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„Strong limit theorems for non-integrable random variables"

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Adam Novotta, BSc

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Contents

Abstract

Die vorliegende Masterarbeit behandelt einige Grenzwertsätze über fast sichere Konvergenz beziehungsweise Divergenz von Summen unabhängiger und identisch verteilter Zufallsvariablen mit schweren „tails". Insbesondere werden Zufallsvariablen betrachtet, bei denen der Erwartungswert unendlich beziehungsweise nicht definiert ist und daher eine Voraussetzung für das starke Gesetz der großen Zahlen nicht erfüllt ist. Konkret werden Integralkriterien für das asymptotische Verhalten von Quotienten solcher Summen besprochen.

1 Introduction

In this master thesis we will present some theorems about limit theorems of random variables, where the expected value of the random variables is not defined or not integrable. That means we will consider sequences $(X_n)_{n\geq 1}$ of i.i.d. random variables, where the requirement of strong law of large numbers is not satisfied. At first we will discuss the strong law large numbers and its history. The idea of probability defined as limiting frequency can be found in one early work of Cardano(1501-1576). But this idea was published after the death of Cardano in the middle of the 17th century. An interesting fact is, that basics of combinatorics and games of chance go back to antiquity and the ideas like Cardano's came so late.

The first remarkable theorem was published in the beginning of the 18th century. Bernoulli has the first version of the limiting-frequency statement. If S_n is the number of successes observed in n independent trials with success probability p, then

$$\frac{S_n}{n} \to p \text{ in probability for } n \to \infty. \tag{1.1}$$

It took almost two hundred years, that we get the first theorem with the almost sure convergence theorem. Emil Borel has nearly the complete proof of the strong law of large numbers for the independent trials with constant probability of success p. After that, Cantelli extended the theorem to general distribution with bounded fourth moments: If $(X_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with $\mathbb{E}[|X_n - \mathbb{E}[X_n]|^4] < \infty$, then

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}[X_i]) \to 0 \text{ a.s. for n} \to \infty. \tag{1.2}$$

In the beginning of the 20th century concepts for the law of large numbers were developed and specified. Therefore in the of the 1920s was necessary to distinguish the two theorems. Convergence *in probability* as in (1.1) has the name *weak law of large numbers* and *almost sure* convergence as in (1.2) is under the name *strong law of large numbers*. At first we will see the development of the weak version.

Khinchin gives two proofs of the weak law of large numbers. If $(X_n)_{n\geq 1}$

is a sequence of i.i.d. random variables with mean μ, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \to \mu \text{ in probability.}$$

After that, Kolgomorov refined this theory: If $(X_n)_{n\geq 1}$ is a sequence of i.i.d. random variables, then there exist constants $(c_n)_{n\geq 1}$ with

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i - c_n \to 0 \text{ in probability.}$$

Kolgomorov has two version of the strong law of large numbers. He presented in a publication the following strong law of large numbers: If $(X_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with mean 0 and variances σ^2 and

$$\sum_{i=1}^{\infty} \frac{\sigma_i^2}{i^2} < \infty,$$

then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \to 0 \text{ a.s.}$$

Of this theorem the converse is also given: if $\sum_{i=1}^{\infty} \frac{\sigma^2}{i^2} = \infty$, then exist a sequence of i.i.d. random variables with mean 0 and variance σ^2, and the convergence fails

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \to \infty \text{ a.s.}$$

The modern era of the probability theory begins with Kolgomorov's book [7]. Kolgomorov presents in his book without the proof: If $(X_n)_{n\geq 1}$ is a sequence of i.i.d. random variables then:

$$\mathbb{E}[|X|] < \infty \text{ and } \mathbb{E}[X] = \mu \Leftrightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \to \mu \text{ a.s.}$$

His idea was to prove the previous statement: For the direct half uses the truncation of the random variables and the Kronecker's lemma to reduce to the first strong law theorem of Kolgomorov. For converse half uses the statement:

$$\frac{X_n}{n} = \frac{S_n}{n} - \frac{n-1}{n}\frac{S_{n-1}}{n-1} \to 0 \text{ a.s.}$$

to get $\mathbb{P}[|X_n| > n\ i.o.] = 0$. By means of Borel-Cantelli lemma he obtained:

$$\sum_{i=1}^{\infty} \mathbb{P}[|X_n| > n] < \infty$$

This implies $\mathbb{E}[|X_1|] < \infty$.

It was long believed that the strong law of large numbers theorem of Kolgomorov is not improvable. But in 1972 Etemadi [5] obtained a new verison of this theorem. He weakened the independence to pairwise independence.

At the same time was developed the generalizations of Kolgomorov's strong law theorems. The first generalization was the Birkhoff's pointwise ergodic theorem. The strong law of large number is a special case of the ergodic theorem. Birkhoff consider stationary sequences of random variables. There is a another main setting, where we obtain the strong law of large numbers as special case of the martingale convergence theorem.

The strong law of large numbers is one of the most important theorem of the probability theory. It is useful if we consider for example a simple random walk. If $(X_n)_{n\geq 1}$ is a sequence of i.i.d. random variable with

$$\mathbb{P}[X_n = 1] = \frac{1}{2} \text{ and } \mathbb{P}[X_n = -1] = \frac{1}{2}\ \forall n \geq 1$$

Since the $\mathbb{E}[|X_n|]$ is finite, we can use the strong law of large numbers for the random walk $S_n = X_1 + \cdots + X_n$:

$$\lim_{n\to\infty} \frac{1}{n} S_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathbb{E}[X_1] = 0 \text{ a.s.}$$

But if we consider the recurrence time to m:

$$T(m) = \inf\{n \in \mathbb{N} : S_n = m\}$$

Let

$$T(m)_0 = 0,\ T(m)_1 := T(m)^1 = \inf\{n \in \mathbb{N} : S_n = m\}$$
$$T(m)^n := inf\{n \geq 1 : S_{n+T(m)_n} = m\}, T(m)_n = T(m)_{n-1} + T(m)^n$$

For the sequence $(T(m)_n)_{n\geq 0}$ we cannot use the strong law of large numbers, because the

$$\mathbb{E}[T(m)] = \infty.$$

We can show that by means of the generating function. There is another more intresting example.

Example 1.1. *If $(X_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with the following distribution:*

$$\mathbb{P}[X = k] = \mathbb{P}[X^+ = k] := \frac{\kappa_+}{k^2} \textit{ for } k \geq 1 \textit{ and}$$
$$\mathbb{P}[X = -k] = \mathbb{P}[X^- = k] := \frac{\kappa_-}{k^2 \log\log k} \textit{ for } k \geq 3$$

where κ_+, $\kappa_- > 0$ such that:

$$\sum_{k=1}^{\infty} \left(\frac{\kappa_+}{k^2} + \frac{\kappa_-}{k^2 \log\log k} \right) = 1.$$

We consider the random walk with jumps X_n:

$$S_0 := 0 \textit{ and } S_n = X_1 + \cdots + X_n.$$

By means of the theorems of the next chapter we well see that:

$$\liminf_{n\to\infty} \frac{S_n}{n} = -\infty \textit{ a.s. and } \limsup_{n\to\infty} \frac{S_n}{n} = \infty \textit{ a.s.}. \tag{1.3}$$

This is remarkable because the tail of X^+ is truly stronger then the tail of X^-

$$\frac{\mathbb{P}[X^- \geq t]}{\mathbb{P}[X^+ \geq t]} \to 0 \textit{ for } t \to \infty. \tag{1.4}$$

The statement (1.3) will be proved in the end of the section 3.

2 Two limit theorems for random variables with heavy tails

Most of the material of this section is taken from a publication of Willian Feller [6]. Let $(X_k)_{k\geq 1}$ be a sequence of independent random variables and $(a_k)_{k\geq 1}$ a positive monotonic increasing sequence. As usual we define the sum of random variables

$$S_n := \sum_{k=1}^{n} X_k,\ n \geq 1.$$

We are interested in the probability of the following event $\mathcal{L}$:

$$\mathcal{L} := \{|S_n| > a_n \ i.o.\}$$

That means $|S_n|$ is greater than a_n for infinitely many n. For the simplicity we shall consider, that X_k are identically distributed. According to the familiar $0-1$ *law*, the probability for such events can be only zero or one.

For the case, where the X_k are individually bounded a theory has already been developed. By means of *law of iterated logarithm* we can decide in any special case whether the probability of $\mathcal{L}$ is zero or one. This theory depends essentially on the central limit theorem. As soon as we have random variables X_k, for which we cannot use the central limit theorem we find ourselves in a different setting.

Theorem 2.1. *Let $(X_k)_{k\geq 1}$ be a sequence of identically distributed independent random variables and $(a_n)_{n\geq 1}$ a monotonic sequence. Suppose that for some $0 < \delta < 1$*

$$\mathbb{E}[|X_k|^{1+\delta}] = \infty \ \forall k \in \mathbb{N}, \tag{2.1}$$

but the first moment exists and

$$\mathbb{E}[X_k] = 0 \ \forall k \in \mathbb{N}.$$

For any sequence a_n for which there exists an ϵ with $0 \leq \epsilon < 1$ such that

$$a_n n^{\frac{-1}{1+\epsilon}} \uparrow \ \textit{and} \ \frac{a_n}{n} \downarrow \tag{2.2}$$

the probability of $\mathcal{L}$:

$$\mathbb{P}\left[|S_n| > a_n i.o.\right] = \begin{cases} 0 \;\; if \;\; \sum_{k=1}^{\infty} \mathbb{P}[|X_1| \geq a_k] < \infty \\ 1 \;\; if \;\; \sum_{k=1}^{\infty} \mathbb{P}[|X_1| \geq a_k] = \infty. \end{cases} \tag{2.3}$$

Proof. Part 1: At first we will assume that (2.3) converges. We define a new random variable of X_k by truncating it and then centering it

$$X_k' = \begin{cases} X_k - \mu_k & \text{if } |X_k| < a_k \\ 0 & \text{if } |X_k| \geq a_k \end{cases} \tag{1}$$

where $\mu_k := \mathbb{E}[X_k \mathbb{1}_{\{|X_k| < a_k\}}]$. Now we can see, that the probability of $X_k \neq X_k' + \mu_k$ is the same as in the terms of (2.3)

$$\sum_{k=1}^{\infty} \mathbb{P}[|X_k| \geq a_k] = \sum_{k=1}^{\infty} \mathbb{P}[X_k \neq X_k' + \mu_k].$$

As the series on the right hand side converges, by means of Borel-Cantelli $A.2$ we get $\mathbb{P}[X_k \neq X_k' + \mu_k \; i.o.] = 0$ and therefore we obtain with probability one

$$\lim_{n\to\infty} (S_n - \sum_{k=1}^{n} X_k' + \mu_k) < \infty,$$

and hence

$$\lim_{n\to\infty} (\frac{S_n}{a_n} - \frac{\sum_{k=1}^{n} X_k' + \mu_k}{a_n}) = 0. \tag{2}$$

Therefore, we have to show that the second fraction of (2) is 0 almost surely. Then we would obtain, that $\lim_{n\to\infty} S_n/a_n$ is 0 almost surely if

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k'}{a_n} = 0 \; a.s., \; and \; \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mu_k}{a_n} = 0 \tag{3}$$

In order to prove the first statement of (3), it suffices according to a straightforward application of the Kronecker's lemma $A.1.$ to show that

$$\sum_{k=1}^{\infty} \frac{1}{a_k} X_k' \text{ converges } a.s.. \tag{4}$$

By definition X'_k has vanishing expected value and according to the Khintchine-Kolmogoroff $A.4$ theorem it suffices to prove that the sum of the variances of $\frac{X_k-\mu_k}{a_k}$ is finite,

$$\sum_{k=1}^{\infty} \frac{1}{a_k^2} \mathbb{E}[(X_k - \mu_k)^2 \mathbb{1}_{\{|X_k|<a_k\}}] < \infty.$$

We are going to simplify the last expression

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{a_k^2} \mathbb{E}[(X_k^2 - 2X_k\mu_k + \mu_k^2)\mathbb{1}_{\{|X_k|\leq a_k\}}] \\ &= \sum_{k=1}^{\infty} \frac{1}{a_k^2} \left[\mathbb{E}[(X_k^2 \mathbb{1}_{\{|X_k|\leq a_k\}}] - 2\mu_k \underbrace{\mathbb{E}[X_k \mathbb{1}_{\{|X_k|\leq a_k\}}]}_{=\mu_k} + \underbrace{\mathbb{P}(|X_1| \leq a_k)]}_{\leq 1} \cdot \mu_k^2 \right] \quad (5) \\ &\leq \sum_{k=1}^{\infty} \frac{1}{a_k^2} \mathbb{E}[(X_k^2 \mathbb{1}_{\{|X_k|\leq a_k\}}] \end{aligned}$$

Without loss of generality $a_0 = 0$ and by assumption that the sequence a_n is nondecreasing, we can split the expected values on $[0, a_n]$ into a sum of expected values on disjoint subintervals $[a_i, a_{i+1})$.

