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„Length Structures and Geodesics on Riemannian Manifolds of Low Regularity"

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Introduction

In this thesis we deal with Riemannian geometry for metrics with low regularity. Our approach will be to rely on concepts from metric geometry such as Length structures and shortest paths as well as regularization and comparison geometry.
In the first chapter we introduce concepts from metric geometry. We define Length structures and Length spaces and the intrinsic metrics with respect to a Length. Further we deal with the variational length of a metric space. We then move on to prove existence of shortest paths under certain conditions on the metric space. Furthermore we give a definition of geodesics in a metric space and prove some properties as well as a Length space version of the Hopf-Rinov theorem. Lastly in this chapter we investigate absolutely continuous paths in metric spaces and generalize the formula "length equals integral of speed".
The second chapter is concerned with one of the prime examples of Length spaces, namely Riemannian manifolds. With the Riemannian arclength and distance any Riemannian manifold with a smooth metric is turned into a Length space. In this section we will generalize this to manifolds with continuous Riemannian metrics. Further we will compare different Length structures on Riemannian manifolds in order to establish a generalization of the arclength to absolutely continuous paths and to rectifiable paths via the variational Length from chapter 1. This will be done first for smooth metrics and then also for continuous ones relying on regularization of the continuous metric and using the smooth result.
Having established that a Riemannian manifold with continuous metric is a Length space, in the third chapter we compare the definition of geodesics, respectively shortest paths in metric spaces, to the definition of geodesic in the Riemannian sense. We begin with a counterexample by Hartman and Wintner [11], refuting a connection between locally shortest paths to solutions of the geodesic equation for metrics of regulatity $C^{1,\alpha}$, for $0 < \alpha < 1$. We then move on to the case of a C^1 metric, where we show that shortest paths solve the geodesic equation and are of class C^2. Further we investigate a paper by Lytchak and Yaman [14], showing that metric space geodesics for C^α metrics are locally uniformly of regularity $C^{1,\beta}$ for $\beta = \frac{\alpha}{2-\alpha}$.
The fourth chapter is concerned with two different approaches ([17], [18] and [23] to showing that the exponential map of a $C^{1,1}$ metric is a bi-Lipschitz homeomorphism on an open neighbourhood of 0. The first approach will involve regularization of the metric and the use of Jacobi fields to help carry the bi-Lipschitz property through the limit of the regularized metrics. The second approach uses a low regularity version of the Inverse Function Theorem ([24] and [25]) and strong differentiability of the exponential map at 0, to obtain the bi-Lipschitz property. Using this, it is possible to formulate a low regularity version of the Gauss Lemma, to establish that locally, geodesics in Riemannian manifolds, are shortest paths.

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Notation

By $\mathrm{int}(U)$ we denote the interior of a set U, and by $\bar{U}$ its closure. For two sets in a topological space the relation $V \subset\subset U$ always means that V is compactly contained in U, i.e. there exists a compact set K such that $V \subseteq \bar{V} \subseteq K \subseteq \mathrm{int}(U) \subseteq U$. The distance function on a metric space is allowed to take infinity as a value. Further we denote the (metric) ball of radius r and center p by $B_r(p)$.
By $\gamma \cup \sigma$ we denote the concatenation of the paths γ and σ. All manifolds in this text are assumed to be Hausdorff and second countable and therefore also paracompact and metrizable. A chart (ψ, U) on a manifold consists of a bijective map $\psi : U \to V$ onto an open subset of $\mathbb{R}^n$. If f is a map between two manifolds, its tangent map is denoted by Tf and the tangent map at a point p by $T_p f$. If possible we always employ Einstein summation convention, i.e. summation is carried out over indices appearing in both upper and lower slots. The evaluation of a vectorfield X at a point p will sometimes be denoted as $X_p := X(p)$. If M is a manifold the tangent bundle of M is denoted by TM and the corresponding (r,s)-tensor bundle by $T^r_s M$ with tensorbundle charts $(T\psi)^r_s$ for a chart ψ of M. On a Riemannian manifold with smooth metric g, we denote its exponential map at a point p by $\exp^g_p$ or $\exp_p$ if it is clear which metric is used. The Riemannian curvature tensor is defined as $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]} Z$, where $\nabla_X Y$ denotes the Levi-Civita connection of the manifold.
The norm on the tangentspace of a Riemannian manifold given by the metric g will always be denoted by $\|.\|_g$. The Euclidean norm on $\mathbb{R}^n$ will be denoted by $\|.\|_e$ or $\|.\|_{e(\mathbb{R}^n)}$ if the dimension is important. By $C(X,Y)$ we denote the class of all continuous mapping between two topological spaces X and Y. Sometimes continuous functions will be called C^0, by $C^{1,\alpha}$ we mean continuously differentiable functions, with locally α-Hölder continuous derivative. If the derivative is locally Lipschitz continuous we will denote the class by $C^{1,1}$. Lastly by $C^\infty_c(\Omega)$ we denote the space of all test functions on $\Omega \subseteq \mathbb{R}^n$, i.e. smooth functions with compact support. By a mollifier we mean a nonnegative function in $C^\infty_c(\mathbb{R}^n)$ with unit integral.

Chapter 1

Some Metric Geometry

In this chapter we introduce length structures and length spaces. We then show that a length structure always provides a metric on a Hausdorff space. We will define a particular length structure on a metric space, induced by the variational length, which will be the object of our studies for the rest of the first section. The second section deals with paths into metric spaces. We will see under what conditions there are shortest paths. In the third we introduce the notion of geodesics on metric spaces, as locally distance preserving paths. We will end the section with a length space version of the theorem of Hopf-Rinow. The final part of this section is an addendum about absolutely continuous paths in metric spaces, which will be needed in chapter 2 and provides a sufficient framework to define a length via the formula „length equals integral of speed" in metric spaces. Throughout the sections 1 and 2 we closely follow [3] chapter 2 and [5] chapters 1 and 2. Section 3 follows mainly [3] and [6]. For the fourth section we follow [1].

1.1 Length Structures and Length spaces

The following definition introduces so-called admissible classes of paths and lengths of such paths. Usually these paths are allowed to be defined on different intervals, where no restrictions on the intervals are supposed, so they might be open or closed or neither, they may be a single point or also all of $\mathbb{R}$. However for simplicity of notation we will sometimes use intervals of the form $[a, b]$ in the definition and throughout this section. Note that we define a path as a continuous map from an interval to the space in question.

1.1.1 Definition. Let M be a topological Hausdorff space and let $\mathcal{A}$ be a subfamily of all continuous maps from arbitrary intervals $I \subseteq \mathbb{R}$ to M. $\mathcal{A}$ respectively its elements are called an *admissible class*, respectively admissible curves, if

($\mathcal{A}$1) $\mathcal{A}$ is closed under restrictions, that means if $\gamma \in \mathcal{A}$, $\gamma : I \to M$, then for any subinterval $J \subseteq I$ the map $\gamma_{|J}$ is still admissible, i.e. $\gamma_{|J} \in \mathcal{A}$.

($\mathcal{A}$2) $\mathcal{A}$ is closed under concatenations, i.e. if $\gamma : [a, b] \to M$ is a path such that for some $a \leq c \leq b$ the paths $\alpha := \gamma_{|[a,c]}$ and $\beta := \gamma_{|[c,b]}$ are admissible, then γ is admissible.

($\mathcal{A}3$) $\mathcal{A}$ is closed under certain (but at least all affine) reparameterizations, depending on the class. By affine reparameterizations we mean homeomorphisms φ of intervals of the form $\varphi(t) = ct + d$ for certain $c, d \in \mathbb{R}$.

Furthermore a map $L : \mathcal{A} \to [0, \infty]$ is called a length for $\mathcal{A}$, if the following conditions are satisfied

($L1$) L is additive, i.e. for γ, α, β as in ($\mathcal{A}2$) it holds that $L(\gamma) = L(\alpha) + L(\beta)$

($L2$) The length of an admissible path continuously varies with the length of the interval where the path is defined. More precisely, if $\gamma : [a, b] \to M$ is admissible and $L(\gamma) < \infty$, then the map $t \mapsto L(\gamma_{|[a,t]})$, $[a, b] \to [0, \infty)$ is continuous.

($L3$) The length is invariant under all reparameterizations considered with $\mathcal{A}$, i.e. $L(\gamma) = L(\gamma \circ \varphi)$ for an admissible path γ and such a reparameterization φ.

The class of reparameterizations for an admissible class is often naturally given by the choice of admissible classes as we will see in the following examples, which will later show up in a wider context.

1.1.2 Example.

(i) The class of all paths is admissible under continuous reparameterizations, i.e. homeomorphisms of intervals.

(ii) The set of all smooth paths into $\mathbb{R}^n$ is not admissible, as it is not closed under concatenations, since break points may occur.

(iii) The class of all piecewise smooth paths into the Euclidean space $\mathbb{R}^n$ (or any Riemannian manifold) is admissible with reparameterizations given by all diffeomorphisms on their domain of definition. One obtains a length for this class by considering $L(\gamma) = \int_I \|\gamma'(t)\| dt$, where in the manifold case $\|.\|$ means the norm in the tangent space of a point, given by the Riemannian metric.

(iv) Let (M, d) be a metric space. Consider for a path $\gamma : [a, b] \to M$ the following expression

$$L_d(\gamma) := \sup \Big\{ \sum_{i=1}^{n} d\big(\gamma(t_{i-1}), \gamma(t_i)\big) \mid n \in \mathbb{N},\ a = t_0 < \ldots < t_n = b \Big\}. \tag{1.1}$$

The class of all paths is an admissible class and L_d a length for it, called the total variation or variational length. This statement will be proven in 1.1.11. Paths in this class, whose lengths are finite are called rectifiable. We will write $V_\sigma(\gamma)$ for the finite variation $\sum_{i=1}^{n} d\big(\gamma(t_{i-1}), \gamma(t_i)\big)$ subordinate to a subdivision $\sigma := (t_i)_{i=0}^{n}$, $t_i < t_{i+1}$ of the domain of a path γ.

(v) A path $\gamma : I \to M$ into the metric space (M, d) is called Lipschitz continuous, if its Lipschitz constant is finite, i.e.

$$\mathrm{Lip}(\gamma) := \sup_{t,s \in I, t \neq s} \frac{d\big(\gamma(t), \gamma(s)\big)}{|t - s|} < \infty,$$

it then holds that $d(\gamma(t),\gamma(s)) \leq \mathrm{Lip}(\gamma)|t-s|$ for all $t,s \in I$.
We check that the class of all Lipschitz paths is admissible: In fact ($\mathcal{A}1$) holds, since for restrictions the supremum is taken over a smaller set.
To show ($\mathcal{A}2$) set $I=[a,b]$, $J_1=[a,c]$ and $J_2=[c,b]$. Further take a path $\gamma : I \to M$ such that $\alpha := \gamma_{|J_1}$ and $\beta := \gamma_{|J_2}$ are Lipschitz paths. We need to consider expressions of the form $\frac{d\left(\gamma(t),\gamma(s)\right)}{|t-s|}$. For $t,s \in J_1$ respectively $t,s \in J_2$ these expressions are always smaller than or equal to the Lipschiz constants of α respectively of β. In the remaining cases for t,s assume w.l.o.g. that $t \in J_1\backslash\{c\}$ and $s \in J_2\backslash\{c\}$, then since $\alpha(c)=\beta(c)$

$$\begin{aligned}\frac{d\big(\gamma(t),\gamma(s)\big)}{s-t} &= \frac{d\big(\alpha(t),\beta(s)\big)}{s-t} \leq \frac{d\big(\alpha(t),\alpha(c)\big)+d\big(\beta(c),\beta(s)\big)}{s-t} \\ &\leq \frac{d\big(\alpha(t),\alpha(c)\big)}{c-t}+\frac{d\big(\beta(c),\beta(s)\big)}{s-c} \leq \mathrm{Lip}(\alpha)+\mathrm{Lip}(\beta)\end{aligned}$$

which implies that γ is Lipschitz.
($\mathcal{A}3$) holds for all Lipschitz reparameterizations, i.e. bijective Lipschitz maps between intervals with a Lipschitz continuous inverse, since the composition of Lipschitz maps is Lipschitz. (cf. [3], Prop. 1.4.3, p.9)

1.1.3 Definition (Length Structure)**.**
Let M be a topological Hausdorff space. A *Length Structure* on M is a triple $(M,\mathcal{A},L)$, where $\mathcal{A}$ is an admissible class of paths and L is a length for $\mathcal{A}$ such that L respects the topology of M in the following way:
For any $p \in M$ and any neighbourhood U of p the length of all admissible paths, connecting p with any point in the complement of U, is bounded away from 0, i.e.

$$\inf\left\{L(\gamma)\middle|\gamma \in \mathcal{A} \text{ with } \gamma(a)=p,\ \gamma(b) \in M\backslash U\right\} > 0.$$

We will next see that a length structure induces a metric on a Hausdorff space.

1.1.4 Definition. Let M be a topological Hausdorff space with a length structure $(M,\mathcal{A},L)$, then the *intrinsic metric* of $(M,\mathcal{A},L)$ is defined by

$$d_L(p,q) := \inf_{\gamma \in \mathcal{A}}\left\{L(\gamma)\middle|\gamma : [a,b] \to M,\ \gamma(a)=p,\ \gamma(b)=q\right\}. \tag{1.2}$$

Here we use the convention $\inf \emptyset = \infty$. If it is clear from the context which length structure defines d_L, we will often write d instead. Observe that the intrinsic distance between two points can be infinite. We now justify the name intrinsic metric.

1.1.5 Proposition. If M is a topological Hausdorff space with length structure $(M,\mathcal{A},L)$, then the intrinsic metric d_L is a metric on M, which induces a finer topology than the topology initially given on M.

Proof. To simplyfy notation γ will always be admissible and connect the given points.
First we show that d_L is a metric on M. Let $p \in M$, then $0 \leq d_L(p,p) = \inf\{L(\gamma)\,|\gamma \in \mathcal{A}\} \leq L(t \mapsto p) = 0$ and therefore $d_L(p,p)=0$. Further assume $d_L(p,q)=0$ and $p \neq q$, then there exist disjoint neighbourhoods U_p and U_q of p

and q respectively. By definition of a length structure the infimum of all lengths of paths connecting p and q is bounded away from 0, and therefore $d_L(p,q) > 0$, which contradicts the assumption.
The symmetry of d_L follows from reversing the orientation of the path, i.e. applying the reparameterization $\varphi : [a,b] \to [0, b-a], t \mapsto b-t$. The resulting path is admissible since φ is an affine reparameterization.
For the triangle inequality take $p, q, r \in M$, and observe that the concatenation of paths that connect p to q and q to r also connects p to r. By additivity of L we obtain

$$d_L(p,r) = \inf_{\gamma \in \mathcal{A}} \left\{ L(\gamma) \middle| \, \gamma \text{ is a path from } p \text{ to } r \right\}$$

$$\leq \inf_{\alpha,\beta \in \mathcal{A}} \left\{ L(\alpha \cup \beta) \middle| \, \alpha \text{ from } p \text{ to } q, \, \beta \text{ from } q \text{ to } r \right\} = d_L(p,q) + d_L(q,r).$$

To show the last statement, let $\mathcal{O}$ be the Hausdorff topology on M. To see that the metric topology is finer than $\mathcal{O}$, take any open set $O \in \mathcal{O}$ and $p \in O$ then $\varepsilon_p := \inf\{d_L(p,q) | q \in M \backslash O\} =: d_L(p, M\backslash O) > 0$ since $(M, \mathcal{A}, L)$ is a length structure. Now $B_{\varepsilon_p}(p) \subseteq O$ and therefore O is open in (M, d_L). □

The topology of the space M does in general not coincide with the metric topology induced by the intrinsic metric. Still we want to consider spaces where this is the case.

1.1.6 Definition. A metric space (M,d) with a length structure $(M, \mathcal{A}, L)$ is called a *length space*, if the intrinsic metric coincides with the original one, i.e. $d = d_L$.

If not further specified the length structure corresponding to a length space will always be denoted by $(M, \mathcal{A}, L)$.

1.1.7 Proposition. Any length space (M,d) is locally pathwise connected, i.e. every point in M has a neighbourhood which contains a pathwise connected neighbourhood of that point.

Proof. Let $p \in M$ and U be a neighbourhood of p. There is an $\varepsilon > 0$ such that $B_\varepsilon(p) \subseteq U$. It is enough to show that for any $q \in B_\varepsilon(p)$ there is an admissible path connecting p to q, staying in $B_\varepsilon(p)$, since then for arbitrary points in $B_\varepsilon(p)$ we can take the concatenation of such paths.
Let $q \in B_\varepsilon(p)$, by definition of the intrinsic metric as an infimum there exists an admissible path $\gamma : [a,b] \to M$ with $\gamma(a) = p$, $\gamma(b) = q$ and $d(p,q) \leq L(\gamma) < \varepsilon$. We now show that γ remains entirely in $B_\varepsilon(p)$. To this end take any $t \in [a,b]$ and observe

$$d\big(p, \gamma(t)\big) \leq L(\gamma_{|[a,t]}) \leq L(\gamma) < \varepsilon$$

which means $\gamma(t) \in B_\varepsilon(p)$, and hence M is locally pathwise connected. □

1.1.8 Remark. At first glance it might seem that any length space (M,d) is even (globally) pathwise connected, but this is only the case if the metric is finite. Indeed then for any $p, q \in M$ we have $d := d(p,q) = \inf_{\gamma \in \mathcal{A}}\{L(\gamma)\} \,|\, \gamma$ connects p to $q\} < \infty$. By the definition of the infimum for any $\varepsilon > 0$, there is a path connecting p to q with length less or equal than $d + \varepsilon$. In particular there is a path connecting p and q, so M is pathwise connected.

But not every length space is pathwise connected, take for example the union of two disjoint balls B_1 and B_2 in $\mathbb{R}^n$ with the metric given by

$$d(p,q) := \begin{cases} \|p-q\|_e \text{ if } p,q \in B_1 \text{ or } p,q \in B_2 \\ \infty \text{ otherwise.} \end{cases}$$

This defines a length space with the length of 1.1.2 (iii) or (iv), but it is not pathwise connected.

1.1.9 Definition. A length structure is called *complete*, if any pair of points can be joined by an admissible path, whose length is equal to the distance of the points. A path having this property is called *minimizing*.

1.1.10 Example.

(i) Euclidean space $\mathbb{R}^n$, with the natural metric induced by the Euclidean norm, is a complete length space, when considered with the length structure from 1.1.2(iii).

(ii) If we remove a single point x from $\mathbb{R}^n$, it is still a length space, but no longer complete. Indeed any two points on a straight line through x and on opposite side of x, cannot be connected by a minimizing path. There are however paths connecting them with length arbitrarily close to the distance of the points. These paths are obtained by deforming minimally the affine segment from p to q, so it does not pass through the removed point.

(iii) If an open ball B is removed from $\mathbb{R}^n$, the resulting metric space is no longer a length space. For any two points on an affine segment intersecting the ball, all paths in $\mathbb{R}^n \backslash B$ joining them have length uniformly bounded away from their distance.

(iv) A connected Riemannian manifold (i.e. a smooth manifold with a smooth Riemannian metric) is a length space. We will explicitly deal with such length spaces in Chapter 2.

Obviously a non-pathwise connected space cannot be endowed with a complete length structure. In general complete length spaces are not complete in the metric sense (consider an open ball in $\mathbb{R}^n$).
By the above example Euclidean space with with the length of piecewise smooth (or even piecewise C^1) paths is a length space. We can consider a different length on the same class of paths (actually even on the bigger class of all paths) given by the total variation from (1.1). These lengths coincide in $\mathbb{R}^n$, as is usually shown in elementary differential geometry see e.g. [5], Prop 1.3.1, p.22.

On any metric space (M,d) the class of all paths, with the total variation L_d as length function, gives rise to a length structure by the following

1.1.11 Lemma. Let (M,d) be a metric space, then the triple $(M,\mathcal{C},L_d)$ is a length structure, where $\mathcal{C}$ denotes the class of all paths into M, and L_d is the variational length defined in (1.1).

Proof. Clearly $\mathcal{C}$ is an admissible class. Let $\gamma : [a,b] \to M$ be continuous, we want to show $L_d(\gamma) = L_d(\alpha) + L_d(\beta)$, where $\alpha := \gamma_{|[a,c]}$ and $\beta := \gamma_{|[c,b]}$. For

any subdivision σ of $[a,b]$,we can add the point c and obtain a finer subdivision $\tilde{\sigma}$, therefore $V_\sigma(\gamma) \leq V_{\tilde{\sigma}}(\gamma)$ by the triangle inequality. We can now split this subdivision into subdivisions $\sigma_1 := \tilde{\sigma} \cap [a,c]$ on $[a,c]$ and $\sigma_2 := \tilde{\sigma} \cap [c,b]$ on $[c,b]$, with $V_{\tilde{\sigma}}(\gamma) = V_{\sigma_1}(\alpha) + V_{\sigma_2}(\beta)$. This yields $L_d(\gamma) = L_d(\alpha) + L_d(\beta)$.
Next we show continuity of the map $t \mapsto L_d(\gamma_{|[a,t]})$ for rectifiable paths $\gamma : [a,b] \to M$. Let $\varepsilon > 0$, then there is a subdivision $\sigma := (t_i)_{i=1}^n$ of $[a,b]$ such that $t_i - t_{i-1} < \delta$, for some fixed $\delta > 0$ and such that $L(\gamma) - V_\sigma(\gamma) < \varepsilon$. For every $t, s \in [a,b]$, $s < t$ we can further w.l.o.g. assume that $t, s \in \sigma$. Denote by $\sigma_1 := \sigma \cap [a,s]$ and $\sigma_2 := \sigma \cap [t,b]$. Then it holds that

$$L_d(\gamma_{|[a,s]}) + L_d(\gamma_{|[s,t]}) + L_d(\gamma_{|[t,b]}) = L_d(\gamma) \leq V_\sigma(\gamma) + \varepsilon$$
$$= V_{\sigma_1}(\gamma_{|[a,s]}) + d(\gamma(s),\gamma(t)) + V_{\sigma_2}(\gamma_{|[t,b]}) + \varepsilon$$
$$\leq L_d(\gamma_{|[a,s]}) + d(\gamma(s),\gamma(t)) + L(\gamma_{|[t,b]}) + \varepsilon$$

In summary $L_d(\gamma_{|[s,t]}) \leq d(\gamma(s),\gamma(t)) + \varepsilon$. Since γ is uniformly continuous we can find δ small enough such that $d(\gamma(s),\gamma(t)) < \varepsilon$, implying continuity of $t \mapsto L_d(\gamma_{|[a,t]})$.
Lastly we show that the variational length is invariant under continuous reparameterizations, i.e. homeomorphisms of intervals. If $\tilde{\gamma} : J \to M$ is a affine reparameterization of the path $\gamma : I \to M$ then for every subdivision $\sigma = (t_i)_{i=0}^n$ of J there is a corresponding subdivision $\tilde{\sigma} = (\varphi(t_i))_{i=0}^n$ of I and vice versa, where φ is the reparameterization. Thus $V_\sigma(\gamma) = V_{\tilde{\sigma}}(\tilde{\gamma})$, this implies that the lengths are equal.
It remains to show that the length L_d respects the topology in the sense of 1.1.3. It suffices to show $L_d(\gamma) \geq d(\gamma(a),\gamma(b))$ for a path $\gamma : I \to M$, $a, b \in I$. But indeed $L_d(\gamma_{|[a,b]}) \geq d(\gamma(a),\gamma(b))$ always holds by definition of L_d, since $d(\gamma(a),\gamma(b)) = V_\sigma(\gamma)$ for the trivial subdivision $\{a,b\}$ of $[a,b]$. ☐

This length structure induces an intrinsic metric $\hat{d}$ on M. By Proposition 1.1.5, the topology induced by $\hat{d}$ is finer that the topology initially given. Even more is true:

1.1.12 Proposition. Let (M,d) be a metric space and $\hat{d}$ be the intrinsic metric given by the length structure $(M,\mathcal{C},L_d)$. The following hold:

(i) If γ is a rectifiable path into (M,d), then it is also a rectifiable path into $(M,\hat{d})$ and $L_{\hat{d}}(\gamma) = L_d(\gamma)$.

(ii) The intrinsic metric induced by the length structure $(M,\hat{\mathcal{C}},L_{\hat{d}})$ coincides with $\hat{d}$, where $\hat{\mathcal{C}}$ denotes all paths into $(M,\hat{d})$

Proof. We first show that a rectifiable path $\gamma : [a,b] \to (M,d)$ is continuous into $(M,\hat{d})$. Let $(x_n)_n \subseteq [a,b]$ be a sequence with $x_n \to x$ as $n \to \infty$ and assume w.l.o.g. $x_n \leq x$. Then $d(\gamma(x_n),\gamma(x)) \to 0$ and since γ is rectifiable this implies $L_d(\gamma_{|[x_n,x]}) \to 0$ as $n \to \infty$ by 1.1.1 ($L2$) and 1.1.11. Note that since $\hat{d}$ is the intrinsic metric w.r.t. L_d, we have $\hat{d}(\gamma(x_n),\gamma(x)) \leq L_d(\gamma_{|[x_n,x]})$, so $\hat{d}(\gamma(x_n),\gamma(x)) \to 0$ as $n \to \infty$ and γ is continuous into $(M,\hat{d})$.
It holds that $L_d(\gamma) \geq d(p,q)$, where γ is a path connecting p and q. Since this holds for all paths connecting p and q, we obtain $\hat{d}(p,q) = \inf_\gamma L_d(\gamma) \geq d(p,q)$. This in turn immediately implies $L_{\hat{d}}(\gamma) \geq L_d(\gamma)$.

For the reverse inequality let $\gamma : [a, b] \to M$ be a path as in (i) and $a = t_0 < t_1 < \ldots < t_n = b$ a subdivision of $[a, b]$. Again since $\hat{d}$ is the intrinsic metric w.r.t. the length L_d, it holds that $\hat{d}(\gamma(t_{i-1}), \gamma(t_i)) \leq L_d(\gamma_{|[t_{i-1},t_i]})$. This holds for any subdivision of $[a, b]$ and therefore implies $L_{\hat{d}}(\gamma) \leq L_d(\gamma)$.
For the second part denote the intrinsic metric w.r.t. $(M, \hat{\mathcal{C}}, L_{\hat{d}})$ by $\tilde{d}$ and note that $\tilde{d} \geq \hat{d}$ always holds. Conversely we have for $p, q \in M$ and paths connecting p to q

$$\begin{aligned}\tilde{d}(p, q) &= \inf\{L_{\hat{d}}(\gamma) \,|\, \gamma \text{ continuous into } (M, \hat{d})\} \\ &= \inf\{L_{\hat{d}}(\gamma) \,|\, \gamma \text{ continuous and rectifiable into } (M, \hat{d})\} \\ &\overset{\text{(i)}}{\leq} \inf\{L_d(\gamma) \,|\, \gamma \text{ continuous and rectifiable into } (M, d)\} = \hat{d}(p, q).\end{aligned}$$

□

1.1.13 Remark. While every continuous path into $(M, \hat{d})$ is also continuous into (M, d), where $\hat{d}$ is the intrinsic metric from 1.1.12, the reverse is in general not true. However by 1.1.12 every rectifiable path into a metric space is also continuous (and therefore a path) into the metric space endowed with the intrinsic metric $\hat{d}$.
Observe that for any length structure $(M, \mathcal{A}, L)$, it always holds that $d_L(p, q) \leq L(\gamma)$ as well as $d_L(p, q) \leq L_d(\gamma)$ for any admissible γ connecting p and q.

Of course different length structures can induce the same intrinsic metric on a space. When considering length spaces for example, the intrinsic metric itself induces a length structure (namely all continuous paths with the variational length), which coincides with the original one by the following statement.

1.1.14 Proposition. Let (M, d) be a length space, which stems from the length structure $(M, \mathcal{A}, L)$, i.e. $d = d_L$. If $\hat{d}$ is the intrinsic metric induced by d, i.e. $\hat{d} := d_{L_d}$, then $L_d(\gamma) \leq L(\gamma)$ for $\gamma \in \mathcal{A}$ and $d = \hat{d}$.

Proof. Proceeding similarly as in the proof of Proposition 1.1.12, we obtain $\hat{d} \geq d$. For the converse let $\gamma \in \mathcal{A}$ and consider

$$\sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})) \leq \sum_{i=1}^{n} L(\gamma_{|[t_{i-1},t_i]}) = L(\gamma)$$

for any $n \in \mathbb{N}$ and any subdivision $\{t_i\}_{i=0}^{n}$ of the interval on which γ is defined. This implies $L_d(\gamma) \leq L(\gamma)$ which in turn implies $\hat{d} \leq d$. □

We see that 1.1.12(ii) is a special case of 1.1.14 for the length space $(M, \mathcal{C}, L_{\hat{d}})$. The above results do not imply $L = L_d$ in a length space. We will see conditions for this equality in 1.1.19. One also gets a characterization for length spaces.

1.1.15 Corollary. A metric space (M, d) with the length structure $(M, \mathcal{C}, L_d)$, is a length space if and only if for any two points $p, q \in M$ with $d(p, q) < \infty$ and any $\varepsilon > 0$ there is a path connecting p and q with $L_d(\gamma) < d(p, q) + \varepsilon$.

Proof. Follows by 1.1.12 and the the fact that the intrinsic metric always gives rise to a length space. □

On the other hand, when given an intrinsic metric, is it possible to know which length structure induced it? Obviously $(M, \mathcal{C}, L_d)$ is a candidate, but if another length structure induces the same intrinsic metric, how are lengths of paths related? We will see that the following property of the variational length plays an important role for these questions. We recall the definition of lower semi-continuity:

1.1.16 Definition. A function $f : X \to \mathbb{R} \cup \{\infty\}$, where X is a topological space, is called lower semi-continuous, if for every $a \in \mathbb{R}$ the set $\{x \in X \mid f(x) > a\}$ is open or equivalently $\{x \in X \mid f(x) \leq a\}$ is closed.

1.1.17 Proposition. If M is a metric space, then $f : M \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous if and only if for every sequence $(p_n)_{n\in\mathbb{N}} \subseteq M$ with $p_n \to p \in M$, we have $f(p) \leq \liminf f(p_n)$.

Proof. $(\Rightarrow)$: Let f be lower semi-continuous and take a sequence $(p_n)_n$ such that $p_n \to p \in M$. Let $\varepsilon > 0$ and set $m_\varepsilon := f(p) - \varepsilon$, since f is lower semi-continuous the set $B_\varepsilon := \{x \in M \mid f(x) > m_\varepsilon\}$ is open. Since $p \in B_\varepsilon$ and $p_n \to p$ there exists an index N_ε such that $p_n \in B_\varepsilon$ for all $n \geq N_\varepsilon$, in other words $f(p_n) > f(p) - \varepsilon$, implying $\liminf_{n\to\infty} f(p_n) \geq f(p)$.
$(\Leftarrow)$: We show that any set of the form $A_a := \{x \in M | f(x) \leq a\}$, for $a \in \mathbb{R}$ is closed. To this end let $(p_n)_n \subseteq A_a$ such that $p_n \to p \in M$. Since $p_n \in A_a$ it holds that $f(p_n) \leq a$ for all $n \in \mathbb{N}$ and therefore

$$f(p) \leq \liminf_{n\to\infty} f(p_n) \leq a,$$

hence $p \in A_a$ and A_a is closed. □

1.1.18 Proposition. Let (M, d) be a metric space equipped with the variational length L_d. Then the following hold

(i) If $(\gamma_n : [a, b] \to M)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ is a sequence of paths converging pointwise to a path γ, then $\liminf_{n\to\infty} L_d(\gamma_n) \geq L_d(\gamma)$.

(ii) L_d is a lower semi-continuous functional on the class of all paths $\mathcal{C}([a, b])$, equipped with the topology of uniform convergence.

Proof. (i) Let γ and γ_n be as in (i). For a subdivision $\sigma := (t_i)_{i=0}^k$ of $[a, b]$, since $\gamma_n(t) \to \gamma(t)$ for all (finitely many) $t \in \sigma$, we can choose n large enough such that $d(\gamma(t_i), \gamma_n(t_i)) < \varepsilon$ for all $t_i \in \sigma$. Then

$$V_\sigma(\gamma) = \sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i))$$

$$\leq \sum_{i=1}^{k} \big(d(\gamma(t_{i-1}), \gamma_n(t_{i-1})) + d(\gamma_n(t_i), \gamma_n(t_{i-1})) + d(\gamma(t_i), \gamma_n(t_i))\big)$$

$$\leq k\varepsilon + V_\sigma(\gamma_n) + k\varepsilon = 2k\varepsilon + V_\sigma(\gamma_n), \tag{1.3}$$

which implies $V_\sigma(\gamma_n) \to V_\sigma(\gamma)$ as $n \to \infty$. If $L_d(\gamma) < \infty$, for any $\varepsilon > 0$ there exists a subdivision σ such that $L_d(\gamma) - \varepsilon < V_\sigma(\gamma)$. Using (1.3), this implies $L_d(\gamma) - \varepsilon \leq V_\sigma(\gamma_n) \leq L_d(\gamma_n)$ for sufficiently large n, therefore $\liminf_n L_d(\gamma_n) \geq L_d(\gamma)$ follows.

If γ is not rectifiable, we show $\liminf_n L_d(\gamma_n) = \infty$. In that case it holds that for every $N \in \mathbb{N}$ there exists a subdivision $\sigma_N := (t_0, \ldots, t_{l_N})$ of $[a, b]$ such that $V_{\sigma_N}(\gamma) \geq N + 1$. Again using (1.3) we obtain

$$N+1 \leq V_{\sigma_N}(\gamma) \leq V_{\sigma_N}(\gamma_n) + \sum_{i=1}^{l_N} d(\gamma(t_i), \gamma_n(t_i)) + \sum_{i=1}^{l_N} d(\gamma(t_{i-1}), \gamma_n(t_{i-1})).$$

For n large enough we can bound the sums on the right hand side by 1 and therefore obtain $V_{\sigma_N}(\gamma_n) \geq N$, which implies $\liminf_n L_d(\gamma_n) = \infty$.

(ii) The space $\mathcal{C}$ is a metric space and the uniform limit of a path is again a path. In particular the convergence holds also pointwise. Therefore the statement follows by Prop. 1.1.17 and (i).

□

This yields the following description of the intrinsic metric and its length structure. In the following we will denote the set of certain admissible paths from an interval I into a metric space by $\mathcal{A}(I)$.

1.1.19 Theorem. Let (M, d) be a length space and $(M, \mathcal{A}, L)$ its length structure, i.e. d is the intrinsic metric w.r.t. this structure. Let L be such that, if a sequence of admissible paths $(\gamma_n)_{n\in\mathbb{N}}$, converges pointwise to an admissible path γ, this implies that $\liminf_{n\to\infty} L(\gamma_n) \geq L(\gamma)$. It then holds that $L(\gamma) = L_d(\gamma)$ for all $\gamma \in \mathcal{A}$.
In particular the assumption is fulfilled, if the length $L : \mathcal{A}([a, b]) \to [0, \infty]$ is a lower semi-continuous map, where $\mathcal{A}([a, b])$ is equipped with the topology of uniform convergence.

