

Figure 4.4: Profit curves of all collusion technologies for R_{low} and R_{high}



Notes: Figures 4.4a and 4.4b are plotted in MATLAB for the parameter $N = 1$ and $R_{\text{low}} = \frac{8}{7}$ with the following values for the cost parameters $\theta_1 = 0.8$ and $\theta_2 = 0.7$. Respectively for $R_{\text{high}} = 1.\overline{72}$ with $\theta_1 = 0.95$ and $\theta_2 = 0.55$.

Both figures 4.3 and 4.4 indicate that both the SP and the MD technology have a linear profit curve. Furthermore, the Side Payments technology is the most efficient technology of all four and ensures the highest profit, but in many situations forbidden or not realistic. That is why one takes it more as a standard of comparison than as a realistic alternative. If no compensation payments are allowed, profits are restricted compared to the SP technology, since the whole potential of the market cannot be used.

About the quality in terms of efficiency and profitability of the other technologies, the following statements can be made. The MD method is more efficient than the MS method. That is because all companies could alter the price of the product to raise their profits. This holds only provided that each firm gets equal to its market share a share of consumers at a certain point. Initializing the movement from a point on the profit curve of the MS technology, leads to a certain point on the MD curve, when the market share is as described above and the firms change their price (cf. Schmalensee 1987, page 356) [17].

In the graphs, it can be recognized that the PR technology can be essentially less efficient compared to the other technologies. That is because its curve lies both horizontal and vertical close to the edge of the area with the black dots, where the profits are lower. In the figures it is shown that the PR profit curve can achieve a point on the MS curve, but nevertheless total profits are lower, because the companies in the market request one price for the same product. The differences between these technologies grow with the cost differences, respectively the parameter R, as figures 4.4a and 4.4b indicates. If one would have symmetric costs the SP, MS and MD technology would coincide.

5. Bargaining solution methods

This section will discuss how firms could behave when they are cooperating. In other words, on which explicit profit, in particular on which point from the cooperative payoff region, the firms will agree upon. This agreement is achieved by bargaining between the two types of firms. To determine this, Schmalensee (1987) [17] applied four solution methods for cooperating firms from the axiomatic bargaining theory. These procedures find the actual optimal collusive solution for each of the four different collusion technologies from section 4. This means that these concepts are utilized to every profit possibility frontier of the collusion technologies. All given profit possibility frontiers are explicit and the result of an analytical calculation. In addition, these collusive solutions provide a profit for both types of firms which is no less than a point on which they have agreed in advance if the bargaining process does not succeed. Schmalensee (1987) [17] called this point "the status quo point", but it is also known as "disagreement point" by Binmore (1992) [2]. If the firms cannot agree upon a collusive profit or the collusive profit is smaller than the disagreement point, then they gain the disagreement point. I will use the notation "disagreement point". Following Schmalensee (1987) [17], I have taken the Cournot profit as the disagreement point, for the fact that the collusive profit should be compared to the payoff obtained in Cournot competition. Schmalensee's objective is to find a higher outcome from the collusion technologies solved by the bargaining solution methods than from the Cournot case. These particular methods are introduced below. This means that all of the four solution methods produce a profit which lies above the dashed lines in the figures 4.3 and 4.4 from section 4.5.2.

The following statements are made from Roth (1979) [13]. To apply these bargaining solution methods, the individual rationality of all players has to be assumed, see Roth (1979) [13]. The four axioms of Nash have to be fulfilled, such that the Nash solution method can be used. These are the axioms: Independence of equivalent utility representations, Symmetry, Independence of irrelevant alternatives and Pareto-Optimality, see Roth (1979) [13]. For the Kalai-Smorodinsky (K-S) solution only the first, second and fourth axioms must hold, but additionally individually monotonicity, see Roth (1979) [13]. By the process of determining the four collusion technologies Schmalensee (1987) [17] took care of them and the axioms are satisfied.

The first three solution methods can be applied to all four collusion technologies, while the fourth, namely $W^* = S$ solution, can only be used for the MD technology. That holds, since the $W^* = S$ solution is a special case of the Market Division technology. More details follow below. Furthermore, note that not all collusion technologies can be solved analytically for all solution methods. There are explicit analytical solutions for the SP and MD technologies under all feasible solution methods. While the Proportional Reduction technology is explicitly and analytically solvable for the K-S and the Equal Gain solution (EG), it is only numerically solvable for the Nash method. Note that for $N = 1$ the EG solution cannot be applied to the PR technology. However, the MS technology can only be solved numerical for all three solution methods.

Thomson (1994) [12] described and compared the following bargaining methods. He stated that the solution of Nash, that of Kalai-Smorodinsky and the EG solution are the solution methods from bargaining theory, which are the most important about collusion at his times (cf. Thomson 1994, page 1242) [12].

5.1. Nash solution

The first bargaining method, which is introduced by Schmalensee (1987) [17] is the known Nash bargaining solution. Recall the character Π_i^C for $i = 1,2$ denotes the profit of the particular firm in Cournot equilibrium, which holds as disagreement point. Meanwhile Π_i^* for $i = 1,2$ identifies the profit from cooperation from one of the four different technologies in each case. The Nash solution (1950) [11] reads as follows:

$$\max \left(\Pi_1^* - \Pi_1^C\right) \cdot \left(\Pi_2^* - \Pi_2^C\right)^N. \tag{5.1}$$

This method compares the profit of both types of firms using each collusion technology to the Cournot profit. That is exactly the gain in profit the firms get in using the particular collusion technology instead of Cournot competition.

5.2. Kalai-Smorodinsky solution

In this solution method, the profits of all firms are again collated against the Cournot case. Here the difference is that actually the term which is maximized in Nash's solution is put proportional to the highest but constrained profit of each firm. The profit Π_i^m for $i = 1,2$ describes the profit a firm can obtain at its maximum, under the constraint that the other type of firm does not gain less than the corresponding Cournot profit. The solution derived from Kalai and Smorodinsky (1975) [7] is given by:

$$\frac{\Pi_1^* - \Pi_1^C}{\Pi_1^m - \Pi_1^C} = \frac{\Pi_2^* - \Pi_2^C}{\Pi_2^m - \Pi_2^C}. \tag{5.2}$$

At the certain point in the cooperative payoff region, which is obtained in applying this solution method, the profits of both types of firms have the same portion of their highest outcome (cf. Schmalensee 1987, page 358) [17].

5.3. Equal Gain solution

As well in the third solution method, the collusive payoff is compared to the profit of Cournot competition. In addition, the Equal Gain method is searching for the maximal solution, where the absolute profits of both types of firms compared to the Cournot profits are identical. Note that this must not hold for the individual collusive profit Π_i^* of each firm, respectively the relative payoff. If it is from a point of view of social and economic well-being, the method of equal profits is willingly used (cf. Thomson 1994, page 1243) [12]. Thus, Roth (1979) [13] introduced the following concept:

$$\Pi_1^* - \Pi_1^C = \Pi_2^* - \Pi_2^C. \tag{5.3}$$

Since firms of type 2 have higher costs, their profit is lower than the profit of the low-cost firm, especially in Cournot competition. That is why relatively considered the high-cost firms will get more additionally profit than firm 1.

5.4. $W^* = S$ solution

This last solution method is only feasible for the Market Division technology, since it focuses on an explicit solution of the distribution of the costumer. Remember that firm 1 gets assigned the fraction W^* of customers, while the section $1 - W^*$ is allocated to firms of type 2. This solution concept is simple, the part W^* should now be identical to the market share of the low-cost firm in Cournot equilibrium, namely S from equation (3.7). It is as well a solution that is almost certain considered as "fair" for both types of firms, since each company gets assigned its original market share from Cournot competition, see Sloman (2006) [19]. Hence, each firm of type 2 analogue obtain their share, namely $\frac{1-W^*}{N} = \frac{1-S}{N}$.

This point $W^* = S$ is a so-called "natural focal point", also known as "Schelling point", Schelling (1980) [1960] [15]. That special point is characterized by an "obvious" behavior of the players, respectively firms, if they cannot speak to each other. Obvious in terms of "so obvious that each be sure that the other is sure that it is "obvious" to both of them." (Schelling 1980 [1960], page 54) [15].

This procedure is relatively simple to solve. The value of the market share S is set equal to W^* in the equations (4.40) and (4.41) of both firm's profit. Then the result is simply determined. That is of course a numerical solution.

5.5. Comparison of the collusive solution methods

In this section, the collusive solution methods will be compared to each other. In addition, the performance of each of the four bargaining concepts is discussed as well with regard to the four collusion technologies.

The solution method of Nash and Kalai-Smorodinsky lead to exactly the same results on condition that the cooperative payoff region is linear (cf. Schmalensee 1987, page 358) [17]. As mentioned in section 4.5.2, this holds for the Side Payments and the Market Division technology. Besides in figures 4.3 and 4.4 can be recognized that both of the technologies have a linear profit curve. Furthermore, the solutions of the first three solution methods, namely Nash, K-S and EG, are equivalent, when the Side Payments technology is applied. A direct conclusion out of this is that the profit of both types of firms is from the same amount of dollar, when looking at the absolute profit, see Schmalensee (1987) [17]. That must not hold for the relative profit, see section 5.3.

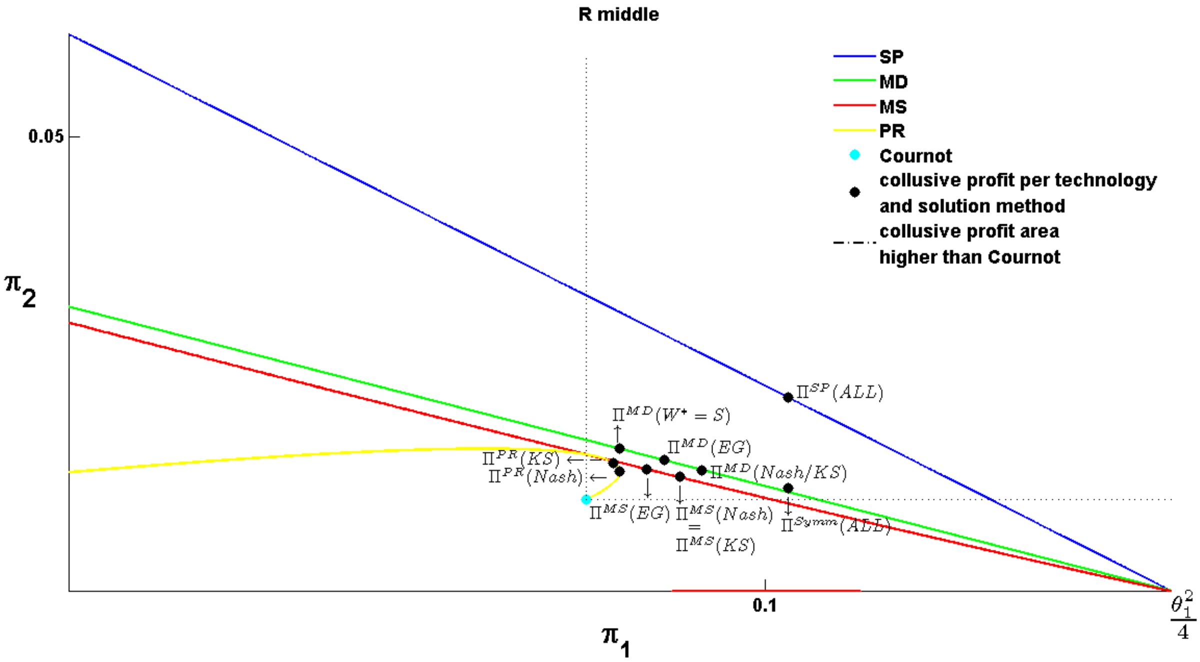
By means of figure 5.1, the performance of the individual solution methods and the corresponding collusion technologies can be well recognized. Based on figure 5.1, it is simple to compare the different bargaining concepts. In figure 5.1, I plotted the profit curves of all four collusion technology for R_{middle} and $N = 1$ and accordingly all solutions of the particular bargaining methods, which can be applied to the specific technology. It is exactly the same graph as in figure 4.3 with the difference that I zoomed out the area above the dashed lines and added the values of every solution. As mentioned above, this is the area, where both types of firms earn more than in Cournot competition.

It can be seen that the Side Payments technology provides the highest profit for both types of firms. No other collusion technology introduced in section 4 can reach its amount of profit. That is because the SP technology can use the whole potential of the market.

In the graph above, it looks like the Nash and the K-S solution method come to the same result in the MS technology. That only holds coincidentally for the parameter R_{low} and R_{middle} for $N = 1$ for the first decimal places. Otherwise these two bargaining concepts take different values. Note that the Nash and the K-S solution do not coincide in the Market Sharing technology.

To compare the profitability of the solution methods of the MS and MD technology the following statements can be made. Due to the fact that the Nash and the K-S solution are located to the right of the EG and the $W^* = S$ solution, with the former solution firm 1 gets

Figure 5.1: Profit curves of all collusion technologies with their solution methods for R_{middle} and $N = 1$



Notes: Figure 5.1 is plotted in MATLAB with the following values for the parameter $R_{\text{middle}} = 1.4$ and $N = 1$. The values of the cost parameters are $\theta_1 = 0.7$ and $\theta_2 = 0.5$.

more profit in absolute terms than firms of type 2. On the other hand, the high-cost firms obtain a higher absolute payoff, if the EG or in the MD technology the $W^* = S$ solution is applied. In the Market Division technology additionally holds that the EG method lies in between the Nash and K-S solution to the right and the $W^* = S$ solution to the left. Therefore, it can be said in general that if the outcome lies more on the left side the high-cost firms get more absolute profit than the low-cost firm. Otherwise, if the solution is suited on the right side firm 1 obtains the higher payoff in absolute terms than firms of type 2.

For the parameter $N = 1$, there cannot be found a solution, if the EG method is applied to the Proportional Reduction technology, see Schmalensee (1987) [17]. For $N > 1$, at least until $N = 5$, the EG solutions can be determined. Especially, if N takes the value 2 the outcome of the low-cost firm will be maximized in the cooperative payoff region of the PR technology.

6. Results of Schmalensee

In this section, the results obtained by Schmalensee's model are presented. In particular, all four solution methods from section 5 are applied to each feasible collusion technology from section 4. Since the Market Sharing technology is not analytically solvable, Schmalensee (1987) [17] chose a completely numeric approach to apply the solution methods to the collusion technologies. However, the solutions of the Nash, K-S and EG methods can be solved in an analytical way for the SP, MD and PR technology with exception of PR the technology, which has to be solved with a numeric approach for the method of Nash. The $W^* = S$ solution can also be derived explicitly, obviously only for the Market division technology.

Note that the profits of each firm are very low, namely between 0 and 1, due to the chosen inverse demand function from equation (2.1) and cost function from equation (3.1). The profit has to be so small, since $\theta_i \in [0,1]$ for $i = 1,2$. That implies that $Q < \theta_i$, which means the profit $\Pi_i = (\theta_i - Q) \cdot q_i$ for $i = 1,2$ cannot take a higher value than 1. Therefore, it should be noticed that the maximum value 1 is unlikely to be achieved. In addition, the profit will not get negative, since then the firm will make losses. Because of the low outcomes, in both tables 6.1 and 6.2 the values of the profit increase are listed in percentage. In fact, these values indicate how much both types of firms gain additionally to Cournot competition in percentage. Remember this simple market demand and cost function is selected to simplify the calculations, in particular, since the MS technology involves more complicated computations.

As mentioned in section 3, the market is described by two parameters R and N. Therefore, Schmalensee (1987) [17] calculated the results for two cases. He varied one parameter while the other one is hold constant. Now the numerical results of the four different bargaining solution methods applied to the collusion technologies are shown in the next two subsections 6.1 and 6.2. A part of the results is verified in simply recalculating, namely in applying the feasible solution methods to the explicit profit possibility frontiers.

6.1. Results with variation in R

Initially, Schmalensee (1987) [17] changed the values of R, while the value $N = 1$ is held constant. In the first case, the value $R_{\text{low}} = 1.143$ is taken which means a relative small cost difference between the firms. Then it follows $R_{\text{middle}} = 1.4$ which is nearly in the middle and $R_{\text{high}} = 1.727$ indicates a relatively high cost difference. The calculations of the values of R can be found in the appendix A.1.

In table 6.1 below, the results of Schmalensee (1987) [17] are presented (cf. Schmalensee 1987, page 360) [17]. Firstly, the difference between the symmetric and the asymmetric costs

Table 6.1: Results for $N = 1$ and with variation of R

Value of R	Technology	Solution	Profit on top of Cournot in % Firm 1	Profit on top of Cournot in % Firms of type 2	Profit Sacrifice
1.000 ($S = 0.50$)	Symmetric	All	12.50 %	12.50 %	0 %
1.143	SP	All	16.67 %	37.50 %	0 %
($S = 0.60$)	MD	Nash/K-S	9.86 %	16.99 %	8.95 %
		EG	8.55 %	19.29 %	9.13 %
		$W^* = S$	6.67 %	22.50 %	9.37 %
	MS	Nash	9.50 %	18.69 %	9.33 %
		K-S	9.50 %	18.69 %	9.33 %
		EG	8.22 %	22.49 %	9.50 %
	PR	Nash	6.67 %	20.14 %	9.97 %
		K-S	6.40 %	21.60 %	9.75 %
1.400	SP	All	12.5 %	112.5 %	0 %
($S = 0.75$)	MD	Nash/K-S	7.17 %	32.91 %	10.42 %
		EG	4.84 %	43.58 %	11.25 %
		$W^* = S$	2.08 %	56.25 %	12.24 %
	MS	Nash	5.81 %	25.18 %	12.05 %
		K-S	5.81 %	25.18 %	12.05 %
		EG	3.78 %	34.00 %	12.82 %
	PR	Nash	2.08 %	31.73 %	14.25 %
		K-S	1.70 %	40.82 %	13.79 %
1.727	SP	All	5.09 %	412.50 %	0 %
($S = 0.90$)	MD	Nash/K-S	3.87 %	105.02 %	4.51 %
		EG	1.94 %	157.31 %	5.66 %
		$W^* = S$	0.28 %	202.50 %	6.56 %
	MS	Nash	2.51 %	48.01 %	6.35 %
		K-S	2.52 %	47.95 %	6.35 %
		EG	0.96 %	78.08 %	7.41 %
	PR	Nash	0.28 %	42.93 %	8.42 %
		K-S	0.21 %	61.27 %	8.27 %

Notes: The results shown in this table 6.1 are the solutions, which Schmalensee (1987) [17] obtained (cf. Schmalensee 1987, page 360) [17]. This outcomes are partly verified in simply recalculating.

case can be recognized. Secondly, between the different technologies applied to the bargaining methods, especially between the SP and the other three technologies a bigger difference can be noticed. As mentioned in section 5.5, there is no difference which bargaining method the firms are using in the SP technology, since the solution methods come to the same results.

Firms using the MS, MD or PR technology have to take losses in profit into account in comparison to the SP technology, which can use the full potential of the market. That is due to the opportunity to reduce the produced quantity, when the firms are colluding. The Side Payments technology can benefit from the whole opportunity. By means of this reduction the cartel members can increase their profits. In other words, in the SP technology the firms are able to maximize the outcomes of the whole industry, since they have the possibility to act as a monopoly in the whole market and only the low-cost firm is producing. In table 6.1, the row with the description 'Profit Sacrifice' gives information about how much more profit is possible, if the firms are allowed to apply the in many cases forbidden SP technology. It describes the loss in profit in percentage of the whole industry. In the case of symmetric costs on the one hand, the firms can benefit from joint profit maximization, on the other hand they cannot profit from the above mentioned opportunity. Nevertheless, in the collusive outcome the payoffs of both types of firms are increased by the same amount. Furthermore, it can be noticed that all solution methods coincide in the symmetric case.

An interesting fact can be found, when looking at the 'Profit Sacrifice'. If the parameter R takes values close to 1 or close to 2 the loss in profit gets smaller, while the deficit for values in the middle of the interval of R is relatively high. Schmalensee (1987) [17] said that the loss reaches its maximum for values, which are in the middle of the interval of R (cf. Schmalensee 1987, page 361) [17].

In table 6.1, it can immediately be recognized that for every collusion technology applied to each solution concept the profit increase of firm 1 is considerably less than the outcome of firms of type 2 compared to Cournot competition. The higher the value of R, which means the higher the cost difference, the higher becomes the difference in this profit increase between the both types of firms. That means the additional profit from collusion compared to that from Cournot competition gets less and less for the low-cost firm until it reaches zero supplementary profit.

What can be said about the performance of the other technologies? In case side payments are forbidden, both types of firms obtain remarkably lower profits. This is because they cannot make use of the opportunity to reduce the quantity, as previously mentioned above. This is due to the fact that in the Side Payments technology, only firm 1 produces goods, whereas in the other three technologies, both types of firms manufacture products. Furthermore, in the MS, MD and PR technologies, it has to be taken into account that firms of type 2 have higher production costs, which directly induce a higher price for the product. Incidentally, it is interesting that in table 6.1 it can be seen that the solutions for the profit of firm 1 according to each solution method have the same sequence for all three values of R. This will not be the case in table 6.2, meaning that the SP technology always provides, as explained above, the highest profit followed by the Nash solution for the MD technology. The lowest profit is given by the K-S solution applied to the PR technology. This is only the case for firm 1.