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{a_k^2} \mathbb{E}[X_k^2 \mathbb{1}_{\{|X_k|<a_k\}}] &= \sum_{k=1}^{\infty} \frac{1}{a_k^2} \sum_{i=1}^{k} \mathbb{E}[X_i^2 \mathbb{1}_{\{a_{i-1}\leq|X_i|<a_i\}}] \\ &= \sum_{i=1}^{\infty} \mathbb{E}[X_i^2 \mathbb{1}_{\{a_{i-1}\leq|X_i|<a_i\}}] \sum_{k=i}^{\infty} \frac{1}{a_k^2} \quad (6) \\ &\underset{X_i^2\leq a_i^2}{\leq} \sum_{i=1}^{\infty} \mathbb{E}[a_i^2 \mathbb{1}_{\{a_{i-1}\leq|X_i|<a_i\}}] \sum_{k=i}^{\infty} \frac{1}{a_k^2}. \end{aligned}$$

The first condition of the conditions (2.2) implies that:

$$\sum_{k=i}^{\infty} \frac{1}{a_k^2} \leq \frac{1}{a_i^2} \cdot i^{\frac{2}{1+\epsilon}} \sum_{k=i}^{\infty} \frac{1}{k^{\frac{2}{1+\epsilon}}} < \frac{3}{1-\epsilon} \cdot \frac{i}{a_i^2}, \quad (7)$$

To see the last inequality, we consider the series:

$$i^{\frac{2}{1+\epsilon}} \sum_{k=i}^{\infty} \frac{1}{k^{\frac{2}{1+\epsilon}}} = \sum_{k=i}^{\infty} \left(\frac{k}{i}\right)^{-\frac{2}{1+\epsilon}} = i \cdot \sum_{k=i}^{\infty} \left(\frac{k}{i}\right)^{-\frac{2}{1+\epsilon}} \frac{1}{i}$$

and this is the Riemann sum of the function $f(x) = x^{-\frac{2}{1+\epsilon}}$ with length $\frac{1}{i}$.

$$
\begin{aligned}
i \cdot \sum_{k=i}^{\infty} \left(\frac{k}{i}\right)^{-\frac{2}{1+\epsilon}} \frac{1}{i} &\leq \int_{i-\frac{1}{i}}^{\infty} x^{-\frac{2}{1+\epsilon}} dx \\
&\leq i \cdot \left(1 + \int_{1}^{\infty} x^{-\frac{2}{1+\epsilon}} dx\right) \\
&\leq i \cdot \left(1 + \lim_{z\to\infty} \left(-\frac{1+\epsilon}{1-\epsilon}\right) x^{-\frac{1-\epsilon}{1-\epsilon}} \Big|_{1}^{z}\right) \\
&= i \cdot \left(1 + 1 \cdot \frac{1+\epsilon}{1-\epsilon}\right) = \frac{2i}{1-\epsilon} < \frac{3}{1-\epsilon}
\end{aligned}
$$

Therefore the last expression of (6) is less than, where the term a_i^2 cancelled.

$$
\frac{3}{1-\epsilon} \sum_{i=1}^{\infty} i\mathbb{E}[\mathbb{1}_{\{a_{i-1} \leq |X_1| < a_i\}}] \tag{8}
$$

We are rewriting the series (8) in order that we can use the assumption((2.3) convergence).

$$
\begin{aligned}
\sum_{i=1}^{n} i\mathbb{E}[\mathbb{1}_{\{a_{i-1} \leq |X_1| < a_i\}}] &= 1 \cdot \mathbb{E}[\mathbb{1}_{\{a_0 \leq |X_1| < a_1\}}] \\
&+ 1 \cdot \mathbb{E}[\mathbb{1}_{\{a_1 \leq |X_1| < a_2\}}] + 1 \cdot \mathbb{E}[\mathbb{1}_{\{a_1 \leq |X_1| < a_2\}}] \\
&\vdots \\
&+ 1 \cdot \mathbb{E}[\mathbb{1}_{\{a_{n-1} \leq |X_1| < a_n\}}] + ... + 1 \cdot \mathbb{E}[\mathbb{1}_{\{a_{n-1} \leq |X_1| < a_{n-1}\}}] \\
&= \sum_{i=0}^{n} \mathbb{E}[\mathbb{1}_{\{a_i \leq |X_1| < a_n\}}]
\end{aligned}
$$

For $n \to \infty$ we obtain:

$$
\begin{aligned}
\sum_{i=1}^{\infty} i\mathbb{E}[\mathbb{1}_{\{a_{i-1} \leq |X_1| < a_i\}}] &= \lim_{n\to\infty} (\mathbb{E}[\mathbb{1}_{\{a_0 \leq |X_1| < a_n\}}] + ... + \mathbb{E}[\mathbb{1}_{\{a_k \leq |X_1| < a_n\}}]) \\
&= \sum_{i=1}^{\infty} \mathbb{E}[\mathbb{1}_{\{a_i \leq |X_1|\}}]
\end{aligned}
$$

Therefore (8) is equal to

$$\frac{3}{1-\epsilon}\sum_{i=1}^{\infty}\mathbb{E}[\mathbb{1}_{\{a_i\le|X_1|\}}] \tag{9}$$

and the series (9) converges by assumption. As the terms of (5) are not exceeded by those of (6), the convergence of (5) and therefore the validity of (4) have been established.

In view of (3) it only remains to prove that

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\mu_k}{a_n}=0 \tag{10}$$

The expected value of X_k is 0 $\forall k\in\mathbb{N}$, hence the expected value on $\{|X_k|<a_k\}^c$ has to be $-\mu_k$ for all $k\in\mathbb{N}$.

$$-\mu_k=\mathbb{E}[X_k\mathbb{1}_{\{a_k\le|X_k|\}}]\Rightarrow\mu_k\le\mathbb{E}[|X_k|\mathbb{1}_{\{a_k\le|X_k|\}}], \tag{11}$$

It follows then from (11) that the following inequalities hold for an arbitrary integer N and for all $n\ge N$,

$$\begin{aligned}|\frac{1}{a_n}\sum_{k=1}^{n}\mu_k| &\le \mathbb{E}[|X_1|]\cdot\frac{N}{a_n}+\frac{1}{a_n}\sum_{k=N}^{n}\mathbb{E}[|X_k|\mathbb{1}_{\{a_k\le|X_k|\}}]\\ &\le \mathbb{E}[|X_1|]\cdot\frac{N}{a_n}+\frac{n}{a_n}\mathbb{E}[|X_n|\mathbb{1}_{\{a_n\le|X_n|\}}]+\frac{1}{a_n}\sum_{k=N}^{n}\mathbb{E}[|X_k|\mathbb{1}_{\{a_k\le|X_k|<a_n\}}]\end{aligned} \tag{12}$$

Now, using the second condition of ((2.2)) for the second term of the right hand side from the last inequality and give us the following estimate:

$$\begin{aligned}\frac{n}{a_n}\mathbb{E}[|X_n|\mathbb{1}_{\{a_n\le|X_n|\}}] &\le \sum_{i=n}^{\infty}\frac{i}{a_i}\mathbb{E}[|X_i|\mathbb{1}_{\{a_i\le|X_i|<a_{i+1}\}}]\ \ (n/a_n\le i/a_i\ \forall i\ge n)\\ &\le \sum_{i=n}^{\infty}i\mathbb{E}[\mathbb{1}_{\{a_i\le|X_1|<a_{i+1}\}}]\\ &= \sum_{i=n}^{\infty}\mathbb{E}[\mathbb{1}_{\{a_i\le|X_1|\}}]\end{aligned}$$

The series on the right hand side tends to zero for $n \to \infty$ since the terms of a convergent series with decreasing terms. It only remains to consider the last expression in (12)

$$\frac{1}{a_n}\sum_{k=N}^{n}\mathbb{E}[|X_k|\mathbb{1}_{\{a_k\leq|X_k|<a_n\}}] \leq \sum_{k=N}^{n}\mathbb{E}[\mathbb{1}_{\{|X_1|\geq a_k\}}]$$

and we see that it becomes arbitrarily small for sufficiently large N.

Therefore, we have shown the claims made in (3). We obtain that

$$\lim_{n\to\infty}\frac{S_n}{a_n} = 0 \ a.s.$$

and therefore

$$\mathbb{P}[|S_n| > a_n \ i.o.] = 0.$$

Part 2:Now we will assume that (2.3) diverges. By means of Borel-Cantelli we know that

$$\sum_{k=1}^{\infty}\mathbb{P}[|X_k| \geq a_k] = \infty \Rightarrow \mathbb{P}[|X_k| > a_k \ i.o.] = 1. \tag{13}$$

The second condition of the (1.2) implies that:

$$\frac{a_{2n}}{2n} < \frac{a_n}{n} \Leftrightarrow a_{2n} > 2a_n$$

Since $\mathbb{P}[|X_1| > a_n]$ are nonincreasing we obtain that:

$$\sum_{n=1}^{\infty}\mathbb{P}[|X_1| > a_n] = \infty \Rightarrow \sum_{n=1}^{\infty}\mathbb{P}[|X_1| > a_{2n}] = \infty$$

We have to find some connection between $|X_n|$ and $|S_n|$

$$\max(|S_n|, |S_{n-1}|) \geq \frac{|X_n|}{2}. \tag{14}$$

To see this, start with $n = 2$ ($max(|b_1+b_2|, |b_1|) \geq |b_2|/2$) and then for $n > 2$ is trivially. From (13), (14) follows that

$$\mathbb{P}[|S_m| > \frac{a_m}{2} \ i.o.] = 1$$

□

Before we start with the second theorem of this chapter, we consider the sequence $(X_k)_{k\geq 1}$ of random variables with $\mathbb{E}[X_k] = \infty$, then we should get

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k \to \infty \text{ a.s.}$$

If we compare the sequence with the sequence $(X_k \wedge m)_{k\geq 1}$ for $m \in \mathbb{N}$, by definition we get $\sum_{k=1}^{n}(X_k \wedge m) \leq \sum_{k=1}^{n} X_k$ for all $n,\ m \in \mathbb{N}$. Additionally we get that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(X_k \wedge m) \to \mathbb{E}[X_1 \wedge m] \text{ a.s.}$$

and by property of X_k we obtain, that $\mathbb{E}[X_k \wedge m] \to \infty$ for $m \to \infty$.

Now we consider sequences $(X_k)_{k\geq 1}$ of random variables with property $\mathbb{E}[|X_k|] = \infty$ for all $k \in \mathbb{N}$.

Theorem 2.2. *Let $(X_k)_{k\geq 1}$ be a sequence of identically distributed independent random variables. If*

$$\mathbb{E}[|X_k|] = \infty \ \forall k \in \mathbb{N} \tag{2.4}$$

then for any positive increasing sequences $(a_n)_{n\geq 1}$ with

$$\frac{a_n}{n} \uparrow \tag{2.5}$$

the probability of $\mathcal{L}$:

$$\mathbb{P}\left[|S_n| > a_n \ i.o.\right] = \begin{cases} 0 \ \textit{if} \ \sum_{k=1}^{\infty} \mathbb{P}[|X_1| \geq a_k] < \infty \\ 1 \ \textit{if} \ \sum_{k=1}^{\infty} \mathbb{P}[|X_1| \geq a_k] = \infty. \end{cases} \tag{2.6}$$

Proof. Part 1. Suppose that:

$$\sum_{k=1}^{\infty} \mathbb{P}[|X_1| \geq a_k] < \infty$$

The steps of the proof are the same as before. One shows that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k'}{a_n} = 0 \ a.s. \text{ and } \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mu_k}{a_n} = 0. \tag{1}$$

The proof of the first statement of (1) is the same as in *Theorem* 2.1. Therefore, it only remains to prove the second statement of (1).

$$
\begin{aligned}
|\frac{1}{a_n}\sum_{k=1}^{n}\mu_k| &\leq \frac{1}{a_n}\sum_{k=N}^{n}\mathbb{E}[|X_k|\mathbb{1}_{\{a_k\geq|X_k|\}}] \\
&= O\left(\frac{Na_N}{a_n}\right) + \frac{n}{a_n}\mathbb{E}[|X_n|\mathbb{1}_{\{a_N\leq|X_n|\leq a_n\}}] \\
&\leq O\left(\frac{Na_N}{a_n}\right) + \sum_{i=N}^{n}\frac{i}{a_i}\mathbb{E}[|X_i|\mathbb{1}_{\{a_{i-1}\leq|X_i|<a_i\}}]
\end{aligned}
$$

We have already shown that the last expression tends to zero. Therefore $P[|S_n| > a_n\ i.o.] = 0$.

