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"Topics in the spectral theory of the $\overline{\partial}^E$-Neumann problem"

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Introduction

This thesis is concerned with questions regarding the spectral theory of the Dolbeault Laplacian with $\overline{\partial}$-Neumann boundary conditions, viewed as a self-adjoint operator acting on the space of square-integrable differential forms on a Hermitian manifold. The associated $\overline{\partial}$-Neumann problem has become an important tool in the function theory of several complex variables. Its inception seems to be rooted in some unpublished works of D. C. Spencer from the 1950s, with major contributions to its foundation due to Kohn [Koh63], Kohn–Nirenberg [KN65], Morrey [Mor58], and Hörmander [Hör65], among others. We refer to [Hör03] for a historical account. The theory is most developed on bounded pseudoconvex domains in $\mathbb{C}^n$, and we mention the monograph [Str10] for an in-depth treatment also containing extensive references.

Before discussing the contents of the thesis and its main results, we shall briefly set up the required notation and concepts in at least some detail. On a complex manifold M, there are complex subbundles $\Lambda^{p,q}T^*M$, with $0 \leq p, q \leq n := \dim_{\mathbb{C}}(M)$, of the bundle $\Lambda^k T^*M \otimes \mathbb{C}$ of complex k-forms on M which are spanned, over the domain of a given chart $(z_1, \ldots, z_n)$ of M, by differential forms of the type

$$dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q}. \tag{1}$$

We denote the space of smooth sections of $\Lambda^{p,q}T^*M$ by $\Omega^{p,q}(M)$. Alternatively, $\Lambda^{p,q}T^*M$ may be constructed from the eigenspaces of the complex structure operator. It turns out that the exterior derivative d sends $\Omega^{p,q}(M)$ to $\Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$, and

$$\overline{\partial}\colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$$

is defined as the part of d that is mapped to $\Omega^{p,q+1}(M)$. In coordinates,

$$\overline{\partial}(f\alpha) = \overline{\partial}f \wedge \alpha = \sum_{\ell=1}^{n} \frac{\partial f}{\partial \overline{z}_\ell}\, d\overline{z}_\ell \wedge \alpha,$$

with α as in (1), so $\overline{\partial}$ generalizes the *Wirtinger derivative* $d/d\overline{z}$ from single variable complex analysis. Moreover, $\overline{\partial}\overline{\partial} = 0$, so we have the *Dolbeault complex*

$$0 \to \Omega^{p,0}(M) \xrightarrow{\overline{\partial}} \Omega^{p,1}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{p,n}(M) \to 0$$

for every $0 \leq p \leq n$. There is also a vector-valued analogue of $\overline{\partial}$: if $E \to M$ is a holomorphic vector bundle, then there are first order differential operators

$$\overline{\partial}^E\colon \Omega^{p,q}(M, E) \to \Omega^{p,q+1}(M, E),$$

with $\Omega^{p,q}(M,E) \coloneqq \Gamma(M, \Lambda^{p,q}T^*M \otimes E)$ the space of smooth E-valued differential forms of bidegree (p,q). If $(\xi_1,\dots,\xi_r)$ is a local holomorphic frame of E over some open subset $U \subseteq M$, then

$$\overline{\partial}^E u = \sum_{j=1}^{r} \overline{\partial}\alpha_j \otimes \xi_j$$

for all $u = \sum_{j=1}^r \alpha_j \otimes \xi_j$, with $\alpha_j \in \Omega^{p,q}(U)$. Clearly, $\overline{\partial}^E\overline{\partial}^E = 0$, so we again end up with a complex of differential operators. The holomorphic sections of E are then precisely the $s \in \Gamma(M,E)$ that satisfy the (homogeneous) *Cauchy–Riemann equation*

$$\overline{\partial}^E s = 0.$$

The *inhomogeneous Cauchy–Riemann equation*

$$\overline{\partial}^E s = u \tag{2}$$

for a given $u \in \Omega^{0,1}(M,E)$ (necessarily satisfying $\overline{\partial}^E u = 0$) is also important in the construction of global holomorphic sections of E with prescribed properties. For example, one may wish for holomorphic functions with prescribed singularities, or to extend holomorphic functions initially defined on a hypersurface to a holomorphic function on a neighborhood of that hypersurface. These questions and more can often be answered by first constructing a "real" solution to the problem, and then correcting it to a holomorphic one using a solution of (2). We refer to textbooks on complex analysis for examples of this principle, for instance [Kra01, chapter 5].

Even better than solving (2) is to do so with estimates. For instance, one can wish for a solution operator $S\colon \operatorname{img}(\overline{\partial}^E) \subseteq \Omega^{0,1}(M,E) \to \Gamma(M,E)$ which is continuous for the L^2 norms induced by a Riemannian metric on M and a Hermitian metric on E. To this end, it is useful to consider $\overline{\partial}^E$ in the sense of distributions, *i.e.*, as an unbounded operator $\overline{\partial}^E_w$ on the Hilbert space $L^2_{p,q}(M,E)$ of square-integrable E-valued (p,q)-forms, with domain those $u \in L^2_{p,q}(M,E)$ such that $\overline{\partial}^E u$, when computed in the sense of distributions, lies in $L^2_{p,q+1}(M,E)$. Since weak solutions of (2) are automatically smooth by interior elliptic regularity, hence holomorphic, this is a natural extension of $\overline{\partial}^E$ to consider. We refer to $\overline{\partial}^E_w$ as the *weak extension* of the differential operator $\overline{\partial}^E$, and $(L^2_{p,\bullet}(M,E), \overline{\partial}^E_w)$ is a prime example of a *Hilbert complex*, *i.e.*, a cochain complex of closed operators between Hilbert spaces. As we will see in section 1.2, it is then fruitful to consider the self-adjoint operator

$$\Box^E_{p,q} \coloneqq \overline{\partial}^E_w \overline{\partial}^{E,*}_w + \overline{\partial}^{E,*}_w \overline{\partial}^E_w \tag{3}$$

on $L^2_{p,q}(M,E)$, for its inverse $N^E_{p,q}\colon \operatorname{img}(\Box^E_{p,q}) \to L^2_{p,q}(M,E)$, the $\overline{\partial}^E$*-Neumann operator*, gives a solution operator to (2) via

$$S^E_{p,q} \coloneqq \overline{\partial}^{E,*}_w N^E_{p,q}.$$

In fact, $S^E_{p,q}$ gives the solution of minimal L^2 norm, and its continuity may be read off from operator theoretic properties of $\Box^E_{p,q}$. More precisely, $S^E_{p,q}$ and $S^E_{p,q+1}$ are continuous if and only if $\Box^E_{p,q}$ has closed range. Equivalently, 0 either doesn't belong to or is an isolated point of

$\sigma_e(\square^E_{p,q})$, the *essential spectrum* of E. Here, the essential spectrum of a self-adjoint operator consists of the points in its spectrum that are *not* isolated eigenvalues of finite multiplicity. One says that a self-adjoint operator has *discrete spectrum* if its essential spectrum is empty. The *compactness* of $S^E_{p,q}$ and $S^E_{p,q+1}$ is equivalent to $N^E_{p,q}$ being compact, which is the same as either $\sigma_e(\square^E_{p,q})$ being empty or containing 0 as its only element. We point out that $\square^E$ is an extension of the elliptic *Dolbeault Laplacian* (or simply *complex Laplacian*) $\bar\partial^E\bar\partial^{E,\dagger} + \bar\partial^{E,\dagger}\bar\partial^E$, where $\bar\partial^{E,\dagger}$ is the formal adjoint of $\bar\partial^E$. As such, general elliptic operator theory gives compactness of $N^E_{p,q}$ when M is compact.

In case M is a (smoothly bounded) domain in a larger manifold, the boundary conditions that are imposed on elements of $\Omega_c(\overline{M}, E)$ by their membership in $\mathrm{dom}(\square^E)$ are called *$\bar\partial$-Neumann boundary conditions*, and $\square^E$ is for this reason sometimes called the *Dolbeault Laplacian with $\bar\partial$-Neumann boundary conditions*. Therefore, the equation $\square^E u = v$ for given $v \in L^2_{\bullet,\bullet}(M, E)$ is really a boundary value problem in disguise, called the *$\bar\partial^E$-Neumann problem*. From a PDE point of view, this problem is analytically delicate because the boundary conditions do not lead to good estimates for its solutions on the boundary and, as a result, one may not expect u to be smooth up to the boundary even if v is (the problem is not “globally regular”). That this can be remedied in at least some cases was first demonstrated by Kohn in [Koh63], where he showed that the problem exhibits a *subelliptic gain* if $M \subseteq \mathbb{C}^n$ is a bounded strongly pseudoconvex domain with smooth boundary, and, as a consequence, global regularity holds for such M. Moreover, the subelliptic estimates that Kohn proved imply, together with Rellich’s theorem, that the $\bar\partial$-Neumann operator is compact in this case. Conversely, it was shown by Kohn and Nirenberg in [KN65] that if $M \subseteq \mathbb{C}^n$ is bounded, pseudoconvex, and has a smooth boundary, then the compactness of the $\bar\partial$-Neumann operator implies global regularity. These results provide another motivation for studying the discreteness of spectrum of $\square^E$. Furthermore, it is also known that the spectrum of $\square^E$ contains geometric information of (the boundary of) M which goes beyond pseudoconvexity. As an example, Fu showed in [Fu08] that, in case $M = \Omega$ is a smoothly bounded and bounded pseudoconvex domain in $\mathbb{C}^2$, the growth of the spectral counting function (*i.e.*, the number of eigenvalues of the complex Laplacian below a given parameter) is related to Ω being of *finite type*.

Overview of the thesis. In chapter 1, we will introduce the basic notions used throughout the thesis: differential operators and their extensions, as well as Hilbert complexes. Chapter 2 deals with the general properties of (nonnegative) self-adjoint extensions of elliptic differential operators. The most important result there is a slight extension of *Persson’s theorem* [Per60] which characterizes the bottom of the essential spectrum of such operators.

Chapter 3 sets up the $\bar\partial^E$-Neumann problem in detail and gives some fundamental properties. One of these is that, under suitable boundary and curvature assumptions, the discreteness of spectrum of $\square^E$ “percolates” up the $\bar\partial^E$-complex: if the spectrum of $\square^E_{p,q}$ is discrete, then the same holds for $\square^E_{p,q+1}$. This is well-known for bounded pseudoconvex domains in $\mathbb{C}^n$, see [Fu08, Proposition 2.2] or [Str10, Proposition 4.5]. It was shown in [Has14] that this also holds for the *weighted $\bar\partial$-Neumann problem* with a plurisubharmonic weight, by which we mean taking $M = \mathbb{C}^n$ and E a trivial line bundle with metric chosen such that the L^2 norm of a function f becomes $\int_{\mathbb{C}^n} |f|^2 e^{-\varphi} d\lambda$, with $\varphi\colon \mathbb{C}^n \to \mathbb{R}$ smooth and plurisubharmonic, and λ the Lebesgue measure. Further information on the weighted problem can be found in [Has14]. Our generalization is the following:

Theorem 3.2.23. *Let $M \subseteq M'$ be a q-Levi pseudoconvex open subset of a Kähler manifold of 1-bounded geometry, with smooth boundary $\partial M \subseteq M'$, and let $E \to \overline{M}$ be a Hermitian holomorphic vector bundle such that $E|_M$ is q-Nakano lower semibounded. If $\square^E_{p,q-1}$ has discrete spectrum, then so does $\square^E_{p,q}$.*

Here, the requirement of M having 1-bounded geometry means that its injectivity radius is positive, and both the Riemann curvature tensor as well as its first covariant derivative are bounded, uniformly on M. General Riemannian manifolds of bounded geometry are discussed in section 4.1. Another area where manifolds with some bounded geometry are of use is the analysis of Schrödinger operators, by which we mean operators of the form $\nabla^\dagger\nabla + V$ for some (metric) connection ∇ on a Hermitian vector bundle F and vector bundle morphism $V\colon F \to F$. It turns out that every differential operator of *Laplace type* is of this form, which in particular applies to (twice) the Dolbeault Laplacian. For Schrödinger operators acting on the sections of a Hermitian *line* bundle $L \to M$, where (M, g) is a Riemannian manifold of 1-bounded geometry, we establish in Theorem 4.2.7 a generalization of a result of Iwatsuka [Iwa86, Theorem 5.2]: if such an operator has a lower semibounded self-adjoint extension with discrete spectrum, then

$$\lim_{x\to\infty} \int_{B(x,r)} (|R^\nabla|^2 + |V|)\, d\mu_g = \infty$$

for $r > 0$ small enough, with $B(x, r)$ the geodesic ball and R^∇ the curvature of ∇. The application to $\square^E$ is then the following:

Theorem 4.3.2. *Let $L \to M$ be a Hermitian holomorphic line bundle over a Kähler manifold of 1-bounded geometry, and let $p \in \{0, n\}$. Assume that*

(i) $\square^L_{p,n}$ has discrete spectrum, or
(ii) for some $0 \le q \le n-1$, L is $(q+1)$-Nakano lower semibounded and $\square^L_{p,q}$ has discrete spectrum.

Then

$$\lim_{x\to\infty} \int_{B(x,r)} |R^L|^2\, d\mu_g = \infty \tag{4.3.3}$$

for all $r > 0$ small enough.

Here, the above Theorem 3.2.23 is used to transfer the discreteness of spectrum of $\square^L_{p,q}$ to that of $\square^L_{p,n}$, where the general result on Schrödinger operators applies. Theorem 4.3.2 generalizes a result that was known in the setting of the weighted $\bar\partial$-Neumann problem on $\mathbb{C}^n$, see [BH17].

In chapter 5, we study the (essential) spectrum of the Dolbeault Laplacian with $\overline{\partial}$-Neumann boundary conditions for product manifolds. Let $E \to M$ and $F \to N$ be Hermitian holomorphic vector bundles over Hermitian manifolds. Then we can form the bundle $E \boxtimes F \to M \times N$, which has fiber $E_x \otimes F_y$ over $(x, y) \in M \times N$. We obtain the following result, which extends work by Chakrabarti [Cha10], who deduced the formula (5.3.3).

Theorem 5.3.1. *Let $E \to M$ and $F \to N$ be Hermitian holomorphic vector bundles over Hermitian manifolds. Then, for $0 \leq p, q \leq \dim_{\mathbb{C}}(M) + \dim_{\mathbb{C}}(N)$,*

$$\sigma(\Box_{p,q}^{E\boxtimes F}) = \bigcup_{\substack{p'+p''=p \\ q'+q''=q}} (\sigma(\Box_{p',q'}^{E}) + \sigma(\Box_{p'',q''}^{F})) \tag{5.3.3}$$

and

$$\sigma_e(\Box_{p,q}^{E\boxtimes F}) = \bigcup_{\substack{p'+p''=p \\ q'+q''=q}} (\sigma_e(\Box_{p',q'}^{E}) + \sigma(\Box_{p'',q''}^{F})) \cup (\sigma(\Box_{p',q'}^{E}) + \sigma_e(\Box_{p'',q''}^{F})), \tag{5.3.4}$$

where p' and q' range over $\{0, \dots, \dim_{\mathbb{C}}(M)\}$, and p'' and q'' range over $\{0, \dots, \dim_{\mathbb{C}}(N)\}$.

Theorem 5.3.1 also has consequences for the compactness of the $\overline{\partial}$-Neumann operator $N_{p,q}^{E\boxtimes F}$, as well as the minimal solution operator $S_{p,q}^{E\boxtimes F}$. It was known to Krantz [Kra88] that the minimal solution operator fails to be compact on the level of $(0, 1)$-forms on the bidisc in $\mathbb{C}^2$, which is the product of two one-dimensional discs. Moreover, Haslinger and Helffer in [HH07] show that this extends to the weighted $\overline{\partial}$-Neumann problem on $\mathbb{C}^n$ if one considers *decoupled weights*, which are functions of the form $\varphi(z_1, \dots, z_n) = \varphi_1(z_1) + \cdots + \varphi_n(z_n)$. The question whether such a product structure is an obstruction for compactness on higher degree forms was left mostly unanswered, but can now be settled as a consequence of Theorem 5.3.1. The proof of Theorem 5.3.1 uses the fact that the Hilbert complex $(L^2_{p,\bullet}(M \times N, E \boxtimes F), \overline{\partial}_w^{E\boxtimes F})$ is equivalent to the direct sum of tensor products of Hilbert complexes of the form $(L^2_{p',\bullet}(M, E), \overline{\partial}_w^{E})$ and $(L^2_{p'',\bullet}(N, F), \overline{\partial}_w^{F})$. Therefore, we will also discuss tensor products of general Hilbert complexes in section 5.1.

The main chapters of this thesis are supplemented by appendices A to C, which provide some of the necessary background on Hermitian and differential geometry as well as on spectral theory. This thesis strives to be self-contained to a large degree, which is why (proofs of) a lot of auxiliary results are also presented.

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CHAPTER 1

Differential operators, Hilbert complexes, and elliptic theory

In this chapter, we review and develop the basic tools needed throughout this thesis. Section 1.1 defines differential operators acting between the sections of smooth vector bundles and discusses their basic properties. As an important class of examples, Dirac type operators will receive special attention in section 1.1.2. Section 1.2 deals with the basic theory of Hilbert complexes. These are (cochain) complexes of closed operators between Hilbert spaces, and they naturally occur when studying closed extensions of complexes of differential operators arising in geometry. The weak extension of the Dolbeault complex is the central example of a Hilbert complex in this thesis. In section 1.3, we will take a closer look at extensions of differential operators to operators on Hilbert spaces of square integrable sections. Sobolev spaces are also introduced in this section, as is part of elliptic regularity theory. Finally, section 1.4 is devoted to the question of whether compactly supported sections are dense in the domains of closed extensions of differential operators. In addition, the essential self-adjointness of first and second order operators is discussed.

1.1. Differential operators

Let M be a smooth manifold of dimension n with (possibly empty) boundary ∂M. All manifolds are assumed to be second countable (thus paracompact, so partitions of unity exist) and of positive dimension, and for simplicity we will always assume that M (and hence ∂M) is oriented. We denote the interior of M by M°. For a (smooth) vector bundle $E \to M$, we denote by $\Gamma(M, E)$ the space of smooth sections of E, by $\Gamma_c(M, E)$ the smooth sections with compact support in M, and by $\Gamma_{cc}(M, E)$ the smooth sections of E with compact support contained in M°. Thus, we may identify $\Gamma_{cc}(M, E)$ with $\Gamma_c(M^\circ, E)$. Similarly, $C_c^\infty(M)$ and $C_{cc}^\infty(M)$ denote the smooth functions (complex valued if this makes sense and is not stated otherwise) on M with compact support and compact support contained in M°, respectively. For more on the (mostly standard) notation used throughout this section, we refer to appendix A.

Suppose now that $E, F \to M$ are two (smooth) vector bundles. A (linear) *differential operator* is a $\mathbb{R}$-linear[1] map $D\colon \Gamma(M, E) \to \Gamma(M, F)$ which is *local* in the sense that $\operatorname{supp}(Ds) \subseteq \operatorname{supp}(s)$ for all sections $s \in \Gamma(M, E)$. By Peetre's theorem, see [Nar73, Theorem 3.3.8] for a proof, this is equivalent to D being represented as a matrix of partial differential operators on an open subset of $\mathbb{R}^n$ in each chart of M and local trivializations of E and F. The *order* of D is the maximal order of operators in the matrix representation in a local trivialization,

[1]Or $\mathbb{C}$-linear if E and F are complex vector bundles.

and this is independent of the specific trivialization. There is also a more algebraic approach to differential operators, see for instance [Nic14, section 10.1] or [Pal65, chapter IV], and we will encounter some of this in the discussion of the principal symbol below. Clearly, every differential operator is uniquely determined by its restriction $D\colon \Gamma_c(M,E) \to \Gamma_c(M,F)$ to the sections with compact support. The composition of two differential operators is again a differential operator. Important examples of differential operators are the exterior derivative

$$d\colon \Omega(M) \to \Omega(M),$$

with $\Omega(M) \coloneqq \Gamma(M, \Lambda T^*M)$ the space of smooth differential forms, and connections

$$\nabla\colon \Gamma(M,E) \to \Omega^1(M,E),$$

see appendix A.1. Moreover, every vector bundle morphism $E \to F$ defines a differential operator.

Assume furthermore that (M,g) is a Riemannian manifold and that E and F carry Hermitian metrics. If $X, Y \in T_xM$, we will often write $\langle X, Y\rangle$ instead of $g(X,Y)$. The Riemannian volume form induced by the metric and the orientation will be denoted by $\mathrm{vol}_g \in \Omega^n(M)$. Then $C_c(M,\mathbb{R}) \to \mathbb{R}$, $f \mapsto \int_M f \,\mathrm{vol}_g$, is a positive linear functional, so that by the Riesz representation theorem, see [Fol99, Theorem 7.2], there is a unique positive Radon measure μ_g of full support on M such that $\int_M f\,\mathrm{vol}_g = \int_M f\, d\mu_g$ for all $f \in C_c(M,\mathbb{C})$. It follows that the boundary ∂M is a set of measure zero for μ_g. The induced volume form on ∂M is denoted by $\mathrm{vol}_{\partial M}$, and the associated measure on ∂M by $\mu_{\partial M}$. We also extend the metric g to a Hermitian form on $TM \otimes_{\mathbb{R}} \mathbb{C}$, denoted by $\langle \bullet, \bullet \rangle$, and we also use the same notation for the (pointwise) Hermitian metrics on E and F. Then

$$\langle\!\langle s,t\rangle\!\rangle_{L^2(M,E)} \coloneqq \int_M \langle s,t\rangle \, d\mu_g \tag{1.1.1}$$

or just $\langle\!\langle s,t\rangle\!\rangle$ defines an inner product on $\Gamma_{cc}(M,E)$, and similarly for $\Gamma_{cc}(M,F)$. The Hilbert space $L^2(M,E)$ is defined as the completion of $\Gamma_{cc}(M,E)$ with respect to $\langle\!\langle \bullet,\bullet\rangle\!\rangle$. As usual, this may be identified with the space of equivalence classes of measurable maps $s\colon M \to E$ such that $s(x) \in E_x$ for almost every $x \in M$ and satisfying $\int_M |s(x)|^2\, d\mu_g(x) < \infty$, and where two such maps are equivalent if and only if they differ on a set of measure zero. For every differential operator $D\colon \Gamma(M,E) \to \Gamma(M,F)$ there is a unique differential operator $D^\dagger\colon \Gamma(M,F) \to \Gamma(M,E)$, called the *formal adjoint*[2] to D, such that

$$\langle\!\langle Ds,t\rangle\!\rangle = \langle\!\langle s, D^\dagger t\rangle\!\rangle \tag{1.1.2}$$

holds for all $s \in \Gamma_{cc}(M,E)$ and $t \in \Gamma_c(M,F)$. Both D and $D^\dagger$ are of the same order. A differential operator D is called *formally self-adjoint* if $E = F$ and $D = D^\dagger$.

[2]The formal adjoint may be constructed via $(D^\dagger)|_{\Gamma_c(M,E)} = (D^*_{cc})|_{\Gamma_c(M,E)}$, where D^*_{cc} is the Hilbert space adjoint of the densely defined operator $D_{cc} \coloneqq D|_{\Gamma_{cc}(M,E)}$. We refer to section 1.3 for more on this. Of course, computations in coordinates using integration by parts also allows to prove existence of $D^\dagger$. We use the notation $D^\dagger$ instead of D^* because the latter will be reserved for *true* adjoints (recall that (1.1.2) only holds if s has compact support in M°, and not on the whole domain of D).

Example 1.1.1. If ∇ is a metric compatible connection on a Hermitian vector bundle $E \to M$, then the formal adjoint of ∇_Z for $Z \in \Gamma(M, TM \otimes_{\mathbb{R}} \mathbb{C})$ satisfies

$$(\nabla_Z)^\dagger = -\nabla_{\overline{Z}} - \operatorname{div}(\overline{Z}), \tag{1.1.3}$$

with $\operatorname{div}(\overline{Z})$ the divergence of $\overline{Z}$ with respect to the Riemannian metric on M. Indeed, for $s, t \in \Gamma_c(M, E)$, we have $\langle \nabla_Z s, t\rangle + \langle s, \nabla_{\overline{Z}} t\rangle = Z\langle s, t\rangle = \operatorname{ins}_Z(d\langle s, t\rangle)$ by (A.1.2), where ins_Z is the insertion operator, see (A.0.1), and hence

$$\begin{aligned}\langle\!\langle \nabla_Z s, t\rangle\!\rangle + \langle\!\langle s, \nabla_{\overline{Z}} t\rangle\!\rangle = \int_M \operatorname{ins}_Z(d\langle s, t\rangle)\operatorname{vol}_g = \int_M d(\langle s, t\rangle) \wedge \operatorname{ins}_Z(\operatorname{vol}_g) = \\ = \int_{\partial M} \langle s, t\rangle\, \iota^*(\operatorname{ins}_Z(\operatorname{vol}_g)) - \int_M \langle s, t\rangle\, d(\operatorname{ins}_Z(\operatorname{vol}_g)) \end{aligned} \tag{1.1.4}$$

by Stokes' theorem, where $\iota\colon \partial M \hookrightarrow M$ is the boundary inclusion. If $s \in \Gamma_{cc}(M, E)$, then the boundary integral vanishes, while in the last term we have $d(\operatorname{ins}_Z(\operatorname{vol}_g)) = \mathcal{L}_Z(\operatorname{vol}_g) = \operatorname{div}(Z)\operatorname{vol}_g$, see [Lee13, p. 423], where $\mathcal{L}_Z$ is the Lie derivative.[3] This shows (1.1.3). Together, (1.1.3) and (1.1.4) imply, for $s, t \in \Gamma_c(M, E)$,

$$\langle\!\langle \nabla_Z s, t\rangle\!\rangle = \langle\!\langle s, (\nabla_Z)^\dagger t\rangle\!\rangle - \int_{\partial M} \langle s, t\rangle \langle Z, \nu\rangle\, d\mu_{\partial M} \tag{1.1.5}$$

where we have used that $\iota^*(\operatorname{ins}_Z(\operatorname{vol}_g)) = -\langle Z, \nu\rangle \operatorname{vol}_{\partial M}$, see [Lee13, Lemma 16.30], with ν the *inward* unit normal vector field to ∂M, and $\mu_{\partial M}$ the measure induced on the boundary. This result will be generalized to arbitrary first order differential operators in Theorem 1.1.8 below. ♦

Example 1.1.2. Let $E \to M$ be a vector bundle over a Riemannian manifold, and suppose ∇^E is a connection on E. Then the *Bochner Laplacian* is the second order differential operator

$$\Delta^E s := -\operatorname{tr}_g\left(\nabla^{T^*M\otimes E}\nabla^E s\right) = -\sum_{j=1}^n (\nabla^{T^*M\otimes E}\nabla^E s)(e_j, e_j)\colon \Gamma(M, E) \to \Gamma(M, E), \tag{1.1.6}$$

where $\{e_j\}_{j=1}^n$ is a local orthonormal frame of TM, the connection $\nabla^{T^*M\otimes E}$ is induced by ∇^E and the Levi–Civita connection ∇^{TM} on TM, and $\operatorname{tr}_g\colon T^*M \otimes T^*M \to \mathbb{R}$ is defined by taking the trace of $A \in T^*M \otimes T^*M$ after identifying it with an element of $\operatorname{End}(TM) \cong T^*M \otimes TM$ by metric duality, *i.e.*, $\operatorname{tr}_g(A) := \operatorname{tr}(X \mapsto (Y \mapsto A(X, Y))^\sharp)$. It is easy to see that

$$\Delta^E = \sum_{j=1}^n \Big(-\nabla^E_{e_j}\nabla^E_{e_j} + \nabla^E_{\nabla^{TM}_{e_j} e_j}\Big) = \sum_{j=1}^n \Big(-\nabla^E_{e_j}\nabla^E_{e_j} - \operatorname{div}(e_j)\nabla^E_{e_j}\Big),$$

where we have used in the last step that

$$\operatorname{div}(X) = \operatorname{tr}(\nabla^{TM} X) = \sum_{k=1}^n \langle \nabla^{TM}_{e_k} X, e_k\rangle,$$

hence $\sum_{j=1}^n \operatorname{div}(e_j)e_j = \sum_{j,k=1}^n \langle \nabla^{TM}_{e_k} e_j, e_k\rangle e_j = -\sum_{j,k=1}^n \langle \nabla^{TM}_{e_k} e_k, e_j\rangle e_j = -\sum_{k=1}^n \nabla^{TM}_{e_k} e_k$.

[3] The *divergence* of Z (with respect to g) may in fact be defined as the function $\operatorname{div}(Z)$ which satisfies $\mathcal{L}_Z(\operatorname{vol}_g) = \operatorname{div}(Z)\operatorname{vol}_g$. Alternatively, $\operatorname{div}(Z) = \operatorname{tr}(\nabla Z)$, with ∇ the Levi–Civita connection.

Now suppose E is Hermitian and that ∇^E is a metric connection. Since $\nabla^E = e^j \otimes \nabla^E_{e_j}$, with $\{e^j\}_{j=1}^n$ the dual frame to $\{e_j\}_{j=1}^n$, we have $\nabla^{E,\dagger} = (\nabla^E_{e_j})^\dagger \operatorname{ins}_{e_j} = (-\nabla^E_{e_j} - \operatorname{div}(e_j)) \operatorname{ins}_{e_j}$, according to (1.1.3). Therefore,

$$\Delta^E = \nabla^{E,\dagger}\nabla^E,$$

where $\nabla^{E,\dagger}$ is the formal adjoint of ∇^E. In particular, Δ^E is formally self-adjoint and nonnegative. ♦

1.1.1. The principal symbol of a differential operator. Let E and F be real or complex vector bundles over M. For $k \in \mathbb{N}_0$, we denote by $\mathrm{PDO}^{(k)}(E,F)$ the set of $\mathbb{R}$- (or $\mathbb{C}$-) linear maps $T\colon \Gamma(M,E) \to \Gamma(M,F)$ such that $T \in \ker(\operatorname{ad}(f_1)\cdots\operatorname{ad}(f_{k+1}))$ for all $f_j \in C^\infty(M)$, $1 \le j \le k+1$, where

$$\operatorname{ad}(f)T \coloneqq [T,f] = Tf - fT\colon \Gamma(M,E) \to \Gamma(M,F)$$

is the commutator of T with the operator of multiplication by f. Then the elements of $\mathrm{PDO}^{(k)}(E,F)$ are differential operators (*i.e.*, local), and $\mathrm{PDO}^{(k)}(E,F)$ is called the set of *differential operators of order at most k*. It turns out that, for $f \in C^\infty(M)$, $s \in \Gamma(M,E)$, and given $x \in M$, the map

$$f \mapsto \frac{1}{k!}((\operatorname{ad}(f)^k D)s)(x)$$

depends *only* on $df(x) \in T^*_xM$ and $s(x) \in E_x$, provided $D \in \mathrm{PDO}^{(k)}(E,F)$. Therefore, it makes sense to define the *k-symbol of D at x*, which is given by

$$\operatorname{Symb}_k(D)(x,\bullet)\colon T^*_xM \to \operatorname{Hom}(E_x,F_x), \quad \operatorname{Symb}_k(D)(x,\xi)e \coloneqq \frac{1}{k!}((\operatorname{ad}(f)^k D)s)(x), \quad (1.1.7)$$

with $f \in C^\infty(M)$ and $s \in \Gamma(M,E)$ satisfying $df(x) = \xi \in T^*_xM$ and $s(x) = e \in E_x$. More abstractly, we can view this as a (smooth) section $\operatorname{Symb}_k(D)$ of $\operatorname{Hom}(\pi^*E,\pi^*F) \to T^*M$, where $\pi\colon T^*M \to M$ is the cotangent bundle of M. One can show that it is equal to

$$\operatorname{Symb}_k(D)(x,\xi)(e) = \frac{1}{k!}D((f-f(x))^k s)(x),$$

with f and s as above. If $\xi \in T^*M$, then we write $\operatorname{Symb}_k(D)(\xi)$ for $\operatorname{Symb}_k(D)(\pi(\xi),\xi)$.

A differential operator D has order $k > 0$ (in the sense used at the beginning of this section) if and only if $D \in \mathrm{PDO}^{(k)}(M,E)$ and $\operatorname{Symb}_k(D)$ does not vanish identically, in which case $\operatorname{Symb}_k(D)$ is called the *principal symbol* of D, and we shall denote it by $\operatorname{Symb}(D)$. The differential operators $\Gamma(M,E) \to \Gamma(M,F)$ of order 0 are given by the vector bundle morphisms $A\colon E \to F$, and clearly

$$\operatorname{Symb}(A)(x,\xi) \coloneqq \operatorname{Symb}_0(A)(x,\xi) = A\colon E_x \to F_x.$$

If $D = \sum_{|\alpha|\le k} a_\alpha(x)\partial^\alpha$ is a (scalar) partial differential operator on $\mathbb{R}^n$, then its k-symbol is given by $\operatorname{Symb}_k(D)(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$ for $\xi \in T^*_xM \cong \mathbb{R}^n$, and is equal to the principal symbol provided not all of the a_α for $|\alpha| = k$ vanish identically. We refer to [Nic14, section 10.1] or [Pal65, chapter IV] for more information on the (principal) symbol.

Definition 1.1.3. A differential operator $D\colon \Gamma(M,E) \to \Gamma(M,F)$ is called *elliptic* if its principal symbol

$$\mathrm{Symb}(D)(x,\xi)\colon E_x \to F_x$$

is a linear isomorphism for every $x \in M$ and $\xi \in T^*_xM \setminus \{0\}$. In particular, E and F must have the same rank.

Remark 1.1.4. Some authors define the k-symbol with an additional factor i^k. Of course, this only immediately makes sense for complex vector bundles, but this modification gets rid of the sign factor in equation (1.1.10) below for the k-symbol of the formal adjoint.

Example 1.1.5. Let ∇ be a connection on E, see appendix A.1. For the first order differential operator $\nabla\colon \Gamma(M,E) \to \Gamma(M,T^*M\otimes E)$, we obtain

$$\mathrm{Symb}(\nabla)(x,\xi)e = ([\nabla,f]s)(x) = (\nabla(fs) - f\nabla s)(x) = (df\otimes s)(x) = \xi\otimes e,$$

again with $f \in C^\infty(M)$ and $s \in \Gamma(M,E)$ such that $df(x) = \xi$ and $s(x) = e$. For a vector field $X \in \Gamma(M,TM)$ we have

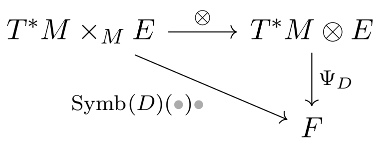
$$\mathrm{Symb}(\nabla_X)(x,\xi)e = ([\nabla_X,f]s)(x) = X(f)(x)e = df(X)(x)e = \xi(X(x))e.$$

Alternatively, we could have obtained this by applying (1.1.9) to $\nabla_X = \mathrm{ins}_X \circ\nabla$, with ins_X the insertion operator. If d^∇ is the exterior covariant derivative associated to ∇, see appendix A.1.1, then $d^\nabla = \varepsilon\circ\nabla^{\Lambda T^*M\otimes E}$ with $\nabla^{\Lambda T^*M\otimes E}$ induced from ∇ and a torsion free connection on TM, and we immediately get

$$\mathrm{Symb}(d^\nabla)(x,\xi)u = \varepsilon(\xi\otimes u) = \xi\wedge u \tag{1.1.8}$$

for all $\xi \in T^*_xM$ and $u \in \Lambda T^*_xM \otimes E_x$. Here, $\varepsilon\colon T^*M\otimes\Lambda T^*M\otimes E \to \Lambda T^*M\otimes E$ is the wedge product map. ♦

Example 1.1.6. Let $D\colon \Gamma(M,E)\to\Gamma(M,F)$ be a first order differential operator. The definition of $\mathrm{Symb}(D)$ shows that the map $T^*_xM \to \mathrm{Hom}(E_x,F_x)$, $\xi\mapsto \mathrm{Symb}(D)(x,\xi)$, is *linear*, hence we obtain a vector bundle morphism $\Psi_D\colon T^*M\otimes E\to F$ given by $\Psi_D(\xi\otimes e) \coloneqq \mathrm{Symb}(D)(\xi)e$.



Suppose that ∇ is a connection on E. Then, for $f\in C^\infty(M)$ and $s\in\Gamma(M,E)$,

$$(D-\Psi_D\circ\nabla)(fs) = \mathrm{Symb}(D)(df)s + fDs - \Psi_D(df\otimes s) - f\Psi_D(\nabla s) = f(D-\Psi_D\circ\nabla)s.$$

This shows that $D = \Psi_D\circ\nabla + Q$ for a vector bundle morphism $Q\colon E\to F$. ♦

If $D\in \mathrm{PDO}^{(k)}(E,F)$ and $D'\in\mathrm{PDO}^{(k')}(F,F')$ is another differential operator, then $D'\circ D$ belongs to $\mathrm{PDO}^{(k+k')}(E,F')$ and we have the *symbol calculus*

$$\mathrm{Symb}_{k'+k}(D'\circ D)(x,\xi) = \mathrm{Symb}_{k'}(D')(x,\xi)\circ\mathrm{Symb}_k(D)(x,\xi)\colon E_x\to F'_x. \tag{1.1.9}$$

In particular, if $A\colon F \to F'$ is a bundle morphism, then $\operatorname{Symb}_0(A)(x,\xi)f = A(f) \in F'_x$ and therefore $\operatorname{Symb}_k(A \circ D)(x,\xi) = A \circ \operatorname{Symb}_k(D)(x,\xi)$. Note that (1.1.9) need not hold for the *principal* symbols. Indeed, if $d^\nabla\colon \Omega(M,E) \to \Omega(M,E)$ is the exterior covariant derivative associated to a connection ∇ on E, then $\operatorname{Symb}(d^\nabla)(x,\xi)\operatorname{Symb}(d^\nabla)(x,\xi)u = \xi \wedge \xi \wedge u = 0$ by (1.1.8), while $d^\nabla d^\nabla u = R^\nabla \wedge_{\mathrm{ev}} u$ for $u \in \Omega(M,E)$, with R^∇ the curvature of ∇, is a zeroth order operator which of course need not be zero.

Suppose that M is Riemannian and that E and F are equipped with Hermitian metrics. If $D \in \mathrm{PDO}^{(k)}(E,F)$, then (1.1.7) immediately implies that

$$\operatorname{Symb}_k(D^\dagger)(x,\xi) = (-1)^k \operatorname{Symb}_k(D)(x,\xi)^* \tag{1.1.10}$$

is the adjoint of $\operatorname{Symb}_k(D)(x,\xi)$, meaning that

$$\langle \operatorname{Symb}_k(D)(x,\xi)e, f\rangle = (-1)^k \langle e, \operatorname{Symb}_k(D^\dagger)(x,\xi)f\rangle$$

holds for all $e \in E_x$ and $f \in F_x$.

Example 1.1.7. Let ∇ be a connection on E. For $\xi, \eta \in T^*_xM$ and $e, \widetilde{e} \in E_x$, we have

$$\langle \operatorname{Symb}(\nabla)(x,\xi)\widetilde{e}, \eta \otimes e\rangle = \langle \xi \otimes \widetilde{e}, \eta \otimes e\rangle = \langle \widetilde{e}, \langle \eta, \xi\rangle e\rangle,$$

by Example 1.1.5, and hence the principal symbol of $\nabla^\dagger$ is given by

$$\operatorname{Symb}(\nabla^\dagger)(x,\xi)(\eta \otimes e) = -\langle \eta, \xi\rangle e = -\operatorname{ins}_{\xi^\sharp}(\eta \otimes e), \tag{1.1.11}$$

where $\xi^\sharp$ is the metric dual. In particular, the principal symbol of the second order operator $\nabla^\dagger \circ \nabla$ is

$$\operatorname{Symb}(\nabla^\dagger \circ \nabla)(x,\xi) = -|\xi|_x^2 \operatorname{id}_{E_x} \tag{1.1.12}$$

for all $\xi \in T^*_xM$. The same holds for the Bochner Laplacian from (1.1.6):

$$\operatorname{Symb}(\Delta^E)(\xi)e = -\operatorname{tr}_g(\operatorname{Symb}(\nabla^{T^*M\otimes E})(\xi)\operatorname{Symb}(\nabla^E)(\xi)e) = -\operatorname{tr}_g(\xi \otimes \xi \otimes e) = -|\xi|^2 e. \tag{1.1.13}$$

Of course, if ∇ is metric compatible, then $\Delta^E = \nabla^\dagger \circ \nabla$ by Example 1.1.2. ♦

We finish this section with the general version of *Green's formula*, which is also known as the general *integration by parts formula*. It can be found, for example, in [Tay11a, Chap. 2, Proposition 9.1], but we give a quick coordinate free proof here, based on (1.1.5).

Theorem 1.1.8. *Let M be an oriented Riemannian manifold with boundary ∂M, and let $E, F \to M$ be Hermitian vector bundles. Let $D\colon \Gamma(M,E) \to \Gamma(M,F)$ be a* first order *differential operator. Then*

$$\langle\!\langle Ds, t\rangle\!\rangle = \langle\!\langle s, D^\dagger t\rangle\!\rangle - \int_{\partial M} \langle \operatorname{Symb}(D)(\nu^\flat)s, t\rangle \, d\mu_{\partial M} \tag{1.1.14}$$

for all $s \in \Gamma_c(M,E)$ and $t \in \Gamma_c(M,F)$, where ν is the inward *unit normal vector field to ∂M and $\nu^\flat$ is its metric dual.*

Proof. Let ∇ be a metric connection on E. By Example 1.1.6, we have $D = \Psi_D \circ \nabla + Q$ for some vector bundle morphism $Q\colon E \to F$. If $U \subseteq M$ is open and $(e_k)_{k=1}^n$ is an orthonormal frame of $TM|_U$, with dual frame $(\xi_k)_{k=1}^n$ of $T^*M|_U$, then on U we have $D = \operatorname{Symb}(D)(\xi_k) \circ \nabla_{e_k} + Q$, with implied summation over k, and using (1.1.10) we compute

$$\langle Ds, t\rangle - \langle s, D^\dagger t\rangle = -\langle \nabla_{e_k} s, \operatorname{Symb}(D^\dagger)(\xi_k)t\rangle + \langle s, (\nabla_{e_k})^\dagger \operatorname{Symb}(D^\dagger)(\xi_k)t\rangle.$$

By (1.1.5), integrating this equation over M gives

$$\begin{aligned}\langle\!\langle Ds, t\rangle\!\rangle - \langle\!\langle s, D^\dagger t\rangle\!\rangle &= -(\langle\!\langle \nabla_{e_k} s, \operatorname{Symb}(D^\dagger)(\xi_k)t\rangle\!\rangle - \langle\!\langle s, (\nabla_{e_k})^\dagger \operatorname{Symb}(D^\dagger)(\xi_k)t\rangle\!\rangle) = \\ &= -\int_{\partial M} \langle \operatorname{Symb}(D)(\xi_k)s, t\rangle \langle e_k, \nu\rangle \, d\mu_{\partial M} = -\int_{\partial M} \langle \operatorname{Symb}(D)(\nu^\flat)s, t\rangle \, d\mu_{\partial M},\end{aligned}$$

for $s \in \Gamma_c(M, E)$ and $t \in \Gamma_c(M, F)$ with support in U. The general case follows by a partition of unity argument. ■

1.1.2. Dirac type operators. Let E be a vector bundle over a Riemannian manifold M. A second order differential operator $D\colon \Gamma(M, E) \to \Gamma(M, E)$ is said to be *of Laplace type* (or *a generalized Laplacian*) if $\operatorname{Symb}(D)(\xi) = -|\xi|^2 \operatorname{id}_E$ for all $\xi \in T^*M$. From (1.1.13), we know that the Bochner Laplacian from Example 1.1.2 is an operator of Laplace type. It turns out that, conversely, any second order differential operator $D\colon \Gamma(M, E) \to \Gamma(M, E)$ of Laplace type can be written in the form

$$D = \Delta^E + V, \tag{1.1.15}$$

where $V\colon E \to E$ is a vector bundle morphism and Δ^E is the Bochner Laplacian associated to a connection ∇^E on E. For a proof, we refer to [Gil08, Lemma 2.1] or [BGV04, Proposition 2.5], or [BB13, Proposition 2.1] in the context of operators of Dirac type (which will be defined momentarily). If E is Hermitian and D is formally self-adjoint, then ∇^E may be chosen to be metric compatible, and this requirement determines the pair (∇^E, V) uniquely. The proof of this last claim is basically contained in the proof of [BB13, Proposition 2.1]. According to Example 1.1.2, we then have $D = \nabla^{E,\dagger}\nabla^E + V$, and V is necessarily also self-adjoint. Because of (1.1.15), Laplace type operators are also sometimes called *generalized Schrödinger operators* or *Schrödinger type operators*. Formulas like (1.1.15) are sometimes called *Weitzenböck (type) formulas* or *Lichnerowicz formulas*.

Definition 1.1.9. A differential operator $D\colon \Gamma(M, E) \to \Gamma(M, E)$ is said to be *of Dirac type* if D^2 is an operator of Laplace type.

In particular, every Dirac type operator is of first order and elliptic. Dirac type operators are closely related to Clifford analysis. A *Clifford module structure* on a vector bundle E over a Riemannian manifold M is a vector bundle morphism $c\colon T^*M \otimes E \to E$ with

$$c(\xi)c(\eta) + c(\eta)c(\xi) = -2\langle \xi, \eta\rangle \operatorname{id}_E \tag{1.1.16}$$

for all $\xi, \eta \in T_x^*M$ and all $x \in M$, where $c(\xi) \in \operatorname{End}(E)$ is defined by $c(\xi) \coloneqq (e \mapsto c(\xi \otimes e))$. The next Proposition shows that Definition 1.1.9 agrees with the definition in [LM89, II.§5]:

Proposition 1.1.10. *Let E be a vector bundle over a Riemannian manifold M. Then the following are equivalent:*

(i) E admits the structure of a bundle of Clifford modules, and
(ii) there exists an operator $D\colon \Gamma(M,E) \to \Gamma(M,E)$ of Dirac type.

In fact, the principal symbol of a Dirac type operator is a Clifford module structure on E and, conversely, $c \circ \nabla$ is of Dirac type for any connection ∇ on E.

Proof. We follow the proof of [Nic14, Proposition 11.1.7]. Suppose that $c\colon T^*M \otimes E \to E$ is a Clifford module structure, and let $\nabla\colon \Gamma(M,E) \to \Gamma(M, T^*M \otimes E)$ be a connection on E. Put $D \coloneqq c \circ \nabla$. Then $\mathrm{Symb}(D)(\xi) = c(\xi \otimes \bullet)$ by (1.1.9) and Example 1.1.5, and hence

$$\mathrm{Symb}_2(D^2)(x,\xi)e = c(\xi, c(\xi, e)) = c(\xi)c(\xi)e = -|\xi|^2 e$$

by (1.1.16) for all $\xi \in T^*M$ and $e \in E$, *i.e.*, D^2 is a Laplace type operator.

Conversely, assume that D is a Dirac type operator on E. Since D is of first order, $(\xi, e) \mapsto \mathrm{Symb}(D)(\xi)e$ is $\mathbb{R}$-bilinear, hence there is a vector bundle morphism $c\colon T^*M \otimes E \to E$, given by $c(\xi \otimes e) = \mathrm{Symb}(D)(\xi)e$. As before, we write $c(\xi) \coloneqq c(\xi \otimes \bullet) = \mathrm{Symb}(D)(\xi)$. By assumption, and using (1.1.9), we have $c(\xi)^2 = \mathrm{Symb}(D^2)(\xi) = -|\xi|^2 \,\mathrm{id}_E$. In particular,

$$c(\xi)c(\eta) + c(\eta)c(\xi) = (c(\xi)+c(\eta))^2 - c(\xi)^2 - c(\eta)^2 = (-|\xi+\eta|^2 + |\xi|^2 + |\eta|^2)\,\mathrm{id}_E = -2\langle \xi, \eta\rangle \,\mathrm{id}_E$$

for all $\xi, \eta \in T^*M$, so c is a Clifford module structure on E. ■

While every Clifford module structure on E gives rise to Dirac type operators, there is no canonical choice of such an operator. We next introduce Dirac bundles in the sense of Lawson and Michelsohn, see [LM89], which do come with a preferred Dirac type operator:

Definition 1.1.11. A *Dirac bundle* (E, M, c, ∇^E) is a Hermitian vector bundle E over a Riemannian manifold M together with a Clifford module structure $c\colon T^*M \otimes E \to E$ and a metric connection ∇^E on E such that

(i) for all $\xi \in T^*M$ and $e_1, e_2 \in E$ over the same basepoint,
$$\langle c(\xi \otimes e_1), e_2\rangle = -\langle e_1, c(\xi \otimes e_2)\rangle,$$
i.e., the endomorphisms $c(\xi) \coloneqq c(\xi \otimes \bullet)$ of E are skew-Hermitian, and

(ii) for all $X \in \Gamma(M, TM)$, $\alpha \in \Omega^1(M)$, and $s \in \Gamma(M,E)$,
$$\nabla^E_X(c(\alpha \otimes s)) = c(\nabla^{T^*M}_X \alpha \otimes s) + c(\alpha \otimes \nabla^E_X s),$$
where ∇^{T^*M} is the (dual of the) Levi–Civita connection. This is equivalent to $\nabla c = 0$, where ∇ is the induced connection on $\mathrm{Hom}(T^*M \otimes E, E)$.

The operator $D^E \coloneqq c \circ \nabla^E$ is called the *Dirac operator* associated to the Dirac bundle (E, M, c, ∇^E), *cf.*, Proposition 1.1.10.

Remark 1.1.12. It can be shown that every bundle of Clifford modules can be made into a Dirac bundle, see [Nic14, Proposition 11.1.65], *i.e.*, one can always find compatible Hermitian metrics and connections. For the proof, one can use the representation theory of spin groups. Moreover, the Dirac operator associated to a Dirac bundle is formally self-adjoint, see [Nic14, Proposition 11.1.66].

If V is a finite dimensional real vector space and $q\colon V \times V \to \mathbb{R}$ is a symmetric bilinear form, then the *Clifford algebra over* (V, q), denoted by $\mathrm{Cliff}(V, q)$, is the associative $\mathbb{R}$ algebra (with unit) generated by V and subject to the relations

$$vw + wv = -2q(v, w)$$

for $v, w \in V$. It may be realized as the quotient of the tensor algebra of V by the ideal generated by $\{v \otimes w + w \otimes v + 2q(v, w) : v, w \in V\}$, and is characterized by the following *universal property*: for every associative $\mathbb{R}$-algebra A with unit 1_A and every linear map $\varphi\colon V \to A$ such that $\varphi(v)\varphi(w) + \varphi(w)\varphi(v) = -2q(v, w)\,1_A$ for all $v, w \in V$, there exists a unique algebra homomorphism $\widehat{\varphi}\colon \mathrm{Cliff}(V, q) \to A$ satisfying $\widehat{\varphi} \circ \iota = \varphi$, where $\iota\colon V \hookrightarrow \mathrm{Cliff}(V, q)$ is the inclusion.

If E is any other real vector space, then any linear map $c\colon V \to \mathrm{End}(E)$ with $c(v)c(w) + c(w)c(v) = -2q(v, w)\,\mathrm{id}_E$ for all $v, w \in V$ extends in a unique way to an algebra homomorphism $\widehat{c}\colon \mathrm{Cliff}(V, q) \to \mathrm{End}(E)$ by the above universal property, *i.e.*, E is made into a module over the algebra $\mathrm{Cliff}(V, q)$. We will continue to denote $\widehat{c}$ simply by c.

Example 1.1.13. Let $(V, \langle \bullet, \bullet \rangle)$ be a finite dimensional Euclidean vector space, and consider the exterior algebra ΛV. For $v \in V$, define $c(v) \coloneqq \varepsilon(v) - \mathrm{ins}_{v^\sharp} \in \mathrm{End}(\Lambda V)$, where $\varepsilon(v)(\alpha) \coloneqq v \wedge \alpha$, and $\mathrm{ins}_{v^\sharp}$ is the insertion operator (using the identification $V \cong V^{**}$). Then $c(v)c(w) + c(w)c(v) = -2(\varepsilon(v) \circ \mathrm{ins}_{w^\sharp} + \mathrm{ins}_{v^\sharp} \circ \varepsilon(w)) = -2\langle v, w \rangle\,\mathrm{id}_V$ for $v, w \in V$, so we obtain an algebra homomorphism $c\colon \mathrm{Cliff}(V, \langle \bullet, \bullet \rangle) \to \mathrm{End}(\Lambda V)$. Define the *symbol map* $\sigma\colon \mathrm{Cliff}(V, \langle \bullet, \bullet \rangle) \to \Lambda V$ by $\sigma(x) \coloneqq c(x)1$, where 1 is the unit in ΛV. Then σ is bijective, see [BGV04, Proposition 3.5], with its inverse $\Lambda V \to \mathrm{Cliff}(V, \langle \bullet, \bullet \rangle)$ being called the *quantization map*. If $\{e_j\}_{j=1}^n$ is an orthonormal basis of V, then the quantization map is given by sending $e_{j_1} \wedge \cdots \wedge e_{j_m}$ to $c_{j_1} \cdots c_{j_m}$, where $c_j \coloneqq \sigma^{-1}(e_j)$ is the element of $\mathrm{Cliff}(V, \langle \bullet, \bullet, \rangle)$ corresponding to e_j. ♦

Using the tools from the theory of principal fiber bundles, one can transfer these objects and results to the setting of vector bundles over Riemannian manifolds. In particular, there is a bundle $\mathrm{Cliff}(T^*M) \to M$ of algebras, with fiber over $x \in M$ precisely the Clifford algebra over $(T^*_x M, \langle \bullet, \bullet \rangle_x)$, and any Clifford module structure on a vector bundle E as above extends uniquely to a vector bundle morphism $c\colon \mathrm{Cliff}(T^*M) \to \mathrm{End}(E)$. Moreover, the Levi–Civita connection on T^*M extends to a connection on $\mathrm{Cliff}(T^*M)$ which is compatible with the multiplication in $\mathrm{Cliff}(T^*M)$.

The quantization map from Example 1.1.13 allows us to define $c(\alpha) \in \Gamma(M, \mathrm{End}(E))$ for every differential form $\alpha \in \Omega(M)$. If $\alpha \otimes A \in \Omega(M, \mathrm{End}(E))$ with $\alpha \in \Omega(M)$ and $A \in \Gamma(M, \mathrm{End}(E))$, then we extend this to $c(\alpha \otimes A) \coloneqq c(\alpha) \circ A \in \Gamma(M, \mathrm{End}(E))$, and hence we obtain a $C^\infty(M)$-linear map $c\colon \Omega(M, \mathrm{End}(E)) \to \Gamma(M, \mathrm{End}(E))$. If $\alpha_1, \ldots, \alpha_k$ are one-forms and $A \in \Gamma(M, \mathrm{End}(E))$, then $c((\alpha_1 \wedge \cdots \wedge \alpha_k) \otimes A) = c(\alpha_1) \circ \cdots \circ c(\alpha_k) \circ A$. In particular, if ∇ is a connection on E with curvature $R^\nabla \in \Omega^2(M, \mathrm{End}(E))$, then it makes sense to form $c(R^\nabla) \in \Gamma(M, \mathrm{End}(E))$. If $\{e_j\}_{j=1}^n$ is an orthonormal basis of T_xM, with dual basis $\{e^j\}_{j=1}^n$, then

$$c(R^\nabla)|_x = \sum_{j<k} c(e^j) \circ c(e^k) \circ R^\nabla(e_j, e_k) = \frac{1}{2} \sum_{j,k=1}^n c(e^j) \circ c(e^k) \circ R^\nabla(e_j, e_k), \tag{1.1.17}$$

where the second equality is due to (1.1.16) and R^∇ being alternating.

From (1.1.15), we know that if D is a Dirac type operator on E, then D^2 may be written as $\Delta^E + V$ for some connection ∇^E on E and some vector bundle morphism $V\colon E \to E$. The following Theorem computes this representation in case D comes from a Dirac bundle:

Theorem 1.1.14 (General Bochner–Weitzenböck formula)**.** *Let (E, M, c, ∇^E) be a Dirac bundle with associated Dirac operator D^E. Then*

$$(D^E)^2 = \Delta^E + c(R^E),$$

where $\Delta^E = \nabla^{E,\dagger}\nabla^E$ is the Bochner Laplacian, and R^E is the curvature of ∇^E.

Proof. See [LM89, Theorem II.8.2] or [Nic14, Theorem 11.1.67]. ■

Example 1.1.15. If M is a Riemannian manifold, then $(M, \Lambda T^*M, c, \nabla)$, with ∇ the Levi–Civita connection and $c(v) := \varepsilon(v) - \mathrm{ins}_{v^\sharp}$, *cf.*, Example 1.1.13, is a Dirac bundle, see [Nic14, Proposition 11.2.1]. The associated Dirac operator is $d + d^\dagger$, which squares to the *Hodge Laplacian* $d^\dagger d + dd^\dagger$. The endomorphism from the Bochner–Weitzenböck formula splits as $c(R^{\Lambda T^*M}) = \bigoplus_{k\geq 0} \mathcal{K}_k$, with $\mathcal{K}_k \in \mathrm{End}(\Lambda^k T^*M)$. One can show that $\mathcal{K}_0 = 0$ and

$$\mathcal{K}_1(\alpha) = \mathrm{Ric}_M(\alpha^\sharp, \bullet)$$

for $\alpha \in \Lambda^1 T^*M$, where $\mathrm{Ric}_M(X, Y) := \sum_{j=1}^{\dim(M)} \langle R^M(X, e_j)e_j, Y\rangle$ is the *Ricci tensor* of M. We refer to [Nic14, section 11.2.1] for a proof. ◆

We will encounter the Bochner–Weitzenböck formula for a Hermitian holomorphic vector bundle over a Kähler manifold in section 3.1.1. The corresponding Dirac operator then squares to twice the Dolbeault Laplacian.

1.2. Hilbert complexes

In this section we will review some of the basics of the theory of Hilbert complexes. For a more in-depth introduction, with focus on different aspects of the theory, see [BL92]. In addition, we will supplement this by adding concepts and results which are standard in the L^2 theory of the $\overline{\partial}$-complex from several complex variables. For a quick primer on the basic concepts of unbounded operator theory, see the beginning of appendix C. By a *Hilbert (cochain) complex* $(H, \mathcal{D}, d)$ (or simply (H, d)) we mean a graded Hilbert space $H = \bigoplus_{i\in\mathbb{Z}} H_i$ with only finitely many nonzero (mutually orthogonal) terms, a dense graded linear subspace

$\mathcal{D} = \bigoplus_{i\in\mathbb{Z}} \mathcal{D}_i$, and a closed linear operator $d\colon \mathcal{D} \to \mathcal{D}$ on H of degree 1 such that $d \circ d = 0$. We therefore obtain the (cochain) complex

$$\cdots \xrightarrow{d_{i-2}} \mathcal{D}_{i-1} \xrightarrow{d_{i-1}} \mathcal{D}_i \xrightarrow{d_i} \mathcal{D}_{i+1} \xrightarrow{d_{i+1}} \cdots$$

with closed and densely defined *differentials* $d_i := d|_{\mathcal{D}_i}\colon \mathcal{D}_i \to \mathcal{D}_{i+1}$. Hilbert complexes were most prominently studied in [BL92], but the concept also appears in some earlier works [GV82; Vas80]. They are useful in order to formalize the basic operator theoretic properties common to boundary value problems for elliptic complexes.

An important operator associated to (H, d) is its *Laplacian*, defined by $\Delta := \bigoplus_{i\in\mathbb{Z}} \Delta_i$ with

$$\Delta_i := d_i^* d_i + d_{i-1} d_{i-1}^* \quad \text{on} \quad \operatorname{dom}(\Delta_i) := \{x \in \mathcal{D}_i \cap \mathcal{D}_i^* : dx \in \mathcal{D}_{i+1}^* \text{ and } d^*x \in \mathcal{D}_{i-1}\},$$

where $\mathcal{D}_i^* \subseteq H_i$ is the domain of d_{i-1}^*, the adjoint of d_{i-1}. This gives the chain complex

$$\cdots \xleftarrow{d_{i-2}^*} \mathcal{D}_{i-1}^* \xleftarrow{d_{i-1}^*} \mathcal{D}_i^* \xleftarrow{d_i^*} \mathcal{D}_{i+1}^* \xleftarrow{d_{i+1}^*} \cdots$$

and we write $\mathcal{D}^* := \bigoplus_{i\in\mathbb{Z}} \mathcal{D}_i^* \subseteq H$ and $d^* := \bigoplus_{i\in\mathbb{Z}} d_i^*$. Each Δ_i is a nonnegative self-adjoint operator on H_i, and it is useful to study the Laplacian in order to gain insight into the solutions of the inhomogeneous d-equation, as we will see below. The fact that the Laplacian is self-adjoint is usually attributed to Gaffney [Gaf55], where the corresponding result for the de Rham complex is found.

We next describe the quadratic form associated to Δ, and refer to appendix C.2 for some background on quadratic forms associated to self-adjoint operators. As a general notation in this thesis, we write $S \subseteq T$ for two (partially defined) operators S and T from one Hilbert space H_1 to another Hilbert space H_2 if $\operatorname{dom}(S) \subseteq \operatorname{dom}(T)$ and $Sx = Tx$ for all $x \in \operatorname{dom}(S)$. In other words, $S \subseteq T$ if and only if $\operatorname{Graph}(S) \subseteq \operatorname{Graph}(T)$, where $\operatorname{Graph}(S) := \{(x, Sx) : x \in \operatorname{dom}(S)\} \subseteq H_1 \times H_2$ is the *graph* of S.

Lemma 1.2.1. *Let (H, d) be a Hilbert complex with Laplacian Δ. Then $d + d^*$ (with domain $\mathcal{D} \cap \mathcal{D}^*$) is self-adjoint, $\Delta = (d + d^*)^2$, and the quadratic form Q_Δ associated to Δ satisfies* $\operatorname{dom}(Q_\Delta) = \mathcal{D} \cap \mathcal{D}^*$ *and*

$$Q_\Delta(x, y) = \langle dx, dy\rangle + \langle d^*x, d^*y\rangle \tag{1.2.1}$$

for $x, y \in \operatorname{dom}(Q_\Delta)$.

Proof. Since $\operatorname{dom}(\Delta) \subseteq \mathcal{D} \cap \mathcal{D}^*$, the operator $d + d^*$ is densely defined and therefore has an adjoint, which satisfies

$$d + d^* = d^* + d^{**} \subseteq (d + d^*)^*.$$

In other words, $d + d^*$ is symmetric. The reverse inclusion is shown in detail in [GMM11, Proposition 2.3]. We have $(d + d^*)^2 \supseteq dd + dd^* + d^*d + d^*d^* = \Delta$ since $dd = d^*d^* = 0$, and hence $\Delta = (d + d^*)^2$ because self-adjoint operators do not have proper self-adjoint extensions. Formula (1.2.1) follows from Example C.2.3 applied to the operator $T = d + d^*$, and the fact that $\langle dx, d^*y\rangle = 0$ for all $x, y \in \operatorname{dom}(Q_\Delta)$. ∎

If $(H, \mathcal{D}, d)$ and $(H', \mathcal{D}', d')$ are Hilbert complexes, then a graded linear map $g\colon H \to H'$ (of degree 0) is called a *morphism of Hilbert complexes* if g is bounded (*i.e.*, $g_i := g|_{H_i}\colon H_i \to H_i'$ is bounded for every $i \in \mathbb{Z}$) and $g \circ d \subseteq d' \circ g$. In particular, $g(\mathcal{D}) \subseteq \mathcal{D}'$. An *isomorphism of Hilbert complexes* is a bijective morphism of Hilbert complexes $g\colon H \to H'$ such that $g \circ d = d' \circ g$ (in the sense of unbounded operators; in particular, $g(\mathcal{D}) = \mathcal{D}'$). A *unitary equivalence* between Hilbert complexes is a unitary isomorphism of Hilbert complexes. Note that if $g\colon (H, d) \to (H', d')$ is such a unitary equivalence, then $d^* \circ g^{-1} = g^{-1} \circ d'^*$ and hence $g \circ \Delta = \Delta' \circ g$, so that the Laplacians are unitarily equivalent.

If $\{(H^j, d^j) : j \in F\}$ is a finite collection of Hilbert complexes, then their *direct sum* is the Hilbert complex $\bigoplus_{j\in F}(H^j, d^j) := (\bigoplus_{j\in F} H^j, \bigoplus_{j\in F} d^j)$. Evidently, its Laplacian is given by $\bigoplus_{j\in F} \Delta^j$, with Δ^j the Laplacian of (H^j, d^j).

The *cohomology* of a Hilbert cochain complex (H, d) is the graded vector space

$$\mathcal{H}(H, d) := \bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i(H, d), \quad \text{where} \quad \mathcal{H}^i(H, d) := \ker(d_i) \,/\, \operatorname{img}(d_{i-1}).$$

The *reduced cohomology* of (H, d) is

$$\overline{\mathcal{H}}(H, d) := \bigoplus_{i\in\mathbb{Z}} \overline{\mathcal{H}}^i(H, d), \quad \text{where} \quad \overline{\mathcal{H}}^i(H, d) := \ker(d_i) \,/\, \overline{\operatorname{img}(d_{i-1})}.$$

In general, the differentials of a Hilbert complex do not have closed range, so that typically only $\overline{\mathcal{H}}(H, d)$ will be a Hilbert space in a natural way. One of the main tools available is the Hodge decomposition, see [BL92, Lemma 2.1]:

Proposition 1.2.2 (Weak Hodge decomposition)**.** *Every Hilbert complex (H, d) induces an orthogonal decomposition*

$$H_i = \ker(\Delta_i) \oplus \overline{\operatorname{img}(d_{i-1})} \oplus \overline{\operatorname{img}(d_i^*)}. \quad (i \in \mathbb{Z})$$

Moreover, the space of harmonic elements*,*

$$\ker(\Delta) = \bigoplus_{i\in\mathbb{Z}} \big(\ker(d_i) \cap \ker(d_{i-1}^*)\big),$$

is canonically isomorphic to $\overline{\mathcal{H}}(H, d)$, in the sense that every equivalence class in $\overline{\mathcal{H}}(H, d)$ has a unique harmonic representative.

Let $P^d\colon H \to H$ denote the orthogonal projection of H onto $\ker(d)$. The *minimal (or canonical) solution operator* to (H, d) is the closed operator

$$S = S(H, d)\colon \operatorname{img}(d) \subseteq H \to \ker(d)^\perp \subseteq H, \quad S(dx) := (\mathrm{id}_H - P^d)x. \tag{1.2.2}$$

This is well-defined since $\ker(d) = \ker(\mathrm{id}_H - P^d)$. We write $S_i = S_i(H, d)\colon \operatorname{img}(d_{i-1}) \subseteq H_i \to H_{i-1}$ for its restriction to H_i. By definition, S gives the norm-minimal solution to the inhomogeneous d-equation,

$$d(Sy) = y \quad \text{and} \quad Sy \perp \ker(d)$$

for $y \in \operatorname{img}(d)$. It is a map of degree -1 (if (H, d) is a cochain complex), and its closedness is an easy consequence of closedness of d. Actually,

$$\operatorname{Graph}(S) = \operatorname{img}((d, (\operatorname{id}_H - P^d)|_{\mathcal{D}})\colon \mathcal{D} \to H \oplus H)$$

and the kernel of the map $(d, (\operatorname{id}_H - P^d)|_{\mathcal{D}})$ is $\ker(d)$. For $x \in \mathcal{D} \cap \ker(d)^\perp$, we have

$$\|dx\|^2 + \|(\operatorname{id}_H - P^d)x\|^2 = \|dx\|^2 + \|x\|^2 \geq \|x\|^2,$$

hence $(d, (\operatorname{id}_H - P^d)|_{\mathcal{D}})$ has closed range and S is a closed operator.

The remaining results of this section are well-known for the (weak extension of the) Dolbeault complex on Hermitian manifolds. As a (non-exhaustive) list of references, we cite [CS01; FK72; Has14; Hör65; KN65; Str10]. For the convenience of the reader, we provide here the proofs of the corresponding results for Hilbert complexes. Note that while most of those references do not consider the case where Δ_i has a nontrivial kernel (since the complex Laplacian on bounded pseudoconvex domains in $\mathbb{C}^n$ is injective), this is easily incorporated into the arguments, see also [ØR14; Rup11].

Lemma 1.2.3. *Let (H, d) be a Hilbert complex. Then $S_i\colon \operatorname{img}(d_{i-1}) \subseteq H_i \to H_{i-1}$ is bounded if and only if d_{i-1} has closed range. In this case we extend S_i to H_i by zero on $\operatorname{img}(d_{i-1})^\perp$.*

Proof. If d_{i-1} has closed range, then S_i is a closed and everywhere defined operator on the Hilbert space $\operatorname{img}(d_{i-1})$, hence bounded by the closed graph theorem. Conversely, if S_i is bounded there exists $C > 0$ such that $\|S(dx)\| \leq C\|dx\|$ for all $x \in \mathcal{D}_{i-1}$. If $x \in \mathcal{D}_{i-1} \cap \ker(d_{i-1})^\perp$, then $S(dx) = x$ and hence $\|x\| = \|S(dx)\| \leq C\|dx\|$, which shows that d_{i-1} has closed range. ∎

The next result shows that the minimal solution operator is closely related to the Laplacian:

Proposition 1.2.4. *Let (H, d) be a Hilbert complex and define*

$$N = N(H, d) \coloneqq \left(\Delta|_{\operatorname{dom}(\Delta) \cap \ker(\Delta)^\perp}\right)^{-1}\colon \operatorname{img}(\Delta) \to H$$

as the inverse of the Laplacian. We write $N_i = N_i(H, d)\colon \operatorname{img}(\Delta_i) \to H_i$ for its restriction to H_i. Then:

(i) $dN = Nd$ on $\mathcal{D} \cap \operatorname{img}(\Delta)$ and $d^*N = Nd^*$ on $\mathcal{D}^* \cap \operatorname{img}(\Delta)$.

(ii) On $\operatorname{img}(d) \cap \operatorname{img}(\Delta)$ we have

$$S = d^*N. \tag{1.2.3}$$

(iii) On $\mathcal{D} \cap d^{-1}(\operatorname{img}(\Delta))$ we have

$$I - P^d = d^*Nd. \tag{1.2.4}$$

Proof. If $x \in \mathcal{D}_i \cap \operatorname{img}(\Delta_i)$, then $x = \Delta_i y$ for some $y \in \operatorname{dom}(\Delta_i) \cap \ker(\Delta_i)^\perp$. It follows that $d_i y \in \operatorname{dom}(\Delta_{i+1})$ and

$$Nd_i x = Nd_i\Delta_i y = Nd_i d_i^* d_i y = N(d_{i+1}^* d_{i+1} + d_i d_i^*) d_i y = d_i y = d_i N x.$$

This shows the first equation in (i), the other one follows similarly. If $x \in \operatorname{img}(d_{i-1}) \cap \operatorname{img}(\Delta_i)$, then

$$x = \Delta_i N_i x = d_i^* d_i N_i x + d_{i-1} d_{i-1}^* N_i x.$$

Because $x \in \ker(d_i)$ and $d_i \circ d_{i-1} = 0$, this implies $d_i^* d_i N_i x \in \ker(d_i) \cap \operatorname{img}(d_i^*) = 0$. Therefore, $x = d_{i-1} d_{i-1}^* N_i x$ and

$$S_i x = (I - P^d) d_{i-1}^* N_i x = d_{i-1}^* N_i x$$

since $\overline{\operatorname{img}(d_{i-1}^*)} = \operatorname{img}(I - P^d) \cap H_{i-1}$. This shows (1.2.3), and (1.2.4) is immediate from the definition of S. ∎

Lemma 1.2.5. *Let (H, d) be a Hilbert complex. Then the following are equivalent:*

(i) $N_i \colon \operatorname{img}(\Delta_i) \to H_i$ *is bounded.*
(ii) Δ_i *has closed range.*
(iii) d_{i-1} *and* d_i *both have closed range.*
(iv) There is $C > 0$ *such that, for all* $x \in \mathcal{D}_i \cap \mathcal{D}_i^* \cap \ker(\Delta_i)^\perp$,

$$\|x\|^2 \le C\big(\|d_i x\|^2 + \|d_{i-1}^* x\|^2\big).$$

(v) $S_i \colon \operatorname{img}(d_{i-1}) \to H_{i-1}$ *and* $S_{i+1} \colon \operatorname{img}(d_i) \to H_i$ *are both bounded.*
(vi) The space $\mathcal{D}_i \cap \mathcal{D}_i^* \cap \ker(\Delta_i)^\perp$ *is a Hilbert space with inner product*

$$(x, y) \mapsto \langle d_i x, d_i y \rangle + \langle d_{i-1}^* x, d_{i-1}^* y \rangle \tag{1.2.5}$$

and the inclusion $j_i \colon \mathcal{D}_i \cap \mathcal{D}_i^* \cap \ker(\Delta_i)^\perp \hookrightarrow H_i$ *is continuous.*

In this case, we extend N_i *by zero on* $\operatorname{img}(\Delta_i)^\perp = \ker(\Delta_i)$.

Proof. Because N_i is closed, (ii)⇒(i). Conversely, suppose N_i is bounded and take $u_j \to u$ with $u_j \in \operatorname{img}(\Delta_i)$. Then $N_i u_j \to v$ for some $v \in H$ and we have $\Delta_i N_i u_j = u_j$. As Δ_i is closed, $v \in \operatorname{dom}(\Delta_i)$ and $\Delta_i v = u$, hence $u \in \operatorname{img}(\Delta_i)$. Thus, (i)⇔(ii).

We now show (ii)⇒(iii), so assume that Δ_i has closed range. For $x \in \mathcal{D}_i \cap \ker(d_i)^\perp \subseteq \ker(\Delta_i)^\perp = \operatorname{img}(\Delta_i)$, we have

$$\|x\|^2 = \langle \Delta_i N_i x, x \rangle = \langle d_i^* d_i N_i x, x \rangle + \langle d_{i-1} d_{i-1}^* N_i x, x \rangle = \langle d_i N_i x, d_i x \rangle \le C \|x\| \|d_i x\|$$

because $d_{i-1} d_{i-1}^* N_i x \in \ker(d_i) \perp x$, and the operators $d_i N_i$ and $d_{i-1}^* N_i$ are bounded on $\operatorname{img}(\Delta_i)$ since

$$\|d_i N_i y\|^2 + \|d_{i-1}^* N_i y\|^2 = \langle \Delta_i N_i y, N_i y \rangle = \langle y, N_i y \rangle \quad (y \in \operatorname{img}(\Delta_i))$$

and N_i is bounded by (i). Therefore, d_i has closed range. Interchanging the roles of d_i and d_{i-1}^*, one shows that the latter operator also has closed range.

Now assume that d_{i-1} and d_i have closed range. It follows that d_i^* also has closed range. If $x \in \mathcal{D}_i \cap \mathcal{D}_i^* \cap \ker(\Delta_i)^\perp = \mathcal{D}_i \cap \mathcal{D}_i^* \cap (\operatorname{img}(d_{i-1}) \oplus \operatorname{img}(d_i^*))$, write $x = x_1 + x_2$ with $x_1 \in \mathcal{D}_i \cap \mathcal{D}_i^* \cap \operatorname{img}(d_{i-1})$ and $x_2 \in \mathcal{D}_i \cap \mathcal{D}_i^* \cap \operatorname{img}(d_i^*)$. There exist $C_1, C_2 > 0$ such that

$$\|x_1\|^2 \le C_1 \|d_{i-1}^* x_1\|^2 = C_1 \|d_{i-1}^* x\|^2 \quad \text{and} \quad \|x_2\|^2 \le C_2 \|d_i x_2\|^2 = C_2 \|d_i x\|^2$$

by our assumptions on d_{i-1} and d_i, and hence

$$\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 \le C(\|d_{i-1}^* x\|^2 + \|d_i x\|^2)$$

with $C := \max\{C_1, C_2\}$. This shows (iii)⇒(iv), and (iv)⇒(ii) is immediate as $\operatorname{dom}(\Delta_i) \subseteq \mathcal{D}_i \cap \mathcal{D}_i^*$ and $\|d_i x\|^2 + \|d_{i-1}^* x\|^2 = \langle \Delta_i x, x\rangle \le \|\Delta_i x\|\|x\|$ for $x \in \operatorname{dom}(\Delta_i)$. The equivalence (iii)⇔(v) follows from Lemma 1.2.3. The equivalence (iv)⇔(vi) is straightforward. For completeness of $\mathcal{D}_i \cap \mathcal{D}_i^* \cap \ker(\Delta_i)^\perp$ with respect to (1.2.5) one uses closedness of d_i and d_{i-1}^*. ■

Proposition 1.2.6. *Let (H, d) be a Hilbert complex, and assume that any of the equivalent statements of Lemma 1.2.5 holds. We extend N_i by zero on $\operatorname{img}(\Delta_i)^\perp$. Then*

(i) $dN = Nd$ on $\mathcal{D}_i$ and $d^*N = Nd^*$ on $\mathcal{D}_i^*$,
(ii) $S_i = d^*N_i$ on H_i,
(iii) $N_i = S_i^* S_i + S_{i+1} S_{i+1}^*$, where S_i and S_{i+1} are the extensions by zero, see Lemma 1.2.3,
(iv) $\max\{\|S_i\|^2, \|S_{i+1}\|^2\}$ is bounded from above by $(\inf(\sigma(\Delta_i) \setminus \{0\}))^{-1}$, the reciprocal of the spectral gap *of Δ_i,*
(v) $N_i = j_i \circ j_i^*$, where $j_i \colon \mathcal{D}_i \cap \mathcal{D}_i^* \cap \ker(\Delta_i)^\perp \to H_i$ is the inclusion from Lemma 1.2.5 and the adjoint is with respect to the inner product (1.2.5),
(vi) $\inf \sigma(\Delta_i) > 0$ if and only if $\ker(\Delta_i) = 0$, and
(vii) $\inf \sigma_e(\Delta_i) > 0$ if and only if $\dim(\ker(\Delta_i)) < \infty$.

Proof. On $\mathcal{D}_i \cap \operatorname{img}(\Delta_i)^\perp \subseteq \ker(d_i)$ we have $dN = 0$ and $Nd = 0$, so $dN = Nd$ holds on $\mathcal{D}_i$, and similarly one shows $d^*N = Nd^*$ on $\mathcal{D}_i^*$.

By the Hodge decomposition, $\operatorname{img}(\Delta_i) = \operatorname{img}(d_{i-1}) \oplus \operatorname{img}(d_i^*)$. Thus, $\operatorname{img}(d_{i-1}) \subseteq \operatorname{img}(\Delta_i)$ and hence $S = d^*N$ on $\operatorname{img}(d_{i-1})$. Since $S|_{\operatorname{img}(d_{i-1})^\perp} = 0$ by definition, it remains to show that d^*N also vanishes on $\operatorname{img}(d_{i-1})^\perp$. As $\operatorname{img}(d_{i-1})^\perp = \ker(\Delta_i) \oplus \operatorname{img}(d_i^*)$ and $N|_{\ker(\Delta_i)} = 0$, we are left with showing that $d^*N|_{\operatorname{img}(d_i^*)} = 0$. Now if $y \in \mathcal{D}_{i+1}^* = \operatorname{dom}(d_i^*)$, then $d^*Nd_i^*y = d^*d_i^*Ny = 0$ by (i). This shows that $S = d^*N$ on H_i.

