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„Dynamic Uncertainty Principle for the Discrete Dirac Operator“

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Mateusz Piorkowski, Bsc

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Abstract

This work explores the so-called 'dynamic uncertainty principle' for the one-dimensional discrete Dirac operator. It says that a solution which decays strong enough at plus or minus infinity at two different times must already vanish. To this end, we will use the spectral theorem to derive a Fourier transform, which allows us to formulate the above problem in terms of holomorphic functions. The main argument in the proof will then use the asymptotic growth of holomorphic functions defined on sectors.

Zusammenfassung

In dieser Arbeit untersuchen wir das sogenannte 'dynamische Unschärfeprinzip' für den eindimensionalen diskreten Dirac-Operator. Dieses besagt, dass eine Lösung, die an zwei Zeitpunkten für plus oder minus unendlich stark abfällt, bereits verschwinden muss. Dazu werden wir mit Hilfe des Spektraltheorems eine Fouriertransformation herleiten, mit der wir das oben genannte Problem auf eine Fragestellung über holomorphe Funktionen zurückführen können. Das Hauptargument in dem Beweis verwendet dann das asymptotische Wachstum holomorpher Funktionen auf Sektoren.

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Contents

1 Preface

We will deal with the following question. Given the one-dimensional discrete Dirac operator (with a potential) $\mathcal{D} = \mathcal{D}_0 + Q$ and a solution $\mathbf{u}(t) \in C^1([0,1], l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z}))$, if $\mathbf{u}(0)$ and $\mathbf{u}(1)$ have sufficient strong decays at $+\infty$, does this already imply that $\mathbf{u} \equiv 0$? This phenomenon goes by the name of 'dynamic uncertainty principle' and we will derive sufficient conditions on $\mathbf{u}$ and the potential Q for the above statement to hold. To this end, we will need results from complex analysis, in particular, the theory of the asymptotic growth of holomorphic functions defined on sectors. All the necessary results will be stated in the next section and we will also prove some auxiliary results, which are not proven in [1].
In the third section, we will give a quick introduction to the theory of Jost solutions of the one-dimensional discrete Dirac operator, which asymptotically correspond to free particles at $\pm\infty$. We will also derive conditions on the decay of the potential, such that the Jost solutions can be also defined for complex eigenvalues. This will be done using a power series ansatz, where care must be taken as the eigenvalue contains a square-root and thus cannot be written as a power series.
In the next section, we proceed to derive the form of the unitary transform given by the spectral theorem. We will show that we can write the unitary transform, which can be thought of as a Fourier transform, in terms of the Jost solutions. This will allow us to associate elements of $l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$ with functions on the upper half of the torus $\mathbb{T}_+$ and by the results of the third section, with holomorphic functions on $\mathbb{D} \setminus [0,1)$. This will be the key relationship with complex analysis.
Finally, we will prove a sufficient condition for the question stated above to have a positive answer. An analogous argumentation has been already carried out for the one-dimensional discrete Schrödinger operator in [2] and [3].
In the Appendix we also derive the explicit form of the spectral measure of the Fourier transform using Weyl-sequences, though it is not required for our main proof.

2 Growth of Holomorphic Functions

In this section, we will deal with the asymptotic growth of holomorphic functions defined on either the entire complex plane or sectors of it. All the definitions and most of the proofs can be found in [1]. Theorems and proofs not found in the book will be given here.

2.1 Growth of Entire Functions

Any holomorphic function $f(z)$ in a domain $D \subseteq \mathbb{C}$ containing 0 can be locally (in a neighbourhood $U \subseteq D$) expanded in its Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad z \in U. \tag{2.1}$$

The function $f(z)$ can be extended to a holomorphic function on $\mathbb{C}$ (in which case the extension is unique), if and only if the Taylor series has an infinite radius of convergence. We call such holomorphic functions 'entire'. If $f(z)$ is entire, then (2.1) holds on all of $\mathbb{C}$. So we see that there is a one-to-one correspondence between entire functions and power series with an infinite radius of convergence. By the Cauchy–Hadamard-formula, the latter condition is equivalent with

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 0. \tag{2.2}$$

Intuitively the above formula says that the sequence c_n must decay quicker then any geometric sequence. We will see that the decay of the Taylor coefficients c_n can tell us a lot about the asymptotic behavior of $f(z)$. To this end we introduce the growth function

$$M_f(r) := \sup\{|f(z)| : |z| = r\} \tag{2.3}$$

for $r \geq 0$. We are now ready to define the order of an entire function.

Definition 2.1. *Let $f(z)$ be an non-constant entire function. Then we define the order ρ of $f(z)$ by*

$$\rho = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}. \tag{2.4}$$

Note that for constant $f(z) \equiv c$, the above expression does not make sense if $c \leq 1$. For non-constant $f(z)$ there is no problem, as $M_f(r)$ must go to infinity as $r \to \infty$ by Liouville's theorem. An equivalent definition of the order ρ of $f(z)$ is the greatest lower bound of all ρ', such that there exists a corresponding $A > 0$ with

$$M_f(r) \overset{as}{<} \mathrm{e}^{Ar^{\rho'}} \tag{2.5}$$

where $\overset{as}{<}$ means that the inequality holds for r large enough. When we talk about functions of order ρ, we will always implicitly assume that the order is finite.

Next we define the type of a function with a fixed order ρ.

Definition 2.2. *Let $f(z)$ be an entire function of order ρ. Then the type σ of $f(z)$ is defined by*

$$\sigma = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^\rho}. \tag{2.6}$$

Similarly to above one can define the type σ of an entire function $f(z)$ as the greatest lower bound of all σ' with

$$M_f(r) \overset{as}{<} \mathrm{e}^{\sigma' r^\rho}. \tag{2.7}$$

A natural question would be how the order or the type of an entire function is encoded in its Taylor coefficients? The next two propositions give us a complete answer.

Proposition 2.3. *Let $f(z)$ be an entire function with a Taylor expansion given by (2.1). Then the order ρ of $f(z)$ is given by the following formula*

$$\rho = \limsup_{n\to\infty} \frac{n\log n}{\log(1/|c_n|)}. \tag{2.8}$$

Proposition 2.4. *Let $f(z)$ be an entire function of order ρ with a Taylor expansion given by (2.1). Then the type σ of $f(z)$ is given by the following formula*

$$\sigma = \frac{1}{\rho\mathrm{e}} \limsup_{n\to\infty} n \sqrt[n]{|c_n|^\rho}. \tag{2.9}$$

Remark 2.5. *As multiplying an entire function with an integer power of z, does not change the order or the type of that function, we can always translate the index n of c_n in the above expressions, without changing the result*

The latter of the two formulas will be of special interest to us later. From now one we will implicitly assume that an entire function of order ρ has a finite type.

2.2 Growth of Holomorphic Functions in a Sector

Next we will look at holomorphic functions defined on sectors. A sector $V(\alpha,\beta) \subset \mathbb{C}$ is given by

$$V(\alpha,\beta) = \{z : \alpha < \arg(z) < \beta\} \tag{2.10}$$

with $\alpha, \beta \in [0, 2\pi]$. In particular we require $z \neq 0$, as otherwise $\arg(z)$ would not be defined. Sometimes will also write $V[\alpha,\beta]$ if we include the boundary rays (note $V[\alpha,\beta]$ is not closed as $0 \notin V[\alpha,\beta]$). For our purposes we will also have to consider sectors intersected with sets of the form $\{z : |z| > s\}$. To this end we also introduce so called generalized sectors

$$V(\alpha,\beta;s) = \{z : \alpha < \arg(z) < \beta, |z| > s\} \tag{2.11}$$

and analogously $V[\alpha,\beta;s]$. Note that $V(\alpha,\beta;0) = V(\alpha,\beta)$.

The definition of the growth function naturally generalizes for holomorphic functions defined on generalized sectors $V(\alpha,\beta;s)$:

$$M_f(r) := \sup\{|f(z)| : |z| = r, \quad z \in V(\alpha,\beta;s)\} \tag{2.12}$$

where $r > s$. Similarly, we generalize the definition of order and type of a holomorphic function. However, to avoid problems with the case that $|M_f(r)| \leq 1$, we define the order by (2.5). Again we will assume that both quantities are smaller than $+\infty$. For $V[\alpha,\beta;s]$ we will assume continuity up to the boundary rays. Note that because there might not be a Taylor representation for $f(z)$, we no longer have the formulas (2.8) and (2.9) for the order and the type of $f(z)$.

The following theorem can be regarded as a generalization of the maximum modulus principle for bounded domains.

Theorem 2.6 (Phragmén–Lindelöf principle). *Let $f(z)$ be a holomorphic function on $V[\alpha,\beta;s]$. Furthermore, assume that there are positive constants C and ρ, such that*

$$M_f(r) \overset{as}{<} C\mathrm{e}^{r^\rho} \tag{2.13}$$

and $\pi/\rho > \beta - \alpha$. If $|f(z)| \leq M$ for some constant M on $\partial V[\alpha,\beta;s]$, then $|f(z)| \leq M$ on $V[\alpha,\beta;s]$.

Proof. Without loss of generality assume that $f(z)$ is holomorphic on $V[-\alpha,\alpha;s]$ with $\alpha < \pi/2\rho$. Choose ρ_1 such that $\rho < \rho_1 < \pi/2\alpha$. and set

$$\phi_\delta(z) := f(z)\mathrm{e}^{-\delta z^{\rho_1}}, \quad \delta > 0. \tag{2.14}$$

The asymptotic inequality yields

$$|\phi_\delta(z)| \overset{as}{<} C\mathrm{e}^{|z|^\rho - \delta|z|^{\rho_1}\cos(\rho_1\alpha)}. \tag{2.15}$$

Since $\rho < \rho_1$ and $\cos(\rho_1\alpha) > 0$, we get that

$$|\phi_\delta(R\mathrm{e}^{\mathrm{i}\theta})| \leq M, \quad -\alpha \leq \theta \leq \alpha \tag{2.16}$$

holds for $R > R_\delta \geq s$. Applying the maximum principle to the function $\phi_\delta(z)$ on the set $\{r\mathrm{e}^{\mathrm{i}\theta} : s < r < R, |\theta| < \alpha\}$ (note that $|\phi_\delta(z)| \leq |f(z)| \leq M$ on $\partial V[\alpha, \beta; s]$), with $R > R_\delta$, we conclude that $|\phi_\delta(z)| \leq M$ on that set. As R tends to infinity, we see that the above inequality holds on $V[-\alpha, \alpha, s]$. As $\delta > 0$ was arbitrary, we obtain $|f(z)| \leq M$ on $V[-\alpha, \alpha, s]$. □

Remark 2.7. *In the literature the Phragmén–Lindelöf principle is usually proven for sectors. The above proof is a straightforward adaptation for generalised sectors. Other version of this principle also exist like the Lindelöf theorem for strips (see [4]). More generally the Phragmén–Lindelöf principle can be regarded as the maximum principle applied to unbounded domains.*

Next we will be interested in the growth of holomorphic functions on half-rays.

Definition 2.8. *Let $f(z)$ be a holomorphic function of order ρ on $V(\alpha, \beta; s)$. Then the function*

$$h_f(\theta) := \limsup_{r\to\infty} \frac{\log|f(r\mathrm{e}^{\mathrm{i}\theta})|}{r^\rho} \tag{2.17}$$

with domain (α, β) is called the indicator function of $f(z)$ with respect to the order ρ.

Note that if σ is the type of $f(z)$, then we must have $h_f(\theta) \leq \sigma$. As we assumed the type always to be finite, $h_f(\theta)$ must always be smaller than $+\infty$. However we cannot exclude the case $h_f(\theta) = -\infty$. Furthermore, for any holomprphic function of order ρ, we can also define its indicator function of order $\rho' > \rho$. In that case the indicator function is always non-positive.

Remark 2.9. *Let $f(z)$ and $g(z)$ be holomorphic functions on a generalized sector of order at most ρ. Then we have*

$$h_{fg}(\theta) \leq h_f(\theta) + h_g(\theta) \tag{2.18}$$

$$h_{f+g}(\theta) \leq \max\{h_f(\theta), h_g(\theta)\} \tag{2.19}$$

for the indicator functions of $f(z)$ and $g(z)$ with respect to the order ρ. These inequalities will turn out to be very useful, when trying to bound the indicator functions of complicated expressions.

Let us compute the indicator function of the function $\mathrm{e}^{(A-\mathrm{i}B)z^\rho}$ on a sector with opening smaller than 2π. We have

$$|f(r\mathrm{e}^{\mathrm{i}\theta})| = \mathrm{e}^{(A\cos(\rho\theta)+B\sin(\rho\theta))r^\rho} \tag{2.20}$$

and so its indicator function is given by

$$H(\theta) := A\cos(\rho\theta) + B\sin(\rho\theta). \tag{2.21}$$

Functions $H(\theta)$ of this form (with finite values) are called sinusoidal or ρ-trigonometric. They are uniquely specified by their values at two points, provided these points are close enough.

Proposition 2.10. *Let θ_1, θ_2 be given with $0 < \theta_2 - \theta_1 < \pi/\rho$. Then the ρ-trigonometric function $H(\theta)$, with $H(\theta_1) = h_1$, $H(\theta_2) = h_2$ is unique and given by the formula*

$$H(\theta) = \frac{h_1 \sin(\rho(\theta_2 - \theta)) + h_2 \sin(\rho(\theta - \theta_1))}{\sin(\rho(\theta_2 - \theta_1))} \tag{2.22}$$

Related to ρ-trigonometric functions, we introduce the concept of ρ-trigonometrical convexity.

Definition 2.11. *A function $h(\theta)$ is called ρ-trigonometrically convex on the closed interval $[\alpha, \beta]$, if for $\alpha \leq \theta_1 < \theta_2 \leq \beta$ and $0 < \theta_2 - \theta_1 < \pi/\rho$ the inequalities*

$$h(\theta_1) \leq h_1, \qquad h(\theta_2) \leq h_2 \tag{2.23}$$

imply the inequality

$$h(\theta) \leq H(\theta), \qquad \theta_1 \leq \theta \leq \theta_2, \tag{2.24}$$

where $H(\theta)$ is the ρ-trigonometric function assuming the values h_1 and h_2 at the points θ_1 and θ_2. It is called ρ-trigonometrically convex in an arbitrary interval, if it is ρ-trigonometrically convex on each closed subinterval. For ρ-trigonometrically convex functions we will allow the value $-\infty$.

The next theorem will be crucial for this paper and follows from the Phragmén–Lindelöf principle.

Theorem 2.12. *Let $f(z)$ be a holomorphic function of order at most ρ on a (closed) generalized sector. Then its indicator function $h_f(\theta)$ with respect to the order ρ is a ρ-trigonometrically convex function.*

As a corollary from (2.22) we get

Corollary 2.13. *Let $h(\theta)$ be a ρ-trigonometrically convex function. Then the fundamental relation given by*

$$h(\theta_1)\sin(\rho(\theta - \theta_2)) + h(\theta)\sin(\rho(\theta_2 - \theta_1)) + h(\theta_2)\sin(\rho(\theta_1 - \theta)) \leq 0 \tag{2.25}$$

holds, with $\theta_1 < \theta < \theta_2$ and $0 < \theta_2 - \theta_1 < \pi/\rho$.