Part 2. Suppose that:

$$\sum_{k=1}^{\infty}\mathbb{P}[|X_1|\geq a_k] = \infty$$

Since $a_k/k\ \uparrow$ we obtain that $na_k \geq a_{nk}$ $\forall n\geq 1$ and $a_k\ \uparrow$.

$$
\begin{aligned}
\sum_{k=1}^{\infty}\mathbb{P}[|X_k|\geq na_k] &\geq \sum_{k=1}^{\infty}\mathbb{P}[|X_k|\geq a_{nk}] \\
&\geq \frac{1}{n}\sum_{m=n}^{\infty}\mathbb{P}[|X_1|\geq a_m]
\end{aligned}
$$

The last observation shows that if the last series is infinite, then we get $\limsup_{n\to\infty}\frac{|X_n|}{a_n} = \infty$ a.s.. Since $max(|S_n|,|S_{n-1}|) \geq |X_n|/2$, it follows that $\limsup_{n\to\infty}\frac{|S_n|}{a_n} = \infty$ a.s..

Finally, we obtain that $P[|S_n| > a_n\ i.o.] = 1$. □

3 Connection between the limit of $\frac{S_n}{n}$ and integral tests

Most of the material of this section is taken from a publication of K. Bruce Erickson [4]. In this chapter we will generalize the theorems of the previous chapter. For example: Feller's strong law of large numbers does not cover the case where

$$\mathbb{E}[a^{-1}(|X_1|)] = \infty.$$

For convenience we begin with the sequence $(a_n)_{n\geq 1}$, where $a_n = n$ for $n \geq 1$. In the second part of this chapter we consider the general case of $(a_n)_{n\geq 1}$.

To consider the previous statement, we have to start with some basic probability theory. If the random variable X in some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ nonnegative, then we can write the expected value as:

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \geq t]dt.$$

Let X be arbitrary random variable. Now we split the random variables X in two terms X^+ and X^-, where $X^+ := \max(X, 0)$ and $X^- := \max(-X, 0)$. Therefore we can write $X = X^+ - X^-$ and we define two functions such that the limits of these functions are the expected value of X^+ and X^-.

$$m_+(x) = \int_0^x P(X^+ \geq t)dt,$$

$$m_-(x) = \int_0^x P(X^- \geq t)dt.$$

If $\mathbb{E}[X]$ defined, that means at least one of them $\mathbb{E}[X^+]$, $\mathbb{E}[X^-]$ has to be finite, then the following holds:

$$\mathbb{P}\left[\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X_1]\right] = 1 \tag{3.1}$$

for $(X_n)_{n\geq 1}$ i.i.d. sequence of random variables.

Now we will see what happens if the expected value not defined:

Theorem 3.1. *Let $(X_i)_{i\geq 1}$ be some sequence of i.i.d. random variables in some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If $\mathbb{E}[X_1^+] = \mathbb{E}[X_1^-] = \infty$, then one of the following alternatives must be hold.*

(i) $\mathbb{P}[\lim_{n\to\infty} \frac{S_n}{n} = \infty] = 1$

(ii) $\mathbb{P}[\lim_{n\to\infty} \frac{S_n}{n} = -\infty] = 1$

(iii) $\mathbb{P}\left[\limsup_{n\to\infty} \frac{S_n}{n} = \infty \text{ and } \liminf_{n\to\infty} \frac{S_n}{n} = -\infty\right] = 1$

We will not prove the previous theorem, but we consider all three points and we shall give the sufficient criterion in the following form of integral test if $m_+(x)$ and $m_-(x)$ are not equal to 0:

$$J_+(X) = \int_0^\infty \frac{x}{m_-(x)} dF(x),$$

$$J_-(X) = \int_0^\infty \frac{|x|}{m_+(|x|)} d(1 - F(-x)),$$

where $F(x)$ the distribution function of X. We know already that for $t \to \infty$ holds: $m_+(t) \to \mathbb{E}[X^+]$, $m_-(t) \to \mathbb{E}[X^-]$ and both of them are nondecreasing. Therefore we obtain the following inequalities:

$$J_+(X) = \infty \Rightarrow \mathbb{E}[X^+] = \infty, \tag{3.2}$$

this statement holds also for $\mathbb{E}[X^-]$ and $J_-(X)$.

Theorem 3.2. *Let $(X_i)_{i\geq 1}$ be some sequence of i.i.d random variables in some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and we have no assumption on $\mathbb{E}[X_1]$.*

(i) $J_+(X) = \infty$ *if and only if* $\mathbb{P}\left[\limsup_{n\to\infty} \frac{S_n}{n} = +\infty\right] = 1$

(ii) $J_-(X) = \infty$ *if and only if* $\mathbb{P}\left[\liminf_{n\to\infty} \frac{S_n}{n} = -\infty\right] = 1$

(iii) $J_-(X) < J_+(X) = \infty$ *if and only if* $\mathbb{P}\left[\lim_{n\to\infty} \frac{S_n}{n} = +\infty\right] = 1$

(iv) $J_+(X) < J_-(X) = \infty$ *if and only if* $\mathbb{P}\left[\lim_{n\to\infty} \frac{S_n}{n} = -\infty\right] = 1$

In combination of the previous theorem and the Hewitt-Savage theorem A.7 we obtain, that if J_+ and J_- are finite, then $\frac{S_n}{n}$ must be bounded with probability 1. But $\frac{S_n}{n}$ is bounded if and only if $\mathbb{E}[X_1] < \infty$. From this and (3.2) we conclude that

$$J_+(X) + J_-(X) < \infty \;\Leftrightarrow\; \mathbb{E}[|X_1|] < \infty. \tag{3.3}$$

The proof of this statement is analytic and similar to the general case $(a_n)_{n\geq 1}$ (*See theorem* 4.1). The ideas of the proofs are the same but the general one contains some technical steps. So we will prove just the general case in the next subsection, where we consider the limits of $\frac{S_n}{a_n}$ for $n \to \infty$.

Proposition 3.1. *Let* $(X_i)_{i\geq 1}$ *be some sequence of i.i.d random variables in some probability space* $(\Omega, \mathcal{A}, \mathbb{P})$. *Assume* $\mathbb{E}[|X_1|] = \infty$. *Then at most one of* J_+, J_- *is finite and*

(i) $\mathbb{P}\left[\lim_{n\to\infty} \frac{S_n}{n} = +\infty\right] = 1$ *if and only if* $J_-(X_1) < \infty$

(ii) $\mathbb{P}\left[\lim_{n\to\infty} \frac{S_n}{n} = -\infty\right] = 1$ *if and only if* $J_+(X_1) < \infty$

(iii) $\mathbb{P}\left[\limsup_{n\to\infty} \frac{S_n}{n} = +\infty \text{ and } \liminf_{n\to\infty} \frac{S_n}{n} = -\infty\right] = 1$ *if and only if* $J_-(X_1) = J_+(X_1) = \infty$

Proof. Suppose that $\mathbb{E}[|X_1|] = \infty$. This implies that $\mathbb{E}[X_1^+]$ or $\mathbb{E}[X_1^-]$ has to be infinite.

(i): Due of (3.3) we see that:

$$J_-(X_1) < \infty \Leftrightarrow J_-(X_1) < \infty \text{ and } J_+(X_1) = \infty. \tag{3.4}$$

By means of theorem 3.2 that the right side of (3.4) is equivalent to

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{S_n}{n} = +\infty\right] = 1.$$

(ii):The theorem is symmetric. We can replace X^+ with X^- and $J_+(X_1)$ with $J_-(X_1)$. Therefore (ii) follows from (i).
(iii):This statement follows from the combination of theorem 3.2 (i) and (ii). □

The proof of the theorem 3.2 is complicated and contains more steps. Therefore we prepare our selves with the following lemmas for the proof. The first lemma is just a preparation for the other lemmas. This lemma contains a statement about the convolution of random variables. You find the introduction in the Appendix chapter under Convolution.

Lemma 3.1. *Let G be any probability distribution concentrated on $[0, \infty)$. Put*

$$U(t) = \sum_{n=0}^{\infty} G^{n^*}(t) \ , \ m(x) = \int_0^x (1 - G(t))dt,$$

where G^n is the n-fold convolution. Then

$$1 \leq m(t)U(t) \leq 2 \text{ for all } t \geq 0 \tag{3.5}$$

and

$$min(1, \frac{a}{2}) \leq \frac{U(at)}{U(t)} \leq max(1, 2a) \tag{3.6}$$

for all $t, a > 0$.

We will not prove this lemma, but in the following corollary we will see how the statement of this lemma related to the function J_+.

Corollary 3.1. *Let G be any probability distribution concentrated on $[0, \infty)$ and let F be any probability distribution. The following integral $\int_0^\infty U(at)dF(t)$ $= \int_0^\infty \sum_{n=0}^\infty G^{n^*}(ax)dF(x)$, either converges for all $a > 0$ or diverges for all $a > 0$, according as $\int_0^\infty x/m(x)dF(x)$ converges or diverges, where $m(x) = \int_0^x (1 - G(t))dt$.*

This corollary is helpful to prove later Lemma. The proof of this corollary contains one step.

Proof. In combination of (3.5) and (3.6), we obtain the following inequalities:

$$\min\left(1, \frac{a}{2}\right) \frac{t}{m(t)} \leq U(t) \min\left(1, \frac{a}{2}\right) \leq U(at)$$
$$\leq \min(1, 2a)U(t) \leq \min(1, 2a)\frac{t}{m(t)}.$$

Therefore we get the following inequalities:

$$\int_0^\infty \min\left(1, \frac{a}{2}\right) \frac{t}{m(t)} dF(x) \le \int_0^\infty U(t) dF(x) \le \int_0^\infty \min(1, 2a) \frac{t}{m(t)} dF(x).$$

□

Lemma 3.2. *Let $(X_i)_{i\ge 1}$ be some sequence of i.i.d random variables in some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $a > 0$ be fixed and put $A_0 = \Omega$ certain event, $A_1 := \{X_1 > 0\}$ and $A_n = \{X_1^- + \cdots + X_{n-1}^- < aX_n^+\}$, $n > 1$*

(i) If $\sum_{n=0}^\infty \mathbb{P}[A_n] < \infty$, then

$$\limsup_{n\to\infty} \frac{X_n^+}{X_1^- + \cdots + X_n^-} \le \frac{1}{a} \;\; a.s.. \tag{3.7}$$

(ii) If $\sum_{n=0}^\infty \mathbb{P}[A_n] = \infty$, then

$$\limsup_{n\to\infty} \frac{X_n^+}{X_1^- + \cdots + X_n^-} \ge \frac{1}{a} \;\; a.s.. \tag{3.8}$$

Proof. Part 1: Assume that $\sum_{n=0}^\infty \mathbb{P}[A_n] < \infty$. By means of Borel-Cantelli theorem A.2 we obtain that:

$$\mathbb{P}[\{A_n \; i.o.\}] = 0 \;\Rightarrow\; \mathbb{P}\left[\limsup_{n\to\infty} \frac{X_n^+}{X_1^- + \cdots + X_n^-} > \frac{1}{a}\right] = 0.$$

This implies:$\limsup\limits_{n\to\infty} \frac{X_n^+}{X_1^- + \cdots + X_n^-} \le \frac{1}{a}$ a.s..

