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Abstract

Inverse limit spaces of unimodal maps have triggered a substantial amount of mathematical research in the last three decades and the topology of the spaces is almost fully understood. One of the main reasons to study inverse limits is the fact that they present a natural way to model attractors of chaotic dynamical systems and can thus give a valuable insight in the topological structure of the attractors of important dynamical systems.

The question which served as the main motivation for this thesis was posed by the topologist and dynamicist Philip Boyland. Originating from the interest in Dynamical System he asked if there exist planar embeddings of inverse limit spaces of unimodal maps that are not equivalent to the standard two embeddings constructed by Brucks & Diamond and Bruin respectively in the early 1990's.

In this thesis, a construction of uncountably many pairwise non-equivalent planar embeddings of inverse limit spaces of unimodal maps is given. Specifically, for every point in the inverse limit space of a unimodal map we construct a planar embedding of this space which makes the given point accessible from the complement of the space. Furthermore, we partially characterize the accessible points in the constructed embeddings and show that the constructed embeddings are unlike the already known ones in the sense that the natural shift homeomorphism cannot be extended to the whole plane.

Furthermore, the full characterization of sets of accessible points and the prime end structure of the two standard embeddings is given.

Zusammenfassung

Inverse-Limes-Räume von unimodalen Abbildungen haben in den letzten drei Jahrzehnten eine beträchtliche Menge an mathematischer Forschung ausgelöst und man hat die Topologie dieser Räume fast vollständig verstanden. Einer der Hauptgründe diese Räume zu untersuchen ist die Tatsache, dass sie eine natürliche Weise bieten um Attraktoren chaotischer dynamischen Systeme zu modellieren und so einen guten Einblick in die topologische Struktur der Attraktoren eines wichtigen dynamischen Systems zu geben.

Die Frage, die als Hauptmotivation für diese Arbeit diente, wurde vom Topologen und Dynamiker Philip Boyland gestellt. Aus dem Interesse an dynamischen Systemen fragte er, ob es planare Einbettungen von Inversen-Limes-Räumen von unimodalen Abbildungen gibt, die nicht den zwei Standardeinbettungen entsprechen, die von Brucks und Diamond, bzw. von Bruin in den frühen 1990er Jahren konstruiert wurden.

In dieser Arbeit wird eine Konstruktion von überabzählbar vielen, paarweise nicht zueinander äquivalenten planaren Einbettungen von Inversen-Limes-Räumen von unimodalen Abbildungen gegeben, wobei symbolische Dynamik als Hauptwerkzeug dient. Speziell konstruieren wir für jeden Punkt im Inversen-Limes-Raum einer unimodalen Funktion eine planare Einbettung dieses Raumes, die den gegebenen Punkt aus dem Komplement des Raumes zugänglich macht. Darüber hinaus charakterisieren wir teilweise die zugänglichen Punkte in den konstruierten Einbettungen und zeigen, dass für sie im Gegensatz zu den bereits bekannten Einbettungen der natürliche Links-Shift nicht zu einem Homöomorphismus auf die ganze Ebene ausgedehnt werden kann. Insbesondere für die beiden Standardeinbettungen von Inversen-Limes-Räumen für unimodalen Abbildungen, die von Brucks und Diamond, sowie von Bruin konstruiert wur-

den, wird eine vollständige Charakterisierung der zugänglichen Punkte und die Anzahl der einfachen dichten Kanäle gegeben.

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Chapter 1

Introduction

Throughout the thesis we study *continua*, *i.e.,* compact connected metric spaces. Let $\{K_i\}_{i\in\mathbb{N}_0}$ be a collection of continua and let $f_i : K_i \to K_{i+1}$ be a continuous function for every $i \in \mathbb{N}_0$. The *inverse limit space* is the collection of all possible backward orbits on the ordered collection of the *factor spaces* K_i with the action of the *bonding maps* f_i on the spaces K_i. It turns out that inverse limit spaces defined as above are continua (see *e.g.* [38]) with the *product metric* inherited from the space $\ldots \times K_1 \times K_0$. Moreover, every continuum can be expressed as an inverse limit on compact connected polyhedra as factor spaces with onto continuous bonding maps (see *e.g.* [50]).

Let K be a continuum. A *subcontinuum* of a continuum K is going to be called proper, if it does not equal to K and is not degenerate (*i.e.,* consists of more than one point). The *composant (arc-component)* $\mathcal{U}_x \subset K$ of a point $x \in \mathcal{U}_x$ is the union of all proper subcontinua (arcs) of K containing the point x. We call K *indecomposable* if it cannot be expressed as the union of two proper subcontinua. In that case, K has uncountably many composants and every composant is dense in K (see *e.g.* [50]).

In this thesis we study inverse limit spaces with a single unimodal (maps with one strictly increasing branch and one strictly decreasing branch) interval bonding maps $T : [0,1] \to [0,1]$ where $T(0) = T(1) = 0$. Let c denote the critical point of the unimodal map T and we call the interval $[T^2(c), T(c)]$ the *core* of the map T. From now on we denote the inverse limit

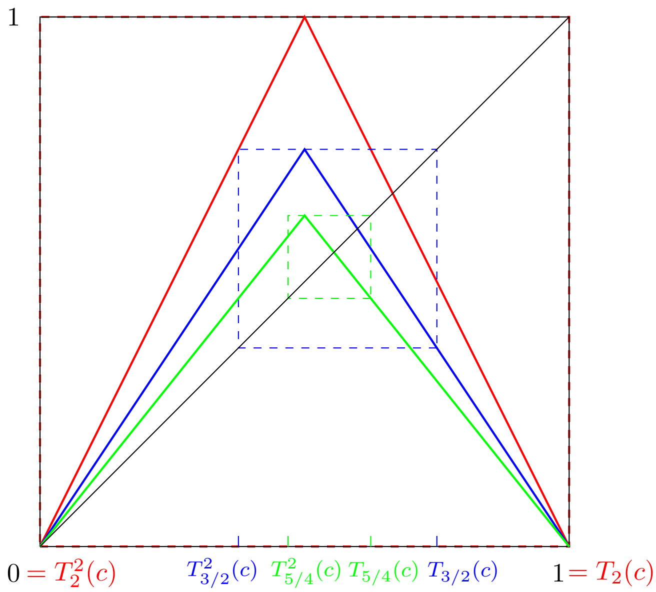

Figure 1.1: Parametrized tent map family; dashed boxes denote the invariant cores of the maps T_s

space with a single unimodal bonding map by $X := \varprojlim([0,1], T)$. We restrict our study on the maps T with $T^2(c) > T(c)$ and thus to the unimodal maps T with two fixed points: 0 and r. Let $T_s(z) := \min\{sz, s(1-z)\}$ for $z \in [0,1]$ be the *tent map family* for $s \in (0,2]$, see Figure 1.

Note that we did not make any assumption about the differentiability of the unimodal map, so our definition allows us to study tent maps as unimodal maps.

Denote the arc-component of $(\dots, 0, 0) \in X$ by $\mathcal{C}$. We can decompose $X = \mathcal{C} \cup X'$, where X' is the *core of* X and $\mathcal{C}$ compactifies on X', see [14]. For a visualisation of an example of X and X' see Figure 6.2 and Figure 6.3 respectively. Let $\mathcal{R}$ denote the arc-component of $(\dots, r, r) \in X'$. Let us give an insight in the structure of the inverse limit spaces of tent maps when we move along the parameter s. For $s \in (0,1)$ the inverse limit space $\varprojlim([0,1], T_s)$ is only an arc and thus not interesting to study. For $s = 1$, $\varprojlim([0,1], T_s)$ is an arc with a ray converging to it and thus also not interesting from topological perspective. For $s \in (1, \sqrt{2}]$ the core $\varprojlim([T_s^2, T_s(c)], T_s)$ is homeomorphic to two copies of cores of $\varprojlim(I, T_{s^2})$ joined at one point and is therefore decomposable. For $s \in (\sqrt{2}, 2]$ the space $\varprojlim([T_s^2, T_s(c)], T_s)$ is indecomposable (see Lemma 7 for the proof of the last statement) and specifically, $\varprojlim(I, T_2)$ is the Knaster continuum, arguably the simplest example of an indecomposable continuum. The Knaster continuum first appeared in the literature in an article by Kuratowski [40], who attributed the idea to Knaster, see Figure 1.2 for a construction of the Knaster continuum.

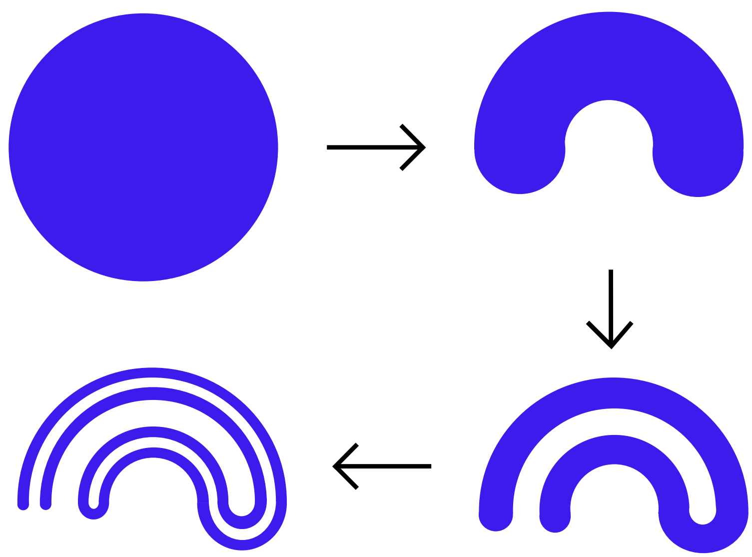

Figure 1.2: Approximations of the Knaster continuum

Inverse limit spaces of unimodal maps received a lot of attention in the last three decades. The main reason that the spaces were under extensive mathematical study is the classification problem that became known as the *Ingram Conjecture* and was posed in the early 1990's:

> **The Ingram Conjecture:** If $1 < s < \tilde{s} \leq 2$, then the inverse limit spaces $\varprojlim([0,1], T_s)$ and $\varprojlim([0,1], T_{\tilde{s}})$ are not homeomorphic.

After series of partial results (see [8, 25, 39, 16, 54, 30, 29, 53]) the Ingram Conjecture was finally answered in the affirmative in 2013 by Barge, Bruin & Štimac in [7]. However, the proof presented in [7] uses the properties of the ray $\mathcal{C}$ which compactifies on the indecomposable core

$\varprojlim([T_s^2(c), T_s(c)], T_s)$. It was proved by Minc in [48] that there exists $2^{\aleph_0}$ many topologically different ways to compactify any continuum with a ray. Therefore, it is natural to restate the Ingram Conjecture to its (still outstanding) core version:

> **The Core Ingram Conjecture:** If $1 < s < \tilde{s} \leq 2$ then the core inverse limit spaces $\varprojlim([T_s^2(c), T_s(c)], T_s)$ and $\varprojlim([T_{\tilde{s}}^2(c), T_{\tilde{s}}(c)], T_{\tilde{s}})$ are not homeomorphic.

In the paper [3] which is not included in this thesis we partially solve The Core Ingram Conjecture. To provide an answer, we study the properties of the above mentioned arc-component $\mathcal{R}$ and conclude that for a dense set of parameters $s, \tilde{s} \in (\sqrt{2}, 2]$ for any homeomorphism $h : \varprojlim([T_s^2(c), T_s(c)], T_s) \to \varprojlim([T_{\tilde{s}}^2(c), T_{\tilde{s}}(c)], T_{\tilde{s}})$ it holds that $h(\mathcal{R}) = \tilde{\mathcal{R}}$, where $\tilde{\mathcal{R}}$ is the arc-component in $\varprojlim([T_{\tilde{s}}^2(c), T_{\tilde{s}}(c)], T_{\tilde{s}})$ analogous to $\mathcal{R}$ containing a fixed point of the map $T_{\tilde{s}}$. We conclude similarly as in [7] that $\mathcal{R}$ topologically completely determines the space $\varprojlim([T_s^2(c), T_s(c)], T_s)$, which partially settles the problem. However, The Core Ingram Conjecture still remains an outstanding problem in general.

A priori, inverse limit spaces are subsets of the infinite dimensional space $\ldots \times K_1 \times K_0$ (in the case when $K_i = [0,1]$ for every $i \in \mathbb{N}_0$ the space $[0,1]^\infty$ is called the *Hilbert cube*). We say that a space $K \subset \ldots \times K_1 \times K_0$ can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$, if there exists a continuous function $g : \ldots \times K_1 \times K_0 \to \mathbb{R}^N$ so that $g(K) = E$ and $E \subset \mathbb{R}^N$ is homeomorphic to $g(K)$. Thus it is natural to ask what is the minimal natural number $N \in \mathbb{N}$ so that $\varprojlim(K_i, f_i)$ can be embedded in $\mathbb{R}^N$, if such N exists.

A continuum is said to be *chainable*, if it admits an ε-mapping on the interval $[0,1]$ for every $\varepsilon > 0$. It can be derived from the paper of Isbell [36] that every chainable continuum can be represented as the inverse limit space on the interval as the unique factor space $\varprojlim([0,1], f_i)$, where the functions $f_i : [0,1] \to [0,1]$ are allowed to vary for $i \in \mathbb{N}_0$. The study of embeddings of chainable continua dates back to 1951 when Bing proved in [15] that every chainable continuum can be embedded in the $\mathbb{R}^2$. However, his proof does not offer any insight what such embeddings look like.

The first explicit class of planar embeddings of X was given by Brucks & Diamond in [23]. Later, Bruin [24] extended this result showing that the embedding of X can be made such that the *natural shift homeomorphism* $\sigma : X \to X$ defined by

$$\pi_i(\sigma(x)) := T(\pi_i(x)) \text{ for every } i \in \mathbb{N}_0 \text{ and } x \in X \tag{1.1}$$

extends to a Lipschitz map on $\mathbb{R}^2$. Here $\pi_i \colon [0,1]^{\mathbb{N}_0} \to [0,1]$ denote the *coordinate projections* for $i \in \mathbb{N}_0$. Both mentioned results are using symbolic dynamics as the main tool in the description of X. Throughout the thesis we will refer to the embeddings of X constructed by Brucks & Diamond [23] and Bruin [24] as the *standard embeddings* of X.

Locally, inverse limit spaces of unimodal maps roughly resemble Cantor sets of arcs. However, every unimodal inverse limit space different from an arc contains at least one point that is locally not homeomorphic to a Cantor sets of arcs. In [6] Barge, Brucks & Diamond proved that in the tent family $\{T_s\}_{s\in(0,2]}$ for a dense G_δ set of slopes $s \in [\sqrt{2}, 2]$, every open neighbourhood of every point in the inverse limit space $\varprojlim([T_s^2(c), T_s(c)], T_s)$ not only contains a homeomorphic copy of the space itself but also homeomorphic copies of every inverse limit space of tent maps as the bonding maps. Thus it would be interesting to see what kind of embeddings in the plane of locally very complicated X are there possible in general.

The main motivation for the thesis are the following questions posed by Philip Boyland on the Continuum Theory and Dynamical Systems Workshop in Vienna in July 2015:

> Can a complicated X be embedded in $\mathbb{R}^2$ in multiple ways? For example, do there exist embeddings of X in $\mathbb{R}^2$ that are not equivalent to the standard embeddings?

Denote two planar embeddings of X by $g_1 \colon X \to E_1 \subset \mathbb{R}^2$ and $g_2 \colon X \to E_2 \subset \mathbb{R}^2$. We say that g_1 and g_2 are *equivalent embeddings* if there exists a homeomorphism $h \colon E_1 \to E_2$ which can be extended to the homeomorphism of the whole plane. For the special case of the full tent map T_2 (*i.e.,* X is the Knaster continuum) Boyland's question was already answered in the affirmative by Mayer [42], Mahavier [43], Schwartz [55] and Dębski & Tymchatyn [28].

Say that X is embedded in the plane. We say that a point $x \in X \subset \mathbb{R}^2$ is *accessible (i.e., from the complement of X)* if there exists an arc $Q = [a, b] \subset \mathbb{R}^2$ such that $x = a$ and $Q \cap X = \{x\}$, see Figure 5.5. We say that a composant (arc-component) $\mathcal{U} \subset X'$ or $\mathcal{U} = \mathcal{C}$ is *accessible*, if $\mathcal{U}$ contains an accessible point. We say that an arc-component $\mathcal{U} \subset X$ is *fully accessible*, if every point from $\mathcal{U}$ is accessible.

For the Knaster continuum Mayer in [42] constructed uncountably many non-equivalent embeddings. Later, Mahavier showed in [43] that for every arc-component (which is also a composant in the Knaster continuum) there exists a planar embedding of the Knaster continuum which makes this arc-component fully accessible. Schwartz [55] extended Mahavier's result and proved that embeddings of X which do not make $\mathcal{C}$ or $\mathcal{R}$ accessible are non-equivalent to the standard embeddings. In this thesis we make use of symbolic dynamics description of X introduced in [23] and [24] and answer on the question of Boyland in the affirmative. For every point $x \in X$ we construct an embedding of X so that x is accessible. Furthermore, if the inverse limit space X with the bonding map T contains an indecomposable subcontinuum we obtain uncountably many non-equivalent embeddings of the space X. In such a way we not only give an answer to Boyland's question but also provide a partial answer to the following question of Mayer posed in early 1980's (also listed as Problem 140 in the paper by Lewis [41]):

Question (Mayer (1983)): Are there uncountably many non-equivalent planar embeddings of every indecomposable chainable continuum?

Furthermore, we provide a partial answer also on the following question:

Question (Nadler, Quinn (1972)): If $x \in K$ is a point in a chainable continuum K, does there exist an embedding of K in the plane, so that x is accessible?

We give the affirmative answer to these two question for all indecomposable continua that can be obtained as the inverse limit spaces of one unimodal bonding map T. In a forthcoming paper [2] we address the last two questions in greater generality.

Inverse limit spaces do not only provide interesting examples in Continuum Theory but they also offer a topological insight in the attractors of some dynamical systems. One of the simplest examples of an attractor with an interesting topological structure is the Knaster continuum $\varprojlim(I, T_2)$, which is the attractor of the Smale's horseshoe map, see [5] for details.
Furthermore, inverse limit spaces can be used as a model to construct attractors of some planar diffeomorphisms, see for example [56, 57, 9]. Of special interest to us is the family of planar diffeomorphisms known as the Hénon map family, defined by $H_{a,b}(x, y) := (1-ax^2+by, x)$ for some $(x, y) \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$. For a large open disk $U \subset \mathbb{R}^2$ and specific set of parameters a

and b, the set $\cap_{i\in\mathbb{N}}^{\infty} H_{a,b}(U)$ is the global attractor of the map $H_{a,b}$. This map was proposed by the astronomer and mathematician Michel Hénon in [32] as the simplest example of a planar map that could exhibit a "strange" attractor, which was confirmed by Benedicks & Carlson in [13]. In [9] Barge & Holte proved that every inverse limit space of a unimodal map T with a periodic critical point c is homeomorphic to some "strange" attractor from the parametrized Hénon map family.

Say that M is a metric space and $f : M \to M$ is a continuous map. On the inverse limit space $\varprojlim(M, f)$ we have an action of the natural shift homeomorphism σ defined as in (1.1). From a dynamical system perspective, the self-homeomorphism $\sigma : \varprojlim(M, f) \to \varprojlim(M, f)$ is the dynamically minimal extension of the map f to a homeomorphism. Assume that the space $\varprojlim(M, f)$ can be embedded in the plane. If σ can be continuously extended from $\varprojlim(M, f)$ to the whole plane this gives rise to a dynamical system with the attractor exactly $\varprojlim(M, f)$. In such a case we obtain a planar homeomorphism σ^{-1} such that its attracting set is exactly the set $\varprojlim(M, f)$, see [12] for details. Boyland, de Carvalho & Hall studied the construction of attractors of planar homeomorphisms using inverse limit spaces in [17]. Because X can be embedded in the plane [15], a natural question that arises in this context and the second question posed on the Continuum Theory and Dynamical Systems Workshop in Vienna by Philip Boyland is:

> Do there exist planar embeddings of X non-equivalent to the standard ones so that the homeomorphism $\sigma : X \to X \subset \mathbb{R}^2$ can be continuously extended to the whole plane?

In this thesis we prove that shift homeomorphism is not extendable to the plane for all embeddings of X that we construct except for the embeddings equivalent to the standard ones. Therefore, the constructed planar embeddings which are not equivalent to the standard embeddings cannot be used for the construction of attractors of planar homeomorphism of the map σ^{-1}. Thus we partially address the last question by Boyland, but in full generality the answer still remains outstanding and we address the problem in a greater generality in a forthcoming paper [2].

Let us provide an outline of the thesis. In Chapter 2 we first review some basic notation and definitions that are needed throughout the thesis. Then we provide an introduction to the kneading theory for unimodal maps. We also recall some preliminary results from the papers by Brucks & Diamond [23] and Bruin [24], which will be the base for all the chapters afterwords.

In Chapter 3 we explicitly construct a class of embeddings of X (as also described in [4]). With the use of symbolic dynamics we provide a representation of the space X in the plane in Section 3.1. We prove in Section 3.2 that such a representation indeed yields a planar embedding of X so that a pre-fixed point from X is accessible. We denote the class of embeddings of tent inverse limit spaces X and their cores X' by $\mathcal{E}$ and refer to them as $\mathcal{E}$-embeddings. Intuitively, every $\mathcal{E}$-embedding of X is represented as a union of uncountably many horizontal segments (called basic arcs) which are aligned along vertically embedded Cantor set with prescribed identifications between some endpoints of basic arcs, see Figures 3.4 and 3.5. An $\mathcal{E}$-embedding of X is then uniquely determined by the left infinite itinerary $L = \dots l_{-2}l_{-1}$, which is a symbolic description of the largest basic arcs among all basic arcs.

In Chapter 4 we study subcontinua of inverse limit spaces X; especially we dedicate our attention to the study of arc-components of X. In particular, we provide the answers to some of the problems asked by Brian Raines in the paper by Tom W. Ingram [37] about open problems in the field of Inverse limit spaces and Dynamical Systems.

In Chapter 5 we restrict our study to the inverse limit spaces of tent maps X. We start the chapter by giving a symbolic characterization of arc-components in X, generalizing the result from the paper by Brucks& Diamond [23]. In Section 5.2, we characterize the possible sets of accessible points in an arc-component of any indecomposable plane non-separating continuum K. In Section 5.3 we briefly introduce Carathéodory's prime end theory and discuss the existence of fourth kind prime ends in special cases of tent map inverse limits. In Section 5.4, we begin our study of embeddings $\mathcal{E}$. We introduce the notion of cylinders of basic arcs and techniques to explicitly calculate their extrema. We show that two $\mathcal{E}$-embeddings of the same space X are equivalent when they are determined by eventually the same left infinite tail L. Given an $\mathcal{E}$-embedding of X, we prove that the arc-component of the top basic arc with symbolic description L (throughout the thesis this arc-component is denoted by $\mathcal{U}_L$) is fully accessible, if the top basic arc is not a spiral point (see Definition 9 and Figure 5.1). However, we also show that $\mathcal{U}_L$ is not necessarily the unique fully accessible arc-component. In the same section we briefly discuss $\mathcal{E}$-embeddings of decomposable continuum X and characterize the set of accessible points up to two points on the corresponding circle of prime ends. From Section 5.5 onwards we study $\mathcal{E}$-embeddings of indecomposable tent cores inverse limit spaces X'. In Section 5.5 we give sufficient conditions on itineraries of L and kneading sequences ν associated with X' so that the $\mathcal{E}$-embeddings of X' allow more than one fully accessible arc-component and give some interesting examples of such embeddings.

We say that $x \in X$ is a *folding point* if for every $\varepsilon > 0$ there exists a neighbourhood U_ε of x which is not homeomorphic to the $C \times (0,1)$, where C is the Cantor set. A point $x \in X$ is called an *endpoint*, if for every two subcontinua $X_1, X_2 \subset X$ such that $x \in X_1 \cap X_2$, either $X_1 \subset X_2$ or $X_2 \subset X_1$. Note that endpoints are also folding points. In Section 5.6 we characterize accessible folding points of $\mathcal{E}$-embeddings when the critical orbit of the tent map is finite. Surprisingly, no endpoints will be accessible in any $\mathcal{E}$-embedding of X' with the exception of a standard Brucks-Diamond embedding. Another surprising phenomenon is the occurrence of *Type 3* folding points (see Definition 23 and Figure 5.13) when the orbit of the third iterate of the critical point is periodic but the critical point itself is not periodic. Such a phenomenon does not occur in the standard embeddings of any tent map inverse limit space. In Section 5.7 we study special class of $\mathcal{E}$-embeddings of X'. We explicitly show that every X' can be embedded with at least two non-degenerate fully accessible arc-components. In a finite orbit, case when we have exactly two fully accessible arc-components we show that there exists an embedding of X' with exactly two simple dense canals.

Chapter 6 starts with a proof that for every $\mathcal{E}$-embedding except for the standard embeddings, the natural shift homeomorphism cannot be extended from the $\mathcal{E}$-embedding of X' to the whole plane. Showing that, we answer on a question posed by Boyland, de Carvalho and Hall in the paper [19] on the page 4. From this point onwards inverse limits that are attractors of orientation preserving and orientation reversing planar homeomorphisms are studied. We conclude the chapter with the complete characterization of sets of accessible points (and thus also the prime end structure of the corresponding circle of prime ends) of the standard two embeddings: Bruin's embedding of X' (Section 6.2) and the Brucks-Diamond embedding of X' (Section 6.3) using symbolic dynamics. In Section 6.2 we show that for Bruin's embedding of X' there is exactly one fully accessible non-degenerate arc-component and no other point from the embedding of X' is accessible, if X' is different from the Knaster continuum. We show that if X' is not the Knaster continuum, then Bruin's embedding of X' has exactly one simple dense canal. In Section 6.3 we explicitly calculate the extrema of cylinders and neighbourhoods of folding points in the second standard embedding and obtain equivalent results as obtained recently by Boyland, de Carvalho and Hall in [19]. Moreover, since the symbolic description makes it possible to distinguish endpoints within the set of folding points, our results extend the classification given in [19].

Chapter 2

Preliminaries

By $\mathbb{N}$ we denote the set of natural numbers and let $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. The *Hilbert cube* is the space $[0,1]^{\mathbb{N}_0}$ equipped with the product metric

$$d(x,y) := \sum_{i \leq 0} 2^i |\pi_i(x) - \pi_i(y)|,$$

where $\pi_i \colon [0,1]^{\mathbb{N}_0} \to [0,1]$ denote the *coordinate projections* for $i \leq 0$ and $x, y \in [0,1]^{\mathbb{N}_0}$.

Let $T : [0,1] \to [0,1]$ be a unimodal map fixing 0 and let c denote the *critical point* of T. We are going to be especially interested in the following parametrized family of maps. The *tent map family* $T_s : [0,1] \to [0,1]$ is defined by $T_s(z) := \min\{sz, s(1-z)\}$ where $z \in [0,1]$ and $s \in (0,2]$. Here $c = \frac{1}{2}$ is the critical point of the map T_s. The *inverse limit space with the bonding map* T is a subspace of the Hilbert cube defined by

$$X := \varprojlim([0,1], T) = \{x \in [0,1]^{\mathbb{N}_0} : T(\pi_i(x)) = \pi_{i+1}(x),\ i \leq 0\}.$$

The space X is a *continuum*, *i.e.,* compact and connected metric space. Define the *shift homeomorphism* as $\sigma \colon X \to X$, $\pi_i(\sigma(x)) := T(\pi_i(x))$ for every $i \leq 0$ and $x \in X$.

The space obtained by restricting the bonding map T to its forward invariant dynamical core $[T^2(c), T(c)]$ is called the *core of the inverse limit space* X and will be denoted by X':

$$X' := \varprojlim([T^2(c), T(c)], T|_{[T^2(c),T(c)]}).$$

Recall that a continuum is *indecomposable*, if it cannot be expressed as a union of two of its proper subcontinua.

In the construction of planar embeddings of spaces X we recall a well-known symbolic description of unimodal inverse limit spaces introduced by Brucks & Diamond in [23]. The space X will be represented by the quotient space $\Sigma_{adm}/\sim$, where $\Sigma_{adm} \subseteq \{0,1\}^{\mathbb{Z}}$ is equipped with the product topology. We first need to recall the kneading theory for unimodal maps. To every $z \in [0,1]$ we assign its forward *itinerary*:

$$\nu(z) := \nu_0(z)\nu_1(z)\dots,$$

where

$$\nu_i(z) := \begin{cases} 0, & T^i(z) \in [0,c), \\ *, & T^i(z) = c, \\ 1, & T^i(z) \in (c,1]. \end{cases}$$

Note that if $\nu_i(z) = *$ for some $i \in \mathbb{N}_0$, then $\nu_{i+1}(z)\nu_{i+2}(z)\dots = \nu(T(c))$. The sequence $\nu := \nu(T(c))$ is called the *kneading sequence of* T and is denoted by $\nu = c_1c_2\dots$, where $c_i := \nu_i(T(c)) \in \{0,*,1\}$ for every $i \in \mathbb{N}$. Observe that if $*$ appears in the kneading sequence, then c is periodic under T, *i.e.,* there exists $n > 0$ such that $T^n(c) = c$ and the kneading sequence is of the form $\nu = (c_1 \dots c_{n-1}*)^\infty$. In this case we adjust the kneading sequence by taking the smallest of $(c_1 \dots c_{n-1}0)^\infty$ and $(c_1 \dots c_{n-1}1)^\infty$ in the parity-lexicographical ordering defined below.

By $\#_1(a_1 \dots a_n)$ we denote the number of ones in a finite word $a_1 \dots a_n \in \{0,1\}^n$; it can be either even or odd.

Choose $\overrightarrow{t} = t_0t_1 \dots \in \{0,1\}^{\mathbb{N}_0}$ and $\overrightarrow{s} = s_0s_1 \dots \in \{0,1\}^{\mathbb{N}_0}$ such that $\overrightarrow{s} \neq \overrightarrow{t}$. Take the smallest $k \in \mathbb{N}_0$ such that $s_k \neq t_k$. Then the *parity-lexicographical ordering* is defined by

$$\overrightarrow{s} \prec \overrightarrow{t} \Leftrightarrow \begin{cases} s_k < t_k \text{ and } \#_1(s_0 \dots s_{k-1}) \text{ is even, or} \\ s_k > t_k \text{ and } \#_1(s_0 \dots s_{k-1}) \text{ is odd.} \end{cases}$$

This ordering is also well-defined on $\{0,*,1\}^{\mathbb{N}_0}$ once we define $0 < * < 1$.
Thus if $(c_1 \dots c_{n-1}0)^\infty \prec (c_1 \dots c_{n-1}1)^\infty$ we modify $\nu = (c_1 \dots c_{n-1}0)^\infty$, otherwise $\nu = (c_1 \dots c_{n-1}1)^\infty$.

Example. If c is periodic of period 3 then the kneading sequence for T is $\nu = (10*)^\infty$. Since $101 \prec 100$ in parity-lexicographical ordering, we modify $\nu = (101)^\infty$.

In the same way we modify itinerary of an arbitrary point $z \in [0,1]$. If $\nu_i(z) = *$ and i is the smallest positive integer with this property then we replace $\nu_{i+1}(z)\nu_{i+2}(z)\dots$ with the modified kneading sequence. Thus $*$ can appear only once in the modified itinerary of an arbitrary point $z \in [0,1]$.

From now onwards we assume that the itineraries of points from $[0,1]$ are modified.

It is a well-known fact (see [47]) that a kneading sequence completely characterizes the dynamics of unimodal map in the sense of the following proposition:

Proposition 1. *If a sequence $s_0s_1\dots \in \{0,*,1\}^{\mathbb{N}_0}$ is the itinerary of a point $z \in [T^2(c), T(c)]$, then*

$$\nu(T^2(c)) \preceq s_k s_{k+1}\dots \preceq \nu = \nu(T(c)), \text{ for every } k \in \mathbb{N}_0. \tag{2.1}$$

Conversely, assume $s_0s_1\dots \in \{0,*,1\}^{\mathbb{N}}$ satisfies (2.1). If there exists $j \in \mathbb{N}_0$ such that $s_{j+1}s_{j+2}\dots = \nu$, and j is minimal with this property, assume additionally that $s_j = *$. Then $s_0s_1\dots$ is realized as the itinerary of some $z \in [T^2(c), T(c)]$.

Definition 1. *We say that a sequence $s_0s_1\dots \in \{0,*,1\}^{\mathbb{N}_0}$ is* admissible*, if it is realized as the itinerary of some $z \in [0,1]$.*

Remark 1. *Note that the Proposition 1 gives conditions on admissible itineraries of points $z \in [T^2(c), T(c)]$. For points $w \in [0, T^2(c)) \cup (T(c), 1]$ admissible itineraries are exactly 0^∞, 10^∞, $0^j s_0s_1\dots$, $10^{j-1}s_0s_1\dots$ where $s_0s_1\dots \in \{0,*,1\}^{\mathbb{N}_0}$ is the itinerary of the point $T^j(w)$ which satisfies the conditions of Proposition 1 for $j := \min\{i \in \mathbb{N}_0 : T^i(w) \in [T^2(c), T(c)]\}$.*

Next we show how to expand the above construction to inverse limit spaces X. Take $x \in X$. Define the itinerary of x as a two-sided infinite sequence

$$\overleftarrow{x}\,.\,\overrightarrow{x} = \bar{x} := \dots x_{-2}x_{-1}.x_0x_1\dots \in \{0,*,1\}^{\mathbb{Z}},$$

where $x_0x_1\dots = \nu(\pi_0(x))$ and

$$x_i = \begin{cases} 0, & \pi_i(x) \in [0,c), \\ *, & \pi_i(x) = c, \\ 1, & \pi_i(x) \in (c,1], \end{cases}$$

for all $i < 0$.

We make the same modifications as above. If $*$ appears for the first time at x_k for some $k \in \mathbb{Z}$, then $x_{k+1}x_{k+2}\ldots = \nu$. If there is no such minimal k, then the kneading sequence is periodic with a period $n \in \mathbb{N}$, $\nu = (c_1c_2\ldots c_{n-1}*)^\infty$ and the itinerary of x is of the form $(c_1\ldots c_{n-1}*)^{\mathbb{Z}}$. Replace $(c_1\ldots c_{n-1}*)^{\mathbb{Z}}$ with the modified itinerary $(c_1c_2\ldots c_{n-1}c_n)^{\mathbb{Z}}$, where $\nu = (c_1\ldots\, c_{n-1}c_n)^\infty$. In this way $*$ can appear at most once in every itinerary. Now we are ready to identify the inverse limit space with a quotient of a space of two-sided sequences consisting of two symbols.

Let $\Sigma := \{0,1\}^{\mathbb{Z}}$ be the space of two-sided sequences equipped with the metric

$$d((s_i)_{i\in\mathbb{Z}}, (t_i)_{i\in\mathbb{Z}}) := \sum_{i\in\mathbb{Z}} \frac{|s_i - t_i|}{2^{|i|}},$$

for $(s_i)_{i\in\mathbb{Z}}, (t_i)_{i\in\mathbb{Z}} \in \Sigma$. We define the shift homeomorphism $\sigma_\Sigma \colon \Sigma \to \Sigma$ as

$$\sigma_\Sigma(\ldots s_{-2}s_{-1}.s_0s_1\ldots) := \ldots s_{-2}.s_{-1}s_0s_1\ldots$$

By $\Sigma_{adm} \subseteq \Sigma$ we denote all $\bar{s} \in \Sigma$ such that either

(a) $s_ks_{k+1}\ldots$ is admissible for every $k \in \mathbb{Z}$, or

(b) there exists $k \in \mathbb{Z}$ such that $s_{k+1}s_{k+2}\ldots = \nu$ and $s_{k-i}\ldots s_{k-1}*s_{k+1}s_{k+2}\ldots$ is admissible for every $i \in \mathbb{N}$.

We abuse the notation and call the two-sided sequences in Σ_{adm} also *admissible*.

Let us define an equivalence relation on the space Σ_{adm}. For sequences $\bar{s} = (s_i)_{i\in\mathbb{Z}}$, $\bar{t} = (t_i)_{i\in\mathbb{Z}} \in \Sigma_{adm}$ we define the relation

$$\bar{s} \sim \bar{t} \Leftrightarrow \begin{cases} \text{either } s_i = t_i \text{ for every } i \in \mathbb{Z}, \\ \text{or if there exists } k \in \mathbb{Z} \text{ such that } s_i = t_i \text{ for all } i \neq k \text{ but } s_k \neq t_k \\ \text{and } s_{k+1}s_{k+2}\ldots = t_{k+1}t_{k+2}\ldots = \nu. \end{cases}$$

It is not difficult to see that this is indeed an equivalence relation on the space Σ_{adm}. Furthermore, every itinerary is identified with at most one different itinerary and the quotient space $\Sigma_{adm}/{\sim}$ of Σ_{adm} is well defined. It was also shown that $\Sigma_{adm}/{\sim}$ is homeomorphic to

X. So in order to embed X in the plane it is enough to embed $\Sigma_{adm}/\sim$ in the plane. For all observations in this paragraph we refer to the paper [23] of Brucks & Diamond (lemmas 2.2-2.4 and Theorem 2.5).