Using the fundamental relation one can show the following proposition.

Proposition 2.14. *Let $h(\theta)$ be a bounded ρ-trigonometrically convex function, i.e., $|h(\theta)| < K$ for $\theta \in (\alpha, \beta)$. Then it is continuous on (α, β) and satisfies in each closed subinterval a Lipschitz condition.*

Remark 2.15. *The indicator function on a closed interval might not be continuous at the boundaries (e.g the indicator function of order 1 of $f(z) = \mathrm{e}^{-z^2}$ on the sector $V[-\pi/4, \pi/4]$). However because of the trigonometrical convexity, we always have*

$$\limsup_{\theta \to \alpha} h_f(\theta) \leq h_f(\alpha) \tag{2.26}$$

if α is a boundary point.

As already mentioned, the indicator function might evaluate to $-\infty$. However in that case its behavior is restricted by the next proposition.

Proposition 2.16. *Let $h(\theta)$ be a ρ-trigonometrically convex function on an interval (α, β). If $h(\theta_1) = -\infty$ for some $\theta_1 \in (\alpha, \beta)$, then $h(\theta) \equiv -\infty$ on (α, β). Furthermore if $h(\theta)$ is the indicator function of $f(z)$ with respect to the order ρ, then the convergence*

$$\frac{\log|f(r\mathrm{e}^{\mathrm{i}\theta})|}{r^\rho} \longrightarrow -\infty \tag{2.27}$$

is uniform on each closed subsector.

Proof. We will show that $h(\theta) = -\infty$ for each $\theta \in (\alpha, \beta)$ satisfying the condition $\theta_1 < \theta < \theta_1 + \pi/\rho$. For any $\theta_2 \in (\alpha, \beta)$ with $\theta_1 < \theta < \theta_2 < \theta_1 + \pi/\rho$, we define the ρ-trigonometric function $H_\epsilon(\theta)$, which assumes the values $H_\epsilon(\theta_1) = -1/\epsilon$ and $H_\epsilon(\theta_2) = \max\{-1/\epsilon, h(\theta_2)\}$. As $h(\theta)$ is ρ-trigonometrically convex, we have $h(\theta) \le H_\epsilon(\theta), \theta_1 \le \theta \le \theta_2$. Taking the limit of ϵ going to zero, we obtain $h(\theta) = -\infty$ for $\theta_1 < \theta < \theta_2$. Iterating this procedure we get the result for each $\theta \in (\theta_1, \beta)$ and by a similar argument for each $\theta \in (\alpha, \theta_1)$.
For the second claim it is enough to show uniform convergence on an arbitrary generalised subsector $V[\gamma, \delta; s]$, with $\delta - \gamma < \pi/\rho$. For any $c > 0$, there is are constants $K_c > 0, R_c > s$, such that

$$|f(r\mathrm{e}^{\mathrm{i}\gamma})| \le K_c \mathrm{e}^{-cr^\rho}, \qquad |f(r\mathrm{e}^{\mathrm{i}\delta})| \le K_c \mathrm{e}^{-cr^\rho} \tag{2.28}$$

for all $r \ge R_c$. Let $H_c(\theta) = A\cos(\rho\theta) + B\sin(\rho\theta)$ be the ρ-trigonometrical function with $H_c(\gamma) = H_c(\delta) = c$ and define $g(z) := f(z)\mathrm{e}^{(A-\mathrm{i}B)z^\rho}$. We might increase K_c if necessary to obtain $|g(z)| \le K_c$ on the boundary of $V[\gamma, \delta; R_c]$ and by the Phragmén–Lindelöf principle we obtain $|g(z)| \le K_c$ on the whole generalised subsector. Thus we conclude

$$\frac{\log|g(z)|}{r^\rho} = \frac{\log|f(r\mathrm{e}^{\mathrm{i}\theta})|}{r^\rho} + H_c(\theta) \le \frac{\log K_c}{r^\rho} \tag{2.29}$$

on $V[\gamma, \delta; R_c]$. Finally it is clear by formula (2.22), that by letting c go to ∞, $H_c(\theta)$ goes uniformly to ∞ on $[\gamma, \delta]$. Thus we conclude that for any $M > 0$, we can choose c large enough such that $H_c(\theta) > M$ on $V[\gamma, \delta]$, and then choose $r_M \ge R_c$ large enough such that $H_c(\theta) > M + \log K_c/r_M^\rho$ to obtain

$$\frac{\log|f(r\mathrm{e}^{\mathrm{i}\theta})|}{r^\rho} < -M \tag{2.30}$$

for all $r \ge r_M$, which proves the claim. □

Lemma 2.17. *Let $f(z)$ be a holomorphic function of order at most ρ on the generalised sector $V[\alpha, \beta; s]$ with opening greater or equal to π/ρ. If $h_f(\alpha) = -\infty$, then $f(z)$ is identically* 0.

Proof. First note that by (a slight adaptation) of the last proposition, $h_f(\theta) \equiv -\infty$ on $[\alpha, \beta)$. By making a transformation of the form $z \to z^\sigma \mathrm{e}^{\mathrm{i}\theta}$, we can assume that the generalised sector is of the form $V[-\pi/2, \pi/2; s] =: V$ and consequently $\rho \ge 1$. We can also assume that $f(z)$ is bounded for $|z| = s$, as otherwise we can increase s. Note that we then have that the function $f(z)\overline{f(\overline{z})}$ is also holomorphic in V and its indicator function is equal to $-\infty$ on $[\alpha, \beta]$. As showing $f(z) \equiv 0$ is equivalent to showing that $f(z)\overline{f(\overline{z})} \equiv 0$ we can assume that $h_f(\theta) = -\infty$ for $\theta \in [\alpha, \beta]$. Now we define a family of functions

$$g_\lambda(z) := f(z)\mathrm{e}^{\lambda(z-s)} \tag{2.31}$$

indexed by the real positive parameter λ. Notice that

$$|g_\lambda(z)| \leq |f(z)| \tag{2.32}$$

on the boundary of V. Now it follows from the assumptions made that

$$\sup_{z\in\partial V} |g_\lambda(z)| \leq \sup_{z\in\partial V} |f(z)| = c < \infty \tag{2.33}$$

and that

$$\lim_{|z|\to\infty} |g_\lambda(z)| = 0 \tag{2.34}$$

uniformly by the last proposition (as $h_{g_\lambda}(\theta) \equiv -\infty$). Together these imply by the maximum principle, that

$$|g_\lambda(z)| \leq c. \tag{2.35}$$

(Note that the conditions for the Phragmén–Lindelöf principle are not satisfied.) If now $f(\tilde{z}) \neq 0$ with $\Re(\tilde{z}) > s$ in the interior of the sector V, then

$$\lim_{\lambda\to\infty} |f(\tilde{z})\mathrm{e}^{\lambda(\tilde{z}-s)}| = \lim_{\lambda\to\infty} |g_\lambda(\tilde{z})| = \infty \tag{2.36}$$

which is the required contradiction. $\square$

Note that by the example from remark 2.15 it is not enough to assume that $h_f(\theta) = -\infty$ for some θ in the interior of $[\alpha, \beta]$. However if the opening is strictly greater than π/ρ, then $h_f(\theta) = -\infty$ for some $\theta \in (\alpha, \beta)$ already implies $f(z) \equiv 0$. As a corollary we get.

Corollary 2.18. *Let $f(z)$ be a non-zero holomorphic function of order at most ρ in the generalised sector $V[\alpha, \beta; s]$ and let $h_f(\theta)$ be the corresponding indicator function. Then*

$$h_f(\phi) + h_f(\phi + \pi/\rho) \geq 0, \qquad \alpha \leq \phi < \phi + \pi/\rho \leq \beta. \tag{2.37}$$

Proof. Note that we must have $\beta - \alpha \geq \pi/\rho$, otherwise the statement is trivial. It follows from the last lemma that we can exclude the case of $h_f(\phi)$ or $h_f(\phi + \pi/\rho)$ being equal to $-\infty$. We will now show (2.37) for any ρ-trigonometrically convex function $h(\theta)$ with $h(\theta) > -\infty$ on $[\alpha, \beta]$. By rewriting the fundamental relation (2.25) with $\theta_1 = \phi + \tau$, $\theta = \phi + \pi/2\rho$, $\theta_2 = \phi + \pi/\rho$. We obtain

$$h(\phi + \pi/2\rho)\sin(\pi - \tau/\rho) \leq h(\phi + \tau) + h(\phi + \pi/\rho)\sin(\pi/2 - \tau/\rho). \tag{2.38}$$

Now letting $\tau \to 0$ we get by remark (2.15)

$$0 \leq h(\phi) + h(\phi + \pi/\rho). \tag{2.39}$$

Finally, let us consider the case of $h_f(\theta) = -\infty$ for some (and hence all) $\theta \in (\alpha, \beta)$ but $h_f(\alpha), h_f(\beta) > -\infty$ (this is only possible if $\beta - \alpha \leq \pi/\rho$, hence assume $\beta - \alpha = \pi/\rho$). Then the only thing we need to show is that $h_f(\alpha) + h_f(\beta) \geq 0$. If not, the holomorphic function $k(z) = f(z)\overline{f(\overline{z})}$ (assuming for the sake of simplicity $\rho = 1$, $\alpha = -\pi/2, \beta = \pi/2$) has indicator function $h_k(\theta) = -\infty$ for $\theta \in (-\pi/2, \pi/2)$ and $h_k(-\pi/2) = h_k(\pi/2) \leq h_f(-\pi/2) + h_f(+\pi/2) < 0$. As showing that $f(z) \equiv 0$ is equivalent to showing $k(z) \equiv 0$, we will assume that the indicator function of $f(z)$ is negative on the boundary rays. In particular, $f(z)$ must be bounded on the boundary of our generalized sector. As, the indicator function is everywhere negative, we can define the indicator function $h_{f,\rho'}(\theta)$ of arbitrary order $\rho' > 0$ (as the value $+\infty$ is not attained). Taking some $\rho' < 1$ we then have $h_{f,\rho'}(\theta) \equiv -\infty$ on $[-\pi/2, \pi/2]$. Now by proposition 2.16, we see that $|f(z)| \to 0$ uniformly as $z \to \infty$. By defining a family of functions $g_\lambda(z) := f(z)\mathrm{e}^{\lambda(z-s)}$ for $\lambda > 0$, we see that $|g_\lambda(z)| \le |f(z)| \le c$ on the boundary, and $h_{g_\lambda}(\theta) = h_f(\theta)$. As before we can argue that $g_\lambda(z)$ goes uniformly to 0 as $z \to \infty$. Now, as in 2.17, we conclude by the maximum modulus principle that $|g_\lambda(z)| \le c$ on the whole generalized sector, and by letting $\lambda \to \infty$, we get the required contradiction unless $f(z) \equiv 0$.

□

3 Discrete one-dimensional Dirac Operator

In this section we introduce the discrete version of the one-dimensional Dirac operator and summarize its properties. The necessary background can be found in [2] and the theory dealing with the continuous version in [5]. The one-dimensional discrete Dirac operator has the form

$$\mathrm{i}\dot{\mathbf{w}}(t) = \mathcal{D}\mathbf{w}(t) = (\mathcal{D}_0 + Q)\mathbf{w}(t) \tag{3.1}$$

$$\mathbf{w}(t,n) = (x(t,n), y(t,n)) \in \mathbb{C}^2, \qquad n \in \mathbb{Z}$$

where the operator $\mathcal{D}_0$ is the free Dirac operator

$$\mathcal{D}_0 := \begin{pmatrix} m & d \\ d^* & -m \end{pmatrix}, \qquad m > 0 \tag{3.2}$$

with $(dx)(n) = ((1 - S^+)x)(n) = x(n) - x(n+1)$, and the matrix potential $Q = Q(n) = (q_n^{ij})_{i,j=1,2}$ is symmetric with $q_n^{12} = q_n^{21} \neq -1$. The last condition can be motivated by the observation that for $q_n^{12} = -1$, we could not recursively solve the underlying difference equation.

As we regard the potential Q as a perturbation of the much simpler free Dirac operator (3.2), we need to consider decay properties of the coefficients q_n^{ij}. To this end we introduce the weighted spaces of complex valued sequences $l^p_\sigma = l^p_\sigma(\mathbb{Z})$, $\sigma \in \mathbb{R}$ with the norm

$$\|u\|_{l^p_\sigma} = \begin{cases} (\sum_{n\in\mathbb{Z}} (1+|n|)^{p\sigma} |u(n)|^p)^{1/p}, & p \in [1, \infty) \\ \sup_{n\in\mathbb{Z}} (1+|n|)^\sigma |u(n)|, & p = \infty. \end{cases} \tag{3.3}$$

Setting $\sigma = 0$ we just retrieve the usual l^p spaces. Furthermore we set $\mathbf{l}^p_\sigma := l^p_\sigma \oplus l^p_\sigma$.

3.1 Spectral Properties

Given the symmetry of the potential, the operator $\mathcal{D} = \mathcal{D}_0 + Q$ is self-adjoint on $\mathbf{l}^2(\mathbb{Z})$. To examine its spectral properties we will have to first look at the eigenvalue equation

$$\mathcal{D}_0\mathbf{w} = \lambda\mathbf{w} \tag{3.4}$$

of the free Dirac operator. However, we will also allow for solutions not necessarily in our Hilbert space $\mathbf{l}^2(\mathbb{Z})$. It is easy to see that by specifying $\mathbf{w}(n) = (x(n), y(n))$

for one value of $n \in \mathbb{Z}$, we get a unique solution of (3.4) for any $\lambda \in \mathbb{C}$. This implies that the dimension of an eigenspace is always equal to two. Moreover, applying the Fourier transform

$$\hat{\mathbf{w}}(z) = \sum_{n\in\mathbb{Z}} \mathbf{w}(n) z^n \tag{3.5}$$

the operator $\mathcal{D}_0$ maps to multiplication by the matrix

$$\hat{\mathcal{D}}_0 = \begin{pmatrix} m & 1 - z^{-1} \\ 1 - z & -m \end{pmatrix} \tag{3.6}$$

Since the eigenvalues of this matrix are given by $\pm\lambda(z) = \pm\sqrt{m^2 + 2 - z - z^{-1}}$, which are real for $|z| = 1$, we obtain the following result.

