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„A minimal extension of the Standard Model with classical scale-invariance“

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Gabriel Sommer, BSc

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Abstract

In this thesis, we study classically scale-invariant versions of the Standard Model (SM) with additional scalar degrees of freedom. There, spontaneous symmetry breaking (SSB) and the generation of all particles' masses is induced by radiative corrections in the S. Coleman-E. Weinberg effective potential. Having just the particle spectrum of the SM, the SM cannot be described as a scale-invariant theory with SSB à la S. Coleman and E. Weinberg because of the large mass of the top-quark. One way to circumvent this problem is adding an arbitrary number of scalar singlets to the particle content of the SM. We consider models where either one real or complex scalar singlet is added and we construct tree-level scalar potentials in a classically scale-invariant way. We calculate the full effective potential up to one-loop order by means of the perturbative approach introduced by E. Gildener and S. Weinberg. In each model, we add an arbitrary number of right-handed neutrino fields with the aim to describe neutrinos as massive particles via the seesaw-mechanism. In the final step, we investigate if the considered models can be described in perturbation theory. From our calculations, we find that the last model in this thesis, where we add one complex scalar X singlet to the particle content of the SM and construct a scalar tree-level potential which is invariant under $X \leftrightarrow X^*$, accounts for both the introduction of massive neutrinos and the validity of perturbation theory with at least one massive scalar field with a mass in the TeV-range.

Zusammenfassung

In dieser Arbeit behandeln wir klassisch skaleninvariante Versionen des Standardmodells der Elementarteilchenphysik mit zusätzlichen skalaren Freiheitsgraden. Spontane Symmetriebrechung und die Massen aller Elementarteilchen werden durch Quantenkorrekturen im effektiven Potential nach S. Coleman und E. Weinberg hervorgerufen. Mit dem Teilchenspektrum des Standardmodells ist es nicht möglich es als skaleninvariante Theorie mit Symmetriebrechung nach S. Coleman und E. Weinberg zu beschreiben, da das top-Quark eine sehr schwere Masse besitzt. Ein Weg dieses Problem zu umgehen und das Standardmodell als skaleninvariante Theorie zu beschreiben, ist das Hinzufügen einer beliebigen Anzahl skalarer Singuletts zum Teilchenspektrum des Standardmodelles. Wir betrachten Modelle, in denen lediglich ein reelles oder komplexes skalares Singulett hinzugefügt wird und konstruieren skalare Potentiale mit klassischer Skaleninvarianz. Wir berechnen das effektive Potential auf Einschleifenniveau nach der störungstheoretischen Herangehensweise von E. Gildener und S. Weinberg. Darüberhinaus fügen wir in jedem Modell eine beliebige Anzahl rechtshändiger Neutrinofelder hinzu, mit dem Ziel Neutrinomassen mit Hilfe des Seesaw-Mechanismus zu erklären. Abschließend untersuchen wir, ob die betrachteten Modelle störungstheoretisch beschrieben werden können. Aufgrund unserer Berechnungen ist das als letztes in dieser Arbeit betrachtete Modell – in dem wir ein komplexes skalares Singulett X hinzufügen und ein Potential betrachten, das invariant unter $X \leftrightarrow X^*$ ist – geeignet, Neutrinos als massive Teilchen zu beschreiben und es sind alle Rechnungen auch störungstheoretisch durchführbar, wobei zumindest ein Skalarfeld eine schwere Masse im TeV-Bereich besitzt.

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Contents

1 Introduction

The *Standard Model of elementary particle physics* (SM) [1–3] is a renormalizable gauge field theory that contains all elementary particles known so far and describes the interactions between them. The origin of the masses of all the charged fermions, the gauge bosons and the Higgs boson itself is described by the Higgs mechanism [4–9]. With the discovery of the Higgs boson − as one of the particles contained in the SM − at ATLAS [10] and CMS [11] in July 2012, the minimal version of the SM is completed.

The SM has, both from its theoretical formulation and from experimental data, a great success in predicting and describing the building blocks of matter we know today. But although it seems to be a self-consistent theory in its mathematical formulation, nowadays we know that it can not describe all phenomena in nature and so physics beyond the Standard Model (BSM) seems to exist. One obvious reason is that neutrino-oscillations were observed in experiments [12–14] and therefore at least two of the three neutrino flavors included in the SM have to be massive. This is in contradiction to the minimal version of the SM where they are described as massless particles.

Considering the SM as a quantum field theory, an astonishing property of the SM-Lagrangian is the fact that all terms except the explicit mass term $\mu^2\Phi^\dagger\Phi$ in the Higgs potential have operator dimension 4. Thus this term is the only one which breaks scale-invariance. Describing extensions of the SM as a classically scale-invariant theory is under intensive consideration nowadays and investigated in many current works [15–44]. But this is not possible with the SM gauge-group and the known particle content due to the mass values of the top-quark and the Higgs boson. To solve this issue, i) one can extend the gauge-group of the SM, ii) one can add an arbitrary number of real or complex scalar fields to the particle content of the SM without extending its gauge-group or iii) one can both introduce new particles and extend the gauge-group. In this thesis, we will only tackle the second approach which is known as minimal scale-invariant extensions of the SM.

In the entire thesis, only models with one additional real or complex scalar degree of freedom will be considered. Apart from the fact that the SM can be described as a scale-invariant theory by just adding one scalar degree of freedom, we can give an explanation for massive neutrinos via the seesaw-mechanism of type 1 [45–48]. By adding an arbitrary number of right-handed neutrinos to the particle content of the SM that couple to the additional scalar singlet, these couplings are of Majorana-type. Thus, when the scalar singlet gets a nonvanishing vacuum expectation value (VEV) and spontaneous symmetry breaking (SSB) occurs, Majorana-massterms for the neutrinos will be generated. Furthermore by having both left-handed and right-handed neutrinos, the construction of Yukawa couplings to the Higgs doublet is possible resulting in

Dirac-massterms after SSB.

Moreover, in absence of an explicit scalar mass term, electroweak SSB and the generation of all SM particles' masses will be induced by radiative corrections to the tree-level scalar potential. We will calculate the Coleman-Weinberg (CW) effective potential up to one-loop order [49].

This thesis is organized as follows: In chapter 2, we review the most important aspects of the SM needed for this work, like the Higgs mechanism, the occurrence of electroweak SSB and the introduction of Yukawa couplings of the fermions to the Higgs boson and describe how the charged fermions, gauge bosons and the Higgs boson receive their masses after SSB. Then in chapter 3 going already to BSM-physics, we proceed with the seesaw-mechanism of type 1. There, we will add an arbitrary number of right-handed neutrino singlets and at least one scalar singlet (being both singlets under the SM gauge group) to the particle content of the SM.

After that, in chapter 4, we review the CW-mechanism of effective potentials at one-loop order giving an explanation of electroweak SSB in classically scale-invariant theories. The calculations are done in dimensional regularization [50, 51] and give the one-loop effective potential for scalar, gauge boson and fermion loop-corrections writing all terms in the modified minimal subtraction scheme ($\overline{\text{MS}}$-scheme) [52, 53]. In chapter 5, we proceed by summarizing the main aspects of the work of Gildener and S. Weinberg (GW) [54] which makes us of with the CW-potential and gives an explanation which condition on all particles' masses included in a considered model has to be fulfilled so that a specific model can be described perturbatively.

After all these theoretical aspects, we show the calculations and give the results of all the different models we worked with, where the SM is considered as a classically scale-invariant theory with an extended scalar sector. In addition to the new scalar degrees of freedom that will be introduced, we will add right-handed neutrinos in each model to describe neutrinos as massive particles. In chapter 6, we discuss a model with one additional real scalar singlet compared to the particle content of the SM. In chapter 7, we discuss a model with one additional complex scalar field, where the scalar potential is invariant under a $U(1)$-transformation of that field. In this model, there is again one more physical scalar compared to the SM and due to that $U(1)$-symmetry which is already broken at one-loop order there will occur a real Goldstone boson, the Majoron. The Majoron stays massless after SSB in all orders of perturbation theory. In chapter 8, we discuss a model with one additional complex scalar field X, where the effective potential is invariant under the discrete symmetry $X \leftrightarrow X^*$. In this case, we have three heavy scalar particles. To conclude, we summarize our results in chapter 9 and give some prospects for future work.

2 The Standard Model of Particle Physics

The Standard Model (SM) of elementary Particle Physics – a quantum field theory (QFT) in its mathematical formulation – includes all fundamental particles (known so far) shown in figure 2.1 and describes their interactions among each other. The matter how we know it on earth is mostly built from the first family of quarks (up- and down-quarks), being the constituents of protons and neutrons, and together with the electron, as one constituent living in the first family of the leptons, they make up atoms. The remaining fermions (quarks and leptons) living in the other two generations are mostly produced in accelerators, or they are radiated from stars where the energy-scale is much higher than in everyday world.

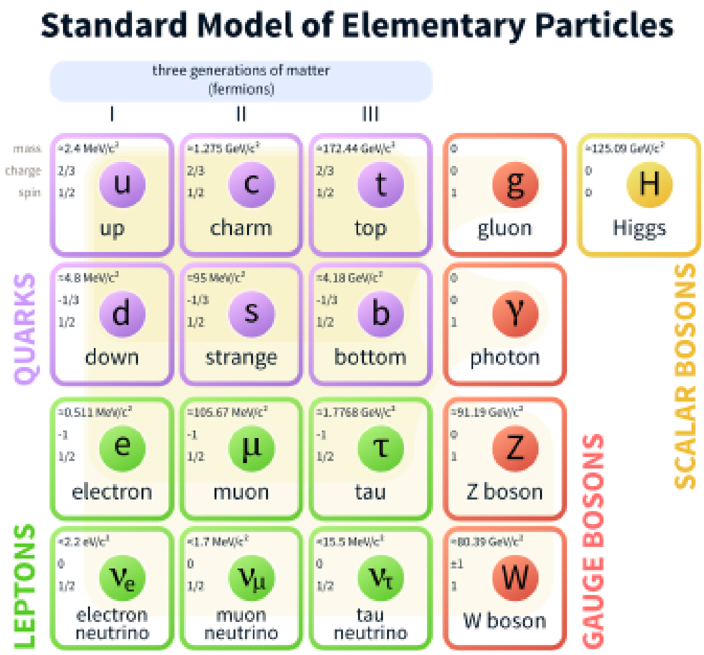

Figure 2.1: All particles included in the SM (taken from [55])

This chapter gives a phenomenogical overview of the SM, but not a complete mathematical description. Of course, one can find a more detailed description of the SM in many textbooks nowadays, as in [56–58].

2.1 Building up the SM-Lagrangian

The SM is a renormalizable quantum field theory based on local gauge invariance, given by the gauge group,

$$\mathrm{G_{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y \,. \tag{2.1}$$

Gauge invariance means that the Lagrangian of the SM is constructed such that it is invariant, if the fields (fermion and scalar fields) transform under its group (2.1), and local corresponds to the fact that these transformations have to depend on space and time. Each fermion, the three generations of left- and right-handed leptons $(E_L)_i^{\mathrm{T}} = \begin{pmatrix}\nu_L & e_L\end{pmatrix}_i$, $(e_R)_i$ and quarks $(Q_L)_i^{\mathrm{T}} = \begin{pmatrix}u_L & d_L\end{pmatrix}_i$, $(u_R)_i$ and $(d_R)_i$, and the scalar field Φ of the SM have $SU(2)_L$ and $U(1)_Y$ quantum numbers shown in table 2.1. The hypercharge Y is defined by $Q = \tau_3 + Y$, with Q being the electric charge and τ_3 being the third generator of $SU(2)_L$.

	$SU(2)_L$	$U(1)_Y$
$(u_L)_i$	$\frac{1}{2}$	$\frac{1}{6}$
$(d_L)_i$	-$\frac{1}{2}$	$\frac{1}{6}$
$(u_R)_i$	0	$\frac{2}{3}$
$(d_R)_i$	0	-$\frac{1}{3}$
$(\nu_L)_i$	$\frac{1}{2}$	-$\frac{1}{2}$
$(e_L)_i$	-$\frac{1}{2}$	-$\frac{1}{2}$
$(e_R)_i$	0	-1
$(\nu_R)_i$	0	0
$\Phi = \begin{pmatrix}\Phi^+ \\ \Phi^0\end{pmatrix}$	-$\frac{1}{2}$	$\frac{1}{2}$

Table 2.1: The quantum numbers of τ_3 for $SU(2)_L$ and of Y for $U(1)_Y$, of the fermions and of the Higgs boson in the SM

A necessary component to build up the SM-Lagrangian is the covariant derivative reading

$$D_\mu = \partial_\mu - ig'YB_\mu - ig\tau_a A_\mu^a - ig_s T_A G_\mu^A \,, \tag{2.2}$$

where Y is the generator of $U(1)_Y$, τ_a $(a = 1, 2, 3)$ are the three generators of $SU(2)_L$ given by

$$\tau_a = \begin{cases} \frac{1}{2}\sigma_a & \text{for } SU(2)_L \text{ doublets} \\ 0 & \text{for } SU(2)_L \text{ singlets} \end{cases} \tag{2.3}$$

with σ_a being the Pauli-matrices and T_A $(A = 1, ..., 8)$ are the eight generators of $SU(3)_C$ given by

$$T_A = \begin{cases} \frac{1}{2}\lambda_A & \text{for } SU(3)_C \text{ triplets} \\ 0 & \text{for } SU(3)_C \text{ singlets} \end{cases} \tag{2.4}$$

with λ_A being the Gellmann-matrices. $SU(3)_C$ stands for the strong interaction and its gauge fields are the eight gluons. $SU(2)_L \times U(1)_Y$ stands for the weak and electromagnetic interactions and its gauge fields are the A_μ^a and B_μ. The three Pauli-matrices and the eight Gellman-matrices are listed here:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.5}$$

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\ \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \tag{2.6}$$

The SM-Lagrangian is given by

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_f + \mathcal{L}_{Higgs} + \mathcal{L}_Y \,, \tag{2.7}$$

with $\mathcal{L}_{gauge}$ including the kinetic terms and the triple and quartic selfinteractions of A_μ^a and G_μ^A,

$$\mathcal{L}_{gauge} = -\frac{1}{4}\, G_{\mu\nu}^A\, G^{\mu\nu A} - \frac{1}{4}\, W_{\mu\nu}^i\, W^{\mu\nu i} - \frac{1}{4}\, B_{\mu\nu}\, B^{\mu\nu} \,, \tag{2.8}$$

with

$$G_{\mu\nu}^A = \partial_\mu G_\nu^A - \partial_\nu G_\mu^A - g_s f_{ABC} G_\mu^B G_\nu^C \,, \tag{2.9}$$

$$W_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g\epsilon_{abc} A_\mu^b A_\nu^c \,, \tag{2.10}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \,, \tag{2.11}$$

where f_{ABC} are the structure constants of $SU(3)_C$, $[T_A, T_B] = if_{ABC}T^C$, and ϵ_{ijk} (the Levi-Civita symbol in three dimensions) are the structure constants of $SU(2)_L$, $[\tau_a, \tau_b] = i\epsilon_{abc}\tau^c$.

The Lagrangian $\mathcal{L}_f$ includes the kinetic term for all fermions and the interactions between fermions and gauge bosons as well and reads

$$\mathcal{L}_f = i \sum_{\psi} \overline{\psi} \not{D} \psi \,, \tag{2.12}$$

with $\psi = \{(Q_L)_i, (u_R)_i, (d_R)_i, (E_L)_i, (e_R)_i, (\nu_R)_i\}$. Explicit fermion mass terms cannot be included in the SM-Lagrangian, since they would violate the required gauge-invariance. The Lagrangian in the Higgs sector $\mathcal{L}_{Higgs}$ and the Yukawa-Lagrangian $\mathcal{L}_Y$ will be discussed in sections 2.2 and 2.3.

2.2 The Higgs-mechanism

Adding explicit mass terms for gauge bosons and for fermions would violate the required gauge-invariance of the SM-Lagrangian. The Higgs doublet of the SM generates gauge boson masses dynamically due to the kinetic term in the Lagrangian and it will also be possible to add Yukawa-couplings of the fermions to the Higgs field so that they will get massive after SSB.

2.2.1 Introducing one complex scalar doublet

The Lagrangian in the scalar sector of the SM is given by

$$\mathcal{L}_{Higgs} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - \mathrm{V}(\Phi) \qquad \text{with} \qquad \mathrm{V}(\Phi) = \mu^2\, \Phi^\dagger \Phi + \lambda\, (\Phi^\dagger \Phi)^2, \tag{2.13}$$

with Φ being a complex scalar doublet,

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_4 + i\varphi_3 \end{pmatrix}, \tag{2.14}$$

where all the fields φ_i ($i = 1, 2, 3, 4$) are real. The doublet Φ interacts with the electroweak part, $SU(2)_L \times U(1)_Y$, of the SM with its quantum numbers given in table 2.1. The quartic selfcoupling λ has to be positive ($\lambda > 0$), so that $\mathrm{V}(\Phi)$ in (2.13) is bounded from below, and μ^2 is chosen to be negative ($\mu^2 < 0$), since only then $\mathrm{V}(\Phi)$ has a nonvanishing minimum away from the origin. As Φ only acts on the $SU(2)_L \times U(1)_Y$-part in the SM, its covariant derivarite D_μ reads

$$D_\mu = \partial_\mu - igA^a_\mu \tau_a - i\frac{g'}{2} B_\mu. \tag{2.15}$$

Without loss of generality, the VEV of Φ can be chosen to be

$$\langle 0|\Phi|0\rangle = \langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \langle 0|\varphi_4|0\rangle = v \quad \text{and} \quad \langle 0|\varphi_i|0\rangle = 0 \quad \text{for} \quad i = 1, 2, 3\,, \tag{2.16}$$

where v is positive and reads $v = \sqrt{-\frac{\mu^2}{\lambda}}$. With this choice the VEV is not gauge-invariant although the Lagrangian $\mathcal{L}_{Higgs}$ is. One says that electroweak SSB occurs.

Since the scalar doublet interacts only with $SU(2)_L \times U(1)_Y$, this part of the gauge group has to be broken. It breaks down to $U(1)_{em}$,

$$SU(2)_L \times U(1)_Y \;\rightarrow\; U(1)_{em}\,, \tag{2.17}$$

the electromagnetic interaction.

The general transformation of the Higgs doublet reads

$$\Phi \rightarrow \Phi' = \exp\left\{-i(\vec{\alpha}(x)\cdot\vec{\tau} + \beta(x)Y)\right\}\Phi\,. \tag{2.18}$$

Writing the Higgs doublet in unitary gauge, which is a specific choice of (2.18), the expansion of Φ around its VEV is given by

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x)\end{pmatrix}\,, \tag{2.19}$$

with $h(x)$ being a real scalar field.

One can find the mass m_h of the scalar field $h(x)$ by inserting (2.19) into V(Φ),

$$\mathrm{V}(h) = -\frac{\mathrm{m}_h^4}{16\lambda} + \frac{\mathrm{m}_h^2}{2}\,h^2 + \sqrt{\frac{\lambda}{2}}\,\mathrm{m}_h\,h^3 + \frac{\lambda}{4}\,h^4 \qquad \text{with} \qquad \mathrm{m}_h^2 = 2\lambda v^2. \tag{2.20}$$

The other three scalar fields appearing in (2.14) are would-be Goldstone bosons and are eaten up by the three gauge bosons $W^{\pm}$ and Z^0, when electroweak SSB occurs.

The generation of the masses of $W^{\pm}$ and Z^0 is explained by *Goldstone's Theorem* [59, 60]: It says that for every broken generator of a continuous symmetry in gauge theories, a massless Goldstone boson comes into being. This breakdown is induced by the VEV of a scalar field. Then, via the dynamics of the Goldstone bosons, the gauge bosons get massive. In the case of the SM, three generators are broken via the breakdown shown in (2.17); the mathematical description of this fact is shown in the next paragraphs (section 2.2.2).