We have $d_{i-1}^* N_i x \in \mathcal{D}_{i-1}$ and $d_i N_i x \in \mathcal{D}_{i+1}^*$ for $x \in H_i$, and therefore

$$\begin{aligned} N_i x &= N_i \Delta_i N_i x \\ &= (N_i d_{i-1})(d_{i-1}^* N_i)x + (N_i d_i^*)(d_i N_i)x \\ &= (d_{i-1} N_i)(d_{i-1}^* N_i)x + (d_i^* N_i)(d_i N_i)x \end{aligned}$$

by (i). Applying $S_i^* = d_{i-1} N_{i-1}$ and $S_{i+1}^* = d_i N_i$ shows (iii).

Concerning (iv), we have, by (iii) and with $\lambda_0 := \inf(\sigma(\Delta_i) \setminus \{0\})$,

$$\|S_i x\|^2 + \|S_{i+1}^* x\|^2 = \langle (S_i^* S_i + S_{i+1} S_{i+1}^*)x, x\rangle = \langle N_i x, x\rangle \le \|N_i\|\|x\|^2 = \lambda_0^{-1}\|x\|^2$$

for all $x \in H_i$. Therefore,

$$\|S_i\|^2 = \sup_{x \in H_i \setminus \{0\}} \frac{\|S_i x\|^2}{\|x\|^2} \le \lambda_0^{-1}$$

and

$$\|S_{i+1}\|^2 = \|S_{i+1}^*\|^2 = \sup_{x \in H_i \setminus \{0\}} \frac{\|S_{i+1}^* x\|^2}{\|x\|^2} \leq \lambda_0^{-1}.$$

We now show (v). Let $K_i := \mathcal{D}_i \cap \mathcal{D}_i^* \cap \ker(\Delta_i)^\perp$. By Lemma 1.2.5, K_i is a Hilbert space with inner product $Q(x,y) := \langle d_i x, d_i y\rangle + \langle d_{i-1}^* x, d_{i-1}^* y\rangle$. Let $x \in H_i$. Then $j_i^* x \in K_i$, and for $y = y_1 + y_2 \in \operatorname{dom}(\Delta_i)$ with $y_1 \in \ker(\Delta_i)$ and $y_2 \perp \ker(\Delta_i)$ we have

$$\langle \Delta_i y, j_i^* x\rangle = \langle d_i y_2, d_i j_i^* x\rangle + \langle d_{i-1}^* y_2, d_{i-1}^* j_i^* x\rangle = Q(y_2, j_i^* x) = \langle j_i y_2, x\rangle.$$

Thus, $y \mapsto \langle \Delta_i y, j_i^* x\rangle$ is continuous on $\operatorname{dom}(\Delta_i)$, so $j_i^* x \in \operatorname{dom}(\Delta_i^*) = \operatorname{dom}(\Delta_i)$ and hence $j_i \circ j_i^*$ maps H_i to $\operatorname{dom}(\Delta_i) \cap \ker(\Delta_i)^\perp \subseteq H_i$.

Because $\ker(\Delta_i) \subseteq \operatorname{img}(j_i)^\perp = \ker(j_i^*)$, we have $(j_i \circ j_i^*)|_{\ker(\Delta_i)} = 0 = N_i|_{\ker(\Delta_i)}$. Let $y = y_1 + y_2 \in \mathcal{D}_i \cap \mathcal{D}_i^*$ with $y_1 \in \ker(\Delta_i)$ and $y_2 \perp \ker(\Delta_i)$. If $x \in \operatorname{img}(\Delta_i)$, then

$$\langle \Delta_i (j_i \circ j_i^*) x, y\rangle = Q(j_i^* x, y_2) = \langle x, j_i y_2\rangle = \langle x, y_2\rangle = \langle x, y\rangle,$$

hence $\Delta_i (j_i \circ j_i^*)|_{\operatorname{img}(\Delta_i)} = \operatorname{id}_{\operatorname{img}(\Delta_i)}$ since $\mathcal{D}_i \cap \mathcal{D}_i^*$ is dense in H_i (it contains the domain of the self-adjoint operator Δ_i). Now let $x \in \operatorname{dom}(\Delta_i) \cap \ker(\Delta_i)^\perp$. Then

$$\langle (j_i \circ j_i^*) \Delta_i x, y\rangle = \langle \Delta_i x, (j_i \circ j_i^*) y_2\rangle = Q(x, j_i^* y_2) = \langle x, y_2\rangle = \langle x, y\rangle$$

and hence $(j_i \circ j_i^*) \Delta_i|_{\operatorname{dom}(\Delta_i) \cap \ker(\Delta_i)^\perp} = \operatorname{id}_{\operatorname{dom}(\Delta_i) \cap \ker(\Delta_i)^\perp}$. This shows that

$$(j_i \circ j_i^*)|_{\operatorname{img}(\Delta_i)} = \left(\Delta_i|_{\operatorname{dom}(\Delta_i) \cap \ker(\Delta_i)^\perp}\right)^{-1} \colon \operatorname{img}(\Delta_i) \to H_i$$

and therefore $N_i = j_i \circ j_i^*$.

The orthogonal decomposition $H_i = \ker(\Delta_i) \oplus \operatorname{img}(\Delta_i)$ induces a unitary equivalence of Δ_i with the self-adjoint operator

$$0 \oplus \Delta_i|_{\operatorname{img}(\Delta_i) \cap \operatorname{dom}(\Delta_i)} \colon \ker(\Delta_i) \oplus (\operatorname{img}(\Delta_i) \cap \operatorname{dom}(\Delta_i)) \to \ker(\Delta_i) \oplus \operatorname{img}(\Delta_i),$$

hence $\sigma(\Delta_i) \setminus \{0\} = \sigma(\Delta_i|_{\operatorname{img}(\Delta_i) \cap \operatorname{dom}(\Delta_i)})$ and $\sigma_e(\Delta_i) \setminus \{0\} = \sigma_e(\Delta_i|_{\operatorname{img}(\Delta_i) \cap \operatorname{dom}(\Delta_i)})$ since $0 \notin \sigma(\Delta_i|_{\operatorname{img}(\Delta_i)})$ by the boundedness of $N_i|_{\operatorname{img}(\Delta_i)}$. Moreover, $0 \in \sigma(\Delta_i)$ (resp. $0 \in \sigma_e(\Delta_i)$) if and only if $\ker(\Delta_i) \neq 0$ (resp. $\dim(\ker(\Delta_i)) = \infty$). This immediately gives (vi) and (vii). ■

Remark 1.2.7. Concerning items (vi) and (vii) of Proposition 1.2.6, one even has that $\inf \sigma(\Delta_i) > 0$ (resp. $\inf \sigma_e(\Delta_i) > 0$) if and only if the conditions of Lemma 1.2.5 are satisfied and $\ker(\Delta_i) = 0$ (resp. $\dim(\ker(\Delta_i)) < \infty$). This is Proposition 2.2 (resp. Proposition 2.3) of [Fu10].

We are interested in determining whether N and S are compact operators. Recall that the *essential spectrum* $\sigma_e(T)$ of a normal operator T on a Hilbert space is the set of complex numbers which are accumulation points of its spectrum or eigenvalues of infinite multiplicity. We refer to appendix C.1 for the precise definition and more information on $\sigma_e(T)$.

Proposition 1.2.8. *Let (H,d) be a Hilbert complex and assume that $i \in \mathbb{Z}$ is such that any of the equivalent statements of Lemma 1.2.5 holds. Then the following are equivalent:*

(i) $N_i \colon H_i \to H_i$ is compact.

(ii) $S_i \colon H_i \to H_{i-1}$ *and* $S_{i+1} \colon H_{i+1} \to H_i$ *are both compact.*
(iii) $j_i \colon \mathcal{D}_i \cap \mathcal{D}_i^* \cap \ker(\Delta_i)^\perp \hookrightarrow H_i$ *from Lemma 1.2.5 is compact.*
(iv) $\sigma_e(\Delta_i) \subseteq \{0\}$.

Proof. If S_i and S_{i+1} are compact (in particular: bounded), then (iii) of Proposition 1.2.6 shows that N_i is also compact. Conversely, S_i and S_{i+1} are compact as soon as N_i is since both $S_i^* S_i$ and $S_{i+1} S_{i+1}^*$ are nonnegative operators. Indeed, if A and B are bounded nonnegative operators on a Hilbert space $(K, \langle \bullet, \bullet \rangle)$ with $A \leq B$ and B compact, then the compact operator $B^{1/2}$ satisfies

$$\|A^{1/2}x\|^2 = \langle Ax, x\rangle \leq \langle Bx, x\rangle = \|B^{1/2}x\|^2$$

for every $x \in K$. Since $B^{1/2}x_j \to 0$ in K for every weak null sequence x_j, we see that $A^{1/2}$ is compact, and hence A is also compact. Now apply this to $S_i^* S_i \leq N_i$ and $S_{i+1} S_{i+1}^* \leq N_i$. Since $N_i = j_i \circ j_i^*$ by Proposition 1.2.6, it is clear that N_i is compact if and only if j_i is.

We know that $\sigma_e(\Delta_i) \setminus \{0\} = \sigma_e(\Delta_i|_{\mathrm{img}(\Delta_i) \cap \mathrm{dom}(\Delta_i)})$, see the proof of item (vii) of Proposition 1.2.6. But N_i is compact if and only if $N_i|_{\mathrm{img}(\Delta_i)}$ is, and this is the case if and only if $\Delta_i|_{\mathrm{img}(\Delta_i) \cap \mathrm{dom}(\Delta_i)}$ has compact resolvent, *i.e.*, $\sigma_e(\Delta_i|_{\mathrm{img}(\Delta_i) \cap \mathrm{dom}(\Delta_i)}) = \emptyset$. ■

1.3. Strong and weak extensions of differential operators

Let M be a Riemannian manifold, and let $E, F \to M$ be Hermitian vector bundles over M. This data allows us to define the Hilbert spaces $L^2(M, E)$ and $L^2(M, F)$ of square-integrable sections of E and F, respectively, with inner product (1.1.1). If $D \colon \Gamma(M, E) \to \Gamma(M, F)$ is a (linear) differential operator, then it makes sense to ask whether the linear map

$$D_{cc} := D|_{\Gamma_{cc}(M,E)} \colon \Gamma_{cc}(M, E) \to L^2(M, F)$$

extends to a closed operator from $L^2(M, E)$ to $L^2(M, F)$, and if so, in how many different ways this is possible.

Let D_{cc}^* denote the Hilbert space adjoint of the densely defined operator D_{cc}. The definition of $D^\dagger$ implies that $(D^\dagger)_{cc} \subseteq D_{cc}^*$ (recall that this means $\mathrm{Graph}((D^\dagger)_{cc}) \subseteq \mathrm{Graph}(D_{cc}^*)$), and the general theory of unbounded operator then states that D_{cc} is closable since its adjoint is densely defined. To save on notational clutter, we shall say that a linear operator $A \colon \mathrm{dom}(A) \subseteq L^2(M, E) \to L^2(M, F)$ is an *extension of* D (or "A *extends* D") if $D_{cc} \subseteq A$, and a *closed extension of* D if, in addition, A is closed.

Definition 1.3.1. The *strong extension* (or *minimal closed extension*) of D, denoted by D_s, is the closure of $D_{cc} \colon \Gamma_{cc}(M, E) \subseteq L^2(M, E) \to L^2(M, F)$, and the *weak extension* (or *maximal closed extension*) of D is $D_w := (D^\dagger)_{cc}^*$.

Since $D_{cc} = (D^{\dagger\dagger})_{cc} \subseteq (D^\dagger)_{cc}^*$, the operator D_w really is an extension of D. Both the strong and weak extensions of D are closed, and hence $D_s \subseteq D_w$. It holds that $D_w = ((D^\dagger)_s)^*$, since a densely defined operator and its closure have the same adjoint. This immediately implies

$$(D^\dagger)_w = (D_s)^* \quad \text{and} \quad (D^\dagger)_s = (D_w)^*. \tag{1.3.1}$$

By the definition of the formal adjoint of $D^\dagger$, we have $\langle\!\langle (D^\dagger)_{cc} t, s \rangle\!\rangle = \langle\!\langle t, Ds \rangle\!\rangle$ for $t \in \Gamma_{cc}(M, F)$ and $s \in \Gamma_c(M, E)$, thus $\Gamma_c(M, E) \subseteq \operatorname{dom}((D^\dagger)^*_{cc}) = \operatorname{dom}(D_w)$ and $D_w|_{\Gamma_c(M,E)} = D|_{\Gamma_c(M,E)}$. In particular, every extension A of D with $A \subseteq D_w$ satisfies

$$A|_{\operatorname{dom}(A)\cap\Gamma_c(M,E)} = D|_{\operatorname{dom}(A)\cap\Gamma_c(M,E)}. \tag{1.3.2}$$

Remark 1.3.2. If $\varphi\colon \Gamma_c(M, E) \to \mathbb{C}$ is a linear functional, then we may define the functional $D\varphi\colon \Gamma_c(M, F) \to \mathbb{C}$ by $(D\varphi)(s) := \varphi(D^\dagger s)$, and this definition yields a continuous operator $D\colon \mathcal{D}'(M^\circ, E) \to \mathcal{D}'(M^\circ, F)$ between the spaces of *distributional sections* of $E|_{M^\circ}$ and $F|_{M^\circ}$, which are defined by $\mathcal{D}'(M^\circ, E) := (\Gamma_c(M^\circ, E))'$, the dual space, and similarly for $\mathcal{D}'(M^\circ, F)$. Here, $\Gamma_c(M^\circ, E)$ is equipped with the topology induced by the seminorms

$$s \mapsto \max_{j\le k} \|\nabla^j s\|_{L^\infty(K,(T^*M)^{\otimes j}\otimes E)},$$

where $k \in \mathbb{N}$ and K runs through the compact subsets of M°, and where ∇ is any given connection on E and ∇^j denotes the j^{th} covariant derivative, see section 1.3.1. Every $t \in L^1_{\text{loc}}(M, E)$ defines a distribution via $s \mapsto \int_M \langle s, t \rangle \, d\mu_g = \langle\!\langle s, t \rangle\!\rangle$, and this gives an embedding of $L^1_{\text{loc}}(M, E)$ into $\mathcal{D}'(M^\circ, E)$. In particular, we may view $L^p(M, E)$ as a subspace of $\mathcal{D}'(M^\circ, E)$ for $1 \le p \le \infty$. For more details on distributions on manifolds, we refer to [Gro+01, chapter 3]. The definitions may also be generalized to the case where neither M nor E comes equipped with metrics.

As its name suggests, the weak extension admits a description in terms of the distributional action of D:

Proposition 1.3.3. *The weak extension of D satisfies*

$$\operatorname{dom}(D_w) = \{s \in L^2(M, E) : Ds \in L^2(M, F) \text{ in the sense of distributions}\}$$

and $D_w s = Ds$ for $s \in \operatorname{dom}(D_w)$, where Ds is the distributional derivative.

Proof. By definition, the domain of $D_w = (D^\dagger)^*_{cc}$ consists of all $s \in L^2(M, E)$ such that there exists $t \in L^2(M, F)$ with $\langle\!\langle s, D^\dagger u \rangle\!\rangle = \langle\!\langle t, u \rangle\!\rangle$ for all $u \in \operatorname{dom}((D^\dagger)_{cc}) = \Gamma_{cc}(M, F)$. This is equivalent to having $Ds \in L^2(M, F)$ in the sense of distributions, and $Ds = t = D_w s$ in this case. ∎

It is clear that D_s is the smallest extension of D_{cc} to a closed operator from $L^2(M, E)$ to $L^2(M, F)$. The weak extension D_w is maximal in the sense that it is the largest extension of D whose adjoint extends $D^\dagger$, as the next Proposition shows.

Proposition 1.3.4. *Let $D\colon \Gamma(M, E) \to \Gamma(M, F)$ be a differential operator. An extension A of D satisfies $A \subseteq D_w$ if and only if $\Gamma_{cc}(M, F) \subseteq \operatorname{dom}(A^*)$. In this case, A^* is an extension of $D^\dagger$.*

Proof. If A is an extension of D with $A \subseteq D_w$, then $(D^\dagger)_{cc} \subseteq (D^\dagger)_s = (D_w)^* \subseteq A^*$, and hence $\Gamma_{cc}(M,F) \subseteq \mathrm{dom}(A^*)$. Conversely, suppose that $\mathrm{dom}(A^*)$ contains $\Gamma_{cc}(M,F)$. Let $s \in \mathrm{dom}(A) \subseteq L^2(M,E)$ and compute the action of D on the distribution s: if $t \in \Gamma_{cc}(M,F)$, then $t \in \mathrm{dom}(A^*)$ and hence

$$(Ds)(t) = \langle\!\langle s, D^\dagger t\rangle\!\rangle = \langle\!\langle s, (D^\dagger)_w t\rangle\!\rangle = \langle\!\langle s, A^* t\rangle\!\rangle = \langle\!\langle As, t\rangle\!\rangle,$$

where we have used (1.3.1) to see that $A^* \subseteq D_{cc}^* = (D_s)^* = (D^\dagger)_w$. This means that the distribution Ds is identified with the section $As \in L^2(M,F)$, therefore $s \in \mathrm{dom}(D_w)$ and $(D_w)|_{\mathrm{dom}(A)} = A$.

Since A^* is a restriction of $(D^\dagger)_w$ and $\Gamma_{cc}(M,F) \subseteq \mathrm{dom}(A^*)$, we have $(D^\dagger)_{cc} \subseteq A^* \subseteq (D^\dagger)_w$, so A^* is an extension of the formal adjoint of D. ■

Remark 1.3.5. If A is a *symmetric* extension of a (necessarily formally self-adjoint) differential operator $D\colon \Gamma(M,E) \to \Gamma(M,E)$, then $D_{cc} \subseteq A \subseteq A^*$, so $\Gamma_{cc}(M,E) \subseteq \mathrm{dom}(A) \subseteq \mathrm{dom}(A^*)$ and hence the condition of Proposition 1.3.4 is always satisfied. Thus, A is a restriction of D_w. This comes as no surprise, since $D_w = (D^\dagger)_w = (D_s)^* = D_{cc}^*$ and all symmetric extensions of a symmetric operator on a Hilbert space are restrictions of its adjoint.

1.3.1. Sobolev spaces. Let $E \to M$ be a vector bundle over a manifold with (possibly empty) boundary ∂M. Suppose that connections ∇^{TM} and ∇^E are chosen on TM and E, respectively. Denote by $\nabla^{E,j}\colon \Gamma(M,E) \to \Gamma(M,(T^*M)^{\otimes j} \otimes E)$ the j^{th} *covariant derivative*, which is defined as follows: we have induced connections $\widetilde{\nabla}^{E,i}$ on $(T^*M)^{\otimes i} \otimes E$ for $i \geq 1$, and we let $\nabla^{E,j}s := \widetilde{\nabla}^{E,j-1}\widetilde{\nabla}^{E,j-2}\cdots\widetilde{\nabla}^{E,1}\nabla^E s$ for $j \geq 1$. Viewing $\Gamma(M,(T^*M)^{\otimes j} \otimes E)$ as the space of $C^\infty(M)$-multilinear maps $\Gamma(M,TM)^{\times j} \to \Gamma(M,E)$, this may also be defined inductively as $(\nabla^{E,1}u)(X) := \nabla^E_X u$ and

$$(\nabla^{E,j+1}u)(X_0,\dots,X_j) := \nabla^E_{X_0}((\nabla^{E,j}u)(X_1,\dots,X_j)) - \\ - \sum_{i=1}^{j} (\nabla^{E,j}u)(X_1,\dots,X_{i-1},\nabla^{TM}_{X_0}X_i,X_{i+1},\dots,X_j),$$

see [Lee09, section 12.8]. We also set $\nabla^{E,0} := \mathrm{id}_{\Gamma(M,E)}$.

Let $(\nabla^{E,j})_w$ denote the weak extension of the differential operator $\nabla^{E,j}$, see section 1.3. Suppose now that (M,g) is a Riemannian manifold and that E carries a Hermitian metric. For $k \in \mathbb{N}_0$ and $u \in \Gamma_c(M,E)$, define by

$$\|u\|_{H^k(M,E)} := \left(\sum_{j=0}^{k} \|\nabla^{E,j}u\|^2_{L^2(M,(T^*M)^{\otimes j}\otimes E)}\right)^{1/2} \tag{1.3.3}$$

the k^{th} *order Sobolev norm* (see [BB12]).

Definition 1.3.6. The *Sobolev space* $H^k(M,E)$ is defined as $\bigcap_{j=0}^k \mathrm{dom}((\nabla^{E,j})_w)$ and therefore consists of all sections in $L^2(M,E)$ with distributional covariant derivatives up to order k also being square integrable.

We may view $H^k(M,E)$ as the domain of the closed operator $((\nabla^{E,1})_w,\dots,(\nabla^{E,k})_w)$ from $L^2(M,E)$ to $\bigoplus_{j=1}^k L^2(M,(T^*M)^{\otimes j}\otimes E)$, and hence it is a Hilbert space under the graph norm of this operator, which is given by the square root of $u\mapsto \|u\|^2+\sum_{j=1}^k \|(\nabla^{E,j})_w u\|^2$. Evidently, this norm extends (1.3.3). The *Sobolev space* $H_0^k(M,E)$ is the completion of $\Gamma_{cc}(M,E)$ with respect to (1.3.3), thus a closed subspace of $H^k(M,E)$. It follows that $H_0^1(M,E) = \mathrm{dom}((\nabla^E)_s)$. Put

$$H^k_{\mathrm{loc}}(M,E) := \{u\in L^2_{\mathrm{loc}}(M,E) : \varphi u \in H^k_0(M,E) \text{ for all } \varphi\in C^\infty_{cc}(M)\}.$$

Obvious extensions to Sobolev spaces based on $L^p(M,E)$ instead of $L^2(M,E)$ are available, but we will not need them here. In general, the above Sobolev spaces depend on the choice of metrics and connections, although this is suppressed in our notation. If M is *compact* (possibly with boundary), then any of these choices produce equivalent norms, so that the Sobolev spaces and their topologies only depend on the vector bundle $E\to M$. Using interpolation methods and duality, one can define Sobolev spaces $H^s(M,E)$, $H^s_0(M,E)$, and $H^s_{\mathrm{loc}}(M,E)$ for every $s\in\mathbb{R}$, see [Tay11a, chapter 4].

An important consequence of the *Sobolev embedding theorems* is that sections which belong to every Sobolev space are smooth in the interior of M. In fact,

$$\bigcap_{k=0}^\infty H^k_{\mathrm{loc}}(M,E) = \Gamma(M^\circ,E), \tag{1.3.4}$$

see [BB12, p. 17].

Remark 1.3.7. For general M and E, the spaces $\Gamma(M^\circ,E)\cap H^k(M,E)$ and $\Gamma_{cc}(M,E)$ need *not* be dense in $H^k(M,E)$, and their closures may also be different. If $k\geq 2$ and M is of $(k-2)$-bounded geometry, see section 4.1, then all these spaces are dense in $H^k(M,E)$ by [Eic88, Proposition 1.6].

On the other hand, interior elliptic regularity implies that $\Gamma(M^\circ,E)\cap H^1(M,E)$ is dense in $H^1(M,E)$, without any additional assumptions: by Remark 1.3.5, the self-adjoint operator $A := (\nabla^E)^*_w(\nabla^E)_w$ on $L^2(M,E)$ is a restriction of $(\nabla^{E,\dagger}\nabla^E)_w$, with $\nabla^{E,\dagger}\nabla^E$ being elliptic by (1.1.12). By Corollary 1.3.10 below, A has a core consisting of sections which are smooth on M°, and it follows that this is also a core for the associated quadratic form Q_A. By Example C.2.3, $\mathrm{dom}(Q_A) = H^1(M,E)$ as Hilbert spaces, so the claim follows. In case (M,g) is complete (possibly with boundary), then Proposition 1.4.11 will show that $\Gamma_c(M,E)$ is dense in $H^1(M,E)$.

Theorem 1.3.8 (Rellich–Kondrachov theorem)**.** *Let $s\geq 0$ and $t>0$.*

(i) If M is compact, possibly with (smooth) boundary, then the inclusion $H^{s+t}(M,E)\hookrightarrow H^s(M,E)$ is compact.

(ii) If $U\subseteq M^\circ$ is open and with compact closure, then the inclusion $H_0^{s+t}(U,E)\hookrightarrow H_0^s(U,E)$ is compact.

Proof. The statement can be reduced to the corresponding result about Sobolev spaces on subsets of $\mathbb{R}^n$. We refer to [Tay11a, Proposition 4.4] for the details. ∎

1.3.2. Elliptic operator theory. On a closed (*i.e.*, compact and without boundary) manifold M, differential operators of order k extend to bounded linear maps from $H^s(M,E)$ to $H^{s-k}(M,F)$ for all $s \in \mathbb{R}$. If the operator is elliptic, then these extensions are Fredholm, with index independent of s, see [LM89, Theorem 5.2]. This is no longer true for open manifolds or manifolds with boundary, but elliptic operators on them still enjoy some nice properties, some of which we will list below.

Theorem 1.3.9 (Interior elliptic regularity)**.** *Let M be a smooth manifold (possibly with boundary), and suppose $D\colon \Gamma(M,E) \to \Gamma(M,F)$ is an elliptic differential operator of order k. If $u \in \mathcal{D}'(M^\circ, E)$ is such that $Du \in H^s_{\mathrm{loc}}(M,F)$, then $u \in H^{s+k}_{\mathrm{loc}}(M,E)$ and for all open subsets $U, V \subseteq M$ with $U \subset\subset V \subset\subset M^\circ$ and all $t < s+k$, there is $C > 0$, independent of u, such that*

$$\|u\|_{H^{s+k}(U,E)} \leq C\big(\|Du\|_{H^s(V,F)} + \|u\|_{H^t(V,E)}\big).$$

For a proof, see [Tay11a, Theorem 5.11.1] or [Nic14, section 10.3.2], or [Eva10, section 6.3.1] for a treatment of interior regularity for elliptic operators on $\mathbb{R}^n$. An immediate application of this is the regularity of sections in the domain of the weak extension of an elliptic operator:

Corollary 1.3.10. *Let $D\colon \Gamma(M,E) \to \Gamma(M,F)$ be a k^{th} order elliptic differential operator, and suppose A is an extension of D with $D_s \subseteq A \subseteq D_w$.*[4] *Then* $\mathrm{dom}(A) \subseteq H^k_{\mathrm{loc}}(M,E)$, *and if A is closed, then it has a core consisting of sections which are smooth on M°.*

Proof. The proof follows [Bei17, Proposition 2.1]. If $u \in \mathrm{dom}(A) \subseteq \mathrm{dom}(D_w)$, then $Du \in L^2(M,F)$ in the sense of distributions, hence $u \in H^k_{\mathrm{loc}}(M,E)$ by Theorem 1.3.9. Now suppose that A is closed. It follows that A^*A is self-adjoint and satisfies $A^*A \supseteq (D_w)^* D_s = (D^\dagger)_s D_s$, hence is a self-adjoint extension of the elliptic differential operator $D^\dagger D$ of order $2k$. From Remark 1.3.5, we deduce that A^*A is a restriction of $(D^\dagger D)_w$ and, again by Theorem 1.3.9, $\mathrm{dom}(A^*A) \subseteq H^{2k}_{\mathrm{loc}}(M,E)$. Iterating this procedure, we see that $\mathrm{dom}((A^*A)^j) \subseteq H^{2jk}_{\mathrm{loc}}(M,E)$ for $j \geq 1$. But $\bigcap_{j\geq 1} \mathrm{dom}((A^*A)^j) \subseteq \bigcap_{j \geq 1} H^{2jk}_{\mathrm{loc}}(M,E) \subseteq \Gamma(M^\circ, E)$, see (1.3.4), is a core for A^*A (see the argument in [BL92, p. 98]), and $\mathrm{dom}(A^*A)$ in turn is densely included in $\mathrm{dom}(A)$, see Example C.2.3. ∎

Lemma 1.3.11. *Let $D\colon \Gamma(M,E) \to \Gamma(M,F)$ be an elliptic differential operator and $\varphi \in C^\infty_{cc}(M)$. Then the operator of multiplication by φ maps* $\mathrm{dom}(D_w)$ *to* $\mathrm{dom}(D_s)$*.*

Proof. By Corollary 1.3.10, D_w has a core consisting of sections which are smooth on M°. Let $s \in \mathrm{dom}(D_w)$, and choose $s_k \in \Gamma(M^\circ, E) \cap \mathrm{dom}(D_w)$ with $s_k \to s$ in $\mathrm{dom}(D_w)$. Then $\varphi s_k \in \Gamma_{cc}(M,E) \subseteq \mathrm{dom}(D_s)$ and $\varphi s_k \to \varphi s$ in $L^2(M,E)$. Moreover,

$$\|D_s(\varphi s_k) - D_s(\varphi s_j)\| \leq \|\varphi(Ds_k - Ds_j)\| + \|[D,\varphi](s_k - s_j)\|.$$

[4] In particular, this is true for self-adjoint A, see Remark 1.3.5.

Choose relatively compact open subsets $U \subset\subset V \subseteq M^\circ$ such that $\operatorname{supp}(\varphi) \subseteq U$. If d is the order of D, then $[D, \varphi]$ has order $d-1$ and vanishes outside of $\operatorname{supp}(\varphi)$, hence there is a constant $C > 0$ such that $\|[D, \varphi]u\| \leq C\|u\|_{H^{d-1}(U,E)}$ for all $u \in \Gamma_{cc}(U, E)$. By the interior elliptic regularity estimates from Theorem 1.3.9, we therefore have

$$\|D_s(\varphi s_k) - D_s(\varphi s_j)\| \leq \|\varphi\|_{L^\infty}\|Ds_k - Ds_j\| + \widetilde{C}(\|Ds_k - Ds_j\| + \|s_k - s_j\|)$$

for some constant $\widetilde{C} > 0$ and all $j, k \geq 1$. We conclude that $(\varphi s_k)_{k\in\mathbb{N}}$ is Cauchy in $\operatorname{dom}(D_s)$, hence convergent, and the limit must agree with φs by the convergence in $L^2(M, E)$. ■

Corollary 1.3.12. *Let A be an extension of an elliptic differential operator D with $D_s \subseteq A \subseteq D_w$. If $s \in \operatorname{dom}(A)$, then $fs \in \operatorname{dom}(A)$ for all functions $f \in C^\infty(M)$ that are constant outside of a compact subset of M°.*

Proof. We can assume that $f|_{M\setminus K} = 1$ for some compact $K \subseteq M^\circ$. Write $f = 1 - \varphi$ with $\varphi \in C^\infty_{cc}(M)$. Then $\varphi s \in \operatorname{dom}(D_s) \subseteq \operatorname{dom}(A)$ by Lemma 1.3.11, and therefore also $fs = s - \varphi s \in \operatorname{dom}(A)$. ■

If D is a first order operator, then the ellipticity assumption in Lemma 1.3.11 can be disposed of:

Lemma 1.3.13. *Let $D\colon \Gamma(M, E) \to \Gamma(M, F)$ be a* first order *differential operator, and let $s \in \operatorname{dom}(D_w)$ have compact support contained in M°. Then $s \in \operatorname{dom}(D_s)$.*

Proof. See [GL02, Lemma 2.1]. The proof can be done similarly to Proposition 1.4.11 below, by using trivializations of E and F to translate the problem to a first order differential operator on $\mathbb{R}^n$, and then applying Friedrichs' lemma. ■

1.3.3. Complexes of differential operators. The main examples of Hilbert complexes are (closed extensions of) complexes of differential operators arising in differential geometry. By this we mean a sequence of differential operators

$$0 \to \Gamma(M, E_0) \xrightarrow{d_0} \Gamma(M, E_1) \xrightarrow{d_1} \Gamma(M, E_2) \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} \Gamma(M, E_n) \to 0$$

with smooth vector bundles E_i over a smooth manifold M, and such that $d_{i+1} \circ d_i = 0$ for all i. We will assume that the order of all the nonzero d_i coincide and are at least one, and denote such a complex simply by $(E_\bullet, d)$. Suppose that M is Riemannian and that all E_i, $0 \leq i \leq n$, are Hermitian vector bundles, so that we may form the formal adjoints of the d_i and consider the spaces $L^2(M, E_i)$ of square-integrable sections of E_i. The complex is called *elliptic* if all the *Laplacians*

$$\Delta_i^E := d_{i-1}d_{i-1}^\dagger + d_i^\dagger d_i \colon \Gamma(M, E_i) \to \Gamma(M, E_i)$$

are elliptic differential operators, where $d_i^\dagger\colon \Gamma(M, E_{i-1}) \to \Gamma(M, E_i)$ denotes the formal adjoint of d_i. For more details on elliptic complexes, see [AB67], [Nic14, section 10.4.3], or

[Tay11b, chapter 12]. If we denote by d, $d^\dagger$, and Δ^E the operators considered on $\Gamma(M,E) = \bigoplus_{i=0}^n \Gamma(M,E_i)$, with $E \coloneqq \bigoplus_{i=0}^n E_i$, then we have

$$\Delta^E = (d+d^\dagger)^2 \colon \Gamma(M,E) \to \Gamma(M,E).$$

Let k_i denote the order of d_i, and put $A \coloneqq \mathrm{Symb}(d_i)(\xi)$ and $B \coloneqq \mathrm{Symb}(d_{i-1})(\xi)$ for fixed $\xi \in T^*M$. By (1.1.9) and (1.1.10), we have

$$\mathrm{Symb}_{k_{i-1}+k_i}(\Delta_i^E)(\xi) = BB^* + A^*A.$$

Because $(E_\bullet, d)$ is a complex, $AB = 0$ by (1.1.9), and hence $\mathrm{img}(B) \subseteq \ker(A) = \mathrm{img}(A^*)^\perp$, so that $\mathrm{Symb}_{k_{i-1}+k_i}(\Delta_i^E)(\xi) \neq 0$ unless $A = B = 0$. Therefore, Δ_i^E has order $k_{i-1}+k_i$, and its principal symbol at ξ is $BB^* + AA^*$.

Proposition 1.3.14. *A complex $(E_\bullet, d)$ of differential operators between Hermitian vector bundles over a Riemannian manifold is elliptic if and only if the principal symbol sequence*

$$0 \to \pi^*E_0 \xrightarrow{\mathrm{Symb}(d_0)} \pi^*E_1 \xrightarrow{\mathrm{Symb}(d_1)} \pi^*E_2 \xrightarrow{\mathrm{Symb}(d_2)} \cdots \xrightarrow{\mathrm{Symb}(d_{n-1})} \pi^*E_n \to 0 \tag{1.3.5}$$

is exact away from the zero section of T^*M, where $\pi\colon T^*M \to M$ is the cotangent bundle of M. This means that $\mathrm{img}(\mathrm{Symb}(d_{i-1})(x,\xi)) = \ker(\mathrm{Symb}(d_i)(x,\xi))$ *as subspaces of* $(E_i)_x$ *for all $x \in M$ and* $0 \neq \xi \in T_x^*M$.

Proof. Fix $x \in M$ and $\xi \in T_x^*M \setminus \{0\}$, and let $A\colon (E_i)_x \to (E_{i+1})_x$ and $B\colon (E_{i-1})_x \to (E_i)_x$ be as above. The sequence (1.3.5) is exact at π^*E_i if and only if

$$0 = \ker(A)/\,\mathrm{img}(B) \cong \ker(A) \cap \mathrm{img}(B)^\perp = \ker(A) \cap \ker(B^*) = \ker(A^*A + BB^*),$$

which is equivalent to $A^*A + BB^* = \mathrm{Symb}(\Delta_i^E)(\xi)$ being bijective since $(E_i)_x$ is finite dimensional. ■

Note that in case the complex only consists of a single nontrivial operator, *i.e.*, we have the sequence $0 \to \Gamma(M,E) \xrightarrow{d_0} \Gamma(M,F) \to 0$, then ellipticity of this complex is equivalent to d_0 being elliptic, since exactness of the corresponding symbol sequence (1.3.5) is the same as $\mathrm{Symb}(d_0)(\xi)$ being invertible for all $\xi \neq 0$.

A choice of closed extensions $(d_H)_i$ of d_i that produces a Hilbert complex $(L^2(M,E_\bullet), d_H)$ and satisfies $d_s \subseteq d_H \subseteq d_w$ is called an *ideal boundary condition* for $(E_\bullet, d)$. Here, we make the agreement that $L^2(M,E_i) = 0$ for $i \notin \{0,\dots,n\}$, *i.e.*, $E_i = M \times \{0\} \to M$ for these i. Such ideal boundary conditions always exist, for the strong and weak extensions themselves, see section 1.3, give rise to ideal boundary conditions, *cf.*, [BL92, Lemma 3.1]. Thus, we have the Hilbert complexes

$$(L^2(M,E_\bullet), d_s) \quad \text{and} \quad (L^2(M,E_\bullet), d_w),$$

and we write $d_{i,w}$ for $(d_i)_w = (d_w)_i$, and similarly for $d_{i,s}$. The self-adjoint extension of Δ^E induced by the Hilbert complex $(L^2(M,E_\bullet), d_w)$ is sometimes called the *Gaffney extension* of

Δ^E. By definition, it is the operator

$$\Delta_G^E \coloneqq (d_w)^* d_w + d_w (d_w)^* \tag{1.3.6}$$

on $L^2(M, E_\bullet)$ with domain

$$\operatorname{dom}(\Delta_G^E) \coloneqq \{x \in \operatorname{dom}(d_w) \cap \operatorname{dom}((d_w)^*) : d_w x \in \operatorname{dom}((d_w)^*) \text{ and } (d_w)^* x \in \operatorname{dom}(d_w)\}.$$

1.4. Density of compactly supported sections and essential self-adjointness

Let $E, F \to M$ be Hermitian vector bundles over a Riemannian manifold, possibly with boundary. Suppose $D \colon \Gamma(M, E) \to \Gamma(M, F)$ is a differential operator, and A is a closed extension of D to an operator from $L^2(M, E)$ to $L^2(M, F)$. In this section, we want to study whether sections with compact support are dense in $\operatorname{dom}(A)$ for the graph norm. Put differently: does A have a core consisting of sections with compact support? The results will be for *first order* operators, and we will also discuss the related issue of essential self-adjointness for both first order and some second order operators.

1.4.1. Complete Riemannian manifolds.

By a *complete Riemannian manifold* (M, g) we mean a connected manifold M, possibly with boundary, together with Riemannian metric g such that (M, d_g) is a complete metric space, where

$$d_g(x, y) \coloneqq \inf_\gamma \int_0^1 \sqrt{g(\dot\gamma(t), \dot\gamma(t))}\, dt \tag{1.4.1}$$

is the *Riemannian distance* between $x, y \in M$, with the infimum being taken over all smooth paths $\gamma \colon [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$. The topology defined on M by d_g coincides with the original one, see [Lee13, Theorem 13.29].

The metric space (M, d_g) is a *length space*, *i.e.*, a metric space in which the distance between two points is given by the infimum of the lengths of continuous paths connecting them. A generalization of the *theorem of Hopf–Rinow* says that a locally compact length space (X, d) is complete if and only if its compact subsets are exactly the closed and bounded ones, see [Gro99, p. 9] or [Pap14, Theorem 2.1.15]. Such a space is then automatically *geodesic*, meaning that for any two points $x, y \in X$ there exists a geodesic path connecting them, *i.e.*, an isometry $\gamma \colon [a, b] \to X$ with $\gamma(a) = x$ and $\gamma(b) = y$, see [Pap14, p. 71].

The following Lemma is a standard characterization of complete Riemannian manifolds. The proof is adapted from [Dem02, Lemma 12.1].

Lemma 1.4.1. *Let (M, g) be a connected Riemannian manifold, possibly with boundary. Then the following are equivalent:*

(i) (M, g) is complete.
(ii) There exists a smooth proper function $\psi \colon M \to [0, \infty)$ with $|d\psi| \leq 1$.
(iii) There is a sequence $(\chi_k)_{k\in\mathbb{N}}$ of functions in $C_c^\infty(M, [0,1])$ with $(\chi_{k+1})|_{\operatorname{supp}(\chi_k)} = 1$ and $|d\chi_k| \leq 2^{-k}$ for all $k \in \mathbb{N}$, and such that $(\operatorname{supp}(\chi_k))_{k\in\mathbb{N}}$ is a compact exhaustion of M.

Proof. Assume that (M, g) is complete. For fixed $x_0 \in M$, the function $\widetilde{\delta}\colon (M, d_g) \to [0, \infty)$, $\widetilde{\delta}(x) := \frac{1}{2} d_g(x, x_0)$ is Lipschitz with $|d\widetilde{\delta}| \le \frac{1}{2}$ almost everywhere, hence there is a smooth function $\delta\colon M \to [0, \infty)$ such that $|d\delta| \le 1$ and $|\delta - \widetilde{\delta}| \le 1$. Let $\rho\colon \mathbb{R} \to [0, 1]$ be smooth with $\rho(t) = 1$ for $t \le 1$, $\rho(t) = 0$ for $t \ge 2$, and $|\rho'| \le 2$. Put $\chi_k(x) := \rho(2^{-k-1}\delta(x))$ for $x \in M$ and $k \in \mathbb{N}$. Then $\chi_k\colon M \to [0, 1]$ is smooth and satisfies $|d\chi_k| \le 2^{-k}$ and $\operatorname{supp}(\chi_k) \subseteq \delta^{-1}([0, 2^k])$. On this set $\chi_{k+1} = 1$, and $\operatorname{supp}(\chi_k)$ is a closed subset of M contained in $\delta^{-1}([0, 2^k]) \subseteq \widetilde{\delta}^{-1}([0, 2^k + 1]) = \{x \in M : d_g(x, x_0) \le 2(2^k + 1)\}$. Since the length space (M, d_g) is complete, the closed balls are compact by the Hopf–Rinow theorem, and hence $\operatorname{supp}(\chi_k)$ is also compact. The construction also immediately implies that the supports of χ_k form a compact exhaustion of M. Thus, $(\chi_k)_{k\in\mathbb{N}}$ has the properties required in (iii).

If $(\chi_k)_{k\in\mathbb{N}}$ is as in (iii), then the function $\psi := \sum_{k=1}^{\infty} 2^k(1 - \chi_k)\colon M \to [0, \infty)$ is smooth, proper, and satisfies $|d\psi| \le 1$.

Suppose finally that $\psi\colon M \to [0, \infty)$ is smooth, proper, and satisfies $|d\psi| \le 1$. If $\gamma\colon [0, 1] \to M$ is a smooth path with $\gamma(1) = x$ and $\gamma(0) = y$, then

$$\psi(x) - \psi(y) = \psi(\gamma(1)) - \psi(\gamma(0)) = \int_\gamma d\psi = \int_0^1 d\psi(\dot{\gamma}(t))\, dt.$$

It follows that $|\psi(x) - \psi(y)| \le \big(\sup_{z\in M} |d\psi(z)|\big) \int_0^1 |\dot{\gamma}(t)|\, dt$, hence $|\psi(x) - \psi(y)| \le d_g(x, y)$ for all $x, y \in M$. Since ψ is proper, any closed bounded set for d_g is therefore compact, and since every Cauchy sequence in M is bounded, completeness follows. ■

1.4.2. Density of sections with compact support. Recall that if A is a closed operator on a Hilbert space, then $\operatorname{dom}(A)$ is a Hilbert space if equipped with the *graph norm* $x \mapsto (\|x\|^2 + \|Ax\|^2)^{1/2}$, and any dense subspace of it is called a *core* for A. In particular, if $W \subseteq \operatorname{dom}(A)$ is a core for A, then the closure of $A|_W$ equals A. We also refer to appendix C.2 for more information. In this section, we discuss sufficient conditions for closed extensions of differential operators to have a core consisting of sections with compact support.

Definition 1.4.2. Let $D\colon \Gamma(M, E) \to \Gamma(M, F)$ be a first order differential operator. We say that an extension A of D *satisfies the Leibniz rule (with respect to $C_c^\infty(M)$)* if $fs \in \operatorname{dom}(A)$ and

$$A(fs) = fAs + \operatorname{Symb}(D)(df)s \tag{1.4.2}$$

for all $s \in \operatorname{dom}(A)$ and $f \in C_c^\infty(M)$.

Theorem 1.4.3. *Let A be a closed extension of a first order differential operator D satisfying the Leibniz rule* (1.4.2)*. Suppose that (M, g) is complete and that the principal symbol of D satisfies*

$$|\operatorname{Symb}(D)(\xi)| \le C|\xi| \tag{1.4.3}$$

for some constant $C > 0$ and all $\xi \in T^*M$. If $W \subseteq \operatorname{dom}(A)$ is a core for A, then $\{\varphi s : \varphi \in C_c^\infty(M),\ s \in W\}$ is also a core for A. In particular, the compactly supported elements are dense in $\operatorname{dom}(A)$.

Proof. We slightly modify the proof of [BB12, Theorem 3.3], where the statement is shown for D_w, *cf.*, Corollary 1.4.7 below. Let $s \in \operatorname{dom}(A)$. Since W is a core for A, we find $s_k \in W$ with $s_k \to s$ in $\operatorname{dom}(A)$. By the completeness of (M, g), there exists a sequence $(\chi_k)_{k \in \mathbb{N}}$ of functions in $C_c^\infty(M, [0,1])$ with $(\chi_{k+1})|_{\operatorname{supp}(\chi_k)} = 1$, and $|d\chi_k| \leq 2^{-k}$ for all $k \in \mathbb{N}$, see Lemma 1.4.1. Then $\chi_k s_k$ has compact support and is an element of $\operatorname{dom}(A)$ by assumption. By the dominated convergence theorem, $\|\chi_k s - s\| \to 0$ and $\|\chi_k As - As\| \to 0$. It follows that $\chi_k s_k \to s$ in $L^2(M, E)$, and

$$\begin{aligned}
&\|A(\chi_k s_k) - As\| \\
&\qquad \leq \|A(\chi_k s_k) - A(\chi_k s)\| + \|A(\chi_k s) - As)\| \\
&\qquad \leq \|\chi_k A(s_k - s)\| + \|\operatorname{Symb}(D)(d\chi_k)(s_k - s)\| + \|\chi_k As - As\| + \|\operatorname{Symb}(D)(d\chi_k)s\| \\
&\qquad \leq \|As_k - As\| + \frac{C}{2^k}\|s_k - s\| + \|\chi_k As - As\| + \frac{C}{2^k}\|s\|
\end{aligned}$$

also converges to zero as $k \to \infty$. Therefore, $\chi_k s_k \to s$ in $\operatorname{dom}(A)$, which proves the claim. ■

Remark 1.4.4. A more sophisticated condition is given in [BB12, Theorem 1.2]: if M is a connected Riemannian manifold which admits a complete Riemannian metric h such that

$$|\operatorname{Symb}(D)(\xi)| \leq C(\operatorname{dist}_{d_h}(x, \partial M))|\xi|_h$$

for all $x \in M$ and $\xi \in T_x^* M$, with $C\colon [0, \infty) \to \mathbb{R}$ a positive, continuous, monotonically increasing function satisfying

$$\int_0^\infty \frac{1}{C(r)}\, dr = \infty,$$

then compactly supported elements of $\operatorname{dom}(D_w)$ are a core for D_w. After a conformal change of metric, this case is reduced to (1.4.3).

Example 1.4.5. Let $D\colon \Gamma(M, E) \to \Gamma(M, E)$ be a formally self-adjoint differential operator of Dirac type, see section 1.1.2. For instance, this is the case if D is the Dirac operator associated to a Dirac bundle, *cf.*, Remark 1.1.12. By definition, D^2 is of Laplace type, hence

$$|\operatorname{Symb}(D)(\xi)|^2 = |\operatorname{Symb}(D)(\xi)^* \operatorname{Symb}(D)(\xi)| = |\operatorname{Symb}(D^\dagger D)(\xi)| = |\operatorname{Symb}(D^2)(\xi)| = |\xi|^2 \tag{1.4.4}$$

for $\xi \in T^* M$. Therefore, (1.4.3) is satisfied. ♦

Of course, the value of Theorem 1.4.3 depends on the number of extensions of D for which the Leibniz rule can be established. (Note that the support of f in (1.4.2) may intersect the boundary.) The next Proposition gives us something to work with:

Proposition 1.4.6. *Let A be an extension of a first order differential operators D with $D_{cc} \subseteq A \subseteq D_w$ and satisfying the Leibniz rule* (1.4.2). *Then the closure $\overline{A}$ satisfies the Leibniz rule, and the extension A^* of $D^\dagger$, see Proposition 1.3.4, also has this property,* i.e., *$fs \in \operatorname{dom}(A^*)$ and*

$$A^*(fs) = fA^* s + \operatorname{Symb}(D^\dagger)(df)s$$

for all $s \in \mathrm{dom}(A^*)$ *and* $f \in C_c^\infty(M)$.

Proof. Let $s \in \mathrm{dom}(A^*)$, $f \in C_c^\infty(M,\mathbb{R})$, and $t \in \mathrm{dom}(A)$. Then $ft \in \mathrm{dom}(A)$ and $A(ft) = fAt + \mathrm{Symb}(D)(df)t$ by assumption, and

$$\langle\!\langle fs, At\rangle\!\rangle = \langle\!\langle s, fAt\rangle\!\rangle = \langle\!\langle s, A(ft) - \mathrm{Symb}(D)(df)t\rangle\!\rangle =$$
$$= \langle\!\langle A^*s, ft\rangle\!\rangle + \langle\!\langle \mathrm{Symb}(D^\dagger)(df)s, t\rangle\!\rangle = \langle\!\langle fA^*s + \mathrm{Symb}(D^\dagger)(df)s, t\rangle\!\rangle.$$

This implies $fs \in \mathrm{dom}(A^*)$ and $A^*(fs) = fA^*s + \mathrm{Symb}(D^\dagger)(df)s$. The closure of A is given by $\overline{A} = A^{**}$, so the claim for $\overline{A}$ follows immediately. ■

Corollary 1.4.7. *Let D be a first order differential operator. Then D_s and D_w satisfy the Leibniz rule* (1.4.2)*.*

Proof. Clearly, D_{cc} and $(D^\dagger)_{cc}$ both satisfy the Leibniz rule, so Proposition 1.4.6 implies that $D_s = \overline{D_{cc}}$ and $D_w = ((D^\dagger)_{cc})^*$ also have this property. ■

Remark 1.4.8. (i) As a consequence of Proposition 1.4.6, an extension A of D with $A \subseteq D_w$ satisfies the Leibniz rule if and only if $fs \in \mathrm{dom}(A)$ for all $f \in C_c^\infty(M)$ and $s \in \mathrm{dom}(A)$. Indeed, we have $fs \in \mathrm{dom}(D_w)$ and

$$A(fs) = D_w(fs) = fD_ws + \mathrm{Symb}(D)(df)s = fAs + \mathrm{Symb}(D)(df)s$$

automatically holds in this case, so (1.4.2) is satisfied.

(ii) One can also consider the Leibniz rule with respect to other spaces of functions on M. For instance, every extension A of D with $D_s \subseteq A \subseteq D_w$ satisfies the Leibniz rule with respect to $C_{cc}^\infty(M)$, the space of smooth functions on M with compact support in M°. Indeed, if $s \in \mathrm{dom}(A)$ and $f \in C_{cc}^\infty(M,\mathbb{R})$, then $fs \in \mathrm{dom}(D_w)$ by Proposition 1.4.6 and hence $fs \in \mathrm{dom}(D_s)$ by Lemma 1.3.11. In particular, $fs \in \mathrm{dom}(A)$, and the same argument as above shows that A satisfies the Leibniz rule with respect to $C_{cc}^\infty(M)$. Note that if M has no boundary, then $C_c^\infty(M) = C_{cc}^\infty(M)$, hence (1.4.2) is satisfied for all extensions A of D lying between D_s and D_w.

Remark 1.4.9. The proof of Proposition 1.4.6 also works if we replace $C_c^\infty(M)$ by the space of bounded smooth functions $f\colon M \to \mathbb{R}$ such that $x \mapsto |\mathrm{Symb}(D)(x, df(x))|$ is bounded on M. If D satisfies the symbol bound (1.4.3), then bounded smooth Lipschitz functions have this property. In particular, D_w and D_s satisfy the Leibniz rule with respect to these functions.

Example 1.4.10. Let $(E_\bullet, d)$ be an elliptic complex of first order differential operators, see section 1.3.3, and consider the Hilbert complex $(L^2(M, E_\bullet), d_w)$. The operator $d_w + d_w^* = d_w + (d^\dagger)_s$ is a self-adjoint extension of $d + d^\dagger$, see Lemma 1.2.1, with domain $\mathrm{dom}(d_w) \cap \mathrm{dom}(d_w^*) = \mathrm{dom}(d_w) \cap \mathrm{dom}((d^\dagger)_s)$. Note that $d + d^\dagger$ is also of first order, since otherwise

$$\mathrm{Symb}(d)(\xi) = -\,\mathrm{Symb}(d^\dagger)(\xi) = -\,\mathrm{Symb}(d)(\xi)^*,$$

which would imply $\ker(\mathrm{Symb}(d)(\xi)) = \mathrm{img}(\mathrm{Symb}(d)(\xi))^\perp$, hence contradicts $\mathrm{Symb}(d)(\xi)^2 = 0$. It follows that $\mathrm{Symb}(d+d^\dagger) = \mathrm{Symb}(d)+\mathrm{Symb}(d^\dagger)$. Let $f \in C_c^\infty(M)$ and $s \in \mathrm{dom}(d_w + d_w^*)$. Then $fs \in \mathrm{dom}(d_w + d_w^*)$ by Corollary 1.4.7, and

$$(d_w + d_w^*)(fs) = f d_w s + \mathrm{Symb}(d)(df)s + \\ + f(d^\dagger)_s s + \mathrm{Symb}(d^\dagger)(df)s = f(d_w + d_w^*)s + \mathrm{Symb}(d+d^\dagger)(df)s,$$

so $d_w + d_w^*$ satisfies the Leibniz rule. Similarly, one shows that this is also true for $d_s + d_s^*$. The same argument can also be used to show that if $(L^2(M, E_\bullet), d_H)$ is an ideal boundary condition for $(E_\bullet, d)$ such that d_H satisfies the Leibniz rule (1.4.2), then the self-adjoint operator $d_H + d_H^*$ also has this property. ◆

For the weak extension of a first order operator, we can use Friedrichs' lemma to sharpen Theorem 1.4.3 and show that even $\Gamma_c(M, E)$ is always a core:

Proposition 1.4.11. *Let (M, g) be a complete Riemannian manifold, possibly having a boundary, and let $D\colon \Gamma(M, E) \to \Gamma(M, F)$ be a first order differential operator satisfying* (1.4.3)*. Then $\Gamma_c(M, E)$ is a core for D_w.*

Proof. The proof is an adaptation of the proof of item (ii) of [Str10, Proposition 2.3], where the statement is shown for the $\overline{\partial}$-operator on a bounded domain in $\mathbb{C}^n$. By Theorem 1.4.3 and Corollary 1.4.7, the compactly supported elements are dense in $\mathrm{dom}(D_w)$. So let $u \in \mathrm{dom}(D_w)$ have compact support. By a partition of unity argument, we may suppose that $\mathrm{supp}(u)$ is contained in the relatively compact domain U of a chart $\chi\colon U \xrightarrow{\cong} V$ of M over which E and F are trivial. Thus, we have vector bundle isomorphisms $\Phi\colon E|_U \to V \times \mathbb{C}^{\mathrm{rank}(E)}$ and $\Psi\colon F|_U \to V \times \mathbb{C}^{\mathrm{rank}(F)}$ covering χ, with $U \subseteq M$ open and V open in $\mathbb{R}^n_\leq := \{y \in \mathbb{R}^n : y_n \leq 0\}$. We obtain bijections

$$\Phi^*\colon \Gamma(U, E) \to C^\infty(V, \mathbb{C}^{\mathrm{rank}(E)}) \quad \text{and} \quad \Psi^*\colon \Gamma(U, F) \to C^\infty(V, \mathbb{C}^{\mathrm{rank}(F)})$$

via $\Phi^*(s) := \Phi \circ s \circ \chi^{-1}$ and identifying sections of the trivial bundles with functions, and similarly for Ψ^*. Both Φ^* and Ψ^* extend to continuous operators $\Phi^*\colon L^2(U, E) \to L^2(V, \mathbb{C}^{\mathrm{rank}(E)})$ and $\Psi^*\colon L^2(U, F) \to L^2(V, \mathbb{C}^{\mathrm{rank}(F)})$, and with continuous inverses. Here, $L^2(V, \mathbb{C}^{\mathrm{rank}(E)})$ and $L^2(V, \mathbb{C}^{\mathrm{rank}(F)})$ are defined by using Lebesgue measure on $V \subseteq \mathbb{R}^n$.

Consider the first order differential operator $\widetilde{D} := \Psi^* \circ D \circ (\Phi^*)^{-1}\colon C^\infty(V, \mathbb{C}^{\mathrm{rank}(E)}) \to C^\infty(V, \mathbb{C}^{\mathrm{rank}(F)})$. The coefficients of $\widetilde{D}$ are smooth on V, so we can extend $\widetilde{D}$ to a first order differential operator acting on $C^\infty(\mathbb{R}^n, \mathbb{C}^{\mathrm{rank}(E)})$. We put $\widetilde{u} := \Phi^* u \in L^2(V, \mathbb{C}^{\mathrm{rank}(E)})$ and denote by $\widetilde{u}_0 \in L^2(\mathbb{R}^n, \mathbb{C}^{\mathrm{rank}(E)})$ its extension to $\mathbb{R}^n$ by zero.

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi \, d\lambda = 1$, and $\mathrm{supp}(\varphi) \subseteq \mathbb{R}^n_>$. Here, λ is Lebesgue measure on $\mathbb{R}^n$, and $\mathbb{R}^n_> := \{y \in \mathbb{R}^n : y_n > 0\}$, with $\mathbb{R}^n_<$ defined similarly. Set $\varphi_\varepsilon(y) := \varepsilon^{-n}\varphi(y/\varepsilon)$ for $\varepsilon \in (0,1]$ and $y \in \mathbb{R}^n$. By Friedrichs' lemma, see [MM07, Lemma 3.1.3],

$$\lim_{\varepsilon \to 0} \|\widetilde{D}(\varphi_\varepsilon * \widetilde{u}_0) - \varphi_\varepsilon * (\widetilde{D}\widetilde{u}_0)\|_{L^2(\mathbb{R}^n, \mathbb{C}^{\mathrm{rank}(F)})} = 0, \tag{1.4.5}$$

where $*$ denotes (component-wise) convolution on $\mathbb{R}^n$, and $\widetilde{D}$ is the distributional derivative. Moreover,

$$(\varphi_\varepsilon * (\widetilde{D}\widetilde{u}_0 - (\widetilde{D}\widetilde{u})_0))(x) = \int_{\mathbb{R}^n} \varphi_\varepsilon(y)(\widetilde{D}\widetilde{u}_0 - (\widetilde{D}\widetilde{u})_0)(x-y)\, d\lambda(y) = 0 \quad \text{for } x \in \mathbb{R}^n_{\leq} \tag{1.4.6}$$

because $\operatorname{supp}(\varphi_\varepsilon) \subseteq \mathbb{R}^n_{>}$ and $\widetilde{D}\widetilde{u}_0 = (\widetilde{D}\widetilde{u})_0$ on $\mathbb{R}^n_{<}$. Put $u_\varepsilon := (\Phi^*)^{-1}(\varphi_\varepsilon * \widetilde{u}_0)|_V$. For ε small enough, we have $\operatorname{supp}((\varphi_\varepsilon * \widetilde{u}_0)|_V) \subset\subset V$, so that $u_\varepsilon \in \Gamma_c(U,E)$ and hence $(u_\varepsilon)_0 \in \Gamma_c(M,E)$. Now

$$\begin{aligned}\|u - (u_\varepsilon)_0\|_{L^2(M,E)} &= \|u - u_\varepsilon\|_{L^2(U,E)} \leq C\|\Phi^* u - \Phi^* u_\varepsilon\|_{L^2(V,\mathbb{C}^{\operatorname{rank}(E)})} \\ &= C\|\widetilde{u} - \varphi_\varepsilon * \widetilde{u}_0\|_{L^2(V,\mathbb{C}^{\operatorname{rank}(E)})} \leq C\|\widetilde{u}_0 - \varphi_\varepsilon * \widetilde{u}_0\|_{L^2(\mathbb{R}^n,\mathbb{C}^{\operatorname{rank}(E)})}\end{aligned}$$

converges to 0 as $\varepsilon \to 0$, with $C > 0$ depending on the geometry of U and $E|_U$, and

$$\begin{aligned}&\|D_w u - D_w (u_\varepsilon)_0\|_{L^2(M,F)} \\ &\quad = \|D_w u - D_w u_\varepsilon\|_{L^2(U,F)} \\ &\quad \leq C'\|\Psi^*(Du) - \Psi^*(Du_\varepsilon)\|_{L^2(V,\mathbb{C}^{\operatorname{rank}(F)})} \\ &\quad = C'\|\widetilde{D}\widetilde{u} - \widetilde{D}((\varphi_\varepsilon * \widetilde{u}_0)|_V)\|_{L^2(V,\mathbb{C}^{\operatorname{rank}(F)})} \\ &\quad = C'\|(\widetilde{D}\widetilde{u})_0 - (\widetilde{D}((\varphi_\varepsilon * \widetilde{u}_0)|_V))_0\|_{L^2(\mathbb{R}^n,\mathbb{C}^{\operatorname{rank}(F)})} \\ &\quad \leq C'\|(\widetilde{D}\widetilde{u})_0 - \varphi_\varepsilon * (\widetilde{D}\widetilde{u})_0\|_{L^2(\mathbb{R}^n,\mathbb{C}^{\operatorname{rank}(F)})} + C'\|\varphi_\varepsilon * (\widetilde{D}\widetilde{u})_0 - \widetilde{D}(\varphi_\varepsilon * \widetilde{u}_0)\|_{L^2(\mathbb{R}^n,\mathbb{C}^{\operatorname{rank}(F)})} \\ &\quad = C'\|(\widetilde{D}\widetilde{u})_0 - \varphi_\varepsilon * (\widetilde{D}\widetilde{u})_0\|_{L^2(\mathbb{R}^n,\mathbb{C}^{\operatorname{rank}(F)})} + C'\|\varphi_\varepsilon * (\widetilde{D}\widetilde{u}_0) - \widetilde{D}(\varphi_\varepsilon * \widetilde{u}_0)\|_{L^2(\mathbb{R}^n,\mathbb{C}^{\operatorname{rank}(F)})},\end{aligned}$$

where $C' > 0$ depends on the geometry of U and $F|_U$. Clearly, the first term converges to zero as $\varepsilon \to 0$, and the second one does so because of (1.4.5). Thus, $(u_\varepsilon)_0 \to u$ in $\operatorname{dom}(D_w)$ as $\varepsilon \to 0$, as required. ■

1.4.3. Essential self-adjointness. Suppose now that $E = F$ and that D is formally self-adjoint (not necessarily of first order), in which case it also makes sense to ask how many self-adjoint extension of D there are. If $\overline{D_{cc}} = D_s$ is self-adjoint, then D (more precisely: D_{cc}) is called *essentially self-adjoint (on $\Gamma_{cc}(M,E)$)*. If this is so, then D_s is the only self-adjoint extension of D, since $D_s \subseteq A = A^* \subseteq (D_s)^* = D_s$ for any self-adjoint extension A of D. Moreover, $D_s = (D_s)^* = (D^\dagger)_w = D_w$, and D_s is the only closed symmetric extension of D_{cc}, see Remark 1.3.5. Conversely, if $D_s = D_w$, then D is essentially self-adjoint, since $(D_s)^* = (D_w)^* = (D^\dagger)_s = D_s$. Essential self-adjointness is also equivalent to D_w being symmetric, since then $D_w \subseteq (D_w)^* \subseteq D_s$, hence $(D_s)^* \subseteq (D_w)^* = D_s$, and we have equality because D_s is also symmetric.

Theorem 1.4.12. *Let M be a complete Riemannian manifold* without *boundary, and suppose $D\colon \Gamma(M,E) \to \Gamma(M,F)$ is a first order differential operator. If the principal symbol of D satisfies* (1.4.3)*, then $D_s = D_w$ and the second order operator $D^\dagger D\colon \Gamma(M,E) \to \Gamma(M,E)$ is essentially self-adjoint.*

Variants of Theorem 1.4.12 appear in various pieces of literature, for instance in [Gaf51; Gaf54] for the Hodge Laplacian, in [AV65] for the Dolbeault Laplacian, and [Wol73] for the square of the Dirac operator. A proof of the result as stated above can be found in [Alb08, Theorem 2.13]. Combining Theorem 1.4.12 with the discussion above, we have:

Corollary 1.4.13. *Let M be a complete Riemannian manifold without boundary, and suppose $D\colon \Gamma(M,E) \to \Gamma(M,E)$ is a first order, formally self-adjoint differential operator satisfying* (1.4.3). *Then both D and D^2 are essentially self-adjoint. In particular, this is true for Dirac type operators.*

An extension of this is the following theorem from [Che73, Theorem 2.2]:

Theorem 1.4.14 (Chernoff)**.** *Let M be a complete Riemannian manifold without boundary, and suppose $D\colon \Gamma(M,E) \to \Gamma(M,E)$ is a first order, formally self-adjoint differential operator. Put*

$$c(r) \coloneqq \sup\{|\operatorname{Symb}(D)(x,\bullet)| : x \in M \text{ with } d_g(x,x_0) = r\},$$

where $x_0 \in M$ is any reference point. If $\int_0^\infty \frac{1}{c(r)}\,dr = +\infty$, then D^k is essentially self-adjoint for any integer $k \geq 1$.

In particular, Chernoff's theorem says that the symbol bound (1.4.3) is sufficient for *all* powers of D to be essentially self-adjoint.

Let $D\colon \Gamma(M,E) \to \Gamma(M,E)$ be a formally self-adjoint differential operator of Laplace type. Recall from section 1.1.2 that D can be written in the form $D = \nabla^\dagger\nabla + V$ for a unique metric connection ∇ and endomorphism V of E. In the following, we list some sufficient conditions from [BMS02] for D to be essentially self-adjoint:

Theorem 1.4.15 ([BMS02, Corollary 2.9])**.** *Let (M,g) be a complete Riemannian manifold* without *boundary, $E \to M$ be a Hermitian vector bundle, and $D = \nabla^\dagger\nabla + V$ a formally self-adjoint differential operator of Laplace type, as above. If $V \geq -q$ in the sense of quadratic forms, where $q\colon M \to [1,\infty)$ is a smooth function such that $q^{-1/2}\colon (M,d_g) \to \mathbb{R}$ is (globally) Lipschitz and with the property that $\int_\gamma \frac{ds}{\sqrt{q}} = \infty$ for every curve γ in M going to infinity, then D is essentially self-adjoint.*

Theorem 1.4.16 ([BMS02, Theorem 2.13])**.** *Let M, E, and D be as in Theorem 1.4.15. If D is lower semibounded, then D is essentially self-adjoint.*

We point out that the results of [BMS02] are far more general than the ones we present here. In particular, they hold for potentials with very weak regularity.

CHAPTER 2

The essential spectrum of self-adjoint elliptic differential operators

In this chapter, we consider (nonnegative) self-adjoint extensions A of general elliptic differential operators on a Riemannian manifold M, possibly having a boundary. Section 2.1 will first set up the notation used throughout this chapter, including A_U, which denotes the restriction of A to an open subset U of M, defined by using the quadratic form associated to A (see appendix C.2 for the basics on quadratic forms on Hilbert spaces). The highlight of section 2.1 is the *decomposition principle*, which states that one can restrict A to complements of compact subsets of M° without changing the essential spectrum. In section 2.2, the bottom of the essential spectrum of such operators is considered. One of the key results there is Theorem 2.2.8, a generalization of a theorem of Persson, and it states that $\inf \sigma_e(A)$ is the limit of the net $K \mapsto \inf \sigma(A_{M\setminus K})$, where K runs through the compact subsets of M°, directed by inclusion. The results in this sections are not fundamentally new, but we have taken care to keep them as general as possible. For instance, we shall not make the often used assumption for A to have a core of smooth sections with compact support (although this will be satisfied in our applications).

2.1. The decomposition principle

Let (M, g) be a Riemannian manifold with (possibly empty) boundary ∂M, and let $E \to M$ be a (complex) Hermitian vector bundle. Suppose $D\colon \Gamma(M, E) \to \Gamma(M, E)$ is a formally self-adjoint differential operator of order at least one, and let

$$A\colon\ \mathrm{dom}(A) \subseteq L^2(M, E) \to L^2(M, E)$$

be a lower semibounded self-adjoint extension of D, by which we mean $D_{cc} \subseteq A$, *cf.*, section 1.1. We denote by Q_A the quadratic form associated to A, see appendix C.2. In order to formulate the results of the following sections, it will be convenient to restrict A to open subsets of M:

Lemma 2.1.1. *Let $U \subseteq M$ be an open subset. Then the quadratic form $\widetilde{Q}_{A,U}$ on $L^2(U, E)$ with $\mathrm{dom}(\widetilde{Q}_{A,U}) := \{s|_U : u \in \mathrm{dom}(Q_A)$ and $\mathrm{supp}(s) \subseteq U\}$ and $\widetilde{Q}_{A,U}(s, s) := Q_A(s_0, s_0)$ for $s \in \mathrm{dom}(\widetilde{Q}_{A,U})$ is closable, where $s_0 \in L^2(M, E)$ denotes the extension of s by zero.*

Proof. We need to show that if $u_k \in \mathrm{dom}(\widetilde{Q}_{A,U})$ is a sequence with $u_k \to 0$ in $L^2(U, E)$ and such that for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ with $|\widetilde{Q}_{A,U}(u_k - u_j, u_k - u_j)| \leq \varepsilon$ for $j, k \geq N$, then also $\widetilde{Q}_{A,U}(u_k, u_k) \to 0$ as $k \to \infty$, see [Sch12, Proposition 10.3]. These assumptions on

u_k imply that $((u_k)_0)_{k\in\mathbb{N}}$ is Cauchy in $\mathrm{dom}(Q_A)$. Since Q_A is closed, there is $t \in \mathrm{dom}(Q_A)$ with $(u_k)_0 \to t$ in $\mathrm{dom}(Q_A)$, and as also $(u_k)_0 \to 0$ in $L^2(M, E)$ by assumption, we have $t = 0$. Now

$$\widetilde{Q}_{A,U}(u_k, u_k) = Q_A((u_k)_0, (u_k)_0) \to Q_A(t, t) = 0$$

as $k \to \infty$, so $\widetilde{Q}_{A,U}$ is closable. ∎

Definition 2.1.2. For $U \subseteq M$ an open subset, we define the quadratic form $Q_{A,U}$ as the closure of the quadratic form $\widetilde{Q}_{A,U}$ from Lemma 2.1.1. The self-adjoint operator associated to $Q_{A,U}$ is denoted by A_U.

Note that the open subset U in Definition 2.1.2 is allowed to intersect ∂M. We think of A_U as being obtained by putting Dirichlet boundary conditions on $\partial U \setminus \partial M$, and keeping the original boundary conditions on $\partial M \cap U$. By (C.2.3), and since $\mathrm{dom}(\widetilde{Q}_{A,U})$ is dense in $\mathrm{dom}(Q_{A,U})$, the operator A_U is given by

$$\begin{aligned}\mathrm{dom}(A_U) = \{s \in \mathrm{dom}(Q_{A,U}) : \text{there is } u_s \in L^2(U, E) \text{ such that} \\ Q_{A,U}(s, t) = \langle\!\langle u_s, t\rangle\!\rangle_{L^2(U,E)} \text{ for all } t \in \mathrm{dom}(\widetilde{Q}_{A,U})\},\end{aligned} \quad (2.1.1)$$

and $A_U s := u_s$ for $s \in \mathrm{dom}(A_U)$.

Proposition 2.1.3. *Let $U \subseteq M$ be open. Then $\{s|_U : s \in \mathrm{dom}(A)$ and $\mathrm{supp}(s) \subseteq U\}$ is contained in $\mathrm{dom}(A_U)$, and $A_U(s|_U) = (As)|_U$. In particular, A_U is an extension of the differential operator $D_U := D|_{\Gamma(U,E)} \colon \Gamma(U, E) \to \Gamma(U, E)$, again in the sense that $(D_U)_{cc} \subseteq A_U$. By Remark 1.3.5, A_U is a restriction of $(D_U)_w$.*

Proof. Let $s \in \mathrm{dom}(A)$ with $\mathrm{supp}(s) \subseteq U$. Then $s \in \mathrm{dom}(Q_A)$, hence $s|_U \in \mathrm{dom}(\widetilde{Q}_{A,U})$ by definition. Moreover, for $t \in \mathrm{dom}(\widetilde{Q}_{A,U})$, we have

$$Q_{A,U}(s|_U, t) = Q_A((s|_U)_0, t_0) = Q_A(s, t_0) = \langle\!\langle As, t_0\rangle\!\rangle_{L^2(M,E)} = \langle\!\langle (As)|_U, t\rangle\!\rangle_{L^2(U,E)}.$$

It now follows from (2.1.1) that $s|_U \in \mathrm{dom}(A_U)$ and $A_U(s|_U) = (As)|_U$. ∎

Lemma 2.1.4. *Let $U, V \subseteq M$ be open subsets with $U \subseteq V$. If $u \in \mathrm{dom}(Q_{A,U})$, then $u_0 \in L^2(V, E)$ belongs to $\mathrm{dom}(Q_{A,V})$, and $Q_{A,U}(u, v) = Q_{A,V}(u_0, v_0)$ for all $u, v \in \mathrm{dom}(Q_{A,U})$. In particular, $\inf \sigma(A_U) \geq \inf \sigma(A_V)$.*

Proof. Let $s_k \in \mathrm{dom}(Q_A)$ be a sequence with $\mathrm{supp}(s_k) \subseteq U$ and $s_k|_U \to u$ in $\mathrm{dom}(Q_{A,U})$. Then the definition of $Q_{A,U}$ implies that $k \mapsto s_k$ is Cauchy in $\mathrm{dom}(Q_A)$, hence convergent to some $s_\infty \in \mathrm{dom}(Q_A)$. Moreover,

$$Q_{A,U}(u, u) = \lim_{k\to\infty} \widetilde{Q}_{A,U}(s_k|_U, s_k|_U) = \lim_{k\to\infty} Q_A(s_k, s_k) = Q_A(s_\infty, s_\infty).$$

Similarly, since $\mathrm{supp}(s_k) \subseteq V$, we have $s_k|_V \to u_0$ in $L^2(V, E)$ and $k \mapsto s_k|_V$ is Cauchy in $\mathrm{dom}(Q_{A,V})$, hence convergent to u_0 in $\mathrm{dom}(Q_{A,V})$. Thus,

$$Q_{A,V}(u_0, u_0) = \lim_{k\to\infty} \widetilde{Q}_{A,V}(s_k|_V, s_k|_V) = \lim_{k\to\infty} Q_A(s_k, s_k) = Q_A(s_\infty, s_\infty) = Q_{A,U}(u, u).$$

By the polarization identity (C.2.1), we get equality also away from the diagonal.

The inequality about the bottom of the spectra follows from the fact that $\inf \sigma(A_U)$ is the largest lower bound of $Q_{A,U}$, for we have

$$Q_{A,U}(s,s) = Q_{A,V}(s_0,s_0) \geq (\inf \sigma(A_V))\|s_0\|^2 = (\inf \sigma(A_V))\|s\|^2$$

for all $s \in \mathrm{dom}(Q_{A,U})$, hence $\inf \sigma(A_U) \geq \inf \sigma(A_V)$. ■

The next Theorem is central for what follows. It shows that the essential spectrum of a self-adjoint *elliptic* differential operator depends only on the situation at infinity and near the boundary ∂M. We have adapted the proof of [Bär00, Proposition 1] to our situation. Other sources with similar statements or for certain classes of operators include [Eic88, Proposition 4.9], [Eic07, Proposition 1.4], and [MM07, Proposition 3.2.4]. The minor difference between the decomposition principle of Bär and ours is that we do not assume that $\Gamma_c(M,E) \cap \mathrm{dom}(A)$ is a core for A. Recall (for example, from Proposition C.1.3) that a number $\lambda \in \mathbb{C}$ belongs to the essential spectrum $\sigma_e(T)$ of a normal operator T if and only if there exists a sequence $x_k \in D_0$, where D_0 is any core of T, with $x_k \to 0$ weakly, $\liminf_{k\to\infty} \|x_k\| > 0$, and $(T-\lambda)x_k \to 0$. Such a sequence is called a *singular Weyl sequence* for (T,λ).

Theorem 2.1.5 (Decomposition principle)**.** *Let A be a lower semibounded self-adjoint extension of an elliptic differential operator as above. Then*

$$\sigma_e(A_U) = \sigma_e(A_{U\setminus K})$$

for all $U \subseteq M$ open and $K \subseteq U^\circ$ compact, where $U^\circ = U \setminus \partial M$ is the interior of U as a manifold with boundary $U \cap \partial M$.

Proof. Since A_U is a self-adjoint extension of an elliptic differential operator, it has a core consisting of smooth sections, *i.e.*, elements of $\Gamma(U^\circ, E)$, see Corollary 1.3.10. In particular, $\Gamma(U^\circ, E) \cap \mathrm{dom}(A_U)$ is a core for A_U, and a similar statement holds for $A_{U\setminus K}$.

Let $s_k \in \Gamma(U^\circ \setminus K, E) \cap \mathrm{dom}(A_{U\setminus K})$, $k \geq 1$, be a singular Weyl sequence for $(A_{U\setminus K}, \lambda)$. Clearly, $(s_k)_0 \to 0$ weakly and $\liminf_{k\to\infty} \|(s_k)_0\| > 0$. Moreover,

$$A_{U\setminus K} s_k = A_{U\setminus K}(((s_k)_0)|_{U\setminus K}) = (A_U(s_k)_0)|_{U\setminus K}$$

by Proposition 2.1.3, hence $A_U(s_k)_0 = (A_{U\setminus K} s_k)_0$ because $\mathrm{supp}((s_k)_0) \subseteq U \setminus K$ and differential operators do not increase the support of sections. Therefore, $\|A_U(s_k)_0 - \lambda(s_k)_0\| = \|(A_{U\setminus K}s_k)_0 - \lambda(s_k)_0\| = \|A_{U\setminus K}s_k - \lambda s_k\| \to 0$ as $k \to \infty$, showing that $\sigma_e(A_{U\setminus K}) \subseteq \sigma_e(A_U)$.

Conversely, suppose that $\lambda \in \sigma_e(A_U)$. Let $(u_k)_{k\geq 1}$ be a corresponding singular Weyl sequence for (A_U, λ) contained in $\Gamma(U^\circ, E) \cap \mathrm{dom}(A_U)$. Let K' and K'' be compact manifolds with boundary such that $K \subseteq (K')^\circ \subseteq K' \subseteq (K'')^\circ \subseteq K'' \subseteq U^\circ$. By the elliptic estimates from Theorem 1.3.9, there is $C > 0$ (independent of k) such that

$$\begin{aligned}\|u_k|_{K''}\|_{H^d(K'',E)} &\leq C\big(\|A_U u_k\|_{L^2(U,E)} + \|u_k\|_{L^2(U,E)}\big) \leq \\ &\leq C\big(\|A_U u_k - \lambda u_k\|_{L^2(U,E)} + (|\lambda|+1)\sup_k \|u_k\|\big), \quad (2.1.2)\end{aligned}$$

where d is the order of D. Every weakly convergent sequence is bounded, and (2.1.2) shows that $\{u_k|_{K''}\}_{k\geq 1}$ is bounded in the Sobolev space $H^d(K'', E)$. Because K'' is compact, Rellich's theorem (see Theorem 1.3.8) implies that a subsequence, which we still denote by $(u_k|_{K''})_{k\geq 1}$, converges in $H^{d-1}(K'', E)$. Call this limit u_∞. Choose $\varphi \in C_c^\infty(M, [0,1])$ with $\varphi|_{K'} = 1$, and $\varphi|_{M\setminus K''} = 0$. Since $(u_k)_{k\geq 1}$ is weakly null, we have $\|\varphi u_\infty\|^2 = \lim_{k\to\infty}\langle\!\langle \varphi u_k, \varphi u_\infty\rangle\!\rangle = \lim_{k\to\infty}\langle\!\langle u_k, \varphi^2 u_\infty\rangle\!\rangle = 0$, and hence $u_\infty|_{K'} = 0$ almost everywhere. Thus, $u_k|_{K'} \to 0$ in $L^2(K', E)$ and, in particular, $\|u_k\|_{L^2(U\setminus K', E)} \geq \liminf_{j\to\infty}\|u_j\|/2 > 0$ for sufficiently large k.