Proof. The inequality $L_d \leq L$ holds for any length structure and was proved in 1.1.14.
To see the reverse inequality, first note that, since for a path $\gamma : [a, b] \to M$ of finite length, for $\varepsilon > 0$ there exists $\delta > 0$ such that for a sufficiently fine subdivision $\{t_i\}_{i=0}^n$ of $[a, b]$ (take $t_i - t_{i-1} < \delta$), it holds that

$$d(\gamma(t_i), \gamma(t_{i+1})) < \varepsilon, \tag{1.4}$$

for all $i = 0, \ldots, n-1$. Since d is the intrinsic metric w.r.t. L there exist paths $\alpha_i : [t_i, t_{i+1}] \to M$ such that $\alpha_i(t_i) = \gamma(t_i)$, $\alpha_i(t_{i+1}) = \gamma(t_{i+1})$ and $L(\alpha_i) \leq d(\gamma(t_i), \gamma(t_{i+1})) + \frac{\varepsilon}{n}$. We set α_ε to be the concatenation of all α_i, then it holds that

$$L(\alpha_\varepsilon) = \sum_{i=0}^{n-1} L(\alpha_i) \leq \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) + \varepsilon \leq L_d(\gamma) + \varepsilon. \tag{1.5}$$

Further for $t \in [a, b]$ we have for some i, $t_i \leq t \leq t_{i+1}$ and by (1.4) it holds that $d(\gamma(t), \gamma(t_i)) < \varepsilon$. Therefore since $\alpha_i(t_i) = \gamma(t_i)$, we obtain

$$\begin{aligned} d(\gamma(t), \alpha_\varepsilon(t)) &\leq d(\gamma(t), \gamma(t_i)) + d(\alpha_i(t_i), \alpha_\varepsilon(t)) \\ &\leq \varepsilon + L(\alpha_{i|[t_i,t]}) \leq \varepsilon + d(\gamma(t_i), \gamma(t_{i+1})) + \frac{\varepsilon}{n} \leq 3\varepsilon \end{aligned}$$

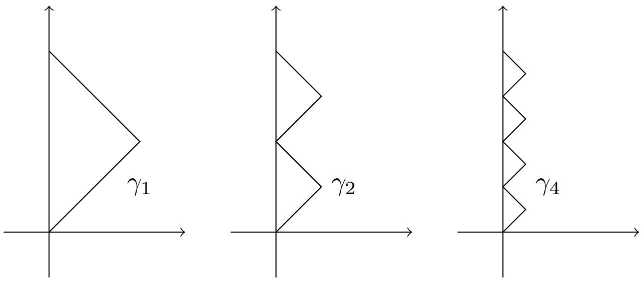
and therefore $\alpha_\varepsilon(t) \to \gamma(t)$ for all $t \in [a, b]$. Now the assumption on L and (1.5) imply

$$L(\gamma) \leq \liminf_{\varepsilon\to 0} L(\alpha_\varepsilon) \leq L_d(\gamma)$$

□

Not every length is lower semi-continuous as can be see by the following example.

1.1.20 Example. Consider $\mathbb{R}^2$ with the length given by the Minkowski-metric, so for a path $\gamma : I \to \mathbb{R}^2$ we set $L(\gamma) = \int_I \|\gamma'(t)\|_g \, ds$ where $\|x\|_g = (|-x_1^2+x_2^2|)^{\frac{1}{2}}$, for $x = (x_1, x_2) \in \mathbb{R}^2$. Set $\gamma : [0,1] \to \mathbb{R}^2, \gamma(t) = (0,t)$. Then $L(\gamma) = 1$, since $\|\gamma(t)\|_g = 1$ for all $t \in [0,1]$. Observe that paths on the diagonals, i.e. the subspaces generated by $(\pm 1, 1)$, have length zero. We will use this fact to construct a sequence of paths converging to γ, but for which the length function violates the lower semi-continuity. Let $n \in \mathbb{N}$ and set $x_i = (0, \frac{i}{n})$ for $0 \leq i \leq n$. We want to move along affine lines with a slope of $45°$ to the left and then to the right. Set $\beta_i : [\frac{i}{n}, \frac{2i+1}{2n}] \to \mathbb{R}^2$, $\beta_i(t) = (t - \frac{i}{n}, t)$ and $\tilde{\beta}_i : [\frac{2i+1}{2n}, \frac{i+1}{n}] \to \mathbb{R}^2$, $\tilde{\beta}_i(t) = (-t + \frac{i+1}{n}, t)$. We set $\gamma_n = \beta_0 \cup \tilde{\beta}_0 \ldots \cup \tilde{\beta}_{n-1}$, see the figure below.



We obtain

$$L(\gamma_n) = \sum_{i=0}^{n-1} (L(\beta_i) + L(\tilde{\beta}_i)).$$

Since $\beta_i'(t) = (1,1)$ we obtain $\|\beta_i'(t)\|_g = (|-1+1|)^{\frac{1}{2}} = 0$ for all $t \in [0,1]$ and analogously $\|\tilde{\beta}_i'(t)\|_g = 0$, which implies $L(\gamma_n) = 0$. On the other hand $\gamma_n \to \gamma$ uniformly (in the topology induced by the natural metric on $\mathbb{R}^2$). In summary $0 = \liminf L(\gamma_n) < L(\gamma) = 1$, so that L is not lower semi-continuous. Note that by „rounding off the corners“ this example can be modified in such a way, that all paths γ_n of the sequence are smooth and still satisfy $\liminf_n L(\gamma_n) = 0$.
This shows in particular that the above defined length does not give rise to a length structure on the Minkowski space, since two different points can be joined by a path of length 0. This problem occurs in general when considering lengths on Lorentzian manifolds.

1.2 Shortest Paths in Metric Spaces

In the definition of admissible classes we considered reparameterizations of paths. When talking about paths, we mean the maps and not their images, so it is of some importance to deal with reparameterizations. For example the unit circle in $\mathbb{R}^2$ is the image of many different paths. They may do several „laps “or just a single one, they also may differ in „speed“or have different orientation. We would consider these paths to be different, whereas their images are not. If we change the parameter by a strictly increasing change of variables the resulting path will run through the same points in the same order. One could consider equivalence classes of paths related by appropriate reparameterizations, for example homeomorhisms. If we also want to allow for paths which are constant on some subintervals (i.e. they stop for a while) this concept is too restrictive.

1.2.1 Definition. Let M be a metric space and $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$ paths. γ_1 is said to be a monotonous reparameterization of γ_2, if there exists a nondecreasing, surjective map $\varphi : I_1 \to I_2$ such that $\gamma_1 = \gamma_2 \circ \varphi$.

1.2.2 Remark. The map φ from Definition 1.2.1 is necessarily continuous, since it is surjective and monotonous, it is however not required to be a homeomorphism. The above introduced notion does not directly lead to an equivalence relation, since the existence of an inverse is not guaranteed. We can however define an equivalence relation in the following way: two paths $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$ are equivalent if and only if there is an interval J and changes of variable $\varphi_i : J \to I_i$, $i = 1, 2$, such that $\gamma_1 \circ \varphi_1 = \gamma_2 \circ \varphi_2$. For more details confer [3], p.44 Remark 2.5.2 and Exercise 2.5.3.

1.2.3 Lemma. Monotonous reparameterizations leave the length L_d of paths invariant.

Proof. Let $\gamma_1 : [a, b] \to M$ and $\gamma_2 : [c, d] \to M$ be paths and γ_1 be a monotonous reparameterization of γ_2 by $\varphi : [a, b] \to [c, d]$. We show $L_d(\gamma_1) \geq L_d(\gamma_2)$. Let $\tau := (t_i)_{i=0}^n$ be a subdivision of $[a, b]$, then $\sigma := (s_i)_{i=0}^n$, where $s_i = \varphi(t_i)$, is a subdivision of $[c, d]$, since φ is monotonous. We obtain $V_\tau(\gamma_1) = V_\sigma(\gamma_2)$, since $\gamma_1(t_i) = \gamma_2 \circ \varphi(t_i) = \gamma_2(s_i)$ for all i. Taking the supremum over all subdivisions of $[a, b]$ yields $L_d(\gamma_1) \geq V_\sigma(\gamma_2)$, which in turn yields $L_d(\gamma_1) \geq L_d(\gamma_2)$. The inverse inequality follows similarly by defining, for a subdivision $\sigma := (s_i)_{i=0}^n$ of $[c, d]$, a subdivision $\tau := (t_i)_{i=0}^n$ of $[a, b]$ by choosing some $t_i \in \varphi^{-1}(s_i)$. $\square$

Definition 1.2.1 now allows for a rather convenient notion.

1.2.4 Definition. A path $\gamma : [a, b] \to M$ is called parameterized by arclength, if for all $t, s \in [a, b]$, $s \leq t$ it holds that $L_d(\gamma_{|[s,t]}) = t - s$.

Informally speaking this means that the path is traversed with unit speed. Observe that in general the map $t \mapsto L_d(\gamma_{|[a,t]})$ is not differentiable, but when parameterized by arclength, we obtain $\frac{d}{dt} L_d(\gamma_{|[a,t]}) = 1$.

1.2.5 Remark. It is sometimes convenient to reparameterize a path on the unit interval $[0, 1]$. Such a parameterization can in gerneral not be parameterized by arclength, since its length would have to be equal to 1. However we want a concept that still ensures the path is run through with constant speed. Therefore we say a path $\gamma : [0, 1] \to M$ is parameterized proportionally to arclength, if it is either constant or it is a reparameterization of a path parameterized by arclength on an interval $[a, b]$, via the affine homeomorphism $\varphi : [0, 1] \to [a, b]$, $\varphi(x) = ((b - a)x + a)$. It can easily be seen that such a path is $L_d(\gamma)$-Lipschitz, cf. [5], Prop. 1.2.9, p.21.

1.2.6 Remark. When dealing with metric spaces and the length structure given by all paths with the variational length L_d, the above reparameteritations, are also reparameterizations in the sense of definition 1.1.1.

1.2.7 Example. We recall the definition of parameterization by arclength usually given when considerning paths in $\mathbb{R}^n$. We consider on $\mathbb{R}^n$ the class of all regularly parameterized curves, i.e. continuously differentiable maps $\gamma : I \to \mathbb{R}^n$, such that their derivative does not vanish on any $t \in I$, with reparameterization given by C^1-diffeomorphisms of the domains of definition. Their length is defined by $L(\gamma) = \int_I \|\gamma'(t)\|_{\mathbb{R}^n} dt$. In this context a parameterization by arclength is usually defined by requiring that $\|\gamma'(t)\|_{\mathbb{R}^n} = 1$ for all $t \in I$. This also fits into our context, since then $L(\gamma_{|[s,t]}) = t - s$. One can show the existence of such reparameterizations, see for example [4], p.2 Lemma 1.1.4.

We will now show the existence of parameterizations by arclength in a metric space, for the variational length L_d. Since our paths are allowed to stop, the first thing we want to do in order to obtain a parameterization by arclength, is to eliminate those subintervals on which the path is constant. We start with

1.2.8 Lemma. Let $\gamma : [a, b] \to M$ be a rectifiable path. For each $s \in [0, L_d(\gamma)]$ there is a unique point $p \in M$ and a closed subinterval $I_s \subseteq [a, b]$, such that $\gamma(t) = p$ and $L_d(\gamma_{|[a,t]}) = s$ for all $t \in I_s$. In particular we may define the map $\alpha : [0, L_d(\gamma)] \to M$ with $\alpha(s) := \gamma(t)$ for any $t \in I_s$.

Proof. The map $t \mapsto L_d(\gamma_{|[a,t]})$ is continuous and non-decreasing. By the mean value theorem for every $0 \le s \le L_d(\gamma)$, there is a $t \in [a, b]$ with $s = L_d(\gamma_{|[a,t]})$. Now for $a \le t_1 \le t_2 \le b$ such that $L_d(\gamma_{[a,t_1]}) = L_d(\gamma_{[a,t_2]})$ we obtain, by additivity of the length that $L_d(\gamma_{[t_1,t_2]}) = L_d(\gamma_{[a,t_2]}) - L_d(\gamma_{[a,t_1]}) = 0$. Using $L_d(\gamma_{|[t,s]}) \ge d(\gamma(t), \gamma(s)) \ge 0$, this implies that γ is constant on the interval $[t_1, t_2]$. Further the set I_s of all t such that $L_d(\gamma_{|[a,t]}) = s$, is an interval since otherwise $t \mapsto L_d(\gamma_{|[a,t]})$ could not be non-decreasing. I_s is closed by continuity of $t \mapsto L_d(\gamma_{|[a,t]})$. Since γ is constant on I_s, p is unique. □

1.2.9 Lemma. Under the assumptions of Lemma 1.2.8, the map α defined there, is Lipschitz continuous with Lipschitz constant 1. Further α is a monotonous reparamerterizaion of γ with the change of parameter $\varphi : [a, b] \to [0, L_d(\gamma)]$, $\varphi(t) = L_d(\gamma_{|[a,t]})$.

Proof. Take $s_1, s_2 \in [0, L_d(\gamma)]$, $s_1 \le s_2$ and let $t_1, t_2 \in [a, b]$ such that $s_i = L_d(\gamma_{|[a,t_i]})$, $i = 1, 2$. Then $\alpha(s_i) = \gamma(t_i)$ per definition, we obtain

$$d(\alpha(s_1), \alpha(s_2)) = d(\gamma(t_1), \gamma(t_2)) \le L_d(\gamma_{|[t_1,t_2]}) =$$

$$L_d(\gamma_{|[a,t_2]}) - L_d(\gamma_{|[a,t_1]}) = s_2 - s_1,$$

so α is 1-Lipschitz. Furthermore the map φ is increasing and surjective and the uniqueness of p in Lemma 1.2.8 implies $\gamma = \alpha \circ \varphi$ □

We are now ready to prove

1.2.10 Proposition. Let γ be a rectifiable path into a metric space, then there exists a monotonous reparameterization φ of γ, such that $\gamma \circ \varphi$ is parameterized by arclength.

Proof. Let $\gamma : [a, b] \to M$ be a rectifiable path. Denote by α the map associated to γ, cf. 1.2.8, with reparamterization φ from 1.2.9. We claim that for all $s \in [0, L_d(\gamma)]$ it holds that $s = L_d(\alpha_{|[0,s]})$. Indeed by 1.2.9 $\alpha_{|[0,s]}$ arises as a monotonous reparameterization of $\gamma_{|[a,t]}$ for some $t \in [a, b]$. Therefore those paths have the same length by 1.2.3. Since $L_d(\gamma_{|[a,t]}) = s$ the claim follows. This implies for $0 \leq s_1 \leq s_2 \leq L_d(\gamma)$

$$s_2 - s_1 = L_d(\alpha_{|[0,s_2]}) - L_d(\alpha_{|[0,s_1]}) = L_d(\alpha_{|[s_1,s_2]}),$$

so α is parameterized by arclength. $\square$

Let us show that parameterization by arclength respects concatenations of paths.

1.2.11 Proposition. Let γ be the concatenation of two paths γ_1 and γ_2, which are parameterized by arclength, then also γ is parameterized by arclength.

Proof. Let γ_1 be defined on $[a,b]$ and γ_2 be defined on $[b,c]$. Take $t,s \in [a,c]$, $t < s$ if $t,s \in [a,b]$ respectively $t,s \in [b,c]$, then $L_d(\gamma_{|[t,s]}) = L_d(\gamma_{i|[t,s]}) = s - t$ for $i = 1$ resp. $i = 2$. In the case $a \leq t \leq b \leq s \leq c$, we calculate as follows

$$s - t = (s-b) + (b-t) = L_d(\gamma_{2|[b,s]}) + L_d(\gamma_{1|[t,b]}) =$$

$$L_d(\gamma_{[t,b]}) + L_d(\gamma_{|[b,s]}) = L_d(\gamma_{[t,s]}).$$

$\square$

1.2.12 Remark. The above definitions and results are all formulated in terms of the variational length L_d. Analogously one could define parameterization by arclength for any length L on an admissible class. In the following we however need the length function to be lower semi-continuous. By Theorem 1.1.19, if a length L is lower semi-continuous, then it is equal to the variational length, induced by the metric associated to L, anyway.

Since we want to eventually find an appropriate definition of geodesics in metric spaces, we now come to the closely related notion of paths of minimal length.

1.2.13 Corollary. Let $\gamma_n : [a,b] \to M$ be a sequence of paths converging uniformly to a path $\gamma : [a,b] \to M$ such that their length is uniformly bounded, i.e. $L_d(\gamma_n) \leq M < \infty$ for all n, then γ has finite length $L_d(\gamma) \leq M$.

Proof. Since L_d is lower semi-continuous it holds by 1.1.18 that

$$L_d(\gamma) \leq \liminf_n L_d(\gamma_n) \leq M.$$

$\square$

Our goal will be to obtain conditions under which, for any given two points, there is a path of minimal length between them. We want to be able to obtain, from a sequence of paths with uniformly bounded length, a converging subsequence. This requires a version of the theorem of Arzela and Ascoli for metric spaces. We recall the definition of a uniformly equicontinuous family.

1.2.14 Definition. Let (X, d_X) and (Y, d_Y) be metric spaces. A family $\mathcal{F} \subseteq C(X,Y)$ of maps is called uniformly equicontinuous, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in X$ with $d_X(x,y) < \delta$ and for all $f \in \mathcal{F}$ it holds that $d_Y(f(x), f(y)) < \varepsilon$.

1.2.15 Example. We say a family $\mathcal{F}$ of maps between metric spaces (X, d_X) and (Y, d_Y) is uniformly α-Hölder ($0 < \alpha \leq 1$), if for all $f \in \mathcal{F}$ there is a constant $C > 0$ such that $d_Y(f(x), f(y)) \leq C d_X(x, y)^\alpha$. For $\alpha = 1$ this yields Lipschitz continuity with a uniform Lipschitz constant. A family of such maps is uniformly equicontinuous.
We will later use the fact, that a family of paths $(\gamma_i)_{i \in I}$, all parameterized proportionally to arclength , whose length is uniformly bounded, i.e. $\sup\{L_d(\gamma_i) \mid i \in I\} =: M < \infty$, is uniformly equicontinuous. Indeed this follows from the above, since such paths γ are M-Lipschitz.

There are various versions of the theorem by Arzela and Ascoli valid for different classes of spaces. A natural condition on the target space would be compactness. This, however will be too restrictive for us and we will show the theorem for metric spaces with the Heine-Borel property, i.e. where closed and bounded sets are compact. In such spaces every bounded sequence has a convergent subsequence (since bounded sets are relatively compact), also such spaces are seperable. Examples of metric spaces with the Heine-Borel property are $\mathbb{R}^n$ or complete Riemannian manifolds, by the theorem of Hopf-Rinow, see e.g. [8] Theorem 8.16, p.137.

1.2.16 Theorem. Let Y be a seperable metric space and X be a metric space with the Heine-Borel property. Let $(f_n)_{n \in \mathbb{N}}$ be a uniformly equicontinuous sequence of maps from Y to X, which is poitwise bounded, i.e. $(f_n(y))_{n \in \mathbb{N}}$ is bounded in X for every $y \in Y$. Then $(f_n)_{n \in \mathbb{N}}$ has a subsequence, converging uniformly on compact subsets of Y, to a map f. Furthermore f is uniformly continuous.

Proof. Denote by d_X and d_Y the metrics on X and Y respectively. Let $D = \{x_1, x_2, \ldots\}$ be a countable, dense subset of Y. Our argument will include Cantor's diagonal process. First note that, by assumption $f_n(x_1)$ is bounded and therefore, since X has the Heine-Borel property, $(f_n(x_1))$ has a convergent subsequence, which we, by slight abuse of notation, denote by $(f_{n_1}(x_1))$. Further $(f_n(x_2))$ is bounded and therefore also $(f_{n_1}(x_2))$ is bounded and possesses a convergent subsequence. Iterating this procedure we obtain for every $k \in \mathbb{N}$ a subsequence (f_{n_k}) of $(f_{n_{k-1}})$ such that $(f_{n_k}(x_i))$ converges for $1 \leq i \leq k$. Now, denoting by (f_{n_n}) the diagonal sequence, we have found a subsequence of (f_n), converging pointwise at x_i for every $i \in \mathbb{N}$. From now on we will therefore w.l.o.g. assume, that the given sequence $(f_n)_n$ converges for all $x_i \in D$. Note that Y and therefore D might by finite, but in this case the statement follows immediately.
We now proceed to prove convergence of (f_n) for every $x \in Y$. Let $\varepsilon > 0$, by uniform equicontinuity of (f_n), there is a $\delta > 0$ such that $d_X(f_n(x), f_n(y)) \leq \varepsilon$, whenever $d_Y(x, y) \leq \delta$ and for all $n \in \mathbb{N}$. Let $y \in Y$ arbitrary, by denseness of D, there exists $x \in D$ such that $d_Y(x, y) \leq \delta$. Further, since $f_n(x)$ converges, there is an integer $N \in \mathbb{N}$ such that $d_X(f_n(x), f_m(x)) \leq \varepsilon$ for all $n, m \geq N$. We obtain

$$d_X(f_n(y), f_m(y)) \leq d_X(f_n(y), f_n(x)) + d_X(f_n(x), f_m(x))$$
$$+ d_X(f_m(x), f_m(y)) \leq 3\varepsilon$$

So $(f_n(y))$ is a Cauchy sequence in X. Since X has the Heine-Borel property, $(f_n(y))$ has a convergent subsequence and is therefore itself convergent. We

now define $f : Y \to X$, $f(x) := \lim_{n\to\infty} f_n(x)$. Let us show that this map is uniformly continuous. Let $\varepsilon > 0$ and choose δ as above, then for $x, y \in Y$ such that $d_Y(x, y) < \delta$ and for every $n \in \mathbb{N}$, we get $d_X(f_n(x), f_n(y)) < \varepsilon$. Since this holds for every $n \in \mathbb{N}$ we obtain $d_X(f(x), f(y)) \leq \varepsilon$ and f is uniformly continuous.
It remains to show the claimed uniform convergence of (f_n) to f on compact subsets of Y. To this end let $K \subseteq Y$ be compact, then the set $D \cap K$ is finite. For every $x \in K$ there exists a $z \in D \cap K$ such that $d_Y(x, z) < \delta$ (again for the above δ). Since $D \cap K$ is finite, there exists an integer $M \in \mathbb{N}$ such that for all $n \geq M$

$$\max_{y \in D\cap K} \left\{ d_X(f_n(y), f(y)) \right\} < \varepsilon.$$

Therefore we obtain for $x \in K$ and $n \geq M$

$$d_X(f(x), f_n(x)) \leq d_X(f(x), f(z)) + d_X(f(z), f_n(z)) + d_X(f_n(z), f_n(x)) \leq 3\varepsilon,$$

implying uniform convergence of $(f_n)_n$ on K. $\square$

We obtain some consequences for paths into appropriate spaces.

1.2.17 Corollary. Let M be a compact metric space. Further for every $n \in \mathbb{N}$ let $\gamma_n : [0, 1] \to M$ be a path parameterized proportionally to arclength such that $L_d(\gamma_n) \leq C$ for all $n \in \mathbb{N}$ and some $C > 0$. There then exists a subsequence on $(\gamma_n)_{n\in\mathbb{N}}$, converging uniformly to a path γ with $L_d(\gamma) \leq C$.

Proof. Since γ_n is C-Lipschitz with C independent of n, the sequence (γ_n) is uniformly equicontinuous. By compactness of M each γ_n is pointwise bounded. By 1.2.16 there exists a subsequence converging uniformly to a path γ. By lower semi-continuity of L_d we obtain $L_d(\gamma) \leq \liminf L_d(\gamma_n) \leq C$. $\square$

Note that for a sequence of paths with uniformly bounded length, one can w.l.o.g. always assume them to be paramerterized proportionally to arclength in the above way. Theorem 1.2.16 helps us to obtain shortest paths in metric spaces.

1.2.18 Definition. Let $(M, \mathcal{A}, L)$ be a length structure and $p, q \in M$. An admissible path $\gamma : [a, b] \to M$ is called shortest path between p and q, if its length is minimal under all paths in $\mathcal{A}$ connecting p and q, i.e. $L(\gamma) \leq L(\sigma)$ for all paths $\sigma \in \mathcal{A}$ that connect p to q.

1.2.19 Proposition. Let (M, d) be a length space stemming from the length structure $(M, \mathcal{A}, L)$, then the following hold:

(i) A path $\gamma : [a, b] \to M$ joining two points of finite distance, is a shortest path if and only if its length is equal to the distance between its endpoints.

(ii) For shortest paths $\gamma \in \mathcal{A}$ the lengths L_d and L are equal.

(iii) If a sequence of shortest paths γ_n between p_n and q_n $(n \in \mathbb{N})$ converges pointwise to a path γ for $n \to \infty$, then γ is a shortest path between $p := \lim_{n\to\infty} p_n$ and $q := \lim_{n\to\infty} q_n$.

Proof. (i) Since M is a length space and for $p := \gamma(a)$ and $q := \gamma(b)$ we have $d(p,q) < \infty$. For every $\varepsilon > 0$ there exists a path σ from p to q with $L(\sigma) < d(p,q) + \varepsilon$. If γ is a shortest path from p to q it has to fulfill $L(\gamma) \leq L(\sigma) \leq d(p,q) + \varepsilon$ for any $\varepsilon > 0$, thus $L(\gamma) = d(p,q)$.
The reverse implication holds trivially by definition of the intrinsic metric.

(ii) By 1.1.14, $d(p,q) = d_{L_d}(p,q)$ for all $p, q \in M$. For a shortest path γ as in (i) it then holds that

$$L(\gamma) = d(p,q) = d_{L_d}(p,q) \leq L_d(\gamma),$$

and since $L_d \leq L$ always holds in length spaces, we arrive at $L(\gamma) = L_d(\gamma)$.

(iii) Since all γ_n are shortest paths by (ii), $L(\gamma_n) = L_d(\gamma_n)$. By 1.1.14 and 1.1.18 we obtain

$$\begin{aligned} d(p,q) &= d_{L_d}(p,q) \leq L_d(\gamma) \leq \liminf_{n\to\infty} L_d(\gamma_n) \\ &\leq \liminf_{n\to\infty} L(\gamma_n) = \liminf_{n\to\infty} d(p_n, q_n) = d(p,q). \end{aligned}$$

□

We will mostly be concerned with shortest paths in a metric space w.r.t. the structure $(M, \mathcal{C}, L_d)$, if no other length structure is mentioned shortest paths will always be relative to this length structure. To prove the existence of shortest paths in spaces with the Heine-Borel property, we need the following lemma.

1.2.20 Lemma. Let M be a metric space with the Heine-Borel property, further let $\gamma_n : [0,1] \to M$ be a sequence of paths parameterized proportional to arclength with $L_d(\gamma_n) \leq C$ for all $n \in \mathbb{N}$ and some $C > 0$. If the set $\{\gamma_n(0) \mid n \in \mathbb{N}\}$ is bounded in M, then there is a subsequence of $(\gamma_n)_{n\in\mathbb{N}}$, converging uniformly to a path γ with $L_d(\gamma) \leq C$.

Proof. Since M has the Heine-Borel property, every bounded set has an accumulation point, so by passing to a subsequence of (γ_n), we can w.l.o.g. assume convergence of $(\gamma_n(0))$. We set $p := \lim_{n\to\infty} \gamma_n(0)$. There then is a constant $R \geq 0$ such that $d(\gamma_n(0), p) \leq R$ for all n. Further for all $n \in \mathbb{N}$ and all $t \in [0,1]$ we estimate

$$d(p, \gamma_n(t)) \leq d(p, \gamma_n(0)) + d(\gamma_n(0), \gamma_n(t)) \leq R + L_d(\gamma_n)t \leq R + C$$

and therefore $\{\gamma_n(t) \mid n \in \mathbb{N}\}$ is bounded for all t. The statement now follows from 1.2.16 and the lower semi-continuity of the length 1.1.18. □

1.2.21 Proposition. Let M be a metric space with the Heine-Borel Property and let $p, q \in M$. If there is a rectifiable path connecting p and q, then there is a shortest path from p to q.

Proof. We will show that there is a path, whose length is equal to the infimum of the lengths of all paths connecting p and q. Set $l := \inf \big\{ L_d(\sigma) \mid \sigma \text{ a path form } p \text{ to } q \big\} < \infty$ and let $(\gamma_n)_n$ be a sequence of paths from p to q such that $L_d(\gamma_n) \to l$ as $n \to \infty$ (which exists by the definition of l as an infimum). We can assume all γ_n to have finite length, be parameterized proportionally to arclength and fulfill $\gamma_n(0) = p$ and $\gamma_n(1) = q$. By Lemma 1.2.20 there exists a subsequence of (γ_n) converging uniformly to a path γ. This path also connects p to q. By lower semi-continuity of L_d we obtain $L_d(\gamma) \leq \liminf L_d(\gamma_n) = l$, but by the definition of l also $L_d(\gamma) \geq l$, so $L_d(\gamma) = l$. $\square$

Next we want to identify another class of spaces with the Heine-Borel property, we will need the following lemma.

1.2.22 Lemma. Let M be a length space and $p, q \in M$. If $\alpha, \beta \geq 0$ are such that $\alpha + \beta \geq d(p, q)$, then for every $\varepsilon > 0$, there exists $x \in M$ such that $d(p, x) \leq \alpha$ and $d(q, x) \leq \beta + \varepsilon$.

Proof. Let $\varepsilon > 0$, since M is a length space there is a path $\gamma : [a, b] \to M$ form p to q such that $L(\gamma) \leq d(p, q) + \varepsilon$. Without loss of generality let $\alpha \leq L(\gamma)$ and γ be parameterized by arclength. Set $x := \gamma(a + \alpha)$. It follows that $d(p, x) \leq L(\gamma_{|[a,a+\alpha]}) = \alpha$. Further

$$d(q, x) \leq L(\gamma_{|[a+\alpha,b]}) = L(\gamma) - \alpha \leq d(p, q) + \varepsilon - \alpha \leq \beta + \varepsilon,$$

proving the lemma. $\square$

1.2.23 Theorem. Let (M, d) be a complete (in the metric sense), locally compact length space, then M has the Heine-Borel property, i.e. bounded and closed subsets of M are compact.

Proof. It suffices to show that closed balls are compact. We will us the fact that in a complete metric space closed, precompact sets are compact. Further we will use that a set is precompact, if there exists an ε-mesh for any $\varepsilon > 0$. An ε-mesh of a set K is a finite cover by sets of diameter less or equal than ε, in particular ε-balls are admissible.
Let $p \in M$ be arbitrary. We claim that, if for all $0 < r < \rho$ the ball $\overline{B_r(p)}$ is compact, then $\overline{B_\rho(p)}$ is compact. Let us prove this now:
Let $\varepsilon > 0$ be small enough such that $\rho - \frac{\varepsilon}{3} > 0$ and let $r > 0$ be such that $\rho - \frac{\varepsilon}{3} < r < \rho$. Since $B := \overline{B_r(p)}$ is compact, there are finitely many $x_1, \ldots, x_n \in B$ such that $B \subseteq \bigcup_{i=1}^n B_{\frac{\varepsilon}{3}}(x_i)$. Let $q \in \overline{B_\rho(p)}$, since M is a length space, by 1.2.22 there is a point $z \in M$ such that $d(p, z) \leq \rho - \frac{\varepsilon}{3}$ and $d(q, z) \leq \frac{2\varepsilon}{3}$. This means $z \in B_r(p)$ and therefore $z \in B_{\frac{\varepsilon}{3}}(x_i)$ for some $i \in \{1, \ldots, n\}$. Further the second inequality defining z implies $d(q, x_i) \leq d(q, z) + d(x_i, z) \leq \varepsilon$. So $\overline{B_\rho(p)} \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i)$, which implies that $\overline{B_\rho(p)}$ is precompact (since contained in any ε-mesh), and therefore compact. This proves the claim.
Set $R := \sup\{r > 0 \,|\, \overline{B_r(p)} \text{ is compact}\}$, since M is locally compact, $R > 0$. Assume that $R < \infty$. For any $q \in M$ there is some $r_q > 0$ such that $B_{r_q}(q)$ is compact. Since $B' := \overline{B_R(p)} \subseteq \bigcup_{q \in B'} B_{\frac{r_q}{2}}(q)$ and since B' is compact, there is a finite set $F \subseteq B'$ such that $B' \subseteq \bigcup_{q \in F} B_{\frac{r_q}{2}}(q)$. Set $r_0 := \min_{q \in F}\{\frac{r_q}{2}\}$, then $r_0 > 0$. Further, since M is a length space, for any $z \in \overline{B_{R+r_0}(p)}$ by 1.2.22, there is a point $y \in M$ such that $d(p, y) \leq R$ and $d(y, z) \leq r_0$. This implies that $y \in B'$ and $y \in B_{\frac{r_q}{2}}(q)$ for some $q \in F$. The fact that $d(z, y) \leq r_0 \leq \frac{r_q}{2}$ implies

$$d(z, q) \leq d(z, y) + d(q, y) \leq \frac{r_q}{2} + \frac{r_q}{2} = r_q,$$

so $z \in B_{r_q}(q)$. We thus obtain $\overline{B_{R+r_0}(p)} \subseteq \bigcup_{q \in F} B_{r_q}(q)$. As a finite union of compact sets, the set $\bigcup_{q \in F} \overline{B_{r_q}(q)}$ is itself compact. This means that $\overline{B_{R+r_0}(p)}$ is compact, since it is closed and contained in a compact set. We have arrived at a contradiction to the assumption, so $R = \infty$ and the theorem is proved. $\square$

1.2.24 Remark. By the above theorem and 1.2.21, in a complete, locally compact length space M any two points $p, q \in M$ with $d(p, q) < \infty$ can be joined by a shortest path. Therefore if the metric on M is finite, i.e. $d(p, q) < \infty$ for all $p, q \in M$, M is also complete as a length space in the sense of 1.1.9.

1.3 Geodesics in Metric Spaces

There are different ways to define geodesics in a metric space via the notion of shortest paths. If we want geodesics to be (globally) shortest paths, which are a little easier to handle, we would have to exclude the very natural case of great circles on the sphere. A great circle would cease to be a geodesic as soon as its image contains antipoldal points. We follow in this chapter [5] where geodesics are defined as globally distance preserving as well as [3] and [6] for the local case.

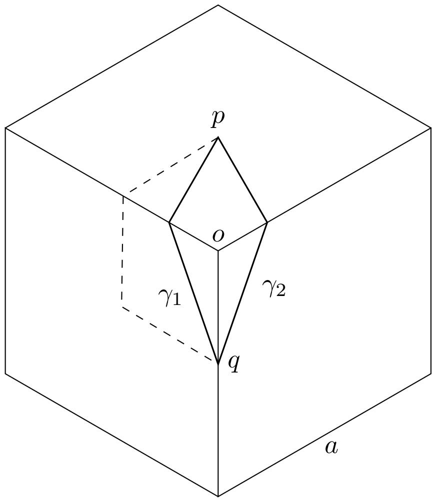
1.3.1 Definition. Let M be a metric space. A path $\gamma : I \to M$ is called a *geodesic*, if it is locally distance preserving from I to M. By that we mean for every $t_0 \in \operatorname{int}(I)$ there exists a neighbourhood $J \subseteq I$ of t_0, such that for all $t, s \in J$ we have $d(\gamma(t), \gamma(s)) = |s - t|$.

1.3.2 Remark. (i) Note that the set J in the above definition can be replaced by a compact neighbourhood of t_0. Further J can always be chosen as an interval.

(ii) Neither existence nor uniqueness of geodesics is guaranteed, not even in length spaces.

(iii) Limits of geodesics are not necessarily geodesics. A study of limits of globally distance preserving paths is done in [5], chapter 2.3.

1.3.3 Example. Consider in $\mathbb{R}^3$ the surface C of a cube and equip it with the intrinsic metric induced by the Euclidean distance. This means the distance of two points is equal to the (Euclidean) length of the shortest polygon on C connecting them. Straight lines not running through a vertex are geodesics, however there are points around the vertices, which can be joined by more than one geodesic, as the figure below illustrates.



Here p lies in the center of the upper side of the cube with side length a, and q lies in the middle of the centered edge. The paths γ_1 and γ_2 meet the edges at distance $\frac{a}{4}$ from o. An easy calculation shows that both paths have length $\frac{a}{2}\sqrt{5}$ and are shortest paths form p to q.
Note that the „straight line“ from q to o and then to p is not a geodesic nor a shortest path from q to p, since it has length $\frac{a}{2}(1+\sqrt{2}) > \frac{a}{2}\sqrt{5}$ In fact any geodesic approaching a vertex can not be continued as a geodesic across that vertex. This shows that the limit of a sequence of geodesics is not a geodesic in general.

Let us investigate the connection of geodesic and shortest path.