2.2.2 Masses of the gauge bosons

After SSB, the masses of the gauge bosons are generated dynamically from the kinetic term $(D_\mu\Phi)^\dagger(D^\mu\Phi)$ in the Higgs-Lagrangian (2.13). The relevant terms are those that contain the generators of $SU(2)$ and $U(1)$. Evaluating $(D_\mu\Phi)^\dagger(D^\mu\Phi)$ at the VEV of the Higgs doublet

(2.16), one obtains

$$\begin{aligned}(D_\mu\Phi)^\dagger(D^\mu\Phi) &\sim \frac{1}{2}\begin{pmatrix}0 & v\end{pmatrix}(-igA^a_\mu\tau_a - i\frac{g'}{2}B_\mu)^\dagger(-igA^a_\mu\tau_a - i\frac{g'}{2}B_\mu)\begin{pmatrix}0\\ v\end{pmatrix}\\ &= \frac{1}{2}\frac{v^2}{4}\left[g^2(W^+_\mu)^2 + g^2(W^-_\mu)^2 + (g^2+(g')^2)(Z^0_\mu)^2\right],\end{aligned} \tag{2.21}$$

where the mass eigenfields were introduced via

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \mp iA^2_\mu) \qquad \text{and} \qquad Z^0_\mu = \frac{1}{\sqrt{g^2+(g')^2}}(gA^3_\mu - g'B_\mu) \tag{2.22}$$

and their masses can be calculated to be

$$\mathrm{M}_W = \frac{g\,v}{2}, \qquad \text{and} \qquad \mathrm{M}_Z = \frac{\sqrt{g^2+(g')^2}\;v}{2}. \tag{2.23}$$

The remaining gauge boson, the photon γ, remains massless and is defined to be orthogonal to Z^0_μ,

$$A_\mu = \frac{1}{\sqrt{g^2+(g')^2}}(gB_\mu + g'A^3_\mu) \qquad \text{with} \qquad M_\gamma = 0. \tag{2.24}$$

Having identified the mass-eigenfields, it is common to rewrite D_μ from (2.15) in dependence of these fields rather than in dependence of A^1_μ, A^2_μ, A^3_μ and B_μ. The covariant derivative for the leptons living in a general $SU(2)_L \times U(1)_Y$-representation takes the following form [56],

$$\begin{aligned}D_\mu &= \partial_\mu - ig\tau_a A^a_\mu - ig'YB_\mu\\ &= \partial_\mu - i\,\frac{g}{\sqrt{2}}\left(W^+_\mu\tau^+ + W^-_\mu\tau^-\right) - i\,\frac{1}{\sqrt{g^2+(g')^2}}\left(g^2\tau^3 - (g')^2Y\right)Z^0_\mu\\ &\quad - i\,\frac{gg'}{\sqrt{g^2+(g')^2}}\left(\tau^3+Y\right)A_\mu,\end{aligned} \tag{2.25}$$

with $\tau^\pm$ defined to be $\tau^\pm = \tau_1 \pm i\,\tau_2$.

From the last term of this equation, one can read off the coupling of the photon to the leptons. Defining the operator of the electric charge as $Q = \tau^3 + Y$ and electric charge as $e = \frac{gg'}{\sqrt{g^2+(g')^2}}$, one obtains

$$D_\mu \sim -i\,\frac{gg'}{\sqrt{g^2+(g')^2}}\left(\tau^3+Y\right)A_\mu = -i\,e\,Q\,A_\mu, \tag{2.26}$$

which is the same covariant derivative as for QED if setting $Q = -1$ as it is valid for electrons.

With the definition of the *Fermi constant* G_F,

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8\,\mathrm{M}_W^2}\,, \tag{2.27}$$

together with the mass of $W^\pm$, one achieves for the VEV of the Higgs field,

$$v^2 = \frac{1}{\sqrt{2}G_F}\,. \tag{2.28}$$

With the experimental value of G_F [61], one gets $v = 246\,\mathrm{GeV}$.

2.3 Implementation of fermion masses

An explicit fermion mass term $m_f\,\overline{\psi}\psi = m_f\,(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L)$ with ψ being any fermionic field is forbidden, since ψ_L and ψ_R live in different representations of $SU(2)_L \times U(1)_Y$ and thus the term would spoil the gauge-invariance.

In the SM, fermion masses are generated after SSB via Yukawa-interactions of the fermions to the Higgs doublet,

$$\mathcal{L}_Y = \mathcal{L}_{lepton} + \mathcal{L}_{quark}\,, \tag{2.29}$$

with

$$\mathcal{L}_{lepton} = -(\bar{E}_L)_i\,\lambda_\ell^{ij}\,\Phi\,(e_R)_j + \text{h.c.}\,, \tag{2.30}$$

$$\mathcal{L}_{quark} = -(\bar{Q}_L)_i\,\lambda_d^{ij}\,\Phi\,(d_R)_j - (\bar{Q}_L)_i\,\lambda_u^{ij}\,\tilde{\Phi}\,(u_R)_j + \text{h.c.}\,. \tag{2.31}$$

Here, $(e_R)_j$ are the three right-handed charged lepton fields, $(u_R)_j$ and $(d_R)_j$ are the three right-handed up-quark and down-quark fields. $(E_L)_i$ and $(Q_L)_i$ denote the three left-handed lepton and quarks doublets,

$$(E_L)_i = \begin{pmatrix}\nu_L\\ e_L\end{pmatrix}_i \quad \text{and} \quad (Q_L)_i = \begin{pmatrix}u_L\\ d_L\end{pmatrix}_i\,. \tag{2.32}$$

All the three Yukawa-coupling matrices λ_ℓ^{ij}, λ_d^{ij} and λ_u^{ij} are in general complex 3×3-matrices and in the last term the Higgs doublet is implemented via $\tilde{\Phi} = (i\sigma_2)\Phi^*$. In that way, all the terms in $\mathcal{L}_Y$ (2.29) are invariant under the SM gauge-group.

2.3.1 Charged leptons

After SSB, the Yukawa-Lagrangian expressed in unitary gauge for the charged leptons reads

$$\mathcal{L}_{lepton} = -(\bar{E}_L)_i \frac{\lambda_\ell^{ij}}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix} (e_R)_j \, + \text{h.c.} \tag{2.33}$$

$$= -\frac{\lambda_\ell^{ij}}{\sqrt{2}} (\bar{e}_L)_i \, (v+h) \, (e_R)_j \, + \text{h.c.}\,. \tag{2.34}$$

From here, one can read off the mass matrix of the charged leptons as

$$\mathcal{M}_\ell^{ij} = \frac{1}{\sqrt{2}} \lambda_\ell^{ij} \, v\,. \tag{2.35}$$

For diagonalizing the Yukawa-coupling matrix λ_ℓ^{ij}, one uses two unitary matrices U_ℓ and W_ℓ such that the matrix $\hat{\lambda}_\ell = U_\ell^\dagger \, \lambda_\ell \, W_\ell$ and thus

$$\hat{\mathcal{M}}_\ell = U_\ell^\dagger \, \mathcal{M}_\ell \, W_\ell \tag{2.36}$$

is diagonal with only real and positive entries, denoted with M_ℓ^i. Together with that diagonalization, the physical mass-eigenfields of the leptons are given by

$$\hat{e}_L^i = (U_\ell^\dagger)^{ij} (e_L)_j \quad \text{and} \quad \hat{e}_R^i = (W_\ell^\dagger)^{ij} (e_R)_j \tag{2.37}$$

and the Lagrangian reads

$$\mathcal{L}_{lepton} = -(\bar{\hat{e}}_L)_i \, \hat{\mathcal{M}}_\ell^{ij} \, (\hat{e}_R)_i \left(1 + \frac{h}{v}\right) \, + \text{h.c.} \tag{2.38}$$

$$= -M_\ell^i \, (\bar{\hat{e}}_L^i \, \hat{e}_R^i \, + \, \bar{\hat{e}}_R^i \, \hat{e}_L^i) \left(1 + \frac{h}{v}\right)\,. \tag{2.39}$$

From here, one can also read off the couplings of the leptons to the Higgs boson, namely

$$g_{Lhh,i} = \frac{M_\ell^i}{v}\,. \tag{2.40}$$

2.3.2 Quarks

The quark sector can be treated in analogy as it was done in the leptonic case. After SSB, the Yukawa-Lagrangian expressed in unitary gauge for the quarks reads

$$\mathcal{L}_{quark} = -(\bar{Q}_L)_i \frac{\lambda_d^{ij}}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix} (d_R)_j - (\bar{Q}_L)_i \frac{\lambda_u^{ij}}{\sqrt{2}} \begin{pmatrix} v+h \\ 0 \end{pmatrix} (u_R)_j + \text{h.c.} \tag{2.41}$$

$$= -\frac{\lambda_d^{ij}}{\sqrt{2}} (\bar{d}_L)_i (v+h) (d_R)_j - \frac{\lambda_u^{ij}}{\sqrt{2}} (\bar{u}_L)_i (v+h) (u_R)_j + \text{h.c.}\,. \tag{2.42}$$

From here, one can read off the mass matrices of the down-type and up-type quarks with

$$\mathcal{M}_d^{ij} = \frac{1}{\sqrt{2}} \lambda_d^{ij} v \quad \text{and} \quad \mathcal{M}_u^{ij} = \frac{1}{\sqrt{2}} \lambda_u^{ij} v\,. \tag{2.43}$$

As in the leptonic case, one needs two unitary matrices for diagonalizing the mass-matrix again, but this time one has two different mass-matrices for down-type and up-type quarks and one has to treat them separately, namely

$$\hat{\mathcal{M}}_d = U_d^\dagger \mathcal{M}_u W_d \quad \text{and} \quad \hat{\mathcal{M}}_u = U_u^\dagger \mathcal{M}_u W_u \tag{2.44}$$

are both diagonal with real and positive entries, denoted with M_d^i and M_u^i. Together with these diagonalizations, the physical mass-eigenstates of the quarks are given by

$$\hat{d}_L^i = (U_d^\dagger)^{ij} (d_L)_j\,, \quad \hat{d}_R^i = (W_d^\dagger)^{ij} (d_R)_j\,, \tag{2.45}$$

$$\hat{u}_L^i = (U_u^\dagger)^{ij} (u_L)_j\,, \quad \hat{u}_R^i = (W_u^\dagger)^{ij} (u_R)_j\,, \tag{2.46}$$

and the Lagrangian reads

$$\mathcal{L}_{quark} = -\left\{ (\bar{\hat{d}}_L)_i \hat{\mathcal{M}}_d^{ij} (\hat{d}_R)_i + (\bar{\hat{u}}_L)_i \hat{\mathcal{M}}_u^{ij} (\hat{u}_R)_i \right\} \left(1 + \frac{h}{v}\right) + \text{h.c.} \tag{2.47}$$

$$= -\left\{ M_d^i (\bar{\hat{d}}_L^i \hat{d}_R^i + \bar{\hat{d}}_R^i \hat{d}_L^i) + M_u^i (\bar{\hat{u}}_L^i \hat{u}_R^i + \bar{\hat{u}}_R^i \hat{u}_L^i) \right\} \left(1 + \frac{h}{v}\right)\,. \tag{2.48}$$

From here, one can also read off the couplings of the quarks to the Higgs boson, namely

$$g_{qhh,i} = \frac{M_d^i}{v} \tag{2.49}$$

for the three down-type quarks, and

$$g_{qhh,i} = \frac{M_u^i}{v} \tag{2.50}$$

for the three up-type quarks.

2.3.3 The issue of neutrinos

In the minimal version of the SM, neutrinos are described as massless particles without possessing couplings of Yukawa-type like charged leptons and quarks as discussed in this section. But from neutrino oscillation measurements [12–14], one knows that they have to be massive particle states even if the masses are very tiny. In the next chapter, we will discuss the seesaw-mechanism, a way how they can be described as massive particle states.

3 Neutrino Masses in BSM physics

As mentioned before, neutrinos are described as massless particles within the minimal version of the SM. By means of an enlargement of the particle content of the SM by an arbitrary number n_R of right-handed neutrino fields and one scalar singlet X, being both singlets under the SM gauge-group G_{SM} (2.1), it is possible to write down both Yukawa couplings of the neutrinos to the Higgs doublet and couplings of Majorana-type to the additional scalar singlet X. After SSB occurs and X gets a nonvanishing VEV, either due to an explicit scalar mass term in the Lagrangian or due to radiative corrections to the scalar potential, an explanation for light neutrino masses is given by the seesaw-mechanism of type 1 which was first studied independently by some authors in [45–48].

3.1 Dirac- and Majorana-Lagrangian for neutrinos

Adding n_R right-handed neutrino singlets,

$$(\nu_R)_j\,, \quad j = \{1, 2, ..., n_R\}\,, \tag{3.1}$$

to the particle spectrum of the SM, one can write down a Yukawa-interaction term to the Higgs doublet as follows [62],

$$\mathcal{L}_D = -\sum_{i=1}^{3}\sum_{j=1}^{n_R} \overline{(E_L)_i}\,(\Delta)_{ij}\,\tilde{\Phi}\,(\nu_R)_j\; + \text{h.c.}\,, \tag{3.2}$$

where $(\Delta)_{ij}$ is a complex-valued $(3 \times n_R)$-matrix and $\tilde{\Phi} = i\sigma_2\Phi^*$.

Moreover by adding a complex scalar singlet $X = \varphi_5 + i\varphi_6$ to the particle spectrum, one can with both the right-handed neutrinos and X construct an explicit term with couplings of Majorana-type with [62]

$$\begin{aligned}\mathcal{L}_M &= -\frac{1}{2}\sum_{j_1,j_2=1}^{n_R} \overline{(\nu_R)^c_{j_1}}\,(\Gamma_\nu)_{j_1 j_2}\,X\,(\nu_R)_{j_2}\; + \text{h.c.} \\ &= \frac{1}{2}\sum_{j_1,j_2=1}^{n_R} (\nu_R)^{\mathrm{T}}_{j_1}\,C^{-1}\,(\Gamma_\nu)_{j_1 j_2}\,X\,(\nu_R)_{j_2}\; + \text{h.c.}\,.\end{aligned} \tag{3.3}$$

Here $(\Gamma_\nu)_{j_1 j_2}$ is a $(n_R \times n_R)$-matrix with, in general, complex entries and C is the charge conjugation matrix with the following properties [63],

$$C^{\mathrm{T}} = -C, \quad C^{\dagger} = C^{-1}, \quad C\,\gamma_\mu^{\mathrm{T}}\,C^{-1} = -\gamma_\mu\,.$$

In (3.3) we have used that [63]

$$\overline{(\nu_R)^c_{j_1}} = -(\nu_R)^{\mathrm{T}}_{j_1}\,C^{-1} \tag{3.4}$$

holds and that $(\nu_R)^c_{j_1}$ is a left-handed particle state defined by [63]

$$(\nu_R)^c_{j_1} = C\,(\overline{\nu_R})^{\mathrm{T}}_{j_1}\,. \tag{3.5}$$

After SSB and when both the scalar doublet Φ and the scalar singlet X get nonvanishing VEVs, $\langle\Phi\rangle$ (2.16) and $\langle X\rangle$, the neutrino mass termes read

$$\mathcal{L}_D = -\sum_{i=1}^{3}\sum_{j=1}^{n_R}\overline{(\nu_L)_i}\,(M_D)_{ij}\,(\nu_R)_j \;+\;\text{h.c.}\,, \tag{3.6}$$

$$\mathcal{L}_M = \frac{1}{2}\sum_{j_1,j_2=1}^{n_R}(\nu_R)^{\mathrm{T}}_{j_1}\,C^{-1}\,(M_R)_{j_1j_2}\,(\nu_R)_{j_2} \;+\;\text{h.c.}\,, \tag{3.7}$$

with the neutrino mass matrices M_D and M_R

$$(M_D)_{ij} = \frac{1}{\sqrt{2}}\,(\Delta)_{ij}\,v \quad \text{and} \quad (M_R)_{j_1j_2} = (\Gamma_\nu)_{j_1j_2}\langle X\rangle\,. \tag{3.8}$$

Due to the anticommutation relation of fermion fields and the knowledge that the charge conjugation matrix C is an antisymmetric matrix, it follows that $(\nu_R)^{\mathrm{T}}_{j_1}\,C^{-1}\,(\nu_R)_{j_2} = (\nu_R)^{\mathrm{T}}_{j_2}\,C^{-1}\,(\nu_R)_{j_1}$ and thus that the Majorana-massmatrix M_R has to be symmetric:

$$\begin{aligned}
\mathcal{L}_M &= \frac{1}{2}\sum_{j_1,j_2=1}^{n_R}(\nu_R)^{\mathrm{T}}_{j_1}\,C^{-1}\,(M_R)_{j_1j_2}\,(\nu_R)_{j_2} \;+\;\text{h.c.}\\
&= \frac{1}{2}\sum_{j_1,j_2=1}^{n_R}(\nu_R)^{\mathrm{T}}_{j_2}\,C^{-1}\,(M_R)_{j_1j_2}\,(\nu_R)_{j_1} \;+\;\text{h.c.}\\
&= \frac{1}{2}\sum_{j_1,j_2=1}^{n_R}(\nu_R)^{\mathrm{T}}_{j_1}\,C^{-1}\,(M_R)_{j_2j_1}\,(\nu_R)_{j_1} \;+\;\text{h.c.}\;.
\end{aligned}$$

At the first equality we have used the relation from above, $(\nu_R)^{\mathrm{T}}_{j_1}\,C^{-1}\,(\nu_R)_{j_2} = (\nu_R)^{\mathrm{T}}_{j_2}\,C^{-1}\,(\nu_R)_{j_1}$, and at the second equality we have just renamed respectively exchanged the indices.

Defining a $(n_R + 3)$-component right-handed neutrino field with [62]

$$\omega_R = \begin{pmatrix} (\nu_L)^c \\ \nu_R \end{pmatrix} , \tag{3.9}$$

one can add the Dirac (3.6) and the Majorana (3.7) Lagrangian gaining (suppressing the family indices of the fields) [62]

$$\begin{aligned} \mathcal{L}_D + \mathcal{L}_M &= -\bar{\nu}_L \, M_D \, \nu_R + \frac{1}{2} \, \nu_R^{\mathrm{T}} \, C^{-1} \, M_R \, \nu_R \, + \text{ h.c.} \\ &= \frac{1}{2} \, \omega_R^{\mathrm{T}} \, C^{-1} \, M_{D+M} \, \omega_R \, + \text{ h.c.} \\ &= -\frac{1}{2} \, \overline{(\omega_R)^c} \, M_{D+M} \, \omega_R \, + \text{ h.c.} \end{aligned} \tag{3.10}$$

with the $(3 + n_R) \times (3 + n_R)$-dimensional mass matrix M_{D+M} given by [62]

$$M_{D+M} = \begin{matrix} & 3 & n_R \\ 3 & 0 & M_D \\ n_R & M_D^{\mathrm{T}} & M_R \end{matrix} \quad \text{i.e.} \quad M_{D+M} = \begin{pmatrix} 0 & M_D \\ M_D^{\mathrm{T}} & M_R \end{pmatrix} . \tag{3.11}$$

For every symmetric matrix with complex entries, like M_{D+M}, it is possible to make a transformation such that

$$\hat{M}_{D+M} = U^{\mathrm{T}} \, M_{D+M} \, U \tag{3.12}$$

is a diagonal matrix with only non-negative entries, where U being a unitary matrix. With that same transformation, one obtains the tree-level mass-eigenfields of the neutrinos by [62]

$$\hat{\omega}_R = U^{\dagger} \, \omega_R \, . \tag{3.13}$$

After that transformation, the Lagrangian (3.10) reads

$$\begin{aligned} \mathcal{L}_D + \mathcal{L}_M &= -\frac{1}{2} \, \overline{(\omega_R)^c} \, M_{D+M} \, \omega_R \, + \text{ h.c.} = \frac{1}{2} \, \omega_R^{\mathrm{T}} \, C^{-1} \, M_{D+M} \, \omega_R \, + \text{ h.c.} \\ &= \frac{1}{2} \, \hat{\omega}_R^{\mathrm{T}} \, U^{\mathrm{T}} \, C^{-1} \, U^* \, \hat{M}_{D+M} \, U^{\dagger} \, U \, \hat{\omega}_R \, + \text{ h.c.} \\ &= \frac{1}{2} \, \hat{\omega}_R^{\mathrm{T}} \, C^{-1} \, \hat{M}_{D+M} \, \hat{\omega}_R \, + \text{ h.c.} = -\frac{1}{2} \, \overline{(\hat{\omega}_R)^c} \, \hat{M}_{D+M} \, \hat{\omega}_R \, + \text{ h.c.} \, . \end{aligned} \tag{3.14}$$

Defining the vector N as [62, 63]

$$N = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_{3+n_R} \end{pmatrix} = \hat{\omega}_R + (\hat{\omega}_R)^c \,, \tag{3.15}$$

where all fields ν_k satisfy

$$(\nu_k)^c = \nu_k \qquad \forall\, k = 1, \ldots, 3 + n_R \,, \tag{3.16}$$

the Lagrangian can be written as [62, 63]

$$\mathcal{L}_D + \mathcal{L}_M = -\frac{1}{2}\,\overline{N}\,\hat{M}_{D+M} N = -\frac{1}{2}\sum_{k=1}^{3+n_R} m_k\,\overline{\nu_k}\,\nu_k \,. \tag{3.17}$$

This shows that all the mass-eigenfields coming from a Dirac-Majorana-Lagrangian are in general Majorana fields, being spin $\frac{1}{2}$ particle and having no nontrivial charges. The particles and antiparticles are exactly the same.

