MASTERARBEIT / MASTER´S THESIS

Titel der Masterarbeit / Title of the Master´s Thesis

„Absolute Continuity of Poisson Processes“

verfasst von / submitted by

Adrian Maksymowicz BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2018 / Vienna 2018

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von / Supervisor:	Assoz.-Prof. Privatdoz. Mag. Dr. Roland Zweimüller

Contents

Abstract

This thesis covers the connection between absolute continuity of Poisson processes and absolute continuity of their intensity measures. In the first chapter we define Poisson processes and prove various important results, which we will use later on. However the main focus lies in the second chapter, where we will show that two Poisson processes are mutually absolutely continuous if and only if their intensity measures are mutually absolutely continuous and the Hellinger distance between said intensity measures is finite. In case of mutual absolutely continuity we will also get a nice representation for the Hellinger distance between the Poisson measures. These results were originally obtained in a paper by Y.Takahashi. Extending the original work a little, we try to keep the setting as general as possible and only assume that our intensity measures are σ-finite and nonatomic. The space we are working with will only be equipped with a σ-algebra, but no topology.

Abstract (German version)

Diese Arbeit behandelt den Zusammenhang zwischen absoluter Stetigkeit von Poisson Prozessen und deren Intensitätsmaßen. Im ersten Kapitel werden wir Poisson Prozesse definieren und einige wichtige Resultate beweisen, welche wir im weiteren Verlauf der Arbeit verwenden werden. Das Hauptaugenmerk der Arbeit liegt jedoch im zweiten Kapitel, in dem wir zeigen werden, dass zwei Poisson Prozesse genau dann zueinander absolut stetig sind, wenn deren Intensitätsmaße zueinander absolut stetig sind und die Hellingerdistanz zwischen genannten Intensitätsmaßen endlich ist. Im Falle von absoluter Stetigkeit werden wir auch eine einfache Formel für die Berechnung der Hellingerdistanz zwischen Poisson Maßen herleiten. Diese Resultate wurden bereits in einem Paper von Y.Takahashi bewiesen. Aufbauend auf diese Arbeit, versuchen wir dabei das Setting so allgemein wie möglich zu halten und fordern lediglich, dass unsere Intensitätsmaße σ-endlich und nicht atomar sind. Unser Grundraum wird nur mit einer σ-algebra ausgestattet sein und keiner Topologie.

1 Introduction

Poisson processes are a class of very important stochastic processes. Intuitively one can think of them as a random set of points on some space $\mathbb{X}$. These points are Poisson distributed with some intensity measure λ, meaning if we take a measurable subset of our space $B \subseteq \mathbb{X}$, the number of points in this set is Poisson distributed with parameter $\lambda(B)$. The points in one such set also do not influence the number of points in another disjoint set. In other words the number of points in one set is independent from the number of points in another disjoint set, or in short Poisson processes have independent increments.

Poisson processes have many useful properties. For example they behave nicely under measurable maps. That is if we take a measurable map f and apply it to a Poisson process, we get another Poisson process and the intensity measure of the new process is simply $f(\lambda)$. Similarly we can discard all the Points in some measurable set $C \subseteq \mathbb{X}$ and still end up with a Poisson process. Another nice property is that we can take the sum of countably many such processes and again receive a Poisson process.

Our goal is to prove a theorem, which states that under some minor assumptions two Poisson processes are similar in a measure theoretic sense, i.e. absolutely continuous, if and only if their intensity measures are absolutely continuous and the Hellinger distance between the intensity measures is finite. The theorem was first proven by Yoichiro Takahashi, as can be seen in [2]. The main focus of this paper was to generalize this result and among others omit the need for a topology on our space.

However to do this, we need to first rigorously define Poisson processes and prove some important statements, that will aid us, when proving the desired theorem. This will be the object of the following chapter.

2 Poisson Processes

In this section we want to define Poisson processes and give some useful properties of such processes. Details can be found in [1] chapters 1-5.

2.1 Point Processes

We start by defining point processes, which essentially are random, at most countable collections of points in some space $\mathbb{X}$. In order to have a very general definition of such processes, we want to define them as random counting measures. A more intuitive, but therefore also less general approach can be found in [4].

Take some measurable space $(\mathbb{X}, \mathcal{X})$ and let $\mathbf{N}_{<\infty}(\mathbb{X}) \equiv \mathbf{N}$ be the space of measures μ with $\mu(B) \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ for all measurable sets $B \in \mathcal{X}$. Now let $\mathbf{N}(\mathbb{X}) \equiv \mathbf{N}$ denote the space of all measures that can be written as a countable sum of measures from $\mathbf{N}_{<\infty}$.

Note that not only Dirac measures δ_x, for some $x \in \mathbb{X}$ lie in $\mathbf{N}$, but also countable sums of dirac measures. That is for some $k \in \overline{\mathbb{N}}_0 := \mathbb{N} \cup \{\infty\}$

$$\mu = \sum_{n=1}^{k} \delta_{x_n}.$$

Of course there are also elements of $\mathbf{N}$ which do not have such a representation for general $\mathbb{X}$. To see this, assume $\mathbb{X} = [0, 1]$ and let $\mathcal{X}$ be the σ-algebra generated by all finite subsets of $\mathbb{X}$. Now for $B \in \mathcal{X}$, consider the measure

$$\mu(B) = \begin{cases} 0, & \text{if} \quad B \text{ is finite} \\ 1, & \text{if} \quad B \text{ is infinite} \end{cases} .$$

Clearly $\mu \in \mathbf{N}$, but one cannot write it as a sum of Dirac measures.