1.3.4 Proposition. Let M be a metric space, and $\gamma : I \to M$ a geodesic, then γ is locally a shortest path.

Proof. Let $a, b \in I$ be such that $\gamma_{|[a,b]}$ is distance preserving and set $p := \gamma(a)$, $q := \gamma(b)$. Assume there were a path $\sigma : [0,1] \to M$ with $\sigma(0) = p$, $\sigma(1) = q$ and $L_d(\sigma) < L_d(\gamma_{|[a,b]})$. By assumption on γ we have $d(p,q) = b - a$ and $d(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in [a,b]$. Therefore

$$L_d(\gamma_{|[a,b]}) = \sup\{\sum_{i=1}^{N} d(\gamma(t_i), \gamma(t_{i-1})) \,|\, a = t_0 < \ldots < t_N = b,\, N \in \mathbb{N}\}$$

$$= \sup\{\sum_{i=1}^{N} (t_i - t_{i-1}) \,|\, a = t_0 < \ldots < t_N = b,\, N \in \mathbb{N}\} = b - a.$$

We obtain $b - a = L_d(\gamma_{|[a,b]}) > L_d(\sigma) \geq d(p,q) = b - a$, a contradiction. $\square$

We further obtain

1.3.5 Proposition. Let M be a metric space and $\gamma : [a,b] \to M$ be a geodesic, then $L_d(\gamma_{|[t,s]}) = s - t$, for $t, s \in [a,b]$, $t \leq s$. In particular geodesics are always parameterized by arclength.

Proof. By [5], Proposition 1.1.10, p. 14. we have $L_d(\gamma) = \lim_{|\sigma|\to 0} V_\sigma(\gamma)$, where σ is a subdivision of $[a,b]$ with maximum mesh size $|\sigma|$ and $V_\sigma(\gamma)$ is the corresponding variation. For any $t,s \in [a,b]$, $t \leq s$ there exist finitely many $t_i \in [t,s]$ such that $\gamma_{|[t_i,t_[t+1]}$ is distance preserving. By taking ever finer subdivisions σ of $[t,s]$ with $|\sigma| \to 0$, we obtain $L_d(\gamma_{|[t,s]}) = s - t$ by the proof of 1.3.4 and additivity of length. $\square$

One major difference between the metric space definition and the definition of geodesics in (Semi)-Riemannian geometry is, that linear reparameterizations of geodesics in metric spaces are in general not locally distance preserving, as speed may change. Only reparameterizations of the form $t \mapsto \pm t + c$, $c \in \mathbb{R}$ are again geodesics. We call a linear reparameterization of a geodesic an affine geodesic. In analogy to Riemannian geometry a path that has a reparameterization as a geodesic, is called a pregeodesic.
We now give a short account of globally distance preserving maps as treated in [5], chapters 2.2-2.4.
In the above we have seen that geodesics are parameterized by arclength, we now investigate under which conditions paths parameterized by arclength are globally distance preserving paths.

1.3.6 Proposition. Let M be a metric space, $\gamma : [a,b] \to M$ a path parameterized by arclength, then the following are equivalent:

(i) γ is globally distance preserving, i.e. $d(\gamma(t),\gamma(s)) = |t-s|$ for all $t,s \in [a,b]$,

(ii) for all $t,s \in [a,b]$ with $a \leq t \leq s \leq b$ the following holds
$$d(\gamma(a),\gamma(s)) = d(\gamma(a),\gamma(t)) + d(\gamma(t),\gamma(s)),$$

(iii) $L_d(\gamma) = d(\gamma(a),\gamma(b))$.

In particular if any of these conditions hold γ is a geodesic. Further shortest paths in length spaces, which are parameterized by arclength, are geodesics and even globally distance preserving.

Proof. (i)$\Rightarrow$(ii): Let γ be globally distance preserving, then for all t,s as in (ii), we have
$$d(\gamma(a),\gamma(s)) = s - a = s - t + t - a = d(\gamma(s),\gamma(t)) + d(\gamma(t),\gamma(a))$$
(ii)$\Rightarrow$(iii): For any subdivision $\{t_i\}_{i=0}^N$ of $[a,b]$ we obtain by repeatedly applying (ii):
$$\sum_{i=1}^N d(\gamma(t_i),\gamma(t_{i-1})) = d(\gamma(a),\gamma(b)).$$
Taking now the supremum over all subdivisions we obtain $L_d(\gamma) = d(\gamma(a),\gamma(b))$.
(iii) $\Rightarrow$ (i): For $a \leq t \leq s \leq b$ we have
$$L_d(\gamma) = d(\gamma(a),\gamma(b)) \leq d(\gamma(a),\gamma(t)) + d(\gamma(t),\gamma(s)) + d(\gamma(s),\gamma(b))$$
$$\leq L_d(\gamma_{|[a,t]}) + L_d(\gamma_{|[t,s]}) + L_d(\gamma_{|[s,b]}) = L_d(\gamma),$$

where we have used additivity of L_d and the fact that length of a path is greater than the distance of its endpoints with respect to the intrinsic metric of that length, confer 1.1.13. We therefore have equality in each line in the above calculation. This implies that for $t, s \in [a, b]$, $t \leq s$ we have $d(\gamma(t), \gamma(s)) = L_d(\gamma_{|[t,s]}) = s - t$, since γ is parameterized by arclength. Therefore we arrive at (i).
The addendum about shortest paths in length spaces holds, since 1.2.19(i)-(ii) implies (iii). □

We can deduce from this the following corollary for (non-globally distance preserving) geodesics.

1.3.7 Corollary. Let M be a metric space and $\gamma : [a, b] \to M$ be a path parameterized by arclength. The following are equivalent:

(i) γ is a geodesic,

(ii) for all $t \in (a, b)$ there exists a compact interval $[c, d] \subseteq [a, b]$ containing t in its interior such that for $c \leq s_1 \leq s_2 \leq d$, it holds that

$$d(\gamma(c), \gamma(s_2)) = d(\gamma(c), \gamma(s_1)) + d(\gamma(s_1), \gamma(s_2)),$$

(iii) for all $t \in (a, b)$ there exists a compact interval $[c, d] \subseteq [a, b]$ containing t in its interior such that $L_d(\gamma_{|[c,d]}) = d(\gamma(c), \gamma(d))$.

In particular in length spaces locally shortest paths parameterized by arclength are geodesics.

Proof. The stated equivalence will follow from the one in 1.3.6 if we restrict the path in question to suitable intervals.
Let γ be a geodesic, then for $t \in (a, b)$ by definition of a geodesic there exists an interval $[c, d] \subseteq [a, b]$ containing t as an interior point such that $\gamma_{|[c,d]}$ is globally distance preserving. By 1.3.6 this is implies (ii) and (iii).
(ii) is equivalent to (iii) with the same interval $[c, d]$ by 1.3.6 by restricting γ to $[c, d]$.
The implication $(iii) \Rightarrow (i)$ follows, since for all t the existence of the interval $[c, d]$ guarantees by 1.3.6 that γ is distance preserving on $[c, d]$.
Also the addendum follows as in 1.3.6, since for a locally shortest path γ, around every point exists a compact interval J containing that point in its interior, such that $\gamma_{|J}$ is a shortest path and (iii) holds on J by 1.2.19. □

1.3.8 Remark. By the above result and 1.3.4 paths in length spaces, parameterized by arclength are locally shortest paths if and only if they are geodesics. Compare this e.g. to [3] where geodesics in length spaces are defined as locally shortest paths w.r.t. L.

We now deal with the existence of geodesics resp. distance preserving paths.

1.3.9 Corollary. Let M be a length space with the Heine-Borel property, then for all $p, q \in M$ with $d(p, q) < \infty$, there exists a geodesic γ joining these points. Futhermore γ is a shortest path and globally distance preserving.

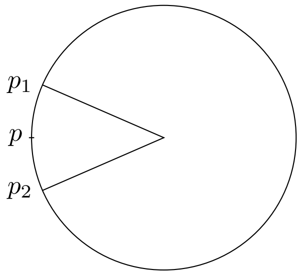
Proof. By 1.2.21 there exists a shortest path γ form p to q with $L(\gamma) = d(p, q)$. By 1.2.10 there exists a reparameterization of γ by arclength. By 1.3.6 this reparameterization is a geodesic and globally distance preserving. □

In [5], chapter 2.4 a metric space is called geodesic space, if any pair of points can be joined by a globally distance preserving geodesic. Such spaces are length spaces by the next result.

1.3.10 Corollary. Let M be a geodesic space, then it is a complete length space in the sense of Definition 1.1.9.

Proof. Let $p, q \in M$ and $\gamma : [a, b] \to M$ be a globally distance preserving geodesic joining them, with $\gamma(a) = p$, $\gamma(b) = q$. We have by 1.3.6 $d(p, q) = L_d(\gamma)$ and M is a complete length space. □

1.3.11 Remark. If we omit globally distance preserving, there can be no analogue of the above result, as can be seen by the unit circle S^1 with one point p removed. Equip $S^1\backslash\{p\}$ with the length metric of S^1, i.e. the distance of two points is equal to the length of the shorter arc (in S^1) spanned by them. Any two points p_1, p_2 in this space can be joined by a geodesic, but points close to p but on opposite sides can only be joined by a geodesic of length greater π, see the figure below, so $S^1\backslash\{p\}$ is not a length space.



On Riemannian manifolds (with smooth Riemannian metrics) the existence of Riemannian geodesics is guaranteed at least locally in normal neighbourhoods and such radial geodesics are locally shortest paths and even unique. Thus radial geodesics are affine geodesics in the sense of 1.3.1. We will prove now a similar result in locally compact length spaces, to do this we need

1.3.12 Lemma. Let M be a length space, $p \in M$ and $r > 0$. For any two points $x, y \in B_r(p)$, there is a path from x to y of length less than $2r$. Furthermore any such path is contained in $B_{2r}(p)$.

Proof. Since $d(x, y) < 2r$ and since M is a length space there exists a path $\gamma : [0, 1] \to M$ of length less than $2r$ from x to y. Assume $\gamma([0, 1])$ is not contained in $B_{2r}(p)$. Then there exists $t \in [0, 1]$ such that $d(\gamma(t), p) \geq 2r$. Using the triangle inequality, this leads to

$$d(x, \gamma(t)) \geq d(\gamma(t), p) - d(p, x) > r$$

and analogously $d(y, \gamma(t)) > r$. Combining these estimates we obtain

$$L(\gamma) = L(\gamma_{|[0,t]}) + L(\gamma_{|[t,1]}) \geq d(x, \gamma(t)) + d(y, \gamma(t)) > 2r,$$

a contradiction. □

1.3.13 Theorem. Let M be a locally compact length space. For every $p \in M$ exists $r > 0$ such that any two points $x, y \in B_r(p)$ can be joined by a geodesic γ, which is contained in and distance preserving into $B_{2r}(p)$.

Proof. Let $p \in M$, since M is locally compact there exists $r > 0$ such that $B_{2r}(p)$ is relatively compact, hence has the Heine-Borel property. Since M is a length space, by the above Lemma there exists a rectifiable path form x to y contained in $B_{2r}(p)$. By 1.2.21, there exists a shortest path $\gamma : [a, b] \to B_{2r}(p)$ form x to y which can can assume to be parameterizred by arclength. By 1.2.19 $L(\gamma) = L_d(\gamma) = d(x, y) < 2r$ and by 1.3.6 γ is a globally distance preserving geodesic. □

We now state and proof a refinement of theorem 1.2.23 credited to Hopf-Rinow-Cohn-Vossen.

1.3.14 Theorem. Let M be a locally compact length space, then the following conditions are equivalent

(i) M is (metrically) complete,

(ii) M has the Heine-Borel property (i.e. closed, bounded sets are compact),

(iii) every geodesic $\gamma : [0, a) \to M$ can be extended to a continuous path $\bar{\gamma} : [0, a] \to M$,

(iv) there exists $p \in M$ such that every shortest path $\gamma : [0, a) \to M$ with $\gamma(0) = p$ can be extended to a continuous path $\bar{\gamma} : [0, a] \to M$.

Proof. The implication (i)⇒(ii) was shown in 1.2.23.
(ii)⇒(i) holds in any metric space: Let $(x_n)_n$ be a Cauchy sequence in M, then $(x_n)_n$ is bounded and therefore by assumption relatively compact. This means $(x_n)_n$ has an accumulation point and since it is Cauchy it converges to this point, hence M is complete.
(i)⇒(iii): Let $\gamma : [0, a) \to M$ be a geodesic and let $(t_i)_{i\in\mathbb{N}} \subseteq [0, a)$ be a sequence converging to a. Then $(\gamma(t_i))_i$ is a Cauchy sequence in M, since $d(\gamma(t_i), \gamma(t_j)) = |t_i - t_j|$ for i, j large enough since γ is a geodesic. By (i) $(\gamma(t_i))_i$ has a limit point q and by setting $\bar{\gamma}(t) := \begin{cases} \gamma(t) \text{ for } t \in [0, a) \\ q \text{ for } t = a \end{cases}$ we obtain the desired continuous path.

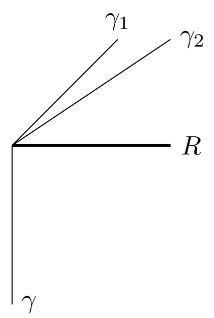
(iii)⇒(iv): Holds since via parameterization by arclength a shortest path is a geodesic by 1.3.6.
It remains to show (iv)⇒(ii). We set $R := \sup\{r \geq 0 \,|\, \overline{B_r(p)} \text{ is compact}\}$, where p is as in (iv). $R > 0$ since M is locally compact. If $R = \infty$ then (ii) is proved. Assume $R < \infty$, we claim that the open ball $B_R(p)$ is precompact. To see this, by [7], Theorem 13.11, p.170 it suffices to show, that every sequence in $B_R(p)$ has a subsequence converging in M. Let (x_i) be such a sequence and set $r_i := d(p, x_i) < R$. We may assume $r_i \to R$, since otherwise (x_i) would be contained in a compact ball with radius less than R and would therefore have a convergent subsequence by definition of R. Since for small $\varepsilon > 0$ the closed ball $\overline{B_{r_i+\varepsilon}(p)}$ is compact (and therfore has the Heine-Borel property), there exists a shortest path γ_i from p to x_i by 1.2.21. Let γ_i be parameterized by arclength such that $\gamma_i : [0, r_i] \to M$ for all $i \in \mathbb{N}$. Since the lengths of these paths are uniformly bounded (by R) the family γ_i is uniformly equicontinuous. By theorem 1.2.16, the sequence $(\gamma_{i|[0,r_1]})$ has a uniformly converging subsequence (γ_{i_k}). The sequence $(\gamma_{i_k|[0,r_2]})_{k\in\mathbb{N}}$ by the same arguments again has a uniformly

converging subsequence. Iterating this procedure we obtain diagonal sequence (for simplicity again denoted by) $(\gamma_i)_i$ of paths $\gamma_i : [0, r_i] \to M$. For every $t \in [0, R)$ for large enough i $(\gamma_i(t))$ is defined and converges in M. By 1.2.19(iii) the map defined by $\gamma : [0, R) \to M$, $t \mapsto \lim_{i\to\infty} \gamma_i(t)$ is a shortest path, since it is a limit of shortest paths. By (iv) this path can be extended to a path $\bar{\gamma} : [0, R] \to M$. Since $\bar{\gamma}$ is a continuous extension and since $r_i \to R$, we obtain $x_i = \gamma_i(r_i) \to \bar{\gamma}(R)$ and thus the claim is proved.
By this claim $B := \overline{B_R(p)}$ is compact. To arrive at a contradiction we want to show existence of a compact ball around p with radius greater than R. We proceed similarly as in the proof of 1.2.23. Since M is locally compact, for every $q \in B$ there is a $r(q) > 0$ such that $B_q := B_{r(q)}(q)$ is relatively compact. Since $B \subseteq \bigcup_{q\in B} B_q$ by compactness of B there exists a finite set $F \subseteq B$ such that $B \subseteq \bigcup_{q\in F} B_q \subseteq \bigcup_{q\in F} \overline{B_q}$. As in the proof of 1.2.23, by using that M is a length space and 1.2.22, we obtain for $r_0 = \min_{q\in F} \frac{r(q)}{2}$, that

$$\overline{B_{R+r_0}(p)} \subseteq \bigcup_{q\in F} \overline{B_q},$$

so $\overline{B_{R+r_0}(p)}$ is compact since it is closed and contained in a finite union of compact sets, a contradiction. Note that in this last part we may proceed as in 1.2.23, since the there assumed completeness is not used in this part. □

1.3.15 Remark. Note that for locally compact length spaces any of the equivalent conditions of 1.3.14 imply that points with finite distance can be joined by a globally distance preserving geodesic, by 1.3.9.
The path $\bar{\gamma}$ in 1.3.14 (iii) is also a geodesic, since the definition of geodesics is only concerned with interior points of the domain of definition. However when prolonging geodesics beyond a single point problems may arise, as can be seen in the following example presented in [5], 2.2.3 p.51. Consider $\mathbb{R}^2 \backslash ((0, 1) \times \{0\})$ and equip it with the metric from $\mathbb{R}^2$. The straight line from $(0, -1)$ to the origin can be prolonged in various ways as a geodesic, by any straight line as in the figure below.



γ ... initial geodesic
γ_1 ... first prolongation
γ_2 ... second prolongation
R ... removed segment

1.3.16 Remark. The above results provide a solution to the minimizing problem

$$\min\{L(\gamma) \,|\, \gamma \text{ is a path that connects } p \text{ to } q\}. \tag{1.6}$$

In [1], section 4.4 another, intrinsic formulation of this problem is treated, namely

$$\min\{H^1(C) \,|\, p, q \in C,\ C \text{ closed and connected}\}. \tag{1.7}$$

Here H^1 denotes the one dimensional Hausdorff measure. This formulation does not involve paths, but only their images. We aim to find, for two points, a „one dimensional set“, closed and connected containing these points. In [1], 4.4.20, p.78 it is shown that (1.7) has a solution C under the conditions of 1.3.14 and if $d(p, q) < \infty$. Further the problems (1.6) and (1.7) are equivalent, in particular the set C can be parameterized as a shortest path.

1.4 Absolutely continuous paths

In this section we discuss some auxiliary results and absolutely continuous paths, which will be used in chapter 2, when comparing different lengths on Riemannian manifolds.
In Riemannian geometry the arclength of e.g. a piecewise C^1 path γ is given by $\int \|\gamma'(t)\| \, dt$, where the norm is taken with respect to a Riemannian metric. Here γ' represents velocity of the path. In a metric space there is no sense of direction and hence there can be no analogue of γ'. However to calculate the length, we only need the speed $\|\gamma'\|$ of the path. This will lead us to the so-called metric derivative of a path. Such a derivative will in general not exist for all paths. In $\mathbb{R}^n$ the largest class of paths, which are differentiable (almost everywhere) and for which an integral expression as above makes sense, are the absolutely continuous paths. In $\mathbb{R}^n$ this notion has different, but equivalent formulations, a measure theoretic approach and an ε-δ definition. In metric spaces we need to distinguish between them. We use the definitions of [2], section 3.5.

1.4.1 Definition. Let $I \subseteq \mathbb{R}$ be an interval and (M, d) a metric space. A path $\gamma : I \to M$ is called

(i) *measure absolutely continuous*, if there exists $l \in L^1(I)$ such that for all $a, b \in I$, $a \le b$ it holds that

$$d(\gamma(a), \gamma(b)) \le \int_a^b l(t) \, dt, \tag{1.8}$$

(ii) *metric absolutely continuous*, if $\forall \varepsilon > 0 \, \exists \delta > 0$ such that for all $n \in \mathbb{N}$ and any collection of pairwise disjoint intervals $[a_i, b_i] \subseteq I$, $i = 1, \ldots, n$, it holds that $\sum_{i=1}^n d(\gamma(a_i), \gamma(b_i)) < \varepsilon$, if $\sum_{i=1}^n (b_i - a_i) < \delta$.

Here $L^1(I)$ denotes the Lebesgue-integrable functions on I. We will sometimes abbreviate measure absolutely continuous by mac.

For functions $\mathbb{R} \to \mathbb{R}$, we know that these two definitions coincide (see e.g. [9], Corollary 3.33, p. 105.) In chapter 2 we will see that they even coincide on connected Riemannian manifolds. In general however, only the following is true

1.4.2 Lemma. Let M be a metric space, then any measure absolutely continuous path is metric absolutely continuous.

Proof. Let $\gamma : [0, 1] \to M$ be measure absolutely continuous and let l be as in 1.4.1 (i). We define $F(s) := \int_0^s l(t) \, dt$, which is absolutely continuous (into $\mathbb{R}$), therefore F is also metric absolutely continuous. For any collection of pairwise disjoint subintervals $[a_i, b_i]$, $i = 1, \ldots, n$ it thus holds that

$$\sum_{i=1}^n d(\gamma(a_i), \gamma(b_i)) \le \sum_{i=1}^n \int_{[a_i, b_i]} l(t) \, dt \le \sum_{i=1}^n |F(b_i) - F(a_i)|.$$

It follows that γ is metric absolutely continuous. $\square$

We will now introduce the metric derivative of a path, it serves as a replacement for the speed $\|\gamma'\|$ of a path.

1.4.3 Definition. Let $\gamma : [a, b] \to M$ be a path into a metric space M. The metric derivative at a point $t \in (a, b)$ is defined as

$$|\dot{\gamma}|(t) := \lim_{h \to 0} \frac{d(\gamma(t+h), \gamma(t))}{|h|},$$

whenever it exists.

Note that $|\dot{\gamma}|$ need not exist at any point. For differentiable paths into $\mathbb{R}^n$ the metric derivative (w.r.t. the Euclidean distance) exists everywhere and agrees with the derivative, but for example the path $t \mapsto |t|$ is not differentiable at 0, but its metric derivative at 0 exist and equals 1.
We now check for which classes of paths the metric derivative exists and the formula length equals integral of speed holds.

1.4.4 Proposition. Let M be a metric space and $\gamma : [a, b] \to M$ a path. If γ is Lipschitz continuous, then it is measure absolutely continuous. Further for measure absolutely continuous paths the metric derivative exists almost everywhere in $[a, b]$ and $|\dot{\gamma}| \in L^1([a, b])$. Finally the metric derivative is the minimal $L^1([a, b])$ function such that

$$d(\gamma(t), \gamma(s)) \leq \int_t^s |\dot{\gamma}|(t)\, dt, \qquad \text{for all } t, s \in [a, b],\ t \leq s. \tag{1.9}$$

This means that $\||\dot{\gamma}|\|_{L^1([a,b])} \leq \|l\|_{L^1([a,b])}$ for any $l \in L^1([a, b])$ such that (1.8) holds.

Proof. Let us abbreviate $I := [a, b]$. We will start the proof for Lipschitz and mac paths simultaneously and show that any Lipschitz path is mac by using the first part of the proof. Since in any case γ is continuous, $\gamma(I)$ is compact and hence separable. Let $(x_n)_n$ be a dense sequence in $\gamma(I)$. By 1.4.2 mac paths are metric absolutely continuous, note that Lipschitz paths are also metric absolutely continuous. The map $p \mapsto d(p, q)$ is 1-Lipschitz for fixed $q \in M$ and the composition of 1-Lipschitz maps with metric absolutely continuous maps is again metric absolutely continuous. Therefore the functions

$$\varphi_n : I \to \mathbb{R},\ t \mapsto d(\gamma(t), x_n)$$

are metric absolutely continuous. As absolutely continuous functions into $\mathbb{R}$ they are differentiable almost everywhere. Since countable unions of nullsets are null, at almost every point all φ_n are simultaneously differentiable so we can define

$$\varphi(t) := \sup_{n \in \mathbb{N}} |\varphi_n'(t)| \text{ for almost every } t \in I. \tag{1.10}$$

We will now show that φ is integrable and that $\varphi(t) = |\dot{\gamma}|(t)$ almost everywhere. Consider

$$\liminf_{h \to 0} \frac{d\big(\gamma(t+h), \gamma(t)\big)}{|h|} \geq \liminf_{h \to 0} \frac{|\varphi_n(t+h) - \varphi_n(t)|}{|h|} = |\varphi_n'(t)|$$

for almost every $t \in I$ and for all $n \in \mathbb{N}$, where the inequality is due to the reversed triangle inequality. This implies

$$\liminf_{h\to 0} \frac{d\big(\gamma(t+h),\gamma(t)\big)}{|h|} \geq \varphi(t) \text{ for a.e. } t \in I. \tag{1.11}$$

We claim $d(\gamma(t+h),\gamma(t)) = \sup_{n\in\mathbb{N}} |\varphi_n(t+h)-\varphi_n(t)|$. Indeed we have by the reversed triangle inequality

$$|\varphi_n(t+h)-\varphi_n(t)| = |d(\gamma(t+h),x_n) - d(\gamma(t),x_n)| \leq d(\gamma(t+h),\gamma(t)),$$

for all $n \in \mathbb{N}$ and thus $d(\gamma(t+h),\gamma(t)) \geq \sup_{n\in\mathbb{N}} |\varphi_n(t+h)-\varphi_n(t)|$. By denseness of $(x_n)_n$ there is a subsequence $(x_{n_k})_k$ such that $\lim_{k\to\infty} x_{n_k} = \gamma(t)$, this implies

$$\sup_{n\in\mathbb{N}} |\varphi_n(t+h)-\varphi_n(t)| \geq \lim_{k\to\infty} |\varphi_{n_k}(t+h)-\varphi_{n_k}(t)|$$

$$= \lim_{k\to\infty} |d(\gamma(t+h),x_{n_k}) - d(\gamma(t),x_{n_k})| = d(\gamma(t+h),\gamma(t)).$$

Thus we have proved the claim.

If γ is Lipschiz, then $\mathrm{Lip}(\varphi_n) \leq \mathrm{Lip}(\gamma)$ for all $n \in \mathbb{N}$ by [3], Proposition 1.4.3 (2), p. 9, since $p \mapsto d(p,q)$ is 1-Lipschitz. This leads to $|\varphi_n'(t)| \leq \mathrm{Lip}(\gamma)$ almost everywhere and for all n, so $|\varphi(t)| \leq \mathrm{Lip}(\gamma)$ almost everywhere and $\varphi \in L^\infty(I) \subseteq L^1(I)$, since $I = [a,b]$ is bounded.

If γ is mac, then by definition there exists $l \in L^1(I)$ such that (1.8) holds. (1.11) now implies for every Lebesgue point (i.e. points of the Lebesgue set, cf. [9], chapter 3.4, p.95-98) $t \in I$ of l

$$0 \leq \varphi(t) \leq \liminf_{h\to 0} \frac{1}{|h|} d(\gamma(t+h),\gamma(t)) \leq \liminf \frac{1}{|h|} \int_t^{t+h} l(r)\,dr = l(t). \tag{1.12}$$

Since almost every point in I is a Lebesgue point w.r.t. l, we obtain $\varphi \in L^1(I)$. In both cases (γ Lipschitz or mac), applying the fundamental theorem of calculus to φ_n leads to

$$\begin{aligned} d(\gamma(t+h),\gamma(t)) &= \sup_{n\in\mathbb{N}} |\varphi_n(t+h)-\varphi_n(t)| \\ \leq \sup_{n\in\mathbb{N}} \left| \int_t^{t+h} \varphi_n'(r)\,dr \right| &\leq \int_t^{t+h} \sup_{n\in\mathbb{N}} |\varphi_n'(r)|\,dr = \int_t^{t+h} \varphi(r)\,dr < \infty. \end{aligned} \tag{1.13}$$

This shows in particular that any Lipschitz path is mac. Further (1.13) leads to

$$\limsup_{h\to 0} \frac{1}{|h|} d(\gamma(t+h),\gamma(t)) \leq \varphi(t),$$

for every Lebesgue point t of φ. In summary

$$\limsup_{h\to 0} \frac{1}{|h|} d(\gamma(t+h),\gamma(t)) \leq \varphi(t) \leq \liminf_{h\to 0} \frac{1}{|h|} d(\gamma(t+h),\gamma(t)),$$

for almost every $t \in I$. Hence $|\dot\gamma| = \varphi$ in $L^1(I)$. The statement about minimality follows from (1.12). $\square$

With the help of this result we can now derive an integral expression for L_d for mac paths.

1.4.5 Corollary. Let $\gamma : [0,1] \to M$ be a measure absolutely continuous path, then

$$L_d(\gamma) = \int_0^1 |\dot\gamma|(t)\,dt.$$

Proof. Let $\gamma : [0,1] \to M$ be a mac path. We denote $\tilde L(\gamma) := \int_0^1 |\dot\gamma|(t)\,dt$. To show $L_d \le \tilde L$ take any subdivision $0 = t_0 < \ldots < t_n = 1$, then by (1.9)

$$\sum_{i=1}^n d\big(\gamma(t_{i-1}),\gamma(t_i)\big) \le \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |\dot\gamma|(\tau)\,d\tau = \int_0^1 |\dot\gamma|(\tau)\,d\tau,$$

which implies $L_d(\gamma) \le \tilde L(\gamma)$.
To see the converse take any $0 < \varepsilon < 1$ and $n \in \mathbb{N}$ such that $n \ge \frac{1}{\varepsilon}$ and set $h := \frac{1}{n}$ and $t_i = \frac{i}{n}$. So $h \le \varepsilon$ and therefore

$$\frac{1}{h}\int_0^{1-\varepsilon} d\big(\gamma(t+h),\gamma(t)\big)dt \le \frac{1}{h}\int_0^{1-h} d\big(\gamma(t+h),\gamma(t)\big)\,dt$$

$$= \frac{1}{h}\int_0^h \sum_{i=1}^{n-1} d\big(\gamma(\tau+t_i),\gamma(\tau+t_{i-1})\big)d\tau \le \frac{1}{h}\int_0^h L_d(\gamma)d\tau = L_d(\gamma).$$

This and the Lemma of Fatou imply, that

$$\int_0^{1-\varepsilon} |\dot\gamma|(t)\,dt = \int_0^{1-\varepsilon} \liminf_{h\to 0} \frac{d\big(\gamma(t+h),\gamma(t)\big)}{|h|}\,dt$$

$$\le \liminf_{h\to 0} \frac{1}{|h|}\int_0^{1-\varepsilon} d\big(\gamma(t+h),\gamma(t)\big)\,dt \le L_d(\gamma)$$

holds for any $\varepsilon > 0$ and therefore $L_d(\gamma) \ge \tilde L(\gamma)$, thus $L_d(\gamma) = \tilde L(\gamma)$. □

1.4.6 Remark.

(i) Note that in a metric space the length $\tilde L$ together with the class of mac paths is a length structure by the above corollary.

(ii) Every path parameterized by arclength is 1-Lipschitz and therefore its metric derivative exists almost everywhere. Further for such a path $\gamma : [a,b] \to M$, we have

$$t - a = L_d(\gamma_{|[a,t]}) = \int_a^t |\dot\gamma|(r)\,dr.$$

Differentiating this equation yields $|\dot\gamma|(t) = 1$ for almost every t.

(iii) For a geodesic γ the metric derivative exists everywhere and equals 1, since for $h > 0$ small enough $\frac{d(\gamma(t+h),\gamma(t))}{|h|} = \frac{h}{h} = 1$.

Chapter 2

Length structures on Riemannian manifolds

In this chapter we deal with a specific class of metric spaces/length structures, namely Riemannian manifolds, i.e. smooth manifolds with a Riemannian metric. A smooth Riemannian metric is a smooth $(0, 2)$-tensorfield, which assigns to every point p of the manifold a symmetric, positive definite bilinear form $g_p = g(p)$ on the tangent space T_pM of this point. We start out with manifolds with smooth Riemannian metrics and investigate certain length structures on them. In the second part of the chapter we deal with the situation of continuous Riemannian metrics g, by that we mean that the map $p \mapsto g_p = g(p)$ is merely continuous, not smooth. The crucial difference to the smooth case is, that the exponential map will not be available as a tool, in fact for general continuous metrics we cannot even formulate the geodesic equations. A more detailed study of geodesics for metrics of regularity between continuous and $C^{1,1}$ as well as the exponential map for $C^{1,1}$ metrics is done in chapters 3 and 4.
In this chapter we presuppose a basic knowledge of Riemannian geometry including results on the exponential map, normal neighbourhoods, Riemannian distance etc., confer e.g. [10], chapters 3 and 5.

Throughout this and the next chapters for questions of low regularity we only lower the regularity of the Riemannian metric and assume the manifold to be smooth, which is no restriction in view of [26], Theorem 2.9 (SO AUS EUREM PAPER ÜBERNOMMEN FINDE DAS THEOREM DORT ABER NICHT!!!!)

2.1 Smooth Riemannian metrics

In this section, if not explicitly stated otherwise a Riemannian manifold M will always be equipped with a smooth Riemannian metric denoted by g.
The length usually considered in Riemannian geometry is the arclength, we define it for piecewise smooth paths, but the same definition can be extended without problems to piecewise C^1 paths or even absolutely continuous paths. As we will see later, the resulting length structures are equivalent. From now on we shall denote the class of all piecewise smooth paths into M by $\mathcal{A}_\infty$.

2.1.1 Definition. Let $\gamma : [a,b] \to M$ be a piecewise smooth path into a Riemannian manifold M. Its *arclength* is defined by

$$L(\gamma) := \int_a^b \|\gamma'(t)\|_g \, dt,$$

where $\|\gamma'(t)\|_g = (g(\gamma'(t), \gamma'(t))^{\frac{1}{2}}$.

Let us collect a few results from Riemannian geometry on the arclength.

2.1.2 Remark. For the arclength the following hold (see e.g. [12], Lemma 2.3.2. p. 50)

(i) L is invariant under monotone reparameterizations

(ii) If $\|\gamma'(t)\|_g \neq 0$ for all t, then γ has a reparameterization h, such that $\|(\gamma \circ h)'(t)\|_g = 1$ for all t.

(iii) By additivity of the integral L is additive.

(iv) Continuity of $t \mapsto L(\gamma_{|[a,t]})$ also follows form the properties of the integral.

In summary, L is a length by the criteria in 1.1.1.

In fact, we have

2.1.3 Proposition. Let M be a Riemannian manifold, then the triple $(M, \mathcal{A}_\infty, L)$ is a length structure.

Proof. It only remains to show, that the structure is compatible with the topology of the manifold, but this follows from the existence of a neighbourhood basis of so-called ε-neighbourhoods around a point cf. [12], Proposition 2.3.6, p. 53. $\square$

We can now equip a Riemannian manifold with the intrinsic metric w.r.t. this structure. Another classical result then is the following (for the proof see [12], theorem 2.3.9. p. 54)

2.1.4 Theorem. Let M be a connected Riemannian manifold, then the intrinsic metric with respect to the length structure $(M, \mathcal{A}_\infty, L)$ induces the manifold topology on M.

By the above theorem a connected Riemannian manifold can be turned into a metric space. Next we want to compare the given length structure $(M, \mathcal{A}_\infty, L)$ to the metric one $(M, \mathcal{C}, L_d)$. By 1.1.14 the intrinsic metric induced by $(M, \mathcal{A}_\infty, L_d)$ equals d. This suggests the idea that L_d might serve as a possible extension of the arclength to continuous paths, since L_d makes sense for such paths. In this section we subsequently want to establish the claim $L = L_d$ for an appropriate class of paths, starting with piecewise smooth ones. Later we will even show this for manifolds with continuous Riemannian metrics. For most of this chapter we closely follow [2].