3.2 Neutrino Masses via seesaw-mechanism of type 1

To make the diagonalization of M_{D+M} explicit, one decomposes the matrix U via [63, 64]

$$U = W\,V = \begin{pmatrix} \sqrt{\mathbb{1}_3 - BB^\dagger} & B \\ -B^\dagger & \sqrt{\mathbb{1}_{n_R} - B^\dagger B} \end{pmatrix} \mathrm{diag}(V_L\,, V_R) \,. \tag{3.18}$$

Here, all three matrices W, V_L and V_R are unitary matrices with W being a $(3 + n_R) \times (3 + n_R)$-matrix, V_L being 3×3 and V_R being $n_R \times n_R$ [64]. In W, B can be expanded in powers of $\frac{M_D}{M_R}$ and reads $B = M_D^*\,(M_R^*)^{-1}$ at lowest order [63] independently what forms M_D and M_R exactly take. This approximation is possible, since the basic and important assumption of the seesaw-mechanism is that the scale of M_D is much smaller than the scale of M_R. With this assumption, the matrix W is given by [63]

$$\begin{aligned} W \simeq \begin{matrix} \\ 3 \\ n_R \end{matrix} & \begin{matrix} \quad 3 & \qquad\qquad n_R \\ \end{matrix} \\ & \begin{pmatrix} \mathbb{1}_3 - \frac{1}{2}\,BB^\dagger & B \\ -B^\dagger & \mathbb{1}_{n_R} - \frac{1}{2}\,B^\dagger B \end{pmatrix} \\ = & \begin{pmatrix} \mathbb{1}_3 - \frac{1}{2}\,M_D^*\,(M_R^*)^{-1}\,M_R^{-1}\,M_D^{\mathrm{T}} & M_D^*\,(M_R^*)^{-1} \\ -M_R^{-1}\,M_D^{\mathrm{T}} & \mathbb{1}_{n_R} - \frac{1}{2}\,M_R^{-1}\,M_D^{\mathrm{T}}\,M_D^*\,(M_R^*)^{-1} \end{pmatrix} \end{aligned} \tag{3.19}$$

at lowest order.

The procedure of the diagonalization is done in two consecutive steps. The matrix W brings M_{D+M} in block-diagonal form and with the matrix V one achieves the diagonal matrix with the mass-eigenvalues being on the diagonal.

As first step, bringing the mass-matrix M_{D+M} in block-diagonal form via separating the small from the large scale, one obtains [63, 64]

$$W^{\mathrm{T}} M_{D+M} W \simeq \begin{array}{c} \\ 3 \\ n_R \end{array} \!\!\begin{array}{c} \begin{array}{cc} 3 & \quad n_R \end{array} \\ \begin{pmatrix} \mathcal{M}_{light} & 0 \\ 0 & \mathcal{M}_{heavy} \end{pmatrix} \end{array}, \tag{3.20}$$

with $\mathcal{M}_{light}$ and $\mathcal{M}_{heavy}$ given by [62, 63, 65]

$$\mathcal{M}_{light} = -\mathbf{M}_D(\mathbf{M}_R)^{-1}\mathbf{M}_D^{\mathrm{T}} \quad \text{and} \quad \mathcal{M}_{heavy} = \mathbf{M}_R \,. \tag{3.21}$$

In this block-diagonal matrix (3.20), $\mathcal{M}_{heavy}$ is given up to order $\mathcal{O}(\frac{M_D^2}{M_R})$ and the three remaining matrices are accurate up to order $\mathcal{O}(\frac{M_D^3}{M_R^2})$ [66].

In the second step, the diagonalization of the two remaining submatrices $\mathcal{M}_{light}$ and $\mathcal{M}_{heavy}$ gives [64]

$$V_R^{\mathrm{T}}\, \mathcal{M}_{heavy}\, V_R = \text{diag}(M_{\nu,1}, ..., M_{\nu,n_R})\,, \tag{3.22}$$

$$V_L^{\mathrm{T}}\, \mathcal{M}_{light}\, V_L = \text{diag}(M_{\nu,n_R+1}, M_{\nu,n_R+2}, M_{\nu,n_R+3})\,. \tag{3.23}$$

This mechanism we considered and summarized here is called seesaw-mechanism and is based on the assumption that the scale of the Majorana-massterm M_R is much larger than that of the Dirac-massmatrix M_D and thus the scale of electroweak symmetry breaking. In addition, there is the assumption that the total lepton number L is violated by the term with M_R. The scale of M_R can be rather low given by the TeV scale or higher in the range of the grand unification scale given by $\sim 10^{15}$ GeV or even as high as the Planck scale with $\sim 10^{19}$ GeV in dependence which model is under consideration [63].

An attractive feature of the Seesaw-mechanism (of type 1) is the fact that the masses of the three light neutrinos are arising in a natural way via formula (3.21) in dependence of the two given mass scales in a specific model considered. In (3.21), one finds the explanation that these masses are by orders smaller than the masses of all the fundamental charged fermions, e.g. $\frac{m_\nu}{m_e} = 10^{-6}$ with m_e being the mass of the electron [67].

3.2.1 Example of one left- and one right-handed neutrino field

For illustration and giving the simplest example, let us consider the case of having just one left-handed ν_L and one right-handed ν_R neutrino field. Then after SSB, the Dirac-Majorana massterm reads [63]

$$\mathcal{L}_D + \mathcal{L}_M = -\overline{\nu_L}\, m_D\, \nu_R - \frac{1}{2}\, \overline{(\nu_R)^c}\, m_R\, \nu_R + \text{h.c.} = \frac{1}{2}\, (\omega_R)^{\mathrm{T}}\, C^{-1}\, M_{D+M}\, \omega_R + \text{h.c.}\,, \quad (3.24)$$

where the scale of m_R is assumed to be much bigger than that of m_D, $m_D \ll m_R$, and here both m_D and m_R are, in general, complex numbers. The full 2×2 mass-matrix M_{D+M} reads

$$M_{D+M} = \begin{pmatrix} 0 & m_D \\ m_D & m_R \end{pmatrix} \quad (3.25)$$

and the two component vector ω_R is given by

$$\omega_R = \begin{pmatrix} (\nu_L)^c \\ \nu_R \end{pmatrix} . \quad (3.26)$$

The two mass eigenvalues of M_{D+M} can be calculated to be

$$m_{light} = \sqrt{\frac{m_R^2}{4} + m_D^2} - \frac{m_R}{2} \;\approx \frac{m_D^2}{m_R}\,, \quad (3.27)$$

$$m_{heavy} = \sqrt{\frac{m_R^2}{4} + m_D^2} + \frac{m_R}{2} \;\approx m_R\,, \quad (3.28)$$

using the approximation $m_D \ll m_R$.

To diagonalize M_{D+M} (3.25), one uses the unitary matrix

$$U = \begin{pmatrix} -i\,\cos\theta & \sin\theta \\ i\,\sin\theta & \cos\theta \end{pmatrix} \quad (3.29)$$

with $\sin\theta = \sqrt{\frac{m_{light}}{m_{light}+m_{heavy}}} \approx \frac{m_D}{m_R}$ and $\cos\theta > 0$. Then one obtains from equation (3.12)

$$\hat{M}_{D+M} = U^{\mathrm{T}}\, M_{D+M}\, U = \text{diag}(m_{heavy}, m_{light}) \quad (3.30)$$

showing that one has one light and one heavy neutrino − both being Majorana particles − with the masses given in (3.27) and (3.28).

4 Effective Potential at one-loop order: Coleman-Weinberg Potential

In this chapter, we introduce the basics and give an overview on the computation of effective potentials V_{eff} for a renormalizable field theory and we show the results of calculating them up to one-loop order: the CW-Potential [49]. Considering only tree-level and one-loop contributions to the effective potential V_{eff}, it is called a semiclassical or quasiclassical approximation [49, 68]. In this procedure, there is the possibility that the symmetry is broken spontaneously − in models where one does not expect such a breakdown from considering the tree-level potential − due to quantum corrections [49] instead of having a negative Higgs term in the Lagrangian. That means considering classically scale-invariant models.

A great advantage of the semiclassical approximation is to look at all possible vacuum states simultaneously. While we are doing this, we can compute all higher order corrections to the effective potential V_{eff}, which corresponds to a diagramatic loop expansion [49]. The minima of V_{eff} then give the true vacua of the considered theory.

The derivation we will show in the following sections is valid for all classically scale-invariant (non-)Abelian renormalizable field theories. After explaining what classical scale-invariance means (section 4.1), we will consider a quantum field theory with just one single real scalar field $\varphi(x)$ (making the notation simpler) implemented by the Lagrangian $\mathcal{L}(\varphi, \partial_\mu\varphi)$ in sections 4.2 and 4.3. Moreover, for later convenience, all the formulae in section 4.2 are written in Euclidean space.

4.1 Models with classical scale-invariance

Under a scale transformation [69], all space time points transform with the same parameter ρ, namely

$$x_\mu \to x'_\mu = \rho\, x_\mu \tag{4.1}$$

and simultaneously the momenta transform with

$$p_\mu \to p'_\mu = \frac{1}{\rho}\, p_\mu . \tag{4.2}$$

At the same time, physical parameters like explicit (tree-level) masses and couplings λ_{ijkl}

remain unchanged. Thus in (classically) scale-invariant theories, all the particles get their masses merely due to their couplings to the scalar fields after SSB, as shown for gauge fields in section 2.2.2 and fermions in section 2.3.

In infinitesimal form, such a scale transformation reads [69]

$$x_\mu \to x'_\mu = (1+\varepsilon)\, x_\mu \tag{4.3}$$

and simultaneously any field $k(x)$ transforms with [69]

$$k(x) \to k'(x') = (1 - d_i\varepsilon)\, k(x), \tag{4.4}$$

where the number d_i is the (mass-)dimension of the field, $[k] = [mass]^{d_i}$. Thus:

$$d_i = 1 \quad \text{for scalar fields,} \tag{4.5}$$

$$d_i = \frac{3}{2} \quad \text{for fermionic fields,} \tag{4.6}$$

$$d_i = 1 \quad \text{for gauge fields.} \tag{4.7}$$

Writing it not in the infinitesimal form, scalar fields $\varphi_i(x)$, fermionic fields $\psi_j(x)$ and gauge fields $A_l^\mu(x)$ transform in the following form:

$$\varphi_i(x) \to \frac{1}{\rho}\, \varphi_i(x), \tag{4.8}$$

$$\psi_j(x) \to \frac{1}{\rho^{\frac{3}{2}}}\, \psi_j(x), \tag{4.9}$$

$$A_l^\mu(x) \to \frac{1}{\rho}\, A_l^\mu(x). \tag{4.10}$$

A theory given by the Lagrangian $\mathcal{L}$ is said to be classically scale-invariant if its action $S = \int \mathrm{d}^4x\, \mathcal{L}$ is invariant under the transformations in (4.8) to (4.10). This is exactly then the case, if all terms in $\mathcal{L}$ have operator dimension equal to 4. As a matter of fact, the only term in the SM-Lagrangian whose term is not 4-dimensional, and therefore breaks scale-invariance, is the Higgs mass term $\mu^2\, \Phi^\dagger\Phi$ in (2.13).

4.1.1 Example: Massless φ^4-theory

To illustrate the classical scale-invariance given at tree-level which then breaks down already at one-loop order, let us consider a massless φ^4-theory with just one single real scalar field φ. Its Lagrangian is given by

$$\mathcal{L}(\varphi, \partial_\mu\varphi) \;=\; (\partial^\mu\varphi)(\partial_\mu\varphi) + \frac{\lambda}{4!}\,\varphi^4. \tag{4.11}$$

The action reads

$$S = \int \mathrm{d}^4x \left[(\partial^\mu \varphi)(\partial_\mu \varphi) + \frac{\lambda}{4!}\,\varphi^4 \right] \tag{4.12}$$

and is obviously invariant under

$$x \to \rho\, x, \quad \mathrm{d}^4x \to \rho^4\, \mathrm{d}^4x, \tag{4.13}$$

$$\varphi \to \frac{1}{\rho}\,\varphi, \quad \partial_\mu \to \frac{1}{\rho}\,\partial_\mu. \tag{4.14}$$

But the effective potential at one-loop order (for the form of the result see section 4.4.1 and for more details of the calculation [49]) reads

$$\mathrm{V}_{\mathrm{eff}}(\varphi) \;=\; \frac{\lambda}{4!}\,\varphi^4 + \frac{\lambda^2\,\varphi^4}{256\,\pi^2}\left(\ln\frac{\frac{\lambda}{2}\,\varphi^2}{\mu^2} - \frac{3}{2}\right) \tag{4.15}$$

and breaks scale-invariance being there at tree-level, since

$$\int \mathrm{d}^4x\,\varphi^4\,\ln\varphi^2 \quad\to\quad \int \mathrm{d}^4x\,\varphi^4\,\left(\ln\varphi^2 - \ln\rho^2\right). \tag{4.16}$$

4.2 Derivation of the effective potential

To describe a quantum field theory completely, one can consider Z$[f]$, the generating functional of all Green's functions. It is defined as

$$\mathrm{Z}[f] \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^4x_1 \cdots \mathrm{d}^4x_n \,\langle\, 0\,|\,\mathrm{T}\,[\hat\varphi(x_1)\cdots\hat\varphi(x_n)]\;|\,0\,\rangle\, f(x_1)\cdots f(x_n), \tag{4.17}$$

where each of the coefficients $\langle\, 0\,|\,\mathrm{T}\,[\hat\varphi(x_1)\cdots\hat\varphi(x_n)]\;|\,0\,\rangle$, for a fixed number n, is a n-point Green's functions with different space-time point x_1 to x_n. Here, $f(x)$ is an external source, which has to vanish at large distances, $f(x) \to 0$ for $|x| \to \infty$, so that all the integrals in (4.17) converge [68]. It is introduced to the Lagrangian by coupling $\varphi(x)$ to an external field $f(x)$ via [49, 70]

$$\mathcal{L}'(\varphi, \partial_\mu\varphi, f) \;=\; \mathcal{L}(\varphi, \partial_\mu\varphi) + \mathcal{L}_f(\varphi, f) \;=\; \mathcal{L}(\varphi, \partial_\mu\varphi) - \varphi(x)f(x). \tag{4.18}$$

The generating functional Z$[f]$ is the transition amplitude from the groundstate $|0\rangle$ to the groundstate $|0\rangle$ under influence of $f(x)$. On the one hand, Z$[f]$ can be written as the exponential of W$[f]$ [49, 68],

$$\mathrm{Z}[f] \;=\; \langle\, 0\,|\,0\,\rangle_f \;=\; \exp\{-\mathrm{W}[f]\}, \tag{4.19}$$

with

$$-\mathrm{W}[f] \;=\; \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}^4x_1 \cdots \mathrm{d}^4x_n \; \underbrace{\langle\, 0\,|\, \mathrm{T}\, [\hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n)] \;|\, 0\,\rangle_c}_{\mathrm{G}_c^{(n)}(x_1,\, ...,\, x_n)} \; f(x_1) \cdots f(x_n) \tag{4.20}$$

being the generating functional of all connected Green's functions $\mathrm{G}_c^{(n)}(x_1,\, ...,\, x_n)$. On the other hand, one can write Z[f] as a functional of the time-ordered exponential [68, 70]

$$\mathrm{Z}[f] \;=\; \langle\, 0\,|\, 0\,\rangle_f \;=\; \langle\, 0\,|\mathrm{T}\left[\exp\left\{\int \mathrm{d}^4x \;\hat{\varphi}(x) f(x)\right\}\right] |\, 0\,\rangle. \tag{4.21}$$

In path integral representation, the connected n-point Green's functions $\mathrm{G}_c^{(n)}(x_1,\, ...,\, x_n)$ can be expressed as [68]

$$\mathrm{G}_c^{(n)}(x_1,\, ...,\, x_n) \;=\; \frac{1}{\mathcal{N}} \int [d\varphi]\, \varphi(x_1) \cdots \varphi(x_n)\; \exp\{-\, S_{cl}(\varphi)\} \tag{4.22}$$

with the normalization defined as $\mathcal{N} = \int [d\varphi] \exp\{-S_{cl}\}$ and $S_{cl} = \int \mathrm{d}^4x\, \mathcal{L}(\varphi, \partial_\mu \varphi)$ being the classical action of the theory. Furtheron, one can write Z[f] in the path integral representation as [68]

$$\mathrm{Z}[f] \;=\; \exp\{-\mathrm{W}[f]\} \;=\; \frac{1}{\mathcal{N}} \int [d\varphi]\; \exp\left\{-\int \mathrm{d}^4x \;\mathcal{L}'(\varphi, \partial_\mu\varphi, f)\right\}. \tag{4.23}$$

The term in the denominator is chosen such that the generating functional Z[f] is normalized to Z[0]=1. Due to the change in the Lagrangian (4.18), the model gets a perturbation from the term $\mathcal{L}_f(\varphi, f)$. In (4.23), W[$f$] gives the response of the model to that perturbation and represents an average value of the classical action, given by $\exp\{-\int \mathrm{d}^4x\, \mathcal{L}(\varphi, \partial_\mu\varphi)\}$ [68].

The functional derivative of W[f] with respect to $f(x)$ [49],

$$\varphi_c(x) \;=\; \frac{\delta \mathrm{W}[f]}{\delta f(x)} \;=\; \left.\frac{\langle\, 0\,|\, \hat{\varphi}(x)\,|\, 0\,\rangle}{\langle\, 0\,|\, 0\,\rangle}\right|_f, \tag{4.24}$$

is called the classical field $\varphi_c(x)$. The *effective action* $\Gamma[\varphi_c]$, then defined as a functional $-$ in dependence of $\varphi_c(x)$ $-$ via the Legendre-transformation, reads [49, 70]

$$\Gamma[\varphi_c] \;=\; \mathrm{W}[f] \;+\; \int \mathrm{d}^4x\, f(x)\varphi_c(x). \tag{4.25}$$

Similarly to W[f] and Z[f], the *effective action* can be expanded in powers of $\varphi_c(x)$ [49],

$$\Gamma[\varphi_c] \;=\; \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}^4x_1 \cdots \mathrm{d}^4x_n\, \Gamma^{(n)}(x_1,\, ...,\, x_n)\, \varphi_c(x_1) \cdots \varphi_c(x_n), \tag{4.26}$$

where the coefficients $\Gamma^{(n)}(x_1,\, ...,\, x_n)$ turn out to be the sum of all n-point one-particle irreducible (1PI) Feynman graphs. Not expanded in $\varphi_c(x)$ itself, but in momenta thereof, the *effective action* reads [49]

$$\Gamma[\varphi_c] \,=\, \int \mathrm{d}^4x \,\left[\mathrm{V}_{\mathrm{eff}}(\varphi_c) \,+\, \frac{1}{2}(\partial_\mu\varphi_c)(\partial^\mu\varphi_c)\,\mathrm{Z}(\varphi_c) \,+\, ...\right], \tag{4.27}$$

with the term independent of the momenta being the *effective potential* $\mathrm{V}_{\mathrm{eff}}(\varphi_c)$. Considering (4.26) and (4.27) in the limit of vanishing external momenta, the sum af all n-point 1PI Green's functions can be computed by taking the n-th derivative of $\mathrm{V}_{\mathrm{eff}}(\varphi_c)$ [49].

4.3 Loop-Expansion of the effective potential

If we know the structure of the effective potential $\mathrm{V}_{\mathrm{eff}}$ of the considered theory, we can give information about SSB. An exact calculation of $\mathrm{V}_{\mathrm{eff}}$ is impossible by evaluating all Feynman-diagramms, since one would need to consider an infinite number of loops and one would have to calculate an infinite number of diagramms at each loop order. Thus, we need an approximation method, namely considering $\mathrm{V}_{\mathrm{eff}}$ per loop order; first summing up all tree-diagramms, then summing up those with one loop, then those with two loops and so forth and so on.

Introducing the parameter $\hbar$ to the Lagrangian $\mathcal{L}'$ in (4.18) via [49]

$$\mathcal{L}'(\varphi, \partial_\mu\varphi, f) \,\rightarrow\, \frac{1}{\hbar}\,\mathcal{L}'(\varphi, \partial_\mu\varphi, f), \tag{4.28}$$

while treating $\mathcal{L}'(\varphi, \partial_\mu\varphi, f)$ as independent of $\hbar$, one can use as the desired approximation method [49, 68]

$$\mathrm{W}[f] \,=\, \sum_\ell \hbar^\ell \mathrm{W}^{(\ell)}[f], \tag{4.29}$$

with ℓ being the number of closed loops in each 1PI Feynman-diagramm and $\mathrm{W}^{(\ell)}[f]$ being all the contribution to $\mathrm{W}[f]$ at loop order ℓ.