Proposition 3.1. *Let $\mathcal{D}_0$ be the one-dimensional discrete Dirac operator (3.2) on the Hilbert space $\mathbf{l}^2(\mathbb{Z})$. Then $\mathcal{D}_0$ has a purely absolutely continuous spectrum given by $\sigma(\mathcal{D}_0) = [-\sqrt{m^2+4}, -m] \cup [m, \sqrt{m^2+4}]$.*

We will denote the set $(-\sqrt{m^2+4}, -m) \cup (m, \sqrt{m^2+4})$ by Γ. Thus, the spectrum is given by the closure $\overline{\Gamma}$. In order to study the spectral properties of $\mathcal{D}$, we need to know what happens to the spectrum of $\mathcal{D}_0$ under the perturbation Q. A partial answer is given in the following

Proposition 3.2 (Weyl)**.** *The essential spetrum of a self-adjoint operator is invariant under compact perturbations.*

The proof and further generalizations can be found in [6]. In Hilbert spaces the set of all compact operators is the closure (in the operator norm) of finite rank operators. Thus we get a simple criterion for compactness of Q, namely Q is compact if and only if q_n^{ij} vanishes at $\pm\infty$ for $i, j = 1, 2$. The following stronger result can be found in [2].

Theorem 3.3. *Let $\mathcal{D} = \mathcal{D}_0 + Q$ be a one-dimensional discrete Dirac operator on the Hilbert space $\mathbf{l}^2(\mathbb{Z})$. If for the coefficients of the potential we have $q_n^{ij} \in l_2^1(\mathbb{Z})$, $i, j = 1, 2$, then the spectrum of $\mathcal{D}$ has a purely absolutely continuous part consisting of $[-\sqrt{m^2+4}, -m] \cup [m, \sqrt{m^2+4}] = \overline{\Gamma}$, and a finite number of simple eigenvalues in $\mathbb{R} \setminus \overline{\Gamma}$.*

We now turn to the study of fundamental solutions associated to (3.4) with $\lambda \in \mathbb{C}$. To this end we introduce a new variable z related to λ via

$$z(\lambda) := \frac{1}{2}\left(m^2 + 2 - \lambda^2 - \sqrt{(m^2 + 2 - \lambda^2)^2 - 4}\right). \tag{3.7}$$

Here the square root is chosen with a branch cut along Γ and with the branch of the root such that $|z(\lambda)| < 1$ for $z \in \mathbb{C} \setminus \overline{\Gamma}$. Note that $z(.)$ is analytic in $\mathbb{C} \setminus \overline{\Gamma}$ and maps this domain to the open unit disc. We can choose the values along the branch cut such that $z(\lambda) := \lim_{\varepsilon\downarrow 0} z(\lambda + \mathrm{i}\varepsilon)$, $\lambda \in \overline{\Gamma}$. Observe $z(-\lambda) = z(\lambda)$ for $\lambda \in \mathbb{C} \setminus \overline{\Gamma}$, $z(\pm m) = 1$, and $z(\pm\sqrt{m^2+1}) = -1$. Moreover, $|z(\lambda)| = 1$ if and only if $\lambda \in \overline{\Gamma}$ and $z(\lambda)^{-1} = \frac{1}{2}\left(m^2 + 2 - \lambda^2 + \sqrt{(m^2+2-\lambda^2)^2 - 4}\right)$. A partial inverse can be defined on $\mathbb{D} \setminus [0, (m^2 + 2 - \sqrt{(m^2+2)^2 - 4})/2] \cup \mathbb{T}_+$ (with $\mathbb{T}_+ := \{z \in S^1 : \Im(z) > 0\}$) and is given by

$$\lambda(z) := \sqrt{m^2 + 2 - z - z^{-1}} \tag{3.8}$$

with $\Re(\lambda) > 0$ (note that $\mathbb{T}_+$ gets mapped to $(m, \sqrt{m^2+4})$). Now we can take a fundamental system of solutions $\mathbf{w}_0^\pm(z)$ of (3.4) with $z = z(\lambda)$ and $\Re(\lambda) > 0$, given by

$$\mathbf{w}_0^\pm(z,n) = \begin{pmatrix} 1 \\ \alpha_\pm(z) \end{pmatrix} z^{\pm n} \tag{3.9}$$

with

$$\alpha_\pm(z) := \frac{1 - z^{\mp 1}}{\lambda(z) + m} = \frac{\lambda(z) - m}{1 - z^\pm}. \tag{3.10}$$

Note that $\mathbf{w}_0^\pm(z,n)$ is exponentially decaying (increasing) as $n \to \pm\infty$ ($n \to \mp\infty$) for $\lambda(z) \in \mathbb{C} \setminus \overline{\Gamma}$, while $\mathbf{w}_0^\pm(z,n)$ is bounded for $\lambda(z) \in \overline{\Gamma}$.

Analogously we can try to define Jost solutions $\mathbf{w}^\pm(z)$ of the perturbed equation

$$\mathcal{D}\mathbf{w} = \lambda \mathbf{w} \tag{3.11}$$

which asymptoticaly look like the unperturbed ones

$$\mathbf{w}^\pm(z,n) = \begin{pmatrix} x^\pm(z,n) \\ y^\pm(z,n) \end{pmatrix} \to \begin{pmatrix} 1 \\ \alpha_\pm(z) \end{pmatrix} z^{\pm n}, \qquad n \to \pm\infty. \tag{3.12}$$

If the potential coefficients are in $l_1^1(\mathbb{Z})$, then the Jost solutions can be extended to holomorphic functions on $\mathbb{D} \setminus [0,1)$ (for simplicity we shall work with $\mathbb{D} \setminus [0,1)$ instead of $\mathbb{D} \setminus [0, 1/2(m^2 + 2 - \sqrt{(m^2+2)^2 - 4}])$. The Jost solutions will still satisfy the asymptotic boundary condition (3.12). We will prove their existence purely algebraically by a power series ansatz in the next section. Requiring even stronger decay conditions on the potential, we will be even able to extend the domain of definition to all of $\mathbb{C}_- := \mathbb{C} \setminus [0,\infty)$.

What remains to be done in this section, is to study the relations between the four solutions $\mathbf{w}^\pm(z)$ and $\mathbf{w}^\pm(z^{-1})$ corresponding to a single eigenvalue $\lambda(z)$ for a fixed $z \in \mathbb{D} \setminus [0,1)$. Remember that the eigenspaces for a fixed eigenvalue are always two dimensional, and so there has to be some linear relation between these Jost solutions. To examine these relations we introduce the Wronskian determinant of two solutions $\mathbf{w}_1$ and $\mathbf{w}_2$ by

$$W_n(\mathbf{w}_1, \mathbf{w}_2) := \begin{vmatrix} x_1(n) & x_2(n) \\ y_1(n+1) & y_2(n+1) \end{vmatrix}. \tag{3.13}$$

If $\mathbf{w}_1$ and $\mathbf{w}_2$ are both solutions of (3.11) with the same eigenvalue and the potential Q is symmetric, then a straightforward calculation shows that the Wronskian determinant is constant. It is equal to 0 if and only if both eigenvectors are linearly dependent. As we will deal only with such cases the subscript n of W will be omitted. In the case of two Jost solution with the same superscript $\pm$, the Wronskian determinant can by computed by letting n go to $\pm\infty$

$$W(\mathbf{w}^\pm(z), \mathbf{w}^\pm(z^{-1})) = \begin{vmatrix} z^n & z^{-n} \\ \alpha_\pm(z) z^n & \alpha_\pm(z^{-1}) z^{-n} \end{vmatrix} = \pm \frac{z^{-1} - z}{\lambda(z) + m}. \tag{3.14}$$

So we see two Jost solutions with the same superscript are linearly independent (and hence span the whole space of solutions for a spectral paramter $\lambda(z)$), whenever $z \neq \pm 1$. Recall that the latter case corresponds to the edges of the continuous spectrum, i.e. $\lambda \in \{m, \sqrt{m^2+4}\}$. From the linear independence we can infer, that there are unique functions $a_\pm(z)$, $b_\pm(z)$ with domain $\mathbb{D} \setminus [0,1)$, such that

$$\mathbf{w}^\pm(z) = a_\mp(z)\mathbf{w}^\mp(z^{-1}) + b_\mp(z)\mathbf{w}^\mp(z). \tag{3.15}$$

By calculating the Wronskian determinants for $z \neq \pm 1$ we obtain

$$a_+(z) = a_-(z) = \frac{W(z)}{W(\mathbf{w}^-(z^{-1}), \mathbf{w}^-(z))} = \frac{W(z)(\lambda(z)+m)}{z^{-1}-z} \tag{3.16}$$

$$b_\pm(z) = \pm\frac{W^\pm(z)}{W(\mathbf{w}^-(z), \mathbf{w}^-(z^{-1}))} = \pm\frac{W^\pm(z)(\lambda(z)+m)}{z^{-1}-z} \tag{3.17}$$

with

$$W(z) := W(\mathbf{w}^+(z), \mathbf{w}^-(z)), \qquad W^\pm(z) := W(\mathbf{w}^\mp(z), \mathbf{w}^\pm(z^{-1})). \tag{3.18}$$

For $a_+(z) = a_-(z)$ we will simply write $a(z)$. As the Jost solutions can be continuously extended to $\overline{\mathbb{D}} \setminus [0,1]$, the same holds true for $a(z)$ and $b(z)$ except for $z = -1$ (note however that the continuous extension of $\lambda(z)$ is no longer injective). The coefficients $a(z)$ and $b_\pm(z)$ and their analytic properties will be important in the last section.

Remark 3.4. *Because $\overline{\mathbf{w}^\pm(z)} = \mathbf{w}^\pm(z^{-1})$ for $|z| = 1$ (which will be shown in the next section), it follows that $a(z^{-1}) = \overline{a(z)}$, and $b_\pm(z^{-1}) = \overline{b_\pm(z)}$. Now applying (3.15) twice we end up with*

$$W(\mathbf{w}^+(z), \mathbf{w}^+(z^{-1})) = (|a(z)|^2 - |b(z)|^2)W(\mathbf{w}^-(z^{-1}), \mathbf{w}^-(z)) \tag{3.19}$$

which with (3.14) implies that $|a(z)|^2 - |b(z)|^2 = 1$.

Finally note that all of this can be analogously done for the other branch of the square root, i.e for eigenvalues λ with negative real part. The resulting Jost solutions will be denoted by $\tilde{\mathbf{w}}^\pm(z,n)$.

3.2 Power Series Representation of Jost Solutions

We make the following ansatz for the Jost solution $\mathbf{w}^+(z,n) = (x(z,n), y(z,n))$ with the eigenvalue $\lambda(z)$:

$$\begin{pmatrix} x(z,n) \\ y(z,n) \end{pmatrix} = \frac{-mz^n}{1-z}\begin{pmatrix} K^x(z,n) \\ K^y(z,n) \end{pmatrix} + \frac{\lambda(z)z^n}{1-z}\begin{pmatrix} L^x(z,n) \\ L^y(z,n) \end{pmatrix} \tag{3.20}$$

where again we set $\lambda(z) = \sqrt{2+m^2-z-z^{-1}}$ and $K^{x,y}(z,n)$, $L^{x,y}(z,n)$ (we will use the notation x, y whenever some property holds when substituting both x and y) are assumed to be holomorphic around 0 and thus can expressed as their Taylor series

$$\begin{pmatrix} K^x(z,n) \\ K^y(z,n) \end{pmatrix} = \sum_{j=0}^{\infty}\begin{pmatrix} K^x_j(n)z^j \\ K^y_j(n)z^j \end{pmatrix} \tag{3.21}$$

$$\begin{pmatrix} L^x(z,n) \\ L^y(z,n) \end{pmatrix} = \sum_{j=0}^{\infty}\begin{pmatrix} L^x_j(n)z^j \\ L^y_j(n)z^j \end{pmatrix}. \tag{3.22}$$

The choice of prefactors $-mz^n/(1-z)$ and $\lambda(z)z^n/(1-z)$ is motivated by the asymptotic behaviour of the Jost solution. If we now plug our ansatz (3.20) into the eigenvector equation (3.11), we get a system of two coupled difference equations

$$(\mathcal{D}_0 + Q)\begin{pmatrix} -mK^x(z,n) \\ -mK^y(z,n) \end{pmatrix} = \overbrace{(2+m^2-z-z^{-1})}^{\lambda(z)^2}\begin{pmatrix} L^x(z,n) \\ L^y(z,n) \end{pmatrix} \tag{3.23}$$

$$(\mathcal{D}_0 + Q)\begin{pmatrix} L^x(z,n) \\ L^y(z,n) \end{pmatrix} = -m\begin{pmatrix} K^x(z,n) \\ K^y(z,n) \end{pmatrix}. \tag{3.24}$$

Here, we have seperated terms with and without $\lambda(z)$, which is necessary, as all other terms are by assumption holomorphic around 0 but $\lambda(z)$ is not meromorphic around 0. Now eliminating the $K^{x,y}(z,n)$ from the above equations we obtain

$$(\mathcal{D}_0 + Q)^2\begin{pmatrix} L^x(z,n) \\ L^y(z,n) \end{pmatrix} = \lambda(z)^2\begin{pmatrix} L^x(z,n) \\ L^y(z,n) \end{pmatrix}. \tag{3.25}$$

By comparing Taylor coefficients on both sides the above equation can be rewritten as

$$L^x_{j+1}(n)r_{n+1} = L^x_{j+1}(n+1) - L^x_{j-1}(n+2)r_{n+2} + L^x_{j-1}(n+1) \tag{3.26}$$

$$+L^x_j(n+1)(2mq^{11}_{n+1} + (q^{11}_{n+1})^2 + r^2_{n+1} - 1) + L^y_j(n+1)r_{n+1}(q^{11}_{n+1} + q^{22}_{n+1})$$

$$-L^y_{j-1}(n+2)(q^{11}_{n+1} + q^{22}_{n+2})$$

and

$$L^y_{j+1}(n)r_n = L^y_{j+1}(n+1) - L^y_{j-1}(n+2)r_{n+1} + L^y_{j-1}(n+1) \tag{3.27}$$

$$+L^y_j(n+1)(-2mq^{22}_{n+1} + (q^{22}_{n+1})^2 + r^2_{n+1} - 1) + L^x_j(n+1)r_{n+1}(q^{11}_{n+1} + q^{22}_{n+1})$$

$$-L^x_{j+1}(n)(q^{11}_n + q^{22}_{n+1})$$

where we have set $r_n := 1 + q^{12}_n$. To simplify calculation further we make the following definitions

$$A(n) := \prod_{k=n}^{\infty} r_k \tag{3.28}$$

$$N^x_j(n) := A(n+1)L^x_j(n) \tag{3.29}$$

$$N^y_j(n) := A(n)L^y_j(n) \tag{3.30}$$

$$Q^x_j(n) := N^x_{j-1}(n+1)(1 - r^2_{n+1}) + N^x_j(n)(2mq^{11}_n + (q^{11}_n)^2 + r^2_n - 1) \tag{3.31}$$

$$-N^y_j(n)(q^{11}_n + q^{22}_n) - N^y_{j-1}(n+1)(q^{11}_n + q^{22}_{n+1})$$

$$Q_j^y(n) := N_{j-1}^y(n+1)(1-r_n^2) + N_j^y(n)(-2mq_n^{22} + (q_n^{22})^2 + r_n^2 - 1) \tag{3.32}$$

$$+N_j^x(n) r_n^2 (q_n^{11} + q_n^{22}) + N_{j+1}^x(n-1)(q_n^{22} + q_{n-1}^{11})$$

Here, we require $q_n^{ij} \in l^1(\mathbb{N})$ for $A(n)$ to be well-defined. With these the equations (3.26), (3.27) become

$$N_{j+1}^x(n) = N_{j+1}^x(n+1) + N_{j-1}^x(n+1) - N_{j-1}^x(n+2) + Q_j^x(n+1) \tag{3.33}$$

$$N_{j+1}^y(n) = N_{j+1}^y(n+1) + N_{j-1}^y(n+1) - N_{j-1}^y(n+2) + Q_j^y(n+1). \tag{3.34}$$

We can find a sequence $c(n)$ depending only on the mass m and the potential Q satisfying

$$Q_j^x(n) \le c(n) \max\{|N_{j-1}^x(n+1)|, |N_j^x(n)|, |N_j^y(n)|, |N_{j-1}^y(n+1)|\} \tag{3.35}$$

and

$$Q_j^y(n) \le c(n) \max\{|N_{j-1}^y(n+1)|, |N_j^y(n)|, |N_j^x(n)|, |N_{j+1}^x(n-1)|\}. \tag{3.36}$$

Notice that $1 - r_n^2 = -2q_n^{12} - (q_n^{12})^2$, which implies that we can make $c(n)$ decay arbitrary fast as $n \to +\infty$, by imposing the right decay properties on the potential. In particular if $q_n^{ij} \in l_\sigma^p(\mathbb{N})$ for $i,j = 1,2$, we can choose $c(n) \in l_\sigma^p(\mathbb{N})$. With the above notation, we can now formulate the following result.

