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„Stability conditions on quivers and semistable non-commutative curve counting"

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Arkadij Bojko, BSc

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Betreut von / Supervisor:	Univ.-Prof. Ludmil Katzarkov, PhD
Mitbetreut von / Co-Supervisor:	Dr. George Dimitrov

Abstract

The recently introduced concept of (semi-stable) non-commutative curve counting is examined for the derived category of the acyclic triangular quiver. First, we carefully recall and introduce all the notions necessary for the final results, these include the definition of stability conditions on triangulated categories. After reminding the reader of the previously known result that there are only two non-commutative curves of genus one in the derived category of the acyclic triangular quiver, we construct stability conditions of this category such that all the possible combinations of these non-commutative curves of genus one become semi-stable.

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1 Introduction

The notion of stability conditions on triangular categories as introduced by T. Bridgeland in [5] has its roots in mirror symmetry. To understand these motivations we need to sum up some basic ideas.

A Calabi-Yau manifold X is a Kaehler manifold with a Ricci flat metric. This is equivalent to the local existence of a holomorphic $(n,0)$ form $\Omega^{n,0}$ such that its associated volume form is equal up to a constant to the volume form on X. A special Lagrangian in X is a Lagrangian submanifold with the pullback of $\Omega^{n,0}$ being equal to the volume form up to some phase $\pi\phi$. A σ-model on a Calabi-Yau manifold X specifies an N=2 super-conformal field theory (or SCFT) in the moduli space $\mathcal{M}$ of SCFT. For every SCFT one associates two different topological twists, the topological conformal field theories (TCFT) of model A and B. These are given by Calabi-Yau A_∞ categories, where for the case of the A-model given for a CY manifold X the category is the derived category of the Fukaya category for the simplectic form on X. The B-model is represented by an enhancement of the derived category of coherent sheaves. The general structure of a Fukaya category consists of the objects being the Lagrangians in X with some additional data and the morphisms constructed from the Floer complexes with the Floer differential. In the moduli space $\mathcal{M}$ at the point corresponding to the CY manifold X there are two foliations which originate from keeping the A-model or the B-model of the TCFT constant while varying the SCFT.

The mirror map coming from the symmetry of the N=2 super-conformal algebra induces a map on the moduli space $\mathcal{M}$ which has the effect of interchanging the A and B models of the TCFT. This implies especially that the derived categories corresponding to these topological twists are equivalent. Further, let X be CY and $\mathcal{M}_{\mathbb{C}}(X)$ the leaf of the foliation in $\mathcal{M}$ for constant model B (varying the symplectic form) and $\mathcal{M}_K(X)$ the leaf of the foliation for constant model A (varying the complex structure). The mirror map tells us that for the mirror pairs X_1 and X_2 we should have the relations:

$$\mathcal{M}_{\mathbb{C}}(X_1) \cong \mathcal{M}_K(X_2), \qquad \mathcal{M}_{\mathbb{C}}(X_2) \cong \mathcal{M}_K(X_1).$$

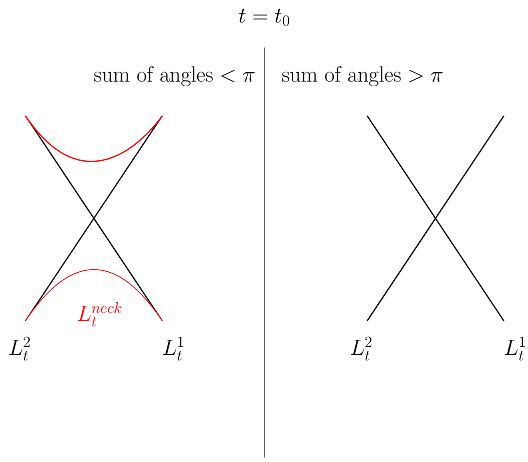
This mirror symmetry has been generalized beyond just the applications to CY manifolds in [1]. An example can be found in the work [2], where it is shown that the mirror to a weighted projective space $\mathbb{C}P^2(a,b,c)$ is the affine hypersurface $X = \{(x,y,z) \in (\mathbb{C}^*)^3 : x^a y^b z^c = 1\}$ with the superpotential $W = x + y + z$.

The motivation to introduce the stability conditions as given in the Definition 18 comes from the goal to describe the space $\mathcal{M}_K(X)$ and the parameters for varying the complex structure. Notice that above we were able to assign a phase to every special Lagrangian in the Fukaya category of the CY manifold. We could form a full subcategory of the Fukaya category $\mathcal{P}(\phi)$ consisting of the special Lagrangians with phase $\pi\phi$. Additionally, we can construct a map Z which acts on every Lagrangian L by:

$$Z(L) = \int_L \Omega^{n,0} \in \mathbb{C}.$$

Varying the complex structure, we end up changing the categories $\mathcal{P}(\phi)$ and the values of Z while keeping the Fukaya category fixed. This corresponds in our picture to moving in the space of stability conditions defined in subsection 4.4.

An especially interesting phenomenon resulting from varying the complex structure is that of wall crossing. The following picture is a heuristic one and was discussed by M. Konsevich in his talks. If $\{X_t\}$ is a family of CY manifolds with the corresponding holomorphic forms $\Omega_t^{n,0}$, where varying the parameter t corresponds to changing the complex structure, then a wall is formed by the parameters t_0 where two rigid special Lagrangians $L^1_{t_0}$ and $L^2_{t_0}$ have the same phase. The rigidness secures the existence of these special Lagrangians in the neighborhood of X_{t_0} in the moduli space $\mathcal{M}$, and these will be labeled L_t^1 and L_t^2 for the corresponding parameter t. If these two Lagrangians intersect for the parameter t_0 transversally and the sum of the angles is π (Maslov index is equal to 1), then one gets a following (simplified) geometric picture:



As the parameter t passes across the wall $t = t_0$ the sum of angles between the Lagrangians changes from less than π to larger than π, and on the "left" side of the wall there is an additional Lagrangian L_t^{neck} that is constructed by connecting the other two Lagrangians through a neck. As one passes through the wall this Lagrangian vanishes. The notion of semistable non-commutative curve counting presented rigorously later is motivated by this picture.

In the following work, we will at first sum up and prove some useful results about triangulated categories and derived categories in Sections 2 and 3. These will be used in Section 4 to give the definition of the spaces of stability conditions for triangulated categories introduced in [5] and to allow us to compute the example of stability conditions on the derived category of the quiver A_1. Sections 5 and 6 are meant to prepare the reader for the methods used for the so called non-commutative curve counting. The non-commutative curve counting will be presented in Section 7 the way it was given for the first time in [9]. We also develop some methods for finding the σ-semistable non-commutative curves for some given stability conditions on a triangulated category. The last section will consider a specific example by addressing the derived category $D^b(Q)$ of the following quiver:

$$\begin{tikzcd} & 3 & \\ 1 \arrow[ur] \arrow[rr] & & 2 \arrow[ul] \end{tikzcd}$$

The non-commutative curve counting invariants for this category are computed in [10]. Here, we recall the result for genus 1 non-commutative curves and study the following question about the semi-stability of genus 1 non-commutative curves.

Question 1. Are there stability conditions in the space of stability conditions of $D^b(Q)$ for each one of the following cases?

1. Neither one of the genus 1 non-commutative curves is σ-semistable.

2. Each one of the genus 1 non-commutative curves is σ-semistable, while the other one isn't.

3. Both non-commutative curves of genus 1 are σ-semistable.

We construct the stability conditions for each of these cases and thus prove the Proposition 12.15 given in [9] with its proof left for future work.

While it is beyond the scope of this work, one would also be interested to find all the walls in the space of stability conditions similar to those described above for Lagrangians and the complex structures, such that when we pass through them, the number of semistable non-commutative curves changes.

2 Triangulated categories

In this work, we use the definition of triangulated categories found for example in [3, p. 239]. We give here the form of the octahedral axiom that we will use later: If one has the following commutative diagram in a triangulated category $\mathcal{T}$

$$\begin{tikzcd} E \arrow[rr] \arrow[rrrr, bend left] & & F \arrow[dl] \arrow[rr] & & G \arrow[dl] \\ & A \arrow[ul, "{[1]}"] & & B \arrow[ul, "{[1]}"] \arrow[ll, "{[1]}"] & \end{tikzcd} \tag{1}$$

with the triangles E, F, A and F, B, G being distinguished and the others being commutative, then there exists an object F' with a diagram

$$\begin{tikzcd} E \arrow[rr] & & G \arrow[dl] \arrow[dd] \\ & F' \arrow[ul, "{[1]}"] \arrow[dr] & \\ A \arrow[uu, "{[1]}"] \arrow[ur] & & B \arrow[ll, "{[1]}"] \end{tikzcd} , \tag{2}$$

such that the upper and lower triangle are again distinguished and the left and right triangle are commutative. The arrows $A \to E$, $E \to G$, $G \to B$ and $B \to A$ are the same in both diagrams.

Remark 1. *Using the axioms of a triangulated category one can also show that if one has a diagram (2), then one can construct a diagram of the form (1), where the arrows $A \to E$ and $G \to B$ will be the same and the arrows $E \to G$ and $B \to A$ will get an additional minus sign in the newly constructed diagram.*

If k is a field, we say that $\mathcal{T}$ is a k-linear category when for any two objects $X, Y \in Ob(\mathcal{T})$, the set $\mathrm{Hom}(X, Y)$ has a structure of a vector space, such that the composition is a bilinear operation. A k-linear functor between two k-linear categories acts as a linear map on the morphism spaces. The translation functor on a k-linear triangular category $\mathcal{T}$ will be most commonly denoted by affixing [1] to the object/morphism (resp. morphism) that is being acted on. From now on every category and functor we will be working with will be assumed to be k-linear unless specified otherwise. If $\mathcal{E}$ is a class of objects in a triangulated category $\mathcal{T}$, then $\langle \mathcal{E} \rangle$ will denote the triangulated subcategory of $\mathcal{T}$ generated by $\mathcal{E}$. Let us begin by stating a lemma that will be used in later sections.

Lemma 2. *If the diagram of distinguished triangles of the form*

$$\begin{tikzcd} E \arrow[rr] & & F \arrow[dl, "g"] \arrow[rr] & & G \arrow[dl] \\ & A \arrow[ul, "{[1]}"] & & B \arrow[ul, "f"', "{[1]}"] & \end{tikzcd} \tag{3}$$

satisfies $g[1] \circ f = 0$, then there exists another diagram of distinguished triangles of the following form:

$$\begin{tikzcd} E \arrow[rr] & & F'' \arrow[dl] \arrow[rr] & & G \arrow[dl] \\ & B \arrow[ul, "{[1]}"] & & A \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Proof. The commutative diagram

$$\begin{tikzcd}
E \arrow[rr] \arrow[rrrr, bend left] & & F \arrow[dl, "g"'] \arrow[rr] & & G \arrow[dl] \\
& A \arrow[ul, "{[1]}"] & & B \arrow[ll, "0", "{[1]}"'] \arrow[ul, "f"', "{[1]}"] &
\end{tikzcd}$$

gives by the octahedral axiom explained above the commutative diagram of distinguished triangles of the form

$$\begin{tikzcd}
E \arrow[rr] & & G \arrow[dl] \arrow[dd] \\
& F' \arrow[ul, "{[1]}"] \arrow[dr] & \\
A \arrow[uu, "{[1]}"] \arrow[ur] & & B \arrow[ll, "0", "{[1]}"']
\end{tikzcd} \; .$$

By the vanishing of $B \to A$ the lower triangle is a biproduct diagram (to recall what a biproduct diagram is, see the proof of Lemma 11). Thus we can interchange A and B in the last diagram and apply Remark 1 to get the following diagram:

$$\begin{tikzcd}
E \arrow[rr] \arrow[rrrr, bend left] & & F'' \arrow[dl, "g"'] \arrow[rr] & & G \arrow[dl] \\
& B \arrow[ul, "{[1]}"] & & A \arrow[ll, "{[1]}"] \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

□

Let us recall now the definition and the properties of t-structures.

Definition 3. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{T}^{\leq 0}$, $\mathcal{T}^{\geq 0}$ its strictly full subcategories. $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is called a t-structure on $\mathcal{T}$ when the following conditions are satisfied:*

1. *$\mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1}$ where $\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]$.*
2. *$\mathcal{T}^{\geq 1} \subset \mathcal{T}^{\geq 0}$ where $\mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n]$.*
3. *For any $X \in Ob(\mathcal{T}^{\leq 0})$ and any $Y \in Ob(\mathcal{T}^{\geq 1})$ one has $Hom(X, Y) = 0$.*
4. *Let $A \in Ob(\mathcal{T})$, then there exists a distinguished triangle $E^{\leq 0} \longrightarrow E \longrightarrow E^{\geq 1} \longrightarrow E^{\leq 0}[1]$ where $E^{\leq 0} \in Ob(\mathcal{T}^{\leq 0})$ and $E^{\geq 1} \in Ob(\mathcal{T}^{\geq 1})$.*

Remark 4. *For the definition of a t-structure, one could start equivalently just from $\mathcal{T}^{\leq 0}$ and define $\mathcal{T}^{\geq 1} = \{Y \in Ob(\mathcal{T}) | Hom(X, Y) = 0 \forall X \in Ob(\mathcal{T}^{\leq 0})\}$ requiring that the 1. and 4. axiom holds. The other two axioms would be an obvious consequence of this definition. Conversely, if the t-structure is given by the Definition 3 and Y is such that $Hom(X, Y) = 0$ for all $X \in Ob(\mathcal{T}^{\leq 0})$, then acting with the functor $Hom(Y^{\leq 0}, -)$ on the distinguished triangle $Y^{\leq 0} \longrightarrow Y \longrightarrow Y^{\geq 1} \longrightarrow Y^{\leq 0}[1]$ and using the axioms 2. and 3. of the above definition, shows that $Y^{\leq 0} = 0$ and thusly $Y \cong Y^{\geq 1}$ and $Y \in Ob(\mathcal{T}^{\geq 1})$. As such, we will label any given t-structure just by its first subcategory.*

For a given t-structure $\mathcal{T}^{\leq 0}$, one can define the truncation functors $\tau_{\leq n} : \mathcal{T} \to \mathcal{T}^{\leq n}$ and $\tau_{\geq n} : \mathcal{T} \to \mathcal{T}^{\geq n}$ as described in [3, p. 279] for any integer n. The defining property is that for any object X and any $n \in \mathbb{Z}$ there exists a distinguished triangle

$$\tau_{\leq n}(X) \longrightarrow X \longrightarrow \tau_{\geq n+1}(X) \longrightarrow \tau_{\leq n}(X)[1] \, . \tag{4}$$

The functors $\tau_{\leq n}$ are right adjoint to the embedding functors of $\mathcal{T}^{\leq n}$, while the functors $\tau_{\geq n}$ are left adjoint to the embedding functors of $\mathcal{T}^{\geq n}$.

Remark 5. *As a direct result of the definition of the truncation functors, one sees that for any object X in $\mathcal{T}^{\leq n}$ the following hold:*

$$\tau_{\leq n}(X) \cong X\,, \qquad \tau_{\geq n+1}(X) = 0\,.$$

For X in $\mathcal{T}^{\geq n+1}$ we also get:

$$\tau_{\leq n}(X) = 0\,, \qquad \tau_{\geq n+1}(X) \cong X$$

.

Additionally, one defines the heart of the t-structure to be the full subcategory $\mathcal{A} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$. It is well known (see e.g.[3, p. 279]) that $\mathcal{A}$ is an abelian category with $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ being an exact triple in $\mathcal{A}$ exactly when there exists a distinguished triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to A[1]$ in $\mathcal{T}$, such that A, B, and C are objects of $\mathcal{A}$.

We will be interested in a special kind of t-structures. A t-structure $\mathcal{T}^{\leq 0}$ is bounded if

$$\mathcal{T} = \bigcup_{i,j\in\mathbb{Z}} \mathcal{T}^{\leq i, \geq j} \tag{5}$$

where $\mathcal{T}^{\leq i,\geq j} = \mathcal{T}^{\leq i} \cap \mathcal{T}^{\geq j}$. In what follows, the t-structures we consider will be bounded. The following proposition will be used repeatedly in this section the proof of which can be found in [4, Proposition 8.1.8].