Hence, we can write $\mathrm{Z}[f]$ as an expansion in powers of $\hbar$ as

$$\mathrm{Z}[f] \,=\, \exp\left(-\frac{1}{\hbar}\,\mathrm{W}[f]\right) \,=\, \exp\left(-\frac{1}{\hbar}\mathrm{W}^{(0)}[f] \,-\, \mathrm{W}^{(1)}[f] \,-\, \hbar\mathrm{W}^{(2)}[f] \,+\, ...\right). \tag{4.30}$$

In the approximation of only considering tree-level graphs, i.e. $\exp\left(-\frac{1}{\hbar}\mathrm{W}^{(0)}[f]\right)$, $\mathrm{Z}[f]$ will fall off exponentially when $\hbar \rightarrow 0$ – a limit reffered to as the classical limit. From (4.30), it is visible that the one-loop quantum corrections give a finite contribution in this limit and all corrections of higher order vanish. This procedure of only considering tree-level and one-loop contributions is called *semiclassical approximation*.

A great advantage of this procedure is that one can compute $\mathrm{V}_{\mathrm{eff}}$ at any loop order without considering one specific vacuum state and since $\mathcal{L}'$ is independent of $\hbar$, the expansion parameter

$\hbar$ is unaffected by redefinition of the scalar fields [49].

4.4 The Coleman-Weinberg Potential

We want to display the one-loop corrections to the tree-level scalar potential V_0 for all possible renormalizable massless gauge field theories. The theory we consider here would consist of fields which are all assumed to be massless at tree-level. The fields are real spinless scalar fields φ^i, fermionic fields ψ^j and real vector fields A^l_μ, where the indices i, j and l numerate the respective fields in each case. For the couplings, we assume quartic self-interactions of the scalar fields, fermion-scalar couplings of *Yukawa-type* and couplings of the vector fields to the scalars which are minimally gauge-invariant. In the following, we will write all scalar fields φ^i into a vector $\vec{\varphi}$ for convenience.

As argued in [49], the effective potential at one-loop order reads

$$\mathrm{V}_{\mathrm{eff}}(\vec{\varphi}) \;=\; \mathrm{V}_0(\vec{\varphi}) + \delta\mathrm{V}^{(1)}(\vec{\varphi}) \;=\; \mathrm{V}_0(\vec{\varphi}) + \mathrm{V}_{\mathrm{S}}(\vec{\varphi}) + \mathrm{V}_{\mathrm{g}}(\vec{\varphi}) + \mathrm{V}_{\mathrm{f}}(\vec{\varphi}), \tag{4.31}$$

where V_0 is the scalar tree-level potential (shown in figure 4.1), V_{S} is the scalar one-loop contribution (shown in figure 4.2), V_{g} is the one-loop contribution due to gauge bosons (shown in figure 4.3) and V_{f} is the one-loop contribution due to fermions (shown in figure 4.4).

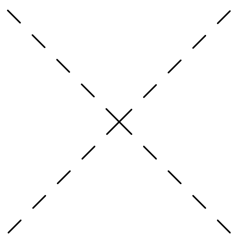

Figure 4.1: Tree-level graph in the scalar-sector contributing to V_0

At tree-level, there is just one single diagram contributing to $\mathrm{V}_0(\vec{\varphi})$, namely the one depicted in figure 4.1.

All the renormalized one-loop contributions written in the $\overline{\mathrm{MS}}$-scheme evaluated at renormalization-scale μ are given in the next paragraphs.

4.4.1 scalar loops

The one-loop contribution in the scalar sector, V_{S}, can be computed by summing up all 1PI graphs (infinite summation) with scalar particles in the loop as depicted in figure 4.2. The result can be computed as [71]

$$\mathrm{V_S}(\vec{\varphi}) \;=\; \frac{1}{64\pi^2}\,\mathrm{Tr}\left\{M_S^4(\vec{\varphi})\left[\ln\frac{M_S^2(\vec{\varphi})}{\mu^2}-\frac{3}{2}\right]\right\}, \tag{4.32}$$

with $M_S^2(\vec{\varphi})$ being a real and symmetric matrix as well as a quadratic function in $\vec{\varphi}$ reading

$$\left(M_S^2(\vec{\varphi})\right)_{ik} \;=\; \frac{\partial^2 \mathrm{V}_0(\vec{\varphi})}{\partial\varphi_i\,\partial\varphi_k}. \tag{4.33}$$

Evaluating (4.33) at the VEV of $\vec{\varphi}$, $\langle\vec{\varphi}\rangle$, one can read off the squared mass matrix of the scalars.

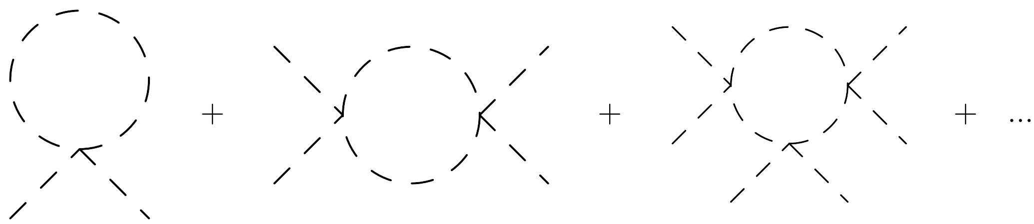

Figure 4.2: 1PI one-loop graphs contributing to V_S (scalar particles running around the loop) up to order n = 6

4.4.2 gauge boson loops

Figure 4.3 shows the first diagramms contributing to V_g. By summing up all 1PI graphs with a gauge boson running around the loop, one arrives at [71]

$$\mathrm{V_g}(\vec{\varphi}) \;=\; \frac{3}{64\pi^2}\,\mathrm{Tr}\left\{M_V^4(\vec{\varphi})\left[\ln\frac{M_V^2(\vec{\varphi})}{\mu^2}-\frac{5}{6}\right]\right\}, \tag{4.34}$$

where $M_V^2(\vec{\varphi})$ is again a real and symmetric matrix and quadratic in $\vec{\varphi}$. For gauge bosons which couple minimally to the scalar fields, it is given by $(M_V^2(\vec{\varphi}))_{lm} \;=\; g_l g_m (T_l\vec{\varphi}\,,\,T_m\vec{\varphi})$, where T_l and T_m are the generators of the underlying gauge groups and g_l and g_m are the coupling constants to the corresponding gauge fields. Inserting the VEV of $\vec{\varphi}$, $\langle\vec{\varphi}\rangle$, $M_V^2(\langle\vec{\varphi}\rangle)$ gives the squared mass matrix of the gauge bosons. The extra factor of 3 compared to the scalar case comes from the polarization degrees of freedom for each gauge boson.

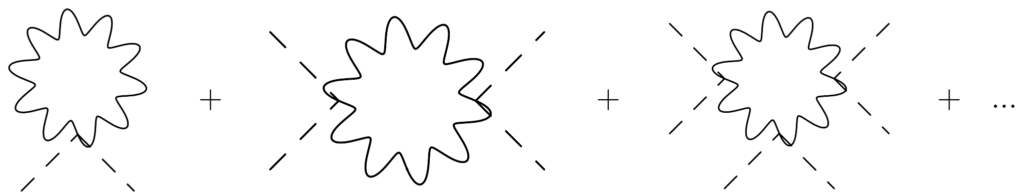

Figure 4.3: 1PI one-loop graphs contributing to V_g (gauge bosons running around the loop) up to order n = 6

4.4.3 fermion loops

In order to find the fermionic one-loop contribution to the effective potential, one considers the most general form of Yukawa-interactions of the fermions to scalars. This part of the Lagrangian reads

$$\mathcal{L} \;=\; \overline{\chi^c} M_\mathrm{f}(\vec{\varphi})\chi \;+\; \text{h.c.}\,, \tag{4.35}$$

where the symmetric matrix $M_\mathrm{f}(\vec{\varphi})$ is a linear function in $\vec{\varphi}$ and contains the Yukawa-couplings. If evaluated at $\langle\vec{\varphi}\rangle$, $M_\mathrm{f}(\langle\vec{\varphi}\rangle)$, it is the fermion-mass matrix. Furtheron, χ denotes the right-handed fermionic field vector and it can be decomposed as follows,

$$\chi = \begin{pmatrix} (\psi_L)^c \\ \psi_R \end{pmatrix}\,. \tag{4.36}$$

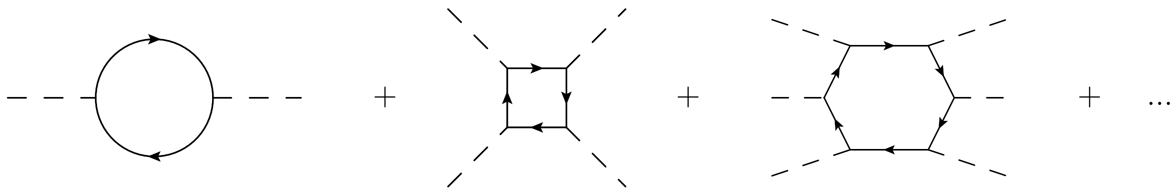

Figure 4.4: 1PI one-loop graphs contributing to V_f (fermions running around the loop) up to order n = 6

Since the trace of an odd number of gamma matrices vanishes, only diagramms (shown in figure 4.4) with an even number of internal fermion lines resulting in an even number of external scalar states contribute to the GW-potential. In this case, the result reads [71]

$$\mathrm{V}_\mathrm{f}(\vec{\varphi}) \;=\; -\frac{c_\mathrm{f}}{64\pi^2}\,\mathrm{Tr}\left\{\left(M_\mathrm{f}(\vec{\varphi})M_\mathrm{f}^\dagger(\vec{\varphi})\right)^2\left[\ln\frac{M_\mathrm{f}(\vec{\varphi})M_\mathrm{f}^\dagger(\vec{\varphi})}{\mu^2}-\frac{3}{2}\right]\right\}, \tag{4.37}$$

where $c_\mathrm{f} = 2$ for Majorana fermions and $c_\mathrm{f} = 4$ for Dirac fermions running around the loop. Each closed fermion loop gets a minus sign, and so we have an extra factor of -1 compared to the scalar case in front of the trace in (4.37).

5 Gildener Weinberg approach to the effective potential

In this chapter, we will follow the discussion of both Eldad Gildener's and Steven Weinberg's work [54] and the work of Lisa Alexander-Nunneley and Apostolos Pilaftsis [15] in order to describe when SSB in field theories with classical scale-invariance occurs.

5.1 SSB in classically scale-invariant theories

In their work [54], Eldad Gildener and Steven Weinberg (GW) work out the procedure of SSB in classically scale-invariant field theories with an arbitrary but fixed number n of weakly-coupled real scalar fields. They describe a method for finding a minimum of $\mathrm{V}_{\text{eff}} = \mathrm{V}_0 + \delta\mathrm{V}$ ($\delta\mathrm{V}$ being the loop-corrections to a given tree-level potential V_0) away from the origin in field space, but in a region where perturbation theory is still applicable.

In the following, we will consider a renormalizable gauge field theory with n real scalar fields φ_i $(i = \{1, 2, ..., n\})$, for convenience often written in an n-dimensional field vector $\vec{\varphi}$. Scale-invariance of the tree-level potential $\mathrm{V}_0(\vec{\varphi})$ is assumed and then the most general renormalizable tree-level potential can be written as

$$\mathrm{V}_0(\vec{\varphi}) = \frac{1}{4!}\lambda_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l, \tag{5.1}$$

where the fully symmetric (under exchange of all its indices) quartic scalar couplings λ_{ijkl} are defined via [54]

$$\left.\frac{\partial^4\,\mathrm{V}_0(\vec{\varphi})}{\partial\varphi_i\,\partial\varphi_j\,\partial\varphi_k\,\partial\varphi_l}\right|_{\vec{\varphi}\sim\mu} = \lambda_{ijkl}(\mu) \tag{5.2}$$

with μ being a renormalization scale with the dimension of a mass.

The tree-level potential in (5.1) has a continuous set of non-trivial local minima along a radially outwards going ray $\vec{\phi}_{flat} = \phi\vec{n}$ (called the „flat-direction") for a certain renormalization scale $\mu = \Lambda_W$. For finding this ray, first one has to search for the minimum on the unit sphere in field space, $\sum_i N_i^2 = 1$ (N_i denotes the component of a unit vector $\vec{N}$ in the radial direction). The non-trivial minimum is achieved at one specific point on the sphere, picking one specific direction $\vec{N} = \vec{n}$, and now one can choose Λ_W such that this minimum becomes zero,

$$\min_{n_\ell n_\ell = 1} \lambda_{ijkl}(\Lambda_W) n_i n_j n_k n_l = 0, \tag{5.3}$$

in the following reffered as to the GW-condition. This defines the „flat-direction“ as the direction going through both the origin and the minimum on the sphere. Because of the structure of the tree-level potential $\mathrm{V}_0 \sim \varphi_i \varphi_j \varphi_k \varphi_l$, V_0 does not vanish only at this point but gets the value 0 along the whole ray $\vec{\phi}_{flat}$ spanned by the vector $\vec{n}$. This equation (5.3) gives just a single constraint to all the λ_{ijkl}. So, implementing (5.3) reduces the number of the free dimensionless parameters by one in favor of introducing one dimensional parameter (a procedure called „*dimensional transmutation*“ [49]), namely $\mu = \Lambda_W$.

To make sure that $\vec{\phi}_{flat} = \phi \vec{n}$ is really a stationary line,

$$\left.\frac{\partial \mathrm{V}_0(\vec{\varphi})}{\partial \varphi_i}\right|_{\vec{N}=\vec{n}} = \frac{1}{3!} \lambda_{ijkl}(\Lambda_W) n_j n_k n_l = 0 \tag{5.4}$$

has to be fulfilled. Moreover, in order for the ray $\vec{\phi}_{flat}$ to form a set of (at least local) minima, the matrix of second derivatives (Hessian matrix) evaluated in the „flat-direction“,

$$(P)_{ij} = \left.\frac{\partial^2 \mathrm{V}_0(\vec{\varphi})}{\partial \varphi_i \, \partial \varphi_j}\right|_{\vec{\varphi}=\vec{n}} = \frac{1}{2} \lambda_{ijkl}(\Lambda_W) n_k n_l, \tag{5.5}$$

has to be positive semi-definit.

Combining (5.4) and (5.5), one immediately sees that the vector $\vec{n}$ is an eigenvector of the matrix (P) with eigenvalue 0. The particle corresponding to the eigenstate lying in the „flat-direction“ is the so-called **scalon** [54].

The full effective potential $\mathrm{V}_{\text{eff}}(\vec{\varphi})$ will get nonvanishing contributions from higher order quantum corrections and especially from one-loop corrections $\delta\mathrm{V}^{(1)}(\vec{\varphi})$ along $\vec{\phi}_{flat}$. These have a great impact on the description of the theory, because the tree-level potential vanishes there. Adding higher-order corrections to $\mathrm{V}_0(\vec{\varphi})$ gives the potential a small curvature along $\vec{\phi}_{flat}$ (in the radial direction) picking out a specific point $\langle\phi\rangle \, \vec{n}$ on the flat direction. In addition, the potential also gets a shift in an arbitrary direction $\delta\vec{\varphi}$ away from the ray. Demanding that $\langle\phi\rangle \, \vec{n} + \delta\vec{\varphi}$ is then the stationary point, one finds

$$0 = \left.\frac{\partial}{\partial \varphi_i} \underbrace{\left[\mathrm{V}_0(\vec{\varphi}) + \delta\mathrm{V}^{(1)}(\vec{\varphi})\right]}_{=\mathrm{V}_{\text{eff}}^{1-loop}(\vec{\varphi})}\right|_{\langle\phi\rangle \, \vec{n} + \delta\vec{\varphi}} = (P)_{ij} \, \delta\varphi_j \, \langle\phi\rangle^2 + \left.\frac{\partial \delta\mathrm{V}^{(1)}(\vec{\varphi})}{\partial \varphi_i}\right|_{\langle\phi\rangle\vec{n}}, \tag{5.6}$$

where $\delta\vec{\varphi}$ is to be assumed as a correction of one-loop order. Contracting (5.6) with $\vec{n}$ using

$(P)_{ij}n_j = 0$, one obtains

$$0 = n_i \, \frac{\partial}{\partial \varphi_i} \, \delta \mathrm{V}^{(1)}(\vec{\varphi}) \bigg|_{\langle \phi \rangle \, \vec{n}} = \frac{\partial \delta \mathrm{V}^{(1)}(\phi \, \vec{n})}{\partial \phi} \bigg|_{\langle \phi \rangle} \tag{5.7}$$

as the condition for finding the value $\langle \phi \rangle$ to one-loop order in perturbation theory.

Along the flat direction $\vec{\phi}_{flat} = \phi \vec{n}$ with the specific renormalization-scale $\mu = \Lambda_W$, the effective potential at one-loop order $\mathrm{V}_{\text{eff}}^{1-loop}(\vec{\phi}_{flat})$ can be written as [54]

$$\mathrm{V}_{\text{eff}}^{1-loop}(\vec{\phi}_{flat}) \;=\; \delta \mathrm{V}^{(1)}(\vec{\phi}_{flat}) \;=\; \mathrm{A}\phi^4 \;+\; \mathrm{B}\phi^4 \ln \frac{\phi^2}{\Lambda_W^2} \tag{5.8}$$

with A and B being two dimensionless constants,

$$\begin{aligned} \mathrm{A} \;=\; \frac{1}{64\pi^2 \langle \phi \rangle^4} \; & \left[3\,\mathrm{Tr} \left\{ M_V^4 \left(\ln \frac{M_V^2}{\langle \phi \rangle^2} - \frac{5}{6} \right) \right\} \;+\; \mathrm{Tr} \left\{ M_S^4 \left(\ln \frac{M_S^2}{\langle \phi \rangle^2} - \frac{3}{2} \right) \right\} \right. \\ & \left. -\, 4\,\mathrm{Tr} \left\{ M_{DF}^4 \left(\ln \frac{M_{DF}^2}{\langle \phi \rangle^2} - \frac{3}{2} \right) \right\} \;-\; 2\,\mathrm{Tr} \left\{ M_{MF}^4 \left(\ln \frac{M_{MF}^2}{\langle \phi \rangle^2} - \frac{3}{2} \right) \right\} \right] \end{aligned} \tag{5.9}$$

and

$$\mathrm{B} \;=\; \frac{1}{64\pi^2 \langle \phi \rangle^4} \, \Big(3\,\mathrm{Tr}\, M_V^4 \;+\; \mathrm{Tr}\, M_S^4 \;-\; 4\,\mathrm{Tr}\, M_{DF}^4 \;-\; 2\,\mathrm{Tr}\, M_{MF}^4 \Big) \tag{5.10}$$

in $\overline{\text{MS}}$ scheme. In these formulae, $M_{V,S,DF,MF}$ evaluated at $\langle \phi \rangle \, \vec{n}$ are the tree-level mass matrices for vector fields, scalar particles, Dirac and Majorana (right-handed neutrinos) fermions. The trace runs over all internal degrees of freedom like flavor and color and is taken over the matrices itself as well.

Searching for the minimum of (5.8) according to (5.7), $\langle \phi \rangle$ is given by

$$\langle \phi \rangle^2 = \Lambda_W^2 \exp \left[- \left(\frac{1}{2} + \frac{\mathrm{A}}{\mathrm{B}} \right) \right]. \tag{5.11}$$

So that this stationary point may be indeed a minimum, the value of B (5.10) has to be positive (B>0). Evaluating (5.8) at the minimum (5.11) in the flat direction, one gets [54]

$$\mathrm{V}_{\text{eff}}^{1-loop}(\langle \phi \rangle \vec{n}) \;=\; \underbrace{\mathrm{V}_0(\langle \phi \rangle \vec{n})}_{=0} + \delta \mathrm{V}^{(1)}(\langle \phi \rangle \vec{n}) \;=\; -\frac{1}{2} \mathrm{B} \langle \phi \rangle^4 < 0 \tag{5.12}$$

which shows that $\mathrm{V}_{\text{eff}}^{1-loop}(\langle \phi \rangle \vec{n})$ is definitely less than $\mathrm{V}(0) = 0$ and so electroweak SSB takes place if this stationary point is really a (local) minimum. The proof that this stationary point (5.11) is a minimum of the one-loop effective potential is given in the section 5.2.

5.2 Masses of the scalars

By means of the masses of the scalar bosons, one can calculate if this stationary point at $\langle\phi\rangle\,\vec{n}+\delta\vec{\varphi}$ is really a minimum or not.

At tree-level, the eigenvalues of the matrix [49, 54]

$$(M_S^2)_{ij} \;=\; \left.\frac{\partial^2 \mathrm{V}_0(\vec{\varphi})}{\partial\varphi_i\partial\varphi_j}\right|_{\langle\phi\rangle\,\vec{n}} \;=\; P_{ij}\langle\phi\rangle^2 \tag{5.13}$$

evaluated in the flat direction give the squared masses of all scalar particles. (5.13) is a symmetric n×n matrix and since (P) (5.5) has only vanishing and positive eigenvalues, the masses of (5.13) are zero or positive as well. The matrix (5.13) contains a set of vanishing eigenvalues corresponding to Goldstone bosons living in any gauge theory with a continuous symmetry and has one vanishing eigenvalue due to the eigenvector $\vec{n}$ corresponding to the scalon (associated as the pseudo-Goldstone boson from the breakdown of classical scale-invariance [15]) and a set of positive eigenvalues due to massive scalar particles.