When looking on the efficiency of the bargaining methods, it can be recognized that the PR technology provides under both feasible solution methods perceivable less profit for the low-cost firm. If the EG solution method is used, firms of type 2 get noticeable more profit than firm 1, when compared to the Nash and the K-S solution method. As mentioned in section 5.5, the Nash and the K-S methods are very close to each other, but not identical, when the MS technology is applied. The solutions obtained by all feasible solutions methods adapted to the MS and MD technology are quite similar. While with the $W^* = S$ solution method the high-cost firms are better off and the low-cost firm is worse off in comparison to the other three bargaining methods. Therefore, it can be concluded that the fraction W^* must be higher than the market share of firm 1, namely S, such that it is more profitable for firm 1, when the Nash, K-S or EG method is applied.

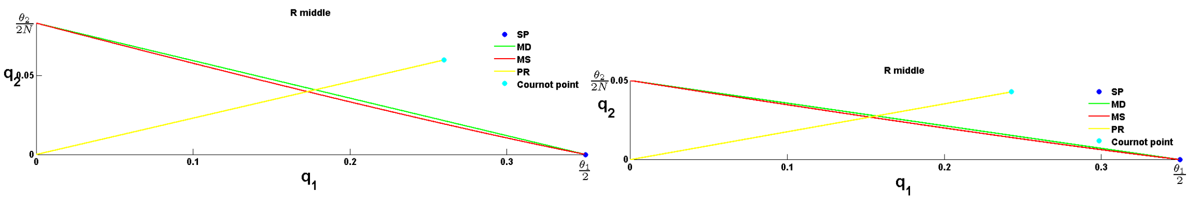
6.2. Results with variation in N

Now Schmalensee's results (1987) [17] are shown, when the parameter R is fixed at the value $R_{\text{middle}} = 1.4$, while there will be a variation in N. The number of high-cost firms N will take the values 1 as before and additionally 3 and 5.

Before the table with the numeric values of the solutions are introduced, the curves of the quantities and profits of the four technologies are shown for higher N, namely for $N = 3$ and $N = 5$, in the following figures 6.1 and 6.2. These graphs serve as an illustration of the performance of the collusion technologies with a higher number of high-cost firms, as well as comparison to the case with only one single firm of type 2. To plot the graphs in MATLAB, in figures 6.1 and 6.2 exactly the same formulas are used as in section 4.5, both for the quantity and the profit curves. The only difference is that in the formulas higher values for N are applied. Note that the following figures are plotted only for the parameter R_{middle}. Additionally, there is only one of the N identical high-cost firms compared to firm 1 in the graph.

Figure 6.1: Quantity curves of all collusion technologies for R_{middle} and $N = 3$ and $N = 5$

(a) $N = 3$ (b) $N = 5$



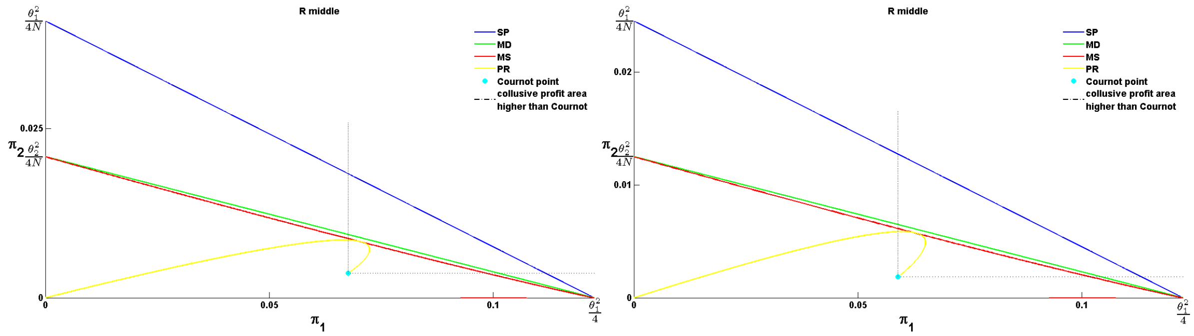
Notes: Figures 6.1a and 6.1b are plotted in MATLAB for the parameter $R_{\text{middle}} = 1.4$ with the following values for the cost parameters $\theta_1 = 0.7$ and $\theta_2 = 0.5$. Additionally, the number of high-cost firms is $N = 3$, respectively $N = 5$.

The values of the x-axis for firm 1 stay the same, since there is only one firm 1 and this does not change. On the contrary, the highest possible value on the y-axis decrease, which means that the quantity for each high-cost firm gets less, due to that the number of firms of type 2 is increasing and they have to split the production. Furthermore, the quantity curves of the three technologies MS, MD and PR become flatter with higher N. Accordingly, the MS curve approximates the MD curve, when the number N of firms of type 2 grows. In addition, the value of the Cournot point of each firm of type 2 falls. But from the point of view of all high-cost firms together the highest possible value for the quantity remain constant. The other properties can be found in section 4.5.1.

In the following figure 6.2 the graphs of the profit curves of all four collusion technologies are plotted in MATLAB for the parameter $R_{\text{middle}} = 1.4$ and one graph each for $N = 3$ and $N = 5$. Here the same fact about the change in values of the quantity curves on both axes holds as well for the profit curves. Similarly, the curves get flatter with an increase in the value N. In addition, the shape of the peak of the profit curve from the PR technology changes. But it cannot be made a high quality statement about the performance of the PR payoff curve, since the form of this curve is completely different from that of the other three technologies. In the same way, the y-coordinate of the Cournot point which corresponds to firms of type 2 decreases with a higher number of high-cost firms, namely N. Again, when looking at all firms of type 2, the highest possible profit, namely that under the SP technology, is identical to the case with $N = 1$. However, with the difference that all N firms of type 2 have to distribute the profit among themselves. The graphs for $N = 1$, the corresponding facts about the comparison of the technologies and more characteristics of the curves are described in section 4.5.2.

Figure 6.2: Profit curves of all collusion technologies for R_{middle} and $N = 3$ and $N = 5$

(a) $N = 3$ (b) $N = 5$



Notes: Figures 6.2a and 6.2b are plotted in MATLAB for the parameter $R_{\text{middle}} = 1.4$ with the following values for the cost parameters $\theta_1 = 0.7$ and $\theta_2 = 0.5$. Additionally, the number of high-cost firms is $N = 3$, respectively $N = 5$.

To illustrate the difference between the four collusion technologies and the applied solution methods, the values of both firm's profits are shown for each collusive solution in table 6.2. This table shows the results of Schmalensee (cf. Schmalensee 1987, page 362) [17].

Similarly as before, first the dissimilarity between the symmetric case and asymmetric case are discussed. In the second place, the performance of the different technologies and their corresponding solutions by means of the bargaining methods are described.

Concerning Cournot competition and symmetric costs it can be said that the higher the number of high-cost firms, the higher the degree of competition grows. Therefore, the outcomes of all firms in the market can be raised by means of collusion. Considering the actual interesting case of asymmetric costs, the higher the number of firms of type 2 gets, the higher the additional profit from collusion becomes for both firms. But especially the profit of the high-cost firms grow even more, because their profits are smaller in Cournot competition. That is due to the fact that the higher N gets, the higher the bargaining power of all firms of type 2 together is raised. Hence this leads to a loss in profit compared to the SP technology, because of collusion, which adjusts the profits more fairly for all firms in the market. This loss even grows with higher N, as can be seen in table 6.2. On the other hand, the increase in profit of firms of type 2 is higher, since with a higher number of high-cost firms, the share in Cournot competition grows. That is owing to the reduction in quantity, see Schmalensee (1987) [17].

A high-quality statement cannot be made as regards the differences between the collusion technologies and the corresponding bargaining methods, since at first it seems that the difference increases, but then some of the values change not according to the sequence in table 6.1.

Table 6.2: Results for $R = 1.4$ and with variation of N

			Profit on top of Cournot in %		
Value of N	**Technology**	**Solution**	**Firm 1**	**Firms of type 2**	**Profit Sacrifice**
1	Symm. ($R = 1$)	All	12.5 %	12.5 %	0 %
($S = 0.75$)	SP	All	12.5 %	112.5 %	0 %
	MD	Nash/K-S	7.17 %	39.21 %	10.42 %
		EG	4.84 %	43.58 %	11.25 %
		$W^* = S$	2.08 %	56.25 %	12.24 %
	MS	Nash	5.81 %	25.18 %	12.05 %
		K-S	5.81 %	25.18 %	12.05 %
		EG	3.78 %	34.00 %	12.82 %
	PR	Nash	2.08 %	31.73 %	14.25 %
		K-S	1.70 %	40.82 %	13.79 %
3	Symm. ($R = 1$)	All	56.25 %	56.25 %	0 %
($S = 0.59$)	SP	All	16.31 %	306.25 %	0 %
	MD	Nash/ K-S	12.47 %	119.52 %	18.58 %
		EG	7.25 %	136.19 %	19.99 %
		$W^* = S$	7.08 %	136.74 %	20.04 %
	MS	Nash	11.32 %	107.28 %	20.30 %
		K-S	11.47 %	106.80 %	20.26 %
		EG	6.52 %	122.48 %	21.60 %
	PR	Nash	7.02 %	104.99 %	22.87 %
		K-S	6.19 %	119.19 %	22.08 %
		EG	6.29 %	118.18 %	22.11 %
5	Symm. ($R = 1$)	All	104.17 %	104.17 %	0 %
($S = 0.53$)	SP	All	15.63 %	493.06 %	0 %
	MD	Nash/ K-S	12.86 %	210.74 %	22.36 %
		EG	7.15 %	229.48 %	23.71 %
		$W^* = S$	10.34 %	219.01 %	22.96 %
	MS	Nash	11.84 %	195.24 %	24.02 %
		K-S	12.14 %	194.25 %	23.95 %
		EG	6.62 %	212.51 %	25.24 %
	PR	Nash	10.05 %	184.56 %	25.68 %
		K-S	9.29 %	196.84 %	25.13 %
		EG	6.62 %	212.50 %	25.24 %

Notes: The results shown in this table 6.2 are the solutions, which Schmalensee (1987) [17] obtained (cf. Schmalensee 1987, page 362) [17]. This outcomes are partly verified in simply recalculating.

In other words, the solutions for the profits of firm 1 had the same sequence for all three values of R. In contrast, this sequence change for different values of N. More precisely, there is not such a particular sequence in this case. But the Side Payments technology is again the most efficient technology. In addition, an increase of the number of high-cost firms gives the impression that the profit of firm 1 grows for lower N. This is due to the reduction of the quantity. But for higher N, it is obvious that the relative outcome gets smaller, because the bargaining power of firms of type 2 grows.

Part II.
Extensions

So far, the problem of collusion and Schmalensee's model became familiar, such that it can be taken a look at some extensions of his approach. In particular, if and how the model of Schmalensee (1987) [17] can be expanded. Some ideas for diversification are described in the following. What will a change of the inverse demand function or the cost function bring with it? Or is it possible to implement the four collusion technologies and accordingly the four bargaining methods, if there are more than two firms with different costs in the market?

7. Variations of Schmalensee's model

Now the model of Richard Schmalensee (1987) [17] will be expanded in the following ways. Firstly, I will change the inverse demand function to a more general one, while everything else remains the same. Then the same thing is done with the cost function. Finally, the duopoly will be extended to a triopoly. In fact, additionally to the low-cost firm, there will be N middle-cost firms and M high-cost firms.

In the following sections 7.1 – 7.3, I will modify Schmalensee's model by recalculating all four collusion technologies with the adaptations and discuss if all solution methods can still be applied. It is examined if the collusion technologies also work with the modifications or if a problem is met. Consider that in this Part II, there will be partly new details and some other parts will stay the same as in Part I. I will describe these differences and similarities. Furthermore, the calculations will be shorter and less detailed than in section 4, since the explanations are similar.

7.1. Inverse demand function

In this section, the focus lies on a modification of the simple market demand function from equation (2.1). The inverse demand function is again linear, but it is more general. Now this new inverse demand function will be applied to all four collusion technologies. Every other facts of Schmalensee's model from section 3 stays the same. That means there are as well two types of firms and the same cost function (3.1) is used. This change just holds for this section 7.1. Thus, the new inverse market demand function is given by:

$$P(Q) = \max_{0 \leq Q < \infty} \{A - b \cdot Q,\ 0\} \quad \text{for } A, b > 0. \tag{7.1}$$

In addition, the condition $A > b \cdot Q - (1 - \theta_i)$ for $i = 1, 2$ has to be satisfied for A, since the profit should be positive.In difference to the original inverse demand function (2.1), there are two new parameters A and b. A variation of the parameter A describes for values above 1 an upward shift on the y-axis, which means on the price, and a shift to the right on the x-axis, which means on the market demand. Conversely an alternation for values below 1 leads to a downward shift for the price (lower prices) and a shift to the left for the market demand. A variation in b characterizes a change in the slope. For values above 1 the slope gets steeper, while for values in between 0 and 1, the slope becomes flatter. Note that for $A = b = 1$ this demand function is identical to the demand function from equation (2.1) in Schmalensee's model.

The corresponding profit functions are given by:

$$\Pi_i(Q) = (A + \theta_i - 1 - b \cdot Q) \cdot q_i \quad \text{for } i = 1, 2. \tag{7.2}$$

Before diving into the recalculations of the collusion technologies, the new values for the Cournot model and the monopoly outcome are shortly determined in the next section.

7.1.1. Cournot competition

Analogous to section 2.1, the aim is to find the maximum value of the profit of each firm i for $i = 1, \ldots, 1 + N$.

$$\max_{0 \leq q_i < \infty} \Pi_i(Q) = \max_{0 \leq q_i < \infty} (A + \theta_i - 1 - b \cdot Q) \cdot q_i \quad \text{for } i = 1, \ldots, 1 + N \tag{7.3}$$

Hence, differentiating leads to the following FOC's:

$$\frac{\partial \Pi_i(Q)}{\partial q_i} = A + \theta_i - 1 - 2b \cdot q_i - b \cdot Q_{-i} \stackrel{!}{=} 0 \quad \text{for } i = 1, \ldots, 1 + N. \tag{7.4}$$

As a result of the first order conditions above, the following reaction functions are obtained:

$$q_i = \begin{cases} \frac{1}{2b} \cdot (A + \theta_i - 1 - b \cdot Q_{-i}) & \text{for} \quad Q_{-i} \leq \frac{1}{b}(A + \theta_i - 1) \\ 0 & \text{otherwise} \end{cases} \quad \text{for } i = 1, \ldots, 1 + N. \tag{7.5}$$

To prove if the maximization above results indeed in a maximum, the subsequent SOC's have to be fulfilled:

$$\frac{\partial^2 \Pi_i}{\partial q_i^2} = -2b < 0. \tag{7.6}$$

Due to the fact that b has to be positive, which means that $-2b$ is negative, the second order condition is satisfied. Furthermore, since all N high-cost firms have identical costs $1-\theta_2$ and produce the same quantity q_2, the symmetric argument can be applied, namely $q_2 = \cdots = q_{1+N}$. Hence solving the system of reaction functions (7.5), yields the following quantities in Cournot equilibrium:

$$q_1^C = \frac{1}{b(2+N)} \cdot [A - 1 + (1+N)\cdot\theta_1 - N\theta_2] \tag{7.7}$$

$$N \cdot q_2^C = \frac{N}{b(2+N)} \cdot [A - 1 + 2\theta_2 - \theta_1]\,. \tag{7.8}$$

Setting these quantities in the profit function (7.2), leads to the subsequent Cournot profit:

$$\Pi_1^C = \frac{1}{b\cdot(2+N)^2} \cdot [A - 1 + (1+N)\cdot\theta_1 - N\theta_2]^2 \tag{7.9}$$

$$N \cdot \Pi_2^C = \frac{N}{b\cdot(2+N)^2} \cdot [A - 1 + 2\theta_2 - \theta_1]^2\,. \tag{7.10}$$

7.1.2. Side Payments

From section 4.1, it is known that if side payments or mergers are allowed, firm 1 undertakes the whole production of the industry, while firms of type 2 shut down their production. This implies that q_2^{SP} is zero and firm 1's profit as a monopolist will be maximized. Thus, the profit of the whole industry is identical to that of firm 1 as a monopolist, namely $\Pi = \Pi_1^M$.

$$\max_{0\le q_1<\infty} \Pi(q_1) = \max_{0\le q_1<\infty} (A + \theta_1 - 1 - b\cdot q_1)\cdot q_1 \tag{7.11}$$

In the appendix B.1 the precise computation of the maximization of a monopoly can be found. Considering that the quantity produced of firm 1 corresponds to the total quantity produced in the industry, the following is obtained:

$$Q^{SP} = q_1^M = \frac{1}{2b}\cdot(A + \theta_1 - 1) = q_1^{SP}. \tag{7.12}$$

Therefore, the explicit solution for the quantity is:

$$\left(q_1^{SP},\, Nq_2^{SP}\right) = \left(\frac{A+\theta_1-1}{2b},\, 0\right). \tag{7.13}$$

The total profit of the industry for the SP technology is explicitly determined by setting the total quantity Q^{SP} from above in the profit function (7.2) for $i = 1$.

$$\Pi_1^M = \Pi^{SP} = \Pi_1^{SP} + N\Pi_2^{SP} = \frac{1}{4b} \cdot (A+\theta_1-1)^2 \tag{7.14}$$

From the equation above, the subsequent profit possibility frontier is derived.

$$\Pi_2^{SP} = \frac{1}{N}\left(\frac{(A+\theta_1-1)^2}{4b} - \Pi_1^{SP}\right) \tag{7.15}$$

The joint profit should be split between the firms. The optimal solution will be found by means of the solution methods from the cooperative payoff region above. Remember that this technology serve as comparison.

7.1.3. Market Sharing

In the Market Sharing technology all firms in the market get assigned particular 'output quotas'. They have to retain their quotas and maximize their profits with respect to this rate, while holding the other type of firm fixed at its quota. Therefore, the quantity of each firm is expressed through its profit and the total quantity:

$$q_i = \frac{\Pi_i}{A+\theta_i-1-b\cdot Q} \quad \text{for } i = 1,2. \tag{7.16}$$

Now the total quantity can be described by:

$$Q = q_1 + N\cdot q_2 = \frac{\Pi_1}{A+\theta_1-1-b\cdot Q} + N\cdot\frac{\Pi_2}{A+\theta_2-1-b\cdot Q}. \tag{7.17}$$

Hence, transforming the equations above yields, these expressions:

$$q_1 = Q - N\cdot\frac{\Pi_2}{A+\theta_2-1-b\cdot Q} \quad \text{and analogue} \tag{7.18}$$

$$q_2 = Q - \frac{\Pi_1}{A+\theta_1-1-b\cdot Q}. \tag{7.19}$$

Now the profit of all N high-cost firms will be expressed as a function of firm 1's profit and vice versa. By inserting q_1 from equation (7.18) and q_2 from equation (7.16) for $i = 2$ into the profit function of firm 1 from equation (7.2) for $i = 1$, the transformed profit of firm 1 below is obtained. This works analogous for firms of type 2. Since some terms immediately cancel out similarly to section 4.2, one step is skipped and the following is obtained:

$$\max_{0\leq Q<\infty} N\Pi_2(\Pi_1) = \max_{0\leq Q<\infty}\left(Q - \frac{\Pi_1}{A+\theta_1-1-b\cdot Q}\right)\cdot(A+\theta_2-1-b\cdot Q) \tag{7.20}$$

$$\max_{0\leq Q<\infty} \Pi_1(\Pi_2) = \max_{0\leq Q<\infty}\left(Q - \frac{N\Pi_2}{A+\theta_2-1-b\cdot Q}\right)\cdot(A+\theta_1-1-b\cdot Q). \tag{7.21}$$

These profits are directly maximized with respect to the total quantity Q, such that the 'output quotas' are fulfilled. Maximization yields the subsequent FOC's:

$$\frac{\partial N\Pi_2(\Pi_1)}{\partial Q} = A+\theta_2-1-2b\cdot Q + \frac{b\Pi_1\cdot(\theta_1-\theta_2)}{(A+\theta_1-1-b\cdot Q)^2} \stackrel{!}{=} 0 \tag{7.22}$$

$$\frac{\partial \Pi_1(\Pi_2)}{\partial Q} = A+\theta_1-1-2b\cdot Q + \frac{bN\Pi_2\cdot(\theta_2-\theta_1)}{(A+\theta_2-1-b\cdot Q)^2} \stackrel{!}{=} 0. \tag{7.23}$$

Converting the equations above and expressing the profit of firm 1, respectively of firms of type 2, leads to these explicit profits below for the MS technology:

$$\Pi_1^{MS} = \frac{(A+\theta_1-1-b\cdot Q)^2\cdot(2b\cdot Q-A-\theta_2+1)}{b\cdot(\theta_1-\theta_2)} \tag{7.24}$$

$$N\Pi_2^{MS} = \frac{(A+\theta_2-1-b\cdot Q)^2\cdot(A+\theta_1-1-2b\cdot Q)}{b\cdot(\theta_1-\theta_2)}. \tag{7.25}$$

Moreover, it can be seen that only the change of the demand function makes the formulas for the Market Sharing technology more complex in comparison to equations (4.16) and (4.17) in section 4.2.