Proposition 6. *Let $m, n \in \mathbb{Z}$.*

1. *If $m \geq n$, then we have $\tau^{\leq m} \circ \tau^{\geq n} \cong \tau^{\geq n} \circ \tau^{\leq m}$.*
2. *If $m < n$, then $\tau^{\leq m} \circ \tau^{\geq n} = 0 = \tau^{\geq n} \circ \tau^{\leq m}$.*
3. *If $m \geq n$, then $\tau^{\leq m} \circ \tau^{\leq n} \cong \tau^{\leq n} \circ \tau^{\leq m} \cong \tau^{\leq n}$ and $\tau^{\geq m} \circ \tau^{\geq n} \cong \tau^{\geq n} \circ \tau^{\geq m} \cong \tau^{\geq m}$.*

Remark 7. *One can show that $\tau^{\leq m}(\mathcal{T}^{\geq n}) \subset \mathcal{T}^{\geq n}$, $\tau^{\geq m}(\mathcal{T}^{\geq n}) \subset \mathcal{T}^{\geq n}$, $\tau^{\leq m}(\mathcal{T}^{\leq n}) \subset \mathcal{T}^{\leq n}$, $\tau^{\geq m}(\mathcal{T}^{\leq n}) \subset \mathcal{T}^{\leq n}$ for all $n, m \in \mathbb{Z}$.*

Using this, the functor $\tau_{\leq m,\geq n} : \mathcal{T} \to \mathcal{T}^{\leq m,\geq n}$ can be defined as $\tau_{\leq m,\geq n} = \tau_{\leq m} \circ \tau_{\geq n} \cong \tau_{\geq n} \circ \tau_{\leq m}$ for all $m \geq n$ in $\mathbb{Z}$. A special case of this functor is when $m = n$ and we call it the n'th homology functor $H^n = \tau_{\leq n,\geq n} : \mathcal{T} \to \mathcal{A}[-n]$. By the result from [3, p. 283], saying that $X \in Ob(\mathcal{T}^{\leq n})$ if and only if $H^i(X) = 0$ whenever $i > n$ (and similarly for $\mathcal{T}^{\geq n}$), we see that

$$Ob(\mathcal{T}^{\leq m,\geq n}) = \left\{X \in Ob(\mathcal{T}) \mid H^i(X) = 0 \quad \forall\, i : i < n \text{ or } i > m\right\}. \tag{6}$$

T. Bridgeland states the lemma [5, Lemma 3.2] without proving it. For completeness, we give the proof here.

Lemma 8. *Let $\mathcal{A} \subset \mathcal{T}$ be a full additive subcategory of the triangulated category $\mathcal{T}$, then $\mathcal{A}$ is a heart of some bounded t-structure $\mathcal{T}^{\leq 0} \subset \mathcal{T}$ if and only if the following two conditions hold:*

1. *If $A \in Ob(\mathcal{A}[i])$ and $B \in Ob(\mathcal{A}[j])$ where $i > j$, then $Hom(A, B) = 0$.*
2. *For any non-zero $E \in Ob(\mathcal{T})$ there exists a sequence $i_1 > i_2 > \ldots > i_n$ and a filtration*

$$\begin{tikzcd}
0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dr] & & E_2 \arrow[r] & \ldots \arrow[r] & E_{n-1} \arrow[rr] & & E \arrow[dl] \\
& A_1 \arrow[ul, "{[1]}"] & & A_2 \arrow[ul, "{[1]}"] & & & & A_n \arrow[ul, "{[1]}"] &
\end{tikzcd} \tag{7}$$

such that $A_l \in Ob(\mathcal{A}[i_l])$ are non-zero for all l and the triangles are distinguished.

Proof. If there is such a bounded t-structure $\mathcal{T}^{\leq 0}$ with its truncation functor and homology functors, such that $\mathcal{A}$ is its heart, then the first condition follows directly from the third axiom of the Definition 3. For any object E we can find such integers i_1 and m with $-i_1 \leq m$, such that $E \in Ob(\mathcal{T}^{\leq m, \geq -i_1})$ and such that m is the largest integer for which $H^m(E)$ is non-zero and $-i_1$ is the least such integer. Using the property (4) we construct

$$\tau_{\leq -i_1}(E) \longrightarrow E \longrightarrow \tau_{\geq -i_1+1}(E) \longrightarrow \tau_{\leq -i_1}(E)[1]\,.$$

From Remark 5, we see that $E \cong \tau_{\geq -i_1}(E)$. Together with the definition of the homology functors, we obtain a distinguished triangle:

$$H^{-i_1}(E) \longrightarrow E \longrightarrow \tau_{\geq -i_1+1}(E) \longrightarrow H^{-i_1}(E)[1] \tag{8}$$

and set $A_1 = H^{-i_1}(E) \in Ob(\mathcal{A}[i_1])$ and $E_2' = \tau_{\geq -i_1+1}(E)$. By Proposition 6 and Remark 7, we see that $E_2' \in Ob(\mathcal{T}^{\leq m, \geq -i_1+1})$. Now we can do the same with E_2', finding the least integer $-i_2$ for which $H^{-i_2}(E_2') \neq 0$. Using the Proposition 6, we see that $H^k(\tau^{\geq m}(E)) \cong H^k(E)$ whenever $k \geq m$, thus the largest integer with non-vanishing homology of E_2' is the same as for E. Combining the distinguished triangle (8) with the corresponding one for E_2', we have:

$$\begin{tikzcd}
A_1 \arrow[rr] & & E \arrow[dl] \arrow[dd] \\
& E_2' \arrow[ul, "{[1]}"'] \arrow[dr] & \\
H^{-i_2}(E_2') \arrow[uu, "{[1]}"] \arrow[ur] & & \tau_{-i_2+1}(E_2') \arrow[ll, "{[1]}"]
\end{tikzcd}\;. \tag{9}$$

Here we again label the new objects, such that $A_2 = H^{-i_2}(E_2') \cong H^{-i_2}(E)$ and $E_3' = \tau_{-i_2+1}(E_2') \cong \tau_{-i_2+1}(E)$. Using Remark 1 on (9), we construct the following diagram with a new object E_2 at the center:

$$\begin{tikzcd}
A_1 \arrow[rr] \arrow[rrrr, bend left] & & E_2 \arrow[rr] \arrow[dl] & & E \arrow[dl] \\
& A_2 \arrow[ul, "{[1]}"] & & E_3' \arrow[ul, "{[1]}"] \arrow[ll, "{[1]}"] &
\end{tikzcd} \tag{10}$$

We can now repeat this step until E_n' lies in $\mathcal{A}[-m]$ and rename it to $A_n = E_n'$ to obtain the diagram (7).

Conversely, if the two conditions hold, define $\mathcal{T}^{\leq 0}$ to be the full subcategory with object being all E, such that $i_n \geq 0$, and the objects of the full subcategory $\mathcal{T}^{\geq 0}$ are such E with $i_1 \leq 0$. If E has factors of the filtration given by A_i as in (7) then $E[-n]$ has factors $A_i[-n]$. This shows that $\mathcal{T}^{\leq n}$ consists of objects with $i_n \geq -n$ and $\mathcal{T}^{\geq n}$ of objects with $i_1 \leq -n$. As such, the axioms 1 and 2 of the Definition 3 follow immediately.

For the axiom 4, consider any object E of the form (7). We now want to merge all the factors A_l where $i_l \leq -1$ into one single factor. The first step is to use the octahedral axiom on the diagram

$$\begin{tikzcd}
E_{n-2} \arrow[rr] \arrow[rrrr, bend left] & & E_{n-1} \arrow[rr] \arrow[dl] & & E \arrow[dl] \\
& A_{n-1} \arrow[ul, "{[1]}"] & & A_n \arrow[ul, "{[1]}"] \arrow[ll, "{[1]}"] &
\end{tikzcd}$$

to obtain

$$\begin{tikzcd}
E_{n-2} \arrow[rr] & & E \arrow[dl] \arrow[dd] \\
& G_1 \arrow[ul, "{[1]}"'] \arrow[dr] & \\
A_{n-1} \arrow[uu, "{[1]}"] \arrow[ur] & & A_n \arrow[ll, "{[1]}"]
\end{tikzcd} .$$

Thus constructing the factor G_1 with filtration consisting of factors A_{n-1} and A_n. In the same way we can apply to octahedral axiom to merge A_{n-2} with G_1 to get G_2 which has a filtration with the factors A_{n-2}, A_{n-1} and A_n. Repeating this until $G = G_k$ is the object with filtration with all factors A_l such that $i_k \leq -1$, we get a distinguished triangle

$$E_s \longrightarrow E \longrightarrow F \longrightarrow E_s[1]\,,$$

where s is the largest integer such that $i_s \geq 0$. We see that E_s is an object of $\mathcal{T}^{\leq 0}$ and G lies in $\mathcal{T}^{\geq 1}$.

Finally, consider $E \in Ob(\mathcal{T}^{\leq 0})$ and $F \in Ob(\mathcal{T}^{\geq 1})$, where E has the form in 7 with $i_n \geq 0$ and F has the filtration

$$\begin{tikzcd}
0 \arrow[rr] & & F_1 \arrow[rr] \arrow[dl] & & F_2 \arrow[r] & \dots \arrow[r] & F_{n-1} \arrow[rr] & & F \arrow[dl] \\
& B_1 \arrow[ul, "{[1]}"] & & B_2 \arrow[ul, "{[1]}"] & & & & B_m \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

where $B_l \in Ob(\mathcal{A}[j_l])$ and $-1 \geq j_1 > j_2 > \ldots > j_m$. We want to show now that $\mathrm{Hom}(E, F) \cong \mathrm{Hom}(A_1, F) \cong \mathrm{Hom}(A_1, B_1) = 0$. Acting with $\mathrm{Hom}(A_i, -)$ on the first distinguished triangle from the right of the above diagram, we obtain an exact sequence:

$$\mathrm{Hom}(A_i, B_m[-1]) \longrightarrow \mathrm{Hom}(A_i, F_{n-1}) \longrightarrow \mathrm{Hom}(A_i, F) \longrightarrow \mathrm{Hom}(A_i, B_m)\,. \tag{11}$$

If $m > 1$ then both the left and right term of the above sequence vanish and we have $\mathrm{Hom}(A_i, F_{n-1}) \cong \mathrm{Hom}(A_n, F)$. Using this argument repeatedly, we obtain $\mathrm{Hom}(A_i, F) \cong \mathrm{Hom}(A_i, B_1) = 0$. If $m = 1$ the statement is a tautology. Now act with the functor $\mathrm{Hom}(-, F)$ to get the exact sequences

$$0 = \mathrm{Hom}(A_i, F) \longrightarrow \mathrm{Hom}(E, F) \longrightarrow \mathrm{Hom}(E_{n-1}, F) \tag{12}$$

showing that $\mathrm{Hom}(E, F)$ is a sub-object of $\mathrm{Hom}(E_{n-1}, F)$. Applying this repeatedly we see that $\mathrm{Hom}(E, F) \subset \mathrm{Hom}(A_1, F) = 0$ and the axiom 3 then follows. □

Remark 9. *If the two conditions of the above lemma hold, then the filtration (7) of an object E is unique up to an isomorphism of the diagrams. To see this, consider a morphism $E \to E'$. Allowing the factors A_i and A'_i of E and E' respectively to be zero if necessary, so that E has a factor in $\mathcal{A}_l$ if and only if E' has one, we get filtration of E and E' of the same length with some steps being extensions by* 0*. The diagram for E' is now the same as (7) with E_i replaced by E'_i and A_i by A'_i. From the proof of Lemma 8, we see that $Hom(E_{n-1}, A'_n) = 0 = Hom(E_{n-1}, A'_n[-1])$. And the exact sequence*

$$Hom(E_{n-1}, A'_n[-1]) \longrightarrow Hom(E_{n-1}, E'_{n-1}) \longrightarrow Hom(E_{n-1}, E') \longrightarrow Hom(E_{n-1}, A'_n)$$

tells us that there is an isomorphism $Hom(E_{n-1}, E'_{n-1}) \cong Hom(E_{n-1}, E')$ given by post-composing with the morphism $E'_{n-1} \to E'$. Thus there is a unique morphism $E_{n-1} \to E'_{n-1}$, such that the compositions $E_{n-1} \to E \to E'$ and $E_{n-1} \to E'_{n-1} \to E'$ are equal. The two morphisms $E \to E'$ and $E_{n-1} \to E'_{n-1}$ can be completed to a morphism of the distinguished triangles by a unique morphism

$A_n \to A_n'$: *The existence of this morphism follows from the axiom of the triangulated category, while the uniqueness follows because of the exact sequence*

$$Hom(E_{n-1}, A_n') \longrightarrow Hom(A_n, A_n') \longrightarrow Hom(E, A_n'),$$

where the vanishing of the first term implies the injectivity of the second arrow. Repeating this argument extends this to a unique morphism of the diagrams of E and E', thus if applied to the identity $E \to E$, the result follows.

Remark 10. *The previous remark together with Lemma 8 imply that there is a one to one correspondence between bounded t-structures and their hearts.*

3 Derived categories

In this work, we are interested in bounded derived categories as examples of triangulated categories. Let $\mathcal{A}$ be an abelian category and $\mathrm{Com}^b(\mathcal{A})$ the category of its bounded complexes. The bounded derived category $D^b(\mathcal{A})$ of $\mathcal{A}$ is a category with a functor $Q : \mathrm{Com}^b(\mathcal{A}) \to D^b(\mathcal{A})$ which maps every quasi-isomorphism to an isomorphism, and every functor that has this property factorizes through $D^b(\mathcal{A})$ with respect to Q. It can be constructed as the localization $D^b(\mathcal{A}) = K^b(\mathcal{A})[S^{-1}]$ of the homotopy category of bounded complexes $K^b(\mathcal{A})$ by the localizing class S of quasi-isomorphisms. The functor $J : \mathcal{A} \to D^b(\mathcal{A})$ that maps every object A to a complex K_A where $K_A^0 = A$ and $K_A^i = 0$ for all nonzero integers i and acts in an obvious way on morphisms, is a fully faithful functor. As such, we will view $\mathcal{A}$ as a subcategory of $D^b(\mathcal{A})$ and its objects as the corresponding 0-complexes.

Using the standard homology functors on $D^b(\mathcal{A})$ which we will denote $H^i : D^b(\mathcal{A}) \to \mathcal{A}$, one can define the the standard t-structure:

$$\begin{aligned} D^{\leq 0}(\mathcal{A}) &= \{X \in Ob\big(D^b(\mathcal{A})\big) \mid H^i(X) = 0 \quad \forall\, i > 0\}, \\ D^{\geq 0}(\mathcal{A}) &= \{X \in Ob\big(D^b(\mathcal{A})\big) \mid H^i(X) = 0 \quad \forall\, i < 0\}. \end{aligned} \tag{13}$$

The proof of it solving the axioms for a t-structure can be found in [3, p. 278, Proposition 3] . The heart of this t-structure is equivalent to $\mathcal{A}$ as a category and contains it, so it is its closure under isomorphisms. We can view the heart as the abelian category itself.

An abelian category is said to be semisimple when its short exact sequences split (all short exact sequences can be completed into biproduct diagrams - we recall what this means in the proof of Lemma 11). In a special case of $\mathcal{A}$ being semisimple, one can simplify its bounded derived category.

Lemma 11. *Let $\mathcal{A}$ be a semisimple abelian category, then its bounded derived category $D^b(\mathcal{A})$ is equivalent to the full subcategory $Com_0^b(\mathcal{A})$ whose objects are the bounded complexes with 0 boundary maps.*

Proof. We construct an equivalence functor of the category $D^b(\mathcal{A})$ and the full subcategory whose objects are the bounded complexes with 0 boundary maps and denote this full subcategory by $Com_0^b(\mathcal{A})$. Let the functor $H^\bullet : D^b(\mathcal{A}) \to Com_0^b(\mathcal{A})$ be defined by connecting to every complex $(K^\bullet, d)$ the complex $(H^\bullet(K), 0)$ of its cohomologies, and acting on morphisms in the obvious way. The equivalence inverse is then the functor $I : Com_0^b(\mathcal{A}) \to D^b(\mathcal{A})$ which maps every complex to itself as viewed in the derived category. We now have that $H^\bullet \circ I = \mathrm{id}_{Com_0^b(\mathcal{A})}$, so we only need to show that $I \circ H^\bullet$ is isomorphic to $\mathrm{id}_{D^b(\mathcal{A})}$. For this, consider the short exact sequences

$$0 \longrightarrow \ker d^n \xrightarrow{i_{K^n}} K^n \xrightarrow{q_{K^n}} \mathrm{im}\, d^n \longrightarrow 0, \tag{14}$$

$$0 \longrightarrow \mathrm{im}\, d^{n-1} \xrightarrow{j_{K^n}} \ker d^n \xrightarrow{p_{K^n}} H^n(K) \longrightarrow 0, \tag{15}$$

given for any K^n where $d^n = i_{K^{n+1}} \circ j_{K^{n+1}} \circ q_{K^n}$. As the category is semisimple, these can be completed into the corresponding biproduct diagrams:

$$\ker\, d^n \underset{}{\overset{i_{K^n}}{\rightleftarrows}} K^n \underset{q_{K^n}}{\overset{r_{K^n}}{\leftrightarrows}} \operatorname{im}\, d^n \tag{16}$$

$$\operatorname{im}\, d^{n-1} \underset{t_{K^n}}{\overset{j_{K^n}}{\rightleftarrows}} \ker d^n \underset{p_{K^n}}{\overset{s_{K^n}}{\leftrightarrows}} H^n(K) \tag{17}$$

Using these and $d^n \circ i_{K^n} = 0$, we can construct for every $K^\bullet$ the morphisms:

$$\begin{aligned} i_{K^\bullet} \circ s_{K^\bullet} &:(H^\bullet(K), 0) \to K^\bullet \\ (i_{K^\bullet} \circ s_{K^\bullet})^n = i_{K^n} \circ s_{K^n} &:H^n(K) \to K^n\,. \end{aligned}$$

To find $H^n(i_{K^\bullet} \circ s_{K^\bullet}) : H^n(K) \to H^n(K)$, notice that the triple in (15) for $H^\bullet(K)$ becomes

$$0 \longrightarrow H^n(K) \xrightarrow{id} H^n(K)\,.$$

The morphism $(i_{K^\bullet} \circ s_{K^\bullet})^n : H^n(K) \to K^n$ factorizes through $\ker\, d^n$ by $s_{K^n} : H^n(K) \to \ker\, d^n$, so we are looking for a morphism represented by the right vertical arrow $H^n(K) \to H^n(K)$ which would make the diagram below commutative.

$$\begin{tikzcd}
\operatorname{im}\, d^{n-1} \arrow[r, "j_{K^n}"] & \ker d^n \arrow[r, "p_{K^n}"] & H^n(K) \\
0 \arrow[u] \arrow[r] & H^n(K) \arrow[u, "s_{K^n}"] \arrow[r, "\mathrm{id}"] & H^n(K) \arrow[u]
\end{tikzcd}$$

This morphism is the identity $\mathrm{id}_{H^n(K)} : H^n(K) \to H^n(K)$ because s_{K^n} was the split monomorphism of the upper exact triple. Thus, we have constructed $H^n(i_{K^\bullet} \circ s_{K^\bullet}) = \mathrm{id}_{H^n(K)}$, which shows that $i_{K^\bullet} \circ s_{K^\bullet}$ is an isomorphism in the derived category.

