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Abstract

The goal of this master thesis is to investigate global and local issues related to gauge fixing. We address global issues associated with the Gribov ambiguity from the perspective of physics as well as mathematics and discuss three different ways of resolving it. Considering local issues, we discuss a modified Faddeev-Popov path integral density for the quantization of Yang-Mills theory in the Feynman gauge. The modification consists of replacing the contributions of Faddeev-Popov ghost fields by multi-point gauge field interactions. By performing an explicit calculation up to the second order in the gauge coupling constant g, we show the equivalence between the usual Faddeev-Popov scheme and its modified version.

Zusammenfassung

Ziel dieser Masterarbeit ist die Untersuchung globaler und lokaler Probleme bei Eichfixierung. Bezüglich der globalen Aspekte behandeln wir die Gribov Ambiguität sowohl von physikalischer als auch mathematischer Perspektive und beschreiben drei verschiedene Lösungsmethoden. Bezüglich der lokalen Aspekte diskutieren wir eine modifizierte Faddeev-Popov-Pfadintegraldichte für die Quantisierung der Yang-Mills-Theorie in der Feynman Eichung. Die Modifikation besteht darin, die Beiträge von Faddeev-Popov Geisterfeldern durch Vielpunkt-Eichfeld-Wechselwirkungen zu ersetzen. Indem wir eine explizite Rechnung bis zu zweiter Ordnung in der Eichkopplungskonstante g durchführen, zeigen wir Äquivalenz zwischen der konventionellen Faddeev-Popov Methode und deren modifierten Version.

Words of Thanks and Appreciation

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Chapter 1

Quantization of Yang-Mills Theory

1.1 Introduction

Non-abelian gauge theories are the essential ingredient in the construction of unified interaction between electromagnetic, strong and weak forces. They were first proposed by Yang and Mills [1] in 1954, as a failed attempt to construct the theory of strong interactions by using the weakly broken isospin symmetry. Today, we know that the correct theory of strong interactions is a non-abelian gauge theory but with a much more sophisticated gauge group. In the 1950's, quantum electrodynamics (QED) was a well-established theory that gave an accurate description of electromagnetic fields and forces. It was a gauge theory with an abelian gauge group $U(1)$. A natural question, which was pursued by Yang, Mills, and others, was whether the generalization of QED to non-abelian gauge theories could be used to describe the weak and strong force. At that time, there was a significant phenomenological obstacle to this idea. It seemed that the short-range nature of these forces completely ruled out a gauge structure due to the massless nature of the Yang-Mills fields. For an interesting tour through the long and twisted history of gauge theory, we refer to [2].

For the weak force, the problem was solved by introducing an additional scalar field into the theory [3–5]. An explanation of the massive nature of Yang-Mills fields, while preserving the gauge structure, was given by Higgs, Englert and Kibble [6–8]. Today, the scalar particle is called the *Higgs boson* and the mechanism by which gauge fields obtain their mass is called the *Higgs mechanism* [1].

For the strong force, the solution was not to add more fields to the theory but by discovering that the theory itself has a unique property called *asymptotic freedom* [10, 11]. This property tells us that at short distances quarks and gluons behave as free particles. At long distances, the interaction strength between gluons and quarks increases such that it confines them within composite hadrons. The phenomenon that quarks and gluons cannot be observed at long distances is called color confinement. Until today, there does not exist an analytic proof of color confinement for any of the non-abelian gauge theories. We will return to the problem of confinement in our discussion of the Gribov ambiguity.

There were also problems regarding the quantization of Yang-Mills theory [12]. Due to the gauge symmetry, the Lagrangian of the Yang-Mills theory is singular. For abelian gauge theories, one can circumvent this problem by using Fermi's method of indefinite metric. Unfortunately, as Feynman [13] observed, this method does not work for gravitational and Yang-Mills theories. He discovered that there were diagrams with closed loops, which do

[1]Depending on the literature, the Higgs mechanism is also called *Brout-Englert-Higgs* mechanism or *Englert-Brout-Higgs-Guralnik-Hagen-Kibble* mechanism or *ABEGHHK'tH* mechanism standing for Anderson, Brout, Englert, Guralnik, Hagen, Higgs, Kibble and 't Hooft [9].

affect unitarity and depended on the choice of the propagator. As a solution to this problem, he proposed modified Feynman rules for the computation of 1-loop diagrams. Faddeev and Popov were the first who found a way of generalizing these rules to arbitrary diagrams [14] thus developing a manifestly covariant quantization method for Yang-Mills theory. In this chapter, we will analyze the problems associated with the quantization of Yang-Mills theories and establish the Faddeev-Popov quantization method.

1.2 Yang-Mills Theory and Faddeev-Popov quantization method

The Lagrange density of a $SU(N)$ gauge theory reads

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}\,[F_{\mu\nu}F_{\mu\nu}], \tag{1.2.1}$$

where $F_{\mu\nu}$ is the *field strength tensor*

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{1.2.2}$$

and A_μ are the *gauge fields*. Vanishing field strength tensor $F_{\mu\nu} = 0$ implies that the gauge field A_μ is pure gauge $A_\mu = (\partial_\mu \Omega)\Omega^{-1}$ and *vice versa*. The elements of the group $SU(N)$ are unitary $N \times N$ matrices Ω with a unit determinant. By using the exponential map, we can write a group element of $SU(N)$ as

$$\Omega = e^{ig\Lambda^a X^a}, \tag{1.2.3}$$

where X^a are the *generators* of the $SU(N)$ gauge group and g is the *coupling constant*. The generators X^a define a Lie algebra through the relation

$$[X^a, X^b] = if^{abc}X^c \tag{1.2.4}$$

where f^{abc} are the structure constants. If the structure constants are zero, then the Lie group is abelian. Gauge indices are labeled by small latin letters $a, b, c, ...$ and they run from 1 up to $N^2 - 1$. In our convention, the generators are hermitian $X^\dagger = X$ with the normalization condition

$$\mathrm{Tr}[X^a X^b] = \frac{1}{2}\delta^{ab}, \tag{1.2.5}$$

which is the normalization condition for generators in the fundamental representation. An important property of the Lie bracket is that it satisfies the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \tag{1.2.6}$$

where $A, B, C \in SU(N)$. In terms of the structure constants, Eq. (1.2.6) takes the form

$$f^{abd}f^{dce} + f^{bcd}f^{dae} + f^{cad}f^{dbe} = 0. \tag{1.2.7}$$

With the adjoint map, we express the gauge fields A_μ and the field strength $F_{\mu\nu}$ in terms of the generators X^a as

$$A_\mu = A^a_\mu X^a \quad \& \quad F_{\mu\nu} = F^a_{\mu\nu}X^a, \tag{1.2.8}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^b_\mu + gf^{abc}A^b_\mu A^c_\nu. \tag{1.2.9}$$

In our convention, the gauge fields A^a_μ are anti-hermitian $(A^a_\mu)^\dagger = -A^a_\mu$. Under the $SU(N)$ gauge group, the gauge fields A_μ transform as

$$A^\Omega_\mu = \Omega A_\mu \Omega^{-1} - \frac{i}{g}(\partial_\mu \Omega)\Omega^{-1}, \tag{1.2.10}$$

where $\Omega = \Omega(x)$ is a local group element of $SU(N)$. A straightforward calculation shows that the field strength tensor $F_{\mu\nu}$ transforms in the adjoint representation of the gauge group

$$F^\Omega_{\mu\nu} = \Omega F_{\mu\nu}\Omega^{-1}. \tag{1.2.11}$$

Although the field strength tensor itself is not invariant under a gauge transformation, the Lagrangian of Yang-Mills theory is. The *action functional* S is defined as the space-time integral over the Lagrangian density

$$S_{\text{inv}}[A] = \int_M d^4x\, \mathcal{L} = \frac{1}{2}\int_M d^4x\, \text{Tr}\left[F_{\mu\nu}F_{\mu\nu}\right] = \frac{1}{4}\int_M d^4x\, F^a_{\mu\nu}(x)F^a_{\mu\nu}(x), \tag{1.2.12}$$

where M is a compact and oriented manifold with the trivial Euclidean metric $\delta_{\mu\nu}$. It represents space-time on which the gauge fields $A_\mu(x)$ are evaluated. Since the field strength tensor $F_{\mu\nu}$ is hermitian $(F^a_{\mu\nu})^\dagger = F^a_{\mu\nu}$, the action S is real. To compute transition functions, we write the partition function Z [15] as

$$Z(J) = \int_{\mathcal{A}} \mathcal{D}A e^{-S_{\text{inv}}[A] + \int_M dx J^a_\mu A^a_\mu}, \tag{1.2.13}$$

where $\mathcal{A}$ is the functional configuration space of the gauge fields. This infinite-dimensional affine space is, in fact, a Hilbert space [16–18]. In general, we are not interested in the gauge fields as single objects but rather in sets of gauge fields that are related by a gauge transformation as these are the physically relevant objects. The set of all gauge fields A_μ that are related by a gauge transformation is called the *orbit* and it is a subspace of the configuration space $\mathcal{A}$. The non-trivial Riemannian structure [19, 20] of the orbits is responsible for the Gribov problem in non-Abelian gauge theories. For the time being, this should serve as an interesting remark as we have not talked talked about the Gribov problem/ambiguity, yet. On the orbit space, the action S_{inv} and the measure $\mathcal{D}A$ are constant, which makes the integral in Eq. (1.2.13) proportional to the volume of the gauge group. The integration over the volume of the group results in an infinite factor, which makes Eq. (1.2.13) ill-defined. An explicit way of seeing this is by computing the gauge field propagator. Taking only the quadratic form of the action, the partition function reduces to

$$\begin{aligned} Z_0(J) &= \int \mathcal{D}A e^{-\frac{1}{4}\int d^dx(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \int d^dx J^a_\mu A^a_\mu} \\ &= \int \mathcal{D}A e^{\frac{1}{2}\int d^dx d^dy A^a_\nu(x) K^{ab}_{\mu\nu}(x-y) A_\mu(y)^b + \int d^dx J^a_\mu A^a_\mu}, \end{aligned} \tag{1.2.14}$$

with $K^{ab}_{\mu\nu}(x-y) = \delta^{ab}\delta^d(x-y)(\partial^2 - \partial_\mu\partial_\nu)$. Eq.(1.2.14) is a Gaussian integral whose formal solution is given by

$$Z_0(J) = (\det K)^{-1}\int \mathcal{D}A e^{-\frac{1}{2}\int d^dx d^dy J^a_\mu (K^{ab}_{\mu\nu})^{-1} J^a_\nu}. \tag{1.2.15}$$

Unfortunately, the inverse of the operator $K_{\mu\nu}$ does not exist. This problem is linked to the existence of eigenvectors of $K_{\mu\nu}$ with zero eigenvalues. Taking $Y_\mu(x) = \partial_\mu \Lambda(x)$, with $\Lambda(x)$ being an arbitrary function of space-time, we get

$$\int dy[\delta^d(x-y)(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)]\partial_\mu\Lambda(y) = 0, \tag{1.2.16}$$

where the derivatives $\partial_\mu \equiv \frac{\partial}{\partial y^\mu}$ are performed with respect to the y coordinate. These zero-modes are in fact gauge transformations of $A_\mu = 0$

$$A_\mu^\Omega = \Omega A_\mu \Omega^{-1} - \frac{i}{g}(\partial_\mu \Omega)\Omega^{-1} = -\frac{i}{g}(\partial_\mu \Omega)\Omega^{-1} = \partial_\mu \Lambda \tag{1.2.17}$$

where we expanded Ω in a power series of the coupling constant g and kept only terms up to the first order. We can avoid the zero modes by eliminating the freedom of performing a gauge transformation. The mathematical procedure by which one accomplishes this is called *gauge fixing*. After gauge fixing, we are left with the space of physically distinguishable configurations known as the quotient space $\mathcal{A}/G$. The gauge fixing constraint is implemented by imposing that the function $F[A]$ on $\mathcal{A}$ vanishes $F[A] = 0$ (Fig.1.1.)

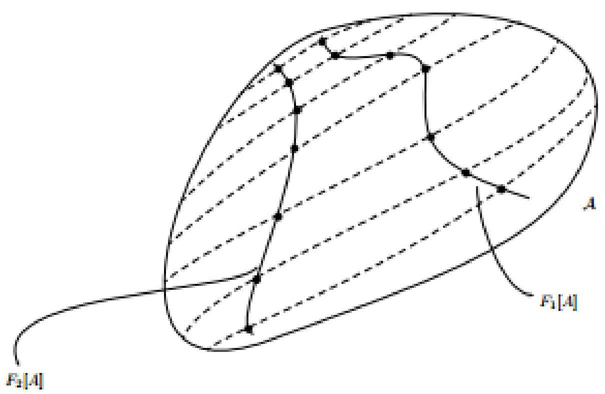


Figure 1.1: Configuration space of the gauge fields. Dashed lines represent all field configuration A^Ω that are connected to A by a gauge transformation. Gauge fields that do not lie on the same line are not connected by a gauge transformation. Functions $F_1[A]$ and $F_2[A]$ represent different "hypersurfaces" or gauge slices. Original figure from [21]

The gauge fixing constraint $F[A] = \{F^a(A_\mu, \partial_\mu A, ...)(x)\,,\, x \in M\}$ is the set of functions on the configuration space $\mathcal{A}$ that depend on the gauge field and its derivatives. For the Lorenz gauge condition, it is written as

$$F^a(A_\mu, \partial_\mu A, ..)(x) = \partial_\mu A_\mu^a(x). \tag{1.2.18}$$

To implement the gauge fixing constrained we separate the integral in Eq. (1.2.13) into the integration over the hypersurface (the gauge slice) defined by $F[A]$ and into the integration over the gauge orbits. For every field $A \in \mathcal{A}$, there is a gauge equivalent field A_F that satisfies $F[A_F] = 0$. This is equivalent to saying that there exists a group element $\Omega \in G$ such that it sends the field A_μ to the gauge slice where $F[A^\Omega] = 0$. This Ω depends on the original A and on the gauge slice F. We denote the group element Ω, that takes a given gauge field onto the gauge slice, as $\Omega_F[A]$. Therefore, for every $A \in \mathcal{A}$ we associate $\Omega_F[A] \in G$ such that $F[A_{\Omega_F[A]}] = 0$. We will assume that the group element $\Omega_F[A]$ is unique. In a nutshell, the Faddeev-Popov method is a way of re-expresses the integral in Eq. (1.2.13) in terms of $\Omega_F[A]$. The trick lies in the fact that we can write unity in a complicated way as

$$1 = \int_G \mathcal{D}[\Omega]\ \delta(F[A_{\Omega_F[A]}])\,|\det[F^{'}[A_{\Omega_F[A]}]]|, \tag{1.2.19}$$

where the measure $\mathcal{D}[\Omega]$ is defined as

$$\mathcal{D}[\Omega] = \prod_{x\in\mathcal{M}} d\mu(\Omega(x)), \tag{1.2.20}$$

with $d\mu$ being the Haar measure on the gauge group $SU(N)$ [22, 23] and the delta function

$$\delta(F[A_{\Omega_F[A]}]) = \prod_{x\in\mathcal{M}} \prod_{a}^{n^2-1} \delta(F^a[A_{\Omega_F[A]}](x)). \tag{1.2.21}$$

A nice property of the Haar measure $d\mu$ is its invariance under a gauge transformations

$$\begin{aligned}\int d\mu(\Omega(x))f(\Omega(x)) &= \int d\mu(\Omega(x))f(\Omega_0^{-1}\Omega(x)) = \int d\mu((\Omega_0^{-1}\Omega(x))f(\Omega(x)) \\ &= \int d\mu(\Omega(x))f(\Omega(x)),\end{aligned} \tag{1.2.22}$$

where we assumed that we have a left group action. A similar condition holds for the right action. For compact, simple, semi-simple and finite Lie groups the left and right invariant measures are equal [24]. The gauge invariance of $d\mu$ implies that an integral over the gauge group is gauge invariant, independent of the fact whether the function $f(\Omega(x))$ is gauge invariant or not.

The quantity $F^{'}[A_{\Omega_F[A]}]$ is called the Faddeev-Popov operator and its matrix elements are defined as

$$F^{'}[A^{\Omega}]^{ab}(x,y) \equiv \frac{\delta(F^a[A^{\Omega}](x))}{\delta\Lambda^b(y)}\Big|_{\Lambda=0}, \tag{1.2.23}$$

where the functional derivation is formally defined as $\delta F[\psi]/\delta\psi(y) \equiv \lim_{\epsilon\to 0}\frac{1}{\epsilon}(F[\psi'] - F[\psi])$ with $\psi' = \psi(x) + \epsilon\delta(x-y)$. To lighten up the notation, we omitted the fact that the group element depends on the gauge field A_μ and the function F. Inserting the unit element defined in Eq. (1.2.19) in the partition function Z yields

$$\begin{aligned}Z &= \int_{\mathcal{A}} \mathcal{D}Ae^{-S_{\text{inv}}[A]} \int_G \mathcal{D}[\Omega]\ \delta(F[A_{\Omega_F[A]}])\,|\det[F^{'}[A_{\Omega_F[A]}]], \\ &= \int_G \mathcal{D}[\Omega] \int_{\mathcal{A}} \mathcal{D}A\, e^{-S_{\text{inv}}[A]}\, \delta(F[A_{\Omega_F[A]}])\,|\det[F^{'}[A_{\Omega_F[A]}]]|.\end{aligned} \tag{1.2.24}$$

For a fixed $\Omega_F[A]$, we get

$$Z = \int_G \mathcal{D}[\Omega] \left(\int_{\mathcal{A}} \mathcal{D}[A_{\Omega_F[A]}]\, e^{-S_{\text{inv}}[A_{\Omega_F[A]}]}\, \delta(F[A_{\Omega_F[A]}])\,|\det[F^{'}[A_{\Omega_F[A]}]]|\right), \tag{1.2.25}$$

where we used the fact that the action S_{inv} and the measure $\mathcal{D}[A]$ are gauge invariant. The integral in bracket is independent of Ω. Hence,

$$Z = \int_G \mathcal{D}[\Omega] \left(\int_{\mathcal{A}} \mathcal{D}[A]\, e^{-S_{\text{inv}}[A]}\, \delta(F[A])\,|\det[F^{'}[A]]|\right), \tag{1.2.26}$$

where we replaced $A_{\Omega_F[A]}$ with A, since it was just a dummy variable. In the end, we were able to factor out the infinite volume factor $\int_G \mathcal{D}[\Omega]$, which made the integral in Eq. (1.2.13) ill-defined. The importance of this becomes apparent when we try to compute expectation values of physical observables f

$$\langle f\rangle = \frac{\int_{\mathcal{A}} \mathcal{D}A\, f[A]\, e^{-S_{\text{inv}}}}{\int_{\mathcal{A}} \mathcal{D}A\, e^{-S_{\text{inv}}}}. \tag{1.2.27}$$

Using the Faddeev-Popov trick, we can rewrite it as

$$\langle f \rangle = \frac{\int_{\mathcal{A}} \mathcal{D}A\, f[A]\, \delta(F[A])\, |\det[F'[A]]|\, e^{-S_{\rm inv}}}{\int_{\mathcal{A}} \mathcal{D}A\, \delta(F[A])\, |\det[F'[A]]|\, e^{-S_{\rm inv}}}, \tag{1.2.28}$$

where we canceled the integration over the gauge group from the numerator and denominator. Thus, we are left with a well-defined partition function. The functional determinant $|\det[F'[A]]|$ is called the Faddeev-Popov determinant and it is equal to the volume of the gauge slice [19, 20]. To proceed, we need to compute the Faddeev-Popov determinant for a specific choice of a gauge fixing function. The standard gauge fixing condition that one chooses at this step is the Lorenz-Feynman-Landau gauge condition

$$\partial_\mu A^a_\mu = \epsilon^a(x), \tag{1.2.29}$$

where $\epsilon : \mathcal{M} \to Lie(G)$ is a function from the space-time manifold M to the Lie algebra of the gauge group. For this gauge choice, the functions $F^a[A]$ are given by

$$F[A]^a(x) = \partial_\mu A^a_\mu - \epsilon^a(x), \tag{1.2.30}$$

Under an infinitesimal gauge transformation, the gauge field A_μ transforms as

$$A^a_\mu(x) \to A^a_\mu(x) - D^{ab}_\mu \Lambda^b(x), \tag{1.2.31}$$

where $D^{ab}_\mu = \delta^{ab}\partial_\mu - f^{abc}A^c_\mu$ is the covariant derivative in the adjoint representation. The matrix elements of the Faddeev-Popov operator $F'[A]$ are given by

$$\begin{aligned} F'[A]^{ab}(x,y) &= \frac{\delta(\partial_\mu(A^a_\mu - D^{ac}_\mu \Lambda^c(x)) - \epsilon^a(x))}{\delta \Lambda^b(y)}\Big|_{\Lambda=0}, \\ &= -[\partial_\mu D_\mu]^{ab}(x,y), \\ &= -[\partial^2 \delta^{ab} - g f^{abc}\partial_\mu A^c_\mu(x)]\, \delta^4(x-y). \end{aligned} \tag{1.2.32}$$

The partition function defined in Eq. (1.2.26), under the assumption that the Faddeev-Popov determinant does not change sign yields

$$Z = \int_{\mathcal{A}} \mathcal{D}[A]\, e^{-S_{\rm inv}[A]}\, \delta(\partial_\mu A^a(x) - \epsilon^a(x))\, \det[-[\partial_\mu D_\mu]^{ab}(x,y)]. \tag{1.2.33}$$

When expanded in a perturbation series, the Faddeev-Popov determinant leads to non-local interactions between the gauge fields A_μ. This is due to the fact that a functional determinant $\det N$ can be written as $\det N = \exp(\mathrm{Tr}[\log N])$. A local form is obtained by introducing anti-commuting, Grassmann variables c and $\bar{c}$

$$\det[-[\partial_\mu D_\mu]^{ab}(x,y)] = \int \mathcal{D}[c,\bar{c}]\, e^{-S_{\rm ghost}[c,\bar{c},A]} \tag{1.2.34}$$

with

$$\begin{aligned} S_{\rm ghost} &= \int d^4x\, d^4y\, \bar{c}^a(x)\, [\partial_\mu D_\mu]^{ab}(x,y)\, c^b(y) \\ &= \int dx\, \bar{c}^a(x)[\partial_\mu(\partial_\mu \delta^{ab} - f^{abc}A^c_\mu(x))]c^b(x) \\ &= \int dx\, [\bar{c}^a(x)\partial^2 c^b(x)\delta^{ab} + f^{abc}\partial_\mu \bar{c}^a(x) c^b(x) A^c_\mu(x)]. \end{aligned} \tag{1.2.35}$$

In the second line, using the delta function $\delta^{(4)}(x-y)$, we were able to eliminate the integration over y. It was shown in [25], that the correct hermiticity assignment for the ghost field is

$$(c(x)^a)^\dagger = c(x)^a \qquad (\bar{c}(x)^a)^\dagger = -\bar{c}(x)^a. \tag{1.2.36}$$

With this hermiticity assignment, the full Lagrangian including the ghost part is hermitian $\mathcal{L}^\dagger = \mathcal{L}$. This is easy to see since

$$(\bar{c}^a(x)\partial^2 c^b(x))^\dagger = (\partial^2 c^b(x))^\dagger \bar{c}^a(x)^\dagger = -\partial^2 c^b(x)\bar{c}^a(x) = \bar{c}^a(x)\partial^2 c^b(x) \tag{1.2.37}$$

and

$$\begin{aligned}(\partial_\mu \bar{c}^a(x) c^b(x) A^c_\mu(x))^\dagger = A^c_\mu(x)^\dagger c^b(x)^\dagger (\partial_\mu \bar{c}^a(x))^\dagger &= -A^c_\mu(x) c^b(x) \partial_\mu \bar{c}^a(x) \\ &= \partial_\mu \bar{c}^a(x) c^b(x) A^c_\mu(x),\end{aligned} \tag{1.2.38}$$

where we used the fact that Grassmann fields anti-commute. Alternatively, we can express the complex Grassmann fields in terms of two real independent real Euclidean Grassmann fields $v^a(x)$ and $u^a(x)$ as

$$c(x)^a = v^a(x) \qquad \bar{c}(x)^a = i u^a(x). \tag{1.2.39}$$

The Faddeev-Popov ghosts are fermionic scalar fields with values in the Lie algebra. They have the same quantum numbers as the gauge fields A_μ, since they belong to the same representation of the gauge group. The term "ghost" is due to Feynman as he used it to point out that these objects do not have a real physical meaning since they violate the standard spin-statistics relation. Because they do not represent physical particles, they do not appear as asymptotic states [26]. For abelian theories $f^{abc} = 0$, the ghost action reduces to

$$S^{\text{Abelian}}_{\text{ghost}} = \int dx\ \bar{c}^a(x)\partial^2 c^a(x), \tag{1.2.40}$$

which means that Faddeev-Popov ghosts completely decoupled from the theory. Inserting Eq. (1.2.34) into the partition function yields

$$Z = \int_{\mathcal{A}} \mathcal{D}[A, c, \bar{c}]\, \delta(\partial_\mu A^a(x) - \epsilon^a(x))\, e^{-S_{\text{inv}}[A] - S_{\text{ghost}}[c,\bar{c},A]}. \tag{1.2.41}$$

Since gauge invariant quantities do not depend on ϵ^a, we average over the auxiliary field by multiplying the partition function with a Gaussian factor $e^{\frac{1}{2\xi}\int d^4x\, \epsilon^a(x)\epsilon^a(x)}$ to bring it into the action

$$Z = \int \mathcal{D}[A, c, \bar{c}]\mathcal{D}[\epsilon]\delta(\partial_\mu A^a(x) - \epsilon^a(x))\, e^{\frac{1}{2\xi}\int d^4x\, \epsilon^a(x)\epsilon^a(x)}\, e^{-S_{\text{inv}}[A] - S_{\text{ghost}}[c,\bar{c},A]}, \tag{1.2.42}$$

where ξ is a constant parameter. Different values of the parameter ξ correspond to different gauges. The Feynman-'t Hooft gauge $\xi = 1$ is well-suited for computational purposes since in this gauge the Feynman rules have the simplest form. When dealing with bound state problems, it is advantageous to work in the Fried-Yennie gauge $\xi = 3$ [27] because many diagrams, which are infrared divergent in other gauges (Feynman or Landau), are infrared finite in the Fried-Yennie gauge [28]. The limit $\xi \to 0$ corresponds to the Landau (Lorenz) gauge condition, which has the advantage of being Lorenz invariant. Evaluating the integral over ϵ gives us the final result

$$Z = \int \mathcal{D}[A, c, \bar{c}] e^{-S_{\text{inv}} - S_{\text{ghost}} - S_{\text{gf}}} \tag{1.2.43}$$

with

$$S_{\text{gf}} = -\frac{1}{2\xi}\int d^4x\, \partial_\mu A^a_\mu \partial_\nu A^a_\nu. \tag{1.2.44}$$