Theorem 3.5. *Let $\mathcal{D}$ be a discrete Dirac operator of the form (3.1). Assume that the potential Q has coefficients from $l_2^1(\mathbb{N})$. Then the Jost solution $\mathbf{w}^+(z,n) = (x(z,n), y(z,n))$, with eigenvalue $\lambda(z) = \sqrt{2 + m^2 - z - z^{-1}}$ are holomorphic in $z \in \mathbb{D} \setminus [0,1)$ and can be continuously extended to $\mathbb{T} \setminus \{1\}$. They are given explicitly by*

$$\begin{pmatrix} x(z,n) \\ y(z,n) \end{pmatrix} = \frac{1}{1-z}(\mathcal{D}_0 + Q + \lambda(z)) \begin{pmatrix} N^x(n,z) z^n / A(n+1) \\ N^y(n,z) z^n / A(n) \end{pmatrix} \tag{3.37}$$

where

$$N^x(n,z) := \sum_{j=0}^{\infty} N_j^x(n) z^j, \qquad N^y(n,z) := \sum_{j=0}^{\infty} N_j^y(n) z^j \tag{3.38}$$

and

$$\lim_{n\to\infty} N_j^x(n) = 0, \qquad \lim_{n\to\infty} N_j^y(n) = \delta_{0j} \tag{3.39}$$

for $j \ge 0$. The $N_j^{x,y}(n)$ satisfy the recursion relations (3.33), (3.34) (taking $N_j^{x,y} = 0$ for $j < 0$). We get the following inequalities

$$|N_j^x(n)| \le (1 + C_+(n+1)) D_{+,j-1}(n+1)^2 C_+(n + \lfloor j/2 \rfloor) \tag{3.40}$$

for $j \ge 0$ and

$$|N_j^y(n)| \le D_{+,j}(n)^2 C_+(n + \lfloor j/2 \rfloor) \tag{3.41}$$

for $j \ge 1$, with

$$C_+(n) := \sum_{k=n}^{\infty} c(k), \qquad D_{+,j}(n) = \prod_{k=1}^{j} (1 + C_+(k+n)). \tag{3.42}$$

and the sequence $c(n) \in l_1^1(\mathbb{N})$ satisfying (3.35), (3.36). The radius of convergence of $N^{x,y}(n,z)$ is at least 1.

Proof. Note that setting $Q \equiv 0$, equation (3.37 gives us the free Jost solutions $\mathbf{w}_0^+(z,n)$. Furthermore, because of the decay of the potential, $c(n)$ can be chosen to be in $l_2^1(\mathbb{N})$, which implies that $C_+(n) \in l_1^1(\mathbb{N})$. In particular, $\lim_{j\to\infty} D_{+,j}(n) := D_{+,\infty}(n)$ is finite. Also $N_{-1}^{x,y}(n) = 0$ implies $Q_{-1}^x(n+1) = 0$ and so (3.33) implies $N_0^x(n) = 0$ for all n. Furthermore $N_0^x(n) = 0$ implies $Q_{-1}^y(n+1) = 0$, and so by (3.34) we conclude that $N_0^y(n) = 1$. We will now prove the existence, uniqueness and the estimates (3.40) and (3.41) by induction with respect to j. The induction base consists of the cases $N_1^{x,y}(n), N_2^{x,y}(n)$ which have to be treated separately as $N_0^y(n)$ does not fulfill (3.41) and is given in the appendix.
Set $j \geq 2$. Let us assume that

$$|N_i^x(n)| \leq (1 + C_+(n+1))D_{+,i-1}(n+1)^2 C_+(n + \lfloor i/2 \rfloor) \tag{3.43}$$

$$|N_i^y(n)| \leq D_{+,i}(n)^2 C_+(n + \lfloor i/2 \rfloor) \tag{3.44}$$

for all values $i \leq j$, and all n (except for $N_0^y(n)$). We can iteratively apply (3.33), to obtain

$$N_{j+1}^x(n) = \lim_{m\to\infty} \Big(N_{j+1}^x(m) + N_{j-1}^x(n+1) - N_{j-1}^x(m+1) + \sum_{k=n+1}^{m} Q_j^x(k) \Big) \tag{3.45}$$

$$= N_{j-1}^x(n+1) + \sum_{k=n+1}^{\infty} Q_j^x(k).$$

From (3.39) it also follows that if a solution with the specified boundary conditions exists, it must be of this form. This manipulation is still formal as we have not shown convergence of the sum. However if the above expression converges, then it will automatically satisfy (3.33). An analog results holds for the y-component

$$N_{j+1}^y(n) = N_{j-1}^y(n+1) + \sum_{k=n+1}^{\infty} Q_j^y(k). \tag{3.46}$$

We will prove convergence by simultaneously proving the estimates (3.40) and (3.41). First note that for $k \geq n+1$ we have

$$\max\{|N_{j-1}^x(k+1)|, |N_j^x(k)|, |N_j^y(k)|, |N_{j-1}^y(k+1)|\} \tag{3.47}$$

$$\leq D_{+,j}(n+1)^2 C_+(n + 1 + \lfloor j/2 \rfloor)$$

and

$$\max\{|N_{i-1}^y(k+1)|, |N_i^y(k)|, |N_i^x(k)|, |N_{i+1}^x(k-1)|\} \tag{3.48}$$

$$\leq (1 + C_+(n+1))D_{+,j}(n+1)^2 C_+(n + \lfloor (j+1)/2 \rfloor).$$

For the x-component we get

$$|N_{j+1}^x(n)| \leq |N_{j-1}^x(n+1)| + \sum_{k=n+1}^{\infty} |Q^x(j,k)| \tag{3.49}$$

$$\leq (1 + C_+(n+2))D_{+,j-2}(n+1)^2 C_+(n + 1 + \lfloor (j-1)/2 \rfloor$$

$$+\sum_{k=n+1}^{\infty} c(k) D_{+,j}(n+1)^2 C_+(n+1+\lfloor j/2 \rfloor)$$

$$\leq (1+C_+(n+2)) D_{+,j-2}(n+1)^2 C_+(n+1+\lfloor (j-1)/2 \rfloor)$$

$$+C_+(n+1) D_{+,j}(n+1)^2 C_+(n+1+\lfloor j/2 \rfloor)$$

$$\leq C_+(n+\lfloor (j+1)/2 \rfloor)((1+C_+(n+2)) D_{+,j-2}(n+2)^2 + C_+(n+1) D_{+,j}(n+1)^2)$$

$$\leq C_+(n+\lfloor (j+1)/2 \rfloor)(D_{+,j-1}(n+2)^2 + C_+(n+1) D_{+,j}(n+1)^2)$$

$$\leq C_+(n+\lfloor (j+1)/2 \rfloor)(1+C_+(n+1)) D_{+,j}(n+1)^2.$$

Similarly for the y-component we get

$$|N_{j+1}^y(n)| \leq |N_{j-1}^y(n+1)| + \sum_{k=n+1}^{\infty} |Q_j^y(k)| \tag{3.50}$$

$$\leq D_{+,j-1}(n+1)^2 C_+(n+1+\lfloor (j-1)/2 \rfloor)$$

$$+C_+(n+1)(1+C_+(n+1)) D_{+,j}(n+1)^2 C_+(n+\lfloor (j+1)/2 \rfloor) \tag{3.51}$$

$$= C_+(n+\lfloor (j+1)/2 \rfloor)(D_{+,j-1}(n+1)^2$$

$$+C_+(n+1)(1+C_+(n+1)) D_{+,j}(n+1)^2)$$

$$\leq C_+(n+\lfloor (j+1)/2 \rfloor)(1+C_+(n+1))^2 D_{+,j}(n+1)^2$$

$$= C_+(n+\lfloor (j+1)/2 \rfloor) D_{+,j+1}(n)^2.$$

Finally we need to show that (3.37) is in fact the Jost solution $\mathbf{w}^+(z)$. In particular we need to show the convergence of the series $N^{x,y}(n,z)$ and that it has the right asymptotic behavior. For the convergence we compute (assuming $z \in \overline{\mathbb{D}}$)

$$\sum_{j=0}^{\infty} |N_j^x(n) z^j| \leq \sum_{j=0}^{\infty} |N_j^x(n)| \leq (1+C_+(n+1)) D_{+,\infty}(n)^2 \sum_{j=0}^{\infty} C_+(n+\lfloor j/2 \rfloor) < \infty \tag{3.52}$$

and analogously for $N^y(n,z)$. Recall that the Jost solution $\mathbf{w}^+(z)$ (to an eigenvalue $\lambda(z)$) satisfy

$$\lim_{n\to\infty} \mathbf{w}^+(z,n) z^{-n} = \begin{pmatrix} 1 \\ \alpha_+(z) \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{\lambda(z)-m}{1-z} \end{pmatrix}. \tag{3.53}$$

This is satisfied if we have

$$\lim_{n\to\infty} \begin{pmatrix} N^x(z,n) \\ N^y(z,n) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.54}$$

Notice that the above equation is plausible in the light of the boundary condition (3.39). By looking at the expression (3.52) we however immediately see that indeed $\lim_{n\to\infty} N^x(z,n) = 0$ even uniformly for all $z \in \overline{\mathbb{D}}$. The analog statement holds for $N^y(z,n)$ where one has to remember that $N_0^y(n) = 1$. $\square$

Remark 3.6. *In the above proof we have used certain monotonicity properties of the $D_{+,j}(n)$ and $C_+(n)$. Namely $D_{+,j}(n)$ is increasing in j, but decreasing in n and $C_+(n)$ is decreasing in n. These facts are useful to remember and will be used without mention in the next section.*

Remark 3.7. *Also note that the speed of convergence of*

$$\lim_{n\to\infty} \mathbf{w}^+(z,n) z^{-n} = \begin{pmatrix} 1 \\ \dfrac{\lambda(z) - m}{1-z} \end{pmatrix} \tag{3.55}$$

depends only on the convergence of $N^{x,y}(n,z)$ and on the prefactors in (3.37). As $N^{x,y}(n,z)$ converge uniformly for $z \in \overline{\mathbb{D}}$ and the prefactors get unbounded for $z \to 0, 1$, we get uniform convergence of the above expression on subsets of $\overline{\mathbb{D}} \setminus [0,1]$ that stay away from $\{0,1\}$.

A natural question would be under which circumstances the domain of the Jost solution $\mathbf{w}^+(z,n)$ can be extended to all of $\mathbb{C}_-$? The next lemma gives an answer.

Lemma 3.8. *If there exists a sequence $c(n)$, $n \in \mathbb{Z}$ satisfying (3.35) and (3.36) and positive constants C_+, ϵ, δ such that*

$$C_+(n) = \sum_{k=n}^{\infty} c(k) \leq \frac{C_+}{(\epsilon n)^{\delta n}} \tag{3.56}$$

for all $n \geq N \in \mathbb{N}$, then the Jost solution $\mathbf{w}^+(z,n)$ can be extended to a holomorphic function on $\mathbb{C}_-$. Furthermore (3.12) still holds uniformly on compact subsets of $\mathbb{C}_-$.

Proof. For the x-component we compute ($N_0^x(n) = 0$)

$$|\sum_{j=1}^{\infty} N_j^x(n) z^j| \leq (1 + C_+(n+1)) D_{+,\infty}(n+1)^2 \sum_{j=1}^{\infty} |C_+(n + \lfloor j/2 \rfloor)| |z|^j \tag{3.57}$$

where $D_{+,\infty}(n+1) := \lim_{j\to\infty} D_{+,j}(n+1)$ which is finite by assumption. By the Cauchy–Hadamard-formula, we conclude that the radius of convergence is infinite. Now we will consider the convergence to the free Jost solution. As the prefactors remain bounded they can be ignored. Taking the limit $n \to \infty$ of the right-hand side of the above equation we obtain

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} |C_+(n + \lfloor j/2 \rfloor)| |z|^j \leq \lim_{n\to\infty} C_+ \sum_{j=1}^{\infty} \left(\frac{|z|^{j/(\delta(n+\lfloor j/2 \rfloor))}}{\epsilon(n + \lfloor j/2 \rfloor)} \right)^{\delta(n+\lfloor j/2 \rfloor)} \tag{3.58}$$

$$\leq C_+ \lim_{n\to\infty} \sum_{j=1}^{\infty} \left(\frac{\max\{1, |z|^{2/\delta}\}}{\epsilon(n + \lfloor j/2 \rfloor)} \right)^{\delta(n+\lfloor j/2 \rfloor)}$$

$$\leq C_+ \lim_{n\to\infty} \sum_{k=2n+1}^{\infty} \left(\frac{\max\{1, |z|^{2/\delta}\}}{\epsilon\lfloor k/2 \rfloor} \right)^{\delta\lfloor k/2 \rfloor} = 0.$$

In a similar way we get

$$\lim_{n\to\infty} |\sum_{j=1}^{\infty} N_j^y(n) z^j| = 0 \tag{3.59}$$

and observing that $N_0^y(n) = 1$ for all n, we obtain (3.12). □

Remark 3.9. *We can find such a sequence $c(n)$ satisfying (3.56) whenever the potential coefficient satisfy an analog estimate with possibly different constants.*

Remark 3.10. *Note that in all the calculations in this section we only used that $\lambda(z)^2 = 2 + m^2 - z - z^{-1}$, i.e the branch of the square root was not relevant. That means that in principle we could define the Jost solutions on some Riemannian surface. However we will stick with our convention and restrict our domain of definitions to $\mathbb{D} \setminus [0, 1)$ or $\mathbb{C}_-$.*

4 Spectral Decomposition

We now turn to the spectral decomposition of the discrete Dirac operator $\mathcal{D}$. The theorems and proof are taken from [5], where they are shown for the continuous case.