An *arc* is a homeomorphic image of an interval $[a,b] \subset \mathbb{R}$. A key fact for constructing embeddings is noting that X is the union of *basic arcs* defined below. Let $\overleftarrow{s} = \dots s_{-2}s_{-1} \in \{0,1\}^{\mathbb{N}}$ be an admissible left-infinite sequence (*i.e.,* every finite subword is admissible). The subset of X:

$$A(\overleftarrow{s}) := \{x \in X : x_i = s_i, \forall i < 0\}$$

is called a *basic arc*. Note that $\pi_0 : A(\overleftarrow{s}) \to [0,1]$ is injective. In [24, Lemma 1] it was observed that $A(\overleftarrow{s})$ is indeed either an arc or degenerate.

Remark 2. *Let $\overleftarrow{s} = \dots s_{-2}s_{-1} \in \{0,1\}^{\infty}$ be an admissible left-infinite sequence. There is a one-to-one correspondence between sequences $\overleftarrow{s}$ and basic arcs $A(\overleftarrow{s})$. When it is clear from the context that we refer to the basic arc $A(\overleftarrow{s})$ we abbreviate notation and write only $\overleftarrow{s}$.*

For every basic arc we define two quantities as follows:

$$\begin{aligned}
\tau_L(\overleftarrow{s}) &:= \sup\{n > 1 : s_{-(n-1)} \dots s_{-1} = c_1c_2 \dots c_{n-1}, \#_1(c_1 \dots c_{n-1}) \text{ odd}\},\\
\tau_R(\overleftarrow{s}) &:= \sup\{n \geq 1 : s_{-(n-1)} \dots s_{-1} = c_1c_2 \dots c_{n-1}, \#_1(c_1 \dots c_{n-1}) \text{ even}\}.
\end{aligned}$$

Remark 3. *For $n = 1$, $c_1c_2 \dots c_{n-1} = \emptyset$ and $\#_1(\emptyset)$ is even. Thus $\tau_R(\overleftarrow{s}) = 1$ if and only if $s_{-(n-1)} \dots s_{-1} \neq c_1c_2 \dots c_{n-1}$ for all $n > 1$.*

These definitions first appeared in [24] in order to study the number of endpoints of inverse limit spaces X. We now adapt two lemmas from [24] that we will use later in the thesis.

Lemma 1. *([24], Lemma 2) Let $\overleftarrow{s} \in \{0,1\}^{\mathbb{N}}$ be admissible such that $\tau_L(\overleftarrow{s}), \tau_R(\overleftarrow{s}) < \infty$. Then*

$$\pi_0(A(\overleftarrow{s})) = [T^{\tau_L(\overleftarrow{s})}(c), T^{\tau_R(\overleftarrow{s})}(c)].$$

If $\overleftarrow{t} \in \{0,1\}^{\mathbb{N}}$ is another admissible left-infinite sequence such that $s_i = t_i$ for all $i < 0$ except for $i = -\tau_R(\overleftarrow{s}) = -\tau_R(\overleftarrow{t})$ (or $i = -\tau_L(\overleftarrow{s}) = -\tau_L(\overleftarrow{t})$), then $A(\overleftarrow{s})$ and $A(\overleftarrow{t})$ have a common boundary point.

Lemma 2. *([24], Lemma 3) If* $\overleftarrow{s} \in \{0,1\}^{\mathbb{N}}$ *is admissible, then*

$$\sup \pi_0(A(\overleftarrow{s})) = \inf\{T^n(c) : s_{-(n-1)} \dots s_{-1} = c_1 \dots c_{n-1}, n \geq 1, \#_1(c_1 \dots c_{n-1}) \text{ even}\},$$
$$\inf \pi_0(A(\overleftarrow{s})) = \sup\{T^n(c) : s_{-(n-1)} \dots s_{-1} = c_1 \dots c_{n-1}, n \geq 1, \#_1(c_1 \dots c_{n-1}) \text{ odd}\}.$$

Example. Take the unimodal map with the kneading sequence $\nu = (101)^\infty$. Then $\overleftarrow{s} = (011)^\infty 010.$ and $\overleftarrow{t} = (011)^\infty 110.$ are admissible, $\tau_L(\overleftarrow{s}) = \tau_L(\overleftarrow{t}) = 3$, $\tau_R(\overleftarrow{s}) = \tau_R(\overleftarrow{t}) = 1$ and $s_i = t_i$ for all $i < 0$ except for $i = -3 = -\tau_L(\overleftarrow{s}) = -\tau_L(\overleftarrow{t})$. By Lemma 2, $\pi_0(A(\overleftarrow{s})) = \pi_0(A(\overleftarrow{t})) = [T^3(c), T(c)]$, and by Lemma 1, $A(\overleftarrow{s})$ and $A(\overleftarrow{t})$ have a common boundary point which is projected to $T^3(c)$, see Figure 2.1. Note that in this example both τ_L and τ_R agree for $\overleftarrow{s}$ and $\overleftarrow{t}$, which need not be the case in general.

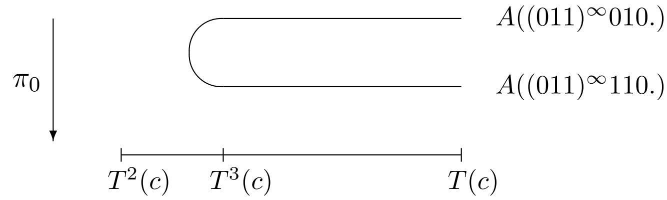

Figure 2.1: Example of two basic arcs having a boundary point in common.

Neighbourhoods of points from X locally resemble Cantor set of arcs. However, this is not always the case. Through the rest of the thesis we will often work with points from X so that every of their neighbourhood is not Cantor set of arcs. We define them below and give symbolic representation of them.

Let us recall the definitions of folding points and endpoints. We say that $x \in X$ is a *folding point* if for every $\varepsilon > 0$ there exists a neighbourhood U_ε of x, which is not homeomorphic to the $C \times (0,1)$, where C is the Cantor set. A point $x \in X$ is called an *endpoint* if for every two subcontinua $X_1, X_2 \subset X$ such that $x \in X_1 \cap X_2$, either $X_1 \subset X_2$ or $X_2 \subset X_1$. Note that endpoints are also folding points.

Let $\omega(c)$ denote the set of accumulation points of the forward orbit of the critical point c by the map T, *i.e.,* :

$$\omega(c) = \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} \{T^j(c)\} \tag{2.2}$$

Proposition 2. *[52, Theorem 2.2] A point $x \in X$ is a folding point if and only if $\pi_n(x) \in \omega(c)$ for every $n \in \mathbb{N}$.*

We have the following symbolic characterization of endpoints in X.

Proposition 3. *[24, Proposition 2] A point $x \in X$ such that $\pi_i(x) \neq c$ for every $i < 0$ is an endpoint of X if and only if $\tau_L(\overleftarrow{x}) = \infty$ and $\pi_0(x) = \inf \pi_0(A(\overleftarrow{x}))$ or $\tau_R(\overleftarrow{x}) = \infty$ and $\pi_0(x) = \sup \pi_0(A(\overleftarrow{x}))$.*

If $\pi_i(x) = c$ for some $i < 0$, then x is an endpoint of X' if and only if $\sigma^i(x)$ is an endpoint. We can apply Proposition 3 to $\sigma^i(x)$ in this case.

Chapter 3

Uncountably many non-equivalent embeddings of X

3.1 Representation of X in the plane

This section is the first step towards embedding X in the plane so that an arbitrary point $x \in X$ becomes accessible. Let $x \in A(\dots l_{-2}l_{-1})$. We present the following ordering on $\{0,1\}^{\mathbb{N}}$ depending on some $L = \dots l_{-2}l_{-1}$ and we work with this ordering from now onwards.

Definition 2. *Let $\overleftarrow{s}, \overleftarrow{t} \in \{0,1\}^{\mathbb{N}}$ and let $k \in \mathbb{N}$ be the smallest natural number such that $s_{-k} \neq t_{-k}$. Then*

$$\overleftarrow{s} \prec_L \overleftarrow{t} \Leftrightarrow \begin{cases} t_{-k} = l_{-k} \text{ and } \#_1(s_{-(k-1)} \dots s_{-1}) - \#_1(l_{-(k-1)} \dots l_{-1}) \text{ even, or} \\ s_{-k} = l_{-k} \text{ and } \#_1(s_{-(k-1)} \dots s_{-1}) - \#_1(l_{-(k-1)} \dots l_{-1}) \text{ odd.} \end{cases}$$

Note that such ordering is well-defined and the left infinite tail L is the largest sequence.

Lemma 3. *Assume $\overleftarrow{s} \prec_L \overleftarrow{u} \prec_L \overleftarrow{t}$ and assume that $s_{-n} \dots s_{-1} = t_{-n} \dots t_{-1}$. Then also $u_{-n} \dots u_{-1} = s_{-n} \dots s_{-1} = t_{-n} \dots t_{-1}$.*

Proof. If $n = 1$ the statement follows easily so let us assume that $n \geq 2$. Assume that there exists $k < n$ such that $u_{-k} \neq s_{-k}$ and take k the smallest natural number with this property.

Assume without loss of generality that $(-1)^{\#_1(s_{-(k-1)}\dots s_{-1})} = (-1)^{\#_1(l_{-(k-1)}\dots l_{-1})}$ (the proof follows similarly when the parities are different). Since $\overleftarrow{s} \prec_L \overleftarrow{u}$ it follows that $u_{-k} = l_{-k}$. Also, $\overleftarrow{u} \prec_L \overleftarrow{t}$ gives $t_{-k} = l_{-k}$. Since $u_{-k} \neq t_{-k}$, we get a contradiction. $\square$

Let $C \subset [0,1]$ be the middle-third Cantor set,

$$C := [0,1] \setminus \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} (\frac{3k+1}{3^m}, \frac{3k+2}{3^m}).$$

Points in C are coded by the left-infinite sequences of zeros and ones. We embed basic arcs in the plane as horizontal lines along the Cantor set and then join corresponding endpoints with semi-circles as in Figure 2.1. The ordering has to be defined in a way that semi-circles neither cross horizontal lines nor each other.

Example. For $L = 1^\infty$, points in C are coded as in Figure 3.1 (a). Note that this is the same ordering as in the paper by Bruin [24]. The ordering obtained by $L = 0^\infty 1$ is the ordering from the paper by Brucks & Diamond [23]. In Figure 3.1 (b) points in Cantor set are coded with respect to $L = \dots 101$.

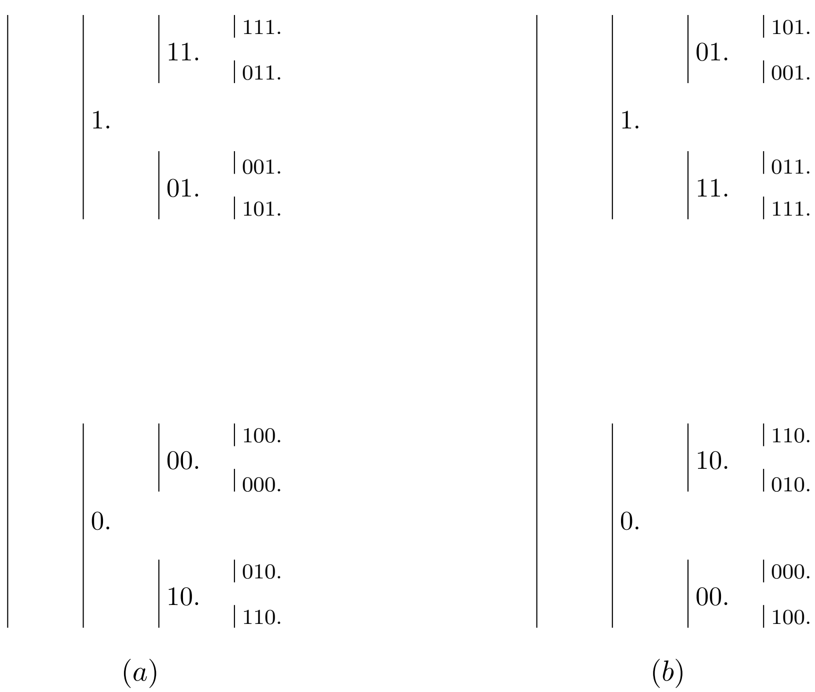

Figure 3.1: Coding the Cantor set with respect to (a) $L = \dots 111$ and (b) $L = \dots 101$.

From now onwards, we assume that $\overleftarrow{s} \in \{0,1\}^{\mathbb{N}}$ is an admissible left-infinite sequence. Define $\psi_L : \{0,1\}^{\mathbb{N}} \to C$ as

$$\psi_L(\overleftarrow{s}) := \sum_{i=1}^{\infty} (-1)^{\#_1(l_{-i}\dots l_{-1}) - \#_1(s_{-i}\dots s_{-1})} 3^{-i} + \frac{1}{2},$$

and we let $C_{adm} := \{\psi_L(\overleftarrow{s}) : \overleftarrow{s}$ admissible left-infinite sequence$\}$ be the subset of "admissible vertical coordinates". Note that $\psi_L(L) = 1$ is the largest point in C_{adm}.

From now onwards let d_e denote the Euclidean distance in $\mathbb{R}^2$.

Remark 4. *Note that if $\overleftarrow{s}, \overleftarrow{t} \in \{0,1\}^{\mathbb{N}}$ are such that $s_{-n} \dots s_{-1} = t_{-n} \dots t_{-1}$, then it holds that $d_e(\psi_L(\overleftarrow{s}), \psi_L(\overleftarrow{t})) \leq 3^{-n}$. If $s_{-n} \neq t_{-n}$, then $d_e(\psi_L(\overleftarrow{s}), \psi_L(\overleftarrow{t})) \geq 3^{-n}$.*

Now we represent X as the quotient space of the subset of $[0,1] \times C_{adm}$. To every point $x \in X$ we will assign either a point or two points in $[0,1] \times C_{adm}$ by the rule (3.1) below. Recall that $\dots x_{-3}x_{-2}x_{-1} = \overleftarrow{x}$ denotes the left-infinite symbolic code of x. Let $\varphi : X \to [0,1] \times C_{adm}$ be defined in the following way:

$$\varphi(x) := \begin{cases} (\pi_0(x), \psi_L((x_i)_{i<0})), & \text{if } x_i \neq * \text{ for every } i < 0, \\ (\pi_0(x), p) \cup (\pi_0(x), q), & \text{if } x_i = * \text{ for some } i < 0, \end{cases} \tag{3.1}$$

where

$$\begin{cases} p = \psi_L(\dots x_{-(i+1)} 0 x_{-(i-1)} \dots x_{-1}), \\ q = \psi_L(\dots x_{-(i+1)} 1 x_{-(i-1)} \dots x_{-1}). \end{cases}$$

Set $Y := \varphi(X) \subset [0,1] \times C_{adm}$. The next step is to identify points from Y in the same way as they are identified in the symbolic representation of X. For $a, b \in Y$:

$$a \sim b \text{ if there exists } x \in X \text{ such that } a, b \in \varphi(x).$$

If $a \neq b \sim a$ we write $\tilde{a} := b$. If $\tilde{a} = b$ and $x \in X$ is such that $a, b \in \varphi(x)$ and $x_{-i} = *$ we say that a and b are *joined at level* i.

Note that $\varphi \colon X \to Y/\!\sim$ is a well-defined map. Equip Y with the Euclidean topology and $Y/\!\sim$ with the standard quotient topology. Let $\pi_C : [0,1] \times C \to C$ and $\pi_{[0,1]} \colon [0,1] \times C \to [0,1]$ denote the *natural projections*. The next proposition is an analogue of Proposition 4 from [24]. We prove it here for the sake of completeness.

Proposition 4. *The map $\varphi\colon X \to Y/\!\sim$ is a homeomorphism.*

Proof. We first prove that $Y/\!\sim$ is a Hausdorff space and because X is compact it is enough to check that φ is a continuous bijection to obtain a homeomorphism between X and $Y/\!\sim$, see *e.g.* Theorem 26.6. in [49].

Take $y \neq y' \in Y$ such that $y \neq \tilde{y}'$. First assume that $|\pi_{[0,1]}(y) - \pi_{[0,1]}(y')| = 0$. Let $\delta := \min\{|\pi_C(y) - \pi_C(y')|, |\pi_C(\tilde{y}) - \pi_C(y')|\}$. Then take $\{w : |\pi_C(y) - \pi_C(w)| \text{ or } |\pi_C(\tilde{y}) - \pi_C(w)| < \delta/3\}$ and $\{w : |\pi_C(y') - \pi_C(w)| \text{ or } |\pi_C(\tilde{y}') - \pi_C(w)| < \delta/3\}$ for open neighbourhoods of y and y' respectively and they are disjoint. Now assume that $\varepsilon := |\pi_{[0,1]}(y) - \pi_{[0,1]}(y')| > 0$. Then $\{w : |\pi_{[0,1]}(y) - \pi_{[0,1]}(w)| < \varepsilon/3\}$ and $\{w : |\pi_{[0,1]}(y') - \pi_{[0,1]}(w)| < \varepsilon/3\}$ are disjoint open neighbourhoods for y and y' respectively, so $Y/\!\sim$ is indeed a Hausdorff space.

Now we prove that φ is continuous. It is enough to prove that for $a \in X$ and a sequence $(x^n)_{n\in\mathbb{N}} \subset X$ such that $\lim_{n\to\infty} x^n = a$ it holds that $\lim_{n\to\infty} \varphi(x^n) = \varphi(a)$. Assume that $\lim_{n\to\infty} x^n = a$. Thus for every $M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ it follows that $x^n_{-M} \dots x^n_M = a_{-M} \dots a_M$. We need to show that for every open $\varphi(a) \in U \subset Y/\!\sim$ there exists $N' \in \mathbb{N}$ such that for every $n \geq N'$ it holds that $\varphi(x^n) \in U$. Let us fix an open set $U \ni \varphi(a)$. If for $x \in X$ there exists $i \in \mathbb{N}$ such that $x_{-i} = *$ then we set $\varphi(x) = \varphi'(x) \cup \varphi''(x)$ where $\varphi'(x) := (\pi_0(x), \psi_L(\dots x_{-(i+1)} 0 x_{-(i-1)} \dots x_{-2}x_{-1}))$ and $\varphi''(x) := (\pi_0(x), \psi_L(\dots x_{-(i+1)} 1 x_{-(i-1)} \dots x_{-2}x_{-1}))$.

Case I: For every $i \in \mathbb{N}$, $a_{-i} \neq *$. If there exists $K \in \mathbb{N}$ such that for every $n \geq K$ it follows that $x^n_{-j} \neq *$ for every $j \in \mathbb{N}$, then there is $N' \geq K$ such that $\varphi(x^n) \in U$ for every $n \geq N'$. Now assume that there exists an increasing sequence $(n_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ such that $x^{n_i}_{-j} = *$ for some $j \in \mathbb{N}$. Then there exist open sets $U_1^{n_i}, U_2^{n_i} \subset Y$ such that $\varphi'(x^{n_i}) \in U_1^{n_i}$ and $\varphi''(x^{n_i}) \in U_2^{n_i}$ and $\varphi^{-1}(U) = U_1^{n_i} \cup U_2^{n_i}$ for every $i \in \mathbb{N}$. Because $x^n \to a$ as $n \to \infty$, by the definition of φ it follows that $\varphi'(x^{n_i}) \to \varphi(a)$ and $\varphi''(x^{n_i}) \to \varphi(a)$ as $i \to \infty$. Thus we again conclude that there exists $N' \in \mathbb{N}$ such that for every $n \geq N'$ it follows that $\varphi(x^n) \in U$.

Case II: Let $K \in \mathbb{N}$ be such that $a_K = *$ and thus $\varphi(a) = \varphi'(a) \cup \varphi''(a)$. Take $M > K$ so that $a_{-M} \dots a_M = x^n_{-M} \dots x^n_{-K} \dots x^n_K \dots x^n_M$ for every $n \geq N$, and so $\varphi(x^n) = \varphi'(x^n) \cup \varphi''(x^n)$. Thus there exist open sets $U_1, U_2 \subset Y$ such that $\varphi'(a) \in U_1$, $\varphi''(a) \in U_2$ and $\varphi^{-1}(U) = U_1 \cup U_2$.

It follows that there exists $N' > N$ such that for every $n > N'$ it holds that $\varphi'(x^n) \in U_1$ and $\varphi''(x^n) \in U_2$ and thus $\varphi(x^n) \in U$. $\square$

Now we are ready to represent X in the plane. This is still not an embedding but it is the first step towards it. Connect identified points in $[0,1] \times C_{adm}$ with semi-circles. Suppose $y \neq y' \in Y$ are joined at level n. By Lemma 1, points y and y' are both endpoints of basic arcs in $[0,1] \times C_{adm}$ and are both right or left endpoints. If $\#_1(c_1 \dots c_{n-1})$ is even (odd), y and y' are right (left) endpoints and we join them with a semi-circle on the right (left), see Figure 2.1.

Proposition 5. *Every semi-circle defined above crosses neither Y nor another semi-circle.*

Proof. **Case I:** Assume that there is a semi-circle oriented to the right which intersects an arc Q in Y. (See Figure 3.2.)

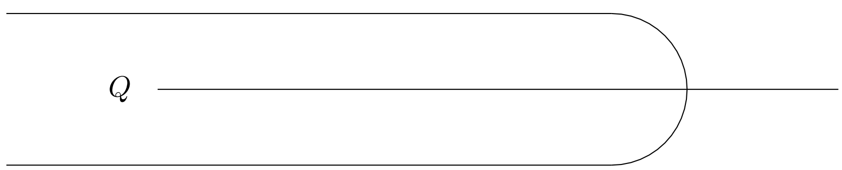

Figure 3.2: Case I in the proof of Proposition 5.

Translated to symbolics, this means that there exist $n \in \mathbb{N}$ and $\overleftarrow{s} \prec_L \overleftarrow{u} \prec_L \overleftarrow{t}$ such that $s_{-(n-1)} \dots s_{-1} = t_{-(n-1)} \dots t_{-1} = c_1 \dots c_{n-1}$, $s_{-n} \neq t_{-n}$ and $\#_1(c_1 \dots c_{n-1})$ is even. By Lemma 3, $u_{-(n-1)} \dots u_{-1} = c_1 \dots c_{n-1}$. By Lemma 2 it follows that $\sup\{\pi_{[0,1]}(Q)\} \leq T^n(c)$, and thus an intersection between the arc Q and a semi-circle cannot occur.

Case II: Assume that we have a crossing of two semi-circles which project to the same point in $[0,1]$, see Figure 3.3.

Assume that there exist $n \in \mathbb{N}$ and $\overleftarrow{s} \prec_L \overleftarrow{u} \prec_L \overleftarrow{t} \prec_L \overleftarrow{v}$ such that $s_i = t_i$ for all $i < 0$ except for $i = -n$ and $s_{-(n-1)} \dots s_{-1} = t_{-(n-1)} \dots t_{-1} = c_1 \dots c_{n-1}$ and $u_i = v_i$ for all $i < 0$ except for $i = -n$ and $u_{-(n-1)} \dots u_{-1} = v_{-(n-1)} \dots v_{-1} = c_1 \dots c_{n-1}$. If $s_{-n} = v_{-n}$, then by Lemma 3 also $t_{-n} = u_{-n} = s_{-n} = v_{-n}$ which contradicts the assumption. It follows that $s_{-n} \neq v_{-n}$, because $\overleftarrow{v}, \overleftarrow{u}$ and $\overleftarrow{t}, \overleftarrow{s}$ are respectively connected by a right semi-circle. Assume without loss of generality that $v_{-n} = 1$ and $s_{-n} = 0$. This gives $t_{-n} = 1$ and $u_{-n} = 0$.

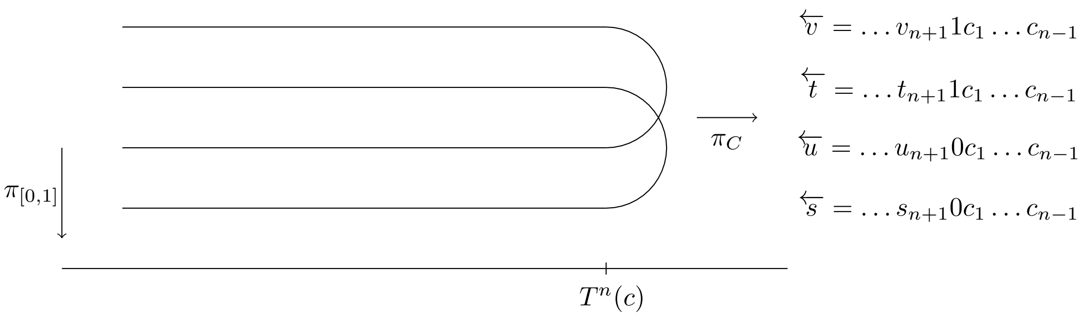

Figure 3.3: Case II in the proof of Proposition 5.

Now take the smallest integer $m > n$ such that $v_{-m} \neq t_{-m}$; this m is also the smallest integer such that $u_{-m} \neq s_{-m}$. By the previous paragraph, if $(-1)^{\#_1(s_{-(m-1)}\dots s_{-1})} = (-1)^{\#_1(u_{-(m-1)}\dots u_{-1})} \neq (-1)^{\#_1(t_{-(m-1)}\dots t_{-1})} = (-1)^{\#_1(v_{-(m-1)}\dots v_{-1})}$, the possibilities for s_{-m}, u_{-m}, t_{-m}, v_{-m} are (depending on the parities of ones): (1) $s_{-m} = 0, u_{-m} = 1, t_{-m} = 1, v_{-m} = 0$, or (2) $s_{-m} = 1, u_{-m} = 0, t_{-m} = 0, v_{-m} = 1$. Both cases lead to a contradiction with $s_{-m} = t_{-m}$ and $u_{-m} = v_{-m}$. $\square$

Thus our ordering gives a representation $Y \cup \{\text{semi-circles}\}$ of X in the plane. Figure 3.4 and Figure 3.5 give two examples of these planar representations.

3.2 Embeddings of X

In this section we show that representations of X constructed in the previous section are indeed embeddings.

Lemma 4. *Let $U \subset \mathbb{R}^2$ be homeomorphic to the open unit disk, and let $W \subset \mathbb{R}$ be a closed set such that $W \times J \subset U$ for some closed interval J. There exists a continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(\{w\} \times J)$ is a point for every $w \in W$, $f(\{w\} \times J) \neq f(\{w'\} \times J)$ for every $w \neq w' \in W$, $f|_{U \setminus W \times J}$ is injective and $f|_{\mathbb{R}^2 \setminus U}$ is the identity.*

Proof. Without loss of generality we can take $U := (-1, 2) \times (-1, 1)$, $J := [-1/2, 1/2]$ and $\min(W) = 0$, $\max(W) = 1$, see Figure 3.6.

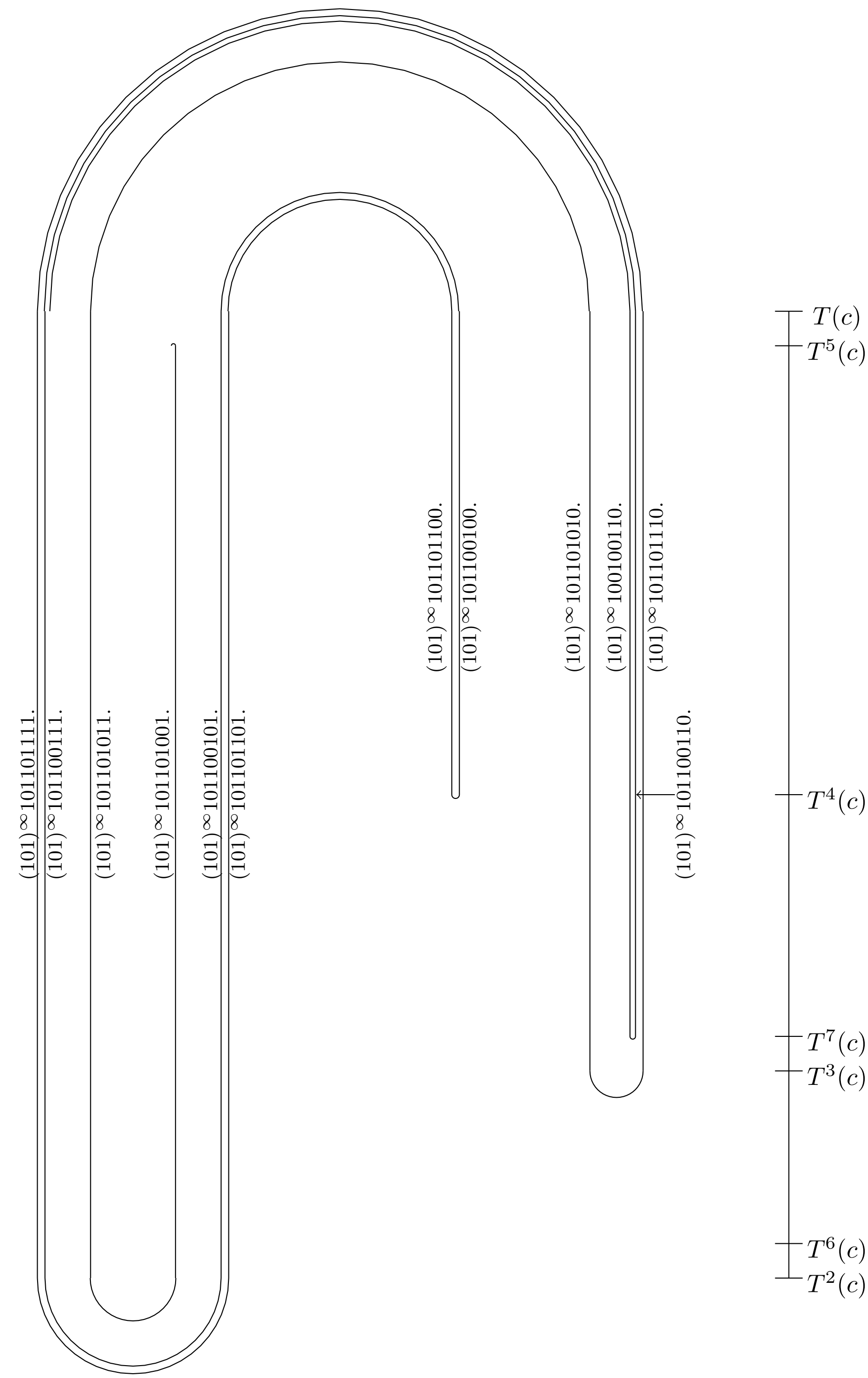

Figure 3.4: Planar representation of an arc in X with the kneading sequence $\nu = 100110010\ldots$. The basic arc coded by $L = 1^\infty$ is the largest.

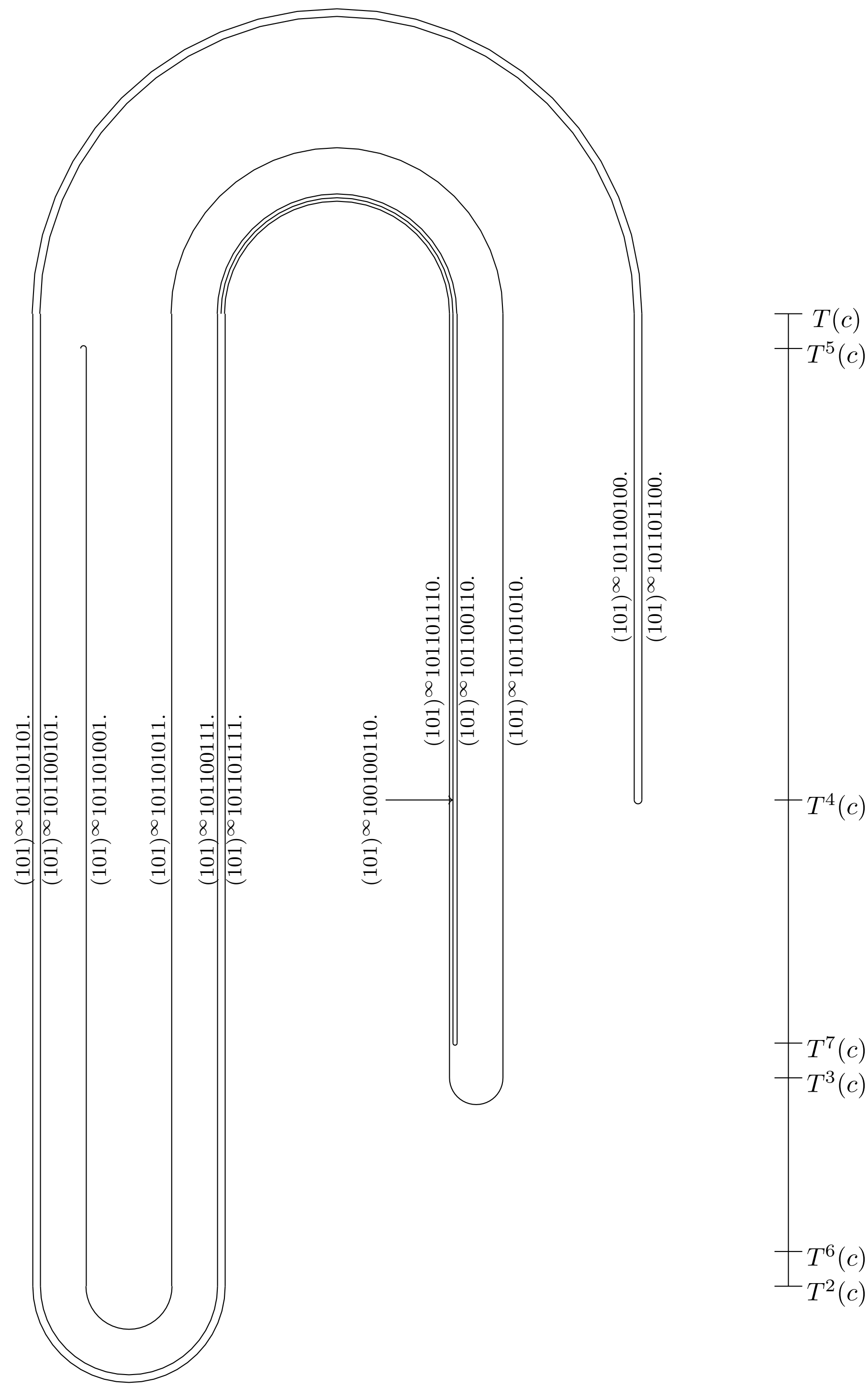

Figure 3.5: Planar representation of the same arc as in Figure 3.4 where the basic arc coded by $L = (101)^\infty$ is the largest.

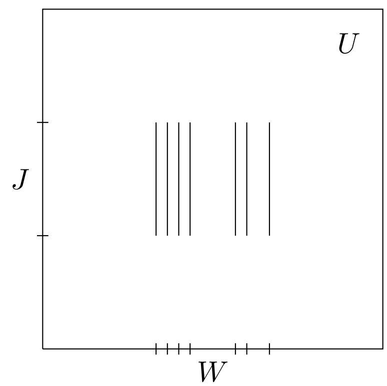

Figure 3.6: Set-up in Lemma 4.

For every $a \in [0,2]$ we define a continuous function $g(a,\cdot) : [-1,1] \to [-1,1]$ as

$$g(a,z) := \begin{cases} (2-a)z + 1 - a, & z \in [-1,-1/2], \\ az, & z \in [-1/2,1/2], \\ (2-a)z + a - 1, & z \in [1/2,1]. \end{cases}$$

Note that $g(a,\cdot)$ is injective for every $a \in [0,2]$, $g(0,z) = 0$ for all $z \in [-1/2,1/2]$, and $g(1,z) = z$ for all $z \in [-1,1]$.

Define $\hat{f} : [-1,2] \times [-1,1] \to [-1,2] \times [-1,1]$ as

$$\hat{f}(z,y) := (z, g(d_e(z,W),y)),$$

where $d_e(z,W) = \inf_{w\in W}\{d_e(z,w)\}$. Note that $z \mapsto d_e(z,W)$ is continuous, so $\hat{f}$ is continuous. Also, $\hat{f}(w,y) = (w, g(0,y)) = (w,0)$ for $(w,y) \in W \times J$ and $\hat{f}$ is injective otherwise. Also note that $\hat{f}$ is the identity on the boundary of $[-1,2] \times [-1,1]$, so $\hat{f}$ can be extended continuously to the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f|_{\mathbb{R}^2\setminus U}$ is the identity. $\square$

Define $W_n \subset \mathbb{R}^2$ to be the set consisting of all semi-circles that join pairs of points at level n. Note that there exists a set $W \subset \mathbb{R}$ such that W_n is homeomorphic to $W \times J$. Observe that W is closed. Indeed, if for a sequence $(\overleftarrow{s}^k)_{k\in\mathbb{N}} \subset \{0,1\}^{\mathbb{N}}$ there exists $m \in \mathbb{N}$ such that $\tau_R(\overleftarrow{s}^k) = m$ for every $k \in \mathbb{N}$ and $\lim_{k\to\infty} \overleftarrow{s}^k = \overleftarrow{s}$, then $\tau_R(\overleftarrow{s}) = m$. The analogous argument holds for τ_L.

Lemma 5. *There exist open sets $U_n \subset \mathbb{R}^2$ such that $W_n \subset U_n$ and for every $n \neq m \in \mathbb{N}$, $U_n \cap U_m = \emptyset$ and* $\operatorname{diam}(U_n) \to 0$ *as $n \to \infty$.*

Proof. We define the set $G_n := \{\psi_L(\overleftarrow{s}) : \overleftarrow{s} \in \{0,1\}^{\mathbb{N}} \text{ admissible}, \ \tau_R(\overleftarrow{s}) = n \text{ or } \tau_L(\overleftarrow{s}) = n\}$ for every $n \in \mathbb{N}$ and let A_n be the smallest interval in $[0,1]$ containing G_n. Note that A_n is closed and $\operatorname{diam}(A_n) \leq 3^{-n}$.