The presence of the gauge fixing term S_{gf} eliminates the zero-modes from the gauge field propagator and therefore creates a well-defined partition functional. The generalization of the gauge fixing term for an arbitrary gauge fixing condition $F^a[A(x)]$ reads

$$S_{\text{gf}} = -\frac{1}{2\xi}\int d^4x\, F^a[A(x)]\, F^a[A(x)], \tag{1.2.45}$$

which leads to the generalized ghost field action

$$S_{\text{ghost}} = \int d^4x\, d^4y\, \bar{c}^a(x) \left[\frac{\delta F^a(A^\Omega_\mu(x))}{\delta \Lambda^b(y)}\right]\Bigg|_{\Lambda=0} c^b(y). \tag{1.2.46}$$

1.3 BRST

Despite the fact that the local gauge invariance of the action is lost, it would be desirable to maintain invariance in an infinitesimal form. It may seem impossible to have a symmetry transformation whose infinitesimal form leaves the action invariant, but its finite version does not. The problem lies in the fact that infinitesimal transformations can be extended to finite ones by repeating the former many times. We call a gauge transformation infinitesimal when expanding $\Omega = e^{ig\Lambda^a X^a}$ in a Taylor series and keeping only $\mathcal{O}(\Lambda)$ terms. Therefore, to have a symmetry whose infinitesimal form mirrors the original local gauge invariance but does not reproduce finite transformation, the relation $(\Lambda^a(x))^2 = 0$ needs to be an exact relation, not an approximate one [25]. We implement this nilpotency condition by regarding Λ^a's as differential forms, which constitute a finite-dimensional Grassmann algebra equipped with an exterior product [23, 29]. This symmetry is called BRST symmetry, and it was formulated by C. Becchi, A. Rouet, R. Stora [30] and I.V. Tyutin [31]. The idea is to use the Faddeev-Popov ghost fields $c^a(x)$ to construct a nilpotent operator δ which characterizes the BRST transformations. The BRST transformations of the fields are given by

$$\delta A^a_\mu = -D^{ab}_\mu c^b \tag{1.3.1}$$

$$\delta c^a = \frac{g}{2} f^{abc} c^b c^c \tag{1.3.2}$$

$$\delta \bar{c}^a = \epsilon^a \tag{1.3.3}$$

$$\delta \epsilon^a = 0. \tag{1.3.4}$$

From the action of the operator δ on the fields we can conclude that BRST symmetry is actually a supersymmetry, since it transforms bosonic fields into fermionic ones and vice versa. The gauge invariant part of the action S_{inv} is trivially invariant under BRST transformations since the ghost field does not affect the original gauge invariance. In general, anything that was gauge invariant will be automatically BRST invariant. The variation of the gauge fixing term for the Landau gauge condition reads

$$\delta \mathcal{L}_{\text{gf}} = \frac{1}{\xi}(\partial_\mu A^a_\mu)(\partial_\nu (D_\nu c)^a). \tag{1.3.5}$$

Under a BRST transformation, the ghost Lagrangian $\mathcal{L}_{\text{ghost}}$ transforms as

$$\delta \mathcal{L}_{\text{ghost}} = (\delta \bar{c}^a)\partial_\mu (D_\mu c)^a - \bar{c}^a \partial_\mu \delta (D_\mu c)^a, \tag{1.3.6}$$

where the variation of the covariant derivative is given by

$$\delta(D_\mu c)^a = \left[\frac{g}{2} f^{abc}\partial_\mu(c^b c^c) + g f^{abc}(\partial_\mu c^c)c^b - g^2 f^{abc} f^{cde} A^e_\mu c^d c^b - \frac{g^2}{2} f^{abc} f^{bde} A^c_\mu c^d c^e\right]. \quad (1.3.7)$$

The first two terms in Eq. (1.3.7) cancel by noting that we can rewrite the second term as

$$\begin{aligned}\frac{1}{2}(\partial_\mu c^{[c})c^{b]} &\equiv \frac{1}{2}(\partial_\mu c^c)c^b - \frac{1}{2}(\partial_\mu c^b)c^c \\ &= \frac{1}{2}(\partial_\mu c^c)c^b + \frac{1}{2}c^c(\partial_\mu c^b) \\ &= \frac{1}{2}\partial_\mu(c^c c^b) = -\frac{1}{2}\partial_\mu(c^b c^c). \end{aligned} \quad (1.3.8)$$

We can also anti-symmetrize the third term in Eq. (1.3.7) as

$$f^{abc} f^{cde} = \frac{1}{2}\left[f^{abc} f^{cde} - f^{adc} f^{cbe}\right] = -\frac{1}{2} f^{bdh} f^{hae}, \quad (1.3.9)$$

where in the last step we used the fact that the structure constants satisfy the Jacobi identity. Thus, we were able to show the cancellation of the last two terms in Eq. (1.3.7). The sum of the remaining terms in Eq. (1.3.5) and Eq. (1.3.6) vanish as they can be written as a total space-time derivative

$$\delta(\mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{gf}}) = \int_M \partial_\mu \left(\frac{1}{\xi}(\partial_\mu A^a_\mu)(D^{ab}_\mu c^b)\right), \quad (1.3.10)$$

which completes the proof that the gauge fixed Lagrangian is invariant under BRST transformations. The final property that we need to check is whether the BRST transformations are nilpotent as required. Acting on the gauge field A^a_μ twice we obtain

$$\begin{aligned}\delta(\delta A^a_\mu) &= \delta(-D^{ab}_\mu c^b) = -\delta(\partial_\mu c^a - g f^{abc} A^c_\mu c^b) \\ &= -(\delta^{ab}\partial_\mu - g f^{abc} A^c_\mu)\delta(c^b) + g f^{abc}(\delta A^c_\mu)c^b \\ &= -D^{ab}_\mu \delta(c^b) - g f^{abc}(\partial_\mu c^c)c^b + g^2 f^{abc} f^{cde} A^e_\mu c^d c^b \\ &= -D^{ab}_\mu \delta(c^b) - \frac{g}{2} f^{abc}\partial_\mu(c^c c^b) - \frac{1}{2} f^{bdh} f^{hae} A^e_\mu c^d c^b \\ &= -D^{ah}_\mu \delta(c^h) + (\delta^{ah}\partial_\mu - g f^{ahe} A^e_\mu)\frac{g}{2} f^{hbc} c^b c^c \\ &= -D^{ah}_\mu \left(\delta(c^h) + \frac{g}{2} f^{hbc} c^b c^c\right), \end{aligned} \quad (1.3.11)$$

which vanishes due to the variation of the ghost field in Eq. (1.3.2). Applying twice the BRST operator δ on the ghost field $c^a(x)$ yields

$$\delta^2 c^a = \frac{g^2}{6}(f^{bce} f^{eda} + f^{cde} f^{eba} + f^{dbe} f^{eca})c^b c^c c^d = 0, \quad (1.3.12)$$

due to the Jacobi identity. It is trivial to prove the nilpontency on $\bar{c}^a$ and ϵ^a. The presence of this new global symmetry is of the utmost importance since it allows us to prove two crucial properties of the Yang-Mills action: renormalizability and unitarity [33–35]. From the perspective of BRST symmetry, the Slavnov-Taylor-identities [36, 37] are interpreted as a consequence of a charge conservation [30, 31], which follows from the invariance of the quantum action $\Gamma[J] = -\ln Z[J]$ with respect to the BRST transformation. Imposing BRST symmetry on Γ one can solve the Slavnov-Taylor identities in an algebraic way and prove the renormalizability of the theory. With the use of the BRST charge $Q^{\mathbf{BRST}}$, which is defined as the space-time integral of the temporal component of the conserved current j^{BRST}_μ,

$$j^{\text{BRST}}_\mu = b^a(D_\mu c)^a - \partial_\mu b^a c^a + i\frac{g}{2} f^{abc}\partial_\mu \bar{c}^a c^b c^c, \quad (1.3.13)$$

we define physical states ψ as

$$Q^{\mathbf{BRST}}\psi = 0, \tag{1.3.14}$$

where ψ must be closed in the BRST cohomology; i.e., $\psi \neq Q^{\mathbf{BRST}}(\text{something})$ and it has to have vanishing ghost number. These conditions are known as the Kugo-Ojima conditions [38, 39]. A theory is unitary if all the physical states ψ have a positive norm and states belonging to the physical state space $\mathcal{H}_{phys}$ stay in $\mathcal{H}_{phys}$ after interacting with each other. Both of these conditions are proven by using the BRST symmetry.

Chapter 2

Global Issues in Gauge fixing

This chapter deals with global issues in gauge fixing, i.e., the Gribov ambiguity. We intend to recall limitations of the Faddeev-Popov procedure and complications associated with the non-perturbative regime of gauge theories. However, we would like to point out that this chapter is mainly a summary of the work done by Gribov [40], Zwanziger [56?] and Capri [68] and no original contributions have been made. Original work of this thesis is presented in the fourth and fifth chapter. Whenever it is possible, we will connect the ideas and methods from this chapter with the work presented in the chapters four and five.

2.1 Gribov Ambiguity

In the derivation of Eq. (1.2.43), we assumed that:

- The Landau gauge condition $\partial_\mu A^a_\mu = 0$ is *ideal*.
- The Faddeev-Popov determinant does not change sign.

A gauge fixing condition is called ideal if the orbits intersect the gauge fixing hypersurface only once. This implies that the gauge field, fulfilling dynamical equations for a given choice of gauge fixing, is unique. The second condition was imposed to rewrite the Faddeev-Popov (FP) determinant in terms of the ghost and anti-ghost fields. The negative sign of the FD determinant would not pose any problems since we could always reabsorb the negative sign in the normalization constant. Problems arise, however, when the FP determinant changes sign as it has to vanish at some point during the transition. In his seminal paper, V. Gribov [40] pointed out that for a given gauge field A^a_μ satisfying the gauge condition $\partial_\mu A_\mu$ there exist equivalent gauge fields A^Ω_μ fulfilling the same condition. The criterion for the existence of such copies reads

$$\begin{aligned} &\partial_\mu A_\mu = 0 \quad \& \quad \partial_\mu A^\Omega_\mu = 0 \\ \Longrightarrow \; &(\partial_\mu \Omega) A_\mu \Omega^{-1} + \Omega A_\mu \partial_\mu ((\Omega^{-1}) - \frac{i}{g}(\partial^2 \Omega)\Omega^{-1} - \frac{i}{g}(\partial_\mu \Omega)\partial_\mu(\Omega^{-1}) = 0, \end{aligned} \tag{2.1.1}$$

which for an infinitesimal transformation reduces to

$$-\partial_\mu D_\mu \Lambda = 0, \tag{2.1.2}$$

where $D_\mu = \partial_\mu + ig[., A_\mu]$. Therefore, we conclude that the existence of infinitesimal copies can be associated with the existence of the zero-modes of the Faddeev-Popov operator.

Generally, there will exist equivalent gauge potentials A_μ and A^Ω_μ satisfying the same gauge fixing conditions. These copies are called Gribov copies named after V. Gribov who pointed out their existence. We can study Eq. (2.1.2) in following form of an eigenvalue equation

$$-\partial_\mu(\partial_\mu \xi + ig[\xi, A_\mu]) = \lambda(A)\xi, \tag{2.1.3}$$

where we replaced the parameter Λ by the field ξ. In this form, we can interpret Eq. (**??**) as a Schroedinger equation where the gauge field A_μ plays the role of the potential [**?**]. For vanishing gauge field $A_\mu = 0$, the eigenvalue equation

$$-\partial^2 \xi = \lambda(0)\xi, \tag{2.1.4}$$

has only[1] positive eigenvalues λ . For small perturbations around $A_\mu = 0$, no Gribov copies will be present. Due to this fact, we are able to get away with the assumptions about uniqueness and positivity in the derivation of the partition function. As long as we stay within the perturbative regime, our assumptions will remain true (Fig.2.1)

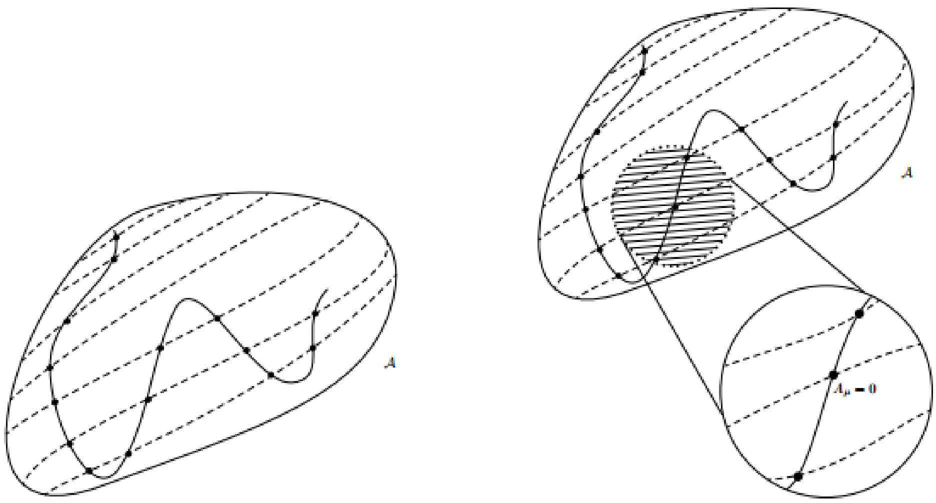


Figure 2.1: Left side: The gauge slice intersects gauge orbits more than once. Right side : Locally, it is always possible to find an ideal gauge fixing around the point $A_\mu = 0$. Original figure from [21]

However, as we go to larger values of the gauge field, the negative contribution from the second part of Eq. (2.1.3) will dominate. For large enough values of the gauge field, one of the eigenvalues say $\lambda_1(A)$, will vanish and as A_μ continues increasing further, it will become negative. For even bigger values of the gauge field, the second eigenvalues say $\lambda_2(A)$, will vanish and become negative as A_μ continues increasing [40, 41]. This pattern continues as the magnitude of the gauge field keeps increasing. With this in mind, we partition the configuration space $\mathcal{A}$ into regions (Fig. 2.2) as Gribov suggested, where C_0 labels the region where the Faddeev-Popov operator has only positive eigenvalues, region C_1 where the Faddeev-Popov operator has one negative eigenvalue and so on.

[1]Apart from the trivial null space solutions when ξ is constant.

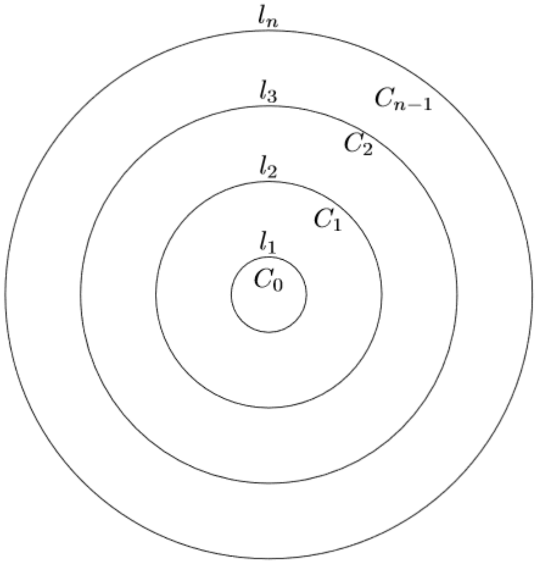


Figure 2.2: Partition of the configuration space in Gribov regions.

The boundary of the region C_0, which we labeled as l_1, contains the first zero eigenvalues of the Faddeev-Popov operator. It means that there exists a normalized [2] zero mode ξ which satisfies

$$\partial_\mu(\partial_\mu \xi + ig[\xi, A_\mu]) = 0. \tag{2.1.5}$$

The region C_0 is called the first Gribov region and is defined as the set of all transverse gauge fields for which the Faddeev-Popov operator is positive definite

$$C_0 = \{A_\mu, \ \partial_\mu A_\mu = 0 \ | \ -\partial_\mu D_\mu > 0 \ \}. \tag{2.1.6}$$

In the first Gribov region, the Landau gauge condition $\partial_\mu A_\mu$ is free of copies at least from the infinitesimal ones. In the next region C_1, the Faddeev-Popov operator has one negative eigenvalue and at the boundary l_2, a second zero eigenvalue reappears [40, 41]. The pattern indicated here appears pretty obvious. In the region C_N, the Faddeev-Popov operator possesses $N-1$ negative eigenvalues, and at the boundary $l+1$, the $N-th$ zero eigenvalue appears.

An alternative formulation of the first Gribov region was worked out in [42–44] by minimizing the L^2 norm of the gauge field along the gauge orbit

$$||A_\mu^\Omega||^2 = \mathrm{Tr}\int dx\, A_\mu^\Omega(x) A_\mu^\Omega(x) = \frac{1}{2}\int dx\, A_\mu^{a,\Omega} A_\mu^{a,\Omega}. \tag{2.1.7}$$

A gauge field A_μ^a will minimizes the functional in Eq.(2.1.7) if it is transverse $\partial_\mu A_\mu^a = 0$

$$\begin{aligned}
\delta||A_\mu||^2 &= \delta\left(\frac{1}{2}\int dx\ A_\mu^a(x) A_\mu^a(x)\right) = \int dx\ (\delta A_\mu^a(x)) A_\mu^a(x) = -\int\ dx\, (D_\mu^{ab}\Lambda^a(x)) A_\mu^a(x) \\
&= -\int\ dx\left(\partial_\mu \delta^{ab}\Lambda^a(x) - g f^{abc} A_\mu^c(x)\Lambda^a(x)\right) A_\mu^a(x) = -\int dx\, (\partial_\mu \Lambda^a(x)) A_\mu^a(x) \\
&= \int\ dx\, \Lambda^a(\partial_\mu A_\mu) = 0,
\end{aligned} \tag{2.1.8}$$

and the Faddeev-Popov operator $-\partial_\mu D_\mu$ is positive

[2] Only normalized solutions can be used to construct Gribov copies.

$$\delta^2||A_\mu||^2 = \delta\left(-\int dx(\partial_\mu\Lambda^a(x))A_\mu^a(x)\right) = \int dx\ \partial_\mu\Lambda(x)^a D_\mu^{ab}\Lambda^b(x)$$
$$= \int dx\ \Lambda^a(x)(-\partial_\mu D_\mu^{ab})(A)\,\Lambda^b(x) > 0. \tag{2.1.9}$$

Therefore, we conclude that the first Gribov region is the set of local minima of the functional in Eq. (2.1.7). Zwanziger and Dell'Antonio proved that every gauge orbit crosses the first Gribov region [45], which is an important result because it tells us that C_0 contains all the physical configurations. It was also proven in [45] that the first Gribov region is convex and bounded in every direction. The former states that given two gauge fields A_μ and B_μ in the first Gribov region then the field $C_\mu = c_1A_\mu + c_2B_\mu$, where $c_1 + c_2 = 1$, is also inside C_0. This is easily proven by using the fact that the Faddeev-Popov operator is linear in the gauge fields

$$\mathcal{F}^{ab}(c_1A_\mu + c_2B_\mu) = c_1\mathcal{F}^{ab}(A_\mu) + c_2\mathcal{F}^{ab}(B_\mu), \tag{2.1.10}$$

where we labeled the Faddeev-Popov operator as $\mathcal{F}^{ab}[A_\mu]$ since the discussion is valid for an arbitrary choice of the gauge fixing condition. To prove the fact that the first Gribov region is bounded in every direction, we recall that

$$\mathcal{F}^{ab}(A_\mu) = \mathcal{F}_0^{ab} + \mathcal{F}_1^{ab}(A_\mu), \qquad \mathcal{F}_0^{ab} = -\partial^2\delta^{ab} \quad \& \quad \mathcal{F}_1^{ab}(A_\mu) = gf^{abc}\partial_\mu A_\mu^c, \tag{2.1.11}$$

where $\mathcal{F}_0^{ab}$ has only positive eigenvalues and $\mathcal{F}_1^{ab}(A_\mu)$ is an skew-symmetric matrix whose trace is zero. Since $\mathcal{F}_1^{ab}(A_\mu)$ is traceless, the sum of all of its eigenvalues is zero. For non-vanishing gauge field, there exists at least one eigenvector ξ^a with a negative eigenvalue

$$\int dx\,\xi^a(x)\mathcal{F}_2(A_\mu)^{ab}\xi(x)^b = \lambda < 0. \tag{2.1.12}$$

Due to the linearity of the Faddeev-Popov operator, any eigenvector of $\mathcal{F}_1^{ab}(A_\mu)$ is also an eigenvector of $\mathcal{F}_1^{ab}(cA_\mu)$ with the eigenvalue $c\lambda$. With this in mind, we have

$$\int dx\,\xi^a(x)\Big(\mathcal{F}(cA_\mu)^{ab}\Big)\xi(x)^b = -\int dx\,\xi^a(x)(\partial^2)\xi(x)^b + c\lambda. \tag{2.1.13}$$

For large enough values of c the Faddeev-Popov operator will no longer be positive indicating that we left the first Gribov region. Throughout this proof, we assumed that the eigenvector ξ^a has a unit norm. From these arguments one concludes that C_0 is bounded in every direction.

2.2 Gribov pendulum

In this section, we are going to investigate the existence of Gribov copies for the simplest possible case. We will be working in 3-dimensional space with the gauge group $SU(2)$ and the gauge fixing condition ∂_iA_i. Furthermore, we are going to restriction ourselves to spherically symmetric gauge fields A_i, $i = 1, 2, 3$, i.e., fields depending only on the unit vector $n_i = x_i/r$, with $r = \sqrt{x_ix_i}$. In his paper [40], Gribov wrote down the most general expression for a spherically symmetric gauge field to be

$$A_i(x) = f_1(r)\frac{\partial\hat{n}}{\partial x_i} + f_2(r)\hat{n}\frac{\partial\hat{n}}{\partial x_i} + f_3(r)\hat{n}n_i, \tag{2.2.1}$$

where $\hat{n} = in_a\sigma_a$. The σ_a are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.2.2}$$

satisfying the following identity

$$\hat{n}^2 = -n_a n_b \sigma_a \sigma_b = -n_a n_b (\delta_{ab} - i\epsilon_{abc}\sigma_c) = -n_a n_a = -1. \tag{2.2.3}$$

We recall from the last section that a gauge field A_i will have Gribov Copies if the following condition is satisfied

$$\frac{\partial A_i^\Omega}{\partial x_i} = \frac{\partial A_i}{\partial x_i}. \tag{2.2.4}$$

The right-hand side of Eq. (2.2.4) reads

$$\begin{aligned}
\frac{\partial A_i}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(f_1(r)\frac{\partial \hat{n}}{\partial x_i} + f_2(r)\hat{n}\frac{\partial \hat{n}}{\partial x_i} + f_3(r)\hat{n}n_i\right) \\
&= f_1'(r)n_i\frac{\partial \hat{n}}{\partial x_i} + f_1(r)\frac{\partial^2 \hat{n}}{\partial x_i^2} + f_2'(r)n_i\hat{n}\frac{\partial \hat{n}}{\partial x_i} + f_2(r)\frac{\partial}{\partial x_i}\left(\hat{n}\frac{\partial \hat{n}}{\partial x_i}\right) + f_3'(r)\hat{n}n_i^2 + f_3(r)\frac{\partial}{\partial x_i}(\hat{n}n_i) \\
&= f_1(r)\frac{\partial^2 \hat{n}}{\partial x_i^2} + f_2(r)\frac{\partial}{\partial x_i}\left(\hat{n}\frac{\partial \hat{n}}{\partial x_i}\right) + f_3'(r)\hat{n} + f_3(r)\hat{n}\frac{\partial n_i}{\partial x_i},
\end{aligned} \tag{2.2.5}$$

where we used the relation $n_i \frac{\partial \hat{n}}{\partial x_i} = 0$. Furthermore, we have that

$$\begin{aligned}
\frac{\partial^2 \hat{n}}{\partial x_i^2} &= \frac{\partial}{\partial x_i}\left(\frac{i}{r}(\sigma_i - (\vec{\sigma}\cdot\vec{n})n_i)\right) = -i\frac{\sigma_i n_i}{r^2} - i\frac{((\vec{\sigma}\cdot\vec{n})n_i)'r - (\vec{\sigma}\cdot\vec{n})n_i r'}{r^2}, \\
&= -\frac{\hat{n}}{r^2} - \frac{i}{r^2}\left(\left(\frac{\partial}{\partial x_i}(\vec{\sigma}\cdot\vec{n})\right)n_i\, r + (\vec{\sigma}\cdot\vec{n})\frac{\partial n_i}{\partial x_i}r - (\vec{\sigma}\cdot\vec{n})n_i^2\right) \\
&= -\frac{\hat{n}}{r^2} - i\frac{(\vec{\sigma}\cdot\vec{n})}{r^2} \\
&= -2\frac{\hat{n}}{r^2},
\end{aligned} \tag{2.2.6}$$

and

$$\left(\frac{\partial \hat{n}}{\partial x_i}\right)^2 = -\frac{1}{r^2}(\sigma_i - (\vec{\sigma}\cdot\vec{n})n_i)(\sigma_i - (\vec{\sigma}\cdot\vec{n})n_i) = -\frac{2}{r^2}, \tag{2.2.7}$$

which is straightforward to obtain by multiplying terms out and using the identity

$$(\vec{\sigma}\cdot\vec{n})\sigma_i = -\sigma_i(\vec{\sigma}\cdot\vec{n}) + 2n_i. \tag{2.2.8}$$

Inserting Eq. (2.2.6) and (2.2.7) into Eq.(2.2.5) leads to the following expression

$$\frac{\partial A_i}{\partial x_i} = \hat{n}\left[\frac{\partial f_3(r)}{\partial r} + \frac{2}{r}f_3(r) - \frac{2}{r^2}f_1(r)\right]. \tag{2.2.9}$$

To evaluate the left-hand-side of Eq. (2.2.4), we recall that under a spherical symmetric gauge transformation

$$\Omega = e^{\frac{1}{2}\alpha(r)\hat{n}} = \cos\left(\frac{\alpha(r)}{2}\right) + \hat{n}\sin\left(\frac{\alpha(r)}{2}\right), \tag{2.2.10}$$

the gauge field transform as

$$A_i^\Omega = \Omega^{-1} A_i \Omega + \Omega^{-1}\partial_i \Omega, \tag{2.2.11}$$

where we rewrote the gauge transformation in a slightly different way to get rid of the extra factors of $\frac{i}{g}$. Inserting the $SU(2)$ group element yields

$$\begin{aligned} A_i^\Omega =& \left(\cos\frac{\alpha}{2} - \hat{n}\sin\frac{\alpha}{2}\right)\left(f_1(r)\frac{\partial\hat{n}}{\partial x_i} + f_2(r)\hat{n}\frac{\partial\hat{n}}{\partial x_i} + f_3(r)\hat{n}n_i\right)\left(\cos\frac{\alpha}{2} + \hat{n}\sin\frac{\alpha}{2}\right) \\ &+ \left(\cos\frac{\alpha}{2} - \hat{n}\sin\frac{\alpha}{2}\right)\left(\left(\hat{n}\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\frac{\alpha'(r)n_i}{2} + \sin\frac{\alpha}{2}\frac{\partial\hat{n}}{\partial x_i}\right), \end{aligned}$$

with $\frac{\partial\alpha(r)}{\partial x_i} = \frac{\partial\alpha(r)}{\partial r}\frac{\partial r}{\partial x_i}$ and $\frac{\partial r}{\partial x_i} = n_i$. Multiplying everything out we obtain

$$A_i^\Omega = \tilde{f}_1(r)\frac{\partial\hat{n}}{\partial x_i} + \tilde{f}_2(r)\hat{n}\frac{\partial\hat{n}}{\partial x_i} + \tilde{f}_3(r)\hat{n}n_i \tag{2.2.12}$$

whereby

$$\begin{aligned} \tilde{f}_1(r) &= \cos\alpha f_1(r) + \sin\alpha\left(f_2(r) + \frac{1}{2}\right) \\ \tilde{f}_2(r) &= \left(f_2(r) + \frac{1}{2}\right)\cos\alpha - f_1(r)\sin\alpha - \frac{1}{2} \\ \tilde{f}_3(r) &= f_3(r) + \frac{1}{2}\frac{\partial\alpha(r)}{\partial r}. \end{aligned} \tag{2.2.13}$$