Furthermore consider the σ-algebra $\mathcal{N}(\mathbb{X}) = \mathcal{N}$, which is generated by the collection of all subsets of $\mathbf{N}$ of the form

$$\{\mu \in \mathbf{N} : \mu(B) = k\}, \quad B \in \mathcal{X}, k \in \mathbb{N}.$$

In other words, $\mathcal{N}$ is the smallest σ-algebra on $\mathbf{N}$ such that the mapping $\mu \to \mu(B)$ is measurable for all $B \in \mathcal{X}$.

Definition 1. *A* point process *on* $\mathbb{X}$ *is a random element* η *of* $(\mathbf{N}, \mathcal{N})$*, that is a measurable mapping* $\eta : \Omega \to \mathbf{N}$ *on some probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*.*

Remark 1. The mapping $\omega \to \eta(\omega, B) := \eta(\omega)(B)$ is a random variable taking values in $\overline{\mathbb{N}}_0$. We will use the notation $\eta(B)$ for this random variable.

Definition 2. *We say a point process* η *on* $\mathbb{X}$ *is a* proper point process *if there exist random elements* $X_1, X_2, ...$ *in* $\mathbb{X}$ *and an* $\overline{\mathbb{N}}_0$*-valued random variable* κ *such that almost surely*

$$\eta = \sum_{n=1}^{\kappa} \delta_{X_n}.$$

In the case $\kappa = 0$ *we interpret this as the zero measure.*

This definition is motivated by our intuition of a point process, that is a random set of points, as stated in the beginning of this chapter. However in many cases it is sufficient to consider only proper point processes. We will see later on that each Poisson process is proper up to equality in distribution. Furthermore one can show that if $\mathbb{X}$ is a Borel subspace of a complete metric space, then any locally finite point process on $\mathbb{X}$ is proper, as can be seen in [1] chapter 6.

Definition 3. *The* intensity measure *of a point process* η *on* $\mathbb{X}$ *is the meausre* λ *definded by*

$$\lambda(B) := \mathbb{E}[\eta(B)], \quad B \in \mathcal{X}.$$

This is indeed a measure. For fixed $\omega \in \Omega$, $\eta(\omega)$ is a measure, so it is nonnegative and maps the empty set to zero. Taking expectations does not change this fact. Finally to check σ-additivity consider some measurable sets $B_1, B_2, ... \in \mathcal{X}$. Then

$$
\begin{aligned}
\lambda\Big(\bigcup_{i=1}^{\infty} B_i\Big) &= \mathbb{E}\Big[\eta\Big(\bigcup_{i=1}^{\infty} B_i\Big)\Big] = \mathbb{E}\Big[\sum_{i=1}^{\infty} \eta(B_i)\Big] \\
&= \sum_{i=1}^{\infty} \mathbb{E}[\eta(B_i)] = \sum_{i=1}^{\infty} \lambda(B_i),
\end{aligned}
$$

where we used the linearity of expectation in the third step.

Definition 4. *The* distribution *of a point process η on $\mathbb{X}$ is the probability measure $\mathbb{P}_\eta$ on $(\mathbf{N}, \mathcal{N})$ given by $A \to \mathbb{P}(\eta \in A)$. If η' is another point process with the same distribution as η we write $\eta \overset{d}{=} \eta'$.*

The following definition gives us a powerful tool when working with point processes and their distributions, as can be seen in the subsequent proposition.

Definition 5. *The* Laplace functional *of a point process η on $\mathbb{X}$ is the mapping $L_\eta : \mathbb{R}_+(\mathbb{X}) \to [0, 1]$ defined by*

$$
L_\eta(u) := \mathbb{E}\Big[exp\Big(-\int u(x)\eta(dx)\Big)\Big], \quad u \in \mathbb{R}_+(\mathbb{X}).
$$

Here $\mathbb{R}_+(\mathbb{X})$ denotes the space of measurable functions on $\mathbb{X}$ taking values in $\mathbb{R}_+ = [0, \infty)$.

Proposition 1. *For point processes η and η' on $\mathbb{X}$ the following assertions are equivalent.*

(i) $\eta \overset{d}{=} \eta'$.

(ii) $(\eta(B_1), ..., \eta(B_m)) \overset{d}{=} (\eta'(B_1), ..., \eta'(B_m))$ *for all $m \in \mathbb{N}$ and all pairwise disjoint $B_1, ..., B_m \in \mathcal{X}$*

(iii) $L_\eta(u) = L_{\eta'}(u)$ *for all $u \in \mathbb{R}_+(\mathbb{X})$*

(iv) *for all $u \in \mathbb{R}_+(\mathbb{X}), \eta(u) \overset{d}{=} \eta'(u)$ as random variables in $\overline{\mathbb{R}}_+$.*

Proof. Can be found in [1] on pages 14-15. □

2.2 Poisson Processes

We are now almost able to define the main subject of this thesis - the Poisson process. Furthermore let us from here on fix an arbitrary measurable space $(\mathbb{X}, \mathcal{X})$ to save space.