By the spectral theorem we know that there exists a unitary transform

$$V = V_1 \oplus ... \oplus V_p : l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z}) \to \bigoplus_{j=1}^{p} L^2(\sigma(\mathcal{D}), d\sigma_j) \tag{4.1}$$

such that

$$(V\mathcal{D}f)(\lambda) = \lambda(Vf)(\lambda) \tag{4.2}$$

Under the assumption $q^{ij} \in l_2^1(\mathbb{Z})$ we know that the spectrum of $\mathcal{D}$ consists of a an absolutely continuous part $\overline{\Gamma} = [-m, -\sqrt{m^2+4}] \cup [m, \sqrt{m^2+4}]$ and a finite number of simple eigenvalues in $\mathbb{R} \setminus \overline{\Gamma}$. Let $\chi_{[k,l]}$ be the multiplication operator with the characteristic function of $\{k, ..., l\}$, i.e

$$\chi_{[k,l]}\mathbf{u}(n) = \begin{cases} \mathbf{u}(n), & k \leq n \leq l \\ 0, & \text{else} \end{cases} \tag{4.3}$$

Note that for a bounded operator B on $\mathbf{l}^2(\mathbb{Z})$, the operator $B\chi_{[k,l]}$ has finite-dimensional range, and in particular defines a Hilbert–Schmidt operator. Furthermore the the Hilbert space $\mathbf{l}^2(\mathbb{Z})$ can be regarded as a measure space. These observations will be very helpful in combination with the following theorem taken from [7] concerning Hilbert–Schmidt operators on L^2-spaces.

Theorem 4.1. *A bounded operator $K : L^2(Y, d\nu) \to L^2(X, d\mu)$, is a Hilbert–Schmidt operator, if and only if it is a Hilbert–Schmidt integral operator, i.e there exists a kernel $k \in L^2(X \times Y, d\mu \otimes d\nu)$, such that*

$$(Kf)(x) = \int_Y k(x,y) f(y) d\nu(y) \tag{4.4}$$

Using the above theorem we can deduce that for each pair $(k,l) \in \mathbb{Z} \times \mathbb{Z}$, with $k \le l$, there exist functions $\mathbf{v}_{j,[k,l]}(\lambda, n) = (v^x_{j,[k,l]}(\lambda, n), v^y_{j,[k,l]}(\lambda, n))$ with

$$(V\chi_{[k,l]}\mathbf{u})(\lambda) = \bigoplus_{j=1}^{p} \sum_{n \in \mathbb{Z}} \langle \mathbf{v}_{j,[k,l]}(\lambda, n), \mathbf{u}(n) \rangle \tag{4.5}$$

It is easily seen that $\mathbf{v}_{j,[k,l]}(\lambda, n) = \mathbf{v}_{j,[k',l']}(\lambda, n)$, ρ_j-a.e for all $n \in [k,l] \cap [k',l']$ and so we can define $\mathbf{v}_j(\lambda, n) := \mathbf{v}_{j,[k,l]}(\lambda, n)$ with $n \in [k,l]$. As V is unitary and hence bounded and $\chi_{[k,l]}\mathbf{u} \to \mathbf{u}$ for $k \to -\infty$, $l \to \infty$ we deduce that also

$$\sum_{n=k}^{l} \langle \mathbf{v}_j(\lambda, n), \mathbf{u}(n) \rangle \to \sum_{n \in \mathbb{Z}} \langle \mathbf{v}_j(\lambda, n), \mathbf{u}(n) \rangle \tag{4.6}$$

in $L^2(\sigma(\mathcal{D}), d\sigma_j)$. In particular the right hand side exists whenever $u(n) \in \mathbf{l}^2(\mathbb{Z})$. Note that because of (4.2) we have have that

$$\sum_{n \in \mathbb{Z}} \langle \mathbf{v}_j(\lambda, n), \mathcal{D}\mathbf{u}(n) \rangle = \lambda \sum_{n \in \mathbb{Z}} \langle \mathbf{v}_j(\lambda, n), \mathbf{u}(n) \rangle. \tag{4.7}$$

As $\mathcal{D}$ is symmetric the above expression is also equal to

$$\sum_{n \in \mathbb{Z}} \langle \mathcal{D}\mathbf{v}_j(\lambda, n), \mathbf{u}(n) \rangle. \tag{4.8}$$

As $\mathbf{u}(n)$ is arbitrary (the domain of $\mathcal{D}$ is $l^2(\mathbb{Z})$), we can conclude that $\mathcal{D}\mathbf{v}_j(\lambda, n) = \lambda \mathbf{v}_j(\lambda, n)$ for σ_j-a.e. In fact, from now on we will assume that $\mathbf{v}_j(\lambda, n)$ is an eigenvector of $\mathcal{D}$ (or 0) for all $\lambda \in \sigma(\mathcal{D})$.
Next we will compute the spectral multiplicity of $\mathcal{D}$. Assume that $p > 2$. Then the set $\{\mathbf{v}_1(\lambda, .), ..., \mathbf{v}_p(\lambda, .)\}$ is linearly dependent for all $\lambda \in \sigma(\mathcal{D})$. This implies that the functions $\{(V_1\mathbf{u})(\lambda), ..., (V_p\mathbf{u})(\lambda)\}$ are also linearly dependent for all $\lambda \in \sigma(\mathcal{D})$ (in fact the same linear relationships hold). Without loss of generality we can assume that the spectral measures σ_j are ordered, i.e σ_j is absolutely continuous with respect to σ_i whenever $i \le j$. Let $M_s := \{\lambda : \mathbf{v}_s(\lambda, .) \in \mathrm{span}_{j \ne s} \mathbf{v}_j(\lambda, .)\}$. This set is measurable as it can be written as the set of all λ such that there exist $j_1, ..., j_k \ne s$, such that $\mathbf{v}_{j_1}(\lambda, .), ..., \mathbf{v}_{j_k}(\lambda, .)$ are linearly independent, but $\mathbf{v}_{j_1}(\lambda, .), ..., \mathbf{v}_{j_k}(\lambda, .), \mathbf{v}_s(\lambda, .)$ are linearly dependent. These conditions can be formulated using determinants. Note that $\sigma(\mathcal{D}) = M_1 \cup ... \cup M_p$, which implies that there is an $s \in \{1, ..., p\}$, such that M_s has positive σ_p-measure (and hence positive σ_s-measure). Thus $(V_j\mathbf{u})(\lambda) = 0$ for all $j \ne s$ implies $(V_s\mathbf{u})(\lambda) = 0$ for $\lambda \in M_s$, which contradicts the surjectivity of V. Thus the spectral multiplicty is at most 2.

Assuming $\lambda = \lambda(z) \in (m, \sqrt{m^2 + 4})$, there are coefficients $t_j^{\pm}(z)$ with $z \in \mathbb{T}_+$ such that

$$\mathbf{v}_j(\lambda, n) = \overline{t_j^+(z)}\mathbf{w}^+(z, n) + \overline{t_j^-(z)}\mathbf{w}^-(z, n) \tag{4.9}$$

with $j = 1$ or $j = 1, 2$ (the complex conjugation will become clear later). Here we use the fact that the Jost solutions $\mathbf{w}^+(z, n)$ and $\mathbf{w}^-(z, n)$ are linearly independent for all eigenvalues in $(m, \sqrt{m^2 + 4})$. This is because linear dependence is equivalent to $W(z) = 0$, but that would imply $a(z) = 0$ for $z \in \mathbb{T}_+$, which contradicts remark 3.4 . By Cramer's rule we have

$$\overline{t_j^+(z)} = \frac{\det(\mathbf{v}_j(\lambda, n)\ \mathbf{w}^-(z, n))}{\det(\mathbf{w}^+(z, n)\ \mathbf{w}^-(z, n))}, \quad \overline{t_j^-(z)} = \frac{\det(\mathbf{w}^+(z, n)\ \mathbf{v}_j(\lambda, n))}{\det(\mathbf{w}^+(z, n)\ \mathbf{w}^-(z, n))}, \tag{4.10}$$

so we see that these coefficients are measurable and square integrable away from $z = 1, -1$ (where the Jost solutions might become linearly dependent). We would like to rewrite the spectral decomposition in terms of the Jost solutions. To this end, we introduce

$$\rho^{++}(z) = \sum_{j=1}^{p} \int_{z_0}^{\lambda(z')} \overline{t_j^+(z')} t_j^+(z') d\sigma_j(\lambda) \tag{4.11}$$

where $p = 1$ or $p = 2$, z, z' are defined implicitly and $z_0 \in \mathbb{T}_+$ is fixed. We define $\rho^{+-}(z)$, $\rho^{--}(z)$ analogously. This defines now a non-decreasing continuous 2×2-matrix-valued function

$$\rho(z) := \begin{pmatrix} \rho^{++}(z) & \rho^{-+}(z) \\ \rho^{+-}(z) & \rho^{--}(z) \end{pmatrix} \tag{4.12}$$

in the sense that for $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$ we have

$$\langle \xi, \rho(I)\xi \rangle = \langle \xi, \sum_{j=1}^{p} \int_I \overline{\mathbf{t}_j(z)} \mathbf{t}_j(z)^T d\sigma_j(\lambda)\xi \rangle = \sum_{j=1}^{p} \int_I |\langle \mathbf{t}_j(z), \xi \rangle|^2 d\sigma_j(\lambda) \geq 0 \tag{4.13}$$

where $\mathbf{t}_j(z) = (t_j^+(z), t_j^-(z))^T$. Next we can define the Hilbert space of $\mathbb{C}^2$-valued functions $L^2(\mathbb{T}_+, d\rho)$ with scalar product given by

$$\langle f(z), g(z) \rangle_\rho := \int_{\mathbb{T}_+} \langle f(z), d\rho(z) g(z) \rangle. \tag{4.14}$$

The $\mathbb{C}^2$-valued functions are given by all pairs $(x(z), y(z))$, such that $x(z)t_j^+(z) + y(z)t_j^-(z)$ are in $L^2(\sigma_{ac}(\mathcal{D})_+, d\sigma_j)$ (where $\sigma_{ac}(\mathcal{D})_+$ is the positive absolutely continuous spectrum), moded out by the usual equivalence relation. All of the above can be analogously done for the Jost solutions $\tilde{\mathbf{w}}^\pm(z, n)$ corresponding to negative eigenvalues. To deal with the eigenvalues we will use the following simple principle.

Remark 4.2. *If* $\mathbf{v}$ *is a (square-summable) eigenvector with eigenvalue* λ *of* $\mathcal{D}$*, and* $\mathbf{w}$ *is a corresponding Jost solution with the same eigenvalue, then if* $\mathbf{w}$ *is square summable at plus or minus infinity, then we already have that* $\mathbf{v}$ *and* $\mathbf{w}$ *are proportional. That is because we can evaluate the Wronskian of* $\mathbf{v}$ *and* $\mathbf{w}$ *at* $\pm\infty$ *and because of the square-summability, it must be equal to* 0*, which is equivalent to* $\mathbf{v}$ *and* $\mathbf{w}$ *being linearly dependent.*

We are now ready to formulate the main theorem of this section.

Theorem 4.3. *Assuming* $q_n^{ij} \in l_2^1(\mathbb{Z})$ *and using the notation above the map*

$$U : \mathbf{l}^2(\mathbb{Z}) \to L^2(\mathbb{T}_+ \cup \sigma_p^-, d\tilde{\rho} + d\mu^-) \oplus L^2(\mathbb{T}_+ \cup \sigma_p^{+,0}, d\rho + d\mu^{+,0}), \tag{4.15}$$

$$\mathbf{u}(n) \mapsto \sum_{n\in\mathbb{Z}} \begin{pmatrix} \langle \tilde{\mathbf{w}}^+(z,n), \mathbf{u}(n)\rangle \\ \langle \tilde{\mathbf{w}}^-(z,n), \mathbf{u}(n)\rangle \end{pmatrix} \oplus \sum_{n\in\mathbb{Z}} \begin{pmatrix} \langle \mathbf{w}^+(z,n), \mathbf{u}(n)\rangle \\ \langle \mathbf{w}^-(z,n), \mathbf{u}(n)\rangle \end{pmatrix} \tag{4.16}$$

where $\sigma_p^-/\sigma_p^{+,0}$ is the set of all negative/non-negative eigenvalues of $\mathcal{D}$ (in the z-domain) and

$$d\mu^- = \begin{pmatrix} d\eta^- & 0 \\ 0 & 0 \end{pmatrix}, \quad d\mu^{+,0} = \begin{pmatrix} d\eta^{+,0} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.17}$$

have support $\sigma_p^-/\sigma_p^{+,0}$, is unitary. Furthermore the inverse is given by

$$U^{-1}(\tilde{\phi}(z) \oplus \phi(z)) = \int_{\mathbb{T}^+\cup\sigma_p^-} \begin{pmatrix} \left\langle \overline{\begin{pmatrix} \tilde{w}_x^+(z,n) \\ \tilde{w}_x^-(z,n) \end{pmatrix}}, (d\tilde{\rho}(z) + d\mu^-(z))\tilde{\phi}(z) \right\rangle \\ \left\langle \overline{\begin{pmatrix} \tilde{w}_y^+(z,n) \\ \tilde{w}_y^-(z,n) \end{pmatrix}}, (d\tilde{\rho}(z) + d\mu^-(z))\tilde{\phi}(z) \right\rangle \end{pmatrix} \tag{4.18}$$

$$+ \int_{\mathbb{T}^+\cup\sigma_p^{+,0}} \begin{pmatrix} \left\langle \overline{\begin{pmatrix} w_x^+(z,n) \\ w_x^-(z,n) \end{pmatrix}}, (d\rho(z) + d\mu^{+,0}(z))\phi(z) \right\rangle \\ \left\langle \overline{\begin{pmatrix} w_y^+(z,n) \\ w_y^-(z,n) \end{pmatrix}}, (d\rho(z) + d\mu^{+,0}(z)\phi(z) \right\rangle \end{pmatrix}.$$

Proof. We know that under the assumption $q^{ij} \in l^1_2(\mathbb{Z})$, $\mathcal{D}$ has an absolutely continuous spectrum $\overline{\Gamma}$ and finitely many eigenvalues in $\mathbb{R}\setminus\overline{\Gamma}$.