Let us prove for any $f : (K^\bullet, d) \to (L^\bullet, c)$ a morphism of complexes in $\mathrm{Com}^b(\mathcal{A})$ the commutativity of the following diagram in $D^b(\mathcal{A})$.

$$\begin{tikzcd}
H^\bullet(K) \arrow[r, "H^\bullet(f)"] \arrow[d, "i_{K^\bullet} \circ s_{K^\bullet}"'] & H^\bullet(L) \arrow[d, "i_{L^\bullet} \circ s_{L^\bullet}"] \\
K^\bullet \arrow[r, "f"] & L^\bullet
\end{tikzcd} \tag{18}$$

Using the diagram

$$\begin{tikzcd}
\ker\, d^n \arrow[r, "i_{K^n}"] \arrow[d, "f'_n"'] & K^n \arrow[d, "f_n"] \\
\ker\, c^n \arrow[r, "i_{L^n}"] & L^n
\end{tikzcd}$$

where f' is uniquely defined such that it is commutative, we get $f_n \circ i_{K^n} \circ s_{K^n} = i_{L_n} \circ f'_n \circ s_{K^n}$ Consider the commutative diagram used to define $H^n(f)$, where the first and second row can be completed into biproducts.

$$\begin{tikzcd}
\operatorname{im}\, d^{n-1} \arrow[r, "j_{K^n}"] \arrow[d] & \ker d^n \arrow[r, "p_{K^n}"] \arrow[d, "f'_n"] & H^n(K) \arrow[d, "H^n(f)"] \\
\operatorname{im}\, c^{n-1} \arrow[r, "j_{L^n}"] & \ker\, c^n \arrow[r, "p_{L^n}"] & H^n(L)
\end{tikzcd}$$

Notice that $i_{L^n} \circ \left(f'_n \circ s_{K^n} - s_{L^n} \circ H^n(f)\right) = i_{L^n} \circ j_{L^n} \circ t_{L^n} \circ f'_n \circ s_{K^n}$ by using $\mathrm{id}_{\ker c^n} = j_{L^n} \circ t_{L^n} + s_{L^n} \circ p_{L^n}$. We can show that this difference is homotopic to 0. Consider the maps

$$u_n = r_{L^{n-1}} \circ t_{L^n} \circ f'_n \circ s_{K^n} : H^n(K) \to L^{n-1}\,.$$

It follows that $c_{n-1} \circ u_n = i_{L^n} \circ j_{L^n} \circ t_{L^n} \circ f'_n \circ s_{K^n}$ which ensures the commutativity of (18) in $K^b(\mathcal{A})$.

A general morphism $K^\bullet \to L^\bullet$ in $D^b(\mathcal{A})$ is represented by the roof $K^\bullet \xleftarrow{q} M^\bullet \xrightarrow{f} L^\bullet$ where q is a quasi-isomorphism. The previous results implies the commutativity of the right and left rectangle of the following diagram:

$$\begin{tikzcd}
H^\bullet(K) \arrow[r, "H^\bullet(q)^{-1}"] \arrow[d, "i_{K^\bullet}\circ s_{K^\bullet}"] & H^\bullet(M) \arrow[r, "H^\bullet(\phi)"] \arrow[d, "i_{M^\bullet}\circ s_{M^\bullet}"] & H^\bullet(L) \arrow[d, "i_{L^\bullet}\circ s_{L^\bullet}"] \\
K^\bullet & M^\bullet \arrow[l, "q"'] \arrow[r, "f"'] & L^\bullet
\end{tikzcd} \tag{19}$$

This completes the proof. □

4 Stability conditions on triangulated categories

4.1 Slicings and stability conditions

In this section, we mainly remind of the definitions and results introduced by T. Bridgeland in [5].

Definition 12. *Let $\mathcal{T}$ be a triangulated category. A slicing $\mathcal{P}$ is a collection of strictly full additive subcategories $\mathcal{P}(\phi)$ given for any $\phi \in \mathbb{R}$, such that the following conditions hold:*

1. $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)\,.$
2. *If $X \in Ob\big(\mathcal{P}(\phi)\big)$ and $Y \in Ob\big(\mathcal{P}(\psi)\big)$ where $\phi > \psi$, then $Hom(X, Y) = 0\,.$*
3. *For any non-zero $E \in Ob(\mathcal{T})$ there exists a sequence $\phi_1 > \phi_2 > \ldots > \phi_n$ and a diagram of distinguished triangles*

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \ldots \arrow[r] & E_{n-1} \arrow[rr] & & E \arrow[dl] \\
& A_1 \arrow[ul, "{[1]}"] & & A_2 \arrow[ul, "{[1]}"] & & & & A_n \arrow[ul, "{[1]}"] &
\end{tikzcd} \tag{20}$$

where $A_i \in Ob\big(\mathcal{P}(\phi_i)\big)$ are non-zero.

Remark 13. *The uniqueness of the diagram (20) for any object E is demonstrated in the same way we illustrated it in Remark 9 for the filtration with respect to a t-structure. We call this diagram the Harder-Narasimhan filtration of E.*

Every object X in $\mathcal{P}(\phi)$ for some ϕ is called semistable, and we write $\phi_{\mathcal{P}}(X) = \phi$. The class of all semistable objects is labeled $\mathcal{P}^{ss}$. Next, by the uniqueness remark above, one defines for any E with the HN filtration given by (20) the quantities $\phi^+_{\mathcal{P}}(E) = \phi_1$ and $\phi^-_{\mathcal{P}}(E) = \phi_n\,.$

Proposition 14. *Let $\mathcal{P}$ be a slicing of a triangulated category $\mathcal{T}$, then $\mathcal{P}(\phi)$ are closed under extensions for all $\phi \in \mathbb{R}$.*

Proof. Let A and A' be in $\mathcal{P}(\phi)$ and E their extension: $A \longrightarrow E \longrightarrow A' \longrightarrow A[1]\,.$ Using the arguments from the proof of Lemma 8, we see that for any two objects X and Y with $\phi^-(X) > \phi^+(Y)$ the space $\mathrm{Hom}(X, Y)$ is trivial. Thus $\phi^+(E) < \phi$ would imply that $A \to E$ vanishes and $A' = A[1] \oplus E$

while simultaneously $\mathrm{Hom}(A[1], A') = 0$, which presents a contradiction. A similar argument applies to the case $\phi^-(E) > \phi$, and we conclude for now that $\phi^-(E) \leq \phi \leq \phi^+(E)$.

Suppose, the filtration of E is given by (20) where we assume now that $\phi(A_n) < \phi$ holds. The composition $A \to E \to A_n$ equals the zero morphism and so the morphism $E \to A_n$ factors via A' and is must itself be 0, which gives a contradiction: If the morphism $E \to A_n$ is zero, then we have $E_{n-1} = E \oplus A_n[-1]$ which has the HN filtration with factors $A_1, \ldots, A_{n-1}, A_n, A_n[-1]$ but also one with factors $A_1, \ldots, A_{n-1}$. From the uniqueness in the Remark 13, we get the contradiction. Assuming that $\phi(E_1) > \phi$ leads to the arrow $E_1 \to E$ being a zero morphism, too. Summarizing our results, we see that $\phi^+(E) = \phi = \phi^-(E)$, and E also lies in $\mathcal{P}(\phi)$. □

If $I \subset \mathbb{R}$ is an interval, then we will denote the extension closure containing all $\mathcal{P}(\phi)$ with $\phi \in I$ by $\mathcal{P}(I)$.

Remark 15. *The subcategory $\mathcal{P}(I)$ can be explicitly given as the full subcategory containing all objects E, such that $\phi^+(E), \phi^-(E) \in I$. Obviously, every such object lies in $\mathcal{P}(I)$. Conversely, let A and B be two objects with $\phi^\pm(A), \phi^\pm(B) \in I$. Using the method from the proof of Proposition 14 yields for any extension $A \longrightarrow E \longrightarrow B \longrightarrow A[1]$ that the following inequalities must hold:*

$$\phi^-(A) \leq \phi^-(E) \leq \phi^+(E) \leq \phi^+(B)\,. \tag{21}$$

Thus we can assert that this full subcategory is closed under extensions, and so it must coincide with $\mathcal{P}(I)$.

If I is the unbounded interval $(-\infty, b]$, $(-\infty, b)$, $[a, \infty)$, or (a, ∞) then the corresponding $\mathcal{P}(I)$ is denoted by $\mathcal{P}(\leq b)$, $\mathcal{P}(< b)$, $\mathcal{P}(\geq a)$, or $\mathcal{P}(> a)$. They are of special interest as they give t-structures on $\mathcal{T}$. Indeed, let $\mathcal{T}^{\leq 0} := \mathcal{P}(\geq a)$ and $\mathcal{T}^{\geq 1} := \mathcal{P}(< a)$ (resp. $\mathcal{T}^{\leq 0} := \mathcal{P}(> a)$ and $\mathcal{T}^{\geq 1} := \mathcal{P}(\leq a)$), then all four axioms can be seen to be true from the previous results. The heart of this t-structure is $\mathcal{P}\big([a, a+1)\big)$ (resp. $\mathcal{P}\big((a, a+1]\big)$) and so it will be an abelian subcategory for any $a \in \mathbb{R}$. We will call $\mathcal{P}\big((0, 1]\big)$ the heart of a t-structure.

In addition to the concept of slicing, we require the definition of a Grothendieck group of a triangulated category.

Definition 16. *Let $\mathcal{T}$ be a triangulated category, its Grothendieck group is an abelian group $K_0(\mathcal{T})$ with a map $i : Ob(\mathcal{T}) \to K_0(\mathcal{T})$, such that $i(B) = i(A) + i(C)$, whenever $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ is a distinguished triangle. Further it has the universal property that if $\phi : Ob(\mathcal{T}) \to H$ is a map into an abelian group solving the same requirement as i, then there exists a unique group homomorphism $\tilde{\phi} : K_0(\mathcal{T}) \to H$, such that $\phi = \tilde{\phi} \circ i$.*

Remark 17. *Notice that up to a group isomorphism, $K_0(\mathcal{T})$ is uniquely given by an abelian group generated by the isomorphism classes of objects of $\mathcal{T}$ with the relations $[B] = [A] + [C]$ for all distinguished triangles. The map i acts by connecting the corresponding class $i(A) = [A]$. We will usually omit writing the brackets or i and simply denote the class by A.*

Now the definition of stability conditions on $\mathcal{T}$ can be given.

Definition 18. *Let $\mathcal{T}$ be a triangulated category. A pair $\sigma = (Z, \mathcal{P})$, where $\mathcal{P}$ is a slicing on $\mathcal{T}$ and $Z : K_0(\mathcal{T}) \to \mathbb{C}$ is a group homomorphism, is said to be a stability condition on $\mathcal{T}$ when for any non-zero $A \in \mathcal{P}^{ss}$ there exists such $m_\sigma(A) \in \mathbb{R}^{>0}$, such that*

$$Z(A) = m_\sigma(A)exp\big(i\pi\phi_{\mathcal{P}}(A)\big)\,. \tag{22}$$

The homomorphism Z is then called the central charge of σ.

For a stability condition $\sigma = (Z, \mathcal{P})$ we will write $\sigma^{ss} = \mathcal{P}^{ss}$ and every $A \in \sigma^{ss}$ will be called σ-semistable. Additionally, we denote $\phi_\sigma(A) = \phi_{\mathcal{P}}(A)$ and $\phi^\pm_\sigma(E) = \phi^\pm_{\mathcal{P}}(E)$ for any non-zero object E. If E is a non-zero object with its HN filtration given by (20) with respect to the slicing $\mathcal{P}$, then we define

$$m_\sigma(E) = \sum_{i=1}^{n} m_\sigma(A_i)\,. \tag{23}$$

One is usually interested in a special case of stability conditions called locally finite. For that, we need to introduce quasi-abelian categories.

Definition 19. *Let $\mathcal{A}$ be an additive category with kernels and cokernels (thus also with pullbacks and pushouts). For any morphism $f : A \to B$, consider the canonical factorization*

$$\ker(f) \longrightarrow A \longrightarrow coim(f) \longrightarrow im(f) \longrightarrow B \longrightarrow coker(f)\,,$$

then f is said to be strict when $coim(f) \to im(f)$ is an isomorphism. We call $\mathcal{A}$ quasi-abelian when every pullback of a strict epimorphism is a strict epimorphism and every pushout of a strict monomorphism is a strict monomorphism.

For a quasi-abelian category $\mathcal{A}$, one defines strict subobjects of an object B to be given by strict monomorphisms $A \to B$ and strict quotients by strict epimorphisms $B \to C$. One says that $\mathcal{A}$ is noetherian (resp. artinian) when every ascending (resp. descending) sequence of strict subobjects stabilizes. If $\mathcal{A}$ is both neotherian and artinian, then it is said to be of finite length. The following result explains why we are interested in this concept.

Lemma 20. *[5, Lemma 4.3] Let $\mathcal{P}$ be a slicing on a triangulated category $\mathcal{T}$ and I an interval of length less than one, then the category $\mathcal{P}(I)$ is quasi-abelian.*

Finally, we are able to specify the stability conditions that we are going to be interested in.

Definition 21. *A stability condition $\sigma = (Z, \mathcal{P})$ on a triangulated category $\mathcal{T}$ is locally finite when there exists such $1/2 > \varepsilon > 0$, such that for all $\phi \in \mathbb{R}$ the quasi-abelian categories $\mathcal{P}\big((\phi - \varepsilon, \phi + \varepsilon)\big)$ are of finite length. The set of locally finite stability conditions on $\mathcal{T}$ will be denoted by $Stab(\mathcal{T})$.*

4.2 Stability conditions on $D^b(A_1)$

Using just these definitions and simple results thus far, let us address an example of the simplest derived category and its stability conditions. For a quiver Q its category of representations $\mathrm{Rep}_k(Q)$ over the field k is an abelian category, and as such we may consider its derived category which we will label $D^b(Q) = D^b\big(\mathrm{Rep}_k(Q)\big)$. The quiver A_1 is the simplest quiver with a single vertex and no arrows. We claim that we can replace the set $\mathrm{Stab}\big(D^b(A_1)\big)$ by a complex plain under a bijection. But before we show this, let us remind the reader of the concept of exact functors between triangulated categories.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are triangulated categories, an exact functor from $\mathcal{T}_1$ to $\mathcal{T}_2$ a pair consisting of an additive functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ and an isomorphism of functors $\mu : F \circ T_1 \xrightarrow{\sim} T_2 \circ F$ where T_i is the shift functor on $\mathcal{T}_i$, and for any distinguished triangle

$$A \longrightarrow B \longrightarrow C \xrightarrow{\;f\;} A[1]$$

in $\mathcal{T}_1$, the triangle

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \xrightarrow{\mu(A)\circ F(f)} F(A)[1]$$

is distinguished in $\mathcal{T}_2$. Moreover, an equivalence is an exact functor (F, μ) whose underlying functor F is an equivalence of categories. If $G : \mathcal{T}_2 \to \mathcal{T}_1$ is the inverse of F then it is a well known fact that there exists a ν such that (G, ν) is an exact equivalence (see [10, Section 3.2] for more details). The following proposition is well known . For the sake of completeness, we give details of the proof here:

Proposition 22. *Let $F : \mathcal{T}_1 \to \mathcal{T}_2$ be an exact equivalence and G its inverse, then it induces a bijection between the sets of locally finite stability conditions $Stab(\mathcal{T}_1)$ and $Stab(\mathcal{T}_2)$.*

Proof. First, notice that F induces an isomorphism of groups $[F] : K_0(\mathcal{T}_1) \to K_0(\mathcal{T}_2)$ by $[F]([A]) = [F(A)]$ with the inverse $[G]$ defined in the same way. This follows directly from the definition of the exact equivalence and the definition of the Grothendieck group. The bijection is then defined by $\sigma = (Z, \mathcal{P}) \mapsto \sigma' = (Z', \mathcal{P}')$, where $Z' = Z \circ [G]$ (or equivalently $Z' \circ [F] = Z$), and $\mathcal{P}'(\phi) = \overline{F(\mathcal{P}(\phi))}$, where the over-line denotes the closure under isomorphisms.

We want to show that $(Z', \mathcal{P}')$ is indeed a locally finite stability condition. Let us start with the axioms of the Definition 12.

1. $\mathcal{P}'(\phi)[1] = \overline{F(\mathcal{P}(\phi))}[1] = \overline{F(\mathcal{P}(\phi)[1])} = \mathcal{P}'(\phi+1)$, where we have used for the second to last equality that $F \circ T_1 \cong T_2 \circ F$.

2. For any $X_i' \in Ob(\mathcal{P}'(\phi_i))$ there exist $X_i \in Ob(\mathcal{P}(\phi_i))$, such that $X_i' \cong F(X_i)$. From F being an equivalence, we get $\mathrm{Hom}(X_1', X_2') = \mathrm{Hom}(X_1, X_2)$ and thus it vanishes if $\phi_1 > \phi_2$.

3. For any $E' \in Ob(\mathcal{T}_2)$ there exists an $E \in Ob(\mathcal{T}_1)$, such that $E' \cong F(E)$. Let E have a filtration with respect to $\mathcal{P}$ given by (7), then by exactness of F, E' has a filtration with the factors $F(A_i)$.