2.1.5 Theorem. Let M be a connected Riemannian manifold, then

$$L(\gamma) = L_d(\gamma), \quad \gamma \in \mathcal{A}_\infty.$$

Proof. ($L_d \leq L$) was already shown for arbitrary length spaces, see the proof of 1.1.14.
($L \leq L_d$): Let $\gamma : [0,1] \to M$ be piecewise smooth and let $t \in [0,1]$ be such that $\gamma'(t)$ exists (i.e. t is not a break point of γ). The exponential map $\exp_{\gamma(t)}$ is a diffeomorphism on a sufficiently small neighbourhood U of $\gamma(t)$. If $\delta > 0$ is such that $\gamma([t-\delta, t+\delta]) \subseteq U$, we obtain

$$\frac{1}{\delta} d(\gamma(t), \gamma(t+\delta)) = \frac{1}{\delta} \left\| \exp_{\gamma(t)}^{-1}(\gamma(t+\delta)) \right\|_g = \left\| \frac{1}{\delta} \exp_{\gamma(t)}^{-1}(\gamma(t+\delta)) \right\|_g, \tag{2.1}$$

where the first equality is due to the fact that in a sufficiently small neighbourhood U the Riemannian distance between points $p, q \in U$ is equal to the radius function $\left\|\exp_p^{-1}(q)\right\|_g$, see e.g. [12], 2.3.3. p.51. Equation (2.1) implies for the metric derivative

$$\begin{aligned} \lim_{\delta \downarrow 0} \frac{d(\gamma(t+\delta), \gamma(t))}{\delta} &= \left\| \frac{d}{d\delta}\Big|_{\delta=0} \exp_{\gamma(t)}^{-1}(\gamma(t+\delta)) \right\|_g \\ &= \left\| (T_0 \exp_{\gamma(t)})^{-1}(\gamma'(t)) \right\|_g = \left\|\gamma'(t)\right\|_g, \end{aligned}$$

since $T_0 \exp_p = id_{T_pM}$, where $T_0 \exp_p : T_0(T_pM) \cong T_pM \to T_pM$. By $L_d \leq L$, we obtain

$$\frac{1}{\delta} d(\gamma(t+\delta), \gamma(t)) \leq \frac{1}{\delta} L_d(\gamma_{|[t,t+\delta]}) \leq \frac{1}{\delta} \int_t^{t+\delta} \left\|\gamma'(s)\right\|_g \, ds. \tag{2.2}$$

Letting $\delta \to 0$, both sides in the above inequality converge to $\left\|\gamma'(t)\right\|_g$. A similar calculation holds for $t-\delta$ instead of $t+\delta$. Noting that $\frac{1}{\delta} L_d(\gamma_{|[t,t+\delta]}) = \frac{1}{\delta}\left(L_d(\gamma_{|[0,t+\delta]}) - L_d(\gamma_{|[0,t]})\right)$, we obtain by letting $\delta \to 0$

$$\frac{d}{dt} L_d(\gamma_{|[0,t]}) = \left\|\gamma'(t)\right\|_g,$$

almost everywhere (i.e. everwhere except at finitely many break points). By the fundamental theorem of calculus we obtain

$$L_d(\gamma) = L_d(\gamma) - L_d(\gamma_{|[0,0]}) = \int_0^1 \frac{d}{dt} L_d(\gamma_{|[0,t]}) \, dt = \int_0^1 \left\|\gamma'(t)\right\|_g \, dt = L(\gamma).$$

□

The fact that the intrinsic metric d w.r.t. the arclength and the class $\mathcal{A}_\infty$ equals the intrinsic metric w.r.t. $(M, \mathcal{A}_\infty, L_d)$ would follow from the above theorem, but in fact as already stated follows also from 1.1.14.

We have now established $L_d = L$ for $\mathcal{A}_\infty$, but the arclength makes sense for a larger class of paths and if L_d is to serve as an extension of L, the lengths should be equal on any class L is defined on. A natural requirement to define the arclength would be differentiability almost everywhere and integrability of the derivative. In short we would like to generalize Theorem 2.1.5 to the class of absolutely continuous paths. As mentioned in section 1.4. in metric spaces there is no unique notion of absolute continuity. The above requirements would suggest measure absolutely continuous, we will however introduce another concept of absolute continuity on manifolds and prove that all notions of absolute

continuity introduced so far coincide on Riemannian manifolds, as is the case in $\mathbb{R}^n$.
Note that differentiability almost everywhere or bounded variation is not a sufficient requirement as can be seen by the graph of the Cantor function $\gamma := (id_{\mathbb{R}}, \Gamma)$, since

$$L(\gamma) = \int_0^1 |\gamma'(t)| \, dt = 1 < \sqrt{2} = d((0,0),(1,1)) \leq L_d(\gamma),$$

where $\Gamma : [0,1] \to [0,1]$ denotes the Cantor function, for more details we suggest [9], p. 38-39.
Recall that a path $\gamma : I \to \mathbb{R}^n$ is called locally absolutely continuous, if the restriction to every compact subinterval of I is absolutely continuous. Note that for such paths $\|\gamma'\|_{\mathbb{R}^n}$ is locally integrable.

We define an absolutely continuous path on a manifold as one that is locally absolutely continuous into $\mathbb{R}^n$ when composed with charts.

2.1.6 Definition. Let M be a connected smooth manifold. A path $\gamma : [a,b] \to M$ is called absolutely continuous, if for every chart (ψ, U) of M, the map

$$\psi \circ \gamma : \gamma^{-1}(\gamma([a,b]) \cap U) \to \psi(U) \subseteq \mathbb{R}^n$$

is locally absolutely continuous. We denote the class of such paths by $\mathcal{A}_{ac}$.

By [2], Proposition 3.4 p. 4, for $M = \mathbb{R}^n$, this notion coincides with the usual definition of absolute continuity in $\mathbb{R}^n$.
In order to make sense of the arclength of such an absolutely continuous path, we need the following

2.1.7 Proposition. Let M be a connected Riemannian manifold with continuous Riemannian metric g and let $\gamma : [0,1] \to M$ be absolutely continuous, then the derivative of γ exists at almost every $t \in [0,1]$ and $\|\gamma'\|_g$ is integrable.

Proof. Let $(\psi = (x^1, \ldots, x^n, U)$ be any chart of M, then $x^i \circ \gamma : \gamma^{-1}(\gamma([0,1]) \cap U) \to \mathbb{R}$ is locally absolutely continuous. Thus $(x^i \circ \gamma)'$ exists almost everywhere and is locally integrable. This now yields

$$\|\gamma'\|_g = \sqrt{g(\gamma', \gamma')} = \left| \sum_{i,j} g_{ij} \frac{d(x^i \circ \gamma)}{dt} \frac{d(x^j \circ \gamma)}{dt} \right|^{\frac{1}{2}}, \tag{2.3}$$

almost everywhere. This implies differentiability of γ almost everywhere and integrability of $\|\gamma'\|_g$ on $[0,1]$ can be seen as follows. Cover the compact set $\gamma([0,1])$ by finitely many chart neighbourhoods U_i, $i = 1, \ldots, m$. For every chart (ψ_i, U_i) the right hand side of (2.3) is integrable on $I_i := \gamma^{-1}(U_i \cap \gamma([0,1]))$, since g is bounded on the compact set $\gamma([0,1])$. So we obtain

$$\int_0^1 \|\gamma'(t)\|_g \, dt = \int_{\bigcup_{i=1}^m I_i} \|\gamma'(t)\|_g \, dt \leq \sum_{i=1}^m \underbrace{\int_{I_i} \|\gamma'(t)\|_g \, dt}_{<\infty} < \infty.$$

$\square$

This Proposition allows us to define the arclength for $\mathcal{A}_{ac}$ as in the smooth case via $L(\gamma) := \int \|\gamma'(t)\|_g \, dt$.
In the case of a smooth Riemannian metric the manifold is a metric space by 2.1.4, therefore we may ask if the metric derivative exists as well for absolutely continuous paths and what the relation to metric/measure absolutely continuous paths is.

2.1.8 Corollary. Let M be a connected Riemannian manifold and $\gamma \in \mathcal{A}_{ac}$, then

$$|\dot{\gamma}|(t) = \|\gamma'(t)\|_g \,,$$

almost everywhere.

Proof. This follows exactly as in the proof of 2.1.5. □

2.2 Continuous Riemannian metrics

So far we have studied the length structures $(M, \mathcal{A}_\infty, L)$, $(M, \mathcal{A}_\infty, L_d)$. The lengths are equal on $\mathcal{A}_\infty$ and they induce the same intrinsic metric d on M. Next we compare them to the length structures $(M, \mathcal{A}_{ac}, L)$ and $(M, \mathcal{A}_{ac}, L_d)$. In this section we deal with Riemannian manifolds with continuous Riemannian metrics and we will compare length structures on such manifolds. First we have to establish that such manifolds are indeed length spaces.

2.2.1 Theorem. Let M be a connected Riemannian manifold with continuous Riemannian metric g , then

(i) $(M, \mathcal{A}_{ac}, L)$ is a length structure and

(ii) the intrinsic metric d w.r.t. this structure induces the topology of M, i.e. (M, d) is a length space.

Proof. That $\mathcal{A}_{ac}$ is an admissible class is clear and that $L(\gamma) = \int_I \|\gamma'\|_g$ is a length is shown as in the case of a smooth Riemannian metric, cf. 2.1.2.
We have to show, that the length structure is compatible with the topology of M. Let $p \in M$ and $(\psi = (x^1, \ldots, x^n, U))$ be a chart around p with $\psi(p) = 0$, where n is the dimesion of M. Since M is locally compact, p has a compact neighbourhood which w.l.o.g. can be chosen as $K := \psi^{-1}(\overline{B_r(0)}) \subseteq U$, for some $r > 0$. We will show that paths leaving $\mathrm{int}(K)$ have length greater than c, for some $c > 0$. On U the Euclidean metric tensor is given by

$$e_U := \delta_{ij} \, dx^i \otimes dx^j.$$

Since e_u and g are non-degenerate they induce isomorphisms $TM \to T^*M, x \mapsto g(x, .)$ and $x \mapsto e_U(x, .)$ respectively. We denote the eigenvalues of the symmetric $(0, 2)$-tensor $e_U^{-1} \circ g$ at a point p by η_i, $i = 1, \ldots, n$, and the corresponding eigenvectors by v_i. This means

$$g_p(v_i, .) = \eta_i \, e_{U\,p}(v_i, .), \quad \forall i = 1, \ldots, n.$$

W.l.o.g. we order the eigenvalues such that $\eta_1 \leq \ldots \leq \eta_n$. Let us define functions, which assign to a ponit $q \in U$ the value $\lambda(q) := \sqrt{\eta_1}$ and $\mu(q) := \sqrt{\eta_n}$. Here η_1 and η_n denote the smallest resp. the largest eigenvalue of the linear map

$(e_U^{-1} \circ g)_q : T_qM \to T_qM$, $v \mapsto (e_U^{-1} \circ g)_q(v, .)$. We can choose eigenvectors v_i of η_i, $i = 1, \dots, n$ such that v_i form a basis of T_qM. Then for any $v = \sum_{i=1}^n s_i v_i$, $s_i \in \mathbb{R}$ we get

$$\begin{aligned} g_p(v, .) &= \sum_{i=1}^n s_i g_p(v_i, .) = \sum_{i=1}^n s_i \eta_i e_{U\,p}(v_i, .) \\ &\leq \eta_n \sum_{i=1}^n s_i e_{U\,p}(v_i, .) = \eta_n e_{U\,p}(v, .), \end{aligned}$$

similarly we can estimate from below using η_1. We thus obtain for $q \in U$ and $v \in T_qM$

$$\lambda(q)\|v\|_{e_U} \leq \|v\|_g \leq \mu(q)\|v\|_{e_U}. \tag{2.4}$$

Since g is continuous, so are $\lambda, \mu : U \to \mathbb{R}^+$, since here the eigenvalues of $((e_U)^{-1} \circ g)_p$ depend continuously on the coefficients of the matrix representation, which in turn depend continuously on p. Thus they attain their maximum and minimum values on the compact set K. We set $\lambda_0 := \min_{q \in K} \lambda(q)$ and $\mu_0 := \max_{q \in K} \mu(q)$, since $e_U^{-1} \circ g$ is positive definite (since g and e_U are) we have $\lambda_0 > 0$. From (2.4) we thus obtain

$$\lambda_0\|v\|_{e_U} \leq \|v\|_g \leq \mu_0\|v\|_{e_U}, \tag{2.5}$$

for $v \in T_qM$ and all $q \in K$.
Now let $y \in M \setminus \mathrm{int}(K)$ be joined to p by an absolutely continuous path $\gamma : [0, 1] \to M$. Let $t_0 \in [0, 1]$ be such that $\gamma(t_0) \in \partial K \cap \gamma([0, 1])$, then it holds that

$$\begin{aligned} 0 < r\lambda_0 = \lambda_0\|(\psi \circ \gamma)(t_0)\|_{e(\mathbb{R}^n)} &= \lambda_0\|\psi(\gamma(t_0)) - \underbrace{\psi(\gamma(0))}_{=0}\|_{e(\mathbb{R}^n)} \\ \leq \int_0^{t_0} \lambda_0\|\gamma'(t)\|_{e_U}\, dt &\overset{(2.5)}{\leq} \int_0^{t_0} \|\gamma'(t)\|_g\, dt = L(\gamma_{|[0,t_0]}) \end{aligned}$$

The first inequality is due to the fact that the Euclidean distance of two points is less or equal to the (arc)length of paths connecting them. We have thus proved (i).
From the above calculation we see that that Euclidean distance $\|\psi(p) - \psi(q)\|_{e(\mathbb{R}^n)}$ of two points in a chart neighbourhood U is less than a multiple of d (where d denotes the metric induced by $(M, \mathcal{A}_{ac}, L)$), i.e. we have $d_{e_U}(p, q) \leq \frac{1}{\lambda_0} d(p, q)$. A similar calculation shows $\mu_0 d_{e_U}(p, q) \geq d(p, q)$ and thus these metrics are equivalent on U. Since the manifold topology is induced by d_{e_U}, we obtain (ii). □

In order to relate their lengths we will approximate absolutely continuous curves by piecewise smooth curves. This is done in the following topology.

2.2.2 Definition. Let M be a connected Riemannian manifold with continuous Riemannian metric g and the intrinsic distance d w.r.t $(M, \mathcal{A}_{ac}, L)$. The variational metric on the class $\mathcal{A}_{ac}(I)$ is defined by

$$D_{ac}(\gamma, \sigma) := \sup_{t \in I} d(\gamma(t), \sigma(t)) + \int_I \left| \|\gamma'(s)\|_g - \|\sigma'(s)\|_g \right| ds,$$

for $\gamma, \sigma : I \to M$ absolutely continuous.

It is easy to see that D_{ac} is a metric on $\mathcal{A}_{ac}$, one can show that, if M is complete, so is $(\mathcal{A}_{ac}(I), D_{ac})$, cf. [2], Proposition 3.19, p.10.
We now prove that the piecewise smooth paths are dense in $\mathcal{A}_{ac}$ with respect to this topology. For a given absolutely continuous path γ, a regularization argument will provide us with an approximating sequence of smooth paths. The endpoints of those paths will however differ from the ones of γ. Without the existence of convex neighbourhoods (due to the nature of g), it is difficult to find appropriate paths to connect the new endpoints with the old by sufficiently short paths. Note that for a smooth Riemannian metric this could be achieved via radial geodesics in convex neighbourhoods. From now on by $\mathcal{A}_{ac}$ we mean $\mathcal{A}_{ac}(I)$ for a given interval I.

2.2.3 Theorem. Let M be a connected Riemannian manifold with continuous Riemannian metric, then $\mathcal{A}_\infty$ is dense in $\mathcal{A}_{ac}$ in the topology induced by D_{ac}.

Proof. Let $\gamma : [0,1] \to M$ be an absolutely continuous path. We can cover the (compact) image $\gamma([0,1])$ by finitely many charts (ψ_i, U_i), $i = 1, \ldots, N$. The charts can be chosen in such a way that $\psi_i(U_i)$ is convex and that $\overline{U_i}$ is compact. Therefore $V := \bigcup_{i=1}^N \bar{U}_i$ is compact in M and by the proof of 2.2.1 the norm $\|.\|_g$ can be estimated by a multiple of the Euclidean norm $\|.\|_{e_V}$. We therefore w.l.o.g. assume them to be equal. Further we can subdivide $[0,1]$ into small enough subintervals $0 = t_0 < t_1 < \ldots < t_N = 1$ such that $\gamma([t_{i-1}, t_i]) \subseteq U_i$, for $i = 1, \ldots, N$. Thus we can calculate in a single chart domain henceforth denoted by U. We set $I_j := [t_{j-1}, t_j]$
Since γ is absolutely continuous, $\|\gamma'\|_g \in L^1([0,1])$ by 2.1.7, so the fundamental theorem of calculus is applicable and for $\eta > 0$ we can find $\delta \in (0, \frac{1}{2}|t_j - t_{j-1}|)$, such that the following hold simultaneously

$$\sup_{\substack{s,t \in I_j \\ |s-t| < 2\delta}} d(\gamma(s), \gamma(t)) < \eta, \tag{2.6}$$

$$\sup_{\substack{s,t \in I_j \\ |s-t| < 2\delta \\ s \le t}} \int_s^t \|\gamma'(r)\|_g \, dr < \eta, \tag{2.7}$$

$$\sup_{\substack{s,t \in I_j \\ |s-t| < 2\delta}} \|\psi(\gamma(s)) - \psi(\gamma(t))\|_{e(\mathbb{R}^n)} < \eta. \tag{2.8}$$

The first is due to uniform continuity if γ, the second is due to the fundamental theorem of calculus and the third is due to uniform continuity of $\psi \circ \gamma$.
Our plan is to smooth the path $\psi \circ \gamma : [0,1] \to \mathbb{R}^n$ by convolution with a mollifier, in order to obtain a smooth approximating path. Let ρ be a mollifier and denote by $(\psi \circ \gamma) * \rho_\varepsilon$ the path obtained by componentwise convolution of $\psi \circ \gamma$ with ρ_ε. This means for $\psi = (x^1, \ldots, x^n)$ we have

$$(\psi \circ \gamma) * \rho_\varepsilon = ((x^i \circ \gamma) * \rho_\varepsilon)_{i=1}^n$$

For sufficiently small ε the path $(\psi \circ \gamma) * \rho_\varepsilon$ will lie in $\psi(U)$ and therefore we can pull it back to a smooth path $\gamma_\varepsilon := \psi^{-1}\big((\psi \circ \gamma) * \rho_\varepsilon\big)$. For $\varepsilon > 0$ small enough we obtain on the interval I_j, the following approximations

$$\sup_{t \in I_j} \|\psi(\gamma(t)) - \psi(\gamma_\varepsilon(t))\|_{e(\mathbb{R}^n)} < \eta, \tag{2.9}$$

$$\|(\psi \circ \gamma_\varepsilon)' - (\psi \circ \gamma)'\|_{L^1(I_j)} < \eta. \tag{2.10}$$

The equivalence of e_U and g from 2.2.1 yields the following approximations on M

$$\sup_{t \in I_j} d(\gamma(t), \gamma_\varepsilon(t)) < \eta, \tag{2.11}$$

$$\int_{I_j} \Big| \|\gamma'\|_g - \|\gamma_\varepsilon'\|_g \Big| < \eta. \tag{2.12}$$

We have found an approximation of γ, however since the endpoints may differ, we need to connect them by sufficiently short smooth paths joining $\gamma_\varepsilon(t_i)$ to $\gamma(t_i)$ for all i. If g were smooth we could cover $\gamma([0,1])$ by geodesically convex chart neighbourhoods and use radial geodesics to connect the endpoints. Since g is merely continuous we cannot proceed like this. However we assumed $\psi(U)$ to be convex and therefore we can join $\psi(\gamma(t_{j-1})) =: \psi(p)$ to $\psi(\gamma_\varepsilon(t_{j-1}+\delta)) =: \psi(q)$ by a straight line $\hat{\nu}_{j-1}$ in $\psi(U)$. It is given by

$$\hat{\nu}_{j-1} : [t_{j-1}, t_{j-1}+\delta] \to \psi(U) \subseteq \mathbb{R}^n, \quad \hat{\nu}_{j-1}(t) = \psi(p) + \frac{t - t_{j-1}}{\delta}(\psi(q) - \psi(p)).$$

Similarly we can connect $\psi(\gamma_\varepsilon(t_j - \delta))$ to $\psi(\gamma(t_j))$ by the straight line $\hat{\mu}_j$. Pulling back these lines to M via ψ to obtain smooth paths $\nu_{j-1} := \psi^{-1} \circ \hat{\nu}_{j-1}$ and $\mu_j := \psi^{-1} \circ \hat{\mu}_j$, connecting the respective points in M. We need to be able to control the length of these paths in order to obtain an approximation of γ by concatenating them with γ_ε. We compute again using $e_U = g$

$$\left\|\nu_{j-1}'(t)\right\|_g = \|\hat{\nu}_{j-1}'(t)\|_{e(\mathbb{R}^n)} = \|\frac{1}{\delta}(\psi(q) - \psi(p))\|_{e(\mathbb{R}^n)}$$

$$\leq \frac{1}{\delta}\big(\underbrace{\|\psi(q) - \psi(\gamma(t_{j-1}+\delta))\|_{e(\mathbb{R}^n)}}_{\substack{<\eta \\ \text{for } \varepsilon \text{ sufficiently small by (2.9)}}} + \underbrace{\|\psi(\gamma(t_{j-1}+\delta)) - \psi(p)\|_{e(\mathbb{R}^n)}}_{\substack{<\eta \\ \text{for } \delta \text{ sufficiently small by(2.8)}}} \big) < \frac{2\eta}{\delta}.$$

We now get

$$L(\nu_{j-1}) = \int_{t_{j-1}}^{t_{j-1}+\delta} \left\|\nu_{j-1}'(t)\right\|_g \, dt \leq \delta \frac{2\eta}{\delta} = 2\eta, \tag{2.13}$$

and analogously $L(\mu_j) \leq 2\eta$. We see that points on ν_{j-1} resp. μ_j are not too far away from the corresponding points on γ. By (2.6) and (2.13), we obtain for $t \in I_j$

$$d(\gamma(t), \nu_{j-1}(t)) \leq d(\gamma(t), \gamma(t_{j-1})) + d(\underbrace{\gamma(t_{j-1})}_{=\nu_{j-1}(t_{j-1})}, \nu_{j-1}(t)) < \eta + 2\eta = 3\eta, \tag{2.14}$$

and similarly for μ_j. We also need to control the second term in the definition of D_{ac}, by (2.7) and (2.13) we get

$$\begin{aligned} &\int_{t_{j-1}}^{t_{j-1}+\delta} \Big| \|\gamma'(t)\|_g - \left\|\nu_{j-1}'(t)\right\|_g \Big| \, dt \\ \leq &\int_{t_{j-1}}^{t_{j-1}+\delta} \|\gamma'(t)\|_g \, dt + \int_{t_{j-1}}^{t_{j-1}+\delta} \left\|\nu_{j-1}'(t)\right\|_g \, dt < \eta + 2\eta = 3\eta. \end{aligned} \tag{2.15}$$

This approximation procedure can be applied to all I_j and ε and δ can be chosen small enough such that the above approximations hold on all (finitely many) subintervals I_j simultaneously. We can finally define our (globally) approximating, piecewise smooth path λ_η by

$$\lambda_\eta(t) := \begin{cases} \nu_{j-1}(t), & t \in [t_{j-1}, t_{j-1} + \delta] \\ \gamma_\varepsilon(t), & t \in [t_{j-1} + \delta, t_j - \delta] \\ \mu_j(t), & t \in [t_j - \delta, t_j]. \end{cases}$$

These paths are indeed the desired approximating paths by the following calculations. We have

$$d(\gamma(t), \lambda_\eta(t)) < 3\eta, \qquad \text{by (2.11) and (2.14),} \tag{2.16}$$

and

$$\begin{aligned}
\int_0^1 \left| \|\gamma'(t)\|_g - \left\|\lambda_\eta'(t)\right\|_g \right| dt &= \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \left| \|\gamma'(t)\|_g - \left\|\lambda_\eta'(t)\right\|_g \right| dt \\
= \sum_{j=1}^N \Big(\int_{t_{j-1}}^{t_{j-1}+\delta} \left| \|\gamma'(t)\|_g - \left\|\lambda_\eta'(t)\right\|_g \right| dt &+ \int_{t_{j-1}+\delta}^{t_j-\delta} \left| \|\gamma'(t)\|_g - \left\|\lambda_\eta'(t)\right\|_g \right| dt \\
&+ \int_{t_j-\delta}^{t_j} \left| \|\gamma'(t)\|_g - \left\|\lambda_\eta'(t)\right\|_g \right| dt \Big) \\
= \sum_{j=1}^N \Big(\underbrace{\int_{t_{j-1}}^{t_{j-1}+\delta} \left| \|\gamma'(t)\|_g - \left\|\nu_{j-1}'(t)\right\|_g \right| dt}_{<3\eta \text{ by (2.15)}} &+ \underbrace{\int_{t_{j-1}+\delta}^{t_j-\delta} \left| \|\gamma'(t)\|_g - \|\gamma_\varepsilon'(t)\|_g \right| dt}_{<\eta \text{ by (2.12)}} \\
&+ \underbrace{\int_{t_j-\delta}^{t_j} \left| \|\gamma'(t)\|_g - \left\|\mu_j'(t)\right\|_g \right| dt}_{<3\eta \text{ by (2.15)}} \Big) < 7N\eta.
\end{aligned}$$

In summary

$$D_{ac}(\gamma, \lambda_\eta) < (3 + 7N)\eta,$$

and thus the theorem is proved. □

This has an important consequence

2.2.4 Corollary. Let M be as in the above theorem, the length function $L : \mathcal{A}_{ac} \to \mathbb{R}$ is Lipschitz continuous w.r.t. D_{ac} and the intrinsic metric d w.r.t. $(M, \mathcal{A}_\infty, L)$ coincides with the intrinsic metric d_{ac} w.r.t. $(M, \mathcal{A}_{ac}, L)$.

Proof. Take $\gamma, \sigma \in \mathcal{A}_{ac}$, then

$$\begin{aligned}
|L(\gamma) - L(\sigma)| &= \left| \int_I \|\gamma'(s)\|_g - \|\sigma'(s)\|_g \, ds \right| \\
&\leq \int_I \left| \|\gamma'(s)\|_g - \|\sigma'(s)\|_g \right| ds \leq D_{ac}(\gamma, \sigma),
\end{aligned} \tag{2.17}$$

which proves the first part of the statement.
$d_{ac} \leq d$ is clear by definition, since $\mathcal{A}_\infty \subseteq \mathcal{A}_{ac}$. On the other hand $d \leq d_{ac}$ follows by the above theorem. Indeed by (2.17), for every $\gamma \in \mathcal{A}_{ac}$ connecting two given points and every $\varepsilon > 0$ we can find a path $\tilde{\gamma} \in \mathcal{A}_\infty$ connecting the same points with $L(\tilde{\gamma}) \leq L(\gamma) + \varepsilon$. This implies $L(\gamma) \geq \inf_{\tilde{\gamma}\in\mathcal{A}_\infty} L(\tilde{\gamma})$, where the infimum is taken over all paths in $\mathcal{A}_\infty$ connecting the given points, and so $d_{ac} \geq d$. □

As another consequence we can now prove the equivalence of all the definitions of absolute continuity on Riemannian manifolds with continuous Riemannian metrics.

2.2.5 Proposition. Let M be a connected Riemannian manifold with continuous Riemannian metric. As a metric space with the intrinsic metric $d = d_{ac}$, the classes of absolutely continuous paths, metric absolutely continuous paths and measure absolutely continuous paths coincide.

Proof. That mac implies metric absolutely continuous was proved for metric spaces in 1.4.2.
Next we show that metric absolutely continuous paths are absolutely continuous (in the manifold sense). Let γ be metric absolutely continuous and let (ψ, U) be a chart on M. ψ is Lipschitz continuous on any set $\gamma([a,b]) \subseteq U$, since it is a diffeomorphism on U, which means

$$\|\psi(\gamma(b)) - \psi(\gamma(a))\|_{e(\mathbb{R}^n)} \leq C d(\gamma(b), \gamma(a)),$$

for some $C > 0$. Since γ is metric absolutely continuous the composition $\psi \circ \gamma$ is locally absolutely continuous into $\mathbb{R}^n$.
Finally let $\gamma \in \mathcal{A}_{ac}(I)$, we show that γ is mac. Set $l := \|\gamma'\|_g \in L^1(I)$, by 2.1.7. Since $d = d_{ac}$, by the above Corollary, for any $a, b \in I$, $a < b$ it holds that

$$d(\gamma(a), \gamma(b)) = d_{ac}(\gamma(a), \gamma(b)) \leq L(\gamma_{|[a,b]}) = \int_a^b l(t)\, dt.$$

□

Our last goal in this section will be to generalize theorem 2.1.5, to manifolds with continuous Riemannian metrics and to show the equivalence of the length structures $(M, \mathcal{A}_{ac}, L)$ and $(M, \mathcal{A}_{ac}, L_d)$ in this case. Several steps are needed. First we will completely clear the case for smooth Riemannian metrics, by establishing $L = L_d$ for absolutely continuous paths. As in the proof of 2.1.5 we would like to apply the fundamental theorem of calculus to L_d. We first need

2.2.6 Lemma. Let M be a Riemannian manifold with continuous Riemannian metric g and let $\gamma : [0,1] \to M$ be absolutely continuous, then $t \mapsto L_d(\gamma_{|[0,t]})$, $[0,1] \to [0,\infty)$ is absolutely continuous.

Proof. Since $\|\gamma'\|_g$ is integrable the function $t \mapsto L(\gamma_{|[0,t]})$ is absolutely continuous. Therefore since $0 \leq L_d(\gamma_{|[0,t]}) \leq L(\gamma_{|[0,t]})$, which holds by $d = d_{ac}$, 1.1.14 and 2.2.1, also $t \mapsto L_d(\gamma_{|[0,t]})$ is absolutely continuous. □

We can now clear the smooth case

2.2.7 Proposition. Let M be a connected Riemannian manifold with smooth Riemannian metric, then $L(\gamma) = L_d(\gamma)$ for all $\gamma \in \mathcal{A}_{ac}$.

Proof. We only need to refine the proof of 2.1.5 to the case of $\mathcal{A}_{ac}$.
($L_d \leq L$) holds by $d = d_{ac}$, 2.2.1 and 1.1.14.
($L \leq L_d$): Let $t \in (0,1)$ such that $\gamma'(t)$ exists, which is the case for almost every $t \in (0,1)$ by 2.1.7. As in the proof of 2.1.5, we obtain

$$\frac{d}{dt} L_d(\gamma_{|[0,t]}) = \|\gamma'(t)\|_g .$$

By 2.2.6, $t \mapsto L_d(\gamma_{|[0,t]})$ is absolutely continuous and we can proceed as in the proof of 2.1.5 to obtain

$$L_d(\gamma) = \int_0^1 \frac{d}{dt} L_d(\gamma_{|[0,1]})\, dt = \int_0^1 \|\gamma'(t)\|_g \, dt = L(\gamma).$$

□

2.2.8 Remark. Since the piecewise smooth paths define the same length structure as the absolutely continuous paths so do all intermediate classes $\mathcal{A}$ with $\mathcal{A}_\infty \subseteq \mathcal{A} \subseteq \mathcal{A}_{ac}$.

In order to extend the above proposition to manifolds with continuous Riemannian metrics, we need a way to circumvent the use of the exponential map. Our strategy will be to approximate a given Riemannian metric by smooth ones. This will help us to extend 2.1.8 to continuous Riemannian metrics in order to apply 1.4.5.

2.2.9 Theorem. Let M be a connected Riemannian manifold with continuous Riemannian metric g. There exists a sequence $(g_n)_{n\in\mathbb{N}}$ of smooth Riemannian metrics on M, such that

(i) $(g_n)_n$ converges uniformly on M to g and

(ii) the induced Riemannian distances converge uniformly to the distance induced by g.

Proof. Let $p \in M$ and let (K_p, ψ) be a chart such that K_p is a compact neighbourhood of p. By convolution of the chart representation $T^0_2\psi \circ g \circ \psi^{-1}$ of g, with a mollifier we obtain a sequence of smooth maps $f_n : \psi(K_p) \to \psi(K_p) \times (\mathbb{R}^n)^0_2$. These maps can be pulled back to M to obtain maps

$$h^p_n := (T^0_2\psi)^{-1} \circ f_n \circ \psi : K_p \to T^0_2 M.$$

The h^p_n (resp. the f_n) can be constructed in such a way that $h^p_n(q)$ is positive definite (since this is an open condition) and symmetric for all $q \in K_p$ and large enough n.
As in the proof of 2.2.1, we can consider the eigenvalues of $(g^{-1} \circ h^p_n)(q)$, which all converge to 1 by construction of h^p_n. By passing to a subsequence of (h^p_n) we can therefore estimate as in 2.2.1 on the compact set K_p to obtain

$$\frac{n-1}{n} \|v\|_g \leq \|v\|_{h^p_n} \leq \frac{n+1}{n} \|v\|_g , \tag{2.18}$$

for $v \in T_qM$ and $q \in K_p$. By [4], theorem 2.3.10, p.29, as a smooth second countable Hausdorff manifold M possesses a smooth partition of unity $\{\alpha_p\}_{p\in M}$ subordinate to the cover $\{\mathrm{int}(K_p)\}$, where K_p are compact chart neighbourhoods as above. It remains to patch the locally defined maps h_n^p together via α_p to obtain

$$g_n := \sum_{p\in M} \alpha_p h_n^p.$$

This map satisfies the estimate (2.18) globally by construction, i.e.

$$\frac{n-1}{n}\left\|v\right\|_g \le \left\|v\right\|_{g_n} \le \frac{n+1}{n}\left\|v\right\|_g, \tag{2.19}$$

for $v \in T_pM$ and $p \in M$ and therefore $g_n \to g$ uniformly on M.
For the second claim let $p, q \in M$. Since d is the intrinsic metric w.r.t. $(M, \mathcal{A}_{ac}, L)$, for every $\varepsilon > 0$ there exists a path γ_ε from p to q with $L(\gamma_\varepsilon) < d(p,q)+\varepsilon$. Using (2.19) we obtain

$$\begin{aligned} d(p,q)+\varepsilon > L(\gamma_\varepsilon) &= \int \left\|\gamma_\varepsilon'(t)\right\|_g\, dt \ge \frac{n}{n+1}\int \left\|\gamma_\varepsilon'(t)\right\|_{g_n} dt \\ &= \frac{n}{n+1}L_n(\gamma_\varepsilon) \ge \frac{n}{n+1}d_n(p,q), \end{aligned}$$

where L_n resp. d_n are the arclength and distance induced by g_n. Therefore

$$\frac{n+1}{n}d(p,q) \ge d_n(p,q), \quad \forall p,q \in M.$$

A similar calculation using the other inequality of (2.19) and $L_n(\gamma_\varepsilon^n) < d_n(p,q)+\varepsilon$ for appropriate paths γ_ε^n, shows

$$\frac{n-1}{n}d(p,q) \le d_n(p,q), \quad \forall p,q \in M.$$

Thus $d_n \to d$ uniformly on $M \times M$. □

Our last preparatory step will be to show the equality of the metric derivative $|\dot\gamma|$ and the norm of the „analytic“ derivative $\left\|\gamma'\right\|_g$.

2.2.10 Proposition. Let M be a Riemannian manifold with continuous Riemannian metric g and let $\gamma : [0,1] \to M$ be absolutely continuous, then

$$\left\|\gamma'(t)\right\|_g = |\dot\gamma|(t), \quad \text{for almost every } t \in [0,1].$$

Proof. We make use of the above theorem to approximate g by smooth metrics g_n and d by the corresponding d_n. By the proof of 2.1.8, for g_n resp. d_n we know

$$|\dot\gamma|_n(t) := \lim_{\delta\to 0}\frac{d_n(\gamma(t+\delta),\gamma(t))}{|\delta|} = \left\|\gamma'(t)\right\|_{g_n}, \tag{2.20}$$

almost everywhere. By theorem 2.2.9, we are allowed to interchange limits and can calculate as follows

$$|\dot\gamma|(t) = \lim_{\delta\to 0}\frac{d(\gamma(t+\delta),\gamma(t))}{|\delta|} = \lim_{\delta\to 0}\lim_{n\to\infty}\frac{d_n(\gamma(t+\delta),\gamma(t))}{|\delta|}$$

$$= \lim_{n\to\infty}\lim_{\delta\to 0}\frac{d_n(\gamma(t+\delta),\gamma(t))}{|\delta|} \overset{(2.20)}{=} \lim_{n\to\infty}|\dot\gamma|_n(t) = \lim_{n\to\infty}\left\|\gamma'(t)\right\|_{g_n} = \left\|\gamma'(t)\right\|_g,$$

almost everywhere. □

In the following theorem we denote the metric arclength of an absolutely continuous path γ by $\tilde{L}(\gamma) := \int |\dot{\gamma}|(t)\, dt$.

2.2.11 Theorem. Let M be a Riemannian manifold with continuous Riemannian metric g, then

$$L(\gamma) = L_d(\gamma) = \tilde{L}(\gamma), \quad \forall \gamma \in \mathcal{A}_{ac}.$$

Proof. By the above proposition $L(\gamma) = \int \|\gamma'(t)\|_g \, dt = \int |\dot{\gamma}|(t)\, dt = \tilde{L}(\gamma)$. Further by 1.4.5 and 2.2.5 we have $\tilde{L}(\gamma) = L_d(\gamma)$ for all $\gamma \in \mathcal{A}_{ac}$. $\square$

Chapter 3

Geodesics in Riemannian Manifolds of low Regularity

In this chapter we deal with manifolds equipped with Riemannian metrics of low regularity and their influence on the regularity of geodesics. By low regularity we mean metrics of differentiability below $C^{1,1}$. We will however, restrict ourselves to the cases for metrics of regularity ranging from continuous to $C^{1,1}$ and provide no details on bounded or distributional metrics.