Part 2: Assume that $\sum_{n=0}^\infty \mathbb{P}[A_n] = \infty$. By means of Hewitt-Savage 0-1 law we obtain that $\mathbb{P}[\{A_n \; i.o.\}]$ is either 0 or 1. Therefore is enough to show that:

$$\mathbb{P}[\{A_n \; i.o.\}] > 0. \tag{1}$$

We know X_i^- is nonnegative $\forall i \ge 0$. Therefore we get

$$A_n \cap A_m \subset A_n \cap \{X_{n+1}^- + \cdots + X_{m-1}^- < aX_m^+\}.$$

for $m > n$. By stationary of $(X_n)_{n\ge 1}$ the $\mathbb{P}[A_{m-n}]$ is the same as $\mathbb{P}[\{X_{n+1}^- + \cdots + X_{m-1}^- < aX_m^+\}]$. Since $A_n \in \sigma(X_1, ..., X_n)$ and $\{X_{n+1}^- + \cdots + X_{m-1}^- < aX_m^+\} \in \sigma(X_{n+1}, ..., X_m)$ we obtain that:

$$\begin{aligned}\mathbb{P}\left[A_n \cap A_m\right] &\leq \mathbb{P}[A_n \cap \{X_{n+1}^- + \cdots + X_{m-1}^- < aX_m^+\}] \\ &= \mathbb{P}[A_n]\mathbb{P}[\{X_{n+1}^- + \cdots + X_{m-1}^- < aX_m^+\}] \\ &= \mathbb{P}[A_n]\mathbb{P}[A_{m-n}].\end{aligned}$$

We define a new random variable $Z_n = \sum_{k=0}^n \mathbb{1}_{A_k}$ the number of A_k which occur up to the time n.

$$\mathbb{E}[Z_n^2] \leq 2\sum_{k=0}^n \mathbb{P}[A_k] \sum_{i=k}^n \mathbb{P}[A_{i-k}] \leq 2\left(\sum_{k=0}^n \mathbb{P}[A_k]\right)^2 = 2\left(\mathbb{E}[Z_n]\right)^2 \tag{2}$$

From this we define a random variable

$$R_n = \frac{Z_n}{\mathbb{E}[Z_n]} \Rightarrow \mathbb{E}[R_n] = 1$$

Assume (1) fails: $\mathbb{P}[A_n\ i.o] = 0$. This means Z_n finite for a.e., but $\mathbb{E}[Z_n] = \sum_{i=1}^n \mathbb{P}[A_n] \to \infty$, therefore $R_n \to 0$ *a.s.*. By means of the definition of R_n and (2) we obtain that

$$\mathbb{E}[R_n^2] = \mathbb{E}\left[\frac{Z_n^2}{\mathbb{E}[Z_n]^2}\right] \leq \frac{1}{\mathbb{E}[Z_n]^2} \cdot 2\mathbb{E}[Z_n]^2 = 2\ \forall n$$

That means R_n has bounded second moments and we know that $R_n \to 0$ *a.s.*. This implies $\mathbb{E}[R_n] \to 0$ and this a contradiction to the definition of R_n, because the $\mathbb{E}[R_n] = 1$. □

Lemma 3.3. *Let $(X_i)_{i\geq 1}$ be some sequence of i.i.d random variables in some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, then $\limsup_{n\to\infty} \left(\frac{X_n^+}{X_1^- + \cdots + X_n^-}\right) 0$ or ∞ with probability 1, according as $J_+(X_1)$ is finite or infinite.*

Proof. Let $A_n = \{X_1^- + \cdots + X_{n-1}^- < aX_n^+\}$ and $G(t) = \mathbb{P}[X_1^- \leq t]$ and $G(t-) = \mathbb{P}[X_1^- < t]$. We start with upper bound of $\mathbb{P}[A_n]$

$$\begin{aligned}&\mathbb{P}\left[X_1^- + \cdots + X_{n-1}^- < aX_n^+\right] = \mathbb{P}[A_n] \\ &= \int_0^\infty \mathbb{P}\left[X_1^- + \cdots + X_{n-1}^- < ay\right] \mathbb{P}[X_n^+ \in dy] \\ &= \int_0^\infty G^{n-1}(ay-) dF(x) \\ &\leq \int_0^\infty G^{n-1}(ay) dF(x).\end{aligned}$$

The lower bound is given by:

$$\mathbb{P}[A_n] \geq \int_0^\infty G^{n-1}(by)dF(x)$$

for all $0 < b < a$. Therefore by means of the corollary 3.1 and lemma 3.2 we obtain that: $\sum_{n=0}^{\infty} \mathbb{P}\left[X_1^- + \cdots + X_{n-1}^- < aX_n^+\right]$ converges or diverges for all $a > 0$ according as J_+ finite or infinite. □

Lemma 3.4. *Let $(X_i)_{i\geq 1}$ be some sequence of i.i.d random variables in some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathbb{P}[X_i \geq 0] < 1$. If*

$$\limsup_{n\to\infty} \frac{X_n^+}{X_1^- + \cdots + X_n^-} = \infty, \tag{3.8}$$

then $\mathbb{E}[X_1^+] = \infty$ and $\limsup\limits_{n\to\infty} \frac{S_n}{n} = \infty$ a.s..

Proof. Assume that $\limsup\limits_{n\to\infty} \frac{X_n^+}{X_1^- + \cdots + X_n^-} = \infty$. This implies that the event $A_n = \{X_1^- + \cdots + X_n^- < \frac{1}{2}X_n^+\}$ takes place with probability one for infinitely many time. For such n we have:

$$\begin{aligned} S_n &= X_n^+ - \left(X_1^- + \cdots + X_n^-\right) + X_1^+ + \cdots + X_{n-1}^+ \\ &\geq 2\left(X_1^- + \cdots + X_n^-\right) - \left(X_1^- + \cdots + X_n^-\right) + X_1^+ + \cdots + X_{n-1}^+ \\ &\geq |X_1| + \cdots + |X_{n-1}|. \end{aligned}$$

Hence $\frac{S_n}{n} \geq \frac{|X_1| + \cdots + |X_{n-1}|}{n}$

$$\limsup_{n\to\infty} \frac{S_n}{n} \geq \liminf_{n\to\infty} \frac{|X_1| + \cdots + |X_{n-1}|}{n}.$$

But we know already

$$\limsup_{n\to\infty} \frac{S_n}{n} \geq \liminf_{n\to\infty} \frac{|X_1| + \cdots + |X_{n-1}|}{n} = \mathbb{E}[|X_1|] \ a.s.$$

and since $X_i^+ \geq X_i \ \forall i \geq 1$.

$$\mathbb{E}[X_1^+] = \lim_{n\to\infty} \frac{X_1^+ + \cdots + X_n^+}{n} \geq \limsup_{n\to\infty} \frac{S_n}{n} \ a.s.$$

From follows that $\mathbb{E}[X_1^+] \geq \mathbb{E}[|X_1|]$, but by assumption $\mathbb{P}[X_1 > 0] > 0$ we obtain that: $\mathbb{E}[X_1^+] \geq \mathbb{E}[|X_1|] = \infty$. From follows that $\frac{S_n}{n}$ converges to infinte a.s. for $n \to \infty$. □

Lemma 3.5. *If* $\mathbb{E}[X_1^+] = \infty$ *and if* $\mathbb{P}[S_n > 0 \ i.o.] > 0$, *then*

$$\limsup_{n\to\infty} \frac{X_n^+}{X_1^- + \cdots + X_n^-} = \infty \ a.s.. \tag{3.9}$$

The previous statement means, that the positive parts of random variables $(X_n)_{n\geq 0}$ are neglectable. You can find the proof of this lemma in [3]. In this thesis we skip the proof, because it is not relevant for the next chapters.

Proof. (Theorem 3.2) At first we may assume that:

$$\mathbb{P}[X_1 < 0] \cdot \mathbb{P}[X_1 > 0] \neq 0,$$

because in another case we get: $\mathbb{P}[X_1 \geq 0] = 1$, then $\mathbb{E}[X_1^+] = \mathbb{E}[X_1] < \infty$ if and only if $J_+ < \infty$. It easy to see, that the theorem is symmetric. That means we can replace X_n with $\tilde{X} = -X$, then implied $\tilde{J}_+ = -J_-$. Therefore (ii) follows form (i), and (iv) follows form (iii). At first we prove the part (i)

Part 1: (i) $(\Rightarrow)$: By means of theorem and lemma 3.3 we know that if: $J_+(X) = \infty$, then we obtain that $\mathbb{P}\left[\limsup\limits_{n\to\infty} \frac{X_+}{X_1^- + \cdots + X_n^+} = \infty\right] = 1$ and by lemma 3.4 we get that:

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{S_n}{n} = \infty\right] = 1.$$

$(\Leftarrow)$: Suppose that $\mathbb{P}\left[\limsup\limits_{n\to\infty} \frac{S_n}{n} = \infty\right] = 1$. We know already that $\mathbb{P}\left[\lim\limits_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X_1]\right] = 1$, therefore $\mathbb{E}[X_1^+]$ has to be infinite, otherwise we would have $\lim\limits_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X_1^+] - \mathbb{E}[X_1^-] \neq \infty$. This assumption implies an other statements: $\mathbb{P}[S_n > 0 \ i.o.] = 1$. Now by lemma 3.3 and 3.5 we obtain that $J_+(X) = \infty$.

Part 2: (iii) $\Rightarrow$: Assume that $J_+(X) = \infty$, $J_-(X) < \infty$. By parts of (i), (ii) we have:

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{S_n}{n} = \infty \text{ and } \liminf_{n\to\infty} \frac{S_n}{n} > -\infty\right] = 1.$$

That implies $\mathbb{E}[X_1^+] = \infty$. Now we split the proof in two cases. At first we consider the case: $\mathbb{E}[X_1^-] < \infty$. by means of (3.1) we obtain that:

$$\mathbb{P}\left[\lim_{n\to\infty} \frac{S_n}{n} = \infty\right] = 1. \tag{1}$$

In the other case: if $\mathbb{E}[X_1^-] = \infty$. We consider the theorem 3.2 and we see that: (ii), (iii) are impossible, therefore we obtain, that (1).

In the other direction: Assume that (1) holds, this implies:

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{S_n}{n} = \liminf_{n\to\infty} \frac{S_n}{n} = \infty\right] = 1.$$

From the previous statement and (i), (ii) follows, that $J_+(X) = \infty$ and $J_-(X) < \infty$. □

After the proof of *theorem* 3.2 we can show the statement (1.3) of *example* 1.1. we start with m_+ and m_-:

$$m_+(t) = \int_0^t \mathbb{P}[X^+ \geq s]ds,\ m_-(t) = \int_0^t \mathbb{P}[X^- \geq s]ds.$$

Order to prove the statement (1.3), we have to show that $J_+(X) = \infty$ and $J_-(X) = \infty$. At first we will show that

$$J_+(X) = \int_0^\infty \frac{t}{m_-(t)} dF(t) = \infty$$

and we know from [4](Remark 3), that

$$\mathbb{E}[X^+] = \infty \Leftrightarrow \int_0^\infty \frac{t}{m_+(t)} dF(t) = \infty.$$

Now we have to consider $t/m_-(t)$:

$$\frac{t}{m_-(t)} = \frac{t}{m_+(t)} \frac{m_+(t)}{m_-(t)} \text{ for t} > 0.$$

By means of 1.4 we obtain, that the second term goes to infinite for $t \to \infty$. That means

$$\frac{t}{m_+(t)} = o\left(\frac{t}{m_-(t)}\right). \tag{3.10}$$

Now we consider the $J_+(X)$ and we will find a lower bound.

$$J_+(X) = \int_0^\infty \frac{t}{m_-(t)} dF(t) \geq \int_c^\infty \frac{t}{m_-(t)} dF(t)$$
$$\geq \int_c^\infty \frac{t}{m_+(t)} dF(t) = \infty.$$

The second inequality holds for some $c > 0$, because (3.10). A last integral of the previous statement is infinite, because the integral $\int_0^c \frac{t}{m_+(t)} dF(t)$ has to be finite. Now we will show that $J_-(X) = \infty$. At we first we consider the $m_+(t)$

$$m_+(n) = \sum_{i=1}^{n} \mathbb{P}[X^+ = k]k = \sum_{i=1}^{n} \frac{\kappa_+}{k^2} k = \kappa_+ \sum_{i=1}^{n} \frac{1}{k} \approx \kappa_+ \log(n).$$

Now we rewrite the $J_-(X)$

$$J_-(X) = \int_0^\infty \frac{t}{m_+(t)} dF(t) = \sum_{n=3}^{\infty} \mathbb{P}[X^- = n] \frac{n}{m_+(n)}$$
$$\approx \sum_{n=3}^{\infty} \frac{\kappa_-}{n^2 \log\log(n)} \frac{n}{\kappa_+ \log(n)}$$
$$= \frac{\kappa_-}{\kappa_+} \sum_{n=3}^{\infty} \frac{1}{n \log(n) \log\log(n)}.$$

Since Abel's series $\sum_{n=2}^{\infty} \frac{1}{n \log(n) \log\log(n)}$ diverges we get that $J_-(X) = \infty$.

4 Limit theorem for $\frac{S_n}{a_n}$

Most of the material of this section is taken from a publication of Yuan Shih Chow and Cun-Hui Zhang [2]. We are studying further the fraction $\frac{S_n}{a_n}$, where S_n is the sum of random variables, where $(a_n)_{n\geq 0}$ is sequence such that $a_0 = 0$ and a_n/n is nondecreasing in n. At first we define a function $a : [0, \infty) \to \mathbb{R}$:

$$a(x) =: \begin{cases} a_n & \text{if } x = n, \\ a_n + (a_{n+1} - a_n)(x - n) & \text{if } n \leq x < n + 1 \end{cases}$$

Let $a^{-1}(\cdot)$ denote the inverse function of $a(\cdot)$ and

$$J_+(a) = \int_0^\infty \min(a^{-1}(x), x/m_-(x))dF(x)$$

$$J_-(a) = \int_0^\infty \min(a^{-1}(x), x/m_+(x))d\,(1 - F(-x))$$

The following theorem gives us a first impression, how the previous integrals and the limits of $\frac{S_n}{a_n}$ are related to each other.