When the potential receives small perturbations $\delta\mathrm{V}^{(1)}(\vec{\varphi})$ due to one-loop corrections, the eigenvalues of the squared mass matrix

$$\left(M_S^2+(\delta M_S^{(1)})^2\right)_{ij}(\vec{\varphi}) \;=\; \frac{\partial^2}{\partial\varphi_i\partial\varphi_j}\left[\mathrm{V}_0(\vec{\varphi})+\delta\mathrm{V}^{(1)}(\vec{\varphi})\right] \tag{5.14}$$

are shifted too. Considering only terms of first order in the perturbation and evaluating it at the stationary point $\langle\phi\rangle\,\vec{n}+\delta\vec{\varphi}$, one gets [54]

$$(\delta M_S^{(1)})^2_{ij}(\vec{\varphi}) \;=\; \left.\frac{\partial^2\,\delta\mathrm{V}^{(1)}(\vec{\varphi})}{\partial\varphi_i\,\partial\varphi_j}\right|_{\langle\phi\rangle\vec{n}} + \lambda_{ijkl}\,n_k\,\delta\varphi_l\,\langle\phi\rangle. \tag{5.15}$$

For the stationary point $\langle\phi\rangle\,\vec{n}+\delta\vec{\varphi}$ to be a local minimum, the matrix (5.15) must not have negative eigenvalues either. The set of vanishing eigenvalues corresponding to the Goldstone bosons remain massless as long as $\delta\mathrm{V}^{(1)}(\vec{\varphi})$ has the same global symmetries as $\mathrm{V}_0(\vec{\varphi})$. The squared masses $(\mathrm{m}_H^2+(\delta\mathrm{m}_H^{(1)})^2)$ of the heavy states remain positive, since only small perturbations are considered. Thus we finally have to calculate the one-loop corrected mass of the scalon. From perturbation theory we know that it is given by the expectation value $\vec{n}^{\mathrm{T}}(\delta M_S^{(1)})^2\vec{n}$ with the eigenvector $\vec{n}$ from tree-level. One receives

$$(\mathrm{m}_S^{(1)})^2 \;=\; (\delta M_S^{(1)})^2_{ij}n_in_j \;=\; \left.\frac{\partial^2\,\delta\mathrm{V}^{(1)}(\vec{\varphi})}{\partial\varphi_i\,\partial\varphi_j}\right|_{\langle\phi\rangle\vec{n}} n_in_j \;=\; \left.\frac{\partial^2\delta\mathrm{V}^{(1)}(\phi\,\vec{n})}{\partial\phi^2}\right|_{\langle\phi\rangle} \tag{5.16}$$

and doing the full calculation, one obtains [54]

$$(\mathrm{m}_S^{(1)})^2 \;=\; 8\mathrm{B}\langle\phi\rangle^2 \tag{5.17}$$

for the squared mass of the scalon at one-loop order. This is certainly positive, since B>0. The scalon is known as the pseudo-Goldstone boson of the spontaneous breakdown of scale-invariance. It is massless at tree-level and gets its mass when scale-invariance is broken due to quantum corrections. So, the matrix in (5.14) has only vanishing and positive eigenvalues too and thus the assumption that the discussed stationary point is indeed a minimum is completed here.

5.3 Example: Scale-invariant version of SM with one scalar doublet

At this point, it has to be noted that the construction of a classically scale-invariant version of the SM with just one scalar doublet fails. Having only one doublet in the SM, there is just a single physical scalar state, namely the Higgs boson with a mass of $\mathrm{m}_h = (125.09 \pm 0.21 \pm 0.11)$ GeV [61], which has to be identified as the scalon.

For the dimensionless constant B (5.10), we get (taking only the t-quark in the fermion sector, since it is so much heavier than all the other fermions)

$$\mathrm{B} = \frac{1}{64\,\pi^2\,\langle\phi\rangle^4}\left(6\,\mathrm{M}_W^4 + 3\,\mathrm{M}_Z^4 - 12\,\mathrm{M}_t^4\right). \tag{5.18}$$

The extra factor of 2 in front of the W-boson mass comes from the fact that W^+ and W^- have the same mass and the extra factor of 3 in front of the t-quark mass is due to color. Thus we obtain for the mass-square of the scalon (5.17) (masses for t-quark, gauge bosons taken from [61])

$$\mathrm{m}_h^2 = \left(\mathrm{m}_S^{(1)}\right)^2 = 8\mathrm{B}\langle\phi\rangle^2 = \frac{1}{8\,\pi^2\,\langle\phi\rangle^2}\underbrace{\left(6\,\mathrm{M}_W^4 + 3\,\mathrm{M}_Z^4 - 12\,\mathrm{M}_t^4\right)}_{\approx\,-\,(319\,\mathrm{GeV})^4} < 0. \tag{5.19}$$

Here, we see that the square of the Higgs mass would get negative, because the t-quark is so much heavier than the gauge bosons.

To solve this problem, there are two possibilities: Either one can enlarge the scalar sector of the SM with additional scalar fields ([15–17, 21–33], making no claim to be complete) and/or one extends the gauge group by (non)-Abelian factors. In the former case, one gets heavy fields already at tree-level which give positive contribution on the right-hand side of (5.18) and in the latter case more gauge boson come into being because of the larger pattern of SSB (giving positive contributions on the right-hand side of (5.18) as well).

In this thesis, we will discuss different scale-invariant versions of the SM where we just add one real or complex scalar singlet X without an extension of its gauge group. In this case the

Lagrangian reads

$$\mathcal{L} = \mathcal{L}_{SM}|_{\mu=0} + (\partial_\mu X)^\dagger (\partial^\mu X) + \mathrm{V}_0\,(\Phi,\, X) \tag{5.20}$$

and in addition $-$ with right-handed neutrino fields $-$ one has to consider the Dirac- and Majorana terms from section 3.1.

6 Minimal scale-invariant version of the SM with one real scalar singlet

6.1 Model-setup: Tree-level potential in the scalar sector

As Krzysztof A. Meissner and Hermann Nicolai [16] proposed, the minimal way to write a classically scale-invariant version of the SM with one additional real scalar field φ_5, being a singlet under $SU(2)_L \times U(1)_Y$ – the electroweak part of the SM gauge group – is by enlarging the tree-level potential in the scalar sector via

$$\mathrm{V}_0\,(\Phi,\,\varphi_5) = \lambda_H(\Phi^\dagger\Phi)^2 + \frac{1}{4}\lambda_S\varphi_5^4 - \lambda_{HS}(\Phi^\dagger\Phi)\varphi_5^2, \tag{6.1}$$

with the complex scalar doublet $\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\varphi_1 + i\varphi_2\\ \varphi_4 + i\varphi_3\end{pmatrix}$ taken from the SM. By demanding classical scale-invariance and only considering terms with operator dimension 4, the tree-level potential (6.1) is the most general one [16], we can consider. Assuming it to be hermitian, all three scalar couplings λ_H, λ_S and λ_{HS} have to be taken real.

Writing (6.1) in terms of all the real fields φ_i (i = 1, 2, 3, 4, 5), the potential reads

$$\mathrm{V}_0\,(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5) \;=\; \frac{1}{4}\lambda_H\left(\sum_{i=1}^{4}\varphi_i^2\right)^2 \;+\; \frac{1}{4}\lambda_S\varphi_5^4 \;-\; \frac{1}{2}\lambda_{HS}\left(\sum_{i=1}^{4}\varphi_i^2\right)\varphi_5^2. \tag{6.2}$$

6.2 GW-condition and the flat direction

To find the GW-condition [54], we have to consider (6.2) on the unit sphere in field space ($\sum\limits_{j=1}^{5}\varphi_j^2 = 1$). There, (6.2) can be parametrized by φ_5, namely

$$\mathrm{V}_0\,(\varphi_5) \;=\; \frac{1}{4}\,\Big[(\lambda_H + \lambda_S + 2\lambda_{HS})\,\varphi_5^4 \;-\; 2\,(\lambda_H + \lambda_{HS})\,\varphi_5^2 + \lambda_H\Big]. \tag{6.3}$$

The minimum on the sphere is found for

$$\varphi_5^2 \;=\; \frac{\lambda_H \,+\, \lambda_{HS}}{\lambda_H \,+\, \lambda_S \,+\, 2\lambda_{HS}} \tag{6.4}$$

and demanding that it is equal to zero,

$$\lambda_{HS}^2(\Lambda_W) \;=\; \lambda_H(\Lambda_W)\,\lambda_S(\Lambda_W) \tag{6.5}$$

has to be fulfilled for a certain renormalization scale Λ_W. With expressions (6.4) and (6.5), the flat direction can be written as

$$\vec{\phi}_{flat} \;=\; \phi \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos\omega \\ \sin\omega \end{pmatrix}}_{=\vec{n}} \quad+\quad \text{Goldstone Boson directions}\,, \tag{6.6}$$

where the mixing-angle ω is given by

$$\cos^2\omega \;=\; \frac{\lambda_S \,+\, \lambda_{HS}}{\lambda_H \,+\, \lambda_S \,+\, 2\lambda_{HS}} \;=\; \frac{\lambda_{HS}}{\lambda_H \,+\, \lambda_{HS}}\,, \tag{6.7}$$

$$\sin^2\omega \;=\; \frac{\lambda_H \,+\, \lambda_{HS}}{\lambda_H \,+\, \lambda_S \,+\, 2\lambda_{HS}} \;=\; \frac{\lambda_H}{\lambda_H \,+\, \lambda_{HS}}\,. \tag{6.8}$$

In the following, we will often use the 2-dimensional vector

$$\vec{n}_{flat} \;=\; \begin{pmatrix} \cos\omega \\ \sin\omega \end{pmatrix} \tag{6.9}$$

restricting ourselves to the relevant subspace of the two physical scalar states.

Since the scalar doublet Φ in the tree-level potential (6.1) behaves with the same transformations as in the SM, one can adjust the VEV of Φ to be real and so the two VEVs (the VEV of φ_5 has to be real anyway) read

$$\langle\Phi\rangle \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \langle\varphi_4\rangle \;=\; v \end{pmatrix} \quad \text{and} \quad \langle\varphi_5\rangle \;=\; v_S, \tag{6.10}$$

where $v = \langle\phi\rangle\cos\omega = 246\,\text{GeV}$ and $v_S = \langle\phi\rangle\sin\omega$. Thus

$$\begin{pmatrix} \langle\varphi_4\rangle \\ \langle\varphi_5\rangle \end{pmatrix} \;=\; \begin{pmatrix} v \\ v_S \end{pmatrix} \;=\; \langle\phi\rangle\,\vec{n}_{flat}. \tag{6.11}$$

6.3 Masses of the three would-be Goldstone bosons

When Φ and φ_5 achieve their VEVs (6.10), electroweak SSB occurs and the three would-be Goldstone boson fields φ_1, φ_2 and φ_3 are eaten up by the three gauge bosons $W^{\pm}$ and Z^0.

The full matrix for the squared masses in the scalar sector ($j, k = 1, ..., 5$) reads

$$\left(\frac{\partial^2 V_0}{\partial\varphi_j \partial\varphi_k}\right) = \begin{pmatrix} 2\lambda_H\,\tilde{\varphi}\tilde{\varphi}^T + [\lambda_H\,\tilde{\varphi}^T\tilde{\varphi} - \lambda_{HS}\,\varphi_5^2]\,\mathbb{1}_4 & -2\lambda_{HS}\varphi_5\tilde{\varphi} \\ -2\lambda_{HS}\varphi_5\tilde{\varphi}^T & 3\frac{\lambda_{HS}^2}{\lambda_H}\varphi_5^2 - \lambda_{HS}\,\tilde{\varphi}^T\tilde{\varphi} \end{pmatrix}, \tag{6.12}$$

where

$$\tilde{\varphi}^T = \begin{pmatrix}\varphi_1, & \varphi_2, & \varphi_3, & \varphi_4\end{pmatrix}. \tag{6.13}$$

To get an expression for the masses of the would-be Goldstone bosons M_{GB}, we multiply the matrix in (6.12) with the 5-dimensional vector $\vec{m} = \begin{pmatrix}\vec{m}'\\0\end{pmatrix}$ ($\vec{m}'$ being 4-dimensional and chosen such that $\tilde{\varphi}^T\vec{m}' = 0$). Then, we obtain,

$$\begin{pmatrix} 2\lambda_H\,\tilde{\varphi}\tilde{\varphi}^T + [\lambda_H\,\tilde{\varphi}^T\tilde{\varphi} - \lambda_{HS}\,\varphi_5^2]\,\mathbb{1}_4 & -2\lambda_{HS}\varphi_5\tilde{\varphi} \\ -2\lambda_{HS}\varphi_5\tilde{\varphi}^T & 3\frac{\lambda_{HS}^2}{\lambda_H}\varphi_5^2 - \lambda_{HS}\,\tilde{\varphi}^T\tilde{\varphi} \end{pmatrix}\begin{pmatrix}\vec{m}'\\0\end{pmatrix} =$$

$$= (\lambda_H\,\tilde{\varphi}^T\tilde{\varphi} - \lambda_{HS}\,\varphi_5^2)\begin{pmatrix}\vec{m}'\\0\end{pmatrix}.$$

Since $\vec{m}'$ is a 4-dimensional vector, the eigenvalue $(\lambda_H\,\tilde{\varphi}^T\tilde{\varphi} - \lambda_{HS}\,\varphi_5^2)$ is three-fold degenerate. Evaluating it at the VEVs (6.10), we achieve the squared masses of the 3 would-be Goldstone bosons with

$$M_{GB}^2 = (\lambda_H\,\tilde{\varphi}^T\tilde{\varphi} - \lambda_{HS}\,\varphi_5^2)\Big|_{v,v_S} = 0. \tag{6.14}$$

6.4 Masses of the two physical scalars

As we worked out in the previous chapter, from the analysis of GW [54], we know that the scalon S has to be one of the two physical scalar mass-eigenstates and it lies in the flat direction $\vec{n}_{flat}$ (6.9) – restricting ourselves to the relevant 2-dimensional subspace. The other state H has to live in a direction orthogonal to (6.9) and is already massive at tree-level. We can expand both φ_4 and φ_5 around their VEVs (6.11) and express them in dependence of the mass-eigenfields S and

H,

$$\begin{pmatrix}\varphi_4\\ \varphi_5\end{pmatrix} = \begin{pmatrix}v \,+\, \varphi_4'\\ v_S \,+\, \varphi_5'\end{pmatrix} = (\langle\phi\rangle \,+\, \mathrm{S})\begin{pmatrix}\cos\omega\\ \sin\omega\end{pmatrix} \,+\, \mathrm{H}\begin{pmatrix}-\sin\omega\\ \cos\omega\end{pmatrix}. \tag{6.15}$$

In order to get the tree-level masses of S and H, we insert (6.15) into (6.2) by neglecting the three would-be Goldstone fields φ_1, φ_2 and φ_3 and extracting only the terms proportional to $\langle\phi\rangle^2$. Doing the calculation, we obtain

$$\mathrm{V}_0 \overset{\mathcal{O}(\langle\phi\rangle^2)}{=} \lambda_{HS}\langle\phi\rangle^2\mathrm{H}^2 \,=\, \frac{1}{2}\underbrace{2\lambda_{HS}\langle\phi\rangle^2}_{=\mathrm{m}_H^2(v,v_S)}\mathrm{H}^2, \tag{6.16}$$

and knowing that V_0 is of the form

$$\mathrm{V}_0 \,=\, \frac{1}{2}\,\mathrm{m}_S^2\,\mathrm{S}^2 \,+\, \frac{1}{2}\,\mathrm{m}_H^2\,\mathrm{H}^2, \tag{6.17}$$

we see that the scalon is really massless at tree-level and the heavy state H has a tree-level mass of $\mathrm{m}_H^2(v, v_S) = 2\lambda_{HS}\langle\phi\rangle^2$.

Having merely one state H with non-vanishing tree-level mass, the one-loop effective potential in the scalar sector evaluated at (6.11) reads

$$\delta\mathrm{V}_S^{(1)}(\langle\phi\rangle\vec{n}_{flat}) \,=\, \frac{1}{64\,\pi^2}\,\left(2\lambda_{HS}\langle\phi\rangle^2\right)^2\left[\ln\frac{2\lambda_{HS}\langle\phi\rangle^2}{\Lambda_W^2} \,-\, \frac{3}{2}\right]. \tag{6.18}$$

6.5 Full effective potential at one-loop order

With the contributions from the gauge bosons and from the fermionic sector (only t-quarks and heavy right-handed neutrinos) included, the full one-loop effective potential evaluated at the vacuum reads

$$\begin{aligned}\delta\,\mathrm{V}^{(1)}(\langle\phi\rangle\vec{n}) \,=\, \frac{1}{64\,\pi^2}\Bigg\{&\left(2\lambda_{HS}\langle\phi\rangle^2\right)^2\left[\ln\frac{2\lambda_{HS}\langle\phi\rangle^2}{\Lambda_W^2} \,-\, \frac{3}{2}\right] \,+\, 6\cdot\left(\frac{1}{4}\,\mathrm{g}^2\,v^2\right)^2\left[\ln\frac{\frac{1}{4}\,\mathrm{g}^2\,v^2}{\Lambda_W^2} \,-\, \frac{5}{6}\right] \,+\\ &+3\cdot\left(\frac{1}{4}\,(\mathrm{g}^2 \,+\mathrm{g}'^2)\,v^2\right)^2\left[\ln\frac{\frac{1}{4}\,(\mathrm{g}^2 \,+\mathrm{g}'^2)\,v^2}{\Lambda_W^2} \,-\, \frac{5}{6}\right] \,-\\ &-12\cdot\left(\frac{1}{2}\,g_t^2\,v^2\right)^2\left[\ln\frac{\frac{1}{2}\,g_t^2\,v^2}{\Lambda_W^2} \,-\, \frac{3}{2}\right] \,-\\ &2\cdot\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4\left[\ln\frac{\mathrm{M}_{\nu,r}^2}{\Lambda_W^2} \,-\, \frac{3}{2}\right]\Bigg\}.\end{aligned} \tag{6.19}$$

From this formula (6.19), we can read off

$$\mathrm{B} \,=\, \frac{1}{64\,\pi^2\,\langle\phi\rangle^4}\,\left(\mathrm{m}_H^4 \,+\, 6\,\mathrm{M}_W^4 \,+\, 3\,\mathrm{M}_Z^4 \,-\, 12\,\mathrm{M}_t^4 \,-\, 2\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4\right) \tag{6.20}$$

and demanding that B has to be strictly positive [54], we get (with the mass-values $\mathrm{M}_t \simeq 173\,\mathrm{GeV}$, $\mathrm{M}_W \simeq 80\,\mathrm{GeV}$ and $\mathrm{M}_Z \simeq 91\,\mathrm{GeV}$ are taken from [61]):

$$\mathrm{m}_H^4 \,-\, 2\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4 \,>\, 12\,\mathrm{M}_t^4 \,-\, 6\,\mathrm{M}_W^4 \,-\, 3\,\mathrm{M}_Z^4 \approx (319\,\mathrm{GeV})^4 \tag{6.21}$$

$$\Rightarrow \quad \mathrm{m}_{H,\,min} \,\simeq\, 319\,\mathrm{GeV}. \tag{6.22}$$

This shows that the heavy state H with a tree-level mass of at least 319 GeV (if the right-handed neutrinos get masses of order $\mathcal{O}(\mathrm{TeV})$, the scalar H has a mass of order $\mathcal{O}(\mathrm{TeV})$ too) can not be identified with the Higgs boson discovered by ATLAS [10] and CMS [11] in 2012 with a mass of $\mathrm{m}_h \,=\, (125.09 \pm 0.21 \pm 0.11)\,\mathrm{GeV}$ [61]. Thus, in the model considered here, the scalon S has to be the observed Higgs particle and thus on the one hand one has

$$\mathrm{m}_h^2 \,=\, \mathrm{m}_S^2 \,=\, (125.09\,\mathrm{GeV})^2, \tag{6.23}$$

and on the other hand one has

$$\mathrm{m}_h^2 \,=\, \mathrm{m}_S^2 \,=\, 8\,\mathrm{B}\,\langle\phi\rangle^2 \,=\, \frac{1}{8\,\pi^2\,\langle\phi\rangle^2}\left(-\,(319\,\mathrm{GeV})^4 \,+\, \mathrm{m}_H^4 \,-\, 2\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4\right). \tag{6.24}$$

Combining the last two formulae and with the knowledge that B>0, one achieves an even higher bound for the mass of the heavy state H:

$$\mathrm{m}_H^4 \,-\, 2\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4 \,>\, 8\,\pi^2\langle\phi\rangle^2\,\mathrm{m}_h^2 \,+ (319\,\mathrm{GeV})^4 \tag{6.25}$$

$$\Rightarrow \quad \mathrm{m}_{H,\,min} \,\simeq\, 540\,\mathrm{GeV}\,. \tag{6.26}$$

Knowing that $\mathrm{m}_H^2 \,=\, 2\lambda_{HS}\langle\phi\rangle^2 \,>\, (540\,\mathrm{GeV})^2$, we achieve a lower bound for λ_{HS} of

$$\lambda_{HS} \,>\, 2.41\,\cos^2\omega. \tag{6.27}$$

To implement heavy right-handed neutrinos with the seesaw mechanism of type 1 [45–48] (as described in section 3.1), we would need the VEV of the additional singlet φ_5, v_S, to be much bigger than the VEV of the doublet Φ, $v = 246\,\mathrm{GeV}$.