3.1 Geodesics for $\mathcal{C}^1$ metrics

By the previous chapters we know that manifolds with continuous Riemannian metrics can be equipped with a metric structure and therefore a sense of shortest paths. We will compare the metric space geodesics (short m-geodesics) with solutions of the geodesic equation.
Various objects in Riemannian Geometry, involve metric, derivatives of the metric and the connection of the metric. If continuous differentiability of the metric is no longer guaranteed, one needs other concepts to make sense of such objects. Already if the metric is no longer smooth its derivative is not in the same differentiability class. Take for example the so-called musical isomorphism between vector fields and one-forms, which takes a vector field X to the one-form ω, with $\omega(Y) := g(X,Y)$, for a vector field Y. If g is not smooth there is no guarantee that ω is smooth. Let us investigate the Christoffel symbols of a metric as another example.

3.1.1 Example. Let M be a smooth manifold, with smooth Riemannian metric g. The Christoffel symbols relative to a chart $(\psi = (x^1, \dots, x^n), U)$, are given as the smooth functions defined via

$$\Gamma^k_{ij}\,\partial_k = \nabla_{\partial_i}\partial_j,$$

where $\partial_i|_p := (T_p\psi)^{-1}(e_i)$ and e_i denotes the i-th standard basis vector in $\mathbb{R}^n$. An application of the Koszul formula (see e.g. [10], theorem 3.11, p.61) yields a representation directly involving the metric

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right). \tag{3.1}$$

Here $g_{ij} =: g(\partial_i, \partial_j)$ denote the local coefficients of the metric and $g^{ij}(p)$ the entries of the inverse of the matrix $(g_{ij}(p))$. Since g is smooth so are all its coefficients and by the inversion formula for matrices also g^{ij}.
So if g is smooth, so is Γ^k_{ij}. If g is no longer smooth, but e.g. only C^1 already a naive approach (not worrying if the Koszul formula or similar results on ∇ are applicable) would only guarantee continuity of the Christoffel symbols. This already has consequences for the theory of geodesics since the geodesic equations

$$\frac{d^2(x^k \circ c)}{dt^2} + \Gamma^k_{ij} \circ c \, \frac{d(x^i \circ c)}{dt} \, \frac{d(x^j \circ c)}{dt} = 0, \quad 1 \leq k \leq n \tag{3.2}$$

for a curve $c : I \to M$, are in general no longer uniquely solvable, since the theorem of Picard-Lindelöf is no longer applicable. In order to apply this theorem the coefficients would have to be (locally) Lipschitz continuous. This suggests unique solvability at regularity $C^{1,1}$. We will see in chapter 4, that in fact this is the case and even the exponential map is still a Bi-Lipschitz Homeomorphism.

Let us now deal with the relation between m-geodesics (geodesics in the sense of 1.3.1) and R-geodesics (solutions to the geodesic equation, paths whose velocity vectorfield is parallel along itself). In the smooth case the notions coincide by the following

3.1.2 Theorem. Let M be a manifold with smooth Riemannian metric, then radial R-geodesics are locally shortest paths and shortest paths are R-geodesics (up to monotone reparameterizations).

Proof. See e.g. [12], Proposition 2.3.6, p. 53 and Corollary 2.3.11, p. 56. □

The proof of this theorem relies heavily on the exponential map, therefore we would expect the result not to hold for metrics of regularity below $C^{1,1}$. In the following we will discuss in detail a counterexample given in [11], for a Riemannian metric of Hölder regularity $C^{1,\alpha}$ for $0 < \alpha < 1$ (cf. 3.2.1 below), showing that the geodesic equations can not always be solved uniquely and that not every R-geodesic is an m-geodesic. We will closely follow [11] and handwritten seminar notes provided by Michael Kunzinger, [13].

Consider the following setup.

3.1.3 Problem. Let $M := \{(x,y) \in \mathbb{R}^2 \,|\, |x| < 1\}$ a strip in the plane and equip it with the Riemannian metric given by

$$g_{ij}(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - |x|^\lambda \end{pmatrix}, \qquad \text{where } 1 < |\lambda| < 2.$$

Is the geodesic equation uniquely solvable for all initial data, are R-geodesics locally shortest paths and is there always a (unique) R-geodesic connecting any two points? If g were smooth, the first two questions could be answered in the positive, the third also if M were complete and connected.

Let us observe a few immediate properties. g is differentiable with derivative of class $(\lambda-1)$ Hölder. Note that the geometry of the strip M differs in such a way from the Euclidean one, that the length $\|.\|_g$ of a vector (x,y) is less than than its Euclidean length if $x \neq 0$. However along the y-axis $\{(x,y)\,|\,x=0\}$, the metric is just the flat metric. Let us compute the Christoffel symbols using (3.1). Since g_{11} and $g_{12}=g_{21}$ have vanishing derivatives, we have

$$\Gamma^1_{11} = \frac{1}{2} g^{1m} \left(\underbrace{\frac{\partial g_{1m}}{\partial x^1}}_{=0} + \underbrace{\frac{\partial g_{1m}}{\partial x^1}}_{=0} - \underbrace{\frac{\partial g_{11}}{\partial x^m}}_{=0} \right) = 0,$$

and similar for Γ^2_{22} and $\Gamma^1_{12} = \Gamma^1_{21}$. In the remaining cases a short calculation shows

$$\Gamma^1_{22}(x,y) = \frac{\lambda}{2}|x|^{\lambda-1}\mathrm{sgn}(x)$$

and

$$\Gamma^2_{12}(x,y) = \Gamma^2_{21}(x,y) = -\frac{\lambda}{2}\frac{|x|^{\lambda-1}\mathrm{sgn}(x)}{1-|x|^\lambda}.$$

We therefore obtain the geodesic equations for a path $\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ as

$$x'' + \underbrace{\Gamma^1_{11}}_{=0} x'x' + 2\underbrace{\Gamma^1_{12}}_{=0} x'y' + \Gamma^1_{22} y'y' = 0$$

and

$$y'' + \underbrace{\Gamma^2_{11}}_{=0} x'x' + 2\Gamma^2_{21} x'y' + \underbrace{\Gamma^2_{22}}_{=0} y'y' = 0.$$

Inserting Γ^k_{ij} from above, the geodesics equations take the following form

$$x'' + \frac{\lambda}{2}|x|^{\lambda-1}\mathrm{sgn}(x)(y')^2 = 0 \tag{3.3}$$

and

$$y'' - \lambda \frac{|x|^{\lambda-1}\mathrm{sgn}(x)}{1-|x|^\lambda} x'y' = \big((1-|x|^\lambda)y'\big)' = 0. \tag{3.4}$$

Equation (3.4) is equivalent to

$$(1-|x|^\lambda)y' = c, \tag{3.5}$$

for a constant c. We now deal with different values of c case by case.
First let $c=0$, this implies $y'=0$, so y is constant and therefore by (3.3) we have $x''=0$, so x is linear in t. In summary $\gamma(t) = \begin{pmatrix} at+b \\ k \end{pmatrix}$ for some $a,b,k \in \mathbb{R}$. These geodesics are straight lines parallel to the y-axis. This seems little surprising, since g equals the Euclidean metric on the y-axis.
In the case $c \neq 0$, we have $y' \neq 0$ along γ and so γ can be parameterized by arclength. Therefore $\|\gamma'\|_g = 1$ and in particular $g_{ij}\left((x_i)', (x_j)'\right) = 1$, where $x_1 = x$ and $x_2 = y$. Using the definition of g_{ij} we obtain

$$(x')^2 + \underbrace{(1-|x|^\lambda)(y')^2}_{=\frac{c^2}{1-|x|^\lambda} \text{ by (3.5)}} = 1,$$

and thus

$$x' = \pm \left(1 - \frac{c^2}{1 - |x|^\lambda}\right)^{\frac{1}{2}}. \tag{3.6}$$

We want to solve the geodesic equation for any point x_0 and any initial velocity, but since g is smooth away from the y-axis we only consider initial values $x_0 = 0$, since anything else is covered by the smooth theory. For $c^2 > 1$ we would have $1 - \frac{c^2}{1-|x|^\lambda} < 0$ and hence there would be no (real) solutions to the equation (3.6). If $c^2 = 1$, since $1 - |x|^\lambda \leq 1$, the only possible (real) solution is $x(t) = 0$ for all t.
Let us deal with the case $c^2 < 1$. For small times $x(t)$ is small and therefore the rigth hand side of (3.6) is C^1. Thus initially there exists (at least locally) a solution x_c of (3.6) with $1 - |x_c|^\lambda \neq 0$. By (3.5) we then obtain

$$y' = \frac{c}{1 - |x_c|^\lambda},$$

thus

$$y(t) = \int_0^t \frac{c}{1 - |x_c(s)|^\lambda}\, ds. \tag{3.7}$$

In this case $\gamma'(0) = (x'(0), y'(0)) = (\pm\sqrt{1-c^2}, c)$ is the initial velocity, thus for every initial velocity in the unit sphere, we obtain a solution of the initial value problem, in short we have shown

3.1.4 Proposition. Let M be as in 3.1.3. For every initial data (x_0, v) with $x_0 \in M$ and $v \in \mathbb{R}^2 = T_{x_0}M$, there is a unique R-geodesic γ with $\gamma(0) = x_0$ and $\gamma'(0) = v$.

Studying the behaviour of geodesics staring at the origin, while letting the initial speed go to $(0, 1)$, we will notice some differences to the case of g being smooth. Since on the y-axis the metric g is the flat metric of $\mathbb{R}^n$, the geodesic with initial speed $(0, 1)$ starting at the origin $x_0 = 0$, is just a straight line. Let us introduce the parameter $\varepsilon \in [0, 1)$ by setting $c = \pm\sqrt{1-\varepsilon}$ and solving the geodesic equation for this c. By (3.5) we have $y'(0) = \frac{c}{1-|x(0)|^\lambda} = c$. So we obtain the initial speed $\gamma'(0) = (\pm\sqrt{\varepsilon}, \pm\sqrt{1-\varepsilon})$ by (3.6). Due to the fact that g depends only on $|x|$, it is sufficient to consider only the geodesics in the upper half plane, i.e. $y'(0) = \sqrt{1-\varepsilon} > 0$. Since $c > 0$ proceeding as above we can assume γ to be parameterized by arclength and obtain the geodesic equations

$$x'(t) = \pm \left(1 - \frac{c^2}{1 - |x(t)|^\lambda}\right)^{\frac{1}{2}} = \pm \left(1 - \frac{1-\varepsilon}{1 - |x(t)|^\lambda}\right)^{\frac{1}{2}} \tag{3.8}$$

$$y'(t) \overset{(3.7)}{=} \frac{c}{1 - |x(t)|^\lambda} = \frac{\sqrt{1-\varepsilon}}{1 - |x(t)|^\lambda}. \tag{3.9}$$

Observe that these equations make sense only if $\frac{1-\varepsilon}{1-|x(t)|^\lambda} < 1$, i.e. $|x(t)| < \sqrt[\lambda]{\varepsilon}$. We denote by $\gamma_{\pm\varepsilon}$ the unique geodesic solution to these equations with initial velocity $(\pm\sqrt{\varepsilon}, \sqrt{1-\varepsilon})$. A short calculation shows that γ_ε reaches the vertical line $\{(x, y) \,|\, x = \sqrt[\lambda]{\varepsilon}\}$ in finite time at

$$y_1 := \int_0^{\sqrt[\lambda]{\varepsilon}} \frac{dy}{dx} = \int_0^{\sqrt[\lambda]{\varepsilon}} \frac{1-\varepsilon}{1 - |x|^\lambda} \left(1 - \frac{1-\varepsilon}{1 - |x|^\lambda}\right)^{-\frac{1}{2}} dx < \infty. \tag{3.10}$$

Let s_0 be that time, i.e. $\gamma_\varepsilon(s_0) = (\sqrt[\lambda]{\varepsilon}, y_1)$.
The velocity of γ at s_0 is vertical, since by (3.8) and (3.9)

$$\gamma'_\varepsilon(s_0) = (0, \frac{1}{\sqrt{1-\varepsilon}}).$$

Note that, since g is independent of y, reflecting γ_ε at $\{(x, y) \,|\, y = y_1\}$ yields a geodesic from $(\sqrt[\lambda]{\varepsilon}, y_1)$ to $(0, 2y_1)$, matching the velocity of γ_ε at s_0. By concatenating these paths we obtain a geodesic from $(0, 0)$ to $(0, 2y_1)$ which we denote by Γ_ε, see figure below.

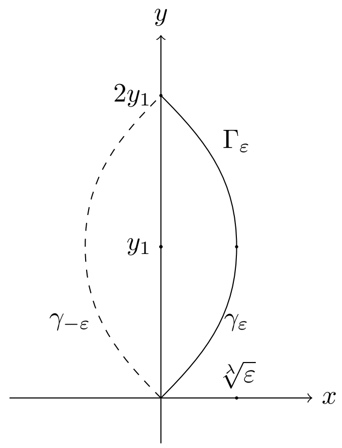


Figure 3.1

By symmetry w.r.t the y-axis, the same arguments lead to a geodesic starting as $\gamma_{-\varepsilon}$ from $(0, 0)$ to $(0, 2y_1)$. Also the curve $t \mapsto (0, t)$ is a geodesic between these points. Further the value y_1 depends on ε in the way described above which can be reformulated as follows

$$y_1(\varepsilon) = \sqrt{1-\varepsilon} \int_0^{\sqrt[\lambda]{\varepsilon}} \frac{1}{\sqrt{(1-|x|^\lambda)(\varepsilon - |x|^\lambda)}} \, dx.$$

In order to show that this construction violates the usual results known in the smooth case, that for close enough points a geodesic connecting them is unique, we have to show that $y_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

3.1.5 Proposition. In the above setting we have $y_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. Let s denote the arclength of γ_ε. For $\varepsilon < \frac{1}{2}$, since $|x(s)|^\lambda \leq \varepsilon$, (3.9) implies that

$$y'(s) = \frac{\sqrt{1-\varepsilon}}{1-|x(s)|^\lambda} \leq \frac{\sqrt{1-\varepsilon}}{1-\varepsilon} = \frac{1}{\sqrt{1-\varepsilon}} \leq \frac{1}{\sqrt{2}} < 2,$$

and therefore by the fundamental theorem of calculus

$$y(s) = y(0) + \int_0^s y'(r) \, dr < 2s.$$

This implies that $y_1 = y(s_0) \leq 2s_0$ and it remains to show $s_0(\varepsilon) \to 0$ as $\varepsilon \to 0$. By (3.8) and substitution we have

$$0 \leq s_0(\varepsilon) = \int_0^{s_0(\varepsilon)} 1\, dt = \int_0^{s_0(\varepsilon)} \frac{1}{x'}\, dx$$

$$= \int_0^{\sqrt[\lambda]{\varepsilon}} \left(1 - \frac{1-\varepsilon}{1-|x|^\lambda}\right)^{-\frac{1}{2}} dx = \int_0^{\sqrt[\lambda]{\varepsilon}} \frac{\sqrt{1-|x|^\lambda}}{\sqrt{\varepsilon - |x|^\lambda}}\, dx \leq \int_0^{\sqrt[\lambda]{\varepsilon}} \frac{1}{\sqrt{\varepsilon - |x|^\lambda}}\, dx$$

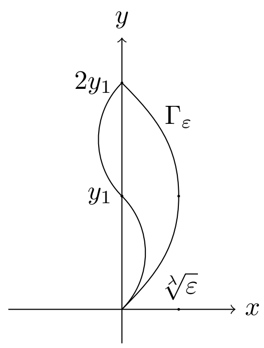
$$= \frac{1}{\sqrt{\varepsilon}} \int_0^{\sqrt[\lambda]{\varepsilon}} \frac{1}{\sqrt{1 - |\frac{x}{\sqrt[\lambda]{\varepsilon}}|^\lambda}}\, dx \overset{(z=\frac{x}{\sqrt[\lambda]{\varepsilon}})}{=} \frac{\sqrt[\lambda]{\varepsilon}}{\sqrt{\varepsilon}} \int_0^1 \frac{1}{\sqrt{1-|z|^\lambda}}\, dz \to 0 \text{ as } \varepsilon \to 0,$$

since $1 < \lambda < 2$ and $\int_0^1 \frac{1}{\sqrt{1-|z|^\lambda}}\, dz < \infty$. □

We have shown

3.1.6 Theorem. R-geodesics in 3.1.3 connecting 0 to any point on the y-axis arbitrarily close to 0 are not unique.

3.1.7 Remark. Two points on the y-axis can even be joined by infinitely many different R-geodesics constructed as above, by connecting 0 to y_1 by a geodesic and then y_1 to $2y_1$ with a matching velocity at $(0, y_1)$ and then iterating this procedure, see figure below.



Let us turn to the relation between R-geodesics, shortest paths and m-geodesics. By 2.2.1 and 1.3.13 for y_1 sufficiently small, there exists a distance (by this we mean the Riemannian distance induced by g) minimizing path form $(0,0)$ to $(0, 2y_1)$. There are multiple R-geodesics between these points. We now claim that the R-geodesic lying on the y-axis is not distance minimizing.

3.1.8 Proposition. The path $\sigma : [0, 2y_1] \to M$, $t \mapsto (0,t)$ is not distance minimizing, i.e. it is not a shortest path from $(0,0)$ to $(0, 2y_1)$, in particular the R-geodesic σ is not an m-geodesic.

Proof. Since g restricted to the y-axis is just the flat metric we obtain $L(\sigma) = 2y_1$. Let Γ_ε be the geodesic path described above and parameterized by arclength. Since $L(\gamma_{\varepsilon|[0,s_0]}) = s_0$, we have $L(\Gamma_{\varepsilon|[0,2s_0]}) = 2s_0$. This in order to prove the first part of the proposition, we only need to show $s_0(\varepsilon) < y_1(\varepsilon)$, for sufficiently small ε. We can calculate as follows, for $x > 0$

$$y_1(\varepsilon) = \int_0^{s_0} y'(t)\,dt \overset{(3.9)}{=} \sqrt{1-\varepsilon}\int_0^{s_0} \frac{1}{1-x^\lambda}\,dt$$

substituting $x = x(t)$ and using (3.8), we obtain

$$\begin{aligned} y_1(\varepsilon) &= \sqrt{1-\varepsilon}\int_0^{\sqrt[\lambda]{\varepsilon}=x(s_0)} \frac{1}{1-x^\lambda}\frac{\sqrt{1-x^\lambda}}{\sqrt{\varepsilon-x^\lambda}}\,dx \\ &= \sqrt{1-\varepsilon}\int_0^{\sqrt[\lambda]{\varepsilon}} \frac{1}{\sqrt{1-x^\lambda}\sqrt{\varepsilon-x^\lambda}}\,dx. \end{aligned}$$

By substituting $s(x) = \frac{\sqrt{\varepsilon-x^\lambda}}{\sqrt{\varepsilon}}$, we get

$$y_1(\varepsilon) = \left(-\frac{2}{\lambda}\right)\sqrt{1-\varepsilon}\sqrt{\varepsilon}\int_{1=s(0)}^{0=s(\sqrt[\lambda]{\varepsilon})} \frac{1}{\sqrt{1-x^\lambda}}\frac{1}{s}\underbrace{\frac{\sqrt{\varepsilon-x^\lambda}}{\sqrt{\varepsilon}}}_{=s} x^{1-\lambda}\,ds,$$

using $x^\lambda = \varepsilon(1-s^2)$ and $x^{1-\lambda} = \varepsilon^{\frac{1}{\lambda}-1}(1-s^2)^{\frac{1}{\lambda}-1}$, we obtain

$$y_1(\varepsilon) = \frac{2\sqrt{1-\varepsilon}\sqrt{\varepsilon}}{\lambda}\varepsilon^{\frac{1}{\lambda}-1}\int_0^1 \left(1-\varepsilon(1-s^2)\right)^{-\frac{1}{2}}(1-s^2)^{\frac{1}{\lambda}-1}\,ds.$$

Therefore

$$\frac{1}{2}\lambda y_1(\varepsilon) = \varepsilon^{\frac{1}{\lambda}-\frac{1}{2}}\sqrt{1-\varepsilon}\int_0^1 \left(1-\varepsilon(1-s^2)\right)^{\frac{1}{2}}(1-s^2)^{\frac{1}{\lambda}-1}\,ds. \tag{3.11}$$

We set

$$\begin{aligned} a(\varepsilon) &:= \frac{1}{2}\lambda\varepsilon^{\frac{1}{2}-\frac{1}{\lambda}}y_1(\varepsilon), \\ b(\varepsilon) &:= \frac{1}{2}\lambda\varepsilon^{\frac{1}{2}-\frac{1}{\lambda}}s_0(\varepsilon). \end{aligned}$$

A similar calculation for s_0 shows

$$\frac{1}{2}\lambda s_0(\varepsilon) = \varepsilon^{\frac{1}{\lambda}-\frac{1}{2}}\int_0^1 \left(1-\varepsilon(1-s^2)\right)^{\frac{1}{2}}(1-s^2)^{\frac{1}{\lambda}-1}\,ds, \tag{3.12}$$

which implies for the limit $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0} a(\varepsilon) = \lim_{\varepsilon\to 0} b(\varepsilon) =: \alpha.$$

We can therefore extend the functions a and b to $\varepsilon = 0$ by α. These functions (denoted again by a and b) are differentiable and

$$a'(0) = -\frac{1}{4}\lambda\int_0^1 (1-s^2)^{\frac{1}{\lambda}}\,ds$$

and

$$b'(0) = -\frac{1}{2}\int_0^1 (1-s^2)^{\frac{1}{\lambda}}\,ds.$$

So $a'(0) > b'(0)$ and since $a(0) = b(0)$ this implies $a(\varepsilon) > b(\varepsilon)$ for small $\varepsilon > 0$. Thus there exists $\varepsilon_0 > 0$ such that $y_1(\varepsilon) > s_0(\varepsilon)$ for all $0 < \varepsilon < \varepsilon_0$ and the path Γ_ε is not a shortest path. That it is also not locally a shortest path follows since it is not a shortest path for all $\varepsilon < \varepsilon_0$, i.e. on every segment of the y-axis from 0 to $y_1(\varepsilon)$ for $\varepsilon < \varepsilon_0$. □

3.1.9 Remark. We have also shown that there is no shortest path starting at $(0, 0)$ in the direction $(0, 1)$.
It can also be shown that the paths Γ_ε resp. $\Gamma_{-\varepsilon}$ are the shortest paths between their endpoints.
In [11] another example is presented, where different (but also distinct from smooth case) results hold: The boundary value problem is locally uniquely solvable (i.e. between close enough points exist unique shortest paths) but the initial value problem is not uniquely solvable, i.e. there is not necessarily a unique geodesic for a given initial point and velocity.

As we have seen the geodesic equations are, in low regularity, not always uniquely solvable. By the results in the previous chapter, if the Riemannian metric is at least continuous, the manifold can be equipped with a metric, compatible with the topology. As we have seen in section 1.3 under certain condition the existence of m-geodesics between two points is guaranteed by 1.3.9. We know also that such paths are always Lipschitz and the metric derivative always exists. Our next goal is to find out under what conditions such paths solve the geodesic equations. It turns out that a continuously differentiable Riemannian metric is sufficient. The proof relies on methods from variational calculus. Let us recall some details.
Let $x : [a, b] \to \mathbb{R}$ a minimizer of the functional

$$L(x) := \int_a^b F(t, x(t), x'(t))\, dt,$$

then for all $\varphi \in C_c^\infty([a, b])$, we have

$$\frac{d}{d\varepsilon}\Big|_0 L(x + \varepsilon\varphi) = 0.$$

So

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\Big|_0 L(x + \varepsilon\varphi) = \frac{d}{d\varepsilon}\Big|_0 \int_a^b F(t, x + \varepsilon\varphi(t), x' + \varepsilon\varphi')\, dt \\ &= \int_a^b \partial_2 F \cdot \varphi + \partial_3 F \cdot \varphi'\, dt \overset{\text{integration}}{\underset{\text{by parts}}{=}} \int_a^b \left(\partial_3 F - \int_a^b \partial_2 F\, ds\right) \varphi'\, dt. \end{aligned} \tag{3.13}$$

If G denotes the derivative w.r.t. t of $\partial_3 F - \int_a^b \partial_2 F\, ds$, then one more integration by parts yields $\int_a^b G \cdot \varphi = 0$, for all $\varphi \in C_c^\infty(\mathbb{R})$ and therefore $G = 0$. Thus $\partial_3 F - \int_a^b \partial_2 F\, ds = c$, for some constant c.
If x takes values in $\mathbb{R}^n$, this calculation can be done componentwise, yielding

$$\partial_{x_i'} F - \int_a^b \partial_{x_k} F(t, x, x')\, dt = c_i \quad 1 \le i \le n, \tag{3.14}$$

where $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and ∂_{x_k} is differentiation w.r.t. the slot of x_k and $\partial_{x'_k}$ w.r.t. the slot of x'_k.
We are now ready to prove

3.1.10 Theorem. Let (M, g) be a Riemannian manifold with a C^1 Riemannian metric g. If $\gamma : [0, b] \to M$ is an m-geodesic, it solves the geodesic equation and is twice continuously differentiable.

Proof. First note that by restricting γ to a small enough interval, it is a shortest path by 1.3.8 and 2.2.1. Thus since solving the geodesic equation is a local property, we can assume γ to be a shortest path.
Note that, since by 1.3.5 γ is parameterized by arclength, it is Lipschitz continuous. Further by 1.4.6(iii) and 2.2.10 we have

$$\|\gamma'(t)\|_g = \left(g_{ij}(\gamma(t))\, \gamma'_i(t)\, \gamma'_j(t)\right)^{\frac{1}{2}} = 1, \qquad \text{for almost every } t \in [0, b],$$

where $\gamma'(t) = \gamma'_j(t) \frac{\partial}{\partial x^j}|_{\gamma(t)}$ and all components are relative to the chart $\psi = (x^1, \dots, x^n)$. The statement is of local nature and therefore we may assume the image of γ to be contained in a single chart domain. Since γ is a shortest path with respect to the Riemannian arclength, it is a minimizer of

$$L(\gamma) = \int_0^b \|\gamma'(t)\|_g \, dt,$$

or in coordinates

$$L(\gamma) = \int_0^b \left(g_{ij}(\gamma(t))\, \gamma'_i(t)\, \gamma'_j(t)\right)^{\frac{1}{2}} \, dt$$

So for $F(t, \gamma, \gamma') := \left(g_{ij}(\gamma)\, \gamma'_i\, \gamma'_j\right)^{\frac{1}{2}}$ we have

$$\partial_{\gamma'_k} F(t, \gamma, \gamma') := \partial_{n+k+1} F(t, \gamma, \gamma')$$

$$= \frac{1}{2} \Bigg(\underbrace{g_{ij}(\gamma(t))\, \gamma'_i(t)\, \gamma'_j(t)}_{=1 \text{ a.e.}} \Bigg)^{-\frac{1}{2}} \cdot \left(g_{kj}(\gamma(t))\gamma'_j(t) + g_{jk}(\gamma(t))\gamma'_j(t)\right)$$

$$= g_{kj}(\gamma(t))\gamma'_j(t),$$

almost everywhere, and

$$\partial_{\gamma_k} F(t, \gamma, \gamma') = \partial_{k+1} F(t, \gamma, \gamma')$$

$$= \frac{1}{2} \left(g_{ij}(\gamma(t))\, \gamma'_i(t)\, \gamma'_j(t)\right)^{-\frac{1}{2}} \, \partial_k g_{ij}(\gamma(t))\, \gamma'_i(t)\, \gamma'_j(t).$$

Since the restriction of γ to any subinterval of $[0, b]$ is also a shortest path, plugging this into (3.14), we obtain for $t \in [0, b]$

$$c_k = g_{kj}(\gamma(t))\gamma'_j(t) - \frac{1}{2} \int_0^t \partial_k g_{ij}(\gamma(s))\, \gamma'_i(s)\, \gamma'_j(s) \, ds.$$

Since γ is Lipschitz and therefore absolutely continuous, by 2.2.5 $x^i \circ \gamma$ is locally absolutely continuous and therefore $\gamma_i' \in L^\infty([0,t])$. So $\gamma_i' \cdot \gamma_j' \in L^\infty([0,t])$. This leads to

$$\gamma_k'(t) = g^{ik}(\gamma(t)) \left(\frac{1}{2} \int_0^t \underbrace{\partial_k g_{ij}(\gamma(s))}_{\in C([0,t])} \underbrace{\gamma_i'(s)\gamma_j'(s)}_{\in L^\infty([0,t])} \, ds + c_k \right). \tag{3.15}$$

Here g^{ij} denote the components of the inverse metric, which are also C^1 by the inversion formula for matrices. So by (3.15), γ_k' is continuous. We now know that the integrand in (3.15) is continuous, so it follows that γ_k' is C^1, and therefore γ is C^2.
Lastly we need to show that γ solves the geodesic equations (3.2). By one more integration by parts of (3.13), we obtain $\frac{d}{dt}\left(\partial_3 F - \int \partial_2 F \, ds\right) = 0$. In our case this leads to

$$\frac{d}{dt}\partial_{\gamma_k'} F(t,\gamma,\gamma') - \partial_{\gamma_k} F(t,\gamma,\gamma') = 0, \quad 1 \le k \le n. \tag{3.16}$$

Let us calculate these expressions by using what we already know from above about the partial derivatives of F.

$$\frac{d}{dt}\partial_{\gamma_k'} F(t,\gamma,\gamma') = \frac{d}{dt}\left(g_{kj}(\gamma(t))\gamma_j'(t)\right)$$

$$= \partial_i g_{kj}\gamma_i'\gamma_j' + g_{lj}\gamma_l'' = \frac{1}{2}\left(\partial_i g_{kj} + \partial_j g_{ik}\right)\gamma_i'\gamma_j' + g_{lj}\gamma_l'',$$

where in the last line we rearranged indices. Plugging into (3.16) and using the derived expressions for the partial derivatives of F, we arrive at

$$\frac{1}{2}\left(\partial_i g_{kj}(\gamma(t)) + \partial_j g_{ik}(\gamma(t)) - \partial_k g_{ij}(\gamma(t))\right)\gamma_i'(t)\gamma_j'(t) + g_{lj}(\gamma(t))\gamma_l''(t) = 0,$$

for all $1 \le k \le n$. Using (3.1) this leads to

$$g_{lj}(\gamma(t))\,\Gamma^l_{ij}(\gamma(t))\,\gamma_i'(t)\gamma_j'(t) + g_{lj}(\gamma(t))\,\gamma_l''(t) = 0.$$

By applying g^{lj} to both side we see that γ solves the geodesic equation.

□

3.1.11 Remark. The above theorem implies, that if the Riemannian metric is C^1, using 1.3.13, between close enough points, there always exists a R-geodesic which is C^2. Further, by 1.3.9, if the manifold has the Heine-Borel property, any two points with finite distance can be joined by a C^2 R-geodesic, which is also a shortest path.

3.2 The case of C^α Riemannian metrics

In this section we deal with α-Hölder continuous Riemannian metrics, for $0 < \alpha \le 1$ and the regularity of m-geodesics, i.e. shortest paths, in such manifolds. We follow the paper [14] by Lytchak and Yaman. The authors consider the situation of Finsler structures, we will however restrict ourselves to the special

case of Riemannian manifolds. Some of our estimates in the following proofs will differ slightly from those in [14]. As before we always assume all manifolds to be smooth (in contrast to [14], where $C^{1,\alpha}$ manifolds are defined) and equip them with C^α Riemannian metric (a detailed definition of this will follow).
In order to properly introduce Hölder continuous Riemannian metrics we first define Hölder continuity between metric spaces.

3.2.1 Definition. Let $f : (M, d_1) \to (N, d_2)$ be a map between metric spaces. For $0 < \alpha \leq 1$,

(i) f is called α-Hölder, if there exists $C > 0$ such that for all $p, q \in M$, it holds that

$$d_2(f(p), f(q)) \leq C d_1(p, q)^\alpha,$$

(ii) f is called locally α-Hölder or C^α, if the restriction to each compact subset is α-Hölder.

(iii) a map $f : U \to \mathbb{R}^m$, where $U \subseteq \mathbb{R}^n$ is open, is called $C^{1,\alpha}$, if it is continuously differentiable and its differential $Df : U \to L(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{n \cdot m}$ is locally α-Hölder.

Note that C^α in our definition always refers to locally α-Hölder. In [14] the authors deal with Finsler structures, i.e. manifolds where at each each point a norm is given in the corresponding tangent space. In their paper they also deal with the convexity type of norms in order to deduce certain estimates. We will only consider the Riemannian case, where the norms stem form an inner product and therefore have a naturally given convexity.

3.2.2 Definition. Let M be a smooth manifold and g a continuous Riemannian metric on M, then g is called C^α, if all the local representations $\psi \circ g \circ (T^0_2\varphi)^{-1}$ of g, are locally α-Hölder. Here ψ denotes a chart of M and $T^0_2\varphi$ denotes a chart of the tensor bundle $T^0_2 M$ corresponding to the chart φ of M.

The statements that will be proved in this chapter are all of local nature and therefore it will be sufficient to do calculations in the image of a chart, i.e. an open subset of $\mathbb{R}^n$. In this case the following result from [14], Lemma 2.2, originally proved in in [15], Lemma 2.1 will be of central importance when deducing additional regularity of m-geodesics.

3.2.3 Lemma. Let $\mathcal{F}$ be a family of locally uniform Lipschitz maps $f_i : U_i \to \mathbb{R}^m$, defined on the open subsets $U_i \subseteq \mathbb{R}^n$, i.e. all f_i are locally Lipschitz continuous and on a fixed compact set every Lipschitz constant can be bounded from above by a common constant. If for every ball $B \subseteq \mathbb{R}^m$ and every i, there is a cover of U_i by open subsets $O_{i,j}$, such that for all $x \in O_{i,j}$ and all $h_i \in \mathbb{R}^n$ such that $x \pm h_i \in O_{i,j}$, we have

$$\|f_i(x + h_i) + f_i(x - h_i) - 2f_i(x)\| \leq C\|h\|^{1+\alpha},$$

for some $C > 0$ and $0 < \alpha \leq 1$, then $\mathcal{F}$ is locally uniformly $C^{1,\alpha}$.

Proof. See [15], Lemma 2.1. □

We will need another technical lemma that describes the convexity of the norm. We will formulate it in terms of a norm stemming from an inner product, for the general statement see [14], Lemma 2.3.

3.2.4 Lemma. Let V be a finite dimensional vector space with norm $|.|$ stemming from the inner product $\langle .,. \rangle$. Let $K, \alpha > 0$, then there exist $\varepsilon, \lambda > 0$ depending only on K and α, such that for all $0 \leq h \leq \varepsilon$ and all $v, w \in B_{h+Kh^{1+\alpha}}(0)$ with $|v+w| \geq 2h - Kh^{1+\alpha}$, we have $|v-w| \leq \lambda h^{1+\frac{\alpha}{2}}$.

3.2.5 Remark. The above lemma can be interpreted in such a way that, if the length of the sum of the vectors v and w adds almost to the diameter of the ball they are contained in, then they can not be too far apart and no „cancellation“ can occur. Further their distance can be controlled relative to the convexity of the norm. In our case the norm is a „2-norm“, so the factor $\frac{\alpha}{2}$ can be explained, since such norms are of convexity type 2. A detailed discussion of norms of different convexity types is given in [16], chapter 1.e.