Now the reaction functions for both types of firms should be determined from the corresponding cubic equation. For q_1 the profit function from equation (7.2) is set in the just calculated profit of firm 1 from equation (7.24) for the MS technology. The determination proceeds analogous to section 4.2 on page 16.

$$(A+\theta_1-1-b\cdot Q)\cdot q_1 = \frac{(A+\theta_1-1-b\cdot Q)^2\cdot(2b\cdot Q-A-\theta_2+1)}{b\cdot(\theta_1-\theta_2)} \tag{7.26}$$

$$(A+\theta_1-1-b\cdot Q)\cdot\left[\frac{(A+\theta_1-1-b\cdot Q)\cdot(2b\cdot Q-A-\theta_2+1)}{b\cdot(\theta_1-\theta_2)} - q_1\right] = 0 \tag{7.27}$$

From the last equation, the first solution of q_1 can be derived by means of $Q = q_1 + N \cdot q_2$.

$$q_1^{(1)} = \frac{1}{b} \cdot (A + \theta_1 - 1 - bN \cdot q_2) \tag{7.28}$$

The determination of the roots of the quadratic part of equation (7.27) can be found in the appendix B.3. It results in:

$$\begin{aligned} q_1^{(2,3)} &= \frac{3\,(A-1) + \theta_1 + 2\theta_2}{4b} - Nq_2 \\ &\pm \frac{1}{4b} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 8bN \cdot q_2 \cdot (\theta_1 - \theta_2) + 2 \cdot (A-1)\,(2\theta_2 - \theta_1) + (A-1)^2}. \end{aligned} \tag{7.29}$$

As well, the reaction function for q_2 is computed analogous to q_1, such that the following formula is achieved:

$$N \cdot (A + \theta_2 - 1 - b \cdot Q) \cdot q_2 = \frac{(A + \theta_2 - 1 - b \cdot Q)^2 \cdot (A + \theta_1 - 1 - 2b \cdot Q)}{b \cdot (\theta_1 - \theta_2)} \tag{7.30}$$

$$(A + \theta_2 - 1 - b \cdot Q) \cdot \left[\frac{(A + \theta_2 - 1 - b \cdot Q) \cdot (A + \theta_1 - 1 - 2b \cdot Q)}{b \cdot (\theta_1 - \theta_2)} - Nq_2 \right]. \tag{7.31}$$

The first solution $q_2^{(1)}$ is derived from the first non-quadratic part of the equation (7.31).

$$q_2^{(1)} = \frac{1}{bN} \cdot (A + \theta_2 - 1 - b \cdot q_1) \tag{7.32}$$

The second and third solutions are derived in the appendix B.3.

$$\begin{aligned} q_2^{(2,3)} &= \frac{3\,(A-1) + 2\theta_1 + \theta_2}{4bN} - \frac{q_1}{N} \\ &\pm \frac{1}{4bN} \cdot \sqrt{(2\theta_1 - \theta_2)^2 - 8b \cdot q_1 \cdot (\theta_1 - \theta_2) + 2 \cdot (A-1)\,(2\theta_1 - \theta_2) + (A+1)^2} \end{aligned} \tag{7.33}$$

Analogous to the MS technology introduced in section 4.2, the third solutions of the reaction functions q_1 and q_2 can be applied. Hence, the explicit reaction functions are given by:

$$\begin{aligned} q_1^{MS} &= \frac{3(A-1)+\theta_1+2\theta_2}{4b} - Nq_2 \\ &- \frac{1}{4b}\cdot\sqrt{(\theta_1-2\theta_2)^2+8bN\cdot q_2\cdot(\theta_1-\theta_2)+2\cdot(A-1)(2\theta_2-\theta_1)+(A-1)^2} \end{aligned} \tag{7.34}$$

$$\begin{aligned} q_2^{MS} &= \frac{3(A-1)+2\theta_1+\theta_2}{4bN} - \frac{q_1}{N} \\ &- \frac{1}{4bN}\cdot\sqrt{(2\theta_1-\theta_2)^2-8b\cdot q_1\cdot(\theta_1-\theta_2)+2\cdot(A-1)(2\theta_1-\theta_2)+(A+1)^2}. \end{aligned} \tag{7.35}$$

With the choice of a slightly more complex demand function (7.1) instead of (2.1), the formulas get longer and thereby the calculations become longer and more complex.

7.1.4. Market Division

The market is divided between the two types of firms. Hence, recall that firm 1 got assigned a fraction W^* of all potential customers, while firms of type 2 were allocated a fragment $1-W^*$ of them. Both firms are selling their products in the same market, but can only access their own section. In its part each type of firm has a monopoly position. Evidently, the products of all firms are sold at their price as monopolist. Therefore, because of the same reasons as in section 4.3 the inverse demand function of firm 1 changes to $P(\frac{q_1}{W^*})$, respectively of firms of type 2 to $P(\frac{N\cdot q_2}{1-W^*})$. The calculations proceed similarly to section 4.3.

$$P_1 = P\left(\frac{q_1}{W^*}\right) = A - b\cdot\frac{q_1}{W^*} \tag{7.36}$$

$$P_2 = P\left(\frac{N\cdot q_2}{1-W^*}\right) = A - b\cdot\frac{N\cdot q_2}{1-W^*} \tag{7.37}$$

Thus, the profit functions are given by:

$$\Pi_1\left(\frac{q_1}{W^*}\right) = \left(A+\theta_1-1-b\cdot\frac{q_1}{W^*}\right)\cdot q_1 \tag{7.38}$$

$$\Pi_2\left(\frac{N\cdot q_2}{1-W^*}\right) = \left(A+\theta_2-1-b\cdot\frac{N\cdot q_2}{1-W^*}\right)\cdot q_2. \tag{7.39}$$

Hence, these profit functions should be maximized.

$$\max_{0\le q_1<\infty}\Pi_1\left(\frac{q_1}{W^*}\right) = \left(A+\theta_1-1-b\cdot\frac{q_1}{W^*}\right)\cdot q_1 \tag{7.40}$$

$$\max_{0\le q_2<\infty}\Pi_2\left(\frac{N\cdot q_2}{1-W^*}\right) = \left(A+\theta_2-1-b\cdot\frac{N\cdot q_2}{1-W^*}\right)\cdot q_2 \tag{7.41}$$

Obviously, the maximization of the equations above yields to the following FOC's:

$$\frac{\partial \Pi_1}{\partial q_1} = A + \theta_1 - 1 - \frac{2b}{W^*} \cdot q_1 \stackrel{!}{=} 0 \tag{7.42}$$

$$\frac{\partial \Pi_2}{\partial q_2} = A + \theta_2 - 1 - \frac{2bN}{1-W^*} \cdot q_2 \stackrel{!}{=} 0. \tag{7.43}$$

Solving the equations above for the quantities q_1, respectively q_2, leads to the explicit values for both quantities:

$$q_1 = \frac{W^*}{2b} \cdot (A + \theta_1 - 1) \tag{7.44}$$

$$N \cdot q_2 = \frac{1-W^*}{2b} \cdot (A + \theta_2 - 1), \text{ respectively} \qquad q_2 = \frac{1-W^*}{2bN} \cdot (A + \theta_2 - 1). \tag{7.45}$$

The explicit reaction function for the MD technology is derived in expressing the fraction W^* from the equation (7.44) and inserting into the equation (7.45) for Nq_2.

$$W^* = \frac{2b}{A + \theta_1 - 1} \cdot q_1 \tag{7.46}$$

$$\Rightarrow q_2 = \frac{1 - \frac{2b}{A+\theta_1-1} \cdot q_1}{2Nb} \cdot (A + \theta_2 - 1) \tag{7.47}$$

Therefore, the explicit reaction function looks like:

$$q_2^{MD} = \frac{A + \theta_2 - 1}{2bN} \cdot \left(1 - \frac{2b \cdot q_1}{A + \theta_1 - 1}\right). \tag{7.48}$$

Furthermore, the explicit profits can be determined from the quantities from equations (4.36) and (4.37).

$$\Pi_1^{MD} = \frac{W^*}{4b} \cdot (A + \theta_1 - 1)^2 \tag{7.49}$$

$$N\Pi_2^{MD} = \frac{1-W^*}{4b} \cdot (A + \theta_2 - 1)^2, \text{ respectively} \qquad \Pi_2^{MD} = \frac{1-W^*}{4bN} \cdot (A + \theta_2 - 1)^2 \tag{7.50}$$

Analogous, the profit possibility frontier is computed to connect the profits of both types of firms. Hence, the fraction W^* is expressed analogue to the case for the quantity:

$$W^* = \frac{4b}{(A + \theta_1 - 1)^2} \cdot \Pi_1. \tag{7.51}$$

Accordingly, the cooperative payoff region is given by:

$$\Pi_2^{MD} = \frac{(A+\theta_2-1)^2}{4bN} \cdot \left(1 - \frac{4b \cdot \Pi_1}{(A+\theta_1-1)^2}\right). \tag{7.52}$$

Finally, the price is examined in setting the explicit quantity q_1, respectively q_2 in the corresponding inverse demand function. From section 4.3, it is known that each type of firm sells their products for the price of having a monopoly.

$$P_1 = \frac{1}{2} \cdot (A - \theta_1 + 1) \tag{7.53}$$

$$P_2 = \frac{1}{2} \cdot (A - \theta_2 + 1) \tag{7.54}$$

That means that the products of each type of firm are sold for different prices. Consequently, the price of the high-cost firms will be higher than that of the low-cost firm, due to their higher costs. This is shown in the following two inequalities, since $\frac{1}{2} \cdot (A - \theta_1 + 1) < \frac{1}{2} \cdot (A - \theta_2 + 1)$, it follows $\theta_1 > \theta_2$, which is true. Keep in mind that because of the different prices, arbitrage can be a problem.

7.1.5. Proportional Reduction

Remember from section 4.4 the PR technology is a simplified version of the Market Sharing technology, which holds the market shares fixed at their values in Cournot. The Cournot quantities further are decreased proportionally. It follows that the market share of the low-cost firm is preserved at its share in Cournot competition, analogous for N high-cost firms:

$$S = \frac{q_1^C}{q_1^C + Nq_2^C} = \frac{A - 1 + (1+N) \cdot \theta_1 - N\theta_2}{(1+N) \cdot A - (1+N) + \theta_1 + N\theta_2} \tag{7.55}$$

$$S_2 = \frac{Nq_2^C}{q_1^C + Nq_2^C} = \frac{N \cdot (A - 1 + 2\theta_2 - \theta_1)}{(1+N) \cdot A - (1+N) + \theta_1 + N\theta_2}. \tag{7.56}$$

Accordingly, the market share is retained tight at the ratio, details see section 4.4.

$$S = \frac{q_1^C}{Q^C} : \frac{Nq_2^C}{Q^C} = S_2 \quad \Rightarrow \quad q_1^C : Nq_2^C \tag{7.57}$$

Evidently, a line from the zero point until the Cournot quantity $\left(q_1^C,\ Nq_2^C\right)$ is characterized with this ratio. Consequently, all points, which satisfy $\left(\lambda \cdot q_1^C,\ \lambda \cdot Nq_2^C\right)$ for $\lambda \in [0,\ 1]$, can be possible quantities, see section 4.4.

$$\left(q_1^{PR},\ Nq_2^{PR}\right) = \left(\lambda \cdot q_1^C,\ \lambda \cdot Nq_2^C\right) \quad \text{for } \lambda \in [0,\ 1] \tag{7.58}$$

$$\left(q_1^{PR},\ Nq_2^{PR}\right) = \frac{1}{b \cdot (2+N)} \cdot \Big(\lambda \cdot (A-1+(1+N)\cdot\theta_1 - N\theta_2),\ \lambda N \cdot (A-1+2\theta_2-\theta_1)\Big) \tag{7.59}$$

Furthermore, the ratio from equation (7.57) implies:

$$q_1^{PR} : N \cdot q_2^{PR} = q_1^C : N \cdot q_2^C. \tag{7.60}$$

Expressing q_2^{PR} from the ratio (7.60) leads to the subsequent explicit reaction function:

$$q_2^{PR} = \frac{q_2^C}{q_1^C} \cdot q_1^{PR} = \frac{A-1+2\theta_2-\theta_1}{A-1+(1+N)\cdot\theta_1 - N\theta_2} \cdot q_1^{PR}. \tag{7.61}$$

The corresponding profits can be written as:

$$\Pi_1^{PR} = (A+\theta_1-1-b\cdot q_1^{PR} - bN\cdot q_2^{PR}) \cdot q_1^{PR} \tag{7.62}$$

$$N\Pi_2^{PR} = (A+\theta_2-1-b\cdot q_1^{PR} - bN\cdot q_2^{PR}) \cdot Nq_2^{PR}. \tag{7.63}$$

If the values for the Cournot quantities are set in the profit functions above, the following is obtained:

$$\Pi_1^{PR} = \left(A+\theta_1-1-\frac{b\lambda}{2+N}(A-1+(1+N)\cdot\theta_1-N\theta_2) - \frac{bN\lambda}{2+N}(A-1+2\theta_2-\theta_1)\right)$$

$$\cdot \frac{\lambda}{2+N} \cdot (A-1+(1+N)\cdot\theta_1 - N\theta_2) \tag{7.64}$$

$$= \frac{\lambda}{(2+N)^2} \cdot (A-1+2\theta_2-\theta_1) \cdot [(2+N)\cdot(A+\theta_1-1) - b\lambda\cdot((1+N)\cdot(A-1)+\theta_1+N\theta_2)] \tag{7.65}$$

$$N\Pi_2^{PR} = \left(A+\theta_2-1-\frac{b\lambda}{2+N}(A-1+(1+N)\cdot\theta_1-N\theta_2) - \frac{bN\lambda}{2+N}(A-1+2\theta_2-\theta_1)\right) \tag{7.66}$$

$$\cdot \frac{bN\lambda}{2+N} \cdot (A-1+2\theta_2-\theta_1) \tag{7.67}$$

$$= \frac{N\lambda}{(2+N)^2} \cdot (A-1+2\theta_2-\theta_1) \cdot [(2+N)\cdot(A+\theta_2-1) - b\lambda\cdot((1+N)\cdot(A-1)+\theta_1+N\theta_2)]. \tag{7.68}$$

7.1.6. Bargaining solution methods

If all relevant axioms are fulfilled, the three solution methods presented in section 5, namely the Nash, the K-S and the EG solutions, can be applied to all of the modified collusion technologies from sections 7.1.2 – 7.1.5. Since the $W^* = S$ solution method is a special case, more precisely a special point, in the Market Division technology, this method can be used without making adaptions.

7.1.7. Graphic representation of the collusion technologies

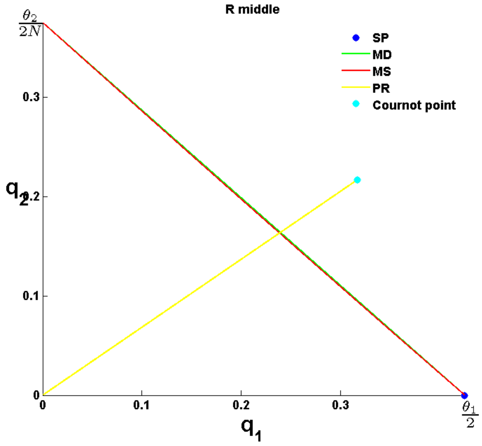
In this section, the graphic illustrations are shown for all collusion technologies with the modified demand function. Therefore, two different values are plotted for the new parameters A and b, as well, for the known parameters R_{middle} and $N = 1$.

In figure 7.1, the quantity curves of the collusion technologies are illustrated with the new demand function. The explicit reaction functions of the respective collusion technology determined in sections 7.1.2 – 7.1.5 were used for the graphic representation. Furthermore, the profit curves of all collusion technologies are shown in figure 7.2. Accordingly, this figures are plotted by means of the equations of the respective profit possibility frontiers of all collusion technologies with the modified demand function.

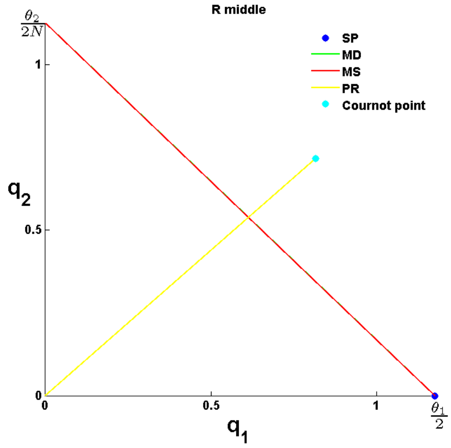
It can be seen that in the figures 7.1 and 7.2 the MS and the MD curves are closer than in the Schmalensee's model in the figures in section 4.5.1 and 4.5.2. In particular, it seems that the curves are identical. However, the shape of the curves and the corresponding performance of the collusion technologies do not differ from the case in Part I.

Figure 7.1: Quantity curves of all collusion technologies with changed demand function

(a) $A = 2,\ b = 2$



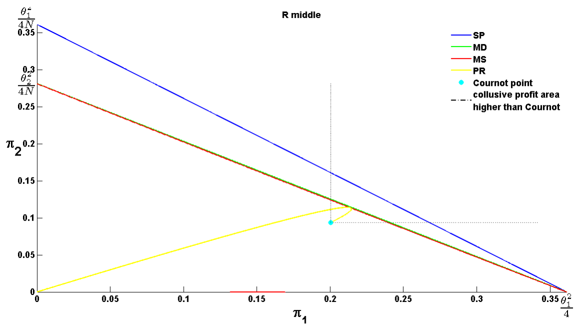
(b) $A = 5,\ b = 2$



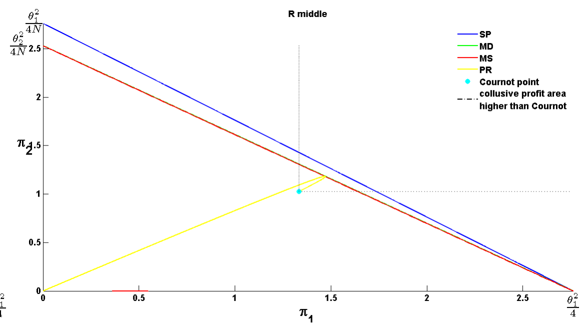
Notes: Figure 7.1a is plotted in MATLAB for $N = 1$ and $A = 2,\ b = 2$, while figure 7.1b is plotted for $A = 5$ and $b = 2$. The values for the parameter $R_{\text{middle}} = 1.4$ are the following: $\theta_1 = 0.7$ and $\theta_2 = 0.5$.

Figure 7.2: Profit curves of all collusion technologies with changed demand function

(a) $A = 2, b = 2$



(b) $A = 5, b = 2$



Notes: Figure 7.2a is plotted in MATLAB for $N = 1$ and $A = 2, b = 2$, while figure 7.2b is plotted for $A = 5$ and $b = 2$. The values for the parameter $R_{\text{middle}} = 1.4$ are the following: $\theta_1 = 0.7$ and $\theta_2 = 0.5$.

7.2. Cost function

To generalize the cost function from equation (3.1), the idea of a generalized modification of the cost function is described in the following. Again, the basic principle is to modify only the cost function while everything else stays the same. Therefore, the demand function (2.1) is not changed. Firstly, the cost parameters θ_i for $i = 1,2$ would be alternated according to equation (3.9) as used in section 2.1. Secondly, the interval of the cost parameters θ_i for $i = 1,2$ should be increased to a broader range of values, namely to $[0,T]$. The parameter T should be a natural number. In addition, individual fixed costs for each type of firm should be introduced by means of the parameter d. Thus, the new cost function would look like:

$$C(q_i) = c_i * q_i + d_i \quad \text{for } i = 1,2. \tag{7.69}$$

Furthermore, the following has to hold for the cost parameters:

$$c_1 < c_2 \quad \text{and} \quad c_i \in [0,T] \quad \text{for } i = 1,2 \tag{7.70}$$

$$d_1 < d_2 \quad \text{and} \quad d_i \in [0,T] \quad \text{for } i = 1,2. \tag{7.71}$$

This way of modifying the cost function is limited to the demand function (2.1), since a negative profit for the firms does not make sense. But this change of the cost function would lead to an even less profit of the firms. That means if this demand function is retained, the cost function cannot be arbitrarily generalized. Hence, it makes sense to generalize the demand function additionally to the cost function, such that all types of firms gain a positive profit.