Identities that have been used in the derivation of the above results are

$$\begin{aligned} \hat{n}\frac{\partial\hat{n}}{\partial x_i} + \frac{\partial\hat{n}}{\partial x_i}\hat{n} &= 0, \\ \hat{n}\frac{\partial\hat{n}}{\partial x_i}\hat{n} - \frac{\partial\hat{n}}{\partial x_i} &= 0. \end{aligned} \tag{2.2.14}$$

Now, for the left-hand side of Eq. (2.2.4), we insert $\tilde{f}_i(r)$. Thus,

$$\begin{aligned} \frac{\partial A_i^\Omega}{\partial x_i} &= \hat{n}\left[\frac{\partial\tilde{f}_3(r)}{\partial r} + \frac{2}{r}\tilde{f}_3(r) - \frac{2}{r^2}\tilde{f}_1(r)\right] \\ &= \hat{n}\left[\frac{\partial f_3(r)}{\partial r} + \frac{1}{2}\frac{\partial^2\alpha}{\partial r^2} + \frac{2}{r}f_3(r) + \frac{1}{r}\frac{\partial\alpha}{\partial r} - \frac{2}{r^2}\left(\cos\alpha f_1(r) + \sin\alpha\left(f_2(r) + \frac{1}{2}\right)\right)\right]. \end{aligned} \tag{2.2.15}$$

The condition for existence of Gribov copies is equivalent to the equation

$$\alpha''(r) + \frac{2}{r}\alpha' - \frac{4}{r^2}\left(\sin\alpha\left(f_2(r) + \frac{1}{2}\right) + f_1(r)(\cos\alpha - 1)\right) = 0, \tag{2.2.16}$$

where $\alpha' \equiv \frac{\partial\alpha}{\partial r}$. It is useful to introduce a new variable $\tau = \ln(r)$ such that Eq. (2.2.16) can be rewritten as

$$\ddot{\alpha} + \dot{\alpha} - (2 + 4f_2)\sin\alpha + 4f_1(1 - \cos\alpha) = 0. \tag{2.2.17}$$

Equation (2.2.17) is called *the Gribov pendulum*. Due to its non-linear nature, the Gribov pendulum equation does not have any closed analytic solutions [47, 48]. Since there are no solutions to the general problem one is forced to simplify the situation by considering approximative cases with appropriate boundary conditions. One way of achieving this is by restricting oneself to transverse gauge fields

$$A_i = \frac{i}{r^2}\epsilon_{ijk}x_j\sigma_k f(r), \tag{2.2.18}$$

where $f_1(r) = f_3(r) = 0$ and $f_2(r) = f(r)$. The transformed gauge field A_i^{Ω} is given by

$$\tilde{A}_i = \left(f(r) + \frac{1}{2}\right)\sin\alpha\frac{\partial\hat{n}}{\partial x_i} + \left(f(r) + \frac{1}{2}(\cos\alpha - 1)\right)\hat{n}\frac{\partial\hat{n}}{\partial x_i} + \frac{1}{2}\alpha'(r)\hat{n}n_i, \tag{2.2.19}$$

and for this case, the Gribov pendulum equation reduces to

$$\ddot{\alpha} + \dot{\alpha} - (2 + 4f(\tau))\sin\alpha = 0, \tag{2.2.20}$$

where $f(\tau)$ is a smooth function. At this point, one imposes so-called strong and weak boundary conditions [49–52] on the gauge fields to find α's that are compatible with the boundary conditions and are solutions of Eq. (2.2.20). For an in depth discussion on this topic and the various solutions one obtains we refer the reader to [41].

Consider the gauge parameter to be infinitesimal for all the values of τ, then Eq. (2.2.20) reduces to

$$\ddot{\alpha}(\tau) + \dot{\alpha}(\tau) - (2 + 4f(\tau))\,\alpha(\tau) = 0. \tag{2.2.21}$$

If we take $\xi = \alpha(r)\hat{n}$ and A_i to be transverse as defined in Eq. (2.3.17) then Gribov's pendulum in its infinitesimal form is equivalent to the following eigenvalue equation

$$\partial_i D_i \xi = 0, \tag{2.2.22}$$

with $D_i = \partial_i + A_i$. From the previous chapter, we know that we can interpret Eq. (2.2.22) as a zero eigenvalue Schroedinger equation with A_i playing the role of the potential. We also know that for a particularly large value of A_i solution to Eq. (2.2.22) exist. In other words, if the transverse field A_i is located ona the boundaries of the first Gribov region, then Eq. (2.2.21) is satisfied. The importance of Eq. (2.2.21) lies in the fact that by solving it we not only prove the existence of Gribov copies, but also find the location of the boundary of the first Gribov region in configuration space.

2.3 Restriction to the first Gribov region

To avoid copies of the gauge field A_μ, Gribov proposed to restrict the domain of integration to the first Gribov region by introducing a new factor $\mathcal{V}(C_0)$ into the partition function

$$\mathcal{Z} = \mathcal{N}\int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\ \mathcal{V}(C_0)\ \delta(\partial_\mu A_\mu)e^{-S_{\text{inv}} - \int d^4x\bar{c}^a\partial_\mu D_\mu^{ab}c^b}. \tag{2.3.1}$$

Inside the first Gribov region C_0, the Faddeev-Popov operator by definition is positive. The ghost propagator, being expressed by the inverse of the Faddeev-Popov operator, thus is expected to be non-singular inside the first Gribov region. An equivalent statement would be that the region where the ghost propagator is non-singular is precisely the first Gribov region C_0. To find $\mathcal{V}(C_0)$, we have to investigate the pole structure of the ghost propagator and try to find a region where it is non-singular. From the form of the renormalized connected ghost propagator ($\mathcal{V}(C_0) = 1$),

$$\langle \bar{c}^a(x) c^b(y) \rangle_c = \delta^{ab} \int \frac{d^4q}{(2\pi)^4} \mathcal{G}(q) e^{iq(x-y)}, \tag{2.3.2}$$

with

$$\mathcal{G}(q) = \underbrace{\frac{1}{q^2}}_{\mathcal{G}_1} \underbrace{\frac{1}{(1 - \frac{11g^2N}{48\pi^2} \log \frac{\Lambda^2}{q^2})^{\frac{9}{44}}}}_{\mathcal{G}_2}, \tag{2.3.3}$$

we see that it has two poles

$$q^2 = 0 \implies \mathcal{G}_1 \to \infty \qquad \& \qquad q_p^2 = \Lambda^2 e^{-\frac{1}{g^2}\frac{48\pi^2}{11N}} \implies \mathcal{G}_2 \to \infty, \tag{2.3.4}$$

where Λ is the UV-cutoff and N is the Casimir of the adjoint representation of the group $SU(N)$

$$f^{acd} f^{bcd} = N \delta^{ab}. \tag{2.3.5}$$

Due to the presence of $\mathcal{V}(C_0)$ in the partition function, the ghost propagator $\mathcal{G}(q)$ can only have singularities at vanishing momenta, since below $q^2 < \Lambda^2 e^{-\frac{1}{g^2}\frac{48\pi^2}{11N}}$, $\mathcal{G}_2$ becomes complex which is an indicator that we left the region C_0. Thus we are left with the singularity at $q^2 = 0$ which indicates that we are on the boundary l_1 of the first Gribov region C_0 [40].

To determine $\mathcal{V}(C_0)$, we are going to compute the color singlet ghost propagator and demand that no singularities for non-vanishing momenta exist. Demanding that the ghost propagator has no poles for non-vanishing momenta is called Gribov's *no-pole* condition. The starting point for implementing Gribov's no-pole condition is the connected, color singlet, ghost two-point function

$$\begin{aligned} \sum_{ab} \frac{\delta^{ab}}{N^2-1} \langle \bar{c}^a(x) c^b(y) \rangle_c &= \mathcal{N} \int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \, \frac{\bar{c}^a(x) c^a(y)}{N^2-1} \, \delta(\partial_\mu A_\mu) \, e^{-(S_{\text{inv}} + S_{ghost})} \\ &= \mathcal{N} \int \mathcal{D}A \, \delta(\partial_\mu A_\mu) \, e^{-S_{\text{inv}}} \mathcal{G}(x,y;A) \end{aligned} \tag{2.3.6}$$

with

$$\mathcal{G}(x,y;A) = \int \mathcal{D}c \mathcal{D}\bar{c} \, \frac{\bar{c}^a(x) c^a(y)}{N^2-1} e^{-S_{\text{ghost}}}. \tag{2.3.7}$$

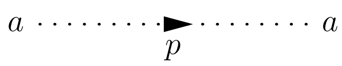
The momentum space representation of Eq. (2.3.7) reads

$$\mathcal{G}(p;A) = \int d^4x d^4y e^{ip(x-y)} \mathcal{G}(x,y;A), \tag{2.3.8}$$

where A_μ is treated as a external classical field. It is important to note that treating the gauge fields as external fields is done up to the second order in perturbation theory. Expanding Eq. (2.3.8) up to second order in perturbation theory yields

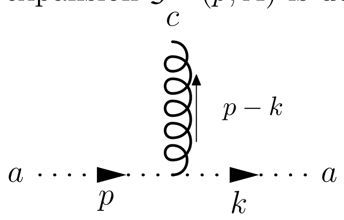
$$\mathcal{G}(p;A) = \mathcal{G}^{(0)}(p;A) + \mathcal{G}^{(1)}(p;A) + \mathcal{G}^{(2)}(p;A), \tag{2.3.9}$$

with $\mathcal{G}^{(0)}(p;A)$ being just the free ghost propagator



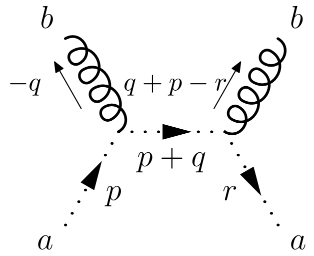
$$= \frac{1}{p^2}. \tag{2.3.10}$$

The first order expansion $\mathcal{G}^{(1)}(p;A)$ is determined by the following Feynman diagram



$$= g\frac{1}{p^2}\frac{1}{q^2}f^{aca}ik_\mu A^c_\mu(p-k) = 0, \tag{2.3.11}$$

which implies that the first order expansion does not contribute to the diagonal element. The second order expansion term $\mathcal{G}^{(2)}(p;A)$ yields



$$= -g^2\frac{N}{N^2-1}\int\frac{d^4q}{(2\pi)^4}\frac{1}{p^2}\frac{1}{r^2}\frac{(p+q)_\mu r_\nu}{(q+p)^2}A^b_\mu(-q)A^b_\nu(q-r+p), \tag{2.3.12}$$

which can be rewritten as

$$\mathcal{G}^{(2)}(p;A) = g^2\frac{N}{N^2-1}\frac{1}{p^4}\int\frac{d^4q}{(2\pi)^4}\frac{(p-q)_\mu p_\nu}{(p-q)^2}A^b_\mu(-q)A^b_\nu(q), \tag{2.3.13}$$

where we used the fact that the incoming momentum p should be equal to the outcoming momentum r. Therefore, the ghost propagator up to the second order in perturbation theory yields

$$\begin{aligned}\mathcal{G}(p;A) &= \frac{1}{p^2}\left(1+\frac{g^2}{V}\frac{N}{N^2-1}\frac{1}{p^2}\int\frac{d^4q}{(2\pi)^4}\frac{(p-q)_\mu p_\nu}{(p-q)^2}A^b_\mu(-q)A^b_\nu(q)\right)\\ &= \frac{1}{p^2}(1+\sigma(p;A)) \underbrace{\approx}_{perturb.approx} \frac{1}{p^2}\frac{1}{(1-\sigma(p;A))}\end{aligned} \tag{2.3.14}$$

with

$$\sigma(p;A) = \frac{g^2}{V}\frac{N}{N^2-1}\frac{1}{p^2}\int\frac{d^4q}{(2\pi)^4}\frac{(p-q)_\mu p_\nu}{(p-q)^2}A^b_\mu(-q)A^b_\nu(q). \tag{2.3.15}$$

In the definition of $\sigma(p;A)$, we introduced an infinite volume factor $1/V$ to preserve the right dimensionality. If the inequality

$$\sigma(p;A) < 1 \tag{2.3.16}$$

holds, then the ghost propagator is finite for non-vanishing momenta. By recalling that in the Landau gauge condition, the gauge fields are transverse $q_\mu A_\mu(p) = 0$, implies that

$$A^a_\mu(-q)A^a_\nu(q) = \omega(A)\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right), \tag{2.3.17}$$

where we determine $\omega(A)$ by contracting both sides with $\delta_{\mu\nu}$

$$\omega(A) = \frac{1}{3}A^a_\rho(-q)A^a_\rho(q) \tag{2.3.18}$$

Inserting Eq. (2.3.17) into Eq. (2.3.15) and using the transversality condition yields

$$\sigma(p;A) = \frac{g^2}{3V}\frac{N}{N^2-1}\frac{p_\mu p_\nu}{p^2}\int \frac{d^4q}{(2\pi)^4}\frac{A^a_\rho(-q)A^a_\rho(q)}{(p-q)^2}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right). \tag{2.3.19}$$

Since $\sigma(p;A)$ reaches its maximum value at $p = 0$, the no-pole condition reduces to

$$\sigma(0;A) < 1, \tag{2.3.20}$$

with

$$\sigma(0;A) = \frac{g^2}{3V}\frac{N}{N^2-1}\lim_{p^2\to 0}\frac{p_\mu p_\nu}{p^2}\left(\int \frac{d^4q}{(2\pi)^4}\frac{A^a_\rho(-q)A^a_\rho(q)}{q^2}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\right). \tag{2.3.21}$$

Due to Lorentz invariance[3] the integral in Eq. (2.3.21) has to be of the form

$$\int \frac{d^4q}{(2\pi)^4}\frac{A^a_\rho(-q)A^a_\rho(q)}{q^2}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) = \mathcal{J}\delta_{\mu\nu}, \tag{2.3.22}$$

where $\mathcal{J}$ is computed by contracting both sides by $\delta_{\mu\nu}$

$$\mathcal{J} = \frac{3}{4}\int \frac{d^4q}{(2\pi)^4}\frac{A^a_\rho(-q)A^a_\rho(q)}{q^2}. \tag{2.3.23}$$

By inserting Eq. (2.3.22) in Eq. (2.3.21) gives us the expression for $\sigma(0;A)$ in the Landau gauge

$$\begin{aligned}\sigma(0;A) &= \frac{g^2}{3V}\frac{N}{N^2-1}\lim_{p^2\to 0}\frac{p_\mu p_\nu}{p^2}\delta_{\mu\nu}\mathcal{J} \\ &= \frac{g^2}{4V}\frac{N}{N^2-1}\int \frac{d^4q}{(2\pi)^4}\frac{A^a_\rho(-q)A^a_\rho(q)}{q^2}.\end{aligned} \tag{2.3.24}$$

We implement the condition $\sigma(0;A) < 1$ in the path integral by writing the factor $\mathcal{V}(C_0)$ as

$$\mathcal{V}(C_0) = \Theta(1 - \sigma(0;A)), \tag{2.3.25}$$

where $\Theta(x)$ is the Heaviside step-function. Rewriting the step-function in its integral form

$$\mathcal{V}(C_0) = \int_{-i\infty+\epsilon}^{i\infty+\epsilon}\frac{d\beta}{(2\pi i\beta)}e^{\beta(1-\sigma(0;A))} \tag{2.3.26}$$

and inserting in the path integral yields

$$\mathcal{Z} = \mathcal{N}\int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\int \frac{d\beta}{(2\pi i\beta)}e^{\beta(1-\sigma(0;A))}e^{-S_{\text{inv}}-S_{gf}}. \tag{2.3.27}$$

[3]Lorentz invariance imposes that the solution to the integral has to have the same Lorentz structure as the integrand. Since the variable q is being integrated out the only possible object that we can write down is the metric tensor itself times a factor that is Lorentz invariant.

The presence of the new factor in the path integral leads to modification of the gauge propagator as $\sigma(0;p)$ is quadratic in the gauge fields. We also note that the term $\beta\sigma(0;A)$ is non-local as it depends on the momentum. Taking only the quadratic part in the gauge fields A_μ gives us

$$\mathcal{Z}_0[J] = \mathcal{N}\int \frac{d\beta}{(2\pi i\beta)}\int \mathcal{D}A \exp\left[-\frac{1}{2}\int \frac{d^4q}{(2\pi)^4}A^a_\mu(q)K^{ab}_{\mu\nu}(q)A^b_\nu(-q) + \int \frac{d^4q}{(2\pi)^4}A^a_\mu(q)J^a_\mu(-q)\right], \tag{2.3.28}$$

with

$$K^{ab}_{\mu\nu}(q) = \delta^{ab}\left(q_\mu q_\nu\left(\frac{1}{\alpha}-1\right) + q^2\delta_{\mu\nu} + \frac{\beta N g^2}{2V(N^2-1)}\frac{1}{q^2}\delta_{\mu\nu}\right), \tag{2.3.29}$$

where at the end of the calculation, we have to take the limit $\alpha \to 0$ to recover the Landau gauge condition [41, 56]. Formally, the solution to the gluon two-point function reads

$$\begin{aligned}\langle A^a_\mu(q)A^b_\nu(p)\rangle &= \frac{\delta^2}{\delta J^a_\mu(-q)\delta J^b_\mu(-p)}\mathcal{Z}_0[J] \\ &= \mathcal{N}\int \frac{d\beta e^\beta}{(2\pi i\beta)}\left(\det K^{ab}_{\mu\nu}\right)^{-\frac{1}{2}}(K^{ab}_{\mu\nu})^{-1}(q)\delta(p+q).\end{aligned} \tag{2.3.30}$$

To compute the factor $(\det K^{ab}_{\mu\nu})^{-\frac{1}{2}}$, we write it as

$$(\det K^{ab}_{\mu\nu})^{-\frac{1}{2}} = e^{-\frac{1}{2}\ln\det K^{ab}_{\mu\nu}} = e^{-\frac{1}{2}\mathrm{Tr}\ln K^{ab}_{\mu\nu}}, \tag{2.3.31}$$

where the trace is taken over the Lie algebra indices, Lorenz indices and all the momenta q. Thus, the object that we need to compute is

$$\mathrm{Tr}\ln K^{ab}_{\mu\nu} = (N^2-1)\mathrm{Tr}\ln\left[q_\mu q_\nu\left(\frac{1}{\alpha}-1\right) + q^2\delta_{\mu\nu} + \frac{\beta N g^2}{2V(N^2-1)}\frac{1}{q^2}\delta_{\mu\nu}\right], \tag{2.3.32}$$

where we already used the fact that the trace over the Lie algebra indices is just N^2-1. In the next step, we are going to write the expression in the bracket as

$$\left[q_\mu q_\nu\left(\frac{1}{\alpha}-1\right) + \left(q^2+\frac{t}{q^2}\right)\delta_{\mu\nu}\right] = \left[\delta_{\mu\rho}\left(q^2+\frac{t}{q^2}\right)\left(\delta_{\rho\nu} + \frac{1}{q^2+\frac{t}{q^2}}\left(\frac{1}{\alpha}-1\right)q_\rho q_\nu\right)\right], \tag{2.3.33}$$

to make use of the property

$$\ln[AB] = \ln[A] + \ln[B]. \tag{2.3.34}$$

Thus,

$$\mathrm{Tr}\ln K^{ab}_{\mu\nu} = (N^2-1)\left(\mathrm{Tr}\ln\left[\delta_{\mu\nu}\left(q^2+\frac{t}{q^2}\right)\right] + \mathrm{Tr}\ln\left[\left(\delta_{\mu\nu} + \frac{1}{q^2+\frac{t}{q^2}}\left(\frac{1}{\alpha}-1\right)q_\mu q_\nu\right)\right]\right), \tag{2.3.35}$$

where we defined

$$t = \frac{\beta N g^2}{2V(N^2-1)}. \tag{2.3.36}$$

Let us examine the first part of Eq. (2.3.35). We notice the argument of the logarithm is a diagonal matrix with $q^2+\frac{t}{q^2}$ being the diagonal elements. Therefore, the logarithm of a diagonal matrix is also a diagonal matrix with the diagonal elements $\ln\left(q^2 + \frac{t}{q^2}\right)$. Taking the trace over the space-times indices and the momenta q yields

$$\mathrm{Tr}\ln\left[\delta_{\mu\nu}\left(q^2 + \frac{t}{q^2}\right)\right] = 4\int \frac{d^4q}{(2\pi)^4}\ln\left(q^2 + \frac{t}{q^2}\right). \tag{2.3.37}$$

For the second term, we Taylor expand the logarithm $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$, take the trace and use the obtained series to reconstruct the logarithm $x - \frac{x^2}{2} + \frac{x^3}{3} - ... = \ln(1+x)$

$$\begin{aligned}\mathrm{Tr}\ln\left[\left(\delta_{\mu\nu} + \frac{1}{q^2+\frac{t}{q^2}}\left(\frac{1}{\alpha}-1\right)q_\mu q_\nu\right)\right] &= \int \frac{d^4q}{(2\pi)^4}\ln\left(1 + \frac{1}{q^2+\frac{t}{q^2}}\left(\frac{1}{\alpha}-1\right)q^2\right)\\ &= \int \frac{d^4q}{(2\pi)^4}\ln\left(\left(\frac{q^4+t}{q^2}\right)^{-1}\left(\frac{t}{q^2}+\frac{q^2}{\alpha}\right)\right).\end{aligned} \tag{2.3.38}$$

Hence,

$$\begin{aligned}\mathrm{Tr}\ln K^{ab}_{\mu\nu} &= (N^2-1)\left(4\int \frac{d^4q}{(2\pi)^4}\ln\left(q^2+\frac{t}{q^2}\right) + \int \frac{d^4q}{(2\pi)^4}\ln\left(\left(\frac{q^4+t}{q^2}\right)^{-1}\left(\frac{t}{q^2}+\frac{q^2}{\alpha}\right)\right)\right)\\ &= (N^2-1)\left(3\int \frac{d^4q}{(2\pi)^4}\ln\left(q^2+\frac{t}{q^2}\right) + \int \frac{d^4q}{(2\pi)^4}\ln\left(\frac{t}{q^2}+\frac{q^2}{\alpha}\right)\right).\end{aligned} \tag{2.3.39}$$

We rewrite the second part of Eq. (2.3.39) as

$$\int \frac{d^4q}{(2\pi)^4}\ln\left(\frac{t}{q^2}+\frac{q^2}{\alpha}\right) = \int \frac{d^4q}{(2\pi)^4}\ln\left(t+\frac{q^4}{\alpha}\right) - \int \frac{d^4q}{(2\pi)^4}\ln(q^2), \tag{2.3.40}$$

where the last term vanishes in dimensional regularization due to the identity

$$\int \frac{d^Dq}{(2\pi)^D}(q^2)^a = 0, \qquad \text{for} \quad a \geq 0. \tag{2.3.41}$$

By plugging the first term in mathematica one can easily convince itself that it is proportional to the gauge fixing parameter α plus contributions that vanish in dimensional regularization due to the above identity. In the Landau gauge $\alpha \to 0$, the factor $(\det K^{ab}_{\mu\nu})^{-\frac{1}{2}}$ reads

$$(\det K^{ab}_{\mu\nu})^{-\frac{1}{2}} = \exp\left[-\frac{3(N^2-1)}{2}V\int \frac{d^4q}{(2\pi)^4}\ln\left(q^2+\frac{1}{q^2}\frac{\beta N g^2}{2V(N^2-1)}\right)\right]. \tag{2.3.42}$$

Inserting the obtained result for $(\det K^{ab}_{\mu\nu})^{-\frac{1}{2}}$ into Eq. (2.3.30) yields

$$\langle A^a_\mu(q)A^b_\nu(p)\rangle = \mathcal{N}\int \frac{d\beta}{2\pi i}e^{f(\beta)}(K^{ab}_{\mu\nu})^{-1}(q)\delta(p+q), \tag{2.3.43}$$

with

$$f(\beta) = \beta - \ln(\beta) - \frac{3(N^2-1)}{2}V\int \frac{d^4q}{(2\pi)^4}\ln\left(q^2+\frac{1}{q^2}\frac{\beta N g^2}{2V(N^2-1)}\right). \tag{2.3.44}$$

To solve the integral of this type one uses the steepest-descent method [53–55]. This method exploits the fact that such integrals are dominated by the contributions from neighborhoods of saddle points. Near the saddle point β_0, we approximate the function $f(\beta)$ by its Taylor series

$$f(\beta) = f(\beta_0) + \frac{1}{2} f''(\beta)(\beta - \beta_0)^2 \quad \text{and} \quad K^{ab}_{\mu\nu}(\beta) \approx K^{ab}_{\mu\nu}(\beta_0). \tag{2.3.45}$$

Therefore,

$$\begin{aligned} \int \frac{d\beta}{2\pi i} e^{f(\beta)} (K^{ab}_{\mu\nu})^{-1}(q)\delta(p+q) &\approx e^{f(\beta_0)} (K^{ab}_{\mu\nu})^{-1}(\beta_0)\delta(p+q) \int \frac{d\beta}{2\pi i} e^{\frac{1}{2} f''(\beta)(\beta-\beta_0)^2} \\ &= e^{f(\beta_0)} (K^{ab}_{\mu\nu})^{-1}(\beta_0)\delta(p+q) \sqrt{\frac{2\pi}{f''(\beta_0)}}. \end{aligned} \tag{2.3.46}$$

The saddle point β_0 is determined by the condition

$$f'(\beta_0) = 0 \implies 1 = \frac{1}{\beta_0} + \frac{3N}{4} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^4 + \gamma^4)} \tag{2.3.47}$$

with

$$\gamma^4 = \frac{\beta_0 N g^2}{2V(N^2-1)}. \tag{2.3.48}$$

The parameter γ is called the Gribov mass parameter and it acts as a infrared regulator. For the Gribov mass parameter to be finite, β_0 has to be proportional to the infinite volume factor $\beta_0 \sim V$. In the infinite volume limit $V \to \infty$, we neglect all the $\frac{1}{\beta_0}$ terms. As was pointed out by Gribov [40], in this limit the Heaviside step function is equivalent the a delta function. The reason for this is the integral representation of the delta function, and the integral presentation of the Heaviside step function differ by the term $\ln(\beta)$. This term in the saddle point approximation goes to $1/\beta_0$, as we saw in the above calculation, and it can be neglected in the thermodynamical (infinite volume) limit $\beta_0 \sim V \to \infty$. From the saddle point condition, we notice that γ is not a free parameter and it is determined by the so-called gap equation

$$1 = \frac{3N}{4} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^4 + \gamma^4)}. \tag{2.3.49}$$

We observe that the integral in Eq. (2.3.49) is divergent and renormalization is needed to solve it. It is straightforward to show that the inverse of $K^{ab}_{\mu\nu}$ at $\beta = \beta_0$ for the Landau gauge condition $\alpha = 0$ is given by

$$K^{ab}_{\mu\nu}(q)^{-1} = \delta^{ab} \left(\frac{q^2}{q^4 + \gamma^4} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \right). \tag{2.3.50}$$

The final expression for the two-point gauge function reads

$$\langle A^a_\mu(q) A^b_\nu(-q) \rangle = \delta^{ab} g^2 \frac{q^2}{q^4 + \gamma^4} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right), \tag{2.3.51}$$

where we chosen $\mathcal{N}$ such that it cancels the term $e^{f(\beta_0)}$. We observe that due to the presence of the Gribov mass parameter γ^4, the gauge propagator is no longer divergent as $q \to 0$. By decomposing the gauge propagator as

$$\langle A^a_\mu(q) A^b_\nu(-q) \rangle = \delta^{ab} \frac{1}{2} \left(\frac{1}{q^2 + i\gamma^2} + \frac{1}{q^2 - i\gamma^2} \right) \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right), \tag{2.3.52}$$

we see that it has two complex poles at $q^2_\pm = \pm i\gamma^2$ implying that at low momentum the gluon propagator has no physical interpretation since it does not possess a positive Kaellen-Lehmann spectral representation. One may interpret this as a signature of confinement. It is remarkable that a possible solution to a technical problem implements such physical behavior.