Now take $\psi \in C_c^\infty(M, [0,1])$ with $\psi|_K = 1$ and $\psi|_{M\setminus K'} = 0$. Put $s_k := (1-\psi)u_k|_{U^\circ\setminus K} \in \Gamma(U^\circ \setminus K, E)$. Then $s_k \in \operatorname{dom}(A_{U\setminus K})$ by Corollary 1.3.12 and Proposition 2.1.3. By the above, $\liminf_{k\to\infty}\|s_k\| > 0$, and since $\langle\!\langle s_k, v\rangle\!\rangle = \langle\!\langle u_k, (1-\psi)v_0\rangle\!\rangle \to 0$ as $k \to \infty$ for all $v \in L^2(U \setminus K, E)$, where v_0 denotes the extension by zero outside of $U \setminus K$, we have $s_k \to 0$ weakly. Moreover, on $U \setminus K$,

$$A_{U\setminus K}s_k = D((1-\psi)u_k) = (1-\psi)Du_k + [D, 1-\psi]u_k,$$

and the differential operator $[D, 1-\psi]$ of order $d-1$ vanishes outside of K'', since ψ is constant there. Therefore, there is a constant $C' > 0$ such that

$$\|[D, 1-\psi]u_k\|_{L^2(U\setminus K, E)} \leq C'\|u_k\|_{H^{d-1}(K'', E)},$$

and this converges to zero by the above argument involving Rellich's theorem. Finally,

$$\|A_{U\setminus K}s_k - \lambda s_k\| \leq \|(1-\psi)(A_U u_k - \lambda u_k)\| + \|[D, 1-\psi]u_k\|_{L^2(U\setminus K, E)} \to 0$$

as $k \to \infty$, so that $(s_k)_{k\geq 1}$ is a singular Weyl sequence for $(A_{U\setminus K}, \lambda)$. Thus, $\lambda \in \sigma_e(A_{U\setminus K})$. ■

Remark 2.1.6. If A satisfies appropriate elliptic estimates also on compact subsets K intersecting the boundary of M, then the decomposition principle also holds for those K. This is considered in [Bär00].

2.2. The bottom of the essential spectrum

In this section, we wish to study the bottom of the essential spectrum of a nonnegative self-adjoint extension of an elliptic differential operator. Of course, most results also apply to lower semibounded operators after straightforward modifications. Before we do this, we prove the following general result about the bottom of the essential spectrum of a nonnegative self-adjoint operator S on a Hilbert space $(H, \langle \bullet, \bullet\rangle)$. Recall that a real number λ satisfies $\lambda \leq \inf\sigma(S)$ if and only if $\lambda\|x\|^2 \leq Q_S(x,x)$ for all $x \in \operatorname{dom}(Q_S)$, because $\inf\sigma(S)$ is the largest lower bound of Q_S, the quadratic form associated to S. In order to characterize the bottom of the *essential* spectrum, this inequality has to be perturbed by compact operators:

Theorem 2.2.1. *Let S be a nonnegative self-adjoint operator on $(H, \langle \bullet, \bullet\rangle)$. Denote by*

$$Q_S\colon \operatorname{dom}(S^{1/2}) \times \operatorname{dom}(S^{1/2}) \to \mathbb{C}, \quad Q_S(x,y) := \langle S^{1/2}x, S^{1/2}y\rangle$$

the quadratic form associated to S, see appendix C.2, and equip $\operatorname{dom}(Q_S) := \operatorname{dom}(S^{1/2})$ *with the inner product* $(x,y) \mapsto \langle x,y\rangle + Q_S(x,y)$*. Then the following are equivalent for* $0 < \lambda_0 \leq \infty$*:*

(i) $\lambda_0 \leq \inf \sigma_e(S)$.

(ii) *For every* $0 < \lambda < \lambda_0$*, there exists a Banach space* $(Z, \|\bullet\|_Z)$ *and a compact linear operator* $T\colon \operatorname{dom}(Q_S) \to Z$ *such that*

$$\lambda\|x\|_H^2 \leq Q_S(x,x) + \|Tx\|_Z^2 \tag{2.2.1}$$

for all $x \in \operatorname{dom}(Q_S)$.

(iii) *For every* $0 < \lambda < \lambda_0$*, the inclusion* $\operatorname{dom}(Q_S) \cap \operatorname{img}(P_S([0,\lambda])) \hookrightarrow H$ *is a compact operator, where* P_S *is the spectral measure associated to* S*, and the first space is viewed as a subspace of* $\operatorname{dom}(Q_S)$.[1]

Proof. Assume first that (i) is true, and let $0 < \lambda < \lambda_0$. Put $Z := \operatorname{img}(P_S([0,\lambda])) \subseteq H$, equipped with the norm of H, and let $P^\lambda := P_S([0,\lambda])\colon H \to Z$ be the orthogonal projection. Then P^λ is compact (it even has finite rank), and so is $T := \sqrt{\lambda}P^\lambda \circ \iota\colon \operatorname{dom}(Q_S) \to Z$, where $\iota\colon \operatorname{dom}(Q_S) \hookrightarrow H$ is the (continuous) inclusion. Moreover,

$$\lambda\|x\|_H^2 = \lambda\|(1-P^\lambda)x\|_H^2 + \lambda\|P^\lambda x\|_H^2 \leq$$
$$\leq Q_S((1-P^\lambda)x, (1-P^\lambda)x) + \lambda\|P^\lambda\iota(x)\|_Z^2 \leq Q_S(x,x) + \|Tx\|_Z^2$$

for $x \in \operatorname{dom}(Q_S)$, where the inequality $\lambda\|y\|_H^2 \leq Q_S(y,y)$ for $y \in \operatorname{img}(1-P^\lambda)$ is due to $\operatorname{img}(1-P^\lambda) = \operatorname{img}(P_S((\lambda,\infty)))$.

Next, we show (ii)$\Rightarrow$(iii). Suppose that $0 < \lambda < \lambda_0$ and choose $\mu \in (\lambda, \lambda_0)$. By (ii), there exists a compact operator $T\colon \operatorname{dom}(Q_S) \to Z$ such that

$$\mu\|y\|_H^2 \leq Q_S(y,y) + \|Ty\|_Z^2 \leq \lambda\|y\|_H^2 + \|Ty\|_Z^2$$

for all $y \in \operatorname{img}(P_S([0,\lambda]))$. Let $(x_j)_{j\in\mathbb{N}}$ be a bounded sequence in $\operatorname{dom}(Q_S) \cap \operatorname{img}(P_S([0,\lambda]))$. We may assume without losing any generality that $(Tx_j)_{j\in\mathbb{N}}$ converges in Z, say $\lim_j Tx_j = z \in Z$. Since also $x_j - x_k \in \operatorname{img}(P_S([0,\lambda]))$, the estimate

$$(\mu-\lambda)^{1/2}\|x_j - x_k\|_H \leq \|T(x_j - x_k)\|_Z \leq \|Tx_j - z\|_Z + \|Tx_k - z\|_Z$$

for $j,k \in \mathbb{N}$ shows that $(x_j)_{j\in\mathbb{N}}$ is Cauchy in H, hence convergent.

Finally, we show that (iii) implies (i). If $\lambda_0 > \inf \sigma_e(S)$, then $\inf \sigma_e(S)$ is finite (*i.e.*, not $+\infty$) and we choose $\lambda \in (\inf \sigma_e(S), \lambda_0)$. Because the rank of $P_S(B_{1/j}(\inf \sigma_e(S)))$ is infinite for all $j \in \mathbb{N}$, we obtain mutually orthogonal (in H) unit vectors x_j contained in $\operatorname{img}(P_S(B_{1/j}(\inf \sigma_e(S)))) \subseteq \operatorname{dom}(Q_S)$. Find $N \geq 1$ such that $B_{1/N}(\inf \sigma_e(S)) \subseteq [0,\lambda]$. Then the bounded sequence $(x_j)_{j\geq N}$ in $\operatorname{img}(P_S([0,\lambda]))$ cannot have a subsequence which converges in H, a contradiction to (iii). Thus, $\lambda_0 \leq \inf \sigma_e(S)$. ■

We point out the two extremal cases of Theorem 2.2.1 in the following corollaries:

Corollary 2.2.2. *Let* S *be a nonnegative self-adjoint operator on* $(H, \langle\bullet,\bullet\rangle)$*. Then the following are equivalent:*

---

[1]Since $\lambda < \infty$, we actually have $\operatorname{img}(P_S([0,\lambda])) \subseteq \operatorname{dom}(Q_S)$, but we use the notation to emphasize that $\operatorname{img}(P_S([0,\lambda]))$ is considered a subspace of $\operatorname{dom}(Q_S)$.

(i) The range of S is closed and $\dim(\ker(S)) < \infty$. *In other words, S is a (possibly unbounded)* Fredholm operator.

(ii) $0 \notin \sigma_e(S)$ *or, equivalently,* $\inf \sigma_e(S) > 0$.

(iii) There exists a Banach space Z, a compact linear operator $T\colon \operatorname{dom}(Q_S) \to Z$, and a constant $C > 0$ such that

$$\|x\|_H^2 \le C\big(Q_S(x,x) + \|Tx\|_Z^2\big)$$

for all $x \in \operatorname{dom}(Q_S)$.

Proof. The equivalence of (i) and (ii) is a standard fact of the spectral theory of self-adjoint operators, *cf.*, Remark 1.2.7. The rest follows easily from Theorem 2.2.1, with $C := 1/\lambda$ for $\lambda \in (0, \inf \sigma_e(S))$. ■

Corollary 2.2.3. *Let S be a nonnegative self-adjoint operator on $(H, \langle \bullet, \bullet \rangle)$. Then the following are equivalent:*

(i) The spectrum of S is discrete, i.e., $\sigma_e(S) = \emptyset$.

(ii) For every $\varepsilon > 0$, there exists a Banach space $(Z, \|\bullet\|_Z)$ and a compact linear operator $T\colon \operatorname{dom}(Q_S) \to Z$ such that

$$\|x\|_H^2 \le \varepsilon\, Q_S(x,x) + \|Tx\|_Z^2$$

for all $x \in \operatorname{dom}(Q_S)$.

(iii) The inclusion $\operatorname{dom}(Q_S) \hookrightarrow H$ *is a compact operator.*

Proof. If $\sigma_e(S) = \emptyset$, then item (ii) of Theorem 2.2.1 holds for every $\lambda > 0$. Let $\varepsilon > 0$ and find λ with $1/\lambda < \varepsilon$. Then $\|x\|_H^2 \le \varepsilon\, Q_S(x,x) + \|\lambda^{-1/2}\, Tx\|_Z^2$ for some compact $T\colon \operatorname{dom}(Q_S) \to Z$, and of course $\lambda^{-1/2}\, T$ is also compact. Conversely, we put $\varepsilon := 1/\lambda$ for a given $\lambda > 0$. Then $\lambda \|x\|_H^2 \le Q_S(x,x) + \|\sqrt{\lambda} Tx\|_Z^2$ with compact $T\colon \operatorname{dom}(Q_S) \to Z$, hence $\inf \sigma_e(S) > \lambda$ by Corollary 2.2.2.

It remains to establish the equivalence of (i) and (iii). This is a standard fact in spectral theory, see [Sch12, Proposition 5.12], but we give here a different proof, based on approximating $\operatorname{dom}(Q_S) \hookrightarrow H$ by the compact inclusions from Theorem 2.2.1. Assume that (i) is true. By Theorem 2.2.1, we have the compact embeddings $\iota_k\colon \operatorname{dom}(Q_S) \cap \operatorname{img}(P_S([0,k])) \hookrightarrow H$ for $k \in \mathbb{N}$. Put $T_k := \iota_k \circ P_S([0,k])\colon \operatorname{dom}(Q_S) \to H$, and denote by $\iota_\infty\colon \operatorname{dom}(Q_S) \hookrightarrow H$ the inclusion map. The T_k, $k \in \mathbb{N}$, are compact operators, and if $x \in \operatorname{dom}(Q_S)$, then

$$\begin{aligned} \|(\iota_\infty - T_k)x\|_H^2 &= \|P_S((k,\infty))x\|_H^2 = \langle P_S((k,\infty))x, x\rangle = \\ &= \int_{(k,\infty)} d\langle P_S(t)x, x\rangle \le \frac{1}{k}\int_{(k,\infty)} (1+t)\, d\langle P_S(t)x, x\rangle \le \frac{1}{k}\big(\|x\|_H^2 + Q_S(x,x)\big). \end{aligned}$$

It follows that $T_k \to \iota_\infty$ as $k \to \infty$ in the operator norm of $\mathscr{L}(\operatorname{dom}(Q_S), H)$, hence ι_∞ is also compact. Conversely, if ι_∞ is compact, then so are all $\iota_k = \iota_\infty|_{\operatorname{img}(P_S([0,k]))}$, and Theorem 2.2.1 implies that $k < \inf \sigma_e(S)$ for all $k \ge 1$, hence $\sigma_e(S) = \emptyset$. ■

Remark 2.2.4. Theorem 2.2.1 and its corollaries are inspired by [BB12, Proposition A.3] and [Str10, Lemma 4.3]. Both of these references deal with *bounded* operators, and the transition to unbounded self-adjoint operators $S \geq 0$ is essentially accomplished by studying the bounded operator $S^{1/2} \colon \operatorname{dom}(Q_S) \to H$ instead. The article [BB12] treats the bounded operator version of Corollary 2.2.2, while [Str10] considers compact operators (*i.e.*, the analogue of Corollary 2.2.3), with the inverse of a positive self-adjoint operator in mind (see Proposition 4.2 therein), and also contains references to some similar statements in the literature.

Consider now a Hermitian vector bundle $E \to M$ over a Riemannian manifold M, and a nonnegative self-adjoint extension A of an elliptic differential operator $D \colon \Gamma(M, E) \to \Gamma(M, E)$. Apart from the decomposition principle in Theorem 2.1.5, one of the main tools used in the rest of this section will be the following simple property of compact subsets of $L^p(M, E)$:

Lemma 2.2.5. *Let $1 \leq p < \infty$. Suppose that E is a Hermitian vector bundle over a Riemannian manifold M, and let $B \subseteq L^p(M, E)$ be a totally bounded (equivalently: relatively compact) subset. Then for every $\varepsilon > 0$ there exists a compact subset $K \subseteq M^\circ$ such that*

$$\int_{M \setminus K} |s|^p \, d\mu_g \leq \varepsilon$$

for all $s \in B$.

Proof. We adopt the proof from [Rup11, Theorem 2.5], where a characterization of the compact subsets of $L^2(M)$ is given, *cf.*, Remark 2.2.6. As B is totally bounded there is, for any given $\varepsilon > 0$, a finite subset $\mathcal{F} \subseteq L^p(M, E)$ with $B \subseteq \bigcup_{t \in \mathcal{F}} \{s \in L^p(M, E) : \|s - t\|_{L^p(M,E)} < \sqrt[p]{\varepsilon}/2\}$. Since M is assumed to be second countable, its interior M° is exhausted by a sequence of compact subsets.[2] Using this, $\mu_g(\partial M) = 0$, and the fact that measures are continuous from below, we see that there exists $K \subseteq M^\circ$ compact such that $\|\chi_{M \setminus K} t\|_{L^p(M,E)} < \sqrt[p]{\varepsilon}/2$ for all $t \in \mathcal{F}$, where we denote by $\chi_\Omega \colon M \to \{0, 1\}$ the *characteristic function* of $\Omega \subseteq M$. If $s \in B$, we find $t \in \mathcal{F}$ such that $\|s - t\|_{L^p(M,E)} < \sqrt[p]{\varepsilon}/2$, and it then follows that

$$\left(\int_{M \setminus K} |s|^p \, d\mu_g \right)^{1/p} \leq \|\chi_{M \setminus K}(s - t)\|_{L^p(M,E)} + \|\chi_{M \setminus K} t\|_{L^p(M,E)} < \frac{\sqrt[p]{\varepsilon}}{2} + \frac{\sqrt[p]{\varepsilon}}{2} = \sqrt[p]{\varepsilon}. \qquad \blacksquare$$

Remark 2.2.6. (i) In particular, if $T \colon X \to L^p(M, E)$ is a compact linear operator from a Banach space X to $L^p(M, E)$, then the image of the unit ball in X under T is totally bounded in $L^p(M, E)$. Lemma 2.2.5 implies that there exists, for every $\varepsilon > 0$, a compact subset $K \subseteq M^\circ$ such that

$$\int_{M \setminus K} |Tx|^p \, d\mu_g \leq \varepsilon \|x\|_X^p \tag{2.2.2}$$

for all $x \in X$.

[2]This is true for second countable, locally compact Hausdorff spaces, see [Lee10, Proposition 4.76].

(ii) One can extend Lemma 2.2.5 to obtain a *characterization* of the compact subsets of $L^p(M,E)$, see [Alb08, Theorem 4.9]:[3] a subset $B \subseteq L^p(M,E)$ is totally bounded if and only if it is bounded, satisfies the conclusion of Lemma 2.2.5, and has the property that $\{s|_K : s \in B\}$ is totally bounded in $L^p(K,E)$ for every compact $K \subseteq M$.

For the spaces $L^p(\mathbb{R}^n)$, the latter property can be replaced by requiring that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p \, d\lambda(x) < \varepsilon$ for all $f \in B$ and $|y| < \delta$, where λ is Lebesgue measure on $\mathbb{R}^n$. In this context, the characterization is sometimes called the *Kolmogorov–Riesz compactness theorem.* A proof can be found in [AF03, Theorem 2.32] or [HH10], with the latter containing comparisons to the Arzelà–Ascoli theorem and some historical notes. When $\mathbb{R}^n$ is replaced by a Riemannian manifold, then one can use (instead of translations) diffeomorphisms which are close to the identity in a specific sense, see [Rup11] for the details.

We are now ready to show our main Lemma for this section:

Lemma 2.2.7. *Let A be a nonnegative self-adjoint operator*[4] *on $L^2(M,E)$. For every $0 < \lambda < \inf \sigma_e(A)$ and $\varepsilon > 0$, there exists a compact subset $K \subseteq M^\circ$ such that*

$$Q_A(s,s) \geq \int_M (\lambda \chi_{M\setminus K} - \varepsilon \chi_K)|s|^2 \, d\mu_g = \lambda \int_{M\setminus K} |s|^2 \, d\mu_g - \varepsilon \int_K |s|^2 \, d\mu_g$$

for all $s \in \operatorname{dom}(Q_A)$, where again $\chi_{M\setminus K}$ and χ_K are the characteristic functions.

Proof. Denote by P_A the spectral measure associated to A, and let

$$0 < \delta < \min\{\inf \sigma_e(A) - \lambda, \varepsilon/2\}.$$

Put $P_0 := P_A([0, \lambda + \delta])$. By Theorem 2.2.1, the inclusion $\operatorname{img}(P_0) \cap \operatorname{dom}(Q_A) \hookrightarrow L^2(M,E)$ is compact and, by (2.2.2), there exists a compact subset $K \subseteq M^\circ$ such that

$$\int_{M\setminus K} \Big(\lambda + \delta + \frac{\varepsilon}{2}\Big)|s|^2 \, d\mu_g \leq \frac{\varepsilon}{2}\|s\|^2 + Q_A(s,s) \tag{2.2.3}$$

for all $s \in \operatorname{img}(P_0) \subseteq \operatorname{dom}(Q_A)$ and

$$\Big(\int_{M\setminus K} |P_0 s|^2 \, d\mu_g\Big)^{1/2} \leq \frac{\delta}{2(\lambda + \delta + \frac{\varepsilon}{2})}\|s\| \tag{2.2.4}$$

for all $s \in L^2(M,E)$. Here, (2.2.3) is possible since $s \mapsto (\frac{\varepsilon}{2}\|s\|^2 + Q_A(s,s))^{1/2}$ is equivalent to the norm on $\operatorname{dom}(Q_A)$, and (2.2.4) works since $P_0 \colon H \to H$ is a finite rank projection. Now,

[3]The result there is only for $L^2(M,E)$, but the proof easily carries over to $L^p(M,E)$.

[4]Note that A need not be an extension of a differential operator.

for $s \in \operatorname{dom}(Q_A)$,

$$\begin{aligned} Q_A(s,s) &= Q_A(P_0 s, P_0 s) + Q_A((1-P_0)s,(1-P_0)s) \geq \\ &\geq \int_{M\setminus K} \Big(\lambda+\delta+\frac{\varepsilon}{2}\Big)|P_0 s|^2\,d\mu_g - \frac{\varepsilon}{2}\int_M |P_0 s|^2\,d\mu_g + (\lambda+\delta)\int_M |(1-P_0)s|^2\,d\mu_g \geq \\ &\geq \langle\!\langle \chi P_0 s, P_0 s\rangle\!\rangle + \langle\!\langle \chi(1-P_0)s,(1-P_0)s\rangle\!\rangle \end{aligned} \tag{2.2.5}$$

with $\chi := (\lambda+\delta+\frac{\varepsilon}{2})\chi_{M\setminus K} - \frac{\varepsilon}{2} = (\lambda+\delta)\chi_{M\setminus K} - \frac{\varepsilon}{2}\chi_K$, and where we have used that $\chi \leq \lambda+\delta$ to estimate the term with $(1-P_0)s$. The right hand side of (2.2.5) is equal to

$$\begin{aligned} &\langle\!\langle \chi s, s\rangle\!\rangle - \langle\!\langle (P_0\chi(1-P_0) + (1-P_0)\chi P_0)s, s\rangle\!\rangle = \\ &\qquad = \langle\!\langle \chi s, s\rangle\!\rangle - \langle\!\langle P_0(\chi+\tfrac{\varepsilon}{2})(1-P_0)s, s\rangle\!\rangle - \langle\!\langle (1-P_0)(\chi+\tfrac{\varepsilon}{2})P_0 s, s\rangle\!\rangle, \end{aligned}$$

where $P_0(1-P_0)=0$ was used in order to replace χ by $\widetilde{\chi} := \chi + \frac{\varepsilon}{2}$. Moreover,

$$\begin{aligned} &-\langle\!\langle P_0\widetilde{\chi}(1-P_0)s, s\rangle\!\rangle - \langle\!\langle (1-P_0)\widetilde{\chi}P_0 s, s\rangle\!\rangle = \\ &\qquad = -\langle\!\langle \widetilde{\chi}s, P_0 s\rangle\!\rangle + \langle\!\langle \widetilde{\chi}P_0 s, P_0 s\rangle\!\rangle - \langle\!\langle P_0 s, \widetilde{\chi}s\rangle\!\rangle + \langle\!\langle \widetilde{\chi}P_0 s, P_0 s\rangle\!\rangle \geq -2\operatorname{Re}\langle\!\langle \widetilde{\chi}s, P_0 s\rangle\!\rangle, \end{aligned}$$

the inequality being due to $\widetilde{\chi} \geq 0$. Now

$$|2\langle\!\langle \widetilde{\chi}s, P_0 s\rangle\!\rangle| = 2(\lambda+\delta+\tfrac{\varepsilon}{2})|\langle\!\langle \chi_{M\setminus K}s, P_0 s\rangle\!\rangle| \leq 2(\lambda+\delta+\tfrac{\varepsilon}{2})\|s\|\|\chi_{M\setminus K}P_0 s\| \leq \delta\|s\|^2$$

by (2.2.4). Putting it all together, we have shown that

$$Q_A(s,s) \geq \langle\!\langle \chi s, s\rangle\!\rangle - \delta\|s\|^2 = \int_M \Big(\lambda\chi_{M\setminus K} - \Big(\frac{\varepsilon}{2}+\delta\Big)\chi_K\Big)|s|^2\,d\mu_g \geq \int_M (\lambda\chi_{M\setminus K} - \varepsilon\chi_K)|s|^2\,d\mu_g \tag{2.2.6}$$

for all $s \in \operatorname{dom}(Q_A)$, as claimed. ■

The next result is the appropriate generalization of *Persson's theorem* [Per60] to our setting, and gives a characterization of the bottom of the essential spectrum. Its proof is now an easy consequence of Theorem 2.1.5 and Lemma 2.2.7.

Theorem 2.2.8. *Let A be a nonnegative self-adjoint extension of an elliptic differential operator acting on the sections of a Hermitian vector bundle $E \to M$ over a Riemannian manifold. Then for every $\lambda < \inf \sigma_e(A)$, there exists a compact subset $K \subseteq M^\circ$ such that $\inf \sigma(A_{M\setminus K}) \geq \lambda$. In fact,*

$$\lim_K \big(\inf \sigma(A_{M\setminus K})\big) = \inf \sigma_e(A), \tag{2.2.7}$$

where the limit is with respect to the net of compact subsets of M°, directed by $K_1 \geq K_2 :\Leftrightarrow K_1 \supseteq K_2$.

Proof. If $\lambda \leq 0$, then we may put $K := \emptyset$. Given $0 < \lambda < \inf \sigma_e(A)$, there exists a compact subset $K \subseteq M^\circ$ such that $Q_A(s,s) \geq \lambda \int_{M\setminus K} |s|^2\,d\mu_g$ for all $s \in \operatorname{dom}(Q_A)$ with $s|_K = 0$, see

Lemma 2.2.7. For $s \in \operatorname{dom}(Q_{A,M\setminus K})$, we have $s_0 \in \operatorname{dom}(Q_A)$ by Lemma 2.1.4, and

$$Q_{A,M\setminus K}(s,s) = Q_A(s_0,s_0) \geq \lambda \int_{M\setminus K} |s_0|^2 \, d\mu_g = \lambda \|s\|^2.$$

Therefore, $\inf \sigma(A_{M\setminus K}) \geq \lambda$. It follows from Lemma 2.1.4 that $K \mapsto \inf \sigma(A_{M\setminus K})$ is an increasing net, so the limit (2.2.7) exists. Since the above holds for every $\lambda < \inf \sigma_e(A)$, we obtain $\lim_K(\inf \sigma(A_{M\setminus K})) \geq \inf \sigma_e(A)$, and by Theorem 2.1.5 we also have

$$\inf \sigma(A_{M\setminus K}) \leq \inf \sigma_e(A_{M\setminus K}) = \inf \sigma_e(A),$$

so that equality holds in (2.2.7). ■

In case $\sigma(A)$ is discrete, we can use Lemma 2.2.7 to construct proper coercivity functions for Q_A, in the following sense:

Theorem 2.2.9. *Let A be a nonnegative self-adjoint extension of an elliptic differential operator acting on the sections of $E \to M$. Then the following are equivalent:*

(i) The spectrum of A is discrete.

(ii) There exists a proper smooth function $\psi\colon M^\circ \to [-1,\infty)$ such that

$$Q_A(s,s) \geq \int_M \psi |s|^2 \, d\mu_g \tag{2.2.8}$$

for all $s \in \operatorname{dom}(Q_A)$.

(iii) There exists a proper measurable function $\psi\colon M^\circ \to [-1,\infty)$ such that (2.2.8) *holds for all $s \in \operatorname{dom}(Q_A)$.*

Proof. Item (ii) is inspired by [Has14; Iwa86; KS02; Rup11], where the construction is done for certain classes of operators, *cf.*, Remarks 2.2.10 and 2.2.11 below. Assume first that A has discrete spectrum. By Lemma 2.2.7, there are compact subsets $K_k \subseteq M^\circ$, $k \in \mathbb{N}$, such that $Q_A(s,s) \geq 2^k k \int_{M\setminus K_k} |s|^2 \, d\mu_g - \int_{K_k} |s|^2 \, d\mu_g$ for all $s \in \operatorname{dom}(Q_A)$. Without loss of generality, we may assume that $(K_k)_{k\in\mathbb{N}}$ forms a compact exhaustion of M°. For $s \in \operatorname{dom}(Q_A)$, we estimate

$$\begin{aligned} Q_A(s,s) = \sum_{k=1}^{\infty} 2^{-k} Q_A(s,s) &\geq \sum_{k=1}^{\infty} \Big(\int_{M\setminus K_k} k|s|^2 \, d\mu_g - 2^{-k} \int_{K_k} |s|^2 \, d\mu_g \Big) \geq \\ &\geq \sum_{k=1}^{\infty} \int_{K_{k+1}\setminus K_k} k|s|^2 \, d\mu_g - \sum_{k=1}^{\infty} 2^{-k}\|s\|^2 = \sum_{k=1}^{\infty} \int_{K_{k+1}\setminus K_k} k|s|^2 \, d\mu_g - \|s\|^2. \end{aligned}$$

Let $\psi_0\colon M^\circ \to [0,\infty)$ be a smooth function with $k-1 \leq \psi_0|_{K_{k+1}\setminus K_k} \leq k$ for $k \geq 1$ and $\psi_0|_{K_1} = 0$. Then ψ_0 is proper, and $\psi \coloneqq \psi_0 - 1\colon M^\circ \to [-1,\infty)$ has the properties sought in items (ii) and (iii).

Clearly, (ii) implies (iii), and if $\psi\colon M^\circ \to [-1,\infty)$ is as in (iii), then for $\lambda > 0$ fixed we put $K \coloneqq \psi^{-1}([-1,\lambda])$. Since ψ is proper, K is compact, and

$$Q_A(s,s) + \|s\|^2 \geq \int_M (\psi+1)|s|^2 \, d\mu_g \geq \int_{M\setminus K} (\psi+1)|s|^2 \, d\mu_g \geq (\lambda+1) \int_{M\setminus K} |s|^2 \, d\mu_g,$$

hence $Q_A(s,s) \geq \lambda \int_{M\setminus K} |s|^2\, d\mu_g - \int_K |s|^2\, d\mu_g$ for all $s \in \mathrm{dom}(Q_A)$. It follows that $\lambda \leq \inf \sigma(A_{M\setminus K})$, therefore $\lambda \leq \inf \sigma_e(A)$ by Theorem 2.2.8. Since λ was arbitrary, $\sigma_e(A) = \emptyset$. ■

Remark 2.2.10. (i) By modifying the definition of K_k in the proof of Theorem 2.2.9 such that $Q_A(s,s) \geq 2^k k \int_{M\setminus K_k} |s|^2\, d\mu_g - \varepsilon \int_{K_k} |s|^2\, d\mu_g$ for $s \in \mathrm{dom}(Q_A)$ and $k \in \mathbb{N}$, we see that, for every $\varepsilon > 0$, there is $\psi\colon M^\circ \to [-\varepsilon, \infty)$ smooth, proper, and satisfying (2.2.8). However, we cannot expect ψ to be nonnegative everywhere in general, since then $0 = Q_A(s,s) \geq \int_M \psi |s|^2\, d\mu_g$ for $s \in \ker(A)$ implies $s|_U = 0$ on the open subset $U := \psi^{-1}((0,\infty)) \subseteq M^\circ$. If M is connected and D is of order two, this would imply $s = 0$ everywhere by a unique continuation principle of Aronszajn, see [Aro57] or [Dem12, p. 333], so that $\ker(A) = 0$.

(ii) A statement similar to Theorem 2.2.9 says the following: if A has closed range in $L^2(M,E)$, then the discreteness of $\sigma(A|_{\ker(A)^\perp})$ is equivalent to the existence of a function $\psi\colon M^\circ \to [0,\infty)$ such that $\psi(x) \to \infty$ as $x \to \infty$ and $Q_A(s,s) \geq \int_M \psi |s|^2\, d\mu_g$ for all $s \in \mathrm{dom}(Q_A) \cap \ker(A)^\perp$. In this case, $\ker(A)$ is allowed to have infinite dimension, but the function ψ can be made nonnegative (and, in fact, bounded from below by $\inf(\sigma(A)\setminus\{0\}) - \varepsilon$). The discreteness of spectrum assumption in this statement is equivalent to the operator $(A|_{\ker(A)^\perp})^{-1}$ being compact on $\mathrm{img}(A)$, and extending to a compact operator on $L^2(M,E)$.

The proof of this is similar, and can be found in [Has14; Rup11]: Since A has closed range in $L^2(M,E)$, there is $C > 0$ such that $Q_A(s,s) \geq C\|s\|^2$ holds for all $s \in \mathrm{dom}(Q_A) \cap \ker(A)^\perp =: W_0$, and one can argue similarly as in the proofs of Lemma 2.2.7 and Theorem 2.2.9 by using the embedding $W_0 \hookrightarrow L^2(M,E)$ instead, where W_0 now has the inner product $(s,t) \mapsto Q_A(s,t)$, *cf.*, Proposition 1.2.8.

Remark 2.2.11. We currently do not know whether it is possible to have a version of Theorem 2.2.9 for the case $\inf \sigma_e(A) < \infty$, *i.e.*, whether there exists a proper smooth function $\psi\colon M^\circ \to [-1, \inf \sigma_e(A))$ with $Q_A(s,s) \geq \int_M \psi |s|^2\, d\mu_g$ for all $s \in \mathrm{dom}(Q_A)$. It follows easily from Lemma 2.2.7 that for every $0 < \lambda < \inf \sigma_e(A)$, there is a smooth function $\psi_\lambda\colon M \to [-1,\lambda]$ such that $Q_A(s,s) \geq \int_M \psi_\lambda |s|^2\, d\mu_g$ for $s \in \mathrm{dom}(Q_A)$ and $\psi_\lambda|_{M\setminus K} = \lambda$ for some compact $K \subseteq M^\circ$. The difficulty of passing to $\inf \sigma_e(A)$ is that this method does not seem to allow the construction of an *increasing* sequence of step functions $\varphi_k\colon M \to \mathbb{R}$ and compact subsets $K_k \subseteq M^\circ$ satisfying $\varphi_k(M \setminus K_k) = \{-1, \lambda_1, \ldots, \lambda_k\}$ for some increasing sequence λ_k of positive reals with limit $\inf \sigma_e(A)$, and such that $Q_A(s,s) \geq \int_M \varphi_k |s|^2\, d\mu_g$ for all $s \in \mathrm{dom}(Q_A)$ and $k \in \mathbb{N}$. The root cause is that the construction of $(K_j)_{j\leq k}$, if done as in Lemma 2.2.7, would depend on the distance of λ_k to $\inf \sigma_e(A)$.

Nevertheless, one might still expect such a ψ to exist, at least for some classes of operators. For example, it is shown in [Iwa86, Lemma 2.1] that this holds for magnetic Schrödinger operators on $\mathbb{R}^n$, but the proof is tailored greatly to the concrete circumstances.

We next wish to understand better the case where $\inf \sigma_e(A) > 0$.

Theorem 2.2.12. *Let A be a nonnegative self-adjoint extension of an elliptic differential operator $D\colon \Gamma(M,E) \to \Gamma(M,E)$ of order at least one. Then the following are equivalent:*

(i) *The range of* A *is closed and* $\dim(\ker(A)) < \infty$. *In other words,* A *is a (unbounded) Fredholm operator.*

(ii) $0 \notin \sigma_e(A)$ *or, equivalently,* $\inf \sigma_e(A) > 0$.

(iii) *There exists a Banach space* $(Z, \|\bullet\|_Z)$, *a compact linear operator* $T\colon \operatorname{dom}(Q_A) \to Z$, *and a constant* $C > 0$ *such that*

$$\|s\|^2 \leq C(Q_A(s,s) + \|Ts\|_Z^2)$$

for all $s \in \operatorname{dom}(Q_A)$.

(iv) *There exists a compact subset* $K \subseteq M^\circ$ *and a constant* $C > 0$ *such that*

$$\|s\|^2 \leq C\Big(Q_A(s,s) + \int_K |s|^2 \, d\mu_g\Big) \tag{2.2.9}$$

holds for all $s \in \operatorname{dom}(Q_A)$. *In case* $\ker(A) = 0$, *one can choose* $K = \emptyset$.

(v) *The quadratic form* Q_A *is* coercive at infinity, *meaning that there is a compact subset* $K \subseteq M^\circ$ *and a constant* $C > 0$ *such that*

$$\|s\|^2 \leq C Q_A(s,s)$$

for all $s \in \operatorname{dom}(Q_A)$ *with* $\operatorname{supp}(s) \subseteq M \setminus K$.

Proof. The equivalence of (i) to (iii) is supplied by Corollary 2.2.2. If (ii) holds, then we choose $\lambda \in (0, \inf \sigma_e(A))$. By Lemma 2.2.7, there exists a compact subset $K \subseteq M^\circ$ such that $Q_A(s,s) \geq \lambda \int_{M\setminus K} |s|^2 \, d\mu_g - \int_K |s|^2 \, d\mu_g$ for all $s \in \operatorname{dom}(Q_A)$. It follows that

$$\lambda \|s\|^2 \leq Q_A(s,s) + (\lambda + 1) \int_K |s|^2 \, d\mu_g$$

for $s \in \operatorname{dom}(Q_A)$, hence (2.2.9) holds with $C := \frac{\lambda+1}{\lambda}$. If $\ker(A) = 0$, then the inequality $\|s\|^2 \leq C Q_A(s,s) = C\|A^{1/2}s\|^2$ is equivalent to $\operatorname{img}(A^{1/2})$ being closed. But if $A^{1/2}$ has closed range, then so does A, hence we may choose $K = \emptyset$. Thus, (ii) implies (iv). Conversely, if $K \subseteq M^\circ$ and $C > 0$ are as in (iv), then

$$\frac{1}{C} \int_{M\setminus K} |s|^2 \, d\mu_g - \frac{1-C}{C} \int_K |s|^2 \, d\mu_g \leq Q_A(s,s)$$

for all $s \in \operatorname{dom}(Q_A)$, so $1/C \leq \inf \sigma(A_{M\setminus K})$. By Theorem 2.2.8, $0 < 1/C \leq \inf \sigma_e(A)$.

We are left with proving the equivalence of (v) with the rest of the statements. It is clear that (iv) implies (v). If (v) holds, then we have $\|s\|^2 \leq C Q_{A,M\setminus K}(s,s)$ for all $s \in \operatorname{dom}(Q_{A,M\setminus K})$ since $\{s|_{M\setminus K} : s \in \operatorname{dom}(Q_A) \text{ and } \operatorname{supp}(s) \subseteq M \setminus K\}$ is a core for $Q_{A,M\setminus K}$ by definition. Now (2.2.7) implies that also $\inf \sigma_e(A) \geq 1/C > 0$, hence (ii) is satisfied. ∎

Remark 2.2.13. One can also replace the proof of the implication (iv)⇒(i) in Theorem 2.2.12 by showing that, for every compact $K \subseteq M^\circ$, the restriction map $T\colon \operatorname{dom}(Q_A) \to L^2(K, E)$, $s \mapsto s|_K$, is a compact operator, so that (iv) implies (iii) (which in turn implies (i)).

Since T clearly is continuous, we only need to show that T is compact on the dense subspace $\mathrm{dom}(A) \subseteq \mathrm{dom}(Q_A)$.[5]

$$\begin{array}{ccccc} \mathrm{dom}(A) & \hookrightarrow & H^1_{\mathrm{loc}}(M,E) & \xrightarrow{(\varphi\bullet)|_N} & H^1_0(N,E) \\ \| & & \text{compact} & & \downarrow (\bullet)|_K \\ \mathrm{dom}(A) & \underset{\text{dense}}{\hookrightarrow} & \mathrm{dom}(Q_A) & \xrightarrow[(\bullet)|_K]{} & L^2(K,E) \end{array}$$

Let $(s_k)_{k\geq 1}$ be a bounded sequence in $\mathrm{dom}(A)$. Then $s_k \in H^1_{\mathrm{loc}}(M,E)$ by Remark 1.3.5 and Corollary 1.3.10. Choosing $\varphi \in C^\infty_{cc}(M)$ such that $\varphi|_K = 1$ and a compact manifold $N \subseteq M$ with boundary containing $\mathrm{supp}(\varphi)$ in its interior, we see that $(\varphi s_k)_{k\geq 1}$ is contained in $H^1_0(N,E)$, and also bounded in $\mathrm{dom}(Q_A)$. By Gårding's inequality, see [Tay11b, Theorem 6.1], $(\varphi s_k)_{k\geq 1}$ is bounded in the Sobolev space $H^{1/2}_0(N,E)$, and Theorem 1.3.8 tells us that we can select a subsequence $(\varphi s_{k_j})_{j\geq 1}$ which converges in $L^2(N,E)$, and hence in $L^2(K,E)$. Since $(\varphi s_{k_j})|_K = s_{k_j}|_K$, the claim follows.

The implication (iv)⇒(i) in Theorem 2.2.12 is also shown in [MM07, Theorem 3.1.8] for the Dolbeault Laplacian $\square^E$. They do this by showing that (2.2.9) implies that every L^2-bounded sequence $(s_k)_{k\geq 1}$ in $\mathrm{dom}(Q_A)$ with $Q_A(s_k,s_k) \to 0$ has a convergent subsequence, and that this in turn implies closedness of $\mathrm{img}(A)$ and $\dim(\ker(A)) < \infty$. Their proof essentially contains the above argument that the restriction map $\mathrm{dom}(Q_A) \to L^2(K,E)$ is compact. In [MM07], (2.2.9) is called a *fundamental estimate*.

Remark 2.2.14. Assume that D is essentially self-adjoint on $\Gamma_{cc}(M,E)$, and let A be its closure. Then $\Gamma_{cc}(M,E)$ is also a core for Q_A, since the inclusion $\mathrm{dom}(A) \hookrightarrow \mathrm{dom}(Q_A)$ is dense and Lipschitz, see appendix C.2. Condition (v) of Theorem 2.2.12 then reduces to the inequality

$$\|s\| \leq C\|Ds\|$$

for all $s \in \Gamma_{cc}(M,E)$ with $\mathrm{supp}(u) \subseteq M\backslash K$. Thus, Theorem 2.2.12 includes Anghel's condition on the Fredholmness of a first order essentially self-adjoint differential operator, see [Ang93, Theorem 2.1], as a special case. For conditions on when first order differential operators are essentially self-adjoint, see section 1.4. We would also like to mention [BB12, Theorem 1.18], where a different approach is presented.

Example 2.2.15. Let A and D be as in Theorem 2.2.12. If $A_{M\backslash K} \geq \varepsilon$ for some compact subset $K \subseteq M^\circ$ and $\varepsilon > 0$ then this implies, by use of (2.2.7), that $\inf \sigma_e(A) \geq \varepsilon$, so A has closed range and $\ker(A)$ is finite dimensional. A typical situation where this is true is the following: Assume that $D = D_0 + V$, where D_0 is some nonnegative differential operator and $V\colon E \to E$ is a vector bundle morphism with the property that $\langle Vs, s\rangle_x \geq \varepsilon |s|^2_x$ for all $x \in M \setminus K$ and $s \in E_x$. We say that V is *positive at infinity*. For instance, one could

[5] If X and Y are normed spaces, $X_0 \subseteq X$ a dense subspace, $T\colon X \to Y$ a continuous linear operator such that $T|_{X_0}$ is compact, then T is also compact. This is because the closed unit ball $U := \{x \in X : \|x\|_X \leq 1\}$ is the closure (in X) of $U \cap X_0$, hence $T(U) = T(\overline{U \cap X_0}) \subseteq \overline{T(U \cap X_0)}$, the latter being a compact subset of Y.

consider cases where the bundle morphism in a Weitzenböck type formula, see (1.1.15), has this property. If $\partial M = \emptyset$, *i.e.*, $M = M^\circ$, then $Q_{A,M\setminus K}(s,s) = \langle\!\langle D_0 s + Vs, s\rangle\!\rangle \geq \varepsilon \|s\|^2$ for all $u \in \operatorname{dom}(A) \cap \Gamma(M,E)$, hence $Q_{A,M\setminus K} \geq \varepsilon$ and therefore also $A_{M\setminus K} \geq \varepsilon$, where we have used that $\operatorname{dom}(A) \cap \Gamma(M,E)$ is a form core for A by Corollary 1.3.10. Thus, the conditions of Theorem 2.2.12 are satisfied. Note that in the presence of a boundary, the situation is made more complicated by boundary integrals occurring in the formula for Q_A. Similarly, the spectrum of A will be discrete if $V(x) \to \infty$ as $x \to \infty$, in the sense that for every $\lambda > 0$ there is a compact subset $K \subseteq M$ such that $\inf \sigma(V(x)\colon E_x \to E_x) \geq \lambda$ for all $x \in M \setminus K$. ♦

Corollary 2.2.16. *Let $N \subseteq M$ be a measurable subset with the property that the restriction map $r_N\colon \operatorname{dom}(Q_A) \to L^2(N,E)$ is compact. If*

$$Q_A(s,s) \geq C \int_{M\setminus N} |s|^2 \, d\mu_g \tag{2.2.10}$$

for some $C > 0$ and all $s \in \operatorname{dom}(Q_A)$, then A is Fredholm.

Proof. If $s \in \operatorname{dom}(Q_A)$, then

$$\|s\|^2 = \int_{M\setminus N} |s|^2 \, d\mu_g + \int_N |s|^2 \, d\mu_g \leq \frac{1}{C} Q_A(s,s) + \int_N |s|^2 \, d\mu_g = \frac{1}{C}\big(Q_A(s,s) + \|\sqrt{C} r_N(s)\|^2\big),$$

hence the claim follows from Theorem 2.2.12. ■

Example 2.2.17. Suppose that $\operatorname{dom}(Q_A)$ is contained in $H^1(M,E)$ and the inclusion is continuous. For example, this is the case if A is the form sum (see Example C.2.4) of $\nabla_w^* \nabla_w$ and V, where $V \geq 0$ is a nonnegative self-adjoint bundle morphism, and ∇ is the connection on E used in defining the Sobolev space $H^1(M,E)$. If $Q_A(s,s) \geq C \int_{M\setminus U} |s|^2 \, d\mu_g$ for all $s \in \operatorname{dom}(Q_A)$, a constant $C > 0$, and an open subset $U \subseteq M$ for which the embedding $H^1(U,E) \hookrightarrow L^2(U,E)$ is compact, then A is a Fredholm operator. This is because the restriction operator $\operatorname{dom}(Q_A) \to L^2(U,E)$ factorizes as

$$\operatorname{dom}(Q_A) \hookrightarrow H^1(M,E) \to H^1(U,E) \hookrightarrow L^2(U,E).$$

Of course, relatively compact U satisfy this property, and this gives (2.2.9), see Remark 2.2.13. ♦

CHAPTER 3

The Dolbeault Laplacian and the $\overline{\partial}^E$-Neumann problem

This chapter deals with the Laplacian of the Dolbeault complex, the *Dolbeault Laplacian* $\square^E$, as well as one of its important self-adjoint extensions, which leads to the $\overline{\partial}^E$*-Neumann problem*. In section 3.1, two formulas for $\square^E$ are presented: the Bochner–Weitzenböck formula, which has at its roots the Clifford module structure of $\Lambda^{p,\bullet}T^*M \otimes E$ and will be used in applying elements of the theory of Schrödinger operators that will be presented in section 4.2, as well as the Bochner–Kodaira–Nakano formula. We will only consider these for Kähler manifolds, although both formulas have their generalizations to arbitrary Hermitian manifolds. In section 3.2, we discuss the aforementioned self-adjoint extension of $\square^E$ and establish some of its properties. Among these is the fact that the discreteness of its spectrum "percolates" up the Dolbeault complex under some natural assumptions on E and M. In this analysis, the Bochner–Kodaira–Nakano formula is used, and we will have to make some bounded geometry assumption on M.

3.1. Dolbeault Laplacian and Weitzenböck type formulas

Let M be a Hermitian manifold, with almost complex structure J and compatible Riemannian metric g, and let $E \to M$ be a Hermitian holomorphic vector bundle. On the complex vector bundle $(TM \otimes_{\mathbb{R}} \mathbb{C}, i)$ we have the Hermitian metric $\langle \bullet, \bullet \rangle$, defined as the sesquilinear extension of g. Together with the Hermitian metric on E, this induces Hermitian forms on the bundles $\Lambda^k T^*M \otimes E$, which we all continue to denote by $\langle \bullet, \bullet \rangle$. On functions, we put $\langle f, g \rangle \coloneqq f\overline{g}$, as usual. These also induce a global inner product on $\Omega_c(M, E)$, the smooth differential forms on M with values in E and with compact support, given by

$$\langle\!\langle u, v \rangle\!\rangle \coloneqq \int_M \langle u, v \rangle \, d\mu_g, \tag{3.1.1}$$

for $u, v \in \Omega^k(M, E)$, and requiring that $\langle\!\langle u, v \rangle\!\rangle = 0$ if u and v have different degree. In (3.1.1), μ_g is the measure on M induced by the metric g. Since M is Hermitian, it follows that the decomposition $\Omega_c(M, E) = \bigoplus_{p,q} \Omega_c^{p,q}(M, E)$ is orthogonal for this inner product. We will frequently make use of local orthonormal frames. Usually, $(w_j)_{j=1}^n$ will denote such a frame for $T^{1,0}M$, with its conjugate frame $(\overline{w}_j)_{j=1}^n$ a local orthonormal frame of $T^{0,1}M$. Moreover, we have the dual coframes $(w^j)_{j=1}^n$ and $(\overline{w}^j)_{j=1}^n$ of $(T^{1,0}M)^*$ and $(T^{0,1}M)^*$, respectively. We also refer to appendix B.2.

Associated to the Dolbeault complex (B.3.5) is the second order differential operator

$$\square^E \coloneqq \overline{\partial}^{E,\dagger}\overline{\partial}^E + \overline{\partial}^E\overline{\partial}^{E,\dagger} = (\overline{\partial}^E + \overline{\partial}^{E,\dagger})^2 \colon \Omega^{\bullet,\bullet}(M, E) \to \Omega^{\bullet,\bullet}(M, E),$$

called the *Dolbeault Laplacian* (or simply *complex Laplacian*), where we denote by

$$\bar\partial^{E,\dagger}\colon \Omega^{\bullet,\bullet}(M,E) \to \Omega^{\bullet,\bullet-1}(M,E)$$

the formal adjoint to $\bar\partial^E$ with respect to (3.1.1). We refer to appendix B for the background on complex differential geometry, including the definition of $\bar\partial^E$. The principal symbol of $\square^E$ reads

$$\mathrm{Symb}(\square^E)(\xi)u = -\operatorname{ins}_{(\xi^\sharp)^{0,1}}(\xi^{0,1}\wedge u) - \xi^{0,1}\wedge \operatorname{ins}_{(\xi^\sharp)^{0,1}}(u) = -\langle \xi^{0,1},\xi^{0,1}\rangle\, u = -\frac{1}{2}|\xi|^2\, u, \quad (3.1.2)$$

for all $\xi \in T_x^*M \subseteq T_x^*M \otimes_{\mathbb{R}} \mathbb{C}$ and $u \in \Lambda^{\bullet,\bullet}T_x^*M \otimes E_x$, see (B.3.6) for the principal symbol of $\bar\partial^E$, from which

$$\mathrm{Symb}(\bar\partial^{E,\dagger})(\xi)u = -\operatorname{ins}_{(\xi^\sharp)^{0,1}}(u) \quad (3.1.3)$$

follows, and (B.2.2) for the last equality in (3.1.2). In (3.1.3), ins_Z for $Z \in TM \otimes_{\mathbb{R}} \mathbb{C}$ is the insertion operator from (A.0.1). It follows from the above that $2\square^E$ is an operator of Laplace type, meaning that its principal symbol is $\mathrm{Symb}(2\square^E)(\xi) = -|\xi|^2 \operatorname{id}_{\Lambda T^*M\otimes E}$, see section 1.1.2. Consequently, $\sqrt{2}(\bar\partial^E + \bar\partial^{E,\dagger})$ is a Dirac type operator.

3.1.1. The Bochner–Weitzenböck formula for the Dolbeault Laplacian. From (3.1.2), we know that $\sqrt{2}(\bar\partial^E + \bar\partial^{E,\dagger})$ is a Dirac type operator in the sense of section 1.1.2. On a *Kähler* manifold (see appendix B.2.1), this is an important example of a Dirac operator associated to a Dirac bundle in the sense of Definition 1.1.11. In fact, we have the following result (see for instance [BGV04, Proposition 3.27]):

Proposition 3.1.1. *Let M be a Kähler manifold, $E \to M$ a Hermitian holomorphic vector bundle, and $0 \le p \le n =: \dim_{\mathbb{C}}(M)$. Then*

$$c_p(\xi)u \coloneqq \sqrt{2}(\xi^{0,1}\wedge u - \operatorname{ins}_{(\sharp\xi^{0,1})}(u))$$

defines a Clifford module structure on $\Lambda^{p,\bullet}T^*M \otimes E$ such that $(\Lambda^{p,\bullet}T^*M \otimes E, M, c_p, \widetilde\nabla)$ is a Dirac bundle, where $\widetilde\nabla$ is the connection induced by the Levi–Civita connection on TM (equivalently: the Chern connection, see Theorem B.2.1 and also Example B.3.5) and the Chern connection on E. The Dirac operator associated to this structure is

$$D^E \coloneqq \sqrt{2}(\bar\partial^E + \bar\partial^{E,\dagger}).$$

Proof. Let $u \in \Lambda^{p,\bullet}T^*M \otimes E$ and $\xi \in T^*M$. Then

$$c_p(\xi)^2 u = -2\operatorname{ins}_{(\sharp\xi^{0,1})}(\xi^{0,1}\wedge u) - 2\xi^{0,1}\wedge \operatorname{ins}_{(\sharp\xi^{0,1})}(u) = -|\xi|^2 u,$$

see (3.1.2), so c_p is a Clifford module structure. We have $c_p(\xi)^* = \sqrt{2}(\operatorname{ins}_{(\xi^{0,1})^\sharp}(u) - \xi^{0,1}\wedge u) = -c_p(\xi)$, hence $c_p(\xi)$ is skew-Hermitian. Finally,

$$\begin{aligned}\widetilde\nabla_X(c_p(\alpha)u)) &= \sqrt{2}\widetilde\nabla_X(\alpha^{0,1}\wedge u - \operatorname{ins}_{(\sharp\alpha^{0,1})}(u)) = \\ &= \sqrt{2}((\nabla_X\alpha)^{0,1}\wedge u + \alpha^{0,1}\wedge\nabla_X u - \operatorname{ins}_{(\sharp(\nabla_X\alpha)^{0,1})}(u) - \operatorname{ins}_{(\sharp\alpha^{0,1})}(\nabla_X u)),\end{aligned}$$

for every $\alpha \in \Omega^1(M)$ and $X \in \Gamma(M, TM)$, which equals $c(\nabla_X \alpha)u + c(\alpha)(\nabla_X u)$, as required. Note that the Kähler assumption entered when we used $\nabla_X(\alpha^{0,1}) = (\nabla_X \alpha)^{0,1}$, *i.e.*, the Levi–Civita connection preserves the splitting $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$, see Theorem B.2.1. Thus, $(\Lambda^{p,\bullet}T^*M \otimes E, M, c_p, \widetilde{\nabla})$ is a Dirac bundle.

It remains to compute $D^E := c_p \circ \widetilde{\nabla}$. Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{1,0}M$. Then we have the orthonormal frame $\{e_k\}_{k=1}^{2n}$ of TM, defined by $e_{2j+1} := \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} := \frac{i}{\sqrt{2}}(w_j - \overline{w}_j)$, see (B.2.3). By (B.3.3), we have $\overline{\partial}^E = \overline{w}^j \wedge \widetilde{\nabla}_{\overline{w}_j}$. Using (3.2.7), (A.2.3) and Proposition A.2.2, we compute

$$\overline{\partial}^{E,\dagger} = -\overline{\star}^{E^*}\, \varepsilon(\overline{w}^j)\widetilde{\nabla}_{\overline{w}_j}\overline{\star}^E = -\operatorname{ins}_{\overline{w}_j} \widetilde{\nabla}_{w_j},$$

with $\overline{\star}^E$ and $\overline{\star}^{E^*}$ the Hodge star operators. Therefore,

$$\begin{aligned}
D^E &= \sum_{k=1}^{2n} c_p(e^k)\widetilde{\nabla}_{e_k} \\
&= \sum_{j=1}^{n} \left(c_p(e^{2j-1})\widetilde{\nabla}_{e_{2j-1}} + c_p(e^{2j})\widetilde{\nabla}_{e_{2j}}\right) \\
&= \sqrt{2}\sum_{j=1}^{n} \frac{1}{2}\big((\varepsilon(\overline{w}^j) - \operatorname{ins}_{\overline{w}_j})\widetilde{\nabla}_{w_j+\overline{w}_j} + i(\varepsilon(\overline{w}^j) + \operatorname{ins}_{\overline{w}_j})\widetilde{\nabla}_{i(w_j-\overline{w}_j)}\big) \\
&= \sqrt{2}\sum_{j=1}^{n} \left(\varepsilon(\overline{w}^j)\widetilde{\nabla}_{\overline{w}_j} - \operatorname{ins}_{\overline{w}_j} \widetilde{\nabla}_{w_j}\right) = \sqrt{2}(\overline{\partial}^E + \overline{\partial}^{E,\dagger}),
\end{aligned}$$

where we refer to (B.2.4) for the expressions of the dual basis $\{e^k\}_{k=1}^{2n}$. ■

It follows from Proposition 3.1.1 and Theorem 1.1.14 that, on a Kähler manifold, we have the Weitzenböck type formula $2\Box^E = (D^E)^2 = \Delta^{\Lambda^{p,\bullet}T^*M\otimes E} + c_p(R^{\Lambda^{p,\bullet}T^*M\otimes E})$ on $\Omega^{p,\bullet}(M, E)$. The curvature term is made explicit in the following Theorem. The case $p = 0$ can also be found in [MM07, Theorem 1.4.7], from where the presentation of this formula is motivated. If the Kähler assumption is dropped, then the zeroth order term becomes more complicated and involves the torsion of the Chern connection on TM, see [MM07] for the details.

Theorem 3.1.2. *For a Hermitian holomorphic vector bundle E over a Kähler manifold M, we have*

$$2\Box^E = \Delta^{\Lambda^{p,\bullet}T^*M\otimes E} + \mathcal{K}^E \tag{3.1.4}$$

on $\Omega^{p,\bullet}(M, E)$, where $\Delta^{\Lambda^{p,\bullet}T^*M\otimes E}$ is the Bochner Laplacian (see Example 1.1.2) associated to the connection induced from the Levi–Civita connection on TM and the Chern connection *on E, and the bundle endomorphism $\mathcal{K}^E := c_p(R^{\Lambda^{p,\bullet}T^*M\otimes E})$ of $\Lambda^{p,\bullet}T^*M\otimes E$ is given by*

$$-\sum_{j=1}^n \mathrm{id}_{\Lambda^{p,\bullet}T^*M}\otimes R^E(w_j,\overline{w}_j)+\sum_{j,k=1}^n\Big\{2\,\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}\otimes R^E(w_j,\overline{w}_k)+ \\ +\operatorname{tr}(R^{T^{1,0}M}(w_j,\overline{w}_k))\big(\,\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}+\varepsilon(w^j)\,\mathrm{ins}_{w_k}\big)- \\ -2\langle R^{T^{1,0}M}(w_j,\overline{w}_k)w_\ell,w_m\rangle\,\varepsilon(w^\ell)\,\mathrm{ins}_{w_m}\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}\Big\},\quad (3.1.5)$$

with $\{w_j\}_{j=1}^n$ a local orthonormal frame of $T^{1,0}M$.

While we will not need the exact form (3.1.5) of $\mathcal{K}^E$, we nonetheless supply a proof here. It will be split into several lemmas.

Lemma 3.1.3. *If M and E are as in Theorem 3.1.2, then*

$$\mathcal{K}^E=-\sum_{j=1}^n R^{\Lambda^{p,\bullet}T^*M\otimes E}(w_j,\overline{w}_j)+2\sum_{j,k=1}^n\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}R^{\Lambda^{p,\bullet}T^*M\otimes E}(w_j,\overline{w}_k).\quad (3.1.6)$$

Proof. Given $(w_j)_{j=1}^n$, let $\{e_k\}_{k=1}^{2n}$ be the orthonormal frame of TM from (B.2.3), and abbreviate $R := R^{\Lambda^{p,\bullet}T^*M\otimes E}$ as well as $c := c_p$. From the defining property of Clifford module structures and the fact that R is alternating, we know that

$$c(e^{2j-1})c(e^{2k})R(e_{2j-1},e_{2k})=c(e^{2k})c(e^{2j-1})R(e_{2k},e_{2j-1})$$

for all j and k. Moreover,

$$R(e_{2j},e_{2k})=R(Je_{2j-1},Je_{2k-1})=R(e_{2j-1},e_{2k-1}),$$

because R is a $(1,1)$-form, see Proposition B.3.3 and Remark B.3.6. By (1.1.17), and using the above symmetries, we have

$$\mathcal{K}^E=\frac12\sum_{j,k=1}^n\Big\{\big(c(e^{2j-1})c(e^{2k-1})+c(e^{2j})c(e^{2k})\big)R(e_{2j-1},e_{2k-1})+2c(e^{2j-1})c(e^{2k})R(e_{2j-1},e_{2k})\Big\}.$$

Note that

$$c(e^{2j-1})=\varepsilon(\overline{w}^j)-\mathrm{ins}_{\overline{w}_j}\quad\text{and}\quad c(e^{2j})=i(\varepsilon(\overline{w}^j)+\mathrm{ins}_{\overline{w}_j}).\quad (3.1.7)$$

Short calculations show that (with implied summation over j and k)

$$\begin{aligned}\big(c(e^{2j-1})c(e^{2k-1})+c(e^{2j})c(e^{2k})\big)R(e_{2j-1},e_{2k-1})&=\\ &=-\big(\,\varepsilon(\overline{w}^j)\,\mathrm{ins}_{\overline{w}_k}+\mathrm{ins}_{\overline{w}_j}\varepsilon(\overline{w}^k)\big)\big(R(w_j,\overline{w}_k)-R(w_k,\overline{w}_j)\big)\\ &=-\big(\,\varepsilon(\overline{w}^j)\,\mathrm{ins}_{\overline{w}_k}+\delta_{jk}-\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}\big)\big(R(w_j,\overline{w}_k)-R(w_k,\overline{w}_j)\big)\\ &=-2\,\varepsilon(\overline{w}^j)\,\mathrm{ins}_{\overline{w}_k}R(w_j,\overline{w}_k)+2\,\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}R(w_j,\overline{w}_k)\end{aligned}$$

and

$$\begin{aligned}
&c(e^{2j-1})c(e^{2k})R(e_{2j-1}, e_{2k}) = \\
&\qquad = i\big(\,\varepsilon(\overline{w}^j)\,\varepsilon(\overline{w}^k) + \varepsilon(\overline{w}^j)\,\mathrm{ins}_{\overline{w}_k} - \mathrm{ins}_{\overline{w}_j}\,\varepsilon(\overline{w}^k) - \mathrm{ins}_{\overline{w}_j}\,\mathrm{ins}_{\overline{w}_k}\,\big)\circ \\
&\qquad\quad \circ\big(-\tfrac{i}{2}R(w_j,\overline{w}_k) - \tfrac{i}{2}R(w_k,\overline{w}_j)\big) \\
&\qquad = \frac{1}{2}\big(\,\varepsilon(\overline{w}^j)\,\mathrm{ins}_{\overline{w}_k} - \delta_{jk} + \varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}\,\big)\big(R(w_j,\overline{w}_k) + R(w_k,\overline{w}_j)\big) \\
&\qquad = -\sum_j R(w_j,\overline{w}_j) + \sum_{j,k}\big(\,\varepsilon(\overline{w}^j)\,\mathrm{ins}_{\overline{w}_k} R(w_j,\overline{w}_k) + \varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j} R(w_j,\overline{w}_k)\big),
\end{aligned}$$

where the terms with $\varepsilon(\overline{w}^j)\,\varepsilon(\overline{w}^k)$ and $\mathrm{ins}_{\overline{w}_j}\,\mathrm{ins}_{\overline{w}_k}$ disappear because $R(w_j,\overline{w}_k) + R(w_k,\overline{w}_j)$ is symmetric in (j,k). Putting these together, we arrive at

$$\mathcal{K}^E = -\sum_j R(w_j,\overline{w}_j) + 2\sum_{j,k}\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j} R(w_j,\overline{w}_k), \tag{3.1.8}$$

which is what we wanted to prove. ∎

Remark 3.1.4. A different way to establish (3.1.6) is to work with the orthonormal basis $\{w_j, \overline{w}_j\}_{j=1}^n$ of $TM \otimes_{\mathbb{R}} \mathbb{C}$ directly. To this end, one has to first complexify the Clifford action, *i.e.*,

$$\widetilde{c}_p(\xi)u := \sqrt{2}\big(\xi^{0,1}\wedge u - \mathrm{ins}_{(\sharp\bar{\xi})^{0,1}}(u)\big)$$

for $\xi \in T^*M \otimes \mathbb{C}$. (Note the complex conjugation in the second term, which makes $\widetilde{c}_p$ complex linear.) The Clifford relations are then

$$\widetilde{c}_p(\xi)\widetilde{c}_p(\eta) + \widetilde{c}_p(\eta)\widetilde{c}_p(\xi) = -2\langle\xi,\overline{\eta}\rangle,$$

and applying (1.1.17) gives (3.1.6) after some computations.

Lemma 3.1.5. *Let M be a Kähler manifold. Then*

$$-R^{\Lambda^{0,\bullet}T^*M}(w_j,\overline{w}_j) + 2\,\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j} R^{\Lambda^{0,\bullet}T^*M}(w_j,\overline{w}_k) = \mathrm{tr}(R^{T^{1,0}M}(w_j,\overline{w}_k))\,\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j} \tag{3.1.9}$$

on $\Lambda^{0,\bullet}T^*M$.

Proof. We abbreviate $R := R^{\Lambda^{0,\bullet}T^*M}$ to avoid unnecessary clutter. We claim that it is enough to show that (3.1.9) holds on $\Lambda^{0,1}T^*M$. To see this, define an endomorphism of $\Lambda^{0,\bullet}T^*M$ by

$$\mathcal{K}_1 := 2\,\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j} R(w_j,\overline{w}_k),$$

which is just the second term in (3.1.9). We first show that $\mathcal{K}_1$ acts as a derivation, in the sense that

$$\mathcal{K}_1(\alpha\wedge\beta) = \mathcal{K}_1\alpha\wedge\beta + \alpha\wedge\mathcal{K}_1\beta \tag{3.1.10}$$

for all $\alpha, \beta \in \Lambda^{0,\bullet}T^*M$. To show (3.1.10), note that $R(w_j, \overline{w}_k)$ and $\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}$ satisfy this derivation rule (see Example A.1.8), hence

$$\begin{aligned}\mathcal{K}_1(\alpha\wedge\beta) - \mathcal{K}_1\alpha\wedge\beta - \alpha\wedge\mathcal{K}_1\beta &= \\ &= 2(\overline{w}^k\wedge \mathrm{ins}_{\overline{w}_j}(\alpha))\wedge R(w_j,\overline{w}_k)\beta + 2R(w_j,\overline{w}_k)\alpha\wedge(\overline{w}^k\wedge\mathrm{ins}_{\overline{w}_j}(\beta)).\end{aligned}\tag{3.1.11}$$

If $\alpha = \alpha_1\wedge\cdots\wedge\alpha_q$ and $\beta = \beta_1\wedge\cdots\wedge\beta_{q'}$ are the wedge products of one-forms, then the right hand side of (3.1.11) is

$$\begin{aligned}\sum_{r=1}^{q}\sum_{s=1}^{q'}(-1)^{r+s+q}2\big((\overline{w}^k\wedge\mathrm{ins}_{\overline{w}_j}(\alpha_r))\wedge R(w_j,\overline{w}_k)\beta_s + R(w_j,\overline{w}_k)\alpha_r\wedge(\overline{w}^k\,\mathrm{ins}_{\overline{w}_j}(\beta_s))\big)\wedge \\ \wedge\alpha_1\wedge\cdots\wedge\widehat{\alpha_r}\wedge\cdots\wedge\alpha_q\wedge\beta_1\wedge\cdots\wedge\widehat{\beta_r}\wedge\cdots\wedge\beta_{q'}.\end{aligned}$$

Therefore, to establish (3.1.10), it suffices to show that the right hand side of (3.1.11) vanishes for $\alpha,\beta\in\Lambda^{0,1}T^*M$. Plugging in $\alpha = \overline{w}^a$ and $\beta = \overline{w}^b$, the right-hand side of this equation is

$$2\overline{w}^k\wedge R(w_a,\overline{w}_k)\overline{w}^b + 2R(w_b,\overline{w}_k)\overline{w}^a\wedge\overline{w}^k = 2\overline{w}^k\wedge\big(R(w_a,\overline{w}_k)\overline{w}^b - R(w_b,\overline{w}_k)\overline{w}^a\big).$$

Now, by (A.1.13) and Remark B.3.6,

$$R(w_a,\overline{w}_k)\overline{w}^b = (R^{T^{0,1}M}(\overline{w}_a,w_k)\overline{w}_b)^\flat = ((R^{TM}(\overline{w}_a,w_k)\overline{w}_b)^{0,1})^\flat,$$

and using the first Bianchi identity (A.1.14) for the Riemann curvature tensor as well as the fact that R^{TM} is a $(1,1)$ form, we find

$$\begin{aligned}R(w_a,\overline{w}_k)\overline{w}^b = -((R^{TM}(w_k,\overline{w}_b)\overline{w}_a)^{0,1})^\flat &= \\ = -(R^{T^{0,1}M}(w_k,\overline{w}_b)\overline{w}_a)^\flat &= -R(\overline{w}_k,w_b)\overline{w}^a = R(w_b,\overline{w}_k)\overline{w}^a.\end{aligned}$$

This shows (3.1.10), and since $-R(w_j,\overline{w}_j)$ and the right-hand side of (3.1.9) clearly also have this derivation property, it suffices to show (3.1.9) only on $\Lambda^{0,1}T^*M$.

Now the left-hand side of (3.1.9), evaluated at $\overline{w}^a$ and expanding $R(w_j,\overline{w}_k)\overline{w}^a$ in the orthonormal basis $\{\overline{w}^\ell\}_{\ell=1}^n$ as $\sum_\ell\langle R(w_j,\overline{w}_k)\overline{w}^a,\overline{w}^\ell\rangle\overline{w}^\ell$, equals

$$\begin{aligned}-\sum_{j,\ell}\langle R(w_j,\overline{w}_j)\overline{w}^a,\overline{w}^\ell\rangle\overline{w}^\ell + 2\sum_{j,k,\ell}\langle R(w_j,\overline{w}_k)\overline{w}^a,\overline{w}^\ell\rangle\,\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}(\overline{w}^\ell) &= \\ = \sum_{j,k}\Big(-\langle R(w_j,\overline{w}_j)\overline{w}^a,\overline{w}^k\rangle\overline{w}^k + 2\langle R(w_j,\overline{w}_k)\overline{w}^a,\overline{w}^j\rangle\overline{w}^k\Big).\end{aligned}\tag{3.1.12}$$

By (A.1.11) and (A.1.14),

$$\begin{aligned}\langle R(w_j,\overline{w}_k)\overline{w}^a,\overline{w}^j\rangle = -\langle R(w_j,\overline{w}_k)\overline{w}_j,\overline{w}_a\rangle = \langle R(\overline{w}_j,w_j)\overline{w}_k,\overline{w}_a\rangle &= \\ = -\langle R(\overline{w}_j,w_j)\overline{w}^a,\overline{w}^k\rangle &= \langle R(w_j,\overline{w}_j)\overline{w}^a,\overline{w}^k\rangle,\end{aligned}$$

so that (3.1.12) is equal to

$$\begin{aligned}
&\sum_{j,k}\langle R(w_j,\overline{w}_j)\overline{w}^a,\overline{w}^k\rangle\overline{w}^k \\
&\qquad = -\sum_{j,k}\langle R(w_j,\overline{w}_j)\overline{w}_k,\overline{w}_a\rangle\overline{w}^k && \text{by (A.1.11)}\\
&\qquad = -\sum_{j,k}\langle R(\overline{w}_k,w_a)w_j,w_j\rangle\overline{w}^k && \text{by pair symmetry, (A.1.15)}\\
&\qquad = \sum_{j,k}\langle R(w_a,\overline{w}_k)w_j,w_j\rangle\overline{w}^k && \text{since } R \text{ is alternating}\\
&\qquad = \sum_{k}\operatorname{tr}(R^{T^{1,0}M}(w_a,\overline{w}_k))\,\overline{w}^k\\
&\qquad = \sum_{j,k}\operatorname{tr}(R^{T^{1,0}M}(w_j,\overline{w}_k))\,\overline{w}^k\wedge\operatorname{ins}_{\overline{w}_j}(\overline{w}^a),
\end{aligned}$$

as claimed. ■

Lemma 3.1.6. *Let M be a Kähler manifold. Then*

$$-\sum_{j=1}^{n} R^{\Lambda^{\bullet,0}T^*M}(w_j,\overline{w}_j) = \sum_{j,k=1}^{n}\operatorname{tr}(R^{T^{1,0}M}(w_j,\overline{w}_k))\,\varepsilon(w^j)\operatorname{ins}_{w_k} \tag{3.1.13}$$

on $\Lambda^{\bullet,0}T^*M$.

Proof. Put $R \coloneqq R^{\Lambda^{\bullet,0}T^*M}$. Since both sides of (3.1.13) satisfy a derivation rule similar to (3.1.7), we only have to show the claim on $\Lambda^{1,0}T^*M$, and there we have

$$\begin{aligned}
-\sum_{m=1}^{n} R(w_m,\overline{w}_m)w^a &= -\sum_{j,m=1}^{n}\langle R(w_m,\overline{w}_m)w^a,w^j\rangle w^j\\
&= \sum_{j,m=1}^{n}\langle R(w_m,\overline{w}_m)w_j,w_a\rangle w^j\\
&= \sum_{m=1}^{n}\langle R(w_j,\overline{w}_a)w_m,w_m\rangle w^j\\
&= \sum_{j,k=1}^{n}\operatorname{tr}(R(w_j,\overline{w}_k))w^j\wedge\operatorname{ins}_{w_k}(w^a)
\end{aligned}$$

similarly as the computation at the end of the proof of Lemma 3.1.5. ■

Proof of Theorem 3.1.2. By Lemma 3.1.3 and Example A.1.7,

$$\begin{aligned}
\mathcal{K}^E = &-\sum_{j=1}^{n}\operatorname{id}_{\Lambda^{\bullet,\bullet}T^*M}\otimes R^E(w_j,\overline{w}_j) + 2\sum_{j,k=1}^{n}\varepsilon(\overline{w}^k)\operatorname{ins}_{\overline{w}_j}\otimes R^E(w_j,\overline{w}_k) -\\
&-\sum_{j=1}^{n} R^{\Lambda^{\bullet,\bullet}T^*M}(w_j,\overline{w}_j)\otimes\operatorname{id}_E + 2\sum_{j,k=1}^{n}\left(R^{\Lambda^{\bullet,\bullet}T^*M}(w_j,\overline{w}_k)\,\varepsilon(\overline{w}^k)\operatorname{ins}_{\overline{w}_j}\right)\otimes\operatorname{id}_E\,.
\end{aligned}$$

If $\alpha \in \Lambda^{\bullet,0}T^*M$ and $\beta \in \Lambda^{0,\bullet}T^*M$, then

$$\begin{aligned} &- R(w_j,\overline{w}_j)(\alpha\wedge\beta) + 2R(w_j,\overline{w}_k)\,\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}(\alpha\wedge\beta) = \\ &\qquad = -R(w_j,\overline{w}_j)(\alpha)\wedge\beta + 2R(w_j,\overline{w}_k)(\alpha)\wedge\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}(\beta) + \\ &\qquad\qquad + \alpha\wedge\big(-R(w_j,\overline{w}_j)\beta + 2R(w_j,\overline{w}_k)\,\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}(\beta)\big). \end{aligned}$$

By Lemma 3.1.5, the last term equals $\mathrm{tr}(R^{T^{1,0}M}(w_j,\overline{w}_k))\,\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}(\alpha\wedge\beta)$, and the first term in the second line is $-R(w_j,\overline{w}_j)(\alpha)\wedge\beta = -\,\mathrm{tr}(R^{T^{1,0}M}(w_j,\overline{w}_k))\,\varepsilon(w^j)\,\mathrm{ins}_{w_k}(\alpha\wedge\beta)$, by Lemma 3.1.6. It remains to prove that

$$2R(w_j,\overline{w}_k)(\alpha)\wedge\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}(\beta) = -2\langle R^{T^{1,0}M}(w_j,\overline{w}_k)w_\ell,w_m\rangle\,\varepsilon(w^\ell)\,\mathrm{ins}_{w_m}\,\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j}(\alpha\wedge\beta),$$

and for this it suffices to show

$$R(w_j,\overline{w}_k)\alpha = -\langle R^{T^{1,0}M}(w_j,\overline{w}_k)w_\ell,w_m\rangle\,\varepsilon(w^\ell)\,\mathrm{ins}_{w_m}(\alpha).$$

As in Lemma 3.1.6, it is enough to show this for $\alpha \in \Lambda^{1,0}T^*M$, and this is revealed to be true by a calculation very similar to the one in the proof of Lemma 3.1.6,

$$R(w_j,\overline{w}_k)w^a = -\sum_{\ell=1}^{n}\langle R(w_j,\overline{w}_k)w_\ell,w_a\rangle w^\ell = -\sum_{\ell,m=1}^{n}\langle R(w_j,\overline{w}_k)w_\ell,w_m\rangle w^\ell\wedge\mathrm{ins}_{w_m}(w^a).$$

This completes the proof. ■

3.1.2. The Bochner–Kodaira–Nakano formula. We can, more generally, ask for useful formulas expressing $\Box^E$ as the sum of some other second order operator of Laplace type (other than a Bochner Laplacian) and a vector bundle morphism. An important example of this is the Bochner–Kodaira–Nakano formula:

Theorem 3.1.7 (Bochner–Kodaira–Nakano formula)**.** *For a Hermitian holomorphic vector bundle E over a Kähler manifold (M,ω), we have*

$$\Box^E = (d^E_{1,0}d^{E,\dagger}_{1,0} + d^{E,\dagger}_{1,0}d^E_{1,0}) + [iR^E\wedge_{\mathrm{ev}},\Lambda], \tag{3.1.14}$$

where $\Lambda\colon \Lambda^{\bullet,\bullet}T^*M\otimes E \to \Lambda^{\bullet-1,\bullet-1}T^*M\otimes E$ is the adjoint to $u\mapsto\omega\wedge u$, the wedge product $\wedge_{\mathrm{ev}}$ is combined with the evaluation map (see appendix A.1.1), and $[\bullet,\bullet]$ is the commutator of endomorphisms.

In (3.1.14), d^E denotes the exterior covariant derivative associated to the Chern connection on E, with $(1,0)$-part $d^E_{1,0}$ (and $(0,1)$-part $\overline{\partial}^E$) and $d^{E,\dagger}_{1,0}$ is its formal adjoint, see appendix B.3 for the details. The proof, see [MM07, Theorem 1.4.11] or [Ohs15, Theorem 2.7], is usually done by applying the so-called *Kähler identities*, which are formulas for the commutators between the operators $\overline{\partial}^E$, $d^E_{1,0}$, $\varepsilon(\omega)$, and their adjoints. For instance, $[\overline{\partial}^E,\Lambda] = id^{E,\dagger}_{1,0}$. In case E is the trivial Hermitian line bundle, one furthermore shows that $2\Box \coloneqq 2(\overline{\partial}\overline{\partial}^\dagger + \overline{\partial}^\dagger\overline{\partial})$ is equal to the *Hodge Laplacian* $dd^\dagger + d^\dagger d$, see for instance [Bal06, Corollary 5.26]. Formula (3.1.14) has an extension to Hermitian manifolds that are not Kähler, with additional torsion terms occurring. This is due to Demailly [Dem86], and a proof can also be found in [MM07, Theorem 1.4.12].