7.3. Triopoly

Now the original model of Schmalensee (1987) [17] from section 3 is expanded from a duopoly to a triopoly. More precisely, there are now three types of firms operating in the market and competing in quantity. Each type of firm has different costs. In the first place, firm 1 is again the only low-cost firm in the market, while in the second place there are N middle-cost firms, which are named firms of type 2. In the third place, there are M firms of type 3, which represent the high-cost firms. Note that N and M are natural numbers. As previously described firm 1 is the low-cost firm, while firms of type 2 are the N middle-cost firms. The M high-cost firms are named firms of type 3. The total number of firms operating in the market is then $1+N+M$, since there is one firm 1 and N firms of type 2 and M firms of type 3. The inverse demand function (2.1) stays the same except that there are now three firms producing goods. It follows that the total quantity sums up to $Q = q_1 + Nq_2 + Mq_3$. The cost and profit function also remain the same except that there are now three functions instead of two.

$$C(q_i) = (1-\theta_i)\cdot q_i \quad \text{and} \quad \theta_i \in [0,1] \quad \text{for } i = 1,2,3 \tag{7.72}$$

$$\Pi_i = (\theta_i - Q)\cdot q_i \quad \text{for } i = 1,2,3 \tag{7.73}$$

The cost parameters θ_i for $i = 1,2,3$ have to fulfill the subsequent inequalities:

$$\theta_3 < \theta_2 < \theta_1 < 2\cdot\theta_2 \quad \text{and} \quad \theta_1 < 2\cdot\theta_3. \tag{7.74}$$

The first and second part of the first inequality makes sure that firm 1 has the lowest costs, all firms of type 2 have the second lowest costs and all firms of type 3 have the highest costs. The third inequality, respectively the last inequality, ensures that the market price of firm 1 having a monopoly, namely $1-\frac{\theta_1}{2}$, is higher than the costs of each middle-cost firm, namely $1-\theta_2$, respectively of each high-cost firm, namely $1-\theta_3$. In addition, the market price of firms of type 2 as a monopolist, namely $1-\frac{\theta_2}{2}$, has to be higher than the costs of each firm of type 3. This is already fulfilled, since the price of firm 1 as monopolist is higher than the costs of the high-cost firms, see the last inequality above. That is due to the fact that if firm 1 decreases its price, the middle-cost firms as well as the high-cost firms would have more costs to produce the good than they could obtain through the purchase.

Furthermore, the joint profit sums now up to:

$$\Pi = \Pi_1 + N\cdot\Pi_2 + M\cdot\Pi_3. \tag{7.75}$$

In this case, a even lower profit gain will be expected, since there are more firms in the market, which have to split the profit by any means. Therefore, it is a more generalized version.

Here there will not be only one parameter R, which should measure the relative difference in costs. Instead, there are now three different parameters R, such that the cost differences between each two firms can be measured. In the following, R_1 measures the relative differences in costs between firm 1 and firms of type 2, as R did in Part I. Additionally, R_2 characterizes the relative gap between the low-cost firms and firms of type 3, while R_3 describes the relative cost difference between firms of type 2 and the high-cost firms.

$$R_1 = \frac{\theta_1}{\theta_2} \quad \text{and} \quad R_2 = \frac{\theta_1}{\theta_3} \quad \text{and} \quad R_3 = \frac{\theta_2}{\theta_3} \tag{7.76}$$

From the interval of the cost parameters θ_i for $i = 1,2,3$, equation (7.74) and equation (7.76), it can be concluded that all three parameters R_i for $i = 1,2,3$ lie in the interval $[1,2]$, as in section 3. That means the same implications as in section 3 follows. Values of R_i for $i = 1,2,3$ close to 1 point out that the different types of firms have nearly identical costs. By way of contrast, values near 2, characterize a relative big difference in costs.

Furthermore, it follows that now the market can be described by these new parameters of R and the number of middle and high-cost firms, namely N and M. In Part I, there were only two parameters that characterized the market, namely R, respectively S, and N. Now there are five parameters, which describe the market, namely R_1, R_2, R_3, N and M. That means the description of the market gets more complex.

Additionally, the parameter S will also be introduced as in Part I. S characterizes again the market share of the low-cost firm in Cournot competition, while S_2 is the market share of one firm of type 2 in Cournot equilibrium and accordingly $S_3 = 1 - S - S_2$ is the share in Cournot competition of one firm of type 3. Note that $S + S_2 + S_3 = 1$. Setting the Cournot quantities from equations (7.91) – (7.93) in the equation for the market share, the following equations are obtained:

$$S = \frac{q_1^C}{Q^C} = \frac{q_1^C}{q_1^C + N \cdot q_2^C + M \cdot q_3^C} = \frac{(1+N+M) \cdot \theta_1 - N\theta_2 - M\theta_3}{\theta_1 + N\theta_2 + M\theta_3} \tag{7.77}$$

$$S_2 = \frac{Nq_2^C}{Q^C} = \frac{Nq_2^C}{q_1^C + N \cdot q_2^C + M \cdot q_3^C} = \frac{N \cdot [(2+M) \cdot \theta_2 - \theta_1 - M\theta_3]}{\theta_1 + N\theta_2 + M\theta_3} \tag{7.78}$$

$$S_3 = \frac{Mq_3^C}{Q^C} = \frac{Mq_3^C}{q_1^C + N \cdot q_2^C + M \cdot q_3^C} = \frac{M \cdot [(2+N) \cdot \theta_3 - \theta_1 - N\theta_2]}{\theta_1 + N\theta_2 + M\theta_3}. \tag{7.79}$$

The parameters R_1 and R_2 can also be characterized with these four parameters. The computation can be found in the appendix B.5.

$$R_1 = \frac{\left(N + \frac{M}{R_3}\right) \cdot (S+1)}{1+N+M-S} \tag{7.80}$$

$$R_2 = \frac{(N \cdot R_3 + M) \cdot (S+1)}{1+N+M-S} \tag{7.81}$$

Now the market can be described by four parameter, namely R_3, S, N and M.

In addition, if the joint profit is maximized with three different firms operating in the market, the same problem as in section 2.2 is faced. The proof can be found in the appendix A.4. For this reason, in the following sections 7.3.2 – 7.3.5 the four collusion technologies introduced in the sections 4.1 – 4.4 are applied to three types of firms. Before this is done, the Cournot model is determined in the next section.

7.3.1. Cournot competition

Similarly to section 2.1 and 7.1.1, the profits of each firm i for $i = 1, \dots, 1+N+M$ are maximized. Remember that in total there are $1+N+M$ firms operating in the market. The demand function (2.1) remains the same as well as the cost function (3.1), which is only adapted to the new number of firms.

$$C(q_i) = (1-\theta_i)\cdot q_i \quad \text{for } i = 1,\dots,1+N+M \tag{7.82}$$

According to the number of firms, the total quantity now sums up to $Q = q_1 + \dots q_{1+N+M}$. Therefore, the profit functions of each firm is given by:

$$\Pi_i = (\theta_i - Q)\cdot q_i \quad \text{for } i = 1,\dots,1+N+M. \tag{7.83}$$

These profit functions are now maximized.

$$\max_{0\leq q_i<\infty} \Pi_i(Q) = \max_{0\leq q_i<\infty} (\theta_i - Q)\cdot q_i \quad \text{for } i = 1,\dots,1+N+M \tag{7.84}$$

Thus, maximization yields the subsequent FOC's:

$$\frac{\partial \Pi_i(Q)}{\partial q_i} = (\theta_i - Q)\cdot q_i \overset{!}{=} 0 \quad \text{for } i = 1,\dots,1+N+M. \tag{7.85}$$

From the first order conditions above, the following reaction functions can be determined:

$$q_i = \begin{cases} \frac{1}{2}\cdot(\theta_i - Q_{-i}) & \text{for} \quad Q_{-i} \leq \theta_i \\ 0 & \text{otherwise} \end{cases} \quad \text{for } i = 1,\dots,1+N+M. \tag{7.86}$$

To prove if the maximization results indeed in a maximum, the subsequent SOC's have to be fulfilled:

$$\frac{\partial^2 \Pi_i}{\partial q_i^2} = -2 < 0. \tag{7.87}$$

Since the second derivatives are negative, the SOC's are satisfied and the quantities q_i for $i = 1+N+M$ definitely are maxima.

In section 7.3, it was mentioned that there are N identical firms of type 2. For this reason, the symmetric argument can be applied, which means that $q_2 = \dots = q_{1+N}$. Consequently, all N firms of type 2 have costs of $(1-\theta_2)\cdot q_2$. Thus, the total quantity sums now up to $Q = q_1 + N\cdot q_2 + q_{2+N} + \dots + q_{1+N+M}$. Furthermore, there are M identical firms of type 3, which means that again the symmetric argument can be used. The quantity produced by each firm of type 3 is then named $q_3 := q_{2+N} = \cdots = q_{1+N+M}$. Accordingly, all M firms of type 3 have costs of $(1-\theta_3)\cdot q_3$. Finally, the total quantity is now $Q = q_1 + N\cdot q_2 + M\cdot q_3$. Therefore, the system of reaction function (7.86) changes to:

$$q_1 = \frac{1}{2}\cdot(\theta_1 - N\cdot q_2 - M\cdot q_3) \tag{7.88}$$

$$q_2 = \frac{1}{2}\cdot(\theta_2 - q_1 - (N-1)\cdot q_2 - M\cdot q_3) \tag{7.89}$$

$$q_3 = \frac{1}{2}\cdot(\theta_3 - q_1 - N\cdot q_2 - (M-1)\cdot q_3)\,. \tag{7.90}$$

Solving the system of the three equations above (7.88) – (7.90), leads to the following explicit quantities in Cournot competition for the triopoly case.

$$q_1^C = \frac{1}{2+N+M}\cdot[(1+N+M)\cdot\theta_1 - N\theta_2 - M\theta_3] \tag{7.91}$$

$$Nq_2^C = \frac{N}{2+N+M}\cdot[(2+M)\cdot\theta_2 - \theta_1 - M\theta_3] \tag{7.92}$$

$$Mq_3^C = \frac{M}{2+N+M}\cdot[(2+N)\cdot\theta_3 - \theta_1 - N\theta_2] \tag{7.93}$$

Correspondingly, the explicit profit functions for each type of firm are given by:

$$\Pi_1^C = \left(q_1^C\right)^2 = \frac{1}{(2+N+M)^2}\cdot[(1+N+M)\cdot\theta_1 - N\theta_2 - M\theta_3]^2 \tag{7.94}$$

$$N\Pi_2^C = N\cdot\left(q_2^C\right)^2 = \frac{N}{(2+N+M)^2}\cdot[(2+M)\cdot\theta_2 - \theta_1 - M\theta_3]^2 \tag{7.95}$$

$$M\Pi_3^C = M\cdot\left(q_3^C\right)^2 = \frac{M}{(2+N+M)^2}\cdot[(2+N)\cdot\theta_3 - \theta_1 - N\theta_2]^2\,. \tag{7.96}$$

7.3.2. Side Payments

Since compensation payments are allowed in this technology, only the most efficient firm is producing goods to maximize the total profit of the industry. The low-cost firm, namely firm 1, can manufacture at the lowest costs. Consequently, only firm 1 produces goods at its monopoly price, while firms of type 2 and firms of type 3 are not producing any goods. Instead, the middle-cost firms and the high-cost firms will get compensation payments. That means the resulting quantity and profit in the Side payments technology for a triopoly is very similar to that of a duopoly. Accordingly, firm 1's monopoly profit will be maximized:

$$\max_{0\leq q_1<\infty} \Pi(q_1) = \max_{0\leq q_1<\infty} (\theta_1 - q_1)\cdot q_1. \tag{7.97}$$

Total industry profit is equal to the profit of firm 1 as a monopolist, namely $\Pi = \Pi_1^M$. Maximization of firm 1's monopoly profit yields the following quantity for the whole industry:

$$Q^{SP} = q_1^M = \frac{\theta_1}{2} = q_1^{SP}. \tag{7.98}$$

It follows that there is an explicit solution for the quantity of all types of firms at the point:

$$\left(q_1^{SP},\, Nq_2^{SP},\, Mq_3^{SP}\right) = \left(\frac{\theta_1}{2},\, 0,\, 0\right). \tag{7.99}$$

The profit is determined in setting Q^{SP} from equation (7.98) in the monopoly profit of firm 1:

$$\Pi^{SP} = \Pi_1^{SP} + N\Pi_2^{SP} + M\Pi_3^{SP} = \Pi_1^M = \frac{\theta_1^2}{4}. \tag{7.100}$$

This is an explicit solution for the joint profit, whereas there is no explicit solution for the individual firms. But there is an explicit cooperative payoff region:

$$\Pi_3^{SP} = \frac{1}{M}\cdot\left(\frac{\theta_1^2}{4} - \Pi_1^{SP} - N\Pi_2^{SP}\right). \tag{7.101}$$

Furthermore, the profit of each type of firm is less compared to the duopoly case, since the total profit in the duopoly case equals the total profit in the triopoly case, namely the monopoly profit of firm 1. The difference in the triopoly case is that the profit has to be split now between three types of firms, which are $1+N+M$ firms in total, instead of two types of firms, namely $1+N$ firms.

7.3.3. Market Sharing

Analogous to section 4.2, all types of firms get assigned particular quotas for the quantities, which they have to hold. It follows that the firms are fixed at their quotas for their quantity. The aim of each type of firm is to maximize its profit according to this quota and the profit of the other types of firms. Similarly to equation (4.6), the individual quantity is expressed through the total quantity Q and the profit Π_i for $i = 1,2,3$.

$$q_i = \frac{\Pi_i}{\theta_i - Q} \quad \text{for } i = 1,2,3 \tag{7.102}$$

Therefore, the total quantity can be characterized by:

$$Q = q_1 + N \cdot q_2 + M \cdot q_3 = \frac{\Pi_1}{\theta_1 - Q} + N \cdot \frac{\Pi_2}{\theta_2 - Q} + M \cdot \frac{\Pi_3}{\theta_3 - Q}. \tag{7.103}$$

Transforming the equations above, leads to the subsequent equations for the quantities:

$$q_1 = Q - N \cdot \frac{\Pi_2}{\theta_2 - Q} - M \cdot \frac{\Pi_3}{\theta_3 - Q} \tag{7.104}$$

$$N \cdot q_2 = Q - \frac{\Pi_1}{\theta_1 - Q} - M \cdot \frac{\Pi_3}{\theta_3 - Q} \tag{7.105}$$

$$M \cdot q_3 = Q - \frac{\Pi_1}{\theta_1 - Q} - N \cdot \frac{\Pi_2}{\theta_2 - Q}. \tag{7.106}$$

The profit of each type of firm is now expressed as a function of the profit of the other two types, analogue to section 4.2.

$$\Pi_1 (N\Pi_2, M\Pi_3) = \left[\theta_1 - \left(Q - \frac{N \cdot \Pi_2}{\theta_2 - Q} - \frac{M \cdot \Pi_3}{\theta_3 - Q} + \frac{N \cdot \Pi_2}{\theta_2 - Q} + \frac{M \cdot \Pi_3}{\theta_3 - Q}\right)\right] \cdot \left(Q - \frac{N \cdot \Pi_2}{\theta_2 - Q} - \frac{M \cdot \Pi_3}{\theta_3 - Q}\right) \tag{7.107}$$

$$N\Pi_2 (\Pi_1, M\Pi_3) = \left[\theta_2 - \left(\frac{\Pi_1}{\theta_1 - Q} + Q - \frac{\Pi_1}{\theta_1 - Q} - \frac{M \cdot \Pi_3}{\theta_3 - Q} + \frac{M \cdot \Pi_3}{\theta_3 - Q}\right)\right] \cdot \left(Q - \frac{\Pi_1}{\theta_1 - Q} - \frac{M \cdot \Pi_3}{\theta_3 - Q}\right) \tag{7.108}$$

$$M\Pi_3 (\Pi_1, N\Pi_2) = \left[\theta_3 - \left(\frac{\Pi_1}{\theta_1 - Q} + \frac{N \cdot \Pi_2}{\theta_2 - Q} + Q - \frac{\Pi_1}{\theta_1 - Q} - \frac{N \cdot \Pi_2}{\theta_2 - Q}\right)\right] \cdot \left(Q - \frac{\Pi_1}{\theta_1 - Q} - \frac{N \cdot \Pi_2}{\theta_2 - Q}\right) \tag{7.109}$$

Hence, the following profits are maximized according to the total quantity.

$$\max_{0\le Q<\infty} \Pi_1\left(N\Pi_2, M\Pi_3\right) = \max_{0\le Q<\infty} \left(Q - \frac{N\cdot\Pi_2}{\theta_2 - Q} - \frac{M\cdot\Pi_3}{\theta_3 - Q}\right)\cdot(\theta_1 - Q) \tag{7.110}$$

$$\max_{0\le Q<\infty} N\Pi_2\left(\Pi_1, M\Pi_3\right) = \max_{0\le Q<\infty} \left(Q - \frac{\Pi_1}{\theta_1 - Q} - \frac{M\cdot\Pi_3}{\theta_3 - Q}\right)\cdot(\theta_2 - Q) \tag{7.111}$$

$$\max_{0\le Q<\infty} M\Pi_3\left(\Pi_1, N\Pi_2\right) = \max_{0\le Q<\infty} \left(Q - \frac{\Pi_1}{\theta_1 - Q} - \frac{N\cdot\Pi_2}{\theta_2 - Q}\right)\cdot(\theta_3 - Q) \tag{7.112}$$

Therefore, the subsequent FOC's are obtained by maximizing the profits above:

$$\frac{\partial\Pi_1\left(N\Pi_2, M\Pi_3\right)}{\partial Q} = \theta_1 - 2Q + N\Pi_2\cdot\frac{\theta_2-\theta_1}{(\theta_2-Q)^2} + M\Pi_3\cdot\frac{\theta_3-\theta_1}{(\theta_3-Q)^2} \stackrel{!}{=} 0 \tag{7.113}$$

$$\frac{\partial N\Pi_2\left(\Pi_1, M\Pi_3\right)}{\partial Q} = \theta_2 - 2Q + \Pi_1\cdot\frac{\theta_1-\theta_2}{(\theta_1-Q)^2} + M\Pi_3\cdot\frac{\theta_3-\theta_2}{(\theta_3-Q)^2} \stackrel{!}{=} 0 \tag{7.114}$$

$$\frac{\partial M\Pi_3\left(\Pi_1, N\Pi_2\right)}{\partial Q} = \theta_3 - 2Q + \Pi_1\cdot\frac{\theta_1-\theta_3}{(\theta_1-Q)^2} + N\Pi_2\cdot\frac{\theta_2-\theta_3}{(\theta_2-Q)^2} \stackrel{!}{=} 0. \tag{7.115}$$

Determining the explicit profit possibility frontiers from the equations (7.113) – (7.115), leads to even longer and more complex formulas than in section 4.2, such that the formulas cannot be presented in an appropriate way. The degree of complexity of the equations increases from equations of the third degree, namely cubic equations, in the duopoly case to equations of the fifth degree in the triopoly case. Following Galois, equations of the fifth degree can generally not be expressed through roots (cf. Muthsam 2006, page 189) [10].