After we obtained the gauge propagator, we turn our attention to the ghost propagator. To compute the ghost propagator restricted to the first Gribov region, we need to contract the external gauge fields in Eq. (2.3.24) and insert the previously obtained gauge propagator in Eq. (2.3.51)

$$\begin{aligned}\sigma(p) &= g^2 \frac{N}{N^2-1}\frac{1}{p^2}\int \frac{d^4q}{(2\pi)^4}\frac{(p-q)_\mu p_\nu}{(p-q)^2}\langle A^b_\mu(-q)A^b_\nu(q)\rangle \\ &= g^2 N \frac{p_\mu p_\nu}{p^2}\int \frac{d^4q}{(2\pi)^4}\frac{q^2}{q^4+\gamma^4}\frac{1}{(p-q)^2}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right), \end{aligned} \tag{2.3.53}$$

where we used the fact that the gauge field is transverse. After we integrated out the gauge fields, we can write the ghost propagator as

$$\mathcal{G}^{ab} = \frac{1}{p^2}\delta^{ab}(1-\sigma(p))^{-1}, \tag{2.3.54}$$

with $\sigma(p)$ defined in Eq. (2.3.53). Using the gap equation and Eq. (2.3.22), we find that

$$Ng^2\frac{p_\mu p_\nu}{p^2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^4+\gamma^4}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) = 1, \tag{2.3.55}$$

which we can use to rewrite $(1-\sigma(p))$ as

$$\begin{aligned}(1-\sigma(p)) &= g^2 N\frac{p_\mu p_\nu}{p^2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^4+\gamma^4}\left(1-\frac{q^2}{(p-q)^2}\right)\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) \\ &= Ng^2\frac{p_\mu p_\nu}{p^2}\Sigma_{\mu\nu}(p), \end{aligned} \tag{2.3.56}$$

with

$$\Sigma_{\mu\nu}(p) = \int \frac{d^4q}{(2\pi)^4}\frac{1}{q^4+\gamma^4}\left(1-\frac{q^2}{(p-q)^2}\right)\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right). \tag{2.3.57}$$

To analyze the infrared behavior of the ghost propagator we expand the integrand for low momenta $p^2 \approx 0$

$$1-\frac{q^2}{(p-q)^2} = 1-\frac{1}{\frac{p^2}{q^2}-2(p\cdot q)+1} \approx \frac{p^2}{q^2} - ... \tag{2.3.58}$$

from which it follows that

$$\begin{aligned}\Sigma_{\mu\nu}(p)_{p\to 0} &\approx p^2\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2}\frac{1}{q^4+\gamma^4}\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) \\ &= \frac{3p^2}{4}\delta_{\mu\nu}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2}\frac{1}{q^4+\gamma^4} \\ &= \frac{3p^2}{4}\delta_{\mu\nu}\frac{2\pi^2}{(2\pi)^4}\int dq\frac{q}{q^4+\gamma^4} \\ &= \delta_{\mu\nu}p^2\frac{3}{128\pi}\frac{1}{\gamma^2}. \end{aligned} \tag{2.3.59}$$

In the IR limit, the ghost propagator is given by the following expression

$$\mathcal{G}(p)_{p\to 0} \approx \frac{128\pi\gamma^2}{3Ng^2}\frac{1}{p^4}, \tag{2.3.60}$$

which indicates that the gluon propagator is enhanced in the IR regime as it diverges faster than before [40, 41]. As a consequence of restricting the integration to the first Gribov region, the ghost propagator acquires an additional pole, which indicates that the region close to the boundary has a significant effect on the ghost propagator [56]. The IR enhancement of the gluon propagator has been confirmed by lattice simulations [57–60] as well as Dyson-Schwinger equations [61].

2.4 The Zwanziger action

An alternative method of implementing the restriction to the first Gribov region was found by Zwanziger [63]. He proposed restricting the path integral by analyzing the eigenvalues $\lambda(A)$ of the Faddeev-Popov operator

$$\mathcal{F}^{ab}(A)\xi^a = \lambda(A)\xi^b, \tag{2.4.1}$$

as a function of the gauge fields. He was able to restrict to path integral to the first Gribov region by demanding that the smallest eigenvalue of the Faddeev-Popov operator is positive $\lambda(A)_{\min} > 0$. In the infinite volume limit, the Zwanziger conditon can be approximated by demanding the trace of all the eigenvalues should be positive [63]

$$\mathrm{Tr}[\lambda(A)] > 0. \tag{2.4.2}$$

The eigenvalues of the Faddeev-Popov operator are calculated using degenerate perturbation theory, where $\mathcal{F}_0^{ab}$ is treated as the unperturbed operator and $\mathcal{F}_1^{ab}$ describing the perturbation. In this section, we are following the work done in [56]. We are going to assume that we are working in a finite periodic box with length L. We expand eigenvectors ξ^a and eigenvalues $\lambda(A)$ in perturbation theory as

$$\xi^a = \sum_{n=0}^{\infty} g^n \xi_n^a \qquad \& \qquad \lambda(A) = \sum_{n=0}^{\infty} g^n \lambda(A)_n, \tag{2.4.3}$$

where ξ_0^a are the eigenstates of the unperturbed operator $\mathcal{F}_0^{ab}$ with the eigenvalue $\lambda_0 \equiv \lambda_{\vec{n}}^{(0)} = (2\pi/L)^2\vec{n}^2$; $\vec{n}$ are the finite momenta with $\vec{n}_0 = (0, 0, ... \pm 1, ..., 0)$ being the lowest non-zero momentum with $\vec{n}_0^2 = 1$. Inserting the perturbative expansions in Eq. (2.4.1) yields

$$\text{Order } g^0 : \implies \mathcal{F}_0^{ab}(A)\xi_0^a = \lambda_0(A)\xi_0^b \tag{2.4.4}$$

$$\text{Order } g^1 : \implies \mathcal{F}_0^{ab}(A)\xi_1^a + \mathcal{F}_1^{ab}(A)\xi_0^a = \lambda_0(A)\xi_1^b + \lambda_1(A)\xi_0^b \tag{2.4.5}$$

$$\text{Order } g^2 : \implies \mathcal{F}_0^{ab}(A)\xi_2^a + \mathcal{F}_1^{ab}(A)\xi_1^a = \lambda_0(A)\xi_2^b + \lambda_1(A)\xi_1^b + \lambda_2(A)\xi_0^b \tag{2.4.6}$$

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where Eq.(2.4.4) is the eigenvalue equation of the unperturbed operator with $\lambda_0 \equiv \lambda_{\vec{n}_0} = (2\pi/L)^2 I$. The higher order equations are solved by acting with projection operator $P_0 = \sum_a^{N^2-1} \xi_{\vec{n}_0}^a\, (\xi_{\vec{n}_0}^a)^\dagger$ on the equations and using the fact that

$$P_0\, \xi_n^a = 0 \qquad \text{for} \quad n \geq 0, \tag{2.4.7}$$

which can be also written as

$$(\xi^a_{\vec{n}_0})^\dagger \xi^b_n = 0. \tag{2.4.8}$$

Therefore, to the first order in perturbation theory we find

$$\lambda_1(A) = (\xi^a_{\vec{n}_0})^\dagger \mathcal{F}^{ab}_1(A)\xi^b_{\vec{n}_0}. \tag{2.4.9}$$

For $\lambda_2(A)$, we have

$$\underbrace{(\xi^a_{\vec{n}_0})^\dagger \mathcal{F}^{ab}_0(A)}_{\lambda_0(\xi^a_{\vec{n}_0})^\dagger}\xi^a_2 + (\xi^a_{\vec{n}_0})^\dagger \mathcal{F}^{ab}_1(A)\xi^a_1 = \lambda_0(A)(\xi^a_{\vec{n}_0})^\dagger \xi^b_2 + \lambda_1(A)(\xi^a_{\vec{n}_0})^\dagger\ \xi^b_1 + \lambda_2(A)(\xi^a_{\vec{n}_0})^\dagger\ \xi^b_{\vec{n}_0} \tag{2.4.10}$$

$$\implies \lambda_2(A) = (\xi^a_{\vec{n}_0})^\dagger \mathcal{F}^{ab}_1(A)\xi^b_1, \tag{2.4.11}$$

as all the other terms vanish due to Eq. (2.4.8). It comes as no surprise that $\lambda_3(A)$ is equal to

$$\lambda_3(A) = (\xi^a_{\vec{n}_0})^\dagger \mathcal{F}^{ab}_1(A)\xi^a_2, \tag{2.4.12}$$

and so on. To determine ξ^a_1, we start from Eq. (2.4.5) and insert the obtained value for $\lambda_1(A)$ which yields

$$\xi^a_1 = (\mathcal{F}^{ab}_0 - \lambda_0 I)^{-1}(P_0 - I)\mathcal{F}^{bc}_1\xi^c_{\vec{n}_0}. \tag{2.4.13}$$

Inserting the above expression ξ^a_1 into Eq. (2.4.10) gives us

$$\begin{aligned}\lambda_2(A) &= -(\xi^a_{\vec{n}_0})^\dagger \mathcal{F}^{ab}_1(A)(\mathcal{F}^{bc}_0 - \lambda_0 I)^{-1}(I - P_0)\mathcal{F}^{cd}_1\xi^d_{\vec{n}_0},\\ &= -(\xi^a_{\vec{n}_0})^\dagger \mathcal{F}^{ab}_1(A)[(\mathcal{F}^{bc}_0)^{-1}]\mathcal{F}^{cd}_1(A)\xi^d_{\vec{n}_0},\end{aligned} \tag{2.4.14}$$

where in the last line we used the fact that in the infinite volume limit $(\mathcal{F}^{bc}_0 - \lambda_0 I)^{-1}(I - P_0)$ is replaced by $(\mathcal{F}^{ab}_0)^{-1}$ [56]. For completeness $\lambda_3(A)$ is given by

$$\lambda_3(A) = (\xi^a_{\vec{n}_0})^\dagger \mathcal{F}^{ab}_1\left[(\mathcal{F}^{bc}_0)^{-1}\mathcal{F}^{cd}_1(A)(\mathcal{F}^{de}_0)^{-1}\right]\mathcal{F}^{ef}_1(A)\xi^f_{\vec{n}_0}, \tag{2.4.15}$$

and $\lambda_4(A)$ by

$$\lambda_4(A) = -(\xi^a_{\vec{n}_0})^\dagger \mathcal{F}^{ab}_1\left[(\mathcal{F}^{bc}_0)^{-1}\mathcal{F}^{cd}_1(A)(\mathcal{F}^{de}_0)^{-1}\mathcal{F}^{ef}_1(A)(\mathcal{F}^{fg}_0)^{-1}\right]\mathcal{F}^{gh}_1(A)\xi^h_{\vec{n}_0}. \tag{2.4.16}$$

Following the same pattern we can generate all the higher order eigenvalues of the Faddeev-Popov operator. Noticing that

$$\mathcal{F}^{-1} = \frac{1}{1 + \mathcal{F}^{-1}_0\mathcal{F}_1}\mathcal{F}^{-1}_0 = (\mathcal{F}^{-1}_0 - \mathcal{F}^{-1}_0\mathcal{F}_1\mathcal{F}^{-1}_0 + \mathcal{F}^{-1}_0\mathcal{F}_1\mathcal{F}^{-1}_0\mathcal{F}_1\mathcal{F}^{-1}_0) \tag{2.4.17}$$

enables us to write all the eigenvalues as the following sum

$$\lambda_m(A) = \sum_{n\geq 2}\lambda_n(A) = -(\xi^a_{\vec{n}_0})^\dagger \mathcal{F}^{ab}_1(A)(\mathcal{F}^{bc})^{-1}\mathcal{F}^{cd}_1(A)\xi^d_{\vec{n}_0}. \tag{2.4.18}$$

To restrict the path integral to the first Gribov region, we have to demand that the trace of all eigenvalues should be positive

$$\mathrm{Tr}[\lambda(A)] = \mathrm{Tr}[\lambda_0(A)] + \mathrm{Tr}[\lambda_1(A)] + \mathrm{Tr}[\lambda_m(A)] \tag{2.4.19}$$

with

$$\text{Tr}[\lambda_0(A)] = \left(\frac{2\pi}{L}\right)^2 2d(N^2-1),$$
$$\text{Tr}[\lambda_1(A)] = in_{0,\mu}\frac{2\pi}{L}\frac{1}{V}\int d^dx g f^{aab}A^b_\mu(x) = 0, \tag{2.4.20}$$

where in the second line we expressed the eigenvectors in the position space basis and used the fact that the perturbed part of the Faddeev-Popov operator expressed in the position basis is just $f^{abc}A^c_\mu\partial_\mu$. The last part of Eq. (2.4.19) reads

$$\text{Tr}[\lambda_m(A)] = -2\left(\frac{2\pi}{L}\right)^2 g^2\frac{1}{V}\int d^dxd^dy f^{abc}A^b_\mu(x)(\mathcal{F}^{-1})^{ad}(x,y)f^{dec}A^e_\mu(y), \tag{2.4.21}$$

where again we inserted two sets of complete position states $1 = \int d^4x\,|x,a\rangle\,\langle x,a|$ and used the fact that in the infinite volume limit $L\to\infty$,

$$\langle x,a|\,\mathcal{F}_1\,|\xi^b_0\rangle = i\frac{2\pi}{L}g f^{abc}n_\mu A^c_\mu(x)L^{-d/2}, \tag{2.4.22}$$

where we omitted the factor $e^{i\frac{2\pi}{L}\vec{n}\cdot\vec{x}}$. The condition $\text{Tr}[\lambda]>0$, reduces to

$$\text{Tr}[\lambda] = 2\left(\frac{2\pi}{L}\right)^2\left(d(N^2-1) - \frac{1}{V}\int d^dxd^dy f^{abc}A^b_\mu(x)(\mathcal{F}^{-1})^{ad}(x,y)f^{dec}A^e_\mu(y)\right) > 0. \tag{2.4.23}$$

As it was for the case of Gribov's no-pole condition, we implement Zwanziger's condition by inserting the Heaviside theta function in the path integral to ensure that the condition is satisfied

$$\mathcal{Z} = \mathcal{N}\int\mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\;\Theta(d(N^2-1)V - H(A))\;\delta(\partial_\mu A_\mu)e^{-S_{\text{inv}}-\int d^4x\bar{c}^a\partial_\mu D^{ab}_\mu c^b}, \tag{2.4.24}$$

where $H(A)$ is the so-called horizon function

$$H(A) = \int d^dxd^dy f^{abc}A^b_\mu(x)(\mathcal{F}^{-1})^{ad}(x,y)f^{dec}A^e_\mu(y). \tag{2.4.25}$$

To lowest order in perturbation theory, the horizon function is given by

$$H_0(A) = \int d^dxd^dy f^{abc}A^b_\mu(x)\left(-\frac{\delta^{ad}}{\partial^2}\delta(x-y)\right)f^{dec}A^e_\mu(y)$$
$$= N\int d^dx A^a_\mu(x)\left(-\frac{1}{\partial^2}\right)A^a_\mu(x), \tag{2.4.26}$$

which in momentum space reads

$$H_0(A) = N\int\frac{d^dp}{(2\pi)^d}A^a_\mu(p)\left(\frac{1}{p^2}\right)A^a_\mu(-p). \tag{2.4.27}$$

We see that the first Gribov region is contained within an infinite dimensional hypersurface

$$\int\frac{d^dp}{(2\pi)^d}A^a_\mu(p)\left(\frac{1}{p^2}\right)A^a_\mu(-p) < Vd(N^2-1). \tag{2.4.28}$$

An interesting property of higher dimensional spheres is that as the dimension of the sphere increases, more and more of its volume gets concentrated at its surface [62]. For us it means that as we go up in the dimension the Gribov gribov region gets more and more concentrated near the first Gribov boundary l_1. In this case, we can replace the theta function in Eq. (2.4.24) with the delta function

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}A\, \delta(\partial_\mu A^a_\mu) \det(\mathcal{F}^{ab})\ \delta(d(N^2-1)V - H(A))\ \delta(\partial_\mu A_\mu) e^{-S_{\text{inv}}}. \tag{2.4.29}$$

Although this reasoning is true for the lowest order case, it is expected to also hold for all orders in perturbation theory. To bring Zwanziger's horizon function into the action, we write the delta function in its integral representation and perform Wick rotation to eliminate the imaginary numbers

$$\delta(x-y) = \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{d\beta}{2\pi i} e^{\beta(x-y)}. \tag{2.4.30}$$

It is also possible to implement Zwanziger's condition by using the theta function and taking the infinite volume $V \to 0$ limit just as we did for the case of Gribov's no-pole condition. Inserting the Wick rotated integral representation of the delta function into the partition function gives us

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}A\, \delta(\partial_\mu A^a_\mu) \det(\mathcal{F}^{ab})\ \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{d\beta}{2\pi i} e^{\beta(d(N^2-1)V - H(A))} e^{-S_{\text{inv}}}, \tag{2.4.31}$$

which can be rewritten in a more compact form as

$$\mathcal{Z} = \mathcal{N} \int \frac{d\beta}{2\pi i} e^{-f(\beta)}, \tag{2.4.32}$$

with

$$f(\beta) = -\ln\left(\left(\int \mathcal{D}A\, \delta(\partial_\mu A^a_\mu) \det(\mathcal{F}^{ab})\ e^{\beta(d(N^2-1)V - H(A))}\ e^{-S_{\text{inv}}} \right) \right). \tag{2.4.33}$$

Just like in the case of Gribov's no-pole condition, to solve the integral we use the saddle point approximation

$$\mathcal{Z} \approx e^{-f(\beta_0)} = \int \mathcal{D}A\, \delta(\partial_\mu A^a_\mu) \det(\mathcal{F}^{ab})\ e^{-\left(S_{\text{inv}} + \beta_0 H(A) - \beta_0 d(N^2-1)V\right)}, \tag{2.4.34}$$

where β_0 is determined by the condition $f'(\beta_0) = 0$, which implies that

$$\begin{aligned} d(N^2-1)V &= \frac{\int \mathcal{D}A\, \delta(\partial_\mu A^a_\mu) \det(\mathcal{F}^{ab})\ H(A)\, e^{-\beta_0 H(A))}\, e^{-S_{\text{inv}}}}{\int \mathcal{D}A\, \delta(\partial_\mu A^a_\mu) \det(\mathcal{F}^{ab})\ e^{-\beta_0 H(A))}\, e^{-S_{\text{inv}}}} \\ d(N^2-1)V &= \langle H(A) \rangle. \end{aligned} \tag{2.4.35}$$

It turns out that the saddle point approximation becomes exact, which was proven by Zwanziger [63**?** , 64] using the equivalence between the canonical and micro-canonical ensembles in the termodynamical limit. Therefore, the path integral reads

$$\mathcal{Z} = \int \mathcal{D}[A]\mathcal{D}[c]\mathcal{D}[\bar{c}] e^{-(S_{\text{YM}} + S_{GZ})}, \tag{2.4.36}$$

where S_{YM} is the gauge invariant Yang-Mills action plus the gauge fixing term and S_{GZ} is the so-called *Gribov-Zwanziger action*

$$S_{GZ} = S_{\text{ghost}} + \gamma^4 H(A) - \gamma^4 V d(N^2-1)), \tag{2.4.37}$$

where we set $\beta_0 = \gamma^4$. It may seem a bit confusing as in the derivation of the Gribov-Zwanziger action we assumed that all configurations get concentrated on the boundary l_1 whereas in perturbation theory only configurations around $A_\mu = 0$ are important. At this point, one could argue that γ is a non-perturbative parameter which is not accessible in perturbation theory. Its presence becomes noticeable when considering regions where perturbation theory fails to give sensible results [56]. Another way one could argue is that in perturbation theory the Faddeev-Popov operator is always positive and we do not have to care about Gribov copies. The horizon function $H(A)$ contains the inverse of the Faddeev-Popov operator which makes it a non-local quantity. Here we are referring to a spatial non-locality because the inverse Faddeev-Popov operator contains the inverse Laplacian which is non-local with respect to the spatial variables. To make the action local, we write the horizon function in the following way

$$e^{-\gamma^4 H(A)} = \det(\mathcal{F})^{d(N^2-1)} \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] e^{-\int d^dx (\bar{\psi}^{ac}_\mu \mathcal{F}^{ab} \psi^{bc}_\mu - \gamma^2 f^{abc} A^a_\mu (\bar{\psi}+\psi)^{bc}_\mu)}, \tag{2.4.38}$$

where we introduced a pair of bosonic fields ψ^{ab}_μ and $\bar{\psi}^{ab}_\mu$. Introducing a pair of fermionic fields ω^{ab}_μ and $\bar{\omega}^{ab}_\mu$ allows us to rewrite the determinant as

$$\det(\mathcal{F})^{d(N^2-1)} = \int \mathcal{D}[\omega]\mathcal{D}[\bar{\omega}] e^{\int d^dx \bar{\omega}^{ac}_\mu \mathcal{F}^{ab} \omega^{bc}_\mu}. \tag{2.4.39}$$

Hence,

$$e^{-\gamma^4 H(A)} = \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\mathcal{D}[\omega]\mathcal{D}[\bar{\omega}] e^{-\int d^dx (\bar{\psi}^{ac}_\mu \mathcal{F}^{ab} \psi^{bc}_\mu - \bar{\omega}^{ac}_\mu \mathcal{F}^{ab} \omega^{bc}_\mu - \gamma^2 f^{abc} A^a_\mu (\bar{\psi}+\psi)^{bc}_\mu))}. \tag{2.4.40}$$

The two pairs of additional fields that we added to localize the horizon function form a BRST doublet

$$\begin{aligned} \delta\psi^{ab}_\mu &= \omega^{ab}_\mu, & \delta\omega^{ab}_\mu &= 0, \\ \delta\bar{\omega}^{ab}_\mu &= \bar{\psi}^{ab}_\mu, & \delta\bar{\psi}^{ab}_\mu &= 0. \end{aligned} \tag{2.4.41}$$

At this point, we perform a shift of the ω field as follows

$$\omega^{ab}_\mu \to \omega^{ab}_\mu - g \int d^dy (\mathcal{F}^{-1})^{ac}(x,y) f^{cde} \partial_\mu [(D^{df}_\nu c^f(y)) \psi^{eb}_\mu], \tag{2.4.42}$$

which leads to an additional term in the local form of the horizon function

$$e^{-\gamma^4 H(A)} = \int \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\mathcal{D}[\omega]\mathcal{D}[\bar{\omega}] e^{-\int d^dx (\bar{\psi}^{ac}_\mu \mathcal{F}^{ab} \psi^{bc}_\mu - \bar{\omega}^{ac}_\mu \mathcal{F}^{ab} \omega^{bc}_\mu + g f^{abe} \bar{\omega}^{ac}_\mu \partial_\mu [(D^{ed}_\nu c^d) \psi^{bc}_\mu] - \gamma^2 f^{abc} A^a_\mu (\bar{\psi}+\psi)^{bc}_\mu))}. \tag{2.4.43}$$

The reason why we performed the shift is that now the first three terms are BRST exact

$$\delta(\bar{\omega}^{ac}_\mu \mathcal{F}^{ab} \psi^{bc}_\mu) = \bar{\psi}^{ac}_\mu \mathcal{F}^{ab} \psi^{bc}_\mu - \bar{\omega}^{ac}_\mu \mathcal{F}^{ab} \omega^{bc}_\mu + f^{abe} \omega^{ac}_\mu \partial_\mu [(D^{ed}_\nu c^d) \psi^{bc}_\mu]. \tag{2.4.44}$$

However, the presence of the last in Eq. (2.4.43) breaks BRST and the breaking is proportional to the Gribov parameter

$$\delta(\gamma^2 f^{abc} A^a_\mu (\bar{\psi}+\psi)^{bc}_\mu))) = \gamma^2 g f^{abc}((-D^{ad}_\mu c^d)(\bar{\psi}+\psi)^{bc}_\mu + A^a_\mu \omega^{bc}_\mu). \tag{2.4.45}$$

The fact that the Gribov-Zwanziger action breaks BRST symmetry comes as no surprise. The Gribov-Zwanziger action was obtained by introducing a boundary in configuration space to eliminate all field configurations that are located outside of it. However, it was proven by Gribov [40], that to every gauge field infinitesimally close to the boundary there exists an equivalent gauge field on the other side of the boundary. These fields are related by an infinitesimal gauge transformation which is similar to a BRST transformation. Eliminating however, all the gauge fields outside the boundary will break this symmetry. In this way, one can intuitively understand why BRST is broken. In the perturbative regime, BRST invariance is maintained, since in this case γ^2 vanishes and with it the term that leads to the breaking. A way of reconciling the BRST symmetry with the Gribov horizon was worked out by M. A. L. Capri et al., in [65].