Theorem 4.1. *Let X be a random variable on some probability space.*

$$J_-(a) + J_+(a) < \infty \;\Leftrightarrow\; \mathbb{E}[a^{-1}\left(|X|\right)] < \infty.$$

Proof. We can split the proof in two assumptions. In this part we will assume that $\mathbb{E}[|X|] < \infty$. In this case we have to consider the following inequality:

$$\mathbb{E}a^{-1}[|X|] \leq a^{-1}(1)\,[\mathbb{E}[|X|] + 1]$$

At first we define a random variable $Y := a^{-1}(|X|)$ and $y = a^{-1}(1)$. This implies $a(y) = 1$. That means the inequality is equivalent to the following inequality:

$$\mathbb{E}[Y] \leq y\,[\mathbb{E}[a(Y)] + a(y)]$$

We split the proof in two disjoints sets. At first we consider the inequality on the set $\{Y \geq y\}$. Since $a(t)/t$ is nondecreasing we obtain that:

$$
\begin{aligned}
\frac{a(Y)}{Y} &\geq \frac{a(y)}{y} \\
a(Y) \cdot y &\geq a(y)Y \\
\mathbb{1}_{\{Y \geq y\}} \cdot a(Y) \cdot y &\geq \mathbb{1}_{\{Y \geq y\}} \cdot a(y) \cdot Y \\
\mathbb{E}[a(Y) \cdot y] &\geq \mathbb{E}[a(y)Y]
\end{aligned}
$$

Now we consider on the sets $\{Y \geq y\}$ and $\{Y < y\}$.

$$
\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[\mathbb{1}_{\{Y \geq y\}} Y] + \mathbb{E}[\mathbb{1}_{\{Y < y\}} Y] \\
&\leq y \mathbb{E}[\mathbb{1}_{\{Y \geq y\}} a(Y)] + y \\
&\leq y \left[\mathbb{E}[a(Y)] + 1\right]
\end{aligned}
$$

This inequality is useful, because the inequality $\mathbb{E}a^{-1}[|X|] \leq a^{-1}(1) \left[\mathbb{E}[|X|] + 1\right]$ implies that $\mathbb{E}a^{-1}[|X|] < \infty$. That means as of now, we have to consider only this case: $\mathbb{E}[|X|] = \infty$. We suppose that:

$$
J_{-}(a) + J_{+}(a) < \infty
$$

We define $H(x) := \mathbb{P}[|X_1| \leq x] = F(x) - F(-x^{-})$ for $x > 0$. Set

$$
h(x) = \min\left(a^{-1}(x), \frac{x}{\int_0^x \mathbb{P}[|X_1| \geq t]dt}\right) \tag{1}
$$

With short calculation we obtain:

$$
\begin{aligned}
\int_0^x \mathbb{P}[|X_1| \geq t]dt &= \int_0^x 1 - (F(t) - F(-t))dt \\
&= \int_0^x 1 - F(t) + F(-t)dt \\
&= \int_0^x 1 - F(t)dt + \int_0^x F(-t)dt = m_{+}(x) + m_{-}(x)
\end{aligned} \tag{2}
$$

By means of the (2) we estimate the integral of $h(x)$ respect to the $H(x)$.

$$\begin{aligned}\int_0^\infty h(x)dH(x) &= \int_0^\infty \min\left(a^{-1}(x), \frac{x}{m_+(x)+m_-(x)}\right) dH(x)\\ &= \int_0^\infty \min\left(a^{-1}(x), \frac{x}{m_+(x)+m_-(x)}\right) dF(x)+ \\ &+ \int_0^\infty \min\left(a^{-1}(x), \frac{x}{m_+(x)+m_-(x)}\right) d(1-F(-x))\\ &\leq J_+(a)+J_-(a) < \infty\end{aligned} \tag{3}$$

The fact that $m_+(a) \leq m_-(a)+m_+(a)$ and $m_-(a) \leq m_-(a)+m_+(a)$ implies the last inequality. We know that $a^{-1}(x)$ is nondecreasing, since $a(x)/x$ is nondecreasing. At first we have noted that m is absolutely continuous on bounded intervals and this implies that

$$m'(x) = 1 - H(x),$$

consequently the function $x \to x/m(x)$,$x > 0$, is absolutely continuous on intervals $[a,b]$ too, $0 < a < b < \infty$ and is nondecreasing because

$$\left[\frac{x}{m(x)}\right]' = \frac{m(x) - x(1-H(x))}{m^2(x)} \geq 0 \text{ a.e..}$$

Therefore we obtain that

$$h(x) \text{ is nondecreasing and } h(x)/x \text{ is nonincreasing}$$

Now we estimate the $m(y)$:

$$\begin{aligned}m(y) &= \int_0^y \mathbb{P}[|X| \geq t]dt\\ &= \int_0^\infty min(t,y)dH(t)\\ &= \int_0^{x_0} min(t,y)dH(t) + \int_{x_0}^\infty min(t,y)dH(t)\\ &\leq x_0 + \int_{x_0}^\infty t\cdot\frac{h(t)}{t}dH(t)\\ &\leq x_0 + y(h(y))^{-1}\int_{x_0}^\infty h(t)dH(t)\end{aligned}$$

Choose x_0 so that $\int_{x_0}^{\infty} h(x)dH(x) \leq 1/2$. Since $\mathbb{E}[|X|] = \infty$ we obtain:

$$h(y)y^{-1} \int_0^y \mathbb{P}[|X| \geq t]dt \leq x_0 \frac{h(y)}{y} + \frac{1}{2} \to \frac{1}{2}$$

and

$$h(y) = a^{-1}(y) \text{ for all large y}$$

This and (3) finish the proof. □

Set $c = \mathbb{E}[X]/(\lim_{n\to\infty} a_n/n)$ if $\mathbb{E}[|X|] < \infty$ and $c = 0$ otherwise. With the combination of theorems 2.2, 4.1, we obtain the following proposition:

Proposition 4.1. *Let $(X_n)_{n\geq 1}$ be a sequence of random variables on some probability space with finite mean. For any sequence $(a_n)_{n\geq 1}$ for which $\frac{a_n}{n}$ is nondecreasing the following holds*

$$J_-(a) + J_+(a) < \infty \;\Leftrightarrow\; \mathbb{P}\left(\lim_{n\to\infty} \frac{S_n}{a_n} = c\right) = 1$$

Proof. Without loss of generality $\mathbb{E}[X] = 0$.

$$\begin{aligned} J_-(a) + J_+(a) < \infty &\Leftrightarrow \mathbb{E}[a^{-1}(|X_1|)] < \infty \\ &\Leftrightarrow \int_0^\infty \mathbb{P}[a^{-1}(|X_1|) > y]dy < \infty \\ &\Leftrightarrow \int_0^\infty \mathbb{P}[|X| > a(y)]dy < \infty \\ &\Leftrightarrow \mathbb{P}[|X_1| > a_n i.o.] = 0 \Leftrightarrow \lim_{n\to\infty} \frac{X_n}{a_n} = 0 \text{ a.s.} \end{aligned}$$

□

To extend the previous theorems, we consider nonnegative random variables. The following Lemma gives us an inequality for truncated expectations of partial sums of i.i.d. nonnegative random variables. The inequality has its own interest and it is useful to study the ratio of two independent nonnegative random walks.

Lemma 4.1. *Let $Y_1, \cdots, Y_n$ be i.i.d. nonnegative random variables. Set $S_n = Y_1 + \cdots + Y_n$ and $m(x) = \int_0^x \mathbb{P}(Y \geq y)dy$. Let $C > 0$ be a constant. Then*

$$\mathbb{E}[\min(S_n, C)] \leq \min(C, n \cdot m(C)) \leq 16\mathbb{E}[\min(S_n, C)]$$

Proof. Part 1: We define $X_i = \min(Y_i, C)$ for $C > 0$ and $S'_n = \sum_{i=1}^n X_i$. We will discuss two cases, $n \cdot m(C) > 3C$ and $n \cdot m(C) = uC > 3C$. Suppose that:$n \cdot m(C) > 3C$

$$\mathbb{E}[S'_n] = n \cdot \mathbb{E}[X_1] = n \cdot m(C) = uC$$

Therefore

$$\begin{aligned}
\mathbb{P}[S_n \leq C] &= \mathbb{P}[-S_n \geq -C] \\
&= \mathbb{P}[\mathbb{E}[S'_n] - S_n \geq \mathbb{E}[S'_n] - C] \\
&= \mathbb{P}[\mathbb{E}[S'_n] - S_n \geq n \cdot m(C) - C] \\
&= \mathbb{P}[\mathbb{E}[S'_n] - S_n \geq (u-1)C] \\
&\leq (u-1)^{-2}C^{-2}n\mathbb{E}(\min(Y, C))^2 \leq \frac{u}{(u-1)^2} < \frac{3}{4}
\end{aligned}$$

Now we can show the second inequality:

$$\mathbb{E}[\min(S_n, C)] \geq C\mathbb{P}[S_n \geq C] \geq \frac{C}{4}$$

For $nm(C) = uC \leq 3C$,

$$\begin{aligned}
nm(C) &= \mathbb{E}[S'_n] \\
&\leq 8\mathbb{E}[min(S_n, C)] + \mathbb{E}[S'_n \mathbb{1}_{\{S'_n > 8C\}}]
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[S'_n \mathbb{1}_{\{S'_n > 8C\}}] &\leq \frac{n\mathbb{E}[min(Y, C)]^2 + nm(C)^2}{8C} \\
&\leq \frac{Cnm(C) + 3Cnm(C)}{8C} = \frac{nm(C)}{2}
\end{aligned}$$

$$nm(C) \leq 16\mathbb{E}[min(S_n, C)]$$

Part 2: The inequality in the other direction is a simple calculation, where you have to calculate the expected value and use the identity property of the random variables X_i. □

We will study the ratio of two independent nonnegative random sequences.

Theorem 4.2. *Let* $(W_n)_{n\geq 1}$ *and* $(V_n)_{n\geq 1}$ *be two independent sequences of i.i.d. nonnegative random variables. Suppose that*

$$\mathbb{E}[W_1] + \mathbb{E}[V_1] + \lim_{n\to\infty} a_n/n = \infty.$$

Then the following statements are equivalent.

(i) $\lim\limits_{n\to\infty} \frac{W_1+...+W_n}{a_n+V_1+...+V_n} = 0$ *a.s.*

(ii) $\limsup\limits_{n\to\infty} \frac{W_1+...+W_n}{a_n+V_1+...+V_n} < \infty$ *a.s.*

(iii) $\limsup\limits_{n\to\infty} \frac{W_n}{a_n+V_1+...+V_n} < \infty$ *a.s.*

(iv) $\sum_{n=1}^{\infty} \mathbb{P}(\delta W_n > a_n + V_1 + ... + V_n) < \infty$ *for some* $\delta > 0$

(v) $\sum_{n=1}^{\infty} \mathbb{P}(\delta W_n > a_n + V_1 + ... + V_n) < \infty$ *for all* $\delta > 0$

(vi) $\int_0^{\infty} \min(a^{-1}(x), \frac{x}{m_V(x)}) d\mathbb{P}(W_1 \leq x) < \infty.$

Proof.

- $(i) \Rightarrow (ii)$: Let $(x_n)_{n\geq 1}$ be an arbitrary nonnegative sequence such that $\lim\limits_{n\to\infty} x_n = 0$. Then we know that $\limsup\limits_{n\to\infty} x_n < \infty$.