Definition 6. *A measure ν on $\mathbb{X}$ is said to be* s-finite *if ν can be written as a countable sum of finite measures.*

Note that every measure $\mu \in \mathbf{N}$ is s-finite. Another nice property of s-finite measures is that, in contrast to σ-finite measures, any countable sum of s-finite measures is again s-finite.

We will also use the fact that each σ-finite measure is indeed also s-finite. To see this one can simply divide the whole space into countably many sets of finite measure, as in the definition for σ-finiteness, and restrict the given measure to each set respectively. The initial measure is then the countable sum of the restricted measures.

Definition 7. *A random variable X is said to have* Poisson distribution $Po(\gamma)$ with parameter $\gamma \geq 0$ *if*

$$\mathbb{P}(X = k) = Po(\gamma; k) := \frac{\gamma^k}{k!} e^{-\gamma}, \quad k \in \mathbb{N}_0.$$

Where we use the convention $0^0 = 1$. We also allow $\gamma = \infty$, in this case $\mathbb{P}(X = \infty) = 1$, which implies $Po(\infty, k) = 0$ for $k \in \mathbb{N}$.

Remark 2. The Laplace transform of a $Po(\gamma)$ distributed random variable X is given by

$$\mathbb{E}[e^{-tX}] = exp[-\gamma(1 - e^{-t})], \quad t \geq 0. \tag{1}$$

Definition 8. *Let λ be an s-finite measure on $\mathbb{X}$. A* Poisson process with intensity measure λ *is a point process η on $\mathbb{X}$ with the following two defining properties*

(i) for every $B \in \mathcal{X}$ the distribution of $\eta(B)$ is Poisson with parameter $\lambda(B)$, that is $\mathbb{P}(\eta(B) = k) = Po(\lambda(B); k)$ for all $k \in \mathbb{N}_0$.

(ii) For every $m \in \mathbb{N}$ and all pairwise disjoint sets $B_1, ..., B_m \in \mathcal{X}$ the random variables $\eta(B_1), ..., \eta(B_m)$ are independent.

In other words a Poisson process is a point process, for which the number of points in a given set has Poisson distribution (if the process is proper) and which has independent increments.

Proposition 2. *Let η and η' be two Poisson processes on $\mathbb{X}$ with the same s-finite intensity measure. Then $\eta \overset{d}{=} \eta'$*

Proof. Follows directly from Proposition 1. □

2.3 Useful Properties of Poisson Processes

In this section we want to briefly mention some important and useful theorems regarding Poisson processes.

Theorem 1. *(Superposition theorem) Let $\eta_i, i \in \mathbb{N}$ be a sequence of independent Poisson processes on $\mathbb{X}$ with intensity measures λ_i. Then*

$$\eta := \sum_{i=1}^{\infty} \eta_i$$

is again a Poisson process with intensity measure $\lambda := \sum_{i=1}^{\infty} \lambda_i$.

Proof. It is easy to see that η is again a point process. So let us begin with checking the first characterising property of the Poisson process. Let $n \in \mathbb{N}$ and $B \in \mathcal{X}$, then the random variable $\xi_n(B) := \sum_{i=1}^{n} \eta_i(B)$ has Poisson distribution with parameter $\sum_{i=1}^{n} \lambda_i(B)$. Of course we have $\xi_n(B) \nearrow \eta(B)$ and thus for all $k \in \mathbb{N}_0$

$$\begin{aligned}
\mathbb{P}(\eta(B) \le k) &= \lim_{n \to \infty} \mathbb{P}(\xi_n(B) \le k) \\
&= \lim_{n \to \infty} \sum_{j=0}^{k} Po\Big(\sum_{i=1}^{n} \lambda_i(B); j\Big) \\
&= \sum_{j=0}^{k} Po\Big(\sum_{i=1}^{\infty} \lambda_i(B); j\Big).
\end{aligned}$$

In other words $\eta(B)$ has distribution $Po(\lambda(B))$. Note that we used the fact that $Po(\gamma, j)$ is continuous in γ for fixed j, as well as the continuity of probability in this computation.

To check the second property let $B_1, ..., B_n \in \mathcal{X}$ be pairwise disjoint sets. Now consider $\eta_i(B_j), 1 \leq j \leq m, i \in \mathbb{N}$. These form a family of independent random variables. Since we can group sums of independent random random variables without losing independence, the random variables $\sum_i \eta_i(B_1), ..., \sum_i \eta_i(B_m)$ are also independent and thus η has independent increments.

□

Definition 9. *Let $\mathbb{V}$ be a probability measure on $\mathbb{N}_0$ and $\mathbb{Q}$ be a probability measure on $\mathbb{X}$. Furthermore let $X_1, X_2, ...$ be a sequence of random elements in $\mathbb{X}$ with distribution $\mathbb{Q}$ and κ be a random variable with distribution $\mathbb{V}$, which is independent of (X_n). Then*

$$\eta := \sum_{k=1}^{\kappa} \delta_{X_k}$$

is called a mixed binomial process with mixing distribution $\mathbb{V}$ and sampling distribution $\mathbb{Q}$.