Summarizing, the Gribov-Zwanziger action in its local form reads

$$\mathcal{Z} = \int \mathcal{D}[A]\mathcal{D}[c]\mathcal{D}[\bar{c}]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\mathcal{D}[\omega]\mathcal{D}[\bar{\omega}]e^{-(S_{\mathrm{YM}}+S_{GZ}^{\mathrm{local}})}, \tag{2.4.46}$$

with

$$\begin{aligned} S_{GZ}^{\mathrm{local}} = S_{\mathrm{ghost}} &+ \int d^dx (\bar{\psi}_\mu^{ac}\mathcal{F}^{ab}\psi_\mu^{bc} - \bar{\omega}_\mu^{ac}\mathcal{F}^{ab}\omega_\mu^{bc} + gf^{abe}\bar{\omega}_\mu^{ac}\partial_\mu[(D_\nu^{ed}c^d)\psi_\mu^{bc}]) \\ &- \int d^dx \gamma^2 f^{abc}A_\mu^a(\bar{\psi}+\psi)_\mu^{bc} - \gamma^4 V d(N^2-1). \end{aligned} \tag{2.4.47}$$

So far we encountered two different ways of restricting the path integral to the first Gribov region. Both of the conditions yield the same results at the lowest non-trivial order. It was proven in [66, 67] that the gap equations arising from Gribov's no pole condition and Zwanziger's horizon condition are equivalent up to two loops. These results are strong indicators that both Gribov's and Zwanziger's approach could be equivalent. Calculating the ghost propagator to all orders in perturbation theory M. A. L. Capri et al., [68] finally were able to prove the full equivalence between Gribov's no-pole and Zwanziger's horizon conditions. Analogously to Eq. (2.3.9) the momentum space representation of the ghost propagator to all orders in perturbation theory reads

$$\begin{aligned} \mathcal{G}^{ab}(p,A) = \frac{1}{p^2}&\left[\delta^{ab}\delta(p-q) + gA_\mu^{ab}(p-q)\frac{iq_\mu}{q^2} + g^2\int\frac{d^dr}{(2\pi)^d}A_\mu^{ac}(p-r)\frac{ir_\mu}{r^2}A_\nu^{cb}(r-q)\frac{iq_\nu}{q^2} + \cdots\right] \\ +\frac{1}{p^2}&\left[\int\frac{d^dq_1}{(2\pi)^d}\cdots\int\frac{d^dq_{n-1}}{(2\pi)^d}A_{\mu_1}^{aa_1}(p-q_1)\frac{iq_{1\mu_1}}{q_1^2}A_{\mu_2}^{a_1a_2}(q_1-q_2)\frac{iq_{2\mu_2}}{q_2^2}\cdots A_{\mu_n}^{a_{n-1}a}(q_{n-1}-q)\frac{iq_{\mu_n}}{q^2}\right] \\ +\frac{1}{p^2}&\left[\cdots\right] \end{aligned} \tag{2.4.48}$$

where we defined $A_\mu^{ab} \equiv f^{abc}A_\mu^c$. In this case, the color-singlet ghost propagator is given by

$$\mathcal{G}(p,A) = \frac{1}{V(N^2-1)}\mathcal{G}^{ab}(p,A) = \frac{1}{p^2}(1+\sigma(p,A)), \tag{2.4.49}$$

where $\delta(p-q)|_{p=q} = V$ and

$$\begin{aligned} \sigma(p,A) = \frac{1}{V(N^2-1)}&\left[g^2\int\frac{d^dr}{(2\pi)^d}A_\mu^{ac}(p-r)\frac{ir_\mu}{r^2}A_\nu^{cb}(r-p)\frac{ip_\nu}{p^2} + \cdots\right] \\ +\frac{1}{V(N^2-1)}&\left[g^n\int\frac{d^dq_1}{(2\pi)^d}\cdot\int\frac{d^dq_{n-1}}{(2\pi)^d}A_{\mu_1}^{aa_1}(p-q_1)\frac{iq_{1\mu_1}}{q_1^2}A_{\mu_2}^{a_1a_2}(q_1-q_2)\frac{iq_{2\mu_2}}{q_2^2}\cdot\cdot A_{\mu_n}^{a_{n-1}a}(q_{n-1}-p)\frac{ip_{\mu_n}}{p^2}\right] \\ +\cdots& \end{aligned} \tag{2.4.50}$$

Considering the $n-th$ term of Eq. (2.4.50), shifting the momentum $q_i \to q_i + p$, for $i = 1, ...n-1$ and using the property that in the Landau gauge condition the gauge fields A_μ are transverse gives us

$$g^n \int \frac{d^d q_1}{(2\pi)^d} \cdots \int \frac{d^d q_{n-1}}{(2\pi)^d} A^{aa_1}_{\mu_1}(-q_1) \frac{ip_{\mu_1}}{(p+q_1)^2} A^{a_1 a_2}_{\mu_2}(q_1 - q_2) \frac{i(p+q_2)_{\mu_2}}{(p+q_2)^2} \cdots A^{a_{n-1}a}_{\mu_n}(q_{n-1}) \frac{ip_{\mu_n}}{p^2}. \tag{2.4.51}$$

Recall that when we compute Gribov's no-pole conditions, we took the limit $p \to 0$ because $\sigma(p, A)$ decreases with increasing p. Therefore, applying this limit to Eq. (2.4.51) yields

$$-\frac{g^n}{d} \int \frac{d^d q_1}{(2\pi)^d} \cdots \int \frac{d^d q_{n-1}}{(2\pi)^d} A^{aa_1}_{\mu}(-q_1) \frac{1}{q_1^2} A^{a_1 a_2}_{\mu_2}(q_1 - q_2) \frac{iq_{2_{\mu_2}}}{q_2^2} \cdots A^{a_{n-1}a}_{\mu}(q_{n-1}). \tag{2.4.52}$$

It is interesting that by introducing the following matrix notation

$$\mathbb{A}^{ab}_{pq} \equiv A^{ab}_\mu(p-q) \frac{iq_\mu}{q^2}, \tag{2.4.53}$$

with the matrix multiplication defined as

$$(\mathbb{A}^2)^{ab}_{pq} = \int \frac{d^d r}{(2\pi)^d} A^{ac}_\mu(p-r) \frac{ir_\mu}{r^2} A^{cb}_\nu(r-q) \frac{iq_\nu}{q^2}, \tag{2.4.54}$$

M. A. L. Capri et al., [68] were able to rewrite $\sigma(0, A)$ in a closed form

$$\sigma(0, A) = -\frac{g^2}{Vd(N^2-1)} \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} A^{ab}_\mu(-p) \frac{1}{p^2} \left(\prod_{n=0}^{\infty} (g\mathbb{A})^n \right)^{bc}_{pq} A^{ca}_\mu(q) \tag{2.4.55}$$

The Faddeev-Popov operator expressed using the same notiation reads

$$\mathcal{F}^{ab}(p, q) = q^2 \left(\delta^{ab}\delta(p-q) - gA^{ab}_\mu(p-q) \frac{iq_\mu}{q^2} \right) = q^2 (\mathbb{I} - \mathbb{A})^{ab}_{pq}, \tag{2.4.56}$$

where $\delta^{ab}\delta(p-q) = \mathbb{I}$. The beauty of this notation lies in the fact that inverse of the Faddeev-Popov operator can be rewritten as

$$\mathcal{F}^{-1}(p, g) = \frac{1}{q^2} [\mathbb{I} - \mathbb{A}]^{-1}(p, q) = \frac{1}{q^2} \left(\prod_{n=0}^{\infty} (g\mathbb{A})^n \right)(p, q). \tag{2.4.57}$$

With this little notational trick, Capri et al. [68], were able to show that the exact expression of $\sigma(0, A)$ is equivalent to Zwanziger's horizon function

$$\sigma(0, A) = -\frac{g^2}{Vd(N^2-1)} \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} A^{ab}_\mu(-p) (\mathcal{F}^{-1})^{ca}_{pq} A^{ca}_\mu(q) = \frac{H(A)}{dV(N^2-1)}. \tag{2.4.58}$$

From Eq. (2.4.58) we see that Zwanziger's horizon function is exactly the ghost propagator at zero momentum. Inserting the above-obtained result in Gribov's no-pole condition yields

$$\sigma(0, A) - 1 < 0 \underbrace{\Longrightarrow}_{2.4.58} (N^2 - 1) - \langle H(A) \rangle > 0, \tag{2.4.59}$$

which precisely represents Zwanziger's horizon condition [68].

Chapter 3

Geometrical Anatomy of Gauge Theory

"In 1975, impressed with the fact that gauge fields are connections on fiber bundles, I drove to the house of Shiing- Shen Chern in El Cerrito, near Berkeley.

[...] *We had much to talk about: friends, relatives, China. When our conversation turned to fiber bundles, I told him that I had finally learned from Jim Simons the beauty of fiber-bundle theory and the profound Chern-Weil theorem. I said I found it amazing that gauge fields are exactly connections on fiber bundles, which the mathematicians developed without reference to the physical world. I added, "this is both thrilling and puzzling, since you mathematicians dreamed up these concepts out of nowhere." He immediately protested, "No, no. These concepts were not dreamed up. They were natural and real."*
Deep as the relationship is between mathematics and physics, it would be wrong, however, to think that the two disciplines overlap that much. They do not. And they have their separate aims and tastes. They have distinctly different value judgements, and they have different traditions. At the fundamental conceptual level they amazingly share some concepts, but even there, the life force of each discipline runs along its own veins". [69] Chen Ning Yang (1979) .

Another way of studying the Gribov ambiguity is from the perspective of differential geometry. There the problem is related to the absence of a global section on principal fiber bundles. Before we can discuss the Gribov ambiguity and the possible solution proposed by Hueffel and Kelnhofer, we have to introduce the concepts that are going to be relevant for us and will be essential for our understanding.

3.1 Bundle Theory

In this section we will rely on the following literature [70–73].

A *bundle* of topological manifold is a triple $(E, \pi, \mathcal{M})$, where E and $\mathcal{M}$ are topological manifolds called the *total and the base space*, respectively. The surjective map $\pi: E \to \mathcal{M}$ that projects the total space E onto $\mathcal{M}$ is called the *projection* map. Since π is a map between two topological spaces, we want it to be continuous. For every point p in the base space, the preimage of the set $\{p\}$ under the map π, $\text{preim}_\pi(\{p\}) \equiv F_p$, is the *fiber* at the point p [70–73] .

If $(E, \pi, \mathcal{M})$ is bundle such that for all the points $p \in \mathcal{M}$

$$\text{preim}_\pi(\{p\}) \equiv F, \tag{3.1.1}$$

then $(E, \pi, \mathcal{M})$ is called *fiber bundle* with typical fiber F [70–73]. For the case that the fibers F have a structure of a vector space, we call that bundle a vector bundle. If the fibers F

carry a Lie group structure then we talk about principal bundles, which are the ones that will be the important one for us. We see that a fiber bundle is a less general object that a bundle. Another essential object that we are going to need is that of a section.

A map $\sigma : M \to E$ is called a *section* of the bundle $(E, \pi, \mathcal{M}$ if

$$\pi \circ \sigma = \mathrm{id}_{\mathcal{M}}, \tag{3.1.2}$$

where $\mathrm{id}_{\mathcal{M}}$ is the identity on the base space $\mathcal{M}$ [70–73]. Eq. (3.1.2) tells us that a map σ is a section if it maps a point $p \in \mathcal{M}$ to the fiber at that point because only in this case π takes the point back to the base point p. Set of all such sections is denoted by $\Gamma(E)$. If the bundle $(E, \pi, \mathcal{M})$ is a vector bundle, then $\Gamma(E)$ is a vector space. For a group bundle, $\Gamma(E)$ is a group. Consider a fiber bundle whose the total space is a product manifold $E = \mathcal{M} \times F$ with F being the fiber. Then the projection map $\pi : \mathcal{M} \times F \to \mathcal{M}$ is defined as

$$\pi(p, f) = p, \tag{3.1.3}$$

where $p \in \mathcal{M}$ and $f \in F$. Only, in this case, a section can be considered or instead constructed from a function $s : \mathcal{M} \to F$.

For two bundles $(E, \pi, \mathcal{M})$ and $(E', \pi', \mathcal{M}')$ and two maps (u, f) defined as

$$u : E \to E' \qquad \text{and} \qquad f : \mathcal{M} \to \mathcal{M}', \tag{3.1.4}$$

the pair of functions (u, f) is called a *bundle morphism* if the following diagram commutes [70–73] i.e., $\pi' \circ u = f \circ \pi$

$$\begin{array}{ccc} E & \xrightarrow{\;u\;} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathcal{M} & \xrightarrow{\;f\;} & \mathcal{M}' \end{array}$$

Two bundles $(E, \pi, \mathcal{M})$ and $(E', \pi', \mathcal{M}')$ are called *isomorphic as bundles* if there exist bundle morphisms (u, f) and (u^{-1}, f^{-1}). Such (u, f) are called *bundle isomorphisms* and they are the relevant structure-preserving maps for bundles [70–73].

The definition of a bundle isomorphism is a strong condition and one usually weakens this to local isomorphism of bundles.

A bundle $(E, \pi, \mathcal{M})$ is called *locally isomorphic* to $(E', \pi', \mathcal{M}')$ if for every $p \in \mathcal{M}$ there exist an open set $\mathcal{U} \subset \mathcal{M}$ that contains that point $p \in \mathcal{U}$ such that the restricted bundle $(\mathrm{preim}_\pi(\mathcal{U}), \pi|_{\mathrm{preim}_\pi(\mathcal{U})}, \mathcal{U})$ is isomorphic to $(E', \pi', \mathcal{M}')$ [70–73].

In the next couple of definition, we are going to introduce the terminology that is heavily used in the literature.

A bundle $(E, \pi, \mathcal{M})$ is called *trivial* if it is isomorphic as a bundle to a product bundle [70–73].

A bundle $(E, \pi, \mathcal{M})$ is *locally trivial* if it is locally isomorphic to some product bundle [70–73].

It is evident that every bundle that is trivial is also locally trivial but not the other way around. A cylinder with the total space $E = I \times S_1$, where I is some open interval, is an example of a trivial fiber bundle. Whereas the Möbius strip is an example of a locally trivial bundle. It is also important to note that for locally trivial bundles, locally any section can be represented as a function from the base space to the fiber. For us, a special kind of fiber bundles will be important the so-called *principal fiber bundles*. Before we can state the definition of a principal fiber bundle, we need to define a Lie group action on a manifold.

If $(G, \bullet)$ is Lie group, and $\mathcal{M}$ a smooth manifold, then a smooth map

$$\triangleright : G \times \mathcal{M} \to \mathcal{M},$$

satisfying

$$\begin{aligned} &(i) \quad e \triangleright m = m \\ &(ii) \quad g_2 \triangleright (g_1 \triangleright m) = (g_2 \bullet g_1) \triangleright m, \end{aligned}$$

is called a *left G-action* on the manifold $\mathcal{M}$, where e is the identity element in G [70–73].

Similarly, we can also define the right action on the manifold which is the relevant one for principal bundles.

The smooth map $\triangleleft$ is called a *right G-action* $\triangleleft : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$ such that

$$\begin{aligned} &(i) \quad m \triangleleft e = m \\ &(ii) \quad (m \triangleleft g_1) \triangleleft g_2 = m \triangleleft (g_1 \bullet g_2). \end{aligned}$$

Whenever we have a left action $\triangleright$ on $\mathcal{M}$, we can define the right action $\triangleleft$ as

$$m \triangleleft g := g^{-1} \triangleright m \tag{3.1.5}$$

To really see that defined the right action, we need check that $m \triangleleft e = m$,

$$m \triangleleft e \underbrace{=}_{\text{by def.}} e^{-1} \triangleright m = e \triangleright m = m, \tag{3.1.6}$$

and

$$\begin{aligned} (m \triangleleft g_1) \triangleleft g_2 = (g_1^{-1} \triangleright m) \triangleleft g_2 = g_2^{-1} \triangleright (g_1^{-1} \triangleright m) = (g_2^{-1} \bullet g_1^{-1}) \triangleright m = (g_1 \bullet g_2)^{-1} \triangleright m \\ = m \triangleleft (g_1 \bullet g_2). \end{aligned} \tag{3.1.7}$$

Therefore, we conclude that $\triangleleft$ is in fact a right G-action.

For any point $p \in \mathcal{M}$, we define its *orbit* $\mathcal{O}_m \subset \mathcal{M}$ under the action of the group as the set [70–73]

$$\mathcal{O}_m \equiv \{q \in M \mid \exists g \in G : \quad g \triangleright m = q\}. \tag{3.1.8}$$

The orbit is set of all points $q \in M$ that we can reach from m by acting with an element $g \in G$ [70–73]. We already encountered the notion of orbits in the context of configuration space of the gauge fields. We say that two points p and q are equivalent $p \sim q$ if there exist a group element $g \in G$ such that $q = g \triangleright p$. In other words, two points are equivalent if they lie on the same orbit. We also saw that for gauge theories the physically important space is the quotient space $\mathcal{M}/\sim\equiv \mathcal{M}/G$ which we called the *orbit space*. Another important notion that we are going to need is that of the *stabilizer*.

For any $m \in M$, we define the *stabilizer* $\mathcal{S}_m \subset G$ as the set of all points that leave the point m invariant [70–73]

$$S_m \equiv \{g \in G \,|\, g \triangleright m = m\}. \tag{3.1.9}$$

Whereas the orbits are a subspace of M, the stabilizer is a subgroup of G.

An action $\triangleright$ is called *free* if separately $\forall m \in M$ the stabilizer at the point is just the identity element $\mathcal{S}_m = \{e\}$ [70–73].

A crucial property of an free action is that the orbits are isomorphic to the group G. Now, that we are equipped with all the necessary tools, we are ready to state the definition of a principal fiber bundle.

A bundle $(E, \pi, \mathcal{M})$ is called a *principal G-bundle* if

$$\begin{aligned}
&(a) \quad E \text{ is a right G} - \text{space} \\
&(b) \quad \triangleleft \text{ is free} \\
&(c) \quad (E, \pi, \mathcal{M}) \simeq_{\text{bundle}} (E, \rho, E/G),
\end{aligned}$$

where the map $\rho: \ E \to E/\mathcal{G}$ is the map that takes an element of E and maps it to its equivalence class [70–73].

An explanation is in order. The first condition states that the total space E has to be a right G-space, which simply means that E is equipped with a right action $\triangleleft$. The second condition states that this right G-action has to be free, which means that every orbit is the group itself or isomorphic to the group. The last condition states that the bundle $(E, \pi, \mathcal{M})$ is considered to be a principal G-bundle if it is as a bundle isomorphic to the bundle $(E, \rho, E/G)$, whose fibers $\rho^{-1}[\epsilon]$[1] $\simeq G$ are the Lie group G.

Let $(P, \pi, \mathcal{M})$ and $(P', \pi', \mathcal{M}')$ be two principal G-bundles and (u, f) be two maps

$$u: P \to P' \qquad \text{and} \qquad f: \mathcal{M} \to \mathcal{M}'. \tag{3.1.10}$$

Then (u, f) is called a *principal bundle morphism* if the following diagram commutes

$$\begin{array}{ccc}
P & \xrightarrow{\ u\ } & P' \\
{\scriptstyle \triangleleft G}\uparrow & & \uparrow{\scriptstyle \triangleleft' G} \\
P & \xrightarrow{\ u\ } & P' \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\
\mathcal{M} & \xrightarrow{\ f\ } & \mathcal{M}'
\end{array},$$

where $\triangleleft$ and $\triangleleft'$ are two different right G- actions [70–73].

We observe that a principal bundle morphism must be first a bundle morphism i.e.,

$$\pi' \circ u = f \circ \pi \tag{3.1.11}$$

[1] With ϵ we labeled the equivalence classes.

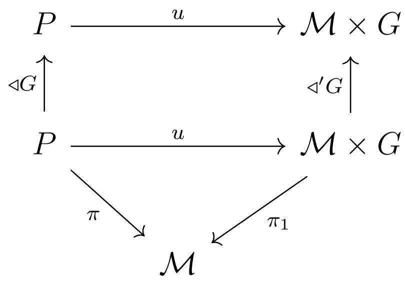
with the additional condition

$$u(m \triangleleft g) = u(m) \triangleleft' g, \quad \forall g \in G \quad \text{and} \quad \forall m \in \mathcal{M}. \tag{3.1.12}$$

A principal G-bundle $(P, \pi, \mathcal{M})$ is called *trivial* if it is isomorphic as a principal G-bundle to the product bundle $(\mathcal{M} \times G, \pi_1, \mathcal{M})$ [70–73]. For completeness the map π_1, just projects to the first factor,i.e., $\pi_1(m, g) = m$. Further, the right group action $\triangleleft'$ is defined as

$$(m, g) \triangleleft' g_1 \equiv (m, g \bullet g_1). \tag{3.1.13}$$

Diagrammatically, a principal bundle G-bundle $(P, \pi, \mathcal{M})$ is called *trivial* if there exist a map u such that the following diagram commutes



Finding a principal bundle map u to prove that a principal bundle is trivial is often very tricky. Luckily, there is a necessary and sufficient criterium which we will state in the following theorem.

Theorem 3.1 A principal G-bundle $(P, \pi, \mathcal{M})$ is trivial if and only if there exists a smooth section $\sigma : \mathcal{M} \to P$ [70–73].

Proof. ($\Rightarrow$) Suppose that $(P, \pi, \mathcal{M})$ is a trivial bundle. Then there exist a principal bundle map u from which we can constrict a section $\sigma : \mathcal{M} \to P$ as follows

$$\sigma(x) := u^{-1}(x, id_G), \tag{3.1.14}$$

where id_G is the identity in the group G. It is not hard to see that $\pi \circ u^{-1}(x, id_G) = \pi_1(x, id_G) = id_{\mathcal{M}}$, since u is a principal bundle map [70].

($\Leftarrow$) Suppose that a section $\sigma : \mathcal{M} \to P$ is given. We choose a point in the principal bundle which lies on some fiber, then by applying the projection map we project the point p down onto the base space, and with the section σ, we lifted it back up to the same fiber. Since the points p and $\sigma(\pi(p))$ lie on the same fiber there exist a group element $\tilde{g}_\sigma(p) \in G$, which depends on the chosen section σ and the point p, such that $\sigma(\pi(p)) \triangleleft \tilde{g}_\sigma(p) = p$. The uniqueness of $\tilde{g}_\sigma(p)$ follows from the property that $\triangleleft$ is a free action. Therefore, we have a map $\tilde{g}_\sigma : P \to G$. We observe that $\forall g \in G$ the following must hold

$$(\sigma(\pi(p)) \triangleleft \tilde{g}_\sigma(p)) \triangleleft g = p \triangleleft g, \tag{3.1.15}$$

and since $\triangleleft$ is a right action we have

$$(\sigma(\pi(p)) \triangleleft (\tilde{g}_\sigma(p) \bullet g) = p \triangleleft g. \tag{3.1.16}$$

We observe that we can rewrite the right hand-side of Eq. (3.1.16) as follows

$$
\begin{aligned}
p \triangleleft g &= \sigma(\pi(p \triangleleft g)) \triangleleft \tilde{g}_\sigma(p \triangleleft g)) \\
&= \sigma(\pi(p))\tilde{g}_\sigma(p \triangleleft g))
\end{aligned} \tag{3.1.17}
$$

where we used that $\pi(p) = \pi(p \triangleleft g)$. From Eq. (3.1.16) we observe that

$$
(\sigma(\pi(p)) \triangleleft (\tilde{g}_\sigma(p) \bullet g) = \sigma(\pi(p))\tilde{g}_\sigma(p \triangleleft g)), \tag{3.1.18}
$$

due to the fact that $\triangleleft$ is a free action we conclude that

$$
(\tilde{g}_\sigma(p) \bullet g) = \tilde{g}_\sigma(p \triangleleft g). \tag{3.1.19}
$$

Now, we are in the position to define a map $u_\sigma : P \to \mathcal{M} \times G$ as

$$
u_\sigma(p) \coloneqq (\pi(p), \tilde{g}_\sigma(p)). \tag{3.1.20}
$$

The last thing we need to show is that such constructed map u_σ is, in fact, a principal bundle map. First, we observe that u_σ is a bundle map $\pi_1 \circ u_\sigma = \pi$ since π_1 will just project the first factor of u_σ out which by our definition is $\pi(p)$. For u_σ to be a principal map the following relation must be true

$$
u_\sigma(p \triangleleft g) = u_\sigma(p) \triangleleft' g, \tag{3.1.21}
$$

where $\triangleleft'$ is the right G-action on the product bundle $(\mathcal{M} \times G, \pi_1, \mathcal{M})$. By definition of u_σ, we have

$$
\begin{aligned}
u_\sigma(p \triangleleft g) &= (\pi(p \triangleleft g), \tilde{g}_\sigma(p \triangleleft g)) \\
&= (\pi(p), \tilde{g}_\sigma(p) \bullet g)),
\end{aligned} \tag{3.1.22}
$$

where we applied Eq. (3.1.19). Using the definition of the right G-action on the product bundle Eq. (3.1.13) yields

$$
(\pi(p), \tilde{g}_\sigma(p) \bullet g)) = (\pi(p), \tilde{g}_\sigma(p)) \triangleleft' g, \tag{3.1.23}
$$

which completes the proof [70].

Therefore, we conclude that we can have a globally well-defined section if and only if the principal G-bundle is trivial. To define the push-forward and pull-back map, we need the definition of the tangent bundle $(T\mathcal{M}, \pi, \mathcal{M})$ and the cotangent bundle $(T^*\mathcal{M}, \pi, \mathcal{M})$.

Given a smooth manifold $\mathcal{M}$ with a curve $\gamma : \mathbb{R} \to \mathcal{M}$ such that at parameter value zero, $\gamma(0) = p$, where $p \in \mathcal{M}$. Then the *directional derivative operator* at the point p along the curve γ is the linear map $X_{\gamma,p}$:

$$
\begin{aligned}
X_{\gamma,p} : \quad & C^\infty(\mathcal{M}) \to \mathbb{R} \\
& f \to (f \circ \gamma)'(0),
\end{aligned} \tag{3.1.24}
$$

where the prime denotes the derivative. In differential geometry, the direction derivative operator $X_{\gamma,p}$ is called the *tangent vector* to the curve γ at the point $p \in \mathcal{M}$ [70–73].

The *tangent vector space* at $p \in \mathcal{M}$ is the set of all tangent vectors $X_{\gamma,p}$ at the point p equipped with the pointwise addition $\oplus_{T_p\mathcal{M}}$ and multiplication $\odot_{T_p\mathcal{M}}$. [70–73]

The *tangent bundle* $(T\mathcal{M}, \pi, \mathcal{M})$ is a bundle over $\mathcal{M}$, whose fiber over a point $p \in M$

is the tangent space at that point. Total space $T\mathcal{M}$ is the disjoint union of the tangent spaces of $\mathcal{M}$ and the projection map π projects each tangent space $T_p\mathcal{M}$ down to p [70–73].

The tangent bundle is an example of a vector bundle.

The vector space dual of the tangent space is called the *cotangent space* $T_p^*\mathcal{M} := (T_p\mathcal{M})^*$ [70–73].

The *cotangent bundle* $(T^*\mathcal{M}, \pi, \mathcal{M})$ is the dual vector bundle to the tangent bundle $(T\mathcal{M}, \pi, \mathcal{M})$ [70–73].