Let M_n denote the midpoint of A_n. If $\#_1(c_1 \dots c_n)$ is odd, let

$$V_n' = \left\{(z,w) \in \mathbb{R}^2 : (z - T^n(c))^2 + (w - M_n)^2 \leq \left(\frac{\operatorname{diam}(A_n)}{2}\right)^2, \ z \leq T^n(c)\right\}$$

be the closed left semi-disc centred around $(T^n(c), M_n)$. Similarly, if $\#_1(c_1 \dots c_n)$ is even, let

$$V_n' = \left\{(z,w) \in \mathbb{R}^2 : (z - T^n(c))^2 + (w - M_n)^2 \leq \left(\frac{\operatorname{diam}(A_n)}{2}\right)^2, \ z \geq T^n(c)\right\}.$$

be the closed right semi-disc centered around $(T^n(c), M_n)$. Note that $W_n \subset V_n'$, $\operatorname{diam}(V_n') \leq 3^{-n}$ and that $d_e(A_n, \psi_L(\overleftarrow{t})) > 3^{-n}$ for all $\psi_L(\overleftarrow{t}) \notin A_n$. Let V_n be the $\frac{\operatorname{diam}(A_n)}{2 \cdot 3}$-neighbourhood of V_n', that is,

$$V_n = \left\{x \in \mathbb{R}^2 : \text{ there exists } y \in V_n' \text{ such that } d_e(x,y) < \frac{\operatorname{diam}(A_n)}{2 \cdot 3}\right\},$$

see Figure 3.7. For every $n \in \mathbb{N}$, the open set

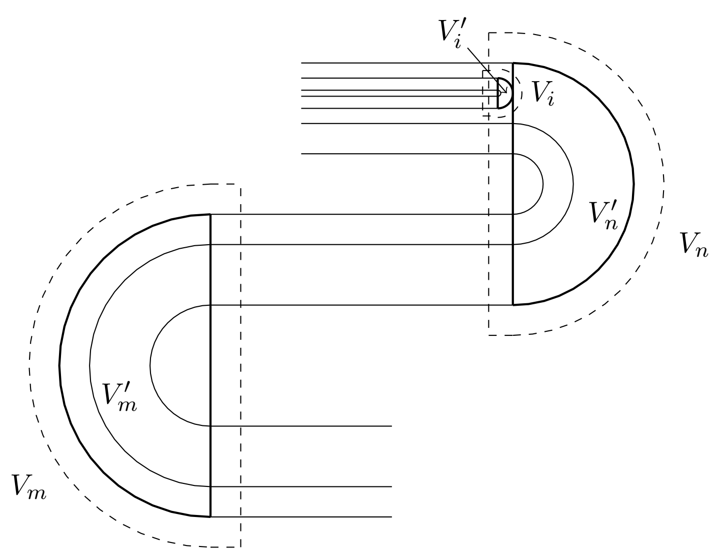

Figure 3.7: Sets constructed in the proof of Lemma 4.

$$U_n := V_n \setminus \overline{\cup_{i>n} V_i}$$

contains W_n, because otherwise there exists an increasing sequence $(i_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ so that points $x^{i_k} \in \{T^{i_k}(c)\} \times G_{i_k}$ and $x := \lim_k x^{i_k} \in W_n$. Since $x^{i_k} \in \{T^{i_k}(c)\} \times G_{i_k}$, the corresponding itinerary satisfies $\tau_R(\overleftarrow{x}^{i_k}) = i_k$, but because $i_k \to \infty$ as $k \to \infty$ this implies that the corresponding itinerary $\overleftarrow{x}$ of the point x satisfies $\tau_R(\overleftarrow{x}) = \infty$, a contradiction.

Note that $\operatorname{diam}(U_n) \le \operatorname{diam}(V_n) \to 0$ as $n \to \infty$. □

Now define a continuous function $f_n\colon \mathbb{R}^2 \to \mathbb{R}^2$ as in Lemma 4 replacing U with U_n and W with W_n. Let $F_n : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $F_n := f_n \circ \ldots \circ f_1$ for every $n \in \mathbb{N}$. We need to show that $F := \lim_{n\to\infty} F_n$ exists and is continuous. It is enough to show the following:

Lemma 6. *Sequence $(F_n)_{n\in\mathbb{N}}$ is uniformly Cauchy.*

Proof. Take $n < m \in \mathbb{N}$ and note that $\sup_{x\in\mathbb{R}^2} d_e(F_m(x), F_n(x)) = \sup_{x\in\mathbb{R}^2} d_e(f_m \circ \ldots \circ f_{n+1} \circ F_n(x), F_n(x)) < \max\{\operatorname{diam}(U_{n+1}), \ldots, \operatorname{diam}(U_m)\} \to 0$ as $n, m \to \infty$. □

Denote by $Z := Y \cup \{\text{semi-circles}\} \subset \mathbb{R}^2$. We want to argue that $F(Z) \subset \mathbb{R}^2$ is homeomorphic to $Y/{\sim}$. Since $F\colon Z \to F(Z)$ is continuous, it follows from [50, Theorem 3.21], that $\{F^{-1}(y) : y \in F(Z)\}$ is a decomposition of Z homeomorphic to $F(Z)$. Note that this decomposition is exactly $Y/{\sim}$.

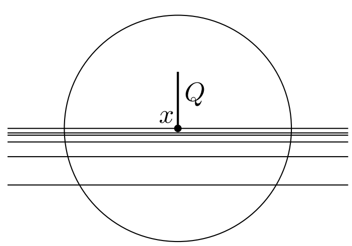

Figure 3.8: Point $x = (\pi_0(x), \psi_L(L))$ is accessible.

Theorem 1. *For every point $x \in X$ there exists an embedding of X in the plane such that x becomes accessible.*

Proof. Assume that the symbolic representation of $x \in X$ is given by $\bar{x} = \ldots l_{-2}l_{-1}.l_0l_1\ldots$. Consider the planar representation Z of X obtained by the ordering on C making $L = \ldots l_{-2}l_{-1}$ the largest (*i.e.*, the point x lies on the largest basic arc). The point x is represented as $(\pi_0(x), 1)$. Take the arc $Q := \{(\pi_0(x), z+1), z \in [0,1]\}$ which is a vertical interval in the plane (see Figure 3.8). Note that $Q \cap Z = \{x\}$. Then $F(Q)$ is an arc such that $F(Q) \cap F(Z) = \{F(x)\}$ which concludes the proof. $\square$

Definition 3. *Let $f : J_1 \to J_1$ and $g : J_2 \to J_2$ where $J_1, J_2 \subset \mathbb{R}$ be given. We say that f and g are* conjugate, *if there exists a homeomorphism $h : J_1 \to J_2$ such that $h \circ f = g \circ h$. If the map h is onto and continuous we say that f and g are* semiconjugate.

Definition 4. *A map $f : J \to J$, where $J \subset \mathbb{R}$ is said to be* locally eventually onto (leo) *if for every interval $J' \subset J$ there exists an $N \in \mathbb{N}$ such that $f^n(J') = J$ for every $n \geq N$.*

Recall that we denote the *core inverse limit space* $\varprojlim([T^2(c), T(c)], T)$ by X'.

Lemma 7. *Let T be a unimodal map such that $s := \exp(h_{top}(T)) > \sqrt{2}$. Then the core inverse limit space X' is indecomposable.*

Proof. The map T is semiconjugate to the tent map T_s, and if T is leo, then the semiconjugacy h is in fact a conjugacy. In this case, X' is indecomposable, see [38]. We give the argument if T is not leo (which also works in the general case). Let p be the orientation reversing fixed point of T, so $h(p) = r = \frac{s}{s+1}$ is the fixed point of T_s. Let $J \ni p$ be a neighbourhood such that $h(J)$ is a non-degenerate neighbourhood of $\frac{s}{s+1}$. Since $s > \sqrt{2}$, T_s is leo, so there is $N \in \mathbb{N}$ such that $T^N(\overline{J_-}) = T^N(\overline{J_+}) = [T^2(c), T(c)]$ for both components $J_\pm$ of $J \setminus \{p\}$. Suppose now by contradiction that $X' = X_1 \cup X_2$ for some proper subcontinua X_1 and X_2 of X'. Hence there exists $n_0 \in \mathbb{N}$ such that the projections $\pi_{n_0}(X_1) \neq [T^2(c), T(c)] \neq \pi_{n_0}(X_2)$. Take $n_1 = n_0 + N$. Since $\pi_{n_1}(X_1)$ and $\pi_{n_1}(X_2)$ are intervals and $\pi_{n_1}(X_1) \cup \pi_{n_1}(X_2) = [T^2(c), T(c)]$, at least one of them, say $\pi_{n_1}(X_1)$, contains at least one of $\overline{J_-}$ or $\overline{J_+}$. But then $\pi_{n_0}(X_1) \supset T^N(\overline{J_-}) \cap T^N(\overline{J_+}) = [T^2(c), T(c)]$, contradicting the definition of n_0. This completes the proof. $\square$

Corollary 1. *Let T be a unimodal interval map with $h_{top}(T) > 0$. Then there are uncountably many non-equivalent embeddings of X in the plane.*

Proof. First assume that $\exp(h_{top}(T)) > \sqrt{2}$. By Lemma 7, X' is indecomposable and thus it has uncountably many pairwise disjoint composants which are dense in X'. By Proposition 7 (we prove it in the following chapter so it does not interfere with the flow of the reading) every subcontinuum $H \subset X'$ contains a ray which is dense in H. Therefore, every composant $\mathcal{U}$ of X' contains a non-degenerate basic arc in X'. We embed X so that this non-degenerate basic arc is the largest. Such embedding of X makes a non-degenerate basic arc of $\mathcal{U}$ accessible.

Assume that $g_1 \colon X \to E_1$ and $g_2 \colon X \to E_2$ are equivalent embeddings, so there exists a homeomorphism $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\tilde{h}(E_1) = E_2$. Mazurkiewicz proved in [44] that the union of accessible composants of an indecomposable continuum in the plane is a countable union of closed sets.

From the prime end theory established by Carathéodory [27] et. al, it follows that there is a one-to-one correspondence between the composants of X' containing more than one accessible point and pairwise disjoint non-degenerate open intervals in the circle of prime ends (see Theorem 4 for details; we shortly introduce the prime end theory in Section 5.3 since we mostly use it later in the thesis). Because there is at most countably many pairwise disjoint non-degenerate open arcs in the circle of prime ends, it follows that X' has at most countably many composants with an accessible non-degenerate arc (see also [45]). That finishes the proof when $\exp(h_{top}(T)) > \sqrt{2}$.

Now assume that $\sqrt{2} \geq \exp(h_{top}(T)) > 1$. The core X' is decomposable and there exists an indecomposable subcontinuum of X' which is homeomorphic to the inverse limit space of a unimodal map with entropy greater than $\log \sqrt{2}$. It follows from the arguments above that we can embed this indecomposable subcontinuum in uncountably many non-equivalent ways; therefore we obtain uncountably many non-equivalent embeddings of X. $\square$

Remark 5. *The results from the prime end theory or the Mazurkiewicz' results are perhaps too strong tools to apply in the proof of Corollary 1. However, to complete the argument we need to know which points are accessible besides those in the arc which is made the largest. Frequently, the only accessible points are the points from the arc-component of a point from the largest basic arc, but there are many exceptions as we shall see in the next sections.*

Chapter 4

Topology of X

4.1 Arc-components of X

A *ray* $R \subset X$ is a continuous one-to-one image of the interval $[0,1)$. A *line* $W \subset X$ is a continuous one-to-one image of the interval $(0,1)$. In this section X is going to denote the inverse limit of a unimodal interval map T and X' inverse limit restricted to the core of T. Recall that we denote by $\mathcal{U}_x$ the arc-component of x in X; *i.e.,* the union of all arcs from X that contain the point x. For the sake of brevity we omit minuses in denoting the coordinate projections; *i.e.,* we write $\pi_i : X \to [0,1]$ for $i \in \mathbb{N}_0$.

In this section we will assume that T is locally eventually onto the interval $[T^2(c), T(c)]$, see Definition 4. Note that if two interval maps f and g are topologically conjugate, its corresponding inverse limit spaces $\varprojlim([0,1], f)$ and $\varprojlim([0,1], g)$ are homeomorphic. Any unimodal map without periodic attractors, wandering intervals or restrictive intervals is topologically conjugate to a tent map restricted to the core $[T_s^2(c), T_s(c)]$, for details see *e.g.* [46]. Furthermore the tent maps T_s for $s > \sqrt{2}$ are locally eventually onto on the interval $[T_s^2(c), T_s(c)]$. Therefore restricting to unimodal maps T which are locally eventually onto the interval $[T^2(c), T(c)]$ is not as restrictive as it might appear.

We first prove some preliminary results which will give a basic insight in the subcontinua of X and will be important later in the chapter.

Lemma 8 (Lemma 1 in [21]). *Let H be a subcontinuum of X. Then for every $i \in \mathbb{N}_0$ the projections $\pi_i(H)$ are intervals. Furthermore, if $c \in \pi_i(H)$ for only finitely many $i \in \mathbb{N}_0$, then H is either a point or an arc.*

Proof. The projections $\pi_i : X \to [0,1]$ are continuous for every $i \in \mathbb{N}$. Continuous images of compact and connected set H are closed and connected in $[0,1]$ and thus either a point or an interval.
To prove the second statement let $N \in \mathbb{N}$ be the maximal natural number so that $c \in \pi_N(H)$. Therefore, we can parametrize the space H with a single variable $t \in \pi_{N+1}(H)$ and thus the statement follows. □

Lemma 9 (Lemma 2 in [21]). *Let T be a unimodal leo map and H be a subcontinuum of X. The subcontinuum H is proper in X or a point if and only if $|\pi_i(H)| \to 0$ as $i \to \infty$.*

Proof. The ($\Leftarrow$) direction follows straightforward since we can find different points from $X \setminus H$. For the ($\Rightarrow$) direction let us assume that $|\pi_i(H)| \nrightarrow 0$ as $i \to \infty$. Let us fix $\varepsilon > 0$. Thus we can find a sequence $(n_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ such that $|\pi_{n_i}(H)| > \varepsilon$ for every $i \in \mathbb{N}$ and thus there exists an interval $J^{n_i} \subset \pi_{n_i}(H)$ such that $|J^{n_i}| > \varepsilon/2$. Because T is assumed to be locally eventually onto the interval $[T^2(c), T(c)]$ there exists $N(n_i) \in \mathbb{N}$ so that $T^{N(n_i)}(J^{n_i}) = [T^2(c), T(c)]$. There exists $M \in \mathbb{N}$ so that $n_M - N(n_M) > 0$ and therefore $\pi_{n_M - N(n_M)}(J^{n_i}) \supset [T^2(c), T(c)]$. Therefore, $\pi^{-1}_{n_M - N(n_M)}(J) = X'$ and thus for every $i > n_M - N(n_M)$ the projections $\pi_i(H) = [T^2(c), T(c)]$ as well, a contradiction with H being a proper subcontinuum of X. □

Let $H \subset X$ be a proper subcontinuum and let $\{n_i\}_{i\in\mathbb{N}} \subset \mathbb{N}_0$ be its *critical projections*; *i.e.,* $c \in \pi_n(H)$ if and only if $n \in \{n_i\}_{i\in\mathbb{N}}$.

Observation 1. *Let $n \in \mathbb{N}_0$. Then $\sigma^{-n}(H) \subset X$ is homeomorphic to $H \subset X$.*

By Observation 1 only the asymptotic behavior of critical projections is important. Thus we can assume without loss of generality that $n_1 = 1$.

Let $J = [e, e'] \subset X$ be an arc. Let us denote by $\mathrm{Bd}(J) = \{e, e'\}$.

Observation 2. *Since $c \in \pi_{n_i}(H)$ it follows that $T(c) \in \pi_{n_i-1}(H)$ and thus $T^{n_i-n_{i-1}}(c) \in \mathrm{Bd}(\pi_{n_{i-1}}(H))$ for every $i \geq 2$.*

Definition 5. *Let $T\colon [0,1] \to [0,1]$ be a continuous interval map. The* lap *of T is a maximal interval of monotonicity of T and a* branch *of T is an image of a lap. We say that T is* long-branched, *if there exists $\delta > 0$ such that the length of all branches of T^n is larger than δ for all $n \in \mathbb{N}$.*

Proposition 6 (Proposition 3 in [21]). *If the map T is long-branched, then the only proper subcontinua of X' are arcs.*

Proof. Assume by contradiction that a subcontinuum $H \subset X'$ with critical projections $\{n_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ is not an arc. By Lemma 8 the set $\{n_i\}_{i\in\mathbb{N}}$ needs to be infinite. By Observation 2 there exists $N(i) \in \mathbb{N}$ so that $[c, T^{N(i)}(c)] \subset \pi_{n_i}(H)$. Since T is long-branched, there exists $\delta > 0$ so that $|T^{N(i)}(c) - c| > \delta$ for every $i \in \mathbb{N}$. However, this is by Lemma 9 in contradiction with H being a proper subcontinuum of X'. □

Definition 6. *For $i \geq 1$ let R_{n_i} denote the closure of component of $\pi_{n_i}(H) \setminus \{c\}$ such that $T^{n_i-n_{i-1}}(R_{n_i}) = \pi_{n_{i-1}}(H)$. Denote by L_{n_i} the closure of the other component of $\pi_{n_i}(H) \setminus \{c\}$. If both components map with $\pi_{n_{i-1}}$ to $\pi_{n_{i-1}}(H)$, then denote by R_{n_i} the component that contains the point $T^{n_{i+1}-n_i}(c)$ as a boundary point.*

For $K \subset \mathbb{R}^2$ we denote by $\mathrm{Cl}(K)$ the closure of K in $\mathbb{R}^2$.

Proposition 7 (Proposition 1 in [21]). *Any subcontinuum $H \subset X$ is either a point or it contains a dense line,* i.e., *there exists a line $W \subset H$ such that* $\mathrm{Cl}(W) = H$, *if we restrict on a subcontinuum H.*

Proof. Assume that H is not degenerate. Define $\gamma_1 : [0,1] \to \pi_1(H)$ so that $\gamma_1(1) = T^{n_2-n_1}(c)$. Set $a_2 = 0$ and $b_2 = 1$. Let $-1 < \ldots a_3 \leq a_2 < b_2 \leq b_3 \ldots < 2$. Assume that $\gamma_j : [a_{j+1}, b_{j+1}] \to \pi_j(H)$ have been already defined for every $j < i$. We define $\gamma_i : [a_{i+1}, b_{i+1}] \to \pi_{n_i}(H)$ to be a continuous bijection so that $T^{n_i-n_{i-1}} \circ \gamma_i|_{[a_{i+1},b_{i+1}]} = \gamma_{i-1}|_{[a_i,b_i]}$ where $\gamma_i([a_i,b_i]) = R_{n_i}$ and $\gamma_i : [a_{i+1}, b_{i+1}] \setminus [a_i, b_i] \to L_{n_i}$ is linear and onto for $i \geq 2$. Furthermore let a_i and b_i be chosen so that $|b_{i+1} - b_i| < \frac{1}{2^i}$ and $|a_{i+1} - a_i| = 0$ or $|a_{i+1} - a_i| < \frac{1}{2^i}$ and $|b_{i+1} - b_i| = 0$

for every $i \in \mathbb{N}$. Denote by $a = \lim_{i\to\infty} a_i$ and $b = \lim_{i\to\infty} b_i$. For $z \in (a,b)$ and $m \geq 1$ set $i_0 = \min\{i | z \in [a_{i+1}, b_{i+1}] \text{ and } m \leq n_i\}$. Then, by construction $T^{n_i - m} \circ \gamma_i(z) = T^{n_{i_0}} \circ \gamma_{i_0}(z)$ for $i \geq i_0$. Thus we can define $\Phi : (a,b) \to H$ by coordinates $\Phi_m := \lim_{i\to\infty} T^{n_i - m} \circ \gamma_i$. By the construction, Φ is one-to-one and continuous and thus $\Phi((a,b))$ is a line. Furthermore, for every $m \in \mathbb{N}$ it follows that $\pi_m(\Phi((a,b))) = \pi_m(H)$ and thus $\Phi((a,b))$ is dense in H. $\square$

A specific case of Proposition 7 is when we take $H = X'$. Since in that case we take for $R_m = [c, c_1]$ it follows that $\Phi((a,b))$ from the proof of Proposition 7 equals $\mathcal{R}$, the arc-component of the fixed point $\rho = (\ldots, r, r)$ of T. Therefore, we obtain the following corollary.

Corollary 2. *Arc-component $\mathcal{R}$ of ρ is a dense line in X'.*

Proposition 8. *Let T be a unimodal leo map. The arc-component of ρ in $\mathcal{R}$ is also the composant of ρ in X'.*

Proof. By Corollary 2 it holds that $\rho := (\ldots, r, r) \in \mathcal{R} \subset X'$ is a dense line in X'. Assume that there exists a subcontinuum $H \subset X'$, where $\mathcal{R} \subset H$ and H is a proper subcontinuum of X'. Because H is closed in X' and $\mathcal{R}$ is dense in X' it follows that $H = X$. Thus the composant of $\rho \subset X$ in X is $\mathcal{R}$. $\square$

Recall that $\omega(c)$ as in (2.2) denotes the set of all accumulation points of the critical orbit. It is possible that $\omega(c) = [T_s^2(c), T_s(c)]$. For that case Barge, Brucks and Diamond observed in [6] that for every point $x \in \varprojlim([T_s^2(c), T_s(c)], T_s)$ a homeomorphic copy of every inverse limit with single tent map appearing in the parametrized tent family can be found in every neighbourhood of point x. In the paper of Tom W. Ingram the following question was posed (Problem 5 in [37]):

> **Question (Raines)**: Suppose that T_s is such that $\omega(c) = [T_s^2(c), T_s(c)]$. Say that $\mathcal{U}_x$ is a composant of $x \in \varprojlim([T_s^2(c), T_s(c)], T_s)$. Does $\mathcal{U}_x$ contain a copy of every continuum that arises as an inverse limit space of a tent family core?

The next corollary in special case answers the question of Raines in the negative, since for every $s \in (\sqrt{2}, 2]$ the map T_s is leo.

Corollary 3. *For every unimodal leo map T there exists a point $x \in X'$ such that the arc-component $\mathcal{U}_x$ is also a composant of x in X' and $\mathcal{U}_x$ is a line.*

Proof. By Proposition 8 the arc-component of $\mathcal{U}_\rho$ is also the composant of ρ and $\mathcal{R}$ is a continuous image of the real line. $\square$

Now we make some general observations about allowed types of arc-components in X'.

Say that there exist $n \in \mathbb{N}$ so that $T^n(c) = c$, *i.e.,* the critical orbit is periodic. Then the inverse limit space X' has its corresponding kneading sequence $\nu = (c_1 \dots c_n)^\infty$. Thus it follows that there exists a point $e \in X'$ such that the itinerary of e equals $\bar{e} = (c_1 \dots c_n)^\infty.(c_1 \dots c_n)^\infty$. By Proposition 3 it follows that e is an endpoint of X'. By Proposition 6 the only proper subcontinua of X' where c is periodic are arcs. Because we have one-to-one coding for arc-components when c is periodic (see Corollary 2.10 in [23], for its extension see Remark 12) we obtain that e is the only endpoint on $\mathcal{U}_e$. Therefore $\mathcal{U}_e$ is a ray in X'.

Now we comment that arc can also be the arc-component of a point from X'. In the paper from Brucks & Bruin [21] it is observed that the topologists $\sin(1/x)$-continuum appears as a subcontinuum of X', see Figure 4.1. Note that for every point $y \in A \subset X$ such that $A = \mathrm{Cl}(\sin(1/x)) \setminus \sin(1/x)$, the arc-component of y is an arc.

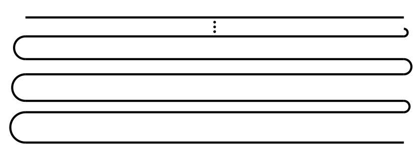

Figure 4.1: A $\sin(1/x)$-continuum appears as a subcontinuum in X.

The following statement from [6] will turn out to be very important for the rest of the section. We state what appears to be a slightly stronger version of Theorem 4 in [6]. However, note that the proof works analogously as in the mentioned paper for all $s \in (\sqrt{2}, 2]$ such that $\mathrm{Orb}(c)$ of T_s is dense in $[T_s^2(c), T_s(c)]$.

Proposition 9 (Theorem 4 in [6])**.** *Assume that $s \in (\sqrt{2}, 2]$ is such that $\mathrm{Orb}(c)$ of T_s is dense in $[T_s^2(c), T_s(c)]$. Given any subcontinuum $H \subset \varprojlim([T_{\tilde{s}}^2(\tilde{c}), T_{\tilde{s}}(\tilde{c})], T_{\tilde{s}})$ where $\tilde{s} \in (1, 2]$ there exists a subcontinuum of $\varprojlim([T_s^2(c), T_s(c)], T_s)$ which is homeomorphic to H.*

Let us rephrase Proposition 9. There exists a G_δ set of parameters in the parametrized tent map family that admit copies of every other inverse limit space of the parametrized family. The following proposition interprets Proposition 9 in a different setting.

Proposition 10. *Say that* $\mathrm{Orb}(c)$ *is dense in* $[T_s^2(c), T_s(c)]$. *Then there exist points* $x \in \varprojlim([T_s^2(c), T_s(c)], T_s)$ *such that* $\mathcal{U}_x$ *is degenerate,* i.e., $\mathcal{U}_x = \{x\}$.

Proof. From Proposition 9 it follows that there exists a dense G_δ set $S \subset [\sqrt{2}, 2]$ such that for every tent map inverse limit $\varprojlim([T_s^2(c), T_s(c)], T_s)$ with parameter from S there exists a dense set of points x such that there exist subcontinua $H_i \subset \varprojlim([T_s^2(c), T_s(c)], T_s)$, $H_{i+1} \subset H_i$ for every $i \in \mathbb{N}$, $\mathrm{diam}(H_i) \to \infty$ as $i \to \infty$ and $\cap_{i\in\mathbb{N}} H_i = \{x\}$. Without loss of generality we can take H_i indecomposable for every $i \in \mathbb{N}$. From Lemma 8 in [6], it follows that such x is an endpoint of $\varprojlim([T_s^2(c), T_s(c)], T_s)$.
Assume there is a non-degenerate arc $x \in A \subset \varprojlim([T_s^2(c), T_s(c)], T_s)$. Since x is an endpoint, either $H_i \subset A$ or $A \subset H_i$ for every $i \in \mathbb{N}$. Because H_i are indecomposable, it follows that $H_i \nsubseteq A$ for every $i \in \mathbb{N}$. Thus $A \subset \cap_{i\in\mathbb{N}} H_i$ which is a contradiction with A being non-degenerate. □

Therefore, we obtain the following proposition as a simple consequence of Proposition 10.

Proposition 11. *Let* $\mathcal{U}_x \subset X'$ *be the arc-component of* $x \in X'$. *Then exactly one of the following holds:*

- $\mathcal{U}_x$ *is a point,*
- $\mathcal{U}_x$ *is a ray,*
- $\mathcal{U}_x$ *is a line,*
- $\mathcal{U}_x$ *is an arc,*

and for every type of $\mathcal{U}_x$ *there exist* X' *so that* $\mathcal{U}_x$ *occurs in* X'.

Proof. From Corollary 3.3 in [31] it follows that an arc-component $\mathcal{U}_x \subset X'$ of an arbitrary point $x \in X$ is either a ray, a line, an arc or a point. From the discussion above it follows that there exist points in X' such that its arc-components are either a ray, a line or an arc. It follows from Proposition 10 that there also exist degenerate arc-components in X'. $\square$

Definition 7. *If the arc-component of a point $x \in X'$ is degenerate we call it a* nasty point*. The set of all nasty points is denoted by Ξ.*

Proposition 12. *Every $x \in \Xi$ is an endpoint.*

Proof. A nasty point x is not contained in any arc from X. If both $\tau_L(\overleftarrow{x}) < \infty$ and $\tau_R(\overleftarrow{x}) < \infty$ then by Lemma 2 $\pi_0(A(\overleftarrow{x}))$ is not degenerate. Thus $\tau_L(\overleftarrow{x}) = \infty$ or $\tau_R(\overleftarrow{x}) = \infty$. By Proposition 3 the point x is an endpoint. $\square$

Remark 6. *If* $\mathrm{Orb}(c)$ *is dense in $[T^2(c), T(c)]$, then Ξ is dense in X', as it follows from Proposition 9. Furthermore, the cardinality of Ξ is $2^{\aleph_0}$. By Proposition 9 it holds that every point x from Ξ in an arbitrary small neighbourhood contains points from Ξ which are different from x and since Ξ is dense in X it holds that Ξ has no isolated points. However, Ξ is not closed in Euclidean topology and thus not perfect and therefore not the Cantor set. To see that Ξ is not closed, observe that* e.g. *we can construct a sequence of points from Ξ converging to $\rho \in X'$.*

To continue with a study of subcontinua and possible arc-components of X' we need some definitions and preliminary results.

Definition 8. *A continuum K is called an* arc+ray *continuum, if $K = R \cup A$, where A is an arc, R is a ray and $Cl(R) \setminus R = A$.*

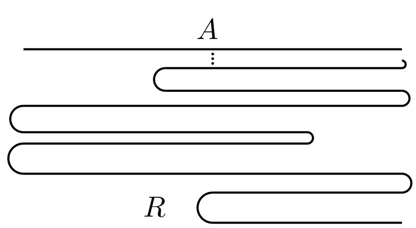

Figure 4.2: Arc+ray continua $K = R \cup A$ appear as subcontinua of X'.

Remark 7. *When the orbit of c is finite, with (pre)period $n \in \mathbb{N}$, there exist exactly n folding points (see [25]). They are contained in different arc-components which are permuted by the* *shift homeomorphism. If the orbit of c is periodic, then the folding points are endpoints (see [11]).*

Let us observe which arc-components occur in X' in special cases.

Remark 8. *It follows from Proposition 6 that when T is long-branched every proper subcontinuum of X' is an arc and thus for every $x \in X'$ arc-component of x is either a ray or a line. Specifically, unimodal maps T with periodic or non-recurrent critical orbit are long-branched. When c is recurrent and the only proper subcontinua of X' are arcs, ray+arcs continua (such cases are described in [21], see Corollary 1 and Corollary 3) it follows from the fact that composants of indecomposable continua X' are dense in X', that the only possible arc-components in X' are arcs, lines and rays.*

Proposition 13. *If $\mathrm{Orb}(c)$ of a unimodal map T is infinite and not dense in $[T^2(c), T(c)]$, then the set of all folding points (and endpoints) is not dense in X.*

Proof. By the assumptions $\omega(c)$ is the Cantor set and at most countable set of points so there is $x \in X$ such that $\pi_i(x) \in [c_2, c_1] \setminus \omega(c)$ for every $i \in \mathbb{N}$. By Proposition 2 there exists a neigbourhood of x homeomorphic to the Cantor set of arcs. Points in that neighbourhood are not folding points, so folding points (and thus also endpoints) are not dense in X. □

Recall that an indecomposable continuum consists of uncountably many pairwise disjoint composants which are dense in K.

Lemma 10. *Let K be an indecomposable chainable continuum. For every $x \in K$ there exists a nested sequence of non-degenerate subcontinua $\{H_i\}_{i\in\mathbb{N}} \subset K$ such that $\cap_{i\in\mathbb{N}} H_i = \{x\}$.*

Proof. Since K is indecomposable, the composant of x is dense in K and thus there exists a proper continuum $K_1 \subset K$ such that $x \in K_1$. Let the set $\{H_\lambda\}_{\lambda\in\Lambda}$ consist of all proper subcontinua of K containing x. The set $H := \cap_{\lambda\in\Lambda} H_\Lambda$ is a continuum. If $H = \{x\}$, we are done since the intersection can be taken nested.

Assume by contradiction that H is a non-degenerate continuum. Then H is indecomposable, because otherwise we could find a nondegenerate continuum $H' \subset H$ such that $H' \neq H_\lambda$ for all $\lambda \in \Lambda$. If H is indecomposable, the composant of x is dense in H so there is a continuum $x \in H'' \not\subset H$, a contradiction. □

We say that a continuum K is *arc-connected*, if for any two points $x \neq y \in K$ there exists an arc within K with endpoints x and y. Now we give a characterization of nasty points in X.

Theorem 2. *A point $x \in X$ is a nasty point if and only if there exists a nested sequence of a non-degenerate subcontinua $\{H_i\}_{i \in \mathbb{N}} \subset X$ such that $\cap_{i \in \mathbb{N}} H_i = \{x\}$ and H_i is not arc-connected for every $i \in \mathbb{N}$.*

Proof. By Lemma 10 there exists a nested sequence of non-degenerate subcontinua $H_i \subset X$ such that $\{x\} = \cap_{i \in \mathbb{N}} H_i$. If H_i is arc-connected for some $i \in \mathbb{N}$, then there exist an arc $x \in A \subset X$ and thus x is not a nasty point.
Conversely, assume by contradiction that there is a sequence as in the statement and that there is an arc $A \subset X$ such that $x \in A$. Since x is an endpoint of X by Proposition 12 and $A \subset H_i$ for every $i \in \mathbb{N}$, we get a contradiction. □

Remark 9. *In Theorem 3 of [21] the spaces $\varprojlim([T_s^2(c), T_s(c)], T_s)$ are studied for the special cases when the critical orbit of T_s is recurrent and $\omega(c)$ is the Cantor set. If the assumptions from Theorem 3 in [21] are satisfied (we will not state the conditions here since they are in the language of cutting times which we do not introduce here) then all proper subcontinua of $\varprojlim([T_s^2(c), T_s(c)], T_s)$ are either arcs, arc+rays continua or are homeomorphic to an inverse limit space of tent maps with finite critical orbit. Specifically, for any countable collection $\mathcal{F}$ of core tent inverse limit spaces with finite critical orbit an inverse limit space of tent map is constructed that has exactly points, arcs, arc+rays continua or continua homeomorphic to an inverse limit space from $\mathcal{F}$. It follows from Proposition 6 that all proper subcontinua of $\varprojlim([T_s^2(c), T_s(c)], T_s)$ are arcs. Thus there exists no nasty points in X' if $|\mathcal{F}|$ is finite. It holds that $H_i \cap H_j = \emptyset$ for every $H_i \neq H_j \in \mathcal{F}$. Therefore, even for $|\mathcal{F}| = \infty$ it holds by Theorem 2 that $\varprojlim([T_s^2(c), T_s(c)], T_s)$ contains no nasty points.*

In this section we mostly dedicated our attention to the special kind of endpoints. The next section will be dedicated to the structure of endpoints in X in general.
Despite the fact that we have proven the existence of nasty points, our knowledge about them

is not complete. For instance, in the beginning of the following chapter we will relate symbolic descriptions with arc-components (at most three per one arc-component as we shall see) but we are still unable to give exact symbolic descriptions of nasty points.

Problem 1. Give a symbolic characterization of nasty points in X.

Let us comment on this problem. By Theorem 2, a point from X is a nasty point if it can be expressed as a nested intersection of not arc-connected subcontinua of X. Since a symbolic description of points in X is determined by subcontinua in which they lie, the natural way to approach this problem is to study the symbolics of subcontinua of X that contain nasty points.

Problem 2. Give necessary conditions on the critical point c so that the corresponding inverse limit space X contains nasty points.

Proposition 10 assures the existence of nasty points when $\mathrm{Orb}(c)$ is dense in $[T^2(c), T(c)]$. Remark 8 and Remark 9 describe cases when there are no nasty points in X. However, when c is recurrent and $\omega(c)$ is the Cantor set the complete characterization of subcontinua of X has not been given in the literature yet. Thus it is possible that there exist X that have as subcontinua nested intersections of other unimodal inverse limit spaces with recurrent critical orbit and $\omega(c)$ being the Cantor set. This relates with the following problem stated by Raines, which is (to our knowledge) still open.

Problem 3 (Raines). (Problem 7 in [37]) Let $\varprojlim([T_s^2(c), T_s(c)], T_s)$ be such that $\mathrm{Orb}(c)$ of T_s is recurrent and $\omega(c)$ is the Cantor set. Classify all possible proper subcontinua of $\varprojlim([T_s^2(c), T_s(c)], T_s)$.

4.2 The structure of endpoints of X

This section will lead to the answer on another question posed by Raines in [37].

Remark 10. *Let T_s be a tent map for $s \in (\sqrt{2}, 2]$. Thus, T_s is not renormalizable. Note that $\omega(c)$ is always compact. If $\mathrm{Orb}(c)$ is infinite and c is recurrent, then it follows that there* *are no isolated points in* $\omega(c)$ *and* $\omega(c)$ *is thus perfect. Therefore, in the case when* $\mathrm{Orb}(c)$ *is infinite and recurrent either* $\omega(c) = [c_2, c_1]$ *or* $\omega(c)$ *is the Cantor set.*

In a paper by Ingram [37] the following problem has been raised (Problem 6):

> **Problem (Raines):** Suppose T is a unimodal map with critical orbit c. Give necessary and sufficient conditions on the critical point c so that $\varprojlim([0,1], T)$ contains a copy of every continuum that arises as an inverse limit space in a core tent map family.

The following proposition answers the last problem posed by Raines for the tent inverse limits for $s \in (\sqrt{2}, 2]$.

Proposition 14. *Let* $s \in (\sqrt{2}, 2]$*. Then the inverse limit space* $\varprojlim([0,1], T_s)$ *contains a copy of every inverse limit space from the parametrized tent map family if and only if* $\omega(c) = [T_s^2(c), T_s(c)]$.