- $(ii) \Rightarrow (iii)$: Since W_n is nonnegative, we can see that:

$$\frac{W_n}{a_n + V_1 + \cdots + V_n} \leq \frac{W_1 + \cdots + W_n}{a_n + V_1 + \cdots + V_n} \quad \forall n \in \mathbb{N}$$

- $(iii) \Rightarrow (iv)$: There are some constants $\delta > 0$ and $k < \infty$ such that:

$$\mathbb{P}\left[\cap_{n=k}^{\infty} \left(\delta W_n > a_n + V_1 + \cdots + V_n\right)\right] \leq 1 - \delta < 1. \tag{1}$$

  At first we define the following events for the simplicity:

$$A_n := \left[\delta W_n > a_n + V_1 + \cdots + V_n\right].$$

  Our aim is to find an overset of $A_n \cap A_{n+m}$, where the sets are independent.

$$A_n \cap A_{m+n} \subset A_n \cap \left[\delta W_{m+n} > a_n + V_m + \cdots + V_{n+m}\right]$$

The independence and stationary implies the following inequality:

$$\mathbb{P}\left[A_n \cap A_{n+m}\right] \leq \mathbb{P}[A_n]\mathbb{P}[A_m].$$

Now we have to generalize the previous statement and then we can use our assumption (1)

$$\mathbb{P}\left[A_n \cap \cup_{j=1}^{\infty} A_{n+jk}\right] \leq \mathbb{P}[A_n]\mathbb{P}\left[\cup_{j=1}^{\infty} A_{jk}\right] \leq \mathbb{P}[A_n](1-\delta).$$

By means of set operation we get the following estimate of $\delta\mathbb{P}[A_n]$.

$$\delta\mathbb{P}[A_n] \leq \mathbb{P}\left[A_n \cap \left(\cup_{j=1}^{\infty} A_{(n+j)k}\right)^c\right]$$

The previous statement gives us

$$\delta \sum_{n=1}^{N} \mathbb{P}\left[A_{nk+i}\right] \leq \sum_{n=1}^{N} \mathbb{P}\left[A_{nk+i} \cap \left(\cup_{j=1}^{\infty} A_{(n+j)k+i}\right)^c\right] \leq 1$$

That means, that for all $\delta > 0$ exist some $N \in \mathbb{N}$. Therefore

$$\sum_{n=1}^{\infty} \mathbb{P}(\delta W_n > a_n + V_1 + ... + V_n) < \infty \text{ for some } \delta > 0$$

- $(iv) \Rightarrow (v)$: We already saw that $a(2n) \geq 2a(n)$, and with a short calculation we obtain

$$\mathbb{P}[2\delta W_{2n+1} > a_{2n+1} + V_1 + \cdots + V_{2n+1}] \\ \leq \mathbb{P}[2\delta W_{2n} > a_{2n} + V_1 + \cdots + V_{2n}].$$

Therefore

$$\sum_{n=1}^{\infty}\mathbb{P}(2\delta W_n > a_n + V_1 + ... + V_n) \\ \leq 1 + 2\sum_{n=1}^{\infty}\mathbb{P}(2\delta W_{2n} > a_{2n} + V_1 + ... + V_{2n}) \\ \leq 1 + 4\sum_{n=1}^{\infty}\mathbb{P}(\delta W_n > a_n + V_1 + ... + V_n)$$

- $(v) \Rightarrow (vi) \Rightarrow (iv)$: Let stopping time $T := inf\{k : S_k > C\}$, $S_k = \sum_{i=1}^k V_i$, $S'_k = \sum_{i=1}^k \min(V_i, C)$ for $C > 0$. Then, from Wald's lemma follows that

$$\int_0^C \mathbb{P}[V_1 \geq t]dt\mathbb{E}[\min(T,n)] = \mathbb{E}[S'_{min(T,n)}]$$
$$\Leftrightarrow \mathbb{E}[\min(T,n)] = \frac{\mathbb{E}[S'_{min(T,n)}]}{\int_0^C \mathbb{P}[V_1 \geq t]dt} = \frac{\mathbb{E}[S'_{min(T,n)}]}{m(C)}$$

Now we have to calculate the expected value of the stopping time $\min(T, n)$ for $n \in \mathbb{N}$

$$\mathbb{E}[min(T,n)] = 1 + \sum_{k=1}^{n} \mathbb{P}[V_1 + \cdots + V_n \leq C]$$

To see this, we have to start with $n = 2$ as a usual expected value.

$$\begin{aligned} \mathbb{E}[min(T,2)] &= 1 \cdot \mathbb{P}[V_1 > C] + 2 \cdot \mathbb{P}[V_1 \leq C] \\ &= 1 \cdot (1 - \mathbb{P}[V_1 \leq C]) + 2 \cdot \mathbb{P}[V_1 \leq C] \\ &= 1 + \mathbb{P}[V_1 \leq C] \end{aligned}$$

Therefore, by Lemma 1.1:

$$\begin{aligned} &\sum_{n=1}^{\infty} \mathbb{P}[W_n > a_n + V_1 + \cdots + V_n] \\ &\quad \leq \int_0^\infty \sum_{a_1 \leq a_n < x} \mathbb{P}[x > V_1 + \cdots + V_n] d\mathbb{P}[W_1 \leq x] \\ &\quad \leq 2 \int_0^\infty \min\left(a^{-1}(x), \frac{x}{\int_0^x \mathbb{P}[V_1 \geq t]dt}\right) d\mathbb{P}[W_1 \leq x] \\ &\quad \leq 32 + 32 \int_0^\infty \sum_{a_1 \leq a_n < x} \mathbb{P}[x \geq V_1 + \cdots + V_n] d\mathbb{P}[W_1 \leq x] \\ &\quad \leq 32 + 32 \int_0^\infty \sum_{n=1}^{\infty} \mathbb{P}[2x > a_n + V_1 + \cdots + V_n] d\mathbb{P}[W_1 \leq x] \\ &\quad = 32 + 32 \sum_{n=1}^{\infty} \mathbb{P}[2W_1 > a_n + V_1 + \cdots + V_n] d\mathbb{P}[W_1 \leq x]. \end{aligned}$$

- $(iv) \Rightarrow (i)$. The first step of the proof is the same as in $(iv) \Rightarrow (v)$.

$$\sum_{n=1}^{\infty} \mathbb{P}(2\delta(W_1 + W_2) > a_n + V_1 + ... + V_n)$$
$$\leq 1 + 2\sum_{n=1}^{\infty} \mathbb{P}(2\delta(W_1 + W_2) > a_{2n} + V_1 + ... + V_{2n})$$
$$\leq 1 + 2\sum_{n=1}^{\infty} \mathbb{P}(\delta W_1 > a_n + V_1 + ... + V_n)$$
$$+ 2\sum_{n=1}^{\infty} \mathbb{P}(\delta W_2 > a_n + V_1 + ... + V_n) < \infty.$$

Let $T(M) := \inf\{n : a_n + V_1 + \cdots V_n + M \geq \delta(W_1 + W_2)\}$. Now we choose M such that $\mathbb{E}[T(M)] < 2$. Set

$$T_1 = T^{(1)} = T(M),$$
$$T^{(n)} = \inf\{j : a_j + V_{k+1} + \cdots + V_{k+j} + M \geq \delta(W_{2n-1} + W_{2n})\}$$

on $\{T_{n-1} = k\}$, and

$$T_n = T_{n-1} + T^{(n)},\ n = 2, 3, ...$$

Then, as per the strong law of large numbers there exists an integer-valued random variable N such that

$$T_n = T^{(1)} + \cdots + T^{(n)} < 2n - 1 \text{ for any } n \geq N$$

Since $\frac{a_n}{n}$ is nondecreasing,

$$a(T^{(1)} + \cdots + T^{(n)}) \leq a(T_n) < a(2n-1) \text{for any } n \geq N$$
$$\delta(W_1 + \cdots + W_{2n}) \leq nM + a(2n-1) + V_1 + \cdots + V_{2n-1}\ n \geq N$$

By the condition that $\mathbb{E}[W_1] + \mathbb{E}[V_1] + \lim_{n\to\infty} \frac{a_n}{n} = \infty$,

$$\lim_{n\to\infty} \frac{nM}{a_n + V_1 + \cdots + V_n + W_1 + \cdots + W_n} = 0\ a.s.$$

Hence

$$\lim_{n\to\infty} \frac{W_1 + \cdots + W_n}{a_n + V_1 + \cdots + V_n} \leq \frac{1}{\delta}\ a.s.$$

Since δ is arbitrary, follows (i).

□

At this moment we consider the ratio of the positive and negative contributions X^- of the random walks. By means of the previous theorem we can show that if we know something about the random variable $|X|$ or the limes of $\frac{a_n}{n}$, then there are two possible outcomes of the integral test $J_+(a)$.

Theorem 4.3. *Suppose that* $\mathbb{E}[|X|] = \infty$ *or* $\lim_{n\to\infty} \frac{a_n}{n} = \infty$. *Then, one of the following alternatives must prevail:*

(i) $J_+(a) = \infty$ *and* $\mathbb{P}\left[\limsup_{n\to\infty} \frac{X_n^+}{a_n+X_1^-+\cdots+X_n^-} = \infty\right] = 1$

(ii) $J_+(a) < \infty$ *and* $\mathbb{P}\left[\lim_{n\to\infty} \frac{X_1^++\cdots+X_n^+}{a_n+X_1^-+\cdots+X_n^-} = 0\right] = 1$

To prove the previous theorem we need the following lemma about stopping time. You find the Introduction to the stopping times and finite times in Appendix chapter under *Stopping time*.

Lemma 4.2. *If* $(X_n)_{n\geq 1}$ *is a sequence of i.i.d. random variables in some probability space* $(\Omega, \mathcal{A}, \mathbb{P})$*,* T *is a finite stopping time with respect to the* $\{\mathcal{F}_n\}$ *where* $\mathcal{F}_n = \sigma(X_1, ..., X_n)$*, then* $\mathcal{F}_T$ *and* $\sigma(X_T, ...)$ *are independent and* $(X_{T+n})_{n\geq 1}$ *are i.i.d. with the same distribution as* X_1.

The proof of this lemma contains the usual steps as a proof of theorems with stopping time statements. That means we start the proof with some set $A \in \mathcal{F}_T$ and show the same equality for Ω.

Proof. Let $\lambda_1, ..., \lambda_n \in \mathbb{R}$ and $A \in \mathcal{F}_T$

$$\mathbb{P}[A \cap_{i=1}^n [X_{T+i} < \lambda_i]] = \sum_{k=1}^{\infty} \mathbb{P}[A \cap [T = k] \cap_{i=1}^n [X_{k+i} < \lambda_i].$$

By assumption, $A \cap [T = k] \in \mathcal{F}_k$ for all $k \geq 1$, and we get

$$\sum_{k=1}^{\infty} \mathbb{P}[A \cap [T = k] \cap_{i=1}^n [X_{k+i} < \lambda_i]] = \sum_{k=1}^{\infty} \mathbb{P}[A \cap [T = k]]\mathbb{P}[\cap_{i=1}^n [X_{k+i} < \lambda_i]].$$

Hence

$$\begin{aligned}\mathbb{P}[A \cap_{i=1}^n [X_{T+i} < \lambda_i]] &= \sum_{k=1}^{\infty} \mathbb{P}[A \cap [T = k]] \prod_{i=1}^{n} \mathbb{P}[X_i < \lambda_i] \\ &= \mathbb{P}[A] \prod_{i=1}^{n} \mathbb{P}[X_i < \lambda_i]\end{aligned}$$

and now we choose $A = \Omega$ and we know that

$$\mathbb{P}[\cap_{i=1}^{n} X_{T+i} < \lambda_i] = \prod_{i=1}^{n} \mathbb{P}[X_i < \lambda_i], \text{ for } 1 \leq i \leq n.$$

Since n arbitrary integer , $(X_{T+n})_{n\geq 1}$ i.i.d sequence of random variables with the same distribution as X_1. Therefore we obtain that

$$\mathbb{P}[A \cap_{i=1}^{n} [X_{T+i} < \lambda_i]] = \mathbb{P}[A] \prod_{i=1}^{n} \mathbb{P}[X_{T+i} < \lambda_i]$$

Since $\lambda_1, ..., \lambda_n$ are arbitrary we get $\mathcal{F}_T$ and $\sigma(X_{T+1}, X_{T+2}, ...)$ are independent. □

Proof. (theorem 4.3):

We may assume that $\mathbb{P}[X > 0] \neq 0$ and $\mathbb{P}[X < 0] \neq 0$. Let

$$T_0 = S_0 = 0,\ T_1 := T^{(1)} = \inf\{k \geq 1 : X_k > 0\}$$
$$T^{(n)} = \inf\{k \geq 1 : X_{k+T_n} > 0\},\ T_n = T_{n-1} + T^{(n)},\ n \geq 2,$$

and

$$W_n = X_{T_n},\ V_n = -\left(S_{T_n} - S_{T_{n-1}} - W_n\right),\ n \geq 1.$$

The random variables V_n, W_n are nonnegative, therefore we can hope to use the theorem 2.4. But first we will consider the sequences $(V_n)_{n\geq 1}$, $(W_n)_{n\geq 1}$. Since $T^{(n)}$ are copies of $T^{(1)}$, one can show that (V_n, W_n) are i.i.d..

$$R_n := (V_n, W_n) = \left(T^{(n)} = m, -(X_{T_{n-1}+1} + X_{T_{n-1}+2} + \cdots + X_{T_n-1}), X_{T_n}\right)$$

It is easily seen from the definition,that T_n is a finite stopping time and that R_n $\mathcal{F}_{T_n}$-measurable and hence also the $(R_1, ..., R_n)$. By the lemma 4.2 we can see that $\sigma(X_{T_n+1}, X_{T_n+2}...)$ is independent of $\mathcal{F}_{T_n}$ $n \geq 1$, and that the $T^{(n+1)}$ and $(X_{T_n+1}, ..., X_{T_{n+1}})$ are $\sigma(X_{T_n+1}, X_{T_n+2}...)$-measurable. Now we need the combination of the previous statements and then we obtain that: $\mathcal{F}_{T_n}$and $\sigma(R_{n+1})$ are independent for $n \geq 1$. Therefore has to be R_{n+1} independent of $(R_1, ..., R_n)$ and so we get that the sequence $(R_n)_{n\geq 1}$ is independent. It remains to show that R_n are identically distributed.