Now, Z' is obviously a group homomorphism, and for any A' in $\mathcal{P}'(\phi)$ take an A in $\mathcal{P}(\phi)$, such that $A' \cong F(A)$. This gives us that $Z'(A') = Z'(F(A)) = Z(G \circ F(A)) = Z(A)$. We conclude that σ' is a stability condition. To complete the proof, we only need to show that if $\mathcal{P}((\phi-\varepsilon, \phi+\varepsilon))$ are the quasi-abelian finite length categories for some $\varepsilon > 0$ and all ϕ, then the isomorphism closure of their image is again a quasi-abelian finite length category corresponding to $\mathcal{P}'((\phi-\varepsilon, \phi+\varepsilon))$. It is obvious that the isomorphism closure of the image coincides with this subcategory. Next, every distinguished triangle which lies completely in $\mathcal{P}'((\phi-\varepsilon, \phi+\varepsilon))$ corresponds to one in $\mathcal{P}((\phi-\varepsilon, \phi+\varepsilon))$, and so every strict subobject corresponds to a strict subobject in $\mathcal{P}((\phi-\varepsilon, \phi+\varepsilon))$. Thus both the ascending and descending sequence of subobjects must stabilize as it does in $\mathcal{P}((\phi-\varepsilon, \phi+\varepsilon))$.

We have shown that $\sigma' \in \mathrm{Stab}(\mathcal{T}_2)$, but acting with G on it in the same way, we will get back σ. This proves that the above described map is a bijection. □

Lemma 23. *For the triangulated category $D^b(A_1)$ one has a bijection*

$$Stab(D^b(A_1)) \longleftrightarrow \mathbb{C}\,. \tag{24}$$

Let k be the object in $D^b(A_1)$ which is the simple representation of A_1 connecting to the vertex the field k itself, then k is semistable for any stability condition σ. A bijection from the left side to the right side is then given by

$$\sigma = (Z, \mathcal{P}) \mapsto \log\left(|Z(k)|\right) + i\pi\phi_\sigma(k)\,. \tag{25}$$

Proof. We notice that $D^b(A_1) = D^b(Vect_k)$ is a derived category of a semisimple abelian category $Vect_k$ of k vector spaces. In the view of Lemma 11 one concludes that $\mathrm{Com}_0^b(Vect_k)$ is an equivalent full subcategory of $D^b(A_1)$. It inherits the translation functor and the distinguished triangles from the derived category and with this structure becomes an equivalent triangulated category to $D^b(A_1)$. We may replace $D^b(A_1)$ by $\mathrm{Com}_0^b(Vect_k)$ when trying to find the stability conditions on $D^b(A_1)$, since the set of stability conditions does not change under equivalence of triangulated categories. From now on, we write $\mathcal{T} = \mathrm{Com}_0^b(Vect_k)$.

Using that every direct sum gives a distinguished triangle

$$A \to A \oplus B \to B \to A[1]\,,$$

we see that the classes represented by these complexes in the Grothendieck group of $\mathcal{T}$ solve $[A \oplus B] = [A] + [B]$. Any complex in $\mathcal{T}$ can be written as a direct sum of $k[i]$. Especially, we get that

$$[K^\bullet] = \sum_i (-1)^i \dim(K^i)[k].$$

Thus $[k]$ is the generator of $K_0(\mathcal{T})$ and any $Z : K_0(\mathcal{T}) \to \mathbb{C}$ is uniquely given by $Z(k)$.

Now we want to show that only complexes which have zero terms everywhere but at i'th position for some $i \in \mathbb{Z}$ (or i-complexes) can form the subcategories $\mathcal{P}(\phi)$ of any slicing $\mathcal{P}$ on $\mathcal{T}$. Assume that there is a $\phi \in \mathbb{R}$, such that $\mathcal{P}(\phi)$ contains a complex $K^\bullet$ with $K^a \neq 0$ and $K^{a+n} \neq 0$ for some $n > 0$ and $a \in \mathbb{Z}$. By the definition of a slicing $K^\bullet[n]$ lies in $\mathcal{P}(\phi + n)$, but also $\mathrm{Hom}(K^\bullet[n], K^\bullet) \neq 0$, which is a contradiction. Additionally, using the same arguments one can show that

(i) only the i-complexes for a given $i \in \mathbb{Z}$ can be in $\mathcal{P}(\phi)$ for a given $\phi \in \mathbb{R}$

(ii) if $\mathcal{P}(\phi) \neq \{0\}$ then $\mathcal{P}(\phi') \neq \{0\}$ if and only if $(\phi' - \phi) \in \mathbb{Z}$.

A final step in finding all possible slicings is to show that the complex k is semi-stable with respect to any slicing $\mathcal{P}$. To do so, assume the opposite. Then k has a Harder-Narasimhan filtration of length larger than or equal to 2. Any 0-complex is a direct sum

$$K^\bullet = \bigoplus_{i=1}^{\dim K^0} k \,.$$

The direct sum of two distinguished triangles is again a distinguished triangle. That is, if $A_1 \to B_1 \to C_1 \to A_1[1]$ and $A_2 \to B_2 \to C_2 \to A_2[1]$ are distinguished then so is

$$A_1 \oplus A_2 \to B_1 \oplus B_2 \to C_1 \oplus C_2 \to A_1[1] \oplus A_2[1] \,.$$

Thus we can take the corresponding direct sum of the HN sequence of k to be the HN filtration of length larger than or equal to two of the 0-complex $K^\bullet$. As a result, any 0-complex is not semi-stable, and especially any i-complex is not semi-stable. This tells us then that $\mathcal{P}(\phi) = \{0\}$ for all $\phi \in \mathbb{R}$, but this would not yield a HN sequence for any non-zero complex.

Combining the previous results, we can say that any stability condition $\sigma = (Z, \mathcal{P})$ on D is given uniquely by $|Z(k)| \neq 0$ and $\phi_{\mathcal{P}}(k)$, since then $Z(k) = |Z(k)| e^{i\pi\phi_{\mathcal{P}}(k)}$ and $\mathcal{P}$ has the form:

$$\mathcal{P}(\psi) = \begin{cases} \{i\text{-complexes}\} & \text{if } \psi = \phi_{\mathcal{P}}(k) + i \\ \{0\} & \text{otherwise} \end{cases} .$$

These are indeed stability conditions by construction for any given $|Z(k)| \in \mathbb{R}^{>0}$ and $\phi_{\mathcal{P}}(k) \in \mathbb{R}$. Take now the subcategories $\mathcal{P}(\phi - 1/2, \phi + 1/2)$ for all ϕ. They are either trivial or correspond to a single $\mathcal{P}(\phi')$ for some $\phi' \in (\phi - 1/2, \phi + 1/2)$. But then it is either $\{0\}$ or $Vect_k$ and as such is finite length. Thus the map $(Z, \mathcal{P}) \mapsto \log\big(|Z(k)|\big) + i\pi\phi_{\mathcal{P}}(k)$ is indeed a bijection from $\mathrm{Stab}(\mathcal{T})$ to $\mathbb{C}$. $\square$

Remark 24. *One should note that the equivalence of $D^b(A_1)$ and $Com_0^b(Vect_k)$ is not necessary for the proof and the same goes for the statement that exact equivalences give same sets of stability conditions. It would've been enough to work out that every object of $D^b(A_1)$ is isomorphic to a direct sum of $k[i]$.*

4.3 Stability conditions in terms of hearts of bounded t-structures

In this subsection, we discuss a method for finding stability conditions which relies on working with hearts of bounded t-structures. For this, we first need to introduce stability functions on abelian categories and their Harder-Narasimhan property.

The Grothendieck group $K_0(\mathcal{A})$ of an abelian category $\mathcal{A}$ is defined by replacing the distinguished triangles in Definition 16 by short exact sequences. Let us also denote by $\mathbb{H}$ the upper half-plane of $\mathbb{C}$: $\mathbb{H} = \{re^{i\pi\phi} \,|\, r > 0, 0 < \phi \leq 1\}$. If $z \in \mathbb{H}$, then $\arg_{(0,1]}(z)$ is the unique real number $\phi \in (0, 1]$ such that $z = |z|e^{i\pi\phi}$.

Definition 25. *A stability function on an abelian category $\mathcal{A}$ is a group homomorphism $Z : K_0(\mathcal{A}) \to \mathbb{C}$, such that for any non-zero object A of $\mathcal{A}$ its image $Z(A)$ lies in $\mathbb{H}$. For a given stability function Z on $\mathcal{A}$ and a non-zero $A \in Ob(\mathcal{A})$, we will write $\phi_Z(A) := arg_{(0,1]}\big(Z(A)\big)$. Such an object A is called semistable with respect to Z when for any non-zero subobject $A' \to A$ we have the inequality $\phi_Z(A') \leq \phi_Z(A)$.*

Notice that the definition of a semistable object is equivalent to saying that for any non-zero quotient $A \to A''$ one has $\phi_Z(A'') \geq \phi_Z(A)$. This implies that if A and B are semistable, then there exists a non-zero morphism $A \to B$ if and only if $\phi_Z(B) \geq \phi_Z(A)$, which follows from writing out the factorization of the morphism in the abelian category. A Harder-Narasimhan filtration of a nonzero object E is a finite sequence of subobjects

$$0 = E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_{n-1} \subset E\,, \tag{26}$$

such that every factor $A_j = E_j/E_{j-1}$ is semistable and $\phi_Z(A_1) > \phi_Z(A_2) > \ldots > \phi_Z(A_n)$. This filtration is again unique up to isomorphisms, if it exists.

Definition 26. *A stability function Z on an abelian category $\mathcal{A}$ is said to have the Harder-Narasimhan property, when for every non-zero object, there exists a Harder-Narasimhan filtration.*

The following condition describes cases for which a stability function has the Harder-Narasimhan property.

Proposition 27. *[5, Proposition 2.4] Let $\mathcal{A}$ be an abelian category with a stability function Z, then Z has the Harder-Narasimhan property if the following two conditions hold:*

1. *There is no infinite sequence of subobjects*

$$\ldots \subset E_{i+1} \subset E_i \subset \ldots \subset E_1 \subset E_0$$

   with $\phi_Z(E_{i+1}) > \phi_Z(E_i)$.

2. *There is no infinite sequence of quotients*

$$F_0 \twoheadrightarrow F_1 \twoheadrightarrow \ldots \twoheadrightarrow F_j \twoheadrightarrow F_{j+1} \twoheadrightarrow \ldots$$

   with $\phi_Z(F_{j+1}) < \phi_Z(F_j)$.

Remark 28. *Notice that this particularly implies that for a finite length abelian category every stability function has the HN-property.*

An important result proven in [5, Proposition 5.3] states that if $\mathcal{T}$ is a triangulated category, then there is the following bijection:

$$\left\{ \begin{array}{c} \text{Stability} \\ \text{conditions on } \mathcal{T} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Pairs consisting of} \\ \text{a heart of a bounded t-structure and} \\ \text{a stability function on it with the HN property} \end{array} \right\} \tag{27}$$

The central charges of stability conditions and the stability functions are related in this bijection by using that $K_0(\mathcal{A})$ can be identified with $K_0(\mathcal{T})$ when $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{T}$ (this can be seen from Lemma 8). The bijection assigns to every stability condition $\sigma = (Z, \mathcal{P})$ its heart $\mathcal{P}\big((0,1]\big)$ and the corresponding stability function with the HN property on it. In the reversed direction, it assigns to every heart of a bounded t-structure $\mathcal{A}$ a slicing $\mathcal{P}$, such that $\mathcal{P}(\phi)$ is the full additive subcategory of semistable objects in $\mathcal{A}$ with respect to its stability function with the argument ϕ for all $\phi \in (0,1]$.

4.4 Spaces of stability conditions

Here we want to summarize the results from [5] stating that $\mathrm{Stab}(\mathcal{T})$ can be endowed with a topology and a complex structure. We show that then $\mathrm{Stab}\big(D^b(A_1)\big)$ is biholomorphic to $\mathbb{C}$.

Firsly, one needs to define for any $(Z, \mathcal{P}) = \sigma \in \mathrm{Stab}(\mathcal{T})$ the generalized norm

$$||\cdot||_\sigma : \mathrm{Hom}\big(K_0(\mathcal{T}), \mathbb{C}\big) \to [0, \infty]\,,$$

such that for any $U \in \mathrm{Hom}\big(K_0(\mathcal{T}, \mathbb{C})\big)$

$$||U||_\sigma = \sup\left\{\frac{|U(E)|}{|Z(E)|} : E \in \sigma^{ss}\right\}. \tag{28}$$

Additionally, one defines a generalized metric d on the set of locally finite slicings. Let $\mathcal{P}$ and $\mathcal{Q}$ be such two slicings of $\mathcal{T}$, then:

$$d(\mathcal{P}, \mathcal{Q}) = \sup\left\{|\phi^-_{\mathcal{P}}(E) - \phi^-_{\mathcal{Q}}(E)|, |\phi^+_{\mathcal{P}}(E) - \phi^+_{\mathcal{Q}}(E)| : E \text{ nonzero object in } \mathcal{T}\right\}. \tag{29}$$

The sets

$$B_\varepsilon(\sigma) = \left\{(W, \mathcal{Q}) = \tau \in \mathrm{Stab}(\mathcal{T}) : ||W - Z||_\sigma < \sin(\pi\varepsilon) \text{ and } d(\mathcal{P}, \mathcal{Q}) < \varepsilon\right\} \tag{30}$$

when taken for all $\sigma \in \mathrm{Stab}(\mathcal{T})$ and $1/8 > \varepsilon > 0$, form a basis of a topology on $\mathrm{Stab}(\mathcal{T})$ (see [5, p. 335]. If Σ is a connected component of $\mathrm{Stab}(\mathcal{T})$ with the above topology, then the generalized norms $||\cdot||_\sigma$ are related for all $\sigma \in \Sigma$, and so the subspace V_Σ of all the $U \in \mathrm{Hom}(K_0(\mathcal{T}, \mathbb{C})$ with a finite norm $||U||_\sigma$ for some $\sigma \in \Sigma$ is uniquely given for Σ. The main result now states:

Theorem 29. *[5, Theorem 1.2] Let $\mathcal{T}$ be a triangulated category, Σ a connected component of $Stab(\mathcal{T})$ with its associated complex subspace $V_\Sigma \subset Hom\big(K_0(\mathcal{T}), \mathbb{Z}\big)$, then the projection of a stability condition onto its central charge $\Sigma \to V_\Sigma, (Z, \mathcal{P}) \mapsto Z$ is a local homeomorphism and thus induces a unique complex structure on Σ .*

Now that we see that the connected components of the space of stability conditions are complex manifolds, we want to apply this to the example $D^b(A_1)$. For this, we will rely on the following result.

Proposition 30. *[5, Proposition 8.1] Let $\mathcal{T}$ be a triangulated category. The topology on the space of stability conditions $Stab(\mathcal{T})$ is induced by the generalized metric $q_{\mathcal{T}}$, where for all $\sigma = (Z, \mathcal{P}), \tau = (W, \mathcal{Q}) \in Stab(\mathcal{T})$ one defines*

$$q_{\mathcal{T}}(\sigma, \tau) = \sup\left\{|\phi^+_\sigma(E) - \phi^+_\tau(E)|, |\phi^-_\sigma(E) - \phi^-_\tau(E)|, \left|\log \frac{m_\sigma(E)}{m_\tau(E)}\right| : 0 \neq E \in Ob(\mathcal{T})\right\}. \tag{31}$$

Remark 31. *Proposition 22 can be extended to state that any exact equivalence $F : \mathcal{T}_1 \to \mathcal{T}_2$ with its inverse G induces a biholomorphism between $Stab(\mathcal{T}_1)$ and $Stab(\mathcal{T}_2)$. Using the bijection $\sigma \mapsto \sigma'$ constructed in its proof, one sees immediately that $q_{\mathcal{T}_2}(\sigma', \tau') = q_{\mathcal{T}_1}(\sigma, \tau)$. Thus this map is a homeomorphism. Restrict it to be a homeomorphism between the connected component Σ and the connected component Σ' which is its image. In the local charts induced by the local biholomorphism from Theorem 29 this map takes the form $Z \mapsto Z \circ [G]$. So, it is simply a restriction of an isomorphism of vector spaces and thus a biholomorphism to its image.*

Using this we obtain the main result of this subsection.