For the (Dirac-)fermion couplings not to deviate much from the SM-values, $\cos\omega$ must not deviate too much from 1 (seeing in formulae (6.9), (6.10) and (6.11)). This deviation has to be in the range [72–74],

$$0.9 < \cos\omega < 1. \tag{6.28}$$

Even if we take the smallest value for $\cos\omega$ respectively the highest possible value for ω, we see from (6.9)-(6.11), that v_S is certainly smaller than v. This finally shows that, in this model, we can not implement heavy right-handed neutrinos, since v_S is far too small.

6.6 Would the model be perturbatively valid?

For scalar field theories which have only φ^4-selfinteractions among themselves,

$$\mathrm{V}_0 = \frac{\lambda_{ijkl}}{4!}\,\varphi_i\varphi_j\varphi_k\varphi_l, \tag{6.29}$$

to be perturbatively valid, all the couplings λ_{ijkl} has to fulfill

$$\lambda_{ijkl} \gg \frac{\lambda_{ijkl}^2}{(4\pi)^2} \quad \rightarrow \quad \lambda_{ijkl} \ll (4\pi)^2 \sim 144. \tag{6.30}$$

This can be seen from comparing the order of the couplings from tree-level and one-loop diagrams with both two incoming and two outgoing scalar particles, where the contribution from one-loop order has to be much smaller than that from tree-level.

Considering our tree-level potential (6.2) again, we have

$$\begin{aligned} \mathrm{V}_0 &= \frac{1}{4}\lambda_H\left(\sum_{i=1}^{4}\varphi_i^2\right)^2 + \frac{1}{4}\lambda_S\varphi_5^4 - \frac{1}{2}\lambda_{HS}\left(\sum_{i=1}^{4}\varphi_i^2\right)\varphi_5^2 \\ &= \frac{1}{4!}\left[6\,\lambda_H\left(\sum_{i=1}^{4}\varphi_i^2\right)^2 + 6\lambda_S\varphi_5^4 - 12\lambda_{HS}\left(\sum_{i=1}^{4}\varphi_i^2\right)\varphi_5^2\right], \end{aligned} \tag{6.31}$$

leading to the following upper bounds for the couplings,

$$0 < \lambda_H \ll 24, \quad 0 < \lambda_S \ll 24 \quad \text{and} \quad |\lambda_{HS}| \ll 12, \tag{6.32}$$

where positivity of λ_H and λ_S is demanded for the potential to be bounded from below. Comparing the lower bound for λ_{HS} (6.27) with these constraints (6.32), one sees that this scalar model could, in principle, be described perturbatively.

To conclude, the only problem why we have to exclude this model is that implementation of heavy neutrinos is not possible here.

7 Minimal scale-invariant version of the SM with one complex scalar singlet and a global $U(1)$-symmetry

7.1 Model-setup

In this chapter, we study and give the results for a scale-invariant version of the SM with one additional complex scalar singlet field X, carrying a (modified) lepton number ℓ, proposed by Krzysztof A. Meissner and Hermann Nicolai in [17]. Considering only terms with operator-dimension 4 and supposing that the tree-level potential in the scalar sector $\mathrm{V}_0\,(\Phi,\,X)$ is invariant under an additional global $U(1)_\ell$ transformation of X ($X \to \exp{(i\ell)}X$), the tree-level potential reads [17]

$$\mathrm{V}_0\,(\Phi,\,X) = \lambda_H(\Phi^\dagger\Phi)^2 + \lambda_X(X^*X)^2 - 2\lambda_{HX}(\Phi^\dagger\Phi)(X^*X) \tag{7.1}$$

with

$$\Phi \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix}\varphi_1 \,+\, i\varphi_2\\ \varphi_4 \,+\, i\varphi_3\end{pmatrix} \quad \text{and} \quad X \;=\; \frac{1}{\sqrt{2}}\,(\varphi_5 \,+\, i\varphi_6). \tag{7.2}$$

Demanding that (7.1) is hermitian, the three quartic scalar couplings λ_H, λ_X and λ_{HX} have to be taken real.

7.2 GW-condition and the flat direction

Considering (7.1) just on the unit sphere $\sum_{i=1}^{6}\varphi_i^2 = 1$, we obtain again the GW-condition [54]

$$\lambda_{HX}^2(\Lambda_W) \;=\; \lambda_H(\Lambda_W)\,\lambda_X(\Lambda_W). \tag{7.3}$$

This characterizes the flat-direction given by

$$\vec{n}^{\mathrm{T}} \;=\; \Big(0,\;\; 0,\;\; 0,\;\; \cos\omega,\;\; \sin\omega,\;\; 0\Big)\,, \tag{7.4}$$

where the mixing angle ω is defined via

$$\cos^2\omega = \frac{\lambda_X + \lambda_{HX}}{\lambda_H + \lambda_X + 2\lambda_{HX}} = \frac{\lambda_{HX}}{\lambda_H + \lambda_{HX}}, \tag{7.5}$$

$$\sin^2\omega = \frac{\lambda_H + \lambda_{HX}}{\lambda_H + \lambda_X + 2\lambda_{HX}} = \frac{\lambda_H}{\lambda_H + \lambda_{HX}}. \tag{7.6}$$

Since the tree-level potential (7.1) is symmetric under exchange of φ_5 and φ_6, due to the global $U(1)_\ell$ transformation, it is possible to rotate the singlet X in a way, so that it achieves a real VEV. Then, the VEVs of both Φ and X read

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} = \frac{1}{\sqrt{2}}\langle\phi\rangle\cos\omega \quad \text{and} \quad \langle X\rangle = \frac{v_X}{\sqrt{2}} = \frac{1}{\sqrt{2}}\langle\phi\rangle\sin\omega, \tag{7.7}$$

hence $v = \langle\phi\rangle\cos\omega$ and $v_X = \langle\phi\rangle\sin\omega$.

7.3 Scalar masses

To obtain all the masses of the scalar particles, we consider the matrix of squared masses in the scalar sector ($i, j = 1, ..., 6$),

$$\left(\frac{\partial^2 \mathrm{V}_0}{\partial\varphi_i\partial\varphi_j}\right) =$$

$$= \begin{pmatrix} 2\lambda_H\,\tilde{\varphi}\tilde{\varphi}^{\mathrm{T}} + [\lambda_H\,\tilde{\varphi}^{\mathrm{T}}\tilde{\varphi} - \lambda_{HX}\,\tilde{X}^{\mathrm{T}}\tilde{X}]\,\mathbb{1}_4 & -2\lambda_{HX}\tilde{\varphi}\tilde{X}^{\mathrm{T}} \\ -2\lambda_{HX}\tilde{X}\tilde{\varphi}^{\mathrm{T}} & 2\,\frac{\lambda_{HX}^2}{\lambda_H}\,\tilde{X}\tilde{X}^{\mathrm{T}} + \left[\frac{\lambda_{HX}^2}{\lambda_H}\,\tilde{X}^{\mathrm{T}}\tilde{X} - \lambda_{HX}\,\tilde{\varphi}^{\mathrm{T}}\tilde{\varphi}\right]\mathbb{1}_2 \end{pmatrix}, \tag{7.8}$$

with $\tilde{\varphi}^{\mathrm{T}} = \begin{pmatrix}\varphi_1, & \varphi_2, & \varphi_3, & \varphi_4\end{pmatrix}$ and $\tilde{X}^{\mathrm{T}} = \begin{pmatrix}\varphi_5, & \varphi_6\end{pmatrix}$.

Aside from the three-fold degenerate eigenvalue of the three would-be Goldstone bosons which again reads

$$M_{GB}^2 = \lambda_H\,\tilde{\varphi}^{\mathrm{T}}\tilde{\varphi} - \lambda_{H_X}\,\tilde{X}^{\mathrm{T}}\tilde{X} \tag{7.9}$$

giving zero, when evaluated at the VEVs (7.7), and

$$\mathrm{m}_H^2(v, v_X) = 2\lambda_{HX}\langle\phi\rangle^2, \tag{7.10}$$

$$\mathrm{m}_S^2(v, v_X) = 0 \tag{7.11}$$

to the same mass-eigenstates as they were defined in (6.15), there exists a 6^{th} eigenvalue (with eigenvector $\vec{m}_1 = \begin{pmatrix} \vec{0} \\ \vec{m}'_1 \end{pmatrix}$, $\vec{m}'_1$ being 2-dimensional with $\tilde{\chi}^{\mathrm{T}}\vec{m}'_1 = 0$, and $\vec{0}$ being 4-dimensional), namely one associated with the Majoron [75, 76], given by

$$M^2_{Maj} = \frac{\lambda^2_{HX}}{\lambda_H}\,\tilde{X}^{\mathrm{T}}\tilde{X} - \lambda_{HX}\,\tilde{\varphi}^{\mathrm{T}}\tilde{\varphi}, \tag{7.12}$$

vanishing, when evaluated at the VEVs (7.7).

7.3.1 The majoron

The majoron is a real Goldstone-boson with vanishing mass which couples dominantly to all the neutrinos, even though weak, and hardly to the matter particles in the SM [75]. This massless Goldstone-boson comes from the spontaneous breakdown of the global $U(1)_\ell$, when X achieves its VEV (7.7). The connection of this global symmetry breakdown to the (modified) lepton number ℓ is understood as follows.

Introducing right-handed neutrinos to our model, we can construct Majorana mass-terms which violate lepton number ($\Delta L = 2$), where the Majorana mass-matrix is proportional to the scalar singlet X. Then having a non-vanishing VEV of $X - \sin\omega \neq 0$ in (7.7) $-$ leads simultaneously to the spontaneous breakdown of $U(1)_\ell$ and the violation of the modified lepton number ℓ [75].

7.4 Full effective potential at one-loop order

After adding the contributions from the gauge bosons and the fermions, no additional heavy physical scalar fields have to be taken into account if compared to (6.19) and so the effective potential $\mathrm{V_{eff}}$ at one-loop order evaluated at the vacuum is exactly the same as in the previous model (6.19), namely

$$\begin{aligned}\delta\,\mathrm{V}^{(1)}(\langle\phi\rangle\vec{n}) = \frac{1}{64\,\pi^2}\Bigg\{&\Big(2\lambda_{HX}\langle\phi\rangle^2\Big)^2\left[\ln\frac{2\lambda_{HX}\langle\phi\rangle^2}{\Lambda_W^2} - \frac{3}{2}\right] + \\ &+ 6\cdot\left(\frac{1}{4}\,\mathrm{g}^2\,v^2\right)^2\left[\ln\frac{\frac{1}{4}\,\mathrm{g}^2\,v^2}{\Lambda_W^2} - \frac{5}{6}\right] + \\ &+ 3\cdot\left(\frac{1}{4}\,(\mathrm{g}^2 + \mathrm{g}'^2)\,v^2\right)^2\left[\ln\frac{\frac{1}{4}\,(\mathrm{g}^2 + \mathrm{g}'^2)\,v^2}{\Lambda_W^2} - \frac{5}{6}\right] - \\ &- 12\cdot\left(\frac{1}{2}\,g_t^2\,v^2\right)^2\left[\ln\frac{\frac{1}{2}\,g_t^2\,v^2}{\Lambda_W^2} - \frac{3}{2}\right] - \\ &- 2\cdot\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}^4_{\nu,r}\left[\ln\frac{\mathrm{M}^2_{\nu,r}}{\Lambda_W^2} - \frac{3}{2}\right]\Bigg\}.\end{aligned} \tag{7.13}$$

From here, we again achieve

$$\mathrm{B} \;=\; \frac{1}{64\,\pi^2\,\langle\phi\rangle^4}\left(\mathrm{m}_H^4 \;+\; 6\,\mathrm{M}_W^4 \;+\; 3\,\mathrm{M}_Z^4 \;-\; 12\,\mathrm{M}_t^4 \;-\; 2\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4\right) \tag{7.14}$$

and from the relation $\mathrm{m}_S^2 \;=\; 8\,\mathrm{B}\,\langle\phi\rangle^2$ and the knowledge that the scalon has again to be identified with the Higgs boson with a mass of $\mathrm{m}_h \;=\; (125.09 \pm 0.21 \pm 0.11)\,\mathrm{GeV}$, we obtain a minimal mass-value for the heavy state H with

$$\mathrm{m}_{H,\,min} \;=\; 540.1\,\mathrm{GeV} \tag{7.15}$$

and thus

$$\lambda_{HX} \;>\; 2.41\,\cos^2\omega. \tag{7.16}$$

7.4.1 Perturbative validity and right-handed neutrinos?

Neclecting the three would-be Goldstone boson fields, the tree-level potential in dependence of the real scalar fields φ_4, φ_5 and φ_6 reads

$$\mathrm{V}_0 \;=\; \frac{1}{4}\lambda_H\left(\sum_{i=1}^{4}\varphi_i^2\right)^2 \;+\; \frac{1}{4}\lambda_X\left(\sum_{j=5}^{6}\varphi_j^2\right)^2 \;-\; \frac{1}{2}\lambda_{HX}\left(\sum_{i=1}^{4}\varphi_i^2\right)\left(\sum_{j=5}^{6}\varphi_j^2\right) \tag{7.17}$$

$$=\; \frac{1}{4!}\left[6\lambda_H\varphi_4^4 \;+\; 6\lambda_X(\varphi_5^2+\varphi_6^2)^2 \;-\; 12\lambda_{HX}\varphi_4^2(\varphi_5^2+\varphi_6^2)\right] \tag{7.18}$$

and so we achieve for the ranges of the couplings, where perturbation theory is valid,

$$0 < \lambda_H \ll 24, \quad 0 < \lambda_X \ll 24 \quad \text{and} \quad |\lambda_{HX}| \ll 12, \tag{7.19}$$

again.

Thus, due to exactly the same reasons as in section 6.6, this model could be explained in perturbation theory too. But the reason, why we exclude this model from further considerations is that one can not add right-handed neutrinos achieving a mass from the VEV of X, v_X. This is the case, since the VEV of Φ is again bigger than that of v_X.

8 Scale-invariant version of SM with one complex scalar singlet X and the discrete symmetry $X \leftrightarrow X^*$

In this chapter, we discuss again a classically scale-invariant version of the SM with one additional complex scalar-singlet X which was considered in works of Arsham Farzinnia et. al. [28–30], Arsham Farzinnia [31] and furtheron in [32] and [33] with different motivations. But this time, the tree-level potential is not protected under a globally $U(1)$-transformation of X, but it is invariant under the discrete transformation $X \leftrightarrow X^*$.

In all the papers [28–31], they assumed the VEV of X to be real. But this is a wrong assumption and not representing the general case: Because of the absence of a $U(1)$-symmetry the field X can not be rotated in a way such that its VEV becomes real.

8.1 Model-Setup: Tree-level potential in the scalar sector

The tree-level potential in the scalar sector considered in [28–33] reads

$$\begin{aligned} \mathrm{V}_0(\Phi,\, X) = &\frac{\lambda_1}{6}(\Phi^\dagger\Phi)^2 + \frac{\lambda_2}{6}|X|^4 + \lambda_3(\Phi^\dagger\Phi)|X|^2 + \frac{\lambda_4}{2}(\Phi^\dagger\Phi)\left(X^2 + (X^*)^2\right) \\ &+ \frac{\lambda_5}{12}\left(X^2 + (X^*)^2\right)|X|^2 + \frac{\lambda_6}{12}\left(X^4 + (X^*)^4\right) \end{aligned} \tag{8.1}$$

with

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\varphi_1 + i\varphi_2 \\ \varphi_4 + i\varphi_3\end{pmatrix} \qquad \text{and} \qquad X = \frac{1}{\sqrt{2}}(\varphi_5 + i\varphi_6),$$

where all the $\{\varphi_i\}$ (i = 1, 2, 3, 4, 5, 6) are real scalar fields. The tree-level potential (8.1), being invariant under the transformation $X \leftrightarrow X^*$ ($\varphi_5 \leftrightarrow \varphi_5$, $\varphi_6 \leftrightarrow -\varphi_6$) and X being a singlet under the SM gauge-group (2.1), is hermitian and so all the couplings $\{\lambda_i\}$ (i = 1, 2, 3, 4, 5, 6) have to be taken real.

Writing (8.1) in terms of the real fields $\{\varphi_i\}$, the potential reads

$$\mathrm{V}_0 = \frac{\lambda_1}{24}(\sum_{j=1}^{4}\varphi_j^2)^2 + \frac{\lambda_2}{24}(\varphi_5^2+\varphi_6^2)^2 + \frac{\lambda_3}{4}(\sum_{j=1}^{4}\varphi_j^2)(\varphi_5^2+\varphi_6^2) + \frac{\lambda_4}{4}(\sum_{j=1}^{4}\varphi_j^2)(\varphi_5^2-\varphi_6^2)$$
$$+\frac{\lambda_5}{24}(\varphi_5^2-\varphi_6^2)(\varphi_5^2+\varphi_6^2) + \frac{\lambda_6}{24}(\varphi_5^2-\varphi_6^2)^2 - \frac{\lambda_6}{6}\varphi_5^2\varphi_6^2. \tag{8.2}$$

Introducing new variables x and y,

$$x = \varphi_5^2+\varphi_6^2, \qquad \text{and} \qquad y = \varphi_5^2-\varphi_6^2, \tag{8.3}$$

or vice versa

$$\varphi_5^2 = \frac{x+y}{2} \qquad \text{and} \qquad \varphi_6^2 = \frac{x-y}{2}, \tag{8.4}$$

we can write the potential (8.2) in terms of x, y and z as

$$\mathrm{V}_0(x,y,z) = \frac{1}{24}\left[\lambda_1 z^2 + 6z(\lambda_3 x + \lambda_4 y) + (\lambda_2-\lambda_6)x^2 + \lambda_5 xy + 2\lambda_6 y^2\right], \tag{8.5}$$

where we have defined z as $z = \sum_{j=1}^{4}\varphi_j^2$.

8.2 GW-condition and the flat direction

As next step, we want to determine the GW-condition [54] to get a relation between the couplings at a certain energy-scale Λ_W. To do that, we have to consider (8.2) on the unit sphere in field space ($\sum_{i=1}^{6}\varphi_i^2 = x+z = 1$). Thus expressing the variable z in terms of x, $z = 1-x$, we have

$$\mathrm{V}_0(x,y) = \frac{1}{24}\left(\lambda_1 + 2a_1x + 2a_2y + 2a_3xy + a_4x^2 + a_5y^2\right) \tag{8.6}$$

where

$$\begin{aligned} a_1 &= 3\lambda_3-\lambda_1,\\ a_2 &= 3\lambda_4,\\ a_3 &= \frac{1}{2}\lambda_5 - 3\lambda_4,\\ a_4 &= \lambda_1+\lambda_2-6\lambda_3-\lambda_6,\\ a_5 &= 2\lambda_6. \end{aligned} \tag{8.7}$$

To obtain the GW-condition, one has to differentiate V_0 with respect to both x and y (now we are working just on a two-dimensional subspace of the full field space) and set the expressions equal to zero,

$$\left.\frac{\partial \mathrm{V}_0}{\partial x}\right|_{x=x_0,\, y=y_0} = a_1 + a_4 x_0 + a_3 y_0 = 0, \tag{8.8}$$

$$\left.\frac{\partial \mathrm{V}_0}{\partial y}\right|_{x=x_0,\, y=y_0} = a_2 + a_3 x_0 + a_5 y_0 = 0, \tag{8.9}$$

and hence the solutions for x_0 and y_0 read

$$x_0 = \frac{a_1 a_5 - a_2 a_3}{a_3^2 - a_4 a_5} \qquad \text{and} \qquad y_0 = \frac{a_2 a_4 - a_1 a_3}{a_3^2 - a_4 a_5}. \tag{8.10}$$

The values of x_0 and y_0 represent the VEVs of x and y from which the VEVs of Φ and X can be attained. Since the scalar tree-level potential (8.1) has just a discrete (but no continuous) symmetry, the potential is not symmetric under exchange of φ_5 and φ_6 and therefore φ_5 and φ_6 can not be rotated in a manner such that X achieves a real VEV as it was claimed in [28–31]. In the general case, X achieves a complex VEV $-$ a fact which will be assumed in all our calculations in the following.