7.3.4. Market Division

In the case of a duopoly, as presented in Part I, identical customers were assigned to each type of firm. More precisely, firm 1 was allocated a section W^* of them. Now, there are three different types of firms in the market. For this reason, the market has to be divided into three fractions. In the triopoly case firm 1 is allotted a fragment W_1^*, while all N middle-cost firms are assigned the part W_2^* of customers. Finally, all M firms of type 3 are allocated a fraction $1 - W_1^* - W_2^*$. Since each type of firm can only access its assigned part of the market, it follows that each type of firm can only operate in its fraction, namely $\frac{1}{W_1^*}$ for firm 1, $\frac{1}{W_2^*}$ for firms of type 2 and $\frac{1}{1-W_1^*-W_2^*}$ for firms of type 3. For more details, how the inverse demand function for the Market Division technology is derived, see section 4.3. That means the inverse demand functions are given by the following equations:

$$P_1 = P\left(\frac{q_1}{W_1^*}\right) = 1 - \frac{q_1}{W_1^*} \tag{7.116}$$

$$P_2 = P\left(\frac{N \cdot q_2}{W_2^*}\right) = 1 - \frac{N \cdot q_2}{W_2^*} \tag{7.117}$$

$$P_3 = P\left(\frac{M \cdot q_3}{1 - W_1^* - W_2^*}\right) = 1 - \frac{M \cdot q_3}{1 - W_1^* - W_2^*}. \tag{7.118}$$

Thus, the corresponding profit functions look like:

$$\Pi_1\left(\frac{q_1}{W_1^*}\right) = \left(\theta_1 - \frac{q_1}{W_1^*}\right) \cdot q_1 \tag{7.119}$$

$$\Pi_2\left(\frac{N \cdot q_2}{W_2^*}\right) = \left(\theta_2 - \frac{N \cdot q_2}{W_2^*}\right) \cdot q_2 \tag{7.120}$$

$$\Pi_3\left(\frac{M \cdot q_3}{1 - W_1^* - W_2^*}\right) = \left(\theta_3 - \frac{M \cdot q_3}{1 - W_1^* - W_2^*}\right) \cdot q_3. \tag{7.121}$$

As in section 4.3, the profit functions are only dependent on the produced quantity of the own type of firm. The aim of each firm is to maximize its own profit.

$$\max_{0 \le q_1 < \infty} \Pi_1\left(\frac{q_1}{W_1^*}\right) = \max_{0 \le q_1 < \infty} \left(\theta_1 - \frac{q_1}{W_1^*}\right) \cdot q_1 \tag{7.122}$$

$$\max_{0 \le q_2 < \infty} \Pi_2\left(\frac{N \cdot q_2}{W_2^*}\right) = \max_{0 \le q_2 < \infty} \left(\theta_2 - \frac{N \cdot q_2}{W_2^*}\right) \cdot q_2 \tag{7.123}$$

$$\max_{0 \le q_3 < \infty} \Pi_3\left(\frac{M \cdot q_3}{1 - W_1^* - W_2^*}\right) = \max_{0 \le q_3 < \infty} \left(\theta_3 - \frac{M \cdot q_3}{1 - W_1^* - W_2^*}\right) \cdot q_3 \tag{7.124}$$

Maximization yields the following FOC's:

$$\frac{\partial \Pi_1}{\partial q_1} = \theta_1 - \frac{2}{W_1^*} \cdot q_1 \stackrel{!}{=} 0 \tag{7.125}$$

$$\frac{\partial \Pi_2}{\partial q_2} = \theta_2 - \frac{2N}{W_2^*} \cdot q_2 \stackrel{!}{=} 0 \tag{7.126}$$

$$\frac{\partial \Pi_3}{\partial q_3} = \theta_3 - \frac{2M}{1 - W_1^* - W_2^*} \cdot q_3 \stackrel{!}{=} 0. \tag{7.127}$$

Therefore, the explicit quantities for each type of firm are given by:

$$q_1 = \frac{W_1^*}{2} \cdot \theta_1 \tag{7.128}$$

$$q_2 = \frac{W_2^*}{2N} \cdot \theta_2, \text{ respectively} \qquad Nq_2 = \frac{W_2^*}{2} \cdot \theta_2 \tag{7.129}$$

$$q_3 = \frac{1 - W_1^* - W_2^*}{2M} \cdot \theta_3, \text{ respectively} \qquad Mq_3 = \frac{1 - W_1^* - W_2^*}{2M} \cdot \theta_3. \tag{7.130}$$

Since the quantities are immediately derived from the FOC's, the reaction function for firms of type 3 are determined separately. Hence, the fraction W_1^*, respectively W_2^*, is expressed from equation (7.128), respectively from equation (7.129).

$$W_1^* = \frac{2}{\theta_1} \cdot q_1 \tag{7.131}$$

$$W_2^* = \frac{2N}{\theta_2} \cdot q_2 \tag{7.132}$$

The explicit reaction function of one high-cost firm is obtained in setting the values for W_1^* and W_2^* from above in the explicit quantity q_3 from equation (7.130):

$$q_3^{MD} = \frac{\theta_3}{2M} \cdot \left(1 - \frac{2}{\theta_1} \cdot q_1 - \frac{2N}{\theta_2} \cdot q_2\right). \tag{7.133}$$

Furthermore, the quantities from equations (7.128) – (7.130) are inserted into the profit functions of the corresponding type of firm to calculate the subsequent explicit profits.

$$\Pi_1^{MD} = \frac{W_1^*}{4} \cdot \theta_1^2 \tag{7.134}$$

$$\Pi_2^{MD} = \frac{W_2^*}{4N} \cdot \theta_2^2, \text{ respectively} \qquad N\Pi_2^{MD} = \frac{W_2^*}{4} \cdot \theta_2^2 \tag{7.135}$$

$$\Pi_3^{MD} = \frac{1 - W_1^* - W_2^*}{4M} \cdot \theta_3^2, \text{ respectively} \qquad M\Pi_3^{MD} = \frac{1 - W_1^* - W_2^*}{4} \cdot \theta_3^2 \tag{7.136}$$

To determine the cooperative payoff region, the fragment W_1^* is derived from equation (7.134), while W_2^* is deduced from equation (7.135). As a result of setting these fragments in the profit of one firm of type 3, the following explicit cooperative payoff region is given by the third equation below.

$$W_1^* = \frac{4}{\theta_1^2} \cdot \Pi_1^{MD} \tag{7.137}$$

$$W_2^* = \frac{4N}{\theta_2} \cdot \Pi_2^{MD} \tag{7.138}$$

$$\Pi_3^{MD} = \frac{\theta_3^2}{4M} \cdot \left(1 - \frac{4}{\theta_1^2} \cdot \Pi_1^{MD} - \frac{4N}{\theta_2} \cdot \Pi_2^{MD}\right) \tag{7.139}$$

Finally, the market price of all three types of firms are determined.

$$P_1 = 1 - \frac{q_1}{W_1^*} = 1 - \frac{\theta_1}{2} \tag{7.140}$$

$$P_2 = 1 - \frac{N \cdot q_2}{W_2^*} = 1 - \frac{\theta_2}{2} \tag{7.141}$$

$$P_3 = 1 - \frac{M \cdot q_3}{1 - W_1^* - W_2^*} = 1 - \frac{\theta_3}{2} \tag{7.142}$$

From the equations above can be seen that each type of firm charges its own monopoly price. That means that each type of firm demand a different price. As a consequence, a mechanism, which can prevent arbitrage, should also be found in the triopoly case.

7.3.5. Proportional Reduction

In the Proportional Reduction technology the market shares are retained at the share in Cournot competition, as aforementioned in section 4.4. Then the quantities of all firms are lowered proportional. Therefore, the market share of firm 1 is preserved at the Cournot share of firm 1, namely $S = \frac{q_1^C}{Q^C}$, see equation (7.77), while the share of firms of type 2 is maintained at $S_2 = \frac{Nq_2^C}{Q^C}$, see equation (7.78). In addition, firms of type 3's market share is retained at the share in Cournot of firms of type 3, which is $S_3 = \frac{Mq_3^C}{Q^C}$ from equation (7.79). Hence, market shares have to be preserved at the following ratio:

$$S : S_2 : S_3 \quad \Rightarrow \quad q_1^C : Nq_2^C : Mq_3^C. \tag{7.143}$$

Evidently, this ratio characterizes a linear function in three dimensions. More precisely, it describes a plane from the origin to the Cournot point $\left(q_1^C,\, Nq_2^C,\, Mq_3^C\right)$. Thus, all values are feasible, for which the ratio (7.143) holds and which lie between the origin and the point of the Cournot quantities. This are all points, which satisfy the following $\left(\lambda \cdot q_1^C,\, \lambda \cdot Nq_2^C,\, \lambda \cdot Mq_3^C\right)$

for $\lambda \in [0,1]$. All feasible quantities in the PR technology are given by:

$$\left(q_1^{PR},\, Nq_2^{PR},\, Mq_3^{PR}\right) = \left(\lambda \cdot q_1^C,\, \lambda \cdot Nq_2^C,\, \lambda \cdot Mq_3^C\right) \quad \text{for } \lambda \in [0,1]. \tag{7.144}$$

Since the formulas are long, each coordinate is separately presented:

$$q_1^{PR} = \frac{\lambda}{2+N+M} \cdot [(1+N+M)\cdot\theta_1 - N\theta_2 - M\theta_3] \tag{7.145}$$

$$Nq_2^{PR} = \frac{N\cdot\lambda}{2+N+M} \cdot [(2+M)\cdot\theta_2 - \theta_1 - M\theta_3] \tag{7.146}$$

$$Mq_3^{PR} = \frac{M\cdot\lambda}{2+N+M} \cdot [(2+N)\cdot\theta_3 - \theta_1 - N\theta_2]. \tag{7.147}$$

Furthermore, the subsequent ratio has to hold for the quantities:

$$q_1^{PR} : Nq_2^{PR} : Mq_3^{PR} = q_1^C : Nq_2^C : Mq_3^C. \tag{7.148}$$

This is a problem of a triple ratio, which can also be described by:

$$\frac{q_1^{PR}}{q_1^C} = \frac{q_2^{PR}}{q_2^C} = \frac{q_3^{PR}}{q_3^C}. \tag{7.149}$$

Consequently, the profits functions look like:

$$\Pi_1^{PR} = \left(\theta_1 - q_1^{PR} - N\cdot q_2^{PR} - M\cdot q_3^{PR}\right)\cdot q_1^{PR} \tag{7.150}$$

$$N\Pi_2^{PR} = \left(\theta_2 - q_1^{PR} - N\cdot q_2^{PR} - M\cdot q_3^{PR}\right)\cdot Nq_2^{PR} \tag{7.151}$$

$$M\Pi_2^{PR} = \left(\theta_2 - q_1^{PR} - N\cdot q_2^{PR} - M\cdot q_3^{PR}\right)\cdot Mq_3^{PR}. \tag{7.152}$$

In setting the Cournot quantities in the corresponding profit function, the following profits for each type of firm is obtained:

$$\Pi_1^{PR} = \frac{\lambda}{(2+N+M)^2}\cdot[(2+N+M)\cdot\theta_1 - \lambda\cdot(\theta_1 + N\cdot\theta_2 + M\cdot\theta_3)] \cdot [(1+N+M)\cdot\theta_1 - N\theta_2 - M\theta_3] \tag{7.153}$$

$$N\Pi_2^{PR} = \frac{N\cdot\lambda}{(2+N+M)^2}\cdot[(2+N+M)\cdot\theta_2 - \lambda\cdot(\theta_1 + N\cdot\theta_2 + M\cdot\theta_3)] \cdot [(2+M)\cdot\theta_2 - \theta_1 - M\theta_3] \tag{7.154}$$

$$M\Pi_2^{PR} = \frac{M\cdot\lambda}{(2+N+M)^2}\cdot[(2+N+M)\cdot\theta_2 - \lambda\cdot(\theta_1+N\cdot\theta_2+M\cdot\theta_3)]$$
$$\cdot[(2+N)\cdot\theta_3-\theta_1-N\theta_2]\,. \tag{7.155}$$

7.3.6. Bargaining solution methods

In this section, the question is discussed if the solution methods from section 5 can be extended to more than two players, especially for three players.

According to Krishna and Serrano (1996) [9], the Nash solution method introduced in section 5.1 can be expanded to more players, namely to n players, without the need of a modification (cf. Krishna and Serrano 1996, page 61) [9]. That means the bargaining method of Nash can be applied to the triopoly case.

Can the K-S solution method be applied to bargaining games with n players? Roth (1979) [14] ascertained that the method of Kalai-Smorodinsky does not always satisfy the axiom of Pareto optimality, and thus the generalization to n players cannot be done easily (cf. Roth 1979, page 60). However, Karos, Muto and Rachmilevitch (2017) [8] and others tried to solve this problem. The strategy of Karos, Muto and Rachmilevitch (2017) [8] is to use a "efficiency-free axiomatization", in which the K-S and the EG method are considered, and to apply not such strong axioms (cf. Karos, Muto and Rachmilevitch 2017, page 2) [8].

Furthermore, the $W^* = S$ solution method, which is only applicable for the Market Division technology, can be expanded to more players. In the triopoly case, this would mean that the fraction W_1^*, which was assigned to firm 1, now equals the market share of firm 1 in Cournot competition, which is S from equation (7.77). Expressed in formulas, the following holds $W_1^* = S$. Consequently, firms of type 2's fragment is equated with their share in Cournot equilibrium. Thus, the fraction of one middle-cost firm is then identical to its share: $\frac{W_2^*}{N} = \frac{S_2}{N}$. In this method, the fragment of a single firm of type 3 equal its market share: $\frac{1-W_1^*-W_2^*}{N} = \frac{1-S-S_2}{N}$.

8. Conclusion

To sum up the objective and the results of this research, first a few words should be said about what can be learned from Part I, particularly from Schmalensee's approach (1987) [17]. Then, whether new insights can be obtained from the extensions of Schmalensee's model in Part II.

The symmetric cost case makes sense in theory, but not that much in reality, since two different firms would rarely have identical costs in the real world. This research was carried out to determine if there is another way of making collusion possible, because the maximization of joint profits, while assuming asymmetric costs, leads to an algebraic problem. Moreover, if compensation payments are not allowed, the production performed in collusion has to be inefficient.

Therefore, the simple model introduced by Schmalensee (1987) [17] is described in full detail to make it easier to follow for other interested researchers. Furthermore, this simple model with asymmetric costs better reflects reality. To find whether there is another possible way for firms to collude in quantity, four collusive technologies that originate from Schmalensee (1987) [17] are analysed. These technologies provide a cooperative payoff region, from which a respective optimal solution is found. For that reason, four solution methods from the axiomatic bargaining theory are applied to the collusion technologies, which were chosen from Schmalensee (1987) [17]. An entirely numeric approach was selected from Schmalensee (1987) [17], due to the fact that not all collusion technologies were analytically solvable for every solution method, especially for the Market Sharing technology. Following this, Schmalensee's results (1987) [17] were discussed. The main result found that if side payments are not allowed, because of competition laws or other restraints, collusion can be possible, but will most probably not be very profitable for the low-cost firm. In other words, in the absence of the Side Payments technology, firm 1's additional profit from collusion is relatively small in comparison to that of the high-cost firms. This is due to the already high profit of the low-cost firm in Cournot competition, with which the profit of both firms in collusion is compared to. Therefore, it is even more advantageous for the high-cost firms, since in comparison to firm 1 their profit in Cournot competition is relatively low.

In Part II, some extensions are presented. First, the inverse demand function and then the cost function is expanded. Furthermore, the duopoly was extended to a triopoly. In addition to firm 1, were added N middle-cost firms and M high-cost firms. In the expansion to the triopoly, it was observed that the complexity of the problem increases with only one additional type of firm. This was especially the case for the Market Sharing technology, which became too complex.

8.1. Discussion

To conclude, there are two respective parameters, with which the market can be described. These parameters are either the market share of firm 1 (S) and the number of high-cost firms (N), or the relative difference in costs (R) and N. There are some differences in the profitability of collusion for each firm, when the two parameters are varied. The most important ones will be explained. With a higher cost difference R, the low-cost firm has a greater advantage from Cournot competition. It is the same, when firm 1 has a large market share, since R can be expressed through N and S. In comparison, the high-cost firms benefit from a larger number of high-cost firms, for the reason that the bargaining power raises with the number of high-cost firms.

Due to the reason that the low-cost firm does not achieve much more profit from collusion, it is a debatable point whether firm 1 will agree upon such a collusive outcome. Moreover, without making an analysis of stability, it can be concluded that especially under the MD, MS and the PR technology the low-cost firm would most likely not make such a collusive agreement, since in comparison to the high-cost firms firm 1 attains much less profit additionally to Cournot competition. In contrast to that, firms of type 2 have a high increase in profit from collusion. The difference becomes even greater with an increased number of high-cost firms and higher cost difference.

It can be discussed whether the PR technology does make sense, since a quality statement cannot be made about its performance compared to the other technologies. That can be seen in the completely different curves of the PR technology in section 4.5. However, it is a good and simply way to set 'output quotas' and is thus a simplification of the more complex MS.

8.2. Prospect

Which improvements can be made for further research, such that a collusive outcome can be achieved and it reflects reality in a good way?

Firstly, an analysis of stability can be made, because in this study stability of the collusive outcome is not proven. But in reality it will be an issue, if the collusive outcome is stable against cheating, since the cartel will make sure that each firm holds on to the assigned 'output quotas'. Therefore, some types of punishments can also be introduced.

Secondly, the cost function can be generalized. For that reason, a change of the inverse demand function is needed in addition to a modification of the cost function. That means the demand and the cost function are simultaneous modified such that the firms earn a positive profit.

Thirdly, maybe a new technology can be found, which assigns ‘output quotas’ such that discrepancies in the difference of additional profit can be better reflected.

Finally, it may be possible to find a completely new approach.

Appendix A

A.1. Values of R

Schmalensee (1987) [17] employed three different values for the parameter R. Here the determination of these three values, which I call $R_{\text{low}}, R_{\text{middle}}$ and R_{high} according to its level of relative difference in costs, are shown. This computations are presented, since these different values of R are needed both for the graphic representation of the collusion technologies, which concerns all graphs in this thesis, and to calculate the results of the solution methods shown in table 6.1 and 6.2, because an entirely numeric approach is chosen. Schmalensee only used all three values for $N = 1$, except R_{middle} is also applied to $N = 3$ and $N = 5$. He also stated the dependent value for S. Accordingly, these values are named S_R^N. In particular, these are $S_{\text{low}}^{N=1}$, $S_{\text{middle}}^{N=1}$, $S_{\text{high}}^{N=1}$, followed by $S_{\text{middle}}^{N=3}$ and $S_{\text{middle}}^{N=5}$. From section 3, the following equations are known (3.5), (3.8) and (3.6), respectively (3.7). These equations are shortly repeated:

$$R = \frac{\theta_1}{\theta_2} \tag{A.1.1}$$

$$R = \frac{N \cdot (1+S)}{N+1-S} \tag{A.1.2}$$

$$S = \frac{q_1^C}{q_1^C + Nq_2^C} = \frac{(1+N)\cdot\theta_1 - N\theta_2}{\theta_1 + N\theta_2}. \tag{A.1.3}$$

Besides Schmalensee (1987) [17] provided different values for the parameters R and S. Given each pair of these values, the values for the cost parameters θ_1 and θ_2 are determined. Therefore, each value, which satisfy the following equations and inequality, is a valid cost parameter for the corresponding value of R and S.

$$\theta_1 = R \cdot \theta_2 \tag{A.1.4}$$

$$S_R^N = \frac{(1+N)\cdot\theta_1 - N\theta_2}{\theta_1 + N\theta_2} \quad \text{and} \quad \theta_2 < \theta_1 < 2\theta_2 \tag{A.1.5}$$

To compute both cost parameters, the parameter θ_2 is iterated over the interval $(0,1)$ in 0.05 steps. In addition, it has to be verified, if the equations and the inequality (A.1.4) and (A.1.5) from above are satisfied. Why the open interval $(0,1)$ is used and not the closed one $[0,1]$ is easily explained. To be exact, the cost parameter θ_2 must not reach the value 0, since then the formula $R = \frac{\theta_1}{\theta_2}$ cannot be applied, because that would mean to divide by zero. On the other hand, θ_2 cannot take the value 1, since then the equation $\theta_2 < \theta_1$ is not fulfilled, because θ_1 cannot reach a value above 1. Analogue, also θ_1 could be taken in steps of 0.05 and then the values for θ_2 could be determined.

The conditions (A.1.4) and (A.1.5) are fulfilled for many values. All these values could be used for the plots of all graphs in this thesis and the calculations of the results in section 6. For simplicity and to have accurate calculations, the most precise values are chosen, which are presented in table A.1. If different values than in table A.1 are selected, which satisfy the equations (A.1.4) and (A.1.5), all figures plotted in this thesis will keep its shape. This is due to the fact that the proportion of the respective R in equation (A.1.4) does not change. I tested this and plotted graphs with some other values, which fulfill the equations (A.1.4) and (A.1.5), and the shape always stayed the same, just the range of values changed.

Some of the values of R and S selected by Schmalensee are not exact, which means they are rounded. The exact value can be determined from equation (A.1.2) or respectively from equation (A.1.3).

In conclusion, the values, which are used in this thesis, are summed up in the following table A.1:

Table A.1: Summary of the used values

R	S	N	θ_1	θ_2
$R_{\text{low}} = 1.\overline{142857} = \frac{8}{7}$	$S_{\text{low}}^{N=1} = 0.6 = \frac{3}{5}$	1	$0.8 = \frac{4}{5}$	$0.7 = \frac{7}{10}$
$R_{\text{middle}} = 1.4 = \frac{7}{5}$	$S_{\text{middle}}^{N=1} = 0.75 = \frac{3}{4}$	1	$0.7 = \frac{7}{10}$	$0.5 = \frac{1}{2}$
$R_{\text{high}} = 1.\overline{72} = \frac{19}{11}$	$S_{\text{high}}^{N=1} = 0.9 = \frac{9}{10}$	1	$0.95 = \frac{19}{20}$	$0.55 = \frac{11}{20}$
$R_{\text{middle}} = 1.4 = \frac{7}{5}$	$S_{\text{middle}}^{N=3} = 0.5\overline{90} = \frac{13}{22}$	3	$0.7 = \frac{7}{10}$	$0.5 = \frac{1}{2}$
$R_{\text{middle}} = 1.4 = \frac{7}{5}$	$S_{\text{middle}}^{N=5} = 0.53125 = \frac{17}{32}$	5	$0.7 = \frac{7}{10}$	$0.5 = \frac{1}{2}$

To be precise, Schmalensee (1987) [17] published the following values. For $N = 1$ and small values of R, he chose $R = 1.143$ and $S = 0.60$, while he selected $R = 1.4$ and $S = 0.75$ for values of R in the middle of its interval. In addition, Schmalensee adopted $R = 1.727$ and $S = 0.90$ for a relative big difference in costs and $N = 1$. Furthermore, for $R = 1.4$ he chose on the one hand $S = 0.59$ and $N = 3$ and on the other hand $S = 0.53$ and $N = 5$.