Theorem 32. *The space of stability conditions $Stab\big(D^b(A_1)\big)$ is connected and biholomorphic to $\mathbb{C}$.*

Proof. Let us first find the topology on $\mathbb{C}$ induced by the bijection (25). If σ and τ are stability conditions mapped to $x_1 + iy_1$ and $x_2 + iy_2$ in $\mathbb{C}$ under the map, the metric distance between them is given by

$$q_{\mathcal{T}}(\sigma, \tau) = \sup_{0 \neq K^\bullet \in D}\left\{|y_1 - y_2|, \left|\log \frac{m_\sigma(K^\bullet)}{m_\tau(K^\bullet)}\right|\right\}.$$

But we can simplify the second term using that

$$m_\sigma(K^\bullet) = \sum_i \dim K^i |Z(k)| = e^{x_1} \sum_i \dim K^i .$$

Thus, we get the following induced metric on $\mathbb{C}$:

$$q(x_1 + iy_1, x_2 + iy_2) = q_{\mathcal{T}}(\sigma, \tau) = \max\left\{ |y_1 - y_2|, |x_1 - x_2| \right\}.$$

This metric is equivalent to the standard one and gives the standard topology on $\mathbb{C}$, therefore (25) is a homeomorphism. We now have the commutative diagram

$$\begin{tikzcd} \mathrm{Stab}(\mathcal{T}) \arrow[rr, "\xi"] \arrow[dr, "p"'] & & (\mathbb{C}, g) \arrow[dl, "\exp"] \\ & \mathbb{C} & \end{tikzcd}$$

where ξ is the map (25), $(\mathbb{C}, g)$ is the complex plane with standard topology and complex structure g induced by ξ, and p is the composition of $(Z, \mathcal{P}) \mapsto Z$ and $Z \mapsto Z(k)$. The arrow labeled exp is the standard exponential map on $\mathbb{C}$ which we see is a local biholomorphism with respect to g by the commutativity. There is a unique complex structure on $\mathbb{C}$ with the standard topology that makes exp into a local biholomorphism which is the standard complex structure. These results tell us that

$$\mathrm{Stab}\big(D^b(A_1)\big) \cong \mathrm{Stab}(\mathcal{T}) \cong \mathbb{C}.$$

□

5 Exceptional objects and mutations

5.1 Exceptional objects in derived categories of quiver representations

Let $\mathcal{T}$ be a triangulated category and X, Y its two objects, then we will write $\mathrm{Hom}^i(X, Y) = \mathrm{Hom}(X, Y[i])$ and $\hom^i(X, Y) = \dim_k \left(\mathrm{Hom}^i(X, Y)\right)$. An object E is said to be exceptional when $\hom^i(E, E) = 0$ except when $i = 0$, and $\hom^0(E, E) = 1$. An exceptional collection of length $n + 1$ is a collection of exceptional objects $\mathcal{E} = (E_0, E_1, \ldots, E_n)$, such that $\hom^l(E_i, E_j) = 0$ whenever $i > j$ and $l \in \mathbb{Z}$. Additionally, one calls it full when it generates $\mathcal{T}$ as a triangulated subcategory, and strong when $\hom^l(E_i, E_j) = 0$ for all $0 \le i, j \le n$ and $l \neq 0$.

One defines an equivalence relation on exceptional objects by $E \sim F$ if and only if there exists an $i \in \mathbb{Z}$, such that $E \cong F[i]$. Similarly, two exceptional collections $\mathcal{E} = (E_0, \ldots, E_n)$ and $\mathcal{F} = (F_0, \ldots, F_n)$ of length $n+1$ are equivalent and we write $\mathcal{E} \sim \mathcal{F}$ if and only if $E_i \sim F_i$ for all $0 \le i \le n$. An exceptional collection of length 2 is called an exceptional pair.

As we will be working with derived categories of representations on quivers, we will have the following lemma at our disposal.

Lemma 33 ([6])**.** *Let Q be an acyclic quiver with q vertices and $D^b(Q)$ the derived category of its representations, then every exceptional object lies in $\mathrm{Rep}_k(Q)[i]$ for some $i \in \mathbb{Z}$. Moreover, every full exceptional collection has length q.*

For further discussion we need the Euler form of a finite quiver Q. Let Q_1 be its set of vertices which for us is going to be $\{1, 2, \ldots, q\}$ for some $q \in \mathbb{N}$ and Q_2 its set of arrows. If $\alpha \in Q_2$, then $s(\alpha)$ denotes the vertex where it starts and $f(\alpha)$ the one where it ends. One now has a bilinear map $\langle -, - \rangle : \mathbb{Z}^q \times \mathbb{Z}^q \to \mathbb{Z}$ called the Euler form defined by

$$\langle x, y \rangle = \sum_{i \in Q_1} x_i y_i - \sum_{\alpha \in Q_2} x_{s(\alpha)} y_{f(\alpha)}. \tag{32}$$

Additionally, for every representation X in $\mathrm{Rep}_k(Q)$, we have the dimension vector

$$\underline{\dim}(X) = \big(\dim(X_1), \dim(X_2), \ldots, \dim(X_q)\big). \tag{33}$$

The following result will become important in the last section.

Lemma 34. *[7, p. 8] Let Q be a finite acyclic quiver, then* $\mathrm{Rep}_k(Q)$ *is a hereditary category, that is, for any two of its objects X and Y, $hom^i(X,Y)$ vanishes whenever $i \neq 0,1$. If X and Y are representations, then*

$$\langle \underline{\dim}(X), \underline{\dim}(Y)\rangle = \hom(X,Y) - \hom^1(X,Y)\,. \tag{34}$$

Thus especially for any exceptional representation E, we have $\langle \underline{\dim}(E), \underline{\dim}(E)\rangle = 1$.

Solutions of the equation $\langle \alpha, \alpha\rangle = 1$ are called real roots. When $\langle \alpha, \alpha\rangle \leq 0$, we say that the α is an imaginary root.

Lemma 35. *[14] Let Q be a finite acyclic quiver, then the dimension vector of any indecomposable representation of Q is a real or imaginary root of its Euler form.*

Lemma 36. *[7, p. 13] Let Q be a finite quiver with q vertices and no cycles. There exists at most one representation X (up to isomorphisms) with a given dimension vector $x \in \mathbb{Z}^q$, such that $hom^1(X,X) = 0$.*

5.2 Mutations of exceptional pairs

Left and right mutations of exceptional pairs allow us to construct new exceptional pairs and give an action on the equivalence classes of exceptional pairs. From now on, we will assume that the triangulated categories we are working with are proper. That is, the sum $\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^i(X,Y)$ is finite dimensional over k for any two objects $X, Y \in \mathcal{T}$.

Let (E,F) be an exceptional pair of objects in $\mathcal{T}$, then one constructs

$$\mathrm{Hom}^\bullet(E,F) \otimes_k E = \bigoplus_{i\in\mathbb{Z}} E[-i]^{\hom^i(E,F)}\,. \tag{35}$$

The space of morphisms from this object into F can be expressed as:

$$\begin{aligned}\mathrm{Hom}\big(\mathrm{Hom}^\bullet(E,F)\otimes_k E, F\big) &\cong \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(E[-i],F)^{\hom^i(E,F)}\\ &\cong \bigoplus_{i\in\mathbb{Z}} \mathrm{End}_k\Big(\mathrm{Hom}(E,F[i])\Big)\,.\end{aligned} \tag{36}$$

We can now choose the respective identities in $\mathrm{End}_k\Big(\mathrm{Hom}(E,F[i])\Big)$ for all i which determines an element in $\mathrm{Hom}(\mathrm{Hom}^\bullet(E,F)\otimes_k E, F)$ up to an isomorphism. This morphism will be called the canonical morphism can-$_{E,F}$. There exists a distinguished triangle induced by $\mathrm{can}_{E,F}$:

$$L_E F \longrightarrow \mathrm{Hom}^\bullet(E,F)\otimes_k E \xrightarrow{\mathrm{can}_{E,F}} F \longrightarrow L_E F[1]\,. \tag{37}$$

The resulting object $L_E F$ is called the left mutation of the exceptional pair (E,F). The pair $(L_E F, E)$ can be shown to be exceptional again.

Lemma 37. *Let (E,F) be an exceptional pair of objects and $L_E F$ its left mutation then $(L_E F, E)$ is also an exceptional pair. Additionally, if (E,F) is full (resp. strong) then so is $(L_E F, E)$.*

Proof. The second part of the statement follows from the defining equation (37) which shows that F is an extension of $\mathrm{Hom}^\bullet(E,F)\otimes_k E$ and $L_E F[1]$.

Next, consider the functor $\mathrm{Hom}^l(E,-)$ acting on $\mathrm{can}_{E,F}$. Notice that when we defined $\mathrm{can}_{E,F}$ we have used the isomorphism $\mathrm{End}_k(V) \cong V^{\dim(V)}$ which is given with respect to a basis $\{v_i\}$ of V, and to any $A \in \mathrm{End}_k(V)$ it connects $(Av_1, Av_2, \ldots)$. So, for the identity on $\mathrm{Hom}(E,F[i])$ we get $(\phi_1, \phi_2, \ldots)$ where ϕ_j form a basis of $\mathrm{Hom}(E,F[i])$. The map

$$\mathrm{Hom}^l(E, \mathrm{can}_{E,F}) : \mathrm{Hom}^l(E, \mathrm{Hom}^\bullet(E,F)\otimes_k E) \to \mathrm{Hom}^l(E,F)$$

can now be shown to be an isomorphism. First, we notice that

$$\begin{aligned}\mathrm{Hom}^l(E, \mathrm{Hom}^\bullet(E,F) \otimes_k E) &\cong \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^{l-i}(E,E)^{\hom^i(E,F)} \\ &\cong \mathrm{Hom}(E,E)^{\hom^l(E,F)}\,.\end{aligned}$$

Any element of this space of morphisms is then expressed as a vector of scalar multiples of the identity on E: $(\lambda_1 \mathrm{id}_E, \lambda_2 \mathrm{id}_E, \ldots)$. Composing this with $\mathrm{can}_{E,F}[l]$, the result becomes the sum $\sum_{i=1}^{\hom^l(E,F)} \lambda_i \phi_i$ where $\{\phi_i\}$ is the basis of $\mathrm{Hom}^l(E,F)$. So, the linear homomorphism induced by the composition is an isomorphism. By the exactness of the sequence

$$\begin{aligned}0 \to \mathrm{Hom}(E, L_E F) \to \mathrm{Hom}(E, \mathrm{Hom}^\bullet(E,F) \otimes_k E) \xrightarrow{\sim} \mathrm{Hom}(E,F) \to \ldots \\ \ldots \to \mathrm{Hom}^l(E, L_E F) \to \mathrm{Hom}^l(E, \mathrm{Hom}^\bullet(E,F) \otimes_k E) \xrightarrow{\sim} \mathrm{Hom}^l(E,F) \to \ldots\end{aligned}$$

we see that $\mathrm{Hom}^l(E, L_E F) = 0$ for all $l \in \mathbb{Z}$. We can use this result when applying the functor $\mathrm{Hom}(-, L_E F)$ to get an exact sequence

$$0 \to \mathrm{Hom}^l(L_E F, L_E F) \to \mathrm{Hom}^{l+1}(F, L_E F) \to 0$$

for all l. Next, one uses $\mathrm{Hom}^l(F,E) = 0$ and the functor $\mathrm{Hom}(F,-)$ to obtain for all l the exact sequence

$$0 \to \mathrm{Hom}^l(F,F) \to \mathrm{Hom}^{l+1}(F, L_E F) \to 0\,.$$

Together these two sequences show that $L_E F$ is an exceptional object, because $\mathrm{Hom}^l(L_E F, L_E F) \cong \mathrm{Hom}^l(F,F)$ for all l.

Finally if (E,F) is strong, then we apply the functor $\mathrm{Hom}(-,E)$ to the triangle (37), giving us that $\mathrm{Hom}^l(L_E F, E) \cong \mathrm{Hom}(E,E)^{\hom^{-l}(E,F)}$ which concludes the proof. □

Similarly, one defines the right mutation $R_F E$ using the object

$$\mathrm{Hom}^\bullet(E,F)^* \otimes_k F = \bigoplus_{i\in\mathbb{Z}} F[i]^{\hom^i(E,F)}$$

and the distinguished triangle

$$E \xrightarrow{\mathrm{can}^*_{E,F}} \mathrm{Hom}^\bullet(E,F)^* \otimes_k F \longrightarrow R_F E \longrightarrow E[1]\,, \tag{38}$$

where one uses that $\mathrm{Hom}(E, \mathrm{Hom}^\bullet(E,F)^* \otimes_k F)$ is isomorphic to the right hand side of (36), and $\mathrm{can}^*_{E,F}$ is again represented by the identities in $\mathrm{End}_k\big(\mathrm{Hom}^i(E,F)\big)$. Slightly altering the proof of the Lemma 37, shows that $(F, R_E F)$ is again an exceptional pair (full or strong if (E,F) is). Mutations can be also generalized to exceptional collections of length $n+1$. Let $\mathcal{E} = (E_0, \ldots, E_n)$ be an exceptional collection, then we will denote

$$\begin{aligned}L_i\mathcal{E} &= (E_0, \ldots, L_{E_i} E_{i+1}, E_i, \ldots, E_n)\,, \\ R_i\mathcal{E} &= (E_0, \ldots, E_{i+1}, R_{E_{i+1}} E_i, \ldots, E_n)\,.\end{aligned}$$

These are again exceptional collections of the same length. From [8, Corollary 1.6] one knows that R_i and L_i are inverse to each other, and that $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$, and $L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1}$. Thus left and right mutations define an action of a braid group on exceptional collections of a given length. Moreover one has the statement:

Theorem 38. *[6] If Q is an acyclic quiver with n vertices, then any orbit of the action of the braid group on full exceptional collections of its derived category $D^b(Q)$ described above intersects every equivalence class of the full exceptional collections of length n.*

This result allows one to find all exceptional collections of the maximal length and all exceptional objects in derived categories of quiver-representations.

5.3 The helix in $D^b\big(K(l)\big)$

Firstly, let us give a general definition.

Definition 39. *Let $\mathcal{T}$ be a triangulated proper category with an exceptional pair (E_0, E_1) then the helix of $\mathcal{T}$ generated by (E_0, E_1) is a sequence of exceptional objects $(E_i)_{i\in\mathbb{Z}}$, defined iteratively by:*

$$E_{i+1} = R_{E_i} E_{i-1} \qquad \text{for } i \geq 1 \tag{39}$$

$$E_{i-1} = L_{E_i} E_{i+1} \qquad \text{for } i \leq 0\,. \tag{40}$$

Remark 40. *From Lemma 37, we know that in the helix $(E_i)_{i\in\mathbb{Z}}$ every object is exceptional and every two neighboring objects form an exceptional pair. Moreover if (E_0, E_1) is a strong or full exceptional pair, then so are (E_i, E_{i+1}) with $hom^0(E_i, E_{i+1}) = hom^0(E_0, E_1)$.*

If the objects of two exceptional pairs are related by a common shift, then the helices they generate behave in the same way:

Lemma 41. *If $F_0 \cong E_0[n]$ and $F_1 \cong E_1[n]$ for some $n \in \mathbb{Z}$, $(F_i)_{i\in\mathbb{Z}}$ the helix generated by (F_0, F_1) and $(E_i)_{i\in\mathbb{Z}}$ the helix generated by (E_0, E_1), then $F_i \cong E_i[n]$ for all $i \in \mathbb{Z}$.*

Proof. We will be showing here that this holds for $i \leq 1$ by induction, because the ramaining cases are treated similarly. Assume that for some i we have $F_i \cong E_i[n]$ and $F_{i+1} \cong E_{i+1}[n]$. From the definition of $\mathrm{can}_{E_i[n],E_{i+1}[n]}$ it follows that one can choose it such that $\mathrm{can}_{E_i[n],E_{i+1}[n]} = \mathrm{can}_{E_i,E_{i+1}}[n]$. This gives us the distinguished triangle

$$L_{F_i}(F_{i+1}) \longrightarrow \mathrm{Hom}^\bullet(E_i, E_{i+1}) \otimes_k E_i[n] \xrightarrow{\mathrm{can}_{E_i,E_{i+1}}[n]} E_{i+1}[n] \longrightarrow L_{F_i} F_{i+1}[1]\,.$$

Which tells us that $F_{i-1} = L_{F_i}(F_{i+1}) \cong L_{E_i}(E_{i+1})[n] = E_{i-1}[n]$. □

Lemma 42. *Let $F : \mathcal{T}_1 \to \mathcal{T}_2$ be a fully faithful exact functor and $(E_i)_{i\in\mathbb{Z}}$ a helix in $\mathcal{T}_1$ generated by the exceptional pair (E_0, E_1), then $\big(F(E_i)\big)_{i\in\mathbb{Z}}$ is a helix in $\mathcal{T}_2$ generated by $\big(F(E_0), F(E_1)\big)$.*

Proof. An exceptional object is mapped to an exceptional object under F, as it is fully faithful and exact. In fact, any exceptional collection is mapped to an exceptional collection. Let F_i be that $i'th$ object of the helix generated by $\big(F(E_0), F(E_1)\big)$, we will show by induction that for $i \leq 1$ we have $F(E_i) \cong F_i$.

Assume that for some $i \leq 0$ this holds for $F(E_i)$ and $F(E_{i+1})$, then we have the following distinguished triangle for F_{i-1}.

$$F_{i-1} \longrightarrow \mathrm{Hom}^\bullet\big(F(E_i), F(E_{i+1})\big) \otimes_k F(E_i) \xrightarrow{\mathrm{can}_{F(E_i),F(E_{i+1})}} F(E_{i+1}) \longrightarrow F_{i-1}[1]$$

But because F is fully faithful and additive, the second term from the left is the image of $\mathrm{Hom}(E_i, E_{i+1}) \otimes_k E_i$ under it. Moreover, one can choose $\mathrm{can}_{F(E_i),F(E_{i+1})}$ in such a way that it is given by $F\big(\mathrm{can}_{E_i,E_{i+1}}\big)$. We know then that there is a distinguished triangle with F_{i-1} replace by $F(E_{i-1})$ which completes the induction. For $i \geq 0$ one uses the same arguments. □

Let $K(l)$ denote the Kronecker quiver with l arrows: $1 \underset{\vdots}{\rightrightarrows} 2$. By Lemma 33, we know that full exceptional collections have length 2, and by [6, Lemma 1], one can complete any exceptional object to an exceptional pair in $D^b\big(K(l)\big)$. Combining this with Theorem 38, tells us that finding any exceptional pair will allow us to construct all exceptional pairs and objects up to equivalences by constructing the helix and then taking all the neighboring pairs.