By means of lemma A.1 there are sets $C_m \in (\mathbb{R}^m, \mathcal{B}^m)$ we have for λ_1, $\lambda_2 \in \mathbb{R}$ for all $n \geq 1$ and for some $m \geq 1$:

$$
\begin{aligned}
q_n &:= \mathbb{P}[T^{(n)} = m, V_n < \lambda_1, W_n < \lambda_2] \\
&= \mathbb{P}[T^{(n)} = m, -(X_{T_{n-1}+1} + X_{T_{n-1}+2} + \cdots + X_{T_n - 1}) < \lambda_1, X_{T_n} < \lambda_2] \\
&= \mathbb{P}[T^{(n)} = m, \sum_{i=1}^{m-1} X_{T_{n-1}+i} < \lambda_1, X_{T_{n-1}+m} < \lambda_2] \\
&= \mathbb{P}[(X_{T_{m-1}+1}, X_{T_{n-1}+2}, ..., X_{T_{n-1}+m}) \in C_m, \sum_{i=1}^{m-1} X_{T_{n-1}+i} < \lambda_1, X_{T_{n-1}+m} < \lambda_2] \\
&= \mathbb{P}[(X_1, X_2, ..., X_m) \in C_m, \sum_{i=1}^{m-1} X_i < \lambda_1, X_m < \lambda_2] \\
&= \mathbb{P}[T^{(1)} = m, \sum_{i=1}^{m-1} X_i < \lambda_1, X_m < \lambda_2] \\
&= \mathbb{P}[T^{(1)} = m, V_1 < \lambda_1, W_1 < \lambda_2] = q_1.
\end{aligned}
$$

Since m arbitrary,

$$\mathbb{P}[V_1 < \lambda_1, W_1 < \lambda_2] = \mathbb{P}[V_n < \lambda_1, W_n < \lambda_2].$$

Therefore we obatain that $(R_n)_{n \geq 1}$ are i.i.d. random variables. Now we have to find some connection between $J_+(a)$ and the integral test of V respect to the W. At first we calculate the distribution of W_1 and a lower bound of $\mathbb{E}[W_1]$:

$$\mathbb{P}[W_1 > t] = \frac{\mathbb{P}[X > t]}{\mathbb{P}[X > 0]}, \ \mathbb{E}[W_1] \geq \mathbb{E}[X_1^+], \tag{1}$$

We can write V_1 as:

$$V_1 = \sum_{n=1}^{T_1} X_n^- \geq X_1^-$$

That implies , $\mathbb{E}[V_1] \geq \mathbb{E}[X_1^-]$. We calculate an upper bound of the integral:

$$
\begin{aligned}
\int_0^x \mathbb{P}[V_1 \geq t] dt = \mathbb{E}[min(V_1, x)] &\leq \mathbb{E}[\sum_{n=1}^{T} min(X_n^-, x)] \\
&= \mathbb{E}[T]\mathbb{E}[min(X_1^-, x)],
\end{aligned} \tag{2}
$$

Our assumption was that X_i are independent. Therefore the probability of $\{T_1 > n\}$ is equal to nth power of $\mathbb{P}[X_1 \le 0]$.

$$\mathbb{P}[T_1 > n] = (\mathbb{P}[X_1 \le 0])^n .$$

That implies:

$$\mathbb{E}[T_1] < \infty. \tag{3}$$

The statements (1), (2), (3) lead us to the observation:

$$\begin{aligned} &J_+(a) < \infty \Leftrightarrow \\ \int_0^\infty \min &\left(a^{-1}(x), \frac{x}{\int_0^x \mathbb{P}[V_1 \ge t]dt} \right) d\mathbb{P}[W_1 \le x] < \infty. \end{aligned} \tag{4}$$

Let k be an integer with $\mathbb{E}[T_1 > k]$. As per strong law of large numbers

$$\mathbb{P}[T_n \ge kn \; i.o.] = 0.$$

Now we will see, how the X_n^+, X_n^- and V_n, W_n are related to each other.

$$\begin{aligned} S_{T_n} &= S_{T_{n-1}} + (W_n - V_n) \\ S_{T_n - 1} &= S_{T_{n-1}} - V_n \end{aligned}$$

We consider the following limit:

$$\begin{aligned} \limsup_{n\to\infty} \frac{X_n^+}{a_n + X_1^- + \cdots + X_n^-} &\ge \limsup_{n\to\infty} \frac{W_n}{a(T_n) + V_1 + \cdots + V_n} \\ &\ge \limsup_{n\to\infty} \frac{W_n}{a(kn) + V_1 + \cdots + V_{kn}} \\ &\ge \limsup_{n\to\infty} \frac{W_{n+i}}{a(kn+i) + V_1 + \cdots + V_{kn+i}} \quad \forall i \\ &= \limsup_{n\to\infty} \frac{W_n}{a_n + V_1 + \cdots + V_n} \quad a.s. \end{aligned}$$

From this statement and theorem 1.4 and (4) follows that:

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{X_n^+}{a_n + X_1^- + \cdots + X_n^-} = \infty \right] = 1 \text{ if } J_+(a) = \infty$$

Part 2. In the other case the calculation is similar to Part 1:

$$\begin{aligned}\limsup_{n\to\infty}\frac{X_1^+ + \cdots + X_n^+}{a_n + X_1^- + \cdots + X_n^-} &\leq \limsup_{n\to\infty}\frac{W_1+\cdots+W_n}{a(T_{n-1}) + V_1 + \cdots + V_{n-1}}\\ &\leq \limsup_{n\to\infty}\frac{W_2+\cdots+W_n}{a_{n-1} + V_2 + \cdots + V_{n-1}}\\ &= \limsup_{n\to\infty}\frac{W_1+\cdots+W_n}{a_n + V_1 + \cdots + V_n}\ \ a.s.\end{aligned}$$

From this statement and from theorem 1.4 and (4) follows that:

$$\mathbb{P}[\limsup_{n\to\infty}\frac{X_1^+ + \cdots + X_n^+}{a_n + X_1^- + \cdots + X_n^-} = 0] = 1$$

if $J_+(a) < \infty$.

□

In the next theorem we consider the following case: at least one of $J_+(a)$, $J_-(a)$ is infinite. With these cases we can extend theorem 3.2.

Theorem 4.4. *Let $(X_n)_{n\geq 1}$ be a i.i.d. sequence in some probability space, then we obtain that:*

(i) $J_+(a) = \infty$ *if and only if* $\mathbb{P}\left[\limsup_{n\to\infty}\frac{S_n}{a_n} = \infty\right] = 1$

(ii) $J_-(a) < J_+(a) = \infty$ *if and only if*
$\mathbb{P}\left[\liminf_{n\to\infty}\frac{S_n}{a_n} = \liminf_{n\to\infty}\left(\frac{|X_1|+\cdots+|X_n|}{a_n}\right)\right] = 1$ *and*
$\mathbb{P}\left[\limsup_{n\to\infty}\frac{S_n}{a_n} = \infty\right] = 1$

To prove the previous theorem, we need some lemmas. The statement and the proof of the following lemma are similar to the lemma 3.4.

Lemma 4.3. *If*

$$\limsup_{n\to\infty}\frac{X_n^+}{a_n + X_1^- + \cdots + X_n^-} = \infty \tag{4.1}$$

then $\mathbb{E}[X_1^+] = \infty$ *and* $\limsup_{n\to\infty}\frac{S_n}{a_n} = \infty$ *a.s..*

We record the following proposition easy consequence of theorem 4.3.

Proposition 4.2. *Let $(X_i)_{i\geq 1}$ be some sequence of i.i.d. random variables in some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Suppose that $\mathbb{E}[|X_1|] + \lim\limits_{n\to\infty} \frac{a_n}{n} = \infty$. Then $J_+(a) < \infty$ and*

$$\frac{S_n}{n} = \frac{-\left(\sum_{i=1}^n X_i^-\right)}{a_n}(1 + o(1)) + o(1) \ a.s.$$

This statement equivalent to:

$$\sum_{i=1}^n X_i^+ = o\left(-\sum_{i=1}^n X_i^-\right) + o(1). \tag{4.2}$$

Proposition 4.3. *Suppose that $\lim \frac{a_n}{n} = \infty$, Then it is impossible for any random walk, that*

$$-\infty < \liminf \frac{S_n}{a_n} < 0.$$

Proof. (Theorem 4.4)

Part 1(i) ($\Rightarrow$): Suppose $J_+(a) = \infty$. By means of theorem 4.3 and lemma 4.3 we obtain that:

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{S_n}{a_n} = \infty\right] = 1. \tag{1}$$

($\Leftarrow$): Suppose that $\mathbb{P}\left[\limsup\limits_{n\to\infty} \frac{S_n}{a_n} = \infty\right] = 1$. This implies a statement: $\mathbb{E}[X_1^+] = \infty$ or $\lim\limits_{n\to\infty} \frac{a_n}{n} = \infty$. By (1) we get, that Theorem 4.3 (*ii*) is impossible and therefore $J_+(a) = \infty$.

Part 2 (ii) $\Rightarrow$: At first we change X^+ with X^- and by (*i*) we get that

$$J_-(a) < \infty \ \Leftrightarrow \mathbb{P}\left[\liminf_{n\to\infty} \frac{S_n}{a_n} > -\infty\right] = 1.$$

Since $J_+(a) = \infty$ we obtain that $\lim\limits_{n\to\infty} \frac{a_n}{n} = \infty$ or $\mathbb{E}[|X_1|] = \infty$. Therefore if $J_-(a) < \infty$ we get:

$$\sum_{i=1}^n X_i^- = o\left(+\sum_{i=1}^n X_i^+\right) + o(1).$$

Finally we obtain that:

$$\mathbb{P}\left[\liminf_{n\to\infty}\frac{S_n}{a_n}=\liminf_{n\to\infty}\left(\frac{|X_1|+\cdots+|X_n|}{a_n}\right)\right]=1.$$

$\Leftarrow$: At first suppose that $\mathbb{P}\left[\limsup_{n\to\infty}\frac{S_n}{a_n}=\infty\right]=1$ we get $J_+(a)=\infty$. Now we suppose that: $\mathbb{P}\left[\liminf_{n\to\infty}\frac{S_n}{a_n}=\liminf_{n\to\infty}\left(\frac{|X_1|+\cdots+|X_n|}{a_n}\right)\right]=1$ and this implies:

$$\liminf_{n\to\infty}\left(\frac{|X_1|+\cdots+|X_n|}{a_n}\right)>-\infty \text{ a.s.}$$

and by means of (i) we obtain that $J_-(a)<\infty$. □

A Appendix

A.1 Analysis

Theorem A.1. *(Kronecker lemma) Let* $(x_n)_{n\geq 1}$ *a sequence with* $x_n \in \mathbb{R}$ $\forall n \geq 1$ *and a sequence* $(a_n)_{n\geq 1}$ *with* $0 < a_n \leq a_{n+1}$ $\forall n \geq 1$ *and* $\lim_{n\to\infty} a_n = \infty$. *If the sequence* $(s_n)_{n\geq 1}$

$$s_n := \sum_{k=1}^{n} \frac{x_k}{a_k}$$

converges, then follows:

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} x_k = 0.$$

Proof. There exist $s \in \mathbb{R}$ such that $s = \lim_{n\to\infty} s_n$ since the sequence s_n is convergent. We will use the summation by parts

$$\frac{1}{a_n} \sum_{k=1}^{n} x_k = s_n - \frac{1}{a_n} \sum_{k=1}^{n-1} (a_{k+1} - a_k) s_k. \tag{1}$$

We split the series of right hand side (1) in two sums and we use usual trick $(a = b + (a - b))$. Pick $\epsilon > 0$ and we can choose $N \in \mathbb{N}$ such that s_n is $\epsilon - close$ to s for all $k > N$.

$$\begin{aligned}
&\frac{1}{a_n} \sum_{k=1}^{N-1} (a_{k+1} - a_k) s_k - \frac{1}{a_n} \sum_{k=N}^{n-1} (a_{k+1} - a_k) s_k \\
&= \frac{1}{a_n} \sum_{k=1}^{N-1} (a_{k+1} - a_k) s_k - \frac{1}{a_n} \sum_{k=N}^{n-1} (a_{k+1} - a_k) s - \frac{1}{a_n} \sum_{k=N}^{n-1} (a_{k+1} - a_k)(s_k - s) \\
&= \frac{1}{a_n} \sum_{k=1}^{N-1} (a_{k+1} - a_k) s_k - \frac{1}{a_n} (a_n - a_N) s - \frac{1}{a_n} \sum_{k=N}^{n-1} (a_{k+1} - a_k)(s_k - s).
\end{aligned}$$

Now, we let n go to infinity and we see that the first term go to 0 since $\frac{1}{a_n}$ go to zero and the series is finite . The second term converges to s. Since the $(a_k)_{k\geq 1}$ sequence is increasing, the last term is bounded by $\epsilon(a_n - a_N)/a_n \leq \epsilon$.