Proposition 3. *If $\mathbb{V} = Po(\gamma)$ for some $\gamma \geq 0$, then η is a Poisson process with intensity measure $\gamma\mathbb{Q}$.*

Proof. Let κ and (X_n) be as in the definition above. Consider a sequence $B_1, ..., B_m \in \mathcal{X}$ of pairwise disjoint sets and assume without loss of generality, that $\bigcup_{i=1}^m B_i = \mathbb{X}$, as we could simply add the complement otherwise. Now let $k_1, ..., k_m \in \mathbb{N}_0$ and $k = \sum_{i=1}^m k_i$. Then

$$\begin{aligned}
&\mathbb{P}(\eta(B_1) = k_1, ...\eta(B_m) = k_m) \\
&= \mathbb{P}(\kappa = k)\mathbb{P}\Big(\sum_{j=1}^{k} 1\{x_j \in B_1\} = k_1, ..., \sum_{j=1}^{k} 1\{x_j \in B_m\} = k_m\Big) \\
&= \frac{\gamma^k}{k!} e^{-\gamma} \frac{k!}{k_1!, ..., k_m!} \mathbb{Q}(B_1)^{k_1} ... \mathbb{Q}(B_m)^{k_m} \\
&= \prod_{j=1}^{m} \frac{(\gamma\mathbb{Q}(B_j))^{k_j}}{k_j!} e^{-\gamma\mathbb{Q}(B_j)},
\end{aligned}$$

where we used the fact that the probabilities in the second row are Poisson and multinomial respectively. The last equivalence is simply a reordering of the line above. Now if we sum over $k_2, ..., k_m$ we get that $\eta(B_1)$ is Poisson distributed with parameter $\gamma\mathbb{Q}(B_1)$ and analogously for $\eta(B_2), ..., \eta(B_m)$. Thus $\eta(B_1), ..., \eta(B_m)$ are independent and we have shown both defining properties of Poisson processes. $\square$

Theorem 2. *(Existence theorem) Let λ be an s-finite measure on $\mathbb{X}$. Then there exists a Poisson process on $\mathbb{X}$ with intensity measure λ.*

Proof. The statement is trivial for $\lambda(\mathbb{X}) = 0$, so let us first assume that $0 < \lambda(\mathbb{X}) < \infty$. Let $(X_n) \in \mathbb{X}$ be a sequence of independent random elements with distribution $\lambda(.)/\lambda(\mathbb{X})$ and κ be a random variable with distribution $Po(\lambda(\mathbb{X}))$, which is independent of (X_n). Now let η be the mixed binomial process with mixing distribution $Po(\lambda(\mathbb{X}))$ and sampling distribution $\lambda(.)/\lambda(\mathbb{X})$. Then by Proposition 3, η is a Poisson process with intensity measure λ.

Finally suppose $\lambda(\mathbb{X}) = \infty$. Consider a sequence (λ_i) of measures on $(\mathbb{X}, \mathcal{X})$ with $0 < \lambda_i(\mathbb{X}) < \infty$ for each $i \in \mathbb{N}$ and set $\lambda = \sum_{i=1}^{\infty} \lambda_i$. Furthermore let (η_i) be a sequence of Poisson processes with intensity measure λ_i respectively. Note that the first part of the proof ensures that these exist. Now by the superposition Theorem, $\eta = \sum_{i=1}^{\infty} \eta_i$ is a Poisson process with intensity measure λ. $\square$

Corollary 1. *Let λ be an s-finite measure on $\mathbb{X}$. Then there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, supporting random elements $X_1, X_2, ...$ of $\mathbb{X}$ and $\kappa \in \overline{\mathbb{N}}_0$, such that*

$$\eta := \sum_{n=1}^{\kappa} \delta_{X_n} \tag{2}$$

is a Poisson process with intensity measure λ. In other words every Poisson process is proper up to equality in distribution.

Proof. Assume that $\lambda(\mathbb{X}) = \infty$, as the other case follows immediately from Proposition 3. As in the proof above, consider a sequence (λ_i) of measures on $(\mathbb{X}, \mathcal{X})$ with $0 < \lambda_i(\mathbb{X}) < \infty$ and set $\gamma_i = \lambda_i(\mathbb{X})$ and $\mathbb{Q}_i = \lambda_i/\gamma_i$. Now take $(\Omega, \mathcal{F}, \mathbb{P})$ as the product of the spaces $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, which again are products of the spaces $(\Omega_{ij}, \mathcal{F}_{ij}, \mathbb{P}_{ij})$, where $\Omega_{i0} = \mathbb{N}_0$, $\mathbb{P}_{i0} = Po(\gamma_i)$ and $(\Omega_{ij}, \mathcal{F}_{ij}, \mathbb{P}_{ij}) = (\mathbb{X}, \mathcal{X}, \mathbb{Q})$ for $j \geq 1$. On this space we can define independent random elements κ_i with distribution $Po(\gamma_i)$ and X_{ij} with distribution $\mathbb{Q}_i$. Now we can simply use the construction in the proof above to define (X_n) and κ to arrive at (2). □

We have already mentioned that the Laplace functional is a powerful tool for analysing point processes. In the case of Poisson processes one gets the following characterisation.