One can take the pair (s_0, s_1), where $s_0[1]$ is the irreducible representation with the dimension vector $(1, 0)$ and s_1 with dimension vector $(0, 1)$. Using Lemma 34, one immediately infers that $\hom^j(s_1, s_0) = 0$ for all j, and $\hom^i(s_0, s_1) = 0$ whenever $i \neq 0$ and otherwise $\hom^0(s_0, s_1) = l$. Starting from this, we construct the helix $(s_i)_{i\in\mathbb{Z}}$ of $D^b\big(K(l)\big)$ generated by (s_0, s_1).

Any two neighboring objects form a full strong exceptional pair, such that $\hom^0(s_i, s_{i+1}) = l$. Additionally, every exceptional object in $D^b\big(K(l)\big)$ is equivalent to s_i for some $i \in \mathbb{Z}$, and every exceptional pair is equivalent to (s_j, s_{j+1}) for some $j \in \mathbb{Z}$. Some further properties can be shown:

Lemma 43. *[9, Lemma 7.5] Let $l \geq 2$ and $(s_i)_{i\in\mathbb{Z}}$ the helix of $D^b\big(K(l)\big)$ generated by (s_0, s_1), then the following statements hold:*

1. *No two objects of the helix are isomorphic.*
2. *For all $i \leq 0$, s_i lies in $Rep_k\big(K(l)\big)[-1]$.*
3. *For all $i \geq 1$, s_i lies in $Rep_k\big(K(l)\big)$.*
4. *If $i \leq j$, then $hom^l(s_i, s_j) = 0$ for all $l \neq 0$ and $hom^0(s_i, s_j) \neq 0$.*
5. *If $i > j + 1$, then $hom^l(s_i, s_j) = 0$ for all $l \neq 1$ and $hom^1(s_i, s_j) \neq 0$.*

This immediately gives the following corollary.

Corollary 44. *If everything is given as in the above lemma, then no two exceptional objects of the helix are equivalent with respect to $\sim$. Further, the only exceptional pairs one can construct from the objects in the helix are the ordered pairs (s_i, s_{i+1}).*

Proof. Let the objects s_i and s_j be equivalent for some $i, j \in \mathbb{Z}$, then $s_i \cong s_j[r]$ for some $r \in \mathbb{Z}$. We can assume that $i \leq j$ (otherwise $s_j \cong s_i[-r]$). We see now that $\hom^0(s_i, s_j[r]) \neq 0$ implies $i = j$ and $r = 0$ by the above lemma. $\square$

6 Left and right orthogonals, semiorthogonal decomposition

Let $\mathcal{E}$ be a class of objects in a triangulated category $\mathcal{T}$, its right orthogonal $\mathcal{E}^\perp$ and left orthogonal ${}^\perp\mathcal{E}$ are defined by:

$$\begin{aligned} \mathcal{E}^\perp &= \{X \in \mathcal{T} : \mathrm{Hom}^i(E, X) = 0 \,\forall E \in \mathcal{E},\, i \in \mathbb{Z}\}, \\ {}^\perp\mathcal{E} &= \{X \in \mathcal{T} : \mathrm{Hom}^i(X, E) = 0 \,\forall E \in \mathcal{E},\, i \in \mathbb{Z}\}. \end{aligned} \tag{41}$$

Both of these classes are easily seen to be closed under the shift functor and extensions so they form triangulated subcategories of $\mathcal{T}$.

Definition 45. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{T}_i$ for $i = 1, \dots, n$ be its triangulated subcategories, such that $\mathcal{T} = \langle \mathcal{T}_1, \dots, \mathcal{T}_2 \rangle$. One says that $\mathcal{T} = \langle \mathcal{T}_1, \dots, \mathcal{T}_n \rangle$ is a semi-orthogonal decomposition of $\mathcal{T}$ when for any $X_i \in Ob(\mathcal{T}_i)$ and $X_j \in Ob(\mathcal{T}_j)$ the space of morphisms $Hom^l(X_i, X_j)$ is trivial for all l whenever $i > j$.*

One can equivalently says that $\mathcal{T}_j$ lie in the right orthogonal $\mathcal{T}_i^\perp$ whenever $i < j$. Using the following proposition, one can state much more.

Proposition 46. *Let $\mathcal{T} = \langle \mathcal{T}_1, \dots, \mathcal{T}_n \rangle$ be a semiorthogonal decomposition of $\mathcal{T}$, then every object E in $\mathcal{T}$ has a unique up to isomorphisms diagram of distinguished triangles*

$$\begin{tikzcd} 0 \arrow[rr] & & E_n \arrow[dl] \arrow[rr] & & E_{n-1} \arrow[dl] \arrow[r] & \dots \arrow[r] & E_2 \arrow[rr] & & E \arrow[dl] \\ & A_n \arrow[ul, "{[1]}"] & & A_{n-1} \arrow[ul, "{[1]}"] & & & & A_1 \arrow[ul, "{[1]}"] & \end{tikzcd} \tag{42}$$

where A_i lie in $\mathcal{T}_i$.

Proof. Every object E can be written as a finite number of extensions by objects in the categories $\mathcal{T}_i$. It is enough to show that these extensions can be reordered correctly and merged into one step if necessary. So consider the diagram (3) where A is an object of $\mathcal{T}_i$ and B of $\mathcal{T}_{i+j}$. If $j > 0$ then by Lemma 2 we can interchange the order of A and B. If $j = 0$, one can merge them into one factor in $\mathcal{T}$ in a single distinguished triangle using the octahedral axiom.

The uniqueness of this diagram follows by the same arguments as used in the Remark 9. $\square$

Corollary 47. *Let $\mathcal{T} = \langle \mathcal{T}_1, \ldots, \mathcal{T}_n \rangle$ be a semiorthogonal decomposition of the triangulated category $\mathcal{T}$, then the following hold for any $1 \leq l < n$:*

$$\langle \mathcal{T}_1, \ldots, \mathcal{T}_l \rangle = \langle \mathcal{T}_{l+1}, \ldots, \mathcal{T}_n \rangle^{\perp},$$
$$\langle \mathcal{T}_{l+1}, \ldots, \mathcal{T}_n \rangle = {}^{\perp}\langle \mathcal{T}_1, \ldots, \mathcal{T}_l \rangle .$$

This also extends to full exceptional collections.

Corollary 48. *Let $(E_1, \ldots, E_n)$ be a full exceptional collection of $\mathcal{T}$, then the following hold for any $1 \leq l < n$:*

$$\langle E_1, \ldots, E_l \rangle = \langle E_{l+1}, \ldots E_n \rangle^{\perp},$$
$$\langle E_{l+1}, \ldots E_n \rangle = {}^{\perp}\langle E_1, \ldots, E_l \rangle .$$

Proof. Because it is a full exceptional collection, we have that $\mathcal{T} = \big\langle \langle E_1 \rangle, \ldots, \langle E_n \rangle \big\rangle$ is a semiorthogonal decomposition of $\mathcal{T}$. Additionally, one can replace E_i by $\langle E_i \rangle$ in the equation without changing the generated triangulated subcategories. Now we apply the previous corollary to conclude the result. □

In the last section we will need some additional results connected to the Grothendieck group.

Lemma 49. *Let $\mathcal{T} = \langle \mathcal{T}_1, \ldots, \mathcal{T}_n \rangle$ be a semiorthogonal decomposition of the triangulated category $\mathcal{T}$ with $\mathcal{T}_i$ being its full triangulated subcategories. The Grothendieck group of $\mathcal{T}$ is a direct sum*

$$K_0(\mathcal{T}) = \bigoplus_{i=1}^{n} K_0(\mathcal{T}_i) .$$

Sketch of Proof. One can define functors $\xi_i : \mathcal{T} \to \mathcal{T}_i$ for all $i = 1, \ldots, n$ in the following way: If E is an object with the unique form (42) then we define $\xi_i(E) = A_i$ and for any morphism $E \to E'$, we can construct unique morphisms $A_i \to A_i'$ as it was done in the Remark 9. One chooses ξ_i, such that $\xi_i(E[1]) = A_i[1]$. The exactness of this functor then follows easily. These functors induce homomorphisms $[\xi_i] : K_0(\mathcal{T}) \to K_0(\mathcal{T}_i)$.

Next we have the inclusions $\mathrm{inc}_i : \mathcal{T}_i \to \mathcal{T}$ which are obviously exact. We notice that $[\xi_i]$ and $[\mathrm{inc}_i]$ give together a biproduct diagram for $K_0(\mathcal{T})$ with factors $K_0(\mathcal{T}_i)$. □

Corollary 50. *If $\mathcal{E} = (E_0, \ldots, E_n)$ is a full exceptional collection in the triangulated category $\mathcal{T}$, then $K_0(\mathcal{T})$ is a free abelian group generated by $[E_i]$ and thus of degree $n+1$.*

Proof. This follows from $\mathcal{T} = \big\langle \langle E_0 \rangle, \ldots, \langle E_n \rangle \big\rangle$ being a semiorthogonal decomposition and the previous lemma. In $\langle E_i \rangle$ every object corresponds to a finite number of extensions by shifts of E_i, so its Grothendieck group is the free abelian group generated by $[E_i]$ (see Remark 69). □

7 Non-commutative curve counting

In the subsection 4.2, we have recalled the definition of exact functors between triangulated categories. Here we will remind the reader of some general facts about exact functors.

Let F_i be two exact functors $F_i : \mathcal{T}_1 \to \mathcal{T}_2$ with their isomorphisms $\mu_i : F_i \circ T_1 \xrightarrow{\sim} T_2 \circ F_i$, then a graded natural transformation $\alpha : F_1 \to F_2$ is a morphism of functors, such that the following diagram is commutative:

$$\begin{tikzcd}
F_1(A[1]) \arrow[r, "\alpha(A[1])"] \arrow[d, "\mu_1(A)"] & F_2(A[1]) \arrow[d, "\mu_2(A)"] \\
F_1(A)[1] \arrow[r, "\alpha(A)[1]"] & F_2(A)[1]
\end{tikzcd} \tag{43}$$

If there exists such a natural graded transformation between F_1 and F_2 which is an isomorphism between the underlying functors, we write $F_1 \simeq F_2$ which defines an equivalence relation on the exact functors from $\mathcal{T}_1$ to $\mathcal{T}_2$. Further, one can define composition $F \circ G$ of two exact functors F and G in such a way, such that the underlying functor of this composition is the composition of the underlying functors (see [10, Section 3.2]).

The equivalence relation $\simeq$ behaves nicely under the composition of exact functors: Let $F_i : \mathcal{T}_1 \to \mathcal{T}_2$ and $G_i : \mathcal{T}_2 \to \mathcal{T}_3$ be exact functors, such that $F_1 \simeq F_2$ and $G_1 \simeq G_2$, then $G_1 \circ F_1 \simeq G_2 \circ F_2$. Especially, for any two exact functors F and G which can be composed, we have the well defined composition of their equivalence classes $[G]_{\simeq} \circ [F]_{\simeq} := [G \circ F]_{\simeq}$ (for further details see [10, Section 3.2]).

The group of exact auto-equivalences on a triangulated category $\mathcal{T}$ denoted by $\mathrm{Aut}(\mathcal{T})$ corresponds to the set of classes of exact auto-equivalence functors on $\mathcal{T}$ with the group structure given by composition. We can now recall the concept of non-commutative curve counting introduced in [9]. This is a special case of the more general non-commutative counting in [9, Definition 12.7] and in [10, Section 4].

Recall that all categories and all functors are k-linear.

Definition 51. *From now on, we will denote $D^b\big(K(l+1)\big)$ by $N\mathbb{P}^l$. Let $\mathcal{T}$ be a triangulated category and $\Gamma \subset Aut(\mathcal{T})$ a subgroup of the group of exact auto-equivalences on $\mathcal{T}$. The set of fully faithful exact functors $N\mathbb{P}^l \to \mathcal{T}$ solving some property P is labeled by $C'_{l,P}(\mathcal{T})$ for all $l \geq 0$. Consider the following equivalence relation on $C'_{l,P}(\mathcal{T})$:*

$$F_1 \sim F_2 \iff F_1 \circ \alpha \simeq \beta \circ F_2 \tag{44}$$

for some $[\alpha] \in Aut\big(N\mathbb{P}^l\big)$ and $[\beta] \in \Gamma$. Then one writes $C^{\Gamma}_{l,P}(\mathcal{T}) = C'_{l,P}(\mathcal{T})/\sim$. An element of $C^{\Gamma}_{l,P}(\mathcal{T})$ is called a non-commutative curve of genus l with property P and modulo Γ in $\mathcal{T}$.

If P is an empty property then we will write $C'_l(\mathcal{T}) = C'_{l,P}(\mathcal{T})$, and $C^{\Gamma}_l(\mathcal{T}) = C^{\Gamma}_{l,P}(\mathcal{T})$. Furthermore, in the special case when $\Gamma = \big\{[\mathrm{id}_{\mathcal{T}}]\big\}$ we will also neglect writing Γ in the superscript. For such Γ, one has an equivalent way of describing the non-commutative curves. The following proposition is a particular case of [10, Lemma 4.5].

Proposition 52. *Let $\mathcal{T}$ be a triangulated category, then there is a bijection*

$$C_l(\mathcal{T}) \to \left\{ \mathcal{D} \subset \mathcal{T} : \begin{array}{l} \mathcal{D} \text{ full triangulated subcategory of } \mathcal{T} \\ \text{s.t. there exists an exact equivalence} \\ F : N\mathcal{P}^l \to \mathcal{D} \end{array} \right\} \tag{45}$$

given by $[F] \mapsto \overline{F(N\mathbb{P}^l)} =: \mathrm{Im}(F)$.

Proof. In the proof of [10, Lemma 4.5] it is stated without proof that $\mathrm{Im}(F)$ is a triangulated subcategory, which we will show here. It is closed under the translation functor, because there is an A in $N\mathbb{P}^l$ for its every object X, such that $X \cong F(A)$. Thus $X[1] \cong F(A[1])$. Let

$$X \xrightarrow{u} Y \longrightarrow Z \longrightarrow X[1]$$

be a distinguished triangle in $\mathcal{T}$ for which X and Y lie in $\mathrm{Im}(F)$. The cone Z will also be its object: Take A and B, such that $F(A) \cong X$ and $F(B) \cong Y$ and the distinguished triangle isomorphic to the first one

$$F(A) \xrightarrow{v} F(B) \longrightarrow Z' \longrightarrow F(A)[1] \ .$$

There is now a distinguished triangle $A \xrightarrow{F^{-1}(v)} B \longrightarrow C \longrightarrow A[1]$ where $F(C)$ will be again a cone of the arrow v and so isomorphic to Z' and Z. $\square$

Sometimes we will mean by $C_l(\mathcal{T})$ the codomain of this bijection. Recall that $N\mathbb{P}^l$ is generated by the full strong exceptional pair (s_0, s_1) where $\hom^0(s_0, s_1) = l + 1$. This motivates the following result.

Proposition 53. *[10, Proposition 5.5] Let $\mathcal{T}$ be a derived category of quiver-represenations for some acyclic quiver Q. It's full triangulated subcategory $\mathcal{D}$ has an exact equivalence $F : N\mathbb{P}^l \to \mathcal{D}$ if and only if it is generated by some strong exceptional pair (E_0, E_1) with $hom^0(E_0, E_1) = l + 1$.*

Remark 54. *The bijection (45) can be equivalently given by $[F] \mapsto \langle F(s_0), F(s_1)\rangle$, because $\langle F(s_0), F(s_1)\rangle = \mathrm{Im}(F)$.*

Following Definition 51, the concept of σ-semistability on non-commutative curves was introduced in [9]. We present the definition given there.

Definition 55. *Let everything be given as in Definition 51, $\sigma \in Stab(\mathcal{T})$, and $(s_i)_{i\in\mathbb{Z}}$ the helix of $N\mathbb{P}^l$ described in section 5.3, then one defines the set*

$$C'_{l,P,\sigma}(\mathcal{T}) = \left\{F \in C'_{l,P} \mid F(s_i) \;\; \sigma\text{-semistable for infinitely many } i \in \mathbb{Z}\right\}. \tag{46}$$

Using the equivalence relation from the Definition 51 we write $C^{\Gamma}_{l,P,\sigma} = C'_{l,P,\sigma}/\sim$. The elements of $C^{\Gamma}_{l,P,\sigma}$ are called the σ-semistable non-commutative curves of genus l with property P and modulo Γ in $\mathcal{T}$.

Again, we will leave out the subscript P when P is an empty property and the superscript Γ when Γ contains only the identity. We want to classify now which functors give semistable non-commutative curves.