A.2. Derivation of the parameter S

Remember the parameter S characterizes the market share of firm 1 in Cournot equilibrium. The first equation just states the sentence directly above in formula, see section 3.

$$S = \frac{q_1^C}{Q^C} = \frac{q_1^C}{q_1^C + N \cdot q_2^C} \tag{A.2.1}$$

$$= \frac{\frac{1}{2+N} \cdot [(1+N) \cdot \theta_1 - N\theta_2]}{\frac{1}{2+N} \cdot [(1+N) \cdot \theta_1 - N\theta_2] + \frac{N}{2+N} \cdot (2\theta_2 - \theta_1)} \tag{A.2.2}$$

$$= \frac{(1+N) \cdot \theta_1 - N\theta_2}{\theta_1 + N\theta_2} \tag{A.2.3}$$

$$= \frac{\frac{\theta_1}{\theta_2} \cdot (1+N) - N}{N + \frac{\theta_1}{\theta_2}} \tag{A.2.4}$$

$$\overset{\frac{\theta_1}{\theta_2}=R}{=} \frac{R \cdot (1+N) - N}{N+R} \tag{A.2.5}$$

In the second step, will be to set the Cournot quantities from equation (3.16) and (3.17) are set in the first formula. Then the equation is simplified. In the next-to-last step, a small trick is used. The whole expression above and below the fraction line is divided by θ_2 to obtain the term $\frac{\theta_1}{\theta_2}$. This is exactly the ratio of $R = \frac{\theta_1}{\theta_2}$. In the last step, the formula of R from above is just switched with its character R.

Now the objective is to express the parameter R as a formula of the parameters S and N. The following equations show the easy transformations:

$$S \cdot (N+R) = R \cdot (1+N) - N \tag{A.2.6}$$

$$SN + N = R \cdot (N + 1 - S) \tag{A.2.7}$$

$$R = \frac{N \cdot (S+1)}{N+1-S}. \tag{A.2.8}$$

The last equation is exactly the equation (3.8) stated at the end of section 3.

Finally, the range of values of S is determined. Remember the interval of R, which is $1 < R < 2$ and the equation (3.8) from section 3, respectively (A.2.8). First I derive the lower bound of S and afterwards the upper bound. This is done by taking the lower bound of R and setting it smaller than the new formula of R:

$$1 < \frac{N \cdot (S+1)}{N+1-S} \tag{A.2.9}$$

$$N+1-S < N \cdot (S+1) \tag{A.2.10}$$

$$\frac{1}{N+1} < S. \tag{A.2.11}$$

In the next steps, S is expressed to obtain the lower bound of S. By the way the sign does not change with the multiplication, because the term $N+1-S$ is positive. That is, since N is a natural number, which does not take a value below 1 and the market share S cannot reach a value above 1, because firm 1 cannot have more than 100% market share. Hence, there is already a upper bound for the parameter S. For completeness, the calculation of the upper bound is shown:

$$\frac{N \cdot (S+1)}{N+1-S} < 2 \tag{A.2.12}$$

$$N \cdot (S+1) < 2 \cdot (N+1-S) \tag{A.2.13}$$

$$S < 1. \tag{A.2.14}$$

To sum up, the parameter S lies inbetween $\frac{1}{N+1} < S < 1$.

A.3. Monopoly

Here it will be derived how a firm behaves as a monopolist in the market, particularly the monopoly quantity and profit. Remember that this demand function (2.1) and this cost function (3.1) is used. With the assumption that firm 1, respectively N high-cost firms, has a monopoly position in the market, the subsequent profit function results in:

$$\Pi_1(Q) = \Pi_1(q_1) = (\theta_1 - q_1) \cdot q_1, \text{ respectively} \tag{A.3.1}$$

$$N\Pi_2(Q) = N\Pi_2(Nq_2) = (\theta_2 - Nq_2) \cdot Nq_2. \tag{A.3.2}$$

Certainly, the total quantity of the industry is in this case just the quantity of firm 1, respectively N high-cost firms, as the only firm, respectively firms, in the market. Thus, $Q = q_1$, respectively $Q = Nq_2$. Now the goal of the monopolist will be to maximize its own profit:

$$\max_{0 \leq q_1 < \infty} \Pi_1(q_1) = (\theta_1 - q_1) \cdot q_1, \text{ respectively} \tag{A.3.3}$$

$$\max_{0 \leq q_2 < \infty} N\Pi_2(Nq_2) = (\theta_2 - Nq_2) \cdot Nq_2. \tag{A.3.4}$$

Therefore, the derivation of the profit is the following:

$$\frac{\partial \Pi_1(q_1)}{\partial q_1} = \theta_1 - 2q_1 \stackrel{!}{=} 0, \text{ respectively} \tag{A.3.5}$$

$$\frac{\partial N\Pi_2(Nq_2)}{\partial q_2} = N \cdot (\theta_2 - 2Nq_2) \stackrel{!}{=} 0. \tag{A.3.6}$$

In expressing q_i for $i = 1,2$, respectively Nq_2, the monopoly quantity for the respective firm are obtained:

$$q_1^M = \frac{\theta_1}{2}, \text{ respectively} \tag{A.3.7}$$

$$Nq_2^M = \frac{\theta_2}{2}, \text{ respectively} \qquad q_2^M = \frac{\theta_2}{2N}. \tag{A.3.8}$$

The monopoly profit can directly be determined in setting the monopoly quantity in the profit function (A.3.1).

$$\Pi_1^M = \left(\theta_1 - \frac{\theta_1}{2}\right) \cdot \frac{\theta_1}{2} = \frac{\theta_1^2}{4} = (q_1^M)^2, \text{ respectively} \tag{A.3.9}$$

$$N\Pi_2^M = \left(\theta_2 - \frac{\theta_2}{2}\right) \cdot \frac{\theta_2}{2} = \frac{\theta_2^2}{4} = \left(Nq_2^M\right)^2, \text{ respectively } \Pi_2^M = \frac{\theta_2^2}{4N} = N \cdot \left(q_2^M\right)^2 \tag{A.3.10}$$

Additionally, the monopoly price, characterized by P_i for $i = 1,2$, is calculated by means of the demand function (2.1).

$$P_i^M = 1 - Q^M = 1 - \frac{\theta_i}{2} \quad \text{for } i = 1,2 \tag{A.3.11}$$

A.4. Problem of Collusion with three different firms

The calculations in this section are analogue to section 2.2. The only difference is that there are now three different firms competing against each other in the market. Thus there is one single low-cost firm, N middle-cost firms and M high-cost firms operating in the market. Hence the joint profit sums up to $\Pi = \Pi_1 + N \cdot \Pi_2 + M \cdot \Pi_3$. Accordingly the total quantity is

$Q = q_1 + N \cdot q_2 + M \cdot q_3$. Since there are three firms with different costs, I assume $c_1 \neq c_2 \neq c_3$. Without loss of generality I can also suppose $c_1 < c_2 < c_3$. The known demand function (2.1) and cost function (2.2) is used.

$$\Pi = (1-Q-c_1)\cdot q_1 + N\cdot(1-Q-c_2)\cdot q_2 + M\cdot(1-Q-c_3)\cdot q_3, \quad \text{where } c_1 < c_2 < c_3 \quad \text{(A.4.1)}$$

As in the case with two different firms, here it is also important to assume asymmetric costs. Now the joint profit will be maximized by taking the derivation respective to the quantities q_i for $i = 1,2,3$.

$$\frac{\partial \Pi(Q)}{\partial q_1} = 1 - 2q_1 - 2Nq_2 - 2Mq_3 - c_1 \stackrel{!}{=} 0 \quad \text{(A.4.2)}$$

$$\frac{\partial \Pi(Q)}{\partial q_2} = 1 - 2q_1 - 2Nq_2 - 2Mq_3 - c_2 \stackrel{!}{=} 0 \quad \text{(A.4.3)}$$

$$\frac{\partial \Pi(Q)}{\partial q_3} = 1 - 2q_1 - 2Nq_2 - 2Mq_3 - c_3 \stackrel{!}{=} 0 \quad \text{(A.4.4)}$$

To achieve the reaction function for q_i for $i = 1,2,3$, these equations above have to be solved for each q_i for $i = 1,2,3$:

$$\iff q_1 = \begin{cases} \frac{1}{2}\cdot(1 - 2Nq_2 - 2Mq_3 - c_1) & \text{for} \quad Nq_2 + Mq_3 \leq \frac{1-c_1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{(A.4.5)}$$

$$\iff q_2 = \begin{cases} \frac{1}{2N}\cdot(1 - 2q_1 - 2Mq_3 - c_2) & \text{for} \quad q_1 + Mq_3 \leq \frac{1-c_2}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{(A.4.6)}$$

$$\iff q_3 = \begin{cases} \frac{1}{2M}\cdot(1 - 2q_1 - 2Nq_2 - c_3) & \text{for} \quad q_1 + Nq_2 \leq \frac{1-c_3}{2} \\ 0 & \text{otherwise} \end{cases}. \quad \text{(A.4.7)}$$

This solution system consisting of the three equations directly above is normally solved by setting q_i in Q_{-i}. But trying to solve the three equations above, lead to an elimination of the quantities q_i for $i = 1,2,3$ and the following equation is obtained:

$$\iff c_1 = c_2 = c_3 \quad \lightning. \quad \text{(A.4.8)}$$

Thus, the same result as in section 2.2 is achieved with the same conclusion that the problem cannot be solved in this way.

A.5. MS: Second order conditions

The second order condition from equations (4.14) and (4.15) are determined to ensure that the profits of both types of firms are maximized. With the chain rule the subsequent SOC's are obtained:

$$\frac{\partial^2 N\Pi_2(\Pi_1)}{\partial Q^2} = -2-2\cdot\frac{\Pi_1\cdot(\theta_1-\theta_2)}{(\theta_1-Q)^3} < 0 \tag{A.5.1}$$

$$\frac{\partial^2 \Pi_1(\Pi_2)}{\partial Q^2} = -2-2\cdot\frac{N\Pi_2\cdot(\theta_2-\theta_1)}{(\theta_2-Q)^3} < 0. \tag{A.5.2}$$

Both equations above are smaller than zero, since the profit $\Pi_i = (\theta_i - Q)\cdot q_i$ for $i = 1,2$ should be positive. For that reason, also the expression $\theta_i - Q$ for $i = 1,2$ does not get negative. In addition, from equation (3.2) it follows that $\theta_1 < \theta_2$. Furthermore, the second SOC from above is negative, since $N\Pi_2$, accordingly $\theta_2 - Q$, cannot take values above 1, because of $\theta_i \in [0,1]$. That means the SOC's are satisfied and the equations (4.16) and (4.17) take a maximum value.

A.6. MS: determination of the roots of the reaction functions

In this section, the calculations of the zeros of the reaction functions q_1 form equation (4.19) on page 16 and q_2 from equation (4.23) on page 17 are shown. Starting with the computations of q_1 from equation (4.19), the following is given:

$$(\theta_1 - q_1 - Nq_2)\cdot\left[\frac{(\theta_1 - q_1 - Nq_2)\cdot(2\cdot(q_1+Nq_2)-\theta_2)}{\theta_1-\theta_2} - q_1\right] = 0. \tag{A.6.1}$$

This is a cubic equation, which implicates that it has three solutions. In the following the superscript (1) – (3) characterizes each of these solutions. The cubic equation in the form $ax^3+bx^2+cx+d=0$ looks like:

$$\begin{aligned}2\cdot q_1^3 + q_1^2\cdot(6Nq_2-3\theta_1-2\theta_2) + q_1\cdot\left(6N^2q_2^2 - 7N\theta_1q_2 - 3N\theta_2q_2 + \theta_1^2 + 3\theta_1\theta_2\right)\\ + \left(2N^3q_2^3 - 4N^2\theta_1q_2^2 - N^2\theta_2q_2^2 + 2N\theta_1^2q_2 + 2N\theta_1\theta_2q_2 - \theta_1^2\theta_2\right) = 0.\end{aligned} \tag{A.6.2}$$

The equation above is obtained by simply expanding equation (A.6.1). If the Horner schema or the polynomial division is applied to equation (A.6.2) with the 'guessed' zero point (A.6.4), the results can be verified and the equation (A.6.1) is obtained. Therefore, the first solution

$q_1^{(1)}$ from equation (A.6.1) is already known, namely the 'guessed zero' point:

$$\theta_1 - q_1^{(1)} - Nq_2 = 0 \tag{A.6.3}$$

$$q_1^{(1)} = \theta_1 - Nq_2. \tag{A.6.4}$$

Now the quadratic part of the equation (A.6.1) has to be solved, which can easily be done with the small solution formula. Hence, the other two solutions $q_1^{(2,3)}$ will be determined from the following equation:

$$\left(\theta_1 - q_1^{(2,3)} - Nq_2\right) \cdot \left(2q_1^{(2,3)} + 2Nq_2 - \theta_2\right) - q_1^{(2,3)} \cdot (\theta_1 - \theta_2) = 0. \tag{A.6.5}$$

Thus, the equation above (A.6.5) is transformed into the form $x^2 + px + q = 0$. In this equation and only in this case q is not the quantity.

$$\left(q_1^{(2,3)}\right)^2 \cdot (-2) + q_1^{(2,3)} \cdot [2\theta_1 - 2Nq_2 + \theta_2 - 2Nq_2 - \theta_1 + \theta_2] + \left[2N\theta_1 q_2 - \theta_1\theta_2 - 2N^2q_2^2 + N\theta_2 q_2\right] = 0 \tag{A.6.6}$$

$$\left(q_1^{(2,3)}\right)^2 + \frac{q_1^{(2,3)}}{2} \cdot [-\theta_1 - 2\theta_2 + 4Nq_2] + \frac{1}{2} \cdot \left[\theta_1\theta_2 + 2N^2q_2^2 - Nq_2 \cdot (2\theta_1 + \theta_2)\right] = 0. \tag{A.6.7}$$

The last transformation above is done, such that the small solution formula[1] can be applied.

$$q_1^{(2,3)} = -\frac{\frac{1}{2} \cdot (-\theta_1 - 2\theta_2 + 4N \cdot q_2)}{2} \pm \sqrt{\frac{\frac{1}{4} \cdot (-\theta_1 - 2\theta_2 + 4N \cdot q_2)^2}{4} - \frac{1}{2} \cdot \left[\theta_1\theta_2 + 2N^2q_2^2 - Nq_2 \cdot (2\theta_1 + \theta_2)\right]} \tag{A.6.8}$$

$$= \frac{\theta_1 + 2\theta_2}{4} - \frac{4Nq_2}{4} \pm \sqrt{\frac{1}{16} \cdot \left[\theta_1^2 + 4\theta_2^2 + 4\theta_1\theta_2 - 8N\theta_1 q_2 - 16N\theta_2 q_2 - 8\theta_1\theta_2 + 8Nq_2 \cdot (2\theta_1 + \theta_2)\right]} \tag{A.6.9}$$

$$= \frac{\theta_1 + 2\theta_2}{4} - Nq_2 \pm \frac{1}{4} \cdot \sqrt{\theta_1^2 + 4\theta_2^2 - 4\theta_1\theta_2 + 8Nq_2 \cdot (-\theta_1 - 2\theta_2 + 2\theta_1 + \theta_2)} \tag{A.6.10}$$

[1]Recall the small solution formula for the equation $x^2 + px + q = 0$ is $x^{(1,2)} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$.

$$q_1^{(2,3)} = \frac{\theta_1 + 2\theta_2}{4} \pm \frac{1}{4} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 8N \cdot q_2 \cdot (\theta_1 - \theta_2)} - Nq_2 \tag{A.6.11}$$

The equations above are the second and third solution of the reaction function q_1.

Then, the roots of the reaction function q_2 from equation (4.23) are derived. These three roots will as well be marked with the superscript (1) – (3). Again, this is not an exponent. Additionally, the cubic equation in the form $ax^3 + bx^2 + cx + d = 0$ is given by:

$$q_2^3 \cdot \left(2N^3\right) + q_2^2 \cdot \left(6N^2 q_1 - 2N^2\theta_1 - 3N^2\theta_2\right) + q_2 \cdot \left(6Nq_1^2 - 3N\theta_1 q_1 - 7N\theta_2 q_2 + 3N\theta_1\theta_2 + N\theta_2^2\right)$$
$$+ \left(2q_1^3 - \theta_1 q_1^2 - 4\theta_2 q_1^2 + 2\theta_1\theta_2 q_1 + 2\theta_2^2 q_1 - \theta_1\theta_2^2\right) = 0. \tag{A.6.12}$$

The 'guessed zero point' was already determined in section 4.2, such that only the computation of the quadratic part of the reaction function q_2 from equation (4.23) is missing, which is in the square brackets. For that reason, the quadratic equation (A.6.13) below is transformed into the form $x^2 + px + q = 0$ to apply the small solution formula.

$$\left(\theta_2 - q_1 - Nq_2^{(2,3)}\right) \cdot \left(\theta_1 - 2q_1 - 2Nq_2^{(2,3)}\right) - Nq_2^{(2,3)} \cdot (\theta_1 - \theta_2) = 0 \tag{A.6.13}$$
$$\left(q_2^{(2,3)}\right)^2 \cdot \left(2N^2\right) + q_2^{(2,3)} \cdot [4Nq_1 - 2N\theta_1 - N\theta_2] + \left[\theta_1\theta_2 - \theta_1 q_1 - 2\theta_2 q_1 + 2q_1^2\right] = 0 \tag{A.6.14}$$
$$\left(q_2^{(2,3)}\right)^2 + \frac{q_2^{(2,3)}}{2N} \cdot [4q_1 - 2\theta_1 - \theta_2] + \frac{1}{2N^2} \cdot \left[\theta_1\theta_2 - \theta_1 q_1 - 2\theta_2 q_1 + 2q_1^2\right] = 0 \tag{A.6.15}$$

Now the small solution formula[1] is used to determine the second and third solution $q_2^{(2,3)}$.

$$q_2^{(2,3)} = -\frac{\frac{1}{2N} \cdot (4q_1 - 2\theta_1 - \theta_2)}{2} \pm \sqrt{\frac{\frac{1}{4N^2} \cdot (4q_1 - 2\theta_1 - \theta_2)^2}{4} - \frac{1}{2N^2} \cdot \left(\theta_1\theta_2 - \theta_1 q_1 - 2\theta_2 q_1 + 2q_1^2\right)} \tag{A.6.16}$$
$$= \frac{2\theta_1 + \theta_2 - 4q_1}{4N} \pm \sqrt{\frac{1}{16N^2} \cdot \left(4\theta_1^2 + \theta_2^2 + 4\theta_1\theta_2 - 16\theta_1 q_1 - 8\theta_2 q_1 - 8\theta_1\theta_2 + 8\theta_1 q_1 + 16q_1\theta_2\right)} \tag{A.6.17}$$
$$= \frac{2\theta_1 + \theta_2}{4N} - \frac{q_1}{N} \pm \frac{1}{4N} \cdot \sqrt{4\theta_1^2 + \theta_2^2 - 4\theta_1\theta_2 + 8q_1 \cdot (\theta_2 - \theta_1)} \tag{A.6.18}$$
$$= \frac{2\theta_1 + \theta_2}{4N} \pm \frac{1}{4N} \cdot \sqrt{(2\theta_1 - \theta_2)^2 - 8 \cdot q_1 \cdot (\theta_1 - \theta_2)} - \frac{q_1}{N} \tag{A.6.19}$$

The last equation describes the second and third solutions of the reaction function q_2.

A.7. MS: exclusion of the 2[nd] solutions of the reaction functions

Here I want to discuss the elimination of some of the solutions from the Market Sharing technology in section 4.2. First the elimination of the second solutions of both reaction functions, namely $q_1^{(2)}$ and $q_2^{(2)}$, is shown, since then the calculations of the exclusion of the first solutions get shorter. In the following should be proved that $q_1^{(2)}$ and $q_2^{(2)}$ are not convex, by means of an easy counterexample. If the solutions are not convex, they cannot be solved with the collusive solution methods, see more details in section 4.2. Before this is done, I recall shortly the definition of convexity. The definition of a convex function according to Heuser (2009) [6] is the following:

Definition A.1. *" Let a function f be convex in the interval I, if for each of the two points $x_1, x_2 \in I$ and for all $\lambda \in (0,1)$ holds:*

$$f\left((1-\lambda)\cdot x_1+\lambda\cdot x_2\right)\leq(1-\lambda)\cdot f(x_1)+\lambda\cdot f(x_2)\,.\text{"} \tag{A.7.1}$$

(Heuser 2009, page 282) [6][2].

For the counterexample, it is enough to find one λ for which the equation (A.7.1) above is not fulfilled. First the function $q_1^{(2)}$ from equation (4.21) is observed in the relevant interval $\left[0, \frac{\theta_2}{2N}\right]$ and then $q_2^{(2)}$ from equation (4.25) in the interval $\left[0, \frac{\theta_1}{2}\right]$. At these points there is an intersection with the axis in question.