Theorem 3. *Let λ be an s-finite measure on $\mathbb{X}$ and let η be a point process on $\mathbb{X}$. Then η is a Poisson process with intensity measure λ if and only if*

$$L_\eta(u) = exp\Big[-\int(1-e^{-u(x)})\lambda(dx)\Big], \quad u \in \overline{\mathbb{R}}_+(\mathbb{X}).$$

Proof. Let us first assume that η is a Poisson process with intensity measure λ. We want to prove this part via measure theoretic induction. Therefore consider the simple function $u := c_1 B_{C_1} + ...c_m 1_{B_m}$ for some $m \in \mathbb{N}$, $c_1, ..., c_m \in (0, \infty)$ and $B_m \in \mathcal{X}$ pairwise disjoint. Now

$$\begin{aligned}
L_\eta(u) &= \mathbb{E}\Big[exp\Big(-\int u(x)\eta(dx)\Big)\Big] = \mathbb{E}\Big[exp\Big(-\sum_{i=1}^m c_i\eta(B_i)\Big)\Big] \\
&= \mathbb{E}\Big[\prod_{i=1}^m exp(-c_i\eta(B_i))\Big] = \prod_{i=1}^m \mathbb{E}\Big[exp(-c_i\eta(B_i))\Big] \\
&= \prod_{i=1}^m exp(-\lambda(B_i)(1-e^{-c_i}) = exp\Big(-\sum_{i=1}^m \lambda(B_i)(1-e^{-c_i}\Big) \\
&= exp\Big(-\sum_{i=1}^m \int_{B_i}(1-e^{-u})d\lambda\Big) = exp\Big(-\int_{\mathbb{X}}(1-e^{-u})d\lambda\Big).
\end{aligned}$$

Note that in this calculation we used the complete independence to pull the product out of the expectation and Remark 2 to compute the expectation.

Now for general $u \in \mathbb{R}_+(X)$, we can find a sequence u_n of simple functions such that $u_n \nearrow u$ as $n \to \infty$ and by using the monotone convergence theorem, as well as the dominated convergence theorem multiple times we get

$$\begin{aligned}L_\eta(u) &= \mathbb{E}\Big[exp\Big(-\int u(x)\eta(dx)\Big)\Big] = \mathbb{E}\Big[exp\Big(-\int \lim_{n\to\infty} u_n(x)\eta(dx)\Big)\Big]\\ &= \lim_{n\to\infty} \mathbb{E}\Big[exp\Big(-\int u_n(x)\eta(dx)\Big)\Big] = \lim_{n\to\infty} exp\Big(-\int_{\mathbb{X}}(1-e^{-u_n})d\lambda\Big)\\ &= exp\Big(-\int_{\mathbb{X}}(1-e^{-u})d\lambda\Big)\end{aligned}$$

Finally the prove the other direction assume that (3) holds and let η' be a Poisson process with intensity measure λ. By the first part of the proof we already know that the Laplace functional of η' is of the form (3), that is $L_\eta(u) = L_{\eta'}(u)$. Now by Proposition 1 this implies that $\eta \overset{d}{=} \eta'$, which means that η is indeed a Poisson process, by Proposition 2 .

□

Theorem 4. *(Mecke equation) Let λ be an s-finite measure on $\mathbb{X}$ and let η be a point process on $\mathbb{X}$. Then η is a Poisson process with intensity measure λ if and only if*

$$\mathbb{E}\Big[\int f(x,\eta)\eta(dx)\Big] = \int \mathbb{E}[f(x,\eta+\delta_x)]\lambda(dx) \tag{3}$$

for all $f \in \overline{\mathbb{R}}(\mathbb{X}\times\mathbf{N})$.

Proof. To begin with, let us first note that the mapping $(x,\mu) \to \mu(B) + 1_B(x)$ is measurable for every $B \in \mathcal{X}$. Let us first prove the "if" direction. So assume that (3) holds. Consider some disjoint measurable sets $B_1, ..., B_m \in \mathcal{X}$ of finite measure and $k_1, ..., k_m \in \overline{\mathbb{N}}_0$, where $k_1 \geq 1$. Define the function

$$f(x,\mu) = 1_{B_1}(x)\prod_{i=1}^m 1\{\mu(B_i) = k_i\}, \quad (x,\mu) \in \mathbb{X}\times\mathbf{N}.$$

Then

$$\begin{aligned}\mathbb{E}[\int f(x,\eta)\eta(dx)] &= \mathbb{E}\Big[\eta(B_1)\prod_{i=1}^m 1\{\eta(B_i = k_i)\}\Big]\\ &= k_1\mathbb{P}\Big(\bigcap_{i=1}^m \{\eta(B_i = k_i)\}\Big).\end{aligned}$$

On the other hand however we have

$$\int \mathbb{E}[f(x, \eta + \delta_x)] = \int 1_{B_1}(x)\mathbb{P}(\eta(B_1) = k_1 - 1, \eta(B_2) = k_2, ..., \eta(B_m) = k_m).$$

Putting this together yields

$$k_1\mathbb{P}\Big(\bigcap_{i=1}^{m}\{\eta(B_i = k_i)\}\Big) = \lambda(B_1)\mathbb{P}\Big(\eta(B_1) = k_1, \bigcap_{i=2}^{m}\{\eta(B_i) = k_i\}\Big). \tag{4}$$

Now if $\mathbb{P}\Big(\bigcap_{i=2}^{m}\{\eta(B_i) = k_i\}\Big) = 0$, then it is trivial that the two events $\eta(B_1 = k_1)$ and $\bigcap_{i=2}^{m}\{\eta(B_i) = k_i\}$ are independent. So assume that said event has positive probability and denote

$$\pi_k = \mathbb{P}\Big(\bigcap_{i=2}^{m}\{\eta(B_i) = k\}\Big), \quad k \in \mathbb{N}_0.$$