Proof. By Proposition 10, if $\omega(c) = [T_s^2(c), T_s(c)]$, then $\varprojlim([0,1], T_s)$ contains a copy of every inverse limit space in the core tent map family.
Conversely, by Remark 10 we only need to prove that in the case when for T_s the $\omega(c)$ is the Cantor set and c is recurrent we can not find every inverse limit space of the core tent map family in $\varprojlim([0,1], T_s)$. Let $\varprojlim([0,1], T_s)$ be a tent inverse limit space so that $\omega(c)$ is the Cantor set and c is recurrent. Assume that there exists $H \subset \varprojlim([0,1], T_s)$ so that H is homeomorphic to $\varprojlim([T_{\tilde{s}}^2(\tilde{c}), T_{\tilde{s}}(\tilde{c})], T_{\tilde{s}})$ where $\tilde{s} \in [\sqrt{2}, 2]$ and critical orbit $\tilde{c}$ is dense in $[T_s^2(\tilde{c}), T_s(\tilde{c})]$. Since it follows from Proposition 2 that every point from H is a folding point, there exists an arc $A \subset \tilde{\mathcal{R}} \subset H \subset \varprojlim([T_s^2(c), T_s(c)], T_s)$ such that every $x \in A$ is a folding point. Therefore, there exists an arc $\pi_0(A) \subset [T_s^2(c), T_s(c)]$ with $|\pi_0(A)| > 0$ and thus $\pi_0(A) \subset \omega(c)$. Since the Cantor set is nowhere dense, we have a contradiction. □

Now let $a \in (1, \sqrt{2}]$. For $\sqrt{2} < a^m \leq 2$ with $m \in \{2, 2^2, 2^3, \ldots\}$ the map T_a is m-times renormalizable by Proposition 3.4.26 from [22]. Moreover, there exists a unique $s \in (\sqrt{2}, 2]$ such that T_a^m and T_s are conjugate and therefore $\varprojlim([0,1], T_s)$ is homeomorphic to $\varprojlim([0,1], T_a^m)$, see Remark 10.1.10 from [22]. Therefore, since infinitely renormalizable maps do not exist in the tent map family we obtain the following corollary.

Corollary 4. *Let $s \in (1, \sqrt{2}]$. Then the inverse limit space $\varprojlim([0,1], T_s)$ contains a copy of every inverse limit space from the parametrized tent map family if and only if $\omega(c)$ contains finite union of arcs.*

Since for $s \in [0,1]$ it holds that $\varprojlim([0,1], T_s)$ does not contain indecomposable continua, Proposition 14 and Corollary 4 solve the quoted problem by Raines for the case of tent inverse limits.

The following corollary answers the quoted problem by Raines in general.

Theorem 3. *Let T be a unimodal map. The inverse limit space $\varprojlim([0,1], T)$ contains a copy of every inverse limit space from the parametrized tent map family, if and only if $\omega(c)$ contains an interval.*

Proof. The if direction is an immediate consequence of Proposition 14 and Corollary 4.
For the only if direction; if for a unimodal map T the $\omega(c)$ contains an interval, then the forward orbit of this interval is a cycle of intervals and hence a renormalization, see *e.g.* [46]. □

We continue with the study of endpoints of X.

Remark 11. *By Corollary 2 from [24] it follows that in the case when c is recurrent and $\omega(c)$ is the Cantor set that X has uncountably many endpoints which densely fill the Cantor set, but away from this Cantor set, X is locally homeomorphic to the Cantor set of arc.*

The following proposition follows implicitly from the proof of Corollary 2 in [24]. We prove it here for the sake of completeness.

Proposition 15. *If* $\mathrm{Orb}(c)$ *is infinite and c is recurrent then the space X has uncountably many endpoints.*

Proof. Since c is recurrent, for every $k \in \mathbb{N}$ there exist countably many $n \in \mathbb{N}$ such that $c_1 \dots c_n = c_1 \dots c_{n-k} c_1 \dots c_k$. Chose the sequence $(i_j)_{j\in\mathbb{N}}$ such that $c_1 \dots c_{i_{j+1}} = c_1 \dots c_{i_{j+1}-i_j}$ $c_1 \dots c_{i_j}$ for every $j \in \mathbb{N}$. Then for the basic arc given by the itinerary

$$\overleftarrow{s} := \dots c_1 \dots c_{i_j},$$

it holds for every $j \in \mathbb{N}$ that $\tau_L(\overleftarrow{s}) = \infty$ or $\tau_R(\overleftarrow{s}) = \infty$. Since the sequence $(i_j)_{j\in\mathbb{N}}$ can be chosen in countably many ways for every $j \in \mathbb{N}$, it follows that there are uncountably many basic arcs containing at least one endpoint. $\square$

Let us sum up the knowledge about the structure of endpoints in X'. We can draw a one-to-one correspondence between the structure of $\omega(c)$ and the structure of endpoints in X'. If c is not recurrent, there exist no endpoints in X'. If c is recurrent and $\mathrm{Orb}(c)$ is finite (and thus also $\omega(c)$ finite) then there are $|\mathrm{Orb}(c)|$ endpoints in X. If c is recurrent and $\mathrm{Orb}(c)$ is infinite we have by Proposition 15 uncountably many endpoints; if $\omega(c)$ is the Cantor set, then set of endpoints in X is homeomorphic to the Cantor set by Remark 11, if $\omega(c) = [T^2(c), T(c)]$, then endpoints lie dense in X by Remark 6.

Another problem one can ask in this context is to characterize unimodal inverse limit spaces in which every folding point is an endpoint. Specifically, Remark 8 implies that for T being long-branched and c recurrent, every folding point from X is an endpoint.
On the other hand, it follows from Proposition 2 that when $\omega(c) = [T_s^2(c), T_s(c)]$ every point in X is a folding point. However, by Proposition 8 none of the points from $\mathcal{R}$ are endpoints, and therefore the set of endpoints is not equal to the set of folding points in this case. The problem is not settled in general yet, see a paper by Alvin [1].

Problem 4 (Alvin). Give conditions on the critical point c of a unimodal map so that all the folding points contained in X are endpoints.

Chapter 5

Accessible points of $\mathcal{E}$-embeddings of X

In the rest of the thesis we work with tent maps for slopes $s \in (\sqrt{2}, 2]$ and when there is no need to specify the slope we set for brevity $T := T_s$. We denote from now onwards by X the tent inverse limit space and by X' its core. For the sake of brevity we omit minuses in symbolic descriptions of left infinite tails and write $\overleftarrow{s} = \dots s_2 s_1 \in \{0,1\}^{\infty}$.

5.1 Symbolic coding of arc-components

We want to describe the sets of accessible points of embedded X, focusing primarily on the fully accessible arc-components. Since the approach in this study is mostly symbolic, we need to obtain a symbolic description of an arc-component in X. Recall that $\mathcal{U}_x$ denotes the arc-component of $x \in X$.

Definition 9. *We say that a point $x \in X$ is a* spiral point *if there exists a ray $R \subset X$ such that x is the endpoint of R and $[x, y] \subset R$ contains infinitely many basic arcs for every $x \neq y \in R$.*

Proposition 16. *If $x \in X$ is a spiral point, then $A(\overleftarrow{x})$ is degenerate and x is an endpoint of X.*

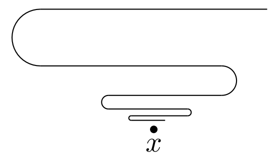

Figure 5.1: Point $x \in X$ is a spiral point.

Proof. Assume that $A(\overleftarrow{x})$ is not degenerate. Note that x is not in the interior of $A(\overleftarrow{x})$ since then $R \cup A(\overleftarrow{x})$ is a triod. Without loss of generality assume x is the right endpoint of $A(\overleftarrow{x})$. If $\tau_R(A(\overleftarrow{x})) < \infty$, then by Lemma 1 there exists $y \in X$ such that $A(\overleftarrow{y})$ and $A(\overleftarrow{x})$ are connected by a semi-circle. If $A(\overleftarrow{y})$ is non-degenerate, then X again contains a triod. If $A(\overleftarrow{y})$ is degenerate, then $y = x$ is an endpoint of X, which is not possible since x is contained in the interior of an arc $A(\overleftarrow{x}) \cup R$. Therefore, $A(\overleftarrow{x})$ is degenerate.
Since $A(\overleftarrow{x})$ is degenerate it follows from Lemma 1 that $\tau_L(\overleftarrow{x}) = \infty$ or $\tau_R(\overleftarrow{x}) = \infty$. Thus, since $x_0 = \inf \pi_0(A(\overleftarrow{x})) = \sup \pi_0(A(\overleftarrow{x}))$, it follows by Proposition 3 that point x is an endpoint of X. $\square$

The following corollary follows directly from Proposition 16 since a spiral point cannot be contained in the interior of an arc.

Corollary 5. *Non-degenerate arc-components in X are:*

- *lines (*i.e., *continuous images of $\mathbb{R}$) with no spiral points,*
- *rays (continuous images of $\mathbb{R}^+$), where only the endpoint can be a spiral point,*
- *arcs, where only endpoints can be spiral points.*

Remark 12. *Let $y \neq w \in X$. By Lemma 1, $A(\overleftarrow{y})$ and $A(\overleftarrow{w})$ are connected by finitely many basic arcs if and only if there exists $k \in \mathbb{N}$ such that $\dots y_{k+1} y_k = \dots w_{k+1} w_k$. We say that y and w have the same* tail. *Thus every arc-component is determined by its tail with the exception of (one or two) spiral points with different tails. This generalizes the symbolic representation of arc-components for finite critical orbit c given in [23] on arbitrary tent inverse limit space X.*

5.2 General results about accessibility

Definition 10. *We say that a continuum $K \subset \mathbb{R}^2$* does not separate the plane *if $\mathbb{R}^2 \setminus K$ is connected.*

The following proposition is a special case of Theorem 3.1. in [20].

Proposition 17. *Let $K \subset \mathbb{R}^2$ be a non-degenerate indecomposable continuum that does not separate the plane and let $Q = [x, y] \subset K$ be an arc. If x and y are accessible, then Q is fully accessible.*

Proof. Assume by contradiction that arc Q is not fully accessible. Because $x, y \in K$ are both accessible there exists a point $w \in \mathbb{R}^2 \setminus K$ and arcs $Q_x := [x, w], Q_y := [y, w] \subset \mathbb{R}^2$ such that $(x, w], (y, w] \subset \mathbb{R}^2 \setminus K$.

Note that $Q \cup Q_x \cup Q_y =: S$ is a simple closed curve in $\mathbb{R}^2$, see Figure 5.2. Thus $\mathbb{R}^2 \setminus S = S_1 \cup S_2$ where S_1 and S_2 are open sets in $\mathbb{R}^2$ such that $\partial S_1 = \partial S_2 = S$. Specifically S_1 contains no accumulation points of S_2 and vice versa. Denote by $K_1 := K \cap \mathrm{Cl}(S_1)$, $K_2 := K \cap \mathrm{Cl}(S_2)$. Note that K_1, K_2 are subcontinua of K and $K_1, K_2 \neq \emptyset$. Because Q is not fully accessible it follows that $K_1, K_2 \neq K$. Furthermore $K_1 \cup K_2 = K$, which is a contradiction with K being indecomposable. $\square$

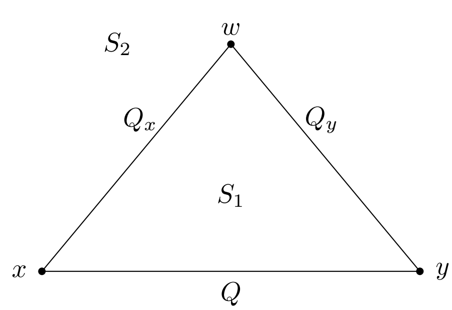

Figure 5.2: Simple closed curve from the proof of Theorem 17.

Corollary 6. *Let K be an indecomposable planar continuum that does not separate the plane and let $\mathcal{U}$ be an arc-component of a point from K. There are four possibilities regarding the accessibility of $\mathcal{U}$:*

- *$\mathcal{U}$ is fully accessible.*
- *There exists an accessible point $u \in \mathcal{U}$ such that one component of $\mathcal{U} \setminus \{u\}$ is not accessible, and the other one is fully accessible.*
- *There exist two (not necessarily different) accessible points $u, v \in \mathcal{U}$ such that $\mathcal{U} \setminus [u, v]$ is not accessible and $[u, v] \subset \mathcal{U}$ is fully accessible.*
- *$\mathcal{U}$ is not accessible.*

Proof. By Proposition 17, the set of accessible points in $\mathcal{U}$ is connected. To see it is closed, take a sequence $(x_i)_{i\in\mathbb{N}}$ of accessible points in $\mathcal{U}$ such that $\lim_{i\to\infty} x_i =: x \in \mathcal{U}$. Let $w \in \mathbb{R}^2 \setminus K$ and let $Q_i \subset \mathbb{R}^2$ be arcs with endpoints x_i and w and such that $Q_i \cap K = x_i$ for every $i \in \mathbb{N}$. Denote by S_i the bounded open set in $\mathbb{R}^2$ with boundary $Q_1 \cup Q_i \cup [x_1, x_i]$, where $[x_1, x_i] \subset \mathcal{U}$. Note that $K \cap S_i = \emptyset$ for every $i \in \mathbb{N}$, since otherwise K is decomposable by analogous arguments as in the proof of Proposition 17. Then also $K \cap (\cup_{i\in\mathbb{N}} S_i) = \emptyset$. Since x is contained in the boundary of $\cup_{i\in\mathbb{N}} S_i$, which is arc-connected (*i.e.,* any two distinct points from the boundary of $\cup_{i\in\mathbb{N}} S_i$ can be connected by an arc within the space), we conclude that x can be accessed with a ray from $\cup_{i\in\mathbb{N}} S_i \subset \mathbb{R}^2 \setminus K$. $\square$

Remark 13. *Note that it follows from the third item of Corollary 6 that there can exists an endpoint $u = v \in \mathcal{U}$ which is accessible and every $x \in \mathcal{U} \setminus \{u\}$ is not accessible. For instance such embeddings for Knaster continuum are described in [55] and the endpoint is the only accessible point in the arc-component $\mathcal{C}$. In the course of this thesis we show that all cases from Corollary 6 indeed occur in some embeddings of tent inverse limit spaces.*

5.3 Basic notions from the prime end theory

In this section we briefly recall Carathéodory's prime end theory. Although the focus of this thesis is not on the characterization of prime ends of studied embeddings of continua, we will include the study of prime ends of some interesting examples throughout the thesis. A general study of prime ends of standard planar embeddings appears at the end of the thesis.

Definition 11. *Let $K \subset \mathbb{R}^2$ be a plane non-separating continuum. A* crosscut *of $\mathbb{R}^2 \setminus K$ is an arc $Q \subset \mathbb{R}^2$ which intersects K only in its endpoints. Note that $K \cup Q$ separates the plane into two components, one bounded and the other unbounded. Denote the bounded component by B_Q. A sequence $\{Q_i\}_{i\in\mathbb{N}}$ of crosscuts is called a* chain*, if the crosscuts are pairwise disjoint,* $\operatorname{diam} Q_i \to 0$ *as $i \to \infty$ and $B_{Q_{i+1}} \subset B_{Q_i}$ for every $i \in \mathbb{N}$. We say that two chains $\{Q_i\}_{i\in\mathbb{N}}$ and $\{R_i\}_{i\in\mathbb{N}}$ are* equivalent *if for every $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $B_{R_j} \subset B_{Q_i}$ and for every $j \in \mathbb{N}$ there exists $i' \in \mathbb{N}$ such that $B_{Q_{i'}} \subset B_{R_j}$. An equivalence class $[\{Q_i\}_{i\in\mathbb{N}}]$ is called a* prime end*. A basis for the natural topology on the set of all prime ends consists of sets $\{[\{R_i\}_{i\in\mathbb{N}}] : B_{R_i} \subset B_Q$ for all $i\}$ for all crosscuts Q. The set of prime ends equipped with the natural topology is a topological circle, called the* circle of prime ends*, see* e.g. *Section 2 in [20].*

Definition 12. *Let $P = [\{R_i\}_{i\in\mathbb{N}}]$ be a prime end. The* principal set of P *is* $\Pi(P) = \{\lim Q_i :$ *$\{Q_i\}_{i\in\mathbb{N}} \in P$ is convergent$\}$ and the* impression of P *is $I(P) = \cap_i \mathrm{Cl}(B_{R_i})$. Note that both $\Pi(P)$ and $I(P)$ are subcontinua in X' and $\Pi(P) \subseteq I(P)$. We say that P is of the*

1. first kind *if $\Pi(P) = I(P)$ is a point.*

2. second kind *if $\Pi(P)$ is a point and $I(P)$ is non-degenerate.*

3. third kind *if $\Pi(P) = I(P)$ is non-degenerate.*

4. fourth kind *if $\Pi(P) \subsetneq I(P)$ are non-degenerate.*

Theorem 4 (Iliadis [35])**.** *Let K be a plane non-separating indecomposable continuum. The circle of prime ends corresponding to K can be decomposed into open intervals and their boundary points such that every open interval J uniquely corresponds to a composant of K which is accessible in more than one point and $I(e) \subsetneq K$ for every $e \in J$. For the boundary points e it holds that $I(e) = K$.*

Proposition 18. *Let K be a plane non-separating continuum such that every proper subcontinuum of K is an arc and such that every composant contains at most one folding point. Then $\Pi(P)$ is degenerate or equal to K for every prime end P. Specially, there exist no prime ends of the fourth kind.*

Proof. Assume there exists a prime end P such that $\Pi(P)$ is non-degenerate and not equal to K. Then $\Pi(P) = [a, b]$ is an arc in K. We claim that both a and b are folding points. Assume that there exists $\varepsilon > 0$ such that $B(a, \varepsilon) \cap K = C \times [0, 1]$, where C is the Cantor set and $B(a, \varepsilon)$ denotes the open planar ball of radius ε around the point a. Since $a \in \Pi(P)$, there exist a chain of crosscuts $\{Q_i\}_{i\in\mathbb{N}} \in P$ such that $Q_i \to a$ as $i \to \infty$. Note that $Q_i \in B(a, \varepsilon)$ for large enough i, so the endpoints of Q_i are contained in $C \times [0, 1]$ and the interior of Q_i does not intersect K. Therefore, it is possible to translate every Q_i along $[0, 1]$ and find a point $z \notin [a, b]$ in the arc-component of $[a, b]$ for which there exists a chain of crosscuts $\{R_i\}_{i\in\mathbb{N}}$ equivalent to $\{Q_i\}_{i\in\mathbb{N}}$ such that $R_i \to z$ as $i \to \infty$, see Figure 5.3. This contradicts the assumption, *i.e.*, point a is a folding point. The proof for the point b is analogous. We conclude that there exists a composant with at least two folding points, which is a contradiction. □

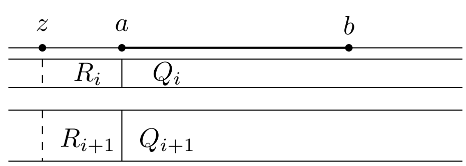

Figure 5.3: Translating the chain of crosscuts along $[0, 1]$ in Proposition 18.

Definition 13. *Let K be a plane non-separating continuum. A prime end P such that $\Pi(P)$ is non-degenerate but different than K is called an* infinite canal. *A third kind prime end P such that $\Pi(P) = I(P) = K$ is called a* simple dense canal.

We obtain the following corollary, which we use later in the thesis for discussing the prime end structure of $\mathcal{E}$-embeddings of X' when the critical orbit is finite.

Corollary 7. *Let K be an indecomposable plane non-separating continuum such that its every subcontinuum is an arc and every composant contains at most one folding point. Then the circle of prime ends corresponding to K can be partitioned into open intervals and their endpoints. Open intervals correspond to accessible open arcs in K. The endpoints of open intervals are the second or the third kind prime ends for which the impression is K. The second kind prime end corresponds to an accessible folding point in K and the third kind prime end corresponds to a simple dense canal in K.*

Question. If X' is the core of a tent map inverse limit, is there a planar embedding $\varphi\colon X' \to \mathbb{R}^2$ such that $\varphi(X')$ has fourth kind prime end?

5.4 An Intro to the study of accessible points of $\mathcal{E}$-embeddings

By Corollary 6, if $x \in \mathcal{U}_x \subset X$ is accessible it does not a priori follow that every point from $\mathcal{U}_x$ is accessible, see *e.g.* Figure 5.4. Recall that $X = \mathcal{C} \cup X'$. In this chapter we study the sets of accessible points of embeddings of either X or X' and the two cases substantially differ as we shall see in this section. In the rest of the thesis we are concerned only with embeddings of the cores X'.

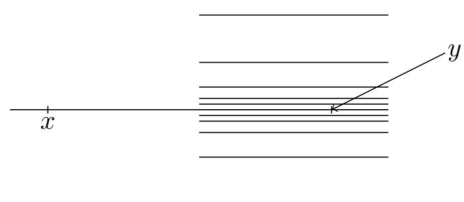

Figure 5.4: Point x is accessible from the complement while point y which has neighbourhood of Cantor set of arcs is not.

We will denote the smallest admissible left-infinite tail in X' with respect to $\prec_L$ by S. The arc-component of points from L (S) will be denoted from now onwards by $\mathcal{U}_L$ ($\mathcal{U}_S$). The following examples show that $\mathcal{U}_L$ and $\mathcal{U}_S$ do not necessarily coincide. Later in this section we will especially be concerned with the accessibility of $\mathcal{U}_L$ and $\mathcal{U}_S$.

Example 1. Assume that the kneading sequence is given by $\nu = (101)^\infty$. Embed X' in the plane according to the ordering in which $L = (01)^\infty$ is the largest. Note that the smallest sequence is then $S = (10)^\infty \not\subset \mathcal{U}_L$.

Example 2. Take the kneading sequence $\nu = 1001(101)^\infty$. Embed X' in the plane according to the ordering in which $L = ((001)(001101))^\infty$ is the largest. The smallest is then $S = ((100)(101100))^\infty \not\subset \mathcal{U}_L$. Note that in comparison with the previous example this time $S \neq \sigma^k(L)$ for every $k \in \mathbb{N}$.

Definition 14. *Let ν be a kneading sequence. For any admissible finite word $a_n \dots a_1 \in$*

$\{0,1\}^n$ *define the* cylinder $[a_n \dots a_1]$ *as*

$$[a_n \dots a_1] := \{\overleftarrow{s} = \dots s_{n+2}s_{n+1}a_n \dots a_1 : \overleftarrow{s} \text{ is an admissible left infinite sequence}\}.$$

Lemma 11. *If $a_n \dots a_1$ is admissible, then $[a_n \dots a_1]$ is not an empty set.*

Proof. Say that $1a_n \dots a_1$ is not admissible. In that case $1a_n \dots a_1 \succ c_1 \dots c_{n+1}$, so $a_n \dots a_1 \prec c_2 \dots c_{n+1}$, which is a contradiction with $a_n \dots a_1$ being admissible. Note that the left infinite tail $1^\infty a_n \dots a_1$ is admissible, which concludes this proof. $\square$

Definition 15. *Assume X is embedded in the plane with respect to $L = \dots l_2 l_1$ and take an admissible finite word $a_n \dots a_1$. The* top *of the cylinder $[a_n \dots a_1]$ is the left infinite sequence denoted by $L_{a_n\dots a_1} \in [a_n \dots a_1]$ such that $L_{a_n\dots a_1} \succeq_L \overleftarrow{s}$, for all $\overleftarrow{s} \in [a_n \dots a_1]$. Analogously we define the* bottom *of the cylinder $[a_n \dots a_1]$, denoted by $S_{a_n\dots a_1}$, as the smallest left infinite sequence in $[a_n \dots a_1]$ with respect to the order $\preceq_L$.*

Remark 14. *Note that each cylinder is a compact set (as a subset of the plane). Thus for admissible finite words $a_n \dots a_1$ there always exist $L_{a_n\dots a_1}$ and $S_{a_n\dots a_1}$ (they can be equal).*

Lemma 12. *Assume X is embedded in the plane with respect to L. For every admissible finite word $a_n \dots a_1$ the arcs $A(L_{a_n\dots a_1})$ and $A(S_{a_n\dots a_1})$ are fully accessible.*

Proof. Take a point $x \in A(L_{a_n\dots a_1})$ and denote by $p_x = \psi(L_{a_n\dots a_1})$ the point in the Cantor set C corresponding to the y-coordinate of x. Then the arc

$$Q = \left\{\left(\pi_0(x), p_x + \frac{z}{2\cdot 3^{n+1}}\right), z \in [0,1]\right\}$$

has the property that $Q \cap X = \{x\}$, see Figure 5.5. When $x \in A(S_{a_n\dots a_1})$, we can analogously construct the arc Q' such that $Q' \cap X = \{x\}$ and conclude that x is accessible. $\square$

From Lemma 12 it follows specially that $A(L)$ and $A(S)$ in Example 1 and Example 2 are fully accessible as they are the largest and the smallest arcs respectively among all the arcs in embedding of X' determined by L.

The following proposition is the first step in determining the set of accessible points of $\mathcal{E}$-embeddings.

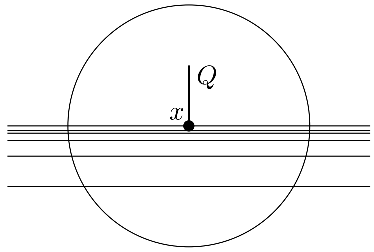

Figure 5.5: Point at the top of the cylinder $[a_n \dots a_1]$ is accessible by an arc Q.

Proposition 19. *Take $L = \dots l_2 l_1$ and construct the embedding of X with respect to L. Then every point in X with the same symbolic tail as L is accessible. If $A(L)$ is not a spiral point, then $\mathcal{U}_L$ is fully accessible.*

Proof. Take a point $x \in X$, where $\overleftarrow{x} = \dots x_2 x_1$ and there exists $n > 0$ such that $\dots x_{n+2} x_{n+1} = \dots l_{n+2} l_{n+1}$. If $\#_1(x_n \dots x_1)$ and $\#_1(l_n \dots l_1)$ have the same parity, then $\dots l_{n+2} l_{n+1} x_n \dots x_1 = L_{x_n \dots x_1}$ and it is equal to the $S_{x_n \dots x_1}$ otherwise. Lemma 12, Corollary 6 and Remark 12 conclude the proof. □

Definition 16. *Let $\varphi, \psi \colon K \to \mathbb{R}^2$ be two embeddings of a continuum K in the plane. We say that the embeddings are* equivalent *if the homeomorphism $\psi \circ \varphi^{-1} \colon \varphi(K) \to \psi(K)$ can be extended to a homeomorphism of the plane.*

By φ_L we denote the $\mathcal{E}$-embedding of X so that the arc $A(L)$ is the largest among all basic arcs. In the following proposition we observe that given two left-infinite sequences L^1, L^2 with eventually the same tail, we get equivalent embeddings.

Proposition 20. *Let $L^1 = \dots l_2^1 l_1^1$ and $L^2 = \dots l_2^2 l_1^2$ be such that there exists $n \in \mathbb{N}$ so that for every $k > n$ it holds that $l_k^1 = l_k^2$. Then the embeddings φ_{L^1} and φ_{L^2} of X are equivalent.*

Proof. If $\#_1(l_n^1 \dots l_1^1)$ and $\#_1(l_n^2 \dots l_1^2)$ are of the same (different) parity, then for every admissible $\overleftarrow{x} = \dots x_2 x_1$ and $\overleftarrow{y} = \dots y_2 y_1$ such that $x_n \dots x_1 = y_n \dots y_1$ it follows that $\overleftarrow{x} \prec_{L^1} \overleftarrow{y}$ if and only if $\overleftarrow{x} \prec_{L^2} \overleftarrow{y}$ ($\overleftarrow{x} \succ_{L^2} \overleftarrow{y}$).
We conclude that $\varphi_{L^2} \circ \varphi_{L^1}^{-1} \colon \varphi_{L^1}(X) \to \varphi_{L^2}(X)$ preserves (reverses) the order in every n-cylinder $[a_n \dots a_1]$. There exists a planar homeomorphism h so that $h|_{\varphi_{L^1}(X)} = \varphi_{L^2}(X)$ and h

permutes n-cylinders from the order determined by L^1 to the order determined by L^2, which concludes the proof. □

Now we briefly comment on $\mathcal{E}$-embeddings of X (including the ray $\mathcal{C}$). For the rest of the section assume that X is not the Knaster continuum (since then $X = X'$, *i.e.,* $\mathcal{C}$ is contained in the core X'). Let X be embedded in the plane with respect to $L = \ldots l_2 l_1 \neq 0^\infty l_n \ldots l_1$ for every $n \in \mathbb{N}$. The case when $\mathcal{E}$-embedding is equivalent to $L = 0^\infty 1$ (the Brucks-Diamond embedding from [23]) will be studied in Section 6.3.

Remark 15. *When we study X (*i.e., *including the arc-component $\mathcal{C}$), there exist cylinders $[a_n \ldots a_1]$ where $a_n \ldots a_1$ is not an admissible word, but there is $k \in \{1, \ldots, n-1\}$ such that $a_k \ldots a_1$ is admissible, $a_k = 1$ and $a_n \ldots a_{k+1} = 0^{n-k}$. In that case, $[a_n \ldots a_1]$ contains only one basic arc, that is $[a_n \ldots a_1] = \{0^\infty a_n \ldots a_1\}$ and $L_{a_n \ldots a_1} = S_{a_n \ldots a_1} = 0^\infty a_n \ldots a_1$.*

Remark 16. *The arc-component $\mathcal{C}$ is isolated (when X is not the Knaster continuum), and thus it is fully accessible in any $\mathcal{E}$-embedding of X.*

Proposition 21. *Take an admissible left-infinite sequence $\overleftarrow{a} = \ldots a_2 a_1$ such that $A(\overleftarrow{a}) \not\subset \mathcal{C}$ and $a_n \neq l_n$ for infinitely many $n \in \mathbb{N}$. Then there exist sequences $(\overleftarrow{s_i})_{i\in\mathbb{N}}$ and $(\overleftarrow{t_i})_{i\in\mathbb{N}}$ such that $A(\overleftarrow{s_i}), A(\overleftarrow{t_i}) \subset \mathcal{C}$, $\overleftarrow{s_i}, \overleftarrow{t_i} \to \overleftarrow{a}$ as $i \to \infty$ and $\overleftarrow{s_i} \prec_L \overleftarrow{a} \prec_L \overleftarrow{t_i}$.*

Proof. First note that the assumption $A(\overleftarrow{a}) \not\subset \mathcal{C}$ is indeed needed since by Remark 16, $\mathcal{C}$ is isolated and thus the statement of the proposition does not hold for basic arcs from $\mathcal{C}$; thus assume $A(\overleftarrow{a}) \not\subset \mathcal{C}$.

Let $(N_i)_{i\in\mathbb{N}}$ be the sequence of natural numbers such that $a_n \neq l_n$ for $n \in \{N_i : i \in \mathbb{N}\}$. Since $a_n \neq l_n$ for infinitely many $n \in \mathbb{N}$ such sequence $(N_i)_{i\in\mathbb{N}}$ indeed exists. Denote by

$$\overleftarrow{t_i} := 0^\infty a^*_{N_{2i-1}} a_{N_{2i-1}-1} \ldots a_1$$

$$\overleftarrow{s_i} := 0^\infty a^*_{N_{2i}} a_{N_{2i}-1} \ldots a_1$$

for every $i \in \mathbb{N}$. By contradiction, if a sequence $\overleftarrow{t_i}$ is not admissible it holds that $1 a_{N_{2i-1}-1} \ldots a_1 \succ_L \nu$. Thus, $a_{N_{2i-1}-1} \ldots a_1 \prec \overrightarrow{c_2}$ which is a contradiction with $a_{N_{2i-1}-1} \ldots a_1$ being an admissible word. Thus $\overleftarrow{t_i}$ is admissible sequence and proof goes analogously for $\overleftarrow{s_i}$. Note that $A(\overleftarrow{t_i}), A(\overleftarrow{s_i}) \subset \mathcal{C}$ for every $i \in \mathbb{N}$.

Since $\#_1(a_{N_{2i-1}-1}\dots a_1)$ and $\#_1(l_{N_{2i-1}-1}\dots l_1)$ are of the same parity (the sequences differ on even number of entries) and $\#_1(a_{N_{2i}-1}\dots a_1)$ and $\#_1(l_{N_{2i}-1}\dots l_1)$ are of different parity (the sequences differ on odd number of entries), it holds that $\overleftarrow{s_i} \prec_L \overleftarrow{a} \prec_L \overleftarrow{t_i}$ for every $i \in \mathbb{N}$. $\square$

Combining Proposition 19 with Proposition 21 we obtain that only basic arcs from $\mathcal{U}_L$ or $\mathcal{C}$ can be tops or bottoms of cylinders of $\mathcal{E}$-embeddings of X. Thus we obtain the following corollary.

Corollary 8. *If $A(L)$ is not a spiral point, then $\varphi_L(X)$ has exactly two non-degenerate fully accessible arc-components, namely $\mathcal{U}_L$ and $\mathcal{C}$ (however in the embedding by Brucks-Diamond it holds that $\mathcal{C} = \mathcal{U}_L$). If $A(L)$ is non-degenerate, there are two remaining points on the circle of prime ends and they correspond either to an infinite canal in X or to a folding point. If $A(L)$ is degenerate then there are no infinite canals in X.*

The following statements are going to be used often throughout the thesis to determine that an arc-component is fully accessible.

Definition 17. *Let $\overleftarrow{s} = \dots s_2 s_1$ be an admissible left-infinite sequence. If $\tau_R(\overleftarrow{s}) < \infty$, the tail $\overleftarrow{r(s)} = \dots s_{\tau_R(\overleftarrow{s})+1} s^*_{\tau_R(\overleftarrow{s})} s_{\tau_R(\overleftarrow{s})-1} \dots s_1$ is called* the right neighbour of $\overleftarrow{s}$ *and if $\tau_L(\overleftarrow{s}) < \infty$, the tail $\overleftarrow{l(s)} = \dots s_{\tau_L(\overleftarrow{s})+1} s^*_{\tau_L(\overleftarrow{s})} s_{\tau_L(\overleftarrow{s})-1} \dots s_1$ is called* the left neighbour of $\overleftarrow{s}$.

Proposition 22. *Embed X' in the plane with respect to L. Assume $\overleftarrow{s}$ is at the bottom (top) of some cylinder. If $\overleftarrow{r(s)}$ is not at the top (bottom) of any cylinder, then $A(\overleftarrow{r(s)})$ contains an accessible folding point, see Figure 5.6. Analogous statement holds for $\overleftarrow{l(s)}$.*

Proof. If $\overleftarrow{r(s)}$ is not the top (bottom) of any cylinder, then there exist left-infinite admissible sequences $\overleftarrow{x_i} \succ_L \overleftarrow{r(s)}$ ($\overleftarrow{x_i} \prec_L \overleftarrow{r(s)}$) such that $\overleftarrow{x_i} \to \overleftarrow{r(s)}$ as $i \to \infty$. If $\tau_R(\overleftarrow{x_i}) = \infty$ for infinitely many $i \in \mathbb{N}$, we have found a folding point in $A(\overleftarrow{r(s)})$. So assume without loss of generality that $\tau_R(\overleftarrow{x_i}) < \infty$ for all $i \in \mathbb{N}$. If $\overleftarrow{s} \succ_L \overleftarrow{r(x_i)}$ ($\overleftarrow{s} \prec_L \overleftarrow{r(x_i)}$) for infinitely many $i \in \mathbb{N}$ we get a contradiction with $\overleftarrow{s}$ being the top (bottom) of some cylinder. But then $\overleftarrow{r(x_i)} \prec_L \overleftarrow{r(s)}$ ($\overleftarrow{r(x_i)} \succ_L \overleftarrow{r(s)}$) for all but finitely many $i \in \mathbb{N}$ which gives a folding point in $A(\overleftarrow{r(s)})$ again. $\square$

The following corollary follows directly from Proposition 22.

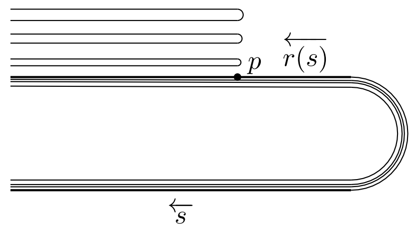

Figure 5.6: Setup of Proposition 22, where p is a folding point.

Corollary 9. *Let $\mathcal{U} \subset X'$ be an arc-component which contains no folding points and let X' be $\mathcal{E}$-embedded. If there exists a basic arc from $\mathcal{U}$ that is fully accessible, then $\mathcal{U}$ is fully accessible.*

Remark 17. *When we embed only the core X', there can exist accessible points in $X' \setminus \mathcal{U}_L$, see* e.g. *Example 1 and Example 2. In these two examples $\mathcal{U}_S \neq \mathcal{U}_L$ and points from $A(S)$ are accessible. In some cases $\mathcal{U}_S$ is fully accessible (see Lemma 21 from Section 5.7), but that is not always the case. In Section 5.6.2 we explicitly construct examples in which the arc-component $\mathcal{U}_S$ is only partially accessible.*

From Lemma 12 it follows that the points at the top or bottom of cylinders are accessible. If a point which is not at the top or bottom of any cylinder has a neighbourhood homeomorphic to the Cantor set of arcs, we can conclude that is not accessible. However, the accessibility of folding points needs to be studied separately, since it is not straightforward to determine if they are accessible or not in a given embedding, see for example Figure 5.7. Thus we need to do a detailed study on conditions for a folding point to be accessible. For instance, in special embeddings of the Knaster continuum in [55] the endpoint is always accessible.