Proof. Let $q_1^{(2)}$ be convex in the interval $\left[0, \frac{\theta_2}{2N}\right]$, if for each two points $x_1, x_2 \in \left[0, \frac{\theta_2}{2N}\right]$ and for all $\lambda \in (0,1)$ holds:

$$q_1\left((1-\lambda)\cdot x_1+\lambda\cdot x_2\right)\leq(1-\lambda)\cdot q_1(x_1)+\lambda\cdot q_2(x_1)\,. \tag{A.7.2}$$

It is incidental to choose the values on the border $x_1 = 0$ and $x_2 = \frac{\theta_2}{2N}$ for q_2 to have an easier calculation. In the following, the inequality above (A.7.2) should be proved or rather disproved. Now the value of the function $q_1^{(2)}(q_2)$ on this two particular points x_1 and x_2 is determined.

$$q_1(0)=\frac{\theta_1+2\theta_2}{4}+\frac{1}{4}\cdot\sqrt{(\theta_1-2\theta_2)^2+0}-0 \tag{A.7.3}$$

$$=\begin{cases}\frac{\theta_1}{2}\\ \theta_2\end{cases} \tag{A.7.4}$$

[2]This definition is translated from (Heuser 2009, page 282) [6].

The function $q_1^{(2)}(q_2)$ at the point $x_2 = \frac{\theta_2}{2N}$ is:

$$q_1\left(\frac{\theta_2}{2N}\right) = \frac{\theta_1 + 2\theta_2}{4} + \frac{1}{4} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 8N \cdot \frac{\theta_2}{2N} \cdot (\theta_1 - \theta_2)} - N \cdot \frac{\theta_2}{2N} \tag{A.7.5}$$

$$= \frac{\theta_1 + 2\theta_2 - 2\theta_2}{4} + \frac{1}{4} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 4 \cdot \theta_2 \cdot (\theta_1 - \theta_2)} \tag{A.7.6}$$

$$= \frac{\theta_1}{4} + \frac{1}{4} \cdot \sqrt{\theta_2^2} \tag{A.7.7}$$

$$= \begin{cases} \frac{\theta_1}{2} \\ 0. \end{cases} \tag{A.7.8}$$

Lastly, the value of the function in the following point $(1-\lambda) \cdot 0 + \lambda \cdot \frac{\theta_2}{2N}$ is calculated.

$$q_1\left((1-\lambda) \cdot 0 + \lambda \cdot \frac{\theta_2}{2N}\right) =$$

$$= \frac{\theta_1 + 2\theta_2}{4} + \frac{1}{4} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 8N \cdot \lambda \cdot \frac{\theta_2}{2N} \cdot (\theta_1 - \theta_2)} - N\lambda \cdot \frac{\theta_2}{2N} \tag{A.7.9}$$

$$= \frac{\theta_1 + 2\theta_2 - 2\lambda \cdot \theta_2}{4} + \frac{1}{4} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 4\lambda \cdot \theta_2 \cdot (\theta_1 - \theta_2)} \tag{A.7.10}$$

$$= \frac{\theta_1 + 2\theta_2 \cdot (1 - 2\lambda)}{4} + \frac{1}{4} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 4\lambda \cdot \theta_2 \cdot (\theta_1 - \theta_2)} \tag{A.7.11}$$

Then the determined values from above are set in the inequality from equation (A.7.2). In this case, the first solutions of the not unique solutions are taken.

$$\frac{\theta_1 + 2\theta_2 \cdot (1 - 2\lambda)}{4} + \frac{1}{4} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 4\lambda \cdot \theta_2 \cdot (\theta_1 - \theta_2)} \leq (1-\lambda) \cdot \frac{\theta_1}{2} + \lambda \cdot \frac{\theta_1}{2} \tag{A.7.12}$$

$$\frac{1}{4} \cdot (2\theta_2 \cdot (1 - 2\lambda) - \theta_1) + \frac{1}{4} \cdot \underbrace{\sqrt{(\theta_1 - 2\theta_2)^2 + 4\lambda\theta_2 \cdot (\theta_1 - \theta_2)}}_{>0} \leq 0 \tag{A.7.13}$$

In a counterexample just one value for λ has to be found to lead to a contradiction. Without loss of generality choose $\lambda = 0.5$. The numerical values for R_{middle} which mean $\theta_1 = 0.7$ and

$\theta_2 = 0.5$ lead to the following result of the inequality above:

$$\frac{1}{4} \cdot (-\theta_1) + \frac{1}{4} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 2\lambda \cdot \theta_2 \cdot (\theta_1 - \theta_2)} \leq 0 \tag{A.7.14}$$

$$-\theta_1 + \sqrt{\theta_1^2 - 2\theta_1\theta_2 + 2\theta_2^2} \leq 0 \tag{A.7.15}$$

$$-0.7 + 1.1358 \leq 0 \quad ↯ \tag{A.7.16}$$

The value above is rounded to 4 decimal places. It is strictly positive, which means the inequality is not fulfilled. That leads to a contradiction. Thus it is proven that the function $q_1^{(2)}$ and therefore the solution $q_1^{(2)}$ is not convex. The first part of the proof is done.

Furthermore, analogue let the function $q_2^{(2)}$ be convex in the interval $\left[0, \frac{\theta_1}{2}\right]$, if for each two points $y_1, y_2 \in \left[0, \frac{\theta_1}{2}\right]$ and for all $\lambda \in (0,1)$ holds:

$$q_2\left((1-\lambda) \cdot y_1 + \lambda \cdot y_2\right) \leq (1-\lambda) \cdot q_2(y_1) + \lambda \cdot q_2(y_1). \tag{A.7.17}$$

Without loss of generality, the values on the border $y_1 = 0$ and $y_2 = \frac{\theta_1}{2}$ are analogue selected for q_1. Then the inequality above (A.7.17) is disproved. Firstly, the value of the function $q_2^{(2)}(q_1)$ on the two specific points y_1 and y_2 is determined.

$$q_2(0) = \frac{2\theta_1 + \theta_2}{4N} + \frac{1}{4N} \cdot \sqrt{(2\theta_1 - \theta_2)^2 - 0} - 0 \tag{A.7.18}$$

$$= \frac{2\theta_1 + \theta_2}{4N} \pm \frac{1}{4N} \cdot (2\theta_1 - \theta_2) \tag{A.7.19}$$

$$= \begin{cases} \frac{\theta_1}{N} \\ \frac{\theta_2}{2N} \end{cases} \tag{A.7.20}$$

Secondly, the value of the function at the point $y_2 = \frac{\theta_1}{2}$ is calculated:

$$q_2\left(\frac{\theta_1}{2}\right) = \frac{2\theta_1 + \theta_2}{4N} + \frac{1}{4N} \cdot \sqrt{(2\theta_1 - \theta_2)^2 - 8 \cdot \frac{\theta_1}{2} \cdot (\theta_1 - \theta_2)} - \frac{1}{N} \cdot \frac{\theta_1}{2} \tag{A.7.21}$$

$$= \frac{2\theta_1 + \theta_2 - 2\theta_1}{4N} + \frac{1}{4N} \cdot \sqrt{(2\theta_1 - \theta_2)^2 - 4\theta_1 \cdot (\theta_1 - \theta_2)} \tag{A.7.22}$$

$$= \frac{\theta_2}{4N} + \frac{1}{4N} \cdot \sqrt{\theta_2^2} \tag{A.7.23}$$

$$= \begin{cases} \frac{\theta_2}{2N} \\ 0. \end{cases} \tag{A.7.24}$$

Thirdly, the determination of the function $q_1^{(2)}$ in the point $(1-\lambda)\cdot 0+\lambda\cdot\frac{\theta_1}{2}$ follows:

$$q_2\left((1-\lambda)\cdot 0+\lambda\cdot\frac{\theta_1}{2}\right)= \tag{A.7.25}$$

$$=\frac{2\theta_1+\theta_2}{4N}+\frac{1}{4N}\cdot\sqrt{(2\theta_1-\theta_2)^2-8\lambda\cdot\frac{\theta_1}{2}\cdot(\theta_1-\theta_2)}-\lambda\cdot\frac{\theta_1}{2N} \tag{A.7.26}$$

$$=\frac{2\theta_1\cdot(1-2\lambda)+\theta_2}{4N}+\frac{1}{4N}\cdot\sqrt{(2\theta_1-\theta_2)^2-4\lambda\cdot\theta_1\cdot(\theta_1-\theta_2)}. \tag{A.7.27}$$

Lastly, the just determined values are set in the inequality (A.7.17). The second solution is taken for the non explicit values.

$$\frac{2\theta_1(1-2\lambda)+\theta_2}{4N}+\frac{1}{4N}\cdot\sqrt{(2\theta_1-\theta_2)^2-4\lambda\theta_1(\theta_1-\theta_2)}\leq(1-\lambda)\frac{\theta_2}{2N}+\lambda\cdot 0 \tag{A.7.28}$$

$$\underbrace{2\theta_1}_{>0}(1-2\lambda)+\underbrace{\theta_2}_{>0}(1-2\lambda)+\underbrace{\sqrt{(2\theta_1-\theta_2)^2-4\lambda\theta_1(\theta_1-\theta_2)}}_{>0}\leq 0 \tag{A.7.29}$$

When choosing $\lambda=0.5$, the following is obtained:

$$\sqrt{2\theta_1^2+2\theta_1\theta_2+\theta_2^2}\leq 0 \quad ↯. \tag{A.7.30}$$

It is easy to assert that the value under the root is strictly positive for $\lambda=0.5$. That means the inequality above leads to a contradiction, from which it follow that $q_2^{(2)}$ is not convex. □

The proof shows that $q_1^{(2)}$ and $q_2^{(2)}$ are not convex and hence can be excluded.

A.8. MS: exclusion of the 1st solutions of the reaction functions

Now some more eliminations of the solutions from the Market Sharing technology are shown. The second solutions are already ruled out in section A.7, because they are not convex. The first solutions cannot be eliminated in this way, since a linear function, which both solutions $q_1^{(1)}$ and $q_1^{(2)}$ are, are concave and convex at the same moment. That means for lines the equation (A.7.1) holds with an equal sign instead of a 'smaller as' sign. But of course they are not strictly convex or concave. I will not prove this, since I will not need it here. Though, it is easy to prove in setting two points in the definition from equation (A.7.1) and change the 'smaller as' sign to an equal sign. Therefore, it remains to discuss the exclusion of the first solutions. Hence, these solutions can be ruled out because of a contradiction in the following calculation. First, the solution $q_1^{(1)}$ from equation (4.20) is set in the two remaining solutions of q_2, namely $q_2^{(1)}$ and $q_2^{(3)}$. Starting with the first solution $q_2^{(1)}$ from equation (4.24), recall the just mentioned solutions, which should be eliminated:

$$q_1^{(1)} = \theta_1 - N \cdot q_2 \tag{A.8.1}$$

$$q_2^{(1)} = \frac{1}{N} \cdot (\theta_2 - q_1) \tag{A.8.2}$$

$$\Leftrightarrow \quad q_1^{(1)} = \theta_1 - N \cdot \frac{1}{N} \cdot (\theta_2 - q_1) = \theta_1 - \theta_2 + q_1 \tag{A.8.3}$$

$$\Leftrightarrow \quad \theta_1 = \theta_2 \quad \lightning. \tag{A.8.4}$$

This solutions would require identical costs, but at the beginning asymmetric costs are assumed. That is a contradiction. As well, if the solution $q_2^{(1)}$ is set in $q_1^{(1)}$, analogue the following is obtained:

$$\Leftrightarrow \quad q_2^{(1)} = \frac{1}{N} \cdot (\theta_2 - (\theta_1 - N \cdot q_2)) = \frac{1}{N} \cdot (\theta_2 - \theta_1) + q_2 \tag{A.8.5}$$

$$\Leftrightarrow \quad \theta_2 = \theta_1 \quad \lightning. \tag{A.8.6}$$

Consequently, this solution pair can be excluded. Now also the solution pair $q_1^{(1)}$ and $q_2^{(3)}$ should be eliminated. Recall the third solution of q_2 from equation (4.25):

$$q_2^{(3)} = \frac{2\theta_1 + \theta_2}{4N} - \frac{1}{4N} \cdot \sqrt{(2\theta_1 - \theta_2)^2 - 8 \cdot q_1 \cdot (\theta_1 - \theta_2)} - \frac{q_1}{N}. \tag{A.8.7}$$

In the first step, the solution $q_1^{(1)}$ is set in to the other solution $q_2^{(3)}$:

$$q_2^{(3)} = \frac{2\theta_1 + \theta_2}{4N} - \frac{1}{4N} \cdot \sqrt{(2\theta_1 - \theta_2)^2 - 8 \cdot (\theta_1 - N \cdot q_2) \cdot (\theta_1 - \theta_2)} - \frac{\theta_1 - N \cdot q_2}{N} \tag{A.8.8}$$

$$= q_2 + \frac{1}{4N} \cdot \left[2\theta_1 + \theta_2 - 4\theta_1 - \sqrt{(2\theta_1 - \theta_2)^2 - 8 \cdot \left(\theta_1^2 + \theta_1\theta_2 + Nq_2 \cdot (\theta_2 - \theta_1)\right)} \right] \tag{A.8.9}$$

$$\Leftrightarrow \quad \sqrt{(2\theta_1 - \theta_2)^2 - 8\theta_1^2 + 8 \cdot \theta_1\theta_2 - 8N \cdot q_2 \cdot (\theta_2 - \theta_1)} = \theta_2 - 2\theta_1 \tag{A.8.10}$$

$$(2\theta_1 - \theta_2)^2 - 8\theta_1^2 + 8 \cdot \theta_1\theta_2 - 8N \cdot q_2 \cdot (\theta_2 - \theta_1) = (-1)^2 \cdot (2\theta_1 - \theta_2)^2 \tag{A.8.11}$$

$$8N \cdot q_2 \cdot (\theta_2 - \theta_1) = 8 \cdot \left(\theta_1\theta_2 - \theta_1^2\right) \tag{A.8.12}$$

$$q_2 = \frac{\theta_1 \cdot (\theta_2 - \theta_1)}{(\theta_2 - \theta_1)} = \frac{\theta_1}{N}. \tag{A.8.13}$$

Furthermore, to determine q_1, the solution $q_2 = \frac{\theta_1}{N}$ is inserted in $q_1^{(1)}$:

$$q_1^{(1)} = \theta_1 - N \cdot \frac{\theta_1}{N} = 0. \tag{A.8.14}$$

The proof, if the solution is unique, is given by:

$$\begin{aligned} q_2^{(3)} &= \frac{2\theta_1 + \theta_2}{4N} - \frac{1}{4N} \cdot \sqrt{(2\theta_1 - \theta_2)^2 - 0} - \frac{0}{N} && \text{(A.8.15)} \\ &= \frac{1}{4N} \cdot (2\theta_1 + \theta_2) \mp (2\theta_1 + \theta_2) && \text{(A.8.16)} \\ &= \begin{cases} \frac{\theta_2}{2N} \\ \frac{\theta_1}{N}. \end{cases} && \text{(A.8.17)} \end{aligned}$$

This means that there is no unique solution, since $(2\theta_1 - \theta_2)^2 = (-1)^2 \cdot (\theta_2 - \theta_1)^2$. In addition, the profits for both firms for the values $q_1 = 0$ and $q_2 = \frac{\theta_1}{N}$ are determined by setting in the equations (4.16) and (4.17).

$$\begin{aligned} \Pi_1^{MS} &= \frac{(\theta_1 - Q)^2 \cdot (2Q - \theta_2)}{\theta_1 - \theta_2} && \text{(A.8.18)} \\ &= \frac{\left(\theta_1 - 0 - N \cdot \frac{\theta_1}{N}\right)^2 \cdot \left(2 \cdot 0 + 2N \cdot \frac{\theta_1}{N} - \theta_2\right)}{\theta_1 - \theta_2} && \text{(A.8.19)} \\ &= 0 && \text{(A.8.20)} \\ N\Pi_2^{MS} &= \frac{(\theta_2 - Q)^2 \cdot (\theta_1 - 2Q)}{\theta_1 - \theta_2} && \text{(A.8.21)} \\ &= \frac{\left(\theta_2 - 0 - N \cdot \frac{\theta_1}{N}\right)^2 \cdot \left(\theta_1 - 2 \cdot 0 - 2N \cdot \frac{\theta_1}{N}\right)}{\theta_1 - \theta_2} && \text{(A.8.22)} \\ &= \frac{(-1)^2 \cdot (\theta_1 - \theta_2)^2 \cdot (-\theta_1)}{\theta_1 - \theta_2} && \text{(A.8.23)} \\ &= (-\theta_1) \cdot (\theta_1 - \theta_2) < 0 \quad \lightning && \text{(A.8.24)} \end{aligned}$$

Firm 1 would have no profit at all, while firms of type 2 would have a negative profit, since $\theta_2 < \theta_1$ from equation (3.2). It is assumed that both types of firms have strictly positive profit, otherwise they would not use the MS technology. In addition, it would not be an optimal solution. For completion, also the profits for $q_1 = 0$ and $q_2 = \frac{\theta_2}{2N}$ are calculated:

$$\Pi_1^{MS} = \frac{\left(\theta_1 - 0 - N \cdot \frac{\theta_2}{2N}\right)^2 \cdot \left(2 \cdot 0 + 2N \cdot \frac{\theta_2}{2N} - \theta_2\right)}{\theta_1 - \theta_2} = 0 \tag{A.8.25}$$

$$N\Pi_2^{MS} = \frac{\left(\theta_2 - 0 - N \cdot \frac{\theta_2}{2N}\right)^2 \cdot \left(\theta_1 - 2 \cdot 0 - 2N \cdot \frac{\theta_2}{2N}\right)}{\theta_1 - \theta_2} \tag{A.8.26}$$

$$= \frac{\left(\frac{\theta_2}{2}\right)^2 \cdot (\theta_1 - \theta_2)}{\theta_1 - \theta_2} = \frac{\theta_2^2}{4}. \tag{A.8.27}$$

This would not be an optimal solution for firm 1. That means the solution pair $q_1^{(1)}$ and $q_2^{(3)}$ can be eliminated. In section 5 the particular optimal solution should be found by three of the solution methods via maximization of the relevant cooperative payoff region. If the profit is already zero, it cannot be maximized to find the better solution than zero. Thus, it is assumed that both types of firms have a strictly positive profit possibility frontier.

Finally, the solution pair $q_1^{(3)}$ and $q_2^{(1)}$ should be ruled out. Recall that $q_1^{(3)}$ is determined in equation (4.21).

$$q_1^{(3)} = \frac{\theta_1 + 2\theta_2}{4} - \frac{1}{4} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 8Nq_2 \cdot (\theta_1 - \theta_2)} - Nq_2 \tag{A.8.28}$$

Now the solution $q_1^{(3)}$ is inserted into the other solution $q_2^{(1)}$.

$$q_2^{(1)} = \frac{1}{N} \cdot \left[\theta_2 - \frac{\theta_1 + 2\theta_2}{4} + \frac{1}{4} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 8Nq_2 \cdot (\theta_1 - \theta_2)} + Nq_2\right] \tag{A.8.29}$$

$$= q_2 + \frac{1}{4N} \cdot \left[4\theta_2 - \theta_1 - 2\theta_2 + \sqrt{(\theta_1 - 2\theta_2)^2 + 8Nq_2 \cdot (\theta_1 - \theta_2)}\right] \tag{A.8.30}$$

$$\Leftrightarrow \quad \theta_1 - 2\theta_2 = \sqrt{(\theta_1 - 2\theta_2)^2 + 8Nq_2 \cdot (\theta_1 - \theta_2)} \tag{A.8.31}$$

$$(\theta_1 - 2\theta_2)^2 = (\theta_1 - 2\theta_2)^2 + 8Nq_2 \cdot (\theta_1 - \theta_2) \tag{A.8.32}$$

$$q_2 = 0 \tag{A.8.33}$$

Furthermore, the new solution $q_2 = 0$ is set in the reaction function $q_1^{(3)}$.

$$\Leftrightarrow \quad q_1^{(3)} = \frac{\theta_1 + 2\theta_2}{4} - \frac{1}{4} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 0} - 0 \tag{A.8.34}$$

$$= \frac{1}{4} \cdot [\theta_1 + 2\theta_2 \mp (\theta_1 - 2\theta_2)] \tag{A.8.35}$$

$$= \begin{cases} \theta_2 \\ \frac{\theta_1}{2} \end{cases} \tag{A.8.36}$$

Hence, the solution above is not unique, because of $(\theta_1 - 2\theta_2)^2 = (-1)\cdot(2\theta_2 - \theta_1)^2$. Moreover, the profits of firm 1 and firms of type 2 are calculated for the values $q_1 = \theta_2$ and $q_2 = 0$:

$$\Pi_1^{MS} = \frac{(\theta_1 - \theta_2 - N\cdot 0)^2 \cdot (2\theta_2 - 2N\cdot 0 - \theta_2)}{\theta_1 - \theta_2} \tag{A.8.37}$$

$$= \theta_2 \cdot (\theta_1 - \theta_2) \tag{A.8.38}$$

$$N\Pi_2^{MS} = \frac{(\theta_2 - Q)^2 \cdot (\theta_1 - 2Q)}{\theta_1 - \theta_2} \tag{A.8.39}$$

$$= \frac{(\theta_2 - \theta_2 - N\cdot 0)^2 \cdot (\theta_1 - 2\theta_2 - 2N\cdot 0)}{\theta_1 - \theta_2} \tag{A.8.40}$$

$$= 0. \tag{A.8.41}$$

Then analogue the gains for the values $q_1 = \frac{\theta_1}{2}$ and $q_2 = 0$ are determined:

$$\Pi_1^{MS} = \frac{\left(\theta_1 - \frac{\theta_1}{2} - N\cdot 0\right)^2 \cdot \left(2\frac{\theta_1}{2} + 2N\cdot 0 - \theta_2\right)}{\theta_1 - \theta_2} \tag{A.8.42}$$

$$= \frac{\left(\frac{\theta_1}{2}\right)^2 \cdot (\theta_1 - \theta_2)}{\theta_1 - \theta_2} = \left(\frac{\theta_1}{2}\right)^2 \tag{A.8.43}$$

$$N\Pi_2^{MS} = \frac{\left(\theta_2 - \frac{\theta_1}{2} - N\cdot 0\right)^2 \cdot \left(\theta_1 - 2\frac{\theta_1}{2} - 2N\cdot 0\right)}{\theta_1 - \theta_2} = 0. \tag{A.8.44}$$

This would not be an optimal solution for firms of type 2. That means this solution pair $q_1^{(3)}$ and $q_2^{(1)}$ can be excluded because of the same reasons as the solution pair $q_1^{(1)}$ and $q_2^{(3)}$ above. In summary, both first solutions $q_1^{(1)}$ and $q_2^{(1)}$ can be ruled out.