Let $\mathcal{M}$ and $\mathcal{N}$ be two smooth manifolds with a smooth map $\phi : \mathcal{M} \to \mathcal{N}$. Then the *push-forward* map $\phi_* : T\mathcal{M} \to T\mathcal{N}$ is defined by

$$\phi_*(X)f := X(f \circ \phi), \tag{3.1.25}$$

where $f \in C^\infty(\mathcal{N})$ and $X \in T\mathcal{M}$ [70–73]. Diagrammatically, we have

$$\begin{array}{ccccc} T\mathcal{M} & \xrightarrow{\phi_*} & \mathcal{N} & & \\ \downarrow{\scriptstyle \pi_{T\mathcal{M}}} & & \downarrow{\scriptstyle \pi_{T\mathcal{N}}} & & \\ \mathcal{M} & \xrightarrow{\phi} & \mathcal{N} & \xrightarrow{f} & \mathbb{R} \end{array}$$

Given two smooth manifolds $\mathcal{M}$ and $\mathcal{N}$ with a smooth map $\phi : \mathcal{M} \to \mathcal{N}$. Then the *pull-back* map $\phi^* : T^*\mathcal{N} \to T^*\mathcal{M}$ is defined by

$$\phi^*(\omega)X := \omega(\phi_* X), \tag{3.1.26}$$

where $\omega \in T^*\mathcal{N}$ and $X \in T\mathcal{M}$. The mnemonic: "Vectors are pushed-forward and covectors are pulled-back" can be useful to remember these two maps. [70–73]

Given a principal $\mathcal{G}$-bundle $P \xrightarrow{\pi} \mathcal{M}$, we define a *connection* on P as a Lie-algebra valued one-form $\omega : \Gamma(TP) \to T_eG$ satisfying

$$\begin{array}{ll} (i) & \omega(X^A) = A \\ (ii) & (\triangleleft\, g)^*\omega = Ad_{g^{-1}*}(\omega), \end{array}$$

where X^A are the vector fields generated by a Lie algebra element $A \in T_eG$ and $Ad_{g*} : T_eG \to T_eG$ is induced by the adjoin map $Ad_g : G \to G$ with $h \to g \bullet h \bullet g^{-1}$. [70–73]

In practice we want to restrict our attention to a local patch $\mathcal{U}_i \subseteq \mathcal{M}$ on the base manifold, which in physics represents space-time. Therefore, for calculational purposes we consider the following bundle $(P, \pi, \mathcal{U})$.

If $(P, \pi, \mathcal{U})$ is a principal bundle with a local section $\sigma_i : \mathcal{U}_i \to P$. Then, we define the *Yang-Mills field* $A_i : \Gamma(T\mathcal{U}_i) \to T_eG$ as [70–73]

$$A_i := \sigma_i^*\omega. \tag{3.1.27}$$

Choosing a section σ_i by which we represent the connection 1-form on space-time corresponds to picking a particular gauge. It is important to note that ω lives in the principal bundle, whereas the Yang-Mills in the local chart in the base manifold. We also know, that given a local section we can define a local trivialisation of the principal bundle, called the *canonical local trivialisation* $h_i : \mathcal{U}_i \times G \to P$ by

$$h_i(m, g) \coloneqq \sigma_i(m) \triangleleft g. \tag{3.1.28}$$

Having a local trivialization enables us to define the local representation of the connection-1 form as $h_i^*\omega_{(m,g)} : T_{(m,g)}(\mathcal{U}_i \times G) \cong T_m\mathcal{U}_i \oplus T_gG \to TP$. The Yang-Mills field A_i and the local representation $h_i^*\omega_{(m,g)}$ are related by the following equation

$$h_i^*\omega_{(m,g)}(X, \gamma) = Ad_{g^{-1}*}(A_i(X)) + \Xi_g(\gamma), \tag{3.1.29}$$

where $X \in T_m\mathcal{U}_i$, $\gamma \in T_gG$ and $\Xi_g(\gamma) : \quad T_gG \to T_eG$ is the so-called *Mauer-Cartan form*, which is a Lie algebra valued one-form on the group G [70–73]

The following diagrammatic representation could be useful to avoid any confusion.

$$
\begin{array}{ccccccc}
 & & \mathcal{U}_i \times G & \xrightarrow{h_i} & P & & \\
 & & \uparrow{\scriptstyle \triangleleft G} & & \uparrow{\scriptstyle \triangleleft G} & & \\
h_i^*\omega & \overset{\textit{lives on}}{\dashrightarrow} & \mathcal{U}_i \times G & \xrightarrow{h_i} & P & \overset{\textit{lives on}}{\dashleftarrow} & \omega \\
 & & \downarrow{\scriptstyle \pi_1} & & {\scriptstyle\pi}\downarrow \; \uparrow{\scriptstyle\sigma_i} & & \\
 & & \mathcal{U}_i & \xrightarrow{\text{id}} & \mathcal{U}_i & \overset{\textit{lives on}}{\dashleftarrow} & A_i = \sigma_i^*\omega
\end{array},
$$

Given two overlapping local patches $\mathcal{U}_i$ and $\mathcal{U}_j$ with two sections $\sigma_i : \mathcal{U}_i \to P$ and $\sigma_j : \mathcal{U}_j \to P$, which are related as $\sigma_j = \sigma_i \triangleleft g$. The two Yang-Mills fields $A_i = \sigma_i^*\omega$ and $A_j = \sigma_j^*\omega$ are related by

$$A_j = g^{-1} A_i \, g + g^{-1} dg, \tag{3.1.30}$$

where d is the exterior derivative. In components $A = A_\mu dx^\mu$ and $d = \partial_\mu dx^\mu$, we obtain the familiar gauge transformation of the gauge field

$$A_\mu^g = g^{-1} A_\mu g + g^{-1}\partial_\mu g. \tag{3.1.31}$$

For completeness, we define the *the curvature 2-form* $\Omega : \Gamma(TP) \otimes \Gamma(TP) \to T_eG$ on the principal bundle P as the covariant derivative of the connection 1-form ω

$$\Omega \coloneqq D\omega \in \Lambda^2(P) \otimes T_eG, \tag{3.1.32}$$

where $\Lambda^p(P)$ is the set of all the p-forms in P and the covariant derivative of a q-form ϕ is defined as

$$D\phi(X_1, ..., X_{r+1}) \coloneqq d_P\phi(hor(X_1), ..., hor(X_{r+1})), \tag{3.1.33}$$

where $hor(X_i)$ is the horizontal component of X_i with $d_p\phi$ representing the exterior derivative on P.Given a section σ_i *Yang-Mills field strength* is defined as the pull-back of the curvature 2-form Ω

$$F_i \equiv \text{Riem}_i = \sigma_i^*\omega \in \Lambda^2(\mathcal{U}_i) \otimes T_eG. \tag{3.1.34}$$

To recall: "We started with a principal $\mathcal{G}$-bundle over the local chart $\mathcal{U} \subseteq \mathcal{M}$: $P \xrightarrow{\pi} \mathcal{U}$ equipped with a local section $\sigma : \mathcal{U} \to \mathcal{U} \times \mathcal{G}$. The first structure we defined was the connection one-form ω on the principal bundle. Then restricting ourselves to a local patch we introduce the Yang-Mills field $A := \sigma^*\omega \in \Omega^1(\mathcal{U}) \otimes T_e\mathcal{G}$ as the pull-back of the section σ_i. The second structure we defined was the curvature $\Omega := D\omega$, which when restricted to a local patch gave us the Yang-Mills field strength $F := \sigma^*\Omega \in \Lambda^2(\mathcal{U}) \otimes T_e\mathcal{G}$". All of this can be represented by the following bundle diagram

$$
\begin{array}{ccccc}
 & & P & & \\
 & & \uparrow {\scriptstyle \triangleleft G} & & \\
\Lambda^2(P) \otimes T_eG \ni D\omega := \Omega & \xrightarrow{\textit{lives on}} & P & \xleftarrow{\textit{lives on}} & \omega \in \Lambda^1(P) \otimes T_eG \\
 & & {\scriptstyle \pi} \downarrow \uparrow {\scriptstyle \sigma} & & \\
\Lambda^2(\mathcal{U}) \otimes T_eG \ni Riem/\mathcal{F} = \sigma^*\Omega & \xrightarrow{\textit{lives on}} & \mathcal{U} \subseteq \mathcal{M} & \xleftarrow{\textit{lives on}} & \sigma^*\omega =: \Gamma/\mathcal{A} \in \Lambda^1(\mathcal{U}) \otimes T_eG
\end{array}
$$

3.2 The Gribov ambiguity from the differential geometric perspective.

We already know that for non-trivial principal bundles there does not exist a global section. As is turns out the set of all irreducible connection 1-forms on a principal G-bundle $(P, \pi, \mathcal{M})$ form themselves a principal G-bundle $(\mathcal{A}, \pi, \mathcal{A}/\mathcal{G} := \mathcal{M})$, where $\mathcal{A}$ is the space of all irreducible connections on P and the gauge group $\mathcal{G}$ is defined as all the vertical automorphism on P reduced by the center of G. The fact that we cannot find a global section, or in other words, that we cannot globally choose a gauge, is precisely the Gribov ambiguity. However, it was shown by Mitter and Viallet in [74], that there exist a locally finite open cover $\mathcal{U} = \{\mathcal{U}_i\}$ of $\mathcal{M}$ together with a set of background gauge fields $\{A_0^i\}$ so that we can define a family of local sections Γ_i of $\mathcal{A} \to \mathcal{M}$ with

$$\Gamma_i = \{B \in \pi^{-1}(\mathcal{U}_i) \mid D^*_{A_0^i}(B - A_0^i) = 0\}. \tag{3.2.1}$$

Here $D^*_{A_0^i}$ is the adjoint operator of $D_{A_0^i}$, $\mathcal{U}_i \subset \mathcal{M}$ and $A_0^i \in \mathcal{A}$ is a background gauge field. Once we have a section we can define the canonical local trivialisation $\chi_i : \Gamma_i \times \mathcal{G} \to \pi^{-1}(\mathcal{U}_i)$ as

$$\chi_i(B, \Omega) := B^\Omega = \Omega^{-1} B \Omega + \Omega^{-1} d\Omega, \tag{3.2.2}$$

where $\Omega \in \mathcal{G}$ and $B \in \Gamma_i$. Now, on this product bundle we define the expectation value of gauge invariant observables $f \in C^\infty(\mathcal{A})$ on a local patch as

$$\langle f \rangle = \frac{\int_{\Gamma_i \times \mathcal{G}} \sqrt{\det G_i} e^{S_i^{\text{tot}}} \chi_i^* f}{\int_{\Gamma_i \times \mathcal{G}} \sqrt{\det G_i} e^{S_i^{\text{tot}}}}, \tag{3.2.3}$$

where G_i is the induced metric on the $\Gamma_i \times G$, whose determinant is given by

$$\det G_i = \nu (\det \mathcal{F}_i)^2 (\det \Delta_{A_0^i})^{-1}, \tag{3.2.4}$$

where $\mathcal{F}_i = D^*_{A_0^i} D_B$ is the Faddeev-Popov operator and $\Delta_{A_0^i} = D^*_{A_0^i} D_{A_0^i}$ is the covariant Laplacian. Furthermore, the action S_i^{tot} reads

$$S_i^{\text{tot}} = \chi_i S_{\text{inv}} + \pi^*_{\mathcal{G}} S_G, \tag{3.2.5}$$

where S_{inv} is the gauge invariant Yang-Mills action expressed in terms of the constrained field B, $S_G \in C^\infty(\mathcal{G})$ is an arbitrary function on G such that

$$\int_{\mathcal{G}} D\Omega \;\; e^{-S_G[\Omega]} < \infty, \tag{3.2.6}$$

with $\pi_G : \Gamma_i \times \mathcal{G} \to \mathcal{G}$. The form of the action S_i^{tot}, was derived in [75] by generalizing the stochastic quantization method of Parisi and Wu [76]. By modifying the drift and the diffusion term of the stochastic process, in such a way that the expectation values of gauge invariant observables where unchanged, they were able to take the equilibrium limit of the Fokker-Planck distribution which gave the modified Yang-Mills action. For all the details we refer the reader to [75] and the references within. To obtain global expectation values, Hueffel and Kelnhofer [77] proposed to take the sum of Eq. (3.2.3) over all the patches. The resulting equation reads

$$\langle f \rangle = \frac{\sum_i \int_{\Gamma_i \times \mathcal{G}} \sqrt{\det G_i}\, e^{S_i^{\text{tot}}} \, \chi_i^*(f \pi^* \gamma_i)}{\sum_i \int_{\Gamma_i \times \mathcal{G}} \sqrt{\det G_i}\, e^{S_i^{\text{tot}}} \, \chi_i^* \pi^* \gamma_i}, \tag{3.2.7}$$

where γ_i is a partition of unity on the base space $\mathcal{M}$. The global expectation value of f given by (3.2.7) can be proven to be independent of the choice of the background gauge fields $A_0^{(i)}$, of the choice of the partition of unity γ_i and of the choice of the locally finite cover U_i. This approach is an alternative solution to the Gribov problem in the language of differential geometry and one can, in principle, compute globally well-defined expectation values of gauge invariant observables.

Chapter 4

Local Issues in Gauge Fixing

4.1 Yang-Mills theory without Faddeev-Popov ghost fields

This part of the thesis addresses local issues associated with the gauge fixing. Aspects such as Gribov copies, the first Gribov region and fiber bundle techniques, which were introduced in the previous chapters, will be now of minor interest to us. Here we will be working in the perturbative regime where the gauge fixing condition is ideal. However, some of the methods learned before might be of future use, as both the Zwanziger's horizon function (see Eq. 2.4.25) and the action S_G (see Eq. (4.1.25)) are sharing similar structures. The goal of this section is to eliminate the Faddeev-Popov ghost fields from the theory by introducing specific finite contributions of the pure gauge degrees of freedom.

When restricted to a local patch Eq.(3.2.7) takes the form

$$\langle f \rangle = \frac{\int \mathcal{D}B \;\det \mathcal{F}_{\mathcal{B}}\, e^{-S_{\mathrm{inv}}[B]}\, f(B)}{\int \mathcal{D}B \det \mathcal{F}_{B}\, e^{-S_{\mathrm{inv}}[B]}} \frac{\int \mathcal{D}\Omega\, e^{-S_{\mathcal{G}}[\Omega]}}{\int \mathcal{D}\Omega\, e^{-S_{\mathcal{G}}[\Omega]}}, \tag{4.1.1}$$

where $f(B)$ is the gauge invariant quantity that we are interested in computing, B_μ is the constrained field satisfying the Lorentz gauge condition $\partial_\mu B_\mu = 0$ and $\det \mathcal{F}_{\mathcal{B}} = \det \partial_\mu D_\mu$ is the determinant of the Faddeev-Popov operator in the Lorentz gauge. We recall that the additional term $S_G[\Omega]$ was characterized by the fact that the integral $\int \mathcal{D}\Omega\, e^{-S_{\mathcal{G}}[\Omega]}$ is finite. However, we can always cancel the contributions from the integral over the gauge group, as gauge invariant quantities are independent of them. Canceling them leads to the usual Faddeev-Popov formula, which when written in terms of the constrained gauge fields B_μ, reads

$$\langle f \rangle = \frac{\int \mathcal{D}B \;\det \mathcal{F}_{\mathcal{B}}\, e^{-S_{\mathrm{inv}}[B]}\, f(B)}{\int \mathcal{D}B \det \mathcal{F}_{B}\, e^{-S_{\mathrm{inv}}[B]}}. \tag{4.1.2}$$

On the other hand, we can perform a transformation from the fields (B_μ, Ω) to the unconstrained fields A_μ. In terms of the unconstrained fields A_μ, the expectation value of f reads

$$\langle f \rangle = \frac{\int \mathcal{D}A\, e^{-S_{\mathrm{inv}}[A] - S_{\mathcal{G}}[\Omega(A)]}\, f(A)}{\int \mathcal{D}A\, e^{-S_{\mathrm{inv}}[A] - S_{\mathcal{G}}[\Omega(A)]}}. \tag{4.1.3}$$

It is important to note that by converting to the original gauge fields A_μ the Faddeev-Popov determinant has been eliminated by the Jacobian of the field transformation. Of course, this transformation would not make sense in the standard Faddeev-Popov path integral prescription as the resulting integration would be divergent and therefore ill-defined. The roll of the action $S_{\mathcal{G}}$ is twofold. First, it has to eliminate the gauge redundancy and second it has to reproduce the effects of the FP ghost fields. In our paper [78], we suggested to specify $S_{\mathcal{G}}$ as

$$S_{\mathcal{G}}[\Omega(A)] = \frac{1}{g^2}\int d^4x \mathrm{Tr}\left((\partial_\mu\theta(A)_\mu)(\partial_\nu\theta(A)_\nu)^\dagger\right), \tag{4.1.4}$$

where

$$\theta(A)_\mu = (\partial_\mu\Omega(A)^{-1})\,\Omega(A) \tag{4.1.5}$$

is defined in terms of the group element $\Omega(A)$ fulfilling

$$0 = \partial_\mu(\Omega(A)^{-1}\,A_\mu\,\Omega(A)) - \frac{i}{g}\partial_\mu((\partial_\mu\Omega(A)^{-1})\Omega(A)). \tag{4.1.6}$$

We are trying to find the group element $\Omega[A]$ that takes a gauge field A_μ to the gauge fixing surface. Solutions to Eq. (4.1.6) can only be obtained by expanding $\Omega(A)$ in a power series $\Omega = e^{iv} = 1 + iv - \frac{v^2}{2}$ where $v = v_1 g + v_2 g^2 + \mathcal{O}(g^3)$ and calculating the coefficients $v_1, v_2, ...v_n$ order by order in the perturbation theory. Since we are going to compute gauge invariant quantities up to $\mathcal{O}(g^2)$ the only relevant coefficients are v_1 and v_2. To make the calculations more transparent we rearrange the terms in Eq. (4.1.6) as

$$\underbrace{\partial_\mu(\Omega(A)^{-1}\,A_\mu\,\Omega(A))}_{\text{LHS}} = \underbrace{\frac{i}{g}\partial_\mu((\partial_\mu\Omega(A)^{-1})\Omega(A))}_{\text{RHS}}\,. \tag{4.1.7}$$

The left-hand-side (LHS) can be computed with the use of the Baker-Campbell-Hausdorff formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + ..., \tag{4.1.8}$$

where setting $A = -vi$ and $B = A_\mu$ leads to

$$\begin{aligned} \text{LHS} &= \partial_\mu\left(A_\mu - i[v, A_\mu]\right) \\ &= \partial_\mu A_\mu - ig\partial_\mu([v_1, A_\mu]). \end{aligned} \tag{4.1.9}$$

To compute the right-hand-side (RHS) we expand the group element in a power series $\Omega = 1 + iv_1 g + g^2(iv_2 - \frac{g^2}{2}v_1^2)$ and insert it in Eq. (4.1.7)

$$\text{RHS} = \partial^2 v_1 + g\left(\partial^2 v_2 + \frac{i}{2}\partial_\mu[\partial_\mu v_1, v_1]\right), \tag{4.1.10}$$

where we used the shorthand notation $\partial^2 := \partial_\mu\partial_\mu$. Therefore, we find that the coefficients v_1 and v_2 are given by

$$\partial^2 v_1 = \partial_\mu A_\mu \tag{4.1.11}$$

and

$$\partial^2 v_2 = i\partial_\mu\left(\frac{1}{2}[v_1, \partial_\mu v_1] - [v_1, A_\mu]\right). \tag{4.1.12}$$

Here the inverse of the Laplacian is defined by its action on a test function $f(x)$ as

$$\frac{1}{\partial^2}f(x) := \int d^4y\, G_0(x, y) f(y), \tag{4.1.13}$$

where $G_0(x, y)$ represents the unique Green's function.

We recall that restricting the path integral to the first Gribov region led to the breaking of BRST symmetry. Due to this fact, one is forced to consider non-local gauge invariant transverse field $\partial_\mu A^\Omega_\mu = 0$, which are determined by minimizing the functional $\mathrm{Tr} \int dx\, A^\Omega_\mu A^\Omega_\mu$ along the gauge orbit (see Eq. (2.1.7)). The reader should be familiar with this expression as it is the functional that was used in determining the first Gribov region. The group element $\Omega(A)$ that ensures the condition $\partial_\mu A^\Omega_\mu = 0$, is determined from Eq. (4.1.6) following the same steps as we did above. The obtained transverse gauge field A^Ω_μ, can be written in the following nice form

$$A^\Omega_\mu = \left(\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}\right)\left(A_\nu - ig[v_1, A_\nu] + \frac{ig}{2}[v_1, \partial_\nu v_1]\right) + \mathcal{O}(A^3). \qquad (4.1.14)$$

This was the starting point for M. A. L. Capri et al., [65] in deriving an exact nilpotent non-perturbative BRST transformation. It is our intention however to use $\Omega(A)$ in order to determine $S_G[\Omega]$ and not A^Ω_μ.

Once we obtained the coefficients v_1 and v_2, we can insert them into Eq. (4.1.4) and start calculating the contributions the gauge fixing action $S_G = S^0_G + S^1_G + S^2_G + \ldots$ order by order in perturbation theory. However, we observe that the action in Eq.(4.1.4) has in front of the integral a factor of $\frac{1}{g^2}$ and as we are interested in computing $S_{\mathcal{G}}$ up to the second order, we would have to expand the integrand up to the fourth order in perturbation theory. But fortunately for us, we can use Eq. (4.1.7) to rewrite S_G as

$$S_G[\Omega(A)] = \int d^4x \mathrm{Tr}[(\partial_\mu(\Omega(A)^{-1}\, A_\mu\, \Omega(A)))(\partial_\nu(\Omega(A)^{-1}\, A_\nu\, \Omega(A)))]. \qquad (4.1.15)$$

It is easy to see that at the zeroth order expansion, we obtain the standard gauge fixing term of Yang-Mills theory in the Feynman gauge

$$S^0_G = \int d^4x\, \mathrm{Tr}\left[(\partial_\mu A_\mu)^2\right] = \int d^4x\, (\partial_\mu A^a_\mu)(\partial_\nu A^b_\nu)\mathrm{Tr}\,[X^a X^b] = \frac{1}{2}\int d^dx\, (\partial_\mu A^a_\mu)^2 \qquad (4.1.16)$$

The first order expansion in the coupling constant g leads to a new triplic gauge field interaction term

$$\begin{aligned} S^1_G &= -ig \int d^4x\, \mathrm{Tr}\left[(\partial_\mu A_\mu)\, \partial_\nu [v_1, A_\nu]\right] + h.c. \\ &= -ig \int d^4x\, \mathrm{Tr}\left[(\partial_\mu A_\mu)\, \partial_\nu \left[\frac{\partial_\rho A_\rho}{\partial^2}, A_\nu\right]\right] + h.c. \\ &= -ig \int d^4x\, \left[(\partial_\mu A^a_\mu)\, \partial_\nu \left(\frac{\partial_\rho A^b_\rho}{\partial^2} A^c_\nu\right)\right] \mathrm{Tr}\left(X^a [X^b, X^c]\right) + h.c \\ &= -ig \int d^4x\, \left[(\partial_\mu A^a_\mu)\, \partial_\nu \left(\frac{\partial_\rho A^b_\rho}{\partial^2} A^c_\nu\right)\right] (if^{bcd})\, \mathrm{Tr}[X^a X^d] + h.c. \\ &= -gf^{abc} \int d^4x\, \left[(\partial_\rho \partial_\mu A^a_\mu)\left(\frac{\partial_\nu A^b_\nu}{\partial^2} A^c_\rho\right)\right]. \end{aligned} \qquad (4.1.17)$$

The second order expansion in g provides us with new quartic gauge field interaction terms

$$\begin{aligned} S^2_G = &-ig^2 \int d^4x\, \mathrm{Tr}\left((\partial_\mu A_\mu)\, \partial_\nu [v_2, A_\nu]\right) \\ &- \frac{1}{2} g^2 \int d^4x\, \mathrm{Tr}\left((\partial_\mu [v_1, A_\mu])\; (\partial_\nu [v_1, A_\nu])\right) \\ &- \frac{1}{2} g^2 \int d^4x\, \mathrm{Tr}\left((\partial_\mu A_\mu)\, \partial_\nu [v_1, [v_1, A_\nu]]\right) + h.c.. \end{aligned} \qquad (4.1.18)$$

Like in the previous equation, we need to evaluate the trace over the Lie algebra indices to derive the appropriate Feynman rules. For the first term in Eq. (4.1.18), we have

$$-ig^2\int d^4x\,\mathrm{Tr}\Big((\partial_\mu A_\mu)\,\partial_\nu[v_2,A_\nu]\Big) = ig^2\int d^4x\,\mathrm{Tr}\Big((\partial_\mu A_\mu)\,\partial_\nu[A_\nu,v_2]\Big)$$
$$= ig^2\int d^4x\,\mathrm{Tr}\left((\partial_\mu A_\mu)\,\partial_\nu\left[A_\nu, i\frac{\partial_\tau}{\partial^2}\left(\frac{1}{2}[v_1,\partial_\tau v_1]-[v_1,A_\tau]\right)\right]\right), \tag{4.1.19}$$

where in the last line we inserted Eq. (4.1.12). The first part of Eq. (4.1.19) reads

$$-\frac{g^2}{2}\int d^4x\,(\partial_\mu A^a_\mu)\partial_\nu\left(A^b_\nu\frac{\partial_\tau}{\partial^2}\left(\frac{\partial_\rho A^c_\rho}{\partial^2}\partial_\tau\left(\frac{\partial_\sigma A^d_\sigma}{\partial^2}\right)\right)\right)\mathrm{Tr}\Big(X^a[X^b,[X^c,X^d]\Big), \tag{4.1.20}$$

whereas for the second part we obtain

$$g^2\int d^4x\,(\partial_\mu A^a_\mu)\partial_\nu\left(A^b_\nu\frac{\partial_\sigma}{\partial^2}\left(\frac{\partial_\rho A^c_\rho}{\partial^2}A^d_\sigma\right)\right)\mathrm{Tr}\Big(X^a[X^b,[X^c,X^d]\Big). \tag{4.1.21}$$

Evaluating the traces reads

$$g^2 f^{abe}f^{cde}\int d^4x(\partial_\nu\partial_\mu A^a_\mu)A^b_\nu\left(\frac{\partial_\sigma}{\partial^2}\left(\frac{\partial_\rho A^c_\rho}{\partial^2}A^d_\sigma\right)-\frac{1}{2}\frac{\partial_\tau}{\partial^2}\left(\frac{\partial_\rho A^c_\rho}{\partial^2}\partial_\tau\left(\frac{\partial_\sigma A^d_\sigma}{\partial^2}\right)\right)\right), \tag{4.1.22}$$

where we multiplied the result by 2 to account for contributions coming from the hermitian conjugate part. For the second of Eq. (4.1.18), we have

$$-\frac{g^2}{2}\int d^4x\,\mathrm{Tr}\Big((\partial_\mu[v_1,A_\mu])\,(\partial_\nu[v_1,A_\nu])\Big) = -\frac{g^2}{2}\int d^4x\,\partial_\nu\left(\frac{\partial_\mu A^a_\mu}{\partial^2}A^b_\nu\right)\partial_\sigma\left(\frac{\partial_\rho A^c_\rho}{\partial^2}A^d_\sigma\right)$$
$$\times\quad \mathrm{Tr}\Big([X^a,X^b][X^c,X^d]\Big)$$
$$= \frac{g^2}{2}f^{abe}f^{cde}\int d^4x\,\partial_\nu\left(\frac{\partial_\mu A^a_\mu}{\partial^2}A^b_\nu\right)\partial_\sigma\left(\frac{\partial_\rho A^c_\rho}{\partial^2}A^d_\sigma\right), \tag{4.1.23}$$

where again the factor 1/2 coming from $\mathrm{Tr}[X^aX^b]=\delta^{ab}/2$ was canceled by the hermitian conjugate part. The final part of Eq. (4.1.18) reads

$$-\frac{g^2}{2}\int d^4x\,\mathrm{Tr}\Big((\partial_\mu A_\mu)\,\partial_\nu[v_1,[v_1,A_\nu]]\Big) = -\frac{g^2}{2}\int d^4x\,(\partial_\mu A^a_\mu)\partial_\sigma\left(\frac{\partial_\nu A^b_\nu}{\partial^2}\frac{\partial_\rho A^c_\rho}{\partial^2}A^d_\sigma\right)\mathrm{Tr}\Big(X^a[X^b,[X^c,X^d]]\Big)$$
$$= -\frac{g^2}{2}f^{abe}f^{cde}\int d^4x(\partial_\sigma\partial_\mu A^a_\mu)\left(\frac{\partial_\nu A^b_\nu}{\partial^2}\frac{\partial_\rho A^c_\rho}{\partial^2}A^d_\sigma\right). \tag{4.1.24}$$

In principle, one could compute S_G up to any order in g which will lead to higher and higher multi-point gauge field interaction terms.