5.5 Tops/bottoms of finite cylinders

In this section we study the symbolics of tops/bottoms of cylinders depending on an $\mathcal{E}$-embedding of X' and we restrict to cases where $L \neq 0^\infty l_n \dots l_1$ for all $n \in \mathbb{N}$.

For $t \in \{0,1\}$, we denote by $t^* = 1 - t$. For $A = a_1 \dots a_n \in \{0,1\}^n$ denote by ${}^*\!A = a_1^* a_2 \dots a_n$, $A^* = a_1 \dots a_{n-1} a_n^*$ and ${}^*\!A^* = a_1^* a_2 \dots a_{n-1} a_n^*$.

Definition 18. *Let ν be a kneading sequence. We say that a finite word $a_1 \dots a_n \in \{0,1\}^n$ is*

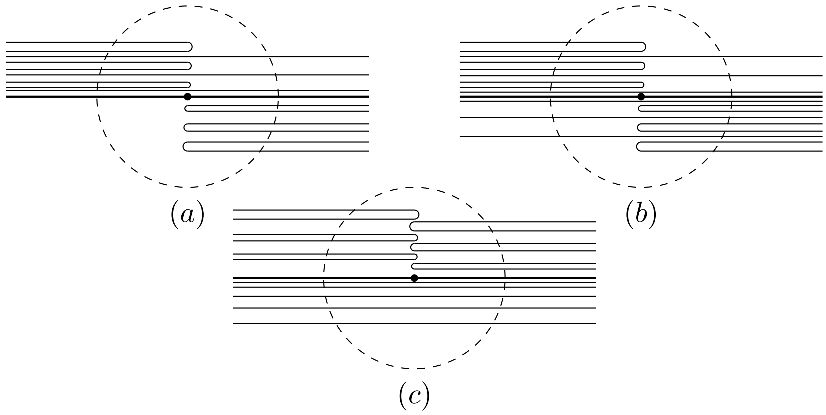

Figure 5.7: Neighbourhoods of folding points. In Case (a) and (c) folding point is accessible, while in Case (b) it is not.

irreducibly non-admissible *if it is not admissible and* $a_2 \dots a_n$ *is admissible.*

Definition 19. *Fix a kneading sequence* ν*. We say that a finite cylinder* $B = [b_n \dots b_1]$ *of length* $n \in \mathbb{N}$ alters $L = \dots l_2 l_1$*, if there exist words* $(A_i)_{i \in \mathbb{N}}$ *such that* $\dots A_3 A_2 A_1 = \dots l_{n+2} l_{n+1}$ *and the words* $A_1 B$ *and* $A^*{}_i{}^*A^*{}_{i-1} \dots {}^*A^*{}_2{}^*A_1 B$ *are irreducibly non-admissible for every* $i \geq 2$.

Proposition 23. *If a finite cylinder* B *alters the admissible sequence* L *then* L_B *or* S_B *has different tail than* L.

Proof. Assume B alters L with words A_i as in the definition. If $\#_1(B) - \#_1(l_n \dots l_1)$ is even, then $L_B = \dots {}^*A^*{}_i{}^*A^*{}_{i-1} \dots {}^*A^*{}_2{}^*A_1 B$. The sequence L_B differs from L on infinitely many places. If $\#_1(B) - \#_1(l_n \dots l_1)$ is odd, then $S_B = \dots {}^*A^*{}_i{}^*A^*{}_{i-1} \dots {}^*A^*{}_2{}^*A_1 B$. $\square$

The following example shows that there exist $\mathcal{E}$-embeddings of X' such that none of the extrema of certain cylinders are contained in $\mathcal{U}_L$.

Example 3. Let $\nu = (100111011)^\infty$ and $L = (001)^\infty 11$. Note that $S_{10} = (100)^\infty(101)10 \subset \mathcal{U}_{L_{10}}$ and $L_{10} = (100)^\infty 10 \subset \mathcal{U}_{L_{10}}$. Therefore, $L_{10}, S_{10} \not\subset \mathcal{U}_L$.

In Example 3 both extrema belong to the same arc-component. This is not necessarily always the case, see *e.g.* Example 4 below.

Proposition 24. *If a finite cylinder B is such that $L_B \not\subset \mathcal{U}_L$ or $S_B \not\subset \mathcal{U}_L$, then there exists a finite cylinder B' such that B' alters L.*

Proof. Assume $\#_1(B) - \#_1(l_n \dots l_1)$ is even and $L_B \not\subset \mathcal{U}_L$. Then obviously $B' = B$ alters L. Similarly, if $\#_1(B) - \#_1(l_n \dots l_1)$ is odd and $S_B \not\subset \mathcal{U}_L$. So assume $\#_1(B) - \#_1(l_n \dots l_1)$ is even and $S_B \not\subset \mathcal{U}_L$. Then $l^*_{n+1}B$ alters L, if $l^*_{n+1}B$ is admissible. If $l^*_{n+1}B$ is not admissible, there exists $i \in \mathbb{N}$ such that $l^*_{n+i} \dots l_n B$ is admissible, since otherwise $S_B = L_B$, which is a contradiction. The word $l^*_{n+i} \dots l_n B$ alters L. Analogously if $\#_1(B) - \#_1(l_n \dots l_1)$ is odd and $L_B \not\subset \mathcal{U}_L$. □

Example 4. Let $\nu = 1001(101)^\infty$ and $L = ((001)(001101))^\infty$. Then it holds that $S = S_0 = ((100)(101100))^\infty \not\subset \mathcal{U}_L$. So $B = 0$ alters L and words A_i are divided by brackets.

Next we show there exist $\mathcal{E}$-embeddings with more than two accessible arc-components.

Proposition 25. *Assume that ν starts with some finite words $\nu = 1B\dots = 1ABA\dots$, where B^* and ABA^* are irreducibly non-admissible. The embedding of X' with respect to $L = (BA)^\infty$ contains at least three tails which are extrema of cylinders.*

Proof. Note that $S = ({}^*B^{*}{}^*ABA^*)^\infty$. Take any admissible word D such that $|D| = |A|$ and such that $\#_1(D) - \#_1(A)$ is even. Then $S_D = ({}^*ABA^{*}{}^*B^*)^\infty D$ and therefore we found three different tails which are extrema of cylinders. □

The following example shows that it is indeed possible to satisfy the conditions of Proposition 25.

Example 5. Take $\nu = 1001100100111\dots$, $B = 001$, $A = 0011$ and $L = (BA)^\infty$ which is easily checked to be admissible. For D take *e.g.* $D = 1111$. Note that $S = ({}^*B^{*}{}^*ABA^*)^\infty$ and $S_D = ({}^*ABA^{*}{}^*B^*)^\infty D$ and thus we obtain three accessible basic arcs with different tails. If we take *e.g.* $\nu = (10011001001111)^\infty$, since by Remark 7 the only folding points are endpoints and there are no spiral points in X', it follows by Lemma 22 that there are three fully accessible non-degenerate dense arc-components. Moreover, none of those arc-components contains an endpoint so they are all lines. We will return to this particular example in Section 5.6, Example 9.

5.6 Accessible folding points

In this section we study accessibility of folding points which are not at the top or the bottom of any cylinder.

5.6.1 Accessible endpoints

Let us fix X' and the $\mathcal{E}$-embedding depending on L. Recall that we denote by $\mathcal{U}_L$ the arc-component of $x \in A(L) \subset X'$. By Proposition 19, every point with the same symbolic tail as L is accessible.

The following remark is a direct consequence of Proposition 3.

Remark 18. *If $e \in X'$ is an endpoint, then there exists a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ such that $\bar{e} = \dots e_{n_i+1} c_1 \dots c_{n_i} . c_{n_i+1} \dots = \dots e_{n_i+1}\nu$ for every $i \in \mathbb{N}$.*

In this section we work with the concept of an endpoint being capped which is defined below. See Figure 5.8.

Definition 20. *Let $e \in X'$ be an endpoint with $\tau_L(\overleftarrow{e}) = \infty$ ($\tau_R(\overleftarrow{e}) = \infty$). We say that a point e is* capped from the left (right), *if there exist sequences of admissible itineraries $(\overleftarrow{y}^i)_{i\in\mathbb{N}}, (\overleftarrow{w}^i)_{i\in\mathbb{N}} \subset \{0,1\}^\infty$ such that $\overleftarrow{y}^i, \overleftarrow{w}^i \to \overleftarrow{e}$ as $i \to \infty$, $\overleftarrow{y}^i \prec_L \overleftarrow{e} \prec_L \overleftarrow{w}^i$ for every $i \in \mathbb{N}$ and arcs $A(\overleftarrow{y}^i)$ and $A(\overleftarrow{w}^i)$ are joined on the left (right).*

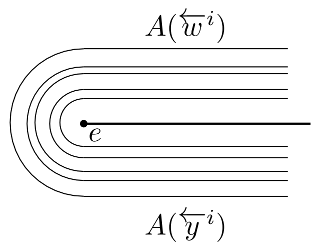

Figure 5.8: Endpoint e is capped from the left.

Remark 19. *If $e \in X'$ is a right (left) endpoint which is not capped from the right (left), then e is accessible by a horizontal arc in the plane. Note that if $\overleftarrow{e}$ lies on an extremum of a cylinder (which holds if* e.g. *e has the same symbolic tail as L), then e is not capped.*

Remark 20. *Let $\nu = 10^\infty$, i.e., $X = X'$ is a Knaster continuum and let L be arbitrary. Note that any two points $x, y \in X'$ that are $\varepsilon > 0$ close to the point $\bar{0}$ and are identified have the form $x_k x_{k-1} \dots x_1 = y_k y_{k-1} \dots y_1 = 10^{k-1}$ for some $k \in \mathbb{N}$. It follows that either $\overleftarrow{x}, \overleftarrow{y} \prec_L \overleftarrow{0}$ or $\overleftarrow{x}, \overleftarrow{y} \succ_L \overleftarrow{0}$, depending on the parity of $\#_1(l_{k-1} \dots l_1)$. Therefore, the endpoint $\bar{0} \in X'$ is not capped and thus always accessible in $\mathcal{E}$-embeddings of the Knaster continuum, see Figure 5.9.*

From now on we assume in this subsection that X' is not the Knaster continuum and thus $\nu \neq 10^\infty$.

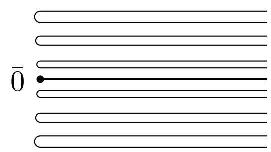

Figure 5.9: Neighbourhood of the end-point $\bar{0}$ of the Knaster continuum ($\nu = 10^\infty$) in an $\mathcal{E}$-embedding.

It is well known (see *e.g.* [11]) that X' contains endpoints if and only if the critical point c of map T is recurrent (*i.e.*, $T^n(c)$ get arbitrary close to c as $n \to \infty$).

Definition 21. *Fix a kneading sequence ν and let $e \in X'$ be an endpoint and thus $\tau_L(\overleftarrow{e}) = \infty$ ($\tau_R(\overleftarrow{e}) = \infty$). A sequence $(m_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ is called the* complete sequence for *e, if for every $n \in \mathbb{N}$ such that $e_n \dots e_1 = c_1 c_2 \dots c_n$ and $\#_1(c_1 c_2 \dots c_n)$ is odd (even) there exist $i \in \mathbb{N}$ such that $m_i = n$.*

From $\tau_L(\overleftarrow{e}) = \infty$ (or $\tau_R(\overleftarrow{e}) = \infty$) it follows that the sequence $(m_i)_{i\in\mathbb{N}}$ indeed exists.
The main result in this subsection is that every endpoint of X' (where X' is not the Knaster continuum) which is not contained in $\mathcal{U}_L$ is capped in an $\mathcal{E}$-embedding of X' which is not equivalent to Brucks-Diamond embedding from [23]. In the proof of Theorem 5 we construct an increasing subsequence $(n_i)_{i\in\mathbb{N}} \subset (m_i)_{i\in\mathbb{N}}$ and basic arcs $A(\overleftarrow{x}^{O(i)}), A(\overleftarrow{x}^{I(i)}) \subset \mathcal{R} \subset X'$ such that

$$\overleftarrow{x}^{O(i)} = 1^\infty a_k^i \dots a_1^i 0 c_1 c_2 \dots c_{n_i} \qquad \overleftarrow{x}^{I(i)} = 1^\infty a_k^i \dots a_1^i 1 c_1 c_2 \dots c_{n_i}. \tag{5.1}$$

and $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{O(i)}$ for some admissible word $a_k^i \dots a_1^i \in \{0,1\}^k$. Note that the arcs $A(\overleftarrow{x}^{O(i)})$ and $A(\overleftarrow{x}^{I(i)})$ are joined by left (right) semi-circle. Here

$\mathcal{R}$ denotes the arc-component of the point $\bar{1}$ which is a dense line in X' independently on the choice of ν (see Proposition 1 in [21]).

Remark 21. *Let $e \in X'$ be an endpoint and thus $\tau_L(\overleftarrow{e}) = \infty$ ($\tau_R(\overleftarrow{e}) = \infty$). Then $\#_1(c_1 \dots c_{m_i})$ is odd (even) and $\#_1(c_1 \dots c_{m_{i+1}-m_i})$ is even (even) for every $i \in \mathbb{N}$.*

Definition 22. *For $\nu = c_1c_2\dots$ we define*

$$\kappa := \min\{i-2 : i \geq 3, c_i = 1\}.$$

Remark 22. *Definition 22 says that the beginning of the kneading sequence is $\nu = 10^{\kappa}1\dots$. If $\kappa = 1$, since we restrict to non-renormalizable tent maps, we can conclude even more, namely that $\nu = 10(11)^n0\dots$, for some $n \in \mathbb{N}$.*

Remark 23. *Fix the kneading sequence ν. Assume that $a_{n-1}\dots a_1 \in \{0,1\}^{n-1}$ is admissible but $a_n \dots a_1 \in \{0,1\}^n$ is not. Then $a_n \dots a_1 \prec c_2 \dots c_{n+1}$.*

Lemma 13. *Let ν be an admissible kneading sequence. A word $c_2 \dots c_n^*$ is not admissible if and only if either $\#_1(c_2 \dots c_n)$ is odd or there exists $k \in \{3, \dots, n\}$ such that $c_k \dots c_n = c_2 \dots c_{n-k+2}$ and $\#_1(c_k \dots c_n)$ is odd.*

Proof. Assume that $c_2 \dots c_n^*$ is not admissible, so there exists $i \in \{2, \dots, n\}$ such that $c_i \dots c_n^*$ is not admissible. Take the largest such index i and note that $c_i \dots c_n = c_2 \dots c_{n-i+2}$ and $c_2 \dots c_{n-i+2}^* \prec c_2 \dots c_{n-i+2}$. Let us assume by contradiction that $\#_1(c_2 \dots c_{n-i+2})$ is even. If $c_{n-i+2} = 0$ ($c_{n-i+2} = 1$) it follows that $\#_1(c_2 \dots c_{n-i+1})$ is even (odd) and in both cases $c_2 \dots c_{n-i+2}^* \succ c_2 \dots c_{n-i+2}$ and thus $c_2 \dots c_{n-i+2}^*$ is admissible, a contradiction. □

Lemma 14. *Let ν be an admissible kneading sequence and let $(m_i)_{i\in\mathbb{N}}$ be the complete sequence for an endpoint $e \in X'$. Then for every natural number $k \geq 3$ and $j \in \{0, \dots, m_i\}$, the word $c_k \dots c_{m_{i+1}-m_i}^* c_1c_2 \dots c_j$ is admissible for every $i \in \mathbb{N}$. Specifically, if $j = 0$, we set $c_1 \dots c_j = \emptyset$.*

Proof. Assume by contradiction that there exists $k \geq 3$ and $j \in \mathbb{N}_0$ such that the word $c_k \dots c_{m_{i+1}-m_i}^* c_1c_2 \dots c_j$ is not admissible and assume that k is the largest and j is the smallest such index. By the choice of k and j every proper subword of $c_k \dots c_{m_{i+1}-m_i}^* c_1c_2 \dots c_j$ is admissible. Thus $c_k \dots c^*_{m_{i+1}-m_i} c_1 c_2 \dots c_j = c_2 c_3 \dots c_{m_{i+1}-m_i-k}$ $c_{m_{i+1}-m_i-k+1} \dots c^*_{m_{i+1}-m_i-k+j+1}$ and $\#_1(c_2 c_3 \dots c^*_{m_{i+1}-m_i-k+j+1})$ is even by Lemma 13. Furthermore, Lemma 13 implies that $\#_1(c_k \dots c^*_{m_{i+1}-m_i} = c_2 \dots c_{m_{i+1}-m_i-k-1})$ is even.

If $j = 1$, then both $\#_1(c_k \dots c^*_{m_{i+1}-m_i})$ and $\#_1(c_k \dots c^*_{m_{i+1}-m_i} c_1)$ are even, which is impossible.

If $j \geq 2$, it follows by Lemma 13 that $\#_1(c_2 \dots c_j)$ is odd. Thus $c_2 \dots c^*_j = c_{m_{i+1}-m_i-k+1}$ $\dots c_{m_{i+1}-m_i-k+j+1}$ is not admissible, which is a contradiction, since $c_2 c_3 \dots c^*_j$ is a subword of ν.

Let $c_1 \dots c_j$ be an empty word. Then $c_k \dots c_{m_{i+1}-m_i} = c_2 c_3 \dots c_{m_{i+1}-m_i-k} c_{m_{i+1}-m_i-k+1}$ and $\#_1(c_2 c_3 \dots c_{m_{i+1}-m_i-k} c_{m_{i+1}-m_i-k+1})$ is odd. Let l be the maximal natural number such that $c_{m_{i+1}-m_i+1} \dots c_{m_{i+1}-m_i+l} = c_{m_{i+1}-m_i-k+2} \dots c_{m_{i+1}-m_i-k+l+1}$, *i.e.*,

$$c_k \dots c_{m_{i+1}-m_i+l} = c_2 c_3 \dots c_{m_{i+1}-m_i-k+l+1}$$

and $c_{m_{i+1}-m_i+l+1} \neq c_{m_{i+1}-m_i-k+l+2}$. Such l indeed exists since (m_i) is complete. Note that $c_{m_{i+1}-m_i-k+2} \dots c_{m_{i+1}-m_i-k+l+1} = c_1 \dots c_l$ and $\#_1(c_1 \dots c_{l+1})$ is odd by Lemma 13. Thus $\#_1(c_1 \dots c_l c^*_{l+1})$ is even and we conclude that $\#_1(c_2 \dots c_{m_{i+1}-m_i-k+l+2})$ is odd. Since $c_2 \dots c^*_{m_{i+1}-m_i-k+l+2} = c_k \dots c_{m_{i+1}-m_i+l+1}$ is admissible, we get a contradiction. □

The main idea of the proof of the following theorem is illustrated in Example 6.

Theorem 5. *Let $e \in X'$ be an endpoint such that $\tau_R(\overleftarrow{e}) = \infty$ ($\tau_L(\overleftarrow{e}) = \infty$) and let $L = \dots l_2 l_1 \neq 0^\infty l_n \dots l_1$ be admissible and $\nu \neq 10^\infty$. If L and $\overleftarrow{e}$ have different tails, then e is capped from the right (left).*

Proof. Let $(m_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ be the complete sequence for an endpoint e where $\tau_R(\overleftarrow{e}) = \infty$. The proof works analogously if $\tau_L(\overleftarrow{e}) = \infty$. We will find infinitely many $i \in \mathbb{N}$ such that $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ (or with reversed inequalities) and arcs $\overleftarrow{x}^{O(i)}$ and $\overleftarrow{x}^{I(i)}$ are joined by a semi-circle on the right.

Fix some $i \in \mathbb{N}$ and let $M_1(i) > m_i + 1$ be the smallest natural number such that $l_{M_1(i)} = e^*_{M_1(i)}$. Note that such $M_1(i)$ exists, otherwise $\overleftarrow{e}$ and L have the same tail.

Assume that $M_1(i) = m_{i+1}$. Note that then $e_{M_1(i)-1} \dots e_{m_i+1} = c_2 \dots c_k = 0^\kappa 1 c_{\kappa+2} \dots c_k$ ($c_{\kappa+2} \dots c_k$ can be empty) and $e_{M_1(i)} = 1$. Then $l_{M_1(i)} \dots l_{m_i+1} = 0^{\kappa+1} 1 c_{\kappa+3} \dots c_k$, which is not admissible.

Assume that $M_1(i) = m_{i+1} + 1$. By the paragraph above $M_1(i+1) \neq m_{i+2}$. If $M_1(i+1) = m_{i+2}+1$, then $l_{m_{i+2}} \dots l_{m_{i+1}+2} l_{m_{i+1}+1} = c_1 \dots c_{m_{i+2}-m_{i+1}+1} c^*_{m_{i+2}-m_{i+1}}$ which is not admissible since $c_1 \dots c_{m_{i+2}-m_{i+1}}$ is even by Remark 21. So either $M_1(i) \in \{m_i+2, \dots m_{i+1}-1\}$ or there is $k \in \mathbb{N}$ such that $M_1(i+k) = M_1(i)$. Note that there are infinitely many $i \in \mathbb{N}$ such that $M_1(i) \in \{m_i+2, \dots m_{i+1}-1\}$ and from now on we work with such $i \in \mathbb{N}$.

If both of the following sequences are admissible, we set:

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{M_1(i)} e_{M_1(i)-1} \dots e_{m_i+2} 0 e_{m_i} \dots e_1,$$

$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{M_1(i)} e_{M_1(i)-1} \dots e_{m_i+2} 1 e_{m_i} \dots e_1.$$

a) Assume that $\#_1(e_{m_i} \dots e_1)$ and $\#_1(l_{m_i} \dots l_1)$ have the same parity and e_{m_i+1} $= l_{m_i+1} = 0$.
Then it follows that $\overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$. Because $l_{M_1(i)-1} \dots l_{m_i+2} = e_{M_1(i)-1} \dots e_{m_i+2}$ the parities of $\#_1(e_{M_1(i)-1} \dots e_1)$ and $\#_1(l_{M_1(i)-1} \dots l_1)$ are the same and because $l_{M_1(i)} = e^*_{M_1(i)}$ it follows that $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e}$.

b) Assume that $\#_1(e_{m_i} \dots e_1)$ and $\#_1(l_{m_i} \dots l_1)$ have the same parity and $e_{m_i+1} = l_{m_i+1} = 1$. Then it follows that $\overleftarrow{e} \succ_L \overleftarrow{x}^{O(i)}$. Because $l_{M_1(i)-1} \dots l_{m_i+2} = e_{M_1(i)-1} \dots e_{m_i+2}$ the parities of $\#_1(e_{M_1(i)-1} \dots e_1)$ and $\#_1(l_{M_1(i)-1} \dots l_1)$ are the same and because $l_{M_1(i)} = e^*_{M_1(i)}$ it follows that $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e}$.

c) Assume that $\#_1(e_{m_i} \dots e_1)$ and $\#_1(l_{m_i} \dots l_1)$ have the same parity and $e_{m_i+1} = 1 \neq 0 = l_{m_i+1}$.
Then $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e}$. Since the parities of $\#_1(e_{M_1(i)-1} \dots e_1)$ and $\#_1(l_{M_1(i)-1} \dots l_1)$ are different and $l_{M_1(i)} = e^*_{M_1(i)}$, it follows that $\overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$.

d) Assume that $\#_1(e_{m_i} \dots e_1)$ and $\#_1(l_{m_i} \dots l_1)$ have the same parity and $e_{m_i+1} = 0 \neq 1 = l_{m_i+1}$.
Then $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e}$. Since the parities of $\#_1(e_{M_1(i)-1} \dots e_1)$ and $\#_1(l_{M_1(i)-1} \dots l_1)$ are different and $l_{M_1(i)} = e^*_{M_1(i)}$, it follows that $\overleftarrow{e} \succ_L \overleftarrow{x}^{O(i)}$.

Note that if $\#_1(e_{m_i}\dots e_1)$ and $\#_1(l_{m_i}\dots l_1)$ are of different parities, then all the inequalities in cases a), b), c) and d) are reversed and we use analogous arguments to conclude that either $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{O(i)}$.

Now assume that one of $e^*_{M_1(i)}e_{M_1(i)-1}\dots e_{m_i+2}e^{(*)}_{m_i+1}\dots e_1$ is not admissible (where $s^{(*)}$ means either s^* or s). Then we set $x^{O(i)}_{M_1(i)} = x^{I(i)}_{M_1(i)} = e_{M_1(i)}$. If $e_{M_1(i)+1} = l_{M_1(i)+1}$, then we set $x^{O(i)}_{M_1(i)+1} = x^{I(i)}_{M_1(i)+1} = e^*_{M_1(i)+1}$ and we argue that $e^*_{M_1(i)+1}e_{M_1(i)}\dots e_{m_i+2}e^{(*)}_{m_i+1}\dots e_1 = e^*_{M_1(i)+1}10^{\kappa-1}1\dots e_1$ are admissible words. Indeed, word $e_{M_1(i)}\dots e_{m_i+2}e^{(*)}_{m_i+1}\dots e_1$ is admissible by Lemma 14. If $e^*_{M_1(i)+1}10^{\kappa-1}1\dots$ were not admissible, then $T^3(c) > T^4(c)$ which is a contradiction with T being non-renormalizable. So the following sequences are admissible:

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{M_1(i)+1}e_{M_1(i)}\dots e_{m_i+2}0e_{m_i}\dots e_1,$$

$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{M_1(i)+1}e_{M_1(i)}\dots e_{m_i+2}1e_{m_i}\dots e_1,$$

and $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{O(i)}$.

Assume that $e^*_{M_1(i)+1} = l_{M_1(i)+1}$. Set $x^{O(i)}_{M_1(i)+1} = x^{I(i)}_{M_1(i)+1} = e_{M_1(i)+1}$. Then the words $e_{M_1(i)+1}e_{M_1(i)}\dots e^{(*)}_{m_i+1}e_{m_i}\dots e_1$ are admissible by Lemma 14, if $M_1(i)+1 \neq m_{i+1}-1$.
Now say that $M_1(i) = m_{i+1}-2$. By the assumption in the beginning of this paragraph, at least one of the words $e^*_{m_{i+1}-2}e_{m_{i+1}-3}\dots e_{m_i+1}e^{(*)}_{m_i+1}\dots e_1$ is not admissible.
a) Say $\nu = 10^\kappa 1\dots$, where $\kappa > 1$. By Lemma 14, $e^*_{m_{i+1}-2}e_{m_{i+1}-3}\dots = c^*_3c_4c_5\dots = 10^{\kappa-2}1\dots$ is always admissible, a contradiction.
b) Say that $\nu = 10(11)^n0\dots$. Then $e^*_{m_{i+1}-2}e_{m_{i+1}-3}\dots = 0(11)^{n-1}10\dots$ is again always admissible, because $\#_1(0(11)^{n-1}1)$ is odd, a contradiction.

Thus caps have been constructed except in the following case:
(one of) $e^*_{M_1(i)}e_{M_1(i)-1}\dots e_{m_i+2}e^{(*)}_{m_i+1}\dots e_1$ is not admissible and $e^*_{M_1(i)+1} = l_{M_1(i)+1}$.

For $j > 1$ denote by $M_j(i)$ the smallest $k \in \mathbb{N}$ such that $k > M_{j-1}(i)$ and $e^*_k = l_k$. By the previous paragraph, it follows that $M_2(i) < m_{i+1}-1$. Take the largest $N \in \mathbb{N}$ such that $M_N(i) < m_{i+1}-1$. Note that for odd $j \in \{1,\dots N\}$ and

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{M_j(i)}e_{M_j(i)-1}\dots e_{m_i+2}0e_{m_i}\dots e_1,$$

$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{M_j(i)} e_{M_j(i)-1} \dots e_{m_i+2} 1 e_{m_i} \dots e_1,$$

if follows that $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{O(i)}$. The conclusion follows from the fact that $\#_1(l_{M_j(i)-1} \dots l_{m_i+2})$ and $\#_1(e_{M_j(i)-1} \dots e_{m_i+2})$ are of the same parity since j is odd.

Assume that for every odd $j \in \{1, \dots, N\}$ we have that $1^\infty e^*_{M_j(i)} e_{M_j(i)-1} \dots e_{m_i+2} e^{(*)}_{m_i+1} e_{m_i} \dots e_1$ are not admissible. If $M_{j+1}(i) > M_j(i) + 1$, we set:

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{M_j(i)+1} e_{M_j(i)} \dots e_{m_i+2} 0 e_{m_i} \dots e_1,$$

$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{M_j(i)+1} e_{M_j(i)} \dots e_{m_i+2} 1 e_{m_i} \dots e_1,$$

and argue that both are admissible as in preceding paragraphs. Calculations as above give $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{O(i)}$.

The situation left to consider is when $1^\infty e^*_{M_j(i)} e_{M_j(i)-1} \dots e_{m_i+2} e^{(*)}_{m_i+1} e_{m_i} \dots e_1$ are not admissible and $M_{j+1}(i) = M_j(i) + 1$ for every odd $j \in \{1, \dots, N\}$. Note that N must be even. Otherwise $1^\infty e^*_{m_{i+1}-2} e_{m_{i+1}-3} \dots e_{m_i+2} e^{(*)}_{m_i+1} e_{m_i} \dots e_1$ are not admissible and we have already argued that this is not possible.

Thus we conclude that L is of the form:

$$\dots e_{M_N(i)+1} e^*_{M_N(i)} e^*_{M_N(i)-1} e_{M_N(i)-2} \dots e_{M_1(i)+2} e^*_{M_1(i)+1} e^*_{M_1(i)} e_{M_1(i)-1} \dots e_{m_i+2} l_{m_i+1} \dots l_1.$$

Note that $\#_1(e^*_{M_N(i)} e^*_{M_N(i)-1} e_{M_N(i)-2} \dots e_{M_1(i)+2} e^*_{M_1(i)+1} e^*_{M_1(i)} e_{M_1(i)-1} \dots e_{m_i+2})$ is of same parity as $\#_1(e_{M_N(i)} e_{M_N(i)-1} e_{M_N(i)-2} \dots e_{M_1(i)+2} e_{M_1(i)+1} e_{M_1(i)} e_{M_1(i)-1} \dots e_{m_i+2})$, because the changes in L compared with $\overleftarrow{e}$ always appear in pairs (as two consecutive letters). We set

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{m_{i+1}-1} e_{m_{i+1}-2} \dots e_{m_i+2} 0 e_{m_i} \dots e_1,$$

$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{m_{i+1}-1} e_{m_{i+1}-2} \dots e_{m_i+2} 1 e_{m_i} \dots e_1,$$

and note that

$\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ (or with reversed inequalities). Also note that $\overleftarrow{x}^{I(i)}$ and $\overleftarrow{x}^{O(i)}$ set in such a way are always admissible by Lemma 14 and since $e_{m_{i+1}-1} = c_2 = 0$.

We have constructed the sequence corresponding to basic arc with the following properties: $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{O(i)}$, $\overleftarrow{x}^{O(i)}$ and $\overleftarrow{x}^{I(i)}$ are joined on the right and $\overleftarrow{x}^{O(i)}, \overleftarrow{x}^{I(i)} \to \overleftarrow{e}$ as $i \to \infty$. Since that can be done for infinitely many $i \in \mathbb{N}$, this concludes the proof. $\square$

Example 6. Let X' be the inverse limit space with the corresponding kneading sequence $\nu = (100111101011010111)^{\infty}$. Let us study the cappedness of the endpoint $e \in X'$ with the itinerary $\bar{e} = (100111101011010111)^{\infty}.(100111101011010111)^{\infty}$ in an embedding determined by $L = (010111110011100111)^{\infty}$. It follows that $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e}$, because $\#_1(100111101011010111)$ and $\#_1(010111110011100111)$ are both even. Note that $M_1(i) := m_i + 5$ is the smallest index strictly greater than $m_i + 1$ such that $e^*_{M_1(i)} = l_{M_1(i)}$. We obtain the following situation:

$$\dots(1\mathbf{0}01111\mathbf{01}011\mathbf{01}0111)(100111101011010111)^i = \overleftarrow{e}$$

$$\dots(0\mathbf{1}01111\mathbf{10}011\mathbf{10}0111)(010111110011100111)^i = L$$

$$1^{\infty}(1\mathbf{1}01111\mathbf{01}011\mathbf{01}0110)(100111101011010111)^i = \overleftarrow{x}^{O(i)}$$

$$1^{\infty}(1\mathbf{1}01111\mathbf{01}011\mathbf{01}0111)(100111101011010111)^i = \overleftarrow{x}^{I(i)}$$

where we denoted with bold the letters of $\overleftarrow{e}$ and L which differ for indices larger than m_i. Note that $M_3(i) = m_i + 10$ but the word $00110 = e^*_{m_i+10}e_{m_i+9}\dots e_{m_i+6}$ is not admissible and thus we need to set $x^{O(i)}_{M_3(i)} = x^{I(i)}_{M_3(i)} = e_{M_3(i)} = 1$. Note that $M_5(i) = m_i + 17 = m_{i+1} - 1$. Thus we set $x^{O(i)}_{M_5(i)} = x^{I(i)}_{M_5(i)} = e^*_{M_5(i)}$. Because $\#_1(e_{m_i+16}\dots e_1)$ and $\#_1(l_{m_i+16}\dots l_1)$ are of the same parity we obtain that $\overleftarrow{e} \prec_L \overleftarrow{x}^{I(i)}$. Lemma 14 again ensures that every subword of $\overleftarrow{x}^{O(i)}$ is admissible. Therefore points $x^{O(i)}, x^{I(i)} \in X'$ cap the point e from the right.

If an endpoint e is capped, we still cannot conclude that it is not accessible, see *e.g.* Figure 5.10. However, if we know that the length of basic arcs arbitrary close to $\overleftarrow{e}$ has a lower bound, *i.e.,* map T is long branched (recall Definition 5), the conclusion follows.

Remark 24. *Note that if the critical point of T is periodic, then T is long-branched.*

Corollary 10. *Assume $T \neq T_2$ is long-branched and let $e \in X'$ be an endpoint of X'. Assume X' is embedded in the plane with respect to L where $A(L) \not\subset \mathcal{C}$. If $\overleftarrow{e}$ and L have different tails, then e is not accessible.*

Proof. By the long-branchedness of the bonding map T it holds that in a sufficiently small neighbourhood of endpoint e every point $e \neq x \in A(\overleftarrow{e})$ has a neighbourhood homeomorphic to the Cantor set of arcs. Since $e \in X'$ is capped by Theorem 5 the proof follows. $\square$

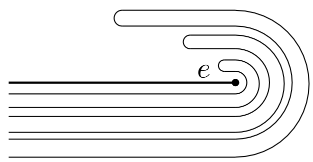

Figure 5.10: Neighbourhood of an endpoint $e \in X'$. Note that e is capped but also accessible.

We merge the knowledge from this and the preceding section and give some interesting examples of embeddings of some X'.

Example 7. Let $\nu = (101)^\infty$ and let $L = (01^k)^\infty$ for any $k \geq 2$. Take an admissible $B = a_n \dots a_1 \in \{0,1\}^n$ for some $n \in \mathbb{N}$. If $l_{n+1}B$ is not admissible, then $\dots l_{n+3}l^*_{n+2}l^*_{n+1}B$ is admissible by the choice of k and since every non-admissible word for $\nu = (101)^\infty$ contains 00. Tail L is thus not altered by B for every finite admissible word B (recall Definition 19). Therefore, it follows that $L_{a_n \dots a_1} = S_{a_n \dots a_1} \subset \mathcal{U}_L$. We conclude that $\mathcal{U}_L$ is fully accessible and it is the only non-degenerate accessible set. By Corollary 10, endpoints of X' are not accessible. The remaining point on the circle of prime ends corresponds to the simple dense canal.

Example 8. Let $\nu = (101)^\infty$ and let $L = (01)^\infty$. Note that $S = (10)^\infty \not\subset \mathcal{U}_L$ and $S = S_0$. Thus, $B = 0$ alters L (recall Definition 19; here $A_1 = 0$, $A_i = 01$ for all $i \geq 2$). Since ν is periodic, it follows from Corollary 9 that both $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible. As in the example above (see also Lemma 21) we can show that no other point from X' is accessible. We conclude that there are two simple dense canals with shores $\mathcal{U}_L$ and $\mathcal{U}_S$.

Example 9. Take $\nu = (10011001001111)^\infty$, $B = 001$, $A = 0011$, $C = 1111$ and $L = (BA)^\infty$ as in Example 5. Recall that at least three arc-components (which are dense lines) are fully accessible. Further calculations show that no other tail can be the top or the bottom of a cylinder. By Corollary 10 endpoints from X' are not accessible. Therefore, the remaining three points on the circle of prime ends correspond to three simple dense canals with shores from pairwise different fully accessible arc-components which are lines. In comparison, the kneading sequence from this example has height 2/7 (see the Definition 24) and belongs to the rational interior case, so the Brucks-Diamond embedding of X' contains 7 fully accessible arc-components which are shores of 7 simple dense canals (see Section 6.3 in this thesis or [19]).

5.6.2 Accessible folding points when ν is preperiodic

In this subsection we assume that $\nu = c_1 \dots c_k(c_{k+1} \dots c_{k+n})^\infty$ and that $c_k \neq c_{k+n}$, since otherwise also $\nu = c_1 \dots c_{k-1}(c_k \dots c_{k+n-1})^\infty$. By Remark 7 the space X' contains n folding points which are not endpoints with symbolic descriptions:

$$\sigma^i((c_{k+1} \dots c_{k+n})^\infty.(c_{k+1} \dots c_{k+n})^\infty)$$

for $i \in \{1, \dots n\}$. In this subsection we study the accessibility of folding points that are not contained in extrema of cylinders in $\mathcal{E}$-embeddings of X' when ν is preperiodic.