The goal of this section is to obtain and understand an integrated form of (3.1.14) in the case where M is an open subset of a larger manifold M', with smooth boundary $\partial M \subseteq M'$. We emphasize again that, unlike in chapter 2, M does *not* contain its boundary. The closure $\overline{M}$ of M inside M' is then a smooth manifold with boundary.

Definition 3.1.8. Suppose that $U \subseteq \overline{M}$ is a (relatively) open subset. We define

$$B_M(U,E) := \{u \in \Omega_c(U,E) : \mathrm{ins}_{(\nu^{0,1})}(u)|_{\partial M \cap U} = 0\}, \tag{3.1.15}$$

where ν is a unit normal vector field to ∂M, and we write $\nu^{0,1} = \frac{1}{2}(\nu + iJ\nu)$ for its component in $T^{0,1}M$. We denote by $B_M^{p,q}(U,E)$ the forms of bidegree (p,q) in $B_M(U,E)$.

Remark 3.1.9. The spaces $B_M(U,E)$ are closed under the multiplication with elements of $C^\infty(U)$, and if $u \in B_M(U,E)$, then also $\mathrm{ins}_X(u) \in B_M(U,E)$ for every vector field $X \in \Gamma(U, TM \otimes \mathbb{C})$, since insertion operators anticommute, *i.e.*, $\mathrm{ins}_X \circ \mathrm{ins}_{(\nu^{0,1})} = -\mathrm{ins}_{(\nu^{0,1})} \circ \mathrm{ins}_X$.

The Levi form of a hypersurface. Let S be a (real) hypersurface of M', *i.e.*, a submanifold of codimension one. For $x \in S$, put $H_xS := T_xS \cap J(T_xS)$. It is the part of T_xS that is invariant under the ambient complex structure J of M', and it is referred to as the *complex tangent space of S at x*. Since S has codimension one, it turns out that $x \mapsto \dim(H_xS)$ is necessarily constant, and $HS := \bigcup_{x\in S} H_xS$ is a (real) vector subbundle of TS, with rank $2n-2$, where n is the complex dimension of M'. The eigenbundle of the restriction of J to $HS \otimes_{\mathbb{R}} \mathbb{C}$ associated to the eigenvalue $+i$ is

$$H^{1,0}S = (T^{1,0}M')|_S \cap (TS \otimes_{\mathbb{R}} \mathbb{C}). \tag{3.1.16}$$

Note that the complex rank of $HS \otimes_{\mathbb{R}} \mathbb{C}$ is $2n-2$, hence that of $H^{1,0}S$ is $n-1$. We can now define the extrinsic Levi form as in [Bog91, p. 160]:

Definition 3.1.10. The *(extrinsic) Levi form* of a hypersurface $S \subseteq M'$ is defined as

$$\mathscr{L}_S \colon \Gamma(S, H^{1,0}S) \times \Gamma(S, H^{1,0}S) \to \Gamma(S, NS \otimes_{\mathbb{R}} \mathbb{C}), \quad (X,Y) \mapsto -\frac{1}{2i}\pi_{NS\otimes_{\mathbb{R}}\mathbb{C}}(J[X,\overline{Y}]),$$

with $NS \to S$ the normal bundle, and $\pi_{NS\otimes_{\mathbb{R}}\mathbb{C}} \colon (TM' \otimes_{\mathbb{R}} \mathbb{C})|_S \to NS \otimes_{\mathbb{R}} \mathbb{C}$ the projection.

Note that $\mathscr{L}_S$ is actually tensorial, *i.e.*, sesquilinear over $C^\infty(S,\mathbb{C})$. For instance, we have

$$\mathscr{L}_S(fX,Y) = f\mathscr{L}_S(X,Y) + \overline{Y}(f)\frac{1}{2i}\pi_{NS\otimes_{\mathbb{R}}\mathbb{C}}(JX) = f\mathscr{L}_S(X,Y)$$

because $H^{1,0}S \perp NS \otimes_{\mathbb{R}} \mathbb{C}$ in $(TM' \otimes_{\mathbb{R}} \mathbb{C})|_S$. Therefore, $\mathscr{L}_S(X,Y)(x)$ only depends on the values of X and Y at $x \in S$, and we may view it as a vector-valued quadratic form

$$\mathscr{L}_S \colon H^{1,0}S \times_S H^{1,0}S \to NS \otimes_{\mathbb{R}} \mathbb{C}.$$

Suppose that S is orientable. If ν is a unit normal vector field (*i.e.*, a unit norm section of NS), then $\mathscr{L}_S(X,Y) = -\frac{1}{2i}\langle J[X,\overline{Y}], \nu\rangle\nu$. After making the choice of an orientation of S (*i.e.*, picking a particular unit normal vector field ν), the Levi form evidently contains the same information as the quadratic form

$$(X,Y)\mapsto\langle\mathscr{L}_S(X,Y),\nu\rangle=-\frac{1}{2i}\langle J[X,\overline{Y}],\nu\rangle \tag{3.1.17}$$

on $H^{1,0}S$. The Levi form may also be defined intrinsically and generalizes to (and is important in the study of) *abstract CR manifolds*, which are pairs $(M,\mathbb{L})$ with M a smooth manifold and $\mathbb{L}$ an involutive (its local sections are closed under the Lie bracket) subbundle of $TM\otimes_{\mathbb{R}}\mathbb{C}$ such that $\mathbb{L}_x\cap\overline{\mathbb{L}_x}=0$ for all $x\in M$. For a (real) hypersurface S in a complex manifold, the pair $(S,H^{1,0}S)$ is a CR manifold. We refer to the literature for details on the definition and more properties, *e.g.*, [Bog91].

Suppose now again that M is an open subset of M' with smooth boundary ∂M. Then ∂M is orientable with the *inward* pointing unit vector field to ∂M given by $\nu=-(d\rho)^\sharp$, where $\sharp\colon T^*M\to TM$ is the musical isomorphism and $\rho\in C^\infty(M',\mathbb{R})$ is a *defining function* for M, *i.e.*, $M=\rho^{-1}((-\infty,0))$ and $|d\rho|=1$ on $\partial M=\rho^{-1}(\{0\})$. It follows from (3.1.16) that, in holomorphic coordinates $(z_1,\dots,z_n)$ around $x\in\partial M$, a vector $W=\sum_{j=1}^n w_j\frac{\partial}{\partial z_j}\big|_x\in T_x^{1,0}M'$ belongs to $H_x^{1,0}(\partial M)$ if and only if

$$\sum_{j=1}^n\frac{\partial\rho}{\partial z_j}(x)w_j=\partial\rho(W)=d\rho(W)=0, \tag{3.1.18}$$

see also [Bog91, Lemma 2, p. 100]. In terms of the defining function, the Levi form of ∂M reads

$$\mathscr{L}_{\partial M}(X,Y)=(\partial\bar\partial\rho(X,\overline{Y}))\nu=-(\partial\bar\partial\rho(X,\overline{Y}))(d\rho)^\sharp, \tag{3.1.19}$$

see [Bog91, section 10.3], since

$$\begin{aligned}
\langle J[X,\overline{Y}],\nu\rangle &= -d\rho(J[X,\overline{Y}]) && \text{since } \nu=-(d\rho)^\sharp\\
&= -(Jd\rho)([X,\overline{Y}])\\
&= -i\partial\rho([X,\overline{Y}])+i\bar\partial\rho([X,\overline{Y}]) && \text{since } d=\partial+\bar\partial\\
&= -i\big(-d\partial\rho(X,\overline{Y})+X(\partial\rho(\overline{Y}))-\overline{Y}(\partial\rho(X))\big)\\
&\qquad +i\big(-d\bar\partial\rho(X,\overline{Y})+X(\bar\partial\rho(\overline{Y}))-\overline{Y}(\bar\partial\rho(X))\big) && \text{by (A.1.5)}\\
&= -2i\partial\bar\partial\rho(X,\overline{Y})+i\overline{Y}\partial\rho(X)+iX(\bar\partial\rho(\overline{Y}))\\
&= -2i\partial\bar\partial\rho(X,\overline{Y}),
\end{aligned}$$

where in the last step we have used (3.1.16) and that $\partial\rho$ and $\bar\partial\rho$ annihilate $T(\partial M)\otimes_{\mathbb{R}}\mathbb{C}$. We can extend the Levi form to act on (p,q)-forms in the following way:

Definition 3.1.11. For $u,v\in B_M(\overline{M},E)$, we define $\mathscr{L}(u,v)\colon\partial M\to\mathbb{C}$ by

$$\mathscr{L}(u,v):=\sum_{j,k=1}^{n-1}\langle\mathscr{L}_{\partial M}(\xi_k,\xi_j),\nu\rangle\langle\bar\xi^j\wedge\operatorname{ins}_{\bar\xi_k}(u),v\rangle, \tag{3.1.20}$$

where ν is the *inward* pointing unit normal vector field to ∂M and $\{\xi_j\}_{j=1}^{n-1}$ is a local orthonormal frame of $H^{1,0}(\partial M)$.

Note that if $\{\xi_j\}_{j=1}^{n-1}$ is as in Definition 3.1.11, then $\{\xi_j(x)\}_{j=1}^{n-1} \cup \{\nu^{1,0}(x)\}$ is an orthonormal basis of $T_x^{1,0}M'$, see (3.1.16). Moreover, $\mathrm{ins}_{\overline{\nu^{1,0}}}(u) = \mathrm{ins}_{\nu^{0,1}}(u) = 0$ for $u \in B_M(\overline{M}, E)$ by definition, hence, using (3.1.19),

$$\mathscr{L}(u,v) = \sum_{j,k=1}^{n} \partial\overline{\partial}\rho(w_k, \overline{w}_j)\langle \overline{w}^j \wedge \mathrm{ins}_{\overline{w}_k}(u), v\rangle \tag{3.1.21}$$

holds for every local orthonormal frame $\{w_j\}_{j=1}^{n}$ of $(T^{1,0}M')|_{\partial M}$ (see [MM07, Definition 1.4.20]). By construction, this is independent of the particular defining function used. If $\{w_j\}_{j=1}^{n}$ is an orthonormal frame of $T^{1,0}M'|_{\partial M}$ over $V \subseteq \partial M$ such that $\partial\overline{\partial}\rho(w_k, \overline{w}_j) = s_j\delta_{jk}$ for some $s_j \colon V \to \mathbb{R}$, and if $u(x) = (w^{j_1} \wedge \cdots \wedge w^{j_p} \wedge \overline{w}^{k_1} \wedge \cdots \wedge \overline{w}^{k_q})(x)$ at $x \in V$, then

$$\mathscr{L}(u,u)(x) = \big(s_{k_1}(x) + \cdots + s_{k_q}(x)\big)|u(x)|^2,$$

as is easily seen from (3.1.21).

Definition 3.1.12. An open subset $M \subseteq M'$ of a Hermitian manifold with smooth boundary $\partial M \subseteq M'$ is called *Levi pseudoconvex at* $x \in \partial M$ if $\langle \mathscr{L}_{\partial M}(X,X), \nu\rangle \geq 0$ for all $X \in H_x^{1,0}(\partial M)$, where $\mathscr{L}_{\partial M}$ is the Levi form of ∂M, see Definition 3.1.10, and ν is the *inward* pointing unit normal vector field to ∂M. If M is Levi pseudoconvex at every $x \in \partial M$, then M is called *Levi pseudoconvex.*

In other words, M is Levi pseudoconvex if and only if the quadratic form $\langle \mathscr{L}_{\partial M}(\bullet,\bullet), \nu\rangle$ on $H^{1,0}(\partial M)$ from (3.1.17) is positive semidefinite for the choice of orientation provided by the inward pointing unit normal. If $\rho\colon M' \to \mathbb{R}$ is a defining function for M, then looking at (3.1.19) we see that this is the same as having an everywhere nonnegative lower bound of $(X,Y) \mapsto \partial\overline{\partial}\rho(X,\overline{Y})$ on $H^{1,0}(\partial M)$. Using the coordinate description (3.1.18) of $H^{1,0}(\partial M)$, this is the case if and only if

$$\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \overline{z}_k}(x) w_j \overline{w}_k \geq 0, \quad (w_1,\ldots,w_n) \in \mathbb{C}^n \text{ with } \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(x) w_j = 0$$

for all $x \in \partial M$ and arbitrary holomorphic coordinates $(z_1,\ldots,z_n)$ of M' around x. Often, this is the way Levi pseudoconvexity is introduced in the first place, see for instance [Str10, p. 22] or [CS01, Definition 3.4.1].

Remark 3.1.13. There are also other notions of pseudoconvexity for complex manifolds M. One of them is the existence of a smooth function $\psi\colon M \to \mathbb{R}$ such that $i\partial\overline{\partial}\psi$ is a Kähler metric (*i.e.*, positive definite) and with the property that $\{x \in M : \psi(x) \leq c\}$ is compact for all $c \in \mathbb{R}$. In other words, ψ is a strictly plurisubharmonic exhaustion function. It was shown by Grauert in [Gra58] that this is equivalent to M being a *Stein manifold.* This means that M is holomorphically convex, and local holomorphic coordinates can be obtained as restrictions of global holomorphic maps from M to $\mathbb{C}^n$, with n the complex dimension of M. The detailed definition and more about Stein manifolds can be found in the literature, for example in [Hör90]. The above notion of pseudoconvexity agrees with Levi pseudoconvexity in case M is a bounded domain in $\mathbb{C}^n$ with smooth boundary (see textbooks on several complex variables, for instance [Kra01]), but fails for smoothly bounded domains in general manifolds. We refer to the survey articles [Nar78; Sib17] for more on this subject.

Equivalent to Levi pseudoconvexity is the condition $\mathscr{L}(\alpha,\alpha) \geq 0$ for all $\alpha \in B^{0,1}_M(\overline{M},\mathbb{C})$, as can be seen from (3.1.20). This makes it easy to generalize this notion:

Definition 3.1.14. An open subset $M \subseteq M'$ of a Hermitian manifold with smooth boundary $\partial M \subseteq M'$ is called *q-Levi pseudoconvex* if $\mathscr{L}(\alpha,\alpha) \geq 0$ holds for all $\alpha \in B^{0,q}_M(\overline{M},\mathbb{C})$.

Remark 3.1.15. Note that, according to this definition, every smoothly bounded open subset $M \subseteq M'$ is n-Levi pseudoconvex, with n the complex dimension of M'. Indeed, every $\alpha \in B^{0,n}_M(\overline{M},\mathbb{C})$ must vanish on ∂M, for if $\{w_1,\dots,w_{n-1},\sqrt{2}\nu^{1,0}\}$ is an orthonormal frame of $T^{1,0}M'$ and $\alpha = f\,\overline{w}_1 \wedge \cdots \wedge \overline{w}_{n-1} \wedge \nu^{0,1}$ for some smooth function f, then $\operatorname{ins}_{\nu^{0,1}}(\alpha)|_{\partial M} = 0$ means $f|_{\partial M} = 0$, hence $\alpha|_{\partial M} = 0$.

Strong (q-) Levi pseudoconvexity is defined in a similar manner, by requiring the corresponding strict inequalities to hold. As in Remark 3.2.16, if $\mathscr{L}(\alpha,\alpha) \geq 0$ for $\alpha \in B^{0,1}_M(\overline{M},\mathbb{C})$, then this inequality continues to hold for $\alpha \in B^{p,q}_M(\overline{M},\mathbb{C})$, with $q \geq 1$. Following the same reasoning, if M is q-Levi pseudoconvex, then it is also q'-Levi pseudoconvex for every $q' \geq q$. It is easy to see that M is q-Levi pseudoconvex for $q \leq n-1$ if and only if the sum of the first q eigenvalues of the Hermitian form (3.1.17) (with respect to $\langle \bullet, \bullet \rangle$) are nonnegative.

The global Bochner–Kodaira–Nakano formula. We will now derive an integrated form of the Bochner–Kodaira–Nakano formula (3.1.14), where the occurring boundary integral will feature the quadratic form $\mathscr{L}$ from Definition 3.1.11. In the presented generality, it is available in [MM07, Theorem 1.4.21], from where our proof is mostly taken. We mention that Ma and Marinescu consider not only the Kähler case, but general Hermitian manifolds, and the integrated formula on these manifolds will again feature torsion terms. The formula is simplest on (n,q) forms, so we will discuss this case first. Put

$$Q^E(u,v) \coloneqq \langle\!\langle \bar\partial^E u, \bar\partial^E v \rangle\!\rangle + \langle\!\langle \bar\partial^{E,\dagger} u, \bar\partial^{E,\dagger} v \rangle\!\rangle$$

for $u, v \in B_M(\overline{M},E)$. Later, we will extend Q^E to become the quadratic form associated to a self-adjoint extension of $\square^E$.

Theorem 3.1.16 (Global Bochner–Kodaira–Nakano formula)**.** *Let E be a Hermitian holomorphic vector bundle over a Kähler manifold M', and let $M \subseteq M'$ be an open subset with smooth boundary $\partial M \subseteq M'$ in case $M \neq M'$. Then*

$$Q^E(u,u) = \|d^{E,\dagger}_{1,0} u\|^2 + \langle\!\langle iR^E \wedge_{\mathrm{ev}} \Lambda u, u \rangle\!\rangle + \int_{\partial M} \mathscr{L}(u,u)\, d\mu_{\partial M} \tag{3.1.22}$$

holds for all $u \in B^{n,\bullet}_M(\overline{M},E)$.

Remark 3.1.17. For smoothly bounded open subsets of $\mathbb{C}^n$, equation (3.1.22) is also referred to as the *Morrey–Kohn–Hörmander formula*, see [Str10, Proposition 2.4] or [CS01, Proposition 4.3.1]. Original works include [Hör65; Mor58], but see [Str10] for extensive references.

In order to show (3.1.22), we need a Lemma:

Lemma 3.1.18. *Let $E \to M$ be a complex vector bundle over a smooth manifold, with connection ∇^E and induced exterior covariant derivative d^E. If $\{e_j\}_{j=1}^m$ is a local frame of $TM \otimes_{\mathbb{R}} \mathbb{C}$, orthonormal with respect to a Hermitian metric $\langle \bullet, \bullet \rangle$ on $TM \otimes_{\mathbb{R}} \mathbb{C}$, and $X \in \Gamma(M, TM \otimes_{\mathbb{R}} \mathbb{C})$ is a complex vector field on M, then*

$$d^E \circ \operatorname{ins}_X + \operatorname{ins}_X \circ d^E = \langle \nabla^{TM}_{e_j} X, e_k \rangle \, \varepsilon(e^j) \circ \operatorname{ins}_{e_k} + \widetilde{\nabla}_X, \tag{3.1.23}$$

where ∇^{TM} is any torsion free connection on TM, extended complex linearly to $TM \otimes_{\mathbb{R}} \mathbb{C}$, and $\widetilde{\nabla}$ is the induced connection on $\Lambda T^*M \otimes E$.

If M is oriented Riemannian, ∇^{TM} is the Levi–Civita connection, $\langle \bullet, \bullet \rangle$ is the induced Hermitian metric on $TM \otimes_{\mathbb{R}} \mathbb{C}$, the bundle E is Hermitian with compatible connection ∇^E, then also

$$d^{E,\dagger} \circ \varepsilon(\alpha) + \varepsilon(\alpha) \circ d^{E,\dagger} = -(e_j(\alpha(e_k)) - \alpha(\nabla^{TM}_{e_j} e_k)) \operatorname{ins}_{e_j} \circ \varepsilon(e^k) - \widetilde{\nabla}_{\overline{\alpha^\sharp}}. \tag{3.1.24}$$

for every one-form $\alpha \in \Omega^1(M, \mathbb{C})$.

Remark 3.1.19. If $\alpha \in \Omega(M)$ and $s \in \Gamma(M, E)$, then a simple computation using the definition of d^E and *Cartan's formula*

$$d \circ \operatorname{ins}_X + \operatorname{ins}_X \circ d = \mathcal{L}_X \quad \text{on } \Omega(M),$$

with $\mathcal{L}_X$ the Lie derivative, shows that (3.1.23) applied to $\alpha \otimes s$, with $\alpha \in \Omega(M)$ and $s \in \Gamma(M, E)$, is equal to

$$(d^E \circ \operatorname{ins}_X + \operatorname{ins}_X \circ d^E)(\alpha \otimes s) = \mathcal{L}_X(\alpha) \otimes s + \alpha \otimes \nabla^E_X s.$$

We derive (3.1.23) in order to easily compute its adjoint formula (3.1.24).

Proof of Lemma 3.1.18. We have $d^E = \varepsilon \circ \widetilde{\nabla} = \varepsilon(e^j) \circ \widetilde{\nabla}_{e_j}$, see (A.1.6). Therefore,

$$d^E \circ \operatorname{ins}_X + \operatorname{ins}_X \circ d^E = \varepsilon(e^j) \circ (\widetilde{\nabla}_{e_j} \circ \operatorname{ins}_X - \operatorname{ins}_X \circ \widetilde{\nabla}_{e_j}) + \operatorname{ins}_X(e^j) \widetilde{\nabla}_{e_j}.$$

Now

$$\widetilde{\nabla}_{e_j} \circ \operatorname{ins}_X - \operatorname{ins}_X \circ \widetilde{\nabla}_{e_j} = ins(\nabla^{TM}_{e_j} X) = \langle \nabla^{TM}_{e_j} X, e_k \rangle \operatorname{ins}_{e_k}$$

and

$$\operatorname{ins}_X(e^j) \widetilde{\nabla}_{e_j} = \langle X, e_j \rangle \widetilde{\nabla}_{e_j} = \widetilde{\nabla}_{\langle X, e_j \rangle e_j} = \widetilde{\nabla}_X.$$

This shows (3.1.23). For M as in (3.1.24) we have available the Hodge star operator, see appendix A.2. Propositions A.2.1 and A.2.3 imply that, on $\Omega^l(M, E^*)$,

$$\overline{\star}^{E^*} \circ d^{E^*} \circ \operatorname{ins}_{\alpha^\sharp} + \overline{\star}^{E^*} \circ \operatorname{ins}_{\alpha^\sharp} \circ d^{E^*} = \\ = (-1)^l d^{E,\dagger} \circ \overline{\star}^{E^*} \circ \operatorname{ins}_{\alpha^\sharp} + (-1)^l \varepsilon(\alpha) \circ \overline{\star}^{E^*} \circ d^{E^*} = \\ = -d^{E,\dagger} \circ \varepsilon(\alpha) \circ \overline{\star}^{E^*} - \varepsilon(\alpha) \circ d^{E,\dagger} \circ \overline{\star}^{E^*}.$$

By (3.1.23) and Propositions A.2.1 and A.2.2, this equals,

$$-\langle \nabla^{TM}_{e_j} \alpha^\sharp, e_k \rangle\, \varepsilon(e^j) \circ \operatorname{ins}_{e_k} \circ \overline{\star}^{E^*} - \widetilde{\nabla}_{\alpha^\sharp} \circ \overline{\star}^{E^*} = -\overline{\star}^{E^*} \overline{\langle \nabla^{TM}_{e_j} \alpha^\sharp, e_k \rangle}\, \operatorname{ins}_{e_j} \circ \varepsilon(e^k) - \overline{\star}^{E^*} \circ \widetilde{\nabla}_{\overline{\alpha^\sharp}}.$$

Finally, $\overline{\langle \nabla^{TM}_{e_j} \alpha^\sharp, e_k \rangle} = \langle e_k, \nabla^{TM}_{e_j} \alpha^\sharp \rangle = e_j(\langle e_k, \alpha^\sharp \rangle) - \langle \nabla^{TM}_{e_j} e_k, \alpha^\sharp \rangle = e_j(\alpha(e_k)) - \alpha(\nabla^{TM}_{e_j} e_k)$. ∎

Proof of Theorem 3.1.16. We proceed as in the proof of [MM07, Theorem 1.4.21]. Note that for $u, v \in \Omega_c(\overline{M}, E)$, we have

$$\langle\!\langle \overline{\partial}^{E,\dagger} u, v \rangle\!\rangle = \langle\!\langle u, \overline{\partial}^E v \rangle\!\rangle - \int_{\partial M} \langle \operatorname{Symb}(\overline{\partial}^{E,\dagger})(\nu^\flat) u, v \rangle \, d\mu_{\partial M} = \\ = \langle\!\langle u, \overline{\partial}^E v \rangle\!\rangle + \int_{\partial M} \langle \operatorname{ins}_{(\nu^{0,1})}(u), v \rangle \, d\mu_{\partial M} \quad (3.1.25)$$

by Green's formula (1.1.14) and because the principal symbol of $\overline{\partial}^{E,\dagger}$ is $\xi \mapsto -\operatorname{ins}_{(\xi^\sharp)^{0,1}}$, see (3.1.3) Therefore, $\|\overline{\partial}^{E,\dagger} u\|^2 = \langle\!\langle \overline{\partial}^E \overline{\partial}^{E,\dagger} u, u \rangle\!\rangle$ if $u \in B^{n,q}_M(M, E)$. For $u \in B^{n,q}_M(\overline{M}, E)$, Theorem 3.1.7 and (1.1.14) imply

$$\begin{aligned} Q^E(u,u) &= \langle\!\langle \overline{\partial}^E u, \overline{\partial}^E u \rangle\!\rangle + \langle\!\langle \overline{\partial}^{E,\dagger} u, \overline{\partial}^{E,\dagger} u \rangle\!\rangle \\ &= \langle\!\langle \square^E u, u \rangle\!\rangle + \int_{\partial M} \langle \operatorname{Symb}(\overline{\partial}^{E,\dagger})(\nu^\flat)(\overline{\partial}^E u), u \rangle \, d\mu_{\partial M} \\ &= \langle\!\langle d^E_{1,0} d^{E,\dagger}_{1,0} u, u \rangle\!\rangle + \langle\!\langle i R^E \wedge_{\mathrm{ev}} \Lambda u, u \rangle\!\rangle + \int_{\partial M} \langle \operatorname{Symb}(\overline{\partial}^{E,\dagger})(\nu^\flat)(\overline{\partial}^E u), u \rangle \, d\mu_{\partial M} \\ &= \|d^{E,\dagger}_{1,0} u\|^2 + \langle\!\langle i R^E \wedge_{\mathrm{ev}} \Lambda u, u \rangle\!\rangle + \\ &\qquad + \int_{\partial M} \big(\langle \operatorname{Symb}(\overline{\partial}^{E,\dagger})(\nu^\flat)(\overline{\partial}^E u), u \rangle - \langle \operatorname{Symb}(d^E_{1,0})(\nu^\flat)(d^{E,\dagger}_{1,0} u), u \rangle \big) \, d\mu_{\partial M}. \end{aligned} \quad (3.1.26)$$

It remains to compute the boundary integral. We have $\operatorname{Symb}(d^E_{1,0})(\nu^\flat) u = (\nu^\flat)^{1,0} \wedge u = -\partial\rho \wedge u$, hence the integrand in (3.1.26) is

$$\big\langle (-\operatorname{ins}_{\nu^{0,1}} \circ \overline{\partial}^E + \varepsilon(\partial\rho) \circ d^{E,\dagger}_{1,0}) u, u \big\rangle.$$

Denote by $\Pi_{p,q} \colon \Lambda^{\bullet,\bullet} T^*M \otimes E \to \Lambda^{p,q} T^*M \otimes E$ the projections. Since $\overline{(\partial\rho)^\sharp} = -\overline{\nu^{1,0}} = -\nu^{0,1}$, Lemma 3.1.18 implies that, for every local orthonormal frame $\{e_j\}_{j=1}^{2n}$ of $TM \otimes_{\mathbb{R}} \mathbb{C}$,

$$(-\operatorname{ins}_{\nu^{0,1}} \circ \overline{\partial}^E + \varepsilon(\partial\rho) \circ d^{E,\dagger}_{1,0}) u$$

$$
\begin{aligned}
&= (-\operatorname{ins}_{\nu^{0,1}} \circ \Pi_{n,q+1} \circ d^E + \varepsilon(\partial\rho) \circ \Pi_{n-1,q} \circ d^{E,\dagger})u \\
&= -\Pi_{n,q}(\operatorname{ins}_{\nu^{0,1}} \circ d^E - \varepsilon(\partial\rho) \circ d^{E,\dagger})u \\
&= -\Pi_{n,q}\Big(- d^E \circ \operatorname{ins}_{\nu^{0,1}} + d^{E,\dagger} \circ \varepsilon(\partial\rho) + \widetilde{\nabla}_{\nu^{0,1}} + \widetilde{\nabla}_{-\nu^{0,1}} \\
&\qquad + \langle \nabla_{e_j} \nu^{0,1}, e_k\rangle\, \varepsilon(e^j) \circ \operatorname{ins}_{e_k} + \big(e_j(\partial\rho(e_k)) - \partial\rho(\nabla_{e_j} e_k)\big) \operatorname{ins}_{e_j} \circ \varepsilon(e^k)\Big)u \\
&= -\Pi_{n,q}\Big(\langle \nabla_{e_j} \nu^{0,1}, e_k\rangle\, \varepsilon(e^j) \circ \operatorname{ins}_{e_k} + \big(e_j(\partial\rho(e_k)) - \partial\rho(\nabla_{e_j} e_k)\big) \operatorname{ins}_{e_j} \circ \varepsilon(e^k)\Big)u,
\end{aligned}
$$

where in the last line we have used $\operatorname{ins}_{\nu^{0,1}}(u) = 0$ and $\partial\rho \wedge u = 0$ since $u \in B^{n,q}_M(\overline{M}, E)$. Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{1,0}M$. Then $\{e_k\}_{k=1}^{2n} = \{w_j, \overline{w}_j\}_{j=1}^n$ is a local orthonormal frame of $TM \otimes_{\mathbb{R}} \mathbb{C}$, and

$$
\begin{aligned}
\Pi_{n,q}\langle \nabla_{e_j} \nu^{0,1}, e_k\rangle\, \varepsilon(e^j) \operatorname{ins}_{e_k}(u) &= \Pi_{n,q}\langle \nabla_{e_j} \nu^{0,1}, \overline{w}_k\rangle\, \varepsilon(e^j) \operatorname{ins}_{\overline{w}_k}(u) \\
&= \langle \nabla_{\overline{w}_j} \nu^{0,1}, \overline{w}_k\rangle\, \varepsilon(\overline{w}^j) \operatorname{ins}_{\overline{w}_k}(u)
\end{aligned}
$$

as well as

$$
\big(e_j(\partial\rho(e_k)) - \partial\rho(\nabla_{e_j} e_k)\big) \operatorname{ins}_{e_j} \varepsilon(e^k)u = \big(e_j(\partial\rho(w_k)) - \partial\rho(\nabla_{e_j} w_k)\big) \operatorname{ins}_{e_j} \varepsilon(w^k)u = 0
$$

since $\partial\rho$ is a $(1,0)$-form and $\varepsilon(w^k)u = 0$. Finally,

$$
\begin{aligned}
&- \langle \nabla_{\overline{w}_j} \nu^{0,1}, \overline{w}_k\rangle \\
&\quad = -\langle \overline{w}^k, (\nabla_{\overline{w}_j} \nu^{0,1})^\flat\rangle && \text{since } \langle X, Y\rangle = \langle Y^\flat, X^\flat\rangle \\
&\quad = \langle \overline{w}^k, \nabla_{w_j} \overline{\partial}\rho\rangle && \text{since } \nabla_Z X^\flat = (\nabla_{\overline{Z}} X)^\flat \\
&\quad = \overline{(\nabla_{w_j} \overline{\partial}\rho)(\overline{w}_k)} && \text{since } \langle \alpha, X^\flat\rangle = \langle X, \alpha^\sharp\rangle = \alpha(X) \\
&\quad = (\nabla_{\overline{w}_j} \partial\rho)(w_k) && \text{since all operations are } \mathbb{C}\text{-linear} \\
&\quad = \overline{w}_j(\partial\rho(w_k)) + \partial\rho(\nabla_{\overline{w}_j} w_k) && \text{by the definition of } \nabla_{\overline{w}_j}\partial\rho \\
&\quad = \overline{w}_j(\partial\rho(w_k)) + \partial\rho([\overline{w}_j, w_k] + \nabla_{w_k} \overline{w}_j) && \text{since } \nabla \text{ is torsion free} \\
&\quad = \overline{w}_j(\partial\rho(w_k)) + w_k(\partial\rho(\overline{w}_j)) + \partial\rho([\overline{w}_j, w_k]) && \text{since } \partial\rho(\overline{w}_j) = \partial\rho(\nabla_{w_k}\overline{w}_j) = 0 \\
&\quad = (d\partial\rho)(\overline{w}_j, w_k) && \text{by (A.1.5)} \\
&\quad = (\overline{\partial}\partial\rho)(\overline{w}_j, w_k) = (\partial\overline{\partial}\rho)(w_k, \overline{w}_j), && \text{since } d = \partial + \overline{\partial} \text{ and } \overline{\partial}\partial = -\partial\overline{\partial}
\end{aligned}
$$

which completes the proof. ■

Remark 3.1.20. On $\Lambda^{n,\bullet} T^*M \otimes E$, the operator $iR^E \wedge_{\mathrm{ev}} \Lambda$ which occurs in (3.1.22) has the form

$$
iR^E \wedge_{\mathrm{ev}} \Lambda u = R^E(w_j, \overline{w}_k)\, \varepsilon(\overline{w}^k) \operatorname{ins}_{\overline{w}_j}(u), \tag{3.1.27}
$$

with $\{w_j\}_{j=1}^n$ a local orthonormal frame of $T^{1,0}M$. To see this, note first that the Kähler form ω has the local expression

$$\begin{aligned}\omega &= \frac{1}{2}\sum_{l,m=1}^{2n} \omega(e_l, e_m)\, e^l \wedge e^m \\ &= \frac{1}{2}\sum_{j=1}^{n}\sum_{m=1}^{2n} \Big(g(Je_{2j}, e_m)\, e^{2j} \wedge e^m + g(Je_{2j-1}, e_m)\, e^{2j-1} \wedge e^m \Big) \\ &= \frac{1}{2}\sum_{j=1}^{n}\sum_{m=1}^{2n} \Big(-g(e_{2j-1}, e_m)\, e^{2j} \wedge e^m + g(e_{2j}, e_m)\, e^{2j-1} \wedge e^m \Big) \\ &= \sum_{j=1}^{n} e^{2j-1} \wedge e^{2j} \\ &= i\sum_{j=1}^{n} w^j \wedge \overline{w}^j,\end{aligned}$$

with $\{e_l\}_{l=1}^{2n}$ as in (B.2.3), and where we have used that $Je_{2j} = J^2 e_{2j-1} = -e_{2j-1}$. Consequently,

$$\Lambda = -i\sum_{j=1}^{n} \mathrm{ins}_{\overline{w}_j}\, \mathrm{ins}_{w_j} = i\sum_{j=1}^{n} \mathrm{ins}_{w_j}\, \mathrm{ins}_{\overline{w}_j}\,. \tag{3.1.28}$$

Similarly, one checks that $R^E = R^E(w_j, \overline{w}_k)\, w^j \wedge \overline{w}^k$. Now

$$iR^E \wedge_{\mathrm{ev}} \Lambda = R^E(w_m, \overline{w}_k)\, \varepsilon(w^m)\, \varepsilon(\overline{w}^k)\, \mathrm{ins}_{\overline{w}_j}\, \mathrm{ins}_{w_j} = R^E(w_j, \overline{w}_k)\, \varepsilon(\overline{w}^k)\, \mathrm{ins}_{\overline{w}_j}$$

after (anti-)commuting the exterior products and the insertion operators, and using that $\varepsilon(w^m)\, \mathrm{ins}_{w_j} = \delta_{mj} - \mathrm{ins}_{w_j}\, \varepsilon(w^m)$.

Example 3.1.21. Let $L \to M'$ be a Hermitian holomorphic line bundle over a complete Kähler manifold, and let $M \subseteq M'$ be a q-Levi pseudoconvex open subset with smooth boundary. The curvature R^L may be identified with a real $(1,1)$-form on M, see Example B.3.8, and if $x \in M$, then by Remark B.3.4 (applied to a unit norm element of L_x) one can find an orthonormal basis $\{w_j\}_{j=1}^n$ of $T_x^{1,0}M$ such that $R^L(w_j, \overline{w}_k) = s_j(x)\, \delta_{jk}\, \mathrm{id}_L$ for some numbers $s_j(x) \in \mathbb{R}$, which we order such that $s_1 \le \cdots \le s_n$. Consequently, if $u = w^1 \wedge \cdots \wedge w^n \wedge \overline{w}^{j_1} \wedge \cdots \wedge \overline{w}^{j_q}$, then

$$iR^L \wedge_{\mathrm{ev}} \Lambda u = (s_{j_1}(x) + \cdots + s_{j_q}(x))u$$

by (3.1.27), hence

$$\langle iR^L \wedge_{\mathrm{ev}} \Lambda u, u\rangle \ge (s_1(x) + \cdots + s_q(x))|u|^2$$

for all $u \in \Lambda^{n,q} T_x^* M \otimes L_x$. This and (3.1.22) implies

$$Q^L(u,u) = \|d_{1,0}^{L,\dagger} u\|^2 + \int_M \langle iR^L \wedge_{\mathrm{ev}} \Lambda u, u\rangle\, d\mu_g + \int_{\partial M} \mathscr{L}(u,u)\, d\mu_{\partial M} \tag{3.1.29}$$

$$\ge \int_M (s_1 + \cdots + s_q)|u|^2\, d\mu_g \tag{3.1.30}$$

for all $u \in B_M^{n,q}(\overline{M}, L)$. ♦

General bidegrees. The global Bochner–Kodaira–Nakano formula (3.1.22) has an extension to (p,q) forms for $0 \le p \le n$, with a term involving the curvature of $T^{1,0}M$ occurring. Consider the morphism of complex vector bundles

$$\Phi\colon (\Lambda^{n,\bullet}T^*M \otimes E) \otimes \Lambda^{n-p,0}TM \to \Lambda^{p,\bullet}T^*M \otimes E, \quad \Phi(v \otimes \xi) := (-1)^{(n-p)(n-p-1)/2}\operatorname{ins}_\xi(v)$$

for $v \in \Lambda^{n,\bullet}T^*M \otimes E$ and $\xi \in \Lambda^{n-p,0}TM$, and where the insertion operator is extended to $\Lambda TM \otimes_{\mathbb{R}} \mathbb{C}$ via $\operatorname{ins}_{(\xi_1\wedge\cdots\wedge\xi_k)} := \operatorname{ins}_{\xi_1} \circ \cdots \circ \operatorname{ins}_{\xi_k}$. Let $\{w_j\}_{j=1}^n$ be an orthonormal basis of $T_x^{1,0}M$, with dual basis $\{w^j\}_{j=1}^n$ of $(T_x^{1,0}M)^*$. Then it is easy to see that

$$u = \sum_{|J|=n-p}{}' (-1)^{(n-p)(n-p-1)/2}\operatorname{ins}_{w_J}(w^J \wedge u) = \sum_{|J|=n-p}{}' \Phi((w^J \wedge u) \otimes w_J)$$

for all $u \in \Lambda^{p,\bullet}T_x^*M \otimes E_x$, where as usual the primed sum means that the summation is done over all increasing maps $J\colon \{1,\dots,n-p\} \to \{1,\dots,n\}$, *i.e.*, all subsets of $\{1,\dots,n\}$ of cardinality $n-p$, and $w^J := w^{J(1)} \wedge \cdots \wedge w^{J(n-p)}$, with analogous definition for w_J. Thus, Φ is bijective (its domain and codomain are vector bundles with the same rank), and its inverse is given by

$$\Psi_p^E(u) := \Phi^{-1}(u) = \sum_{|J|=n-p}{}' (w^J \wedge u) \otimes w_J \in \Lambda^{n,\bullet}T_x^*M \otimes (E_x \otimes \Lambda^{n-p,0}T_xM). \tag{3.1.31}$$

From this, it is immediate that Ψ_p^E is an isometry.

Suppose that $\xi \in \Gamma(U, \Lambda^{n-p,0}TM)$, $\varpi \in \Gamma(U, \Lambda^{n,0}T^*M) = \Omega^{n,0}(U)$, and $s \in \Gamma(U,E)$ are holomorphic on an open subset $U \subseteq M$. Then, if $\alpha \in \Omega^{0,\bullet}(U)$, we have

$$\begin{aligned}
(\Psi_p^E)^{-1}(\overline{\partial}^{E\otimes\Lambda^{n-p,0}TM}((\varpi\wedge\alpha)\otimes s\otimes\xi)) &= (\Psi_p^E)^{-1}((-1)^n(\varpi\wedge\overline{\partial}\alpha)\otimes s\otimes\xi)\\
&= \sigma(-1)^n\operatorname{ins}_\xi((\varpi\wedge\overline{\partial}\alpha)\otimes s)\\
&= \sigma(-1)^n\operatorname{ins}_\xi(\varpi)\wedge\overline{\partial}\alpha\otimes s\\
&= \sigma(-1)^{n-p}\,\overline{\partial}(\operatorname{ins}_\xi(\varpi\wedge\alpha))\otimes s\\
&= (-1)^{n-p}\,\overline{\partial}^E((\Psi_p^E)^{-1}((\varpi\wedge\alpha)\otimes s\otimes\xi)),
\end{aligned}$$

where $\sigma := (-1)^{(n-p)(n-p-1)/2}$, and where we have used that $\overline{\partial}^{\Lambda^nT^*M}\varpi = 0$ if and only if $\overline{\partial}\varpi = 0$ as a $(n,1)$-form on U. Since any $u \in \Omega^{n,\bullet}(M, E\otimes\Lambda^{n-p,0}TM)$ can locally be written as a linear combination of sections of the form $(\varpi\wedge\alpha)\otimes s\otimes\xi$, it follows that $\overline{\partial}^{E\otimes\Lambda^{n-p,0}TM}\circ\Psi_p^E = (-1)^{n-p}\Psi_p^E\circ\overline{\partial}^E$, and because Ψ_p^E is an isometry, it also intertwines the formal adjoints of $\overline{\partial}^E$ and $\overline{\partial}^{E\otimes\Lambda^{n-p,0}TM}$. It is also clear that Ψ_p^E maps $B_M^{p,\bullet}(\overline{M},E)$ to $B_M^{n,\bullet}(\overline{M}, E\otimes\Lambda^{n-p,0}TM)$. Therefore,

$$\begin{aligned}
Q^E(u,u) &= \|(\overline{\partial}^E + \overline{\partial}^{E,\dagger})u\|^2 = \|\Psi_p^E(\overline{\partial}^E + \overline{\partial}^{E,\dagger})u\|^2 =\\
&= \|(\overline{\partial}^{E\otimes\Lambda^{n-p,0}TM} + \overline{\partial}^{E\otimes\Lambda^{n-p,0}TM,\dagger})\Psi_p^E u\|^2 = Q^{E\otimes\Lambda^{n-p,0}TM}(\Psi_p^E u, \Psi_p^E u)
\end{aligned} \tag{3.1.32}$$

for all $u \in B_M^{p,\bullet}(\overline{M},E)$.

Corollary 3.1.22. *Let $M \subseteq M'$ be an open subset of a Kähler manifold, with smooth boundary ∂M in case $M \neq M'$, and let $E \to M'$ be a Hermitian holomorphic vector bundle. For any open subset $U \subseteq \overline{M}$, and any $u \in B^{p,\bullet}_M(U, E)$, we have*

$$Q^E_U(u,u) = \|d^{E\otimes\Lambda^{n-p,0}TM,\dagger}_{1,0}\widetilde{u}\|^2 + \langle\!\langle iR^{E\otimes\Lambda^{n-p,0}TM} \wedge_{\mathrm{ev}} \Lambda\widetilde{u}, \widetilde{u}\rangle\!\rangle + \int_{\partial M\cap U} \mathscr{L}(u,u)\, d\mu_{\partial M} \quad (3.1.33)$$

with $\widetilde{u} := \Psi^E_p(u) \in B^{n,\bullet}_M(U, E \otimes \Lambda^{n-p,0}TM)$.

Proof. If we keep denoting by $u \in \Omega^{p,\bullet}_c(\overline{M}, E)$ its extension by zero outside $\operatorname{supp}(u) \subseteq U$, then it is clear that $u \in B^{p,\bullet}_M(\overline{M}, E)$. By Lemma 2.1.4, (3.1.32), and Theorem 3.1.16, we have

$$Q^E_U(u,u) = \|d^{E\otimes\Lambda^{n-p,0}TM,\dagger}_{1,0}\widetilde{u}\|^2 + \langle\!\langle iR^{E\otimes\Lambda^{n-p,0}TM} \wedge_{\mathrm{ev}} \Lambda\widetilde{u}, \widetilde{u}\rangle\!\rangle + \int_{\partial M} \mathscr{L}(\widetilde{u},\widetilde{u})\, d\mu_{\partial M}.$$

From (3.1.31), it is immediate that $\operatorname{ins}_{\overline{w}_k}(\Psi^E_p(u)) = (-1)^{n-p}\Psi^E_p(\operatorname{ins}_{\overline{w}_k}(u))$, and (3.1.21) implies $\mathscr{L}(\widetilde{u},\widetilde{u}) = \mathscr{L}(u,u)$. Therefore, the last term above equals $\int_{\partial M\cap U} \mathscr{L}(u,u)\, d\mu_{\partial M}$, since u has compact support in U. ■

To avoid cluttering of (3.1.33), we have moved the computation of the curvature term to its own Proposition below. As in the case of (3.1.4), we will not make use of its precise form, but shall provide a proof anyways.

Proposition 3.1.23. *Let $(w_j)^n_{j=1}$ be a local orthonormal frame of $T^{1,0}M$. The curvature term in* (3.1.33) *equals*

$$\langle\!\langle iR^{E\otimes\Lambda^{n-p,0}TM} \wedge_{\mathrm{ev}} \Lambda\widetilde{u}, \widetilde{u}\rangle\!\rangle = \langle\!\langle \{R^E(w_j,\overline{w}_k) + \operatorname{tr}(R^{T^{1,0}M}(w_j,\overline{w}_k)) - \\ - \langle R^{T^{1,0}M}(w_j,\overline{w}_k)w_\ell, w_m\rangle\, \varepsilon(w^\ell) \circ \operatorname{ins}_{w_m}\}\, \varepsilon(\overline{w}^k) \circ \operatorname{ins}_{\overline{w}_j}(u), u\rangle\!\rangle, \quad (3.1.34)$$

with implicit summation over j, k, ℓ, and m.

Remark 3.1.24. For $p = n$, we recover (3.1.27), while for $p = 0$, we end up with

$$\langle\!\langle (\varepsilon(\overline{w}^k) \circ \operatorname{ins}_{\overline{w}_j}) \otimes \big(R^E(w_j,\overline{w}_k) + \operatorname{tr}(R^{T^{1,0}M}(w_j,\overline{w}_k))\operatorname{id}_E\big)u, u\rangle\!\rangle = \\ = \langle\!\langle (\varepsilon(\overline{w}^k) \circ \operatorname{ins}_{\overline{w}_j}) \otimes R^{E\otimes K^*_M}(w_j,\overline{w}_k)u, u\rangle\!\rangle$$

where $K_M := \Lambda^n(T^{1,0}M)^* = \det((T^{1,0}M)^*)$ is the *canonical line bundle* over M, whose dual $K^*_M = \Lambda^n T^{1,0}M$ has curvature

$$R^{K^*_M}(X,Y) = R^{\det(T^{1,0}M)}(X,Y) = \operatorname{tr}(R^{T^{1,0}M}(X,Y)) \in \Omega^2(M, \operatorname{End}(K^*_M)) \cong \Omega^2(M,\mathbb{C}),$$

see Example A.1.8. This is exactly the curvature term in the global Bochner–Kodaira–Nakano formula as presented in [MM07, Theorem 1.4.21].

Proof of Proposition 3.1.23. By Remark 3.1.20, we have

$$iR \wedge_{\mathrm{ev}} \Lambda = \varepsilon(\overline{w}^k)\operatorname{ins}_{\overline{w}_j} \otimes R(w_j,\overline{w}_k) \quad (3.1.35)$$

on $\Lambda^{n,q}T^*M \otimes (E \otimes \Lambda^{n-p,0}TM)$, where we have abbreviated $R := R^{E\otimes\Lambda^{n-p,0}TM}$, and Example A.1.7 shows that

$$R(w_j, \overline{w}_k) = R^E(w_j, \overline{w}_k) \otimes \mathrm{id}_{\Lambda^{n-p,0}TM} + \mathrm{id}_E \otimes R^{\Lambda^{n-p,0}TM}(w_j, \overline{w}_k).$$

Clearly,

$$(\Psi_p^E)^{-1}\big(\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j} \otimes R^E(w_j, \overline{w}_k) \otimes \mathrm{id}_{\Lambda^{n-p,0}TM}\big)\Psi_p^E(u) = \big(\varepsilon(\overline{w}^k)\,\mathrm{ins}_{\overline{w}_j} \otimes R^E(w_j, \overline{w}_k)\big)u,$$

which gives the curvature term involving E in (3.1.34). Note that, by Example A.1.8,

$$\begin{aligned}
&R^{\Lambda^{n-p,0}TM}(w_j, \overline{w}_k)(w_{J(1)} \wedge \cdots \wedge w_{J(n-p)}) \\
&\quad= \sum_{r=1}^{n-p} w_{J(1)} \wedge \cdots \wedge w_{J(r-1)} \wedge \big(R^{T^{1,0}M}(w_j, \overline{w}_k) w_{J(r)}\big) \wedge w_{J(r+1)} \wedge \cdots \wedge w_{J(n-p)} \\
&\quad= \sum_{r=1}^{n-p} \sum_{m=1}^{n} \langle R^{T^{1,0}M}(w_j, \overline{w}_k) w_{J(r)}, w_m\rangle \times \\
&\qquad\qquad \times w_{J(1)} \wedge \cdots \wedge w_{J(r-1)} \wedge w_m \wedge w_{J(r+1)} \wedge \cdots \wedge w_{J(n-p)} \\
&\quad= \sum_{r=1}^{n-p} \sum_{m=1}^{n} (-1)^{r-1} \langle R^{T^{1,0}M}(w_j, \overline{w}_k) w_{J(r)}, w_m\rangle\, w_m \wedge w_{J_r}
\end{aligned}$$

with $J_r\colon \{1, \dots, n-p-1\} \to \{1, \dots, n\}$ the increasing map defined by omitting $J(r)$, *i.e.*, $J_r(i) := J(i)$ for $i < r$, and $J_r(i) := J(i+1)$ for $i \geq r$. We have, with $v \in \Lambda^{0,\bullet}T^*M \otimes E$ and $u = w^K \wedge v$ for some increasing $K\colon \{1, \dots, p\} \to \{0, \dots, n\}$,

$$\begin{aligned}
&{\sum_{|J|=n-p}}' \sum_{r=1}^{n-p} \sum_{m=1}^{n} (-1)^{r-1}\, \mathrm{ins}_{w_m \wedge w_{J_r}}(w^J \wedge u) = \\
&\qquad= \sum_{r=1}^{n-p} \sum_{m=1}^{n} (-1)^{r-1}\, \mathrm{ins}_{w_m \wedge (K^c)_r}(w^{K^c} \wedge u) \\
&\qquad= \sum_{r=1}^{n-p} \sum_{m=1}^{n} \mathrm{ins}_{w_m \wedge w_{(K^c)_r}}(w^{K^c(r)} \wedge w^{(K^c)_r} \wedge u) \\
&\qquad= \sum_{r=1}^{n-p} \sum_{m=1}^{n} (-1)^{n-p-1}\, \mathrm{ins}_{w_m}\, \varepsilon(w^{K^c(r)})\, \mathrm{ins}_{w_{(K^c)_r}}(w^{(K^c)_r} \wedge u) \\
&\qquad= \sum_{r=1}^{n-p} \sum_{m=1}^{n} \sigma\, \mathrm{ins}_{w_m}\, \varepsilon(w^{K^c(r)}) u \\
&\qquad= \sum_{\ell=1}^{n} \sum_{m=1}^{n} \sigma\, \mathrm{ins}_{w_m}\, \varepsilon(w^{\ell}) u \\
&\qquad= \sum_{\ell=1}^{n} \sum_{m=1}^{n} \sigma\big(\delta_{\ell,m} - \varepsilon(w^{\ell})\, \mathrm{ins}_{w_m}\big) u
\end{aligned}$$

where $\sigma := (-1)^{(n-p)(n-p-1)/2}$ is as before and $K^c\colon \{1, \dots, n-p\} \to \{1, \dots, n\}$ is the complement of K, *i.e.*, the increasing map with $\mathrm{img}(K^c) = \mathrm{img}(K)^c$. By linearity, the above computation is valid for $u \in \Lambda^{p,\bullet}T^*M \otimes E$. Putting the above together, we arrive at

$$\begin{aligned}
&(\Psi_p^E)^{-1}\big(\varepsilon(\overline{w}^k)\operatorname{ins}_{\overline{w}_j} \otimes \mathrm{id}_E \otimes R^{\Lambda^{n-p,0}TM}(w_j,\overline{w}_k)\big)\Psi_p^E(u) = \\
&\quad = (\Psi_p^E)^{-1} \sum_{|J|=n-p}{}' w^J \wedge \big(\varepsilon(\overline{w}^k)\operatorname{ins}_{\overline{w}_j}(u)\big) \otimes \big(R^{\Lambda^{n-p,0}TM}(w_j,\overline{w}_k)w_J\big) \\
&\quad = \sum_{|J|=n-p}{}' \sigma \operatorname{ins}_{(R^{\Lambda^{n-p,0}TM}(w_j,\overline{w}_k)w_J)}(w^J \wedge (\varepsilon(\overline{w}^k)\operatorname{ins}_{\overline{w}_j}(u)) \\
&\quad = \sum_{|J|=n-p}{}' \sum_{r=1}^{n-p}\sum_{m=1}^{n} \sigma(-1)^{r-1}\langle R^{T^{1,0}M}(w_j,\overline{w}_k)w_{J(r)}, w_m\rangle \operatorname{ins}_{w_m \wedge w_{J_r}}(\varepsilon(\overline{w}^k)\operatorname{ins}_{\overline{w}_j}(u)) \\
&\quad = \sum_{\ell=1}^{n}\sum_{m=1}^{n}\langle R^{T^{1,0}M}(w_j,\overline{w}_k)w_\ell, w_m\rangle\big(\delta_{\ell,m} - \varepsilon(w^\ell)\operatorname{ins}_{w_m}\big)\,\varepsilon(\overline{w}^k)\operatorname{ins}_{\overline{w}_j}(u) \\
&\quad = \Bigg(\operatorname{tr}(R^{T^{1,0}M}(w_j,\overline{w}_k)) - \\
&\qquad\qquad - \sum_{\ell=1}^{n}\sum_{m=1}^{n}\langle R^{T^{1,0}M}(w_j,\overline{w}_k)w_\ell, w_m\rangle\,\varepsilon(w^\ell)\operatorname{ins}_{w_m}\Bigg)\,\varepsilon(\overline{w}^k)\operatorname{ins}_{\overline{w}_j}(u).
\end{aligned}$$

This explains the terms containing the curvature of $T^{1,0}M$ in (3.1.34) and finishes the proof. ■

3.2. The $\overline{\partial}^E$-Neumann problem

Suppose that $M \subseteq M'$ is an open subset of a larger Hermitian manifold (M', J, g), with boundary $\partial M \subseteq M'$ of class C^∞ in case $M \neq M'$. Assume further that $(\overline{M}, g)$ is complete (in the sense of section 1.4.1) and let $E \to \overline{M}$ be a Hermitian holomorphic vector bundle.[1] We emphasize that M does *not* include its own boundary, as opposed to the notation of chapter 2 where M was a smooth manifold *with* boundary, since this is not customary for complex manifolds. We will slightly abuse our notation and denote the Dolbeault Laplacian $\square^{E|_M}$ for $E|_M \to M$ simply by $\square^E\colon \Omega(M,E) \to \Omega(M,E)$. This notation will also be extended to a certain self-adjoint extension of $\square^E$:

Definition 3.2.1. The *Dolbeault Laplacian with $\overline{\partial}$-Neumann boundary conditions* is the self-adjoint operator

$$\square^E := \overline{\partial}^E_w \overline{\partial}^{E,*}_w + \overline{\partial}^{E,*}_w \overline{\partial}^E_w \tag{3.2.1}$$

on $L^2_{\bullet,\bullet}(M,E)$, where $\overline{\partial}^E_w$ is the weak extension of $\overline{\partial}^E$, see section 1.3, and $\overline{\partial}^{E,*}_w$ is the Hilbert space adjoint of $\overline{\partial}^E_w$. Thus, for each $1 \le p \le \dim_{\mathbb{C}}(M)$, the restriction of $\square^E$ to $L^2_{p,\bullet}(M,E)$ is the Gaffney extension of the elliptic (by (3.1.2)) complex $(\Lambda^{p,\bullet}T^*M \otimes E, \overline{\partial}^E)$, see (1.3.6). Its quadratic form will be denoted by Q^E. By Lemma 1.2.1, it is given by

$$Q^E(u,v) = \langle\!\langle \overline{\partial}^E_w u, \overline{\partial}^E_w v \rangle\!\rangle + \langle\!\langle \overline{\partial}^{E,*}_w u, \overline{\partial}^{E,*}_w v \rangle\!\rangle$$

for $u, v \in \operatorname{dom}(Q^E) = \operatorname{dom}(\overline{\partial}^E_w) \cap \operatorname{dom}(\overline{\partial}^{E,*}_w)$.

[1]This means that E is defined in some open neighborhood of $\overline{M}$ and holomorphic on this neighborhood.

If $U \subseteq \overline{M}$ is (relatively) open, then we denote by $\Box^E_U$ the self-adjoint operator $(\Box^E)_U$ on $L^2_{\bullet,\bullet}(M\cap U, E) = L^2_{\bullet,\bullet}(U, E)$, see Definition 2.1.2, with associated quadratic form $Q^E_U \coloneqq Q_{\Box^E,U}$. We write $\Box^E_{p,q}$ and $\Box^E_{U,p,q}$ for the restrictions of $\Box^E$ and $\Box^E_U$ to $L^2_{p,q}(M, E)$ and $L^2_{p,q}(M \cap U, E)$, respectively.

Remark 3.2.2. Note that the quadratic form $Q^{E|_{M\cap U}}$ is an extension of Q^E_U, in the sense that $\{u|_{M\cap U} : u \in \operatorname{dom}(Q^E_U)\} \subseteq \operatorname{dom}(Q^{E|_{M\cap U}})$ and

$$Q^{E|_{M\cap U}}(u|_{M\cap U}, u|_{M\cap U}) = Q^E_U(u, u) \tag{3.2.2}$$

for all $u \in \operatorname{dom}(Q^E_U)$. Intuitively, this is because Q^E_U requires Dirichlet boundary conditions on $\partial U \cap M^\circ$, while the self-adjoint operator associated to $Q^{E|_{M\cap U}}$ only requires the weaker $\bar\partial$-Neumann boundary conditions, see Remark 3.2.5.

To formally show (3.2.2), let $u \in \operatorname{dom}(Q^E_U)$. Then u_0, defined as the extension of u to $\overline{M}$ by zero, see Lemma 2.1.1, belongs to $\operatorname{dom}(Q^E) = \operatorname{dom}(\bar\partial^E_w) \cap \operatorname{dom}(\bar\partial^{E,\dagger}_s)$, and clearly $u|_{M\cap U} = (u_0)|_{M\cap U} \in \operatorname{dom}(\bar\partial^{E|_{M\cap U}}_w)$. For all $k \in \mathbb{N}$, we find $v_k \in \Omega_{cc}(U, E)$ such that $v_k \to u$ in $L^2_{\bullet,\bullet}(U, E)$ and $\|\bar\partial^{E,\dagger}_s u_0 - \bar\partial^{E,\dagger} v_k\| \le \frac{1}{k}$.[2] Since $(\bar\partial^{E,\dagger} v_k)|_{M\cap U} = \bar\partial^{E|_{M\cap U},\dagger}(v_k|_{M\cap U})$, it follows that $(v_k|_{M\cap U})_{k\in\mathbb{N}}$ is Cauchy in $\operatorname{dom}(\bar\partial^{E|_{M\cap U},\dagger}_s)$, hence converges to $u|_{M\cap U}$ in this space due to the convergence in $L^2_{\bullet,\bullet}(M \cap U, E)$. Thus, $u|_{M\cap U}$ belongs to $\operatorname{dom}(\bar\partial^{E|_{M\cap U},\dagger}_s)$, and

$$\bar\partial^{E|_{M\cap U},\dagger}_s(u|_{M\cap U}) = \lim_k (\bar\partial^{E,\dagger} v_k)|_{M\cap U} = (\bar\partial^{E,\dagger}_s u_0)|_{M\cap U},$$

so that $Q^{E|_{M\cap U}}(u|_{M\cap U}, u|_{M\cap U}) = Q^E(u_0, u_0) = Q^E_U(u, u)$, as claimed.

Proposition 3.2.3. *If $U \subseteq \overline{M}$ is open, then the space $B_M(U, E)$ from Definition 3.1.8 is a form core for $\Box^E_U$.*

Proof. We shall use the known fact that $B_M(\overline{M}, E)$ is a form core for $\Box^E$ if $\overline{M}$ is compact,[3] the proof of which requires careful use of mollifiers. By (1.4.4), Example 1.4.10, and Theorem 1.4.3 (note that at the beginning of this section, we have assumed $\overline{M}$ to be complete), we know that the elements of $\operatorname{dom}(\bar\partial^E_w + \bar\partial^{E,*}_w)$ with compact support in $\overline{M}$ are dense in $\operatorname{dom}(Q^E)$. If $u \in \operatorname{dom}(Q^E)$ has compact support, choose a compact manifold with boundary $X \subseteq \overline{M}$ such that $\operatorname{supp}(u) \subseteq V \coloneqq (\partial M \cap X) \cup X^\circ$, an open subset of $\overline{M}$. Then $u|_V \in \operatorname{dom}(Q^E_V) \subseteq \operatorname{dom}(Q^{E|_{V\cap M}}) = \operatorname{dom}(Q^{E|_{X^\circ}})$, see (3.2.2), and by the aforementioned result for compact manifolds, there exist $v_k \in B_X(X, E)$ with $v_k \to u|_X$ as $k \to \infty$ in $\operatorname{dom}(Q^{E|_{X^\circ}})$. Let $\varphi \in C^\infty_c(\overline{M}, [0, 1])$ with $\varphi|_{\operatorname{supp}(u)} = 1$. Then $\varphi v_k \in \Omega_c(\overline{M}, E)$ and $\operatorname{ins}_{(\nu^{0,1})}(\varphi v_k) = \varphi \operatorname{ins}_{(\nu^{0,1})}(v_k) = 0$ on $\partial M \cap \partial X$, and $\varphi v_k = 0$ on $\partial M \setminus \partial X$ anyways, so

[2] This may be done by first approximating u_0 by $\widetilde{v}_k \in \operatorname{dom}(Q^E)$ with $\operatorname{supp}(v_k) \subseteq U$, and then approximating each $\widetilde{v}_k$ by elements of $\Omega_{cc}(U, E)$.

[3] The statement can be found in [MM07, Lemma 3.5.1], where a reference is made to [Hör65, Proposition 1.2.4]. A proof for M a domain in $\mathbb{C}^n$ can also be found in [Str10, Proposition 2.3].

$\varphi v_k \in B_M(\overline{M}, E)$. By (1.3.2) and since $\sqrt{2}(\bar\partial^E + \bar\partial^{E,\dagger})$ is a Dirac type operator, see section 1.1.2 and (3.1.2), we have

$$\begin{aligned} Q^E(\varphi(v_k - v_j), \varphi(v_k - v_j)) &\leq \\ &\leq 2(\| \operatorname{Symb}(\bar\partial^E + \bar\partial^{E,\dagger})(d\varphi)(v_k - v_j)\|^2 + \|\varphi(\bar\partial^E + \bar\partial^{E,\dagger})(v_k - v_j)\|^2) \leq \\ &\leq \|d\varphi\|^2_{L^\infty(M,T^*M)} \|v_k - v_j\|^2_{L^2_{\bullet,\bullet}(X^\circ,E)} + 2Q^{E|_{X^\circ}}(v_k - v_j, v_k - v_j). \end{aligned} \tag{3.2.3}$$

Thus, $(\varphi v_k)_{k\in\mathbb{N}}$ is Cauchy in $\operatorname{dom}(Q^E)$, hence convergent, and the limit agrees with u by the convergence in $L^2_{\bullet,\bullet}(M, E)$. This shows the claim for $U = \overline{M}$.

Now let $U \subseteq \overline{M}$ be an arbitrary open subset. By the definition of Q^E_U, it suffices to show that every $u|_U$ with $u \in \operatorname{dom}(Q^E)$ and $\operatorname{supp}(u) \subseteq U$ can be approximated in the norm of $\operatorname{dom}(Q^E_U)$ by elements of $B_M(U, E)$. By the above, we obtain $u_k \in B_M(\overline{M}, E)$ with $u_k \to u$ in $\operatorname{dom}(Q^E)$. Let $\varphi \in C^\infty(M, [0, 1])$ be such that $\operatorname{supp}(\varphi) \subseteq U$ and $\varphi|_{\operatorname{supp}(u)} = 1$. Clearly, $\varphi u_k|_U \in B_M(U, E)$, and a computation as in (3.2.3) again gives convergence of $\varphi u_k|_U$ to $u|_U$ in $\operatorname{dom}(Q^E_U)$. ■

Proposition 3.2.4. *Let $M \subseteq M'$ and E be as above. Then*

(i) $\Omega_c(\overline{M}, E) \cap \operatorname{dom}(\bar\partial^{E,*}_w) = B_M(\overline{M}, E)$,

(ii) $\Omega_c(\overline{M}, E) \cap \operatorname{dom}(Q^E) = B_M(\overline{M}, E)$, *and*

(iii) $\Omega_c(\overline{M}, E) \cap \operatorname{dom}(\Box^E) = \{u \in B_M(\overline{M}, E) : \bar\partial^E u \in B_M(\overline{M}, E)\}$.

Moreover, $\bar\partial^{E,*}_w = \bar\partial^{E,\dagger}$ *on* $B_M(\overline{M}, E)$ *and* $\Box^E = \bar\partial^E\bar\partial^{E,\dagger} + \bar\partial^{E,\dagger}\bar\partial^E$ *on* $\Omega_c(\overline{M}, E) \cap \operatorname{dom}(\Box^E)$.

Remark 3.2.5. Item (iii) of Proposition 3.2.4 says that the smooth (on $\overline{M}$) elements u belonging to $\operatorname{dom}(\Box^E)$ satisfy *$\bar\partial$-Neumann boundary conditions* on ∂M, *i.e.*,

$$\operatorname{ins}_{(\nu^{0,1})}(u)|_{\partial M} = 0 \quad \text{and} \quad \operatorname{ins}_{(\nu^{0,1})}(\bar\partial^E u)|_{\partial M} = 0. \tag{3.2.4}$$

Therefore, the equation $\Box^E u = v$ is really a boundary value problem in disguise, called the *$\bar\partial^E$-Neumann problem.*

Proof of Proposition 3.2.4. From (1.3.2), it is clear that $\bar\partial^{E,*}_w$ and $\Box^E$ agree with the respective differential operators on the intersection of $\Omega_c(\overline{M}, E)$ with their domains.

We show the rest of the statements by following the arguments of [FK72, Propositions 1.3.2]. For $u, v \in \Omega_c(\overline{M}, E)$, we have

$$\langle\!\langle \bar\partial^{E,\dagger} u, v\rangle\!\rangle = \langle\!\langle u, \bar\partial^E v\rangle\!\rangle + \int_{\partial M} \langle \operatorname{ins}_{(\nu^{0,1})}(u), v\rangle \, d\mu_{\partial M},$$

see the computation in (3.1.25). If $u \in B_M(\overline{M}, E)$, then the boundary term vanishes, so that $u \in \operatorname{dom}((\bar\partial^E|_{\Omega_c(\overline{M},E)})^*) = \operatorname{dom}(\bar\partial^{E,*}_w)$, since $\Omega_c(\overline{M}, E)$ is a core for $\bar\partial^E_w$ by Proposition 1.4.11, and the closure of a densely defined operator has the same adjoint as the original operator. Conversely, if $u \in \Omega_c(\overline{M}, E) \cap \operatorname{dom}(\bar\partial^{E,*}_w)$, then

$$\langle\!\langle u, \bar\partial^E v\rangle\!\rangle = \langle\!\langle \bar\partial^{E,*}_w u, v\rangle\!\rangle = \langle\!\langle \bar\partial^{E,\dagger} u, v\rangle\!\rangle = \langle\!\langle u, \bar\partial^E v\rangle\!\rangle + \int_{\partial M} \langle \operatorname{ins}_{(\nu^{0,1})}(u), v\rangle \, d\mu_{\partial M}$$

for all $v \in \Omega_c(\overline{M}, E)$. This means that $\int_{\partial M} \langle \mathrm{ins}_{(\nu^{0,1})}(u), v\rangle \, d\mu_{\partial M} = 0$ for all $v \in \Omega_c(\overline{M}, E)$, hence $\mathrm{ins}_{(\nu^{0,1})}(u)|_{\partial M} = 0$. This shows (i).

Since $\Omega_c(\overline{M}, E) \subseteq \mathrm{dom}(\overline{\partial}^E_w)$, see (1.3.2), we also have $\Omega_c(\overline{M}, E) \cap \mathrm{dom}(Q^E) = \Omega_c(\overline{M}, E) \cap \mathrm{dom}(\overline{\partial}^E_w) \cap \mathrm{dom}(\overline{\partial}^{E,*}_w) = B_M(\overline{M}, E)$, so that (ii) follows. Similarly, (iii) is an easy consequence of (i). ■

Definition 3.2.6. The closed operator

$$N^E := \bigoplus_{p=0}^{n} N(L^2_{p,\bullet}(M, E), \overline{\partial}^E_w)$$

on $L^2_{\bullet,\bullet}(M, E)$ from Proposition 1.2.4 is called the $\overline{\partial}^E$-*Neumann operator*. We denote its restriction to $L^2_{p,q}(M, E)$ by $N^E_{p,q}$.

Thus, $N^E \colon \mathrm{img}(\Box^E) \to L^2_{\bullet,\bullet}(M, E)$ is defined as the inverse to $\Box^E|_{\mathrm{dom}(\Box^E) \cap \ker(\Box^E)^\perp}$. If $\mathrm{img}(\Box^E_{p,q})$ is closed in $L^2_{p,q}(M, E)$, see Lemma 1.2.5 for general conditions equivalent to this, then we extend $N^E_{p,q}$ as $N^E_{p,q} \oplus 0$ to a bounded operator on $L^2_{p,q}(M, E) = \mathrm{img}(\Box^E_{p,q}) \oplus \mathrm{img}(\Box^E_{p,q})^\perp$.

The general Proposition 1.2.4 shows that the $\overline{\partial}^E$-Neumann operator is important if one wants to study the solutions $u \in \mathrm{dom}(\overline{\partial}^E_w)$ of the inhomogeneous equation

$$\overline{\partial}^E_w u = v, \tag{3.2.5}$$

with $v \in \mathrm{img}(\overline{\partial}^E_w) \subseteq \ker(\overline{\partial}^E_w)$ given. In fact, $S^E := \overline{\partial}^{E,*}_w N^E$ on $\mathrm{img}(\overline{\partial}^E_w) \cap \mathrm{img}(\Box^E)$, where

$$S^E \colon \mathrm{img}(\overline{\partial}^E_w) \to L^2_{\bullet,\bullet}(M, E)$$

is the *canonical (or minimal) solution operator* to the $\overline{\partial}^E$-equation, which is defined as giving the solution to (3.2.5) of minimal norm, see (1.2.2) for the precise definition. Again, we denote by $S^E_{p,q}$ the restriction to $L^2_{p,q}(M, E)$. In case N^E is a bounded operator, then so is S^E and the equality $S^E = \overline{\partial}^{E,*}_w N^E$ holds on all of $L^2_{\bullet,\bullet}(M, E)$, see Proposition 1.2.6. Moreover, $N^E_{p,q}$ is compact if and only if $S^E_{p,q}$ and $S^E_{p,q+1}$ are, see Proposition 1.2.8. In the case where $N^E_{p,q}$ is bounded, item (iv) of Proposition 1.2.6 implies that we can, given $v \in \mathrm{img}(\overline{\partial}^E_w) \cap L^2_{p,q}(M, E)$, always find $u = S^E v \in \mathrm{dom}(\overline{\partial}^E_w) \cap L^2_{p,q-1}(M, E)$ such that

$$\int_M |u|^2 \, d\mu_g \le \frac{1}{\inf(\sigma(\Box^E_{p,q}) \setminus \{0\})} \int_M |v|^2 \, d\mu_g. \tag{3.2.6}$$

For extensive surveys of the L^2 theory of $\overline{\partial}$, with a focus on bounded pseudoconvex domains in $\mathbb{C}^n$, see [CS01; Str10].

Very simple conditions for the boundedness and compactness of the $\overline{\partial}^L$-Neumann operator, with L a line bundle, are given in the next Proposition:

Proposition 3.2.7. *Suppose $L \to M'$ is a Hermitian holomorphic line bundle, and $M \subseteq M'$ is a Levi pseudoconvex open subset with smooth boundary. Let $s_j \colon M \to \mathbb{R}$ be as in Example 3.1.21, and take $1 \le q \le n$. Then we have:*

(i) $\square^L_{n,q}$ *is a Fredholm operator (equivalently:* $N^L_{n,q}$ *is bounded and* $\dim(\ker(\square^L_{n,q})) < \infty$*, see Remark 1.2.7) if*

$$\liminf_{M\ni x\to\infty}(s_1(x)+\cdots+s_q(x))>0.$$

(ii) $\square^L_{n,q}$ *has discrete spectrum (equivalently:* $N^L_{n,q}$ *is compact and* $\dim(\ker(\square^L_{n,q})) < \infty$*) if*

$$\lim_{M\ni x\to\infty}(s_1(x)+\cdots+s_q(x))=+\infty.$$

Here, the limits have to be understood as x leaving every compact subset of M, i.e.*, either going to infinity or approaching the boundary of M.*

Proof. By Proposition 3.2.3, the space $B^{n,q}_M(\overline{M},L)$ is a form core for $\square^L_{n,q}$. Now the statement follows from (3.1.29) and Theorem 2.2.8, see also Theorems 2.2.9 and 2.2.12. ∎

An analogous result to Proposition 3.2.7 for $\square^L_{0,q}$ is obtained by replacing L with $L\otimes K^*_M$, see Remark 3.1.24. The conditions in Proposition 3.2.7 are not sharp, of course. For example, if $\Omega\subseteq\mathbb{C}^n$ is a bounded (Levi) pseudoconvex open subset, then $\square^L_{p,q}$ is automatically Fredholm for L the trivial Hermitian line bundle (where $s_j=0$ for all j) and $q\geq 1$, see for instance [Str10, Proposition 2.7].

3.2.1. L^2 Serre duality. Let $E\to M$ be a Hermitian holomorphic vector bundle over a Hermitian manifold. Since complex manifolds come with an orientation, we have the Hodge star operator $\overline{\star}^E\colon \Lambda T^*M\otimes E\to\Lambda T^*M\otimes E^*$, see appendix A.2 for the general theory, and this operator maps $\Lambda^{p,q}T^*M\otimes E$ to $\Lambda^{n-p,n-q}T^*M\otimes E^*$, where n is the complex dimension of M. As $\dim_{\mathbb{R}}(M)=2n$ is even, we obtain $(\overline{\star}^E)^{-1}=(-1)^{p+q}\overline{\star}^{E^*}$ on $\Lambda^{p,q}T^*M\otimes E^*$. It follows from Proposition A.2.3 that also

$$\overline{\star}^E\circ\overline{\partial}^{E,\dagger}=(-1)^{p+q}\overline{\partial}^{E^*}\circ\overline{\star}^E \quad\text{and}\quad \overline{\star}^E\circ d^{E,\dagger}_{1,0}=(-1)^{p+q}d^{E^*}_{1,0}\circ\overline{\star}^E \tag{3.2.7}$$

on $\Omega^{p,q}(M,E)$. It follows that

$$\overline{\partial}^{E,\dagger}=-\overline{\star}^{E^*}\circ\overline{\partial}^{E^*}\circ\overline{\star}^E \quad\text{and}\quad d^{E,\dagger}_{1,0}=-\overline{\star}^{E^*}\circ d^{E^*}_{1,0}\circ\overline{\star}^E.$$

The following result can be found in [CS12], where an in-depth account of L^2 Serre duality is given.

Theorem 3.2.8 (L^2 Serre duality)**.** *Let $E\to M$ be a Hermitian holomorphic vector bundle over a Hermitian manifold. The Hodge star operator*

$$\overline{\star}^E\colon L^2_{\bullet,\bullet}(M,E)\to L^2_{n-\bullet,n-\bullet}(M,E^*)$$

restricts to antiunitary mappings $\operatorname{dom}(\overline{\partial}^{E,*}_w)\to\operatorname{dom}(\overline{\partial}^{E^*}_s)$ *and* $\operatorname{dom}(\overline{\partial}^{E,*}_s)\to\operatorname{dom}(\overline{\partial}^{E^*}_w)$*, all equipped with the graph norms, and satisfies*

(i) $\overline{\star}^E\circ\overline{\partial}^{E,*}_s=(-1)^{p+q}\overline{\partial}^{E^*}_w\circ\overline{\star}^E$,

(ii) $\overline{\star}^E\circ\overline{\partial}^{E,*}_w=(-1)^{p+q}\overline{\partial}^{E^*}_s\circ\overline{\star}^E$, *and*

(iii) $\overline{\star}^E\circ\square^E=\square^{E^*}_s\circ\overline{\star}^E$

on $L^2_{p,q}(M,E)$, where $\square^{E^*}_s$ is the Laplacian of the Hilbert complex $(L^2_{n-p,\bullet}(M,E^*),\overline{\partial}^{E^*}_s)$.

Proof. On the level of differential operators, we have (3.2.7). Therefore,

$$\|\bar\star^E u\|^2 + \|\bar\partial^{E^*}\bar\star^E u\|^2 = \|u\|^2 + \|\bar\partial^{E,\dagger} u\|^2$$

for all $u \in \Omega_c(M,E)$. Since $\Omega_c(M,E)$ is dense in $\mathrm{dom}((\bar\partial^{E,\dagger})_s)$ by definition, it follows that $\bar\star^E$ maps $\mathrm{dom}(\bar\partial^{E,*}_w) = \mathrm{dom}((\bar\partial^{E,\dagger})_s)$ to $\mathrm{dom}(\bar\partial^{E^*}_s)$, and it is easy to see that then also $\bar\partial^{E^*}_s \circ \bar\star^E = (-1)^{p+q}\bar\star^E \circ \bar\partial^{E,*}_w$ must hold. This shows (ii), and (i) follows from (ii) applied to E^* instead of E, and taking adjoints (note that $\bar\star^E$ is self-adjoint up to a sign factor by (A.2.2)). Finally, (iii) is a straightforward consequence of (i) and (ii). ■

Corollary 3.2.9. *If M is a complete Hermitian manifold and $E \to M$ is a Hermitian holomorphic vector bundle, then $\bar\star^E \circ \Box^E = \Box^{E^*} \circ \bar\star^E$.*

Proof. Since M is complete, the Dolbeault Laplacian $\Box^E$ is essentially self-adjoint on $\Omega_c(M,E)$, see Corollary 1.4.13, hence its self-adjoint extension $\Box^E_s$ from Theorem 3.2.8 coincides with (3.2.1). ■

3.2.2. A property of the essential spectrum of $\Box^E_{p,\bullet}$. Let again $E \to M$ be a Hermitian holomorphic vector bundle over a Kähler manifold. The goal of this section is to show that, under certain pseudoconvexity assumptions on the manifold and positivity of curvature requirements, the discreteness of spectrum of $\Box^E$ "percolates" up the Dolbeault complex, in the sense that if $\Box^E_{p,q}$ has discrete spectrum, then the same holds true for $\Box^E_{p,q+1}$. This property is well-known in the case of a bounded pseudoconvex domain M in $\mathbb{C}^n$, see [Fu08, Proposition 2.2] or [Str10, Proposition 4.5]. Moreover, this holds also for the weighted $\bar\partial$-equation on $\mathbb{C}^n$, and where the weight is plurisubharmonic, see [Has14]. Recall from Example B.3.7 that this latter case may be obtained by choosing E to be the trivial line bundle on $\mathbb{C}^n$, but with nontrivial Hermitian metric. For a general vector bundle, this condition will have to be replaced by a curvature condition.