Proposition 56. *Let $\mathcal{T}$ be a triangulated category, $F \in C'_l(\mathcal{T})$ for some $l \geq 1$ and (E_0, E_1) a full strong exceptional pair generating $\mathrm{Im}(F)$, such that $hom^0(E_0, E_1) = l$ (such a pair exists, because it can be given by $\big(F(s_0), F(s_1)\big)$), then construct the helix $(E_i)_{i\in\mathbb{Z}}$ generated by (E_0, E_1). The functor F is an element of $C'_{l,\sigma}(\mathcal{T})$ if and only if the there are infinitely many objects of the helix $(E_i)_{i\in\mathbb{Z}}$ which are σ-semistable.*

Proof. By Lemma 42, F takes takes the helix generated by (s_0, s_1) to the helix generated by $\big(F(s_0), F(s_1)\big)$. Also, an object X in $\mathrm{Im}(F)$ is exceptional if and only if it is isomorphic to some $F(A)$ where A is exceptional and a pair (E_0, E_1) has the assumed properties if and only if it is isomorphic to an image of some pair (A_0, A_1) with these properties in $N\mathbb{P}^l$. We know from Section 5.3 that every such pair must be given by $(s_i[p], s_{i+1}[p])$ for some $i, p \in \mathbb{Z}$. Thus from Lemma 41, the helix generated by (E_0, E_1) is the shift by $[p]$ of the helix generated by $(F(s_i), F(s_{i+1}))$ which is simply the image of the standard helix under F. The statement then follows directly from the definition of $C'_{l,\sigma}(\mathcal{T})$. ☐

Combining Proposition 53 with Proposition 56 we get:

Corollary 57. *Let $\mathcal{T}$ be given as in Proposition 53, then one has a bijection for $l \geq 1$:*

$$C_{l,\sigma}(\mathcal{T}) \to \left\{ \mathcal{D} \subset \mathcal{T} : \begin{array}{l} \text{$\mathcal{D}$ full triangulated subcategory of $\mathcal{T}$} \\ \text{generated by a strong exceptional pair} \\ \text{(E_0, E_1) where $hom^0(E_0, E_1) = l + 1$} \\ \text{and the helix generated by it has} \\ \text{infinitely many σ-semistable objects} \end{array} \right\} \tag{47}$$

given by $F \mapsto \mathrm{Im}(F)$.

Let us give another result that is useful whenever one knows all exceptional objects in $\mathcal{T}$ and thus in $\mathrm{Im}(F)$.

Definition 58. *Let $\mathcal{T}$ be a triangulated category and $\sigma \in Stab(\mathcal{T})$. We call an equivalence class with respect to $\sim$ of exceptional objects σ-semistable when one and thus all of its exceptional objects are σ-semistable. A full triangulated subcategory $\mathcal{D} \subset \mathcal{T}$ with infinitely many classes of exceptional objects is σ-semistable when it contains infinitely many σ-semistable equivalence classes of exceptional objects.*

Proposition 59. *Let $\mathcal{T}$ be a triangulated category. For $l \geq 1$, there is a bijection*

$$C_{l,\sigma}(\mathcal{T}) \to \left\{ \mathcal{D} \subset \mathcal{T} : \begin{array}{c} \mathcal{D}\ \sigma\text{-semistable full triangulated} \\ \text{subcategory of } \mathcal{T} \text{ with an exact} \\ \text{equivalence } F : N\mathcal{P}^l \to \mathcal{D} \end{array} \right\} \tag{48}$$

given by $[F] \mapsto Im(F)$ for all $l \geq 1$.

Proof. A functor $F \in C_l'$ lies in $C_{l,\sigma}'$ if and only if $\mathrm{im}(F)$ is σ-semitable: Any two exceptional objects $X \cong F(A)$ and $Y \cong F(B)$ are equivalent in $\mathrm{im}(F)$ if any only if A and B are equivalent exceptional objects. Thus F maps every class of exceptional objects in $N\mathbb{P}^l$ bijectively to the classes of exceptional objects in $\mathrm{im}(F)$. From Corollary 44 we conclude the statement. $\square$

8 Non-commutative curve counting for the acyclic triangular quiver

In this section we will be working with the derived category $\mathcal{T} = D^b(Q)$ of the following quiver:

$$\begin{tikzcd} & 3 & \\ 1 \arrow[ur] \arrow[rr] & & 2 \arrow[ul] \end{tikzcd} \tag{49}$$

It was proven in [10] that $C_1(\mathcal{T}) = 2$. We will additionally show that there exist such stability conditions in $\mathrm{Stab}(\mathcal{T})$ such that every of the following cases happens:

1. No non-commutative curve is semistable.
2. Each of the non-commutative curves is semistable while the other isn't.
3. Both curves are semistable.

8.1 Exceptional objects in $D^b(Q)$

Let $\underline{\dim}(E) = (x, y, z)$ be the dimension vector of an exceptional representation in $\mathrm{Rep}_k(Q)$, then we know from Lemma 34 that it is a real root, and so it must solve the equation

$$x^2 + y^2 + z^2 - xy - xz - yz = 1\,. \tag{50}$$

To solve this equation for non-negative integers, we can assume x to be the smallest integer. Then $y = x + a$ and $z = x + b$ for some non-negative integers a, b. Plugging this into (50) one gets $a^2 - ab + b^2 = 1$ and thus $(a - b)^2 + ab = 1$. Both terms on the left hand side of the last equation are positive, so the only possible solutions are when $a = b = 1$ or $a = 1, b = 0$ or $a = 0, b = 1$. Thus the only allowed dimension vectors become:

$$\begin{array}{ccc} (m+1, m, m) & (m, m+1, m+1) & (m, m, m+1) \\ (m+1, m+1, m) & (m, m+1, m) & (m+1, m, m+1) \end{array} \tag{51}$$

From this, one also sees that the imaginary roots of this Euler form are (m, m, m) for $m \in \mathbb{Z}$.

One has the following additional restriction on the dimension vectors of an exceptional representation.

Lemma 60. *[11, Lemma 2.1] Let $m \geq 1$, then there is no exceptional representation with the dimension vectors $(m, m+1, m)$ and $(m+1, m, m+1)$.*

For the remaining dimension vectors the corresponding unique exceptional representations can be found: Let $\pi_{\pm}^m : k^{m+1} \to k^m$ and $j_{\pm}^m : k^m \to k^{m+1}$ be linear maps such that

$$\pi_+^m(x_1, \ldots, x_{m+1}) = (x_1, \ldots, x_m) \qquad \pi_-^m(x_1, \ldots, x_{m+1}) = (x_2, \ldots, x_{m+1})$$
$$j_+^m(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0) \qquad j_-^m(x_1, \ldots, x_m) = (0, x_1, \ldots, x_m)\,.$$

Proposition 61. *[11, Proposition 2.2] The exceptional objects of $\mathcal{T}$ up to equivalence are:*

$$E_1^m = \begin{tikzcd} & k^m & \\ k^{m+1} \arrow[ur, "\pi_+^m"] \arrow[rr, "\pi_-^m"] & & k^m \arrow[ul, "id"'] \end{tikzcd} \qquad E_2^m = \begin{tikzcd} & k^{m+1} & \\ k^m \arrow[ur, "j_+^m"] \arrow[rr, "j_-^m"] & & k^{m+1} \arrow[ul, "id"'] \end{tikzcd} \qquad E_3^m = \begin{tikzcd} & k^{m+1} & \\ k^m \arrow[ur, "j_+^m"] \arrow[rr, "id"] & & k^m \arrow[ul, "j_-^m"'] \end{tikzcd}$$

$$E_4^m = \begin{tikzcd} & k^m & \\ k^{m+1} \arrow[ur, "\pi_+^m"] \arrow[rr, "id"] & & k^{m+1} \arrow[ul, "\pi_-^m"'] \end{tikzcd} \qquad M = \begin{tikzcd} & 0 & \\ 0 \arrow[ur] \arrow[rr] & & k \arrow[ul] \end{tikzcd} \qquad M' = \begin{tikzcd} & k & \\ k \arrow[ur] \arrow[rr] & & 0 \arrow[ul] \end{tikzcd}$$

where m goes over all non-negative integers.

We construct some embeddings that we will use in the proofs in 8.2.

Lemma 62. *In $\mathcal{T}$, the following maps are embeddings:*

1. *$E_3^m \to E_2^m$ given by $(id_{k^m}, j_-^m, id_{k^{m+1}})$ for all $m \geq 0$.*
2. *$M' \to E_1^m$ given by $(i_{m+1}, 0, i_m)$ where $i_m : k \to k^m$ maps x to $(x, 0, \ldots, 0)$.*
3. *$E_2^{m-1} \to E_3^m$ given by $(j_-^{m-1}, id_{k^m}, j_-^m)$ for all $m \geq 1$.*
4. *$M \to E_4^m$ given by $(0, i_{m+1}, 0)$ for all $m \geq 0$.*

Following [12], we denote

$$a^m = \begin{cases} E_1^{-m} & m \leq 0 \\ E_2^{m-1}[1] & m \geq 1 \end{cases}, \qquad b^m = \begin{cases} E_4^{-m} & m \leq 0 \\ E_3^{m-1}[1] & m \geq 1 \end{cases}. \tag{52}$$

From the table in [11, Proposition 2.4], one gets a statement about the exceptional collections formed by these objects.

Proposition 63. *[12, Corollary 3.12 and Remark 3.14] The exceptional pairs of $\mathcal{T}$ up to equivalence are*

$$(a^m, a^{m+1}), \quad (b^m, b^{m+1}), \quad (a^m, b^{m+1}), \quad (b^m, a^m)$$
$$(a^m, M), \quad (M, b^m), \quad (M', a^m), \quad (b^m, M').$$

The full exceptional collections (triples) of $\mathcal{T}$ up to equivalences are

$$(a^m, a^{m+1}, M). \quad (M', a^m, a^{m+1}), \quad (b^m, b^{m+1}, M'), \quad (M, b^m, b^{m+1}),$$
$$(a^m, M, b^{m+1}), \quad (b^m, M', a^m), \quad (a^m, b^{m+1}, a^{m+1}), \quad (b^m, a^m, b^{m+1}).$$

Here m goes over all integers. If (E_1, E_2) is one of these exceptional pairs then $hom^p(E_1, E_2)$ is non-vanishing for a unique $p \in \mathbb{Z}$. The pairs for which it takes the value $hom^p(E_1, E_2) = 2$ are only (a^m, a^{m+1}) and (b^m, b^{m+1}) for all $m \in \mathbb{Z}$.

In [10, Proposition 6.1], one determined all the non-commutative curves in $\mathcal{T}$. We recall here what the non-commutative curves of genus one are.

Proposition 64. *There are two non-commutative curves of genus 1 as the elements of $C_1(\mathcal{T})$. In terms of the bijection (45) one of the curves is given by the triangulated subcategory $\langle a^m, a^{m+1}\rangle$ and the other by $\langle b^m, b^{m+1}\rangle$ for any $m \in \mathbb{Z}$.*

Proof. Applying Proposition 52 and Proposition 53 to $\mathcal{T}$ and using Proposition 63, we conclude that $\langle a^m, a^{m+1}\rangle$ and $\langle b^m, b^{m+1}\rangle$ are the only possible elements of $C_1(\mathcal{T})$. From Corollary 48 and Proposition 63, we see moreover that

$$\langle a^m, a^{m+1}\rangle = \langle M\rangle^{\perp} \quad \text{and} \quad \langle b^m, b^{m+1}\rangle = \langle M'\rangle^{\perp} \quad \text{for all } m \in \mathbb{Z}\,.$$

Because (a^m, b^m, a^{m+1}) is a full exceptional collection, it can't hold that $\langle b^m, b^{m+1}\rangle = \langle a^m, a^{m+1}\rangle$, because then we would get $\langle a^m, a^{m+1}\rangle = \mathcal{T}$ contradicting Lemma 33. $\square$

Remark 65. *Notice that the only exceptional objects in $\langle a^m, a^{m+1}\rangle$ are the a^i for all $i \in \mathbb{Z}$. If it contained any other exceptional objects then it would be $\mathcal{T}$ itself: Assume that it contains M or M', then we can obtain the exceptional triples (a^m, a^{m+1}, M) or (M, a^m, a^{m+1}) in this subcategory. Similarly as in the above proof we conclude that $\langle a^m, a^{m+1}\rangle = \mathcal{T}$ which by Lemma 33 gives us a contradiction. In the proof of the previous proposition we have shown that no b^k lies in $\langle a^m, a^{m+1}\rangle$. A similar statement holds for the second non-commutative curve $\langle b^m, b^{m+1}\rangle$.*

8.2 Semistable non-commutative curves of genus 1 in $D^b(Q)$

Firstly, we will need to develop some tools that will be used to construct stability conditions.

Definition 66. *An exceptional collection $\mathcal{E} = (E_0, \ldots, E_n)$ is Ext-exceptional when $hom^l(E_i, E_j) = 0$ for all integers $l \leq 0$ and $i \neq j$.*

For a class of objects $\mathcal{E}$ in a triangulated category $\mathcal{T}$ we denote by $\hat{\mathcal{E}}$ the extension closure in $\mathcal{T}$ of $\mathcal{E}$.

Lemma 67. *[13, Lemma 3.14] Let $\mathcal{E} = (E_0, \ldots, E_n)$ be a full Ext-exceptional collection in a triangulated category $\mathcal{T}$, then $\hat{\mathcal{E}}$ is a heart of some bounded t-structure of $\mathcal{T}$.*

Remark 68. *With the notation taken from the previous lemma, every exceptional object E_i becomes simple in $\hat{\mathcal{E}}$. Assume the contrary and let A be a proper nonzero subobject of E_i, then some E_j is a subobject of A and the morphisms $E_j \to A \to E_i$ vanishes unless $i = j$ where it is an isomorphism. Thus we get the contradiction $A \cong E_i$. This also shows that every simple object is isomorphic to some E_i.*

Remark 69. *If we apply this result to the case where $\mathcal{E} = (E)$ is a single exceptional object, we see that $\hat{E}$ is a heart of a bounded t-structure. And due to $hom^i(E, E) = 0$ whenever $i \neq 0$, it follows that any object X in $\hat{E}$ is a direct product of copies of E. Using Lemma 8, one concludes that every object X in $\langle E\rangle$ is a direct product of finitely many objects from the class $\{E[i]\}_{i\in\mathbb{Z}}$.*

One can combine previous results to give a method for constructing stability conditions on a triangulated category $\mathcal{T}$. If $\mathcal{E} = \langle E_0, \ldots, E_n\rangle$ is a full Ext-exceptional collection, then using the bijection (27) one can construct from stability functions with HN-property on $\hat{\mathcal{E}}$ (which by 67 is a heart of a bounded t-structure) new stability conditions. Moreover, if $\hat{\mathcal{E}}$ is finite length, then by Proposition 27 every stability function has the HN-property. By Remark 68 there are only finitely many simple objects in $\hat{\mathcal{E}}$ and so the induced stability condition is locally finite. For any such $\mathcal{E}$ one denotes the set of stability conditions attained by this method by $\mathbb{H}^{\mathcal{E}}(\mathcal{T}) \subset \mathrm{Stab}(\mathcal{T})$. With the Corollary 50, choosing different stability conditions in $\mathbb{H}^{\mathcal{E}}$ becomes equivalent to assigning different values $Z(E_i)$ in $\mathbb{H} \subset \mathbb{C}$ that the stability function takes on the exceptional objects E_i.

We consider now a simple general example of stability conditions constructed in such a way. We will call a stability condition constructed by the above method *non-entangled* if $\phi_Z(E_{i+1}) > \phi_Z(E_i)$ for all $0 \le i < n$ for given $\mathcal{E}$. Note that due to these conditions, it follows that :

$$\phi_Z(E_{i_2}[j_2]) > \phi_Z(E_{i_1}[j_1]) \iff j_2 > j_1 \text{ or } j_2 = j_1, i_2 > i_1 \,. \tag{53}$$

Proposition 70. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{E} = (E_0, \dots, E_n)$ its full Ext-exceptional collection, where $\hat{\mathcal{E}}$ is a finite length abelian category. Let $\sigma = (\mathcal{P}, Z)$ be a non-entangled stability condition for $\mathcal{E}$, then the slicing $\mathcal{P}$ is given in the following way.*

$$\mathcal{P}(\phi) = \begin{cases} \hat{E}_i[j] & \textit{when } \phi = \phi_Z(E_i) + j \\ \{0\} & \textit{otherwise} \end{cases} \tag{54}$$

Proof. Let us show that $\sigma = (\mathcal{P}, Z)$ indeed gives a stability condition when $\mathcal{P}$ is given by the equation (54). For that, we only need to show that $\mathcal{P}$ is a slicing on $\mathcal{T}$. The first two axioms of the Definition 12 follow immediately. For any object F there is a filtration similar to the one in Proposition 46 with factors A_i that are objects of $\langle E_i \rangle$ and as such are the direct sums of shifts of E_i by Remark 69. So $\mathcal{T}$ is the extension closure of $\{E_i[j]\}_{0 \le i \le n, j \in \mathbb{Z}}$. However, when there is a diagram of the form

$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \arrow[rr] & & C \arrow[dl] \\ & E_{i_1}[j_1] \arrow[ul, "{[1]}"] & & E_{i_2}[j_2] \arrow[ul, "{[1]}"] & \end{tikzcd}$$

where $\phi_Z(E_{i_2}[j_2]) > \phi_Z(E_{i_1}[j_1])$, then the composition $E_{i_2}[j_2] \to B \to E_{i_1}[j_1+1]$ is a zero morphism (by $\mathcal{E}$ being Ext-exceptional), so we see from Lemma 2 that we can interchange the order of the factors. Doing so until any two neighboring factors $E_{i_1}[j_1]$ and $E_{i_2}[j_2]$ have $\phi_Z(E_{i_1}[j_1]) > \phi_Z(E_{i_2}[j_2])$, gives us a HN-filtration of F with respect to $\mathcal{P}$. As σ is a stability condition and its corresponding heart is the abelian category $\hat{\mathcal{E}}$ with the restriction of Z being still given by the values $Z(E_i)$, it follows that σ is indeed the non-entangled stability conditions specified by $Z(E_i)$ for $\mathcal{E}$. □

Using this general discussion we return to the case $\mathcal{T} = D^b(Q)$. From [11, Proposition 2.4] and Proposition 63 we see that $\mathcal{E} = (a_0, M, b_1[-1]) = (E_1^0, M, E_3^0)$ is a full Ext-exceptional collection. Its closure under extensions is the abelian subcategory $\mathrm{Rep}_k(Q) \subset \mathcal{T}$ which is the heart of the standard t-structure.