Then (4) simplifies to

$$k\pi_k = \lambda(B_1)\pi_{k-1}$$

and thus $\pi_\infty = 0$. The only distribution satisfying this is $Po(\lambda(B_1), k) = \pi_k$. In other words $\eta(B_1)$ is Poisson distributed and independent of $\mathbb{P}\Big(\bigcap_{i=2}^{m}\{\eta(B_i) = k_i\}\Big)$. Now use induction to see that $\eta(B_1), ..., \eta(B_m)$ are independent. For general $B \in \mathcal{X}$ of finite measure we can simply write it as a disjoint union of measurable sets and still get our desired result. Lastly for $B \in \mathcal{X}$ of infinite measure we get

$$\mathbb{P}(\eta(B) = k - 1) = 0$$

and thus

$$\mathbb{P}(\eta(B) = \infty) = 1$$

which again is the $Po(\infty)$ distribution.

Now let us prove the "only if" direction. By Corollary 2 we can assume that η is proper. Let us furthermore assume $\lambda(\mathbb{X}) < \infty$. We can then write $\lambda = \gamma\mathbb{Q}$ for some $\gamma \geq 0$ and probability measure $\mathbb{Q}$. By Proposition 3 assume that η is a mixed binomial process with mixing distribution $Po(\gamma)$ and consider some function $f \in \overline{\mathbb{R}}_+(\mathbb{X} \times \mathbf{N})$. Then

$$\begin{aligned}
\mathbb{E}\Big[\int f(x,\eta)\eta(dx)\Big] &= \sum_{k=1}^{\infty} e^{-\gamma}\frac{\gamma^k}{k!}\mathbb{E}\Big[\sum_{i=1}^{k} f(X_i, \delta_{X_i})\Big] \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{k} e^{-\gamma}\frac{\gamma^k}{k!}\mathbb{E}[f(X_i, \delta_{X_i})].
\end{aligned}$$

Now since the X_i are independent with distribution $\mathbb{Q}$, we can also write this as

$$\begin{aligned}
&= e^{-\gamma}\sum_{k=1}^{\infty}\frac{\gamma^k}{k!}\mathbb{E}\Big[k\int f\Big(y, \sum_{i=1}^{k}\delta_{X_i} + \delta_y\Big)\mathbb{Q}(dy)\Big], \quad y \in \mathbb{X} \\
&= e^{-\gamma}\gamma\sum_{k=1}^{\infty}\frac{\gamma^{k-1}}{(k-1)!}\int \mathbb{E}\Big[f\Big(y, \sum_{i=1}^{k}\delta_{X_i} + \delta_y\Big)\mathbb{Q}(dy)\Big] \\
&= \int \mathbb{E}[f(y, \eta + \delta_y)\lambda(dy)].
\end{aligned}$$

Finally let us assume that $\lambda(\mathbb{X}) = \infty$. As in the proof of the existence Theorem consider proper independent Poisson processes η_i of finite total measure such that $\eta = \sum_{i=1}^{\infty}\eta_i$. Now since we can group independent processes without losing independence, consider $\xi_i = \sum_{j\leq i}\eta_j$ and $\chi_i \sum_{j\geq i+1}\eta_j$, where obviously $\xi_i \nearrow \eta$. Then by the monotone convergence Theorem

$$\begin{aligned}
\mathbb{E}\Big[\int f(x,\eta)\eta(dx)\Big] &= \mathbb{E}\Big[\int \lim_{i\to\infty} f(x, \xi_i + \chi_i)\xi_i(dx)\Big] \\
&= \lim_{i\to\infty}\mathbb{E}\Big[\int f(x, \xi_i + \chi_i)\xi_i(dx)\Big].
\end{aligned}$$

Now using the previous part of the proof yields

$$= \lim_{i\to\infty} \int \mathbb{E}[f(x, \xi_i + \chi_i + \delta_x)]\lambda_i'(dx)$$
$$= \lim_{i\to\infty} \int \mathbb{E}[f(x, \eta + \delta_x)]\lambda_i'(dx),$$

where $\lambda_i' = \sum_{j=1}^{i} \lambda_j$. Finally using Fatou's lemma we get

$$= \int \mathbb{E}[f(x, \eta + \delta_x)]\lambda(dx),$$

which completes the proof. □

Remark 3. We will only use this theorem for f that do not depend on η and also typically not as a characterisation, but as a way to compute the expecation on the left-hand side.

Theorem 5. *(Mapping theorem) Let η be a point process on $\mathbb{X}$ with intensity measure λ, let $(\mathbb{Y}, \mathcal{Y})$ be a measurable space and let $T : \mathbb{X} \to \mathbb{Y}$ be a measurable mapping. Then the push-forward $T(\eta) := \eta \circ T^{-1}$ is a point process with intensity measure $T(\lambda)$. If η is a Poisson process, then $T(\eta)$ is again a Poisson process.*

Proof. Let us first observe that $T(\mu) \in \mathbf{N}$ for any $\mu \in \mathbf{N}$. Which is easy to see, since if we can write $\mu = \sum_{j=1}^{\infty} \mu_j$, then $T(\mu) = \sum_{j=1}^{\infty} T(\mu_j)$ and since all μ_j are $\mathbb{N}_0$-valued, so are the $T(\mu_j)$.