The proofs rely on the fact that, if $\{X^j\}_{j=1}^n$ is a (constant, since we are still on $\mathbb{C}^n$) orthonormal frame field for $T^{0,1}M$, then the isometry $L^2_{p,q}(M,E) \to L^2_{p,q-1}(M,E)^{\oplus n}$ given by $u \mapsto \frac{1}{\sqrt{q}}(\mathrm{ins}_{X^j}(u))_{j=1}^n$ restricts to a bounded operator from $\mathrm{dom}(Q^E_{p,q})$ to $\mathrm{dom}(Q^E_{p,q-1})^{\oplus n}$, assuming the previously mentioned pseudoconvexity and curvature assumptions hold. The problem is that, if M is a Hermitian manifold, we do not have global frames for $T^{0,1}M$ available, so we have to use local frames and patch the results together. Moreover, the derivatives of the frame elements will have to be controlled. This patching procedure works if M is of 1-*bounded geometry* in the sense of section 4.1, as we will see below.

As in the beginning of section 3.2, we will assume that M' is a Kähler manifold, $M \subseteq M'$ an open subset with (empty in case $M = M'$) smooth boundary $\partial M \subseteq M'$ such that $\overline{M}$ is complete, and $E \to \overline{M}$ a Hermitian holomorphic vector bundle. Let $\mathscr{V}$ be a collection of (relatively) open subsets of $\overline{M}$, and put $U := \bigcup_{V\in\mathscr{V}} V$. The first step is to show that if $\mathscr{V}$ is nice enough, then we can control $\inf\sigma(\Box^E_U)$ by knowing about the bottom of the spectra of the $\Box^E_V$, see Proposition 3.2.11 below, and where $\Box^E_U$ and $\Box^E_V$ are defined as in Definition 2.1.2. What we are doing here is showing that certain spectral properties of $\Box^E$ *localize*, similarly as it is done in [Str10, Proposition 4.4] (but since we are working on possibly unbounded manifolds, we will need some control on the geometry). But first, a Lemma:

Lemma 3.2.10. *Let $\mathscr{V}$ and U be as above. Suppose that φ_V, $V \in \mathscr{V}$, is a family of functions in $C^\infty(U,[0,1])$ such that $\mathscr{V}$ and $(\varphi_V)_{V\in\mathscr{V}}$ have the following properties:*

*(i) There exists a number $N > 0$ such that $\bigcap_{V\in I} V \neq \emptyset$ implies $|I| \leq N$ for all subsets $I \subseteq \mathscr{V}$ (*i.e.*, $\mathscr{V}$ has uniformly finite intersection multiplicity).*

(ii) The functions φ_V^2 form a partition of unity subordinate to $\mathscr{V}$, i.e., $\operatorname{supp}(\varphi_V) \subseteq V$ *for $V \in \mathscr{V}$ and $\sum_{V\in\mathscr{V}} \varphi_V^2 = 1$ on U.*

(iii) $\gamma \coloneqq \sup_{V\in\mathscr{V}} \|d\varphi_V\|^2_{L^\infty(U,T^*U)} < \infty$.

If $u \in \operatorname{dom}(Q_U^E)$, then $(\varphi_V u)|_V \in \operatorname{dom}(Q_V^E)$ for all $V \in \mathscr{V}$, and

$$\sum_{V\in\mathscr{V}} Q_V^E(\varphi_V u, \varphi_V u) \leq \gamma N \|u\|^2 + 2Q_U^E(u,u). \tag{3.2.8}$$

Conversely, if $u \in \operatorname{dom}(Q_V^E)$, then $(\varphi_{V'} u_0)|_U \in \operatorname{dom}(Q_U^E)$ for all $V' \in \mathscr{V}$, and

$$\sum_{\{V'\in\mathscr{V}: V'\cap V\neq\emptyset\}} Q_U^E(\varphi_{V'} u_0, \varphi_{V'} u_0) \leq \gamma N \|u\|^2 + 2Q_V^E(u,u), \tag{3.2.9}$$

where $u_0 \in L^2_{\bullet,\bullet}(U\cap M, E)$ is the extension of u by zero outside of V.

Proof. Let $u \in \operatorname{dom}(Q_U^E)$ and $V \in \mathscr{V}$. Then $u_0 \in \operatorname{dom}(Q^E)$ by Lemma 2.1.4, thus $\varphi_V u_0 \in \operatorname{dom}(Q^E)$ by Example 1.4.10, and therefore $(\varphi_V u)|_V \in \operatorname{dom}(Q_V^E)$ since $\operatorname{supp}(\varphi_V u) \subseteq V$. Moreover,

$$\begin{aligned} Q_V^E(\varphi_V u, \varphi_V u) = Q^E(\varphi_V u_0, \varphi_V u_0) = \|(\overline{\partial}_w^E + \overline{\partial}_w^{E,*})(\varphi_V u_0)\|^2 \leq \\ \leq \|d\varphi_V\|^2_{L^\infty(M,T^*M)} \|u|_V\|^2 + 2\|\varphi_V(\overline{\partial}_w^E + \overline{\partial}_w^{E,*})(u_0)\|^2, \end{aligned}$$

see the computation in (3.2.3), and Remark 1.4.9 for the validity of the Leibniz rule. Adding these estimates, we arrive at

$$\sum_{V\in\mathscr{V}} Q_V^E(\varphi_V u, \varphi_V u) \leq \sum_{V\in\mathscr{V}} (\gamma\|u|_V\|^2 + 2\|\varphi_V(\overline{\partial}_w^E + \overline{\partial}_w^{E,*})(u_0)\|^2) = \gamma \sum_{V\in\mathscr{V}} \|u|_V\|^2 + 2Q_U^E(u,u). \tag{3.2.10}$$

Note that assumption (i) implies that $\mathscr{V}$ is at most countable, for if we take a countable basis for the topology of U, then we may assume that each basis element is contained in a single $V \in \mathscr{V}$, hence intersects at most N elements of $\mathscr{V}$. Fix a bijection $\mathbb{N} \to \mathscr{V}$, $k \mapsto V_k$.[4] With the finite Borel measure $\nu(A) \coloneqq \int_A |u|^2 \, d\mu_g$ on U, we have

$$\sum_{V\in\mathscr{V}} \|u|_V\|^2 = \sum_{k=1}^\infty \nu(V_k) = \nu(U) + \sum_{j<k} \nu(V_j \cap V_k) \leq N\nu(U) = N\|u\|^2,$$

with $N \coloneqq \max_{k\in\mathbb{N}} \#\{j \in \mathbb{N} : V_j \cap V_k \neq \emptyset\}$, the maximal number of intersections that elements of $\mathscr{V}$ have amongst each other. Together with (3.2.10), this shows (3.2.8).

[4]In the case where $\mathscr{V}$ is finite, we may add countably many empty sets to obtain a bijection $\mathbb{N} \to \mathscr{V}$.

Now let $V, V' \in \mathscr{V}$ and $u \in \operatorname{dom}(Q_V^E)$. As before, one argues that $(\varphi_{V'} u_0)|_U \in \operatorname{dom}(Q_U^E)$, and virtually the same computation shows that (3.2.9) also holds. ∎

Proposition 3.2.11. *Let $\mathscr{V}$ and $(\varphi_V)_{V\in\mathscr{V}}$ be as in Lemma 3.2.10. Then*

$$\frac{1}{2}\big(\inf\sigma(\Box_U^E) - \gamma N\big) \le \inf_{V\in\mathscr{V}} \inf\sigma(\Box_V^E) \le 2\inf\sigma(\Box_U^E) + \gamma N. \tag{3.2.11}$$

Proof. By (3.2.8), we have, for $V \in \mathscr{V}$,

$$\Big(\inf_{V\in\mathscr{V}} \inf\sigma(\Box_V^E)\Big)\|u\|^2 \le \sum_{V\in\mathscr{V}} (\inf\sigma(\Box_V^E))\|\varphi_V u\|^2 \le$$
$$\le \sum_{V\in\mathscr{V}} Q^E(\varphi_V u, \varphi_V u) \le \gamma N\|u\|^2 + 2Q_U^E(u,u)$$

for all $u \in \operatorname{dom}(Q_U^E)$. Since $\inf\sigma(Q_U^E)$ is the largest lower bound for Q_U^E, the right hand part of (3.2.11) follows. The same reasoning applied to (3.2.9) gives

$$(\inf\sigma(\Box_U^E))\|u\|^2 = \sum_{V'\in\mathscr{V}} (\inf\sigma(\Box_U^E))\|\varphi_{V'} u_0\|^2 \le$$
$$\le \sum_{V'\in\mathscr{V}} Q_U^E(\varphi_{V'} u_0, \varphi_{V'} u_0) \le \gamma N\|u\|^2 + 2Q_V^E(u,u),$$

hence $\inf\sigma(\Box_V^E) \ge \frac{1}{2}(\inf\sigma(\Box_U^E) - \gamma N)$ for all $V \in \mathscr{V}$. ∎

Next, we take, for each $V \in \mathscr{V}$, a suitable orthonormal frame of $T^{0,1}V$ to bound the bottom of the spectrum of $\Box_{V,p,q}^E$ in terms of $\inf\sigma(\Box_{V,p,q-1}^E)$. The results from Proposition 3.2.11 then allow us to transfer these bounds to $\Box_U^E$, with $U := \bigcup\mathscr{V}$. Again, we outsource some of the computations to a Lemma:

Lemma 3.2.12. *Let $U \subseteq M$ be open and suppose that $(w_j)_{j=1}^n$ is an orthonormal frame of $T^{1,0}U$. Then*

$$\sum_{j=1}^n \big|d_{1,0}^{E,\dagger}(\operatorname{ins}_{\overline{w}_j}(u))\big|^2 \le 2q\big|d_{1,0}^{E,\dagger}u\big|^2 + \Big(2n\max\{|\nabla\overline{w}_k|^2 : 1 \le k \le n\}\Big)|u|^2$$

pointwise on U for every $u \in \Omega^{p,q}(U,E)$.

Proof. Let X be a complex vector field on M. By the derivation rule for the exterior covariant derivative (see (A.1.4)), we have

$$\operatorname{ins}_X \circ d_{1,0}^{E,\dagger} = (d_{1,0}^E \circ \varepsilon(X^\flat))^\dagger = \varepsilon(\partial(X^\flat))^\dagger - (\varepsilon(X^\flat)\circ d_{1,0}^E)^\dagger = \varepsilon(\partial(X^\flat))^\dagger - d_{1,0}^{E,\dagger}\circ\operatorname{ins}_X,$$

where $\varepsilon(\alpha)\colon \Lambda^{\bullet,\bullet}T^*M \otimes E \to \Lambda^{\bullet,\bullet}T^*M \otimes E$ is exterior multiplication with $\alpha \in \Lambda^{\bullet,\bullet}T^*M$. Therefore,

$$\sum_{j=1}^n \big|d_{1,0}^{E,\dagger}(\operatorname{ins}_{\overline{w}_j}(u))\big|^2 \le 2\sum_{j=1}^n \big|\operatorname{ins}_{\overline{w}_j}(d_{1,0}^{E,\dagger}(u))\big|^2 + 2\sum_{j=1}^n |\,\varepsilon(\partial\overline{w}^j)^\dagger u|^2$$
$$\le 2q|d_{1,0}^{E,\dagger}u|^2 + 2\sum_{j=1}^n |\,\varepsilon(\partial\overline{w}^j)^\dagger|^2|u|^2$$

by Lemma B.2.3, where $|\,\varepsilon(\partial\overline{w}^j)^\dagger|$ denotes the (fiberwise) operator norm. Now

$$|\,\varepsilon(\partial\overline{w}^j)^\dagger| = |\,\varepsilon(\partial\overline{w}^j)| \le |\partial\overline{w}^j| \le |\nabla\overline{w}^j| = |\nabla\overline{w}_j|,$$

which finishes the proof. ■

Proposition 3.2.13. *Let $\mathscr{V}$ be a collection of open subsets of $\overline{M}$, and $(\varphi_V)_{V\in\mathscr{V}}$ a family of functions with the properties (i) to (iii) as in Lemma 3.2.10. Suppose that, in addition,*

(iv) for every $V \in \mathscr{V}$, there exists an orthonormal frame $(X_j^V)_{j=1}^n$ of $T^{0,1}V$ such that

$$\kappa := \sup_{V\in\mathscr{V}} \max_{1\le j\le n} \|\nabla X_j^V\|_{L^\infty}^2 < \infty.$$

Assume that $0 \le p \le n$, $1 \le q \le n$, and

$$\langle\!\langle iR^{E\otimes\Lambda^{n-p,0}TM} \wedge_{\mathrm{ev}} \Lambda\Psi_p^E(u), \Psi_p^E(u)\rangle\!\rangle \ge c\|u\|^2 \quad \text{and} \tag{3.2.12}$$

$$\int_{\partial M\cap U} \mathscr{L}(u,u)\, d\mu_{\partial M} \ge 0 \tag{3.2.13}$$

for some constant $c \in \mathbb{R}$ and all $u \in B_M^{p,q}(U,E)$, where $U := \bigcup\mathscr{V}$ and Ψ_p^E is as in (3.1.31) and $\mathscr{L}$ is defined in Definition 3.1.11. Then there exists a constant $C = C(n,q,N,\gamma,\kappa,c) > 0$ such that

$$\inf\sigma(\square^E_{U,p,q-1}) \le C + 8\inf\sigma(\square^E_{U,p,q}). \tag{3.2.14}$$

Proof. Let $V \in \mathscr{V}$. The orthonormal frame $(X_j^V)_{j=1}^n$ from our assumption (iv) induces an isometry $L^2_{p,q}(V,E) \to L^2_{p,q-1}(V,E)^{\oplus n}$, given by $u \mapsto \frac{1}{\sqrt{q}}(\mathrm{ins}_{X_j^V}(u))_{j=1}^n$, see Lemma B.2.3. By the global Bochner–Kodaira–Nakano formula (3.1.33) we have, for every $u \in B_M^{p,q}(V,E)$,

$$\frac{1}{q}\sum_{j=1}^n Q_V^E(\mathrm{ins}_{X_j^V}(u), \mathrm{ins}_{X_j^V}(u)) = \frac{1}{q}\sum_{j=1}^n \Big(\big\|d_{1,0}^{E\otimes\Lambda^{n-p,0}TM,\dagger}(\mathrm{ins}_{X_j^V}(\widetilde{u}))\big\|^2 + \\ + \langle\!\langle iR^{E\otimes\Lambda^{n-p,0}TM} \wedge_{\mathrm{ev}} \Lambda(\,\mathrm{ins}_{X_j^V}(\widetilde{u})), \mathrm{ins}_{X_j^V}(\widetilde{u})\rangle\!\rangle + \int_{\partial M\cap V} \mathscr{L}(\mathrm{ins}_{X_j^V}(u), \mathrm{ins}_{X_j^V}(u))\, d\mu_{\partial M}\Big),$$

where $\widetilde{u} := \Psi_p^E(u)$, which by using Lemma B.2.3 as well as Lemma 3.2.12 (recall from (3.1.35) the local formula for $iR^{E\otimes\Lambda^{n-p,0}TM} \wedge_{\mathrm{ev}} \Lambda$) can be estimated from above by

$$2\|d_{1,0}^{E\otimes\Lambda^{n-p,0}TM,\dagger}\widetilde{u}\|^2 + \frac{q-1}{q}\langle\!\langle iR^{E\otimes\Lambda^{n-p,0}TM} \wedge_{\mathrm{ev}} \Lambda\widetilde{u}, \widetilde{u}\rangle\!\rangle + \\ + \frac{q-1}{q}\int_{\partial M\cap V} \mathscr{L}(u,u)\, d\mu_{\partial M} + \frac{2n}{q}\Big(\max\{\|\nabla X_j^V\|_{L^\infty}^2 : 1 \le j \le n\}\Big)\|u\|^2. \tag{3.2.15}$$

By our assumption (3.2.13), we have

$$\frac{q-1}{q}\int_{\partial M\cap V} \mathscr{L}(u,u)\, d\mu_{\partial M} \le 2\int_{\partial M\cap V} \mathscr{L}(u,u)\, d\mu_{\partial M}, \tag{3.2.16}$$

and (3.2.12) yields

$$\frac{q-1}{q}\langle\!\langle iR^{E\otimes\Lambda^{n-p,0}TM} \wedge_{\mathrm{ev}} \Lambda\widetilde{u}, \widetilde{u}\rangle\!\rangle \le 2\langle\!\langle iR^{E\otimes\Lambda^{n-p,0}TM} \wedge_{\mathrm{ev}} \Lambda\widetilde{u}, \widetilde{u}\rangle\!\rangle + |c|\Big(2 - \frac{q-1}{q}\Big)\|u\|^2. \tag{3.2.17}$$

Using (iv), and again applying (3.1.33), we see that (3.2.15) is dominated by

$$2Q_V^E(u,u)+\Big(|c|\Big(2-\frac{q-1}{q}\Big)+\frac{2n\kappa}{q}\Big)\|u\|^2=2Q_V^E(u,u)+\frac{|c|(q+1)+2n\kappa}{q}\,\|u\|^2$$

Put $\widetilde C\coloneqq\frac{|c|(q+1)+2n\kappa}{q}$. It follows that

$$(\inf\sigma(\square^E_{V,p,q-1}))\|u\|^2=\frac1q\sum_{j=1}^n(\inf\sigma(\square^E_{V,n,q-1}))\|\operatorname{ins}_{X_j^V}(u)\|^2\le$$
$$\le\frac1q\sum_{j=1}^nQ_V^E(\operatorname{ins}_{X_j^V}(u),\operatorname{ins}_{X_j^V}(u))\le 2Q_V^E(u,u)+\widetilde C\|u\|^2.$$

By Proposition 3.2.3, $B_M^{p,q}(V,E)$ is a form core for $\square^E_{p,q,V}$, hence

$$\inf\sigma(\square^E_{V,p,q-1})\le 2\inf\sigma(\square^E_{V,p,q})+\widetilde C.$$

Together with (3.2.11), this implies

$$\frac12\big(\inf\sigma(\square^E_{U,p,q-1})-\gamma N\big)\le\inf_{V\in\mathscr V}\inf\sigma(\square^E_{V,p,q-1})\le$$
$$\le\widetilde C+2\inf_{V\in\mathscr V}\sigma(\square^E_{V,p,q})\le\widetilde C+2\big(\gamma N+2\inf\sigma(\square^E_{U,p,q})\big),$$

hence

$$\inf\sigma(\square^E_{U,p,q-1})\le 2\widetilde C+5\gamma N+8\inf\sigma(\square^E_{U,p,q}),$$

as claimed. ■

By applying Theorem 2.2.8 and the above results, we obtain also a bound for the bottom of the *essential* spectrum of $\square^E_{p,q}$:

Theorem 3.2.14. *Let M' be a Kähler manifold, $M\subseteq M'$ an open subset with smooth boundary $\partial M\subseteq M'$ such that $\overline M$ is complete, $E\to M'$ a Hermitian holomorphic vector bundle, $p\ge 0$, and $q\ge 1$. Let $\mathscr V$ be an open cover of $\overline M$ with the properties (i) to (iii) of Lemma 3.2.10 and (iv) of Proposition 3.2.13. Suppose that*

$$\big\langle\!\big\langle iR^{E\otimes\Lambda^{n-p,0}TM}\wedge_{\mathrm{ev}}\Lambda\Psi_p^E(u),\Psi_p^E(u)\big\rangle\!\big\rangle\ge c\|u\|^2\quad\textit{and}\tag{3.2.18}$$
$$\int_{\partial M}\mathscr L(u,u)\,d\mu_{\partial M}\ge 0\tag{3.2.19}$$

for some constant $c\in\mathbb R$ and all $u\in B_M^{p,q}(\overline M\setminus K,E)$, with $K\subseteq M$ a compact subset and where Ψ_p^E is as in (3.1.31)*. Then*

$$\inf\sigma_e(\square^E_{p,q-1})\le C+8\inf\sigma_e(\square^E_{p,q}),$$

with the constant C computed as in the proof of Proposition 3.2.13. In particular, if $\square^E_{p,q-1}$ has discrete spectrum, then so does $\square^E_{p,q}$.

Of course, (3.2.18) continues to hold for $u \in B^{p,q}_M(\overline{M}, E)$ by continuity, but the constant may be worse than what one gets by restricting to forms with support outside some compact set.

Proof. Let $\lambda < \inf \sigma_e(\Box^E_{p,q-1})$. By Theorem 2.2.8, there exists a compact subset $K_0 \subseteq M$ such that $\inf \sigma(\Box^E_{U,p,q-1}) \geq \lambda$, with $U := \overline{M} \setminus K_0$. Without loss of generality, $K \subseteq K_0$. Let $\mathscr{V}' := \{V \cap U : V \in \mathscr{V}\}$. Then $\mathscr{V}'$ still has the properties required by Proposition 3.2.13, and with $\bigcup \mathscr{V}' = U$. It follows that

$$\lambda \leq \inf \sigma(\Box^E_{U,p,q-1}) \leq C + 8 \inf \sigma(\Box^E_{U,p,q}),$$

hence $\lambda \leq C + 8 \inf \sigma_e(\Box^E_{p,q})$, again by Theorem 2.2.8, and the claim follows. ■

Positivity of vector bundles. The requirement (3.2.19) is satisfied precisely if M is q-Levi pseudoconvex at all points of ∂M. One way to make sure that (3.2.18) is satisfied is by requiring that the curvature of $E \otimes \Lambda^{n-p,0}TM$ is semipositive in the sense of Nakano [Nak55]:

Definition 3.2.15. A Hermitian holomorphic vector bundle $E \to M$ is called *Nakano semipositive at* $x \in M$ if

$$\sum_{j,k,\alpha,\beta} \langle R^E(\tfrac{\partial}{\partial z_j}, \tfrac{\partial}{\partial \bar z_k}) e_\alpha, e_\beta\rangle \, u_{j,\alpha} \overline{u_{k,\beta}} \geq 0 \tag{3.2.20}$$

for all $u = \sum_{j,\alpha} u_{j,\alpha} \frac{\partial}{\partial z_j} \otimes e_\alpha \in T^{1,0}_x M \otimes E_x$, where $(z_1, \ldots, z_n)$ are holomorphic coordinates of M around x and $\{e_\alpha\}_\alpha$ is an orthonormal basis of E_x. If the inequality (3.2.20) is strict for $u \neq 0$, then E is called *Nakano positive at* x. Moreover, E is called *Nakano (semi) positive* if it has the corresponding property at all points $x \in M$. Similarly, the concept of *Nakano (semi) negativity* is defined.

Remark 3.2.16. (i) It is easy to see that E is Nakano semipositive at x if and only if

$$\sum_{j,k=1}^{n} \langle \big(\varepsilon(\overline{w}^k) \circ \mathrm{ins}_{\overline{w}_j}\big) \otimes R^E(w_j, \overline{w}_k) u, u\rangle \geq 0 \tag{3.2.21}$$

holds for all $u \in \Lambda^{0,1}T^*_x M \otimes E_x$ or, equivalently, all $u \in \Lambda^{\bullet,1}T^*_x M \otimes E_x$, where $\{w_j\}_{j=1}^n$ is an orthonormal basis of $T^{0,1}_x M$. On $\Lambda^{n,1}T^*_x M \otimes E_x$, (3.2.21) is the same as

$$\langle iR^E \wedge_{\mathrm{ev}} \Lambda u, u\rangle \geq 0, \tag{3.2.22}$$

see Remark 3.1.20.

(ii) The inequality (3.2.21) continues to hold for $u \in \Lambda^{p,q}T^*_x M \otimes E_x$. This can be seen by induction: if (3.2.21) is true on $\Lambda^{p,q-1}T^*_x M \otimes E_x$ with $q \geq 2$, then also

$$\sum_{j,k=1}^{n} \langle (\varepsilon(\overline{w}^k) \circ \mathrm{ins}_{\overline{w}_j}) \otimes R^E(w_j, \overline{w}_k) u, u\rangle =$$

$$= \frac{1}{q-1} \sum_{m=1}^{n} \sum_{j,k=1}^{n} \langle ((\varepsilon(\overline{w}^k) \circ \mathrm{ins}_{\overline{w}_j}) \otimes R^E(w_j, \overline{w}_k)) \,\mathrm{ins}_{\overline{w}_m}(u), \mathrm{ins}_{\overline{w}_m}(u)\rangle \geq 0$$

for all $u \in \Lambda^{p,q} T_x^* M \otimes E_x$ by Lemma B.2.3. If (3.2.21) is strict for nonzero $(p, q-1)$-forms, then this will also hold for (p, q)-forms (if $u = w^J \wedge \overline{w}^K \otimes e \neq 0$ with $|J| = p$, $|K| = q$, and $e \in E_x$, then at least one of $\mathrm{ins}_{\overline{w}_m}(u)$ will also be nonzero).

Remark 3.2.17. There is also the notion of *Griffiths (semi) positive* vector bundles [Gri69], where the defining inequality (3.2.20) only needs to hold on *elementary* tensors, *i.e.*, for all $u = Z \otimes e$, with $Z \in T_x^{1,0} M$ and $e \in E_x$. Thus, E is Griffiths semipositive at $x \in M$ if and only if

$$\langle R^E(Z, \overline{Z}) e, e \rangle \geq 0$$

holds for all $Z \in T_x^{1,0} M$ and $e \in E_x$. In the language of Remark B.3.4, this means that $s_j(e) \geq 0$ for all $1 \leq j \leq n$ and all $e \in E_x$. Letting $Z = \frac{1}{2}(X - iJX)$, see (B.1.1), this is seen to be equivalent to

$$\langle i R^E(X, JX) e, e \rangle \geq 0 \tag{3.2.23}$$

for all $X \in T_x M$ and $e \in E_x$. The strict inequalities give the corresponding concept of Griffiths positivity, and Griffiths (semi) negative vector bundles are defined in a similar fashion.

Evidently, the condition of Griffiths (semi) positivity is formally weaker than Nakano (semi) positivity, and there are examples of Griffiths semipositive bundles that are not Nakano semipositive, see [Dem12, Example VII.6.8].[5] On the other hand, the two concepts clearly coincide for line bundles, which are then simply called *(semi) positive*. By (3.2.23), a positive line bundle $L \to M$ defines a Kähler metric on M via $g(X, Y) \coloneqq i R^L(X, JY) \in \mathrm{End}(L) \cong \mathbb{C}$. The fact that this is Kähler can be seen with the second Bianchi identity (A.1.10) or by looking at R^L in a trivialization of L, see Example B.3.8.

Since $R^{E^*}(X, Y) = -(R^E(X, Y))^*$, see Example A.1.5, a Hermitian holomorphic vector bundle is Griffiths (semi) positive if and only if its dual E^* is Griffiths (semi) negative, but this is not true for Nakano (semi) positivity, see again [Dem12, Example VII.6.8].

Example 3.2.18. If $L \coloneqq M \times \mathbb{C} \to M$ is the trivial line bundle with metric $|(x, v)|^2 = |v|^2 e^{-\varphi(x)}$ for a smooth function $\varphi \colon M \to \mathbb{R}$, see Example B.3.7, then the curvature of L is $R^L = \partial \bar{\partial} \varphi$. Therefore, L being semipositive is equivalent to φ being *plurisubharmonic*, while L is positive exactly when φ is *strictly plurisubharmonic*. ♦

For more examples and properties of (Nakano) positive vector bundles, we refer to textbooks on complex geometry, for instance [Dem12; Ohs15]. In light of (3.2.18) and Remark 3.2.16, we define:

Definition 3.2.19. A Hermitian holomorphic vector bundle $E \to M$ is called *q-Nakano lower semibounded* if there is $c \in \mathbb{R}$ such that

$$\sum_{j,k=1}^{n} \langle (\varepsilon(\overline{w}^k) \circ \mathrm{ins}_{\overline{w}_j}) \otimes R^E(w_j, \overline{w}_k) u, u \rangle \geq c|u|^2$$

[5] However, Griffiths positivity of E implies Nakano positivity of $E \otimes \det(E)$, see [DS80] or [Dem12, Theorem VII.8.1].

holds for all $u \in \Lambda^{0,q}T^*_xM \otimes E_x$ and all $x \in M$. Equivalently,

$$\langle iR^E \wedge_{\mathrm{ev}} \Lambda u, u\rangle \geq c|u|^2$$

for all $u \in \Lambda^{n,q}T^*M \otimes E$, see Remark 3.1.20. The largest c for which this holds is denoted by $\mathrm{Nak}_q(E)$. If E is q-Nakano lower semibounded for all q (equivalently: for $q = 1$), then we call it simply *Nakano lower semibounded.*

Example 3.2.20. We always have

$$\begin{aligned}
&\sum_{j,k=1}^{n} \langle (\varepsilon(\overline{w}^k) \circ \mathrm{ins}_{\overline{w}_j}) \otimes R^E(w_j, \overline{w}_k)u, u\rangle \geq \\
&\qquad \geq -\sum_{j,k=1}^{n} |(\varepsilon(\overline{w}^k) \circ \mathrm{ins}_{\overline{w}_j}) \otimes R^E(w_j, \overline{w}_k)||u|^2 \\
&\qquad \geq -\sum_{j,k=1}^{n} |R^E(w_j, \overline{w}_k)||u|^2 && \text{by Lemma B.2.3} \\
&\qquad \geq -n\Big(\sum_{j,k=1}^{n} |R^E(w_j, \overline{w}_k)|^2\Big)^{1/2} |u|^2 && \text{by Hölder's inequality} \\
&\qquad \geq -n|R^E||u|^2.
\end{aligned}$$

Thus, if R^E is bounded (*i.e.*, the function $x \to |R^E|_x$ is bounded on M), then E is Nakano lower semibounded. In particular, if (M, J, g) is a Kähler manifold of 0-bounded geometry, see section 4.1, then TM (hence also $T^{1,0}M$) is a Nakano lower semibounded vector bundle. ♦

Example 3.2.21. If $L \to M$ is a Hermitian holomorphic line bundle, and $s_j\colon M \to \mathbb{R}$, $1 \leq j \leq n$ are as in Example 3.1.21, then L is q-Nakano lower semibounded if and only if $s_1 + \cdots + s_q \geq c$ for some $c \in \mathbb{R}$. ♦

Example 3.2.22. Using Example A.1.7, it is easy to see that if $E \to M$ and $F \to M$ are two q-Nakano lower semibounded vector bundles, then the tensor product $E \otimes F \to M$ is again q-Nakano lower semibounded, with $\mathrm{Nak}_q(E \otimes F) = \mathrm{Nak}_q(E) + \mathrm{Nak}_q(F)$. ♦

The assumptions (i) to (iii) of Lemma 3.2.10 and (iv) of Proposition 3.2.13 are satisfied if M' is a Kähler manifold of 1-bounded geometry. Riemannian manifolds of bounded geometry will be discussed in section 4.1, but we will state the corresponding result here:

Theorem 3.2.23. *Let $M \subseteq M'$ be a q-Levi pseudoconvex open subset of a Kähler manifold of* 1*-bounded geometry, with smooth boundary $\partial M \subseteq M'$, and let $E \to \overline{M}$ be a Hermitian holomorphic vector bundle such that $E|_M$ is q-Nakano lower semibounded. If $\Box^E_{p,q-1}$ has discrete spectrum, then so does $\Box^E_{p,q}$.*

Proof. From Lemma 4.1.11, we know that there are geodesic balls $\{B(x_k, r) : k \in \mathbb{N}\}$ that cover M' and with the properties required by Theorem 3.2.14. Intersecting these balls with $\overline{M}$, we obtain a cover of $\overline{M}$ with the same properties. Since M is q-Levi pseudoconvex, (3.2.19) is satisfied. By Examples 3.2.20 and 3.2.22, the bundle $E|_M \otimes \Lambda^{n-p,0}TM$ is q-Nakano lower semibounded, hence (3.2.18) also holds true. We conclude the proof by applying Theorem 3.2.14. ■

Remark 3.2.24. (i) If $M \subseteq M'$ is bounded (hence with compact closure, since M' is complete), then the curvature condition on $E|_M$ in Theorem 3.2.23 is of course vacuous.

(ii) If M is a (Levi) pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, and $E \to M$ is a Nakano semipositive vector bundle, then retracing the proof of Proposition 3.2.13, we find

$$\frac{1}{q}\sum_{j=1}^{n} Q^E(\mathrm{ins}_{X_j}(u), \mathrm{ins}_{X_j}(u)) \leq Q^E(u,u)$$

for all $u \in B_M^{p,q}(M,E)$, with $(X_j)_{j=1}^n$ some constant global orthonormal frame of $T^{0,1}M \cong M \times \mathbb{C}^n$, since all the terms involving estimates of the derivatives of X_j do not appear. Consequently, the Fredholmness of $\Box^E$ (*i.e.*, whether $\Box^E$ has a spectral gap) also percolates up the $\overline{\partial}^E$-complex in this case. This is included in the orginal result of Fu [Fu08, Proposition 2.2].

Using L^2 Serre duality, one immediately obtains a result similar to Theorem 3.2.23, valid for complete Kähler manifolds:

Corollary 3.2.25. *Let M be a (complete) Kähler manifold of* 1*-bounded geometry, and let $E \to M$ be a Hermitian holomorphic vector bundle such that E^* is $(n-q)$-Nakano lower semibounded. If $\Box^E_{p,q+1}$ has discrete spectrum, then so does $\Box^E_{p,q}$.*

Proof. By Theorem 3.2.8 and our assumption, $\Box^{E^*}_{n-p,n-q-1}$ has discrete spectrum. From Theorem 3.2.23, it follows that $\Box^{E^*}_{n-p,n-q}$ also has discrete spectrum, and applying Theorem 3.2.8 again, we see that $\Box^E_{p,q}$ also has this property. ■

CHAPTER 4

Applications of magnetic Schrödinger operator theory

Let (M, g) be a (oriented) Riemannian manifold, and let $E \to M$ be a Hermitian vector bundle. Then every connection ∇ on E and section $V \in \Gamma(M, \mathrm{End}(E))$ defines an elliptic differential operator

$$H_{\nabla,V} := \Delta^E + V : \Gamma(M, E) \to \Gamma(M, E),$$

called a *generalized Schrödinger operator*, where Δ^E is the Bochner Laplacian from Example 1.1.2. This is an operator of Laplace type and, conversely, any Laplace type operator is of this form, see section 1.1.2. In case E is a line bundle, ∇ is a metric connection, and V is self-adjoint, operators of the form $H_{\nabla V}$ are sometimes called *magnetic Schrödinger operators*. The reason for this terminology is that, if $M = \mathbb{R}^n$ and $E = M \times \mathbb{C} \to M$ for the moment, with sections of E being identified with complex valued functions on $\mathbb{R}^n$, then every metric connection on E is of the form $\nabla = d + ia$ for some real 1-form $a = \sum_{j=1}^n a_j \, dx_j$ on $\mathbb{R}^n$. Moreover, $\mathrm{End}(E)$ is trivial for any line bundle E, hence we may identify V with a function on M. Therefore, $H_{\nabla,V}$ is of the form

$$-\sum_{j=1}^{n} \left(\frac{d}{dx_j} + ia_j \right)^2 + V,$$

which is the quantum Hamiltonian of a particle moving in an electric field V and magnetic field $B := da$, the latter being the curvature of ∇. In this setting, a is called the *magnetic vector potential*. In this chapter, we will study spectral properties of $H_{\nabla,V}$ when E is a (possibly nontrivial) line bundle. Most of the results will need the base manifold M to have some form of bounded geometry, which is why we will study those manifolds in section 4.1. The theory will be applied, in section 4.3, to the Dolbeault Laplacian on complete Kähler manifolds (again with some bounded geometry assumptions) on top degree forms with values in a Hermitian holomorphic line bundle.

4.1. Riemannian manifolds of bounded geometry

Let (M, g) be a Riemannian manifold. In this section, we will only consider the case where M has no boundary. For $p \in M$, we denote by $\exp_p : \mathscr{D}_p \subseteq T_pM \to M$ the *(Riemannian) exponential map*, defined by $\exp_p(v) := \gamma_v(1)$, where γ_v is the unique geodesic starting at p and with initial velocity v, and $\mathscr{D}_p$ is the set of vectors for which this is possible, *i.e.*, those $v \in T_pM$ with the property that γ_v is defined at least on $[0, 1]$. Then $\mathscr{D}_p$ is open in T_pM, the exponential map $\exp_p$ is smooth, and in fact a diffeomorphism on a neighborhood of $0 \in T_pM$ by the inverse function theorem, since $T_0 \exp_p : T_0(T_pM) \cong T_pM \to T_pM$ is the identity map. The curves $\gamma(t) \coloneqq \exp_p(tv)$ are geodesics for $t \in [0,1]$ and $v \in \mathscr{D}_p$, with initial velocity $\dot\gamma(0) = v$. The *injectivity radius of* (M,g) *at a point* $p \in M$ is the supremum of all $r > 0$ such that $\exp_p$ restricts to a diffeomorphism on $B_{T_pM}(0,r)$, where $B_{T_pM}(0,r)$ is the open ball in (T_pM, g_p) around 0 and with radius r. The image of this ball under $\exp_p$ is then $B(p,r) \coloneqq \{q \in M : d_g(p,q) < r\}$, the open ball for the Riemannian distance from (1.4.1). The *injectivity radius of* (M,g), denoted by $r_{\mathrm{inj}}(M,g)$, is the infimum over all injectivity radii at points $p \in M$. For proofs of the above facts and more about the exponential map, see [Lee09, chapter 13].

Definition 4.1.1. A connected Riemannian manifold (M,g) is said to be of k*-bounded geometry* if its injectivity radius $r_{\mathrm{inj}}(M,g)$ is positive, and there exist constants $C_j > 0$ such that $|\nabla^j R^M| \le C_j$ for all $0 \le j \le k$, where $\nabla^j R^M$ is the j^{th} covariant derivative of the Riemannian curvature tensor of M, see section 1.3.1. If (M,g) is of k-bounded geometry for all $k \in \mathbb{N}$, then it is said to be of *bounded geometry*.

Remark 4.1.2. (i) All Riemannian manifolds of k-bounded geometry are complete due to the bound on the injectivity radius, see [Eic08, Proposition 2.2].

(ii) We want to point out that there is also a concept of bounded geometry for manifolds with boundary, see [Sch01].

(iii) There is also a notion of bounded geometry for vector bundles: a Hermitian (or Riemannian) vector bundle $E \to M$ with metric connection ∇ is called a *Hermitian (Riemannian) vector bundle of* k*-bounded geometry* if M is a Riemannian manifold of k-bounded geometry, and the curvature of ∇ satisfies $|\nabla^j R^\nabla| \le C_j$ for all $0 \le j \le k$, uniformly on M. Again, E is said to be of *bounded geometry* if this holds for all $k \in \mathbb{N}$. Most prominently, the tangent bundle as well as all tensor bundles of a manifold of bounded geometry (with the Levi–Civita connection) are Riemannian vector bundles of bounded geometry [Eld13, p. 45].

Manifolds of bounded geometry come with a nice cover by open subsets, namely the geodesic balls $B(p,r)$ for fixed $r < r_{\mathrm{inj}}(M,g)$ small enough, see Proposition 4.1.10 below, where it will also be shown that there are nice partitions of unity and local frames of the tangent bundle $TM \to M$ adapted to (a refinement of) this cover. Recall that any choice of orthonormal basis $\{e_j\}_{j=1}^n$ of T_pM, with $p \in M$ fixed, gives rise to a chart of M via

$$B(p,r) \to B_{\mathbb{R}^n}(0,r) \subseteq \mathbb{R}^n, \quad q \mapsto (\exp_p \circ \tau)^{-1}(q),$$

where $\tau\colon \mathbb{R}^n \to T_pM$ is the isometry $\tau(t_1,\dots,t_n) \coloneqq t_1e_1 + \cdots + t_ne_n$. These charts are called *(Riemannian) normal coordinates*. Lemma 4.1.6 will show that the *distortion* of normal coordinates can be uniformly bounded on a manifold of 0-bounded geometry. Since proofs of this seem to be hard to find, we shall provide one here. As a preparation, we need some prerequisites, including the Rauch comparison theorem, which we will discuss in section 4.1.1.

Sectional curvature. Let $p \in M$ and $X, Y \in T_pM$ be two linearly independent vectors. Then the quantity

$$K(\Pi) \coloneqq \frac{\langle R^M(X,Y)Y, X\rangle}{|X|^2|Y|^2 - \langle X, Y\rangle^2}$$

depends only on the two-dimensional subspace $\Pi \coloneqq \operatorname{span}(\{X, Y\})$ of T_pM, and is called the *sectional curvature* of M associated with Π. Thus, K can be viewed as a (smooth) function on the 2-Grassmannian bundle over M. One can show (see [Lee97, Proposition 8.8]) that $K(\Pi)$ is the Gaussian curvature[1] of the two-dimensional submanifold $S_\Pi \coloneqq \exp_p(\Pi \cap V) \subseteq M$ at $p \in S_\Pi$, where $V \subseteq T_pM$ is any neighborhood of 0 such that $\exp_p \colon V \to \exp_p(V)$ is a diffeomorphism. Note that S_Π is the set of points reached after unit time by geodesics emanating from p with initial velocities in $\Pi \cap V$. If $M \subseteq \mathbb{R}^3$ is a two-dimensional submanifold, then Gauss's *Theorema Egregium* states that $K(T_pM)$ is equal to the product of the principal curvatures (*i.e.*, the eigenvalues of the shape operator) of M at $p \in M$, see for instance [Lee97, section 8]. The sectional curvatures actually fully determine the Riemann curvature tensor, see [Lee97, Lemma 8.9].

Example 4.1.3. One can show that any complete, simply-connected Riemannian manifold with constant sectional curvature is isometric to one of these three model spaces, called *space forms*:

(i) The Euclidean space $\mathbb{R}^n$ has zero curvature, hence also constant sectional curvature $K = 0$.

(ii) The sphere $\partial B(0, R) \subseteq \mathbb{R}^n$ of radius $R > 0$ with its induced metric has constant sectional curvature $1/R^2$.

(iii) If $R > 0$, then the *hyperbolic space* $\mathbb{H}^n_R$ may be defined as taking the upper half-space $\{x \in \mathbb{R}^n : x_n > 0\}$ and equipping it with the metric

$$\frac{R^2}{x_n^2}\left((dx_1)^{\otimes 2} + \cdots + (dx_n)^{\otimes 2}\right).$$

It has constant sectional curvature $-1/R^2$.

For a proof, see [Lee97, Theorem 11.12]. ♦

Jacobi fields. Recall that a (smooth) *vector field along a curve* $\gamma \colon I \to M$, with $I \subseteq \mathbb{R}$ an interval, is a (smooth) map $X \colon I \to TM$ such that $X(t) \in T_{\gamma(t)}M$ for all $t \in I$. In other words, X defines a section of the pullback bundle $\gamma^*TM \to I$. A prime example of this is the derivative $\dot\gamma$ of the curve γ. The Levi–Civita connection on TM gives a connection $\gamma^*\nabla \colon \Gamma(I, \gamma^*TM) \to \Omega^1(I, \gamma^*TM)$, and on I we have the constant vector field $e \colon I \to TI \cong I \times \mathbb{R}$, $e(t) = (t, 1)$. Thus, we may define *covariant differentiation along* γ as

$$\Gamma(I, \gamma^*TM) \to \Gamma(I, \gamma^*TM), \quad X \mapsto X' \coloneqq (\gamma^*\nabla)_e X = \operatorname{ins}_e((\gamma^*\nabla)X).$$

[1] The *Gaussian curvature* of a Riemannian 2-manifold S at $x \in S$ is defined as $\langle R^S(\xi, \eta)\eta, \xi\rangle/(|\xi|^2|\eta|^2 - \langle \xi, \eta\rangle^2)$ for any basis $\{\xi, \eta\}$ of T_xS, see [Lee97, p. 144].

This construction satisfies the Leibniz rule

$$(fX)' = f'X + fX'$$

for all $f \in C^\infty(I, \mathbb{R})$, and is metric compatible in the sense that

$$\langle X, Y\rangle' = \langle X', Y\rangle + \langle X, Y'\rangle$$

for all vector fields X, Y along γ, where f' and $\langle X, Y\rangle'$ are just the usual derivatives of functions from I to $\mathbb{R}$. By definition, a curve $\gamma\colon I \to M$ is geodesic if and only if $(\dot\gamma)' = 0$. The above also allows us to define second (and higher) covariant derivatives of X along γ. If $X' = 0$, then X is called *parallel*. The length of any parallel field is constant, since $\frac{d}{dt}|X(t)|^2 = 2\langle X', X\rangle = 0$.

Similarly, one may define sections of a vector bundle $E \to M$ along γ, and if a connection on E is chosen, one also obtains a derivative operator $\Gamma(I, \gamma^*E) \to \Gamma(I, \gamma^*E)$, $\sigma \mapsto \sigma'$, see [Lee09, section 12.3].

A vector field J along a *geodesic* γ is said to be a *Jacobi field* if

$$J'' + R^M(J, \dot\gamma)\dot\gamma = 0, \tag{4.1.1}$$

where $R^M \in \Omega^2(M, \mathrm{End}(TM))$ is the Riemann curvature tensor of M. The *Jacobi equation* (4.1.1) may be solved uniquely if appropriate initial data is given: for every $X_0, Y_0 \in T_{\gamma(t_0)}M$, there is a unique Jacobi field J along γ such that $J(t_0) = X_0$ and $J'(t_0) = Y_0$, see [Lee97, Proposition 10.4]. Jacobi fields are related to variations of geodesics. To illustrate some of this, suppose that $\gamma(t) = \exp_p(tv)$ is a radial geodesic, and consider the map $\Gamma(s, t) := \exp_p(t(v + sY_0))$, defined for $t \in [0, 1]$ and $|s|$ small enough. Put

$$J(t) := \tfrac{d}{ds}\Gamma(s, t)|_{s=0} = t(T_{tv}\exp_p)(Y_0). \tag{4.1.2}$$

Then Γ is what is called a *variation through geodesics*, and by general considerations, see [Lee97, Theorem 10.2], J is a Jacobi field along γ, and we have $J(0) = 0$ as well as $J'(0) = (T_0 \exp_p)(Y_0) = Y_0$.

A vector field X along a curve γ is called *tangential* if $X(t)$ is a multiple of $\dot\gamma(t)$ for all $t \in I$, and *normal* if $\langle X(t), \dot\gamma(t)\rangle = 0$ for all $t \in I$. If J is a tangential Jacobi field along a geodesic γ, then $J'' = 0$ by (4.1.1), hence $J(t) = (at + b)\dot\gamma(t)$ with some $a, b \in \mathbb{R}$ for all $t \in I$ by the uniqueness of Jacobi fields. Regarding normal Jacobi fields, one has the following result for manifolds of constant sectional curvature:

Lemma 4.1.4. *If M has constant sectional curvature $C \in \mathbb{R}$, and $\gamma\colon I \to M$ with $0 \in I$ is a unit speed geodesic, then the normal Jacobi fields along γ with $J(0) = 0$ are given by $J(t) = u(t)E(t)$, where E is a normal vector field along γ with $E' = 0$, and*

$$u(t) := \begin{cases} t, & C = 0 \\ \frac{1}{\sqrt{C}}\sin\left(\sqrt{C}t\right), & C > 0 \\ \frac{1}{\sqrt{-C}}\sinh\left(\sqrt{-C}t\right), & C < 0. \end{cases}$$

Proof. See [Lee97, Lemma 10.8]. The proof uses the fact that on manifolds with constant sectional curvature $C \in \mathbb{R}$, the Riemann curvature tensor has the simple form

$$R^M(X,Y)Z = C(\langle Y,Z\rangle X - \langle X,Z\rangle Y),$$

so that the Jacobi equation for a normal field along γ becomes $J'' + CJ = 0$. ■

Conjugate points. If $\gamma\colon [a,b] \to M$ is a geodesic in M, then $q := \gamma(b)$ is said to be *conjugate to* $p := \gamma(a)$ *along* γ if there is a Jacobi field J along γ with $J(a) = 0 = J(b)$ but $J \neq 0$. Conjugate points describe the failure of the Riemannian exponential map to be a diffeomorphism: if $p \in M$, $v \in \mathscr{D}_p$, and $q := \exp_p(v)$, then $\exp_p$ is a local diffeomorphism around v if and only if q is not conjugate to p along the geodesic $\gamma(t) := \exp_p(tv)$, $t \in [0,1]$, see [Lee97, Proposition 10.11]. On a sphere S in $\mathbb{R}^n$ of radius R, the exponential map is a diffeomorphism on $B_{T_pS}(0,\pi R)$ for any $p \in S$, hence geodesics with length less than πR have no conjugate points.

4.1.1. The Rauch comparison theorem. We are now ready to formulate Rauch's comparison theorem:

Theorem 4.1.5 (Rauch comparison theorem)**.** *Let M and N be Riemannian manifolds of the same dimension, $\gamma\colon [0,T] \to M$ and $\sigma\colon [0,T] \to N$ be unit speed geodesics, and J and W be Jacobi fields along γ and σ, respectively. Assume that*

(i) $J(0) = 0$ *and* $W(0) = 0$,

(ii) $|J'(0)| = |W'(0)|$,

(iii) $\langle \dot\gamma(0), J'(0)\rangle = \langle \dot\sigma(0), W'(0)\rangle$,

(iv) γ *has no conjugate points on* $[0,T]$, *and*

(v) for all $t \in [0,T]$ and any two-dimensional subspaces $\Pi^M_{\gamma(t)} \subseteq T_{\gamma(t)}M$ and $\Pi^N_{\sigma(t)} \subseteq T_{\sigma(t)}N$ containing $\dot\gamma(t)$ and $\dot\sigma(t)$, respectively, we have

$$K(\Pi^M_{\gamma(t)}) \geq K(\Pi^N_{\sigma(t)}).$$

Then σ has no conjugate points on $[0,T]$, and $|J(t)| \leq |W(t)|$ for all $t \in [0,T]$.

Proof. See [Car92, Theorem 2.3]. The statement of the theorem can also be found in [Lee97, Theorem 11.9], with the small difference that it is stated there only for normal Jacobi fields. But if J and W have tangential parts $J_{\|}$ and $W_{\|}$, respectively, then $J_{\|}(t) = (at+b)\dot\gamma(t)$ and $W_{\|}(t) = (ct+d)\dot\sigma(t)$. By the initial condition (i), $b = d = 0$, and by (iii) we have $a = c$, so that $|J_{\|}(t)| = |W_{\|}(t)|$ for all $t \in [0,T]$. Therefore, it suffices to only consider normal fields. Of course, (iii) is vacuous in this case, since $0 = \langle \dot\gamma, J\rangle' = \langle \dot\gamma, J'\rangle$ for normal J, and similarly for W. ■

Our application of Theorem 4.1.5 is to obtain uniform two-sided bounds on the derivative of the Riemannian exponential map, given global bounds on the sectional curvature. An explicit statement of the following Lemma can be found in [Roe88, Lemma 2.2]. As a proof seems to be hard to find, we shall provide it here.

Lemma 4.1.6. *Let (M, g) be a Riemannian manifold with positive injectivity radius and sectional curvatures uniformly bounded from above and below,*

$$-C \leq K(\Pi) \leq C \tag{4.1.3}$$

with $C > 0$ and for all two-dimensional subspaces $\Pi \subseteq T_pM$ and every $p \in M$. Then there exist $0 < r < r_{\text{inj}}(M, g)$ and $C_1, C_2 > 0$ such that

$$\frac{1}{C_2}|X| \leq |(T_v \exp_p)X| \leq C_1|X| \tag{4.1.4}$$

for all $p \in M$, all $0 \neq v \in B_{T_pM}(0, r)$, and all $X \in T_pM$.

Proof. The proof will work by comparing M to the spaces with constant sectional curvatures $\pm C$. We take

$$0 < r < \min\left\{r_{\text{inj}}(M, g), \pi/(2\sqrt{C})\right\}, \tag{4.1.5}$$

the reason for which will become apparent in the proof. Let J be the Jacobi field along the unit speed geodesic $\gamma\colon [0, r] \to M$, $\gamma(t) := \exp_p(tv/|v|)$, such that $J(0) = 0$ and $J'(0) = X$. By (4.1.2), we have $J(t) = t(T_{tv/|v|} \exp_p)(X)$. Because $r < r_{\text{inj}}(M, g)$, the geodesic γ does not have any conjugate points.

We first show the upper bound in (4.1.4). Consider the hyperbolic space $N := \mathbb{H}^n_{1/\sqrt{C}}$, with n the dimension of M. By Example 4.1.3, N has constant sectional curvature $-C$. Let $\sigma\colon [0, r] \to N$ be any unit speed geodesic, and W a Jacobi field along σ with $W(0) = 0$ and $W'(0)$ chosen such that $|W'(0)| = |X|$ and $\langle \dot\sigma(0), W'(0)\rangle = \langle v, X\rangle$. Then J and W satisfy the requirements of Theorem 4.1.5, hence

$$t|(T_{tv/|v|} \exp_p)X| \leq |W(t)|$$

for $t \in [0, r]$. Write $W = W_\parallel + W_\perp$, with $W_\parallel$ a tangential field along σ, and $W_\perp$ normal. Then $W_\parallel(t) = (at + b)\dot\sigma(t)$ and $W_\perp(t) = \frac{1}{\sqrt{C}} \sinh(\sqrt{C}t)E(t)$ with $a, b \in \mathbb{R}$ and E a normal field along σ satisfying $E' = 0$, see Lemma 4.1.4. We have $W_\parallel(0) = 0$, hence $b = 0$ since $\dot\sigma(0) \neq 0$ (geodesics can't change their speed), and $W_\parallel'(0) = a\dot\sigma(0)$ as well as $W_\perp'(0) = E(0)$. Therefore,

$$|X|^2 = |W'(0)|^2 = |W_\parallel'(0)|^2 + |W_\perp'(0)|^2 = a^2 + |E(0)|^2.$$

In particular, $|a| \leq |X|$ and $|E(0)| \leq |X|$, hence also $|E(t)| \leq |X|$ because E is parallel. Combining this, we arrive at

$$t^2|(T_{tv/|v|} \exp_p)X|^2 \leq |W_\parallel(t)|^2 + |W_\perp(t)|^2 \leq t^2|X|^2 + \frac{1}{C}\sinh^2(\sqrt{C}t)|X|^2.$$

Dividing by t^2, we find

$$|(T_{tv/|v|} \exp_p)X|^2 \leq \left(1 + \frac{\sinh^2(\sqrt{C}t)}{(\sqrt{C}t)^2}\right)|X|^2$$

for $t \in (0, r]$. Plugging in $t = |v|$, and using that $s \mapsto \sinh^2(s)/s^2$ is increasing on $[0, \infty)$, we see that

$$|(T_v \exp_p)X| \leq \left(1 + \frac{1}{Cr^2}\sinh^2(\sqrt{C}r)\right)^{1/2}|X|.$$

To prove the lower bound in (4.1.4), we consider the sphere $N := \partial B(0, 1/\sqrt{C}) \subseteq \mathbb{R}^{n+1}$ instead, with constant sectional curvature C. As before, we choose a unit speed geodesic $\sigma\colon [0, r] \to N$ and a Jacobi field W along σ with $W(0) = 0$ and such that $|W'(0)| = |X|$ and $\langle \dot\sigma(0), W'(0)\rangle = \langle v, X\rangle$. Note that σ has no conjugate points, hence Theorem 4.1.5 is applicable (the roles of J and W now being interchanged!) and yields

$$|W(t)| \le t|(T_{tv/|v|} \exp_p)X|$$

for $t \in [0, r]$. As before, and using Lemma 4.1.4, we have

$$W_\|(t) = at\dot\sigma(t) \quad \text{and} \quad W_\perp(t) = \frac{1}{\sqrt{C}} \sin(\sqrt{C}t)E(t),$$

with E a parallel normal field along σ, and $|X|^2 = a^2 + |E(t)|^2$. Therefore,

$$t^2|(T_{tv/|v|} \exp_p)X|^2 \ge |W_\|(t)|^2 + |W_\perp(t)|^2 = a^2t^2 + \frac{1}{C} \sin^2 (\sqrt{C}t)|E(t)|^2$$

holds for $t \in [0, r]$. Dividing by t^2, we arrive at

$$|(T_{tv/|v|} \exp_p)X|^2 \ge a^2 + \left(\frac{\sin (\sqrt{C}t)}{\sqrt{C}t}\right)^2 |E(t)|^2$$

for $t \in (0, r] \subseteq (0, \pi/(2\sqrt{C})]$. On this interval, we have $\sin(\sqrt{C}t)/(\sqrt{C}t) \ge \sin(\pi/2)/(\pi/2) = 2/\pi$, hence

$$|(T_{tv/|v|} \exp_p)X| \ge \frac{2}{\pi}|X|.$$

It remains to plug in $t = |v|$. ■

Remark 4.1.7. In Lemma 4.1.6, we didn't *really* need the injectivity radius to be positive, in the sense that if (M, g) is a Riemannian manifold (without boundary) that satisfies the sectional curvature bound (4.1.3), then the proof shows that there are constants $C_1, C_2 > 0$ such that (4.1.4) holds for all $p \in M$, $r > 0$ sufficiently small, $v \in B_{T_pM}(0, r)$, and $X \in T_pM$. Here, r sufficiently small means that r be less than the injectivity radius of (M, g) at p, and satisfies $r < \pi/(2\sqrt{C})$, see (4.1.5). The point is that the constants are still independent of p, although of course the radius r for which it even makes sense to talk about normal coordinates on a ball with that radius around p will vary with p.

4.1.2. Properties of manifolds of bounded geometry. We will now state the properties of manifolds of bounded geometry that we will use in the sequel. These concern the existence of bump functions with uniform properties, as well as covers by geodesic balls and associated partitions of unities, also enjoying uniform estimates.

Remark 4.1.8. (i) Using Lemma 4.1.6 and Remark 4.1.7, it is easy to see that if (M, g) has uniformly bounded sectional curvature, then the coefficients g^p_{ij} of the metric in normal coordinates $\varphi_p := (\exp_p \circ \tau)^{-1}|_{B(0,r)}$ around a sufficiently small ball $B(p, r)$ are bounded from above and below, independent of p. Indeed, for $y \in B_{\mathbb{R}^n}(0, r)$, the coefficients $g^p_{ij}(y)$ are just the components of the bilinear form $(\tau^* \exp_p^* g)(y)$ on $\mathbb{R}^n$ with respect to the standard basis of $\mathbb{R}^n$. We have, with $v, w \in \mathbb{R}^n$,

$$\begin{aligned}|(\exp_p^* g)(y)(v,w)| = |g_{\exp_p(y)}(T_y \exp_p(v), T_y \exp_p(w))| \leq \\ \leq |g_{\exp_p(y)}||T_y \exp_p|^2|v||w| = \sqrt{n}|T_y \exp_p|^2|v||w|,\end{aligned}$$

where n is the dimension of M, where we have used that $|g_{\exp_p(y)}|^2 = \sum_{j,k=1}^{\dim(M)} |g(e_j, e_k)|^2 = \dim(M)$, with $\{e_j\}_{j=1}^{\dim(M)}$ an orthonormal basis of $T_{\exp_p(y)}M$. Therefore, $|(\tau^* \exp_p^* g)(y)| \leq \sqrt{n}|T_y \exp_p|^2$. Similarly, we have the lower bound

$$|(\exp_p^* g)(y)| \geq \frac{\sqrt{n}}{|(T_y \exp_p)^{-1}|^2}.$$

(ii) It is harder to argue that this also holds for derivatives of the metric coefficients: in [Kau76], it was shown that if $|R^M| \leq C_0$ and $|\nabla R^M| \leq C_1$, then also the Christoffel symbols with respect to normal coordinates (of sufficiently small radius) are bounded, uniformly in $p \in M$. Equivalently, the derivatives of the metric coefficients in such coordinates are also uniformly bounded. This was extended to arbitrary derivatives by Eichhorn in [Eic91, Corollary 2.6]: if (M, g) is open and complete and satisfies $|\nabla^j R^M| \leq C_j$ for $0 \leq j \leq k$, then the derivatives of order up to k of the metric coefficients in normal coordinates around $p \in M$, and with sufficiently small radius r, are also bounded, uniformly in p.

(iii) There is also a corresponding result for vector bundles, see [Eic91, Theorem 3.2]. Assume that (M, g) is of k-bounded geometry, and that $E \to M$ is a Hermitian vector bundle, equipped with a metric connection. Suppose moreover that E is of k-bounded geometry too, in the sense of Remark 4.1.2. Then there is $r > 0$ and constants $\widetilde{C}_\gamma > 0$ such that

$$|\partial^\gamma \Gamma_{i\beta}^\alpha| \leq \widetilde{C}_\gamma \tag{4.1.6}$$

for all multiindices $|\gamma| \leq k-1$, all $1 \leq \alpha, \beta \leq \operatorname{rank}(E)$, and all $1 \leq i \leq \dim(M)$. Here, $\Gamma_{i\beta}^\alpha$ are the connection coefficients of the connection on E with respect to a *synchronous framing*, *i.e.*, with respect to an orthonormal frame $(\xi_1^p, \ldots, \xi_N^p)$ of $E|_{B(p,r)}$ obtained by parallel transporting an orthonormal basis of E_p along the radial geodesics in $B(p, r)$. Thus, $\sum_{\beta=1}^n \Gamma_{i\beta}^\alpha \xi_\alpha^p = \nabla_{\partial_i}^E \xi_\beta^p$ or, equivalently, $\Gamma_{i\beta}^\alpha = \langle \nabla_{\partial_i}^E \xi_\beta^p, \xi_\alpha^p \rangle$, and the point is that the estimates (4.1.6) are again uniform in $p \in M$.

Lemma 4.1.9. *Let (M, g) be a Riemannian manifold of 0-bounded geometry. There exists $r \in (0, r_{\mathrm{inj}}(M, g))$ and a constant $C > 0$ with the following property: for all $p \in M$, there exists a smooth function $f_p \colon M \to [0, 1]$ such that*

(i) $\operatorname{supp}(f_p) \subseteq B(p, r)$,
(ii) $\|df_p\|_{L^\infty(M, T^*M)} \leq C$, *and*
(iii) $\int_M |f_p|^2 \, d\mu_g \geq 1/C$.

Proof. Take $r \in (0, r_{\mathrm{inj}}(M, g))$ small enough such that the conclusion of Lemma 4.1.6 holds, and such that the coefficients of the metric in normal coordinates on $B(p, r)$ are uniformly bounded, independent of p, see Remark 4.1.8. Let $f \in C_c^\infty(B_{\mathbb{R}^n}(0,r),[0,1])$ be any nonzero function, and put $f_p \coloneqq f \circ \varphi_p^{-1} \colon B(p,r) \to [0,1]$, where $\varphi_p \coloneqq (\exp_p \circ \tau)|_{B_{\mathbb{R}^n}(0,r)}$ and $\tau \colon \mathbb{R}^n \to T_pM$ is an isometry such that φ_p is orientation preserving. Then f_p has compact support in $B(p,r)$, and we extend it by zero to all of M. For $X \in T_xM$, we have

$$|df_p(X)| = |(T_x f_p)X| = |T_{\varphi_p^{-1}(x)} f \circ T_x \varphi_p^{-1} X| \leq |df(\varphi_p^{-1}(x))| |(T_x(\exp_p^{-1}))X| \leq C_2 \|df\|_{L^\infty} |X|$$

by Lemma 4.1.6, hence $\|df_p\|_{L^\infty} \leq C_2 \|df\|_{L^\infty}$. Moreover,

$$\begin{aligned} \int_M |f_p|^2 \operatorname{vol}_g &= \int_{B_{\mathbb{R}^n}(0,r)} (|f_p|^2 \circ \varphi_p)\, \varphi_p^* \operatorname{vol}_g \\ &= \int_{B_{\mathbb{R}^n}(0,r)} |f(y)|^2 \det(g_{ij}^p(y))^{1/2}\, d\lambda(y) \geq \widetilde{C} \|f\|^2_{L^2(B_{\mathbb{R}^n}(0,r))} \end{aligned}$$

independent of p, with λ the Lebesgue measure, and where g_{ij}^p are the metric coefficients with respect to the normal coordinate chart φ_p, and the constant $\widetilde{C}$ is a lower bound on $\det(g_{ij}^p)^{1/2}$, *cf.*, Remark 4.1.8. ■

Recall that if $E \to M$ is a vector bundle with connection ∇ and $\gamma \colon [a,b] \to M$ is a smooth curve, then *parallel transport along* γ is defined as the linear map $P_\gamma \colon E_{\gamma(a)} \to E_{\gamma(b)}$ given by $P_\gamma(u) \coloneqq \sigma_{\gamma,u}(b)$, where $\sigma_{\gamma,u}$ is the unique parallel (in the sense that $(\sigma_{\gamma,u})' = 0$) section of E along γ such that $\sigma_{\gamma,u}(a) = u$. Then P_c is a linear isomorphism and if E is equipped with a Hermitian metric and ∇ is a metric connection, then P_γ will be an isometry. Note that P_γ commutes with parallel endomorphisms of E, *i.e.*, with those endomorphisms $A \in \Gamma(M, \operatorname{End}(E))$ satisfying $\nabla_X(As) = A(\nabla_X s)$ for all $X \in \Gamma(M, TM)$ and $s \in \Gamma(M,E)$.

Proposition 4.1.10. *Let (M,g) be a noncompact manifold of 1-bounded geometry. Then there exists $r_0 \in (0, r_{\mathrm{inj}}(M,g))$ such that for all $0 < r < r_0$ there is*

*(i) a countable cover $\{B(p_k,r)\}_{k \geq 1}$ of M by geodesic balls, and a number $N > 0$ such that $\bigcap_{k\in J} B(p_k,r) \neq \emptyset$ implies $|J| \leq N$ for all subsets $J \subseteq \mathbb{N}$ (*i.e.*, the cover has uniformly finite intersection multiplicity),*

(ii) a sequence of functions $\varphi_k \in C^\infty(M,[0,1])$ such that $\operatorname{supp}(\varphi_k) \subseteq B(p_k,r)$, $\sum_{k=1}^\infty \varphi_k^2 = 1$, and with $\sup_{k\in\mathbb{N}} \|d\varphi_k\|_{L^\infty} < \infty$, and

(iii) for every $k \in \mathbb{N}$, an orthonormal frame $(\xi_1^k, \dots, \xi_n^k)$ of $TM|_{B(p_k,r)}$ with

$$\sup_{k,j} \sup_{x \in B(p_k,r)} |\nabla \xi_j^k|_x < \infty.$$

Proof. For (i) and (ii), see [Eld13, Lemma 2.16 and Corollary 2.18], [Shu92, Lemma 1.2 and Lemma 1.3], or [Kaa13, Lemma 2.4]. Pick an orthonormal basis $(e_1^k, \dots, e_n^k)$ of $T_{p_k}M$, and denote by $(\xi_1^k, \dots, \xi_n^k)$ the frame of $TM|_{B(p_k,r)}$ that is obtained by parallel transporting the basis of $T_{p_k}M$ along the radial geodesics in $B(p_k,r)$. In other words, $\xi_j^k(x) = P_{\gamma_{k,x}}(e_j^k)$, with $\gamma_{k,x}\colon [0,1] \to M$, $t \mapsto \exp_{p_k}(t \exp_{p_k}^{-1}(x))$. Then

$$|\nabla \xi_\alpha^k|_x = \sup_{|X|=1} |\nabla_X \xi_\alpha^k|_x \leq \sup_{|X|=1} \sum_{i=1}^n |X^i||\nabla_{\partial_i} \xi_\alpha^k|_x \leq \\ \leq \sup_{|X|=1} \sum_{i,\beta} |X^i||\Gamma_{i\beta}^\alpha(x)\xi_\beta^k|_x \leq \sup_{|X|=1} \sum_{i,\beta} |X^i||\Gamma_{i\beta}^\alpha(x)|, \quad (4.1.7)$$

where $\Gamma_{i\beta}^\alpha$ are the Christoffel symbols corresponding to the trivialization of $TM|_{B(p_k,r)}$ induced by the frame $(\xi_1^k, \dots, \xi_n^k)$ and the normal coordinates, and $X = X^i\partial_i$ with ∂_i the normal coordinate vector fields. By the discussion about bundles of bounded geometry in Remark 4.1.8, $|\Gamma_{i\beta}^\alpha(x)|$ is bounded by constants uniform in $x \in B(p_k, r)$, $k \in \mathbb{N}$, and $\alpha \in \{1, \dots, n\}$. Let $|\bullet|_e$ denote the Euclidian norm on $\mathbb{R}^n$. If $|X| = 1$, then $|g(x)^{1/2}X|_e = 1$, where we view $g(x)$ as the symmetric matrix $(g_{ij}(x))_{i,j}$ (components in normal coordinates on $B(p_k, r)$), and X as the vector $(X^1, \dots, X^n)$. It follows that

$$|X^i| \leq |X|_e = \left|g(x)^{-1/2}g(x)^{1/2}X\right|_e \leq \|g(x)^{-1/2}\|_{\mathscr{L}(\mathbb{R}^n)}\left|g(x)^{1/2}X\right|_e = \|g(x)^{-1/2}\|_{\mathscr{L}(\mathbb{R}^n)} \quad (4.1.8)$$

for $1 \leq i \leq n$, where $\|\bullet\|_{\mathscr{L}(\mathbb{R}^n)}$ is the operator norm. If $|g^{ij}| \leq C_0$ on $B(p_k, r)$ as in Remark 4.1.8, then $\|g(x)^{-1}\|_{\mathscr{L}(\mathbb{R}^n)} \leq \operatorname{tr}(g(x)^{-1}) \leq nC_0$, and hence $\|g(x)^{-1/2}\|_{\mathscr{L}(\mathbb{R}^n)} \leq \sqrt{nC_0}$, uniformly in $x \in B(p_k, r)$, and not depending on k and r. Combining this with (4.1.7) and (4.1.8) finishes the proof. ■

Since Kähler manifolds are also Riemannian manifolds, we may consider Kähler manifolds of bounded geometry. The next result is just a simple adaptation of Proposition 4.1.10 to this case:

Lemma 4.1.11. *Let M be a Kähler manifold of 1-bounded geometry and complex dimension n, and let $\{B(p_k, r)\}_{k\geq 1}$ be a cover of M as in Proposition 4.1.10. Then for every $k \in \mathbb{N}$ there exists an orthonormal frame $(X_1^k, \dots, X_n^k)$ of $T^{1,0}M|_{B(p_k,r)}$ with*

$$\sup_{k,j} \sup_{x\in B(p_k,r)} |\nabla X_j^k|_x < \infty.$$

Moreover, $(\overline{X}_1^k, \dots, \overline{X}_n^k)$ is an orthonormal frame of $T^{0,1}M|_{B(p_k,r)}$ with the same boundedness property.

Proof. Choose an orthonormal basis $(w_1^k, \dots, w_n^k)$ of $T_{p_k}^{1,0}M$. Then $(e_m^k)_{m=1}^{2n}$ from (B.2.3) is an orthonormal basis of $T_{p_k}M$, which we extend to an orthonormal frame $(\xi_1^k, \dots, \xi_{2n}^k)$ of $TM|_{B(p_k,r)}$ as in Proposition 4.1.10. Since M is Kähler, the complex structure J is parallel for the Levi–Civita connection, see Theorem B.2.1. If $x \in B(p_k, r)$ and γ denotes the radial geodesic from p_k to x, then $\xi_m^k = P_\gamma(e_m^k)$, and

$$J(\xi_{2j-1}^k(x) - i\xi_{2j}^k(x)) = JP_\gamma(e_{2j-1}^k - ie_{2j}^k) = P_\gamma J(e_{2j-1}^k - ie_{2j}^k) = \\ = P_\gamma(e_{2j}^k + ie_{2j-1}^k) = i(\xi_{2j-1}^k(x) - i\xi_{2j}^k(x)),$$

since the parallel transport commutes with the parallel endomorphism J. Therefore,

$$X_j^k := \frac{1}{\sqrt{2}}(\xi_{2j-1}^k - i\xi_{2j}^k)$$

defines an orthonormal frame of $T^{1,0}M$ over $B(p_k, r)$, and with the required properties. The claim about $(\overline{X}_1^k, \dots, \overline{X}_n^k)$ is immediate. ■

4.2. Schrödinger operators on line bundles over manifolds of bounded geometry

Let (M, g) be a Riemannian manifold, and let $E \to M$ be a Hermitian vector bundle. Let ∇ be a metric connection on E and $V\colon E \to E$ a self-adjoint bundle endomorphism. We consider the generalized Schrödinger operator

$$H_{\nabla,V} := \nabla^\dagger \nabla + V,$$

and we will always make the assumption that $H_{\nabla,V}$ is lower semibounded. In case M is complete and without boundary, this implies that $H_{\nabla,V}$ is essentially self-adjoint, see Theorem 1.4.16. For $U \subseteq M$ an open subset, define

$$\mathcal{E}_{\nabla,V}(U) := \inf\left\{\frac{\langle\!\langle H_{\nabla,V}s, s\rangle\!\rangle}{\|s\|^2} : s \in \Gamma_c(M,E) \setminus \{0\} \text{ with } \operatorname{supp}(s) \subseteq U\right\}. \tag{4.2.1}$$

Then $\mathcal{E}_{\nabla,V}(U)$ is equal to $\inf \sigma((H_{\nabla|_U,V|_U})_F)$, the bottom of the spectrum of the Friedrichs extension of $H_{\nabla|_U,V|_U}\colon \Gamma_c(U,E) \to \Gamma_c(U,E)$, see Example C.2.2.

Remark 4.2.1. (i) Suppose that $A\colon \operatorname{dom}(A) \subseteq L^2(M,E) \to L^2(M,E)$ is a lower semibounded self-adjoint extension of $H_{\nabla,V}$. Then $\mathcal{E}_{\nabla,V}(U) \geq \inf \sigma(A_U)$ for every open subset U of M, where A_U is defined in Definition 2.1.2. From Theorem 2.2.8, we obtain

$$\lim_K \mathcal{E}_{\nabla,V}(M \setminus K) \geq \inf \sigma_e(A). \tag{4.2.2}$$

(ii) Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of M with $U_n \to \infty$ as $n \to \infty$, meaning that for all compact $K \subseteq M$ there is $n_0 \in \mathbb{N}$ such that $U_n \subseteq M \setminus K$ for all $n \geq n_0$. Then

$$\liminf_{n\to\infty} \mathcal{E}_{\nabla,V}(U_n) \geq \inf \sigma_e(A). \tag{4.2.3}$$

Indeed, let λ be an accumulation point of $n \mapsto \mathcal{E}_{\nabla,V}(U_n)$, with $\lim_{k\to\infty} \mathcal{E}_{\nabla,V}(U_{n_k}) = \lambda$ for some subsequence $k \mapsto U_{n_k}$. Without loss of generality, we can assume that $U_{n_k} \subseteq M \setminus K_k$, where $(K_k)_{k\in\mathbb{N}}$ is an exhaustion of M by compact subsets. It follows from (4.2.2) that

$$\lambda = \lim_{k\to\infty} \mathcal{E}_{\nabla,V}(U_{n_k}) \geq \lim_{k\to\infty} \mathcal{E}_{\nabla,V}(M \setminus K_k) = \lim_K \mathcal{E}_{\nabla,V}(M \setminus K) \geq \inf \sigma_e(A).$$

The following result and its proof are motivated by [Iwa86, Main Theorem] (see also [Shu99, Theorem 6.10]):

Lemma 4.2.2. *Let M be a Riemannian manifold of 1-bounded geometry, $E \to M$ a Hermitian vector bundle, ∇ a connection on E, and V a self-adjoint bundle endomorphism of E. Assume that $H_{\nabla,V}$ is lower semibounded and essentially self-adjoint on $\Gamma_c(M,E)$. Then the following are equivalent:*

(i) The closure of $H_{\nabla,V}$ *has discrete spectrum.*
(ii) $\lim_{k\to\infty} \mathcal{E}_{\nabla,V}(B(x_k,r)) = \infty$ *for all sequences* $x_k \in M$ *with* $x_k \to \infty$ *as* $k \to \infty$ *and all* $r > 0$ *small enough.*

Proof. If the spectrum of $\overline{H_{\nabla,V}}$ is discrete, then clearly condition (ii) holds, see (4.2.3). Conversely, suppose that (ii) is true. We show that there is a proper smooth function $\psi\colon M \to [C,\infty)$, where $C \in \mathbb{R}$ will be determined later, such that $\langle\!\langle H_{\nabla,V}s, s\rangle\!\rangle \geq \int_M \psi |s|^2\, d\mu_g$ for all $s \in \Gamma_c(M,E)$, from which the claim follows by using Theorem 2.2.9 and essential self-adjointness of $H_{\nabla,V}$.