Inserting these values of x_0 and y_0 (8.10) back into the potential (8.6) and demanding that $\mathrm{V}_0(\mathrm{x}_0, \mathrm{y}_0) = 0$, we come to the condition

$$\lambda_1 (a_3^2 - a_4 a_5) = 2 a_1 a_2 a_3 - a_1^2 a_5 - a_2^2 a_4. \tag{8.11}$$

Expressing (8.11) in dependence of just the $\{\lambda_i\}$, we obtain for the GW-condition

$$\left[\lambda_5^2 + 8\lambda_6(\lambda_6 - \lambda_2)\right] \lambda_1 + 36 \left[2\lambda_3^2\lambda_6 - \lambda_3\lambda_4\lambda_5 - \lambda_4^2(\lambda_6 - \lambda_2)\right] = 0 \tag{8.12}$$

with the set of all six λ_i being evaluated at a certain renormalization-scale Λ_W. With the GW-condition (8.12) and (8.4), the flat direction is defined as

$$\vec{\phi}_{flat} = \phi \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{1 - x_0} \\ \sqrt{\frac{x_0 + y_0}{2}} \\ \sqrt{\frac{x_0 - y_0}{2}} \end{pmatrix}}_{=\vec{n}} + \text{ Goldstone Boson directions}\,, \tag{8.13}$$

where the variables x_0, y_0 (8.10) and $(1 - x_0)$ with λ_1 being eliminated by using (8.12) read

$$x_0 \;=\; 6\,\cdot\,\frac{\lambda_4\lambda_5 \;-\; 4\lambda_3\lambda_6}{\mathrm{N}}, \tag{8.14}$$

$$y_0 \;=\; 6\,\cdot\,\frac{\lambda_3\lambda_5 \;+\; 2\lambda_4(\lambda_6 - \lambda_2)}{\mathrm{N}}, \tag{8.15}$$

$$1 - x_0 \;=\; -\,\frac{\lambda_5^2 \;+\; 8\lambda_6(\lambda_6 - \lambda_2)}{\mathrm{N}}, \tag{8.16}$$

with $\mathrm{N} \;=\; 6\lambda_4\lambda_5 \;-\; \lambda_5^2 \;+\; 8\lambda_6(\lambda_2 \;-\; 3\lambda_3 \;-\; \lambda_6)$.

Neglecting the three would-be Goldstone bosons' directions, we will often restrict ourselves to the relevant 3-dimensional subspace and write

$$\vec{n}_{flat} \;=\; \begin{pmatrix} \sqrt{1 - x_0} \\ \sqrt{\frac{x_0 + y_0}{2}} \\ \sqrt{\frac{x_0 - y_0}{2}} \end{pmatrix} \tag{8.17}$$

in our further argumentations.

As already mentioned, the field Φ will achieve a real VEV and χ will achieve a complex VEV and so we write

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \langle\varphi_4\rangle \;=\; v \end{pmatrix} \qquad \text{and} \qquad \langle X\rangle = \frac{1}{\sqrt{2}}\left(\langle\varphi_5\rangle + i\langle\varphi_6\rangle\right) = \frac{1}{\sqrt{2}}\left(v_5 + iv_6\right) \tag{8.18}$$

with v, v_5 and v_6 read (by means of (8.4))

$$v \;=\; \langle\phi\rangle\,\sqrt{1 - x_0}, \qquad v_5 \;=\; \langle\phi\rangle\,\sqrt{\frac{x_0 + y_0}{2}} \qquad \text{and} \qquad v_6 \;=\; \langle\phi\rangle\,\sqrt{\frac{x_0 - y_0}{2}}, \tag{8.19}$$

where the ranges of x_0 and y_0 are

$$0 \;<\; x_0 \;<\; 1 \qquad \text{and} \qquad -\,x_0 \;<\; y_0 \;<\; x_0. \tag{8.20}$$

Hence, we can write

$$\begin{pmatrix} \langle\varphi_4\rangle \\ \langle\varphi_5\rangle \\ \langle\varphi_6\rangle \end{pmatrix} \;=\; \begin{pmatrix} v \\ v_5 \\ v_6 \end{pmatrix} \;=\; \langle\phi\rangle\,\vec{n}_{flat}. \tag{8.21}$$

8.3 Masses of the three would-be Goldstone Bosons

As in the models discussed in the last two chapter (chapters 6 and 7), we again achieve the squared mass-eigenvalues of the would-be Goldstone bosons by considering the 4×4-part of the full matrix in the scalar sector,

$$\left(\frac{\partial^2 \mathrm{V}_0}{\partial\varphi_i \partial\varphi_k}\right) = \frac{\lambda_1}{3}\,(\tilde{\varphi}\tilde{\varphi}^{\mathrm{T}})_{ik} + \underbrace{\left\{\frac{\lambda_1}{6}\tilde{\varphi}^{\mathrm{T}}\tilde{\varphi} + \frac{1}{2}(\lambda_3 x + \lambda_4 y)\right\}}_{=\mathrm{M}^2_{GB}(\tilde{\varphi},x,y)}\delta_{ik} \quad (i,k = 1,2,3,4), \tag{8.22}$$

$$\text{where} \quad \tilde{\varphi}^{\mathrm{T}} = \begin{pmatrix}\varphi_1 & \varphi_2 & \varphi_3 & \varphi_4\end{pmatrix}.$$

Evaluating the squared mass-eigenvalue of the would-be Goldstone bosons $\mathrm{M}^2_{GB}(\tilde{\varphi}, x, y)$ at the VEVs (8.21), we get

$$\mathrm{M}^2_{GB}(v, v_5, v_6) = \frac{\lambda_1}{6}(1 - x_0) + \frac{1}{2}(\lambda_3 x_0 + \lambda_4 y_0) \tag{8.23}$$

and then by eliminating λ_1 by using (8.12), we achieve

$$\mathrm{M}^2_{GB}(v, v_5, v_6) = 0. \tag{8.24}$$

as expected. This eigenvalue is three-fold degenerate and due to it the three gauge bosons $\mathrm{W}^{\pm}$ and Z^0 get their masses after electroweak SSB.

8.4 Masses of the three physical scalars

As in the model taken from [16] discussed in the previous chapter (chapter 6), we again expand the real fields around their VEVs (8.21) for finding the mass-eigenstates. The scalon again lies in the flat direction given in (8.17) and the other two particles lie in the 2-dimensional plane orthogonal to it. The expansion of the fields φ_4, φ_5 and φ_6 around their VEVs reads

$$\begin{pmatrix}\varphi_4\\ \varphi_5\\ \varphi_6\end{pmatrix} = \begin{pmatrix}v + \varphi_4'\\ v_5 + \varphi_5'\\ v_6 + \varphi_6'\end{pmatrix} = (\langle\phi\rangle + \mathrm{S})\,\vec{n}_{flat} + \eta\,\vec{n}_1 + \xi\,\vec{n}_2, \tag{8.25}$$

where the vectors $\vec{n}_1$ and $\vec{n}_2$ have to lie somewhere in the plane orthogonal to $\vec{n}_{flat}$, but they have to be chosen such that they are orthogonal to each other,

$$\vec{n}_1 = \frac{1}{\sqrt{x_0}}\begin{pmatrix}0\\ -\sqrt{\frac{x_0-y_0}{2}}\\ \sqrt{\frac{x_0+y_0}{2}}\end{pmatrix} \quad \text{and} \quad \vec{n}_2 = \frac{1}{\sqrt{x_0}}\begin{pmatrix}-x_0\\ \sqrt{1-x_0}\,\sqrt{\frac{x_0+y_0}{2}}\\ \sqrt{1-x_0}\,\sqrt{\frac{x_0-y_0}{2}}\end{pmatrix}. \tag{8.26}$$

With this choice (8.26), the set of three vectors $\{\vec{n}_{flat}, \vec{n}_1, \vec{n}_2\}$ forms an orthonormal basis.

For the determination of the mass-eigenstates, it is convenient to write the tree-level potential (8.5) in matrix form as

$$\mathrm{V}_0 = \frac{1}{24}\begin{pmatrix} x & y & z \end{pmatrix} \tilde{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{with} \qquad \tilde{A} = \begin{pmatrix} c_1 & \frac{c_3}{2} & 3\lambda_3 \\ \frac{c_3}{2} & c_2 & 3\lambda_4 \\ 3\lambda_3 & 3\lambda_4 & \lambda_1 \end{pmatrix}, \tag{8.27}$$

where the newly introduced parameters c_1, c_2 and c_3 read

$$c_1 = \lambda_2 - \lambda_6, \qquad c_2 = 2\lambda_6 \qquad \text{and} \qquad c_3 = \lambda_5. \tag{8.28}$$

In addition, we can express λ_1, λ_3 and λ_4 in terms of x_0, y_0, c_1, c_2 and c_3 via

$$\lambda_1 = \frac{1}{(1-x_0)^2}\,(c_1 x_0^2 + c_2 y_0^2 + c_3 x_0 y_0), \tag{8.29}$$

$$\lambda_3 = -\frac{1}{6(1-x_0)}\,(2c_1 x_0 + c_3 y_0), \tag{8.30}$$

$$\lambda_4 = -\frac{1}{6(1-x_0)}\,(c_3 x_0 + 2c_2 y_0). \tag{8.31}$$

Originally, we had started with six free parameters λ_1, λ_2, λ_3, λ_4, λ_5 and λ_6 where one of them (choosing λ_1) is fixed at the renormalization-scale Λ_W by the GW-condition (8.12) leaving five of them. For simplifying the notation in the next step, we use $\{x_0, y_0, c_1, c_2, c_3\}$ as a parameter set instead of $\{\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$.

For the determination of the mass-eigenfields, we have to write (8.27) as a function of S, η and ξ rather than x, y and z and consider only terms proportional to $\langle\phi\rangle^2$ (respectively quadratic in the fields S, η and ξ). So, we obtain

$$\mathrm{V}_0(S,\eta,\xi) = \frac{\langle\phi\rangle^2}{2}\begin{pmatrix} S & \eta & \xi \end{pmatrix} \overbrace{\begin{pmatrix} 0 & \vec{0}_2^{\mathrm{T}} \\ \vec{0}_2 & \tilde{P}_{2\times2} \end{pmatrix}}^{=\tilde{P}} \begin{pmatrix} S \\ \eta \\ \xi \end{pmatrix} \tag{8.32}$$

$$\text{with} \quad \tilde{P}_{2\times2} = \frac{1}{3x_0}\begin{pmatrix} c_2\,(x_0^2 - y_0^2) & -\frac{1}{2}\sqrt{\frac{x_0^2-y_0^2}{1-x_0}}\,(c_3 x_0 + 2c_2 y_0) \\ -\frac{1}{2}\sqrt{\frac{x_0^2-y_0^2}{1-x_0}}\,(c_3 x_0 + 2c_2 y_0) & \frac{1}{1-x_0}\,(c_1 x_0^2 + c_2 y_0^2 + c_3 x_0 y_0) \end{pmatrix}, \tag{8.33}$$

which verifies that the scalon S lying in the flat direction has a vanishing mass at tree-level.

Introducing a new parameter via

$$d = \sqrt{\frac{x_0^2 - y_0^2}{1 - x_0}}, \tag{8.34}$$

we can write (8.33) with the set of parameters $\{x_0, \lambda_1, \lambda_4, c_2, d\}$ as

$$\tilde{P}_{2\times2} = \frac{1-x_0}{3x_0} \begin{pmatrix} c_2\, d^2 & 3\,\lambda_4\, d \\ 3\,\lambda_4\, d & \lambda_1 \end{pmatrix}, \tag{8.35}$$

which is more compact and therefore we will use it further on.

With the transformation $\{\varphi_4, \varphi_5, \varphi_6\} \to \{S, \eta, \xi\}$, the squared mass-matrix is not yet in diagonal form, but the scalon S is decoupled from the two other states which are already massive at tree-level. In that way, we reduced the problem of diagonalizing a symmetric 3×3-matrix to that of a 2-dimensional symmetric matrix. To diagonalize the 2×2-matrix $\tilde{P}_{2\times2}$, we use the SO(3)-matrix

$$R' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}, \tag{8.36}$$

and compute the product $R'\tilde{P}R'^{\mathrm{T}}$. So that this product gives a diagonal matrix, the off-diagonal term

$$\cos\alpha\, \sin\alpha\, (\tilde{P}_{22} - \tilde{P}_{33}) + (\cos^2\alpha - \sin^2\alpha)\, \tilde{P}_{23}$$

has to vanish. This is exactly the case, if we define α as

$$\tan(2\alpha) = \frac{2\tilde{P}_{23}}{\tilde{P}_{33} - \tilde{P}_{22}}. \tag{8.37}$$

Writing α as a function of the five parameters $\{x_0, y_0, c_1, c_2, c_3\}$ and $\{x_0, \lambda_1, \lambda_4, c_2, d\}$, it reads

$$\alpha = \frac{1}{2} \arctan\left(\sqrt{x_0^2 - y_0^2}\, \sqrt{1-x_0}\; \frac{c_3 x_0 + 2c_2 y_0}{c_2(x_0^2 - y_0^2)(1-x_0) - (c_1 x_0^2 + c_2 y_0^2 + c_3 x_0 y_0)} \right) \tag{8.38}$$

$$= \frac{1}{2} \arctan\left(\frac{6\lambda_4 d}{\lambda_1 - c_2 d^2} \right). \tag{8.39}$$

Due to this specific transformation $R'\tilde{P}R'^{\mathrm{T}}$, we transform the fields η and ξ in

$$\mathrm{V}_0(S, \eta, \xi) = \frac{\langle\phi\rangle^2}{2} \begin{pmatrix} S & \eta & \xi \end{pmatrix} \tilde{P} \begin{pmatrix} S \\ \eta \\ \xi \end{pmatrix} \tag{8.40}$$

into the two mass-eigenfields, calling them η' and ξ'. Then V_0 reads

$$\mathrm{V}_0(S,\eta',\xi') \;=\; \frac{\langle\phi\rangle^2}{2}\begin{pmatrix} S & \eta' & \xi' \end{pmatrix}[R'\tilde{P}R'^{\mathrm{T}}]_{3\times 3}\begin{pmatrix} S \\ \eta' \\ \xi' \end{pmatrix} \tag{8.41}$$

$$= \frac{1}{2}\left[\mathrm{m}^2_{\eta'}\,(\eta')^2 \,+\, \mathrm{m}^2_{\xi'}\,(\xi')^2\right], \tag{8.42}$$

where the squared masses of η' and ξ' are obtained as

$$\mathrm{m}^2_{\eta'} \;=\; \langle\phi\rangle^2\left(\tilde{P}_{22}\cos^2\alpha \,+\, \tilde{P}_{33}\sin^2\alpha \,-\, 2\tilde{P}_{23}\cos\alpha\sin\alpha\right), \tag{8.43}$$

$$\mathrm{m}^2_{\xi'} \;=\; \langle\phi\rangle^2\left(\tilde{P}_{22}\sin^2\alpha \,+\, \tilde{P}_{33}\cos^2\alpha \,+\, 2\tilde{P}_{23}\cos\alpha\sin\alpha\right), \tag{8.44}$$

and the corresponding mass-eigenfields are

$$\begin{pmatrix} S \\ \eta' \\ \xi' \end{pmatrix} \;=\; R'\begin{pmatrix} S \\ \eta \\ \xi \end{pmatrix} \;=\; \begin{pmatrix} S \\ \cos\alpha\,\eta \,-\, \sin\alpha\,\xi \\ \cos\alpha\,\xi \,+\, \sin\alpha\,\eta \end{pmatrix} \tag{8.45}$$

or vice versa

$$\begin{pmatrix} S \\ \eta \\ \xi \end{pmatrix} \;=\; R'^{\mathrm{T}}\begin{pmatrix} S \\ \eta' \\ \xi' \end{pmatrix} \;=\; \begin{pmatrix} S \\ \cos\alpha\,\eta' \,+\, \sin\alpha\,\xi' \\ \cos\alpha\,\xi' \,-\, \sin\alpha\,\eta' \end{pmatrix}. \tag{8.46}$$

Inserting this transformation (8.46) back into (8.25), we achieve

$$\begin{aligned}
\begin{pmatrix} \varphi_4 \\ \varphi_5 \\ \varphi_6 \end{pmatrix} &= (\langle\phi\rangle \,+\, S)\begin{pmatrix} \sqrt{1-x_0} \\ \sqrt{\frac{x_0+y_0}{2}} \\ \sqrt{\frac{x_0-y_0}{2}} \end{pmatrix} + \frac{1}{\sqrt{x_0}}\left[\eta\begin{pmatrix} 0 \\ -\sqrt{\frac{x_0-y_0}{2}} \\ \sqrt{\frac{x_0+y_0}{2}} \end{pmatrix} + \xi\begin{pmatrix} -x_0 \\ \sqrt{1-x_0}\,\sqrt{\frac{x_0+y_0}{2}} \\ \sqrt{1-x_0}\,\sqrt{\frac{x_0-y_0}{2}} \end{pmatrix}\right] \\
&= (\langle\phi\rangle \,+\, S)\begin{pmatrix} \sqrt{1-x_0} \\ \sqrt{\frac{x_0+y_0}{2}} \\ \sqrt{\frac{x_0-y_0}{2}} \end{pmatrix} + \\
&+ \frac{1}{\sqrt{x_0}}\left[\eta'\begin{pmatrix} x_0\,\sin\alpha \\ -\left(\sqrt{\frac{x_0-y_0}{2}}\,\cos\alpha \,+\, \sqrt{1-x_0}\sqrt{\frac{x_0+y_0}{2}}\,\sin\alpha\right) \\ \sqrt{\frac{x_0+y_0}{2}}\,\cos\alpha \,-\, \sqrt{1-x_0}\sqrt{\frac{x_0-y_0}{2}}\,\sin\alpha \end{pmatrix} + \right. \\
&\qquad \left. +\,\xi'\begin{pmatrix} -x_0\,\cos\alpha \\ \sqrt{1-x_0}\sqrt{\frac{x_0+y_0}{2}}\,\cos\alpha \,-\, \sqrt{\frac{x_0-y_0}{2}}\,\sin\alpha \\ \sqrt{1-x_0}\sqrt{\frac{x_0-y_0}{2}}\,\cos\alpha \,+\, \sqrt{\frac{x_0+y_0}{2}}\,\sin\alpha \end{pmatrix}\right]
\end{aligned} \tag{8.47}$$

for the transformation of the fields $\{\varphi_4,\varphi_5,\varphi_6\}$ to the mass-eigenstates $\{S,\eta',\xi'\}$.