Therefore we obtain for (1):

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} x_k = \lim_{n\to\infty} s_n - (0 - s + 0) = s - s = 0$$

□

A.2 Probability Theory

A.2.1 First auxiliary results

Theorem A.2. *(Borel-Cantelli) Let* $(A_n)_{n\geq 1}$ *be a sequence of events in some probability space* $(\Omega, \mathcal{A}, \mathbb{P})$*. If*

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$$

then

$$\mathbb{P}(\{A_n \ i.o.\}) = 0$$

Proof.

$$\begin{aligned}
\mathbb{P}(\{A_n \ i.o.\}) &= \mathbb{P}(\limsup_{n\to\infty} A_n) \\
&= \lim_{n\to\infty} \mathbb{P}(\cup_{k=n}^{\infty} A_k) \\
&\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k)
\end{aligned}$$

and the tail series converges to zero for $n \to \infty$ under the assumption the series converges. □

Theorem A.3. *(Converse Borel-Cantelli)Let* $(A_n)_{n\geq 1}$ *a sequence of independent events in some probability space* $(\Omega, \mathcal{A}, \mathbb{P})$*. If*

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$$

then

$$\mathbb{P}(\{A_n \ i.o.\}) = 1$$

Proof. First note that

$$\mathbb{P}(\{A_n \ i.o.\}^c) = \mathbb{P}(\liminf_{n\to\infty} A_n^c) = \lim_{n\to\infty} \mathbb{P}(\cap_{k=n}^{\infty} A_k^c)$$

because of independence of A_n hence we have independent of A_n^c. Therefore we see that for every $n \in \mathbb{N}$

$$\begin{aligned}
\mathbb{P}(\cap_{k=n}^{\infty} A_k^c) &= \lim_{N\to\infty} \mathbb{P}(\cap_{k=n}^{N} A_k^c) \\
&= \lim_{N\to\infty} \prod_{k=n}^{N} (\mathbb{P}(1 - A_k)) \\
&\leq \lim_{N\to\infty} \prod_{k=n}^{N} (\exp(-\mathbb{P}(A_k))) \\
&= \lim_{N\to\infty} \exp(\sum_{k=n}^{N} (-\mathbb{P}(A_k))) = 0
\end{aligned}$$

since the series diverges, then the exponent approaches to $-\infty$. □

Theorem A.4. *(Khintchine and Kolgomorov) Suppose X_n are independent random variable with mean zero such that the sum of variances is finite,*

$$\sum_{n=1}^{\infty} Var(X_n) < \infty.$$

Then the sum of the random variables converges almost surely ,

$$\sum_{n=1}^{\infty} X_n \ converges \ a.s..$$

Proof. We define $A_{m,\epsilon} = \{\max_{j>m} |S_j - S_m| \leq \epsilon\}$. Hence

$$\{\sum_{n=1}^{\infty} X_n \ converges\} = \cap_{\epsilon>0} \cup_m A_{m,\epsilon},$$

by Kolgomorov inequality we obtain:

$$\begin{aligned}
\mathbb{P}[\max_{n\geq j>m} |S_j - S_m| > \epsilon] &\leq \frac{Var(S_n - S_m)}{\epsilon^2} \\
&\leq \frac{1}{\epsilon^2} \sum_{i=m+1}^{n} Var(X_i) \leq \frac{1}{\epsilon^2} \sum_{i=m+1}^{\infty} Var(X_i)
\end{aligned}$$

By letting n to infinity and then m to infinity, we have

$$\lim_{m\to\infty} \mathbb{P}[\max_{j>m} |S_j - S_m| > \epsilon] = 0.$$

Then the $\lim_{m\to\infty} \mathbb{P}[A_{m,\epsilon}] = 1$ and so $\mathbb{P}[\cup_{m\geq 1} A_{m,\epsilon}] = 1$ for every $\epsilon > 0$. Hence we obtain

$$\mathbb{P}[\sum_{n=1}^{\infty} X_n \textit{ converges}] = \mathbb{P}[\cap_{\epsilon>0} \cup_{m\geq 1} A_{m,\epsilon}] = 1$$

and almost surely convergence holds for S_n. □

Theorem A.5. *Let τ be a stopping time with respect to an i.i.d. sequence $(X_n)_{n\geq 1}$. If $\mathbb{E}[\tau] < \infty$ and $\mathbb{E}[|X_1|] < \infty$, then*

$$\mathbb{E}\left[\sum_{n=1}^{\tau} X_n\right] = \mathbb{E}[X_1] \cdot \mathbb{E}[\tau]$$

Proof.

$$\sum_{n=1}^{\tau} X_n = [\sum_{n=1}^{\infty} X_n \mathbb{1}_{\{\tau>n-1\}}]$$

Since the sequence $(X_n)_{n\geq 1}$ i.i.d., X_n is independent of $\{X_1, ..., X_{n-1}\}$, so that X_n is independent of event $\{\tau > n-1\}$.

$$\mathbb{E}\left[X_n \mathbb{1}_{\{\tau>n-1\}}\right] = \mathbb{E}[X_1] \cdot \mathbb{E}[\mathbb{1}_{\tau>n-1}]$$

By means of the previous statement we get:

$$\begin{aligned}
\mathbb{E}\left[\sum_{n=1}^{\tau} X_n\right] &= \mathbb{E}[X_1] \left(\sum_{n=1}^{\infty} \mathbb{E}[\mathbb{1}_{\tau>n-1}]\right) \\
&= \mathbb{E}[X_1] \left(\sum_{n=1}^{\infty} \mathbb{P}[\tau > n-1]\right) \\
&= \mathbb{E}[X_1] \left(\sum_{n=0}^{\infty} \mathbb{P}[\tau > n]\right) \\
&= \mathbb{E}[X_1] \cdot \mathbb{E}[\tau]
\end{aligned}$$

□

A.2.2 Convolution

Let X, Y be two independent real valued random variable in some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with density functions $f_1(x)$ and $f_2(x)$. Let $Z = X + Y$ a random variable and we want determinate the distribution function $f_3(x)$ of Z. The convolution $f_3(x) = f_1(x) * f_2(x)$ is given by

$$f_3(x) = \int_{-\infty}^{\infty} f_1(y) \cdot f(x-y)dy.$$

In general case: Let $(\Omega, \mathcal{A}, \mathbb{P}_1)$, $(\Omega, \mathcal{A}, \mathbb{P}_2)$ be a probability spaces, $\mathcal{A} \otimes \mathcal{A}$ $product - \sigma - algebra$ on $\Omega \times \Omega$ and $\mathbb{P}_1 \otimes \mathbb{P}_2$ product measure. The function

$$X : \Omega \times \Omega \to \Omega$$

defined as

$$(x, y) \mapsto x + y$$

is a $\mathcal{A} \otimes \mathcal{A} - \mathcal{A}$ measurable function. The image measure $\mathbb{P}_1 \otimes \mathbb{P}_2$ of X called the convolution of probability measure $\mathbb{P}_1$ and $\mathbb{P}_2$. The convolution is given by

$$\mathbb{P}_1 * \mathbb{P}_2(B) = \mathbb{P}_1 \otimes \mathbb{P}_2((x, y) \in \Omega \times \Omega | x + y \in B).$$

A.2.3 Stopping time

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(\mathcal{A}_n)_{n \geq 1}$ increasing sequence of $sub\, \sigma - algebra$. A random variable $T : \Omega \to \mathbb{N}$ is called *stopping time* if $\{\omega \in \Omega : T(\omega) \leq n\} \in \mathcal{A}_n$ holds for all $n \in \mathbb{N}$(often called $(\mathcal{A}_n) -$ *stopping time*). A stopping time called $finite$ if $\mathbb{P}[T = \infty] = 0$ and $defective$ if $\mathbb{P}[T = \infty] > 0$. Let $(X_n)_{n \geq 1}$ sequence of random variable then $\{X_n, \mathcal{A}_n, n \geq 1\}$ will be called *adapted sequence* if X_n is $\mathcal{A}_n$ measurable for all $n \geq 1$ and $\{X_n, \sigma(X_1, ..., X_n), n \geq 1\}$ is a *adapted sequence* too.

We define

$$X_T(\omega) = X_{T(\omega)}(\omega), \text{ where } X_\infty(\omega) = \liminf_{n \to \infty} X_n(\omega)$$

and

$$\mathcal{A}_\infty = \sigma(\cup_{n=1}^{\infty} \mathcal{A}_n), \ \mathcal{A}_T := \{A : A \in \mathcal{A}_\infty, A \cap [T = n] \in \mathcal{A}_n, \ n \geq 1\}$$

for any adapted sequence $\{X_n, \mathcal{A}_n, n \geq 1\}$ and any $\mathcal{A}_n -$ *stopping time* T.

Lemma A.1. *Let $(X_n)_{n\geq 1}$ be sequence of i.i.d. random variables in some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If T is a $\mathcal{A}_n - stopping$ time, where $\mathcal{A}_n = \sigma(X_1, ...X_n)$, then exist some sequence of $(C_n)_{n\geq 1}$ of disjoint Borel sets of $\mathcal{B}_{\mathbb{R}^n}$, whose corresponding bases B_n are n-dimensional Borel sets $n \geq 1$, such that*

$$\{\omega : T = n\} = \{\omega : (X_1, ..., X_n) \in C_n\}.$$

A.2.4 Zero-one laws

Theorem A.6. *Let $(X_n)_{n\geq 1}$ be sequence of independent random variables in some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Define the tail sigma-field*

$$\mathcal{T} = \cap_{n\geq 1}\sigma(X_{n+1}, X_{n+2}, ...)$$

Then $\mathcal{T}$ is trivial: $\forall A \in \mathcal{T}$ holds $\mathbb{P}[A] \in \{0, 1\}$

Proof. At first we set $\mathcal{T}_m = \sigma(X_{m+1}, X_{m+2}, ...,)$ and $\mathcal{F}_n = \sigma(X_1, ..., X_n)$. Then we obtain that $\mathcal{T}_m$ and $\mathcal{F}_n$ are independent for all $m \geq n$. Hence $\sigma(\cup_n \mathcal{F}_n)$ is also independent of $\mathcal{T}$. Let $A \in \mathcal{T}$, then A has to be in $\sigma(\cup_n \mathcal{F}_n)$. By independent we obtain that :

$$\mathbb{P}[A] = \mathbb{P}[A \cup A] = \mathbb{P}[A]\mathbb{P}[A].$$

Therefore $\mathbb{P}[A] \in \{0, 1\}$ □

Defintion A.1. *An exchangeable sequence $(X_n)_{n\geq 1}$ of random variables is a finite or infinite sequence of random variables such that for any finite permutation of the indices $1, 2...$ the joint probability distribution of the permuted sequence is the same as the joint distribution of the original sequence.*

Theorem A.7. *Let $(X_n)_{n\geq 1}$ be sequence of i.i.d. random variable in some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then the sigma field of exchangeable events $\mathcal{E}$ is trivial.*

Proof. We take $A \in \mathcal{E}$ and we approximate this event by sequence $(A_n)_{n\geq 1}$, where $A_n \in \mathcal{F}_n$ and we know that $\cup_n \mathcal{F}_n$ generated the $\sigma - field$ therefore we get $\mathbb{P}[A \triangle A_n] \to 0$. We write the event $A_n = \{(X_1, ..., X_n) \in B_n\}$ and we set $\tilde{A}_n = \{(X_{n+1}, ..., X_{2n}) \in B_n\}$. By exchangeability $A_n \mapsto \tilde{A}_n$. Then we get $\mathbb{P}[A \triangle A_n] = \mathbb{P}[A \triangle \tilde{A}_n] \to 0$ and therefore $\mathbb{P}[A_n \cap \tilde{A}_n] \to \mathbb{P}[A_n]$. By (X_i) are i.i.d. we get:

$$\mathbb{P}[A_n \cap \tilde{A}_n] = \mathbb{P}[A_n]\mathbb{P}[\tilde{A}_n] = \mathbb{P}[A_n]^2 \to \mathbb{P}[A]^2$$

Hence $\mathbb{P}[A] \in \{0, 1\}$ □

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