Let $Q \subset \mathbb{R}^2$ be an arc. From now onwards let $\mathrm{Int}(Q)$ denote the points from Q, which are not endpoints of Q.

Remark 25. *Let $\nu = c_1 \dots c_k(c_{k+1} \dots c_{k+n})^\infty$ and let $p \in X'$ be a folding point. Then an arc-component of p can contain at most one folding point. Also, since $c_k \neq c_{n+k}$ it holds that $p \in \mathrm{Int}(A(\overleftarrow{p}))$.*

The following lemma restricts the search for the accessible folding points which are not tops/bottoms of cylinders to the case where $\nu = 10(c_3 \dots c_{n+2})^\infty$, *i.e.*, $k = 2$.

Proposition 26. *Assume c is preperiodic and such that $T^3(c)$ is not periodic. Embed X' in the plane with respect to $L \neq 0^\infty l_n \dots l_1$. A folding point $p \in X'$ is accessible if and only if the basic arc $A(\overleftarrow{p})$ is top or bottom of a finite cylinder.*

Proof. Note that $\nu = c_1 \dots c_k(c_{k+1} \dots c_{k+n})^\infty$ where $k > 2$. Take a folding point $p \in X'$ with the symbolic description

$$\bar{p} = (c_{k+1} \dots c_{k+n})^\infty c_{k+1} \dots c_{k+i}.c_{k+i+1} \dots c_{k+n}(c_{k+1} \dots c_{k+n})^\infty$$

and assume it is not on the top or bottom of any cylinder in X'. Denote $\pi_0(A(\overleftarrow{p})) =: [T^l(c), T^r(c)]$. By Remark 25 it holds that $\pi_0(p) \in (T^l(c), T^r(c))$.
Denote by $(p^M)_{M\in\mathbb{N}} \subset X'$ the points with the symbolic description

$$\bar{p}^M := 1^\infty c_1 \dots c_k(c_{k+1} \dots c_{k+n})^M c_{k+1} \dots c_{k+i}.c_{k+i+1} \dots c_{k+n}(c_{k+1} \dots c_{k+n})^\infty$$

Note that points p^M converge to p as $M \to \infty$ and the corresponding basic arcs $A(\overleftarrow{p}^M)$ project to $[T^l(c), T^{k+i+1}(c)]$ (we refer to them as *left*) or $[T^{k+i+1}(c), T^r(c)]$ (referred to as *right*) depending on the parity of M. We will find long basic arcs (*i.e.,* arcs projecting with π_0 also to $[T^l(c), T^r(c)]$) converging to $A(\overleftarrow{p})$ from both sides. Since c is preperiodic there exists a neighbourhood U of $A(\overleftarrow{p})$ which contains only basic arcs which project to $[T^l(c), T^r(c)]$, $[T^l(c), T^{k+i+1}(c)]$ or $[T^{k+i+1}(c), T^r(c)]$ (*i.e.,* only long or left/right arcs).

Assume that all but finitely many long arcs in U are greater than $A(\overleftarrow{p})$. Since $k > 2$, note that for every $M > 0$ basic arcs $1^\infty c_k(c_{k+1} \dots c_{k+n})^M c_{k+1} \dots c_{k+i}$ are long. Since $c_k \neq c_{k+n}$ it holds that both $1^\infty c_k(c_{k+1} \dots c_{k+n})^M c_{k+1} \dots c_{k+i} \succ_L \overleftarrow{p}$ and $\overleftarrow{p}^M \succ_L \overleftarrow{p}$. Thus, it follows that $A(\overleftarrow{p})$ is at the bottom of some cylinder, a contradiction. The proof goes analogously if all but finitely many long arcs are smaller than $A(\overleftarrow{p})$. □

Therefore, by Proposition 26, if we want to find accessible folding points which are not at the top/bottom of any cylinder it is enough to study cases $\nu = 10(c_3 \dots c_{n+2})^\infty$ where $c_{n+2} = 1$.

Remark 26. *Assume c is preperiodic and p is an accessible folding point of an embedding of X'. By Corollary 6 and since every arc-component contains at most one folding point, only the following three cases can occur:*

(1) $\overleftarrow{p}$ is the top or the bottom of some cylinder; then $\mathcal{U}_p$ is fully accessible.

(2) $\overleftarrow{p}$ is not the top or the bottom of any cylinder, but $\overleftarrow{r(p)}$ or $\overleftarrow{l(p)}$ is; then one component of $\mathcal{U}_p \setminus \{p\}$ is fully accessible, and the other component of $\mathcal{U}_p \setminus \{p\}$ is not accessible. See Figure 5.6.

(3) $\overleftarrow{p}$, $\overleftarrow{r(p)}$ *and* $\overleftarrow{l(p)}$ *are not extrema of any cylinder; then* c *is order reversing and* p *is the only accessible point of* $\mathcal{U}_p$*. See Figure 5.7(c).*

Definition 23. *We say that an accessible folding point* p *is accessible of* Type i *if it satisfies the condition* i *from Remark 26 for* $i \in \{1,2,3\}$.

As it turns out, all Types of accessible folding points can occur in $\mathcal{E}$-embeddings. In the following subsections we describe how they can be constructed in preperiodic orbit case (when $T^3(c)$ is periodic) and give examples of such constructions. We will see that the standard Brucks-Diamond embedding does not allow Type 3 folding points for any X' (see Section 6.3).

Folding points of Type 2

First we give examples of X' which cannot be $\mathcal{E}$-embedded with Type 2 folding points. Then we show in general how to construct a Type 2 accessible folding point and give an example of such construction in both the order preserving and the order reversing case.

Lemma 15. *Let* $\nu = 10(c_3 \dots c_{n+2})^\infty$ *and assume that* $c_i^* c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M$ *is admissible for all* $i \in \{3, \dots, n+1\}$ *and for all but finitely many* $M \in \mathbb{N}$*. Then no folding point is Type 2 in any* $\mathcal{E}$*-embedding of* X' *which is non-equivalent to the Brucks-Diamond (*$L = 0^\infty 1$*) embedding.*

Proof. Take a folding point $p \in X'$ with symbolic description $\bar{p} = (c_3 \dots c_{n+2})^\infty.(c_3 \dots c_{n+2})^\infty$. We will try to reconstruct L which embeds p as Type 2 and see that this is not possible.

Assume first that $\#_1(c_3 \dots c_{n+2})$ is odd and for some natural number M we have (the following, possibly with reversed inequalities, needs to be satisfied in order for p to be a Type 2 folding point, see Figure 5.12):

$$\dots 0(c_3 \dots c_{n+2})^M \succ_L \dots 1(c_3 \dots c_{n+2})^M$$

$$\dots c_i^* c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M \prec_L \dots c_i c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M$$

$$\dots 0(c_3 \dots c_{n+2})^{M+k} \prec_L \dots 1(c_3 \dots c_{n+2})^{M+k}$$

$$\dots c_i^* c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+k} \prec_L \dots c_i c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+k}$$

$$\ldots 0(c_3 \ldots c_{n+2})^{M+N} \succ_L \ldots 1(c_3 \ldots c_{n+2})^{M+N}$$

$$\ldots c^*_{n+1} c_{n+2}(c_3 \ldots c_{n+2})^{M+N} \prec_L \ldots c_{n+1} c_{n+2}(c_3 \ldots c_{n+2})^{M+N}$$

for all $i \in \{3, \ldots, n+1\}$ and all $k \in \{1, \ldots, N-1\}$, where natural number $N > 1$ is even. If $\#_1((c_3 \ldots c_{n+2})^M)$ is of the same parity as $\#_1(l_{Mn} \ldots l_1)$, then it follows that $l_{Mn+1} = 0$. If $\#_1((c_3 \ldots c_{n+2})^M)$ is of different parity as $\#_1(l_{Mn} \ldots l_1)$, then $l_{Mn+1} = 1$. In any case, $\#_1(c_{n+2}(c_3 \ldots c_{n+2})^M)$ is of different parity as $\#_1(l_{Mn+1} \ldots l_1)$ so $l_{Mn+2} = c^*_{n+1}$. So $\#_1(c_{n+1}c_{n+2}(c_3 \ldots c_{n+2})^M)$ is of the same parity as $\#_1(l_{Mn+2}l_{Mn+1} \ldots l_1)$ and thus $l_{Mn+3} = c_n$. Continuing further, we get

$$l_{(M+N)n+2} \ldots l_{Mn+2} = c^*_{n+1} c^*_{n+2}(c_3 \ldots c_{n+2})^{N-1} c_3 \ldots c_n c^*_{n+1}.$$

Since $L \subset X'$, it follows that $c^*_{n+1} = 1$, $\#_1(c_3 \ldots c_n)$ is even and the word on the right side of the last equation above is equal to $10(c_3 \ldots c_{n+2})^{N-1} c_3 \ldots c_n c^*_{n+1}$. Note $\#_1(10(c_3 \ldots c_{n+2})^{N-1} c_3 \ldots c_n c_{n+1})$ is even and thus $10(c_3 \ldots c_{n+2})^{N-1} c_3 \ldots c_n c^*_{n+1}$ is not admissible by Lemma 13, a contradiction.

Assume that $\#_1(c_3 \ldots c_{n+2})$ is even. Note that in this case N is not necessarily even, but now the conclusion $c_{n+1} = 0$ implies that $\#_1(c_3 \ldots c_n)$ is odd. We continue with arguments as in the paragraphs above. Since $\#_1(c_3 \ldots c_{n+2})$ is even the word $\#_1(10(c_3 \ldots c_{n+2})^{N-1} c_3 \ldots c_n c_{n+1})$ is even and thus $10(c_3 \ldots c_{n+2})^{N-1} c_3 \ldots c_n c^*_{n+1}$ is by Lemma 13 again not admissible, a contradiction.

Note that the proof works analogously for other folding points from the space X'. □

Next we give examples of preperiodic ν where no folding point can be $\mathcal{E}$-embedded as Type 2, except possibly using the Brucks-Diamond embedding, see Section 6.3, specially the rational endpoint case.

Example 10. The assumptions from Lemma 15 hold for *e.g.* $\nu = 10(0^\alpha 1^\beta)$ for all $\alpha, \beta \in \mathbb{N}$.

The proof of the following lemma follows directly from the statement, see Figure 5.6.

Lemma 16 (Order preserving case)**.** *Let* $\nu = 10(c_3 \ldots c_{n+2})^\infty$, $c_{n+2} = 1$*, and* $\#_1(c_3 \ldots c_{n+2})$ *be even. Let* $\bar{p} = (c_3 \ldots c_{n+2})^\infty c_3 \ldots c_i.c_{i+1} \ldots c_{n+2}(c_3 \ldots c_{n+2})^\infty$ *be a symbolic description of* *a folding point $p \in X'$. Then p is a Type 2 folding point if and only if there exists a natural number M such that*

$$\dots c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i \succ_L \dots c_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i,$$

for all $N \in \mathbb{N}$ and all $j \in \{3, \dots, 1+n\}$ for which $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i$ is admissible, and

$$\dots 0(c_3 \dots c_{n+2})^{M+N'} c_3 \dots c_i \prec_L \dots 1(c_3 \dots c_{n+2})^{M+N'} c_3 \dots c_i,$$

for infinitely many $N' \in \mathbb{N}$, or with reversed inequalities.

We give an example that satisfies the assumptions of Lemma 16.

Example 11 (Type 2, order preserving case)**.** Take $\nu = 10(01101001)^\infty$, $L = (10100101\,11001001)^\infty$ and

$$\bar{p} = (01101001)^\infty 01.101001(01101001)^\infty.$$

Then $\overleftarrow{r(p)}$ is the smallest left-infinite tail so it is the smallest in the cylinder [0]. As the calculations below show, all long basic arcs in small neighbourhood of $A(\overleftarrow{p})$ are below $A(\overleftarrow{p})$ and left arcs are both above and below $A(\overleftarrow{p})$, depending on the parity of period which corresponds with $\overleftarrow{p}$ in the left infinite description of basic arcs, see Figure 5.11.

$$\dots 0(01101001)^{2N} 01 \succ_L \overleftarrow{p},$$

$$\dots 0(01101001)^{2N+1} 01 \prec_L \overleftarrow{p},$$

$$\dots 11(01101001)^{N} 01 \prec_L \overleftarrow{p},$$

$$\dots 101(01101001)^{N} 01 \prec_L \overleftarrow{p},$$

$$\dots 11001(01101001)^{N} 01 \prec_L \overleftarrow{p},$$

$$\dots 001001(01101001)^{N} 01 \prec_L \overleftarrow{p},$$

$$\dots 0101001(01101001)^{N} 01 \prec_L \overleftarrow{p},$$

$$\dots 11101001(01101001)^{N} 01 \prec_L \overleftarrow{p},$$

for all $N \in \mathbb{N}$. Further calculations show that only tails of L and S can appear as the extrema of cylinders. By Proposition 19, the arc-component $\mathcal{U}_L$ is fully accessible and since $\mathcal{U}_L$ contains no folding points, it corresponds to an open interval on the circle of prime ends. The accessible part of $\mathcal{U}_S$ corresponds to a half-open interval on the circle of prime ends, where the endpoint of the half-open interval corresponds to the accessible folding point p. By further calculations we obtain that other folding points are not accessible, so the remaining point on the circle of prime ends corresponds to a simple dense canal with shores being $\mathcal{U}_L$ and $\mathcal{U}_S$.

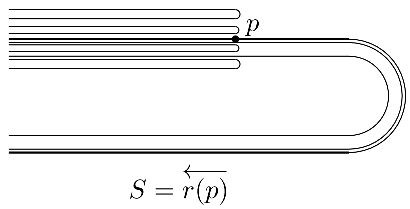

Figure 5.11: Type 2 folding point from Example 11.

The proof of the following lemma follows directly from its assumptions (see Figure 5.12).

Lemma 17 (Order reversing case). *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$, $c_{n+2} = 1$, and $\#_1(c_3 \dots c_{n+2})$ odd. Let $\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i.c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^\infty$ be a symbolic description of a folding point $p \in X'$. Then p is a Type 2 folding point if and only if there exists a natural number M such that*

$$\dots c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i \succ_L \dots c_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i,$$

for all $N \in \mathbb{N}$ and all $j \in \{3, \dots, 1+n\}$ for which $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i$ is admissible, and

$$\dots 0(c_3 \dots c_{n+2})^{M+2N'} c_3 \dots c_i \prec_L \dots 1(c_3 \dots c_{n+2})^{M+N'} c_3 \dots c_i,$$

and

$$\dots 0(c_3 \dots c_{n+2})^{M+2N''+1} c_3 \dots c_i \succ_L \dots 1(c_3 \dots c_{n+2})^{M+N''} c_3 \dots c_i,$$

for infinitely many $N' \in \mathbb{N}$ and all but finitely many $N'' \in \mathbb{N}$, or the whole statement with reversed inequalities.

We give an example that satisfies the assumptions of Lemma 17.

Example 12 (Type 2, order reversing case)**.** Take $\nu = 10(011101001)^\infty$, $L = (0111010010111\,10010)^\infty$ and $\overleftarrow{p} = (011101001)^\infty$. What follows is an easy computation:

$$\ldots 0(011101001)^{2M+1} \prec_L \overleftarrow{p},$$

$$\ldots 0(011101001)^{2M} \succ_L \overleftarrow{p},$$

$$\ldots 11(011101001)^{M} \prec_L \overleftarrow{p},$$

$$\ldots 101(011101001)^{M} \prec_L \overleftarrow{p},$$

$$\ldots 11001(011101001)^{M} \prec_L \overleftarrow{p},$$

$$\ldots 001001(011101001)^{M} \prec_L \overleftarrow{p},$$

$$\ldots 0101001(011101001)^{M} \prec_L \overleftarrow{p},$$

$$\ldots 01101001(011101001)^{M} \prec_L \overleftarrow{p},$$

$$\ldots 111101001(011101001)^{M} \prec_L \overleftarrow{p},$$

for every $M \in \mathbb{N}$. So p is accessible folding point of Type 2. Note that $\overleftarrow{l(p)} = (010010111)^\infty 010\,11 = S_{1011}$, see Figure 5.12. By further symbolic calculations we again conclude that there is one simple dense canal for this embedding of X'.

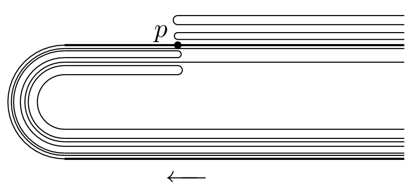

Figure 5.12: Type 2 folding point from Example 12.

Folding points of Type 3

From now onwards we study folding points of Type 3, see Figure 5.13.

Remark 27. *Let $\nu = 10(c_3 \ldots c_{n+2})^\infty$ be such that $\#_1(c_3 \ldots c_{n+2})$ is even and $c_{n+2} = 1$. Then X' does not contain folding points of Type 3.*

The following lemma gives necessary and sufficient symbolic conditions for a folding point to be $\mathcal{E}$-embedded as Type 3, the proof of it again follows directly from its assumptions.

Lemma 18 (Type 3). *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$, $c_{n+2} = 1$, and $\#_1(c_3 \dots c_{n+2})$ odd. Let $\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i . c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^\infty$ be the symbolic description of a folding point $p \in X'$. Then p is a Type 3 folding point if and only if there exists $M > 0$ such that*

$$\dots c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i \prec_L \overleftarrow{p},$$

for all $N \in \mathbb{N}$ and all $j \in \{3, \dots, n+1\}$ for which $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i$ is admissible, and

$$\dots 0(c_3 \dots c_{n+2})^{M+N'} c_3 \dots c_i \succ_L \overleftarrow{p},$$

for infinitely many $N' \in \mathbb{N}$, or with reversed inequalities. See Figure 5.13.

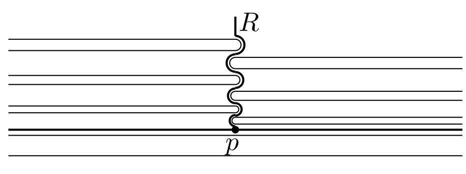

Figure 5.13: Type 3 folding point. Folding point p is accessible from the complement by an arc $R \cup \{p\} \subset \mathbb{R}^2$, where R is a ray.

The following lemma gives conditions on preperiodic, order reversing ν such that no folding point can be $\mathcal{E}$-embedded as Type 3 folding point (except possibly with the Brucks-Diamond embedding studied in detail in Section 6.3).

Lemma 19. *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$ be such that $\#_1(c_3 \dots c_{n+2})$ is odd, $c_{n+2} = 1$ and let $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M c_3 \dots c_i$ be admissible for every $j \in \{3, \dots 1+n\}$ and all $M \in \mathbb{N}$. If $c_{n+1} = 1$ then there exists no L such that folding point $p \in X'$ is of Type 3.*

Proof. Take a folding point $p \in X'$ with the symbolic description

$$\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i . c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^\infty$$

for some $i \in \{3, \dots n+2\}$ and assume that $A(\overleftarrow{p})$ is not at the top or bottom of any cylinder in X'. Since $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M c_3 \dots c_i$ is admissible for every $j \in \{3, \dots n+1\}$ and all $M \in \mathbb{N}$, the same calculations as in the proof of Lemma 15 imply that the only L which satisfies all the conditions from Lemma 18 is

$$L = (c_3 \dots c_n 00)^\infty l_k \dots l_1,$$

for some $l_k \dots l_1$. However, the word $00c_3 \dots c_n$ is not admissible, a contradiction. $\square$

Example 13 (No Type 3 folding point)**.** Note that $\nu = 10(0^\alpha 1^\beta)^\infty$ for $\beta \geq 2$ satisfies the assumptions of Lemma 19. Thus no folding point from the corresponding X' can be embedded as Type 3 folding point using $\mathcal{E}$-embeddings (except maybe Brucks-Diamond). Note that this example also satisfies the assumptions of Lemma 15, so no folding point can be $\mathcal{E}$-embedded as Type 2 either. Thus in these cases a point from X' is accessible if and only if it is on the top or the bottom of some cylinder. So there are $m \in \mathbb{N}$ simple dense canals in $\mathcal{E}$-embeddings of such X', where m is the number of fully accessible arc-components.

The following lemma gives sufficient symbolic conditions on a preperiodic ν such that every folding point can be $\mathcal{E}$-embedded as accessible folding point of Type 3.

Lemma 20. *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$ be such that $\#_1(c_3 \dots c_{n+2})$ is odd and $c_{n+2} = 1$. Assume that $c_{n+1} = 0$ and the tail $(10c_3 \dots c_n)^\infty$ is admissible. For every folding point $p \in X'$ there exists L such that p is of Type 3 in $\varphi_L(X')$.*

Proof. Take a folding point $p \in X'$ with the symbolic description

$$\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i . c_{i+1} \dots c_{n+2} (c_3 \dots c_{n+2})^\infty$$

for some $i \in \{3, \dots n+2\}$. Denote by $\pi_0(A(\overleftarrow{p})) =: [T^l(c), T^r(c)]$ for some $l, r \in \mathbb{N}$.

Let $L = (c_3 \dots c_n c^*_{n+1} c^*_{n+2})^\infty c_3 \dots c_i$. Then

$$\dots 0(c_3 \dots c_{n+2})^m c_3 \dots c_i \succ_L \dots 1(c_3 \dots c_{n+2})^m c_3 \dots c_i,$$

$$\dots c^*_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^m c_3 \dots c_i \prec_L \dots c_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^m c_3 \dots c_i,$$

for every $m \in \mathbb{N}$, every $j \in \{3, \dots n+1\}$ and all admissible $c^*_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^m c_3 \dots c_i$, see Figure 5.13 to visualize the construction. By the assumptions we conclude that $L = (10c_3 \dots c_n)^\infty 10 c_3 \dots c_i$ is indeed admissible. Since $\#_1(c_3 \dots c_{n+2})$ is odd we get pairs of basic arcs joined at a point which projects with π_0 to $\pi_0(p)$, approaching to $A(\overleftarrow{p})$ from above from both left and right side of p, exactly as in Figure 5.13. $\square$

Example 14 (Type 3 folding point)**.** Take $\nu = 10(01101)^\infty$. If we embed X' with respect to admissible $L = (01110)^\infty$, then $\bar{p} = (01101)^\infty.(01101)^\infty$ is an accessible folding point of Type 3, since it satisfies the conditions of Lemma 20. Note that only $\mathcal{U}_L$ can contain the extremum of a cylinder and it corresponds to the circle of prime ends minus a point. The remaining point is the second kind prime end corresponding to the accessible folding point p of Type 3. Specifically, there are no simple dense canals.

5.7 Multiple fully accessible arc-components of X'

In this section we study $\mathcal{E}$-embeddings of an arbitrary X' that allow at least two fully accessible dense arc-components.

Lemma 21. *Let $\nu = 10^\kappa 1\dots$ and embed X' with respect to $L = (0^\kappa 1)^\infty$. The smallest left-infinite tail with respect to $\prec_L$ is $A(S) = A(S_0) = A((10^\kappa)^\infty) \not\subset \mathcal{U}_L$. Moreover, both $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible and dense in X'.*

Proof. First, let us comment that $L = (0^\kappa 1)^\infty$ is admissible. Note that there exists $0 \le \kappa_2 < \kappa$ such that $\nu = 10^\kappa 10^{\kappa_2} 1\dots$, so the word $10^\kappa 10^\kappa$ is indeed admissible.

It is straightforward to calculate S, infinitely many changes occur because $0^{\kappa+1}$ is not admissible, *i.e.,* symbol 0 alters L, see Definition 19.

To prove that $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible, it is enough to show that every basic arc from $\mathcal{U}_L \cup \mathcal{U}_S$ is at the top or the bottom of some cylinder.
Proposition 19 shows that $\mathcal{U}_L$ is fully accessible, since it is a line. Assume that $A(\overleftarrow{x}) \subset \mathcal{U}_S$ and take $k \in \mathbb{N}$ such that $x_{k+i} = s_{k+i}$ for every $i \in \mathbb{N}$ and such that $\kappa+1$ divides k, where $S = \dots s_2 s_1$. Then $\overleftarrow{x} = \dots 10^\kappa 10^\kappa x_k \dots x_1$. Note that if $\#_1(10^\kappa x_k \dots x_1)$ and $\#_1(l_{k+\kappa+1} \dots l_1)$ have the same parity, then $S_{10^\kappa x_k \dots x_1} = \overleftarrow{x}$ and $L_{10^\kappa x_k \dots x_1} = \overleftarrow{x}$ in the other case.

To show that $\mathcal{U}_L$ and $\mathcal{U}_S$ are dense, fix a point $x \in X'$ with backward itinerary $\overleftarrow{x} = \dots x_2 x_1$ and fix $n \in \mathbb{N}$. Denote $\nu = 10^\kappa 10^{\kappa_2} 10^{\kappa_3} 10^{\kappa_4} 1\dots$, where $0 \le \kappa_2 < \kappa$, $0 \le \kappa_3, \kappa_4 \le \kappa$.

If $\kappa_3 > 0$, then there exists $\gamma \geq 0$ so that $A((0^\kappa 1)^\infty 0^{\kappa_2} 1^\gamma x_n \ldots x_1) \subset \mathcal{U}_L$ is admissible. Assume that $\kappa_3 = 0$. If $\kappa_4 < \kappa$, then there exists $\gamma' \geq 0$ so that $A((0^\kappa 1)^\infty 0^{\kappa_2} 110^{\kappa_4+1} 1^{\gamma'} x_n \ldots x_1) \subset \mathcal{U}_L$ is admissible. If $\kappa_4 = \kappa$, then $\nu = 10^\kappa 10^{\kappa_2} 110^\kappa 10^{\kappa_2} 0 \ldots$. Therefore, there exists an appropriate $\gamma'' \geq 0$ so that $A((0^\kappa 1)^\infty 0^{\kappa_2} 110^\kappa 10^{\kappa_2} 11^{\gamma''} x_n \ldots x_1) \subset \mathcal{U}_L$ is admissible. The proof for points from $\mathcal{U}_S$ is analogous. Therefore, there are points from both $\mathcal{U}_L$ and $\mathcal{U}_S$ which are arbitrary close to any $x \in X'$ and thus $\mathcal{U}_L$ and $\mathcal{U}_S$ are dense in X'. □

Theorem 6. *For every X' there exists a planar embedding with two non-degenerate fully accessible dense arc-components.*

Proof. Let $\nu = 10^\kappa 1 \ldots$ and construct $\varphi_L(X')$ with respect to $L = \ldots 0^\kappa 10^\kappa 10^\kappa 1$. Using Lemma 21 we conclude that $\mathcal{U}_S$ and $\mathcal{U}_L$ are fully accessible and dense and the claim follows. □

In a special case when the orbit of c is finite and only $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible we obtain the following corollary.

Corollary 11. *If orbit of the critical point is finite and only $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible, then there exists a planar embedding of X' with two simple dense canals.*

Proof. Take the embedding constructed in Lemma 21. Note that $\mathcal{U}_L$ and $\mathcal{U}_S$ do not contain endpoints for any chosen $\nu = 10^\kappa 1 \ldots$ (since the kneading sequence $\nu = (10^\kappa)^\infty$ does not appear as a kneading sequence in the tent map family) and are thus lines. If ν is periodic, the endpoints of X' are not accessible by Corollary 10. That in combination with Proposition 18 gives two simple dense canals. If ν is preperiodic and $T^3(c)$ is not periodic, the conclusion again follows analogously as above. We only have to argue that Type 3 folding points do not exist for a chosen L. Since L is periodic of period $\kappa+1$, it follows that $\sigma^{\kappa+1} : \varphi_L(X') \to \sigma^{\kappa+1}(\varphi_L(X'))$ is extendable to the whole plane.
Assume that the point $p \in X'$ is a Type 3 folding point. Thus $\sigma^{\kappa+1}(p)$ is also Type 3 folding point. For $\nu = 10(c_3 \ldots c_{n+2})^\infty$, the itineraries of folding points are periodic of period $n \geq \kappa$. Thus $(\kappa+1)|n$. If $\kappa+1 = n$, since $c_{n+2} = 1$ it holds that $c_3 \ldots c_{n+2} = 0^{\kappa-1}11$, which is even, a contradiction with Remark 27. From the circle of prime ends we get that there can be at most two Type 3 accessible folding points and thus $n = 2(\kappa+1)$. Since $\#_1(c_3 \ldots c_{n+2})$ is

odd, it follows that $\dots 0P^{2k+1} \succ_L \dots 1P^{2k+1}$ and $\dots 0P^{2k} \prec_L \dots 1P^{2k}$ for all $k \in \mathbb{N}$, where $P = c_3 \dots c_{n+2}$. That is a contradiction with Lemma 18. $\square$

The following proposition shows that for L as in Lemma 21 and ν of specific form there exist $\mathcal{E}$-embeddings of X' that permit more than two fully accessible arc-components dense in X'. Specifically we improve the upper bound on the number of fully accessible non-degenerate arc-components from three to four (compare to Example 9).

Proposition 27. *Assume ν is of the form $\nu = 10^\kappa 10^{\kappa-1}110\dots$ with $\kappa > 1$. If $L = (0^\kappa 1)^\infty$, then $\varphi_L(X')$ has four fully-accessible dense arc-components.*

Proof. Note that for $L = (0^\kappa 1)^\infty$ and chosen ν it holds that $S = (10^\kappa)^\infty$ and note that for κ even we have $L_{1^{\kappa+1}} = (1110^{\kappa-1}10^{\kappa-1})^\infty 11^{\kappa+1}$ and $S_{01^{\kappa+1}} = (010^{\kappa-1}1110^{\kappa-2})^\infty 01^{\kappa+1}$. For κ odd we get $S_{1^{\kappa+1}} = (1110^{\kappa-1}10^{\kappa-1})^\infty 11^{\kappa+1}$ and $L_{01^{\kappa+1}} = (010^{\kappa-1}1110^{\kappa-2})^\infty 01^{\kappa+1}$. Thus we get at least four different accessible left infinite tails. For the rest of the proof we assume without the loss of generality that κ is even.
To see that $\mathcal{U}_{L_{1^{\kappa+1}}}$ is fully accessible take $\overleftarrow{x} = \dots x_2 x_1 \subset \mathcal{U}_{L_{1^{\kappa+1}}}$ and $n \in \mathbb{N}$ such that $\dots x_{n+2} x_{n+1} = (1110^{\kappa-1}10^{\kappa-1})^\infty$. Note that then $\overleftarrow{x}$ is either the largest or the smallest arc in the cylinder $[1110^{\kappa-1}10^{\kappa-1}x_n \dots x_1]$, depending on the parity of $x_n \dots x_1$. Similarly we show that $\mathcal{U}_{S_{01^{\kappa+1}}}$ is fully accessible.
To see that $\mathcal{U}_{L_{1^{\kappa+1}}}$ is dense in X', fix a point $x \in X'$ with backward itinerary $\overleftarrow{x} = \dots x_2 x_1$ and fix $n \in \mathbb{N}$. Note that $0^\kappa \not\subset L_{1^{\kappa+1}}$ and therefore there exists $\gamma \in \mathbb{N}$ so that $(1110^{\kappa-1}10^{\kappa-1})^\infty 1^\gamma x_n \dots x_1 \subset \mathcal{U}_{L_{1^{\kappa+1}}}$ is admissible. We analogously prove that $\mathcal{U}_{S_{01^{\kappa+1}}}$ is dense in X'. $\square$

The characterization of fully accessible arc-components of $\mathcal{E}$-embeddings of X' (excluding the standard embeddings, see Section 6.2 and Section 6.3) is still outstanding.

Question: Do there exist more than four fully accessible dense arc-components in non-standard (Section 6.2 and Section 6.3) $\mathcal{E}$-embeddings of X'? Specifically, if c is periodic, do there exist $\mathcal{E}$-embeddings of X' so that more than four dense arc-components are fully accessible?

We lack the symbolic techniques to make a general construction that would answer on the preceding question. Later in the thesis we will see that for every $n \in \mathbb{N}$ there exists X' such that the Brucks-Diamond embedding of X' has n fully-accessible dense arc-components. See Section 6.3 and [19] for details.

Chapter 6

X' as attractors of planar homeomorphisms

6.1 Extendability of the shift homeomorphism of $\mathcal{E}$-embeddings

Planar embeddings equivalent to [24] ($L = 1^\infty$) and [23] ($L = 0^\infty 1$) of X make $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{C}$ respectively fully accessible as can be deduced from Proposition 19 and Remark 16 (denote the two special embeddings from now onwards by $\varphi_{\mathcal{R}}$ and $\varphi_{\mathcal{C}}$ respectively). We denote from now onwards by φ_L the planar $\mathcal{E}$-embedding of X or X' determined by the left infinite sequence L. Additionally it can be deduced from Proposition 21 that only remaining accessible points of embeddings of X (if existent) need to be folding points. The embeddings of unimodal inverse limit spaces $\varphi_{\mathcal{C}}$ follow the Barge-Martin construction from [12] of attractors of orientation preserving planar homeomorphisms so σ is extendable to $\mathbb{R}^2$ for these embeddings. Bruin directly showed in [24] that the shift homeomorphism can be extended to the plane for embeddings $\varphi_{\mathcal{R}}$. Now we show that except for the two mentioned standard embedding, σ is not extendable for any $\mathcal{E}$-embedding of X'.

Note that if $\sigma : \varphi_L(X) \to \varphi_L(X)$ is extendable to $\mathbb{R}^2$, then $\sigma|_{\varphi_L(X')} : \varphi_L(X') \to \varphi_L(X')$ is also extendable to $\mathbb{R}^2$.

The following theorem answers the question whether for non-standard $\mathcal{E}$-embeddings the shift homeomorphism is extendable to the whole plane which was posed by Boyland, de Carvalho and Hall in [19].

Theorem 7. *If X' is embedded in the plane with respect to L, where $A(L) \not\subset \mathcal{C}, \mathcal{R}$, then the shift homeomorphism $\sigma\colon \varphi_L(X') \to \varphi_L(X')$ cannot be extended to a homeomorphism of the plane.*

Proof. Let $\nu = c_1c_2\dots$ be a kneading sequence and $A(L) \not\subset \mathcal{C}, \mathcal{R}$ and assume by contradiction that $\sigma\colon \varphi_L(X') \to \varphi_L(X')$ is extendable to $\mathbb{R}^2$. Let $(n_i)_{i\in\mathbb{N}}$ be an increasing sequence in $\mathbb{N}$ such that $l_{n_i+3}l_{n_i+2} = 01$. Since $A(L) \not\subset \mathcal{C}, \mathcal{R}$, the sequence $(n_i)_{i\in\mathbb{N}}$ is indeed well defined. For $i \in \mathbb{N}$ define admissible tails

$$\overleftarrow{x_i} = 1^\infty 1011^{n_i}, \overleftarrow{y_i} = 1^\infty 0111^{n_i}, \overleftarrow{w_i} = 1^\infty 1101^{n_i}.$$

Note that $\overleftarrow{x_i}$ is between $\overleftarrow{y_i}$ and $\overleftarrow{w_i}$ and $\overleftarrow{x_i}1$ is the largest or the smallest among the admissible sequences $\overleftarrow{x_i}1$, $\overleftarrow{y_i}1$ and $\overleftarrow{w_i}1$ because of the chosen $l_{n_i+3}l_{n_i+2} = 01$.
For i large enough, note that $\pi_0(\overleftarrow{x_i}1) = [T^2(c), T(c)]$ so $A(\overleftarrow{x_i}1)$ is a horizontal arc in the plane of length $|T(c)-T^2(c)| =: \delta > 0$. Note also that $\pi_0(\overleftarrow{x_i}) = \pi_0(\overleftarrow{y_i}) = \pi_0(\overleftarrow{w_i}) = [T^2(c), T(c)]$ for i large enough. Let $\overleftarrow{x_i}' = \pi_0^{-1}([c, T(c)]) \cap \overleftarrow{x_i}$, $\overleftarrow{y_i}' = \pi_0^{-1}([c, T(c)]) \cap \overleftarrow{y_i}$ and $\overleftarrow{w_i}' = \pi_0^{-1}([c, T(c)]) \cap \overleftarrow{w_i}$, see Figure 6.1, left picture. Denote by $A_i \subset \mathbb{R}^2$ ($B_i \subset \mathbb{R}^2$) the vertical segment which joins the left (right) endpoints of $\overleftarrow{y_i}'$ and $\overleftarrow{w_i}'$. Note that $\operatorname{diam}(A_i), \operatorname{diam}(B_i) \to 0$ as $i \to \infty$. Also $D = A_i \cup \overleftarrow{y_i}' \cup B_i \cup \overleftarrow{w_i}'$ separates the plane, denote the bounded component of $\mathbb{R}^2 \setminus D$ by $U \subset \mathbb{R}^2$. Note that $\operatorname{Int} \overleftarrow{x_i}' \subset U$.
Now note that $\sigma(\overleftarrow{x_i}') = \overleftarrow{x_i}1$ and similarly for $\overleftarrow{y_i}', \overleftarrow{w_i}'$. Since $\overleftarrow{x_i}1$ is the smallest or the largest among $\overleftarrow{x_i}1, \overleftarrow{y_i}1, \overleftarrow{w_i}1$ and σ is extendable, at least one $\sigma(A_i)$ or $\sigma(B_i)$ has length greater than δ, see Figure 6.1. This contradicts the continuity of σ. $\square$

6.2 Bruin's embeddings $\varphi_{\mathcal{R}}(X')$

In this section we study the core X' as a subset of the plane by Bruin's embedding constructed in [24], *i.e.,* for $L = 1^\infty$. Recall that we denote these embeddings by $\varphi_{\mathcal{R}}(X')$. If the slope $s = 2$ and thus $X' = X$, it follows from Corollary 8 and Remark 20 that $\mathcal{R}$ and $\mathcal{C}$ are both

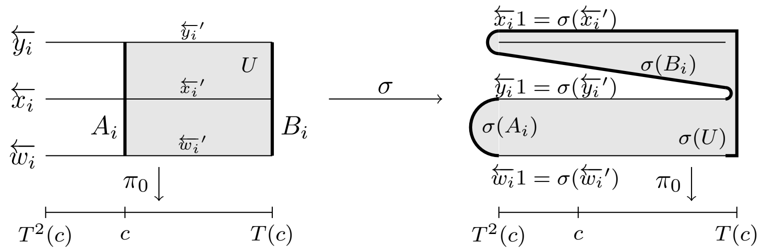

Figure 6.1: Shuffling of basic arcs from the proof of Theorem 7.

fully accessible and since there is no other folding point in the Knaster continuum except the endpoint $\bar{0}$, no other point from $\varphi_{\mathcal{R}}(X')$ is accessible. Thus from the circle of prime ends we conclude that there exists exactly one simple dense canal for $\varphi_{\mathcal{R}}(X')$. Therefore, from now onwards we restrict to cases when $X \neq X'$ (*i.e.,* $s \neq 2$). Bruin showed in [24] that $\sigma : \varphi_{\mathcal{R}}(X') \to \varphi_{\mathcal{R}}(X')$ is extendable to the plane and the extension is an orientation reversing planar homeomorphism.