We recall that in the first two chapters the inverse Faddeev-Popov operator played a significant role. It was the critical part of Zwanziger's horizon function as well as Gribov's no-pole condition. It is indeed possible to rewrite the infinitesimal form of S_G in terms of the inverse of the Faddeev-Popov operator as

$$S_G[\Omega(A)] = \frac{1}{g^2}\int d^4x\mathrm{Tr}\Big((\partial^2(\mathcal{F}_A^{-1}\partial_\mu A_\mu))(\partial^2(\mathcal{F}_A^{-1}\partial_\nu A_\nu))^\dagger\Big), \tag{4.1.25}$$

where we used the fact that the infinitesimal form of Eq. (4.1.7) is given by

$$0 = \partial_\mu A_\mu + \mathcal{F}_A v \implies v = -\mathcal{F}_A^{-1}\partial_\mu A_\mu. \tag{4.1.26}$$

The infinitesimal form of S_G shares a similar structure as Zwanziger's horizon function. Both Eq. (4.1.25) as well as Zwanziger's horizon function are non-local due to the presence of the inverse Faddeev-Popov operator. We hope that with the introduction of auxiliary fields also in our case it will be possible to eliminate the inverse Faddeev-Popov operator and to bring Eq. (4.1.25) into a local form. However, the task of localizing Eq. (4.1.25) might not be straightforward as it was for the horizon function. A difference between Eq. (4.1.25) and Eq. (2.4.25) is that the former is quadratic in $\mathcal{F}^{-1}$ whereas the latter is only linear.

4.2 Feynman Rules

Before we calculate gauge invariant observables in perturbation theory, we have to derive the Feynman rules corresponding to the new multi-point gauge interaction vertices. We start by separating the full action into the free-particle and interaction component, compute $Z_0[J]$ and introduce interactions via the perturbative expansion. For the rest of this theses, we will be working in four-dimensional space-time. The free part of the generating functional $Z_0[J]$, reads

$$Z_0[J] = e^{-\frac{1}{2}\int d^4x d^4y J_\mu(x) D^{ab}_{\mu\nu}(x-y) J_\nu(y)}, \tag{4.2.1}$$

where $D^{ab}_{\mu\nu}(x-y)$ is the gauge propagator in the Feynman gauge

$$D^{ab}_{\mu\nu}(x-y) = \delta^{ab} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{\delta_{\mu\nu}}{k^2}. \tag{4.2.2}$$

The new three-point gauge interaction vertex was given by

$$S_I^{\text{new-3}}[A] = -g f^{abc} \int d^4x \left[(\partial_\rho \partial_\mu A^a_\mu) \left(\frac{\partial_\nu A^b_\nu}{\partial^2} A^c_\rho \right) \right]. \tag{4.2.3}$$

To derive the corresponding Feynman rule we have to compute the following contribution to the partition function $Z[J]$

$$\begin{aligned} -S_I^{\text{new-3}}\left(\frac{\delta}{\delta J}\right) Z_0[J] = &\left[g f^{def} \int d^4x \left(\partial^x_\rho \partial^x_\mu \frac{\delta}{\delta J^d_\mu(x)} \right) \left(\frac{\partial^x_\nu}{\partial^2} \frac{\delta}{\delta J^e_\nu(x)} \right) \left(\frac{\delta}{\delta J^f_\rho(x)} \right) \right] \\ &\times \exp\left[-\frac{1}{2} \int d^4y_1 d^4y_2 J^s_\sigma(y_1) D^{st}_{\sigma\lambda}(y_1-y_2) J^t_\lambda(y_2) \right]. \end{aligned} \tag{4.2.4}$$

It is important to emphasize that not all terms from Eq. (4.2.4) are relevant for us. To understand why this is the case we have to look at the quantity we want to compute. As we said, our goal is to calculate the Feynman rule for the new 3-point gauge interaction vertex and to do so we have to compute the vertex part of

$$\langle 0|T\{A^a_\alpha(y) A^b_\beta(z) A^c_\gamma(w)\}|0\rangle = \left[\left(\frac{\delta}{\delta J^a_\alpha(y)} \right) \left(\frac{\delta}{\delta J^b_\beta(z)} \right) \left(\frac{\delta}{\delta J^c_\gamma(w)} \right) (-S_I^{\text{new-3}} Z_0[J]) \right]_{J=0}. \tag{4.2.5}$$

This implies that in the end when we set $J = 0$, all terms that have a multiplicative J dependence, apart from the one in $Z_0[J]$, drop out. It is easy to convince ourselves that the only term that gives a non-vanishing contribution to Eq. (4.2.5) is

$$-S_I^{\text{new-3}} Z_0[J] = -gf^{def} \int d^4x \left(\int d^4y_2 \partial^x_\rho \partial^x_\mu D^{dt}_{\mu\lambda}(x-y_2) J^t_\lambda(y_2) \right) \left(\int d^4y'_2 \frac{\partial^x_\nu}{\partial^2} D^{et'}_{\nu\lambda'}(x-y'_2) J^{t'}_{\lambda'}(y'_2) \right)$$
$$\times \left(\int d^4y''_2 D^{ft''}_{\rho\lambda''}(x-y''_2) J^{t''}_{\lambda''}(y''_2) \right) Z_0[J]. \tag{4.2.6}$$

The six different possibilities of performing the derivations in Eq.(4.2.5) read

$$\begin{aligned}\langle 0|T\{A^a_\alpha(y)A^b_\beta(z)A^c_\gamma(w)\}|0\rangle = &-gf^{abc} \int d^4x (\partial^x_\rho \partial^x_\mu D_{\mu\alpha}(x-y)) \left(\frac{\partial^x_\nu}{\partial^2} D_{\nu\beta}(x-z) \right) D_{\rho\gamma}(x-w) \\ &- gf^{bac} \int d^4x (\partial^x_\rho \partial^x_\mu D_{\mu\beta}(x-z)) \left(\frac{\partial^x_\nu}{\partial^2} D_{\nu\alpha}(x-y) \right) D_{\rho\gamma}(x-w) \\ &- gf^{cba} \int d^4x (\partial^x_\rho \partial^x_\mu D_{\mu\gamma}(x-w)) \left(\frac{\partial^x_\nu}{\partial^2} D_{\nu\beta}(x-z) \right) D_{\rho\alpha}(x-y) \\ &- gf^{acb} \int d^4x (\partial^x_\rho \partial^x_\mu D_{\mu\alpha}(x-y)) \left(\frac{\partial^x_\nu}{\partial^2} D_{\nu\gamma}(x-w) \right) D_{\rho\beta}(x-z) \\ &- gf^{cab} \int d^4x (\partial^x_\rho \partial^x_\mu D_{\mu\gamma}(x-w)) \left(\frac{\partial^x_\nu}{\partial^2} D_{\nu\alpha}(x-y) \right) D_{\rho\beta}(x-z) \\ &- gf^{bca} \int d^4x (\partial^x_\rho \partial^x_\mu D_{\mu\beta}(x-z)) \left(\frac{\partial^x_\nu}{\partial^2} D_{\nu\gamma}(x-w) \right) D_{\rho\alpha}(x-y).\end{aligned} \tag{4.2.7}$$

Inserting the momentum space representation of the gauge propagators

$$\begin{aligned} D_{\mu\alpha}(x-y) &= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} D_{\mu\alpha}(p), \\ D_{\nu\beta}(x-z) &= \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-z)} D_{\mu\alpha}(q), \\ D_{\rho\gamma}(x-w) &= \int \frac{d^4r}{(2\pi)^4} e^{-ir(x-w)} D_{\mu\alpha}(r), \end{aligned} \tag{4.2.8}$$

yields the following result

$$\begin{aligned}\langle 0|T\{A^a_\alpha(y)A^b_\beta(z)A^c_\gamma(w)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4r}{(2\pi)^4} e^{-ip\cdot y} e^{-iq\cdot z} e^{-ir\cdot w} D^{ad}_{\alpha\mu}(p) D^{be}_{\beta\nu}(q) D^{cf}_{\gamma\rho}(r) \\ \times (2\pi)^4 \delta^4(p+q+r) V^{def}_{\mu,\nu,\rho}(p,q,r),\end{aligned} \tag{4.2.9}$$

with

$$V^{def,\text{new}}_{\mu\nu\rho}(p,q,r) = igf^{def} \left[(p_\mu p_\rho - r_\mu r_\rho) \frac{q_\nu}{q^2} + (r_\rho r_\nu - q_\nu q_\rho) \frac{p_\mu}{p^2} + (q_\nu q_\mu - p_\nu p_\mu) \frac{r_\rho}{r^2} \right]. \tag{4.2.10}$$

The delta function $(2\pi)^4\delta^4(p+q+r)$ came from $\int d^4x e^{-ix\cdot(p+q+r)}$. Now, we turn our attention to the new 4-point gauge interaction vertices. It will turn out to be advantagues to split the new 4-point gauge interaction terms into terms where only two 2 of gauge fields are transverse and the rest. Why this is useful will be evident in the next chapter. The first 4-point gauge interaction term where only two of the gauge fields are transverse comes from Eq. (4.1.23)

$$\frac{g^2}{2} f^{abe} f^{cde} \int d^4x\, \partial_\nu \left(\frac{\partial_\mu A^a_\mu}{\partial^2} A^b_\nu \right) \partial_\sigma \left(\frac{\partial_\rho A^c_\rho}{\partial^2} A^d_\sigma \right) = \frac{g^2}{2} f^{abe} f^{cde} \int d^4x \left(\frac{\partial_\nu \partial_\mu A^a_\mu}{\partial^2} A^b_\nu \right) \left(\frac{\partial_\sigma \partial_\rho A^c_\rho}{\partial^2} A^d_\sigma \right)$$
$$+ g^2 f^{abe} f^{cde} \int d^4x \left(\frac{\partial_\nu \partial_\mu A^a_\mu}{\partial^2} A^b_\nu \right) \left(\frac{\partial_\rho A^c_\rho}{\partial^2} \partial_\sigma A^d_\sigma \right)$$
$$+ \frac{g^2}{2} f^{abe} f^{cde} \int d^4x \left(\frac{\partial_\mu A^a_\mu}{\partial^2} \partial_\nu A^b_\nu \right) \left(\frac{\partial_\rho A^c_\rho}{\partial^2} \partial_\sigma A^d_\sigma \right) \tag{4.2.11}$$

where for now, we are going to be interested only in the first term. Therefore, the first 4-point gauge interaction vertex reads

$$S_I^{\text{new-4.1}}[A] = \frac{g^2}{2} f^{abe} f^{cde} \int d^4x \left(\frac{\partial_\nu \partial_\mu A^a_\mu}{\partial^2} \right) A^b_\nu \left(\frac{\partial_\sigma \partial_\rho A^c_\rho}{\partial^2} \right) A^d_\sigma. \tag{4.2.12}$$

We are going to follow the same steps as we did in the case of the new 3-point gauge interaction vertex. The contribution to the partition function $Z(J)$ is given by

$$-S_I^{\text{new-4.1}} \left(\frac{\delta}{\delta J} \right) Z_0[J] = \left[-\frac{1}{2} g^2 f^{abe} f^{cde} \int d^4x \left(\frac{\partial_\nu \partial_\mu}{\partial^2} \frac{\delta}{\delta J^a_\mu} \right) \frac{\delta}{\delta J^b_\nu} \left(\frac{\partial_\rho \partial_\sigma}{\partial^2} \frac{\delta}{\delta J^c_\rho} \right) \frac{\delta}{\delta J^d_\sigma} \right]$$
$$\times \exp \left[-\frac{1}{2} \int d^4y_1 d^4y_2 J^s_\sigma(y_1) D^{st}_{\sigma\lambda}(y_1 - y_2) J^t_\lambda(y_2) \right]. \tag{4.2.13}$$

As we are interested only in connected 4-point functions

$$\langle 0 | T\{ A^a_\alpha(y) A^b_\beta(z) A^c_\gamma(w) A^d_\delta(v) \} | 0 \rangle = \left[\left(\frac{\delta}{\delta J^a_\alpha(y)} \right) \left(\frac{\delta}{\delta J^b_\beta(z)} \right) \left(\frac{\delta}{\delta J^c_\gamma(q)} \right) \left(\frac{\delta}{\delta J^d_\delta(v)} \right) (-S_I^{\text{new-4.1}} Z_0[J]) \right]_{J=0}, \tag{4.2.14}$$

the only relevant term from Eq. (4.2.13) is

$$-S_I^{\text{new-4.1}} \left(\frac{\delta}{\delta J} \right) Z_0[J] = -\frac{g^2}{2} f^{abe} f^{cde} \int d^4x \left(\int d^4y_2 \frac{\partial^x_\nu \partial^x_\mu}{\partial^2} D^{at}_{\mu\lambda}(x - y_2) J^t_\lambda(y_2) \right)$$
$$\times \left(\int d^4y_2' D^{bt'}_{\nu\lambda'}(x - y_2') J^{t'}_{\lambda'}(y_2') \right)$$
$$\times \left(\int d^4y_2'' \frac{\partial^x_\sigma \partial^x_\rho}{\partial^2} D^{ct''}_{\rho\lambda''}(x - y_2'') J^{t''}_{\lambda''}(y_2'') \right) \left(\int d^4y_2''' D^{dt'''}_{\sigma\lambda'''}(x - y_2''') J^{t'''}_{\lambda'''}(y_2''') \right) \exp[...]. \tag{4.2.15}$$

There will be 24 different ways that the four derivatives $\frac{\delta}{\delta J}$ in Eq. (4.2.14) can act on $-S_I^{\text{new-4.1}} Z_0[J]$. As it would be redundant to write them all out here, we are going to state the result only. Therefore, the solution to the connected 4-point function reads

$$\langle 0 | T\{ A^{a'}_\alpha(y) A^{b'}_\beta(z) A^{c'}_\gamma(w) A^{d'}_\delta(v) \} | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4r}{(2\pi)^4} \frac{d^4s}{(2\pi)^4} e^{-ip\cdot y} e^{-iq\cdot z} e^{-ir\cdot w} e^{-is\cdot v}$$
$$\times D_{\alpha\mu}(p)^{a'a} D_{\beta\nu}(q)^{b'b} D_{\gamma\rho}(r)^{c'c} D_{\delta\sigma}(s)^{d'd} (2\pi)^4 \delta^4(p + q + r + s)$$
$$\times V^{abcd,\,(1)}_{\mu\nu\rho\sigma}(p, q, r, s) \tag{4.2.16}$$

with

$$V^{abcd,\,(1)}_{\mu\nu\rho\sigma}(p,q,r,s) = -\,g^2\left[f^{abe}f^{cde}\left(\frac{p_\mu p_\nu}{p^2}-\frac{q_\mu q_\nu}{q^2}\right)\left(\frac{r_\rho r_\sigma}{r^2}-\frac{s_\rho s_\sigma}{s^2}\right)\right]$$
$$-\,g^2\left[f^{ace}f^{bde}\left(\frac{p_\mu p_\rho}{p^2}-\frac{r_\mu r_\rho}{r^2}\right)\left(\frac{q_\nu q_\sigma}{q^2}-\frac{s_\nu s_\sigma}{s^2}\right)\right]$$
$$-\,g^2\left[f^{ade}f^{bce}\left(\frac{p_\mu p_\sigma}{p^2}-\frac{s_\mu s_\sigma}{s^2}\right)\left(\frac{q_\rho q_\nu}{q^2}-\frac{r_\rho r_\nu}{r^2}\right)\right]. \qquad (4.2.17)$$

The last term for which only two of the gauge fields are transverse comes from the first part of Eq. (4.1.22)

$$g^2 f^{abe}f^{cde}\int d^4x\,(\partial_\nu\partial_\mu A^a_\mu)A^b_\nu\frac{\partial_\sigma}{\partial^2}\left(\frac{\partial_\rho A^c_\rho}{\partial^2}A^d_\sigma\right) = g^2 f^{abe}f^{cde}\int d^4x\,(\partial_\nu\partial_\mu A^a_\mu)A^b_\nu\frac{1}{\partial^2}\left(\frac{\partial_\sigma\partial_\rho A^c_\rho}{\partial^2}A^d_\sigma\right)$$
$$+\,g^2 f^{abe}f^{cde}\int d^4x\,(\partial_\nu\partial_\mu A^a_\mu)A^b_\nu\frac{1}{\partial^2}\left(\frac{\partial_\rho A^c_\rho}{\partial^2}\partial_\sigma A^d_\sigma\right). \qquad (4.2.18)$$

The second 4-point gauge interaction vertex with only two transverse gauge fields is

$$S_I^{\text{new-4.2}} = g^2 f^{abe}f^{cde}\int d^4x\,(\partial_\nu\partial_\mu A^a_\mu)A^b_\nu\frac{1}{\partial^2}\left(\frac{\partial_\sigma\partial_\rho A^c_\rho}{\partial^2}A^d_\sigma\right), \qquad (4.2.19)$$

which leads to

$$\langle 0|T\{A^{a'}_\alpha(y)A^{b'}_\beta(z)A^{c'}_\gamma(w)A^{d'}_\delta(v)\}|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{d^4s}{(2\pi)^4}e^{-ip\cdot y}e^{-iq\cdot z}e^{-ir\cdot w}e^{-is\cdot v}$$
$$\times\,D_{\alpha\mu}(p)^{a'a}D_{\beta\nu}(q)^{b'b}D_{\gamma\rho}(r)^{c'c}D_{\delta\sigma}(r)^{d'd}(2\pi)^4\delta^4(p+q+r+s)$$
$$\times\,V^{abcd,\,(2)}_{\mu\nu\rho\sigma}(p,q,r,s) \qquad (4.2.20)$$

with the following vertex factor

$$V^{abcd,\,(2)}_{\mu\nu\rho\sigma}(p,q,r,s) = -\,g^2 f^{abe}f^{cde}\left[\frac{(p_\mu p_\nu-q_\mu q_\nu)}{(r+s)^2}\left(\frac{r_\rho r_\sigma}{r^2}-\frac{s_\rho s_\sigma}{s^2}\right)+\frac{(r_\rho r_\sigma-s_\rho s_\sigma)}{(p+q)^2}\left(\frac{p_\mu p_\nu}{p^2}-\frac{q_\mu q_\nu}{q^2}\right)\right]$$
$$-\,g^2 f^{ace}f^{bde}\left[\frac{(p_\mu p_\rho-r_\mu r_\rho)}{(q+s)^2}\left(\frac{q_\nu q_\sigma}{q^2}-\frac{s_\nu s_\sigma}{s^2}\right)+\frac{(q_\nu q_\sigma-s_\nu s_\sigma)}{(p+r)^2}\left(\frac{p_\mu p_\rho}{p^2}-\frac{r_\rho r_\mu}{r^2}\right)\right]$$
$$-\,g^2 f^{ade}f^{bce}\left[\frac{(p_\mu p_\sigma-s_\mu s_\sigma)}{(q+r)^2}\left(\frac{q_\nu q_\rho}{q^2}-\frac{r_\nu r_\rho}{r^2}\right)+\frac{(q_\nu q_\rho-r_\nu r_\rho)}{(p+s)^2}\left(\frac{p_\mu p_\sigma}{p^2}-\frac{s_\sigma s_\mu}{s^2}\right)\right]. \qquad (4.2.21)$$

These were the only 4-point gauge interaction vertices where two of the gauge fields are transverse. The reason why we separated them from the others is that these will be the only ones that will lead to non-vanishing contributions for the gauge invariant quantity that we are going to calculate in the next chapter.

For the rest of this section, for completeness, we are going to write out the other 4-point gauge interactions and their corresponding Feynman rules. Combining the second part of Eq. (4.1.22) where we act with the partial derivative ∂_τ on the second term in the bracket

$$-\frac{g^2}{2}f^{abe}f^{cde}\int d^dx(\partial_\nu\partial_\mu A^a_\mu)A^b_\nu\frac{1}{\partial^2}\left(\frac{\partial_\rho A^c_\rho}{\partial^2}\partial_\sigma A^d_\sigma\right), \qquad (4.2.22)$$

and the second term of Eq. (4.2.18)

$$+g^2 f^{abe} f^{cde} \int d^4x\, (\partial_\nu \partial_\mu A_\mu^a) A_\nu^b \frac{1}{\partial^2} \left(\frac{\partial_\rho A_\rho^c}{\partial^2} \partial_\sigma A_\sigma^d \right), \tag{4.2.23}$$

leads to the following Feynman rule

$$\begin{aligned} V_{\mu\nu\rho\sigma}^{abcd,(3)}(p,q,r,s) =& -\frac{g^2 f^{abe} f^{cde}}{2} \left[\frac{(p_\mu p_\nu - q_\mu q_\nu)}{(r+s)^2} \left(\frac{r_\rho s_\sigma}{r^2} - \frac{r_\rho s_\sigma}{s^2} \right) + \frac{(r_\rho r_\sigma - q_\rho q_\sigma)}{(p+q)^2} \left(\frac{p_\mu q_\nu}{p^2} - \frac{p_\mu q_\nu}{q^2} \right) \right] \\ & - \frac{g^2 f^{ace} f^{bde}}{2} \left[\frac{(p_\mu p_\rho - r_\mu r_\rho)}{(q+s)^2} \left(\frac{q_\nu s_\sigma}{q^2} - \frac{q_\nu s_\sigma}{s^2} \right) + \frac{(q_\nu q_\sigma - s_\nu s_\sigma)}{(p+r)^2} \left(\frac{p_\mu r_\rho}{p^2} - \frac{p_\mu r_\rho}{r^2} \right) \right] \\ & - \frac{g^2 f^{ade} f^{bce}}{2} \left[\frac{(p_\mu p_\sigma - s_\mu s_\sigma)}{(q+r)^2} \left(\frac{q_\nu r_\rho}{q^2} - \frac{q_\nu r_\rho}{r^2} \right) + \frac{(q_\nu q_\rho - r_\nu s_\rho)}{(p+s)^2} \left(\frac{p_\mu s_\sigma}{p^2} - \frac{p_\mu s_\sigma}{s^2} \right) \right]. \end{aligned} \tag{4.2.24}$$

The action in Eq. (4.1.24),

$$S_I^{\text{new-3.4}} = -\frac{g^2}{2} f^{abe} f^{cde} \int d^dx (\partial_\sigma \partial_\mu A_\mu^a) \left(\frac{\partial_\nu A_\nu^b}{\partial^2} \frac{\partial_\rho A_\rho^c}{\partial^2} A_\sigma^d \right), \tag{4.2.25}$$

gives us the following Feynman rule

$$\begin{aligned} V_{\mu\nu\rho\sigma}^{abcd,(4)}(p,q,r,s) =& \frac{g^2 f^{abe} f^{cde}}{2} \left[\frac{p_\mu p_\sigma q_\nu r_\rho}{q^2 r^2} - \frac{p_\mu q_\nu q_\sigma r_\rho}{p^2 r^2} - \frac{p_\mu p_\rho q_\nu s_\sigma}{p^2 q^2} + \frac{p_\mu q_\nu q_\rho s_\sigma}{p^2 s^2} - \frac{q_\nu r_\rho r_\mu s_\sigma}{q^2 s^2} + \frac{p_\mu r_\nu r_\rho s_\sigma}{p^2 s^2} \right] \\ & + \frac{g^2 f^{abe} f^{cde}}{2} \left[\frac{s_\mu s_\sigma q_\nu r_\rho}{q^2 r^2} - \frac{p_\mu r_\rho s_\nu s_\sigma}{p^2 r^2} \right] + \frac{g^2 f^{ace} f^{bde}}{2} \left[\frac{p_\mu p_\sigma q_\nu r_\rho}{q^2 r^2} - \frac{p_\mu q_\nu r_\rho r_\sigma}{p^2 q^2} + \frac{p_\mu q_\nu q_\rho s_\sigma}{p^2 s^2} \right] \\ & + \frac{g^2 f^{ace} f^{bde}}{2} \left[-\frac{p_\mu p_\nu r_\rho s_\sigma}{r^2 s^2} - \frac{q_\mu q_\nu r_\rho s_\sigma}{r^2 s^2} + \frac{p_\mu r_\nu r_\rho s_\sigma}{p^2 s^2} + \frac{q_\nu r_\rho s_\mu s_\sigma}{q^2 r^2} - \frac{p_\mu q_\nu s_\rho s_\sigma}{p^2 q^2} \right] \\ & + \frac{g^2 f^{ade} f^{bce}}{2} \left[\frac{p_\mu q_\nu q_\sigma r_\rho}{p^2 r^2} - \frac{p_\mu q_\nu r_\rho r_\sigma}{p^2 q^2} + \frac{p_\mu p_\rho q_\nu s_\sigma}{q^2 s^2} - \frac{p_\mu p_\nu r_\rho s_\delta}{r^2 s^2} - \frac{q_\mu q_\nu r_\rho s_\sigma}{r^2 s^2} + \frac{q_\nu r_\mu r_\rho s_\sigma}{q^2 s^2} \right] \\ & + \frac{g^2 f^{ade} f^{bce}}{2} \left[\frac{p_\mu r_\rho s_\nu s_\sigma}{p^2 r^2} - \frac{p_\mu q_\nu s_\rho s_\sigma}{p^2 q^2} \right] \end{aligned} \tag{4.2.26}$$