First we consider the eigenvalues. Now observe that with our (standard) defintion of the square root, there is a branch cut whenever the argument is negative (which corresponds to a purely imaginary λ). Thus the only eigenvalue which might be problematic is 0. But be letting $z \to (m^2+2)/2 - \sqrt{(m^2+2)^2/4-1} = o$ (which corresponds to $\lambda(z) \to 0$) in (3.37), we see that we can easily extend the domain of definition of $\mathbf{w}^+(z,n)$ to this value. Furthermore by the last remark, we in fact get all eigenvectors by plugging in the corresponding $z \in \mathbb{D}\setminus[0,1)\cup\{o\}$ into either $\mathbf{w}^+(z,n)$ or $\mathbf{w}^-(z,n)$. Next let us consider the absolutely continuous spectrum. We will show that the map

$$T : L^2(\mathbb{T}_+, d\rho) \to \bigoplus_{j=1}^{p} L^2([m, \sqrt{m^2+4}], d\sigma_j), \tag{4.19}$$

$$(x(z), y(z)) \mapsto \bigoplus_{j=1}^{p} t_j^+(z(\lambda))x(z(\lambda)) + t_j^-(z(\lambda))y(z(\lambda))$$

is unitary. First we compute

$$\left\| T\begin{pmatrix} x(z) \\ y(z) \end{pmatrix} \right\|^2 = \sum_{j=1}^{p} \int_{[m,\sqrt{m^2+4}]} |t_j^+(z)x(z) + t_j^-(z)y(z)|^2 d\sigma_j(\lambda) \tag{4.20}$$

$$= \sum_{j=1}^{p} \int_{[m,\sqrt{m^2+4}]} \langle \begin{pmatrix} x(z) \\ y(z) \end{pmatrix}, \overline{\mathbf{t}_j(z)}\mathbf{t}_j(z)^T \begin{pmatrix} x(z) \\ y(z) \end{pmatrix} \rangle d\sigma_j(\lambda)$$

$$= \left\| \begin{pmatrix} x(z) \\ y(z) \end{pmatrix} \right\|_\rho^2.$$

Note that we have $TU_{|\mathbf{l}^2(\mathbb{Z})^+_{ac}} = V_{|\mathbf{l}^2(\mathbb{Z})^+_{ac}}$, where V is the original spectral decomposition and $\mathbf{l}^2(\mathbb{Z})^+_{ac}$ are the absolutely continuous vectors with non-negative spectrum. We thus conclude that T is a surjective isometry, hence unitary. Similarly we can treat the case of the non-positive absolutely continuous spectrum (in the case $m = 0$ there is an overlap at $\lambda = 0$ which is however of measure 0 in the case of the absolutely continuous spectrum). The case of the discrete spectrum is trivial. As $V_{|\mathbf{l}^2(\mathbb{Z})^\pm_{ac}}$ with range $\bigoplus_{j=1}^p L^2(\sigma_{ac}(\mathcal{D})^\pm, d\sigma_j)$ is also unitary, we conclude that $U_{|\mathbf{l}^2(\mathbb{Z})^+_{ac}} = T^{-1}V_{|\mathbf{l}^2(\mathbb{Z})^+_{ac}}$ is unitary. The result for U follows analogously.
It remains to prove the formula for the inverse. By unitarity of U we have

$$\langle U^{-1}(\tilde{\phi}(z) + \phi(z)), \mathbf{u}(n)\rangle = \langle \tilde{\phi}(z) + \phi(z), (U\mathbf{u})(z)\rangle_\rho \tag{4.21}$$

as we have

$$(U\mathbf{u})(z) = \sum_{n\in\mathbb{Z}} \begin{pmatrix} \langle \tilde{\mathbf{w}}^+(z,n), \mathbf{u}(n)\rangle \\ \langle \tilde{\mathbf{w}}^-(z,n), \mathbf{u}(n)\rangle \end{pmatrix} + \sum_{n\in\mathbb{Z}} \begin{pmatrix} \langle \mathbf{w}^+(z,n), \mathbf{u}(n)\rangle \\ \langle \mathbf{w}^-(z,n), \mathbf{u}(n)\rangle \end{pmatrix}. \tag{4.22}$$

Substituting on the right-hand side gives

$$\int_{\mathbb{T}^+\cup\sigma_p^-} \left\langle \tilde{\phi}(z), (d\tilde{\rho}(z) + d\mu^-(z)) \sum_{n\in\mathbb{Z}} \begin{pmatrix} \langle \tilde{\mathbf{w}}^+(z,n), \mathbf{u}(n)\rangle \\ \langle \tilde{\mathbf{w}}^-(z,n), \mathbf{u}(n)\rangle \end{pmatrix} \right\rangle + \tag{4.23}$$

$$\int_{\mathbb{T}^+\cup\sigma_p^{+,0}} \left\langle \phi(z), (d\rho(z) + d\mu^{+,0}(z)) \sum_{n\in\mathbb{Z}} \begin{pmatrix} \langle \mathbf{w}^+(z,n), \mathbf{u}(n)\rangle \\ \langle \mathbf{w}^-(z,n), \mathbf{u}(n)\rangle \end{pmatrix} \right\rangle \tag{4.24}$$

which after interchanging integration and summation (as the scalar product $\langle ., .\rangle_\rho$ is continuous) and using the hermiticity of the measures, equals to

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} \left\langle \int_{\mathbb{T}^+\cup\sigma_p^-} \begin{pmatrix} \left\langle \begin{pmatrix} \overline{\tilde{w}_x^+}(z,n) \\ \overline{\tilde{w}_x^-}(z,n) \end{pmatrix}, (d\tilde{\rho}(z) + d\mu^-(z))\tilde{\phi}(z) \right\rangle \\ \left\langle \begin{pmatrix} \overline{\tilde{w}_y^+}(z,n) \\ \overline{\tilde{w}_y^-}(z,n) \end{pmatrix}, (d\tilde{\rho}(z) + d\mu^-(z))\tilde{\phi}(z) \right\rangle \end{pmatrix}, \mathbf{u}(n) \right\rangle \\ &+ \sum_{n\in\mathbb{Z}} \left\langle \int_{\mathbb{T}^+\cup\sigma_p^{+,0}} \begin{pmatrix} \left\langle \begin{pmatrix} \overline{w_x^+}(z,n) \\ \overline{w_x^-}(z,n) \end{pmatrix}, (d\rho(z) + d\mu^{+,0}(z))\phi(z) \right\rangle \\ \left\langle \begin{pmatrix} \overline{w_y^+}(z,n) \\ \overline{w_y^-}(z,n) \end{pmatrix}, (d\rho(z) + d\mu^{+,0}(z))\phi(z) \right\rangle \end{pmatrix}, \mathbf{u}(n) \right\rangle. \end{aligned} \tag{4.25}$$

□

5 Main Theorem

In this section we will give the proof of our main theorem. Namely that a solution $\mathbf{u}(t)$ that decays fast enough at $+\infty$ at two different times t_1, t_2 (w.l.o.g $t_1 = 0$, $t_2 = 1$) must already be the zero solution (provided that the potential also decays fast enough). The strategy is roughly as follows. We will use the unitary transformation given in the last section, to transform our solution into a time dependent holomorphic function $\Phi(t)$ on a sector. With the help of section 3.2 we will be able to obtain certain bounds on the growth of this function, for the times $t = 0$ and $t = 1$. However, given the simple time evolution in the Fourier space, it will turn out that these growth bounds cannot be satisfied unless $\Phi(0) = \Phi(1) = 0$. As the Fourier transform is unitary, we can conclude that $\mathbf{u}(0) = \mathbf{u}(1) = 0$ and thus $\mathbf{u}(t) = 0$ for all t, as the time evolution is unitary.

We now formulate the desired decay properties of $\mathbf{u}(t,n) = (u^x(t,n), u^y(t,n))$ for $t \in \{0,1\}$, that will hold from now on. We require that there exists a positive constant $\delta > 0$ such that

$$|u^{x,y}(t,n)| \leq C \left(\frac{\mathrm{e}}{2n(2+\delta)} \right)^{2n} \tag{5.1}$$

for all $n \geq 0$ and $t \in \{0,1\}$ (with the convention $(1/0)^0 := 1$). The necessary decay properties of the potential will be formulated using the notation from section 3.2. Apart from $q_n^{ij} \in l_2^1(\mathbb{Z})$ we also require that there exists a sequence $c(n)$ satisfying (3.35) and (3.36) such that

$$C_+(n) \leq \frac{C_+}{n^{4(1+\epsilon)n}} \tag{5.2}$$

for some $\epsilon > 0$ and all n large enough. Note that this condition is stronger than (3.56), and so the $\mathbf{w}^+(z,n)$ can be extended to holomorphic functions on $\mathbb{C}_-$.

Lemma 5.1. *Under the above assumption the Jost solution* $\mathbf{w}^+(z,n)$ *satisfies*

$$\limsup_{|z|\to\infty} \frac{\log|\mathbf{w}^+(z,n)|}{|z|^{1/2}} \leq 0 \tag{5.3}$$

where $|\mathbf{w}^+(z,n)|$ denotes the usual euclidean norm of $\mathbf{w}^+(z,n)$.

Proof. To simplify our reasoning we will make use of the remark 2.5 and 2.9. It allows us to ignore the term

$$\frac{1}{1-z}(Q + \lambda(z)) \tag{5.4}$$

as well as $A(n+1)$, $A(n)$, z^n. This gives us

$$\limsup_{|z|\to\infty} \frac{\log|\mathbf{w}^+(z,n)|}{|z|^{1/2}} \leq \max\{\sigma_{N^x(n,z)}, \sigma_{N^y(n,z)}, \sigma_{N^x(n+1,z)}, \sigma_{N^y(n-1,z)}\}, \tag{5.5}$$

where $\sigma_{N^x(n,z)}$, ... denotes the type of the respective function. Now as $|N_j^{x,y}(n)| \leq D_{+,j}(n)^2 C_+(n + \lfloor j/2 \rfloor)$, we can estimate all j-th Taylor coefficients by $D_{+,j}(n-1)^2 C_+(n-1+\lfloor j/2 \rfloor)$ and then use formula (2.9) to obtain

$$\limsup_{|z|\to\infty} \frac{\log|\mathbf{w}^+(z,n)|}{|z|^{1/2}} \leq \frac{2}{e} \limsup_{j\to\infty} j(D_{+,j}(n-1)^2 C_+(n-1+\lfloor j/2 \rfloor))^{1/2j} \tag{5.6}$$

By assumption the limit $\lim\limits_{j\to\infty} D_{+,j}(n) = D_{+,\infty}(n) < \infty$ exists and is finite. Furthermore by remark (2.5) we can always translate the index j in the expression for the Taylor coefficients. So we obtain

$$\limsup_{|z|\to\infty} \frac{\log|\mathbf{w}^+(z,n)|}{|z|^{1/2}} \leq \frac{2}{\mathrm{e}} \limsup_{j\to\infty} jC_+(\lfloor j/2 \rfloor)^{1/2j} \tag{5.7}$$

$$\leq \frac{2}{\mathrm{e}} \limsup_{j\to\infty} j \frac{C_+^{1/2j}}{\lfloor j/2 \rfloor^{4(1+\epsilon)\lfloor j/2 \rfloor/2j}} = 0.$$

□

As a corollary we get

Corollary 5.2. *The function $b_+(z)$ satisfies*

$$\limsup_{|z|\to\infty} \frac{\log|b_+(z^{-1})|}{|z|^{1/2}} \le 0. \tag{5.8}$$

Proof. By definition we have

$$b_+(z^{-1}) = W^+(z^{-1}) \frac{\lambda(z^{-1}) + m}{z^{-1} - z} \tag{5.9}$$

and $W^+(z^{-1}) = W(\mathbf{w}^-(z^{-1}, n), \mathbf{w}^+(z, n))$ for any $n \in \mathbb{Z}$. The only term that can contribute to an exponential behavior at infinity is $\mathbf{w}^+(z, n)$. Hence the claim follows from lemma 5.1. $\square$

An analogous statement is also true for $a(z)$ an does not even require the last lemma. As will be apparent from the following arguments, we can w.l.o.g. assume that $\mathbf{u}(t, n)$ has a positive spectral measure. Then, as we have seen in the last section, the Fourier transform

$$\begin{pmatrix} \Phi^+(t, z) \\ \Phi^-(t, z) \end{pmatrix} = \sum_{n\in\mathbb{Z}} \begin{pmatrix} \langle \mathbf{w}^+(z, n), \mathbf{u}(t, n)\rangle \\ \langle \mathbf{w}^-(z, n), \mathbf{u}(t, n)\rangle \end{pmatrix} \tag{5.10}$$

completely encodes all information about $\mathbf{u}(t)$. We can now use the equations from section 3.1 to compute

$$\Phi^-(t, z) = \sum_{n\in\mathbb{Z}} \langle \mathbf{w}^-(z, n), \mathbf{u}(t, n)\rangle \tag{5.11}$$

$$= \sum_{n<0} \langle \mathbf{w}^-(z, n), \mathbf{u}(t, n)\rangle + b_+(z) \sum_{n\ge 0} \langle \mathbf{w}^+(z, n), \mathbf{u}(t, n)\rangle + a(z) \sum_{n\ge 0} \langle \mathbf{w}^+(z^{-1}, n), \mathbf{u}(t, n)\rangle$$

$$= A_1(t, z) + b_+(z) A_2(t, z) + a(z) B(t, z^{-1}).$$

By remark 3.7, we have on compact subsets $K \subset \mathbb{D} \setminus [0, 1)$, that $\|\mathbf{w}^-(z, n)\| < M|z|^{-n}$, for $n < 0$ and some $M > 0$ depending on K. We conclude (as $\mathbf{u}(t, n)$ is in $\mathbf{l}^2(\mathbb{Z})$), that $A_1(t, z)$ converges uniformly on K and thus defines a holomorphic function on $\mathbb{D} \setminus [0, 1)$. The same holds for $A_2(t, z)$ and by the last corollary we conclude that $A(t, z) := A_1(t, z) + b_+(z) A_2(t, z)$ is holomorphic on $\mathbb{D} \setminus [0, 1)$ with

$$\limsup_{|z|\to\infty} \frac{\log|A(t, z^{-1})|}{|z|^{1/2}} \le 0. \tag{5.12}$$

Now it remains to study

$$B(t, z) = \sum_{n\ge 0} \langle \mathbf{w}^+(z, n), \mathbf{u}(t, n)\rangle. \tag{5.13}$$

We would like to show that

$$\limsup_{|z|\to\infty} \frac{\log|B(t, z)|}{|z|^{1/2}} \le \frac{1}{2+\delta}, \qquad t \in \{0, 1\}. \tag{5.14}$$