Now let us consider $C \in \mathcal{Y}$, then $T(\eta)(C)$ is a random variable and by using the definition of the intensity measure we get

$$\mathbb{E}[T(\eta)(C)] = \mathbb{E}[\eta(T^{-1}C)] = \lambda(T^{-1}C) = T(\lambda)(C).$$

Also a rather similar computation gives us

$$\begin{aligned}\mathbb{P}(T(\eta(B)) = k) &= \mathbb{P}(\eta(T^{-1}(B)) = k)\\ &= Po(\lambda(T^{-1}(B)); k)\\ &= Po(T(\lambda); k),\end{aligned}$$

that is $T(\eta)(B)$ has Poisson distribution with parameter $T(\lambda)$. The second defining property of the Poisson distribution - the independent increments - follows from the fact that for two disjoint sets $A, B \in \mathcal{X}$, also the preimages $T^{-1}(A)$ and $T^{-1}(B)$ are disjoint and thus

$$\begin{aligned}\mathbb{P}(T(\eta(A)) \cap T(\eta(B))) &= \mathbb{P}(\eta(T^{-1}A) \cap \eta(T^{-1}B)) \\ &= \mathbb{P}(\eta(T^{-1}A))\, \mathbb{P}(\eta(T^{-1}B)) \\ &= \mathbb{P}(T(\eta(A)))\, \mathbb{P}(T(\eta(B))).\end{aligned}$$

$\square$

Another important theorem closely related to the mapping theorem is the restriction theorem, which tells us that we can actually discard all points outside of a given set $B \in \mathcal{X}$ and still end up with a Poisson process. Formally we define the restriction ν_B of a measure ν to the set $B \in \mathcal{X}$ by

$$\nu_B(B') := \nu(B \cap B'), \quad B' \in \mathcal{X}.$$

Theorem 6. *(Restriction theorem) Let η be a Poisson process on $\mathbb{X}$ with s-finite measure λ and let $C_1, C_2, ... \in \mathcal{X}$ be pairwise disjoint. Then the restrictions $\eta_{C_1}, \eta_{C_2}, ...$ are independent Poisson processes with intensity measures $\lambda_{C_1}, \lambda_{C_1}, ...,$ respectively.*

Proof. The only part that we really need to prove is the independence, since the rest is already covered by the mapping theorem, as we could just use the maps 1_{C_i} to get new Poisson processes.

So to prove the independence let us first assume without loss of generality that $\bigcup_i C_i = \mathbb{X}$, otherwise simply add the complement of this union to the sequence. The existence theorem ensures that we can find a sequence η_i, $i \in \mathbb{N}$ of independent Poisson processes with intensity measures λ_{C_i} and by the superposition theorem summing those η_i gives us again a Poisson process with intensity measure $\sum_i \lambda_{C_i} = \lambda$. Thus by Proposition 2 we have $\eta \overset{d}{=} \eta'$. Now for any $k \in \mathbb{N}$ and $f_1, ..., f_k \in \mathbb{R}(\mathbf{N})$

$$\mathbb{E}\Big[\prod_{i=1}^{k} f_i(\eta_{C_i})\Big] = \mathbb{E}\Big[\prod_{i=1}^{k} f_i(\eta'_{C_i})\Big] = \mathbb{E}\Big[\prod_{i=1}^{k} f_i(\eta_i)\Big]$$
$$= \prod_{i=1}^{k} \mathbb{E}[f_i(\eta_i)] = \prod_{i=1}^{k} \mathbb{E}[f_i(\eta_{C_i})].$$

□

2.4 More Tools from Probability Theory

The next theorem can be found in [2] on page 635, where it is given without proof. It is however a very well known theorem that can be found in pretty much every book on probability theory.

Theorem 7. *(Three series theorem) Let $(Z_i)_{i\geq 1}$ be independent random variables. Then*

$$\sum_i Z_i \quad \textit{converges almost surely,}$$

if and only if for every positive constant a the three series

$$\sum_i \mathbb{P}\{|Z_i| > a\}, \quad \sum_i \mathbb{E}[Z_i^{[a]}] \quad \textit{and} \quad \sum_i \mathbb{E}[(Z_i^{[a]})^2]$$

converge, where

$$Z_i^{[a]} := Z_i \quad \textit{if} \quad |Z_i| \leq a \quad \textit{and} \quad Z_i := 0 \quad \textit{otherwise.}$$

Remark 4. For bounded random variables Z_i we actually only need to look at

$$\sum_i \mathbb{E}[Z_i] \quad \text{and} \quad \sum_i \mathbb{E}[Z_i^2]$$

since for big enough a, the probability that $|Z_i| > a$ is zero and also $Z_i^{[a]} = Z_i$.

The last part of this section covers uniform integrability and can be found in [3] on pages 26-30.

Definition 10. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. A subset* $\mathcal{K} \subseteq L^1$ *is called uniformly integrable if*

$$\sup_{f\in\mathcal{K}} \int_{\{|f|>a\}} |f| \, d\mathbb{P} \to \infty \quad as \quad a \to \infty.$$

Remark 5. It follows directly from the definition that every finite subset of L^1 is uniformly integrable.

The following Proposition will be very useful when dealing with uniform integrability in proofs.