A.9. MS: determination of the quantity

The solution pair $q_1^{(3)}$ and $q_2^{(3)}$ is a valid solution, see section 4.2. Now the quantity should be determined from the usable reaction functions. Recall $q_1^{(3)}$ was mentioned in equation (4.21) and $q_2^{(3)}$ in equation (4.25).

$$q_1^{(3)} = \frac{\theta_1 + 2\theta_2}{4} - \frac{1}{4} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 8N \cdot q_2 \cdot (\theta_1 - \theta_2)} - Nq_2 \tag{A.9.1}$$

$$q_2^{(3)} = \frac{2\theta_1 + \theta_2}{4N} - \frac{1}{4N} \cdot \sqrt{(2\theta_1 - \theta_2)^2 - 8q_1 \cdot (\theta_1 - \theta_2)} - \frac{q_1}{N} \tag{A.9.2}$$

For a better demonstration, the equation above for $q_2^{(3)}$ is rearranged. Then set the reaction function $q_1^{(3)}$ in $q_2^{(3)}$.

$$q_2^{(3)} = \frac{2\theta_1 + \theta_2}{4N} - \frac{q_1}{N} - \frac{1}{4N}\sqrt{(2\theta_1 - \theta_2)^2 - 8q_1 \cdot (\theta_1 - \theta_2)} \tag{A.9.3}$$

$$\begin{aligned} q_2^{(3)} &= \frac{2\theta_1 + \theta_2}{4N} - \frac{1}{N}\left[\frac{\theta_1 + 2\theta_2}{4} - \frac{1}{4}\sqrt{(\theta_1 - 2\theta_2)^2 + 8Nq_2 \cdot (\theta_1 - \theta_2)} - Nq_2\right] \\ &- \frac{1}{4N}\sqrt{(2\theta_1 - \theta_2)^2 - 8 \cdot \left[\frac{\theta_1 + 2\theta_2}{4} - \frac{1}{4}\sqrt{(\theta_1 - 2\theta_2)^2 + 8Nq_2 \cdot (\theta_1 - \theta_2)} - Nq_2\right] \cdot (\theta_1 - \theta_2)} \end{aligned} \tag{A.9.4}$$

$$\begin{aligned} &= q_2 + \frac{2\theta_1 + \theta_2}{4N} - \frac{\theta_1 + 2\theta_2}{4N} + \frac{1}{4N} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 8Nq_2 \cdot (\theta_1 - \theta_2)} \\ &- \frac{1}{4N}\sqrt{(2\theta_1 - \theta_2)^2 + \left[-2 \cdot (\theta_1 + 2\theta_2) + 2 \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 8Nq_2 \cdot (\theta_1 - \theta_2)} + 8Nq_2\right] \cdot (\theta_1 - \theta_2)} \end{aligned} \tag{A.9.5}$$

$\Leftrightarrow$

$$\begin{aligned} &\theta_1 - \theta_2 + \sqrt{(\theta_1 - 2\theta_2)^2 + 8Nq_2 \cdot (\theta_1 - \theta_2)} \\ &= \sqrt{2\theta_1^2 - 6\theta_1\theta_2 + 5\theta_2^2 + 8Nq_2 \cdot (\theta_1 - \theta_2) + 2 \cdot (\theta_1 - \theta_2) \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 8Nq_2 \cdot (\theta_1 - \theta_2)}} \end{aligned} \tag{A.9.6}$$

This last equation cannot be solved exactly. That means the solution has to be solved by a numeric approximation. To get a better view, the values for the cost parameters for $R_{\text{middle}} = 1.4$ with $\theta_1 = 0.7$ and $\theta_2 = 0.5$ and $N = 1$ are set in the equation above.

$$0.2 + \sqrt{0.09 + 1.6 \cdot q_2} = \sqrt{0.13 + 1.6 \cdot q_2 + 0.4 \cdot \sqrt{0.04 + 1.6 \cdot q_2}} \tag{A.9.7}$$

Now also the quantity q_1 should be calculated by setting $q_2^{(3)}$ in $q_1^{(3)}$. The order of equation (A.9.1) is changed to have a better representation for the following computing.

$$q_1^{(3)} = \frac{\theta_1 + 2\theta_2}{4} - Nq_2 - \frac{1}{4}\sqrt{(\theta_1 - 2\theta_2)^2 + 8N \cdot q_2 \cdot (\theta_1 - \theta_2)} \tag{A.9.8}$$

$$= \frac{\theta_1 + 2\theta_2}{4} - N\left(\frac{2\theta_1 + \theta_2}{4N} - \frac{1}{4N}\sqrt{(2\theta_1 - \theta_2)^2 - 8q_1 \cdot (\theta_1 - \theta_2)} - \frac{q_1}{N}\right)$$
$$- \frac{1}{4}\sqrt{(\theta_1 - 2\theta_2)^2 + 8N\left[\frac{2\theta_1 + \theta_2}{4N} - \frac{1}{4N}\sqrt{(2\theta_1 - \theta_2)^2 - 8q_1 \cdot (\theta_1 - \theta_2)} - \frac{q_1}{N}\right] \cdot (\theta_1 - \theta_2)} \tag{A.9.9}$$

$$= q_1 + \frac{\theta_1 + 2\theta_2}{4} - \frac{2\theta_1 + \theta_2}{4} + \frac{1}{4}\sqrt{(2\theta_1 - \theta_2)^2 - 8q_1 \cdot (\theta_1 - \theta_2)}$$
$$- \frac{1}{4}\sqrt{(\theta_1 - 2\theta_2)^2 + \left[2 \cdot (2\theta_1 + \theta_2) - 2\sqrt{(2\theta_1 - \theta_2)^2 - 8q_1 \cdot (\theta_1 - \theta_2)} - 8q_1\right] \cdot (\theta_1 - \theta_2)} \tag{A.9.10}$$

$\Leftrightarrow$

$$\theta_2 - \theta_1 + \sqrt{(2\theta_1 - \theta_2)^2 - 8 \cdot q_1 \cdot (\theta_1 - \theta_2)}$$
$$= \sqrt{5\theta_1^2 - 6\theta_1\theta_2 + 2\theta_2^2 - 8q_1 \cdot (\theta_1 - \theta_2) - 2 \cdot (\theta_1 - \theta_2) \cdot \sqrt{(2\theta_1 - \theta_2)^2 - 8 \cdot q_1 \cdot (\theta_1 - \theta_2)}} \tag{A.9.11}$$

This equation also does not have an exact solution. Therefore, it is unsolvable in terms of accurateness. That is why, analogue to the case before a numeric approximation is needed. I show here the equation above for the cost parameters for R_{middle}.

$$-0.2 + \sqrt{0.81 - 1.6 \cdot q_1} = \sqrt{0.85 - 1.6 \cdot q_1 - 0.4 \cdot \sqrt{0.81 - 1.6 \cdot q_1}} \tag{A.9.12}$$

Appendix B

B.1. Inverse demand function: monopoly

Briefly, as well the profit of firm 1 and of N firms of type 2 is derived, when each has a monopoly.

$$\max_{0 \leq q_1 < \infty} \Pi_1(q_1) = \max_{0 \leq q_1 < \infty} (A + \theta_1 - 1 - b \cdot q_1) \cdot q_1 \tag{B.1.1}$$

$$\max_{0 \leq q_2 < \infty} N\Pi_2(Nq_2) = \max_{0 \leq q_2 < \infty} (A + \theta_2 - 1 - bN \cdot q_2) \cdot Nq_2. \tag{B.1.2}$$

By maximizing the monopoly profit, the following FOC's are obtained:

$$\frac{\partial \Pi_1(q_1)}{\partial q_1} = A + \theta_1 - 1 - 2b \cdot q_1 \stackrel{!}{=} 0 \tag{B.1.3}$$

$$\frac{\partial N\Pi_2(Nq_2)}{\partial q_2} = N \cdot (A + \theta_2 - 1 - 2bN \cdot q_2) \stackrel{!}{=} 0. \tag{B.1.4}$$

Thus, the monopoly quantity, the corresponding profit and the price as monopolist of firm 1 is given by:

$$q_1^M = \frac{1}{2b} \cdot (A + \theta_1 - 1) \tag{B.1.5}$$

$$\Pi_1^M = \frac{1}{4b} \cdot (A + \theta_1 - 1)^2 \tag{B.1.6}$$

$$P_1^M = \frac{1}{2} \cdot (A - \theta_1 + 1). \tag{B.1.7}$$

However, when N firms of type 2 would have a monopoly, the quantity, the profit and the prices of N high-cost firms, respectively a single high-cost firm, is:

$$N \cdot q_2^M = \frac{1}{2b} \cdot (A + \theta_2 - 1), \text{ respectively} \qquad q_2^M = \frac{1}{2bN} \cdot (A + \theta_2 - 1) \tag{B.1.8}$$

$$N \cdot \Pi_2^M = \frac{1}{4b} \cdot (A + \theta_2 - 1)^2, \text{ respectively} \qquad \Pi_2^M = \frac{1}{4bN} \cdot (A + \theta_2 - 1)^2 \tag{B.1.9}$$

$$N \cdot P_2^M = \frac{1}{2} \cdot (A - \theta_2 + 1), \text{ respectively} \qquad P_2^M = \frac{1}{2N} \cdot [(2N - 1)A - \theta_2 + 1]. \tag{B.1.10}$$

B.2. Inverse demand function (MS): second order conditions

Here the second order conditions are determined for the Market Sharing technology to prove that the total quantity Q is really a maximum.

$$\frac{\partial^2 \Pi_2(\Pi_1)}{\partial Q^2} = -2 - 2 \cdot \frac{b^2 \Pi_1 \cdot (\theta_1 - \theta_2)}{(A + \theta_1 - 1 - b \cdot Q)^3} < 0 \tag{B.2.1}$$

$$\frac{\partial^2 \Pi_1(\Pi_2)}{\partial Q^2} = -2 - 2 \cdot \frac{b^2 N \Pi_2 \cdot (\theta_2 - \theta_1)}{(A + \theta_2 - 1 - b \cdot Q)^3} < 0 \tag{B.2.2}$$

It follows that the SOC's are satisfied, because of the same reasons as in section A.5.

B.3. Inverse demand function (MS): calculations of the roots

In this section, the roots of the quadratic part of both q_1 and q_2 will be determined. First, the solutions $q_1^{(2)}$ and $q_1^{(3)}$ are derived from the quadratic part of equation (7.27). To apply the small solution formula, the quadratic part of the aforementioned equation is expressed in the form $x^2 + px + q = 0$. Note that q is here not the quantity.

$$(A + \theta_1 - 1 - bq_1 - bNq_2) \cdot (2bq_1 + 2bNq_2 - A - \theta_2 + 1) - q_1 b \cdot (\theta_1 - \theta_2) = 0 \tag{B.3.1}$$

$$\begin{aligned} &\left(q_1^{(2,3)}\right)^2 + \frac{q_1^{(2,3)}}{2b} \cdot (-3A + 3 + 4bNq_2 - \theta_1 - 2\theta_2) + \frac{1}{2b^2} \cdot \left(2b^2 N^2 q_2^2 + 1\right) \\ &+ \frac{1}{2b^2} \cdot \left(bN \cdot q_2 \cdot (3 - 3A - (2\theta_1 + \theta_2)) + A^2 - 2A + (A - 1)(\theta_1 + \theta_2) + \theta_1 \theta_2\right) = 0 \end{aligned} \tag{B.3.2}$$

With the small solution formula applied to the equation above, the subsequent explicit reaction functions are obtained:

$$\begin{aligned} q_1^{(2,3)} = &\frac{3(A-1) + \theta_1 + 2\theta_2}{4b} - Nq_2 \\ &\pm \frac{1}{4b} \cdot \sqrt{(\theta_1 - 2\theta_2)^2 + 8bN \cdot q_2 \cdot (\theta_1 - \theta_2) + 2 \cdot (A-1)(2\theta_2 - \theta_1) + (A-1)^2}. \end{aligned} \tag{B.3.3}$$

Then, the roots of the quadratic part of the equation (7.31) should be computed to find the solutions $q_2^{(2)}$ and $q_2^{(3)}$. Similarly to the calculation above, the quadratic part of the equation (7.31) is transformed into the form $x^2 + px + q = 0$. Note that q is again not the quantity.

$$(A + \theta_2 - 1 - b \cdot q_1) \cdot (A + \theta_1 - 1 - 2b \cdot Q) - b \cdot q_2 \cdot (\theta_1 - \theta_2) = 0 \tag{B.3.4}$$

$$\left(q_2^{(2,3)}\right)^2 + \frac{q_2^{(2,3)}}{2bN} \cdot (-3A + 3 + 4bq_1 - 2\theta_1 - \theta_2) + \frac{1}{2b^2N^2} \cdot \left(2b^2 q_1^2 + 1\right)$$
$$+ \frac{1}{2b^2N^2} \cdot \left(bq_1 \cdot (-3A + 3 - \theta_1 - 2\theta_2) + A^2 - 2A + (A-1) \cdot (\theta_1 + \theta_2) + \theta_1 \theta_2\right) = 0 \quad \text{(B.3.5)}$$

The second and third solutions are:

$$q_2^{(2,3)} = \frac{3(A-1) + 2\theta_1 + \theta_2}{4bN} - \frac{q_1}{N}$$
$$\pm \frac{1}{4bN} \cdot \sqrt{(2\theta_1 - \theta_2)^2 - 8b \cdot q_1 \cdot (\theta_1 - \theta_2) + 2 \cdot (A-1)(2\theta_1 - \theta_2) + (A+1)^2}. \quad \text{(B.3.6)}$$

B.4. Triopoly: monopoly

All M firms of type 3 have a monopoly position in the market. Now the monopoly quantity and profit should be determined. If M firms of type 3 have a monopoly, the following profit function is assumed:

$$M\Pi_3(Q) = M\Pi_3(Mq_3) = (\theta_3 - Mq_3) \cdot Mq_3. \quad \text{(B.4.1)}$$

The profit above is obtained, since the total quantity of the industry is here $Q = Mq_3$. Then the profit is maximized:

$$\max_{0 \le q_3 < \infty} M\Pi_3(Q) = \max_{0 \le q_3 < \infty} (\theta_3 - Mq_3) \cdot Mq_3. \quad \text{(B.4.2)}$$

Accordingly, the FOC is given by:

$$\frac{\partial M\Pi_3(Mq_3)}{\partial q_3} = M(\theta_3 - 2M \cdot q_3) \stackrel{!}{=} 0. \quad \text{(B.4.3)}$$

The superscript M characterizes the quantity of firms of type 3 as a monopolist, while the variable M denotes the number of high-cost firms.

$$Mq_3^M = \frac{\theta_3}{2}, \text{ respectively } \quad q_3^M = \frac{\theta_3}{2M} \quad \text{(B.4.4)}$$

In setting the quantity above in the profit function (B.4.1), the following is obtained:

$$M\Pi_3^M = \frac{\theta_3^2}{4} = \left(Mq_3^M\right)^2, \text{ respectively } \quad \Pi_3^M = \frac{\theta_3^2}{4M} = M \cdot \left(q_3^M\right)^2. \quad \text{(B.4.5)}$$

Finally, the monopoly price is determined.

$$P_3^M = 1 - Mq_3^M = 1 - \frac{\theta_3}{2} \tag{B.4.6}$$

B.5. Triopoly: derivation of the parameter S

The parameter S describes firm 1's market share in Cournot competition and thus is given by:

$$S = \frac{q_1^C}{Q^C} = \frac{q_1^C}{q_1^C + N \cdot q_2^C + M \cdot q_3^C} = \frac{(1+N+M)\cdot\theta_1 - N\theta_2 - M\theta_3}{\theta_1 + N\theta_2 + M\theta_3} \tag{B.5.1}$$

$$= \frac{\frac{\theta_1}{\theta_2}\cdot(1+N+M) - N - M\cdot\frac{\theta_3}{\theta_2}}{\frac{\theta_1}{\theta_2} + N + M\cdot\frac{\theta_3}{\theta_2}} \tag{B.5.2}$$

$$\overset{\frac{\theta_1}{\theta_2}=R}{=} \frac{R_1\cdot(1+N+M) - N - M\cdot\frac{1}{R_3}}{R_1 + N + M\cdot\frac{1}{R_3}}. \tag{B.5.3}$$

The determination follows analogous to section A.2 with a small trick, in which the whole expression is divided through θ_2. To obtain the expression for R_1, the following transformations are done:

$$S\cdot\left(R_1 + N + M\cdot\frac{1}{R_3}\right) = R_1\cdot(1+N+M) - N - M\cdot\frac{1}{R_3} \tag{B.5.4}$$

$$SN + SM\cdot\frac{1}{R_3} + N + M\cdot\frac{1}{R_3} = R_1\cdot(1+N+M-S) \tag{B.5.5}$$

$$R_1 = \frac{\left(N+\frac{M}{R_3}\right)\cdot(S+1)}{1+N+M-S}. \tag{B.5.6}$$

Furthermore, also R_2 should be expressed through the four parameters S, N, M and R_3.

$$S = \frac{(1+N+M)\cdot\theta_1 - N\theta_2 - M\theta_3}{\theta_1 + N\theta_2 + M\theta_3} \tag{B.5.7}$$

$$= \frac{\frac{\theta_1}{\theta_3}\cdot(1+N+M) - N\cdot\frac{\theta_2}{\theta_3} - M}{\frac{\theta_1}{\theta_3} + N\cdot\frac{\theta_2}{\theta_3} + M} \tag{B.5.8}$$

$$\overset{\frac{\theta_1}{\theta_3}=R_2}{=} \frac{R_2\cdot(1+N+M) - N\cdot R_3 - M}{R_2 + N\cdot R_3 + M} \tag{B.5.9}$$

Therefore, the same steps are needed as in the computation of R_1.

$$S \cdot (R_2 + N \cdot R_3 + M) = R_2 \cdot (1 + N + M) - N \cdot R_3 - M \tag{B.5.10}$$

$$SN \cdot R_3 + SM + N \cdot R_3 + M = R_2 \cdot (1 + N + M - S) \tag{B.5.11}$$

$$R_2 = \frac{(N \cdot R_3 + M) \cdot (S + 1)}{1 + N + M - S} \tag{B.5.12}$$

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Kurzfassung

Der Schwerpunkt dieser Masterarbeit basiert auf der Untersuchung, auf welche Art und Weisen Firmen unter asymmetrischen Kosten bezüglich der Stückzahl kolludieren können und, ob Kollusion für alle beteiligten Unternehmen von Vorteil ist. Insbesondere, mit welcher Methode ein kooperativer Ausgang in einem Duopol, ausgehend von einem asymmetrischen Cournot Duopol, erreicht werden kann. Vergleichsweise ist es mit asymmetrischen Kosten schwieriger Kollusion aufrechtzuerhalten als bei symmetrischen Kosten. Dennoch ist dies ein wichtiges Thema, da die Realität besser widerspiegelt wird als bei symmetrischen Kosten. Es ist allgemein bekannt, dass die Maximierung des Gesamtprofits der Industrie unter asymmetrischen Kosten und ohne Kompensationszahlungen nicht plausibel ist. Diese Masterarbeit basiert hauptsächlich auf dem Artikel *"Competitive Advantage and Collusive Optima"* von Schmalensee (1987) [17]. Der Hauptakzent liegt auf der Beschreibung der vier verschiedenen Kollusionsverfahren, von denen die jeweilige optimale Lösung unter Zuhilfenahme von axiomatischer Verhandlungstheorie gefunden wird. Die Ergebnisse zeigen, dass der kooperative Ausgang für das Unternehmen mit den niedrigsten Kosten ernüchternd sein kann. Abschließend wird Schmalensee's Arbeit (1987) [17] auf untrschiedliche Weise erweitert.

Schlüsselwörter. asymmetrische Kosten, Cournot, Duopol, Kartell, Kollusion, Verhandlungstheorie