If M is compact, there is nothing to show due to (4.2.2), so we may assume that M is noncompact. Let $\{B(x_k,r)\}_{k\geq 1}$ be a countable cover of M by geodesic balls as in Proposition 4.1.10, with associated functions $\varphi_k \in C^\infty(M,[0,1])$. Then $x_k \to \infty$ as $k \to \infty$, for if a subsequence would stay in a compact subset of M, it would have a limit point in M, contradicting the fact that this cover has uniformly finite intersection multiplicity. For $s \in \Gamma_c(M,E)$ we have the *localization formula*

$$\langle\!\langle H_{\nabla,V}s, s\rangle\!\rangle = \sum_{k=1}^{\infty} \big(\langle\!\langle H_{\nabla,V}(\varphi_k s), \varphi_k s\rangle\!\rangle - \|d\varphi_k \otimes s\|^2\big),$$

which follows from

$$\begin{aligned}
\langle\!\langle \nabla^\dagger\nabla s, s\rangle\!\rangle &= \sum_{k=1}^{\infty} \operatorname{Re}\langle\!\langle \nabla s, \nabla(\varphi_k^2 s)\rangle\!\rangle \\
&= \sum_{k=1}^{\infty} \operatorname{Re}\langle\!\langle \nabla s, d\varphi_k \otimes (\varphi_k s) + \varphi_k \nabla(\varphi_k s)\rangle\!\rangle \\
&= \sum_{k=1}^{\infty} \operatorname{Re}\langle\!\langle \varphi_k \nabla s, d\varphi_k \otimes s + \nabla(\varphi_k s)\rangle\!\rangle \\
&= \sum_{k=1}^{\infty} \operatorname{Re}\langle\!\langle \nabla(\varphi_k s) - d\varphi_k \otimes s, d\varphi_k \otimes s + \nabla(\varphi_k s)\rangle\!\rangle \\
&= \sum_{k=1}^{\infty} \big(\langle\!\langle \nabla^\dagger\nabla(\varphi_k s), \varphi_k s\rangle\!\rangle - \|d\varphi_k \otimes s\|^2\big).
\end{aligned}$$

Since $\operatorname{supp}(\varphi_k s) \subseteq B(x_k,r)$, we have $\langle\!\langle H_{\nabla,V}(\varphi_k s), \varphi_k s\rangle\!\rangle \geq \mathcal{E}_{\nabla,V}(B(x_k,r))\|\varphi_k s\|^2$, hence

$$\langle\!\langle H_{\nabla,V}s, s\rangle\!\rangle \geq \int_M \sum_{k=1}^{\infty} \Big(\mathcal{E}_{\nabla,V}(B(x_k,r))\varphi_k^2 - |d\varphi_k|^2\Big)|s|^2\, d\mu_g.$$

Let $\psi\colon M \to \mathbb{R}$ denote the function defined by the sum. Then ψ is smooth and maps M to $[C,\infty)$, where $C \coloneqq \inf \sigma(H_{\nabla,V}) - \sup_{k\in\mathbb{N}} \|d\varphi_k\|_{L^\infty}$. Moreover, $\psi\colon M \to [C,\infty)$ is proper: if $\lambda \in \mathbb{R}$, then we find $k_0 \in \mathbb{N}$ such that $\mathcal{E}_{\nabla,V}(B(x_k,r)) \geq \lambda$ for all $k \geq k_0$, *i.e.*, $\psi \geq \lambda - N\gamma$ on $\bigcup_{k\geq k_0} B(x_k,r)$, a set whose complement is bounded, hence with compact closure by the Hopf–Rinow theorem. Here, $N > 0$ is the intersection multiplicity of the cover $\{B(x_k,r)\}_{k\geq 1}$, see Proposition 4.1.10, and $\gamma \coloneqq \sup_{k\in\mathbb{N}} \|d\varphi_k\|_{L^\infty}$. This completes the proof. ∎

Remark 4.2.3. General conditions for the essential self-adjointness of Schrödinger operators on Riemannian manifolds (not only acting on line bundles) can be found in [BMS02], see also Theorem 1.4.15.

In what follows, we are mostly concerned with Schrödinger operators acting on sections of Hermitian *line* bundles, and where the connection is a metric connection. Note that for a line bundle L, the endomorphism bundle $\mathrm{End}(L)$ is trivial via $M \times \mathbb{C} \to \mathrm{End}(L)$, $(x,t) \mapsto t\,\mathrm{id}_{L_x}$. This allows us to identify V canonically with a smooth function on M, and $\Omega^1(M, \mathrm{End}(L))$ with $\Omega^1(M,\mathbb{C})$. We will use the fact that the set of metric connections on a given line bundle $L \to M$ may be described as the affine space $\{\nabla^0 + i\alpha \otimes \mathrm{id}_L : \alpha \in \Omega^1(M,\mathbb{R})\}$ for any given metric connection ∇^0 on L, see (A.1.3).

Lemma 4.2.4 (Gauge invariance)**.** *Let $U \subseteq M$ be a simply connected open subset. Then $\mathcal{E}_{\nabla,V}(U) = \mathcal{E}_{\nabla',V}(U)$ for any two metric connections ∇ and ∇' on $L|_U$ with the same curvature.*

Proof. This is a geometric reinterpretation of the corresponding property of scalar Schrödinger operators on $\mathbb{R}^n$, see for instance [Lei83, Theorem 1.2]. The difference of the two metric connections is a purely imaginary one-form, *i.e.*, $\nabla - \nabla' = i\alpha \otimes \mathrm{id}_L$ with $\alpha \in \Omega^1(U,\mathbb{R})$. Since the curvatures agree, we have $d\alpha = 0$. Indeed, $d^\nabla = d^{\nabla'} + i\,\varepsilon(\alpha)$, and hence

$$R^\nabla \wedge_{\mathrm{ev}} s = d^\nabla(\nabla s) = d^{\nabla'}(\nabla' s + i\alpha \otimes s) + i\alpha \wedge (\nabla' s + i\alpha \otimes s) = R^{\nabla'} \wedge_{\mathrm{ev}} s + i d\alpha \otimes s$$

for all $s \in \Gamma(U,L)$. Because U is simply connected, de Rham's theorem implies that there is $g \in C^\infty(U,\mathbb{R})$ such that $\alpha = dg$. For $s \in \Gamma_c(M,E)$ with support in U, we compute

$$\nabla(e^{-ig}s) = -ie^{-ig} dg \otimes s + e^{-ig}(\nabla' s + i dg \otimes s) = e^{-ig}\,\nabla' s,$$

hence

$$\langle\!\langle H_{\nabla,V}(e^{-ig}s), e^{-ig}s\rangle\!\rangle = \int_M \left(\langle \nabla(e^{-ig})s, \nabla(e^{-ig}s)\rangle + \langle V(e^{-ig}s), e^{-ig}s\rangle\right) d\mu_g = \langle\!\langle H_{\nabla',V}s, s\rangle\!\rangle,$$

and therefore $\mathcal{E}_{\nabla,V}(U) = \mathcal{E}_{\nabla',V}(U)$. ■

The following Lemma extends [Iwa86, Proposition 3.2] to Riemannian manifolds of 0-bounded geometry:

Lemma 4.2.5. *Let M be a Riemannian manifold of 0-bounded geometry. There exists $\rho > 0$ with the following property: if $r \in (0,\rho)$, $x \in M$, and $B \in \Omega^2(\overline{B(x,r)})$ is a closed two-form, then there is $a \in \Omega^1(B(x,r))$ such that $da = B$ and*

$$\|a\|_{L^p(B(x,r),T^*M)} \le C_p(r)\|B\|_{L^p(B(x,r),\Lambda^2 T^*M)}$$

for all $1 \le p \le \infty$, where $C_p(r) > 0$ depends only on p, r, and on the geometry of M, but not *on $x \in M$.*

Proof. Let $\rho > 0$ be such that the distortion of normal coordinates on balls of radius at most ρ is uniformly bounded on M, see Lemma 4.1.6. Take $B \in \Omega^2(B(x,r))$ as in the assumption and put $\widetilde{B} := \varphi_x^* B$, where $\varphi_x := (\exp_x \circ \tau)|_{B_{\mathbb{R}^n}(0,r)}$ are Riemannian normal coordinates, with

$\tau\colon \mathbb{R}^n \to T_xM$ any orthonormal map (*i.e.*, a choice of orthonormal basis of T_xM), chosen in a way that φ_x preserves the orientations. Then $\widetilde{B}$ is an element of $\Omega^2(\overline{B_{\mathbb{R}^n}(0,r)})$, closed by naturality[2] of the exterior derivative, and the construction in [Iwa86, Proposition 3.2] yields $\widetilde{a} \in \Omega^1(B_{\mathbb{R}^n}(0,r))$ such that $da = B$ and

$$\|\widetilde{a}\|_{L^p(B_{\mathbb{R}^n}(0,r),T^*\mathbb{R}^n)} \leq \widetilde{C}_p(r)\|\widetilde{B}\|_{L^p(B_{\mathbb{R}^n}(0,r),\Lambda^2\mathbb{R}^n)}$$

for all $1 \leq p \leq \infty$. This can be achieved by taking

$$\widetilde{a}_y(v) \coloneqq \fint_{B_{\mathbb{R}^n}(0,r)} \int_0^1 \widetilde{B}_{z+t(y-z)}(t(y-z),v)\,dt\,d\lambda(z),$$

with $y \in B_{\mathbb{R}^n}(0,r)$ and $v \in T_y(B_{\mathbb{R}^n}(0,r)) \cong \mathbb{R}^n$, and where $\fint \bullet\, d\lambda$ denotes the average with respect to Lebesgue measure. Define $a \coloneqq (\varphi_x^{-1})^*\widetilde{a}$. Then $da = B$, again by naturality, and we have

$$|\varphi_x^* a|(y) \geq |T_y\varphi_x|^{-1}\,(|a| \circ \varphi_x)(y) \quad \text{and} \quad |\varphi_x^* B|(y) \leq |T_y\varphi_x|^2\,(|B| \circ \varphi_x)(y),$$

where $|T_y\varphi_x|$ is the operator norm. By Lemma 4.1.6 and Remark 4.1.8, there is $C > 0$ such that $1/C \leq |T_y\varphi_x| \leq C$ and $1/C \leq \det(g_{ij}^x(y))^{1/2} \leq C$ uniformly in $y \in B_{\mathbb{R}^n}(0,r)$, and independent of $x \in M$. Here, g_{ij}^x are the metric coefficients with respect to the chart φ_x. Putting this together, we obtain

$$\begin{aligned}
\int_{B(x,r)} |a|^p \operatorname{vol}_g &= \int_{B_{\mathbb{R}^n}(0,r)} (|a|^p \circ \varphi_x)\,\varphi_x^*\operatorname{vol}_g \\
&\leq \int_{B_{\mathbb{R}^n}(0,r)} |T_y\varphi_x|^p\,|\varphi_x^* a|^p(y)\,\det(g_{ij}^x(y))^{1/2}\,d\lambda(y) \\
&\leq C^{p+1} \int_{B_{\mathbb{R}^n}(0,r)} |\varphi_x^* a|^p(y)\,d\lambda(y) \\
&\leq C^{p+1}\widetilde{C}_p(r)^p \int_{B_{\mathbb{R}^n}(0,r)} |\varphi_x^* B|^p(y)\,d\lambda(y) \\
&\leq C^{3p+2}\widetilde{C}_p(r)^p \int_{B_{\mathbb{R}^n}(0,r)} (|B|^p \circ \varphi_x)(y)\,\det(g_{ij}^x(y))^{1/2}\,\lambda(y) \\
&= C^{3p+2}\widetilde{C}_p(r)^p \int_{B(x,r)} |B|^p \operatorname{vol}_g,
\end{aligned}$$

with λ the Lebesgue measure on $\mathbb{R}^n$. Now put $C_p(r) \coloneqq C^{3+2/p}\widetilde{C}_p(r)$. ■

Consider now a local trivialization $\psi\colon p^{-1}(U) \xrightarrow{\cong} U \times \mathbb{C}$ of L over an open subset $U \subseteq M$. Then there is $\alpha_\psi \in \Omega^1(U,\mathbb{C})$ such that

$$((\mathrm{id}_{T^*U} \otimes \psi) \circ \nabla \circ \psi^{-1})f = (d + \alpha_\psi)f$$

for every function $f \in C^\infty(U) = \Gamma(U, U \times \mathbb{C})$. Indeed, the difference of two connections is a one-form (with values in $\mathrm{End}(L)$, which is trivial), and we may use the trivial connection

[2] This means that d commutes with pullbacks.

d on $U \times \mathbb{C} \to U$. For the exterior covariant derivative, this means $(\mathrm{id}_{\Lambda T^*U} \otimes \psi) \circ d^\nabla \circ (\mathrm{id}_{\Lambda T^*U} \otimes \psi^{-1}) = d + \varepsilon(\alpha_\psi)$ on $\Omega(U)$. Note that the curvature of ∇ is on U given by

$$R^\nabla|_U = d\alpha_\psi \otimes \mathrm{id}_L \in \Omega^2(U, \mathrm{End}(L)), \tag{4.2.4}$$

because

$$\begin{aligned} R^\nabla \wedge_{\mathrm{ev}} u &= d^\nabla d^\nabla u \\ &= (\mathrm{id}_{\Lambda T^*U} \otimes \psi^{-1})(d + \varepsilon(\alpha_\psi))^2 (\mathrm{id}_{\Lambda T^*U} \otimes \psi) u \\ &= (\mathrm{id}_{\Lambda T^*U} \otimes \psi^{-1})(d^2 \widetilde{u} + d(\alpha_\psi \wedge \widetilde{u}) + \alpha_\psi \wedge d\widetilde{u} + \alpha_\psi \wedge \alpha_\psi \wedge \widetilde{u}) \\ &= (\mathrm{id}_{\Lambda T^*U} \otimes \psi^{-1})(d\alpha_\psi \wedge \widetilde{u}) \\ &= d\alpha_\psi \wedge u \end{aligned}$$

for $u \in \Omega(U, L)$, where $\widetilde{u} := (\mathrm{id}_{\Lambda T^*U} \otimes \psi) u \in \Omega(U, \mathbb{C})$. While α_ψ depends on the choice of trivialization, this shows that its exterior derivative $d\alpha_\psi$ is a global object. If L carries a Hermitian metric, then there is a smooth function $w_\psi \colon U \to \mathbb{R}$ such that $|\psi^{-1}(y, \lambda)| = |\lambda| e^{-w_\psi(y)}$ for all $(y, \lambda) \in U \times \mathbb{C}$. Indeed, $w_\psi(y) = -\log|\psi^{-1}(y, 1)|$.

Lemma 4.2.6. *Let ∇ be a metric connection on a Hermitian line bundle $L \to M$, and let $V \colon L \to L$ be a self-adjoint vector bundle morphism. Suppose that $U \subseteq M$ is open and contractible,*[3] *and $\varphi \colon M \to [0,1]$ is smooth with $\mathrm{supp}(\varphi) \subseteq U$. Then*

$$\inf_{g \in C^\infty(U, \mathbb{R})} \int_U \left(|\alpha_\psi - dw_\psi + i dg|^2 + |V| \right) d\mu_g \geq \mathcal{E}_{\nabla, V}(U) \|\varphi\|^2_{L^2(M)} - \|d\varphi\|^2_{L^2(M, T^*M)},$$

where ψ is any local trivialization of L over U, and where $\alpha_\psi \in \Omega^1(U, \mathbb{C})$ and $w_\psi \in C^\infty(U, \mathbb{R})$ are as above.

Proof. The proof is a modification of [Iwa86, Lemma 5.1] to accommodate globally non-trivial line bundles. Because U is contractible, $L|_U$ is trivial. Let $\psi \colon p^{-1}(U) \to U \times \mathbb{C}$ be a local trivialization of L, and let $W \colon U \times \mathbb{C} \to U \times \mathbb{C}$ be the vector bundle isomorphism $(y, \lambda) \mapsto (y, e^{-w_\psi(y)} \lambda)$. Then $\psi_0 := W \circ \psi$ is also a local trivialization of L over U, and $|\psi_0^{-1}(y, \lambda)|_L = |\lambda|$. It follows that $(\mathrm{id}_{T^*M} \otimes \psi_0)^{-1} \circ d \circ \psi_0$ is a metric connection on $L|_U$. Since

$$\begin{aligned} \nabla|_U = (\mathrm{id}_{T^*M} \otimes \psi_0)^{-1} \circ (\mathrm{id}_{T^*M} \otimes W) \circ (d + \alpha_\psi) \circ W^{-1} \circ \psi_0 = \\ = (\mathrm{id}_{T^*M} \otimes \psi_0)^{-1} \circ (d + \alpha_\psi - dw_\psi) \circ \psi_0 \end{aligned}$$

and ∇ is a metric connection, we see that $i(\alpha_\psi - dw_\psi) \in \Omega^1(U, \mathbb{R})$. Put

$$s := \psi_0^{-1} \circ (\mathrm{id}_U, \varphi|_U) \colon U \to L,$$

[3] A topological space X is called *contractible* if X is homotopy equivalent to a point. Equivalently, the identity map on X is null-homotopic. All vector bundles over a contractible manifold are trivial, see [Moo01, p. 15].

so that s is a compactly supported section of L over U which extends to a section of L over M by setting it to zero outside of $\operatorname{supp}(\varphi)$. Evidently, $|s|_L^2 = |\varphi|^2 \leq 1$. Moreover, for $g \in C^\infty(U, \mathbb{R})$, the connection $\nabla' := \nabla|_U + idg \otimes \mathrm{id}_L$ on $L|_U$ is metric compatible, and

$$|\nabla' s|^2_{T^*M\otimes L} = |d\varphi + \varphi(\alpha_\psi - dw_\psi + idg)|^2 = \\ = |d\varphi|^2 + |\varphi(\alpha_\psi - dw_\psi + idg)|^2 \leq |d\varphi|^2 + |\alpha_\psi - dw_\psi + idg|^2,$$

since the expression in the parentheses is purely imaginary, and $|\varphi| \leq 1$. Because $ddg = 0$, we have $R^{\nabla'} = R^{\nabla|_U}$, and Lemma 4.2.4 implies

$$\int_U (|\alpha_\psi - dw_\psi + idg|^2 + |V|)\, d\mu_g + \|d\varphi\|^2_{L^2(M,T^*M)} \geq \\ \geq \int_U (|\nabla' s|^2_{T^*M\otimes L} + \langle Vs, s\rangle_L)\, d\mu_g = \langle\!\langle H_{\nabla',V} s, s\rangle\!\rangle \geq \\ \geq \mathcal{E}_{\nabla',V}(U)\, \|s\|^2_{L^2(M,L)} = \mathcal{E}_{\nabla,V}(U)\, \|\varphi\|^2_{L^2(M)}.$$

Since ψ and $g \in C^\infty(U, \mathbb{R})$ were arbitrary, the claim follows. ■

We now show that the appropriate generalization of [Iwa86, Theorem 5.2] continues to hold for Schrödinger operators acting on the sections of line bundles over manifolds of 1-bounded geometry:

Theorem 4.2.7. *Let $L \to M$ be a Hermitian line bundle over a noncompact Riemannian manifold of 1-bounded geometry, and let $H_{\nabla,V} := \nabla^\dagger\nabla + V$ be a generalized Schrödinger operator for a metric connection ∇ and self-adjoint morphism $V\colon L \to L$. Assume that $H_{\nabla,V}$ has a lower semibounded self-adjoint extension with discrete spectrum. Then*

$$\lim_{x\to\infty} \int_{B(x,r)} (|R^\nabla|^2 + |V|)\, d\mu_g = \infty$$

for all $r > 0$.

Proof. It suffices to prove the claim for $r > 0$ small enough, and we take r so that item (ii) of Lemma 4.2.2 and Lemma 4.2.5 work out. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in M with $x_k \to \infty$ as $k \to \infty$. For every $k \in \mathbb{N}$, we find $\varphi_k \in C^\infty(M, [0,1])$ with $\operatorname{supp}(\varphi_k) \subseteq B(x_k, r)$, $\int_M |\varphi_k|^2\, d\mu_g = 1$, and such that $\sup_{k\in\mathbb{N}} \|d\varphi_k\|_{L^\infty(M,T^*M)} < \infty$, see Lemma 4.1.9. Since ∇ is a metric connection, we have $R^\nabla = d\alpha_{\psi_k} \otimes \mathrm{id}_L$ on $B(x_k, r)$ with $i\alpha_{\psi_k} \in \Omega^1(B(x_k, r), \mathbb{R})$ for any choice of local trivializations $\psi_k\colon L|_{B(x_k,r)} \to B(x_k, r) \times \mathbb{C}$, see (4.2.4). By Lemma 4.2.5, there are $a_k \in \Omega^1(B(x_k, r), \mathbb{R})$ with $da_k = id\alpha_{\psi_k}$ and

$$\int_{B(x_k,r)} |R^\nabla|^2\, d\mu_g = \int_{B(x_k,r)} |d\alpha_{\psi_k}|^2\, d\mu_g \geq C \int_{B(x_k,r)} |a_k|^2\, d\mu_g,$$

with $C > 0$ independent of $x \in M$. Since $da_k = id(\alpha_{\psi_k} - dw_{\psi_k})$ and $B(x_k, r)$ is simply connected, there is $g_k \in C^\infty(B(x_k, r), \mathbb{R})$ such that $a_k - i\alpha_{\psi_k} + idw_{\psi_k} = dg_k$, *i.e.*, $a_k =$

$i\alpha_{\psi_k} - idw_{\psi_k} - dg_k$. Using Lemma 4.2.6, we find

$$\int_{B(x_k,r)} (C^{-1}|R^\nabla|^2 + |V|)\, d\mu_g \geq \mathcal{E}_{\nabla,V}(B(x_k,r))\|\varphi_k\|^2_{L^2(M)} - \|d\varphi_k\|^2_{L^2(M,T^*M)}.$$

If A denotes a lower semibounded self-adjoint extension of $H_{\nabla,V}$ with discrete spectrum, then we have $\liminf_{k\to\infty} \mathcal{E}_{\nabla,V}(B(x_k,r)) \geq \inf \sigma_e(A) = \infty$ by (4.2.3), so the claim follows. ■

Remark 4.2.8. On $\mathbb{R}^n$, it is possible to characterize the discreteness of spectrum of operators of the form $-\Delta+V$ (*i.e.*, Schrödinger operators without magnetic field) by considering integrals of $|V|$ over sets which go to infinity, similarly to Theorem 4.2.7. This is done in [Mol53], and uses the concept of Wiener capacity of compact subsets of $\mathbb{R}^n$. There has also been progress to extend this to magnetic Schrödinger operators, see [KMS04; KMS09], but while some of those results are available on manifolds of bounded geometry, it is not clear what their geometric interpretation is, or if they can be generalized to the case of nontrivial line bundles.

4.3. The Dolbeault Laplacian on top degree forms

We now study the operator $\square^E$ on the upper end of the $\overline{\partial}^E$-complex, *i.e.*, on (p,n)-forms. By (3.1.4) and (3.1.5), and using that $\varepsilon(\overline{w}^j)\,\mathrm{ins}_{\overline{w}_k} = \delta_{j,k}$ on $\Lambda^{0,n}T^*M \otimes E$, we see that $\square^E_{0,n}$ has the form

$$2\square^E_{0,n} = \Delta^{\Lambda^{0,n}T^*M\otimes E} + \sum_{j=1}^n \big(\mathrm{id}_{\Lambda^{0,n}T^*M} \otimes R^E(w_j,\overline{w}_j) + \mathrm{tr}(R^{T^{1,0}M}(w_j,\overline{w}_j)\big). \tag{4.3.1}$$

We are interested in deciding from curvature quantities of M and E whether $\square^E_{p,n}$ has discrete spectrum or not. The next simple and well-known Proposition shows that the situation is uninteresting if M is a relatively compact domain in a larger manifold to which E extends. The idea is that $\square^E_{p,n}$ is just a bounded perturbation of the (Bochner) Laplacian with Dirichlet boundary conditions, for which it follows from Rellich's theorem that it has discrete spectrum.

Proposition 4.3.1. *Let $M \subseteq M'$ be a bounded open subset of a complete Kähler manifold, and let $E \to \overline{M}$ be a Hermitian holomorphic vector bundle. Then $\square^E_{p,n}$ has discrete spectrum for $0 \leq p \leq n$.*

Proof. The domain of the quadratic form of $\square^E_{p,n}$ is $\mathrm{dom}(\overline{\partial}^{E,*}_w) = \mathrm{dom}((\overline{\partial}^{E,\dagger})_s)$, hence the space $\Omega^{p,n}_c(M,E)$ of compactly supported (p,n)-forms is a form core for $\square^E_{p,n}$. Since M is relatively compact and E is defined in a neighborhood of $\overline{M}$, the zeroth order term $\mathcal{K}^E$ in (3.1.4) is bounded from below, say by $C \in (-\infty,-1]$, since

$$|\mathcal{K}^E| = |c_p(R^{\Lambda^{p,\bullet}T^*M\otimes E})| \leq \sum_{j<k} |R^{\Lambda^{p,\bullet}T^*M\otimes E}(e_j,e_k)| \leq$$
$$\leq \sqrt{n(2n+1)}\left(\sum_{j<k} |R^{\Lambda^{p,\bullet}T^*M\otimes E}(e_j,e_k)|^2\right)^{1/2} = \sqrt{n(2n+1)}\,|R^{\Lambda^{p,\bullet}T^*M\otimes E}|, \tag{4.3.2}$$

where the first inequality is due to (1.1.16) and (1.1.17) and the second comes from Hölder's inequality. We have

$$2Q^E(u,u) = \langle\!\langle \widetilde{\nabla} u, \widetilde{\nabla} u\rangle\!\rangle + \langle\!\langle \mathcal{K}^E u, u\rangle\!\rangle \geq C\|u\|^2 + \|\widetilde{\nabla} u\|^2$$

for all $u \in \Omega_c^{p,n}(M,E)$, with $\widetilde{\nabla}$ the connection on $\Lambda^{p,n}T^*M \otimes E$ induced from the Levi–Civita connection on TM and the Chern connection on E, as usual. Therefore,

$$\|u\|^2_{H_0^1(M,\Lambda^{p,n}T^*M\otimes E)} = \|u\|^2 + \|\widetilde{\nabla} u\|^2 \leq (1-C)\|u\|^2 + 2Q^E(u,u) \leq (1-C)\|u\|^2_{\operatorname{dom}(Q^E)},$$

and this inequality extends to $u \in \operatorname{dom}(Q^E) \cap L^2_{p,n}(M,E)$ by density, showing that the inclusion $\operatorname{dom}(Q^E) \cap L^2_{p,n}(M,E) \hookrightarrow H_0^1(M,\Lambda^{p,n}T^*M \otimes E)$ is continuous. But by Theorem 1.3.8, the inclusion of this Sobolev space into $L^2_{p,n}(M,E)$ is compact, hence the same is true for $\operatorname{dom}(Q^E) \cap L^2_{p,n}(M,E) \hookrightarrow L^2_{p,n}(M,E)$. By Corollary 2.2.3, $\square^E_{p,n}$ therefore has discrete spectrum. ■

Since $\Lambda^{0,n}T^*M$ and $\Lambda^{n,n}T^*M$ are line bundles, the results from section 4.2 are applicable if E is also a line bundle. We have the following result:

Theorem 4.3.2. *Let $L \to M$ be a Hermitian holomorphic line bundle over a Kähler manifold of 1-bounded geometry, and let $p \in \{0,n\}$. Assume that*

(i) $\square^L_{p,n}$ has discrete spectrum, or

(ii) for some $0 \leq q \leq n-1$, L is $(q+1)$-Nakano lower semibounded and $\square^L_{p,q}$ has discrete spectrum.

Then

$$\lim_{x\to\infty} \int_{B(x,r)} |R^L|^2 \, d\mu_g = \infty \tag{4.3.3}$$

for all $r > 0$ small enough.

Proof. By Theorem 3.1.2, we have $\square^L_{p,n} = \Delta^{\Lambda^{p,n}T^*M\otimes L} + c_p(R^{\Lambda^{p,\bullet}T^*M\otimes L})$, where c_p is the Clifford action on $\Lambda^{p,\bullet}T^*M \otimes L$. By (4.3.2),

$$|c_p(R^{\Lambda^{p,\bullet}T^*M\otimes L})| \leq \sqrt{n(2n+1)}\, |R^{\Lambda^{p,\bullet}T^*M\otimes L}|.$$

Therefore, if the spectrum of $\square^L_{p,n}$ is discrete, Theorem 4.2.7 gives

$$\lim_{x\to\infty} \int_{B(x,r)} \left(|R^{\Lambda^{p,\bullet}T^*M\otimes L}|^2 + \sqrt{n(2n+1)}\, |R^{\Lambda^{p,\bullet}T^*M\otimes L}|\right) d\mu_g = \infty$$

for all $r > 0$ small enough. Now by Hölder's inequality,

$$\int_{B(x,r)} |R^{\Lambda^{p,\bullet}T^*M\otimes L}| \, d\mu_g \leq \sqrt{C}\Big(\int_{B(x,r)} |R^{\Lambda^{p,\bullet}T^*M\otimes L}|^2 \, d\mu_g\Big)^{1/2},$$

with $C := \sup_{x\in M}\mu_g(B(x,r))$.[4] Consequently,

$$\int_{B(x,r)} |R^{\Lambda^{p,\bullet}T^*M\otimes L}|^2\, d\mu_g \to \infty \quad \text{as } x \to \infty,$$

which is the same as (4.3.3) since the curvature of $\Lambda^{p,\bullet}T^*M$ is bounded due to M having 0-bounded geometry. In the case where L is $(q+1)$-Nakano lower semibounded, we use Theorem 3.2.23 to reduce this case to the first one. ∎

A version of Theorem 4.3.2 for the case $M = \mathbb{C}^n$ will appear as joint work with Friedrich Haslinger in [BH17, Theorem 4.1] (see also Corollary 4.3.3 below).

4.3.1. The weighted $\overline{\partial}$-complex on $\mathbb{C}^n$. Consider the trivial line bundle $L := \mathbb{C}^n\times\mathbb{C} \to \mathbb{C}^n$, equipped with a Hermitian metric. According to Example B.3.7, this means that there is a smooth function $\varphi\colon \mathbb{C}^n \to \mathbb{R}$ such that $\langle u, v\rangle = u\overline{v}\, e^{-\varphi}$ for $u, v \in C^\infty(\mathbb{C}^n,\mathbb{C}) \cong \Gamma(\mathbb{C}^n, L)$, and the curvature of the Chern connection on L is given by $\partial\overline{\partial}\varphi \in \Omega^{1,1}(\mathbb{C}^n)$. Therefore, L is Nakano lower semibounded if and only if the *complex Hessian* of φ,

$$H_\varphi(z) := \left(\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}(z)\right)_{j,k=1}^n,$$

is a lower semibounded matrix, with lower bound $c \in \mathbb{R}$ independent of $z \in \mathbb{C}^n$. In particular, this is true if φ is plurisubharmonic, where one can choose $c = 0$. Denote by $s_1(z) \leq \cdots \leq s_n(z)$ the eigenvalues of $H_\varphi(z)$ in increasing order (see also Example 3.1.21). If L is Nakano lower semibounded, then $s_1 \geq c$ (see Example 3.2.21) and

$$\operatorname{tr}(H_\varphi^2) = \sum_{j=1}^n s_j^2 = \left(\sum_{j=1}^n s_j\right)^2 - 2\sum_{j<k} s_j s_k \leq \operatorname{tr}(H_\varphi)^2 - n(n-1)c^2.$$

Moreover, the norm $B \mapsto |B|$ on the $n\times n$ complex matrices is equivalent to the Schatten norm $B \mapsto \operatorname{tr}(|B|^2)^{1/2} = \operatorname{tr}(B^*B)^{1/2}$, and we have

$$|R^L|^2 = |\partial\overline{\partial}\varphi|^2 \leq C\operatorname{tr}(H_\varphi^* H_\varphi) = C\operatorname{tr}(H_\varphi^2) \leq C\operatorname{tr}(H_\varphi)^2 - n(n-1)Cc^2 \tag{4.3.4}$$

for some constant $C > 0$.

Corollary 4.3.3. *Let $\varphi\colon \mathbb{C}^n \to \mathbb{R}$ be smooth and denote by L the trivial line bundle over $\mathbb{C}^n$ with fiber metric $e^{-\varphi}$, as above. If L is Nakano lower semibounded and $\Box^L_{0,q}$ has compact resolvent for some $0 \leq q \leq n$, then*

$$\lim_{z\to\infty}\int_{B(z,1)} (\Delta\varphi)^2\, d\lambda = +\infty \tag{4.3.5}$$

Proof. This follows immediately from Theorem 4.3.2 and (4.3.4), and the fact that $4\operatorname{tr}(H_\varphi) = \Delta\varphi$. ∎

[4]The supremum is finite because in normal coordinates around x and with small enough radius, the metric coefficients g_{ij}^x have uniform two-sided bounds, independent of x, see Remark 4.1.8, hence $\mu_g(B(x,r)) = \int_{B_{\mathbb{R}^n}(0,r)} \det(g_{ij}^x)^{1/2}\, d\lambda$ is also bounded from both sides.

Remark 4.3.4. The condition (4.3.5) is not sufficient for $\Box^L_{0,q}$, with $0 \leq q \leq n-1$, to have discrete spectrum. This can be seen by considering *decoupled* weights, see Remark 5.3.9.

Remark 4.3.5. If $\varphi\colon \mathbb{C}^n \to \mathbb{R}$ is such that $\operatorname{tr}(H_\varphi)$ satisfies the *reverse Hölder condition*,

$$\left(\fint_B \operatorname{tr}(H_\varphi)^r \, d\lambda\right)^{1/r} \leq C \fint_B \operatorname{tr}(H_\varphi)\, d\lambda$$

for some $r \geq 2$, some $C > 0$, and all balls $B \subseteq \mathbb{C}^n$, where $\fint_B \bullet \, d\lambda$ denotes the average over B for the Lebesgue measure, then Hölder's inequality implies that (4.3.5) can be replaced by the formally weaker condition of

$$\lim_{z\to\infty} \int_{B(z,1)} \operatorname{tr}(H_\varphi)\, d\lambda = +\infty. \tag{4.3.6}$$

The class of functions satisfying one of these reverse Hölder conditions equals $A_\infty \coloneqq \bigcup_{p\geq 1} A_p$, where A_p are the *Muckenhoupt classes*, see [Ste93, Theorem 3, p. 212]. Every positive polynomial belongs to A_∞. In fact, $|P|^a \in A_p$ for $p > 1$ if $-1 < ad < p-1$, where d is the degree of P, see [Ste93, 6.5, p 219].

In the proof of Theorem 4.2.7, from which ultimately Theorem 4.3.2 followed, we did not really need M to be of bounded geometry. Rather, we used a sequence of functions $\varphi_k \in C^\infty(M, [0,1])$ that have uniform lower bounds on their L^2 norm and uniform upper bounds on their derivatives, and with $\operatorname{supp}(\varphi_k) \subseteq B(x_k, r)$ for some sequence $x_k \to \infty$ as $k \to \infty$ and fixed $r > 0$. Examples of manifolds where sequences of this kind are *not* available are open subsets Ω of $\mathbb{C}^n$ that are *quasibounded*, that is, they satisfy

$$\lim_{\Omega\ni z\to\infty} \operatorname{dist}(z, \partial\Omega) = 0. \tag{4.3.7}$$

Quasibounded domains often appear as counterexamples in Sobolev space theory, see [AF03, p. 6.9]. Thus, if Ω is *not* quasibounded, then there exists $r > 0$ and a sequence $x_k \in \Omega$ such that $x_k \to \infty$ as $k \to \infty$ and $\overline{B(x_k, r)} \subseteq \Omega$ for all $k \in \mathbb{N}$. Translating a fixed function $\varphi \in C_c^\infty(B(0,r), [0,1])$, we obtain:

Theorem 4.3.6. *Assume that $\Omega \subseteq \mathbb{R}^n$ is open and not quasibounded, and let $L \to \Omega$ be a Hermitian line bundle. Let ∇ be a connection on L, and let $V \in C^\infty(\Omega, \mathbb{R})$ be a function such that the magnetic Schrödinger operator $\nabla^\dagger\nabla + V$ has a lower semibounded self-adjoint extension with discrete spectrum. Then*

$$\lim_{k\to\infty} \int_{B(x_k,r)} (|R^\nabla|^2 + |V|)\, d\lambda = \infty$$

for every sequence $x_k \in \Omega$ and all $r > 0$ such that $x_k \to \infty$ as $k \to \infty$ and $\overline{B(x_k, r)} \subseteq \Omega$ for all $k \in \mathbb{N}$.

Corollary 4.3.7. *Assume that $\Omega \subseteq \mathbb{C}^n$ is open and not quasibounded, and let $L \to \Omega$ be the trivial Hermitian holomorphic line bundle, with metric given by $e^{-\varphi}$ for a smooth function $\varphi\colon \overline{\Omega} \to \mathbb{R}$, see Example B.3.7. Suppose* *(i)* $\square^L_{0,n}$ *has discrete spectrum, or*

(ii) Ω *is smoothly bounded and Levi pseudoconvex,* L *is Nakano lower semibounded, and* $\square^L_{0,q}$ *has discrete spectrum for some* $0 \leq q \leq n$*.*

Then

$$\lim_{k\to\infty} \int_{B(z_k,r)} \operatorname{tr}(H_\varphi)^2 \, d\lambda = \infty$$

for every sequence $z_k \in \Omega$ *and all* $r > 0$ *such that* $z_k \to \infty$ *as* $k \to \infty$ *and* $\overline{B(z_k, r)} \subseteq \Omega$ *for all* $k \in \mathbb{N}$*.*

CHAPTER 5

The (essential) spectrum of the Dolbeault Laplacian on product manifolds

In this chapter we are concerned with the spectral theory of the Laplacian of a *tensor product* of two Hilbert complexes. The Hilbert space of the tensor product of two Hilbert complexes (H, d) and (H', d') is given by the tensor product of graded Hilbert spaces, and the differential is the closure of $\bigoplus_{j+k=i}(d_j \otimes \mathrm{id}_{H'_k} + \sigma_j \otimes d'_k)$, where σ_j is multiplication by $(-1)^j$ on H_j, see section 5.1 for the detailed definitions. Our main result is the following:

Theorem 5.1.3. *Let (H, d) and (H', d') be two Hilbert complexes, with Laplacians Δ and Δ', respectively. If $\widetilde{\Delta}$ denotes the Laplacian of the tensor product Hilbert complex $(H, d) \hat{\circledast} (H', d')$, then*

$$\sigma(\widetilde{\Delta}_i) = \bigcup_{j+k=i} (\sigma(\Delta_j) + \sigma(\Delta'_k)) \tag{5.1.4}$$

and

$$\sigma_e(\widetilde{\Delta}_i) = \bigcup_{j+k=i} (\sigma_e(\Delta_j) + \sigma(\Delta'_k)) \cup (\sigma(\Delta_j) + \sigma_e(\Delta'_k)). \tag{5.1.5}$$

Here, $\sigma(\widetilde{\Delta}_i)$ and $\sigma_e(\widetilde{\Delta}_i)$ are the spectrum and the essential spectrum of $\widetilde{\Delta}_i$, respectively, and we use Minkowski sums in order to add sets of real numbers. In particular, the sum $\sigma_e(\Delta_j) + \sigma(\Delta'_k)$ is meant to be empty if $\sigma_e(\Delta_j)$ is empty. Equations (5.1.4) and (5.1.5) are obtained by first showing that the Laplacian of the tensor product is an appropriate direct sum of the closures of $\Delta_j \otimes \mathrm{id}_{H'_k} + \mathrm{id}_{H_j} \otimes \Delta'_k$, and then computing the (essential) spectrum of these operators by using the Borel functional calculus for strongly commuting tuples of normal operators, see appendix C.1.

The results are motivated by questions arising in the $\overline{\partial}$-Neumann problem on Hermitian manifolds, which is essentially the study of the *(Gaffney extension of the) complex Laplacian*,

$$\Box^E := \overline{\partial}_w^{E,*}\overline{\partial}_w^E + \overline{\partial}_w^E\overline{\partial}_w^{E,*},$$

with $E \to M$ a Hermitian holomorphic vector bundle, $\overline{\partial}_w^E$ the (weak extension of) the Dolbeault operator acting on E-valued differential forms, and $\overline{\partial}_w^{E,*}$ its Hilbert space adjoint with respect to the L^2 inner product induced by the metrics. Since $\overline{\partial}_w^E$ maps (p, q) forms to $(p, q+1)$ forms and squares to zero, we obtain, for every $1 \le p \le \dim_{\mathbb{C}}(M)$, a Hilbert complex which we denote by $(L^2_{p,\bullet}(M, E), \overline{\partial}_w^E)$, with $L^2_{p,q}(M, E)$ being the space of square-integrable (p, q) forms on M with values in E.

The Cauchy–Riemann equations on product domains have been studied previously in [Cha10; CS11; Ehs07; Fu07; Kra88]. In [Cha10], Chakrabarti computes the spectrum of $\Box$ for $M \times N$, the product of two Hermitian manifolds. If we denote, for the moment, the complex Laplacian on the (p, q) forms on $M \times N$ by $\Box^{M\times N}_{p,q}$, then its spectrum according to [Cha10] is

$$\sigma(\Box^{M\times N}_{p,q}) = \bigcup_{\substack{p'+p''=p\\ q'+q''=q}} \left(\sigma(\Box^{M}_{p',q'}) + \sigma(\Box^{N}_{p'',q''})\right).$$

One of our goals was to find a similar formula for the essential spectrum. If we allow for forms with values in Hermitian holomorphic vector bundles, say $E \to M$ and $F \to N$, then the natural bundle to consider over $M \times N$ is $E \boxtimes F := \pi_M^* E \otimes \pi_N^* F$, with $\pi_M \colon M \times N \to M$ and $\pi_N \colon M \times N \to N$ the projections, and it turns out that $\Box^{E\boxtimes F}_{0,\bullet}$ is unitarily equivalent to the Laplacian of the tensor product of the Hilbert complexes $(L^2_{0,\bullet}(M, E), \overline{\partial}^E_w)$ and $(L^2_{0,\bullet}(N, F), \overline{\partial}^F_w)$. Therefore, we obtain

$$\sigma(\Box^{E\boxtimes F}_{0,q}) = \bigcup_{q'+q''=q} \left(\sigma(\Box^{E}_{0,q'}) + \sigma(\Box^{F}_{0,q''})\right) \tag{5.0.1}$$

and

$$\sigma_e(\Box^{E\boxtimes F}_{0,q}) = \bigcup_{q'+q''=q} \left(\sigma_e(\Box^{E}_{0,q'}) + \sigma(\Box^{F}_{0,q''})\right) \cup \left(\sigma(\Box^{E}_{0,q'}) + \sigma_e(\Box^{F}_{0,q''})\right) \tag{5.0.2}$$

from (5.1.4) and (5.1.5). Both equations have their expected analogues for (p, q) forms with $p \neq 0$, but this will require taking an additional direct sum, see Theorem 5.3.1.

We are also interested in questions regarding the compactness of minimal solution operators to the inhomogeneous $\overline{\partial}^E$-equation. Closely related to this is compactness of the $\overline{\partial}$-Neumann operator, which is the inverse of $\Box^E$ (modulo its kernel). Whether the $\overline{\partial}$-Neumann operator is compact can be read off from the essential spectrum of $\Box^E$, and (5.0.2) therefore provides a way to decide compactness for product manifolds in terms of the corresponding property of the factors.

We point out that these above questions have already been investigated for certain special product manifolds. As a standard counterexample, Krantz [Kra88] shows that the minimal solution operator to the $\overline{\partial}$-equation for $(0, 1)$-forms on the unit bidisc in $\mathbb{C}^2$ fails to be compact.

Haslinger and Helffer consider in [HH07, Proposition 4.6] the *weighted* $\overline{\partial}$-problem on $\mathbb{C}^n$, which can be understood as the corresponding problem for the trivial line bundle on $\mathbb{C}^n$ with nontrivial fiber metric: the pointwise norm of a function $f \colon \mathbb{C}^n \to \mathbb{C}$ is then given by $|f|^2 e^{-\varphi}$ for some given smooth function $\varphi \colon \mathbb{C}^n \to \mathbb{R}$. They show that if φ is *decoupled*, $\varphi(z) = \varphi_1(z_1) + \cdots + \varphi_n(z_n)$, and there exist $1 \leq j \leq n$ such that the Bergman space of entire functions on $\mathbb{C}$, square integrable with respect to $e^{-\varphi_j}\lambda$ (with λ the Lebesgue measure), has infinite dimension, then the $\overline{\partial}$-Neumann operator for the weighted problem on $\mathbb{C}^n$ is not compact on $(0, 1)$ forms. The question of whether the conclusion extends to higher degree forms was left unanswered. Indeed, the method of proof seems unsuitable for treating anything but $(0, 1)$ forms, since they basically consider a solution operator for the product complex which only agrees with the minimal one for $(0, 1)$ forms, see the arguments in [CS11]. The deeper reason for this is that the kernel of $\bar\partial$ does not play nicely with respect to the product structure, while L^2 cohomology (the kernel of the Laplacian) does. This is expressed in the Künneth formula (which holds more generally for tensor products of Hilbert complexes, see Proposition 5.1.2). Note that the weighted problem with decoupled weights is covered by our results since, geometrically, it corresponds to considering the line bundle $\bigotimes_{j=1}^n \pi_j^* E_j$ over $\mathbb{C}^n$, where E_j is the trivial line bundle over $\mathbb{C}$ with fiber metric $e^{-\varphi}$, and $\pi_j\colon \mathbb{C}^n \to \mathbb{C}$ the projection onto the j^{th} factor. We will discuss this in more detail in section 5.3.2.

The extension of [HH07, Proposition 4.6] will then be Theorem 5.3.6, where we show that the $\bar\partial$-Neumann operator for the product of n Riemann surfaces (and vector bundles over them) is in fact not compact on $(0,q)$-forms with $0 \le q \le n-1$, provided at least one factor has an infinite dimensional Bergman space.

The results in this chapter have been published in [Ber16].

5.1. Tensor products of Hilbert complexes

For two $\mathbb{Z}$-graded vector spaces $A = \bigoplus_{i\in\mathbb{Z}} A_i$ and $B = \bigoplus_{i\in\mathbb{Z}} B_i$, we denote by $A \circledast B$ their *graded tensor product*, which is the graded vector space

$$A \circledast B = \bigoplus_{i\in\mathbb{Z}} (A \circledast B)_i \quad \text{with} \quad (A \circledast B)_i := \bigoplus_{j+k=i} A_j \otimes B_k. \tag{5.1.1}$$

If H and K are $\mathbb{Z}$-graded Hilbert spaces, and if only finitely many H_i and K_i are nonzero, then we write $H \hat\circledast K$ for the *tensor product of graded Hilbert spaces*,

$$H \hat\circledast K = \bigoplus_{i\in\mathbb{Z}} (H \hat\circledast K)_i \quad \text{with} \quad (H \hat\circledast K)_i := \bigoplus_{j+k=i} H_j \hat\otimes K_k.$$

If A_i with $i \in \mathbb{Z}$ is a sequence of vector spaces, then by $A_\bullet$ we mean the graded vector space $\bigoplus_{i\in\mathbb{Z}} A_i$. In the case where A_i is only defined for a subset of $\mathbb{Z}$, we extend this sequence by zero. We use the same convention for (finitely many) Hilbert spaces, graded vector bundles and sequences of linear operators. Finally, the tensor product of Hilbert complexes is defined as in [BL92]:

Definition 5.1.1. Given two Hilbert complexes $(H, \mathcal{D}, d)$ and $(H', \mathcal{D}', d')$, their *tensor product* complex $(H \hat\circledast H', d \hat\circledast d')$ is given by the tensor product of graded Hilbert spaces and $(d \hat\circledast d')_i$ is the closure of

$$\bigoplus_{j+k=i} \left(d_j \otimes \mathrm{id}_{H'_k} + \sigma_j \otimes d'_k\right) \colon (\mathcal{D} \circledast \mathcal{D}')_i \to (\mathcal{D} \circledast \mathcal{D}')_{i+1}, \tag{5.1.2}$$

where $\sigma_j \colon H_j \to H_j$ is the multiplication by $(-1)^j$. It is straightforward to verify that this again defines a Hilbert complex. Note that the domain of $d \hat\circledast d'$ is, in general, strictly larger than $\mathcal{D} \circledast \mathcal{D}'$. We denote this tensor product complex by $(H, d) \hat\circledast (H', d') := (H \hat\circledast H', d \hat\circledast d')$.

Proposition 5.1.2. *Let (H, d) and (H', d') be two Hilbert complexes, Δ and Δ' their respective Laplacians.*

(i) The Laplacian of $(H,d)\,\hat{\circledast}\,(H',d')$ on $(H\,\hat{\circledast}\,H')_i$ is the closure of

$$\bigoplus_{j+k=i}(\Delta_j\otimes \mathrm{id}_{H'_k}+\mathrm{id}_{H_j}\otimes\Delta'_k)\colon (\mathrm{dom}(\Delta)\circledast\mathrm{dom}(\Delta'))_i\to (H\,\hat{\circledast}\,H')_i. \tag{5.1.3}$$

(ii) If both d and d' have closed range, then so does $d\,\hat{\circledast}\,d'$. Moreover, we have the Künneth formula

$$\mathcal{H}(H\,\hat{\circledast}\,H', d\,\hat{\circledast}\,d')\cong \mathcal{H}(H,d)\,\hat{\circledast}\,\mathcal{H}(H',d'),$$

meaning

$$\mathcal{H}^i(H\,\hat{\circledast}\,H', d\,\hat{\circledast}\,d')\cong \bigoplus_{j+k=i}\mathcal{H}^j(H,d)\,\hat{\otimes}\,\mathcal{H}^k(H',d')$$

for all $i\in\mathbb{Z}$, where the tensor products $\mathcal{H}^j(H,d)\,\hat{\otimes}\,\mathcal{H}^k(H',d')$ are with respect to the natural Hilbert space structure on the cohomology spaces.

Proof. By general principles, $(d\,\hat{\circledast}\,d')^*$ is the adjoint of the operator (5.1.2). It follows that

$$(d\,\hat{\circledast}\,d')_i^*\supseteq \bigoplus_{j+k=i}(d_j^*\otimes\mathrm{id}_{H'_k}+\sigma_j\otimes d_k'^*).$$

If $\widetilde{\Delta}$ denotes the Laplacian of the tensor product complex, then this gives

$$\begin{aligned}\widetilde{\Delta}_i &= (d\,\hat{\circledast}\,d')_i^*(d\,\hat{\circledast}\,d')_i+(d\,\hat{\circledast}\,d')_{i-1}(d\,\hat{\circledast}\,d')_{i-1}^*\supseteq\\ &\supseteq\Big(\bigoplus_{j+k=i}(d_j^*\otimes\mathrm{id}_{H'_k}+\sigma_j\otimes d_k'^*)\Big)\Big(\bigoplus_{j+k=i}(d_j\otimes\mathrm{id}_{H'_k}+\sigma_j\otimes d'_k)\Big)+\\ &\qquad+\Big(\bigoplus_{j+k=i-1}(d_j\otimes\mathrm{id}_{H'_k}+\sigma_j\otimes d'_k)\Big)\Big(\bigoplus_{j+k=i-1}(d_j^*\otimes\mathrm{id}_{H'_k}+\sigma_j\otimes d_k'^*)\Big)\end{aligned}$$

and the latter operator is an extension of

$$\bigoplus_{j+k=i}(\Delta_j\otimes\mathrm{id}_{H'_k}+\mathrm{id}_{H_j}\otimes\Delta'_k+(d_{j-1}^*\sigma_j)\otimes d'_k+(\sigma_{j+1}d_j)\otimes d_{k-1}'^*+(d_j\sigma_j)\otimes d_{k-1}'^*+(\sigma_{j-1}d_{j-1}^*)\otimes d'_k).$$

Since $\sigma_{j+1}d_j=-d_j\sigma_j$ and $\sigma_{j-1}d_{j-1}^*=-d_{j-1}^*\sigma_j$, the cross terms vanish, and because the domain of the whole i^{th} component is $\mathrm{dom}(\Delta_j)\otimes\mathrm{dom}(\Delta'_k)$, the whole expression is equal to the operator (5.1.3) with domain $\bigoplus_{j+k=i}\mathrm{dom}(\Delta_j)\otimes\mathrm{dom}(\Delta'_k)$. It is a general fact that for self-adjoint operators T and S on Hilbert spaces H and K, respectively, the operator $T\otimes\mathrm{id}_K+\mathrm{id}_H\otimes S$ is essentially self-adjoint, see [RS80, Theorem VIII.33]. By the above, $\widetilde{\Delta}_i$ is a self-adjoint extension of (5.1.3) and must therefore equal its closure. This shows (i). For the proof of (ii) we refer to [BL92, Corollary 2.15] or [CS11, Theorem 4.5]. ■

Using Proposition 5.1.2 and the results on the spectra of the (closures of the) operators $\Delta_j\otimes\mathrm{id}_{H'_k}+\mathrm{id}_{H_j}\otimes\Delta'_k$ from appendix C.1, we are now able to show our main result:

Theorem 5.1.3. *Let (H,d) and (H',d') be two Hilbert complexes, with Laplacians Δ and Δ', respectively. If $\widetilde{\Delta}$ denotes the Laplacian of the tensor product Hilbert complex $(H,d)\,\hat{\circledast}$*

(H', d'), *then*

$$\sigma(\widetilde{\Delta}_i) = \bigcup_{j+k=i} (\sigma(\Delta_j) + \sigma(\Delta'_k)) \tag{5.1.4}$$

and

$$\sigma_e(\widetilde{\Delta}_i) = \bigcup_{j+k=i} (\sigma_e(\Delta_j) + \sigma(\Delta'_k)) \cup (\sigma(\Delta_j) + \sigma_e(\Delta'_k)). \tag{5.1.5}$$

Proof. The spectrum of the direct sum of finitely many self-adjoint operators decomposes as the union of the spectra of the individual operators, and the same holds for the essential spectrum. Indeed, let T_n, $1 \leq n \leq N$ be normal operators on Hilbert spaces H_n and denote the spectral measure of T_n by P_n. Put $T \coloneqq \bigoplus_{n=1}^N T_n$ on $\operatorname{dom}(T) \coloneqq \bigoplus_{n=1}^N \operatorname{dom}(T_n) \subseteq \bigoplus_{n=1}^N H_n =: H$ and define $P(M) \coloneqq \bigoplus_{n=1}^N P_n(M)$ for Borel sets $M \subseteq \mathbb{C}$. Then P is a spectral measure, and for $x = (x_n)_{n=1}^N \in \operatorname{dom}(T)$ and $y = (y_n)_{n=1}^N \in H$ we have

$$\int_{\mathbb{C}} t\, d\langle P(t)x, y\rangle = \sum_{n=1}^N \int_{\mathbb{C}} t\, d\langle P_n(t)x_n, y_n\rangle = \sum_{n=1}^N \langle T_n x_n, y_n\rangle = \langle Tx, y\rangle,$$

so that P is the spectral measure associated to T by the spectral theorem, and the (essential) spectrum of T decomposes as the union of the (essential) spectra of the T_n as is easily seen from Definition C.1.2. Now (5.1.4) and (5.1.5) follow from Proposition 5.1.2 and Theorem C.1.8. ■

Remark 5.1.4. Due to our choice of having Hilbert complexes $\mathbb{Z}$-graded and with $H_i = 0$ for $|i|$ large, it may appear at first glance that there are contributions of many "zero" operators in (5.1.4) and (5.1.5), simply by choosing j and k large enough and with opposite sign (so that $j + k = i$). This is not an issue since those zero operators act on the zero Hilbert space, so they are invertible and therefore have *empty* spectrum (and not $\{0\}$!). In fact, in (5.1.4) and (5.1.5), only the terms with $j \in \operatorname{supp}(H, d)$ and $k \in \operatorname{supp}(H', d')$ contribute, where the *support* of a Hilbert complex (H, d) is the finite set

$$\operatorname{supp}(H, d) \coloneqq \{i \in \mathbb{Z} : H_i \neq 0\} = \{i \in \mathbb{Z} : \mathcal{D}_i \neq 0.\}.$$

Evidently, $\operatorname{supp}((H, d) \hat{\circledast} (H', d')) = \operatorname{supp}(H, d) + \operatorname{supp}(H', d')$.

We next give a characterization for the compactness of N for the tensor product complex by using formula (5.1.5). This characterization is simpler and more insightful if the Hilbert complexes are nondegenerate in the following sense:

Definition 5.1.5. A Hilbert complex (H, d) will be called *nondegenerate* if $d_{i-1} \neq 0$ or $d_i \neq 0$ for all $i \in \operatorname{supp}(H, d)$.

Lemma 5.1.6. *Let (H, d) be a nondegenerate Hilbert complex. Then $\sigma(\Delta_i) \not\subseteq \{0\}$ for all $i \in \operatorname{supp}(H, d)$, i.e., $\sigma(\Delta_i)$ is not empty and also not the singleton $\{0\}$.*

Proof. Let $i \in \operatorname{supp}(H, d)$. We have $\Delta_i = 0$ if and only if $d_i = 0$ and $d^*_{i-1} = 0$. Indeed, if $\Delta_i = 0$, then $\operatorname{dom}(\Delta_i) = \ker(\Delta_i) = H_i$ and $\ker(\Delta_i) = \ker(d_i) \cap \ker(d^*_{i-1})$, so $d_i = 0$ and $d^*_{i-1} = 0$. The other implication is obvious. Since the differentials are densely defined and closed, this is equivalent to $d_i = 0$ and $d_{i-1} = 0$. But if $d_i = d_{i-1} = 0$, then $\mathcal{D}_i = 0$ by our non-degeneracy assumption, a contradiction to $i \in \operatorname{supp}(H, d)$. Therefore, $\Delta_i \neq 0$. Since $H_i \neq 0$, we have $\sigma(\Delta_i) \neq \emptyset$. If $\sigma(\Delta_i) = \{0\}$, then $\operatorname{supp}(P_i) = \{0\}$ with P_i the spectral measure associated to Δ_i as in the spectral theorem, and hence $\Delta_i = \int_{\{0\}} \operatorname{id}_{\mathbb{R}} \, dP_i = 0$, a contradiction. It follows that $\sigma(\Delta_i) \neq \{0\}$. ■

Theorem 5.1.7. *Let (H, d) and (H', d') be two Hilbert complexes, with Laplacians Δ and Δ', respectively. Assume that d and d' have closed range (in all degrees). Denote by N the inverse of the Laplacian for $(H, d) \hat{\circledast} (H', d')$ as in Proposition 1.2.4. Then the following are equivalent:*

(i) $N_i \colon (H \hat{\circledast} H')_i \to (H \hat{\circledast} H')_i$ *is a compact operator.*
(ii) $N_i|_{H_j \hat{\otimes} H'_k} \colon H_j \hat{\otimes} H'_k \to H_j \hat{\otimes} H'_k$ *is a compact operator for all $j, k \in \mathbb{Z}$ with $j + k = i$.*
(iii) $\sigma_e(\Delta_j) + \sigma(\Delta'_k) \subseteq \{0\}$ *and* $\sigma(\Delta_j) + \sigma_e(\Delta'_k) \subseteq \{0\}$ *for all $j, k \in \mathbb{Z}$ with $j + k = i$.*

If, in addition, (H, d) and (H', d') are nondegenerate, then the above are also equivalent to:

(iv) $\sigma_e(\Delta_j) = \sigma_e(\Delta'_k) = \emptyset$ *for all $j \in \operatorname{supp}(H, d)$ and $k \in \operatorname{supp}(H', d')$ with $j + k = i$.*
(v) $\sigma_e(\widetilde{\Delta}_i) = \emptyset$, *where $\widetilde{\Delta}$ is the Laplacian for the tensor product complex.*
(vi) For all $j \in \operatorname{supp}(H, d)$ and $k \in \operatorname{supp}(H', d')$ with $j + k = i$,

$$\dim(\mathcal{H}^j(H, d)) < \infty \quad \textit{and} \quad \dim(\mathcal{H}^k(H', d')) < \infty,$$

and the operators

$$N_j(H, d) \colon H_j \to H_j \quad \textit{and} \quad N_k(H', d') \colon H'_k \to H'_k$$

are compact.

Proof. From Proposition 5.1.2 we know that $d \hat{\circledast} d'$ has closed range, hence N_i is a bounded operator for all $i \in \mathbb{Z}$ by Lemma 1.2.5. By Proposition 1.2.8 and (5.1.5), N_i is compact if and only if

$$\sigma_e(\Delta_j) + \sigma(\Delta'_k) \subseteq \{0\} \quad \text{and} \quad \sigma(\Delta_j) + \sigma_e(\Delta'_k) \subseteq \{0\} \tag{5.1.6}$$

for all $j, k \in \mathbb{Z}$ such that $j + k = i$, so (i)⇔(iii). The equivalence (i)⇔(ii) is obvious as the Laplacian of the tensor product complex, and hence also N_i, respects the decomposition $(H \hat{\circledast} H')_i = \bigoplus_{j+k=i} H_j \hat{\otimes} H'_k$.

Now assume that (H, d) and (H', d') are nondegenerate. If both $j \in \operatorname{supp}(H, d)$ and $k \in \operatorname{supp}(H', d')$, then $\sigma(\Delta_j) \nsubseteq \{0\}$ and $\sigma(\Delta'_k) \nsubseteq \{0\}$ by Lemma 5.1.6. It is clear that $\sigma_e(\Delta_j) = \sigma_e(\Delta'_k) = \emptyset$ for j and k as in (iv) implies (5.1.6) for those j and k. If H_j or H'_k is trivial, then (5.1.6) holds since the Laplacian is then the zero operator with empty spectrum. This shows (iv) ⇒ (iii). Conversely, if (iii) holds true, suppose $j \in \operatorname{supp}(H, d)$ and $k \in \operatorname{supp}(H', d')$ with $j + k = i$. Then $\sigma(\Delta_j) \nsubseteq \{0\}$ and $\sigma(\Delta'_k) \nsubseteq \{0\}$ by Lemma 5.1.6 and hence (5.1.6) forces $\sigma_e(\Delta_j) = \sigma_e(\Delta'_k) = \emptyset$.

The equivalence (iv)⇔(v) is clear from (5.1.5) and non-degeneracy. We have $\sigma_e(\Delta_j) = \emptyset$ if and only if $N_j(H, d)$ is compact (so that $\sigma_e(\Delta_j) \subseteq \{0\}$) and $\dim(\ker(\Delta_j)) = \dim(\mathcal{H}^j(H, d)) <$

∞ (so that $0 \notin \sigma_e(\Delta_j)$ by item (vii) of Proposition 1.2.6), and similarly for $\sigma_e(\Delta'_k)$. This shows (iv)$\Leftrightarrow$(vi). ∎

We now provide several immediate corollaries concerning the non-compactness of N and, by Proposition 1.2.8, non-compactness of the minimal solution operators.

Corollary 5.1.8. *Let (H, d) and (H', d') be two nondegenerate Hilbert complexes as in Theorem 5.1.7. Assume that there is $j \in \mathbb{Z}$ such that $N_j(H, d)$ is not compact on H_j. Then*

$$N_{j+k} \colon (H \mathbin{\hat{\circledast}} H')_{j+k} \to (H \mathbin{\hat{\circledast}} H')_{j+k}$$

is not compact either for all $k \in \operatorname{supp}(H', d')$.

Proof. In this case, $j \in \operatorname{supp}(H, d)$ and $\sigma_e(\Delta_j)$ is not empty (it contains values other than 0) by Proposition 1.2.8. Now apply Theorem 5.1.7. ∎

Corollary 5.1.9. *Let (H, d) and (H', d') be two nondegenerate Hilbert complexes as in Theorem 5.1.7. Let Δ and Δ' be their respective Laplacians and denote by $\widetilde{\Delta}$ the Laplacian of the tensor product complex $(H \mathbin{\hat{\circledast}} H', d \mathbin{\hat{\circledast}} d')$.*

(i) If there exists $i \in \mathbb{Z}$ such that

$$\dim(\ker(\widetilde{\Delta}_i)) = \dim(\mathcal{H}^i(H \mathbin{\hat{\circledast}} H', d \mathbin{\hat{\circledast}} d')) = \infty,$$

then $N_i \colon (H \mathbin{\hat{\circledast}} H')_i \to (H \mathbin{\hat{\circledast}} H')_i$ is not compact.

(ii) If there exists $j \in \mathbb{Z}$ such that

$$\dim(\ker(\Delta_j)) = \dim(\mathcal{H}^j(H, d)) = \infty,$$

then $N_{j+k} \colon (H \mathbin{\hat{\circledast}} H')_{j+k} \to (H \mathbin{\hat{\circledast}} H')_{j+k}$ is not compact for all $k \in \operatorname{supp}(H', d')$.

Proof. In the first case $0 \in \sigma_e(\widetilde{\Delta}_i)$, while $j \in \operatorname{supp}(H, d)$ and $0 \in \sigma_e(\Delta_j)$ in the second case. Now apply Theorem 5.1.7. ∎

5.2. Tensor products of complexes of differential operators

Consider two complexes of differential operators, say (E, d^E) and (F, d^F) over manifolds M and N, respectively. We proceed similarly to the construction of the tensor product of Hilbert complexes in order to obtain a complex of differential operators on $M \times N$. Set

$$(E \circledast F)_i := \bigoplus_{j+k=i} E_j \boxtimes F_k,$$

where $E_j \boxtimes F_k := (\pi_M^* E_j) \otimes (\pi_N^* F_k)$, with $\pi_M \colon M \times N \to M$ and $\pi_N \colon M \times N \to N$ the projections, is a vector bundle over $M \times N$ with fiber $(E_j)_x \otimes (F_k)_y$ over $(x, y) \in M \times N$. If M and N are Riemannian and all vector bundles are Hermitian, then $M \times N$ and $(E \circledast F)_i$ are also equipped with metrics in a canonical way.

By $\Gamma_c(M, E_\bullet)$ we denote the $\mathbb{Z}$-graded vector space $\bigoplus_j \Gamma_c(M, E_j)$. Similarly, we define the space $\Gamma_c(N, F_\bullet)$. Their graded tensor product $\Gamma_c(M, E_\bullet) \circledast \Gamma_c(N, F_\bullet)$ is then defined as in (5.1.1). The following Lemma can be found in [BL92, p. 110]:

Lemma 5.2.1. *If (E, d^E) and (F, d^F) are complexes of differential operators, then there exists a unique complex of differential operators*

$$d_i^{E\circledast F}\colon \Gamma_c(M\times N,(E\circledast F)_i)\to \Gamma_c(M\times N,(E\circledast F)_{i+1})$$

such that the diagram

$$\begin{array}{ccccccc}
\cdots \xrightarrow{d^E\circledast d^F} & (\Gamma_c(M,E_\bullet)\circledast\Gamma_c(N,F_\bullet))_i & \xrightarrow{d^E\circledast d^F} & (\Gamma_c(M,E_\bullet)\circledast\Gamma_c(N,F_\bullet))_{i+1} & \xrightarrow{d^E\circledast d^F}\cdots \\
 & \downarrow{\scriptstyle \iota_i} & & \downarrow{\scriptstyle \iota_{i+1}} & \\
\cdots \xrightarrow{d^{E\circledast F}} & \Gamma_c(M\times N,(E\circledast F)_i) & \xrightarrow{d^{E\circledast F}} & \Gamma_c(M\times N,(E\circledast F)_{i+1}) & \xrightarrow{d^{E\circledast F}}\cdots
\end{array}$$

commutes, where $d^E\circledast d^F$ is given by

$$\bigoplus_{j+k=i}(d_j^E\otimes \mathrm{id}_{\Gamma_c(N,F_k)}+\sigma_j\otimes d_k^F)\colon (\Gamma_c(M,E_\bullet)\circledast\Gamma_c(N,F_\bullet))_i\to(\Gamma_c(M,E_\bullet)\circledast\Gamma_c(N,F_\bullet))_{i+1}, \tag{5.2.1}$$

with $\sigma_j\colon \Gamma_c(M,E_j)\to\Gamma_c(M,E_j)$ the multiplication by $(-1)^j$, and

$$\iota_i\colon (\Gamma_c(M,E_\bullet)\circledast\Gamma_c(N,F_\bullet))_i\to\Gamma_c(M\times N,(E\circledast F)_i)$$

is the canonical inclusion given by $\iota_i(s\otimes t)(x,y) := s(x)\otimes t(y)$ for $s\in\Gamma_c(M,E_j)$ and $t\in\Gamma_c(N,F_k)$. If (E,d^E) and (F,d^F) are elliptic complexes, then so is $(E\circledast F, d^{E\circledast F})$.

In the proof of Lemma 5.2.1, one uses the fact that, via ι_i, the space $(\Gamma_c(M,E_\bullet)\circledast\Gamma_c(N,F_\bullet))_i$ is sequentially dense in $\Gamma_c(M\times N,(E\circledast F)_i)$ for the usual LF-topology on this space.

Example 5.2.2. Let M and N be smooth manifolds, and consider their de Rham complexes

$$d_j^M\colon \Omega_c^j(M)\to\Omega_c^{j+1}(M)\quad\text{and}\quad d_k^N\colon\Omega_c^k(N)\to\Omega_c^{k+1}(N),$$

where $\Omega_c^j(M) := \Gamma_c(M,\Lambda^jT^*M)$ and similarly for $\Omega_c^k(N)$, so that $E_j=\Lambda^jT^*M$ and $F_k=\Lambda^kT^*N$ in the language of Lemma 5.2.1. Since the cotangent bundle of the product $M\times N$ splits as $T^*(M\times N)\cong\pi_M^*(T^*M)\oplus\pi_N^*(T^*N)$, we get

$$\Lambda^iT^*(M\times N)\cong\bigoplus_{j+k=i}\pi_M^*(\Lambda^jT^*M)\otimes\pi_N^*(\Lambda^kT^*N)=\bigoplus_{j+k=i}(\Lambda^jT^*M)\boxtimes(\Lambda^kT^*N) \tag{5.2.2}$$

from the properties of the exterior algebra functor, hence $(E\circledast F)_i$ is the vector bundle of i-forms on $M\times N$, and

$$d_i^{E\circledast F}\colon\Gamma_c(M\times N,\Lambda^iT^*(M\times N))\to\Gamma_c(M\times N,\Lambda^{i+1}T^*(M\times N))$$

is the de Rham differential for the product manifold, since this obviously extends (5.2.1) by the Leibniz rule for the exterior derivative. Note that when accounting for the isomorphism (5.2.2), the map $\iota_i\colon\bigoplus_{j+k=i}\Omega_c^j(M)\otimes\Omega_c^k(N)\to\Omega_c^i(M\times N)$ is given by $\iota_i(\omega\otimes\eta)=\pi_M^*\omega\wedge\pi_N^*\eta$. ♦

Example 5.2.3. Let M and N be complex manifolds, $E \to M$ and $F \to N$ two holomorphic vector bundles, and consider, for fixed $1 \leq p' \leq \dim_{\mathbb{C}}(M)$ and $1 \leq p'' \leq \dim_{\mathbb{C}}(N)$, the Dolbeault complexes

$$\overline{\partial}^E_{p',\bullet}\colon \Omega_c^{p',\bullet}(M,E) \to \Omega_c^{p',\bullet+1}(M,E) \quad \text{and} \quad \overline{\partial}^F_{p'',\bullet}\colon \Omega_c^{p'',\bullet}(N,F) \to \Omega_c^{p'',\bullet+1}(N,F),$$

where $\Omega_c^{p',q'}(M,E) \coloneqq \Gamma_c(M, \Lambda^{p',q'}T^*M \otimes E)$ denotes the space of compactly supported smooth (p',q') forms on M with values in E. One might expect the resulting tensor product complex on $M \times N$ to be the $\overline{\partial}^{E\boxtimes F}$-complex, with $E \boxtimes F \coloneqq \pi_M^* E \otimes \pi_N^* F$, restricted to those $(p'+p'',q)$ forms which are sections of

$$\pi_M^*(\Lambda^{p',0}T^*M) \otimes \pi_N^*(\Lambda^{p'',0}T^*N) \otimes \Lambda^{0,\bullet}T^*(M\times N) \otimes (E \boxtimes F). \tag{5.2.3}$$

This is true up to a sign factor. Consider the cochain complex

$$\overline{\partial}^E_{p',\bullet} \circledast (-1)^{p'}\overline{\partial}^F_{p'',\bullet}\colon \Omega_c^{p',\bullet}(M,E) \circledast \Omega_c^{p'',\bullet}(M,F) \to \Omega_c^{p',\bullet}(M,E) \circledast \Omega_c^{p'',\bullet}(M,F)$$

as in (5.2.1), and the dense inclusions (for the LF-topology)

$$\iota_q^{p',p''}\colon (\Omega_c^{p',\bullet}(M,E) \circledast \Omega_c^{p'',\bullet}(N,F))_q \to \bigoplus_{q'+q''=q} \Gamma_c(M\times N, (\Lambda^{p',q'}T^*M\otimes E) \boxtimes (\Lambda^{p'',q''}T^*N \otimes F)) \tag{5.2.4}$$

given, as in Lemma 5.2.1, by $\iota_q^{p',q'}(\omega \otimes \eta) \coloneqq \pi_M^*\omega \otimes \pi_N^*\eta$. We denote the right hand side of (5.2.4) by $\Omega_c(E,F)_q^{p',p''}$. Note that this may be identified with the space of smooth compactly supported sections of (5.2.3). According to the bundle isomorphism

$$\Lambda^{p,q}T^*(M\times N) \otimes (E\boxtimes F) \cong \bigoplus_{\substack{p'+p''=p\\ q'+q''=q}} (\Lambda^{p',q'}T^*M \otimes E) \boxtimes (\Lambda^{p'',q''}T^*N \otimes F),$$

the full space of (p,q) forms decomposes as $\Omega_c^{p,q}(M\times N, E\boxtimes F) \cong \bigoplus_{p'+p''=p} \Omega_c(E,F)_q^{p',p''}$. Now for $\omega \in \Omega_c^{p',q'}(M,E)$ and $\eta \in \Omega_c^{p'',q''}(N,F)$ with $p'+p''=p$ and $q'+q''=q$, we have $\iota_q^{p',p''}(\omega\otimes\eta) \in \Omega_c(E,F)_q^{p',p''}$ and, with $\overline{\partial}^{E\boxtimes F}$ being understood as up to the above isomorphism,

$$\overline{\partial}^{E\boxtimes F}_{p,q}(\iota_q^{p',p''}(\omega\otimes\eta)) = \pi_M^*(\overline{\partial}^E_{p',q'}\omega) \otimes \pi_N^*\eta + (-1)^{q'}\pi_M^*\omega \otimes \pi_N^*\big((-1)^{p'}\overline{\partial}^F_{p'',q''}\eta\big) \in \Omega_c(E,F)_{q+1}^{p',p''}$$

because the total degree of ω is $p'+q'$, and this is precisely $\iota_q^{p',p''}((\overline{\partial}^E_{p',\bullet} \circledast (-1)^{p'}\overline{\partial}^F_{p'',\bullet})(\omega\otimes\eta))$.

$$\begin{array}{ccc} (\Omega_c^{p',\bullet}(E) \circledast \Omega_c^{p'',\bullet}(F))_q & \xrightarrow{\overline{\partial}^E_{p',\bullet} \circledast (-1)^{p'}\overline{\partial}^F_{p'',\bullet}} & (\Omega_c^{p',\bullet}(E) \circledast \Omega_c^{p'',\bullet}(F))_{q+1} \\ \downarrow{\scriptstyle \iota_q^{p',p''}} & & \downarrow{\scriptstyle \iota_{q+1}^{p',p''}} \\ \Omega_c^{p'+p'',q}(M\times N, E\boxtimes F) & \xrightarrow{\overline{\partial}^{E\boxtimes F}} & \Omega_c^{p'+p'',q+1}(M\times N, E\boxtimes F) \end{array}$$

By Lemma 5.2.1, the restriction of $\overline{\partial}^{E\boxtimes F}$ to $\Omega_c(E,F)_\bullet^{p',p''}$ is the unique complex of differential operators extending $\overline{\partial}^E_{p',\bullet} \circledast (-1)^{p'}\overline{\partial}^F_{p'',\bullet}$ via $\iota_\bullet^{p',p''}$. Note that the situation is somewhat simpler (as simple as in Example 5.2.2) if one only considers $(0,q)$ forms. ♦

We now extend the above situation to the level of Hilbert complexes obtained from (E, d^E) and (F, d^F). First note that the inclusions ι_i extend to a unitary isomorphism of graded Hilbert spaces

$$\hat{\iota} := \bigoplus_i \hat{\iota}_i \colon L^2(M, E_\bullet) \mathbin{\hat{\circledast}} L^2(N, F_\bullet) \xrightarrow{\cong} L^2(M \times N, (E \circledast F)_\bullet),$$

where $L^2(M, E_\bullet) := \bigoplus_j L^2(M, E_j)$, and similarly for $L^2(N, F_\bullet)$ and $L^2(M \times N, (E \circledast F)_\bullet)$. The next result is Lemma 3.6 in [BL92]:

Lemma 5.2.4. *Let (E, d^E) and (F, d^F) be complexes of differential operators with Hermitian bundles over Riemannian manifolds, and $(E \circledast F, d^{E\circledast F})$ their tensor product as in Lemma 5.2.1. Then the diagram*

$$\begin{array}{ccccccc}
\cdots \xrightarrow{d_w^E \hat{\circledast} d_w^F} & \operatorname{dom}((d_w^E \mathbin{\hat{\circledast}} d_w^F)_i) & \xrightarrow{d_w^E \hat{\circledast} d_w^F} & \operatorname{dom}((d_w^E \mathbin{\hat{\circledast}} d_w^F)_{i+1}) & \xrightarrow{d_w^E \hat{\circledast} d_w^F} \cdots \\
& \cong \big\downarrow \hat{\iota}_i & & \cong \big\downarrow \hat{\iota}_{i+1} & \\
\cdots \xrightarrow{d_w^{E\circledast F}} & \operatorname{dom}(d_{i,w}^{E\circledast F}) & \xrightarrow{d_w^{E\circledast F}} & \operatorname{dom}(d_{i+1,w}^{E\circledast F}) & \xrightarrow{d_w^{E\circledast F}} \cdots
\end{array}$$

commutes, where d_w^E, d_w^F and $d_w^{E\circledast F}$ denote the (differentials of the) Hilbert complexes of the weak extensions of d^E, d^F and $d^{E\circledast F}$, respectively, and $d_w^E \mathbin{\hat{\circledast}} d_w^F$ is the differential of the tensor product Hilbert complex, see Definition 5.1.1. In other words, $\hat{\iota}$ is a unitary equivalence between $(L^2(M, E_\bullet), d_w^E) \mathbin{\hat{\circledast}} (L^2(N, F_\bullet), d_w^F)$ and $(L^2(M \times N, (E \circledast F)_\bullet), d_w^{E\circledast F})$. An analogous statement holds for the strong extensions.

In particular, Lemma 5.2.4 implies that the Gaffney extension of the $d^{E\circledast F}$-Laplacian, which is the Laplacian of the Hilbert complex $(L^2(M \times N, (E \circledast F)_\bullet), d_w^{E\circledast F})$, see (1.3.6), is a self-adjoint extension of the $d^{E\circledast F}$-Laplacian on $L^2(M \times N, (E \circledast F)_\bullet)$ that is unitarily equivalent to the Laplacian of the Hilbert complex $(L^2(M, E_\bullet) \mathbin{\hat{\circledast}} L^2(N, F_\bullet), d_w^E \mathbin{\hat{\circledast}} d_w^F)$. As a consequence, the two Laplacians share all of their spectral and operator theoretic properties.

5.3. Applications to the $\overline{\partial}^{E\boxtimes F}$-complex

We will now apply the general theory developed in the previous sections to the $\overline{\partial}$-Neumann problem. For a Hermitian manifold M and a Hermitian holomorphic vector bundle E over M, we consider, for fixed $1 \leq p \leq \dim_{\mathbb{C}}(M)$, the Dolbeault complex

$$\overline{\partial}^E \colon \Omega_c^{p,\bullet}(M, E) \to \Omega_c^{p,\bullet+1}(M, E), \tag{5.3.1}$$

see appendix B, and its Laplacians

$$\square_{p,q}^E := \overline{\partial}^{E,\dagger} \overline{\partial}^E + \overline{\partial}^E \overline{\partial}^{E,\dagger} \colon \Omega_c^{p,q}(M, E) \to \Omega_c^{p,q}(M, E),$$

where $\overline{\partial}^{E,\dagger}$ is the formal adjoint to $\overline{\partial}^E$. Recall from section 3.2 that this operator has a self-adjoint extension, called the Dolbeault Laplacian with $\overline{\partial}$-Neumann boundary conditions, given by

$$\square_{p,q}^E := \overline{\partial}_w^{E,*} \overline{\partial}_w^E + \overline{\partial}_w^E \overline{\partial}_w^{E,*} \colon \operatorname{dom}(\square_{p,q}^E) \subseteq L_{p,q}^2(M, E) \to L_{p,q}^2(M, E)$$

with domain

$$\mathrm{dom}(\Box^E_{p,q}) \coloneqq \{u \in \mathrm{dom}(\bar\partial^E_w) \cap \mathrm{dom}(\bar\partial^{E,*}_w) : \bar\partial^E_w u \in \mathrm{dom}(\bar\partial^{E,*}_w) \text{ and } \bar\partial^{E,*}_w \in \mathrm{dom}(\bar\partial^E_w)\},$$

where we denote by $\bar\partial^E_w$ the weak extension of (5.3.1) to a closed operator from $L^2_{p,q}(M,E)$ to $L^2_{p,q+1}(M,E)$, see section 1.3, and we write $\bar\partial^{E,*}_w$ for its Hilbert space adjoint $(\bar\partial^E_w)^* = (\bar\partial^{E,\dagger})_s$, see (1.3.1). As usual, $L^2_{p,q}(M,E) \coloneqq L^2(M, \Lambda^{p,q}T^*M \otimes E)$ denotes the space of square-integrable (p,q) forms on M with values in E. In this way, we obtain a Hilbert complex $(L^2_{p,\bullet}(M,E), \bar\partial^E_w)$ with Laplacian $\Box^E_{p,\bullet}$ for every $1 \le p \le \dim_{\mathbb{C}}(M)$.