Corollary 71. *Let σ be a non-entangled stability condition on $\mathcal{T}$ for $\mathcal{E} = (E_1^0, M, E_3^0)$, then $\#C_{1,\sigma}(\mathcal{T}) = 0$.*

Proof. By Proposition 70 the objects in σ^{ss} are isomorphic to $(E_1^0[i])^p$, $(M[j])^q$, or $(E_3^0[k])^r$ for some integers i, j, k, p, q, r. By Remark 65, we know that the only exceptional objects of the curve $\langle a_m, a_{m+1} \rangle$ for some $m \in \mathbb{Z}$ are the shifts of a_i for all $i \in \mathbb{Z}$. By Proposition 59, we only need to show that there are only finitely many a_i which are σ-semistable. But we see that the only a_i that belongs to σ^{ss} is $a_0 = E_1^0$, as no other element of $\{a_m\}_{m \in \mathbb{Z}}$ is isomorphic to the direct sums above (follows for example by looking at the dimensions). Thus the curve $\langle a_m, a_{m+1} \rangle$ is not σ-semistable. In an identical way we get the same statement for the second curve $\langle b_m, b_{m+1} \rangle$. □

To get stability conditions σ in $\mathrm{Stab}(\mathcal{T})$ such that the non-commutative curves become σ-semistable we need to "entangle" the order slightly. The simplest case is choosing $\theta := \phi_Z(E_0^1) = \phi_Z(M) = \phi_Z(E_3^0)$. The corresponding stability condition $\sigma = (\mathcal{P}, Z)$ then has the slicing

$$\mathcal{P}(\phi) = \begin{cases} \mathrm{Rep}_k(Q)[j] & \text{when } \phi = \theta + j \\ \{0\} & \text{otherwise} \,. \end{cases} \tag{55}$$

As such all exceptional objects a_m and b_m are σ-semistable and $\#C_{1,\sigma}(\mathcal{T}) = 2$. We will now construct more interesting examples.

Firstly, if X is an object in $\operatorname{Rep}_k(Q)$ with the dimension vector (m_1, m_2, m_3), then for any $Z : K_0(\mathcal{T}) \to \mathbb{C}$ the following holds:

$$Z(X) = m_1 Z(E_1^0) + m_2 Z(M) + m_3 Z(E_3^0). \tag{56}$$

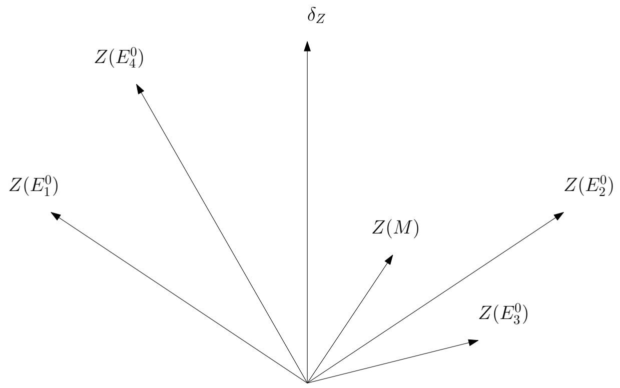
This follows because every object is a finite extension with the factors being E_1^0, M, and E_3^0, and the dimension vectors of these are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Thus especially if we use the notation $\delta_Z = Z(E_1^0) + Z(M) + Z(E_3^0)$ we have:

$$Z(E_1^m) = Z(E_1^0) + m\delta_Z\,, \quad Z(E_2^m) = Z(M) + Z(E_3^0) + m\delta_Z\,, \quad Z(E_3^m) = Z(E_3^0) + m\delta_Z$$
$$Z(E_4^m) = Z(E_1^0) + Z(M) + m\delta_Z\,, \quad Z(M') = Z(E_1^0) + Z(E_3^0)\,.$$

The values of Z on the remaining indecomposable representations are:

$$m\delta_Z\,, \qquad Z(M) + m\delta_Z\,, \qquad Z(M') + m\delta_Z\,.$$

Proposition 72. *Let $\sigma = (\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{E}}$ be a stability condition on $\mathcal{T}$ where $\mathcal{E} = (E_1^0, M, E_3^0)$, such that the values of Z take the following form in the complex plane:*



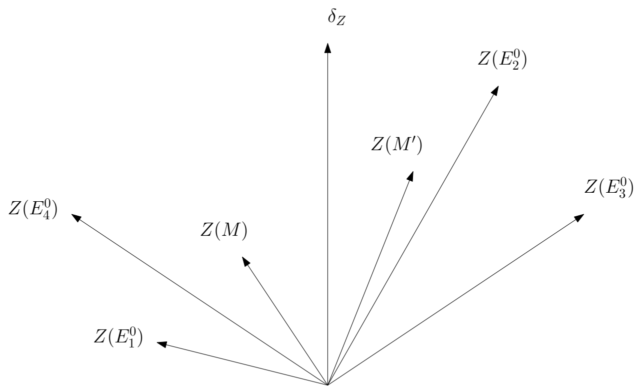
Then $\#C_{1,\sigma}(\mathcal{T}) = 1$, and $\langle b^m, b^{m+1}\rangle$ is σ-semistable.

Proof. It was shown in the proof of [11, Lemma 3.15] that E_4^m are σ-semistable for all $m \geq 0$. This tells us that the curve $\langle b^m, b^{m+1}\rangle$ is σ-semistable.

Notice that there exists such $N > 0$, such that the argument of $Z(E_3^m)$ is larger than the argument of $Z(E_2^m)$ for all $m \geq N$: Choose an $N > 0$, such that $\phi_Z(E_3^m) > \phi_Z(M)$ whenever $m \geq N$. From $Z(E_2^m) = Z(E_3^m) + M$, we obtain for every $m \geq N$ that $\phi_Z(E_3^m) > \phi_Z(E_2^m)$. By Lemma 62, there is an injective morphism $E_3^m \to E_2^m$. This shows that E_2^m is not σ-semistable for all $m \geq N$, as it has a non-zero subobject with greater phase . For the objects E_1^m, one also sees by drawing that the argument of $Z(M')$ is larger than the arguments of $Z(E_1^m)$ for all $m \geq 1$. Moreover, we see from Lemma 62 that there is an injective morphism $M' \to E_1^m$. We conclude that only finitely many objects a^m can be σ-semistable and so the non-commutative curve $\langle a^m, a^{m+1}\rangle$ is not semi-stable. $\square$

Similarly, we now construct a stability condition that makes the other non-commutative curve semistable.

Proposition 73. *Let $\sigma = (\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{E}}$ be a stability condition on $\mathcal{T}$, such that the values of Z take the following form in the complex plane:*



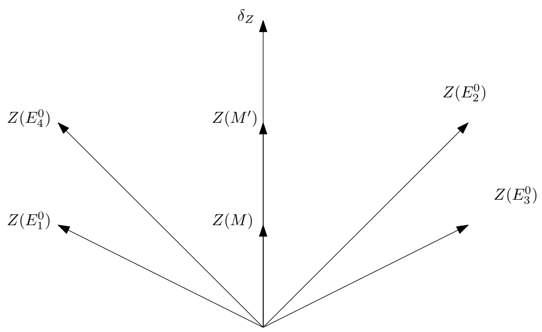
Then $\#C_{1,\sigma}(\mathcal{T}) = 1$*, and* $\langle a^m, a^{m+1}\rangle$ *is* σ*-semistable.*

Proof. Firstly, let us show that E_1^m are σ-semistable for all $m \geq 0$. From the bijection (27), we know that $X \in \mathrm{Rep}_k(Q)$ is σ-semistable if and only if it is semistable with respect to the stability function on $\mathrm{Rep}_k(Q)$. Similarly as it was done in [11, Lemma 3.15], it suffices to show that for any indecomposable subobject $A \subset X$ the argument $\arg_{(0,1]}\big(Z(A)\big)$ is less than $\arg_{(0,1]}\big(Z(X)\big)$ to conclude that X is semistable. From the figure, we know that the indecomposable objects with the value of Z on them from the set $\big\{l\delta_Z, Z(M') + l\delta_Z, Z(M) + Z(E_3^0) + l\delta_Z : l \geq 0\big\}$ will have a smaller phase than E_1^m. Next, for the indecomposable object A with the dimension vectors $(l, l+1, l)$ and the value of Z given by $l\delta_Z + Z(M)$, we can show that there is no embedding $A \to E_1^m$: If such an embedding existed, it would require that the compositions $k^{l+1} \to k^m \xrightarrow{\mathrm{id}} k^m = k^{l+1} \to k^l \to k^m$ are equal, where the left side of the equation is monic. This gives a contradiction. Same argument shows that no E_4^l can be embedded in E_1^m. Finally, looking at the [11, Proposition 2.4], we use that $\hom(E_1^l, E_1^m) = 0$ whenever $0 \leq l < m$ to infer that E_1^m are σ-semistable for all $m \geq 0$. As a result this implies that $\langle a^m, a^{m+1}\rangle$ is a σ-semistable non-commutative curve of genus 1.

To show that $\langle b^m, b^{m+1}\rangle$ is not σ-semistable, we notice from the figure, that there exists such an $N > 0$, such that $\phi_Z(E_2^{m-1})$ is larger than $\phi_Z(E_3^m)$ whenever $m \geq N$: Choose an $N > 0$, such that $\phi_Z(E_2^{m-1}) > \phi_Z(M')$ for all $m \geq N$. From $Z(E_3^m) = Z(E_2^{m-1}) + Z(M')$, we see that for all $m \geq N$ one has $\phi_Z(E_2^{m-1}) > \phi_Z(E_3^m)$. However, we also see from Lemma 62 that there is an embedding $E_2^{m-1} \to E_3^m$. Thus E_3^m is not semistable whenever $m \geq N$ as it has a non-zero subobject with greater phase. Lastly, for any m large enough we observe that the argument of $Z(E_4^m)$ is less than that of $Z(M)$ and there is an embedding $M \to E_4^m$. Only finitely many objects b^m can be σ-semi-stable, and so the curve $\langle b^m, b^{m+1}\rangle$ is not semi-stable. $\square$

We would like to show that there exist non-trivial stability conditions (in the sense that the slicing is non-trivial) with two semi-stable curves of genus 1. Looking at the stability conditions discussed in [12, Lemma 7.5(e)], we see that they satisfy this requirement. In the next proposition, we demonstrate a special case.

Proposition 74. *Let* $\sigma = (\mathcal{P}, Z) \in \mathbb{H}^{\mathcal{E}}$ *be a stability condition on* $\mathcal{T}$*, such that the values of* Z *take the following form in the complex plane (notice especially that* $\phi_Z(M) = \phi_Z(M')$*):*



Then $\#C_{1,\sigma}(\mathcal{T}) = 2$, that is, $\langle a^m, a^{m+1}\rangle$ and $\langle b^m, b^{m+1}\rangle$ are σ-semistable.

Proof. To show that E_1^m are all σ-semistable, we only need to show that no E_4^n can be embedded in them, but we already know that from the proof of Proposition 73. Similarly, we know from the proof of [11, Lemma 3.15] that no E_1^n can be embedded in E_4^m. Thus all the objects E_1^m and E_4^m are σ-semistable and the curves $\langle a^m, a^{m+1}\rangle$ and $\langle b^m, b^{m+1}\rangle$ are semistable. □

Consequently, we have proved what we have set out to show at the beginning of this section.

Theorem 75. *There exists a stability condition $\sigma \in Stab(\mathcal{T})$ for each of the following cases:*

1. $\#C_{1,\sigma}(\mathcal{T}) = 0$.
2. $\#C_{1,\sigma}(\mathcal{T}) = 1$ *and* $\langle a^m, a^{m+1}\rangle$ *is σ-semistable.*
3. $\#C_{1,\sigma}(\mathcal{T}) = 1$ *and* $\langle b^m, b^{m+1}\rangle$ *is σ-semistable.*
4. $\#C_{1,\sigma}(\mathcal{T}) = 2$.

Appendices

A Summary

In this work we addressed the topic of (semistable) non-commutative curve counting, that was introduced for the first time by G. Dimitrov and L. Katzarkov in [9]. The non-commutative curves correspond to certain equivalence classes of exact fully faithful functors on triangulated categories. The semistable non-commutative curves depend on stability conditions of the triangulated category considered. First, we repeated the definition of stability conditions on triangulated categories and reminded the reader that locally finite stability conditions of a triangulated category $\mathcal{T}$ form a complex manifold that is denoted by $\mathrm{Stab}(\mathcal{T})$. The first result of the thesis states that this manifold for the derived category of representation of the quiver A_1 is biholomorphic to $\mathbb{C}$.

After recalling some properties of triangulated categories, derived categories and t-structure in 2 and 3 and using the results about exceptional objects and quivers from 5 and 6, we give the definition of (semistable) non-commutative curves in triangulated categories in Section 7. At the end of this section and in the subsection 8.2, we discuss a method to construct stability conditions and to determine the corresponding semistable noncommutative curves. This method relies on the bijection (27). As such, we view the stability conditions as stability functions on hearts of bounded t-structures generated as extension closures by full Ext-exceptional collections. Under the bijection 48 one also associates corresponding full triangulated subcategories to the semistable NC curves. Applying this method we were able to examine the semistable NC curves with genus 1 of the following quiver:

$$\begin{tikzcd} & 3 & \\ 1 \arrow[ur] \arrow[rr] & & 2 \arrow[ul] \end{tikzcd} ,$$

where we have used the statements about exceptional objects from the Subsection 8.1. After reminding the reader that there are exactly two NC curves with genus 1 in the derived category of this quiver which we labeled $\langle a^m, a^{m+1}\rangle$ and $\langle b^m, b^{m+1}\rangle$, we have proved the Theorem 75 which is the main result of the thesis and states that all possible combinations of the NC curves become σ-semistable for the right choice of a locally finite stability condition σ. Conclusively, we have proven the Proposition 12.15 from [9].

B Zusammenfassung

Diese Arbeit hat das neu eingeführte Thema von (semistabilen) nicht-kommutativen Kurven, das zum ersten Mal in der Arbeit von G. Dimitrov und L. Katzarkov [9] besprochen wurde, behandelt. Die nichtkommutativen Kurven ensprechen bestimmten Äquivalenzklassen von exakten volltreuen Funktoren in triangulierten Kategorien. Die semistabilen nicht-kommutativen Kurven hängen von den Stabilitätkondizionen der triangulierten Kategorie ab. Zuerst wiederholen wir die Definition von Stabilitätkondizionen von einer triangulierten Kategorie. Lokal endliche Stabilitätkondizionen der Kategorie $\mathcal{T}$ bilden eine komplexe Manigfaltigkeit, die wir mit $\mathrm{Stab}(\mathcal{T})$ bezeichnen. Einer unserer erster Resultate sagt, dass diese Manigfaltigkeit für die derivierte Kategorie der Repräsentationen des Köchers A_1 biholomorph zu $\mathbb{C}$ ist.

Mit den Ergebnissen aus den Kapiteln 2 und 3, wo die Eigenschaften von triangulierten Kategorien, derivierten Kategorien und t-Strukturen hergeleitet und wiederholt werden, und den Resultaten über ausgezeichnete Objekte und Köcher in 5 und 6 geben wir die Definition der (semistabilen) nicht-kommutativen Kurven in triangulierten Kategorien in Kapitel 7. Am Ende dieses Kapitels und im Unterkapitel 8.2 geben wir eine Methode um Stabilitätkondizionen zu konstruieren und die dazugehörige semistabilen nichtkommutativen Kurven zu bestimmen. Diese Methode basiert auf der Bijektion (27). Wir betrachten die Stabilitätkondizionen als Stabilitätfunktionen auf Herzen von beschränkten t-Strukturen, die durch volle Ext-ausgezeichnete Sammlungen erzeugt werden. Den semistabilen NK Kurven werden die dazugehörige volle triangulierte Unterkategorien unter der Bijektion (48) zugeordnet. Mithilfe von dieser Methode untersuchen wir die semistabilen nicht-kommutativen Kurven mit Genus 1 des folgenden Köchers:

$$\begin{tikzcd} & 3 & \\ 1 \arrow[ur] \arrow[rr] & & 2 \arrow[ul] \end{tikzcd} ,$$

wobei wir die Aussagen über ausgezeichnete Objekte, die in dem Unterkapitel 8.1 angegeben werden, verwenden. Zuerst erinnern wir den Leser daran, dass es in der derivierten Kategorie dieses Köchers genau 2 nicht-kommutative Kurven mit Genus 1 gibt, die wir mit $\langle a^m, a^{m+1}\rangle$ und $\langle b^m, b^{m+1}\rangle$ bezeichnen. Dann beweisen wir den Hauptresultat dieser Arbeit:

Theorem 76. *Es gibt solche lokal endliche Stabilitätkondizionen σ des oben angegeben Köchers, sodass alle unten angegebenen Fälle vorkommen:*

1. $\#C_{1,\sigma}(\mathcal{T}) = 0$.
2. $\#C_{1,\sigma}(\mathcal{T}) = 1$ *und die NK Kurve* $\langle a^m, a^{m+1}\rangle$ *ist* σ*-semistabil.*
3. $\#C_{1,\sigma}(\mathcal{T}) = 1$ *und die NK Kurve* $\langle b^m, b^{m+1}\rangle$ *ist* σ*-semistabil.*
4. $\#C_{1,\sigma}(\mathcal{T}) = 2$.

Wobei $\#C_{1,\sigma}(\mathcal{T})$ hier die Anzahl der σ-semistabilen NK Kurven mit Genus 1 bezeichnet.

Wir haben also die Proposition 12.15 in [9] bewiesen.

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