8.5 Full effective potential at one-loop order

With the contributions from the gauge bosons and the fermions (taking only t-quarks and right-handed neutrinos) included, the full one-loop effective potential evaluated in the flat direction reads

$$\begin{aligned}\delta\,\mathrm{V}^{(1)}(\langle\phi\rangle\vec{n}) \;=\; \frac{1}{64\,\pi^2}\,\Bigg\{ &\mathrm{m}_{\eta'}^4\left[\ln\frac{\mathrm{m}_{\eta'}^2}{\Lambda_W^2}-\frac{3}{2}\right]+\mathrm{m}_{\xi'}^4\left[\ln\frac{\mathrm{m}_{\xi'}^2}{\Lambda_W^2}-\frac{3}{2}\right]+\\ &+6\cdot\mathrm{M}_W^4\left[\ln\frac{\mathrm{M}_W^2}{\Lambda_W^2}-\frac{5}{6}\right]+3\cdot\mathrm{M}_Z^4\left[\ln\frac{\mathrm{M}_Z^2}{\Lambda_W^2}-\frac{5}{6}\right]-\\ &-12\cdot\mathrm{M}_t^4\left[\ln\frac{\mathrm{M}_t^2}{\Lambda_W^2}-\frac{3}{2}\right]-2\cdot\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4\left[\ln\frac{\mathrm{M}_{\nu,r}^2}{\Lambda_W^2}-\frac{3}{2}\right]\Bigg\}.\end{aligned} \tag{8.48}$$

From here, one can read off

$$\mathrm{B} \;=\; \frac{1}{64\,\pi^2\,\langle\phi\rangle^4}\left(\mathrm{m}_{\eta'}^4+\mathrm{m}_{\xi'}^4+6\,\mathrm{M}_W^4+3\,\mathrm{M}_Z^4-12\,\mathrm{M}_t^4-2\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4\right) \tag{8.49}$$

and demanding that B has to be strictly positive, B>0 [54], one gets

$$\mathrm{m}_{\eta'}^4+\mathrm{m}_{\xi'}^4-2\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4 \;>\; 12\,\mathrm{M}_t^4-6\,\mathrm{M}_W^4-3\,\mathrm{M}_Z^4\approx(319\,\mathrm{GeV})^4\,. \tag{8.50}$$

Thus the minimal value for the sum of $\mathrm{m}_{\eta'}^4$ and $\mathrm{m}_{\xi'}^4$ is given by

$$(\mathrm{m}_{\eta'}^4+\mathrm{m}_{\xi'}^4)_{min} \;=\; (319\,\mathrm{GeV})^4\,. \tag{8.51}$$

8.6 Which scalar is the observed Higgs boson?

For implementing n_R right-handed neutrinos via the seesaw mechanism of type 1 [45–48] (as explained in chapter 3), the VEV of the scalar singlet, $\langle X\rangle$, has to be much bigger than that of the Higgs doublet, $v=\langle\phi\rangle\sqrt{1-x_0}$. The ratio of

$$\langle X\rangle \;=\; \sqrt{\frac{v_5^2+v_6^2}{2}} \;=\; \langle\phi\rangle\sqrt{\frac{x_0}{2}} \quad\text{and}\quad v \;=\; \langle\phi\rangle\sqrt{1-x_0} \tag{8.52}$$

is given by the relation

$$\frac{\langle X\rangle}{v} \;=\; \frac{1}{\sqrt{2}}\sqrt{\frac{x_0}{1-x_0}}\,. \tag{8.53}$$

Knowing that $v = 246\,\text{GeV}$ and by assuming $\langle X \rangle$ to have at least a value of $1\,\text{TeV}$, we achieve a range for x_0 of

$$0.97 \,<\, x_0 \,<\, 1 \tag{8.54}$$

and therefore we get

$$\langle \phi \rangle \,>\, 1.42\,\text{TeV}\,. \tag{8.55}$$

Assuming c_1, c_2 and c_3 to be small but different from zero and y_0 having a fixed nonvanishing value satisfying $|y_0| < x_0$, we can consider α (8.38) only as a function of x_0. Taking the limit $x_0 \to 1$, we get

$$\alpha(x_0) \xrightarrow{x_0 \to 1} 0\,. \tag{8.56}$$

Looking at the expansion of the Higgs doublet around its VEV in dependence of the mass eigenstates (8.47), it reads

$$\Phi \,=\, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \,+\, \sqrt{1-x_0}\;S + \sqrt{x_0}\,\sin\alpha\;\eta' \,-\, \sqrt{x_0}\,\cos\alpha\;\xi' \end{pmatrix} \tag{8.57}$$

and thus we can identify ξ' as the mass eigenstate of the physical Higgs particle with a mass of $\text{m}_h = (125.09 \pm 0.21 \pm 0.11)$ GeV [61] discovered at the LHC in 2012 [10, 11], since just the term $\kappa_f \,=\, -\sqrt{x_0}\,\cos\alpha$ (responsible for the coupling of the fermions to the Higgs field in the SM) standing in front of this state ξ' has an absolute value slightly different from one. With the findings from above, $\sqrt{1-x_0}$ is not much different from zero and $\sqrt{x_0}\,\sin\alpha$ is for sure much smaller than $\sqrt{x_0}\,\cos\alpha$. Therefore the states S and η' couple much weaker to the fermions than the state ξ'.

For the absolute value of the coupling constant $\kappa_f \,=\, -\sqrt{x_0}\,\cos\alpha$, we take [72–74] a range of

$$0.9 \,<\, \sqrt{x_0}\,\cos\alpha \,<\, 1 \tag{8.58}$$

obtaning a range for $\cos\alpha$ of

$$\frac{0.9}{\sqrt{x_0}} \,<\, \cos\alpha \,<\, \frac{1}{\sqrt{x_0}}\,. \tag{8.59}$$

With the values for x_0 (8.54) and not allowing $\cos\alpha$ to be larger than one, we get

$$\begin{aligned} 0.84 \,<\, \cos^2\alpha \,<\, 1 \qquad &\text{for} \quad x_0 \,=\, 0.97\,, \\ 0.81 \,<\, \cos^2\alpha \,<\, 1 \qquad &\text{for} \quad x_0 \,=\, 1. \end{aligned}$$

In the final calculation, we will always take

$$0.84 \,<\, \cos^2\alpha \,<\, 1\,, \tag{8.60}$$

since then the range for α is certainly satisfied no matter which explicit value is chosen for x_0.

Since we have identified the state ξ' as the physical Higgs particle, we obtain from (8.44),

$$\begin{aligned}\mathrm{m}_h^2 \,=\, \mathrm{m}_{\xi'}^2 \,=\, (125.09\,\mathrm{GeV})^2 \,&=\, \langle\phi\rangle^2 \left(\tilde{P}_{22}\sin^2\alpha \,+\, \tilde{P}_{33}\cos^2\alpha \,+\, 2\tilde{P}_{23}\cos\alpha\sin\alpha\right)\\ &=\, \frac{v^2}{3x_0}\left(c_2\,d^2 \,+\, (\lambda_1 - c_2\,d^2)\,\cos^2\alpha \,+\, 6\,\lambda_4\,d\,\cos\alpha\,\sin\alpha\right)\,.\end{aligned} \tag{8.61}$$

Due to the fact that the minimal value for the sum of $\mathrm{m}_{\eta'}^4$ and $\mathrm{m}_{\xi'}^4$ is given by (8.51), we have

$$\mathrm{m}_{\eta'} > 317\,\mathrm{GeV} \tag{8.62}$$

and from the squared mass of the mass eigenstate η' (8.43), we get

$$\begin{aligned}(317\,\mathrm{GeV})^2 < \mathrm{m}_{\eta'}^2 \,&=\, \langle\phi\rangle^2 \left(\tilde{P}_{22}\cos^2\alpha \,+\, \tilde{P}_{33}\sin^2\alpha \,-\, 2\tilde{P}_{23}\cos\alpha\sin\alpha\right)\\ &=\, \frac{v^2}{3x_0}\left(\lambda_1 \,+\, (c_2\,d^2 - \lambda_1)\,\cos^2\alpha \,-\, 6\,\lambda_4\,d\,\cos\alpha\,\sin\alpha\right)\,.\end{aligned} \tag{8.63}$$

If we express λ_4 in (8.61) in terms of the other parameters and inserting it into (8.63), we get

$$3\,x_0\left[\left(\frac{317}{246}\right)^2 + \left(\frac{125.09}{246}\right)^2\right] < \lambda_1 \,+\, c_2\,d^2\,. \tag{8.64}$$

Therefore, we have

$$\begin{aligned}\lambda_1 \,+\, c_2\,d^2 > 5.76 \qquad &\text{for} \quad x_0 = 1\,,\\ \lambda_1 \,+\, c_2\,d^2 > 5.58 \qquad &\text{for} \quad x_0 = 0.97\,.\end{aligned}$$

In the following, we will adjust the values of λ_1, c_2 and d such that

$$\lambda_1 \,+\, c_2\,d^2 > 5.76 \tag{8.65}$$

is certainly fulfilled.

For achieving an upper bound for c_2, we consider the relevant term in (8.2),

$$\frac{\lambda_6}{24}\left(\varphi_5^2 - \varphi_6^2\right)^2 \,-\, \frac{\lambda_6}{6}\,\varphi_5^2\varphi_6^2 \,=\, \frac{\lambda_6}{24}\left(\varphi_5^4 + \varphi_6^4 - 6\varphi_5^2\varphi_6^2\right), \tag{8.66}$$

seeing that $|6\lambda_6|$ has to be much smaller than $(4\pi)^2$ for the model to be perturbatively valid and thus c_2 ($c_2 = 2\lambda_6$) has to be in the following range,

$$0 \, < \, c_2 \, < \, \frac{16}{3}\,\pi^2 \, \approx \, 16\,\pi\,. \tag{8.67}$$

In addition, knowing that y_0 is a real number, one gets an upper bound for d in dependence of x_0,

$$d \leq \frac{x_0}{\sqrt{1 - x_0}} \tag{8.68}$$

and it is assumed to be positive from formula (8.34).

8.6.1 Specific examples for model to be perturbatively valid

Following the same argumentation as in sections 6.6 and 7.4.1, and making the rough approximation $\pi \approx 3$ for the upper bounds, we get as ranges for λ_1 to λ_6 in this model,

$$0 < \lambda_1 \ll 144\,, \quad 0 < \lambda_2 \ll 72\,, \quad |\lambda_3| \ll 24\,, \quad |\lambda_4| \ll 24\,, \quad |\lambda_5| \ll 144\,, \quad |\lambda_6| \ll 24\,. \tag{8.69}$$

Thus, for the model to be carried out perturbatively, we have to choose values for x_0, λ_1, $\cos^2\alpha$, d and c_2 in the ranges found above, listed for clearness again,

$$0.97 < x_0 < 1\,, \tag{8.70}$$

$$d \leq \frac{x_0}{\sqrt{1 - x_0}}\,, \tag{8.71}$$

$$0.84 < \cos^2\alpha < 1\,, \tag{8.72}$$

$$0 < c_2 < 48\,, \tag{8.73}$$

$$\lambda_1 \, + \, c_2\, d^2 > 5.76\,, \tag{8.74}$$

such that the ranges for λ_1 to λ_6 are satisfied. Moreover, the final parameters have to fulfill $\det(\tilde{P}_{2\times 2}) > 0$ with $\tilde{P}_{2\times 2}$ from formula (8.35), so

$$\lambda_1 c_2 - 9\lambda_4^2 > 0\,. \tag{8.75}$$

Example 1

One reasonable set for the input parameters $\{x_0,\, \lambda_1,\, \alpha,\, d,\, c_2\}$ is

$$x_0 \,=\, 0.98\,, \quad \lambda_1 \,=\, 10\,, \quad \alpha \,=\, 0.20\,, \quad d \,=\, 6.50\,, \quad c_2 \,=\, 5.34 \tag{8.76}$$

leading to values for $\{\lambda_2,\, \lambda_3,\, \lambda_4,\, \lambda_5,\, \lambda_6\}$ with

$$\lambda_2 \,=\, 3.22\,, \tag{8.77}$$
$$\lambda_3 \,=\, 0.74\,, \tag{8.78}$$
$$\lambda_4 \,=\, -2.34\,, \tag{8.79}$$
$$\lambda_5 \,=\, -3.42\,, \tag{8.80}$$
$$\lambda_6 \,=\, 2.67\,, \tag{8.81}$$

seeing that all values are in a region where perturbation theory can certainly be carried out.

With these value, the determinant of the mass matrix in (8.35) is positive. We obtain

$$\lambda_1 c_2 - 9\lambda_4^2 = 4.23\,.$$

Moreover, we have chosen the values such that the Higgs mass (8.61) is reproduced with

$$\mathrm{m}_h = 125.11\,\mathrm{GeV}\,, \tag{8.82}$$

and the mass (8.63) of the second heavy scalar particle can be calculated to be

$$\mathrm{m}_{\eta'} = 2.20\,\mathrm{TeV}\,. \tag{8.83}$$

With $x_0 \,=\, 0.98$, we get $\langle\phi\rangle \,=\, 1.74\,\mathrm{TeV}$ and with all the parameters, we achieve for the squared one-loop mass of the scalon,

$$\begin{aligned}(\mathrm{m}_S^{(1)})^2 \,&=\, 8\mathrm{B}\langle\phi\rangle^2 \,=\, \frac{1}{8\,\pi^2\,\langle\phi\rangle^2}\left(\mathrm{m}_h^4 + \mathrm{m}_{\eta'}^4 + 6\,\mathrm{M}_W^4 + 3\,\mathrm{M}_Z^4 - 12\,\mathrm{M}_t^4 - 2\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4\right)\\ &=\, 97.78\,(\mathrm{TeV})^2 - 8.38\cdot 10^{-9}\,(\mathrm{GeV})^{-2}\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4\,.\end{aligned} \tag{8.84}$$

If we do not add heavy neutrinos to the model, the one-loop mass of the scalon is

$$\mathrm{m}_S^{(1)} \,=\, 312.69\,\mathrm{GeV} \tag{8.85}$$

and if there are heavy neutrinos, it is even smaller.

Example 2

Another set for reasonable input parameters $\{x_0,\, \lambda_1,\, \alpha,\, d,\, c_2\}$ is

$$x_0 \;=\; 0.99\,, \quad \lambda_1 \;=\; 15\,, \quad \alpha \;=\; 0.30\,, \quad d \;=\; 9.519\,, \quad c_2 \;=\; 1.65 \tag{8.86}$$

leading to values for $\{\lambda_2,\, \lambda_3,\, \lambda_4,\, \lambda_5,\, \lambda_6\}$ with

$$\lambda_2 \;=\; 0.92\,, \tag{8.87}$$
$$\lambda_3 \;=\; 0.39\,, \tag{8.88}$$
$$\lambda_4 \;=\; -1.61\,, \tag{8.89}$$
$$\lambda_5 \;=\; -0.81\,, \tag{8.90}$$
$$\lambda_6 \;=\; 0.83\,, \tag{8.91}$$

seeing that all values are in a region where perturbation theory can certainly be carried out.

With these value, the determinant of the mass matrix in (8.35) is positive. We obtain

$$\lambda_1 c_2 - 9\lambda_4^2 = 1.39\,.$$

Moreover, we have chosen the values such that the Higgs mass (8.61) is reproduced with

$$\mathrm{m}_h = 125.02\,\mathrm{GeV}\,, \tag{8.92}$$

and the mass (8.63) of the second heavy scalar particle can be calculated to be

$$\mathrm{m}_{\eta'} = 1.83\,\mathrm{TeV}\,. \tag{8.93}$$

With $x_0 \;=\; 0.99$, we get $\langle\phi\rangle \;=\; 2.46\,\mathrm{TeV}$ and with all the parameters, we achieve for the squared one-loop mass of the scalon,

$$\begin{aligned}(\mathrm{m}_S^{(1)})^2 \;=\; 8\mathrm{B}\langle\phi\rangle^2 \;&=\; \frac{1}{8\,\pi^2\,\langle\phi\rangle^2}\left(\mathrm{m}_h^4 + \mathrm{m}_{\eta'}^4 + 6\,\mathrm{M}_W^4 + 3\,\mathrm{M}_Z^4 - 12\,\mathrm{M}_t^4 - 2\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4\right)\\ &=\; 48.89\,(\mathrm{TeV})^2 - 4.18\cdot 10^{-9}\,(\mathrm{GeV})^{-2}\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4\,.\end{aligned} \tag{8.94}$$

If we do not add heavy neutrinos to the model, the one-loop mass of the scalon is

$$\mathrm{m}_S^{(1)} \;=\; 221.10\,\mathrm{GeV} \tag{8.95}$$

and if there are heavy neutrinos, it is even smaller.

Example 3

Another set for reasonable input parameters $\{x_0,\, \lambda_1,\, \alpha,\, d,\, c_2\}$ is

$$x_0 \,=\, 0.999\,, \quad \lambda_1 \,=\, 30\,, \quad \alpha \,=\, 0.108\,, \quad d \,=\, 10\,, \quad c_2 \,=\, 24.87 \tag{8.96}$$

leading to values for $\{\lambda_2,\, \lambda_3,\, \lambda_4,\, \lambda_5,\, \lambda_6\}$ with

$$\lambda_2 \,=\, 34.76\,, \tag{8.97}$$
$$\lambda_3 \,=\, 8.51\,, \tag{8.98}$$
$$\lambda_4 \,=\, -8.99\,, \tag{8.99}$$
$$\lambda_5 \,=\, -47.13\,, \tag{8.100}$$
$$\lambda_6 \,=\, 12.44\,, \tag{8.101}$$

seeing that all values are in a region where perturbation theory can certainly be carried out.

With these value, the determinant of the mass matrix in (8.35) is positive. We obtain

$$\lambda_1 c_2 - 9\lambda_4^2 = 19.47\,.$$

Moreover, we have chosen the values such that the Higgs mass (8.61) is reproduced with

$$\mathrm{m}_h = 124.99\,\mathrm{GeV}\,, \tag{8.102}$$

and the mass (8.63) of the second heavy scalar particle can be calculated to be

$$\mathrm{m}_{\eta'} \,=\, 7.13\,\mathrm{TeV}\,. \tag{8.103}$$

With $x_0 \,=\, 0.999$, we get $\langle\phi\rangle \,=\, 7.78\,\mathrm{TeV}$ and with all the parameters, we achieve for the squared one-loop mass of the scalon,

$$\begin{aligned}(\mathrm{m}_S^{(1)})^2 \,&=\, 8\mathrm{B}\langle\phi\rangle^2 \,=\, \frac{1}{8\,\pi^2\,\langle\phi\rangle^2}\left(\mathrm{m}_h^4 + \mathrm{m}_{\eta'}^4 + 6\,\mathrm{M}_W^4 + 3\,\mathrm{M}_Z^4 - 12\,\mathrm{M}_t^4 - 2\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4\right)\\ &=\, 540.26\,(\mathrm{TeV})^2 - 4.18\cdot 10^{-10}\,(\mathrm{GeV})^{-2}\sum_{r=1}^{\mathrm{n}_R}\mathrm{M}_{\nu,r}^4\,.\end{aligned} \tag{8.104}$$

If we do not add heavy neutrinos to the model, the one-loop mass of the scalon is

$$\mathrm{m}_S^{(1)} \,=\, 735.03\,\mathrm{GeV} \tag{8.105}$$

and if there are heavy neutrinos, it is even smaller.

Summary

In the last of these three examples, the values of the couplings − formula (8.97) to (8.101) − seem to be a bit high, but comparing them with the ranges given in (8.69), nevertheless it seems to fit well. Therefore all three examples given in this section seem to be viable to be carried out in perturbation theory, where the second example has the best prediction since the couplings are really small.

9 Conclusions and Prospects

In this thesis, three versions of the SM with classical scale-invariance were considered, where $-$ compared to the particle content of the minimal version of the SM $-$ scalar degrees of freedom and an arbitrary number of right-handed neutrinos were added but leaving the gauge group of the SM unchanged. In absence of an explicit scalar mass term in the Lagrangian, all particles' masses arised via quantum corrections to the effective potential. In all three models, one real or complex scalar singlet was added and moreover an arbitrary number of right-handed neutrino fields was introduced to give an explanation for massive neutrinos via the seesaw-mechanism of type 1. In the first two models considered, in chapters 6 and 7, it was not possible to include massive neutrinos via the seesaw-mechanism, since from our calculations in the scalar sector, we found that the VEV of the Higgs doublet is much larger than that of the scalar singlet. Therefore, we discarded these models from further calculations with the aim to describe a version of the SM with additional scalar fields and the realization of neutrino masses.

In the last model considered, in chapter 8, we did not come to any contradictions in our calculations and therefore this model seems viable to account for both an additional heavy scalar state with a mass in the TeV-range and for giving an explanation for light neutrino masses via the seesaw-mechanism realized with a symmetry breaking scale in the TeV-range. In addition, one has the scalon as another scalar state having a smaller mass of order $\mathcal{O}(10^2\,\text{GeV})$ or even smaller if there are heavy neutrinos in the TeV-range included. If there really exist physical scalar particles and neutrinos with masses in the TeV-range in nature and if our model can come to fruition, has to be proven in future experiments like at the LHC.

In this potentially viable model, we added one complex scalar singlet X and the scalar potential we investigated was invariant under the discrete transformation $X \leftrightarrow X^*$. In our calculations, we concentrated on the scalar sector: First we found the tree-level mass-eigenstates and the corresponding mass-eigenvalues of all the physical scalar particles. Then we calculated the scalar effective potential up to one-loop order with the quantum corrections containing scalar, gauge boson, t-quark and neutrino loops. The calculations were performed in dimensional regularization and the results were given in the $\overline{\text{MS}}$-scheme. At one-loop order we got an inequality in terms of the masses of all particles contained in that model. Having the mass-eigenvalues of all the scalar states contained in the model in dependence of the six dimensionless couplings λ_1, λ_2, λ_3, λ_4, λ_5, λ_6 and claiming that we had heavy neutrinos with masses of order $\mathcal{O}$(TeV) as well, we then came to a parameter range of the six couplings with the observation that we could adjust them being small enough so that the model is perturbatively valid.

In our calculations, we found three examples where all the couplings $\{\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6\}$ have values where perturbation theory is valid and the mass of the heavy scalar state lies between 1.8 TeV and 7.1 TeV. The mass range of the heavy scalar state might be enlarged by making a scan over the full parameter ranges of all the six couplings. We found the masses by just trying and setting in values for the couplings. In this way, we did not find any values where the mass of the scalar goes to (much) heavier masses than 7.1 TeV and we do not think that this model can be described perturbatively with a massive scalar state with a mass of around 10 TeV or even heavier.

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