Theorem 8. *Say that $X \neq X'$. In embeddings $\varphi_{\mathcal{R}}(X')$ the arc-component $\mathcal{R}$ is fully accessible and no other point from $\varphi_{\mathcal{R}}(X')$ is accessible. There exists one simple dense canal for every $\varphi_{\mathcal{R}}(X')$.*

Proof. For embeddings given by Bruin in [24] it holds that $L = 1^\infty$ and thus $\mathcal{U}_L = \mathcal{R}$.
We will explicitly calculate the top and bottom of an admissible cylinder $[a_n \dots a_1]$ for $n \in \mathbb{N}$.

If $\#_1(a_n \dots a_1)$ equals (does not equal) the parity of natural number n, then $L_{a_n \dots a_1} = 1^\infty a_n \dots a_1$ ($S_{a_n \dots a_1} = 1^\infty a_n \dots a_1$), since $1^\infty a_n \dots a_1$ is always admissible by Lemma 11. Also, $S_{a_n \dots a_1} = 1^\infty 01^k a_n \dots a_1$ ($L_{a_n \dots a_1} = 1^\infty 01^k a_n \dots a_1$), where $k \in \mathbb{N}_0$ is the smallest non-negative integer such that $01^k a_n \dots a_1$ is admissible.
Assume by contradiction that such k does not exists. Then $01^i a_n \dots a_1 \prec c_2 c_3 \dots$ for every $i \in \mathbb{N}_0$. Since the word 01^i is always admissible, it follows that $c_2 c_3 \dots = 01^i$ for every $i \in \mathbb{N}_0$, *i.e.,* $\nu = 101^\infty$ and the unimodal interval map which corresponds to this kneading sequence ν is renormalizable, a contradiction.

Note that every $1^\infty a_n \dots a_1$ is realized as an extremum of a cylinder, namely $1^\infty a_n \dots a_1$ $= L_{a_n \dots a_1}$ if $\#_1(a_n \dots a_1)$ equals the parity of n and $1^\infty a_n \dots a_1 = S_{a_n \dots a_1}$ if $\#_1(a_n \dots a_1)$ and

n are of different parity.

Note that if there was an accessible non-degenerate arc $Q \subset \varphi_{\mathcal{R}}(X')$ which is not the top or the bottom of any cylinder, then, since σ is extendable, also every shift of Q is accessible. But σ expands arcs, so there exists $i \in \mathbb{N}$ such that $\sigma^i(Q)$ contains a basic arc which is an extremum of a cylinder and thus $\sigma^i(Q)$ is a subset of $\varphi_{\mathcal{R}}(\mathcal{R})$. Therefore, also $Q \subset \varphi_{\mathcal{R}}(\mathcal{R})$. We conclude that $\varphi_{\mathcal{R}}(\mathcal{R})$ corresponds to the circle of prime ends minus a point. The remaining prime end P is either of the second, third, or fourth kind.

Assume first by contradiction that P is of the second kind, *i.e.,* it corresponds to an accessible folding point. Since $\sigma : \varphi_{\mathcal{R}}(X') \to \varphi_{\mathcal{R}}(X')$ is extendable to the plane, it follows that P needs to correspond to accessible point ρ (since $\bar{\rho} = \ldots 11.11 \ldots$ is the only σ-invariant itinerary of a point in X'). However, $A(1^\infty)$ is the top or the bottom of a cylinder, so ρ corresponds to a first kind prime end on the circle of prime ends, a contradiction.

Therefore, the remaining point P on the circle of prime ends is either of the third or the fourth kind. Since $\mathcal{R}$ is dense in X' (see Proposition 1 from [21]) and $\varphi_{\mathcal{R}}(\mathcal{R})$ bounds the canal in $\varphi_{\mathcal{R}}(X')$ it follows that $\Pi(P) = \varphi_{\mathcal{R}}(X')$ and thus $I(P) = \Pi(P) = \varphi_{\mathcal{R}}(X')$. Thus there exists one simple dense canal for every $\varphi_{\mathcal{R}}(X')$. $\square$

6.3 Brucks-Diamond embeddings $\varphi_{\mathcal{C}}(X')$

In this section we study the core X' as the subset of the plane by the Brucks-Diamond embedding $\varphi_{\mathcal{C}}$ constructed in [23], *i.e.,* for $L = 0^\infty 1$. If the slope $s = 2$, *i.e.,* $X = X'$ is the Knaster continuum, it follows from Corollary 8 and Remark 20 that $\mathcal{U}_L = \mathcal{C}$ is fully accessible and that no other point from $\varphi_{\mathcal{C}}(X')$ is accessible (observe the circle of prime ends). Specifically, there is no simple dense canal.

Thus we restrict to cases when $X \neq X'$ (*i.e.,* $s \neq 2$). Embeddings $\varphi_{\mathcal{C}}(X')$ can be viewed as global attractors of orientation preserving planar homeomorphisms as described by Barge and Martin in [12]. Therefore, $\sigma\colon \varphi_{\mathcal{C}}(X') \to \varphi_{\mathcal{C}}(X')$ can be extended to a planar homeomorphism. For $\varphi_{\mathcal{C}}(X)$ the set of accessible points is $\mathcal{C}$ and it forms an infinite canal which is dense in the core. However, if $\mathcal{C}$ is stripped off, the set of accessible points and the prime ends of $\varphi_{\mathcal{C}}(X')$ become very interesting, compare Figure 6.2 and Figure 6.3.

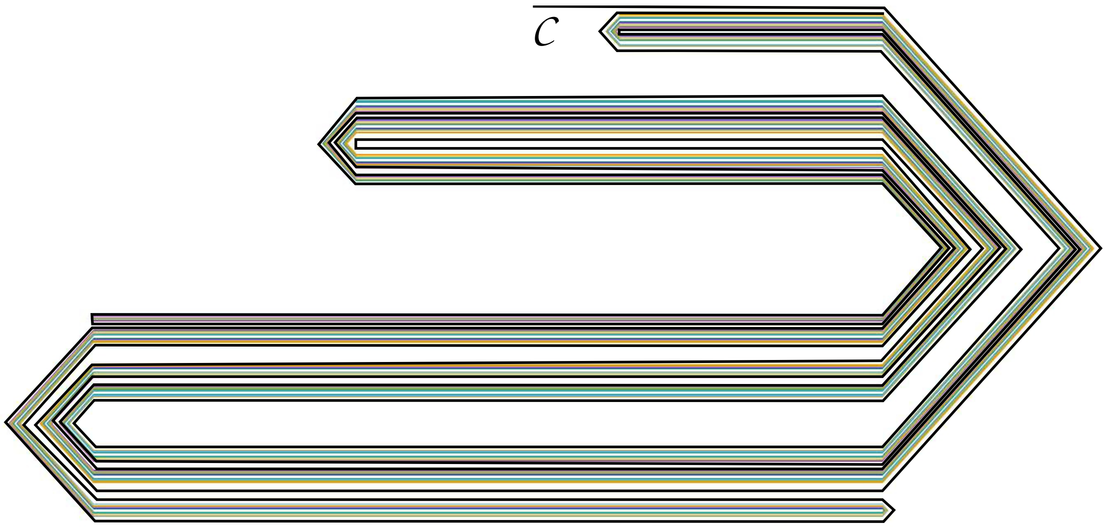

Figure 6.2: Space X for $\nu = (1001)^\infty$ embedded with $L = 01^\infty$.

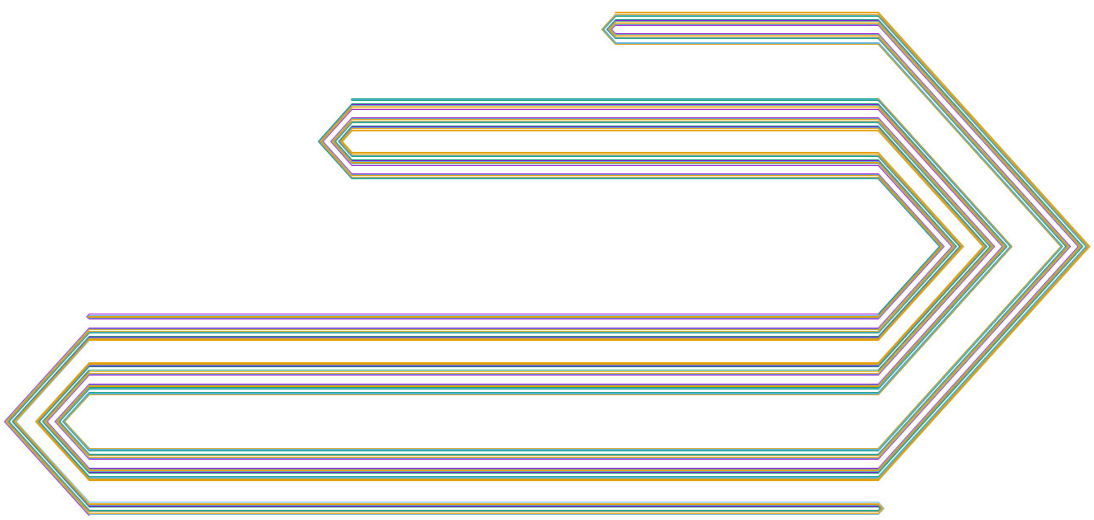

Figure 6.3: Space X' for $\nu = (1001)^\infty$ embedded with $L = 01^\infty$. Four arc-components $\mathcal{U}(e_0)$, $\mathcal{U}(e_1)$, $\mathcal{U}(e_2)$, $\mathcal{U}(e_3)$ of four endpoints $e_i \in X'$ for $i \in \{0, 1, 2, 3\}$ are fully accessible.

Recently Boyland, de Carvalho and Hall gave in [19] a complete characterization of prime ends for embeddings $\varphi_{\mathcal{C}}$ of unimodal inverse limits satisfying certain regularity conditions which hold also for tent map inverse limits with indecomposable cores. In this section we obtain an analogous characterization of accessible points as in [19] using symbolic computations. What this sections adds to the results from [19] is the characterization of types of accessible folding points, specially in the irrational height case (see the definitions below). By knowing the exact symbolic description of points in X' we can determine whether they are folding points or not, and if they are, whether they are endpoints of X'. The classification of accessible sets differentiates (as in [19]) according to the *height* of the kneading sequence which we introduce shortly in this section (for more details see [34]). Throughout this section the order $\prec_L$ corresponds with the standard parity-lexicographical order $\prec$.

We denote by L' the left infinite itinerary which is the largest admissible sequence in the embedding X' for $L = 0^\infty 1$ (as in [23]) after $\mathcal{C}$ is removed. Therefore we need to find which basic arc of X' is the closest to the basic arc $A(0^\infty 1)$. This was calculated in [18].

Definition 24. *Let $q \in (0, \frac{1}{2})$. For $i \in \mathbb{N}$ define*

$$\kappa_i(q) = \begin{cases} \lfloor \frac{1}{q} \rfloor - 1, & \text{if } i = 1, \\ \lfloor \frac{i}{q} \rfloor - \lfloor \frac{i-1}{q} \rfloor - 2, & \text{if } i \geq 2. \end{cases}$$

If q is irrational, we say that the kneading sequence

$$\nu = 10^{\kappa_1(q)} 110^{\kappa_2(q)} 110^{\kappa_3(q)} 11 \dots$$

has height q *or that it is of* irrational type*. If $q = \frac{m}{n}$, where m and n are relatively prime, we define*

$$c_q = 10^{\kappa_1(q)} 110^{\kappa_2(q)} 11 \dots 110^{\kappa_m(q)} 1,$$

$$w_q = 10^{\kappa_1(q)} 110^{\kappa_2(q)} 11 \dots 110^{\kappa_m(q)-1}.$$

By $\hat{a}$ we denote the reverse of a word a, so $\hat{w}_q = 0^{\kappa_m(q)-1} 110^{\kappa_{m-1}(q)} 11 \dots 110^{\kappa_1(q)} 1$. We say that a kneading sequence has rational height q *if $(w_q 1)^\infty \preceq \nu \preceq 10(\hat{w}_q 1)^\infty$. Denote by* $\mathrm{lhe}(q) := (w_q 1)^\infty$, $\mathrm{rhe}(q) := 10(\hat{w}_q 1)^\infty$. *If $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$ we say that ν is of* rational interior type*, and* rational endpoint type *otherwise. Every kneading sequence that appears in the tent map family is either of rational endpoint, rational interior or irrational type, see Lemma 8 and Lemma 9 in [18] (for further information see also [34]).*

Remark 28. *The values of $\kappa_i(q)$ can be obtained in the following way (see Lemma 2.5 in [34] for details). Draw the graph Γ_ζ of the function $\zeta\colon \mathbb{R} \to \mathbb{R}$, $\zeta(z) = qz$. Then $\kappa_i(q) = N_i - 2$, where N_i is the number of intersections of the graph Γ_ζ with vertical lines $z = N$, $N \in \mathbb{N}_0$ in the segment $[i-1, i]$, see Figure 6.4. Note that it automatically follows that the word $\kappa_1(q)\kappa_2(q)\dots\kappa_m(q)$ is a palindrome and thus c_q is a palindrome. Furthermore, for every $i \in \mathbb{N}$ either $\kappa_i(q) = \kappa_1(q)$ or $\kappa_i(q) = \kappa_1(q) - 1$.*

Remark 29. *Assume $q = m/n$ is rational with m and n being relatively prime. Take $k \in \{1, \dots, n-1\}$ such that $\lceil kq \rceil - kq$ obtains the smallest value; such k is unique, since m and n are relatively prime. Denote by $K = \lceil kq \rceil$ and note that for every $i \in \{1, \dots, k\}$ the line that joins $(0,0)$ with (k, K) intersects a vertical line in $[i-1, i]$ if and only if qz intersects a vertical line in $[i-1, i]$. Thus $\kappa_1(q)\dots\kappa_K(q)$ is a palindrome; it is the longest palindrome among $\kappa_1(q)\dots\kappa_i(q)$ for $i < m$. By studying the line which joins (k, K) with (n, m) we conclude that $\kappa_{K+1}(q)\dots\kappa_{m-1}(q)(\kappa_m(q)-1)$ is also a palindrome, see Figure 6.4. Thus for every rational q there exist palindromes Y, Z such that $c_q = Y1Z01$.*

Remark 30. *Note that $\{\kappa_i(q)\}_{i\geq 1}$ is a Sturmian sequence for irrational q and thus there exist infinitely many palindromic prefixes of increasing length (see* e.g. *[33], Theorem 5) which are of even parity. This can also be concluded by studying the rational approximations of q. Namely, if $k \in \mathbb{N}$ is such that $\lceil iq \rceil - iq$ achieves its minimum in $i = k$ for all $i \in \{1, \dots, k\}$, then the word $\kappa_1(q)\dots\kappa_k(q)$ is a palindrome. Note that $10^{\kappa_1(q)}11\dots110^{\kappa_k(q)}1$ is also a palindrome and it is an even word. By choosing better rational approximations of q from above, we see that k can be taken arbitrary large, and thus the beginning of c_q consists of arbitrary long even palindromes.*

Lemma 22. *Let $q = \frac{m}{n}$. Then there exists $N \in \mathbb{N}$ such that $\sigma^N(\mathrm{rhe}(q)) = \mathrm{lhe}(q)$.*

Proof. Recall that $\mathrm{lhe}(q) = (w_q 1)^\infty$, $\mathrm{rhe}(q) = 10(\hat{w}_q 1)^\infty$, where $c_q = w_q 01$. By Remark 29, there exist palindromes Y, Z such that $c_q = Y1Z01$, so $w_q = Y1Z$. It follows that $\mathrm{lhe}(q) = (Y1Z1)^\infty$ and $\mathrm{rhe}(q) = 10(Z1Y1)^\infty$ which finishes the proof. □

Remark 31. *The height of a kneading sequence is the rotation number of the natural mapping on the circle of prime ends. We will only need symbolic representation of the height of a kneading sequence here; for a more detailed study of height see [34].*

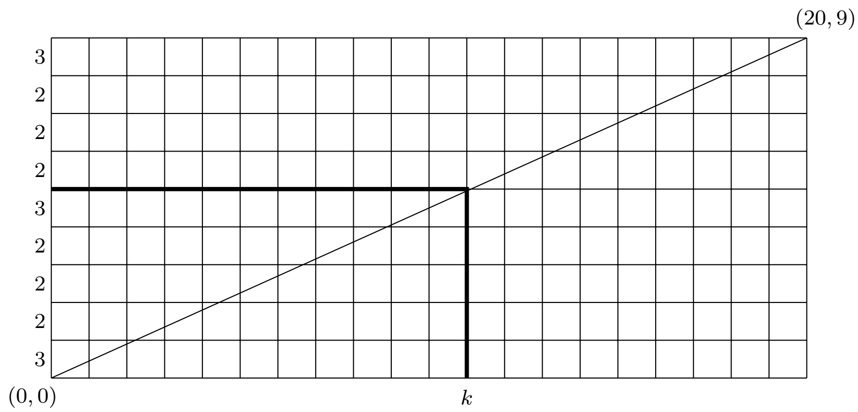

Figure 6.4: Calculating $\kappa_i(q)$ by counting the intersections of the line qz with vertical lines over integers. The picture shows the values N_i for $q = \frac{9}{20}$. It follows that $c_q = 10111111110111111101 = (101111111101)1(111111)01 = Y1Z01$. The decomposition into palindromes Y, Z follows since $\lceil \frac{9}{20}k \rceil - \frac{9}{20}k$ obtains its minimum for $k = 11 = \lfloor \frac{5}{q} \rfloor$ (bold line in the figure).

Definition 25. *Given an infinite sequence $\overrightarrow{x} = x_1x_2x_3\ldots$, we denote in this section its reverse by $\overleftarrow{x} = \ldots x_3x_2x_1$.*

Lemma 23 ([18], Lemma 13)**.** *Let X' be embedded with $\varphi_{\mathcal{C}}$. Denote by L' the largest admissible basic arc in X' and by ν the kneading sequence corresponding to X'. Then,*

$$L' = \begin{cases} \overleftarrow{\mathrm{rhe}(q)}, & \text{if } \mathrm{lhe} \prec \nu \preceq \mathrm{rhe}(q), \\ \overleftarrow{\nu}, & \text{if } q \text{ is irrational or } \nu = \mathrm{lhe}(q). \end{cases}$$

6.3.1 Irrational height case

Assume that q is irrational and note that the map T is then long-branched (since the kneading map is bounded, see [26]). Therefore, every proper subcontinuum is a point or an arc (see Proposition 3 in [21]) and consequently, every composant is an arc-component and thus either a line or a ray (every composant of X' is dense in X' so an arc cannot be a composant of X'). We will show that the basic arc $A(L')$ (which is fully accessible) contains an endpoint of X'. Furthermore, we will prove that the basic arc adjacent to $A(L')$ is not an extremum of a cylinder, and thus contains a folding point which is not an endpoint. Therefore, the ray $\mathcal{U}_{L'}$ is partially accessible; only a compact arc $Q \subset \mathcal{U}_{L'}$ is fully accessible and $\mathcal{U}_{L'} \setminus Q$ is not accessible. Since σ is extendable, also $\sigma^i(Q)$ is accessible for every $i \in \mathbb{Z}$. Later in this subsection we show that no other non-degenerate arc except of $\sigma^i(Q)$ for every $i \in \mathbb{Z}$ is fully accessible. From the circle of prime ends we then see that there is still a Cantor set of points remaining to be associated to either accessible points or infinite canals of $\varphi_{\mathcal{C}}(X')$. We prove that the remaining points on the circle of prime ends correspond to accessible endpoints of $\varphi_{\mathcal{C}}(X')$ and are thus second kind prime ends. Moreover, we prove that every endpoint from $\varphi_{\mathcal{C}}(X')$ is accessible. This is an extension of Theorem 4.46 from [19]. In this subsection the usage of variables m and n should not be confused with the values in the fraction $q = \frac{m}{n}$ which will be used in the rational height case later in this chapter.

Lemma 24. *If ν is of irrational type, then $\tau_R(L') = \infty$ and $A(L')$ is non-degenerate.*

Proof. If ν is of irrational type, then the bonding map T is long-branched, so every basic arc in X' is non-degenerate, *i.e.,* $A(L')$ is also non-degenerate.
To prove the first claim, first note that by Lemma 23 it holds that $L' = \overleftarrow{\nu}$. Remark 30 implies that there exist infinitely many even palindromes of increasing length at the beginning of ν. Thus there exists a strictly increasing sequence $(m_i)_{i\in\mathbb{N}}$ such that $l'_{m_i} \dots l'_1 = c_1 \dots c_{m_i}$ and $\#_1(c_1 \dots c_{m_i})$ is even for every i. Thus it follows that $\tau_R(L') = \infty$. $\square$

The following remark follows from Remark 15 in [18] and the fact that we restrict our study only on the tent map family.

Remark 32. *If ν is of irrational or rational endpoint type, it holds that $\overleftarrow{t} \in \{0,1\}^\infty$ is admissible (*i.e.*, every subword of $\overleftarrow{t}$ is admissible) if and only if $\overrightarrow{t}$ is admissible (*i.e.*, every subword of $\overrightarrow{t}$ is admissible).*

Lemma 25. *Let ν be either of irrational or rational endpoint type and X' embedded with $\varphi_{\mathcal{C}}$. Then every extremum of a cylinder of $\varphi_{\mathcal{C}}(X')$ belongs to $\sigma^i(L')$ for some $i \in \mathbb{Z}$.*

Proof. Take an admissible finite word $a_n \dots a_1 \in \{0,1\}^n$ and pick the smallest $k \in \{0, \dots, n-1\}$ such that $a_n \dots a_{k+1} = c_{n-k+1} \dots c_2$. If there is no such k we set $k = n$.

Assume first that $k > 1$ and note that $a_k = 1$.

Assume that $\#_1(a_{k-1}\dots a_1)$ is even and let us calculate $L_{a_n\dots a_1}$. If admissible, the word $L'a_{k-1}\dots a_1$ is the largest in the cylinder $[a_n\dots a_1]$. Assume that $L'a_{k-1}\dots a_1$ is not admissible. By Remark 32, since both L' and $a_{k-1}\dots a_1$ are admissible, there exists $i\in\{1,\dots,k-1\}$ such that $a_i\dots a_{k-1}l'_1\dots l'_j$ is not admissible for some $j\geq 1$. If $j\leq n-k+1$, then $a_i\dots a_{k-1}l'_1\dots l'_j$ is a subword of $a_1\dots a_n$ which is not admissible, a contradiction. Assume that $j > n-k+1$. In this case the word $a_i\dots a_{k-1}l'_1\dots l'_j \not\subseteq a_1\dots a_n$ is not admissible, but then $a_i\dots a_n = c_2\dots c_{2+n-i}$ which is a contradiction with k being the smallest such that $a_n\dots a_{k+1} = c_{n-k}\dots c_2$. If $\#_1(a_{k-1}\dots a_1)$ is odd we obtain that $S_{a_n\dots a_1} = L'a_{k-1}\dots a_1$ using analogous arguments as above.

Now assume that $\#_1(a_{k-1}\dots a_1)$ is odd and we calculate $L_{a_n\dots a_1}$. Say that $\#_1(a_n\dots a_k)$ is odd. Therefore, since we want to calculate the largest basic arc in the cylinder $[a_n\dots a_1]$, we need to set $L_{a_n\dots a_1} = \dots 1a_n\dots a_1$, and note that $1a_n\dots a_1$ is always admissible by Lemma 11. Then, knowing that $\#_1(a_n\dots a_k)$ is odd it follows from the special structure of ν in the irrational height case that the kneading sequence starts as $a_k\dots a_n11$ or $a_k\dots a_n0$ and thus the word $a_k\dots a_n10$ is admissible. It follows that $L'a_n\dots a_1$ is admissible and equals to $L_{a_n\dots a_1}$. If $\#_1(a_n\dots a_k)$ is even, it follows from the structure of ν (blocks of ones in ν are of even length) that $a_n = 1$ and $a_k\dots a_n$ ends in odd number of ones. The word $a_k\dots a_n0^{\kappa_1(q)}$ is thus admissible and therefore $L_{a_n\dots a_1} = L'a_{n-1}\dots a_1$. Calculations for $S_{a_n\dots a_1}$ when $\#_1(a_{k-1}\dots a_1)$ is even follow analogously.

Now say that $k=1$. Then $L_{a_n\dots a_1} = L'$. We conclude as in the preceding paragraph that if $\#_1(a_n\dots a_1)$ is even, then $S_{a_n\dots a_1} = L'a_{n-1}\dots a_1$ and if $\#_1(a_n\dots a_1)$ is odd, then $S_{a_n\dots a_1} = L'a_n\dots a_1$.

If $k=0$, then $a_1\dots a_n = c_2\dots c_{n+1}$. So $S_{a_n\dots a_1} = S = \dots c_4c_3c_2$. To calculate $L_{a_n\dots a_1}$, let k' be the smallest natural number such that $a_n\dots a_{k'} = c_{n-k'+1}\dots c_1$. If k' does not exist, set $k' = n+1$. From the structure of ν (blocks of ones in ν are of even length) it follows that $\#_1(a_{k'-1}\dots a_1)$ is odd. The rest of the proof for this case follows the same as in the case for $k>1$. $\square$

Lemma 26. *Assume ν is of irrational type and X' embedded with $\varphi_{\mathcal{C}}$. Then the only basic* *arc from $\mathcal{U}_{L'}$ which is an extremum of a cylinder is $A(L')$.*

Proof. Let $a_n \dots a_1$ be an admissible word for some $n \in \mathbb{N}$. If $n = 1$, note that $L_1 = L' \subset \mathcal{U}_{L'}$ and $L_0, S_0, S_1 \not\subset \mathcal{U}_{L'}$, since ν is not (pre)periodic.

Now assume that $n \geq 2$. Since ν is not (pre)periodic, the proof of Lemma 25 gives that if $L_{a_n \dots a_1}$ or $S_{a_n \dots a_1}$ are contained in $\mathcal{U}_{L'}$, then $a_1 \dots a_n = c_1 \dots c_n$ (since otherwise $L_{a_n \dots a_1}$ or $S_{a_n \dots a_1}$ would be contained in $\sigma^i(\mathcal{U}_{L'})$ for some $i \in \mathbb{Z} \setminus \{0\}$). But then, following the proof of Lemma 25 it holds that $L_{a_n \dots a_1} = L'$ and $S_{a_n \dots a_1} = L' a_n \dots a_1$ or $S_{a_n \dots a_1} = L' a_{n-1} \dots a_1$, depending on the parity of $\#_1(a_n \dots a_1)$. Since $L' a_n \dots a_1 \in \sigma^n(L')$ and $L' a_{n-1} \dots a_1 \in \sigma^{n-1}(L')$ the only extremum of a cylinder in $\mathcal{U}_{L'}$ is $A(L')$. $\square$

Remark 33. *It follows from Lemma 26 that when ν has irrational height, then $\mathcal{U}_{L'}$ is partially accessible. To be more precise, from Proposition 22 it follows that $\overleftarrow{l(L')} = \dots 110^{\kappa_3(q)} 110^{\kappa_2(q)} 11$ $0^{\kappa_1(q)-1} 11$ contains a folding point p and $A(L') \cup [a, p]$ is fully accessible, where a denotes the left endpoint of $\overleftarrow{l(L')}$. It follows from Corollary 6 that no other point from $\mathcal{U}_{L'}$ (which is a ray) is accessible. Since $\sigma : \varphi_{\mathcal{C}}(X') \to \varphi_{\mathcal{C}}(X')$ is extendable to the plane, also $\sigma^i(A(L') \cup [a, p])$ is accessible for every $i \in \mathbb{Z}$. Moreover, those are the only accessible non-degenerate arcs, since σ is extendable and expanding (see the discussion in the proof of Theorem 8). In the lemmas to follow we prove that the remaining Cantor set of points on the circle of prime ends correspond to the endpoints of $\varphi_{\mathcal{C}}(X')$, and that all endpoints of $\varphi_{\mathcal{C}}(X')$ are accessible when ν is of irrational type.*

The following lemma follows directly from the fact that $(\kappa_i(q))_{i \in \mathbb{N}}$ is Sturmian, but we prove it here for the sake of completeness. Say that $q \in (0, \frac{1}{2})$ is irrational. Denote by $\kappa = \kappa_1(q)$, so $\kappa_i(q) \in \{\kappa, \kappa - 1\}$ for every $i \in \mathbb{N}$.

Lemma 27. *Let $q \in (0, \frac{1}{2})$ be irrational. There exists $J \in \mathbb{N}$ such that if $\kappa_i(q) \kappa_{i+1}(q)$ $\dots \kappa_{i+N}(q) \kappa_{i+N+1}(q) = \kappa (\kappa - 1)^N \kappa$, then $N \in \{J, J+1\}$.*

Proof. Let $J \in \mathbb{N}$ be such that $\kappa_2(q) = \dots = \kappa_{J+1}(q) = \kappa - 1$ and $\kappa_{J+2}(q) = \kappa$. So there exists a sequence of J consecutive $(\kappa - 1)$s. Denote by $H_n = \lfloor \frac{n}{q} \rfloor$ for $n \in \mathbb{N}$ and note that the function $g \colon \mathbb{N} \to \mathbb{R}$ given by $g(k) = \lceil kq \rceil - kq$ achieves its minimum on $[0, H_{J+2}]$ in H_{J+2} (since $J + 2$ is minimal index $a > 1$ for which $\kappa_a = \kappa$). If we translate the graph of function $\zeta(z) = qz$ by $+\delta$ where $\delta \in (0, g(H_{J+2})]$, then the sequence of consecutive number of intersections with vertical lines over integers begins again with $(\kappa+2)(\kappa+1)^J(\kappa+2)$. Since g restricted to $[0, H_{J+2})$ achieves its minimum in H_1, if $\delta \in (g(H_{J+2}), g(H_1))$, the sequence corresponding to the number of times the graph of $\zeta + \delta$ intersects vertical lines over integers begins with $(\kappa+2)(\kappa+1)^{J+1}(\kappa+2)$, see Figure 6.5. Fix $i \geq 2$ such that $\kappa_i(q) = \kappa$. Note that then $g(H_{i-1}+1) < g(H_1)$ since otherwise $qH_{i-1} > i-1$ which is a contradiction. So the graph of ζ on $[H_{i-1}+1, \infty)$ can be obtained from the graph of ζ on $[0,\infty)$ by translating it by $+\delta$ for $\delta \in (0, g(H_1))$ which finishes the proof. $\square$

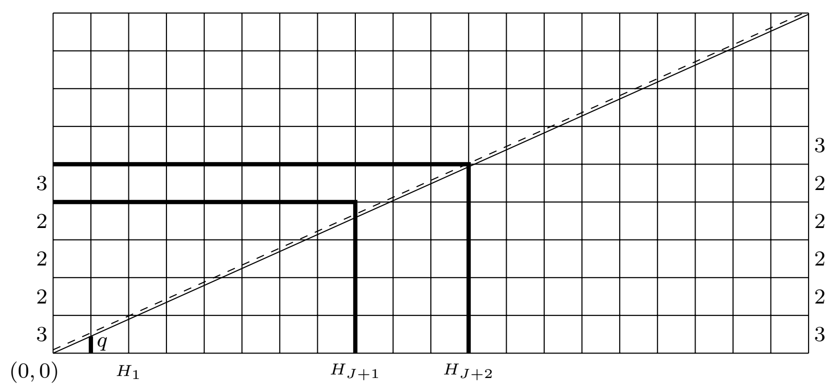

Figure 6.5: The graph of qz for $q \approx 0.4483\ldots$ with the number of intersections with vertical integer lines on the left. The dashed line represents the graph of qz translated by $\delta \in (g(H_{J+2}), g(H_1))$. On the right we count the intersections of the translated graph with vertical integer lines.

Lemma 28. *Let $q \in (0, \frac{1}{2})$ be irrational and $i, N \in \mathbb{N}$ such that $\kappa_{i+1}(q)\ldots\kappa_{i+N}(q) = \kappa_1(q)\ldots\kappa_N(q)$ and $\kappa_{i+N+1}(q) \neq \kappa_{N+1}(q)$. Then $\kappa_1(q)\ldots\kappa_{N+1}(q)$ is a palindrome. Moreover, $\kappa_{i+N+2}(q) = \kappa_1(q)$. If $K \in \mathbb{N}$ is such that $\kappa_{i+N+2}(q)\ldots\kappa_{i+N+K+1}(q) = \kappa_1(q)\ldots\kappa_K(q)$ and $\kappa_{i+N+K+2}(q) \neq \kappa_{K+1}(q)$, then $\kappa_{K+1}(q)\ldots\kappa_1(q)\kappa_{i+N+1}(q)\ldots\kappa_{i+1}(q)$ $= \kappa_1(q)\ldots\kappa_{K+N+1}(q)$.*

Proof. For $i \in \mathbb{N}$ denote by $H_i = \lfloor \frac{i}{q} \rfloor$ and let $f\colon \mathbb{N} \to \mathbb{R}$ be given by $f(z) = zq - \lfloor zq \rfloor$. Note that the graph of $\zeta(z) = qz$ restricted to $[H_i+1, \infty)$ is a translation of the graph of ζ on $[0,\infty)$ by some $\delta > 0$ (see *e.g.* Figure 6.5). The conditions $\kappa_{i+1}(q)\ldots\kappa_{i+N}(q) = \kappa_1(q)\ldots\kappa_N(q)$ and $\kappa_{i+N+1}(q) \neq \kappa_{N+1}(q)$ imply that the global minimum of f on $[H_i, H_{i+N+1}+1]$ is $H_{i+N+1}+1$.

So the graph of $\zeta - f(H_{i+N+1}+1)$ on $[H_i, H_{i+N+1}+1]$ intersects vertical lines over integers the same number of times as ζ except for the point $(H_{i+N+1}+1, i+N+1)$. We conclude that $(\kappa_{i+N+1}(q)+1)\kappa_{i+N}(q)\dots\kappa_{i+1}(q) = \kappa_1(q)\dots\kappa_{N+1}(q)$ which concludes the first part of the proof. To see that $\kappa_{i+N+2}(q) = \kappa_1(q)$ use Lemma 27.

For the last part of the proof assume that $K \in \mathbb{N}$ is such that $\kappa_{i+N+2}(q)\dots\kappa_{i+N+K+1}(q) = \kappa_1(q)\dots\kappa_K(q)$ and $\kappa_{i+N+K+2}(q) \neq \kappa_{K+1}(q)$. That implies that the global minimum of f on $[H_i, H_{i+N+K+2}+1]$ is $H_{i+N+K+2}+1$. Again by translating the graph of ζ on $[H_i, H_{i+N+K+2}+1]$ by $-f(H_{i+N+K+2}+1)$ we conclude the second part of the proof, see Figure 6.6. $\square$

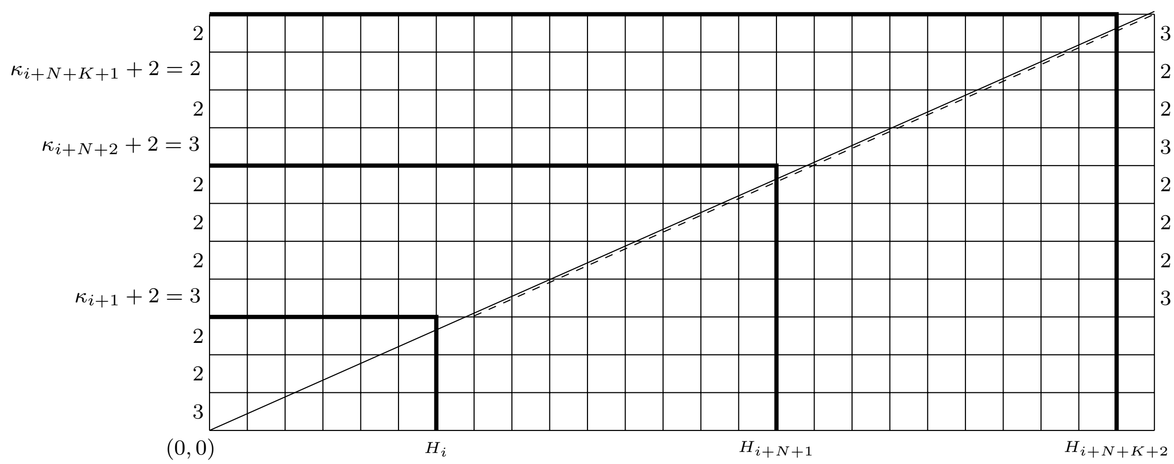

Figure 6.6: Graphic representation of the proof of Lemma 28 for $q \approx 0.443\dots$. The dashed line represents the graph of $\zeta(z) = qz$ on $[H_i+1, H_{i+N+K+2}+1]$ translated by $-f(H_{i+N+K+2}+1)$. On the right side of the grid we count intersections of the dashed line with vertical integer lines.