Proof of 3.2.4. First note that for all $0 < H < \delta$, for $\delta > 0$ small enough and v_0, w_0 with $|v_0|, |w_0| \leq 1$, the inequality $2 - |v_0 + w_0| \leq H$ implies $|v_0 - w_0| \leq C\sqrt{H}$ for $C = \sqrt{10}$. This can be seen by implicitly using the convexity type leading to a rather unintuitive calculation as follows. Assume $|v_0 - w_0| > \sqrt{10H}$, then

$$2 - |v_0 - w_0| \geq 2 \inf\{1 - \frac{|x+y|}{2} \,|\, |x|, |y| \leq 1,\, |x-y| \geq \sqrt{10H}\}$$

$$\geq 2(\frac{1}{8}10H + \mathcal{O}(H^2)) > 2H,$$

for small enough H, a contradiction. Here we have used the parallelogram identity to obtain for $|x|, |y| \leq 1$, $|x-y| \geq \sqrt{10H}$ and H small enough such that no negative terms occur:

$$1 - \frac{|x+y|}{2} = 1 - \frac{1}{2}(2(|x|^2 + |y|^2) - |x-y|^2)^{\frac{1}{2}} \geq 1 - \frac{1}{2}(4 - 10H)^{\frac{1}{2}}$$

$$= 1 - \left(1 - \frac{10H}{4}\right)^{\frac{1}{2}} = 1 - \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(-\frac{10H}{4}\right)^k$$

$$= 1 - 1 - \left(-\frac{1}{2}\frac{10H}{4}\right) + \mathcal{O}(H^2) = \frac{1}{8}10H + \mathcal{O}(H^2).$$

Now we set $v_0 = \frac{v}{h+Kh^{1+\alpha}}$ and $w_0 = \frac{w}{h+Kh^{1+\alpha}}$. It then holds that $|v_0|, |w_0| \leq 1$ and $2 - |v_0 + w_0| \leq 3Kh^{\alpha}$, since

$$2 - |v_0 + w_0| = 2 - \frac{1}{h + Kh^{1+\alpha}}|v+w| \leq 2 - \frac{2h - Kh^{1+\alpha}}{h + Kh^{1+\alpha}}$$

$$= \frac{2h + 2Kh^{1+\alpha} - 2h + Kh^{1+\alpha}}{h + Kh^{1+\alpha}} = \frac{3Kh^{\alpha}}{1 + Kh^{\alpha}} \leq 3Kh^{\alpha}.$$

Choosing h such that $3Kh^{\alpha} \leq \delta$ and such that $h < 1$, the above considerations imply $|v_0 - w_0| \leq C\sqrt{3Kh^{\alpha}} = C\sqrt{3K}h^{\frac{\alpha}{2}}$ and thus we can set $\varepsilon := \left(\frac{\delta}{3K}\right)^{\frac{1}{\alpha}}$ and $\lambda := (K+1)C\sqrt{3K}$ to obtain the claim, since

$$|v-w| = (h + Kh^{1+\alpha})\,|v_0 - w_0| \leq (h + Kh^{1+\alpha})C\sqrt{3Kh^{\alpha}}$$

$$\leq C\sqrt{3K}h^{1+\frac{\alpha}{2}} + KC\sqrt{3K}h^{1+\frac{3}{2}\alpha} \leq (K+1)C\sqrt{3K}h^{1+\frac{\alpha}{2}}$$

□

Let us now set up some notation. Let M be a manifold with a continuous Riemannian metric g, and (ψ, U) a chart of M. We want to measure lengths of vectors in $\mathbb{R}^n$ in relation to g and the chart ψ. For $p \in \psi(U)$ and $v \in \mathbb{R}^n \cong T_p(\psi(U))$, we define

$$|v|_p = \|T_p(\psi)^{-1}(v)\|_{g(\psi^{-1}(p))} = \left(g(\psi^{-1}(p))\left(T_p(\psi)^{-1}v, T_p(\psi)^{-1}v\right)\right)^{\frac{1}{2}}.$$

Note that we can also describe this norm using the push-forward of g under ψ, i.e. $|v|_p = \left((\psi_* g)(v,v)\right)^{\frac{1}{2}}$. Since ψ is bijective and $T_p\psi$ a linear isomorphism, $|.|_p$ is a norm on $\mathbb{R}^n$ which stems from an inner product. In the special case that M is an open subset of $\mathbb{R}^n$, this simplifies to

$$|v|_p = (g(p)(v,v))^{\frac{1}{2}}.$$

As a result we can define different lengths of curves on U as follows

(i) $L(\gamma)$ denotes the arclength with respect to $|.|_{\gamma(t)}$, for an absolutely continuous path γ, i.e.

$$L(\gamma) := \int |\gamma'(t)|_{\gamma(t)}\, dt.$$

The induced metric w.r.t. this length will be denoted by d. Note that by the above definition, if M is an open subset of $\mathbb{R}^n$, this expression is exactly the Riemannian arclength.

(ii) For $p \in U$ fixed, $L_p(\gamma)$ denotes the arclength w.r.t. $|.|_p$, i.e.

$$L_p(\gamma) := \int |\gamma'(t)|_p\, dt,$$

(iii) $L_e(\gamma)$ denotes the Euclidean arclength,

$$L_e(\gamma) := \int \|\gamma'(t)\|_e\, dt.$$

In the following we will compare these lengths and the induced distances to deduce the desired results. In order to do this let us define an auxiliary function.From here on we assume w.l.o.g. that the images of a charts/open sets in $\mathbb{R}^n$ are convex as we often use straight lines connecting points.

3.2.6 Definition. Let U be an open subset of $\mathbb{R}^n$, for $V \subset\subset U$ and $r \geq 0$ we define

$$o_V(r) := \sup_{\substack{p,q\in V \\ \|p-q\|_e \leq r}} \sup_{\|v\|_e \leq 1} \big| |v|_p - |v|_q \big|$$

3.2.7 Remark. For V as above, o_V is finite, continuous and non-decreasing, further $o_V(0) = 0$. For local statements we can always assume U to be relatively compact and therefore o_U to be finite and bounded from above by some $C_2 \geq 2$. Since g is continuous, w.l.o.g. we can choose a domain small enough such that the difference of g to the Euclidean metric is bounded, in other words we can always have

$$\frac{1}{C_2}\|.\|_e \leq |.|_p \leq C_2\|.\|_e, \tag{3.17}$$

on a small enough domain. This in turn implies

$$\frac{1}{C_2} L_e(\gamma) \leq L(\gamma) \leq C_2 L_e(\gamma), \tag{3.18}$$

for an absolutely continuous path γ and thus for the intrinsic distances

$$\frac{1}{C_2} d(p,q) \leq \|p-q\|_e \leq C_2 d(p,q), \tag{3.19}$$

for some $C_2 \geq 2$.

We want to compare now the different lengths of a path.

3.2.8 Lemma. Let $\gamma : [0,b] \to U$ be a Lipschitz path and let U be as above and $\gamma(0) = p \in U$, then

$$|L(\gamma) - L_p(\gamma)| \leq o_U(L_e(\gamma)) L_e(\gamma). \tag{3.20}$$

Further for all $p, q \in U$, we have

$$|L_p(\gamma) - L_q(\gamma)| \leq o_U(\|p-q\|_e) L_e(\gamma). \tag{3.21}$$

Proof. Let γ be parameterized by Euclidean arclength, then

$$L(\gamma) = \int_0^b |\gamma'(t)|_{\gamma(t)}\, dt = \int_0^b |\gamma'(t)|_{\gamma(t)} - |\gamma'(t)|_p + |\gamma'(t)|_p\, dt$$

$$= L_p(\gamma) + \int_0^b \left(|\gamma'(t)|_{\gamma(t)} - |\gamma'(t)|_p \right) dt$$

Since $\|\gamma'(t)\|_e = 1$ for almost every t and by Definition 3.2.6, we get for almost every $t \in [0,b]$

$$\big| |\gamma'(t)|_{\gamma(t)} - |\gamma'(t)|_p \big| \leq o_U(\|p - \gamma(t)\|_e).$$

Again since γ is parameterized by Euclidean arclength we have

$$\| \underbrace{p}_{=\gamma(0)} - \gamma(t)\|_e \leq L_e(\gamma_{|[0,t]}) = \int_0^t \underbrace{\|\gamma'(s)\|_e}_{=1 \text{ a.e.}}\, dt = t.$$

Since o_U is non-decreasing, we obtain

$$|L(\gamma) - L_p(\gamma)| = \left| \int_0^b \left(|\gamma'(t)|_{\gamma(t)} - |\gamma'(t)|_p \right) dt \right|$$

$$\leq \int_0^b o_U(t)\, dt \leq o_U(b) b = o_U(L_e(\gamma)) L_e(\gamma).$$

The second statement is proved the same way, by putting L_q in place of L, since again

$$\big|\, |\gamma'(t)|_q - |\gamma'(t)|_p \big| \leq o_U(\|p-q\|_e),$$

almost everywhere. $\square$

Since $|.|_p$ is a norm on $\mathbb{R}^n$, we know that the intrinsic metric w.r.t. L_p and e.g. the class of all Lipschitz paths, equals the metric induced by the norm. The same clearly holds for $\|.\|_e$. Further we now compare these induced metrics(Note the corrected constants compared to[14] Cor.3.3.)

3.2.9 Corollary. Let U be as above, for all $p, q \in U$, we have

$$|d(p,q) - |p-q|_p| \leq C_2^2 o_U(C_2^2 \|p-q\|_e)\|p-q\|_e,$$

for C_2 from 3.2.7. Further

$$|\,|x-y|_p - |x-y|_q| \leq o_U(\|p-q\|_e)\|x-y\|_e.$$

Proof. Let γ be a path form p to q, by (3.20), we have $L(\gamma) \leq L_p(\gamma) + o_U(L_e(\gamma))L_e(\gamma)$. Now choosing as path $\gamma(t) = qt - (1-t)p$ the straight line from p to q, leads to

$$d(p,q) \leq L(\gamma) \leq L_p(\gamma) + o_U(\|p-q\|_e)\|p-q\|_e$$
$$= |p-q|_p + o_U(\|p-q\|_e)\|p-q\|_e,$$

since $\gamma'(t) = q - p$. In particular $d(p,q) - |p-q|_p \leq C_2^2 o_U(C_2^2\|p-q\|_e)\|p-q\|_e$, since $C_2 \geq 2$.
We now reverse the roles of $|.|_p$ and d. Again by (3.20) and (3.18), we have

$$|p-q|_p \leq L_p(\gamma) \leq L(\gamma) + o_U(L_e(\gamma))L_e(\gamma)$$
$$\leq L(\gamma) + C_2 o_U(C_2 L(\gamma))L(\gamma),$$

for all paths γ from p to q. Since $U \subseteq \mathbb{R}^n$ by 1.3.9 there is a shortest path σ from p to q, w.r.t. L, i.e. $L(\sigma) = d(p,q)$. We can thus calculate, using 3.19

$$|p-q|_p \leq d(p,q) + C_2 o_U(C_2 d(p,q))d(p,q)$$
$$\leq d(p,q) + C_2^2 o_U(C_2^2\|p-q\|_e)\|p-q\|_e,$$

implying the first statement.
The second statement follows in the same way. □

3.2.10 Remark. In [14], a C^α Finsler structure is defined as a $C^{1,\alpha}$ manifold, such that for each chart (ψ, U) and $V \subset\subset \psi(U)$, we have $o_V(r) \leq Cr^\alpha$ for all $r \geq 0$ and some $C = C(V)$. In other words for v with $\|v\|_e \leq 1$, at least locally it holds that $\left|\,|v|_p - |v|_q\right| \leq C\|p-q\|_e^\alpha$.
If (M, g) is a C^α Riemannian manifold, then in particular it is a C^α Finsler structure as defined above. Indeed let (ψ, U) be a chart of M, $V \subset\subset \psi(U)$, $p, q \in V$ and $v \in \mathbb{R}^n$, $\|v\|_e \leq 1$. Then, since g and therefore $(\psi_* g)$ is locally α-Hölder (the α-Hölder property is preserved since ψ is a diffeomorphism)

$$\left||v|_p^2 - |v|_q^2\right| = |(\psi_* g)(p)(v,v) - (\psi_* g)(q)(v,v)|$$

$$= |((\psi_* g)(p) - (\psi_* g)(q))(v,v)| \leq \|(\psi_* g)(p) - (\psi_* g)(q)\|_{\mathbb{R}^{n^2}}\|v\|_e^2$$
$$\leq C\|p-q\|_e^\alpha \|v\|_e^2.$$

In light of (3.17) this implies that (M, g) is α-Finsler.

3.2.11 Corollary. Let (U, g) be a C^α Riemannian manifold, $U \subseteq \mathbb{R}^n$ open and relatively compact. Then

$$|L(\gamma) - L_p(\gamma)| \leq K_1 L(\gamma)^{1+\alpha}$$

for any Lipschitz path γ. Moreover, for $p, q \in U$

$$|\, d(p,q) - |p-q|_p| \leq K_2 d(p,q)^{1+\alpha}.$$

Here $K_1 = C_2^{1+\alpha} C$ and $K_2 = C_2^{3+3\alpha} C$, for the above constants C from 3.2.10 and C_2 from 3.2.7.

Proof. Since (U, g) is a C^α Riemannian manifold, there is a constant C such that $o_U(r) \leq Cr^\alpha$. The first statement follows, since

$$\begin{aligned}|L(\gamma) - L_p(\gamma)| &\overset{(3.20)}{\leq} o_U(L_e(\gamma)) L_e(\gamma) \\ &\overset{(3.18)}{\leq} C_2\, o_U(C_2 L(\gamma)) L(\gamma) \leq C_2^{1+\alpha}\, C\, L(\gamma)^{1+\alpha}.\end{aligned}$$

The second identity follows similarly

$$\begin{aligned}\left|d(p,q) - |p-q|_p\right| &\overset{3.2.9}{\leq} C_2^2 o_U(C_2^2 \|p-q\|_e)\|p-q\|_e \leq C_2^2 C (C_2^2 \|p-q\|_e)^\alpha \|p-q\|_e \\ &\overset{(3.19)}{\leq} C_2^2\, C\, (C_2^3 d(p,q))^\alpha\, C_2\, d(p,q) \leq C_2^{3+3\alpha}\, C\, d(p,q)^{\alpha+1}.\end{aligned}$$

□

We can now proof a first result on the regularity of m-geodesics in C^α Riemannian manifolds, which is the Riemannian version of theorem 1.3 in [14].

3.2.12 Proposition. Let M be a smooth manifold with α-Hölder continuous Riemannian metric g, then m-geodesics (i.e. locally shortest paths) are locally uniformly $C^{1,\frac{\alpha}{2}}$.

Proof. Note that we can assume the m-geodesic to lie in a single chart (ψ, V), since the statement is of local nature, further we can w.l.o.g. assume $\psi(V)$ to be convex. It is therefore enough to consider as M an open convex subset U of $\mathbb{R}^n$. Since $U \subseteq \mathbb{R}^n$, it follows that L equals the Riemannian arclength.
We begin by collecting all estimates comparing the different norms and lengths done so far. By (3.17), (3.19) and 3.2.7, there is a constant $C \geq 2$ such that for all $p \in U$ $\frac{1}{C}\|.\|_e \leq |.|_p \leq C\|.\|_e$, we have $o(r) := o_U(r) \leq Cr^\alpha$ and also $\frac{1}{C} d(p,q) \leq \|p-q\|_e \leq C d(p,q)$. Further for all $p, q \in U$ and Lipschitz paths in U starting at p, we have $|L_p(\gamma) - L(\gamma)| \leq K_1 L(\gamma)^{1+\alpha} \leq C_1 L_p(\gamma)^{1+\alpha}$ for some constant C_1, and therefore by choosing C big enough the estimate holds also for C in place of C_1. Similarly also $|L_p(\gamma) - L(\gamma)| \leq C L(\gamma)^{1+\alpha}$. We can further increase C to replace any occurring constants (e.g. K_1) by C. Equation (3.21) for the straight line from p to q implies $|\,|p-q|_p - |p-q|_q| \leq C\|p-q\|_e^{1+\alpha}$.
Let $0 < h < 1$ and γ a m-geodesic defined on $[-h, h]$. We set $x := \gamma(-h)$, $z := \gamma(h)$ and $m := \frac{1}{2}(x+y)$. A central part in the proof will be to compare and control the distance from the midpoint m on the straight line from x to z, to $y := \gamma(0)$, the midpoint of γ and therefore to see how far away from a straight line (a shortest path in the Euclidean sense) the m-geodesic γ is.

Denote by $\gamma_1 := \gamma_{|[-h,0]}$ and $\gamma_2 := \gamma_{|[0,h]}$. Since the metric induced by $|.|_x$ is intrinsic w.r.t. L_x, we have $|x-y|_x \le L_x(\gamma_1)$. By 1.3.5 and 2.2.11, since γ is a m-geodesic we also have $L(\gamma_1) = h$. By 3.2.11 $L_x(\gamma) - L(\gamma) \le |L_x(\gamma) - L(\gamma)| \le Ch^{1+\alpha}$, so we obtain

$$|x-y|_x \le L_x(\gamma_1) \le L(\gamma_1) + CL(\gamma_1)^{1+\alpha} = h + Ch^{1+\alpha}.$$

A calculation as above shows $|y-z|_y \le L(\gamma_2) + CL(\gamma_2)^{1+\alpha}$. Using 3.2.9 we also have

$$|y-z|_x \le |y-z|_y + C\|x-y\|_e^\alpha \|y-z\|_e.$$

Note that by 1.3.7 by choosing h small enough γ is distance preserving and therefore $d(x,y) = h = d(y,z)$ as well as $d(x,z) = 2h$. Using (3.19), we obtain

$$\begin{aligned}|y-z|_x &\le h + Ch^{1+\alpha} + C^2 d(y,z) C^\alpha d(x,y)^\alpha \\ &= h + Ch^{1+\alpha} + C^{2+\alpha} h^{1+\alpha} \le h + (C + C^3) h^{1+\alpha},\end{aligned}$$

since $C > 1$. By 3.2.11 we obtain

$$d(x,z) - |x-z|_x \le |d(x,z) - |x-z|_x| \le Cd(x,z)^{1+\alpha},$$

and thus

$$|x-z|_x \ge d(x,z) - Cd(x,z)^{1+\alpha} = 2h - C(2h)^{1+\alpha} \ge 2h - 4Ch^{1+\alpha}.$$

We now apply lemma 3.2.4 for $K = \max\{C + C^3, 4C\}$, $v = x - y$, $w = y - z$. By the above we have $|v|_x \le h + Kh^{1+\alpha}$ as well as $|w|_x \le h + Kh^{1+\alpha}$ and $|v+w|_x = |x-z|_x \ge 2h - Kh^{1+\alpha}$. Therefore the lemma implies that $|v-w|_x = |x-y-y+z|_x = |x+z-2y|_x = |2m-2y|_x \le \lambda h^{1+\frac{\alpha}{2}}$ for all $0 \le h \le \varepsilon$ and some λ, where ε, λ are the constants from 3.2.4 depending only on K and α. By the definitions of x, y, z, m, we obtain

$$\frac{1}{C}\|\gamma(-h) + \gamma(h) - 2\gamma(0)\|_e \le |\gamma(-h) + \gamma(h) - 2\gamma(0)|_x \le \lambda h^{1+\frac{\alpha}{2}},$$

and thus

$$\|\gamma(-h) + \gamma(h) - 2\gamma(0)\|_e \le C\lambda h^{1+\frac{\alpha}{2}},$$

and since λ depends only on K and α (and therefore only on C and α), 3.2.3 gives the claim. $\square$

The proof of 3.2.12 works for general Finsler structures with norms of convexity type p and one obtains geodesics of regularity $C^{1,\frac{\alpha}{p}}$. Other than that norms stemming from an inner product are of convexity type 2, no geometry involving the inner product is used in the proof. We will now improve this result to $C^{1,\beta}$ for $\beta = \frac{\alpha}{2-\alpha} \ge \frac{\alpha}{2}$ by using the geometry provided by g. We need some auxiliary estimates

3.2.13 Remark. (i) Let $f : [a,b] \to \mathbb{R}$ be positive and continuous, then

$$\left(\int_a^b f(t)\,dt\right)^2 \le (b-a)\int_a^b f^2(t)\,dt,$$

which follows form the Cauchy-Schwartz inequality for the functions f and the characteristic function of $[a,b]$.

(ii) There exists $\varepsilon > 0$ such that for all $x, a, b > 0$, with $|1-x| + |a| + |b| < \varepsilon$, we have

$$\sqrt{x+a-b} \geq \sqrt{x} + \frac{1}{3}a - b,$$

which follows by Taylor expansion of the square root function.

3.2.14 Theorem. Let (M, g) be a smooth manifold with α-Hölder Riemannian metric g. The m-geodesics are locally uniformly $C^{1,\beta}$, for $\beta = \frac{\alpha}{2-\alpha}$.

Proof. For the same reason as in the proof of 3.2.12, we can assume that M is an open ball U in $\mathbb{R}^n$ with a C^α Riemannian metric. Again we may replace all constants in the estimates of this section by a common constant $C \geq 2$.
Let $\gamma : [-h, h] \to U$ be a m-geodesic for $0 < h < 1$ small. Set $x = \gamma(-h)$, $z = \gamma(h)$, $y = \gamma(0)$ and $m = \frac{x+z}{2}$. W.l.o.g. we may assume $m = 0$ and thus $x = -z$ and we may assume $|.|_0 = \|.\|_e$.
Further we may assume that γ is not constant and set $u = \frac{z}{\|z\|_e}$. By P we denote the 1-dimensional subspace generated by u and by H the to P orthogonal hyperplane, where orthogonal is meant w.r.t. the Euclidean inner product, i.e. $g(0)(p,q) =: \langle p,q\rangle_0 = \langle p,q\rangle_e = 0$, for $p \in P$ and $q \in H$. We want to control the inner product of such vectors at a point a relative to the norm of that point. A straightforward calculation shows for $p, q \in \mathbb{R}^n$ and $a \in U$, that

$$\langle p,q\rangle_a = \frac{1}{4}\left(|p+q|_a^2 - |p-q|_a^2\right).$$

Let $h \in H$ be a Euclidean unit vector, i.e. $\|h\|_e = 1$, note that we have

$$\|u+h\|_e = |u+h|_0 = |u-h|_0 = \|u-h\|_e = \sqrt{2},$$

since $u \perp_e h$. This leads to

$$\langle u,h\rangle_a = \frac{1}{4}\left(|u+h|_a^2 - |u-h|_a^2\right)$$

$$= \frac{1}{4}\left((|u+h|_a^2 - |u+h|_0^2) + (|u-h|_0^2 - |u-h|_a^2)\right).$$

Since $o_U(r) \leq Cr^\alpha$, we have by definition of o_U

$$\big|\,|v|_p - |v|_q\big| \leq o_U(\|p-q\|_e) \leq C\|p-q\|_e^\alpha, \tag{3.22}$$

for all $p, q \in U$ and $v \in \mathbb{R}^n$ with $\|v\|_e = 1$. Note that $\frac{u \pm h}{\sqrt{2}}$ are unit vectors w.r.t. $\|.\|_e$. Using $(s+t)(s-t) = s^2 - t^2$ we expand

$$\left|\left|\frac{u+h}{\sqrt{2}}\right|_a^2 - \left|\frac{u+h}{\sqrt{2}}\right|_0^2\right| = \left|\left|\frac{u+h}{\sqrt{2}}\right|_a + \left|\frac{u+h}{\sqrt{2}}\right|_0\right| \cdot \left|\left|\frac{u+h}{\sqrt{2}}\right|_a - \left|\frac{u+h}{\sqrt{2}}\right|_0\right|$$

$$\overset{(3.22)}{\leq} \left(\left|\frac{u+h}{\sqrt{2}}\right|_a + \left|\frac{u+h}{\sqrt{2}}\right|_0\right) C\|a\|_e^\alpha,$$

and similarly we get

$$\left|\left|\frac{u-h}{\sqrt{2}}\right|_0^2 - \left|\frac{u-h}{\sqrt{2}}\right|_a^2\right| \leq \left(\left|\frac{u-h}{\sqrt{2}}\right|_0 + \left|\frac{u-h}{\sqrt{2}}\right|_a\right) C\|a\|_e^\alpha.$$

We can now further estimate as follows

$$
\begin{aligned}
|\langle u,h\rangle_a| &= \frac{1}{4}\left|\left((|u+h|_a^2-|u+h|_0^2)+(|u-h|_0^2-|u-h|_a^2)\right)\right| \\
&\le \frac{1}{2}\left(\left|\left|\frac{u+h}{\sqrt{2}}\right|_a^2-\left|\frac{u+h}{\sqrt{2}}\right|_0^2\right|+\left|\left|\frac{u-h}{\sqrt{2}}\right|_0^2-\left|\frac{u-h}{\sqrt{2}}\right|_a^2\right|\right) \\
&\le \left(\frac{|u+h|_a}{\sqrt{2}}+\frac{|u+h|_0}{\sqrt{2}}\right)\frac{C}{2}\|a\|_e^\alpha+\left(\frac{|u-h|_0}{\sqrt{2}}+\frac{|u-h|_a}{\sqrt{2}}\right)\frac{C}{2}\|a\|_e^\alpha \\
&= \frac{C}{2\sqrt{2}}\|a\|_e^\alpha\left(|u+h|_a+|u+h|_0+|u-h|_0+|u-h|_a\right) \\
&\le \frac{C}{2\sqrt{2}}\|a\|_e^\alpha\big(2C\underbrace{\|u+h\|_e}_{=\sqrt{2}}+2\|u+h\|_e\big) \\
&\le \frac{C^2+C}{2\sqrt{2}}2\sqrt{2}\|a\|_e^\alpha=(C^2+C)\|a\|_e^\alpha\le 2C^2\|a\|_e^\alpha.
\end{aligned}
$$

Thus

$$
|\langle u,h\rangle_a|\le 2C^2\|a\|_e^\alpha, \tag{3.23}
$$

for $u\in P$, $h\in H$ both Euclidean unit vectors.
We now decompose γ into paths in H and P as $\gamma(t)=f(t)u+v(t)$, for $f:[-h,h]\to\mathbb{R}$ and $v:[-h,h]\to H$. Note that $\gamma(h)=z=\|z\|_e u$ and that $x=-z$ since we assumed $m=\frac{x+z}{2}=0$, thus $f(-h)=-\|z\|_e$ and $f(h)=\|z\|_e$.
We set $\eta(t):=f(t)u$, $\eta:[-h,h]\to P$. By 3.2.12 γ is locally $C^{1,\frac{\alpha}{2}}$, thus so are f and v. For $\varepsilon>0$ we can choose h small enough such that $\|v'(t)\|_e\le\varepsilon$ for all $t\in[-h,h]$. Indeed since v is C^1 and v' is $\frac{\alpha}{2}$-Hölder on $[-h,h]$, the fundamental theorem of calculus yields

$$
v(h)-v(-h)=\int_{-h}^{h}v'(s)\,ds=2hv'(t)+\int_{-h}^{h}(v'(s)-v'(t))\,ds,
$$

where this equation has to be understood componentwise. Thus for some $k\ge 0$

$$
\begin{aligned}
\|v(h)-v(-h)-2hv'(t)\|_e &\le \int_{-h}^{h}\|v'(s)-v'(t)\|_e\,ds \\
&\le \int_{-h}^{h}k|t-s|^{\frac{\alpha}{2}}\,ds\le k\int_{-h}^{h}(2h)^{\frac{\alpha}{2}}\,ds=k(2h)^{1+\frac{\alpha}{2}}.
\end{aligned}
$$

Since $\gamma(h)=z\in P$ and $\gamma(-h)=x\in P$, we have $v(-h)=v(h)=0$, yielding

$$
\|v'(t)\|_e\le k(2h)^{\frac{\alpha}{2}},
$$

so for $0<h\le(\frac{\varepsilon k}{2})^{\frac{2}{\alpha}}$ we have $\|v'(t)\|_e\le\varepsilon$ and also $|v'(t)|_p\le\varepsilon$ by further shrinking h. Since γ is a m-geodesic and C^1, we have $|\gamma'(t)|_{\gamma(t)}=1$ for all $t\in[-h,h]$, this leads to

$$
\begin{aligned}
1 &\ge |f'(t)u|_{\gamma(t)}=|\gamma'(t)-v'(t)|_{\gamma(t)} \\
&\ge |\gamma'(t)|_{\gamma(t)}-|v'(t)|_{\gamma(t)}\ge 1-\varepsilon.
\end{aligned} \tag{3.24}
$$

W.l.o.g. we can assume $f' > 0$ near $-h$, but by (3.24) f' cannot change sign so $f'(t) > 0$ for all $t \in [-h, h]$. Note that (3.24) then implies $1 \geq f'(t)|u|_{\gamma(t)} \geq 1-\varepsilon$ and $1 - f'(t)|u|_{\gamma(t)} \leq \varepsilon$. This means that the acceleration happens almost entirely in P and that the small part not in P, can be controlled by h and α. Further $v'(t)$ should serve as a measurement determining how far γ is deformed from the straight line η. To make this more specific we need a few auxiliary estimates. First note that by 3.2.6 and the fact that g is C^α, we have

$$|u|_{\eta(t)} - |u|_{\gamma(t)} \leq C\|\gamma(t) - \eta(t)\|_e^\alpha = C\|v(t)\|_e^\alpha,$$

since $\|u\|_e = 1$ and thus

$$|u|_{\gamma(t)} \geq |u|_{\eta(t)} - C\|v(t)\|_e^\alpha. \tag{3.25}$$

By (3.17) we have

$$|v'(t)|_{\gamma(t)}^2 \geq \frac{1}{C^2}\|v'(t)\|_e^2. \tag{3.26}$$

Finally

$$-f'(t) = -\underbrace{f'(t)|u|_{\gamma(t)}}_{\leq 1}\frac{1}{|u|_{\gamma(t)}} \geq -\frac{1}{|u|_{\gamma(t)}} \geq -C, \tag{3.27}$$

since $\frac{1}{C} = \frac{1}{C}\|u\|_e \leq |u|_{\gamma(t)}$, leading to $\frac{1}{|u|_{\gamma(t)}} \leq C$. Let us now further determine some estimates on v'.

$$\begin{aligned}
1 = |\gamma'(t)|_{\gamma(t)} &= \left(\langle\gamma'(t), \gamma'(t)\rangle_{\gamma(t)}\right)^{\frac{1}{2}} \\
&\overset{\gamma'(t)=f'(t)u+v'(t)}{=} \left(f'(t)^2|u|_{\gamma(t)}^2 + |v'(t)|_{\gamma(t)}^2 + 2f'(t)\langle u, v'(t)\rangle_{\gamma(t)}\right)^{\frac{1}{2}} \\
&\overset{3.2.13(ii)}{\geq} f'(t)|u|_{\gamma(t)} + \frac{1}{3}|v'(t)|_{\gamma(t)}^2 - 2|f'(t)\langle u, v'(t)\rangle_{\gamma(t)}| \\
&\underset{(3.26)}{\overset{(3.25)}{\geq}} f'(t)\left(|u|_{\eta(t)} - C\|v(t)\|_e^\alpha\right) + \frac{1}{3C^2}\|v'(t)\|_e^2 - 2|f'(t)|\|v'(t)\|_e|\langle u, \frac{v'(t)}{\|v'(t)\|_e}\rangle_{\gamma(t)}| \\
&\overset{(3.23)}{\geq} f'(t)|u|_{\gamma(t)} - f'(t)C\|v(t)\|_e^\alpha + \frac{1}{3C^2}\|v'(t)\|_e^2 - 4C^2 f'(t)\|v'(t)\|_e\|\gamma(t)\|_e^\alpha \\
&\overset{(3.27)}{\geq} f'(t)|u|_{\eta(t)} - C^2\|v(t)\|_e^\alpha + \frac{1}{3C^2}\|v'(t)\|_e^2 - 4C^3\|v'(t)\|_e\|\gamma(t)\|_e^\alpha \\
&\geq f'(t)|u|_{\eta(t)} - C'\|v(t)\|_e^\alpha + \frac{1}{C'}\|v'(t)\|_e^2 - C'\|v'(t)\|_e\|\gamma(t)\|_e^\alpha,
\end{aligned}$$

for $C' = \max\{3C^2, 4C^3, C^2\} = 4C^3$, since $C \geq 2$.
By 1.3.4 we can choose h small enough such that γ is a shortest path from x to z and therefore

$$\int_{-h}^{h} |\gamma'(t)|_{\gamma(t)}\, dt = L(\gamma) \leq L(\eta) = \int_{-h}^{h} f'(t)|u|_{\eta(t)}\, dt,$$

or equivalently

$$\int_{-h}^{h} |\gamma'(t)|_{\gamma(t)} - f'(t)|u|_{\eta(t)}\, dt \leq 0.$$

The above calculation then leads to

$$\int_{-h}^{h}\left(-C'\|v(t)\|_e^\alpha+\frac{1}{C'}\|v'(t)\|_e-C'\|v'(t)\|_e\|\gamma(t)\|_e^\alpha\right)dt\le 0,$$

or equivalently

$$\int_{-h}^{h}\|v'(t)\|_e^2\,dt\le C'^2\int_{-h}^{h}\|v(t)\|_e^\alpha+\|v'(t)\|_e\|\gamma(t)\|_e^\alpha\,dt. \tag{3.28}$$

Note that γ is a distance preserving map and by the proof of 3.2.12, we have

$$\|\gamma(t)\|_e\le\|\gamma(t)-\gamma(0)\|_e+\|\gamma(0)-\underbrace{0}_{=m}\|_e$$

$$\overset{(3.19)}{\le} Cd(\gamma(t),\gamma(0))+d(y,m)\le C|t|+C_3h^{1+\frac{\alpha}{2}}\le C|t|+C_3h\le(C+C_3)h,$$

for $C_3=\frac{C^2}{2}\lambda$ with λ from 3.2.12. Plugging this into (3.28), we get

$$\int_{-h}^{h}\|v'(t)\|_e^2\,dt\le C'^2\int_{-h}^{h}(\|v(t)\|_e^\alpha+\|v'(t)\|_e(C+C_3)^\alpha h^\alpha)\,dt.$$

Thus at least one of the following holds,

$$\int_{-h}^{h}\|v'(t)\|_e^2\le 2C'^2\int_{-h}^{h}\|v(t)\|_e^\alpha\,dt, \tag{3.29}$$

or

$$\int_{-h}^{h}\|v'(t)\|_e^2\le 2C'^2(C+C_3)^\alpha\int_{-h}^{h}\|v'(t)\|_e h^\alpha\,dt, \tag{3.30}$$

which is due to $0\le\frac{s+r}{2}\le\max\{s,r\}$, for positive real numbers s,r.
Set $v_0:=\max_{t\in[-h,h]}\|v(t)\|_e$. Since $v(h)=v(-h)=0$, the fundamental theorem of calculus implies

$$\|v(t)\|_e=\|v(t)-v(-h)\|_e\le\int_{-h}^{t}\|v'(s)\|_e\,ds\le\int_{-h}^{h}\|v'(s)\|_e\,ds, \tag{3.31}$$

for all $t\in[-h,h]$, implying $v_0\le\int_{-h}^{h}\|v'(s)\|_e\,ds$.
Let us first consider the second case (3.30): The Cauchy-Schwartz inequality leads to

$$\left(\int_{-h}^{h}\|v'(t)\|_e^2\,dt\right)^2\le C_4\left(\int_{-h}^{h}\|v'(t)\|_e^2\,dt\right)\left(\int_{-h}^{h}h^{2\alpha}\,dt\right)$$

$$=2C_4\left(\int_{-h}^{h}\|v'(t)\|_e^2\,dt\right)h^{2\alpha+1},$$

for $C_4=4C'^4(C+C_3)^{2\alpha}$. Since $\int_{-h}^{h}\|v'(t)\|_e^2\,dt=0$ only in the trivial case of $\gamma=\eta$, we may divide to obtain

$$\int_{-h}^{h}\|v'(t)\|_e^2\,dt\le 2C_4\,h^{2\alpha+1}.$$

Invoking 3.2.13 (i), we obtain

$$v_0^2 \le \left(\int_{-h}^{h} \|v'(t)\|_e \, dt\right)^2 \overset{3.2.13(i)}{\le} 2h \int_{-h}^{h} \|v'(t)\|_e^2 \, dt \le 4C_4 \, h^{2+2\alpha},$$

and so $v_0 \le 2\sqrt{C_4} h^{1+\alpha}$.
In the first case (3.29), we obtain, again by using 3.2.13(i), that

$$\begin{aligned}\left(\int_{-h}^{h} \|v'(t)\|_e \, dt\right)^2 &\le 2h \, 2C'^2 \int_{-h}^{h} \|v(t)\|_e^\alpha \, dt \\ &\le 4hC'^2 v_0^\alpha 2h = 8C'^2 h^2 v_0^\alpha.\end{aligned}$$

By definition of v_0, we get

$$v_0^2 \le \left(\int_{-h}^{h} \|v'(t)\|_e \, dt\right)^2 \le 8C'^2 h^2 v_0^\alpha,$$

implying

$$v_0 \le C_5 h^{\frac{2}{2-\alpha}},$$

with $C_5 = (8C'^2)^{\frac{1}{2-\alpha}}$.
Set now $\beta := \frac{\alpha}{2-\alpha}$, then $\frac{\alpha}{2} \le \beta \le \alpha$ and $\frac{2}{2-\alpha} = 1 + \beta$, thus also for $0 \le h < 1$ we have $h^{1+\beta} \ge \max\{h^{\frac{2}{2-\alpha}}, h^{1+\alpha}\}$. Hence for $t \in [-h, h]$, we have

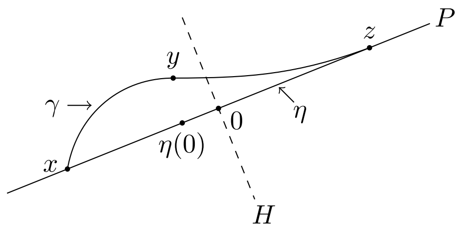
$$\|\gamma(t) - \eta(t)\|_e = \|v(t)\|_e \le v_0 \le C_6 h^{1+\beta}, \tag{3.32}$$

for $C_6 = \max\{2\sqrt{C_4}, C_5\}$. In order to apply 3.2.3, we have to show $\|\gamma(0) - m\|_e \le Lh^{1+\beta}$ for some $L \ge 0$ depending only on β. Since m is the origin, this simplifies to

$$\begin{aligned}\|\gamma(0) - m\|_e &= \|\gamma(0)\|_e = \|v(0) - \eta(0)\|_e \\ &\le \|v(0)\|_e + \|f(0)u\|_e \le C_6 h^{1+\beta} + |f(0)|.\end{aligned}$$

It thus remains to show $|f(0)| \le Lh^{1+\beta}$ for some constant L depending only on C and β. If $f(0) = 0$ there is nothing left to prove, so let us assume $f(0) < 0$, the other case follows similarly. We argue by contradiction, assume $|f(0)| > Lh^{1+\beta}$ for some L which is to be determined. We will show that for sufficiently large L this cannot hold and therefore the converse $|f(0)| \le Lh^{1+\beta}$ has to hold.