The inverse of $\Box^E$, in the sense of Proposition 1.2.4, is customarily called the *$\bar\partial$-Neumann operator* and denoted by N^E. We denote by $N^E_{p,q}$ and $S^E_{p,q}$ the restrictions of N^E and S^E, respectively, to $L^2_{p,q}(M,E)$. By Lemma 1.2.5, $N_{p,q}$ is bounded if and only if $\bar\partial^E_w$ on both $(p,q-1)$ and (p,q) forms has closed range. In this case, the minimal (or canonical) solution operator S^E to the $\bar\partial^E$-equation is also bounded on $L^2_{p,q}(M,E)$ and on $L^2_{p,q+1}(M,E)$, and we have

$$S^E = \bar\partial^{E,*}_w N^E$$

on $L^2_{p,q}(M,E)$ by Proposition 1.2.6. The cohomology of the Hilbert complex $(L^2_{p,\bullet}(M,E), \bar\partial^E_w)$ is the *L^2-Dolbeault cohomology*,

$$\mathcal{H}^{p,q}_{L^2}(M,E) \coloneqq \mathcal{H}^q(L^2_{p,\bullet}(M,E), \bar\partial^E_w) = \ker(\bar\partial^E_w) \cap L^2_{p,q}(M,E) \,/\, \mathrm{img}(\bar\partial^E_w) \cap L^2_{p,q}(M,E),$$

and its reduced cohomology is the *reduced L^2-Dolbeault cohomology*,

$$\overline{\mathcal{H}}^{p,q}_{L^2}(M,E) \coloneqq \overline{\mathcal{H}}^q(L^2_{p,\bullet}(M,E), \bar\partial^E_w) = \ker(\bar\partial^E_w) \cap L^2_{p,q}(M,E) \,/\, \overline{\mathrm{img}(\bar\partial^E_w) \cap L^2_{p,q}(M,E)},$$

which is canonically isomorphic to $\ker(\Box^E_{p,q})$. For instance,

$$A^2(M,E) \coloneqq \ker(\bar\partial^E_w) \cap L^2(M,E) = \ker(\bar\partial^{E,*}_w \bar\partial^E_w) \cap L^2(M,E) \cong \overline{\mathcal{H}}^{0,0}_{L^2}(M,E) \tag{5.3.2}$$

is the space of square-integrable holomorphic sections of E, called the *Bergman space* of $E \to M$. Of course, the cohomology spaces $\mathcal{H}^{p,q}_{L^2}(M,E)$ and $\overline{\mathcal{H}}^{p,q}_{L^2}(M,E)$ coincide if $\bar\partial^E_w$ has closed range in $L^2_{p,q}(M,E)$. Our main result for this section is the following:

Theorem 5.3.1. *Let $E \to M$ and $F \to N$ be Hermitian holomorphic vector bundles over Hermitian manifolds. Then, for $0 \le p,q \le \dim_{\mathbb{C}}(M) + \dim_{\mathbb{C}}(N)$,*

$$\sigma(\Box^{E\boxtimes F}_{p,q}) = \bigcup_{\substack{p'+p''=p \\ q'+q''=q}} (\sigma(\Box^E_{p',q'}) + \sigma(\Box^F_{p'',q''})) \tag{5.3.3}$$

and

$$\sigma_e(\Box^{E\boxtimes F}_{p,q}) = \bigcup_{\substack{p'+p''=p \\ q'+q''=q}} (\sigma_e(\Box^E_{p',q'}) + \sigma(\Box^F_{p'',q''})) \cup (\sigma(\Box^E_{p',q'}) + \sigma_e(\Box^F_{p'',q''})), \tag{5.3.4}$$

where p' and q' range over $\{0,\dots,\dim_{\mathbb{C}}(M)\}$, and p'' and q'' range over $\{0,\dots,\dim_{\mathbb{C}}(N)\}$.

Proof. Fix p' and p'' for the moment and denote by $L^2(E,F)_q^{p',p''}$ the completion of the space $\Omega_c(E,F)_q^{p',p''}$ as in Example 5.2.3, with respect to the induced Hermitian structures. Consider the Hilbert complex $(L^2(E,F)_\bullet^{p',p''}, \overline{\partial}_w^{E\boxtimes F})$, obtained by taking the weak extension of $\overline{\partial}^{E\boxtimes F}$, restricted to $\Omega_c(E,F)_\bullet^{p',p''}$, to a closed operator on $L^2(E,F)_q^{p',p''}$. By Lemma 5.2.1 and Example 5.2.3, we know that this Hilbert complex is unitarily equivalent to

$$(L^2_{p',\bullet}(M,E) \hat{\circledast} L^2_{p'',\bullet}(N,F), \overline{\partial}_w^E \hat{\circledast} (-1)^{p'}\overline{\partial}_w^F),$$

which is the tensor product of $(L^2_{p',\bullet}(M,E), \overline{\partial}_w^E)$ and $(L^2_{p'',\bullet}(N,F), (-1)^{p'}\overline{\partial}_w^F)$, as in Definition 5.1.1. Now for $0 \le p,q \le \dim_{\mathbb{C}}(M) + \dim_{\mathbb{C}}(N)$, we have

$$L^2_{p,q}(M\times N, E\boxtimes F) \cong \bigoplus_{p'+p''=p} L^2(E,F)_q^{p',p''},$$

which is due to the fact that $M \times N$ is Hermitian and hence forms with different bidegree (but same total degree) are orthogonal. It follows that $(L^2_{p,\bullet}(M\times N, E\boxtimes F), \overline{\partial}_w^{E\boxtimes F})$ is unitarily equivalent to the direct sum of Hilbert complexes

$$\bigoplus_{p'+p''=p} (L^2_{p',\bullet}(M,E), \overline{\partial}_w^E) \hat{\circledast} (L^2_{p'',\bullet}(N,F), (-1)^{p'}\overline{\partial}_w^F). \tag{5.3.5}$$

Equations (5.3.3) and (5.3.4) now follow immediately from (5.1.4), (5.1.5) and (5.3.5). Note that the Laplacians of $(L^2_{p'',\bullet}(N,F), (-1)^{p'}\overline{\partial}_w^F)$ and $(L^2_{p'',\bullet}(N,F), \overline{\partial}_w^F)$ coincide and are equal to $\Box^F_{p'',\bullet}$. ■

Since the $\overline{\partial}$-complex is nondegenerate in the sense of Definition 5.1.5, we obtain the following characterization of compactness of the $\overline{\partial}$-Neumann operator from Theorem 5.1.7:

Theorem 5.3.2. *Let $E \to M$ and $F \to N$ be Hermitian holomorphic vector bundles over Hermitian manifolds such that $\overline{\partial}^E$ and $\overline{\partial}^F$ have closed range (in all bidegrees). Then for $0 \le p,q \le \dim_{\mathbb{C}}(M) + \dim_{\mathbb{C}}(N)$, the following are equivalent:*

(i) The $\overline{\partial}$-Neumann operator $N_{p,q}^{E\boxtimes F} \colon L^2_{p,q}(M\times N, E\boxtimes F) \to L^2_{p,q}(M\times N, E\boxtimes F)$ is compact.
(ii) $\sigma_e(\Box_{p,q}^{E\boxtimes F}) = \emptyset$.
(iii) $\sigma_e(\Box^E_{p',q'}) = \sigma_e(\Box^F_{p'',q''}) = \emptyset$ for all $0 \le p',q' \le \dim_{\mathbb{C}}(M)$ and $0 \le p'',q'' \le \dim_{\mathbb{C}}(N)$ with $p'+p''=p$ and $q'+q''=q$.
(iv) For all $0 \le p',q' \le \dim_{\mathbb{C}}(M)$ and $0 \le p'',q'' \le \dim_{\mathbb{C}}(N)$ with $p'+p''=p$ and $q'+q''=q$, the L^2-Dolbeault cohomology spaces

$$\mathcal{H}_{L^2}^{p',q'}(M,E) \quad \text{and} \quad \mathcal{H}_{L^2}^{p'',q''}(N,F)$$

have finite dimension and the $\overline{\partial}$-Neumann operators

$$N^E_{p',q'} \colon L^2_{p',q'}(M,E) \to L^2_{p',q'}(M,E) \quad \text{and} \quad N^F_{p'',q''} \colon L^2_{p'',q''}(N,F) \to L^2_{p'',q''}(N,F)$$

are compact.

Corollary 5.3.3. *Let $E \to M$ and $F \to N$ be Hermitian holomorphic vector bundles over complex manifolds such that $\overline{\partial}^E$ and $\overline{\partial}^F$ have closed range (in all bidegrees).*

(i) If the L^2-Dolbeault cohomology space $\mathcal{H}^{p,q}_{L^2}(M \times N, E \boxtimes F)$ has infinite dimension, then

$$N^{E\boxtimes F}_{p,q} \colon L^2_{p,q}(M \times N, E \boxtimes F) \to L^2_{p,q}(M \times N, E \boxtimes F)$$

is not compact.

(ii) If either of the Bergman spaces

$$A^2(M, E) = L^2(M, E) \cap \mathcal{O}(M, E) \quad \text{or} \quad A^2(N, F) = L^2(N, F) \cap \mathcal{O}(N, F)$$

of holomorphic L^2 sections of E, respectively F, has infinite dimension, then $N^{E\boxtimes F}_{p,q}$ is not compact for all $0 \le p, q \le \dim_{\mathbb{C}}(N)$, respectively $0 \le p, q \le \dim_{\mathbb{C}}(M)$.

Proof. This follows immediate from Corollary 5.1.9, by using $A^2(M, E) = \ker(\Box^E_{0,0}) \cong \mathcal{H}^{0,0}_{L^2}(M, E)$ as in (5.3.2). ■

Remark 5.3.4. (i) We can use higher degree L^2-Dolbeault cohomology spaces of one factor instead of the Bergman spaces as in Corollary 5.3.3 to conclude non-compactness, see Corollary 5.1.9.

(ii) The above results also apply when replacing $\overline{\partial}^E_w$ by the minimal (or strong) extensions (*i.e.*, the closure) of $\overline{\partial}^E \colon \Omega^{p,q}_c(M, E) \to \Omega^{p,q+1}_c(M, E)$, see section 1.3, and similarly for $\overline{\partial}^F_w$. This follows immediately from the fact that Lemma 5.2.4 also holds for the minimal extensions of differential operators.

(iii) We refer to section 2.2 for general conditions on when $\Box^E$ has closed range (and this in turn implies the same property for $\overline{\partial}^E_w$ by Lemma 1.2.5).

Example 5.3.5. Let $E \to M$ and $F \to N$ be as in Theorem 5.3.2, and set $m := \dim_{\mathbb{C}}(M)$ and $n := \dim_{\mathbb{C}}(N)$. Assume that N is compact, so that $\sigma_e(\Box^F_{0,q''}) = \emptyset$ for all $0 \le q'' \le n$ (see for instance Theorem 2.2.8). Then

$$\sigma_e(\Box^{E\boxtimes F}_{0,q}) = \bigcup_{\max\{q-n,0\} \le q' \le m} \left(\sigma_e(\Box^E_{0,q'}) + \sigma(\Box^F_{0,q-q'})\right)$$

by (5.3.4). From Theorem 5.3.2 it follows that $N^{E\boxtimes F}_{0,q}$ is compact if and only if $N^E_{0,q'}$ is compact and $\dim(\mathcal{H}^{0,q'}_{L^2}(M, E)) < \infty$ for all $\max\{q - n, 0\} \le q' \le m$. ♦

5.3.1. Products of Riemann surfaces. The statement of Theorem 5.3.1 can readily be generalized to the product of a finite number of manifolds and vector bundles. We will conclude this section by considering the situation of several one-dimensional factors. Recall that a *Riemann surface* is a complex manifold of dimension one. For simplicity, we will only treat $(0, q)$ forms, and we abbreviate

$$\Box^E_q := \Box^E_{0,q}, \quad S^E_q := S^E_{0,q}, \quad \text{and} \quad N^E_q := N^E_{0,q}.$$

Theorem 5.3.6. *Let M_j, $1 \le j \le n$ with $n \ge 2$ be Hermitian Riemann surfaces and $E_j \to M_j$ Hermitian holomorphic vector bundles, such that $\overline{\partial}^{E_j}$ has closed range for all j. Put*

$$M := M_1 \times \cdots \times M_n \quad \text{and} \quad E := \pi_1^* E_1 \otimes \cdots \otimes \pi_n^* E_n,$$

with $\pi_j\colon M \to M_j$ the projections.

(i) The operator N_0^E is compact if and only if, for all $1 \le j \le n$, the minimal solution operator $S_1^{E_j}$ is compact and $\dim(A^2(M_j, E_j)) < \infty$.

(ii) The operator N_n^E is compact if and only if, for all $1 \le j \le n$, the minimal solution operator $S_1^{E_j}$ is compact and $\dim(\mathcal{H}^{0,1}_{L^2}(M_j, E_j)) < \infty$.

(iii) The operator N_q^E with $q \in \{1, \dots, n-1\}$ is compact if and only if both N_0^E and N_n^E are compact. (Equivalently: $S_1^{E_j}$ is compact for all $1 \le j \le n$ and all factors have finite dimensional L^2-Dolbeault cohomology.)

(iv) If N_0^E is not compact, then N_q^E is also not compact for $q \in \{0, \dots, n-1\}$.

(v) If N_n^E is not compact, then N_q^E is also not compact for $q \in \{1, \dots, n\}$.

(vi) If $N_{q_0}^E$ is not compact for some $q_0 \in \{1, \dots, n-1\}$, then N_q^E is also not compact for all $q \in \{1, \dots, n-1\}$.

(vii) If $S_1^{E_j}$ is not compact for some $1 \le j \le n$, then N_q^E is not compact for all $q \in \{0, \dots, n\}$.

(viii) If there exists $j_0 \in \{1, \dots, n\}$ such that the Bergman space $A^2(M_{j_0}, E_{j_0})$ has infinite dimension, then N_q^E is not compact for all $0 \le q \le n-1$.

Proof of Theorem 5.3.6. The appropriate formula for the essential spectrum of $\Box_i^E$ in the case of several factors is

$$\sigma_e(\Box_q^E) = \bigcup_{\substack{K \in \{0,1\}^n \\ \sum_{j=1}^n K_j = q}} \bigcup_{j=1}^n \Big(\sigma_e(\Box_{K_j}^{E_j}) + \sum_{j' \ne j} \sigma(\Box_{K_{j'}}^{E_{j'}}) \Big), \tag{5.3.6}$$

and compactness of N_q^E is equivalent to $\sigma_e(\Box_q^E) = \emptyset$ by item (v) of Theorem 5.1.7. Concerning (i), we have

$$\sigma_e(\Box_0^E) = \bigcup_{j=1}^n \Big(\sigma_e(\Box_0^{E_j}) + \sum_{j' \ne j} \sigma(\Box_0^{E_{j'}}) \Big),$$

hence $\sigma_e(\Box_0^E) \subseteq \{0\}$ if and only if $\sigma_e(\Box_0^{E_j}) = \emptyset$ for all $1 \le j \le n$. This is the case if and only if all $N_0^{E_j}$ are compact (so that $\sigma_e(\Box_0^{E_j}) \subseteq \{0\}$) and $\dim(A^2(M_j, E_j)) = \dim(\mathcal{H}^{0,0}_{L^2}(M_j, E_j)) < \infty$ (so that $0 \notin \sigma_e(\Box_0^{E_j})$ by item (vii) of Proposition 1.2.6). Because compactness of $N_0^{E_j}$ is equivalent to compactness of both $S_0^{E_j} = 0$ and $S_1^{E_j}$ (see Proposition 1.2.8), (i) follows. For (ii), we use the same argument with the formula

$$\sigma_e(\Box_n^E) = \bigcup_{j=1}^n \Big(\sigma_e(\Box_1^{E_j}) + \sum_{j' \ne j} \sigma(\Box_1^{E_{j'}}) \Big).$$

Note that (i) and (ii) are applications of the several factor version of item (vi) of Theorem 5.1.7. If $q \in \{1, \dots, n-1\}$, then for every $1 \le j \le n$, there are $K \in \{0,1\}^n$ and $K' \in \{0,1\}^n$ which contribute to (5.3.6), and with $K_j = 0$ and $K'_j = 1$. Thus, $\sigma_e(\Box_q^E) = \emptyset$ if and only if $\sigma_e(\Box_0^{E_j}) = \sigma_e(\Box_1^{E_j}) = \emptyset$ for all $1 \le j \le n$, and this is equivalent to N_0^E and N_n^E being compact by the arguments in (i) and (ii). This proves (iii).

Suppose N_0^E is not compact. Then there must exist $j_0 \in \{1, \dots, n\}$ such that $\sigma_e(\Box_0^{E_{j_0}}) \ne \emptyset$. Let $q \in \{0, \dots, n-1\}$ and pick $K \subseteq \{0,1\}^n$ with $\sum_{j=1}^n K_j = q$ and $K_{j_0} = 0$. Then $\sigma_e(\Box_q^E)$

contains the infinite (since $\Box_0^{E_{j'}}$ and $\Box_1^{E_{j'}}$ are unbounded self-adjoint operators) set

$$\sigma_e(\Box_0^{E_{j_0}}) + \sum_{j'\neq j_0} \sigma(\Box_{K_{j'}}^{E_{j'}}),$$

so N_q^E is not compact. This proves (iv), and a similar argument shows (v). If there is $q_0 \in \{1,\dots,n-1\}$ such that $N_{q_0}^E$ is not compact, then one of N_0^E and N_n^E is not compact by (iii), and (vi) follows by combining (iv) and (v). For (vii), combine (i) to (iii).

If j_0 is as in (viii), then $N_0^{E_{j_0}}$ fails to be compact by (i), and hence N_q^E is not compact for $q \in \{0,\dots,n-1\}$ by (iv). ■

Remark 5.3.7. Theorem 5.3.6 holds more generally for the tensor product of n Hilbert complexes of the form $0 \to H_0 \to H_1 \to 0$. Note that in such a complex, $d_0 \colon H_0 \to H_1$ can be an arbitrary densely defined and closed operator.

Of course, the value of Theorems 5.3.2 and 5.3.6 depends on the number of situations in which one can prove or even characterize (non-) compactness of the $\bar\partial$-Neumann or minimal solution operators. One such situation will be discussed in Theorem 5.3.8 below.

5.3.2. The weighted $\bar\partial$-complex on $\mathbb{C}^n$: decoupled weights. Let $\Omega_j \subseteq \mathbb{C}$ for $1 \le j \le n$ be open sets, and consider the trivial line bundles $E_j := \Omega_j \times \mathbb{C} \to \Omega_j$. The choice of a metric on E_j corresponds to picking a function $\varphi_j \in C^\infty(\Omega_j, \mathbb{R})$, with the metric then being determined by $|(z,v)|^2 = |v|^2 e^{-\varphi_j(z)}$ for $(z,v) \in E_j$, see Example B.3.7. Identifying sections of E_j with complex-valued functions on Ω_j, the norm on $L^2(\Omega_j, E_j)$ becomes

$$\|f\|^2_{L^2(\Omega_j,E_j)} = \int_{\Omega_j} |f(z)|^2 e^{-\varphi_j(z)} d\lambda(z),$$

with λ being Lebesgue measure on $\mathbb{C}$. The product manifold $\Omega := \Omega_1 \times \cdots \times \Omega_n$ carries the trivial line bundle $E := \pi_1^* E_1 \otimes \cdots \otimes \pi_n^* E_n \cong \Omega \times \mathbb{C}$, where $\pi_j \colon \Omega \to \Omega_j$ is the projection onto the j^{th} factor, with induced fiber metric

$$|(z_1,\dots,z_n,v)|^2 = |v|^2 e^{-(\varphi_1(z_1)+\cdots+\varphi_n(z_n))},$$

and the integrated norm is

$$\|f\|^2_{L^2(\Omega,E)} = \int_{\Omega} |f(z)|^2 e^{-\varphi(z)} d\lambda(z),$$

where λ is now the Lebesgue measure on $\mathbb{C}^n$ and $\varphi\colon \Omega \to \mathbb{R}$ is

$$\varphi(z_1,\dots,z_n) := (\pi_1^*\varphi_1 + \cdots + \pi_n^*\varphi_n)(z_1,\dots,z_n) = \varphi_1(z_1) + \cdots \varphi_n(z_n).$$

We say that φ is a *decoupled weight*. Hence, under the canonical identification of sections of E with functions on Ω, the Hilbert space $L^2(\Omega, E)$ of square-integrable sections of E is isomorphic to the standard L^2 function space for the measure $e^{-\varphi}\lambda$ on Ω, which we denote by $L^2(\Omega, e^{-\varphi}\lambda)$. An analogous statement holds for the spaces of E-valued (p,q) forms.

Of course, all statements of Theorem 5.3.6 apply in this setting. We denote $\bar\partial_\varphi^* := \bar\partial^{E,*}$ and $\Box^\varphi := \Box^E$ if E is as above. In the following Theorem, we will consider the case where

$\Omega_j = \mathbb{C}$ for all $1 \leq j \leq n$ and all $\Delta\varphi_j$ define nontrivial *doubling measures.* This means that φ_j is not harmonic and there is $C > 0$ such that $\int_{B_{2r}(z)} \Delta\varphi_j \, d\lambda \leq C \int_{B_r(z)} \Delta\varphi_j \, d\lambda$ for all $z \in \mathbb{C}$ and $r > 0$. It is known from [MO09] (or [HH07, Theorem 2.3], with slightly stronger assumptions) that, under these conditions, $S_1^{E_j}$ is compact if and only if

$$\lim_{z\to\infty} \int_{B_1(z)} \Delta\varphi_j \, d\lambda = +\infty \tag{5.3.7}$$

holds. Using this condition and our previous results, we can characterize compactness of the $\overline{\partial}$-Neumann operator in terms of the decoupled weight:

Theorem 5.3.8. *Let $\varphi_j \in C^2(\mathbb{C}, \mathbb{R})$ for $1 \leq j \leq n$ with $n \geq 2$, and set $\varphi(z_1, \ldots, z_n) \coloneqq \varphi_1(z_1) + \cdots + \varphi_n(z_n)$. Assume that all φ_j are subharmonic and such that $\Delta\varphi_j$ defines a nontrivial doubling measure. Then*

(i) $\dim(\ker(\Box^{\varphi}_{0,0})) = \dim(A^2(\mathbb{C}^n, e^{-\varphi})) = \infty$,
(ii) $\ker(\Box^{\varphi}_{0,q}) = 0$ *for* $q \geq 1$,
(iii) $N^{\varphi}_{0,q}$ *is bounded for* $0 \leq q \leq n$,
(iv) $N^{\varphi}_{0,q}$ *with* $0 \leq q \leq n-1$ *is not compact, and*
(v) $N^{\varphi}_{0,n}$ *is compact if and only if*

$$\lim_{z\to\infty} \int_{B_1(z)} \operatorname{tr}(H_\varphi) \, d\lambda = \infty, \tag{5.3.8}$$

where $H_\varphi = (\partial^2\varphi/\partial z_j \partial \overline{z}_k)_{j,k=1}^n$ *is the complex Hessian of* φ*, see section 4.3.1.*

Proof. From [MMO03, Theorem C], it follows from our assumptions on φ that $\overline{\partial}_w$ has closed range in $L^2_{0,1}(\mathbb{C}, e^{-\varphi_j})$ for all j. By Proposition 5.1.2, this also implies that $\overline{\partial}_w$ has closed range in $L^2_{0,q}(\mathbb{C}^n, e^{-\varphi})$ for all $0 \leq q \leq n$, so that the $\overline{\partial}$-Neumann operator is bounded by Lemma 1.2.5.

Moreover, $\ker(\Box^{\varphi_j}_{0,1}) = 0$ for all j. In fact, by (4.3.1) we have

$$2\langle\!\langle \Box^{\varphi_j}_{0,1} u, u \rangle\!\rangle \geq \int_{\mathbb{C}} R^{E_j}(\sqrt{2}\tfrac{\partial}{\partial z}, \sqrt{2}\tfrac{\partial}{\partial \overline{z}}) |u_1 \, d\overline{z}|^2 \, e^{-\varphi_j} d\lambda = \int_{\mathbb{C}} \Delta\varphi_j |u_1|^2 \, e^{-\varphi_j} d\lambda \tag{5.3.9}$$

for all forms $u = u_1 \, d\overline{z} \in \operatorname{dom}(\overline{\partial}^*_{\varphi_j}) \cap \Omega^{0,1}(\mathbb{C})$, where we have used that $R^{E_j} = \partial\overline{\partial}\varphi_j = \frac{1}{4}\Delta\varphi_j \, dz \wedge d\overline{z}$, see Example B.3.7, and that $\sqrt{2}\frac{\partial}{\partial z}$ and $\frac{1}{\sqrt{2}} d\overline{z}$ are orthonormal sections of $T^{1,0}\mathbb{C}^n$ and $(T^{0,1}\mathbb{C}^n)^*$, respectively. If $u \in \ker(\Box^{\varphi_j}_{0,1})$, then u is smooth by elliptic regularity, and if z_0 is such that $\Delta\varphi_j(z_0) > 0$ (note that $\Delta\varphi_j \geq 0$ everywhere by subharmonicity), then $u = 0$ in a neighborhood of z_0 by (5.3.9). But then $u = 0$ everywhere since $\mathbb{C}$ is connected and by using a unique continuation principle of Aronszajn, see [Aro57] or [Dem12, p. 333]. We also have $\ker(\Box^{\varphi}_{0,q}) = 0$ for all $q \geq 1$, either by combining the above with the Künneth formula,

$$\ker(\Box^{\varphi}_{0,q}) \cong \bigoplus_{q_1+\cdots+q_n=q} \ker(\Box^{\varphi_1}_{0,q_1}) \,\hat{\otimes} \cdots \hat{\otimes}\, \ker(\Box^{\varphi_n}_{0,q_n}), \tag{5.3.10}$$

see Proposition 5.1.2, where $\hat{\otimes}$ denotes the Hilbert space tensor product, or directly by using the same argument as above, *i.e.*, applying (3.1.33) to the higher dimensional problem and using that $H_\varphi \neq 0$ at one point.

As a nontrivial doubling measure, $\Delta\varphi_j$ satisfies $\int_{\mathbb{C}} \Delta\varphi_j \, d\lambda = \infty$. Indeed, by [Hei01, Exercise 13.1] or [Chr91, Lemma 2.1], there exists $\alpha > 0$ such that

$$\left(\frac{R}{r}\right)^\alpha \int_{B_r(z)} \Delta\varphi_j \, d\lambda \leq \int_{B_R(z)} \Delta\varphi_j \, d\lambda$$

for all $z \in \mathbb{C}^n$ and $0 < r \leq R$, so just fix $r > 0$, choose z such that $\Delta\varphi_j(z) \neq 0$, and let $R \to \infty$. Consequently, the weighted Bergman space $A^2(\mathbb{C}, e^{-\varphi_j})$ has infinite dimension by [RS06, Theorem 3.2]. This also implies, by (5.3.10), that $\dim(\ker(\Box^\varphi_{0,0})) = \infty$. On the other hand, $\dim(A^2(\mathbb{C}, e^{-\varphi_j})) = \infty$ for at least one $1 \leq j \leq n$ implies that $N^\varphi_{0,q}$ cannot be compact for $0 \leq q \leq n-1$, see Theorem 5.3.6. This finishes the proof of (i) to (iv).

Again by Theorem 5.3.6, $N^\varphi_{0,n}$ is compact if and only if all $N^{\varphi_j}_{0,1}$ are compact, which is the case if and only if (5.3.7) holds for all $1 \leq j \leq n$. It remains to show that this is equivalent to (5.3.8). By a simple scaling argument, the claim is equivalent to

$$\int_{B_1(z_1)\times\cdots\times B_n(z_n)} \operatorname{tr}(H_\varphi)\, d\lambda = \frac{\pi^{n-1}}{4} \sum_{j=1}^n \int_{B_1(z_j)} \Delta\varphi_j \, d\lambda \to \infty \quad \text{as} \quad z = (z_1, \ldots, z_n) \to \infty, \tag{5.3.11}$$

and if (5.3.7) holds for all $1 \leq j \leq n$, then (5.3.11) is also satisfied. Conversely, if (5.3.11) is true, then choosing $z = \zeta e_k$ with $\zeta \in \mathbb{C}$ and e_k the k^{th} standard basis vector of $\mathbb{C}^n$ implies

$$\int_{B_1(\zeta)} \Delta\varphi_k \, d\lambda + \sum_{j\neq k} \int_{B_1(0)} \Delta\varphi_j \, d\lambda \to \infty \quad \text{as} \quad \zeta \to \infty,$$

so that $\lim_{\zeta\to\infty} \int_{B_1(\zeta)} \Delta\varphi_k \, d\lambda = \infty$ since the second term is bounded. This shows (v) and concludes the proof. ∎

Remark 5.3.9. (i) The doubling condition is satisfied if the $\Delta\varphi_j$ belong to A_∞, see [Ste93, p. 196], where we recall from Remark 4.3.5 that A_∞ is the union of the Muckenhoupt classes. As an example, $z \mapsto |z|^\alpha$ is in A_p for $p > 1$ if and only if $-2 < \alpha < 2(p-1)$, and defines a doubling measure for $-2 < \alpha$, cf. [Ste93, 6.4, p. 218]. Since $\Delta|z|^\alpha = \alpha^2|z|^{\alpha-2}$, we see that $\varphi_j(z) = |z|^\alpha$ satisfies the assumptions of Theorem 5.3.8 for all $\alpha \geq 4$ (so that φ is at least C^2).

(ii) Let $n \geq 2$ and consider the weight function

$$\varphi(z) = \sum_{j=1}^n |z_j|^{\alpha_j}, \tag{5.3.12}$$

where $\alpha_j \in \mathbb{R}$, $\alpha_j \geq 4$. Then $\varphi \in C^2(\mathbb{C}^n, \mathbb{R})$ and by the above

$$\lim_{z\to\infty} \operatorname{tr}(H_\varphi(z)) = \lim_{z\to\infty} \frac{1}{4} \sum_{j=1}^n \alpha_j^2 |z_j|^{\alpha_j - 2} = +\infty.$$

Therefore it follows from Theorem 5.3.8 that $N^{\varphi}_{0,q}$ with $0 \le q \le n-1$ is not compact while $N^{\varphi}_{0,n}$ is compact. Hence, the necessary condition (4.3.5) of Corollary 4.3.3 fails to be sufficient for compactness of $N^{\varphi}_{0,q}$ for $0 \le q \le n-1$ in general. Of course, by Remark 4.3.5 and Theorem 5.3.8, the integral condition (4.3.5) is both necessary and sufficient for compactness of $N^{\varphi}_{0,n}$ for plurisubharmonic decoupled weights φ with $\operatorname{tr}(H_\varphi) \in A_\infty$, such as (5.3.12).

(iii) Using a variation of the above decoupled weights, one easily sees that, for $q > n/2$, there is a plurisubharmonic function $\varphi_q \colon \mathbb{C}^n \to \mathbb{R}$, such that $N^{\varphi_q}_{0,k}$ is compact precisely for $q \le k \le n$. Indeed, one may take

$$\varphi_q(z_1, \dots, z_n) := |(z_1, \dots, z_{q-1})|^4 + |(z_q, \dots, z_n)|^4.$$

Then both of the spaces $A^2(\mathbb{C}^{q-1}, e^{-\varphi_1})$ and $A^2(\mathbb{C}^{n-q+1}, e^{-\varphi_2})$, where $\varphi_1 \colon \mathbb{C}^{q-1} \to \mathbb{R}$, $\varphi(z) := |z|^4$, and $\varphi_2 \colon \mathbb{C}^{n-q+1} \to \mathbb{R}$, $\varphi_2(z) := |z|^4$, have infinite dimension by a result of Shigekawa [Shi91, Lemma 3.4]: If $\varphi \colon \mathbb{C}^n \to \mathbb{R}$ is a smooth function such that

$$\lim_{z \to \infty} |z|^2 s_1(z) = +\infty, \tag{5.3.13}$$

where s_1 is the smallest eigenvalue of H_φ, then $A^2(\mathbb{C}^n, e^{-\varphi})$ has infinite dimension.[1] Moreover, the $\overline{\partial}$-Neumann operators $N^{\varphi_j}_{0,q}$ are compact for $q \ge 1$ and $j \in \{1, 2\}$, as is easily deduced by verifying that $\lim_{z\to\infty} s_1(z) = \infty$, which implies the compactness of $N^{\varphi_j}_{0,q}$ by (3.1.29) and Theorem 2.2.8. Since $n - q + 1 < n/2 + 1$ implies $n - q + 1 \le n/2 \le q - 1$, one obtains from Theorem 5.3.2 that $N^{\varphi}_{0,k}$ is compact exactly for $k = q - 1 + j$ with $j \ge 1$, as claimed.

[1] Condition (5.3.13) is not sharp: the function $\varphi(z_1, z_2) = |z_1|^4 + |z_1 z_2|^2$ on $\mathbb{C}^2$ does not satisfy (5.3.13). Nevertheless, the corresponding space $A^2(\mathbb{C}^2, e^{-\varphi})$ is of infinite dimension since it contains all polynomials in z_1, see the computation in [Has14, p. 125].

APPENDIX A

Background on differential geometry

We provide here some of the needed background on differential geometry. As a general assumption, all our manifolds are C^∞ smooth and second countable. The tangent bundle of a manifold M is denoted by TM, and its dual by T^*M. For any smooth map $f\colon M \to N$ between manifolds and $p \in M$, we have the induced tangent maps $T_pf\colon T_pM \to T_{f(p)}N$. If $E \to M$ is any (smooth) vector bundle over M, then we write $\Gamma(M, E)$ for the space of smooth sections of E, with a special notation for $\Omega^k(M, E) \coloneqq \Gamma(M, \Lambda^k T^*M \otimes E)$, the space of smooth k-forms on M with values in E. Equivalently, these are the $C^\infty(M)$-multilinear alternating maps $\Gamma(M, TM)^{\times k} \to \Gamma(M, E)$. We write $\Lambda T^*M \coloneqq \bigoplus_k \Lambda^k T^*M$ and $\Omega(M, E) \coloneqq \Gamma(M, \Lambda T^*M \otimes E) = \bigoplus_k \Omega^k(M, E)$. The bundle of endomorphisms of E is denoted by $\mathrm{End}(E) \to M$, and if $F \to M$ is another vector bundle, then $\mathrm{Hom}(E, F) \to M$ is the bundle of morphisms from E to F.

If $A \in \mathrm{Hom}(E_1 \otimes E_2, F)$ is a morphism of vector bundles, then we will agree that forming the expression $A(s \otimes t)$ with $s \in E_1$ and $t \in E_2$ means that s and t are assumed to be in the fibers of E_1 and E_2 over the same point of M. For a tangent vector $X \in TM$, we have the *insertion operator* $\mathrm{ins}_X\colon \Lambda^\bullet T^*M \to \Lambda^{\bullet-1} T^*M$, defined by

$$\mathrm{ins}_X(u)(Y_1, \ldots, Y_{k-1}) \coloneqq u(X, Y_1, \ldots, Y_k) \tag{A.0.1}$$

for every $u \in \Lambda^k T^*M$. If $\alpha \in \Lambda T^*M$, then we have the exterior multiplication morphism $\varepsilon(\alpha)\colon \Lambda T^*M \to \Lambda T^*M$, defined by

$$\varepsilon(\alpha)u \coloneqq \alpha \wedge u.$$

Both ins_X and $\varepsilon(\alpha)$ extend to E-valued forms by $\mathrm{ins}_X(u \otimes e) \coloneqq \mathrm{ins}_X(u) \otimes e$, and similarly for $\varepsilon(\alpha)$. We shall also sometimes see these two maps as $\mathrm{ins}\colon TM \otimes \Lambda T^*M \otimes E \to \Lambda T^*M \otimes E$ and $\varepsilon\colon T^*M \otimes \Lambda T^*M \otimes E \to \Lambda T^*M \otimes E$ (specializing to the wedge product with one-forms), with $\mathrm{ins}(X \otimes u) \coloneqq \mathrm{ins}_X(u)$ and $\varepsilon(\alpha \otimes u) \coloneqq \varepsilon(\alpha)(u)$.

Most of the material here can be found in standard textbooks on differential geometry, for instance [Lee13] or [Lee09].

A.1. Connections and exterior covariant derivatives

Let M be a smooth manifold and $E \to M$ a smooth real or complex vector bundle. A *(linear) connection* on E is a (real resp. complex) linear map $\nabla\colon \Gamma(M, E) \to \Omega^1(M, E) \coloneqq \Gamma(M, T^*M \otimes E)$, such that $\nabla_X(fs) = X(f)s + f\nabla_X s$ for all $f \in C^\infty(M)$, $X \in \Gamma(M, TM)$, and $s \in \Gamma(M, E)$, where one writes $\nabla_X s \coloneqq \mathrm{ins}_X(\nabla s) = (\nabla s)(X)$. Put differently, one has

$\nabla(fs) = df \otimes s + f\nabla s$ as E-valued 1-forms. Connections on vector bundles $E, F \to M$ give rise to connections on the bundles $E \oplus F$, $E \otimes F$, E^*, $\Lambda^k E$, and so on. In particular, every connection ∇^{TM} on the tangent bundle $TM \to M$ defines a connection on ΛT^*M, satisfying

$$\nabla_X^{\Lambda T^*M}(\alpha \wedge \beta) = \nabla_X^{\Lambda T^*M}\alpha \wedge \beta + \alpha \wedge \nabla_X^{\Lambda T^*M}\beta$$

for all differential forms α and β on M and vector fields X, and

$$(\nabla_X^{T^*M}(\alpha))(Y) = X(\alpha(Y)) - \alpha(\nabla_X^{TM}Y)$$

for $\alpha \in \Omega^1(M)$ and $X, Y \in \Gamma(M, TM)$. This connection is simply the restriction of the one induced on tensor fields.

Example A.1.1. (i) If (M, g) is a Riemannian manifold, then there exists a unique connection ∇ on TM such that $\nabla_X Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \Gamma(M, TM)$ (*i.e.*, this connection is *torsion free*), and such that

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \tag{A.1.1}$$

for all vector fields X, Y, Z on M. The latter property is equivalent to $\nabla g = 0$, where ∇ now also denotes the induced connection on $T^*M \otimes T^*M$, which g is a section of. This is called the *Levi–Civita connection*.

(ii) If E is any (real or) complex vector bundle with Hermitian metric h, then there always exist connections ∇ on E such that

$$X(h(s, t)) = h(\nabla_X s, t) + h(s, \nabla_X t)$$

holds for all (real) vector fields $X \in \Gamma(M, TM)$ and all smooth sections s, t of E. Such connections are called *Hermitian*, *metric compatible*, or *compatible with the Hermitian metric*. If $E \to M$ is a (complex) Hermitian vector bundle with metric connection ∇, then we may also compute covariant derivatives in complex directions. If $Z \in \Gamma(M, TM \otimes_{\mathbb{R}} \mathbb{C})$, then it follows that $i\,\mathrm{Im}(Z)(h(s, t)) = h(i\nabla_{\mathrm{Im}(Z)}s, t) - h(s, i\nabla_{\mathrm{Im}(Z)}t)$, hence

$$Z(h(s, t)) = h(\nabla_Z s, t) + h(s, \nabla_{\overline{Z}} t) \tag{A.1.2}$$

is the extension of (A.1.1) to Hermitian metrics and complex vector fields. ♦

If ∇ and ∇' are two connections on a vector bundle $\pi\colon E \to M$, then their difference $\nabla - \nabla'$ satisfies

$$(\nabla - \nabla')(fs) = df \otimes s + f\nabla s - df \otimes s - f\nabla' s = f(\nabla - \nabla')s,$$

so that $\nabla - \nabla'$ is given by the action of a bundle morphism $E \to \Lambda^1 T^*M \otimes E$, hence there is $A \in \Omega^1(M, \mathrm{End}(E))$ with $(\nabla - \nabla')s = As$. Conversely, given any connection ∇ and any one-form A with values in $\mathrm{End}(E)$, the operator $\nabla + A$ is again a connection on E, so that the set of all connections on E may be described as the affine space

$$\{\nabla + A : A \in \Omega^1(M, \mathrm{End}(E))\}$$

for any given reference connection ∇. If, in addition, E is Hermitian, then it is easy to see that the set of all *metric* connections on E is given by

$$\{\nabla + A : A \in \Omega^1(M, \mathrm{End}(E)) \text{ and } A(X)^* = -A(X) \text{ for all } X \in TM\}, \tag{A.1.3}$$

again with ∇ any given reference metric connection.

Let $\psi\colon \pi^{-1}(U) \to U \times \mathbb{R}^r$ be a local trivialization of E over an open subset $U \subseteq M$. The trivial connection $(f_1, \ldots, f_n) \mapsto (df_1, \ldots, df_r)$ on $U \times \mathbb{R}^r$ defines a connection $(\mathrm{id}_{T^*U} \otimes \psi^{-1}) \circ (d, \ldots, d) \circ \psi$ on $\pi^{-1}(U) \to U$, and it follows from the above that for any connection ∇ on E there exists a one-form $\theta_\psi \in \Omega^1(U, \mathrm{End}(E))$ such that

$$(\nabla s)|_U = ((\mathrm{id}_{T^*U} \otimes \psi^{-1}) \circ (d, \ldots, d) \circ \psi + \theta_\psi) s|_U.$$

We call θ_ψ the *connection form* associated to ∇ and ψ. If $\xi_j\colon U \to \pi^{-1}(U)$, $1 \le j \le r$, is the local frame obtained from ψ, *i.e.*, $\xi_j(x) \coloneqq \psi^{-1}(x, e_j)$ with e_j the j^{th} standard unit vector in $\mathbb{R}^r$, then

$$\nabla_X \xi_j = \theta_\psi(X)\xi_j.$$

A.1.1. Exterior covariant derivatives. Let $E \to M$ be a vector bundle. Every connection ∇ on E extends uniquely to a family of linear operators

$$d^\nabla\colon \Omega^k(M, E) \to \Omega^{k+1}(M, E),$$

called the *exterior covariant derivative* associated to ∇, such that $d^\nabla s = \nabla s$ for $s \in \Gamma(M, E)$ and

$$d^\nabla(\alpha \wedge u) = d\alpha \wedge u + (-1)^k \alpha \wedge d^\nabla u \tag{A.1.4}$$

for all $\alpha \in \Omega(M)$ and $u \in \Omega^k(M, E)$. One can show that

$$\begin{aligned} d^\nabla u(X_0, \ldots, X_k) = &\sum_{0 \le i \le k} (-1)^i \nabla_{X_i}(u(X_0, \ldots, \widehat{X_i}, \ldots, X_k)) + \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} u([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k), \end{aligned} \tag{A.1.5}$$

for $u \in \Omega^k(M, E)$ and vector fields $X_0, \ldots, X_k$, where as usual $\widehat{X_i}$ means that X_i is omitted. If ∇^{TM} is a torsion free connection on M, and ∇^E is a connection on $E \to M$, then these induce a connection $\widetilde{\nabla}$ on $\Lambda^\bullet T^*M \otimes E$, and (A.1.5) implies that $d^{\nabla^E} = \varepsilon \circ \widetilde{\nabla}$, where $\varepsilon\colon T^*M \otimes (\Lambda^\bullet T^*M \otimes E) \to \Lambda^{\bullet+1} T^*M \otimes E$ is the wedge product map, *cf.*, [Lee09, Theorem 12.56]. Note that

$$d^{\nabla^E} = e^j \wedge \widetilde{\nabla}_{e_j} \tag{A.1.6}$$

for every local frame $\{e_j\}_{j=1}^{\dim(M)}$ of TM, and where $\{e^j\}_{j=1}^{TM}$ is the corresponding dual frame of T^*M. In cases where the choice of a connection ∇ on E is implied, we will often write d^E instead of d^∇.

If $u \in \Omega^k(M, E)$ and $v \in \Omega^l(M, E^*)$, then their wedge product $u \wedge_{\mathrm{ev}} v \in \Omega^{k+l}(M)$ is obtained by combining the wedge product on forms with the evaluation morphism $\mathrm{ev}\colon E \otimes E^* \to \mathbb{C}$, $s \otimes t \mapsto t(s)$. We can also define the wedge product $v \wedge_{\mathrm{ev}} u$ in the obvious way, and then we have $u \wedge_{\mathrm{ev}} v = (-1)^{kl} v \wedge_{\mathrm{ev}} u$. If ∇ is a connection on E with induced exterior covariant derivative d^E, and if $d^{E^*}\colon \Omega(M, E^*) \to \Omega(M, E^*)$ denotes the exterior covariant derivative induced by the dual connection on E^*, then this pairing satisfies

$$d(u \wedge_{\mathrm{ev}} v) = d^E u \wedge_{\mathrm{ev}} v + (-1)^k u \wedge_{\mathrm{ev}} d^{E^*} v. \tag{A.1.7}$$

Indeed, first note that if $s \in \Gamma(M, E)$ and $t \in \Gamma(M, E^*)$, then

$$d(t(s))(X) = t(\nabla_X s) + (\nabla_X t)(s) = (d^E s \wedge_{\mathrm{ev}} t)(X) + (s \wedge_{\mathrm{ev}} d^{E^*} t)(X)$$

holds for every vector field X. Thus, the definition of the induced connection on E^* is equivalent to the derivation rule $d(t(s)) = d^E s \wedge_{\mathrm{ev}} t + s \wedge_{\mathrm{ev}} d^{E^*} t$. By linearity, it suffices to establish (A.1.7) for $u = \alpha \otimes s$ and $v = \beta \otimes t$ with $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, $s \in \Gamma(M, E)$, and $t \in \Gamma(M, E^*)$. Then $u \wedge_{\mathrm{ev}} v = t(s)\, \alpha \wedge \beta$ and

$$\begin{aligned}
d(u \wedge_{\mathrm{ev}} v) &= \\
&= d(t(s)\, \alpha \wedge \beta) \\
&= d(t(s)) \wedge \alpha \wedge \beta + t(s)\, d\alpha \wedge \beta + (-1)^k t(s)\, \alpha \wedge d\beta \\
&= (d^E s \wedge_{\mathrm{ev}} t + s \wedge_{\mathrm{ev}} d^{E^*} t) \wedge \alpha \wedge \beta + (d\alpha \otimes s) \wedge_{\mathrm{ev}} (\beta \otimes t) + (-1)^k (\alpha \otimes s) \wedge_{\mathrm{ev}} (d\beta \otimes t) \\
&= (d\alpha \otimes s + (-1)^k \alpha \wedge d^E s) \wedge_{\mathrm{ev}} (\beta \otimes t) + (-1)^k (\alpha \otimes s) \wedge_{\mathrm{ev}} (d\beta \otimes t + (-1)^l \beta \wedge d^{E^*} t) \\
&= d^E(\alpha \otimes s) \wedge_{\mathrm{ev}} (\beta \otimes t) + (-1)^k (\alpha \otimes s) \wedge_{\mathrm{ev}} d^{E^*}(\beta \otimes t) \\
&= d^E u \wedge_{\mathrm{ev}} v + (-1)^k u \wedge_{\mathrm{ev}} d^{E^*} v,
\end{aligned}$$

which is exactly (A.1.7). More generally, if $A\colon E \otimes E' \to E''$ is a bundle morphism, and all three vector bundles come equipped with connections such that

$$\nabla_X (A(s \otimes t)) = A(\nabla_X s \otimes t) + A(s \otimes \nabla_X t) \tag{A.1.8}$$

is valid for all vector fields X and sections s and t of E and E'', respectively, then one shows that

$$d^{E''}(u \wedge_A v) = d^E u \wedge_A v + (-1)^k u \wedge_A d^{E'} v \tag{A.1.9}$$

for all $u \in \Omega^k(M, E)$, $v \in \Omega^l(M, E')$, see [Bal06, p. 5], and where the wedge product $u \wedge_A v \in \Omega^{k+l}(M, E'')$ is defined by using A, *i.e.*, $(\alpha \otimes s) \wedge_A (\beta \otimes t) \coloneqq (\alpha \wedge \beta) \otimes A(s \otimes t)$. The requirement (A.1.8) may be restated as $\nabla A = 0$, with ∇ the induced connection on $\mathrm{Hom}(E \otimes E', E'')$. For example, this assumption is valid for the evaluation morphism $\mathrm{ev}\colon \mathrm{Hom}(E, E') \otimes E \to E'$, where the connection on $\mathrm{Hom}(E, E')$ is the one induced from the connections on E and E', see [Lee09, Proposition 12.62].

A.1.2. Curvature. If ∇ is a connection on E, with exterior covariant derivative d^∇, then we may form the operator

$$d^\nabla \circ d^\nabla \colon \Gamma(M, E) \to \Omega^2(M, E).$$

For a function $f \in C^\infty(M)$ and a section $s \in \Gamma(M,E)$, it follows that

$$(d^\nabla \circ d^\nabla)(fs) = d^\nabla(df \otimes s + f d^\nabla s) = -df \wedge d^\nabla s + df \wedge d^\nabla s + f(d^\nabla \circ d^\nabla)s = f(d^\nabla \circ d^\nabla)s$$

since $d^2 = 0$, hence $d^\nabla \circ d^\nabla$ is actually given by the action of a vector bundle morphism $E \to \Lambda^2 T^*M \otimes E$. It follows that there is a two-form $R^\nabla \in \Omega^2(M, \mathrm{End}(E))$ such that $d^\nabla d^\nabla s = R^\nabla \wedge_{\mathrm{ev}} s$ holds for all $s \in \Gamma(M,E)$, where $\mathrm{ev}\colon \mathrm{End}(E) \otimes E \to E$ is the evaluation map. Since, for $\alpha \in \Omega^k(M)$ and $s \in \Gamma(M,E)$,

$$\begin{aligned}(d^\nabla \circ d^\nabla)(\alpha \otimes s) &= d^\nabla(d\alpha \otimes s + (-1)^k \alpha \wedge d^\nabla s) = \\ &= (-1)^{k+1} d\alpha \wedge d^\nabla s + (-1)^k d\alpha \wedge d^\nabla s + \alpha \wedge (d^\nabla \circ d^\nabla)s = \\ &= \alpha \wedge (R^\nabla \wedge_{\mathrm{ev}} s) = R^\nabla \wedge_{\mathrm{ev}} (\alpha \otimes s),\end{aligned}$$

the equality $(d^\nabla \circ d^\nabla)u = R^\nabla \wedge_{\mathrm{ev}} u$ continues to hold for $u \in \Omega(M,E)$.

Definition A.1.2. The differential form $R^\nabla \in \Omega^2(M, \mathrm{End}(E))$ is called the *curvature (form)* of ∇. The connection ∇ is called *flat* if $R^\nabla = 0$, *i.e.*, if $(\Omega^\bullet(M,E), d^\nabla)$ is a cochain complex.

Remark A.1.3. If s is a section of E and X and Y are two vector fields on M, then (A.1.5) applied to the one-form ∇s implies

$$\begin{aligned}R^\nabla(X,Y)s &= (d^\nabla(\nabla s))(X,Y) \\ &= \nabla_X((\nabla s)(Y)) - \nabla_Y((\nabla s)(X)) - (\nabla s)([X,Y]) \\ &= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.\end{aligned}$$

Remark A.1.4. While d^∇ does not square to zero in general, one has

$$d^\nabla R^\nabla = 0, \tag{A.1.10}$$

where here d^∇ is really the exterior covariant derivative associated to the connection induced on $\mathrm{End}(E)$. It follows from $(d^\nabla)^3 u = d^\nabla(R^\nabla \wedge_{\mathrm{ev}} u) = (d^\nabla R^\nabla) \wedge_{\mathrm{ev}} u + R^\nabla \wedge_{\mathrm{ev}} d^\nabla u$ (by (A.1.9)) and $(d^\nabla)^3 u = (d^\nabla)^2 d^\nabla u = R^\nabla \wedge_{\mathrm{ev}} d^\nabla u$, and is called the *second Bianchi identity*, see [Lee09, section 12.11].

Example A.1.5. Let $E \to M$ be a vector bundle with connection ∇^E. On E^*, we have the dual connection ∇^{E^*}, defined by $(\nabla^{E^*}_X \varphi)(s) \coloneqq X(\varphi(s)) - \varphi(\nabla^E_X s)$ for $s \in \Gamma(M,E)$ and $\varphi \in \Gamma(M,E^*)$. If R^E and R^{E^*} denote the curvatures of ∇^E and ∇^{E^*}, respectively, then a quick computation using Remark A.1.3 shows that

$$(R^{E^*}(X,Y)\varphi)(s) = -\varphi(R^E(X,Y)s),$$

i.e., $R^{E^*}(X,Y) = -(R^E(X,Y))^* \in \mathrm{End}(E^*)$, the dual operator. If E is equipped with a Hermitian metric $\langle \bullet, \bullet \rangle$, and E^* carries the dual metric, defined such that $\langle \iota(s), \iota(t) \rangle_{E^*} =$

$\langle t, s\rangle_E$, with $\iota\colon E \to E^*$ the (conjugate linear) metric isomorphism, then $\iota(s)(t) = \langle t, s\rangle$ and the above implies

$$\langle R^E(X,Y)s, t\rangle_E = -\langle R^{E^*}(X,Y)\iota(t), \iota(s)\rangle_{E^*} \tag{A.1.11}$$

for $s, t \in \Gamma(M, E)$.

Now suppose that E is Hermitian with metric $\langle \bullet, \bullet\rangle$, and ∇^E is a metric connection. Then

$$\langle R^E(X,Y)s, t\rangle = -\langle s, R^E(X,Y)t\rangle \tag{A.1.12}$$

for all $X, Y \in \Gamma(M, TM)$ and $s, t \in \Gamma(M, E)$ since, by Remark A.1.3,

$$\begin{aligned}
\langle R^E(X,Y)s, t\rangle &= \langle \nabla_X\nabla_Y s, t\rangle - \langle \nabla_Y\nabla_X s, t\rangle - \langle \nabla_{[X,Y]}s, t\rangle \\
&= X\langle \nabla_Y s, t\rangle - \langle \nabla_Y s, \nabla_X t\rangle - Y\langle \nabla_X s, t\rangle + \langle \nabla_X s, \nabla_Y t\rangle - \\
&\qquad - [X,Y]\langle s, t\rangle + \langle s, \nabla_{[X,Y]}t\rangle \\
&= XY\langle s, t\rangle - X\langle s, \nabla_Y t\rangle - Y\langle s, \nabla_X t\rangle + \langle s, \nabla_Y\nabla_X t\rangle - \\
&\qquad - YX\langle s, t\rangle + Y\langle s, \nabla_X t\rangle + X\langle s, \nabla_Y t\rangle - \langle s, \nabla_X\nabla_Y t\rangle - \\
&\qquad - [X,Y]\langle s, t\rangle + \langle s, \nabla_{[X,Y]}t\rangle \\
&= \langle s, (\nabla_Y\nabla_X - \nabla_X\nabla_Y + \nabla_{[X,Y]})t\rangle \\
&= -\langle s, R^E(X,Y)t\rangle.
\end{aligned}$$

In particular,

$$R^{E^*}(X,Y)\iota(s) = \iota(R^E(X,Y)s) \tag{A.1.13}$$

by combining the above. ⧫

Remark A.1.6. Let $(E, \langle \bullet, \bullet\rangle)$ be a Hermitian vector bundle, with metric connection ∇^E and associated curvature R^E. For fixed $e \in E_x$, the map $(X, Y) \mapsto \langle R^E(X, \overline{Y})e, e\rangle$ is a Hermitian quadratic form on $T_xM \otimes_{\mathbb{R}} \mathbb{C}$, since

$$\langle R^E(X, \overline{Y})e, e\rangle = -\langle e, R^E(\overline{X}, Y)e\rangle = \overline{\langle R^E(Y, \overline{X})e, e\rangle}$$

by (A.1.12). In particular, there exists an orthonormal $\mathbb{C}$-basis $\{\xi_j\}_{j=1}^n$ of $(T_xM \otimes_{\mathbb{R}} \mathbb{C}, \langle \bullet, \bullet\rangle)$ such that $\langle R^E(\xi_j, \overline{\xi}_k)e, e\rangle = r_j(e)\delta_{jk}$ for all $1 \le j, k \le n$, with $r_j(e) \in \mathbb{R}$.

Example A.1.7. Let $E \to M$ and $F \to M$ be two vector bundles with connections ∇^E and ∇^F, respectively, and let $\nabla^{E\otimes F}$ be the tensor product connection on $E \otimes F$, defined by $\nabla^{E\otimes F}_X(s \otimes t) \coloneqq \nabla^E_X s \otimes t + s \otimes \nabla^E_X t$. Denote by R^E, R^F, and $R^{E\otimes F}$ the curvatures of these connections. Then $R^{E\otimes F} = R^E \otimes \mathrm{id}_F + \mathrm{id}_E \otimes R^F$ in the sense that

$$R^{E\otimes F}(X,Y) = R^E(X,Y) \otimes \mathrm{id}_F + \mathrm{id}_E \otimes R^F(X,Y) \in \Gamma(M, \mathrm{End}(E \otimes F))$$

for all vector fields $X, Y \in \Gamma(M, TM)$. Indeed, for $s \in \Gamma(M, E)$ and $t \in \Gamma(M, F)$, we have

$$\begin{aligned}
R^{E\otimes F} \wedge_{\mathrm{ev}} (s \otimes t) &= d^{E\otimes F}(d^{E\otimes F}(s \otimes t)) = d^{E\otimes F}(d^E s \wedge_\otimes t + s \wedge_\otimes d^F t) = \\
&= (d^E d^E s)\wedge_\otimes t + s\wedge_\otimes(d^F d^F t) = (R^E \wedge_{\mathrm{ev}} s)\wedge_\otimes t + s\wedge_\otimes(R^F \wedge_{\mathrm{ev}} t),
\end{aligned}$$

where $\wedge_\otimes$ is the wedge product combined with the morphism $\mathrm{id}_{E\otimes F}\colon E \otimes F \to E \otimes F$. ⧫

Example A.1.8. Let $E \to M$ be a vector bundle with connection ∇^E and curvature R^E. The induced connection on ΛE is the restriction of the connection on $\mathbb{C} \oplus E \oplus E^{\otimes 2} \oplus \cdots \oplus E^{\otimes \operatorname{rank}(E)}$ to antisymmetric tensors, hence satisfies $\nabla^{\Lambda E} f = df$, $\nabla^{\Lambda E} s = \nabla^E s$, and

$$\nabla^{\Lambda E}_X(\alpha \wedge \beta) = \nabla^{\Lambda E}_X \alpha \wedge \beta + \alpha \wedge \nabla^{\Lambda E}_X \beta$$

for all $f \in C^\infty(M)$, $s \in \Gamma(M, E)$, $X \in \Gamma(M, TM)$, and $\alpha, \beta \in \Gamma(M, \Lambda E)$. The curvature of $\nabla^{\Lambda E}$ then satisfies

$$R^{\Lambda E}(\alpha \wedge \beta) = R^{\Lambda E} \alpha \wedge \beta + \alpha \wedge R^{\Lambda E} \beta$$

for $\alpha, \beta \in \Gamma(M, \Lambda E)$. Consequently, $R^{\Lambda E}$ preserves the grading of ΛE. A special case is the *determinant line bundle of* E, defined by $\det(E) := \Lambda^r E$, with r the rank of E. It's curvature is given by

$$R^{\det(E)}(X, Y) = \operatorname{tr}(R^E(X, Y)).$$

Indeed, let $\{e_j\}_{j=1}^r$ be a basis for E_x, and let $\{\varphi_j\}_{j=1}^r$ be the corresponding dual basis of E_x^*. Then

$$\begin{aligned}
R^{\det(E)}(X, Y)(e_1 \wedge \cdots \wedge e_r) &= \sum_{j=1}^r e_1 \wedge \cdots \wedge e_{j-1} \wedge R^E(X, Y) e_j \wedge e_{j+1} \wedge \cdots \wedge e_r \\
&= \sum_{j=1}^r \sum_{k=1}^r \varphi_k(R^E(X, Y) e_j)\, e_1 \wedge \cdots \wedge e_{j-1} \wedge e_k \wedge e_{j+1} \cdots \wedge e_r \\
&= \sum_{j=1}^r \varphi_j(R^E(X, Y) e_j)\, e_1 \wedge \cdots \wedge e_r \\
&= \operatorname{tr}(R^E(X, Y))(e_1 \wedge \cdots \wedge e_r),
\end{aligned}$$

as claimed. ♦

Definition A.1.9. Let (M, g) be a Riemannian manifold. The curvature form of the Levi–Civita connection on TM is denoted by $R^M \in \Omega^2(M, \operatorname{End}(TM))$ and is called the *Riemann curvature tensor*.

The Riemann curvature tensor enjoys the following additional symmetries:

$$R^M(X, Y)Z + R^M(Y, Z)X + R^M(Z, X)Y = 0 \qquad \text{(first Bianchi identity)} \qquad \text{(A.1.14)}$$

$$g(R^M(X, Y)Z, W) = g(R^M(Z, W)X, Y) \qquad \text{(pair symmetry)} \qquad \text{(A.1.15)}$$

for all $X, Y, Z, W \in TM$, see [Lee09, Theorem 13.19].

A.2. The Hodge star operator

For a Hermitian vector bundle $E \to M$ over an oriented Riemannian manifold of dimension m, the *Hodge star operator* is the unique (conjugate linear) bundle map

$$\bar{\star}^E \colon \Lambda^\bullet T^*M \otimes E \to \Lambda^{m-\bullet} T^*M \otimes E^*$$

such that $\langle u, v\rangle \operatorname{vol}_g = u \wedge_{\mathrm{ev}} \overline{\star}^E v$ holds for all $u, v \in \Lambda^k T^*M \otimes E$, where vol_g is the volume form induced by the orientation and the metric. If $\star\colon \Lambda^\bullet T^*M \otimes \mathbb{C} \to \Lambda^\bullet T^*M \otimes \mathbb{C}$ is the complex linear extension of the real Hodge star operator, then $\overline{\star}^E = (\star \circ \operatorname{conj}) \otimes h$, where conj is complex conjugation on $\Lambda^\bullet T^*M \otimes \mathbb{C}$, and $h\colon E \to E^*$ is the conjugate linear isomorphism induced by the Hermitian metric. It follows that $\overline{\star}^E$ is a fiberwise surjective isometry with inverse given by $(\overline{\star}^E)^{-1} = \sigma^E \circ \overline{\star}^{E^*}$, where $\sigma^E \in \operatorname{End}(\Lambda^\bullet T^*M \otimes E)$ satisfies

$$\sigma^E|_{\Lambda^k T^*M\otimes E} = (-1)^{k(m-k)} \operatorname{id}_{\Lambda^k T^*M\otimes E},$$

see for instance [Lee09, Proposition 9.25]. This also implies

$$\overline{\star}^E\sigma^E = \overline{\star}^E\sigma^E(\sigma^E\overline{\star}^{E^*})\overline{\star}^E = \overline{\star}^E\overline{\star}^{E^*}\overline{\star}^E = \sigma^{E^*}(\sigma^{E^*}\overline{\star}^E)\overline{\star}^{E^*}\overline{\star}^E = \sigma^{E^*} = \sigma^{E^*}\overline{\star}^E \tag{A.2.1}$$

as well as

$$(\overline{\star}^E)^* = (\overline{\star}^E)^{-1} = \sigma^E\overline{\star}^{E^*} \tag{A.2.2}$$

since $\overline{\star}^E$ is antiunitary, so that $\sigma^E\overline{\star}^{E^*}$ is the adjoint of $\overline{\star}^E$.

Proposition A.2.1. *Let X be a vector field on M, and let α be a one-form. Then*

$$\overline{\star}^E(X^\flat \wedge u) = (-1)^k \operatorname{ins}_X(\overline{\star}^E u) \quad \textit{and} \quad \alpha \wedge \overline{\star}^E u = (-1)^{k+1}\overline{\star}^E(\operatorname{ins}_{\alpha^\sharp} u) \tag{A.2.3}$$

holds for all E-valued k-forms u.

Proof. We have

$$\begin{aligned}\langle v, X^\flat \wedge u\rangle \operatorname{vol}_g &= \langle \operatorname{ins}_X v, u\rangle \operatorname{vol}_g = \operatorname{ins}_X(v) \wedge_{\mathrm{ev}} \overline{\star}^E u = \\ &= \operatorname{ins}_X(v \wedge_{\mathrm{ev}} \overline{\star}^E u) - (-1)^{k+1} v \wedge_{\mathrm{ev}} \operatorname{ins}_X(\overline{\star}^E u) = (-1)^k\langle v, (\overline{\star}^E)^{-1} \operatorname{ins}_X(\overline{\star}^E u)\rangle \operatorname{vol}_g\end{aligned}$$

for all $(k+1)$-forms v and k-forms u, where we have used that $v \wedge_{\mathrm{ev}} \overline{\star}^E u = 0$ for dimensional reasons. Therefore, $X^\flat \wedge u = (-1)^k(\overline{\star}^E)^{-1} \operatorname{ins}_X(\overline{\star}^E u)$, and this implies the first formula. Similarly,

$$\begin{aligned}v \wedge_{\mathrm{ev}} (\alpha \wedge \overline{\star}^E u) &= (-1)^{k-1}(\alpha \wedge v) \wedge_{\mathrm{ev}} \overline{\star}^E u = (-1)^{k-1}\langle \alpha \wedge v, u\rangle \operatorname{vol}_g = \\ &= (-1)^{k-1}\langle v, \operatorname{ins}_{\alpha^\sharp}(u)\rangle \operatorname{vol}_g = (-1)^{k-1} v \wedge_{\mathrm{ev}} \overline{\star}^E(\operatorname{ins}_{\alpha^\sharp}(u))\end{aligned}$$

for all $(k-1)$-forms v and all k-forms u. ∎

Proposition A.2.2. *Let $E \to M$ be a Hermitian vector bundle over an oriented Riemannian manifold (M, g), and let ∇^E be a metric connection on E, with dual connection ∇^{E^*}. If $\widetilde{\nabla}^E$ and $\widetilde{\nabla}^{E^*}$ denote the connections on $\Lambda T^*M \otimes E$ and $\Lambda T^*M \otimes E^*$ induced by the Levi–Civita connection on M, and ∇^E and ∇^{E^*}, respectively, then*

$$\widetilde{\nabla}^{E^*}_Z \circ \overline{\star}^E = \overline{\star}^E \circ \widetilde{\nabla}^E_Z$$

for every complex vector field $Z \in \Gamma(M, TM \otimes_{\mathbb{R}} \mathbb{C})$.

Proof. The Riemannian volume form vol_g is parallel for the Levi–Civita connection: for a real vector field X on M, there is $f \in C^\infty(M)$ with $\nabla_X \mathrm{vol}_g = f\,\mathrm{vol}_g$, hence

$$0 = X(1) = X(|\mathrm{vol}_g|^2) = 2\langle \nabla_X \mathrm{vol}_g, \mathrm{vol}_g\rangle = 2f|\mathrm{vol}_g|^2 = 2f,$$

which shows that $\nabla_X \mathrm{vol}_g = 0$, and this continues to hold for complex vector fields. We compute, with $v \in \Omega^k(M,E)$ and $u \in \Omega^{m-k}(M,E)$,

$$\nabla_Z(\langle u,v\rangle\,\mathrm{vol}_g) = Z(\langle u,v\rangle)\,\mathrm{vol}_g = \langle \widetilde{\nabla}^E_Z u, v\rangle\,\mathrm{vol}_g + \langle u, \widetilde{\nabla}^E_{\overline{Z}} v\rangle\,\mathrm{vol}_g = \widetilde{\nabla}^E_Z u \wedge_{\mathrm{ev}} \overline{\star}^E v + u \wedge_{\mathrm{ev}} \overline{\star}^E \widetilde{\nabla}^E_{\overline{Z}} v,$$

see (A.1.2). On the other hand,

$$\nabla_Z(\langle u,v\rangle\,\mathrm{vol}_g) = \nabla_Z(u \wedge_{\mathrm{ev}} \overline{\star}^E v) = \widetilde{\nabla}^E_Z u \wedge_{\mathrm{ev}} \overline{\star}^E v + u \wedge_{\mathrm{ev}} \widetilde{\nabla}^{E^*}_Z \overline{\star}^E v,$$

where the covariant derivatives are compatible with $\wedge_{\mathrm{ev}}$ by (A.1.9). Therefore, $u \wedge_{\mathrm{ev}} \overline{\star}^E \widetilde{\nabla}^E_{\overline{Z}} v = u \wedge_{\mathrm{ev}} \widetilde{\nabla}^{E^*}_Z \overline{\star}^E v$ holds for all u, which implies the result. ■

Proposition A.2.3. *Let $E \to M$ be a Hermitian vector bundle over an oriented Riemannian manifold of dimension m, and let ∇ be a connection on E, with induced exterior covariant derivative $d^E\colon \Omega^\bullet(M,E) \to \Omega^{\bullet+1}(M,E)$, see appendix A.1. Then*

$$\overline{\star}^E \circ d^{E,\dagger} = (-1)^k d^{E^*} \circ \overline{\star}^E \quad \text{and} \quad d^{E^*,\dagger} \circ \overline{\star}^E = (-1)^{k+1} \overline{\star}^E \circ d^E$$

on $\Omega^k(M,E)$, where $d^{E,\dagger}$ is the formal adjoint of d^E, and $d^{E^*}\colon \Omega^\bullet(M,E^*) \to \Omega^{\bullet+1}(M,E^*)$ is the exterior covariant derivative associated to the dual connection on E^*.

Proof. Let $u \in \Omega^k(M,E)$ and $v \in \Omega^{m-k+1}_{cc}(M,E^*)$. Then

$$\begin{aligned}
\langle\!\langle d^{E^*}\overline{\star}^E u, v\rangle\!\rangle &= \int_M (d^{E^*}\overline{\star}^E u) \wedge_{\mathrm{ev}} \overline{\star}^{E^*} v \\
&= \int_M \big(d(\overline{\star}^E u \wedge_{\mathrm{ev}} \overline{\star}^{E^*} v) - (-1)^{m-k}\overline{\star}^E u \wedge_{\mathrm{ev}} d^E \overline{\star}^{E^*} v\big) && \text{by (A.1.7)} \\
&= (-1)^{m-k+1} \int_M \overline{\star}^E u \wedge_{\mathrm{ev}} d^E \overline{\star}^{E^*} v && \text{since } v \text{ has compact support} \\
&= (-1)^{m-k+1} \int_M \overline{\star}^E u \wedge_{\mathrm{ev}} \overline{\star}^{E^*} \sigma^{E^*} \overline{\star}^E d^E \overline{\star}^{E^*} v && \text{since } (\overline{\star}^{E^*})^{-1} = \sigma^{E^*}\overline{\star}^E \\
&= (-1)^{m-k+1} \langle\!\langle \overline{\star}^E u, \sigma^{E^*}\overline{\star}^E d^E \overline{\star}^{E^*} v\rangle\!\rangle && \text{by the definition of } \overline{\star}^{E^*} \\
&= (-1)^{m-k+1} \langle\!\langle \overline{\star}^E u, \overline{\star}^E \sigma^E d^E \overline{\star}^{E^*} v\rangle\!\rangle && \text{by (A.2.1)} \\
&= (-1)^k \langle\!\langle \overline{\star}^E u, \overline{\star}^E d^E \sigma^E \overline{\star}^{E^*} v\rangle\!\rangle && \text{since } \sigma^E d^E = (-1)^{m-1} d^E \sigma^E \\
&= (-1)^k \langle\!\langle u, d^E \sigma^E \overline{\star}^{E^*} v\rangle\!\rangle && \text{since } \overline{\star}^E \text{ is an isometry} \\
&= (-1)^k \langle\!\langle d^{E,\dagger} u, \sigma^E \overline{\star}^{E^*} v\rangle\!\rangle && \text{by the definition of } d^{E,\dagger} \\
&= (-1)^k \langle\!\langle \overline{\star}^E d^{E,\dagger} u, v\rangle\!\rangle && \text{by (A.2.2),}
\end{aligned}$$

which shows the first formula. Applying it to E^* instead of E yields

$$\sigma^{E^*} \circ d^{E^*,\dagger} \circ \overline{\star}^E = \overline{\star}^E \circ (\overline{\star}^{E^*} \circ d^{E^*,\dagger}) \circ \overline{\star}^E = (-1)^{m-k} \overline{\star}^E \circ (d^E \circ \overline{\star}^{E^*}) \circ \overline{\star}^E = (-1)^{k+1} \sigma^{E^*} \circ \overline{\star}^E \circ d^E$$

on $\Omega^k(M, E)$, which implies the second formula. ∎

APPENDIX B

Background on complex and Hermitian geometry

In this appendix, we provide the needed background on Hermitian geometry in a condensed form. In particular, we discuss the splitting of the tangent bundle of a complex manifold induced by its complex structure, the Dolbeault operator $\overline{\partial}^E$, and the basics on Hermitian holomorphic vector bundles, such as the Chern connection and its curvature. More detailed introductions can be found in [Bal06; Huy05; Wel08], which is where most of the content of this appendix was taken from.

B.1. Complex manifolds

An *almost complex manifold* is a smooth manifold M together with a bundle endomorphism $J\colon TM \to TM$ such that $J^2 = -\operatorname{id}_{TM}$. On an almost complex manifold (M, J), the complexified tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$ splits into $T^{1,0}M \oplus T^{0,1}M$, corresponding to the eigenvalues $\pm i$ of the complex linear extension of J to $TM \otimes_{\mathbb{R}} \mathbb{C}$. The maps

$$(TM, J) \to (T^{1,0}M, i),\ X \mapsto X^{1,0} := \frac{1}{2}(X - iJX) \tag{B.1.1}$$

and

$$(TM, J) \to (T^{0,1}M, i),\ X \mapsto X^{0,1} := \frac{1}{2}(X + iJX) \tag{B.1.2}$$

are complex linear and conjugate linear isomorphisms, respectively, where we will always use multiplication by i in the second factor of $TM \otimes_{\mathbb{R}} \mathbb{C}$ as its complex structure. For the bundle of k-forms, we get a splitting into *bidegrees* (p, q),

$$\Lambda^k(TM \otimes_{\mathbb{R}} \mathbb{C})^* = \bigoplus_{p+q=k} \Lambda^p(T^{1,0}M)^* \otimes \Lambda^q(T^{0,1}M)^* = \bigoplus_{p+q=k} \Lambda^{p,q}T^*M,$$

where $\Lambda^{p,q}T^*M := \Lambda^p(T^{1,0}M)^* \otimes \Lambda^q(T^{0,1}M)^*$. Let $\Pi_{p,q}\colon \Lambda(T^*M \otimes_{\mathbb{R}} \mathbb{C}) \to \Lambda^{p,q}T^*M$ denote the projections. One has

$$\sum_{p+q=k} i^{p-q}(\Pi_{p,q}\alpha)(v_1, \dots, v_k) = \alpha(Jv_1, \dots, Jv_k) \tag{B.1.3}$$

for all $\alpha \in \Lambda^k(T^*M \otimes_{\mathbb{R}} \mathbb{C})$ and $v_1, \dots, v_k \in TM \otimes_{\mathbb{R}} \mathbb{C}$, see [Huy05, p. 28]. In particular, a two-form α is of bidegree $(1, 1)$ if and only if $\alpha(v, w) = \alpha(Jv, Jw)$ for all $v, w \in TM \otimes_{\mathbb{R}} \mathbb{C}$. The wedge product extends to bilinear maps

$$\wedge\colon \Lambda^{p,q}T^*M \times \Lambda^{p',q'}T^*M \to \Lambda^{p+p',q+q'}T^*M.$$

We denote by $\Omega^{p,q}(M)$ the smooth sections of $\Lambda^{p,q}T^*M$, and by $\Omega(M,\mathbb{C}) = \bigoplus_{p,q}\Omega^{p,q}(M)$ the space of all smooth complex differential forms. Starting from the exterior derivative, we may define

$$\partial \coloneqq \Pi_{p+1,q} \circ d \colon \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \quad \text{and} \quad \overline{\partial} \coloneqq \Pi_{p,q+1} \circ d \colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M),$$

and we extend them complex linearly to $\Omega(M,\mathbb{C})$.

In general $d \neq \partial + \overline{\partial}$, but we have equality in the case when M is a *complex manifold*. By this, we mean an even dimensional manifold in which the transition functions between charts may be chosen to be biholomorphic, under the usual identification $\mathbb{R}^{2n} \cong \mathbb{C}^n$. If $\varphi\colon U \to V$ is such a chart, with $U \subseteq M$ and $V \subseteq \mathbb{C}^n$ open subsets, then by using its derivative $T\varphi\colon TM|_U \to \mathbb{C}^n$, we obtain a well-defined almost complex structure on M by pulling back the operator of multiplication with i on $\mathbb{C}^n$. In other words, $JX = (T\varphi)^{-1}(i\,T\varphi(X))$ for $X \in TM|_U$. Writing $\varphi = (z_1,\dots,z_n)$, and putting $x_k \coloneqq \mathrm{Re}(z_k)$ and $y_k \coloneqq \mathrm{Im}(z_k)$, we see that

$$J\Big(\frac{\partial}{\partial x_k}\Big) = (T\varphi)^{-1}(i\,e_{2k-1}) = (T\varphi)^{-1}(e_{2k}) = \frac{\partial}{\partial y_k}$$

and

$$J\Big(\frac{\partial}{\partial y_k}\Big) = (T\varphi)^{-1}(i\,e_{2k}) = (T\varphi)^{-1}(-e_{2k-1}) = -\frac{\partial}{\partial x_k}$$

for $1 \le k \le n$, and where $e_j \in \mathbb{R}^{2n}$ are the standard unit vectors. In particular,

$$\frac{\partial}{\partial z_k} \coloneqq \Big(\frac{\partial}{\partial x_k}\Big)^{1,0} = \frac{1}{2}\Big(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\Big) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_k} \coloneqq \Big(\frac{\partial}{\partial x_k}\Big)^{0,1} = \frac{1}{2}\Big(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\Big)$$

are pointwise linearly independent sections of $T^{1,0}M|_U$ and $T^{0,1}M|_U$, respectively. For dimensional reasons, they form a frame of these complex vector bundles, with corresponding dual frames given by

$$dz_k \coloneqq dx_k + i dy_k \quad \text{and} \quad d\overline{z}_k \coloneqq dx_k - i dy_k.$$

With these notations, one has

$$\partial f = \sum_{k=1}^n \frac{\partial}{\partial z_k}(f)\,dz_k \quad \text{and} \quad \overline{\partial} f = \sum_{k=1}^n \frac{\partial}{\partial \overline{z}_k}(f)\,d\overline{z}_k$$

for $f \in C^\infty(U,\mathbb{C})$ and, most importantly, a computation shows that $d = \partial + \overline{\partial}$. Thus, if M is a complex manifold, $0 = d^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2$, and comparison of the bidegrees of the individual terms yields

$$\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

It is remarkable that the converse also holds: if $d = \partial + \overline{\partial}$, then the almost complex manifold (M,J) admits in a unique way the structure of a complex manifold such that J is the almost complex structure induced from the holomorphic charts of that structure, as above. In this case, one says that the almost complex structure is *integrable*. This is the famous *Newlander–Nirenberg theorem* [NN57].