Lemma 29. *If ν is of irrational type or $\nu = lhe(q)$, then every endpoint of $\varphi_{\mathcal{C}}(X')$ is accessible.*

Proof. Let $e \in X'$ be an endpoint and let $\overleftarrow{e}$ denote the left infinite symbolic description of e. Assume that $\tau_R(\overleftarrow{e}) = \infty$ and thus there exists a strictly increasing sequence $(m_i)_{i\in\mathbb{N}}$ such that $c_1\dots c_{m_i} = e_{m_i}\dots e_1$ and $\#_1(e_{m_i}\dots e_1)$ is even. Assume $(m_i)_{i\in\mathbb{N}}$ is the complete sequence for e (see Definition 21).

Assume that for infinitely many $i \in \mathbb{N}$ there exist admissible left infinite itineraries $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e} \prec_L x^{I(i)}$ so that $\overleftarrow{x}^{O(i)}, \overleftarrow{x}^{I(i)} \to \overleftarrow{e}$ as $i \to \infty$, $\overleftarrow{x}^{O(i)}, \overleftarrow{x}^{I(i)}$ differ only at the index $m_i + 1$

and equal $c_1 \dots c_{m_i}$ on the first m_i places (if we are able to construct such $\overleftarrow{x}^{O(i)}, \overleftarrow{x}^{I(i)}$ the arcs will cap the endpoint e which would thus be inaccessible - compare with the proof of Theorem 5). So, $\overleftarrow{x}^{O(i)}$ and $\overleftarrow{x}^{I(i)}$ are of the form:

$$\overleftarrow{x}^{I(i)} = \dots 110^{\kappa_1(q)}110^{\kappa_2(q)}11 \dots 110^{\kappa_j(q)}1.$$

$$\overleftarrow{e} = \dots 110^{\kappa_1(q)}110^{\kappa_2(q)}11 \dots 110^{\kappa_j(q)}1.$$

$$\overleftarrow{x}^{O(i)} = \dots 010^{\kappa_1(q)}110^{\kappa_2(q)}11 \dots 110^{\kappa_j(q)}1.$$

Note first that $0e_{m_i} \dots e_1$ is indeed admissible. Since $\#_1(e_{m_i} \dots e_1)$ is even it holds that $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e}$ for every $i \in \mathbb{N}$. Thus we need to find $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e}$ in order to cap e.
Denote by $J \in \mathbb{N}$ the smallest natural number such that

$$\overleftarrow{e} = \dots 110^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11 \dots 110^{\kappa_2(q)}110^{\kappa_1(q)}110^{\kappa_2(q)}11 \dots 110^{\kappa_j(q)}1.$$

By Lemma 28 it follows that $\kappa_J(q) \dots \kappa_2(q)\kappa_1(q)$ is a palindrome and thus $10^{\kappa_J(q)}11$ $0^{\kappa_{J-1}(q)}11 \dots 110^{\kappa_2(q)}11$ equals the beginning of ν.

We want to find $\overleftarrow{x}^{I(i)} \succ \overleftarrow{e}$. Note that none of $00^{\kappa_2(q)}110^{\kappa_1(q)}, \dots, 00^{\kappa_{J-1}(q)}11 \dots 110^{\kappa_1(q)}$ are admissible. If we set

$$\overleftarrow{x}^{I(i)} = \dots 00^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11 \dots 110^{\kappa_2(q)}110^{\kappa_1(q)}11 \dots 110^{\kappa_j(q)}1,$$

then also

$$\overleftarrow{x}^{O(i)} = \dots 00^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11 \dots 110^{\kappa_2(q)}010^{\kappa_1(q)}11 \dots 110^{\kappa_j(q)}1.$$

But since $100^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11 \dots 110^{\kappa_2(q)}1$ equals the beginning of ν, the word $00^{\kappa_J(q)-1}11$ $0^{\kappa_{J-1}(q)}11 \dots 110^{\kappa_2(q)}0$ is not admissible, a contradiction.
Thus we need to set

$$\overleftarrow{x}^{I(i)} = \dots 110^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11 \dots 110^{\kappa_2(q)}110^{\kappa_1(q)}11 \dots 110^{\kappa_j(q)}1.$$

By Lemma 27 it follows that

$$\overleftarrow{e} = \dots 110^{\kappa_1(q)}110^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11 \dots 110^{\kappa_2(q)}110^{\kappa_1(q)}110^{\kappa_2(q)}11 \dots 110^{\kappa_j(q)}1.$$

Now take the smallest $K \in \mathbb{N}$ such that

$$\overleftarrow{e} = \dots 110^{\kappa_{K+1}(q)-1}110^{\kappa_K(q)}11 \dots 110^{\kappa_1(q)}110^{\kappa_J(q)-1}1$$

$$10^{\kappa_{J-1}(q)}11\dots110^{\kappa_2(q)}110^{\kappa_1(q)}110^{\kappa_2(q)}11\dots110^{\kappa_j(q)}1.$$

By Lemma 28 it follows that $10^{\kappa_{K+1}(q)}11\dots110^{\kappa_1(q)}110^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\dots110^{\kappa_2(q)}11$ is the beginning of ν. Thus we analogously argue that

$$\overleftarrow{x}^{I(i)} = \dots110^{\kappa_{K+1}(q)-1}110^{\kappa_K(q)}11\dots110^{\kappa_1(q)}110^{\kappa_J(q)-1}1$$

$$10^{\kappa_{J-1}(q)}11\dots110^{\kappa_2(q)}110^{\kappa_1(q)}110^{\kappa_2(q)}11\dots110^{\kappa_j(q)}1,$$

which agrees with $\overleftarrow{e}$. Continuing inductively we conclude that $\overleftarrow{x}^{I(i)} = \overleftarrow{e}$. Thus e is not capped. $\square$

Remark 34. *We can expand the definition of Type 3 folding point introduced in the preperiodic orbit case. A point p will be called a Type 3 folding point, if it is not an endpoint, it is accessible, and there is an arc $p \in Q \subset \mathcal{U}_p$ such that $Q \setminus \{p\}$ is not accessible, see Figure 5.13.*

Lemma 30. *If ν is of irrational type or rational endpoint type and X' is embedded with $\varphi_{\mathcal{C}}$, then there are no Type 3 folding points.*

Proof. Assume by contradiction that there is a basic arc $\overleftarrow{x} = \dots x_2x_1$ and an accessible folding point $p \in A(\overleftarrow{x})$ of Type 3. Since p is a folding point by Proposition 2 there exist blocks of symbols of ν of increasing length in $\overleftarrow{x}$.

We claim that if $c_n \dots c_{n+k} = c_m \dots c_{m+k}$ for some $m, n \in \mathbb{N}$ and there exists $i \in \{0, \dots, k\}$ such that $c_{n+i} = 0$, then $\#_1(c_1 \dots c_{n+k}) = \#_1(c_1 \dots c_{m+k})$ (then all the wiggles will accumulate on $A(\overleftarrow{x})$ from exactly one side of p as in Figure 5.11). Indeed, take the largest such index i. Then it follows that $c_n \dots c_{n+i-1} = 1^i$. If i is even (odd) it holds that $\#_1(c_1 \dots c_{n-1})$ is odd (even), which proves the claim.

Therefore, if for $\overleftarrow{x} = \dots x_2x_1$ there exists $i \in \{0, \dots k\}$ such that $c_{n+i} = 0$ and $x_j \dots x_1 = c_n \dots c_{n+k}$ it follows that $A(\overleftarrow{x})$ contains no Type 3 folding point.

Now assume that $\overleftarrow{x} = 1^\infty$. If $\kappa_1(q) > 1$, then $\dots 1101^\alpha \succ_L \overleftarrow{x} \succ_L \dots 1101^{\alpha+1}$ for every odd $\alpha \in \mathbb{N}$ and both $\dots 1101^\alpha$ and $\dots 1101^{\alpha+1}$ project to $[T^2(c), T(c)]$, which is again a contradiction with p being a Type 3 folding point.

If $\kappa_1(q) = 1$, then $\nu = 101^\beta 0\dots$ for some even $\beta \in \mathbb{N}$. Then, basic arcs with symbolic description $1^\infty 01^\gamma$ for every $\gamma > \beta$ project to $[T^2(c), T(c)]$ and we get an analogous conclusion as in the preceding paragraph. □

Lemma 31. *If ν is of irrational type, then there exist no third and fourth kind prime ends corresponding to $\varphi_{\mathcal{C}}(X')$.*

Proof. Since the embedding $\varphi_{\mathcal{C}}(X')$ is realized as an alignment of basic arcs along vertically embedded Cantor set connected with semi-circles, we can study crosscuts which are vertical segments in the plane joining two adjacent cylinders, see Figure 5.3. Note that every infinite canal is realized by such vertical crosscuts. Take two n-cylinders $A = [a_n \dots a_1]$ and $B = [b_n \dots b_1]$ for some $n \in \mathbb{N}$, such that $A \succ_L B$ and A and B are *adjacent* n-cylinders, *i.e.,* there is no n-cylinder D such that $A \succ_L D \succ_L B$. We will show that S_A and L_B have the same tail, *i.e.,* they both belong to $\sigma^i(L')$ for some $i \in \mathbb{Z}$. Since the accessible subsets of $\sigma^i(L')$ are arcs of finite length, it follows immediately that there cannot exist infinite canals for $\varphi_{\mathcal{C}}(X')$.

Take A and B as above and let $m \in \{0, \dots, n-1\}$ be the smallest nonnegative number such that $a_{m+1} \neq b_{m+1}$.
First assume that $\#_1(a_m \dots a_1)$ is odd. Then, $S_A = S_{0a_m\dots a_1}$ and $L_B = L_{1a_m\dots a_1}$, since $A \succ_L B$ are adjacent. Let $k \in \{1, \dots, m-1\}$ be the smallest number such that $c_2 \dots c_{m-k+2} = a_{k+1} \dots a_m 1$, (compare with the proof of Lemma 25). Assume first that such k indeed exists. Since also $c_2 \dots c^*_{m-k+2} \subset S_A$ is admissible, it follows that $\#_1(a_{k+1} \dots a_m)$ is odd. Thus, $\#_1(a_k \dots a_1)$ is even and since $a_k = 1$ it holds that $\#_1(a_{k-1} \dots a_1)$ is odd. As in the proof of Lemma 25, we conclude that $L_{1a_m\dots a_1} = L' 1 a_m \dots a_1$. The same conclusion follows in the case when k does not exist. Note that $k = m$ is not possible. Furthermore, since $\#_1(a_k \dots a_m)$ is odd, it follows from the specific form of ν that $S_{0a_m\dots a_1} = L' 0 a_m \dots a_1$, which is always admissible. Therefore, S_A and L_B have the same left infinite tail.
Now assume that $\#_1(a_m \dots a_1)$ is even. Then $S_A = S_{1a_m\dots a_1}$ and $L_B = L_{0a_m\dots a_1}$, since $A \succ_L B$ are adjacent. Let $k \in \{1, \dots, m-1\}$ again be the smallest number such that $c_2 \dots c_{m-k+1} = a_{k+1} \dots a_m 1$. By analogous arguments as in the preceding paragraph we obtain that $\#_1(a_{k-1} \dots a_1)$ is even and thus as in the proof of Lemma 25, we conclude that $S_{1a_m\dots a_1} = L' 1 a_m \dots a_1$. Furthermore, $L_{0a_m\dots a_1} = L' 0 a_m \dots a_1$ which is always admissible. Again, S_A and L_B have the same left infinite tail. Therefore, it holds that all the canals are

finite, *i.e.,* there exist no third and fourth kind prime ends corresponding to $\varphi_{\mathcal{C}}(X')$. $\square$

The following theorem follows directly from the preceding eight lemmas.

Theorem 9. *If ν is of irrational type and X' is embedded with $\varphi_{\mathcal{C}}$, then there are countably infinitely many partially accessible rays of $\varphi_{\mathcal{C}}(X')$; these are the arc-components which are symbolically described by a tail which is a shift of $\overleftarrow{\nu}$. Each of them contains an endpoint of $\varphi_{\mathcal{C}}(X')$ and the accessible set is a compact arc which contains that endpoint. Furthermore, there exist uncountably many accessible arc-components which are accessible in a single point which is an endpoint of $\varphi_{\mathcal{C}}(X')$. All (uncountably many) endpoints of $\varphi_{\mathcal{C}}(X')$ are accessible.*

6.3.2 Rational endpoint case

Let $q = \frac{m}{n}$. In this subsection we study $\varphi_{\mathcal{C}}(X')$ when ν is either $\mathrm{rhe}(q)$ or $\mathrm{lhe}(q)$. We provide a symbolic proof of Theorem 4.66 from [19].

When $\nu = \mathrm{lhe}(q) = (w_q 1)^\infty$ it follows that $L' = \overleftarrow{\mathrm{lhe}(q)}$. In Remark 29 we argued that there exist palindromes Y, Z such that $\mathrm{lhe}(q) = (Y1Z1)^\infty$, thus $\overleftarrow{\mathrm{lhe}(q)} = (1Z1Y)^\infty$. Note that both Y and Z are even, from which we conclude that $\tau_R(L') = \infty$. Thus the right endpoint of $A(L')$ is also an endpoint of X' and since there are no other folding points on $\mathcal{U}_{L'}$ except of this endpoint, the ray $\mathcal{U}_{L'}$ is a fully accessible. Since σ is extendable to the plane it follows that $\sigma^i(\mathcal{U}_{L'})$ are fully accessible for every $i \in \{0, 1, \dots, n-1\}$ (where n is the period of $\mathrm{lhe}(q)$). Lemma 25 assures that the union of n rays is indeed the complete set of accessible points of $\varphi_{\mathcal{C}}(X')$ for $\nu = \mathrm{lhe}(q)$. Thus the circle of prime ends decomposes into n half-open intervals, where the endpoints represent the endpoints of X'. Summarizing, we have the following theorem:

Theorem 10. *If $\nu = lhe(q)$ for some $q = \frac{m}{n}$, where m and n are relatively prime, then in $\varphi_{\mathcal{C}}(X')$ there exist n fully accessible rays which are symbolically described by a tail which is a shift of $\overleftarrow{\nu}$ and no other point from $\varphi_{\mathcal{C}}(X')$ is accessible. Specifically, there exist no infinite canals in $\varphi_{\mathcal{C}}(X')$.*

When $\nu = \mathrm{rhe}(q)$ it holds by Lemma 23 that $L' = \overleftarrow{\mathrm{rhe}(q)} = (1Y1Z)^\infty 01$. Since Y starts with 1 it holds that there exists a folding point $p \in \mathcal{U}_{L'}$ on a basic arc with itinerary $\overleftarrow{l(L')} = (1Y1Z)^{\infty}11$. Since $\mathrm{rhe}(q)$ is strictly preperiodic it follows that left tail of $\overleftarrow{l(L')}$ always differs from $\overleftarrow{\mathrm{rhe}(q)}$, so Lemma 25 implies that $\overleftarrow{l(L')}$ is not an extremum of any cylinder. Proposition 22 implies that p is Type 2 folding point and consequently $\mathcal{U}_{L'}$ is partially accessible. Moreover, since $\mathcal{U}_{L'}$ contains no other folding points we conclude that one component of $\mathcal{U}_{L'} \setminus \{p\}$ is fully accessible and the other component of $\mathcal{U}_{L'} \setminus \{p\}$ is not accessible. Since σ is extendable, $\sigma^i(\mathcal{U}_{L'})$ are also partially accessible. Lemma 25 implies that the circle of prime ends decomposes into n half-open intervals and their endpoints are representing the accessible folding points of Type 2. Thus we obtain the following theorem:

Theorem 11. *If $\nu = rhe(q)$ for some $q = \frac{m}{n}$, where m and n are relatively prime, then in $\varphi_{\mathcal{C}}(X')$ there exist n partially accessible lines which are symbolically described by a tail which is a shift of $\overleftarrow{\nu}$ and no other point from $\varphi_{\mathcal{C}}(X')$ is accessible. Specifically, there exist no infinite canals in $\varphi_{\mathcal{C}}(X')$.*

6.3.3 Rational interior case

Assume $q = \frac{m}{n}$, where m and n are relatively prime. We will show that in the rational interior case there exist n fully accessible arc-components which are dense lines in X'. We show that folding points which are not lying in the extrema of cylinders are not accessible, so the remaining n points on the circle of prime ends are simple dense canals. That is an analogue of Theorem 4.64 from [19] for tent inverse limits.

Lemma 32 (Theorem 16 in [18])**.** *Suppose that ν is of rational interior type for $q = m/n$, where m and n are relatively prime. Then a sequence $\overleftarrow{t} \in \{0,1\}^{\infty}$ which does not belong to $\mathcal{C}$ is admissible if and only if*

(a) $\sigma^i(\overleftarrow{t}) \preceq \mathrm{rhe}(q)$ *for all* $i \in \mathbb{N}$,

(b) $\sigma^i(\overleftarrow{t}) \preceq \mathrm{lhe}(q)$ *for all* $i \in \mathbb{N}$ *for which* $\sigma^i(\overrightarrow{t}) \succ \sigma^{n+1}(\nu)$.

Lemma 33. *Say that $q = m/n$, where m and n are relatively prime. If $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$, then all the extrema of cylinders of $\varphi_{\mathcal{C}}(X')$ have tails in $\sigma^i(L')$ for some $i \in \mathbb{Z}$.*

Proof. Fix an arbitrary admissible word $b_j \dots b_1 \in \{0,1\}^j$ for some $j \in \mathbb{N}$.

We will calculate the top/bottom of the cylinder $[b_j \dots b_1]$. Assume that $b_j \dots b_1 \succ \sigma^{n+1}(\nu)$ and $\#_1(b_j \dots b_1)$ is even (odd). We first show that if $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ is admissible, then it equals $L_{b_j \dots b_1}(S_{b_j \dots b_1})$. Assume by contradiction with Lemma 32, case (b) that there exists an admissible $\dots x_2 x_1 b_j \dots b_1 \succ \overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ $(\dots x_2 x_1 b_j \dots b_1 \prec \overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1)$. Then $\dots x_2 x_1 \succ \overleftarrow{\mathrm{lhe}(q)}$ $(\dots x_2 x_1 \succ \overleftarrow{\mathrm{lhe}(q)})$. But that combined with $b_j \dots b_1 \succ \sigma^{n+1}(\nu)$ gives by (b) from Lemma 32 that $\dots x_2 x_1 b_j \dots b_1 \succ \overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ is not admissible, a contradiction. Similarly, we show that if $b_j \dots b_1 \preceq \sigma^{n+1}(\nu)$, $\#_1(b_j \dots b_1)$ is even (odd) and $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ is admissible, then it equals $L_{b_j \dots b_1}(S_{b_j \dots b_1})$.

In the next two paragraphs we prove that the sequences of the form $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ in special case to which we restrict later in the proof satisfy conditions (a) and (b) from Lemma 32 and are thus admissible.

If $b_{i+1} \dots b_j$ does not equal the beginning of $\mathrm{rhe}(q)$ for any $i \in \{0, \dots, j-1\}$, then the sequences $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ satisfy (a) from Lemma 32. Assume there is an index $i \in \{0, \dots j-1\}$ such that $b_{i+1} \dots b_j$ is the beginning of $\mathrm{rhe}(q)$ and take the smallest such $i \in \{0, \dots, j-1\}$. Assume $\#_1(b_{i+1} \dots b_j)$ is odd (later in the proof we need only this special case). If $b_{\alpha+1} \dots b_j$ is also the beginning of $\mathrm{rhe}(q)$ for some $\alpha \in \{0, \dots, j-1\}$, where $\alpha \geq i$, then $\#_1(b_{\alpha+1} \dots b_j)$ is also odd. Note that $b_{\alpha+1} \dots b_j 10 \prec \mathrm{rhe}(q)$ for every such α. Thus $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ satisfy condition (a) from Lemma 32.

If for every $i \in \{1, \dots, j\}$ either $b_i \dots b_1 \preceq \sigma^{n+1}(\nu)$ or $b_{i+1} \dots b_j$ is not the beginning of $\mathrm{lhe}(q)$, then $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ satisfy (b) from Lemma 32. Assume there is $i < j$ such that $b_i \dots b_1 \succ \sigma^{n+1}(\nu)$ and $b_{i+1} \dots b_j$ is the beginning of $\mathrm{lhe}(q)$ and take the smallest such index i. If $\#_1(b_{i+1} \dots b_j)$ is odd (as in the paragraph above, later in the proof we need only this special case) and there is $\beta \in \{i, \dots, j-1\}$ such that $b_{\beta+1} \dots b_j$ is also the beginning of $\mathrm{lhe}(q)$, then $\#_1(b_{\beta+1} \dots b_j)$ is also odd and thus $b_{\beta+1} \dots b_j 10 \prec \mathrm{lhe}(q)$ for every such β. We conclude that $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ satisfy condition (b) from Lemma 32.

Recall that $L' = \overleftarrow{\mathrm{rhe}(q)} = (1w_q)^\infty 01$.

Fix an admissible word $a_N \dots a_1 \in \{0,1\}^N$ for some $N \in \mathbb{N}$. Let $k \in \{1, \dots, N\}$ be, if existent,

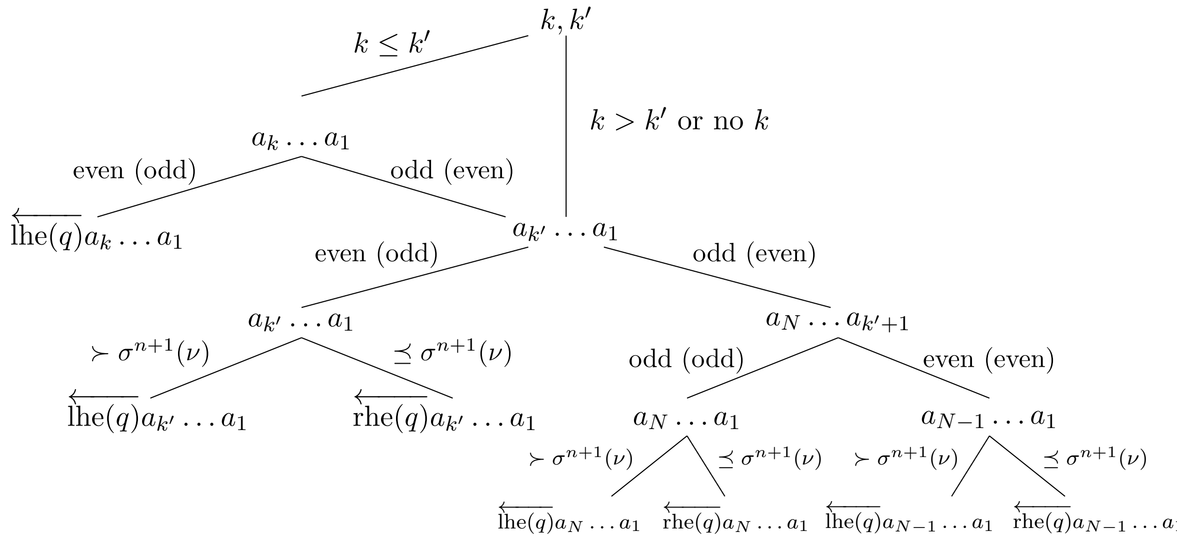

Figure 6.7: Calculating the $L_{a_N \dots a_1}$ and $S_{a_N \dots a_1}$ in the rational interior case. The graph should be read as follows: if we want to calculate $L_{a_N \dots a_1}$ we read the terms outside of the brackets and to calculate $S_{a_N \dots a_1}$ we read the terms inside the brackets. Say we want to calculate $L_{a_N \dots a_1}$ ($S_{a_N \dots a_1}$). We first calculate k and k' and compare them. Say $k > k'$ or k does not exist. We move down the right branch. Next we calculate the parity of $a_{k'} \dots a_1$. Say it is even (odd), then we move down the left branch. If $a_{k'} \dots a_1 \succ \sigma^{n+1}(\nu)$ then $L_{a_N \dots a_1} = \overleftarrow{\mathrm{lhe}(q)}a_{k'} \dots a_1$ ($S_{a_N \dots a_1} = \overleftarrow{\mathrm{lhe}(q)}a_{k'} \dots a_1$) and if $a_{k'} \dots a_1 \preceq \sigma^{n+1}(\nu)$ then $L_{a_N \dots a_1} = \overleftarrow{\mathrm{rhe}(q)}a_{k'} \dots a_1$ ($S_{a_N \dots a_1} = \overleftarrow{\mathrm{rhe}(q)}a_{k'} \dots a_1$).

the smallest index such that $a_k \dots a_1 \succ \sigma^{n+1}(\nu)$ and $a_{k+1} \dots a_N$ is the beginning of $\mathrm{lhe}(q)$. We set $k = N$ when $a_N \dots a_1 \succ \sigma^{n+1}(\nu)$ (then $a_{k+1} \dots a_N = \emptyset$ is the beginning of $\mathrm{lhe}(q)$). Let $k' \in \{0, 1, \dots, N-1\}$ be the smallest index such that $a_{k'+1} \dots a_N$ equals the beginning of $\mathrm{rhe}(q)$. Note that if $a_i = 1$ for some $i \in \{1, \dots, N\}$, then such k' exists. If $a_N \dots a_1 = 0^N$, then $L_{a_N \dots a_1} = \overleftarrow{\mathrm{rhe}(q)}0^N$ and $S_{a_N \dots a_1} = S = (1w_q)^\infty 0$.

If $a_i = 1$ for some $i \in \{1, \dots, N\}$, the diagram in Figure 6.7 provides an algorithm to calculate $L_{a_N \dots a_1}$ ($S_{a_N \dots a_1}$).

To see that the defined sequences are indeed $L_{a_N \dots a_1}$ ($S_{a_N \dots a_1}$) we use the first part of the proof. For example, take the case where the algorithm gives $\overleftarrow{\mathrm{lhe}(q)}a_N \dots a_1$. Since $a_N \dots a_1 \succ \sigma^{n+1}(\nu)$ and $\#_1(a_N \dots a_1 = a_N \dots a_{k'+1}a_{k'} \dots a_1)$ is even (odd), if $\overleftarrow{\mathrm{lhe}(q)}a_N \dots a_1$ is admissible then it equals $L_{a_N \dots a_1}$ ($S_{a_N \dots a_1}$). To see that it satisfies (a), note that $\#_1(a_N \dots a_{k'+1})$ is odd by assumption. To see that is satisfies (b), assume first that there exists k and $k \le k'$. Then $\#_1(a_{k+1} \dots a_{k'})$ is even and thus $\#_1(a_N \dots a_{k+1})$ is odd. If k does not exists, we are done. If $k > k'$, then since $a_{k'+1} \dots a_N$ is the beginning of $\mathrm{rhe}(q)$ and $a_{k+1} \dots a_N$ is the beginning of $\mathrm{lhe}(q)$ it follows that $\#_1(a_{k'+1} \dots a_k)$ is even and thus $\#_1(a_{k+1} \dots a_N)$ is of the same parity as $\#_1(a_{k'+1} \dots a_N)$, which is odd. That finishes the proof in this case. Other cases follow using analogous computations. Note that if $\#_1(a_N \dots a_{k'+1})$ is even, then since $a_{k'+1} \dots a_N$ is the beginning of $\mathrm{rhe}(q)$ it follows that $a_N = 1$ and thus $\#_1(a_{N-1} \dots a_{k'+1})$ is odd (this is needed in the proof of the two cases in the right branch of Figure 6.7). $\square$

Lemma 34. *Say that $q = m/n$, where m and n are relatively prime. If $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$, then every admissible itinerary in $\sigma^i(\mathcal{U}_{L'})$ is realized as an extremum of a cylinder of $\varphi_{\mathcal{C}}(X')$.*

Proof. Assume that $\overleftarrow{x} = \dots x_2 x_1$ is an admissible tail and that there exists $K \in \mathbb{N}_0$ such that $\dots x_{K+2} x_{K+1} = \overleftarrow{\mathrm{lhe}(q)}$ and take K the smallest index with that property. Denote by $\mathrm{lhe}(q) = (w_q 1)^\infty = (y_1 \dots y_n)^\infty$ and note that $\mathrm{rhe}(q) = 10(\hat{w}_q 1)^\infty$ and thus $\sigma^{n+1}(\mathrm{rhe}(q)) = (1\hat{w}_q)^\infty = (y_n \dots y_1)^\infty$. Since $\mathrm{rhe}(q) \succ \nu$ and they agree on the first $n+1$ places (which equal c_q and which is a word of even parity, for details see *e.g.* [19]), it follows that $\sigma^{n+1}(\mathrm{rhe}(q)) \succ \sigma^{n+1}(\nu)$. Let $J \in \mathbb{N}$ be the smallest natural number such that $(y_n \dots y_1)^J \succ \sigma^{n+1}(\nu)$. We study the cylinder $Y = [y_n \dots y_1 (y_n \dots y_1)^J x_K \dots x_1]$. Note that $x_i \dots x_K (y_1 \dots y_n)^{J+1}$ does not agree with the beginning of $\mathrm{lhe}(q)$ for any $i \in \{1, \dots, K\}$. Also $y_i \dots y_n (y_1 \dots y_n)^j$ does not agree with the beginning of $\mathrm{lhe}(q)$ for any $i \in \{2, \dots, n\}$ and any $j \in \mathbb{N}$. Denote by $a_N \dots a_1 = y_n \dots y_1 (y_n \dots y_1)^J x_K \dots x_1$. Let $k \in \{1, \dots, N\}$ be, if existent, the smallest index such that $a_k \dots a_1 \succ \sigma^{n+1}(\nu)$ and $a_{k+1} \dots a_N$ is the beginning of $\mathrm{lhe}(q)$ (compare with the definition of k in the proof of Lemma 33). By the choice of J it follows that k indeed exists and $k \in \{K + Mn : M \in \{0, \dots, J\}\}$. So, if for any $i \in \{0, \dots, K-1\}$ the word $x_{i+1} \dots x_K (y_1 \dots y_n)^{J+1}$ does not equal the beginning of $\mathrm{rhe}(q)$, then Lemma 33 implies that $\overleftarrow{x} = L_Y$ or $\overleftarrow{x} = S_Y$, depending on the parity of $\#(x_K \dots x_1)$.

If there is $\alpha \in \{0, \dots, K-1\}$ such that the word $x_{\alpha+1} \dots x_K (y_1 \dots y_n)^{J+1}$ equals the beginning of $\mathrm{rhe}(q)$, then $x_\alpha \dots x_1 \preceq \sigma^{n+1}(\nu)$ (otherwise Y does not satisfy (b) from Lemma 32 and is thus not admissible). Lemma 33 implies that $\overleftarrow{\mathrm{rhe}(q)} x_\alpha \dots x_1$ equals L_Y or S_Y, depending on the parity of $\#(x_\alpha \dots x_1)$. Since the tails of $\mathrm{rhe}(q)$ and $\mathrm{lhe}(q)$ are shifts of one another and $J \geq 1$ it follows that $\overleftarrow{x} = \overleftarrow{\mathrm{rhe}(q)} x_\alpha \dots x_1$, which concludes the proof. $\square$

Theorem 12. *Say that $q = m/n$, where m and n are relatively prime. If $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$, then in $\varphi_{\mathcal{C}}(X')$ there exist n fully accessible arc-components which are dense lines in X' and n simple dense canals. Moreover, a point from $\varphi_{\mathcal{C}}(X')$ is accessible if and only if it belongs to one of these n lines.*

Proof. Lemma 33 shows that all the extrema of cylinders have tails in $\sigma^i(L')$ for some $i \in \mathbb{Z}$ and Lemma 34 shows that every admissible itinerary in $\sigma^i(\mathcal{U}_{L'})$ is realized as an extremum of a cylinder. Since L' is preperiodic of preperiod n, we obtain n fully accessible lines in $\varphi_{\mathcal{C}}(X')$. Since σ is extendable, no other non-degenerate arc can be accessible. Thus the circle of prime ends can be decomposed into n open intervals and their n endpoints. We claim that the endpoints correspond to simple dense canals.

Assume by contradiction that a folding point $x \in \varphi_{\mathcal{C}}(X')$ is accessible. Then its every shift $\sigma^j(x)$ needs to be accessible for some natural number j which divides n (denoted from now onwards by $j|n$). We conclude that the tail corresponding to the point x must be periodic of period $j|n$, *i.e.,* $\sigma^j(x) = x$. Note that there are no periodic kneading sequences ν of period $j|n$ for $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$ since $\mathrm{lhe}(q)$, $\mathrm{rhe}(q)$ and ν agree on the first $n-1$ places. Thus the basic arc $\overleftarrow{x}$ has $\tau_L(\overleftarrow{x}), \tau_R(\overleftarrow{x})$ finite. Specially, the basic arc $\overleftarrow{x}$ contains no endpoint of X' and x is the only accessible point in $\overleftarrow{x}$ and it thus needs to be Type 3 folding point. Write $\overleftarrow{x} = \dots x_3 x_2 x_1$. Since x is a folding point and not an endpoint, there exist arbitrarily large $M, k_i \in \mathbb{N}$ such that $x_M \dots x_1 = c_{k_i+1} \dots c_{k_i+M}$ and $x_{M+1} \neq c_{k_i}$. Now we proceed similarly as in Proposition 26. Fix a cylinder around $\overleftarrow{x}$ and assume that all long basic arcs in that cylinder lie below (above) $\overleftarrow{x}$. Here long basic arcs $\overleftarrow{y}$ are such that $\pi_0(x) \in \mathrm{Int}\,(\pi_0(\overleftarrow{y}))$.

Specially, for M large enough and when $c_{k_i} c_{k_i+1} \dots c_{k_i+M} \neq c_2 \dots c_{M+2}$, the basic arcs with tails $1^\infty c_{k_i} c_{k_i+1} \dots c_{k_i+M}$ are long (if $M > \tau_L(x), \tau_R(x)$ then $\pi_0(1^\infty c_{k_i} c_{k_i+1} \dots c_{k_i+M}) = [T^{\tau_L(x)}, T^{\tau_R(x)}]$). Basic arcs in the chosen cylinder which do not project to $[T^{\tau_L(x)}, T^{\tau_R(x)}]$ are of the form $\dots {}^0_1 c_1 c_2 \dots c_{k_i} c_{k_i+1} \dots c_{k_i+M}$. Since $c_{k_i} \neq x_{M+1}$, it follows that those arcs are on the same side of $\overleftarrow{x}$ as long arcs $1^\infty c_{k_i} c_{k_i+1} \dots c_{k_i+M}$. Since we assumed that all long basic arcs lie on the same side of $\overleftarrow{x}$ it follows that $\overleftarrow{x}$ is an extremum of a cylinder, a contradiction.

The remaining case is when $c_{k_i} \dots c_{k_i+M} = c_2 \dots c_{M+2}$ for all (but finitely many) $i \in \mathbb{N}$. That is, whenever $x_M \dots x_1$ appears in the kneading sequence, then $x_M \dots x_1 = c_3 \dots c_{M+2}$ and $x_{M+1} \neq c_2 = 0$. However, $\overleftarrow{x}$ is periodic of period $j|n$ and x is a folding point, from which we conclude that $T^3(c)$ is periodic of period $j|n$ and $\overleftarrow{x} = (c_3 \dots c_{n+2})^\infty$. Note that the only kneading sequence $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$ for which $T^3(c)$ is periodic of period $j|n$ is $10(\hat{w}_q 0)^\infty$ which is actually periodic of period n. But there are no periodic kneading sequences ν of period n such that $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$, a contradiction. Thus no folding point $x \in \varphi_{\mathcal{C}}(X')$ is accessible.

We need to show that the n accessible lines $\mathcal{U}^i \subset X'$ for $i \in \{0, \dots, n-1\}$ are indeed dense in X'. It follows from Lemma 33 that the symbolic code of $\mathcal{U}^i$ is eventually $\sigma^i(\overleftarrow{\mathrm{lhe}(q)})$ for $i \in \{0, \dots, n-1\}$. Let $a \in X'$ be a point with the backward itinerary $\overleftarrow{a} = \dots a_2 a_1$. Note that for every natural number β, every $i \in \{0, \dots, n-1\}$ and large enough natural number γ the left infinite sequences $\sigma^i(\overleftarrow{\mathrm{lhe}(q)})1^\gamma a_\beta \dots a_1$ are admissible since they satisfy conditions (a) and (b) from Lemma 32. Thus, sending $\beta \to \infty$ we get a sequence of basic arcs from $\mathcal{U}^i$ converging to $A(\overleftarrow{a})$ such that their π_0-th projections contain $\pi_0(a)$.

Therefore n prime ends $P_1, \dots, P_n$ on the circle of prime ends are either of the third or the fourth kind. Since the shores of the canal are lines which are dense in both directions it follows that $\Pi(P_i) = I(P_i) = \varphi_{\mathcal{C}}(X')$ for every $i \in \{1, \dots, n\}$. Therefore, there are n simple dense canals for every $\varphi_{\mathcal{C}}(X')$. □

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