First we claim that $|x|_x \le h + kh^{1+\alpha}$ for some constant k. The figure below illustrates the relation between the occurring points.



Set $\eta_2 := \eta_{|[0,h]}$. Note that since $f(0) < 0$, $\eta(0)$ lies before 0 on the straight line from x to z. In other words $f(s) = 0$ is only possible for $s > 0$, this means $\eta(s) = 0 = m$. Further note that by 3.2.11, we have $L_x(\eta_2) \leq L(\eta_2) + CL(\eta_2)^{1+\alpha}$. Since $x = -z$,

$$|x|_x = |z|_x = |z - m|_x \leq L_x(\eta_{|[s,h]}) \leq L_x(\eta_2) \leq L(\eta_2) + CL(\eta_2)^{1+\alpha}. \tag{3.33}$$

We now use $h < 1$, as well as several inequalities obtained in this proof to calculate as follows

$$L(\eta_2) = \int_0^h |\eta'(t)|_{\eta(t)}\,dt = \int_0^h f'(t)|u|_{\eta(t)}\,dt$$

$$\overset{(3.25)}{\leq} \int_0^h \big(\underbrace{f'(t)|u|_{\gamma(t)}}_{\leq 1 \text{ by } (3.24)} + Cf'(t)\|v(t)\|_e^\alpha\big)\,dt$$

$$\leq h + C\int_0^h \underbrace{f'(t)}_{\leq C \text{ by } (3.27)} \|v(t)\|_e^\alpha\,dt \overset{(3.32)}{\leq} h + C^2\int_0^h (C_6 h^{1+\beta})^\alpha\,dt$$

$$\leq h + C^2 C_6^\alpha (\underbrace{h^{1+\beta}}_{\leq h})^\alpha \int_0^h 1\,dt \leq h + K_1 h^{1+\alpha},$$

for $K_1 := C^2 C_6^\alpha$. Inserting this into (3.33), we obtain

$$|x|_x \leq h + K_1 h^{1+\alpha} + C(h + K_1 \underbrace{h^{1+\alpha}}_{\leq h})^{1+\alpha}$$

$$\leq h + K_1 h^{1+\alpha} + C(K_1 + 1)^{1+\alpha} h^{1+\alpha} = h + K_2 h^{1+\alpha},$$

for $K_2 := K_1 + C(K_1 + 1)^{1+\alpha}$ and the claim is proved.
We note that

$$|x - \eta(0)|_x = |\eta(-h) - \eta(0)|_x \leq L_x(\eta_{|[-h,0]}) = \int_{-h}^0 |\eta'(t)|_x\,dt$$

$$= \int_{-h}^0 f'(t)|u|_x\,dt = (f(0) - \underbrace{f(-h)}_{=-\|z\|_e})|u|_x \leq f(0)|u|_x + \|z\|_e|u|_x$$

$$= f(0)|u|_x + |z|_x = f(0)|u|_x + |x|_x \leq f(0)|u|_x + h + K_2 h^{1+\alpha}$$

$$\leq h + K_2 h^{1+\alpha} - \underbrace{|f(0)u|_x}_{\geq \frac{1}{C}\|f(0)u\|_e} < h + K_2 h^{1+\alpha} - \frac{L}{C} h^{1+\beta}$$

$$\leq h + \left(K_2 - \frac{L}{C}\right) h^{1+\beta}.$$

This means

$$\begin{aligned} |x - \gamma(0)|_x &\leq |x - \eta(0)|_x + |\eta(0) - \gamma(0)|_x \\ < h + \left(K_2 - \frac{L}{C}\right) h^{1+\beta} + CC_6 h^{1+\beta} &= h - \left(\frac{L}{C} - K_2 - CC_6\right) h^{1+\beta}. \end{aligned} \tag{3.34}$$

On the other hand, since γ is a shortest path we have $h = d(\gamma(-h), \gamma(0)) = d(x, \gamma(0))$ and by 3.2.11

$$\left|d(x,\gamma(0)) - |x-\gamma(0)|_x\right| \leq K d(x,\gamma(0))^{1+\alpha} = K h^{1+\alpha} \leq K h^{1+\beta},$$

for the constant K from 3.2.11, which depends only on C and α. This leads to

$$h - |x-\gamma(0)|_x \leq \left|h - |x-\gamma(0)|_x\right| \leq K h^{1+\beta}$$

and thus

$$|x-\gamma(0)|_x \geq h - K h^{1+\beta}.$$

But by (3.34) we also have

$$|x-\gamma(0)|_x < h - \left(\frac{L}{C} - K_2 - CC_6\right) h^{1+\beta},$$

which gives a contradiction for $\frac{L}{C} - K_2 - CC_6 > K$, i.e. for $L > KC + CK_2 + C^2 C_6$. So by the reasoning above $|f(0)| \leq L h^{1+\beta}$ holds for sufficiently large L and thus

$$\|\gamma(0) - m\|_e \leq C_7 h^{1+\beta},$$

for $C_7 = \max\{C_6, L\}$ and 3.2.3 gives the desired result. □

Chapter 4

The exponential map of a $C^{1,1}$ metric

Previously we have dealt with Riemannian metrics of regularity below $C^{1,1}$, but above C^0. We have seen that in general the geodesic equation is not uniquely solvable and shortest paths are not unique. We have also deduced how much additional regularity m-geodesics gain, if the metric is of a certain regularity class between C^0 and C^1. In this chapter we deal with the borderline case of $C^{1,1}$ metrics and the exponential map in this case. We will not deduce any regularity of m-geodesics. Our result about the exponential map being a bi-Lipschitz homeomorphism, however is still closely related to the subject and shows, among other things, like the Gauss-lemma, the dependence of solutions of the geodesic equation on initial data. A low regularity version of the Gauss-Lemma for Riemannian manifolds then establishes a connection between shortest paths and R-geodesics. In this chapter we will deal with the general case of Semi-Riemannian, instead of Riemannian manifolds, i.e. the metrics are no longer presupposed as positive definite, but only non-degenerate and of constant index. The result that the exponential map for $C^{1,1}$ metrics is a bi-Lipschitz homeomorphism will be proved in the next two sections in two different ways, the first ([17], [18]) using regularization techniques and the second ([23]) using strong differentiability and a low regularity version of the inverse function theorem ([24], [25]).

4.1 The Regularization Approach

In the proof of the main theorem in this section we will use similar arguments as in section 2, namely we will regularize the given metric by convolution and since the statement is local, we will compare the metrics to the Euclidean metric obtained on a chart. Further we will use results comparing solutions of ordinary differential equations to obtain common domains of the exponential maps corresponding to the regularized metrics. An application of the invariance of domains theorem will complete the proof. Throughout this section, if not mentioned otherwise g will always denote a $C^{1,1}$ semi-Riemannian metric.
The main results are proved originally in [17], occasionally we also use results from [18]. We begin however by stating a theorem which is due to J.H.C.

Whitehead and relies on the invariance of domains theorem (see e.g. [19] theorem 2.B.3, p.172). In [20], section 3 a path is defined as a solution $c : I \to U$ to

$$(c^k)'' + \Gamma^k_{ij} \circ c(c^i)'(c^j)' = 0,$$

for locally Lipschitz continuous functions Γ^k_{ij} (note that these are not necessarily the Christoffel symbols of a metric) which are symmetric in i and j. There it is then proved that every point possesses a simple region as a neighbourhood. By simple region we mean an open set such that any two points in this set can be joined by at most one path in the above sense. This implies the following result

4.1.1 Theorem. Let M be a smooth manifold with a $C^{1,1}$ semi-Riemannian metric. For every point $p \in M$, there exists an open neighbourhood U of $0 \in T_pM$ and an open neighbourhood V of p, such that $\exp_p : U \to V$ is a homeomorphism.

Proof. Note that the Christoffel symbols of the metric are locally Lipschitz continuous and therefore the above mentioned result in [20] implies that $\exp_p : (\exp_p)^{-1}(S) \to S$ is continuous and bijective for a simple region S, the invariance of domains theorem now implies that $\exp_p$ is a homeomorphism. $\square$

Let us now discuss improvements of this result to $\exp_p$ being a bi-Lipschitz homeomorphism. Note however that the following deductions do not rely on the above theorem. The result presented hold for the case of Semi-Riemannian metrics, however for Riemannian metrics the proof simplifies to an application of the Rauch comparison theorem, see [17] chapter 3. Since the statement will be of local nature, for all considerations we can assume $M = \mathbb{R}^n$. As in the previous sections we denote the Euclidean inner product by $\langle , \rangle_e$ resp. g_e and the corresponding norm by $\|.\|_e$. Similar as in chapter 2.2 we convolute the given metric with a mollifier ρ, i.e. $\rho \in C^\infty_c(\mathbb{R}^n)$ with unit integral and for $\varepsilon > 0$ we define $\rho_e := \varepsilon^{-n}\rho(\frac{x}{\varepsilon})$. Thus we obtain a net $g_\varepsilon := g * \rho_\varepsilon$ of smooth maps, where the convolution has to be understood componentwise.

4.1.2 Remark. Let us note a few consequences for the regularized metrics:

(i) $g_\varepsilon \to g$ as $\varepsilon \to 0$ in $C^1(M)$, also the second derivatives of g_ε are uniformly bounded on compact sets.

(ii) On a fixed compact set by choosing $\varepsilon_0 > 0$ small enough, for all $0 \leq \varepsilon < \varepsilon_0$, g_ε are semi-Riemannian metrics with the same signature as g and their Riemanian curvature tensors R_ε are bounded uniformly in ε.

In order to deduce properties of the limit g_ε as $\varepsilon \to 0$, we need to find common domains of all the exponential maps corresponding to the metrics g_ε, which will henceforth be denoted by $\exp^{g_\varepsilon}$ resp. $\exp^{g_\varepsilon}_p$ for the exponential map at the point p. We will use a general existence result on ordinary differential equations proved in [21], chapter 10, 10.5.6, p. 289.

4.1.3 Lemma. Let $(X, \|.\|_X)$ be a Banach space, $H \subseteq X$ a convex subset and let $\alpha, k, \mu > 0$. For $F, G \in C(H, X)$, assume that

$$\sup_{x \in H} \|F(x) - G(x)\|_X \leq \alpha.$$

Further let G be Lipschitz continuous on H with $\mathrm{Lip(G)} \leq k$ and F be locally Lipschitz continuous on H. Define

$$\varphi(\xi) := \mu e^{k\xi} + \frac{\alpha}{k}\left(e^{k\xi} - 1\right), \quad \xi \geq 0.$$

If for $x_0 \in H$ and $t \in \mathbb{R}$, u is a solution of

$$u'(t) = G(u(t)), \quad u(t_0) = x_0,$$

defined on $J := (t_0 - b, t_0 + b)$, $b \in \mathbb{R}$, such that for all $t \in J$, we have $\overline{B_{\varphi(|t-t_0|)}(u(t))} \subseteq H$, then for every $y \in H$ with $\|y - x_0\|_X \leq \mu$ there exists a unique solution v of

$$v'(t) = F(v(t)), \quad v(t_0) = y,$$

on J with values in H and even $\|u(t) - v(t)\|_X \leq \varphi(|t - t_0|)$ for $t \in J$.

Next we rewrite the geodesic equations for g as a first order system by setting

$$\begin{aligned} &\frac{d}{dt}c^k(t) =: y^k(t), \\ &\frac{d}{dt}y^k(t) = -\Gamma^k_{g,ij}(c(t))\, y^i(t)\, y^j(t), \end{aligned} \tag{4.1}$$

for $k = 1, \ldots, n$. By $\Gamma^k_{h,ij}$ we denote the Christoffel symbols w.r.t. the metric h. We want to be able to apply the lemma above so let $t_0 = 0$ and $x_0 = (p, 0)$. In order to apply the results to the exponential map, our domain of definition J has to contain $[0, 1]$, say $J = (-b, b)$ for some $b > 1$. Denote by u the constant solution to (4.1) with initial condition $x_0 = (p, 0)$ and for $\delta > 0$ set $H := B_{2\delta}(x_0) \subseteq \mathbb{R}^{2n}$. If g is a $C^{1,1}$ metric, then the Christoffel symbols are Lipschitz functions on the compact set H and by remark 4.1.2 there is a common Lipschitz constant $k > 0$ for both Γ_g and Γ_{g_ε} on H. We can choose $\alpha, \mu > 0$ such that

$$\varphi(b) = \mu e^{bk} + \frac{\alpha}{k}(e^{bk} - a) < \delta.$$

Further we can choose $\varepsilon_0 > 0$ such that for all $0 \leq \varepsilon < \varepsilon_0$, we have

$$\sup_{(x_1,\ldots,x_{2n})\in H} \|\Gamma_g((x_1, \ldots, x_n) - \Gamma_{g_\varepsilon}(x_1, \ldots, x_n)\|_e \leq \alpha.$$

Since u is constant, $\overline{B_{\varphi(|t|)}(u(t))} = \overline{B_{\varphi(|t|)}(x_0)} \subseteq H$ for all $t \in J$. An application of the above lemma yields for $y = (p, w) \in H$, with $\|y - x_0\|_{e(\mathbb{R}^{2n})} = \|w\|_{e(\mathbb{R}^n)} \leq \mu$, a unique solution u_ε on J of

$$\begin{aligned} &\frac{d}{dt}c^k(t) =: y^k(t), \\ &\frac{d}{dt}y^k(t) = -\Gamma^k_{g_\varepsilon,ij}(c(t))\, y^i(t)\, y^j(t), \end{aligned} \tag{4.2}$$

with $u_\varepsilon \in H$ and $u_\varepsilon(0) = y = (p, w)$. Moreover the lemma also provides a unique solution to (4.1) with initial condition y. In particular for $\varepsilon < \varepsilon_0$ and $y = (p, w)$ with $\|w\|_e < \mu$, there exists a unique solution of the geodesic equation defined at least on $[-1, 1]$ and therefore $\exp^g_p$ and $\exp^{g_\varepsilon}_p$ can be defined on the common domain $B^e_\mu(0) := \{w \in \mathbb{R}^n \mid \|w\|_e < \mu\}$.

4.1.4 Remark. From Remark 4.1.2, we obtain for some small $\varepsilon_0 > 0$, that

(i) there exists $k_1 > 0$ such that for the curvature tensor R_ε w.r.t. g_ε and $0 \le \varepsilon < \varepsilon_0$, it holds that $\|R_\varepsilon\|_e \le k_1$ uniformly in ε on $B^e_\mu(0)$. Here $\|.\|_e$ denotes the (Euclidean) mapping norm.

(ii) there exists $k_2 > 0$ such that for $0 \le \varepsilon < \varepsilon_0$, it holds that $\|\Gamma_{g_\varepsilon}\|_e \le k_2$ uniformly in ε on $B^e_\mu(0)$.

We now subsequently shrink the common domain of $\exp^g_p$ and $\exp^{g_\varepsilon}_p$ in order to obtain certain needed properties. A similar and sometimes complementary deduction as in [17] can also be found [18], chapters 3 and 4. Some of the lengthy calculations there will be omitted. First we claim

4.1.5 Lemma. Let all constants be given as in the considerations above and choose $r_1 < \min\left(\frac{1}{2k_2}, \frac{\mu}{2}\right)$, then for all $\varepsilon < \varepsilon_0$, we have

$$\exp^{g_\varepsilon}_p(\overline{B^e_{r_1}(0)}) \subseteq B^e_\mu(p).$$

Proof. Let $\gamma : [0, r_1] \to M$ be a g_ε-geodesic starting at p with $\|\gamma'(0)\|_e = 1$. Set $s_0 := \sup\{s \in [0, r_1] \,|\, \gamma_{|[0,s]} \subseteq B^e_\mu(p)\} > 0$. Assume $s_0 < r_1$. We have

$$\left|\frac{d}{ds}\langle\gamma'(s), \gamma'(s)\rangle_e\right| = 2\left|\langle\frac{\nabla^e}{ds}\gamma'(s), \gamma'(s)\rangle_e\right|.$$

Since the inner product in the above equation is Euclidean and not g_ε, we cannot simply apply the usual differentiation rules of the induced connection w.r.t. g_ε, but rather we have to take the Euclidean induced connection $\frac{\nabla^e}{ds}$. By [18], chapter 3 p. 11-13 and p.15, we can however compare the two and express their difference in terms of their Christoffel symbols as $\|\nabla^e - \nabla^\varepsilon\|_e := \|\Gamma_e - \Gamma_{g_\varepsilon}\|_e = \|\Gamma_{g_\varepsilon}\|_e \le k_2$, by 4.1.4. Let us use $\nabla^\varepsilon_{\gamma'}\gamma' = 0$, to continue the above equation as follows

$$\left|\frac{d}{ds}\langle\gamma'(s), \gamma'(s)\rangle_e\right| = 2\left|\langle\left(\nabla^e_{\gamma'} - \nabla^\varepsilon_{\gamma'}\right)\gamma'(s), \gamma'(s)\rangle_e\right|$$

$$\le 2\|(\nabla^e - \nabla^\varepsilon)_{\gamma'(s)}\gamma'(s)\|_e\|\gamma'(s)\|_e \le 2\|\Gamma_\varepsilon\|_e\|\gamma'(s)\|^3_e \le 2k_2\|\gamma'(s)\|^3_e.$$

We also have

$$2k_2\|\gamma'(s)\|^3_e \ge \left|\frac{d}{ds}\langle\gamma'(s), \gamma'(s)\rangle_e\right| = \frac{d}{ds}\|\gamma'(s)\|^2_e = 2\|\gamma'(s)\|_e\frac{d}{ds}\|\gamma'(s)\|_e.$$

Noting thatγ is a geodesic, $\gamma'(s) \ne 0$ for all $s \in [0, r_1]$. This leads to

$$\left|\frac{d}{ds}\frac{1}{\|\gamma'(s)\|_e}\right| = \left|\frac{1}{\|\gamma'(s)\|^2_e}\frac{d}{ds}\|\gamma'(s)\|_e\right| \le k_2.$$

We obtain for $s \in [0, s_0)$, by using the fundamental theorem of calculus on $\frac{1}{\|\gamma'\|_e}$

$$\Big|\frac{1}{\|\gamma'(s)\|_e} - \underbrace{\frac{1}{\|\gamma'(0)\|_e}}_{=1}\Big| = \left|\int_0^s \frac{d}{d\tau}\left(\frac{1}{\|\gamma'(\tau)\|_e}\right)d\tau\right|$$

$$\le \int_0^s \left|\frac{d}{d\tau}\left(\frac{1}{\|\gamma'(\tau)\|_e}\right)\right| d\tau \le sk_2 \le r_1k_2 < \frac{1}{2},$$

implying

$$\frac{1}{2} \leq \frac{1}{\|\gamma'(s)\|_e} \leq \frac{3}{2}.$$

Therefore, since $\|\gamma'(0)\|_e = 1$ we have

$$\frac{1}{2}\|\gamma'(s)\|_e \leq \|\gamma'(0)\|_e \leq \frac{3}{2}\|\gamma'(s)\|_e,$$

and thus

$$\frac{1}{2}\|\gamma'(0)\|_e \leq \frac{3}{4}\|\gamma'(s)\|_e \leq \|\gamma'(s)\|_e \leq 2\|\gamma'(0)\|_e. \tag{4.3}$$

Denoting the Euclidean length of a curve by L_e, for $s \in [0, s_0)$ this leads to

$$L_e(\gamma_{|[0,s_0]}) = \int_0^{s_0} \|\gamma'(s)\|_e \, ds \leq \int_0^{s_0} 2\|\gamma'(0)\|_e \, ds = 2s_0\|\gamma'(0)\|_e \leq 2r_1 < \mu.$$

This means that $\gamma_{|[0,s_0]}$ lies entirely in the open ball $B^e_\mu(p)$ and thus so does $\gamma_{|[0,s_0+\delta]}$, for some small $\delta > 0$, this stands in contradiction to the definition of s_0, so $s_0 = r_1$. This completes the proof, since then for any $v \in \overline{B^e_{r_1}(0)}$, it holds that $\exp^{g_\varepsilon}_p(v) \in \gamma([0, r_1])$ for a path γ as above.

□

Next we want to find a common domain such that all $\exp^{g_\varepsilon}_p$ for small enough ε are local diffeomorphisms. In preparation for this we need a result on Jacobi fields along g_ε-geodesics. Remember that a vectorfield J along a geodesic γ is called Jacobi field if

$$\nabla_{\gamma'}\nabla_{\gamma'}J(t) = -R(J(t), \gamma'(t))\gamma'(t). \tag{4.4}$$

4.1.6 Lemma. With the constants $r_1, k_1, k_2, \varepsilon_0$ from above, set $c_1 = 2k_2$, $c_2 = 4k_1$ and choose

$$r_2 < \min\left(r_1, \frac{1}{c_1}\log\left(\frac{c_1 + c_2}{c_1/2 + c_2}\right), \frac{1}{2 + c_1}\right).$$

Then for every $\varepsilon < \varepsilon_0$, any g_ε-geodesic $\gamma : [0, r_2] \to M$ with $\gamma(0) = p$ and $\|\gamma'(0)\|_e = 1$, lies entirely in $B^e_\mu(p)$.
Further if J is a g_ε-Jacobi field along γ with $J(0) = 0$ and $\|\nabla^\varepsilon_{\gamma'}J(0)\|_e = 1$, then $\|J(s)\|_e \leq 1$ and $\frac{1}{2} \leq \|\nabla^\varepsilon_{\gamma'}J(s)\|_e \leq 2$ for all $s \in [0, r_2]$.

Proof. The previous lemma implies that γ lies in $B^e_\mu(p)$, since $r_2 < r_1$, also (4.3) implies

$$\max_{s\in[0,r_2]} \|\gamma'(s)\|_e \leq 2. \tag{4.5}$$

Define $s_0 := \sup\{s \in [0, r_2] \,|\, \|J(t)\|_e \leq 1, \forall t \in [0, s]\}$ and assume $s_0 < r_2$. We want to contradict this assumption and therefore prove $s_0 = r_2$, which implies $\|J(t)\|_e \leq 1$ for all $t \in [0, r_2]$.
Since J is a Jacobi field, (4.4) holds, thus by 4.1.2 (ii) and (4.5), we obtain on $[0, s_0]$

$$\left|\frac{d}{ds}\langle\nabla^\varepsilon_{\gamma'}J, \nabla^\varepsilon_{\gamma'}J\rangle_e\right| = 2\left|\langle\nabla^e_{\gamma'}\nabla^\varepsilon_{\gamma'}J, \nabla^\varepsilon_{\gamma'}J\rangle_e\right|$$

$$= 2\Big|\langle \underbrace{\nabla^\varepsilon_{\gamma'}\nabla^\varepsilon_{\gamma'}J(s)}_{=-R_\varepsilon(J(s),\gamma'(s))\gamma'(s)}, \nabla^\varepsilon_{\gamma'}J\rangle_e - \langle \Gamma_{g_\varepsilon}(\nabla^\varepsilon_{\gamma'}J(s),\gamma'(s)), \nabla^\varepsilon_{\gamma'}J(s)\rangle_e\Big|$$

$$\le 2\|R_\varepsilon(J(s),\gamma'(s))\gamma'(s)\|_e\|\nabla^\varepsilon_{\gamma'}J(s)\|_e + 2\|\Gamma_{g_\varepsilon}(\nabla^\varepsilon_{\gamma'}J(s),\gamma'(s))\|_e\|\nabla^\varepsilon_{\gamma'}J(s)\|_e$$

$$\le 2\|R_\varepsilon\|_e \underbrace{\|J(s)\|_e}_{\le 1}(\underbrace{\|\gamma'(s)\|_e}_{\le 2})^2\|\nabla^\varepsilon_{\gamma'}J(s)\|_e + 2\|\Gamma_{g_\varepsilon}\|_e\|\nabla^\varepsilon_{\gamma'}J(s)\|_e^2\|\gamma'(s)\|_e$$

$$\underset{\|R_\varepsilon\|_e\le k_1}{\overset{\|\Gamma_{g_\varepsilon}\|_e\le k_2}{\le}} 8k_1\|\nabla^\varepsilon_{\gamma'(s)}J(s)\|_e + 4k_2\|\nabla^\varepsilon_{\gamma'(s)}J(s)\|_e^2,$$

where again we used the same argument as in the proof of 4.1.5, to substitute the Euclidean induced covariant derivative with the one from g_ε. Again by the same arguments as in 4.1.5, we obtain

$$\left|\frac{d}{ds}\|\nabla^\varepsilon_{\gamma'}J(s)\|_e\right| \le 4k_1 + 2k_2\|\nabla^\varepsilon_{\gamma'}J(s)\|_e = c_1\|\nabla^\varepsilon_{\gamma'}J(s)\|_e + c_2. \tag{4.6}$$

The assumption $\|\nabla^\varepsilon_{\gamma'}J(0)\|_e = 1$, (4.6) and Gronwall's inequailty yield

$$-\frac{c_2}{c_1} + \left(1+\frac{c_2}{c_1}\right)e^{-c_1 s} \le \|\nabla^\varepsilon_{\gamma'}J(s)\|_e \le -\frac{c_2}{c_1} + \left(1+\frac{c_2}{c_1}\right)e^{c_1 s}.$$

The choice of r_2 now implies

$$\frac{1}{2} \le \|\nabla^\varepsilon_{\gamma'}J(s)\|_e \le 2, \tag{4.7}$$

for $s \in [0, s_0]$ and therefore

$$\left|\frac{d}{ds}\|J(s)\|_e\right| = \frac{1}{2}\left|\frac{1}{\|J(s)\|_e}\frac{d}{ds}\langle J(s), J(s)\rangle_e\right| = \frac{1}{\|J(s)\|_e}\left|\langle \nabla^e_{\gamma'}J(s), J(s)\rangle_e\right|$$

$$= \frac{1}{\|J(s)\|_e}\left|\langle \nabla^\varepsilon_{\gamma'}J(s), J(s)\rangle_e - \langle \Gamma_{g_\varepsilon}(J(s),\gamma'(s)), J(s)\rangle_e\right|$$

$$\le 2 + 2k_2,$$

by 4.7, $\|J(s)\|_e \le 1$, $\|\gamma'(s)\|_e \le 2$ and $\|\Gamma_{g_\varepsilon}\|_e \le k_2$ for $s \in [0, s_0]$. By this we obtain

$$\|J(s)\|_e \le (2 + 2k_2)s < \frac{s}{r_2} < 1, \tag{4.8}$$

for all $s \in [0, s_0]$, which is a contradiction for $s = s_0$, by choice of s_0, since we can then find a number $s_1 > s_0$ such that $\|J(s)\|_e \le 1$. □

4.1.7 Lemma. There exists $0 < r_3 < r_2$ such that for all $\varepsilon < \varepsilon_0$, $\exp_p^{g_\varepsilon}$ is a local diffeomorphism on $B^e_{r_3}(0) \subseteq T_pM$.

Proof. Let J be a Jacobi field as is 4.1.6, then

$$\frac{d}{ds}\langle \nabla^\varepsilon_{\gamma'}J(s), J(s)\rangle_e = \underbrace{\langle \nabla^\varepsilon_{\gamma'}\nabla^\varepsilon_{\gamma'}J(s), J(s)\rangle_e}_{=:A} - \underbrace{\langle \Gamma_{g_\varepsilon}(\nabla^\varepsilon_{\gamma'}J(s),\gamma'(s)), J(s)\rangle_e}_{=:B}$$

$$+ \underbrace{\langle \nabla^\varepsilon_{\gamma'}J(s), \nabla^\varepsilon_{\gamma'}J(s)\rangle_e}_{=:C} - \underbrace{\langle \nabla^\varepsilon_{\gamma'}J(s), \Gamma_{g_\varepsilon}(J(s),\gamma'(s))\rangle_e}_{=:D}.$$

We want to control this expression from above and below, which will give us control over the derivative of $\exp_p^{g_\varepsilon}$. Note that $C \geq \frac{1}{4}$ by (4.7). Using 4.1.6, (4.5), (4.7), (4.8) and the definition of a Jacobi field, we get

$$|A| = \left|\langle \nabla^\varepsilon_{\gamma'} \nabla^\varepsilon_{\gamma'} J(s), J(s)\rangle_e\right| = |\langle R_\varepsilon(J(s), \gamma'(s))\gamma'(s), J(s)\rangle_e|$$

$$\leq k_1 \|\gamma'(s)\|_e^2 \|J(s)\|_e^2 \leq 4k_1 \frac{s^2}{r_2^2}.$$

Further

$$|B| = \left|\langle \Gamma_{g_\varepsilon}(\nabla^\varepsilon_{\gamma'} J(s), \gamma'(s), J(s)\rangle_e\right| \leq k_2 \|\gamma'(s)\|_e \|\nabla^\varepsilon_{\gamma'} J(s)\|_e \|J(s)\|_e \leq 4k_2 \frac{s}{r_2},$$

and

$$|D| = \left|\langle \nabla^\varepsilon_{\gamma'} J(s), \Gamma_{g_\varepsilon}(J(s), \gamma'(s))\rangle_e\right| \leq k_2 \|\nabla^\varepsilon_{\gamma'} J(s)\|_e \|J(s)\|_e \|\gamma'(s)\|_e \leq 4k_2 \frac{s}{r_2}.$$

Thus for $r_3 = r_3(r_2, k_1, k_2) < r_2$ small enough, we can control A, B, D and since $C \geq \frac{1}{4}$, it holds that $\frac{d}{ds}\langle \nabla^\varepsilon_{\gamma'} J(s), J(s)\rangle_e$ is positive and bounded from below on $[0, r_3]$. Since (4.7) also provides an upper bound on C, we also obtain that $\frac{d}{ds}\langle \nabla^\varepsilon_{\gamma'} J(s), J(s)\rangle_e$ is bounded from above. We can thus find $c_1 > 0$ such that for all $\varepsilon < \varepsilon_0$ and $s \in [0, r_3]$, we have

$$e^{-c_1} \leq \frac{d}{ds}\langle \nabla^\varepsilon_{\gamma'} J(s), J(s)\rangle_e \leq e^{c_1},$$

implying

$$0 < e^{-c_1} s \leq \langle \nabla^\varepsilon_{\gamma'} J(s), J(s)\rangle_e \leq e^{c_1} s. \tag{4.9}$$

Since $\langle \nabla^\varepsilon_{\gamma'} J(s), J(s)\rangle_e > 0$ we can use the Cauchy-Schwartz inequality to obtain

$$\langle \nabla^\varepsilon_{\gamma'} J(s), J(s)\rangle_e = \left|\langle \nabla^\varepsilon_{\gamma'} J(s), J(s)\rangle_e\right| \leq \|\nabla^\varepsilon_{\gamma'} J(s)\|_e \|J(s)\|_e, \tag{4.10}$$

and using (4.7) we get

$$\frac{s}{r_2} \overset{(4.8)}{\geq} \|J(s)\|_e \overset{(4.10)}{\geq} \frac{\langle \nabla^\varepsilon_{\gamma'} J(s), J(s)\rangle_e}{\|\nabla^\varepsilon_{\gamma'} J(s)\|_e} \overset{(4.7),(4.9)}{\geq} \frac{e^{-c_1}}{2} s.$$

Using this, we can find $c_2 > 0$ such that for $\varepsilon < \varepsilon_0$ and $s \in [0, r_3]$,

$$e^{-c_2} s \leq \|J(s)\|_e \leq e^{c_2} s.$$

Note that any Jacobi field as above, has to be of the form $J(s) = T_{s\gamma'(0)} \exp_p^{g_\varepsilon}(sw)$, for some $w \in T_pM$, $\|w\|_e = 1$, see e.g. [10], Proposition 8.6, p.217, so that for $s \in [0, r_3]$

$$e^{-c_2} \leq \|T_{s\gamma'(0)} \exp_p^{g_\varepsilon}(w)\|_e \leq e^{c_2}.$$

Since we assumed $\|\gamma'(0)\|_e = 1$, we can conclude for all $\varepsilon < \varepsilon_0$, for all $v \in B^e_{r_3}(0)$ and all $w \in T_pM$:

$$e^{-c_2}\|w\|_e \leq \|T_v \exp_p^{g_\varepsilon}(w)\|_e \leq e^{c_2}\|w\|_e. \tag{4.11}$$

This implies that $\exp_p^{g_\varepsilon}$ is a local diffeomorphism on $B^e_{r_3}(0)$ $\square$

In order to show injectivity of $\exp_p^{g_\varepsilon}$, we need to further shrink the radius of the ball that is our common domain of the exponential maps. The following Lemma shows that a ball in T_pM is „dilated“ by the exponential map by a factor less than e^{c_2}, when comparing its radius with a Euclidean ball in M.