B.2. Hermitian manifolds

A complex manifold M together with a Riemannian metric g on M is called a *Hermitian manifold* if g is compatible with the complex structure,

$$g(JX, JY) = g(X, Y)$$

for all vector fields X and Y on M. Associated to such a compatible Riemannian metric is the positive definite Hermitian form

$$h(X, Y) \coloneqq g(X, Y) - ig(JX, Y) \tag{B.2.1}$$

on the complex vector bundle (TM, J). This means that h is sesquilinear (*i.e.*, $h(JX, Y) = ih(X, Y)$ and $h(X, JY) = -ih(X, Y)$), positive definite, and satisfies $h(X, Y) = \overline{h(Y, X)}$. The Riemannian metric g may also be extended to $TM \otimes_{\mathbb{R}} \mathbb{C}$ in a sesquilinear fashion, denoted by $\langle \bullet, \bullet \rangle$, so that

$$\langle X \otimes \lambda, Y \otimes \mu \rangle = \lambda \overline{\mu}\, g(X, Y).$$

It follows that $\langle \bullet, \bullet \rangle$ is a positive definite Hermitian metric on $(TM \otimes_{\mathbb{R}} \mathbb{C}, i)$, and complex conjugation satisfies $\langle \overline{Z_1}, \overline{Z_2} \rangle = \overline{\langle Z_1, Z_2 \rangle} = \langle Z_2, Z_1 \rangle$. The splitting $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ is orthogonal for this inner product, since

$$\langle X - iJX, Y + iJY \rangle = g(X, Y) - ig(JX, Y) - ig(X, JY) - g(JX, JY) = 0.$$

A similar computation shows that

$$\langle X - iJX, Y - iJY \rangle = 2h(X, Y) \tag{B.2.2}$$

for all $X, Y \in TM$, so that $h = 2\langle \bullet, \bullet \rangle$ under the isomorphism $(TM, J) \cong (T^{1,0}M, i)$ from (B.1.1). We will write $|X| \coloneqq \sqrt{\langle X, X \rangle}$ for the pointwise norm induced by $\langle \bullet, \bullet \rangle$, and $|\bullet|_x$ for its value at $x \in M$. Orthogonality in $TM \otimes_{\mathbb{R}} \mathbb{C}$, $T^{1,0}M$, or $T^{0,1}M$, will always taken to be with respect to $\langle \bullet, \bullet \rangle$, unless otherwise specified.

Let $\{w_j\}_{j=1}^n$ be a local frame of $T^{1,0}M$. Then

$$e_{2j-1} \coloneqq \frac{1}{\sqrt{2}}(w_j + \overline{w}_j) \quad \text{and} \quad e_{2j} \coloneqq Je_{2j-1} = \frac{i}{\sqrt{2}}(w_j - \overline{w}_j) \tag{B.2.3}$$

for $1 \leq j \leq n$ defines a frame of $TM \subseteq TM \otimes_{\mathbb{R}} \mathbb{C}$. Denote by $\{w^j\}_{j=1}^n$ the corresponding dual frame of $(T^{1,0}M)^*$. It follows that

$$e^{2j-1} \coloneqq \frac{1}{\sqrt{2}}(w^j + \overline{w}^j) \quad \text{and} \quad e^{2j} \coloneqq \frac{-i}{\sqrt{2}}(w^j - \overline{w}^j) \tag{B.2.4}$$

defines the dual coframe to $\{e_k\}_{k=1}^{2n}$. Note that if $\{w_j\}_{j=1}^n$ is orthonormal in $(T^{1,0}M, \langle \bullet, \bullet \rangle)$, then $\{e_k\}_{k=1}^{2n}$ is a (real) local orthonormal frame of (TM, g). Similarly, we have the *complex* local orthonormal frames $\{e_{2j-1}\}_{j=1}^n$ and $\{e_{2j}\}_{j=1}^n$ of (TM, J, h).

B.2.1. Kähler manifolds. A Hermitian manifold (M, J, g) is called *Kähler* if $d\omega = 0$, where

$$\omega(X, Y) := g(JX, Y) = -\operatorname{Im}(h(X, Y))$$

is called the *Kähler form* associated to g and J. Since

$$\omega(JX, JX) = g(J^2X, JY) = -g(X, JY) = \omega(X, Y),$$

the Kähler form is of bidegree $(1, 1)$, see (B.1.3). An excellent introduction to Kähler manifolds is [Bal06]. The following result gives a few characterizations of the Kähler condition:

Theorem B.2.1. *Let (M, J, g) be a Hermitian manifold, with Kähler form ω as above. The following are equivalent:*

(i) $d\omega = 0$*,* i.e.*,* (M, J, g) *is a Kähler manifold.*
(ii) $\nabla J = 0$*, where* ∇ *is the connection on* $\operatorname{End}(TM)$ *induced by the Levi–Civita connection.*
(iii) The (complexified) Levi–Civita connection preserves the subbundles $T^{1,0}M$ *and* $T^{0,1}M$ *of* $TM \otimes_{\mathbb{R}} \mathbb{C}$*.*
(iv) The Chern connection (see appendix B.3) of the Hermitian holomorphic vector bundle (TM, h)*, with* h *as in* (B.2.1)*, is equal to the Levi–Civita connection.*
(v) For every $x \in M$ *there exists* $\varphi \in C^\infty(U, \mathbb{R})$*, with* U *an open neighborhood of* x*, such that* $\omega|_U = i\partial\bar{\partial}\varphi$*.*

Proof. See [Bal06, Theorem 4.17]. ■

Example B.2.2. Products, submanifolds, and coverings of Kähler manifolds are again Kähler (all equipped with the induced structures). For dimensional reasons, all Hermitian *Riemann surfaces* (one-dimensional complex manifolds) are Kähler, since $d\omega \in \Omega^3(M) = 0$ in this case. The space $\mathbb{C}^n$ with the Euclidean metric is Kähler, and the complex projective spaces $\mathbb{C}P^n$ are compact Kähler manifolds if equipped with the *Fubini–Study metric*, see [Bal06, Examples 4.10] for its construction. ♦

B.2.2. Some exterior algebra identities. For $X \in TM \otimes_{\mathbb{R}} \mathbb{C}$, we denote by $\flat X = X^\flat$ the dual 1-form, defined by $X^\flat := \langle \bullet, X\rangle \in T^*M \otimes_{\mathbb{R}} \mathbb{C}$, and we let $\sharp\colon T^*M \otimes_{\mathbb{R}} \mathbb{C} \to TM \otimes_{\mathbb{R}} \mathbb{C}$, also denoted by $\alpha \mapsto \alpha^\sharp$, be the inverse map. It follows that $\operatorname{ins}_X(\alpha) = \alpha(X) = \langle X, \alpha^\sharp\rangle$. Note that both $X \mapsto X^\flat$ and $\alpha \mapsto \alpha^\sharp$ are *conjugate* linear maps. If $X \in T^{1,0}M$, then $X^\flat$ vanishes on $T^{0,1}M$, so that we may identify $X^\flat$ with an element of $(T^{1,0}M)^* = \Lambda^{1,0}T^*M$, and similarly for $X \in T^{0,1}M$. We define a Hermitian metric on $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ by

$$\langle \alpha, \beta\rangle := \langle \beta^\sharp, \alpha^\sharp\rangle,$$

so that $\sharp$ and $\flat$ become anti-isometries, called the *musical isomorphisms*.

The Hermitian metric $\langle \bullet, \bullet\rangle$ also induces Hermitian metrics on the bundles $\Lambda^{p,q}T^*M$ in the usual way. Since $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ is an orthogonal decomposition, it follows that $\Lambda T^*M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q} \Lambda^{p,q}T^*M$ also has this property. For a complex vector $X \in T_xM \otimes_{\mathbb{R}} \mathbb{C}$, we let $\operatorname{ins}_X \colon \Lambda^\bullet(T_xM \otimes_{\mathbb{R}} \mathbb{C})^* \to \Lambda^{\bullet-1}(T_xM \otimes_{\mathbb{R}} \mathbb{C})^*$ denote the *insertion operator* (or interior product), defined by

$$\operatorname{ins}_X(\alpha)(Y_1, \dots, Y_{k-1}) \coloneqq \alpha(X, Y_1, \dots, Y_{k-1})$$

for $\alpha \in \Lambda^k(T_xM \otimes_{\mathbb{R}} \mathbb{C})^*$. We have the identity $(\operatorname{ins}_X)^* = \varepsilon(X^\flat)$, where the left-hand side denotes the adjoint operator with respect to $\langle \bullet, \bullet \rangle_x$. This operator satisfies the derivation rule

$$\operatorname{ins}_X(\alpha \wedge \beta) = \operatorname{ins}_X(\alpha) \wedge \beta + (-1)^k \alpha \wedge \operatorname{ins}_X(\beta)$$

for all $\alpha \in \Lambda^k(T_xM \otimes_{\mathbb{R}} \mathbb{C})^*$ and $\beta \in \Lambda^\bullet(T_xM \otimes_{\mathbb{R}} \mathbb{C})^*$. In particular,

$$\operatorname{ins}_X \circ\, \varepsilon(Y^\flat) + \varepsilon(Y^\flat) \circ \operatorname{ins}_X = \operatorname{ins}_X(Y^\flat) = \langle X, Y \rangle.$$

If E is a Hermitian vector bundle over M, then the insertion operator extends to

$$\operatorname{ins}_X \colon \Lambda T_x^*M \otimes E_x \to \Lambda T_x^*M \otimes E_x$$

by letting it act as the identity on E_x, and we also obtain an operator $\operatorname{ins}_X \colon \Omega(M, E) \to \Omega(M, E)$ for every smooth vector field $X \in \Gamma(M, TM \otimes_{\mathbb{R}} \mathbb{C})$.

Lemma B.2.3. *Let $x \in M$, and $\{w_j\}_{j=1}^n$ be an orthonormal basis of $T_x^{1,0}M$.*

(i) For all $u, v \in \Lambda^{p,q}T_x^*M \otimes E_x$,

$$\sum_{j=1}^n \langle \operatorname{ins}_{w_j}(u), \operatorname{ins}_{w_j}(v) \rangle = p\langle u, v \rangle \quad \text{and} \quad \sum_{j=1}^n \langle \operatorname{ins}_{\overline{w}_j}(u), \operatorname{ins}_{\overline{w}_j}(v) \rangle = q\langle u, v \rangle.$$

(ii) For all $\xi \in T_xM \otimes \mathbb{C}$ and $u \in \Lambda^{p,q}T_x^*M \otimes E_x$,

$$|\operatorname{ins}_\xi(u)| \le |\xi||u|.$$

(iii) For all $u, v \in \Lambda^{p,q}T_x^*M \otimes E_x$, $\alpha \in T_x^*M$, and $X \in T_xM \otimes_{\mathbb{R}} \mathbb{C}$,

$$\sum_{j=1}^n \big\langle (\alpha \wedge \operatorname{ins}_X) \operatorname{ins}_{\overline{w}_j}(u), \operatorname{ins}_{\overline{w}_j}(v) \big\rangle = (q-1) \big\langle (\alpha \wedge \operatorname{ins}_X) u, v \big\rangle.$$

Proof. For (i), write $u = \sum_{J,K}' \sum_m u_{J,K,m}\, (w^J \wedge \overline{w}^K) \otimes s_m$, with the $s_m \in E_x$ forming an orthonormal basis. Here, the primed sum means that summation is taking place over all increasing maps $J \colon \{1, \dots, p\} \to \{1, \dots, n\}$ and $K \colon \{1, \dots, q\} \to \{1, \dots, n\}$, and $w^J \coloneqq w^{J(1)} \wedge \cdots w^{J(p)}$, with $\overline{w}^K$ defined similarly. Then

$$|u|^2 = {\sum_{J,K}}' \sum_m |u_{J,K,m}|^2 \quad \text{and} \quad \sum_{j=1}^n |\operatorname{ins}_{\overline{w}_j}(u)|^2 = \sum_{j=1}^n {\sum_{J,K}}' \sum_m \chi_K(j) |u_{J,K,m}|^2, \tag{B.2.5}$$

where χ_K is the characteristic function of $\operatorname{img}(K)$. For all K in this sum, there are exactly q of the $j \in \{1, \dots, n\}$ where $\chi_K(j) = 1$. Therefore, $\sum_{j=1}^n |\operatorname{ins}_{\overline{w}_j}(u)|^2 = q|u|^2$. This together with the polarization identity shows the second formula, and the first one follows by a similar argument.

If $\xi \in T_xM \otimes \mathbb{C}$, then $\xi = \sum_{j=1}^n \langle \xi, w_j\rangle w_j + \sum_{j=1}^n \langle \xi, \overline{w}_j\rangle \overline{w}_j$, and hence

$$\operatorname{ins}_\xi(u) = \sum_{j=1}^n \langle \xi, w_j\rangle \operatorname{ins}_{w_j}(u) + \sum_{j=1}^n \langle \xi, \overline{w}_j\rangle \operatorname{ins}_{\overline{w}_j}(u).$$

Since the summands of the two sums are mutually orthogonal, we have

$$|\operatorname{ins}_\xi(u)|^2 = \sum_{j=1}^n |\langle \xi, w_j\rangle|^2 |\operatorname{ins}_{w_j}(u)|^2 + \sum_{j=1}^n |\langle \xi, \overline{w}_j\rangle|^2 |\operatorname{ins}_{\overline{w}_j}(u)|^2 \leq$$

$$\leq \Big(\sum_{j=1}^n |\langle \xi, w_j\rangle|^2 + \sum_{j=1}^n |\langle \xi, \overline{w}_j\rangle|^2\Big)|u|^2 = |\xi|^2|u|^2,$$

where the inequality is due to $|\operatorname{ins}_{w_j}(u)|^2 = \sum'_{J,K}\sum_m \chi_J(j)|u_{J,K,m}|^2 \leq |u|^2$, and similarly for $\operatorname{ins}_{\overline{w}_j}$, *cf.*, (B.2.5). This shows (ii).

Finally, we have

$$\sum_{j=1}^n \big\langle (\alpha\wedge \operatorname{ins}_X)\operatorname{ins}_{\overline{w}_j}(u), \operatorname{ins}_{\overline{w}_j}(v)\big\rangle = \sum_{j=1}^n \big\langle \operatorname{ins}_X(\operatorname{ins}_{\overline{w}_j}(u)), \operatorname{ins}_{\alpha^\sharp}(\operatorname{ins}_{\overline{w}_j}(u))\big\rangle =$$

$$= \sum_{j=1}^n \big\langle \operatorname{ins}_{\overline{w}_j}(\operatorname{ins}_X(u)), \operatorname{ins}_{\overline{w}_j}(\operatorname{ins}_{\alpha^\sharp}(u))\big\rangle = (q-1)\big\langle(\alpha\wedge\operatorname{ins}_X)u, v\big\rangle$$

by (i), and using that the insertion operators anticommute. This shows (iii). ■

B.3. Hermitian holomorphic vector bundles

Let $E \to M$ be a complex vector bundle over a complex manifold M. Then we have $\Omega^1(M,E) = \Omega^{1,0}(M,E) \oplus \Omega^{0,1}(M,E)$, and any connection ∇ on E decomposes as $\nabla = \nabla^{1,0} + \nabla^{0,1}$, with

$$\nabla^{1,0}\colon \Gamma(M,E) \to \Omega^{1,0}(M,E) \quad\text{and}\quad \nabla^{0,1}\colon \Gamma(M,E)\to\Omega^{0,1}(M,E). \tag{B.3.1}$$

It follows that

$$\nabla^{1,0}(fs) = \partial f\otimes s + f\nabla^{1,0}s \quad\text{and}\quad \nabla^{0,1}(fs) = \overline{\partial} f\otimes s + f\nabla^{0,1}s \tag{B.3.2}$$

for all $f \in C^\infty(M,\mathbb{C})$ and $s\in\Gamma(M,E)$. Any linear operators as in (B.3.1) and satisfying (B.3.2) are called *connections of type* $(1,0)$*, respectively of type* $(0,1)$. Since M is a complex manifold, $d = \partial + \overline{\partial}$, and hence the sum of a connection of type $(1,0)$ and a connection of type $(0,1)$ is a connection in the usual sense. If $d^\nabla\colon \Omega(M,E)\to\Omega(M,E)$ is the exterior covariant derivative associated to ∇, then d^∇ splits as $d^\nabla = d^\nabla_{1,0} + d^\nabla_{0,1}$, where

$$d^\nabla_{1,0}(\Omega^{p,q}(M,E)) \subseteq \Omega^{p+1,q}(M,E) \quad\text{and}\quad d^\nabla_{0,1}(\Omega^{p,q}(M,E)) \subseteq \Omega^{p,q+1}(M,E).$$

Moreover, $d^\nabla_{1,0}$ only depends on $\nabla^{1,0}$, and $d^\nabla_{0,1}$ only depends on $\nabla^{0,1}$. Alternatively, one may extend $\nabla^{1,0}$ and $\nabla^{0,1}$ directly as in (A.1.4) to obtain $d^\nabla_{1,0}$ and $d^\nabla_{0,1}$, respectively.

If $\{w_j\}_{j=1}^n$ is a local frame of $T^{1,0}M$, we have $d^\nabla = e^k \wedge \widetilde{\nabla}_{e_k} = w^j \wedge \widetilde{\nabla}_{w_j} + \overline{w}^j \wedge \widetilde{\nabla}_{\overline{w}_j}$ by (A.1.6), where e_k and e^k are as in (B.2.3) and (B.2.4), and $\widetilde{\nabla}$ is the connection induced on $\Lambda T^*M \otimes E$ by the Chern connection on E and any torsion free connection on TM. Therefore,

$$d_{1,0}^\nabla = w^j \wedge \widetilde{\nabla}_{w_j} \quad \text{and} \quad d_{0,1}^\nabla = \overline{w}^j \wedge \widetilde{\nabla}_{\overline{w}_j}. \tag{B.3.3}$$

Assume now that $\pi\colon E \to M$ is a *holomorphic vector bundle*, *i.e.*, E and M are complex manifolds, π is holomorphic, and the local trivializations $E|_U \to U \times \mathbb{C}^{\mathrm{rank}(E)}$ may be chosen to be biholomorphic. A connection ∇ on E is called *compatible with the holomorphic structure* if $\nabla^{0,1}s = 0$ for all local holomorphic sections s of E. Let $(\xi_j)_{j=1}^r$ be a holomorphic frame for E over an open subset $U \subseteq M$, with r the (complex) rank of E. Then any section $s \in \Gamma(M,E)$ may be written over U as $s|_U = \sum_{j=1}^r s_j \xi_j$ for certain functions $s_j \in C^\infty(U,\mathbb{C})$. If ∇ is compatible with the holomorphic structure of E, then

$$(\nabla^{0,1}s)|_U = \sum_{j=1}^r (\overline{\partial} s_j \otimes \xi_j + s_j \nabla^{0,1}\xi_j) = \sum_{j=1}^r \overline{\partial} s_j \otimes \xi_j, \tag{B.3.4}$$

so that the $(0,1)$-part of such a connection does not depend on the specific connection. Conversely, (B.3.4) defines a connection of type $(0,1)$ on E since the transition functions between two holomorphic frames of E are holomorphic. We denote this connection of type $(0,1)$ and its extension to differential forms by

$$\overline{\partial}^E \colon \Omega^{\bullet,\bullet}(M,E) \to \Omega^{\bullet,\bullet+1}(M,E).$$

A (local) section of E is holomorphic if and only if it is annihilated by $\overline{\partial}^E$. If $u = \sum_{j=1}^r \alpha_j \otimes \xi_j$ is an element of $\Omega(U,E)$, with $\alpha_j \in \Omega(U)$ and $(\xi_j)_{j=1}^r$ as above, then

$$\overline{\partial}^E u = \sum_{j=1}^r \overline{\partial}\alpha_j \otimes \xi_j.$$

Note that $\overline{\partial}^2 = 0$ implies $(\overline{\partial}^E)^2 = 0$, so that we obtain the cochain complexes

$$0 \to \Omega^{p,0}(M,E) \xrightarrow{\overline{\partial}^E} \Omega^{p,1}(M,E) \xrightarrow{\overline{\partial}^E} \cdots \xrightarrow{\overline{\partial}^E} \Omega^{p,n}(M,E) \to 0 \tag{B.3.5}$$

for $1 \leq p \leq n$, collectively called the *Dolbeault complex*. Details on the definition and properties of $\overline{\partial}^E$ can be found, for instance, in [Dem12; Dem13; Huy05; MM07; Wel08]. Since $\overline{\partial}^E$ is equal to $\Pi_{p,q+1} \circ d^\nabla$ on $\Omega^{p,q}(M,E)$, where $\Pi_{p,q}\colon \Lambda T^*M \otimes E \to \Lambda^{p,q}T^*M \otimes E$ is the projection, the principal symbol (see section 1.1.1) of $\overline{\partial}^E$ reads

$$\mathrm{Symb}(\overline{\partial}^E)(\xi)u = \Pi_{p,q+1}(\xi \wedge u) = \xi^{0,1} \wedge u \tag{B.3.6}$$

for $u \in \Omega^{p,q}(M,E)$, where $\xi^{0,1} = \frac{1}{2}(\xi + iJ\xi)$ is the component of $\xi \otimes 1 \in T^*M \otimes_{\mathbb{R}} \mathbb{C}$ in $(T^{0,1}M)^*$. Here, the complex structure on T^*M is defined by $(J\alpha)(X) \coloneqq \alpha(JX)$, and one has the isomorphisms $(T^{1,0}M)^* \cong (T^*M)^{1,0}$ and $(T^{0,1}M)^* \cong (T^*M)^{0,1}$, see [Huy05, Lemma 1.2.6].

Proposition B.3.1. *If $(E, \langle \bullet, \bullet \rangle)$ is a Hermitian holomorphic vector bundle, then there exists a unique metric connection on E which is compatible with the holomorphic structure.*

Proof. See for example [Wel08, Theorem 2.1] or [Bal06, Theorem 3.18]. ∎

Definition B.3.2. The connection from Proposition B.3.1 is called the *Chern connection* for $(E, \langle \bullet, \bullet \rangle)$. We write d^E for the exterior covariant derivative associated to it, and R^E for its curvature. By the above, $\overline{\partial}^E = d^E_{0,1}$.

Proposition B.3.3. *The curvature of the Chern connection on* $(E, \langle \bullet, \bullet \rangle)$ *is a* $(1,1)$*-form with values in* $\operatorname{End}(E)$*, i.e.,* $R^E \in \Omega^{1,1}(M, \operatorname{End}(E))$*. Equivalently,* $R^E(JX, JY) = R^E(X, Y)$ *for all* $X, Y \in TM$*, see* (B.1.3).

Proof. See [Bal06, Proposition 3.21] or [Huy05, Proposition 4.3.8]. ∎

Remark B.3.4. As in Remark A.1.6, we can diagonalize the quadratic form $(X, Y) \mapsto \langle R^E(X, \overline{Y})e, e\rangle$ on $T_xM \otimes_{\mathbb{R}} \mathbb{C}$ for fixed $e \in E_x$ to obtain an orthonormal basis $\{\xi_j\}_{j=1}^{2n}$, where n is the complex dimension of M, such that $\langle R^E(\xi_j, \overline{\xi}_k)e, e\rangle = r_j(e)\delta_{jk}$, with $r_j(e) \in \mathbb{R}$. Since $R^E(JX, J\overline{Y}) = R^E(X, \overline{Y})$ by Proposition B.3.3, we may choose this basis to also diagonalize J, meaning that we can find an orthonormal basis $\{w_j\}_{j=1}^n$ of $(T_x^{1,0}M, \langle \bullet, \bullet \rangle)$ such that

$$\langle R^E(w_j, \overline{w}_k)e, e\rangle = s_j(e)\delta_{jk}$$

for some $s_j(e) \in \mathbb{R}$, $1 \le j \le n$. Since $\{\overline{w}_j\}_{j=1}^n$ is then orthonormal in $T_x^{0,1}M$, and

$$\langle R^E(\overline{w}_j, w_k)e, e\rangle = -\langle R^E(w_k, \overline{w}_j)e, e\rangle = -s_k(e)\delta_{jk},$$

we find that $\{r_j(e)\}_{j=1}^{2n} = \{s_j(e)\}_{j=1}^n \cup \{-s_j(e)\}_{j=1}^n$.

Example B.3.5. The tangent bundle $TM \to M$ of a Hermitian manifold (M, J, g) is a particular example of a Hermitian holomorphic vector bundle, with Hermitian metric h given by (B.2.1). The identification $(TM, J, h) \to (T^{1,0}M, i, 2\langle \bullet, \bullet \rangle)$, $X \mapsto \frac{1}{2}(X - iJX)$, is a complex linear isometry, see (B.2.2), and $T^{1,0}M$ is made into a Hermitian holomorphic vector bundle in this way.

Let $\nabla^{T^{1,0}M}$ be the Chern connection for $(T^{1,0}M, \langle \bullet, \bullet \rangle)$. If $Z = Z^{1,0} + Z^{0,1} \in \Gamma(M, TM \otimes_{\mathbb{R}} \mathbb{C})$ with $Z^{1,0} \in T^{1,0}M$ and $Z^{0,1} \in T^{0,1}M$ is a complex vector field, then we put

$$\nabla_X Z \coloneqq \nabla_X^{T^{1,0}M} Z^{1,0} + \overline{\nabla_X^{T^{1,0}M} \overline{Z^{0,1}}}. \tag{B.3.7}$$

It follows that ∇ is a connection on the complex vector bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$ which by construction preserves the splitting $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$. Because $\nabla^{T^{1,0}M}$ is metric compatible, we have

$$X\langle V, W\rangle = \langle \nabla_X^{T^{1,0}M} V, W\rangle + \langle V, \nabla_X^{T^{1,0}M} W\rangle$$

for all $X \in \Gamma(M, TM)$ and $V, W \in \Gamma(M, T^{1,0}M)$, and hence

$$\begin{aligned} &\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \\ &= \langle \nabla_X^{T^{1,0}M} Y^{1,0}, Z^{1,0}\rangle + \langle \overline{Z^{0,1}}, \nabla_X^{T^{1,0}M} \overline{Y^{0,1}}\rangle + \langle Y^{1,0}, \nabla_X^{T^{1,0}M} Z^{1,0}\rangle + \langle \nabla_X^{T^{1,0}M} \overline{Z^{0,1}}, \overline{Y^{0,1}}\rangle = \\ &\qquad = X\langle Y^{1,0}, Z^{1,0}\rangle + X\langle \overline{Z^{0,1}}, \overline{Y^{0,1}}\rangle = X\langle Y, Z\rangle \end{aligned}$$

for complex vector fields Y and Z, where we have used that complex conjugation is an anti-isometry $T^{1,0}M \to T^{0,1}M$. This shows that ∇ is a metric connection on $(TM \otimes_{\mathbb{R}} \mathbb{C}, \langle \bullet, \bullet \rangle)$. If $Y \in \Gamma(M, TM)$ is a real vector field, then $Y^{1,0} = \frac{1}{2}(Y - iJY)$ and $Y^{0,1} = \frac{1}{2}(Y + iJY)$, and therefore

$$\nabla_X Y = \frac{1}{2}\Big(\nabla_X^{T^{1,0}M}(Y - iJY) + \overline{\nabla_X^{T^{1,0}M}\overline{(Y + iJY)}}\Big) = \operatorname{Re}\Big(\nabla_X^{T^{1,0}M}(Y - iJY)\Big)$$

is also real, showing that ∇ restricts to a connection on the subbundle $TM \subseteq TM \otimes_{\mathbb{R}} \mathbb{C}$.

Now let ∇^{TM} be the Chern connection for (TM, h). As we have the isomorphism $(TM, h) \cong (T^{1,0}M, 2\langle \bullet, \bullet \rangle)$ of Hermitian holomorphic vector bundles, given by (B.1.1), the Chern connection $\nabla^{T^{1,0}M}$ of $(T^{1,0}M, \langle \bullet, \bullet \rangle)$ must have the form

$$\nabla_X^{T^{1,0}M}(Y^{1,0}) = \frac{1}{2}\Big(\nabla_X^{T^{1,0}M}(Y - iJY)\Big) = \frac{1}{2}\Big(\nabla_X^{TM} Y - iJ\nabla_X^{TM} Y\Big) = (\nabla_X^{TM} Y)^{1,0} \tag{B.3.8}$$

for $X, Y \in \Gamma(M, TM)$. In particular, $\nabla_X^{TM} Y = \operatorname{Re}(\nabla_X^{T^{1,0}M}(Y - iJY))$, hence ∇^{TM} coincides with the restriction of ∇ to $TM \subseteq TM \otimes_{\mathbb{R}} \mathbb{C}$ as defined above, and ∇ is the $\mathbb{C}$-linear extension of ∇^{TM} to $TM \otimes_{\mathbb{R}} \mathbb{C}$. As stated in Theorem B.2.1, ∇^{TM} equals the Levi–Civita connection if (M, g) is Kähler. Since ∇^{TM} is metric compatible, it satisfies

$$Z(h(X, Y)) = h(\nabla_Z X, Y) + h(X, \nabla_Z Y)$$

for all $X, Y, Z \in \Gamma(M, TM)$, and taking the real part of this equation shows that ∇^{TM} is also compatible with g.

Let $R^{T^{1,0}M}$ and R^{TM} denote the curvatures of $\nabla^{T^{1,0}M}$ and ∇^{TM}, respectively. By (B.3.8) and Remark A.1.3, we have

$$R^{T^{1,0}M}(X, Y)(Z^{1,0}) = \big(R^{TM}(X, Y)Z\big)^{1,0}$$

for all $X, Y, Z \in \Gamma(M, TM)$. ♦

Remark B.3.6. The restriction of ∇ from (B.3.7) to $T^{0,1}M$ is also a metric connection (with respect to $\langle \bullet, \bullet \rangle$), given by

$$\nabla_X^{T^{0,1}M} Y := \overline{\nabla_X^{T^{1,0}M} \overline{Y}}. \tag{B.3.9}$$

Its curvature satisfies

$$R^{T^{0,1}M}(X, Y)W = \overline{R^{T^{1,0}M}(X, Y)\overline{W}} \tag{B.3.10}$$

for $X, Y \in \Gamma(M, TM)$ and $W \in \Gamma(M, T^{0,1}M)$. In particular, this also has bidegree $(1, 1)$. If $Z \in \Gamma(M, TM)$, then we can reformulate (B.3.10) as $R^{T^{0,1}M}(X, Y)(Z^{0,1}) = (R^{TM}(X, Y)Z)^{0,1}$.

Example B.3.7. Let M be a complex manifold, and consider a Hermitian metric $\langle \bullet, \bullet \rangle$ on the trivial line bundle $L := M \times \mathbb{C} \to M$. If $e\colon M \to L$, $e(x) = (x, 1)$ denotes the constant section, then

$$\varphi := -\log \circ \langle e, e \rangle$$

is a smooth, real-valued function on M. Under the identification $\Gamma(M,L) \cong C^\infty(M,\mathbb{C})$, we have

$$\langle f,g\rangle_x = \langle (x,f(x)),(x,g(x))\rangle = f(x)\overline{g(x)}\,\langle e(x),e(x)\rangle = f(x)\overline{g(x)}\,e^{-\varphi(x)}.$$

Conversely, any smooth function $\varphi\colon M \to \mathbb{R}$ gives rise to a Hermitian metric on L in this way. We have $d^L u = du - \partial\varphi \wedge u$ for $u \in \Omega(M,L) \cong \Omega(M,\mathbb{C})$. Indeed,

$$\begin{aligned} d\langle f,g\rangle &= (df \wedge \overline{g} + f \wedge \overline{dg} - f\overline{g} \wedge d\varphi)e^{-\varphi} = \\ &= ((d-\partial\varphi)f \wedge \overline{g} + f \wedge \overline{(d-\partial\varphi)g})e^{-\varphi} = \langle (d-\partial\varphi)f,g\rangle + \langle f,(d-\partial\varphi)g\rangle, \end{aligned}$$

so $(d-\partial\varphi)|_{C^\infty(M,\mathbb{C})}$ is the Chern connection on L, and hence $d^L(\alpha\otimes f) = d\alpha\otimes f + (-1)^k\alpha\wedge(d-\partial\varphi)f = d(f\alpha) - \partial\varphi\wedge(f\alpha)$ for all k-forms α. Consequently, $d^L_{1,0} = \partial - \varepsilon(\partial\varphi)$. The curvature form of L acts on $u \in \Omega(M,\mathbb{C})$ as

$$R^L \wedge_{\mathrm{ev}} u = d^L d^L u = d^2u - d(\partial\varphi\wedge u) - \partial\varphi\wedge du + \partial\varphi\wedge\partial\varphi\wedge u = -(d\partial\varphi)\wedge u = \partial\overline{\partial}\varphi\wedge u,$$

hence $R^L = \partial\overline{\partial}\varphi \in \Omega^{1,1}(M) \cong \Omega^{1,1}(M,\mathrm{End}(L))$. ♦

Example B.3.8. Similarly to Example B.3.7, if L is a general (possibly non-trivial) Hermitian holomorphic line bundle over M, then on any open subset U over which L is trivial, say via $\psi\colon L|_U \to U\times\mathbb{C}$, the metric will take the form

$$\langle s,t\rangle_x = \langle \psi^{-1}(\psi(s(x))),\psi^{-1}(\psi(t(x)))\rangle = \mathrm{pr}_2(\psi(s(x)))\overline{\mathrm{pr}_2(\psi(t(x)))}\,e^{-\varphi(x)},$$

with $\varphi := -\log\circ\langle e,e\rangle\colon U\to\mathbb{R}$ and $e(x) := \psi^{-1}(x,1)$. The curvature equals $R^L|_U = \partial\overline{\partial}\varphi \in \Omega^{1,1}(U) \cong \Omega^{1,1}(U,\mathrm{End}(L))$. In particular, R^L is a closed real $(1,1)$-form. ♦

APPENDIX C

Background on functional analysis

In this appendix, we collect some of the necessary background on the analysis of self-adjoint operators on Hilbert spaces. Fix (complex) Hilbert spaces H_1, H_2, and H_3. The following basic definitions and facts can be found in any textbook which treats unbounded operators, for instance [Sch12; Wei80].

By a *(linear) operator* from H_1 to H_2, we mean a linear map $T\colon \operatorname{dom}(T) \to H_2$, with $\operatorname{dom}(T)$ a linear subspace of H_1, called the *domain* of T. We shall write $T\colon H_1 \rightsquigarrow H_2$ to signify that T may only be partially defined. An *extension* of T is a operator $S\colon H_1 \rightsquigarrow H_2$ such that $\operatorname{dom}(T) \subseteq \operatorname{dom}(S)$ and $S|_{\operatorname{dom}(T)} = T$. In other words, $\operatorname{Graph}(T) \subseteq \operatorname{Graph}(S)$, where $\operatorname{Graph}(T) \coloneqq \{(x, Tx) : x \in \operatorname{dom}(T)\} \subseteq H_1 \times H_2$ is the graph of T, which is why we write $T \subseteq S$ if S extends T. Operator equalities are always understood to mean that both operators have the same graph. If $T_1, T_2\colon H_1 \rightsquigarrow H_2$, then their sum $T_1 + T_2$ is the operator with $\operatorname{dom}(T_1 + T_2) \coloneqq \operatorname{dom}(T_1) \cap \operatorname{dom}(T_2)$ and $(T_1 + T_2)x = T_1x + T_2x$ for $x \in \operatorname{dom}(T_1 + T_2)$. Similarly, if $S\colon H_2 \rightsquigarrow H_3$, then the composition $ST\colon H_1 \rightsquigarrow H_3$ is defined on $\operatorname{dom}(ST) \coloneqq \operatorname{dom}(T) \cap T^{-1}(\operatorname{dom}(S)) \subseteq H_1$ by $(ST)x \coloneqq S(T(x))$. An operator $T\colon H_1 \rightsquigarrow H_2$ is called *densely defined* if $\operatorname{dom}(T)$ is dense in H_1, and *closed* if $\operatorname{Graph}(T)$ is closed in $H_1 \times H_2$. If T is closed and $\operatorname{dom}(T)$ is closed in H_1, then T is bounded by the closed graph theorem, and may be extended to a bounded operator on all of H_1 by setting it to zero on $\operatorname{dom}(T)^\perp \subseteq H_1$. If T is closed, then $\operatorname{dom}(T)$ is a Hilbert space when equipped with the *graph norm* $x \mapsto (\|x\|^2 + \|Tx\|^2)^{1/2}$. Any dense subspace $D \subseteq \operatorname{dom}(T)$ is then called a *(operator) core* for T. Equivalently, T is the closure of $T|_D$. The *range* of an operator $T\colon H_1 \rightsquigarrow H_2$ is its image $\operatorname{img}(T) \coloneqq T(\operatorname{dom}(T)) \subseteq H_2$. The range of a closed operator T is closed if and only if there is $C > 0$ such that $\|Tx\| \geq C\|x\|$ holds for all $x \in \operatorname{dom}(T) \cap \ker(T)^\perp$, see for instance [Hör65, Theorem 1.1.1].

Suppose now that $T\colon H_1 \rightsquigarrow H_2$ is densely defined. Then its *adjoint* $T^*\colon H_2 \rightsquigarrow H_1$ is defined by

$$\operatorname{dom}(T^*) \coloneqq \{y \in H_2 : x \mapsto \langle Tx, y\rangle \text{ is } H_1\text{-continuous on } \operatorname{dom}(T)\}$$

and $T^*y \coloneqq x_y$ for $y \in \operatorname{dom}(T^*)$, where $x_y \in H_1$ is the unique vector (by the Riesz representation theorem) such that $\langle Tx, y\rangle = \langle x, x_y\rangle$ holds for $x \in \operatorname{dom}(T)$. Then T^* is closed, and densely defined if and only if T is closable (*i.e.*, admits a closed extension). In the latter case, $\overline{T} = T^{**}$ is the closure of T (*i.e.*, the smallest closed extension). Moreover, $\ker(T^*) = \operatorname{img}(T)^\perp$. If T is densely defined and closable, then $T^* = (\overline{T})^*$. If $T_1, T_2\colon H_1 \rightsquigarrow H_2$ are densely defined operators such that $T_1 + T_2$ is densely defined, then $T_1^* + T_2^* \subseteq (T_1 + T_2)^*$. If $S\colon H_2 \rightsquigarrow H_3$ and ST is densely defined, then $T^*S^* \subseteq (ST)^*$.

C.1. Spectral theory of strongly commuting normal tuples

By Proposition 5.1.2, the spectrum of the Laplacian for the tensor product of two Hilbert complexes is determined by the closures of the operators $\Delta_j \otimes \mathrm{id}_{H'_k} + \mathrm{id}_{H_j} \otimes \Delta'_k$, with Δ and Δ' being the Laplacians for the individual factors. Hence we are led to consider operators of the form $T \otimes \mathrm{id}_K + \mathrm{id}_H \otimes S$, where T and S are normal operators on Hilbert spaces H and K, respectively. We will make use of the Borel functional calculus for tuples of strongly commuting normal operators.

Let $(H, \langle \bullet, \bullet \rangle)$ be a Hilbert space. A densely defined operator $T\colon H \rightsquigarrow H$ is called *self-adjoint* if $T^* = T$, including domains. More generally, $T\colon H \rightsquigarrow H$ is called *normal* if it is closed and satisfies $T^*T = TT^*$. By [Sch12, Proposition 3.25], this is the case if and only if $\mathrm{dom}(T) = \mathrm{dom}(T^*)$ and $\|Tx\| = \|T^*x\|$ for all $x \in \mathrm{dom}(T)$. Recall that a *spectral measure* on a measurable space (S, Σ) is a strongly countably additive map $P\colon \Sigma \to \mathscr{L}(H)$ with values in the orthogonal projections of a Hilbert space H and such that $P(S) = \mathrm{id}_H$. Thus, $P(\bigcup_{j=1}^{\infty} M_j)x = \sum_{j=1}^{\infty} P(M_j)x$ for every sequence of mutually disjoint sets $M_j \in \Sigma$ and all $x \in H$, with convergence in the norm topology of H. If $x, y \in H$, then $M \mapsto \langle P(M)x, y\rangle$ is a complex measure on S, denoted simply by $\langle Px, y\rangle$. For each measurable $f\colon S \to \mathbb{C} \cup \{\infty\}$ such that f is finite almost everywhere (for P), one defines the *spectral integral* $\int_S f\,dP\colon H \rightsquigarrow H$ as the operator characterized by

$$\mathrm{dom}\left(\int_S f\,dP\right) := \left\{x \in H : \int_S |f|^2\,d\langle Px, x\rangle < \infty\right\}$$

and

$$\left\langle \left(\int_S f\,dP\right)x, y \right\rangle = \int_S f\,d\langle Px, y\rangle$$

for all $x \in \mathrm{dom}(\int_S f\,dP)$ and $y \in H$. Then

(i) $\|(\int_S f\,dP)x\|^2 = \int_S |f|^2\,d\langle Px, x\rangle$ for $x \in \mathrm{dom}(\int_S f\,dP)$,

(ii) $\int_S f\,dP$ is bounded if and only if f is P-essentially bounded, *i.e.*, $f \in L^\infty(S, P)$, with operator norm $\|\int_S f\,dP\| = \|f\|_{L^\infty(S,P)}$,

(iii) $\int_S f\,dP$ is a normal operator, and

(iv) $\int_S \overline{f}\,dP$ is the adjoint of $\int_S f\,dP$. In particular, $\int_S f\,dP$ is self-adjoint if and only if f is real-valued P-almost everywhere.

We refer to [Sch12, section 4.3.2] for this and more. The *spectral theorem* says that normal operators are characterized by spectral integrals: for each normal operator T on H, there is a unique spectral measure $P_T\colon \mathcal{B}(\mathbb{C}) \to \mathscr{L}(H)$ on the Borel subsets of $\mathbb{C}$ such that

$$T = \int_{\mathbb{C}} \mathrm{id}_{\mathbb{C}}\,dP_T.$$

Moreover, $\mathrm{supp}(P_T) := \{z \in \mathbb{C} : P_T(B_\varepsilon(z)) \neq 0 \text{ for all } \varepsilon > 0\}$ is precisely the *spectrum* $\sigma(T)$ of T, hence we may also write $T = \int_{\sigma(T)} \mathrm{id}_{\mathbb{C}}\,dP_T = \int_{\mathbb{C}} \chi_{\sigma(T)}\,dP_T$. The *eigenvalues* of T are the complex numbers λ such that $\ker(T - \lambda\,\mathrm{id}_H) \neq 0$. In terms of the spectral measure, this is equivalent to $P_T(\{\lambda\}) \neq 0$, with $P_T(\{\lambda\})$ being the orthogonal projection onto the associated eigenspace $\ker(T - \lambda \,\mathrm{id}_H)$.

Definition C.1.1. A tuple $T \coloneqq (T_1, \dots, T_n)$ of normal operators on a Hilbert space H is said to be *strongly commuting* if all their spectral projections $P_{T_k}(M)$, for $M \subseteq \mathbb{C}$ Borel measurable and $1 \leq k \leq n$, mutually commute.

Let $T = (T_1, \dots, T_n)$ be a strongly commuting normal tuple on H. The *spectral theorem for strongly commuting normal tuples* (see [Sch12, Theorem 5.21]) gives the existence of a unique spectral measure P on the Borel sets of $\mathbb{C}^n$ such that

$$T_k = \int_{\mathbb{C}^n} z_k \, dP(z_1, \dots, z_n)$$

for all $1 \leq k \leq n$. This measure is called the *joint spectral measure* for the tuple T, and it is in fact the product of the spectral measures of $T_1, \dots, T_n$, in the sense that

$$P(M_1 \times \dots \times M_n) = P_{T_1}(M_1) \cdots P_{T_n}(M_n)$$

for Borel sets $M_k \subseteq \mathbb{C}$, $1 \leq k \leq n$, and where P_{T_k} is the spectral measure of T_k.

Definition C.1.2. Let $T = (T_1, \dots, T_n)$ be a strongly commuting tuple, with joint spectral measure P. The *joint spectrum* of T is the support of P,

$$\sigma(T) \coloneqq \big\{z \in \mathbb{C}^n : P(B_\varepsilon(z)) \neq 0 \text{ for all } \varepsilon > 0\big\},$$

where $B_\varepsilon(z)$ denotes the open ball in $\mathbb{C}^n$ with radius ε and center z. The *joint essential spectrum* of T is

$$\sigma_e(T) \coloneqq \big\{z \in \mathbb{C}^n : \operatorname{rank}(P(B_\varepsilon(z))) = \infty \text{ for all } \varepsilon > 0\big\}.$$

The complement of $\sigma_e(T)$ in $\sigma(T)$ is called the *joint discrete spectrum* of the tuple T,

$$\sigma_d(T) \coloneqq \big\{z \in \mathbb{C}^n : \exists \varepsilon_0 > 0 \text{ such that } 0 < \operatorname{rank}(P(B_\varepsilon(z))) < \infty \text{ for all } \varepsilon \in (0, \varepsilon_0)\big\}.$$

For $n = 1$, these definitions reduce to the usual ones for a single operator. The joint essential spectrum is closed in $\sigma(T)$, and $\sigma_d(T)$ is discrete, but there may be other isolated points in $\sigma(T) \setminus \sigma_d(T)$, namely eigenvalues with associated eigenspace of infinite dimension. Such eigenvalues would then belong to $\sigma_e(T)$.

Proposition C.1.3. *Let $T = (T_1, \dots, T_n)$ be a strongly commuting normal tuple on a Hilbert space H. Suppose that $D \subseteq \bigcap_{k=1}^n \operatorname{dom}(T_k)$ is a core of T_k for every $1 \leq k \leq n$.*[1]

(i) $z \in \sigma(T)$ if and only if there is a sequence $x_j \in D$ such that $\liminf_{j\to\infty} \|x_j\| > 0$ and $\lim_{j\to\infty}(T_k x_j - z_k x_j) = 0$ for all $1 \leq k \leq n$.

(ii) $z \in \sigma_e(T)$ if and only if there is a sequence $x_j \in D$ such that $x_j \to 0$ weakly in H, $\liminf_{j\to\infty} \|x_j\| > 0$, and $\lim_{j\to\infty}(T_k x_j - z_k x_j) = 0$ for all $1 \leq k \leq n$.

[1]Common cores always exist for strongly commuting normal tuples, see [Sch12, Corollary 5.28]. In fact, if P is the joint spectral measure of T, then $\bigcup_{N=1}^\infty \operatorname{img}(P(\{z \in \mathbb{C}^n : |z_k| \leq N \text{ for all } k\}))$ has this property, and is also a core for all T_k^*.

A sequence as in (i) (respectively (ii)) is called a *Weyl sequence* (respectively *singular Weyl sequence*) for (T, z).

Proof. The characterization of the joint spectrum can be found in [Sch12, Proposition 5.24]. Let $z \in \sigma_e(T)$ and denote by P the joint spectral measure of T. By definition, $\operatorname{rank}(P(B_{1/j}(z))) = \infty$ for all $j \geq 1$, so there are unit vectors $y_j = P(B_{1/j}(z))y_j$ for $j \in \mathbb{N}$, and since the rank stays infinite, we can choose $y_j \to 0$ weakly as $j \to \infty$. Clearly, $y_j \in \operatorname{dom}(T_k)$ for all k, and

$$\sum_{k=1}^{n} \|(T_k - z_k)y_j\|^2 = \int_{B_{1/j}(z)} |w - z|^2 \, d\langle P(w)y_j, y_j\rangle \leq 1/j^2,$$

so that $(T_k - z_k)y_j \to 0$ as $j \to \infty$ for all $1 \leq k \leq n$. Now choose $x_j \in D$ such that $\|x_j - y_j\| \leq 1/j$ and $\|T_k x_j - T_k y_j\| \leq 1/j$ for all $1 \leq k \leq n$. Then $(T_k - z_k)x_j \to 0$ as $j \to \infty$ and $\liminf_j \|x_j\| \geq \liminf_j(\|y_j\| - \|x_j - y_j\|) = 1$, as well as $x_j \to 0$ weakly, so $(x_j)_{j \in \mathbb{N}}$ has the desired properties.

To show the converse, we adapt the proof of [Wei80, Theorem 7.24]. Let $x_j \in H$ be a weak null sequence of vectors $x_j \in D \subseteq \bigcap_{k=1}^{n} \operatorname{dom}(T_k)$, and such that $\liminf_{j \to \infty} \|x_j\| > 0$ and $\lim_{j \to \infty}(T_k x_j - z_k x_j) = 0$ for all $1 \leq k \leq n$. Assume that there exists $\varepsilon > 0$ such that $\operatorname{rank}(P(B_\varepsilon(z))) < \infty$, so that the projection $P(B_\varepsilon(z))$ is compact. Then $P(B_\varepsilon(z))x_j \to 0$ in H and

$$\begin{aligned}
\sum_{k=1}^{n} \|(T_k - z_k)x_j\|^2 &= \int_{\mathbb{C}^n} |w - z|^2 \, d\langle P(w)x_j, x_j\rangle \\
&\geq \int_{\mathbb{C}^n \setminus B_\varepsilon(z)} |w - z|^2 \, d\langle P(w)x_j, x_j\rangle \\
&\geq \varepsilon^2 \Big(\int_{\mathbb{C}^n} d\langle P(w)x_j, x_j\rangle - \int_{\mathbb{C}^n} \chi_{B_\varepsilon(z)}(w) \, d\langle P(w)x_j, x_j\rangle \Big) \\
&= \varepsilon^2 \big(\|x_j\|^2 - \|P(B_\varepsilon(z))x_j\|^2\big).
\end{aligned}$$

Thus, there exists $1 \leq k \leq n$ such that $\liminf_{j \to \infty} \|T_k x_j - z_k x_j\| > 0$, a contradiction. ∎

Definition C.1.4. Let $T = (T_1, \ldots, T_n)$ be a strongly commuting normal tuple, with joint spectrum $\sigma(T) \subseteq \mathbb{C}^n$. If $f\colon \sigma(T) \to \mathbb{C} \cup \{+\infty\}$ is an almost everywhere finite Borel measurable function, then we can use the joint spectral measure to define the normal operator

$$f(T) := \int_{\sigma(T)} f \, dP.$$

The assignment $f \mapsto f(T)$ is called the *Borel functional calculus for strongly commuting normal tuples.*

The spectrum of this operator is then the P-essential range of f,

$$\sigma(f(T)) = \{\lambda \in \mathbb{C} : P(f^{-1}(B_\varepsilon(\lambda))) \neq 0 \text{ for all } \varepsilon > 0\},$$

and its essential spectrum is

$$\sigma_e(f(T)) = \{\lambda \in \mathbb{C} : \operatorname{rank}(P(f^{-1}(B_\varepsilon(\lambda)))) = \infty \text{ for all } \varepsilon > 0\}.$$

Both of these formulas follow from the fact that the spectral measure associated to $f(T)$ is $P \circ f^{-1}$, where f^{-1} is the preimage map on the Borel sets of $\mathbb{C}$.

Theorem C.1.5 (Spectral mapping theorem). *Let $T = (T_1, \dots, T_n)$ be a tuple of pairwise strongly commuting normal operators, and let $f \colon \sigma(T) \to \mathbb{C}$ be a continuous function. Then*

$$\sigma(f(T)) = \overline{f(\sigma(T))} \quad \text{and} \quad \sigma_e(f(T)) \supseteq \overline{f(\sigma_e(T))}.$$

If f is also proper *(meaning preimages of compact sets are compact), then*

$$\sigma(f(T)) = f(\sigma(T)) \quad \text{and} \quad \sigma_e(f(T)) = f(\sigma_e(T)).$$

Proof. The spectral mapping theorem for the joint spectrum is well-known and can be found in [Sch12, Proposition 5.25]. The proof of $\sigma_e(f(T)) \supseteq \overline{f(\sigma_e(T))}$ is similar to the corresponding inclusion for the joint spectrum: If $\lambda \in \overline{f(\sigma_e(T))}$ and $\varepsilon > 0$, then there is $z \in \sigma_e(T)$ with $|f(z) - \lambda| < \varepsilon/2$. Since f is continuous, there exists $\delta > 0$ such that $B_\delta(z) \subseteq f^{-1}(B_\varepsilon(\lambda))$. Because z is in the joint essential spectrum, $P(B_\delta(z))$ and hence also $P(f^{-1}(B_\varepsilon(\lambda)))$ has infinite rank, meaning $\lambda \in \sigma_e(f(T))$.

Now let $f \colon \sigma(T) \to \mathbb{C}$ be proper. Then f is a closed map, see [Pal70, Corollary], hence we only have to show $\sigma_e(f(T)) \subseteq f(\sigma_e(T))$. If $\lambda \notin f(\sigma_e(T))$, then we can separate the point $\lambda \in \mathbb{C}$ from the closed set $f(\sigma_e(T))$, so there exists $\varepsilon > 0$ such that $\overline{B_\varepsilon(\lambda)} \cap f(\sigma_e(T)) = \emptyset$. By applying f^{-1}, we find that $f^{-1}(\overline{B_\varepsilon(\lambda)}) \cap \sigma_e(T) = \emptyset$. As f is proper, the set $V := f^{-1}(\overline{B_\varepsilon(\lambda)})$ is a compact subset of $\sigma(T)$ contained in the joint discrete spectrum, implying that $P(V)$ and hence $P(f^{-1}(B_\varepsilon(\lambda)))$ has only finite dimensional range. Therefore, $\lambda \notin \sigma_e(f(T))$. ■

If H and K are Hilbert spaces, then we denote by $H \hat{\otimes} K$ their Hilbert space tensor product, which is the completion of the algebraic tensor product $H \otimes K$ with respect to the usual inner product, defined on elementary tensors by $\langle x \otimes y, x' \otimes y' \rangle := \langle x, x' \rangle \langle y, y' \rangle$. We require a few basic facts about the tensor product of unbounded operators, see [Sch12, section 7.5] for a reference. If T and S are closable linear operators on H and K, respectively, then the induced operators $T \otimes S$ and $T \otimes \mathrm{id}_H + \mathrm{id}_K \otimes S$ on $\mathrm{dom}(T) \otimes \mathrm{dom}(S) \subseteq H \hat{\otimes} K$ are closable. We denote the closure of $T \otimes S$ by $T \hat{\otimes} S$. If both T and S are densely defined and closable, then $(T \hat{\otimes} S)^* = T^* \hat{\otimes} S^*$. Our principal example of a strongly commuting tuple will be the following:

Lemma C.1.6. *Let T and S be normal operators on Hilbert spaces H and K, respectively. Then the operators $T \hat{\otimes} \mathrm{id}_K$ and $\mathrm{id}_H \hat{\otimes} S$ form a strongly commuting normal pair on the Hilbert space $H \hat{\otimes} K$. We have*

$$\sigma(T \hat{\otimes} \mathrm{id}_K, \mathrm{id}_H \hat{\otimes} S) = \sigma(T) \times \sigma(S) \tag{C.1.1}$$

and

$$\sigma_e(T \hat{\otimes} \mathrm{id}_K, \mathrm{id}_H \hat{\otimes} S) = \big(\sigma_e(T) \times \sigma(S)\big) \cup \big(\sigma(T) \times \sigma_e(S)\big). \tag{C.1.2}$$

Proof. The spectral measures of $T \hat{\otimes} \mathrm{id}_K$ and $\mathrm{id}_H \hat{\otimes} S$ are, respectively, given by

$$M \mapsto P_T(M) \hat{\otimes} \mathrm{id}_K \quad \text{and} \quad N \mapsto \mathrm{id}_H \hat{\otimes} P_S(N),$$

where P_T and P_S are the spectral measures of T and S, respectively. Therefore, the joint spectral measure of the pair $(T \mathbin{\hat{\otimes}} \mathrm{id}_K, \mathrm{id}_H \mathbin{\hat{\otimes}} S)$ is given on rectangles $M \times N \subseteq \mathbb{C}^2$ by $P_T(M) \mathbin{\hat{\otimes}} P_S(N)$, and its image is

$$\operatorname{img}(P_T(M) \mathbin{\hat{\otimes}} P_S(N)) = \operatorname{img}(P_T(M)) \mathbin{\hat{\otimes}} \operatorname{img}(P_S(N)).$$

Indeed, we have $\operatorname{img}(P_T(M) \mathbin{\hat{\otimes}} P_S(N)) \supseteq \operatorname{img}(P_T(M) \otimes P_S(N)) = \operatorname{img}(P_T(M)) \otimes \operatorname{img}(P_S(N))$ and hence also $\operatorname{img}(P_T(M) \mathbin{\hat{\otimes}} P_S(N)) \supseteq \operatorname{img}(P_T(M)) \mathbin{\hat{\otimes}} \operatorname{img}(P_S(N))$ since the orthogonal projection $P_T(M) \mathbin{\hat{\otimes}} P_S(N)$ has closed range. The other inclusion is clear[2] as $P_T(M) \otimes P_S(N)$ factors through

$$P_T(M) \otimes P_S(N) \colon H \otimes K \to \operatorname{img}(P_T(M)) \mathbin{\hat{\otimes}} \operatorname{img}(P_S(N)) \hookrightarrow H \mathbin{\hat{\otimes}} K.$$

Now it follows that the image of $P_T(M) \mathbin{\hat{\otimes}} P_S(N)$ is nonzero (resp. infinite dimensional) if and only if both factors are nonzero (resp. at least one of them has infinite dimension and the other is nonzero). Since the products of open discs form a basis for the topology of $\mathbb{C}^2$, the result follows immediately. ∎

Remark C.1.7. The inclusion "$\supseteq$" in (C.1.2) can also be seen by using singular Weyl sequences. Indeed, suppose that $\lambda \in \sigma_e(T)$ and $\mu \in \sigma(S)$. Then there exist sequences of unit vectors $x_n \in \operatorname{dom}(T)$ and $y_n \in \operatorname{dom}(S)$ with $x_n \to 0$ weakly and such that

$$(T - \lambda)x_n \to 0 \quad \text{and} \quad (S - \mu)y_n \to 0$$

as $n \to \infty$. But then the weak null sequence $z_n := x_n \otimes y_n \in \operatorname{dom}(T) \otimes \operatorname{dom}(S)$ satisfies

$$(T \mathbin{\hat{\otimes}} \mathrm{id}_K - \lambda)z_n \to 0 \quad \text{and} \quad (\mathrm{id}_H \mathbin{\hat{\otimes}} S - \mu)z_n \to 0$$

as $n \to \infty$, so that (λ, μ) is in the joint essential spectrum of $(T \mathbin{\hat{\otimes}} \mathrm{id}_K, \mathrm{id}_H \mathbin{\hat{\otimes}} S)$.

Theorem C.1.8. *Let T and S be self-adjoint operators on Hilbert spaces H and K, respectively. Put*

$$A := \int_{\mathbb{R}^2} (t + s) \, dP(t, s),$$

where P is the joint spectral measure of the pair $(T \mathbin{\hat{\otimes}} \mathrm{id}_K, \mathrm{id}_H \mathbin{\hat{\otimes}} S)$. Then A is the closure of the operator $T \otimes \mathrm{id}_K + \mathrm{id}_H \otimes S$ on $H \mathbin{\hat{\otimes}} K$. Moreover,

$$\sigma(A) = \overline{\sigma(T) + \sigma(S)} \quad \textit{and} \quad \sigma_e(A) \supseteq \overline{\sigma_e(T) + \sigma(S)} \cup \overline{\sigma(T) + \sigma_e(S)}. \tag{C.1.3}$$

If, in addition, T and S are lower semibounded, then

$$\sigma(A) = \sigma(T) + \sigma(S) \quad \textit{and} \quad \sigma_e(A) = \big(\sigma_e(T) + \sigma(S)\big) \cup \big(\sigma(T) + \sigma_e(S)\big). \tag{C.1.4}$$

Proof. It is easy to show that A is a self-adjoint extension of $T \otimes \mathrm{id}_K + \mathrm{id}_H \otimes S$. Because $T \otimes \mathrm{id}_K + \mathrm{id}_H \otimes S$ is essentially self-adjoint, see [RS80, Theorem VIII.33], A must be its closure. Equation (C.1.3) follows from Lemma C.1.6 and Theorem C.1.5 by applying it to the function $f \colon \mathbb{C}^2 \to \mathbb{C}$, $f(t, s) := t + s$. If T and S are lower semibounded, then so are $T \mathbin{\hat{\otimes}} \mathrm{id}_K$ and

[2] And in fact true for arbitrary bounded operators on H and K.

$\mathrm{id}_H \hat{\otimes} S$, and hence $\sigma(T \hat{\otimes} \mathrm{id}_K, \mathrm{id}_H \hat{\otimes} S)$ is contained in $[c, \infty) \times [c, \infty)$ for some $c \in \mathbb{R}$. On this set, f is proper and (C.1.4) follows, again, from Lemma C.1.6 and Theorem C.1.5. ■

Remark C.1.9. Of course, the joint spectrum of $(T \hat{\otimes} \mathrm{id}_K, \mathrm{id}_H \hat{\otimes} S)$ and the spectrum of the closure of $T \otimes \mathrm{id}_K + \mathrm{id}_H \otimes S$ are well-known in the literature, see for instance [Sch12, Lemma 7.24] or [RS80, Theorem VIII.33]. However, the corresponding statements regarding their *essential* spectrum, as well as the spectral mapping theorem for $\sigma_e(f(T))$, seem to be new (at least to the knowledge of the author).

C.2. Self-adjoint operators and their quadratic forms

Let $(H_1, \langle \bullet, \bullet \rangle_{H_1})$, $(H_2, \langle \bullet, \bullet \rangle_{H_2})$, and $(H, \langle \bullet, \bullet \rangle)$ be (complex) Hilbert spaces. Recall that a linear operator $T\colon \mathrm{dom}(T) \subseteq H_1 \to H_2$ is closed (meaning that its graph is closed in $H_1 \times H_2$) if and only if $\mathrm{dom}(T)$ is a Hilbert space with respect to the inner product $(x, y) \mapsto \langle x, y \rangle_{H_1} + \langle Tx, Ty \rangle_{H_2}$. This is called the *graph inner product*, and the resulting norm is called the *graph norm*. A symmetric operator T on $(H, \langle \bullet, \bullet \rangle)$ is called *lower semibounded* if there exists $m \in \mathbb{R}$ with $\langle Tx, x \rangle \geq m\|x\|^2$ for all $x \in \mathrm{dom}(T)$, and we write $T \geq mI$ in this case.

By a *quadratic form* on H, we mean a sesquilinear (*i.e.*, conjugate linear in the second argument if complex scalars are used) map $Q\colon \mathrm{dom}(Q) \times \mathrm{dom}(Q) \to \mathbb{C}$ with a linear subspace $\mathrm{dom}(Q) \subseteq H$, called the *domain* of Q. Any quadratic form can be recovered by its restriction to the diagonal of $\mathrm{dom}(Q) \times \mathrm{dom}(Q)$ by the *polarization identity*

$$4Q(x, y) = Q(x+y, x+y) - Q(x-y, x-y) + iQ(x+iy, x+iy) - iQ(x-iy, x-iy). \quad \text{(C.2.1)}$$

An analogous formula holds in the case of real scalars. A quadratic form Q on H is called *Hermitian* if $Q(x, y) = \overline{Q(y, x)}$ for all $x, y \in \mathrm{dom}(Q)$, *densely defined* if $\mathrm{dom}(Q)$ is dense in H, and *lower semibounded* if $Q(x, x) \geq m\|x\|^2$ for all $x \in \mathrm{dom}(Q)$, and we write $Q \geq m$ in this case. A lower semibounded quadratic form $Q \geq m$ is called *closed* if $\mathrm{dom}(Q)$ is a Hilbert space for the inner product $(x, y) \mapsto (1 - m)\langle x, y \rangle + Q(x, y)$, and *closable* if there is a closed quadratic form which extends Q. The smallest closed extension of Q is then called its *closure*. When talking about convergence in $\mathrm{dom}(Q)$, we will always refer to convergence with respect to this inner product.

Definition C.2.1. Let A be a self-adjoint operator on H. Then A defines a densely defined Hermitian quadratic form on H by means of

$$\mathrm{dom}(Q_A) := \left\{ x \in H : \int_{\sigma(A)} |\lambda| \, d\langle P_A(\lambda)x, x \rangle < \infty \right\} \quad \text{and} \quad Q_A(x, y) := \int_{\sigma(A)} \lambda \, d\langle P_A(\lambda)x, y \rangle$$

for $x, y \in \mathrm{dom}(Q_A)$, where P_A is the spectral measure associated to $A = \int_{\sigma(A)} \mathrm{id}_{\mathbb{R}} \, dP_A$ by the spectral theorem.

Often, it is easier to work with Q_A instead of with A directly. If $a, b \in \mathbb{R}$ with $a < b$, then $\operatorname{img}(P_A([a,b])) \subseteq \operatorname{dom}(Q_A)$, and we have

$$Q_A(x,x) = \int_{\sigma(A)} \lambda \, d\langle P_A(\lambda)P_A([a,b])x, P_A([a,b])x\rangle = \int_{\sigma(A)\cap[a,b]} \lambda \, d\langle P_A(\lambda)x, x\rangle$$

for all $x = P_A([a,b])x \in \operatorname{img}(P_A([a,b]))$. Therefore, $a\|x\|^2 \le Q_A(x,x) \le b\|x\|^2$ for those x. An analogous statement holds for $a = -\infty$ or $b = +\infty$. By the functional calculus, we have $\operatorname{dom}(Q_A) = \operatorname{dom}(|A|^{1/2})$ and

$$Q_A(x,y) = \langle U_A|A|^{1/2}x, |A|^{1/2}y\rangle,$$

where U_A is the partial isometry from the polar decomposition $A = U_A|A|$ of A, see [Sch12, Proposition 10.4]. One can show that

$$Q_A(x,y) = \langle Ax, y\rangle \tag{C.2.2}$$

for all $x \in \operatorname{dom}(A)$ and $y \in \operatorname{dom}(Q_A)$, and that A is lower semibounded if and only if Q_A has this property (with the same lower bound, the largest of which is given by $\inf \sigma(A)$). In this case, Q_A is automatically closed. If $A \ge mI$, then we have $\operatorname{dom}(Q_A) = \operatorname{dom}((A - mI)^{1/2})$ and

$$Q_A(x,y) = \langle (A - mI)^{1/2}x, (A - mI)^{1/2}y\rangle + m\langle x, y\rangle,$$

see [Sch12, Proposition 10.5]. If A is *nonnegative*, *i.e.*, $A \ge 0$, then this reduces to $Q_A(x,y) = \langle A^{1/2}x, A^{1/2}y\rangle$. The correspondence between lower semibounded self-adjoint operators and closed densely defined lower semibounded quadratic forms is bijective, meaning that for every such quadratic form Q there is a unique self-adjoint operator A such that $Q = Q_A$, see [Sch12, Theorem 10.7]. The operator A is given by

$$\operatorname{dom}(A) = \big\{x \in \operatorname{dom}(Q) : \text{there is } z_x \in H \text{ such that } Q(x,y) = \langle z_x, y\rangle \text{ for all } y \in \operatorname{dom}(Q)\big\} \tag{C.2.3}$$

and $Ax \coloneqq z_x$ for $x \in \operatorname{dom}(A)$. By the Riesz representation theorem, $\operatorname{dom}(A)$ is therefore the set of all $x \in \operatorname{dom}(Q)$ with the property that $y \mapsto Q(x,y)$ is H-continuous on $\operatorname{dom}(Q)$.

Example C.2.2. If T is a lower semibounded symmetric operator, then the quadratic form $Q_T\colon \operatorname{dom}(T) \times \operatorname{dom}(T) \to \mathbb{C}$, $Q_T(x,y) \coloneqq \langle Tx, y\rangle$, is closable, and the self-adjoint operator associated with its closure is called the *Friedrichs extension* of T, see [Sch12, section 10.4]. We will denote the Friedrichs extension of T by T_F. In particular, $\operatorname{dom}(T)$ is a form core for T_F, but not necessarily an operator core. The bottom of the spectrum of T_F is then given by

$$\inf \sigma(T_F) = \inf \left\{ \frac{\langle Tx, x\rangle}{\|x\|^2} : x \in \operatorname{dom}(T) \setminus \{0\} \right\},$$

since it is equal to the largest lower bound of $\overline{Q_T}$. ♦

As discussed, a core for a closed operator T is a dense subspace of $\operatorname{dom}(T)$ for the topology induced by the graph norm. Similarly, $D_0 \subseteq \operatorname{dom}(Q)$ is called a core for the closed (lower semibounded) quadratic form Q if D_0 is dense in $\operatorname{dom}(Q)$. If $A \geq mI$ is a self-adjoint operator, then $D_0 \subseteq \operatorname{dom}(Q_A)$ is called a *form core* for A if D_0 is a core for Q_A. By (C.2.2), we have

$$\|x\|^2_{\operatorname{dom}(Q_A)} = (1-m)\|x\|^2 + Q_A(x,x) = (1-m)\|x\|^2 + \langle Ax, x\rangle \leq$$
$$\leq (1-m)\|x\|^2 + \|Ax\|\|x\| \leq \left(\frac{3}{2} - m\right)\|x\|^2 + \frac{1}{2}\|Ax\|^2 \leq C\|x\|^2_{\operatorname{dom}(A)} \quad \text{(C.2.4)}$$

for all $x \in \operatorname{dom}(A)$ and with $C \coloneqq \max\{\frac{3}{2} - m, \frac{1}{2}\}$, where $\|\bullet\|_{\operatorname{dom}(Q_A)}$ is the norm on $\operatorname{dom}(Q_A)$, and similarly for $\|\bullet\|_{\operatorname{dom}(A)}$, so that the inclusion $\operatorname{dom}(A) \hookrightarrow \operatorname{dom}(Q_A)$ is continuous. Moreover, $\operatorname{dom}(A)$ is actually a core for Q_A, see [Sch12, Proposition 10.5], hence this inclusion is dense.[3] As the inclusion is even Lipschitz, it follows that any core $D_0 \subseteq \operatorname{dom}(A)$ for A is also a form core for A. Indeed, if $x \in \operatorname{dom}(Q_A)$, then we find $x_k \in \operatorname{dom}(A)$ and $y_k \in D_0$ with $x_k \to x$ in $\operatorname{dom}(Q_A)$ and $\|y_k - x_k\|_{\operatorname{dom}(A)} \leq \frac{1}{k}$ for all $k \in \mathbb{N}$, and hence (C.2.4) gives

$$\|x - y_k\|_{\operatorname{dom}(Q_A)} \leq \|x - x_k\|_{\operatorname{dom}(Q_A)} + \sqrt{C}\|x_k - y_k\|_{\operatorname{dom}(A)} \leq \|x - x_k\|_{\operatorname{dom}(Q_A)} + \frac{\sqrt{C}}{k} \to 0$$

as $k \to \infty$, showing that $y_k \to x$ in $\operatorname{dom}(Q_A)$.

Example C.2.3. Let T be a closed, densely defined operator from H_1 to H_2. Then T^*T is self-adjoint and nonnegative on H_1, and we have $\operatorname{dom}(Q_{T^*T}) = \operatorname{dom}(T)$ and $Q_{T^*T}(x,y) = \langle Tx, Ty\rangle_{H_2}$, so that the Hilbert spaces $\operatorname{dom}(Q_{T^*T})$ and $\operatorname{dom}(T)$ agree. Indeed, $(T^*T)^{1/2} = |T|$ by definition, hence $\operatorname{dom}(Q_{T^*T}) = \operatorname{dom}(|T|) = \operatorname{dom}(T)$, and

$$Q_{T^*T}(x,x) = \|(T^*T)^{1/2}x\|^2 = \||T|x\|^2 = \|Tx\|^2$$

for all $x \in \operatorname{dom}(T)$, see [Sch12, Lemma 7.1] for the last step, and the claim now follows from the polarization identity (C.2.1). In particular, it follows that $\operatorname{dom}(T^*T) \subseteq \operatorname{dom}(T)$ is a dense inclusion, so that the former space is a core for T. One can also use the bijective correspondence between nonnegative self-adjoint operators and nonnegative quadratic forms to show that T^*T is self-adjoint in the first place, see [Sch12, Example 10.5]. ♦

Example C.2.4 (Form sums of operators)**.** Suppose that A and B are two lower semibounded self-adjoint operators on a Hilbert space H. The *operator sum* $A + B$, with domain $\operatorname{dom}(A+B) = \operatorname{dom}(A) \cap \operatorname{dom}(B)$, need not be self-adjoint in general. One obstruction is that the domain of $A + B$ may not be dense in H, so that its adjoint may not even well-defined.

However, under the weaker assumption of $\operatorname{dom}(Q_A) \cap \operatorname{dom}(Q_B)$ being dense in H, the operator sum has a self-adjoint extension, called the *form sum*, which is the operator associated to the quadratic form $(x,y) \mapsto Q_A(x,y) + Q_B(x,y)$, with domain $\operatorname{dom}(Q_A) \cap \operatorname{dom}(Q_B)$, see [Sch12, Proposition 10.22] for the details. In fact, this sum is automatically closed and lower semibounded, and the assumption states that it is densely defined, so the claim follows from

[3] More generally, $\operatorname{dom}(f(A)) \cap \operatorname{dom}(g(A))$ is a common core for both $f(A)$ and $g(A)$ for arbitrary Borel functions $f, g\colon \sigma(A) \to \mathbb{C}$.

the correspondence of quadratic forms with self-adjoint operators. The form sum of A and B is denoted by $A \dot{+} B$. There are examples with $\mathrm{dom}(A + B) = \{0\}$, but the form sum $A \dot{+} B$ being well-defined, see [Sch12, Example 10.10]. ⧫

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Abstract

This thesis is concerned with questions regarding the spectral theory of the Dolbeault Laplacian with $\overline{\partial}$-Neumann boundary conditions, considered as a self-adjoint operator acting on the space of square integrable differential forms on a Hermitian manifold. The corresponding boundary value problem, called the $\overline{\partial}$-Neumann problem, arises naturally in the investigation of the (inhomogeneous) Cauchy–Riemann equations through the methods of (L^2-) Hodge theory. In this way, spectral properties of the Dolbeault Laplacian give information on the solvability of the Cauchy–Riemann equations and, by extension, on the construction of holomorphic functions (or, more generally, sections of holomorphic vector bundles) with prescribed properties. The Dolbeault Laplacian is the Laplacian of the elliptic Dolbeault complex, which generalizes the Wirtinger derivative $d/d\bar{z}$ of single variable complex analysis, and its L^2 realization with $\overline{\partial}$-Neumann boundary conditions corresponds to the weak extension of the Dolbeault complex. Therefore, we also discuss in detail aspects of the spectral theory of self-adjoint extensions of elliptic differential operators in a general setting. For a lot of the results, we consider the $\overline{\partial}$-Neumann problem on Kähler manifolds with some bounded geometry, in order to show that previously known theorems in the setting of (domains in) $\mathbb{C}^n$ continue to hold more generally. One of these is that the discreteness of spectrum of the Dolbeault Laplacian "percolates" up the Dolbeault complex, provided some boundary and curvature assumptions are made. Therefore, necessary conditions for the discreteness of spectrum can be studied on the top end of the Dolbeault complex, where the Laplacian reduces to a somewhat more tractable operator, which we analyze with methods from Schrödinger operator theory. In the last chapter, we consider the $\overline{\partial}$-Neumann problem for the product of two Hermitian manifolds, and describe the (essential) spectrum of the Laplacian in terms of the spectra of the Laplacians on the individual factors.

Zusammenfassung

Diese Dissertation beschäftigt sich mit der Spektraltheorie des komplexen Laplaceoperators mit $\overline{\partial}$-Neumann Randbedingungen, aufgefasst als selbstadjungierter Operator wirkend auf dem Raum der quadratintegrablen Differentialformen einer Hermiteschen Mannigfaltigkeit. Das zugehörige Randwertproblem, das $\overline{\partial}$-Neumann Problem, tritt in natürlicher Weise bei der Behandlung der (inhomogenen) Cauchy–Riemann Gleichungen im Rahmen der (L^2-) Hodge-Theorie auf. Durch diesen Zusammenhang geben spektraltheoretische Eigenschaften des komplexen Laplaceoperators Einsichten in die Lösbarkeit der Cauchy–Riemann Gleichungen und, in weiterer Folge, in die Konstruktion von holomorphen Funktionen (allgemeiner: Schnitten von holomorphen Vektorbündeln) mit vorgeschriebenen Eigenschaften. Zum komplexen Laplaceoperator gehört der elliptische Dolbeault-Komplex, eine Verallgemeinerung der Wirtingerableitung $d/d\bar{z}$ aus der komplexen Analysis einer Veränderlichen, und die L^2-Realisierung mit $\overline{\partial}$-Neumann Randbedingungen entspricht der schwachen Erweiterung des Dolbeault-Komplexes. Aus diesem Grund behandeln wir hier auch Teile der Spektraltheorie von allgemeinen selbstadjungierten Erweiterung von elliptischen Differentialoperatoren auf Mannigfaltigkeiten. Für viele der Resultate betrachten wir das $\overline{\partial}$-Neumann Problem auf Kähler-Mannigfaltigkeiten mit beschränkter Geometrie, was es uns erlaubt, bekannte Sätze über das $\overline{\partial}$-Neumann Problem auf (Gebieten im) $\mathbb{C}^n$ zu verallgemeinern. Einer dieser Sätze besagt, dass sich die Diskretheit des Spektrums des komplexen Laplaceoperators im Dolbeault-Komplex nach oben fortpflanzt falls gewisse Annahmen an die Krümmung und den Rand des betrachteten Gebietes gemacht werden. Daher lassen sich notwendige Bedigungen für die Diskretheit des Spektrums auf dem oberen Ende des Dolbeault-Komplexes formulieren, wo der komplexe Laplaceoperator eine einfachere Form annimmt, die wir mit Hilfe von Methoden der Theorie von Schrödingeroperatoren studieren. Im letzten Kapitel betrachten wir das $\overline{\partial}$-Neumann Problem auf dem Produkt zweier Hermitescher Mannigfaltigkeiten und beschreiben das (wesentliche) Spektrum des komplexen Laplaceoperators durch das Spektrum der Laplaceoperatoren der beiden Faktoren.

Franz Berger

Academic Curriculum Vitæ

Education

since 2014 **Ph.D. studies in Mathematics**, *University of Vienna.*
Working title: "Spectral theory of the $\overline{\partial}^E$-Neumann problem."
Advisor: Prof. Friedrich Haslinger.

2012 – 2014 **Master's studies in Mathematics**, *University of Vienna.*
Master's thesis: "Structure of Toeplitz C^*-algebras for pseudoconvex Reinhardt domains."
Completed on August 26th, 2014.

2010 – 2012 **Bachelor's studies in Mathematics**, *University of Vienna.*
Completed on June 30th, 2012.

2009 – 2012 **Bachelor's studies in Physics**, *University of Vienna.*
Completed on November 28th, 2012.

Publications

2017 **On some spectral properties of the weighted $\overline{\partial}$-Neumann operator**, *with F. Haslinger*, Kyoto Journal of Mathematics, forthcoming.

2016 **Essential spectra of tensor product Hilbert complexes and the $\overline{\partial}$-Neumann problem on product manifolds**, Journal of Functional Analysis, 271.6 (2016), p. 1434–1461.

Talks

Sep. 2017 **19th ÖMG Congress and Annual DMV Meeting**, *Salzburg.*
"Some results on the $\overline{\partial}$-Neumann Laplacian on manifolds with bounded geometry"

Sep. 2016 **Joint mathematical societies (CSASC) meeting**, *Barcelona.*
"Essential spectrum of the complex Laplacian on product manifolds"

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"The essential spectrum of the complex Laplacian on product manifolds"

Teaching

2016 – 2017 **Exercise sessions to the lecture courses "Analysis for physicists I & II"**, *Faculty of Physics*, University of Vienna.
Two classes of *Analysis for physicists I* and three classes of *Analysis for physicists II.*