The Feynman rule for the second last term of Eq. (4.2.11)

$$S_I^{\text{new-4.5}} = g^2 f^{abe} f^{cde} \int d^4x \left(\frac{\partial_\nu \partial_\mu A_\mu^a}{\partial^2} A_\nu^b \right) \left(\frac{\partial_\rho A_\rho^c}{\partial^2} \partial_\sigma A_\sigma^d \right) \tag{4.2.27}$$

is

$$\begin{aligned} V_{\mu\nu\rho\sigma}^{abcd,(5)}(p,q,r,s) =& -g^2 f^{abe} f^{cde} \left[\left(\frac{p_\mu p_\nu}{p^2} - \frac{q_\mu q_\nu}{q^2} \right) \left(\frac{r_\rho s_\sigma}{r^2} - \frac{r_\rho s_\sigma}{s^2} \right) + \left(\frac{r_\rho r_\sigma}{r^2} - \frac{s_\rho s_\sigma}{s^2} \right) \left(\frac{p_\mu q_\nu}{p^2} - \frac{p_\mu q_\nu}{q^2} \right) \right] \\ & - g^2 f^{ace} f^{bde} \left[\left(\frac{p_\mu p_\rho}{p^2} - \frac{r_\mu r_\rho}{r^2} \right) \left(\frac{q_\nu s_\sigma}{q^2} - \frac{q_\nu s_\sigma}{s^2} \right) + \left(\frac{q_\nu q_\sigma}{q^2} - \frac{s_\nu s_\sigma}{s^2} \right) \left(\frac{p_\mu r_\rho}{p^2} - \frac{p_\mu r_\rho}{r^2} \right) \right] \\ & - g^2 f^{ade} f^{bce} \left[\left(\frac{p_\mu p_\sigma}{p^2} - \frac{s_\mu s_\sigma}{s^2} \right) \left(\frac{q_\nu r_\rho}{q^2} - \frac{q_\nu r_\rho}{r^2} \right) + \left(\frac{q_\nu q_\rho}{q^2} - \frac{r_\rho r_\nu}{r^2} \right) \left(\frac{p_\mu s_\sigma}{p^2} - \frac{p_\mu s_\sigma}{s^2} \right) \right] \end{aligned} \tag{4.2.28}$$

The last term of Eq. (4.2.11)

$$S_I^{\text{new-4.6}} = \frac{g^2}{2} f^{abe} f^{cde} \int d^4x \left(\frac{\partial_\mu A_\mu^a}{\partial^2} \partial_\nu A_\nu^b \right) \left(\frac{\partial_\rho A_\rho^c}{\partial^2} \partial_\sigma A_\sigma^d \right), \tag{4.2.29}$$

leads to the following Feynman rule

$$\begin{aligned}V^{abcd,\,(6)}_{\mu\nu\rho\sigma}(p,q,r,s) =& -g^2\left[f^{abe}f^{cde}\left(\frac{p_\mu q_\nu}{p^2}-\frac{p_\mu q_\nu}{q^2}\right)\left(\frac{r_\rho s_\sigma}{r^2}-\frac{r_\rho s_\sigma}{s^2}\right)\right]\\ &-g^2\left[f^{ace}f^{bde}\left(\frac{p_\mu r_\rho}{p^2}-\frac{p_\mu r_\rho}{r^2}\right)\left(\frac{q_\nu s_\sigma}{q^2}-\frac{q_\rho s_\sigma}{s^2}\right)\right]\\ &-g^2\left[f^{ade}f^{bce}\left(\frac{p_\mu s_\sigma}{p^2}-\frac{p_\mu s_\sigma}{s^2}\right)\left(\frac{q_\nu r_\rho}{q^2}-\frac{q_\nu r_\rho}{r^2}\right)\right].\end{aligned} \tag{4.2.30}$$

For completeness, the standard Yang-Mills 3-point gauge interaction vertex is given by

$$V^{abc}_{\mu\nu\rho}(p,q,r) = if^{abc}[(q-r)_\mu\delta_{\nu\rho}+(r-p)_\nu\delta_{\mu\rho}+(p-q)_\rho\delta_{\mu\nu}]. \tag{4.2.31}$$

Chapter 5

Proof of the equivalence between the Faddeev-Popov method and its modified version

5.1 Calculation of $F^a_{\mu\nu}F^a_{\mu\nu}$

In the previous chapter, we saw that the expansion of S_G leads to new interaction terms between the gauge bosons with complicated Feynman rules. The goal of this chapter is to implement the newly obtained Feynman rules and show that they reproduce the same contributions as the standard Faddeev-Popov ghost fields. Expanding $F^a_{\mu\nu}F^a_{\mu\nu}$ up to second order in the coupling constant g yields

$$\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle_{\text{new}} = \int \frac{d^4k}{(2\pi)^4}e^{-ikx}\int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}(2\pi)^4\delta(k-p-p')\langle F^a_{\mu\nu}(p)F^a_{\mu\nu}(p')\rangle_{\text{new}} \tag{5.1.1}$$

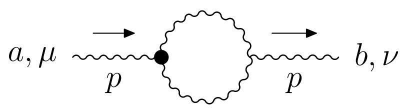
with

$$\langle F^a_{\mu\nu}(p)F^a_{\mu\nu}(p')\rangle_{\text{new}} = (2\pi)^4\delta^4(p+p')\Big[(p^2\delta_{\mu\nu}-p_\mu p_\nu)\frac{1}{p^4}\delta^{ab}\sum_{i=1}^{3}\Pi^{ab,(i)}_{\mu\nu}(p) - igf^{abc}p_\mu I^{abc}_{\nu\mu\nu}(p)\Big]. \tag{5.1.2}$$

The subscript "new" refers to the contributions arising from the new three- and four-point gauge field interaction vertices[1]. The $\Pi^{ab,(i)}_{\mu\nu}(p)$'s are the one particle irreducible Feynman diagrams contributing to the two-point function and $I^{abc}_{\nu\mu\nu}(p)$ is defined as

$$I^{abc}_{\nu\mu\nu}(p) = \int \frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}(2\pi)^4\delta^4(p+q+r)\frac{1}{q^2}\frac{1}{p^2}\frac{1}{r^2}V^{abc}_{\nu\mu\nu}(p,q,r), \tag{5.1.3}$$

The first diagram contributing to $\Pi^{ab}_{\mu\nu}$ which we are considering is



$$= \Pi^{ab,(1)}_{\mu\nu}(p) = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{1}{q^2}\frac{1}{r^2}V^{adc,(\text{new})}_{\mu\sigma\rho}(p,q,r)V^{bcd}_{\nu\rho\sigma}(-p,-r,-q)$$

$$\times (2\pi)^4\delta^4(p+q+r). \tag{5.1.4}$$

[1]The quantity F^2 has been rescaled by a factor of 1/2.

Here the new three-point gauge field interaction vertex is represented by a solid dot whereas the vertex without an unique label corresponds to the standard three-point Yang-Mills vertex. Inserting the corresponding Feynman rules in Eq. (5.1.4) gives us

$$\Pi^{ab,(1)}_{\mu\nu}(p) = g^2 f^{acd} f^{bcd} \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \frac{d^4r}{(2\pi)^4} \frac{1}{q^2} \frac{1}{r^2} N_{\mu\nu} (2\pi)^4 \delta^4(p+q+r), \tag{5.1.5}$$

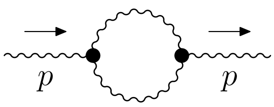
with

$$\begin{aligned} N_{\mu\nu} =& p_\mu p_\nu \left[\frac{r\cdot q - p\cdot q}{q^2} - \frac{r\cdot p - r\cdot q}{r^2} \right] + p_\mu q_\nu \left[\frac{r^2 - q\cdot r}{p^2} - \frac{p\cdot r}{r^2} + \frac{p^2}{q^2} \right] + p_\mu r_\nu \left[\frac{q^2}{p^2} - \frac{p\cdot q}{q^2} + \frac{p^2}{r^2} - \frac{q\cdot r}{p^2} \right] \\ &+ q_\mu q_\nu \left[\frac{r\cdot p}{r^2} \right] - q_\mu r_\nu \left[\frac{r\cdot p}{q^2} + \frac{p\cdot r}{r^2} \right] + r_\mu r_\nu \left[\frac{p\cdot q}{q^2} \right]. \end{aligned} \tag{5.1.6}$$

In principle, we would have to compute the same diagram with the solid dot on the second vertex instead on the first one. As both diagrams lead to the same contribution, we just multiply Eq. (5.1.6) with a factor of 2. Thus, the first contribution to the $\lim_{y\to x} \langle F^a_{\mu\nu}(x) F^a_{\mu\nu}(y) \rangle$ is

$$\begin{aligned} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^4} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \delta^{ab} \Pi^{ab,(1)}_{\mu\nu}(p) = (g f^{abc})^2 \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4r}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p+q+r) \\ \times \frac{p^2 q^2 - (p\cdot q)^2}{p^2 q^2 r^2} \left[-\frac{2}{p^2} \right]. \end{aligned} \tag{5.1.7}$$

The only other possible combination that can arise is the one where both interaction vertices are the new three-point gauge field vertices,

a, μ  b, ν

$$= \Pi^{ab,(2)}_{\mu\nu}(p) = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \frac{d^4r}{(2\pi)^4} \frac{1}{q^2} \frac{1}{r^2} V^{adc,(\text{new})}_{\mu\sigma\rho}(p,q,r) V^{bcd,(\text{new})}_{\nu\rho\sigma}(-p,-r,-q)$$

$$\times (2\pi)^4 \delta^{(4)}(p+q+r). \tag{5.1.8}$$

Inserting the appropriate Feynman rules gives us the following contribution

$$\Pi^{ab,(2)}_{\mu\nu}(p) = g^2 f^{acd} f^{bcd} \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \frac{d^4r}{(2\pi)^4} \frac{1}{q^2} \frac{1}{r^2} M_{\mu\nu} (2\pi)^4 \delta^{(4)}(p+q+r), \tag{5.1.9}$$

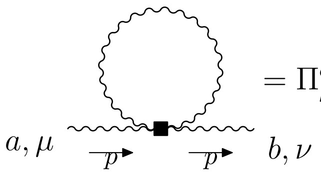
with

$$\begin{aligned} M_{\mu\nu} =& p_\mu p_\nu \left[\frac{(p\cdot r)(q\cdot r)}{q^2 p^2} - 2\frac{(q\cdot p)(r\cdot p)}{q^2 r^2} + 2\frac{(r\cdot q)(p\cdot q)}{p^2 r^2} - \frac{p\cdot q}{p^2} + \frac{p^2}{q^2} - 2\frac{p\cdot r}{p^2} + \frac{p^2}{r^2} \right] \\ &+ p_\mu p_\nu \left[\frac{r^4 - 2(r\cdot q)^2 + q^4}{p^4} \right] + p_\mu q_\nu \left[\frac{(p\cdot r)(q\cdot r)}{p^2 q^2} + \frac{r^2(q\cdot r) - (r\cdot q)q^2}{p^2 r^2} - \frac{p\cdot q}{r^2} \right] \\ &+ p_\mu r_\nu \left[\frac{p\cdot q}{q^2} - \frac{p\cdot r}{q^2} - \frac{(r\cdot q)r^2 - q^2(q\cdot r)}{q^2 p^2} \right] + q_\mu p_\nu \left[\frac{p\cdot r}{r^2} - \frac{p\cdot q}{r^2} + \frac{r^2(q\cdot r) - q^2(r\cdot q)}{p^2 r^2} \right] \\ &+ r_\mu p_\nu \left[\frac{q\cdot r}{p^2} - \frac{p\cdot r}{q^2} - \frac{r^2(q\cdot r)}{q^2 p^2} + \frac{p\cdot q}{q^2} \right] - q_\mu r_\nu + r_\mu r_\nu \left[\frac{r^2}{q^2} \right] - r_\mu q_\nu + q_\mu q_\nu \left[\frac{q^2}{r^2} \right]. \end{aligned} \tag{5.1.10}$$

Thus, the second contribution to $\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle$ from the combination new-new is

$$\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^4}(p^2\delta_{\mu\nu}-p_\mu p_\nu)\delta^{ab}\Pi^{ab,(2)}_{\mu\nu}(p) = (gf^{abc})^2\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p+q+r) \times \frac{p^2q^2-(p\cdot q)^2}{p^2q^2r^2}\left[\frac{q^2+r^2}{p^2r^2}\right]. \tag{5.1.11}$$

Equations (5.1.7) and (5.1.11) are the only contributions to $\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle$ at $\mathcal{O}(g^2)$ coming from the new 3-point gauge field interactions. The story does not end here since we also have contributions from the new 4-point gauge field interactions to $\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle$ which are provided by the following diagram



$$= \Pi^{ab,(3)}_{\mu\nu}(p) = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{1}{q^2}V^{abcd}_{\mu\nu\sigma\rho}(p,-p,q,-q)\delta_{\rho\sigma}\delta^{cd}(2\pi)^4\delta^4(p+q+r). \tag{5.1.12}$$

Here $V^{abcd}_{\mu\nu\sigma\rho}$ is the new 4-point gauge interaction vertices derived in the previous chapter. Inserting the Feynman rules from equations (4.2.17) and (4.2.21) in $\Pi^{ab,(3)}_{\mu\nu}$ leads to

$$\Pi^{ab,(3)}_{\mu\nu}(p) = g^2 f^{acd}f^{bcd}\int \frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{1}{q^2}\frac{1}{r^2}\left(O^{(1)}_{\mu\nu}+O^{(2)}_{\mu\nu}\right)(2\pi)^4\delta^4(p+q+r), \tag{5.1.13}$$

whereby

$$O^{(1)}_{\mu\nu} = -p_\mu p_\nu\left[\frac{r^2}{p^2}\right] + q_\mu p_\nu\left[\frac{r^2(p\cdot q)}{q^2p^2}\right] + p_\mu q_\nu\left[\frac{r^2(p\cdot q)}{p^2q^2}\right] - q_\mu q_\nu\left[\frac{r^2}{q^2}\right]. \tag{5.1.14}$$

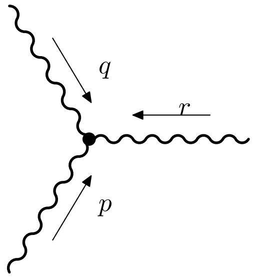
and

$$O^{(2)}_{\mu\nu} = -2p_\mu p_\nu - 2q_\mu q_\nu + 2p_\mu q_\nu\left[\frac{p\cdot q}{p^2}\right] + q_\mu p_\nu\left[\frac{q\cdot p}{p^2}+\frac{p\cdot q}{q^2}\right]. \tag{5.1.15}$$

Throughout the calculation, we used several symmetry arguments one of which was the invariance of the integrand under $p\to -p$, or any of the other momenta. Equation (5.1.13) is the only non-zero contribution to the $\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle$ stemming from $S^{(2)}_G$. All the other contributions from $S^{(2)}_G$ are annihilated by the action of $(p^2\delta_{\mu\nu}-p_\mu p_\nu)$ on $\Pi_{\mu\nu}$'s. Therefore, the full contribution from the new 4-point interactions to $\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle$ at $\mathcal{O}(g^2)$ reads

$$\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^4}(p^2\delta_{\mu\nu}-p_\mu p_\nu)\delta^{ab}\Pi^{ab,(3)}_{\mu\nu}(p) = (gf^{abc})^2\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p+q+r) \times\frac{p^2q^2-(p\cdot q)^2}{p^2q^2r^2}\left[\frac{-2r^2-q^2}{p^2r^2}\right]. \tag{5.1.16}$$

The final Feynman diagram contributing to $I^{abc}_{\nu\mu\nu}$ gives



$$= I^{abc}_{\nu\mu\nu}(p) = \int \frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{1}{p^2}\frac{1}{q^2}\frac{1}{r^2}V^{abc}_{\nu\mu\nu}(p,q,r)(2\pi)^4\delta^4(p+q+r). \tag{5.1.17}$$

Multiplying the vertex factor $V^{abc}_{\nu\mu\nu}(p,q,r)$ with p_μ reads

$$\begin{aligned} p_\mu V^{abc}_{\nu\mu\nu}(p,q,r) &= igf^{abc}p_\mu\left[(p^2-r^2)\frac{q_\mu}{q^2} + ((r\cdot p)r_\mu - (q\cdot p)q_\mu)\frac{1}{p^2} + ((q\cdot r)q_\mu - (r\cdot p)p_\mu)\frac{1}{r^2}\right] \\ &= igf^{abc}\left[(p^2-r^2)\frac{(q\cdot p)}{q^2} + \frac{(r\cdot p)^2-(q\cdot p)^2}{p^2} + \frac{(q\cdot r)(q\cdot p)-(r\cdot p)p^2}{r^2}\right]. \end{aligned} \tag{5.1.18}$$

The second term in Eq. (5.1.18) as well as terms $p^2(p\cdot q)/q^2 - p^2(p\cdot r)/r^2$ vanish since they are anti-symmetric in r and q. Hence, we are left with

$$\begin{aligned} p_\mu V^{abc}_{\nu\mu\nu}(p,q,r) &= igf^{abc}\left[-\frac{r^2(q\cdot p)}{q^2} + \frac{(q\cdot r)(q\cdot p)}{r^2}\right] \\ &= igf^{abc}\left[\frac{r^2q^2+r^2(r\cdot q)}{q^2} - \frac{(r\cdot q)^2+(r\cdot q)q^2}{r^2}\right], \end{aligned} \tag{5.1.19}$$

where we used momentum conservation. We notice that also the terms $(r\cdot q)(r^2/q^2 - q^2/r^2)$ vanish as they are anti-symmetric in r and q whereas the integral is symmetric. Therefore,

$$\begin{aligned} p_\mu V^{abc}_{\nu\mu\nu}(p,q,r) &= igf^{abc}\left[r^2 - \frac{(r\cdot q)^2}{r^2}\right] \\ &= igf^{abc}\left[q^2 - \frac{(p\cdot q)^2}{p^2}\right], \end{aligned} \tag{5.1.20}$$

where in the first term, we interchanged the momentum r with the momentum q and in the second term the momentum r with p. Thus, the final contribution to $\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle$ at $\mathcal{O}(g^2)$ turns out to be

$$-2igf^{abc}\int\frac{d^4p}{(2\pi)^4}p_\nu I^{abc}_{\nu\mu\nu}(p) = (gf^{abc})^2\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p+q+r)\frac{p^2q^2-(p\cdot q)^2}{p^2q^2r^2}\left[\frac{2}{p^2}\right]. \tag{5.1.21}$$

5.2 Faddeev-Popov ghost field contributions in $F^a_{\mu\nu}F^a_{\mu\nu}$

In the first chapter, we spent some time talking about the origin of the ghost fields and the essential role they play in the BRST symmetry. The goal of the fifth chapter is to show that for the gauge invariant quantity $\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle$ the gauge field interactions arising from S_G lead to the same outcome as the ghost fields in the standard Yang-Mills theory. In this section, we are going to take a little detour to the standard Yang-Mills theory where the ghost fields are still present and calculate their contributions to the quantity

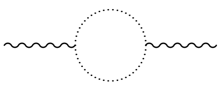
$\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle$ up to the second order in the coupling constant g. Feynman rules for the ghost fields are

$$\langle \bar{c}^a(p)c^b(q)\rangle = i\frac{\delta^{ab}}{p^2}(2\pi)^4\delta(p+q) \tag{5.2.1}$$

with the ghost-gauge field vertex given as

$$\langle \bar{c}^a(p)c^b(q)A^c_\mu(r)\rangle = f^{abc}p_\mu(2\pi)^4\delta^4(p+q+r). \tag{5.2.2}$$

A point that we already mentioned about the ghost fields is that they represent fictitious particles, meaning that in Feynman diagrams they can only appear as internal lines. This observation together with the fact that they only interact with gauge bosons via a ghost-antighost-gauge interaction implies that the only contribution to $\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle$ from the ghost fields comes from the following diagram

a, μ  b, ν

$$= \Pi^{ab,(\mathrm{gh})}_{\mu\nu}(p) = -g^2\int\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{i}{q^2}\frac{i}{r^2}(f^{acd}q_\mu)(-f^{bdc}r_\nu)(2\pi)^4\delta^4(p+q+r), \tag{5.2.3}$$

where we added the minus sign in front of the integral since ghosts are anti-commuting fields. The one-particle irreducible ghost diagram in Eq. (5.2.3) reads

$$\begin{aligned}\Pi^{ab,(\mathrm{gh})}_{\mu\nu}(p) &= g^2 f^{acd}f^{bcd}\int\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{1}{q^2}\frac{1}{r^2}q_\mu r_\nu(2\pi)^4\delta^4(p+q+r)\\ &= -g^2 f^{acd}f^{bcd}\int\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{1}{q^2}\frac{1}{r^2}\Big(q_\mu q_\nu + q_\mu p_\nu\Big)(2\pi)^4\delta^4(p+q+r),\end{aligned} \tag{5.2.4}$$

where we used the fact that $r_\nu = -(p+q)_\nu$. Applying $p^2\mathcal{P}_{\mu\nu} = (p^2\delta_{\mu\nu} - p_\mu p_\nu)$ gives us the ghost contribution to $\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle$ up to the second order in perturbation theory

$$\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle_{\mathrm{gh}} = (gf^{abc})^2\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p+q+r)\frac{p^2q^2-(p\cdot q)^2}{p^2q^2r^2}\Big[-\frac{1}{p^2}\Big]. \tag{5.2.5}$$

5.3 Final Result

To obtain the final result, we need to add all the contributions to $\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle$ arising from S_G at $\mathcal{O}(g^2)$. Thus,

$$
\begin{aligned}
\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle_{\text{new}} &= \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^4}(p^2\delta_{\mu\nu}-p_\mu p_\nu)\delta^{ab}\sum_{i=1}^{3}\Pi^{ab,(i)}_{\mu\nu}(p) - igf^{abc}\int\frac{d^4p}{(2\pi)^4}p_\nu I^{abc}_{\nu\mu\nu}(p) \\
&= (gf^{abc})^2\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p+q+r)\frac{p^2q^2-(p\cdot q)^2}{p^2q^2r^2}B(p,q,r),
\end{aligned}
\tag{5.3.1}
$$

where the function $B(p,q,r)$[2] encodes the different contributions from the various diagrams, see Table I.

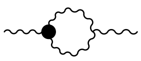
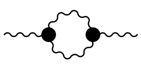
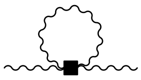
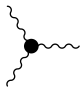
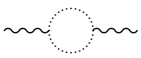
	$-\frac{2}{p^2}$
	$\frac{q^2+r^2}{p^2r^2}$
	$-\left(\frac{2r^2+q^2}{p^2r^2}\right)$
	$\frac{2}{p^2}$
$\sum$	$-\frac{1}{p^2}$
	$-\frac{1}{p^2}$

Table 5.1: Various contributions to $B(p,q,r)$. The solid dot represents the new three-point gauge field interaction vertex and the solid square the new four-point gauge field interaction vertex. Vertices without any special labeling correspond to the standard Yang-Mills ones. In the second last line of the table is the sum of the four diagrams displayed in the first four lines. The bottom line of the table displays the standard Faddeev-Popov ghost field contribution [78].

[2]We want to mention that all the prefactors in front of the two-point function and three-point function in $F^a_{\mu\nu}F^a_{\mu\nu}$ have been accounted for.

Inserting the various contributions from Table 1 into Eq. (5.3.1) yields

$$\begin{aligned}\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle_{\rm new} =&(gf^{abc})^2\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p+q+r)\frac{p^2q^2-(p\cdot q)^2}{p^2q^2r^2}\\ &\times\Big(-\frac{2}{p^2}+\frac{q^2}{p^2r^2}+\frac{1}{p^2}-\frac{2}{p^2}-\frac{q^2}{p^2r^2}+\frac{2}{p^2}\Big)\\ =&(gf^{abc})^2\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p+q+r)\frac{p^2q^2-(p\cdot q)^2}{p^2q^2r^2}\Big[-\frac{1}{p^2}\Big]\end{aligned} \tag{5.3.2}$$

Hence,

$$\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle_{\rm new}-\lim_{y\to x}\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\rangle_{gh}=0, \tag{5.3.3}$$

which proves that, up to second order in the coupling constant g, the modified Yang-Mills action without the Faddeev-Popov ghost fields yields the same result as one would obtain by using the standard Yang-Mills action.

5.4 Outlook

The efficiency of our method improves dramatically by employing software like Mathematica. Notice that within our approach only bosonic variables appear which reduces the computational complexity significantly. This cost reduction could be of importance also for computations in lattice QCD. One of our future projects will be to develop a Mathematica package that implements our approach and makes it accessible to the physics community.

We plan to test our method on various other gauge invariant quantities and for different gauge fixing conditions in order to compare its efficiency and performance with the usual Faddev-Popov procedure. We also intend to use our approach in order to address the issue of Gribov ambiguities because we expect that it will shed new light on this problem.

As P. W. Anderson said: " Physics is the study of symmetries." Therefore, an exciting line of work will be to investigate the new symmetries of the modified Yang-Mills action since the elimination of the ghost fields led to the loss of the BRST symmetry which was a crucial part of proving the renormalizability and unitarity of the theory. With the help of this new symmetry, we plan to develop generalized Slavnov-Tayler identities in order to set up a renormalization program. We hope that this new symmetry could be helpful in explaining some of the problems that we are facing within theoretical and mathematical physics, today.

Chapter 6

Summary

In the first chapter, we reviewed non-abelian gauge theories and their singular structure due to the gauge redundancy of the Lagrangian. A manifestly covariant way of eliminating the gauge degrees of freedom was developed by Faddeev and Popov [14]. As a consequence of fixing the gauge, fictitious particles called Faddeev-Popov ghosts emerge.

In the second chapter, we discussed issues associated with the extension of the gauge fixing procedure beyond the perturbative level. We saw that in the presence of Gribov copies the Faddeev-Popov determinant vanishes, calling for modifications of the Faddeev-Popov method. To avoid the Gribov copies, we had to restrict the path integral to the first Gribov region C_0. By using Gribov's no-pole condition and Zwanziger's horizon function, we were able to implement the restriction, which led to the modification of the IR behavior of the gauge and gluon propagators. In both of the approaches, the gluon propagator gets excluded from the physical spectrum which can be taken as a sign of confinement.

In the third chapter, we looked at the Gribov ambiguity from a more mathematical perspective. In the language of differential geometry, the absence of a global section on a principal bundle is equivalent to the Gribov problem. We looked at Huffel's and Kelnhofer's [77] proposal for dealing with the issue of the Gribov ambiguity in the differential geometric setting. Their idea was to partition the configuration space into local patches, where it is always possible to choose a gauge, and by summing over all patches to obtain the global result.

In the fourth chapter, we saw that by restricting Hüffel's and Kelnhofer's equation for computing global expectation values of gauge invariant observables to a local patch leads to a modification of the Faddeev-Popov path integral density by introducing specific finite contributions of the pure gauge degrees of freedom. The presence of the new degrees of freedom does not affect gauge invariant quantities, and in principle, they can be eliminated. However, Hüffel and Kelnhofer proved in [75] that by keeping the contributions from S_G, it is possible to eliminate the Faddeev-Popov determinant by a field transformation which leads to the quantization of Yang-Mills theory without Faddeev-Popov ghost fields. As a byproduct of eliminating the ghost fields new multi-point gauge interactions appear with complicated Feynman rules.

In the fifth chapter, we were able to show in a specific example that up to the second order in perturbation theory the newly obtained Feynman rules reproduce the same contributions as the Faddeev-Popov ghost fields.

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