4.1.8 Lemma. For all constants from above, we have for $r_4 < e^{-c_2} r_3$, $r_5 < e^{-c_2} r_4$ and $\tilde{r} := e^{c_2} r_4$ and all $\varepsilon < \varepsilon_0$, that

$$\exp_p^{g_\varepsilon}\left(\overline{B^e_{r_5}(0)}\right) \subseteq B^e_{r_4}(p) \subseteq \exp_p^{g_\varepsilon}\left(\overline{B^e_{\tilde{r}}(0)}\right) \subseteq \exp_p^{g_\varepsilon}\left(B^e_{r_3}(0)\right). \tag{4.12}$$

Proof. Note that the last inclusion holds by definition of $\tilde{r}$ and r_4. Take $q \in B^e_{r_4}(p)$ and $\alpha : [0, a] \to M$ a piecewise smooth path from p to q (i.e. $\alpha(0) = p$, $\alpha(a) = q$) of Euclidean length less that r_4. Since $\exp_p^{g_\varepsilon}$ is a local diffeomorphism on $B^e_{r_3}(0)$, for $b > 0$ small enough, there is a unique $\exp_p^{g_\varepsilon}$-lift $\tilde{\alpha} : [0, b] \to B^e_{r_3}(0)$ of $\alpha_{|[0,b]}$ starting at 0. Set $a' := \sup\{b < a \,|\, \tilde{\alpha} \text{ exists on } [0, b]\}$. We claim $a' = a$. Assume $a' < a$ and note that (4.11) implies

$$e^{-2c_2} g_e \leq (\exp_p^{g_\varepsilon})^* g_e \leq e^{2c_2} g_e, \tag{4.13}$$

locally on $B^e_{r_3}(0)$ for $\varepsilon < \varepsilon_0$, where g_e denotes the Euclidean metric and $(.)^*$ denotes the pullback map. Using this we have

$$L_{(\exp_p^{g_\varepsilon})^* g_e}(\tilde{\alpha}_{|[0,a')}) = L_e(\alpha_{|[0,a')}) = \int_0^{a'} \|\alpha'(t)\|_e \, dt \leq r_4,$$

leading to

$$\begin{aligned} L_e(\tilde{\alpha}_{|[0,a')}) &= \int_0^{a'} \left(g_e(\tilde{\alpha}'(t), \tilde{\alpha}'(t)) \, dt\right)^{\frac{1}{2}} \\ &\overset{(4.13)}{\leq} e^{c_2} \int_0^{a'} \left(((\exp_p^{g_\varepsilon})^* g_e)(\tilde{\alpha}'(t), \tilde{\alpha}'(t)) \, dt\right)^{\frac{1}{2}} \\ &= e^{c_2} L_{(\exp_p^{g_\varepsilon})^* g_e}(\tilde{\alpha}_{|[0,a')}) \leq e^{c_2} r_4 = \tilde{r}. \end{aligned}$$

Let us choose a sequence $(a_n) \subseteq [0, a')$ such that $a_n \nearrow a'$, then $\tilde{\alpha}(a_n) \in \overline{B^e_{\tilde{r}}(0)}$. By compactness there exists a subsequence $(\tilde{\alpha}(a_{n_k}))_k$ converging to some point $v \in \overline{B^e_{\tilde{r}}(0)}$. $\exp_p^{g_\varepsilon}$ is a diffeomorphism on some neighbourhood of v by 4.1.7, since $\tilde{r} < r_3$ and by definition of $\tilde{\alpha}$ as a $\exp_p^{g_\varepsilon}$-lift, we have

$$\exp_p^{g_\varepsilon}(v) = \lim_{k \to \infty} \alpha(a_{n_k}) = \alpha(a').$$

This shows that $\tilde{\alpha}$ can be extended past a', contradicting the choice of a'. We have thus shown $a = a'$ and hence also $q = \exp_p^{g_\varepsilon}(\tilde{\alpha}(a)) \in \exp_p^{g_\varepsilon}\left(\overline{B^e_{\tilde{r}}(0)}\right)$, implying the second inclusion.
The see the first inclusion, take $v \in T_pM$ with $\|v\|_e \leq r_5 < r_3$ and set $q := \exp_p^{g_\varepsilon}(v)$. For the radial geodesic $\gamma : [0, 1] \to M$, $t \mapsto \exp_p^{g_\varepsilon}(tv)$ from p to q, we obtain by (4.11)

$$L_e(\gamma_{|[0,s]}) = \int_0^s \|T_{tv} \exp_p^{g_\varepsilon}(v)\|_e \, dt \leq e^{c_2} \|v\|_e \leq e^{c_2} r_5 < r_4.$$

This implies $\sup\{s \in [0, 1] \,|\, \gamma_{|[0,s]} \subseteq B^e_{r_4}(p)\} = 1$ and thus $\gamma(1) = q \in B^e_{r_4}(p)$ and so $\exp_p^{g_\varepsilon}\left(\overline{B^e_{r_5}(0)}\right) \subseteq B^e_{r_4}(p)$.

□

Since in our situation $\exp_p^{g_\varepsilon} : \overline{B^e_{\hat r}(0)} \to \exp_p^{g_\varepsilon}(\overline{B^e_{\hat r}(0)})$, is a surjective local homeomorphism between compact Hausdorff spaces, it is a covering map. This leads to

4.1.9 Lemma. Let $\varepsilon < \varepsilon_0$, then $\exp_p^{g_\varepsilon}$ is a diffeomorphism on $B^e_{r_5}(0)$ onto its image.

Proof. We only have to show that $\exp_p^{g_\varepsilon}$ is injective, hence bijective onto its image. Indeed in this case the inverse exists and since $\exp_p^{g_\varepsilon}$ is a local diffeomorphism on $B^e_{r_3}(0) \supseteq B^e_{r_5}(0)$, it is smooth.
We argue by contradiction, suppose there exist $v_0, v_1 \in B^e_{r_5}(0)$, $v_0 \neq v_1$ and $\varepsilon < \varepsilon_0$, such that $\exp_p^{g_\varepsilon}(v_0) = \exp_p^{g_\varepsilon}(v_1) =: q$. Then we obtain two different geodesics $\gamma_i(t) := \exp_p^{g_\varepsilon}(tv_i)$, $i \in \{0,1\}$, both starting at p with $\gamma_0(1) = \exp_p^{g_\varepsilon}(v_0) = q = \exp_p^{g_\varepsilon}(v_1) = \gamma_1(1)$. The map $(t,s) \mapsto \gamma_s(t) := s\gamma_1(t) + (1-s)\gamma_0(t)$ is a homotopy between γ_0 and γ_1 fixing the endpoints. Further $\gamma_s(t) \in B^e_{r_4}(p)$ for all $t, s \in [0,1]$, since $\exp_p^{g_\varepsilon}\left(\overline{B^e_{r_5}(0)}\right) \subseteq B^e_{r_4}(p)$ by (4.12). Using $B^e_{r_4}(p) \subseteq \exp_p^{g_\varepsilon}\left(\overline{B^e_{\hat r}(0)}\right)$ and the fact that $\exp_p^{g_\varepsilon}$ is a covering map on $\overline{B^e_{\hat r}(0)}$, we obtain a lift of the homotopy to $\overline{B^e_{\hat r}(0)}$. Since $\gamma_i(t) = \exp_p^{g_\varepsilon}(tv_i)$ the lift of γ_i has to be the map $t \mapsto tv_i$ for $i = 0, 1$. These two paths are, however, not homotopic with fixed endpoints, contradicting the assumption. $\square$

Note that (4.11) implies the existence of a uniform Lipschitz constant $c_3 > 0$ for all $\exp_p^{g_\varepsilon}$ with $\varepsilon < \varepsilon_0$ on $B^e_{r_5}(0)$, i.e. for all $u, v \in B^e_{r_5}(0)$ and all $\varepsilon < \varepsilon_0$ we have

$$\|\exp_p^{g_\varepsilon}(u) - \exp_p^{g_\varepsilon}(v)\|_e \leq c_3 \|u - v\|_e. \tag{4.14}$$

In order to obtain a lower bound on the above expression, we use the following lemma proven in [22], 3.2.47 .

4.1.10 Lemma. Let $\Omega \subseteq \mathbb{R}^n$, $\Omega' \subseteq \mathbb{R}^m$ be open, $f \in C^1(\Omega, \Omega')$ and $K \subset\subset \Omega$. There exists $C > 0$ such that $\|f(x) - f(y)\|_{\mathbb{R}^m} \leq C\|x - y\|_{\mathbb{R}^n}$ for all $x, y \in K$. Further C can be chosen as

$$C = C_1 \sup_{x \in L} \{\|f(x)\|_{\mathbb{R}^m} + \|Df(x)\|_{\mathbb{R}^{mn}}\},$$

for any fixed compact neighbourhood L of K in Ω, where C_1 only depends on L.

Let us now state and prove the central theorem of this section.

4.1.11 Theorem. Let M be a smooth manifold equipped with a $C^{1,1}$ semi-Riemannian metric g. For $p \in M$ there exist open neighbourhoods U of $0 \in T_pM$ and V of $p \in M$ such that $\exp_p^{g_\varepsilon} : U \to V$ is a bi-Lipschitz homeomorphism.

Proof. Because the statement is local we may assume $M = \mathbb{R}^n$. Using all constants from previous results in this section, we can choose by 4.1.8 constants r_6, r_7 and $\hat r$ such that $r_7 < r_6 := e^{-c_2}\hat r < \hat r < r_5$ and such that for $\varepsilon < \varepsilon_0$, we have

$$\exp_p^{g_\varepsilon}\left(\overline{B^e_{r_7}(0)}\right) \subseteq B^e_{r_6}(p) \subseteq \exp_p^{g_\varepsilon}\left(\overline{B^e_{\hat r}(0)}\right) \subset\subset \exp_p^{g_\varepsilon}\left(B^e_{r_5}(0)\right).$$

Again (4.11) implies for $\varepsilon < \varepsilon_0$ that

$$e^{-c_2}\|\xi\|_e \leq \|T_q\left(\exp_p^{g_\varepsilon}\right)^{-1}(\xi)\|_e \leq e^{c_2}\|\xi\|_e,$$

for all $q \in \overline{B^e_{r_6}(p)}$ and all $\xi \in T_qM$, since by 4.1.9 $\exp_p^{g_\varepsilon}$ is a diffeomorphism on $\overline{B^e_{r_5}(0)}$. Lemma 4.1.10 with $K = \exp_p^{g_\varepsilon}\left(\overline{B^e_{r_7}(0)}\right)$ implies the existence of some $c_4 > 0$ such that

$$\|\left(\exp_p^{g_\varepsilon}\right)^{-1}(q_1) - \left(\exp_p^{g_\varepsilon}\right)^{-1}(q_2)\|_e \leq c_4^{-1}\|q_1 - q_2\|_e,$$

for all $\varepsilon < \varepsilon_0$ and all $q_1, q_2 \in \exp_p^{g_\varepsilon}\left(B^e_{r_7}(0)\right)$. Using the above and (4.14), we have for all $\varepsilon < \varepsilon_0$ and all $u, v \in B^e_{r_7}(0)$, that

$$c_4\|u - v\|_e \leq \|\exp_p^{g_\varepsilon}(u) - \exp_p^{g_\varepsilon}(v)\|_e \leq c_3\|u - v\|_e.$$

Letting $\varepsilon \to 0$ we obtain for all $u, v \in B^e_{r_7}(0)$, that

$$c_4\|u - v\|_e \leq \|\exp_p^{g}(u) - \exp_p^{g}(v)\|_e \leq c_3\|u - v\|_e.$$

Thus $\exp_p^g$ is a bi-Lipschitz homeomorphism on $U := B^e_{r_7}(0)$. By the invariance of domain theorem $V := \exp_p^g(U)$ is open. □

4.2 Strong Differentiability of the Exponential Map

Around the same time the above theorem was proved in [17], an alternative proof was presented by Minguzzi in [23]. We will now give a short overview on the arguments and theorems used there. In [23] the author proves a more general result about the exponential map for locally Lipschitz sprays on $C^{2,1}$ manifolds (i.e. charts are twice continuously differentiable and the second derivative is locally Lipschitz). The proof of 4.1.11 follows as a special case of [23], Theorem 1.3, p.579, since for $C^{1,1}$ Semi-Riemannian metrics the exponential map for sprays is the exponential map in Semi-Riemannian geometry. The proof of said theorem involves the notion of strong differentiability and a version of the inverse function theorem tailored to strongly differentiable functions, which is similar to Clarke's inverse function theorem for Lipschitz function, see [24] respectively [25].
The theorem also proves the bi-Lipschitz property for the (global) exponential map on some open neighbourhood of the zero section in the tangent bundle of the manifold onto an open neighbourhood of the diagonal in $M \times M$. Let us recall some further details: $\exp : \Omega \subseteq TM \to M \times M$, is defined via $(p, v_p) =: v_p \mapsto (p, \exp_p(v_p)) = (\pi(v_p), \exp_{\pi(v_p)}(v_p))$ on the subset $\Omega := \{v_p \in TM \,|$ the unique geodesic c_{v_p} with $c_{v_p}(0) = p, c'_{v_p}(0) = v_p$ is defined at least on $[0,1]\}$. Let us now introduce the, to the proof essential, notion of strong differentiability of a map.

4.2.1 Definition. Let E, F be Banach spaces and $f : O \subseteq E \to F$, $O \subseteq E$ open. f is called strongly differentiable at $p \in O$, if there exists a bounded linear map $L : E \to F$ such that $\forall \varepsilon > 0\ \exists \delta > 0$ and for $q_1, q_2 \in O$ with $\|q_1 - p\|_E$, $\|q_2 - p\|_E \leq \delta$, it holds that

$$\|f(q_1) - f(q_2) - L(q_1 - q_2)\|_F \leq \varepsilon\|q_1 - q_2\|_E.$$

In this case L is called the strong differential of f at p. If f is strongly differentiable for all $p \in O$, then it is called strongly differentiable on O.

Let us observe some immediate resp. easily shown consequences

4.2.2 Remark. Observe that if the strong differential of a map exists at a point, it is unique and that this map is also Fréchet differentiable at that point. If the Banach spaces in Definition 4.2.1 are finite dimensional, all norms on these spaces are equivalent and the (existence of the) strong differential is independent of the chosen norms $\|.\|_E$, $\|.\|_F$. Let us list, without proof, some useful properties concerning strong differentiability

(i) If f is strongly differentiable at p, then f is Lipschitz continuous on a neighbourhood of p.

(ii) If f is differentiable in a neighbourhood of p and its derivative is continuous at p, then f is strongly differentiable at p.

(iii) Compositions of strongly differentiable maps are strongly differentiable.

(iv) Mixed partial derivatives (obtained by keeping all but one argument constant and then taking the one dimensional strong derivative) coincide whenever they exist, i.e. $\partial_i \partial_j f = \partial_j \partial_i f$. (Where we use the standard notation for derivatives to denote the strong derivative).

(v) If f is strongly differentiable on a subset $A \subseteq E$, then its strong differential is continuous on A w.r.t. the subspace topology.

(vi) By (ii) and (v) we have: f is C^1 on an open subset $O \subseteq E$ if and only if f is strongly differentiable on O.

The definition of strong differentiability has the remarkable property that strong differentiability at a point already implies certain properties for the function in a neighbourhood of that point. Differentiability at a point is a local property, i.e. depends on values in a neighbourhood of that point, but strong differentiability at p forces a function to behave „nicely" in a neighbourhood of p, see (i).

By the above remark strong differentiability on an open set is equivalent to the C^1 property, hence we cannot expect to prove strong differentiability of the exponential map on a neighbourhood of 0 resp. the zero section. However showing the existence of a strong derivative at 0 and its invertiability will supply us with a sufficient condition to deduce existence and Lipschitz continuity of a local inverse. In the smooth setting the result that the exponential map is a diffeomorphism on an open neighbourhood of 0 relies on the inverse function theorem, therefore taking into account 4.2.2 (i), Clarke's inverse function theorem for Lipschitz functions suggests itself. Let us state now the version we will use (see [23], Theorem 1.2.2 p. 578), credited to Leach [25].

4.2.3 Theorem. Let $f : O \to \mathbb{R}^n$ be strongly differentiable at $p \in O$, where $O \subseteq \mathbb{R}^n$ is open and let $L : \mathbb{R}^n \to \mathbb{R}^n$ be its strong differential at p. If L is invertible, then there exist open neighbourhoods N_1 of p and N_2 of $f(p)$ and a function $g : N_2 \to \mathbb{R}^n$, such that $f(N_1) = N_2$, $g(N_2) = N_1$ and $f_{|N_1} \circ g = g \circ f_{|N_1} = id$.
Further both f and g are Lipschitz continuous and g is strongly differentiable at $f(p)$ with strong derivative L^{-1}. Moreover f is (strongly) differentiable at $q \in N_1$ if and only if g is (strongly) differentiable at $f(q)$. In this case the (strong) derivatives are invertible.

We will use this theorem without proof, the interested reader is referred to [24] and [25]. Our plan is to prove the bi-Lipschitz property of the exponential map as follows: By the classical Picard-Lindelöf theorem the first order system with locally Lipschitz continuous coefficients (4.1) has a unique local solution around 0 and the solution exhibits Lipschitz dependence on the initial data, i.e. exp and $\exp_p$ are Lipschitz continuous on a neighbourhood of the zero section resp. of 0. We then aim to prove strong differentiability and invertability of the strong derivative in order to use 4.2.3 to conclude that it has a Lipschitz inverse on a suitable neighbourhood of the diagonal in $M \times M$ resp. of p. In [23] the following theorem is proven to a full extent including also the local existence, uniqueness and Lipschitz dependence of the solutions as in the Picard-Lindelöf theorem using Picard-Iteration. We will however only prove the strong differentiability of exp here, as the rest of the statements are as in the classical case and a proof can be found in [23], Chapter 2.1-2.2. Notation and all constants will however be carried over from the proof in [23], chapter 2. Let us also note that by [23], Theorem 1.2.1, the maximal domain Ω_p of the pointed exponential map is open in T_pM.

4.2.4 Theorem. Let M be a smooth manifold with a $C^{1,1}$ semi-Riemannian metric.

(i) For all $p \in M$ the map $\exp_p : \Omega_p \subseteq T_pM \to M$ is locally Lipschitz continuous and strongly differentiable at 0. In particular $\exp_p$ is a bi-Lipschitz homoemorphism from a star shaped open neighbourhood of 0 in T_pM to an open neighbourhood of p in M.

(ii) There exists and open set $\Omega \subseteq TM$ such that $\exp : \Omega \to M \times M$ is locally Lipschitz continuous. Further exp is strongly differentiable at all points 0_p of the zero section in TM. In particular exp is a bi-Lipschitz homeomorphism from an open neighbourhood of the zero section onto an open neighbourhood of the diagonal in $M \times M$.

Proof. As before we may assume for $p \in M$ and a chart neighbourhood U of p, that U is an open set in $\mathbb{R}^n$ containing the closed ball of radius r around 0, for some $r > 0$ and $p = 0$. Following [23] we introduce the spray $H^k(x, v) := -\Gamma^k_{ij}(x)v_i v_j$, the geodesic equation then becomes

$$\begin{aligned} \frac{dx^i}{dt} &= v^i \\ \frac{dv^i}{dt} &= H^i(x, v). \end{aligned} \tag{4.15}$$

The function H is locally Lipschitz and homogeneous of second degree in v. In particular there are constants $\alpha, \beta > 0$ such that

$$\|H(x_2, v_2) - H(x_1, v_1)\|_e \leq \alpha\|x_2 - x_1\|_e + \beta\|v_2 - v_1\|_e. \tag{4.16}$$

on the set $\overline{B^e_r(0)} \times \{v \mid \|v\|_e \leq 1\}$. We set

$$M := \sup_{x \in \overline{B^e_r(0)}} \sup_{\|v\|_e = 1} \|H(x, v)\|_e.$$

We will derive estimates on solutions of (4.15) with different initial conditions, based on the Picard-Iteration, which will then help us derive strong differentiability of exp.
Let us consider small enough initial conditions, namely for $\delta > 0$ such that

$$\delta < \frac{1}{M}\left(1 - e^{-Mr/2}\right) \le \frac{r}{2},$$

and

$$\frac{\delta}{1-\delta M} \le 1, \quad \frac{\beta\delta}{2(1-\delta M)}\left(1 + \sqrt{1 + 4\frac{\alpha}{\beta^2}}\right) \le 1,$$

choose x_0, v_0 such that $\max\{\|x_0\|_e, \|v_0\|_e\} < \delta$. Using Picard-Iteration one defines the following a sequences of functions for initial conditions x_0, v_0

$$\begin{aligned} x_0^i(t) &\equiv x_0^i \\ v_0^i(t) &\equiv v_0^i \\ x_{k+1}^i(t) &= x_0^i + \int_0^t v_k^i(s)\, ds \\ v_{k+1}^i(t) &= v_0^i + \int_0^t H^i(x_k(s), v_k(s))\, ds. \end{aligned} \tag{4.17}$$

It can be shown that these sequences converge uniformly to the solution $x(t)$ of (4.15), i.e. $x^i \to x$ and $v^i \to x'$. Further due to our choice of δ and the above definition of x_k, by induction we obtain $\|x_k(t)\|_e < r$ and $\|x_k'(t)\|_e = \|v_k(t)\|_e < \frac{\delta}{1-\delta M}$, see [23], p.598.
Further for any $k > \max\{\|v_1\|_e, \|v_2\|_e\}$ due to homogeneity of H in the second slot, (4.16) implies

$$\begin{aligned} \|H(x_2, v_2) - H(x_1, v_1)\|_e &= k^2\|H(x_2, \frac{v_2}{k}) - H(x_1, \frac{v_1}{k})\|_e \\ \le k^2\left(\alpha\|x_2 - x_1\|_e + \frac{\beta}{k}\|v_2 - v_1\|_e\right) &= \alpha k^2\|x_2 - x_1\|_e + \beta k\|v_2 - v_1\|_e. \end{aligned} \tag{4.18}$$

Let us introduce the constants $A := \left(\frac{\delta}{1-\delta M}\right)^2\alpha$ and $B := \frac{\beta\delta}{1-\delta M}$. We can find $D > 0$ such that

$$\frac{A}{D} + B = D, \tag{4.19}$$

by setting

$$D = D(\delta) = \frac{1}{2}\left(B + \sqrt{B^2 + 4A}\right) = \frac{\beta\delta}{2(1-\delta M)}\left(1 + \sqrt{1 + \frac{4\alpha}{\beta^2}}\right). \tag{4.20}$$

Now our choice of δ becomes clearer, as it implies $D \le 1$. Let $x(t), y(t)$ be solutions to 4.15 with initial conditions (x_0, v_0) resp. (y_0, w_0) such that we have $\max\{\|x_0\|_e, \|v_0\|_e\} < \delta$ and $\max\{\|y_0\|_e, \|w_0\|_e\} < \delta$. We want to establish a growth estimate of these solutions w.r.t. the initial conditions using properties of the approximating functions. As above define the approximating functions x_k^i, y_k^i. We can reformulate (4.17) for x and y to obtain

$$x_{k+1}^i(t) - y_{k+1}^i(t) - (x_0^i - y_0^i) - (v_0^i - w_0^i)t = \int_0^t \left(v_k^i(s) - w_k^i(s) - (v_0^i - w_0^i)\right)\, ds,$$

and

$$v_{k+1}^i - w_{k+1}^i - (v_0^i - w_0^i) = \int_0^t \left(H^i(x_k(s), v_k(s)) - H^i(y_k(s), w_k(s))\right) ds,$$

for all $i = 1, \ldots, n$. Taking norms we obtain

$$\begin{aligned} &\|x_{k+1}(t) - y_{k+1}(t) - (x_0 - y_0) - (v_0 - w_0)t\|_e \\ &\leq \int_0^t \|v_k(s) - w_k(s) - (v_0 - w_0)\|_e \, ds, \end{aligned}$$

and since $\frac{\delta}{1-\delta M} > \max\{v_k(s), w_k(s)\}$ for all k, (4.18) is applicable, yielding

$$\|v_{k+1}(t) - w_{k+1}(t) - (v_0 - w_0)\|_e \leq \int_0^t \|H(x_k(s), v_k(s)) - H(y_k(s), w_k(s))\|_e \, ds.$$

$$\leq \int_0^t \left(A\|x_k(s) - y_k(s)\|_e + B\|v_k(s) - w_k(s)\|_e\right) ds$$

$$\begin{aligned} \leq \int_0^t & A\left(\|x_k(s) - y_k(s) - (x_0 - y_0) - (v_0 - w_0)s\|_e + \|x_0 - y_0\|_e + \|v_0 - w_0\|_e s\right) \\ &+ B\left(\|v_k(s) - w_k(s) - (v_0 - w_0)\|_e + \|v_0 - w_0\|_e\right) ds. \end{aligned} \tag{4.21}$$

We now claim that

$$\begin{aligned} &\|x_k(t) - y_k(t) - (x_0 - y_0) - (v_0 - w_0)t\|_e \\ &\leq \max\{D\|x_0 - y_0\|_e, \|v_0 - w_0\|_e\} \left(\frac{e^{Dt} - 1}{D} - t\right), \end{aligned} \tag{4.22}$$

as well as

$$\|v_k(t) - w_k(t) - (v_0 - w_0)\|_e \leq \max\{D\|x_0 - y_0\|_e, \|v_0 - w_0\|_e\} \left(e^{Dt} - 1\right). \tag{4.23}$$

Let us prove this claim by induction. The case $k = 1$ is clear for (4.22), since the left hand side is 0. For (4.23) we get by using (4.18) and (4.19), that

$$\|v_1(t) - w_1(t) - (v_0 - w_0)\|_e \leq \int_0^t \|H(x_0, v_0) - H(y_0, w_0)\|_e \, dt$$

$$\leq \int_0^t A\|x_0 - y_0\|_e + B\|v_0 - w_0\|_e \, dt \leq \max\{D\|x_0 - y_0\|_e, \|v_0 - w_0\|_e\} Dt.$$

Let us assume (4.22) and (4.23) both hold for $0 \leq k \leq m$, then setting $C := \max\{D\|x_0 - y_0\|_e, \|v_0 - w_0\|_e\}$

$$\begin{aligned} &\|x_{m+1}(t) - y_{m+1}(t) - (x_0 - y_0) - t(v_0 - w_0)\|_e \\ &\leq \int_0^t \|v_m(s) - w_m(s) - (v_0 - w_0)\|_e \, ds \\ &\leq C \int_0^t \left(e^{Ds} - 1\right) ds = C\left(\frac{e^{Dt} - 1}{D} - t\right). \end{aligned}$$

For (4.23) we obtain

$$
\begin{aligned}
&\|v_{m+1}(t) - w_{m+1}(t) - (v_0 - w_0)\|_e \\
\overset{(4.21)}{\leq} &\int_0^t \left(AC\left(\frac{e^{Ds}-1}{D} - s\right) + A\|x_0 - y_0\|_e + As\|v_0 - w_0\|_e \right) ds \\
&+ \int_0^t \left(BC(e^{Ds}-1) + B\|v_0 - w_0\|_e \right) ds \\
\leq &\int_0^t \left(C\frac{A}{D}(e^{Ds}-1) - ACs + \frac{A}{D}C + ACs + CB(e^{Ds}-1) + BC \right) ds \\
\overset{(4.19)}{=} &\int_0^t \left(DC(e^{Ds}-1) + DC \right) ds = C\int_0^t De^{Ds}\, ds = C\left(e^{Dt}-1\right),
\end{aligned}
$$

and the claim is proved.
Since the right hand sides of these equations are independent of k, we obtain for the limit $k \to \infty$ (again using that x_k converges uniformly on $[0,1]$ to a solution of (4.15)) the following estimates for the solutions x, y of (4.15), with the respective initial conditions (x_0, v_0) and (y_0, w_0)

$$
\|x(t) - y(t) - (x_0 - y_0) - (v_0 - w_0)t\|_e \leq C\left(\frac{e^{Dt}-1}{D} - t\right). \tag{4.24}
$$

By noting that $v_k \to x'$ and $w_k \to y'$ uniformly on $[0,1]$, we further obtain

$$
\|x'(t) - y'(t) - (v_0 - w_0)\|_e \leq C(e^{Dt}-1).
$$

On $\mathbb{R}^{2n}$ we define $f(x_0, v_0) := (x_0, x(1))$, which equals the coordinate expressing of the exponential map. We claim that its strong derivative exists and is given by

$$
L : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad L = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix},
$$

where I denotes the identity matrix on $\mathbb{R}^n$. Equation (4.24) for $t = 1$ leads to

$$
\begin{aligned}
&\|f(x_0, v_0) - f(y_0, w_0) - L((x_0, v_0) - (y_0, w_0))\|_e \\
&= \|(x_0, x(1)) - (y_0, y(1)) - (x_0 - y_0, x_0 - y_0 + v_0 - w_0)\|_e \\
&= \|(0, x(1) - y(1) - (x_0 - y_0) - (v_0 - w_0))\|_e \leq C\left(\frac{e^D - 1}{D} - 1\right) \\
&\leq \max\{\|x_0 - y_0\|_e, \|v_0 - w_0\|_e\}\left(\frac{e^D - 1}{D} - 1\right),
\end{aligned}
$$

for every $(x_0, v_0), (y_0, w_0)$ with $\max\{\|x_0\|_e, \|v_0\|_e\}, \max\{\|y_0\|_e, \|w_0\|_e\} < \delta$. Note that on $\mathbb{R}^{2n}$, for (x, v) a norm is given by $\|.\|_m := \max\{\|x\|_e, \|v\|_e\}$. By (4.20) we have $\lim_{\delta \to 0} D(\delta) = 0$. This shows that f is strongly differentiable at 0 with strong derivative L, w.r.t. the norm $\|.\|_m$ on $\mathbb{R}^{2n}$. However since the strong derivative is independent of the norm on finite dimensional vector spaces, it is strongly differentiable w.r.t. any norm on $\mathbb{R}^{2n}$. Since for any $p \in M$, there is a chart (ψ, U) around p such that $\psi(p) = 0$, we have that exp is differentiable on all points of the zero section in TM.

Point (ii) now follows from 4.2.3. Fixing the first argument of f the map $v \mapsto f(x, v)$ is still strongly differentiable at 0 with strong derivative I. $\exp_p$ is obtained by composing this map with the strongly differentiable (since smooth) map π_2, the projection onto the second factor. (i) also now follows from 4.2.3. $\square$

4.2.5 Remark. From this result in [23], a version of the Gauss-Lemma is deduced, cf. Theorem 1.3.5. As in the classical case for Riemannian manifolds, this result can be used to show, that a geodesic in a normal neighbourhood N is the unique shortest path in N from the center of the normal neighbourhood to its endpoint in N.
A Lorentzian analog also holds, stating that future-directed causal geodesics are the longest future directed curves from the center of a normal neighbourhood to its endpoints. In [23], this is even done for Finsler and Lorentzian-Finsler structures and absolutely continuous paths. Causality of an absolutely continuous path is defined as usual, but only has to hold for almost every point, i.e. a path γ is causal, if $\gamma'(t)$ is causal for almost every t.

Abstract

In this thesis we deal with Riemannian geometry for metrics with low regularity. Our approach will be to rely on concepts from metric geometry such as Length structures and shortest paths as well as regularization and comparison geometry.
In the first chapter we introduce concepts from metric geometry. We define Length structures and Length spaces and the intrinsic metrics with respect to a Length. Further we deal with the variational length of a metric space. We then move on to prove existence of shortest paths under certain conditions on the metric space. Furthermore we give a definition of geodesics in a metric space and prove some properties as well as a Length space version of the Hopf-Rinov theorem. Lastly in this chapter we investigate absolutely continuous paths in metric spaces and generalize the formula "length equals integral of speed".
The second chapter is concerned with one of the prime examples of Length spaces, namely Riemannian manifolds. With the Riemannian arclength and distance any Riemannian manifold with a smooth metric is turned into a Length space. In this section we will generalize this to manifolds with continuous Riemannian metrics. Further we will compare different Length structures on Riemannian manifolds in order to establish a generalization of the arclength to absolutely continuous paths and to rectifiable paths via the variational Length from chapter 1. This will be done first for smooth metrics and then also for continuous ones relying on regularization of the continuous metric and using the smooth result.
Having established that a Riemannian manifold with continuous metric is a Length space, in the third chapter we compare the definition of geodesics, respectively shortest paths in metric spaces, to the definition of geodesic in the Riemannian sense. We begin with a counterexample by Hartman and Wintner [11], refuting a connection between locally shortest paths to solutions of the geodesic equation for metrics of regulatity $C^{1,\alpha}$, for $0 < \alpha < 1$. We then move on to the case of a C^1 metric, where we show that shortest paths solve the geodesic equation and are of class C^2. Further we investigate a paper by Lytchak and Yaman [14], showing that metric space geodesics for C^α metrics are locally uniformly of regularity $C^{1,\beta}$ for $\beta = \frac{\alpha}{2-\alpha}$.
The fourth chapter is concerned with two different approaches ([17], [18] and [23] to showing that the exponential map of a $C^{1,1}$ metric is a bi-Lipschitz homeomorphism on an open neighbourhood of 0. The first approach will involve regularization of the metric and the use of Jacobi fields to help carry the bi-Lipschitz property through the limit of the regularized metrics. The second approach uses a low regularity version of the Inverse Function Theorem ([24] and [25]) and strong differentiability of the exponential map at 0, to obtain the bi-Lipschitz property. Using this, it is possible to formulate a low regularity version of the Gauss Lemma, to establish that locally, geodesics in Riemannian manifolds, are shortest paths.

Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit Riemannscher Geometrie für Metriken von niedriger Regularität. Unser Zugang beruht auf Methoden der metrischen Geometrie wie Längenstrukturen und kürzesten Wegen sowie Regularisierung und vergleichender Geometrie.
Im ersten Kapitel präsentieren wir Konzepte aus der metrischen Geometrie. Es werden die Begriffe Längenstruktur, Längenraum und intrinsische Metrik definiert, außerdem beschäftigen wir uns mit der Variationslänge einer Kurve. Wir zeigen Existenz von kürzesten Wegen in metrischen Räumen unter gewissen Bedingungen und geben eine Definition von Geodäten in metrischen Räumen. Schließlich beweisen wir eine Längenraumversion des Satzes von Hopf-Rinov. Im letzten Teil dieses Kapitels untersuchen wir absolut stetige Wegen in metrischen Räumen und verallgemeinern die Formel der Länge einer Kurve als Integral der Geschwindigkeit.
Im zweiten Kapitel untersuchen wir eine wichtige Klassen von Längenräumen, nämlich Riemann Mannigfaltigkeiten. Mit der Riemannschen Distanz und Bogenlänge ist eine Riemann Mannigfaltigkeit mit glatter Riemannmetrik ein Längenraum. In diesem Kapitel zeigen wir, das auch Mannigfaltigkeiten mit stetiger Riemannmetrik Längenräume sind. Wir vergleichen außerdem verschiedene Längenstrukturen auf Riemann Mannigfaltigkeiten und verallgemeinern die Bogenlänge auf absolut stetige Wege und via der Variationslänge aus Kapitel 1 auch auf rektifizierbare Wege. Dies zeigen wir zunächst für glatte Metriken und dann auch für stetige indem wir diese regularisieren und die Resultate für den glatten Fall anwenden.
Nachdem wir festgestellt haben das Riemannmannigfaltigkeiten mit stetigen Metriken Längenräume sind, vergleichen wir im dritten Kapitel die Geodäten in metrischen Sinn mit denen im Riemannschen Sinn. Wir beginnen mit einem Gegenbeispiel von Hartman und Wintner [11], welches widerlegt, dass Lösungen der Geodätengleichung lokal kürzeste Wege sind, falls die Metrik nur von Regularität $C^{1,\alpha}$ ist, für $0 < \alpha < 1$. Weiter zeigen wir, dass im Falle einer C^1 Metrik, kürzeste Wege von Differenzierbarkeit C^2 sind und die Geodätengleichung lösen. Im letzten Teil diese Kapitels behandeln wir ein Paper von Lytchak und Yaman [14], welches zeigt, dass Geodäten im metrischen Sinn in Mannigfaltigkeiten mit C^α Metrik, $0 < \alpha < 1$, lokal, gleichmäßig von Regularität $C^{1,\beta}$ sind, für $\beta = \frac{\alpha}{2-\alpha}$.
Im vierten Kapitel beweisen wir auf zwei verschiedene Arten([17], [18] bzw. [23]), dass die Exponentialabbildung einer Mannigfaltigkeit mit $C^{1,1}$ Riemannmetrik, ein bi-Lipschitz Homömorphismus auf einer offenen Umgebung von 0 ist. Die erste Methode beruht auf Regularisierung der Metrik und verwendet Jacobi-Felder um die bi-Lipschitz Eigenschaft auf den Grenzwert der regularisierten Metriken zu übertragen. Die zweite Methode verwenden eine Version des Satzes über inverse Funktionen für Abbildungen niedriger Regularität ([24] und [25]), sowie die starke Differenzierbarkeit der Exponentialabbildung bei 0, um die Bi-Lipschitz Eigenschaft abzuleiten. Unter Verwendung dieser Resultates ist es möglich eine Version des Gauß-Lemmas für $C^{1,1}$ Metriken zu formulieren, welches dazu führt, dass in dieser Situation Geodäten in Riemann Mannigfaltigkeiten lokal kürzeste Wege sind.

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