Again by remark (2.9) it is enough to show this for $B'(t,z)$, with

$$B'(t,z) := \sum_{n\geq 0} \max\{|N^x(n,z)|, |N^y(n,z)|\}|z|^n v(t,n), \tag{5.15}$$

where $v(t,n) \geq 0$ satisfies the same decay property as $\mathbf{u}(t,n)$ for $t \in \{0,1\}$. We can estimate the above expression

$$B'(t,z) \leq B_1'(t,z) + B_2'(t,z) \tag{5.16}$$

with

$$B_1'(t,z) := \sum_{n\geq 0} v(t,n)|z|^n \tag{5.17}$$

and

$$B_2'(t,z) := \sum_{n\geq 0}\sum_{j\geq 1} \max\{|N_j^x(n)|, |N_j^y(n)|\}|z|^{n+j} v(t,n) \tag{5.18}$$

(here we used that $N_0^y(n) = 1$ for all $n \in \mathbb{Z}$). Let us first consider $B_1'(t,z)$. By assumption we have that

$$v(t,n) \leq C \left(\frac{\mathrm{e}}{2n(2+\delta)}\right)^{2n}. \tag{5.19}$$

Thus using formula (2.9) relating the coefficients of a power series with the growth of the function, we conclude

$$\limsup_{|z|\to\infty} \frac{\log(B_1'(t,z))}{|z|^{1/2}} \leq \frac{1}{2+\delta}. \tag{5.20}$$

For $B_2'(t,z)$ we use the estimate (3.41) and obtain

$$B_2'(t,z) \leq \sum_{n\geq 0}\sum_{j\geq 1} D_{+,j}(n)^2 C_+(n + \lfloor j/2 \rfloor)|z|^{n+j} v(t,n) = \sum_{n\geq 0} b(t,n)|z|^n \tag{5.21}$$

with $b(t,n) := \sum_{m=0}^{n-1} D_{+,n-m}(m)^2 C_+(m + \lfloor (n-m)/2 \rfloor) v(t,m)$. Using estimates (5.2) and $D_{+,n-m}(m)^2 \leq D_{+,\infty}(0)^2 < \infty$ we compute

$$D_{+,n-m}(m)^2 C_+(m + \lfloor (n-m)/2 \rfloor) \leq \frac{C'}{(m+\lfloor (n-m)/2 \rfloor)^{4(1+\epsilon)(m+\lfloor (n-m)/2 \rfloor)}}, \tag{5.22}$$

which implies (for $n > 2$)

$$|b(t,n)| \leq \sum_{m=0}^{n-1} \frac{C'}{(m+\lfloor (n-m)/2 \rfloor)^{4(1+\epsilon)(m+\lfloor (n-m)/2 \rfloor)}} \left(\frac{\mathrm{e}}{2m(2+\delta)}\right)^{2m} \leq \tag{5.23}$$

$$\frac{C'}{(n/2-1)^{4(1+\epsilon)(n/2-1)}} \sum_{m=0}^{n-1} \left(\frac{\mathrm{e}}{2m(2+\delta)}\right)^{2m} \leq \frac{C''}{(n/2-1)^{4(1+\epsilon)(n/2-1)}}. \tag{5.24}$$

This implies with (2.9) that

$$\limsup_{|z|\to\infty} \frac{\log(B_2'(t,z))}{|z|^{1/2}} \leq 0. \tag{5.25}$$

Thus we can conclude

$$\limsup_{|z|\to\infty} \frac{\log(\Phi^-(t,z^{-1}))}{|z|^{1/2}} \le \frac{1}{2+\delta}, \qquad t \in \{0,1\}. \tag{5.26}$$

Next we use the time evolution in Fourier space. Note that we have

$$\Phi^-(t,z^{-1}) = e^{-i\lambda(z)t}\Phi^-(0,z^{-1}). \tag{5.27}$$

We will denote the indicator function with respect to the order $1/2$ of $\Phi^-(0,z^{-1})$ and $\Phi^-(1,z^{-1})$ as $h_0(\theta)$ and $h_1(\theta)$ respectively (here use the generalized sector $V(0,2\pi;1)$. From $\Phi^-(1,z^{-1}) = \mathrm{e}^{-\mathrm{i}\lambda(z)}\Phi^-(0,z^{-1})$ we obtain the following relation between $h_0(\theta)$ and $h_1(\theta)$:

$$h_1(\theta) = \limsup_{r\to\infty} \frac{\log|\Phi^-(1,r^{-1}\mathrm{e}^{-\mathrm{i}\theta})|}{r^{1/2}} = \limsup_{r\to\infty} \frac{\log|\mathrm{e}^{-\mathrm{i}\lambda(e^{\mathrm{i}\theta})}\Phi^-(0,r^{-1}\mathrm{e}^{-\mathrm{i}\theta})|}{r^{1/2}} \tag{5.28}$$

$$= \limsup_{r\to\infty} \frac{\log|\Phi^-(0,r^{-1}\mathrm{e}^{-\mathrm{i}\theta}|)}{r^{1/2}} - \cos(\theta/2) = h_0(\theta) - \cos(\theta/2).$$

This holds because

$$\lim_{r\to\infty} \frac{\log|\mathrm{e}^{-\mathrm{i}\lambda(r\mathrm{e}^{\mathrm{i}\theta})}|}{r^{1/2}} = \lim_{r\to\infty} \frac{\Im(\lambda(r\mathrm{e}^{\mathrm{i}\theta}))}{r^{1/2}} = -\cos(\theta/2) \tag{5.29}$$

as we have $\lambda(z) = \sqrt{-z} + o(1) = |z|^{1/2}\mathrm{e}^{(\theta-\pi)\mathrm{i}/2} + o(1)$ for large $|z|$. Now consider the fundamental relation (2.25) with $\theta = \pi$, $\theta_1 = \alpha$ small, $\theta_2 = 2\pi - \alpha$

$$-h_i(\alpha) + h_i(\pi)\frac{\sin((\pi-\alpha)}{\sin((\pi-\alpha)/2)} \le h_i(2\pi-\alpha) \tag{5.30}$$

and $i = 1, 2$. First, let us assume that $h_1(\pi) > -\infty$ and $h_2(\pi) > -\infty$. Then we see that for each $\epsilon' > 0$ there is an $\beta > 0$, such that for all positive $\alpha < \beta$ we get

$$-h_i(\alpha) - \epsilon' \le h_i(2\pi-\alpha). \tag{5.31}$$

In particular because of (5.26), we obtain

$$-\frac{1}{2+\delta} - \epsilon' \le h_i(2\pi-\alpha) \le \frac{1}{2+\delta}. \tag{5.32}$$

This implies that

$$|h_1(2\pi-\alpha) - h_0(2\pi-\alpha)| = |\cos(\pi-\alpha/2)| \le 2\frac{1}{2+\delta} + \epsilon'. \tag{5.33}$$

By choosing ϵ' and then α small enough we get the desired contradiction. Thus we can conclude w.l.o.g that $h_0(\pi) = -\infty$. We will now proceed to show that this already implies that $\Phi^-(0,z^{-1}) \equiv 0$. First we can rewrite $\Phi^-(0,z^{-1})$ as

$$\Phi^-(0,z^{-1}) = f(z) + \lambda(z)g(z) \tag{5.34}$$

where $f(z)$ and $g(z)$ are holomorphic functions on $\mathbb{C} \setminus \overline{\mathbb{D}}$. Recall that $\lambda(z) = \sqrt{m^2+2-z-z^{-1}}$, and that $m^2+2-z-z^{-1} < 0$ on (o,∞) with $o = (m^2 + 2 + \sqrt{(m^2+2)^2-4})/2$. We now define a new sector D with origin at o and an opening $(-\pi/2,\pi/2)$. Note that $2+m^2-z-z^{-1}$ is never positive or 0 on D. Thus we can define using a square root with a branch cut on the positive real axis a new $\tilde{\lambda}(z)$, with $\tilde{\lambda}(z)^2 = m^2 + 2 - z - z^{-1}$, such that $\lambda(z) = \tilde{\lambda}(z)$ on D_-, where $D_- := \{z \in D : \Re(z) < 0\}$ (D_+ is defined analogously). In particular, we can define

$$\tilde{\Phi}^-(z^{-1}) := f(z) + \tilde{\lambda}(z)g(z) \tag{5.35}$$

which again agrees with $\Phi^-(0, z^{-1})$ on D_-. Now by proposition (2.16) we can conclude that the indicator function $\tilde{h}(\theta)$ of $\tilde{\Phi}^-(z^{-1})$ with respect to $1/2$ is equal to $-\infty$ on $(-\pi/2, \pi/2)$. So we can conclude that also the indicator function of $\Phi(0, z) - \tilde{\Phi}(z)$ is equal to $-\infty$ in the sector D_+. However note that $\tilde{\lambda}(z) = -\lambda(z)$ on D_+. This implies that

$$\Phi(0, z^{-1}) - \tilde{\Phi}(z^{-1}) = 2\lambda(z)g(z) \tag{5.36}$$

on D_+. As $\lambda(z)$ can be ignored when calculating the indicator function, as it does not have exponential growth or decay, we can conclude that $g(z)$ has an indicator function $h_g(\theta)$ with respect to the order $1/2$ which is equal to $-\infty$ on $(0, \pi/2)$. As $g(z)$ is holomorphic on $\mathbb{C} \setminus \overline{\mathbb{D}}$, we even get $h_g(\theta) = -\infty$ on $[0, 2\pi]$. By lemma (2.17) this implies that $g(z) \equiv 0$. Now similarly we can conclude that $f(z) \equiv 0$ and so $\Phi^-(0, z) \equiv 0$ for $z \in \mathbb{D} \setminus [0, 1)$. We know just need to show that $\Phi^-(0, z) \equiv 0$ on $\mathbb{T}_+$. To this end, let us define $\mathbf{y}(z, n) := \mathbf{w}^-(z, n) - \mathbf{w}_0^-(z, n)$. Then for any $z \in \mathbb{D} \setminus [0, 1)$ we have

$$\sum_{n<0} \langle \mathbf{w}^-(z, n), \mathbf{u}(t, n)\rangle = \sum_{n<0} \langle \mathbf{w}_0^-(z, n), \mathbf{u}(t, n)\rangle + \langle \mathbf{y}(z, n), \mathbf{u}(t, n)\rangle. \tag{5.37}$$

Note that it follows from $q_n^{ij} \in l_2^1(\mathbb{Z})$, that $c(n)$ can be chosen such that $\sum_{k<n} C_-(k) \in l^1(\mathbb{Z}_-)$. In particular $\sum_{k<n} C_-(k) \in l^2(\mathbb{Z}_-)$, and so it follows from (3.52) that $\mathbf{y}(z, n) \in \mathbf{l}^2(\mathbb{Z}_-)$. Thus,

$$\sum_{n<0} \langle \mathbf{y}(z, n), \mathbf{u}(t, n)\rangle \tag{5.38}$$

evaluates always to a finite number (as it can be regarded as an ordinary inner product between elements in our Hilbert space). By dominated convergence we even have that for $\theta \in (0, 2\pi)$, $\lim_{r\to 1} \mathbf{y}(r\mathrm{e}^{\mathrm{i}\theta}, n) \to \mathbf{y}(\mathrm{e}^{\mathrm{i}\theta}, n)$ in $\mathbf{l}^2(\mathbb{Z}_-)$ and by continuity of the scalar product

$$\lim_{r\to 1+} \sum_{n<0} \langle \mathbf{y}(r\mathrm{e}^{\mathrm{i}\theta}, n), \mathbf{u}(t, n)\rangle \to \sum_{n<0} \langle \mathbf{y}(\mathrm{e}^{\mathrm{i}\theta}, n), \mathbf{u}(t, n)\rangle. \tag{5.39}$$

For the other term in (5.37) we can explicitly compute

$$\sum_{n<0} \langle \mathbf{w}_0^-(z, n), \mathbf{u}(t, n)\rangle = \sum_{n<0} u_x(t, n)\overline{z}^{-n} + \overline{\alpha_-(z)} \sum_{n<0} u_x(t, n)\overline{z}^{-n}. \tag{5.40}$$

Now because $\alpha_-(z)$ is bounded on $\mathbb{T}$, the theory of Fourier series guarantees us that

$$\lim_{r\to 1+} \sum_{n<0} \langle \mathbf{w}_0^-(r\mathrm{e}^{\mathrm{i}\theta}, n), \mathbf{u}(t, n)\rangle \to \sum_{n<0} \langle \mathbf{w}_0^-(\mathrm{e}^{\mathrm{i}\theta}, n), \mathbf{u}(t, n)\rangle \tag{5.41}$$

in $L^2(\mathbb{T}, d\theta)$. In particular, we can find a sequence $r_k \to 1+$, such that the above sum converges pointwise (Lebesgue)-almost everywhere. This allow us to conclude that $\sum_{n<0} \langle \mathbf{w}^-(r_k\mathrm{e}^{\mathrm{i}\theta}, n), \mathbf{u}(t, n)\rangle$ must converge almost everywhere. All the other terms in (5.11) are continuous on $\overline{\mathbb{D}} \setminus [0, 1] \cup \{-1\}$ because $\mathbf{u}(t, n) \in \mathbf{l}^1(\mathbb{N})$. Therefore, we can conclude that $\Phi^-(t, z) \equiv 0$ and by (3.15) that $\Phi^+(t, z) \equiv 0$ on $\overline{\mathbb{D}} \setminus [0, 1] \cup \{-1\}$. This implies, as U is unitary, that $\mathbf{u}(t, n) \equiv 0$.

6 Appendix

6.1 Induction hypothesis

We now give the complete induction base for theorem (3.5). As (3.41) does not hold for $j = 0$, we will have to treat all cases which have a term $N_0^y(n) = 1$ in their recursive definition (3.33) or (3.34) separately. These are $N_1^x(n)$, $N_1^y(n)$, $N_2^x(n)$ and $N_2^y(n)$. The case $N_0^x(n)$ is trivial as it is equal to zero for all $n \in \mathbb{Z}$.

$$|N_1^x(n)| \leq \sum_{k=n+1}^{\infty} |Q_0^x(k)| \leq \sum_{k=n+1}^{\infty} c(k)|N_0^y(k)| \tag{6.1}$$

$$= \sum_{k=n+1}^{\infty} c(k) = C_+(n+1) \leq (1 + C_+(n+1))C_+(n + \lfloor 1/2 \rfloor)$$

Note that $|N_1^x(n)| \leq C_+(n+1)$, which will be used below.

$$|N_1^y(n)| \leq \sum_{k=n+1}^{\infty} |Q_0^y(k)| \leq \sum_{k=n+1}^{\infty} c(k) \max\{|N_0^y(k), N_1^x(k-1)\} \tag{6.2}$$

$$\leq \sum_{k=n+1}^{\infty} c(k)(1 + C_+(n+1)) = C_+(n+1)(1 + C_+(n+1))$$

$$\leq D_{+,1}(n)^2 C_+(n + \lfloor 1/2 \rfloor)$$

Note $|N_1^y(n)|, |N_1^x(n)| \leq C_+(n+1)(1 + C_+(n+1))$ which again will be used below.

$$|N_2^x(n)| \leq 0 + \sum_{k=n+1}^{\infty} |Q_1^x(k)| \tag{6.3}$$

$$\leq \sum_{k=n+1}^{\infty} c(k) \max\{|N_1^x(k)|, |N_1^y(k)|, \overbrace{|N_0^y(k+1)|}^{=1}\}$$

$$\leq \sum_{k=n+1}^{\infty} c(k)(1 + C_+(n+2)(1 + C_+(n+2))$$

$$\leq C_+(n+1)(1 + C_+(n+2)(1 + C_+(n+2))$$

$$\leq C_+(n + \lfloor 2/2 \rfloor)(1 + C_+(n+1))D_1(n+1)^2$$

Because of $|N_2^x(n)| \le C_+(n+1)(1+C_+(n+2))(1+C_+(n+2))$ and $C_+(n+1)(1+C_+(n+1)) \le C_+(n)(1+C_+(n+1)(1+C_+(n+1)))$ we get

$$|N_1^y(n)|, |N_1^x(n)|, |N_2^x(n-1)| \le C_+(n)(1+C_+(n+1)(1+C_+(n+1))) \tag{6.4}$$

which finally gives us

$$|N_2^y(n)| \le \sum_{k=n+1}^{\infty} |Q_1^y(k)| \tag{6.5}$$

$$\le \sum_{k=n+1}^{\infty} c(k) \max\{\overbrace{|N_0^y(k+1)|}^{=1}, |N_1^y(k)|, |N_1^x(k)|, |N_2^x(k-1)|\}$$

$$\le \sum_{k=n+1}^{\infty} c(k)(1+C_+(n)(1+C_+(n+1)(1+C_+(n+1))))$$

$$\le C_+(n+1)(1+C_+(n)(1+C_+(n+1)(1+C_+(n+1))))$$

$$\le C_+(n+1)(1+C_+(n+1)(1+C_+(n+2))^2)$$

$$\le C_+(n+\lfloor 2/2 \rfloor) D_{+,2}(n)^2$$

6.2 Spectral measure

We now give a derivation of the explicit form of the spectral measure $d\rho(z)$ (the result for $d\tilde{\rho}(z)$ is analogous). To this end we define

$$\varphi_{N,k}^+(z,n) := \frac{\chi_{[k,k+N-1]}(n)}{\sqrt{N(1+|\alpha_+(z)|^2)}} \begin{pmatrix} 1 \\ \alpha_+(z) \end{pmatrix} z^n \tag{6.6}$$

and

$$\varphi_{N,k}^-(z,n) := \frac{\chi_{[-k-N+1,-k]}(n)}{\sqrt{N(1+|\alpha_-(z)|^2)}} \begin{pmatrix} 1 \\ \alpha_-(z) \end{pmatrix} z^{-n}. \tag{6.7}$$

We obtain from the expressions above Weyl sequences when we set $k = k_N \ge 0$ and let N go to plus infinity in (6.6) and to minus infinity in (6.7). Next we will compute the unitary transforms $(U\varphi_{N,k}^{\pm}(z))(\xi)$. The reader should keep in mind that we will treat $(U\varphi_{N,k}^{\pm}(z))(\xi)$ as a sequence of distributions on $\mathbb{T}_+$.

$$\sum_{n\in\mathbb{Z}} \begin{pmatrix} \langle \mathbf{w}^+(\xi,n), \varphi_{N,k}^+(z,n)\rangle \\ \langle \mathbf{w}^-(\xi,n), \varphi_{N,k}^+(z,n)\rangle \end{pmatrix} \tag{6.8}$$

$$= \sum_{n\in\mathbb{Z}} \begin{pmatrix} \langle \mathbf{w}^+(\xi,n), \varphi_{N,k}^+(z,n)\rangle \\ \langle a_+(\xi)\mathbf{w}^+(\overline{\xi},n) + b_+(\xi)\mathbf{w}^+(\xi,n)\varphi_{N,k}^+(z,n)\rangle \end{pmatrix}$$

$$= \frac{1+\overline{\alpha_+(\xi)}\alpha_+(z)}{\sqrt{N(1+|\alpha_+(z)|^2)}} \begin{pmatrix} \sum_{n=k}^{k+N-1} (\overline{\xi} z)^n \\ \overline{b_+(\xi)} \sum_{n=k}^{k+N-1} (\overline{\xi} z)^n \end{pmatrix}$$

$$+\frac{1+\overline{\alpha_+(\bar\xi)}\alpha_+(z)}{\sqrt{N(1+|\alpha_+(z)|^2)}}\begin{pmatrix}0\\ \overline{a_+(\bar\xi)\sum_{n=k}^{k+N-1}(\xi z)^n}\end{pmatrix}+er(\xi,z,N,k)$$

$$=\frac{1+\overline{\alpha_+(\xi)}\alpha_+(z)}{\sqrt{1+|\alpha_+(z)|^2}}\begin{pmatrix}(\bar\xi z)^k E_N(\gamma-\theta)\\ \overline{b_+(\xi)}(\bar\xi z)^k E_N(\gamma-\theta)\end{pmatrix}$$

$$+\frac{1+\overline{\alpha_+(\bar\xi)}\alpha_+(z)}{\sqrt{1+|\alpha_+(z)|^2}}\begin{pmatrix}0\\ \overline{a_+(\bar\xi)}(\xi z)^k E_N(\gamma+\theta)\end{pmatrix}+er(\xi,z,N,k)$$

where

$$E_N(\gamma-\theta):=1/\sqrt{N}\sum_{n=0}^{N-1}\mathrm{e}^{\mathrm{i}n(\gamma-\theta)}=\frac{\mathrm{e}^{\mathrm{i}(N-1)(\gamma-\theta)/2}}{\sqrt{N}}\frac{\sin(N(\gamma-\theta)/2)}{\sin((\gamma-\theta)/2)} \tag{6.9}$$

is a variation of the Dirichlet kernel with $\mathrm{e}^{\mathrm{i}\gamma}=z$ and $\mathrm{e}^{\mathrm{i}\theta}=\xi$ and the error term $er(\xi,z,N,k)$ is defined implicitly. Note that the error term stems from using

$$\sqrt{N(1+|\alpha_+(\xi)|^2)}\varphi^+_{N,k}(\xi,n)=\mathbf{w}_0^+(\xi,n)\chi_{[k,k+N-1]}(n) \tag{6.10}$$

instead of $\mathbf{w}^+(\xi,n)\chi_{[k,k+N-1]}(n)$. From remark 3.7 we know that on $\mathbb{T}_+$ the Jost solution $\mathbf{w}^+(\xi,n)$ converges locally uniformly to the free solution $\mathbf{w}_0^+(\xi,n)$ as n goes to plus infinity. With this in mind we conclude that there exists a sequence k_N such that

$$\langle\mathbf{w}^+(\xi,n)-\sqrt{N(1+|\alpha_+(\xi)|^2)}\varphi^+_{N,k_N}(\xi,n),\varphi^+_{N,k_N}(z,n)\rangle_{\mathrm{l}^2(\mathbb{Z})}\to 0 \tag{6.11}$$

as $N\to+\infty$ locally uniformly for all $\xi,z\in\mathbb{T}_+$. Thus, for this choice of k_N, and because $a_+(\xi)$ and $b_+(\xi)$ are locally bounded, the error term goes locally uniformly to 0 on $\mathbb{T}_+$ and will be from now on left out (because we have a Weyl sequence, we can for our purposes restrict everything to some compact subset of $\mathbb{T}_+$ containing z). We have that

$$\lim_{N\to\infty}|(\bar\xi z)^{k_N}E_N(\gamma-\theta)|^2=\lim_{N\to\infty}|E_N(\gamma-\theta)|^2\to 2\pi\delta(\gamma-\theta) \tag{6.12}$$

as a distribution (with respect to the Lebesgue measure). Note that similarly $\lim\limits_{N\to\infty}|E_N(\gamma+\theta)|^2\to 2\pi\delta(\gamma+\theta)$ (in particular, it goes uniformly to 0 on $\mathbb{T}_+$) and thus can be ignored when integrating over compact subsets of $\mathbb{T}_+$. Analogously to before we also get

$$\sum_{n\in\mathbb{Z}}\begin{pmatrix}\langle\mathbf{w}^+(\xi,n),\varphi^-_{N,k_N}(z,n)\rangle\\ \langle\mathbf{w}^-(\xi,n),\varphi^-_{N,k_N}(z,n)\rangle\end{pmatrix}\cong\frac{1+\overline{\alpha_-(\xi)}\alpha_-(z)}{\sqrt{1+|\alpha_-(z)|^2}}\begin{pmatrix}\overline{b_-(\xi)}(\bar\xi z)^k E_N(\gamma-\theta)\\ (\bar\xi z)^k E_N(\gamma-\theta)\end{pmatrix} \tag{6.13}$$

in the distributional sense an N goes to minus infinity. As $|E_N(\gamma-\theta)|^2$ converges to a delta distribution, we can set in the prefactor $\xi=z$, which results in a prefactor $\sqrt{1+|\alpha(z)|^2}$ (we have dropped the $\pm$ subscript as $|\alpha_+(z)|=|\alpha_-(z)|$). By formula (4.11), we see that the entries in $d\rho$ restricted to compact subsets of $\mathbb{T}_+$ are finite complex measures which are absolutely continuous with respect to the Lebesgue measure (as the $d\sigma_j$ are absolutely continuous). By the Radon-Nikodym theorem there must exist a locally integrable matrix-valued function $\tau(z)$ such that

$$d\rho=\begin{pmatrix}\tau^{++}(z) & \tau^{+-}(z)\\ \tau^{-+}(z) & \tau^{--}(z)\end{pmatrix}d\theta \tag{6.14}$$

where $d\theta$ is the Lebesgue measure on $\mathbb{T}_+$. Now a standard result from the theory of distributions tells us that by convolution with an approximate identity, we get convergence in L^1. For functions which are just locally integrable we get a similar result, if we restrict the domain of integration to compact sets. This is justified by the fact that we have a Weyl sequence. We can then always extract a subsequence which converges pointwise almost everywhere. Using that U is unitary, we will be able to derive pointwise equations for the matrix-coefficients of $\tau(z)$ which must hold almost everywhere (with respect to the Lebesgue measure). Recall that

$$\widehat{\varphi^+_{N,k_N}(z)}(\xi) \cong \sqrt{1+|\alpha(z)|^2}\begin{pmatrix} (\bar{\xi}z)^{k_N}E_N(\gamma-\theta) \\ \overline{b_+(\xi)}(\bar{\xi}z)^{k_N}E_N(\gamma-\theta) \end{pmatrix} \tag{6.15}$$

and

$$\widehat{\varphi^-_{N,k_N}(z)}(\xi) \cong \sqrt{1+|\alpha(z)|^2}\begin{pmatrix} \overline{b_-(\xi)}(\bar{\xi}z)^{k_N}E_N(\gamma-\theta) \\ (\bar{\xi}z)^{k_N}E_N(\gamma-\theta) \end{pmatrix}. \tag{6.16}$$

Evaluating the inner product in Fourier-space gives us (for $N \to \infty$ and using the properties of E_N)

$$\lim_{N\to\infty}\int_{\mathbb{T}_+}\langle \widehat{\varphi^+_{N,k_N}(z)}(\xi), d\rho(\xi)\widehat{\varphi^+_{N,k_N}(z)}(\xi)\rangle \tag{6.17}$$

$$= 2\pi(1+|\alpha(z)|^2)(\tau^{++}(z)+|b_+(z)^2|\tau^{--}(z)+b_+(z)\tau^{-+}(z)+\overline{b_+(z)}\tau^{+-}(z)) = 1$$

and

$$\lim_{N\to\infty}\int_{\mathbb{T}_+}\langle \widehat{\varphi^-_{N,k_N}(z)}(\xi), d\rho(\xi)\widehat{\varphi^-_{N,k_N}(z)}(\xi)\rangle \tag{6.18}$$

$$= 2\pi(1+|\alpha(z)|^2)(\tau^{--}(z)+|b_-(z)^2|\tau^{++}(z)+b_-(z)\tau^{+-}(z)+\overline{b_-(z)}\tau^{-+}(z)) = 1.$$

We also get

$$b_-(z)\tau^{++}(z)+\overline{b_+(z)}\tau^{--}(z)+b_-(z)\overline{b_+(z)}\tau^{+-}(z)+\tau^{-+}(z) = 0 \tag{6.19}$$

and

$$\overline{b_-(z)}\tau^{++}(z)+b_+(z)\tau^{--}(z)+\overline{b_-(z)}b_+(z)\tau^{-+}(z)+\tau^{+-}(z) = 0 \tag{6.20}$$

by evaluating the scalar product of $\widehat{\varphi^+_{N,k_N}(z)}(\xi)$ with $\widehat{\varphi^-_{N,k_N}(z)}(\xi)$ (which must asymptotically vanish) in both orders (note that (6.20) is just the complex conjugate of (6.19)). Next notice that because $\overline{\mathbf{w}^\pm(z,n)} = \mathbf{w}^\pm(\bar{z},n)$, we can conclude that $b_-(z) = -\overline{b_+(z)}$ (for $|z|=1$), and thus we can eliminate the term $b_-(z)$ in all four equations (6.17)-(6.20). Written in the matrix notation this gives a linear system with matrix

$$\begin{pmatrix} 2\pi(1+|\alpha(z)|^2) & 2\pi(1+|\alpha(z)|^2)|b_+(z)|^2 & b_+(z) & \overline{b_+(z)} \\ 2\pi(1+|\alpha(z)|^2)|b_+(z)|^2 & 2\pi(1+|\alpha(z)|^2) & -b_+(z) & -\overline{b_+(z)} \\ -\overline{b_+(z)} & \overline{b_+(z)} & 1 & -\overline{b_+(z)}^2 \\ -b_+(z) & b_+(z) & -b_+(z)^2 & 1 \end{pmatrix}. \tag{6.21}$$

The determinant of the above matrix can be calculated to be equal to $2\pi(1+|\alpha(z)|^2)(1+|b_+|^2)(2\pi(1+|\alpha(z)|^2)(1+|b_+(z)|^2)^2+4|b_+(z)|^2))$ which is always positiv and thus non-zero. So there must be a unique solution which can be easily found to be

$$\tau^{++}(z) = \tau^{--}(z) = \frac{1}{2\pi(1+|\alpha(z)|^2)(1+|b_+(z)|^2)} = \frac{1}{2\pi(1+|\alpha(z)|^2)|a(z)|^2} \tag{6.22}$$

$$\tau^{+-} = \tau^{-+} = 0 \tag{6.23}$$

where we have used the formula $1+|b_+(z)|^2 = |a(z)|^2$ from remark 3.4.

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