Proposition 4. *A subset* $\mathcal{K} \in L^1$ *is uniformly integrable if and only if* $\mathcal{K}$ *is* L^1*-bounded and for every* $\varepsilon > 0$*, there is a* $\delta > 0$ *such that*

$$\sup_{f\in\mathcal{K}} \int_A |f| d\mathbb{P} < \varepsilon \tag{5}$$

for any measurable set A *with* $\mathbb{P}(A) < \delta$*.*

Proof. Consider a positive function $f \in L^1_+$, a measurable set A and some $a > 0$, then it is easy to see that

$$\int_A f \, d\mathbb{P} = \int_{A\cap\{f\le a\}} f \, d\mathbb{P} + \int_{A\cap\{f>a\}} f \, d\mathbb{P} \le a \cdot \mathbb{P}(A) + \int_{\{f>a\}} f \, d\mathbb{P}$$

and thus

$$\sup_{f\in\mathcal{K}} \int_A |f| d\mathbb{P} \le a \cdot \mathbb{P}(A) + \sup_{f\in\mathcal{K}} \int_{\{|f|>a\}} |f| d\mathbb{P}. \tag{6}$$

Now first let us assume that $\mathcal{K}$ is uniformly integrable and choose $A = \Omega$, then (6) gives us

$$\sup_{f\in\mathcal{K}} \|f\|_1 \le a + \sup_{f\in\mathcal{K}} \int_{\{|f|>a\}} |f| d\mathbb{P} < \infty.$$

Furthermore to check the second statement for a given $\varepsilon > 0$, choose $a > 0$ so large that

$$\sup_{f\in\mathcal{K}} \int_{\{|f|>a\}} |f| d\mathbb{P} < \frac{\varepsilon}{2}$$

and choose $\delta = \varepsilon/2a$. Then by using (6) again we get

$$\sup_{f\in\mathcal{K}} \int_A |f| d\mathbb{P} < \varepsilon \quad \text{if} \quad \mathbb{P}(A) < \delta.$$

Finally to prove the other direction assume L^1- boundedness as well as (5). Now let us denote $M = \sup_{f\in\mathcal{K}} \int_A |f| d\mathbb{P}$ and choose $a = M/\delta$. A rather similar computation as in the beginning of the proof yields for positive $f \in L^1$

$$\int_A f\, d\mathbb{P} = \int_{A\cap\{f\le a\}} f\, d\mathbb{P} + \int_{A\cap\{f>a\}} f\, d\mathbb{P} \ge \int_{A\cap\{f\le a\}} f\, d\mathbb{P} + a\cdot \mathbb{P}(\{|f|>a\})$$

and thus

$$\sup_{f\in\mathcal{K}} \int_A |f| d\mathbb{P} \ge \sup_{f\in\mathcal{K}} \int_{A\cap\{f\le a\}} f\, d\mathbb{P} + a\cdot \mathbb{P}(\{|f|>a\}).$$

In other words $\sup_{f\in\mathcal{K}} P(\{|f|>a\}) \le M/a = \delta$. If we plug this into (5) we get

$$\int_{\{|f|>a\}} |f| d\mathbb{P} < \varepsilon$$

for all $f \in \mathcal{K}$ and thus

$$\sup_{f\in\mathcal{K}} \int_{\{|f|>a\}} |f| d\mathbb{P} \to 0 \quad \text{as} \quad a\to\infty.$$

$\square$

However with our definition of uniform integrability it is rather difficult to check if a set of functions is uniformly integrable. Fortunately there is a very useful Proposition that helps us with this task.

Proposition 5. *If $\mathcal{K} \subseteq L^1$ and there is a positive increasing funcion ϕ defined on $[0,\infty)$ such that $\lim\limits_{t\to\infty} \frac{\phi(t)}{t} = \infty$ and*

$$\sup_{f\in\mathcal{K}} \int_\Omega (\phi \circ |f|)\, d\mathbb{P} < \infty,$$

then $\mathcal{K}$ is uniformly integrable.

Proof. Let $M = \sup_{f\in\mathcal{K}} \int_\Omega (\phi \circ |f|)\, d\mathbb{P}$, $\varepsilon > 0$ and $a = M/\varepsilon$. Now choose t_0 such that

$$\frac{\phi(t)}{t} \geq a \quad \text{for} \quad t > t_0.$$

If we use that, we see that on the set $\{|f| > t_0\}$ we have

$$|f| \leq \frac{(\phi \circ |f|)}{a}$$

and thus

$$\int_A |f| d\mathbb{P} \leq \frac{1}{a} \int_A (\phi \circ |f|) d\mathbb{P} \leq \frac{M}{a} = \varepsilon.$$

□

Corollary 2. *If $\mathcal{K}$ is L^p bounded for some $p > 1$, then $\mathcal{K}$ is uniformly integrable.*

Proof. Choose $\phi(t) = t^p$ and use the proposition above. □

The reason why uniform integrability is such a useful property is its connection to L^1-convergence as can be seen in the following theorem.

Theorem 8. *If $f_n \subseteq L^1$ and f is a random variable, then the following are equivalent*

(i) (f_n) is uniformly integrable and $f_n \to f$ in probability,

(ii) $f \in L^1$ and f_n converges to f in L^1.

Proof. Let us first assume that (i) holds. The convergence in probability implies that there exists a subsequence f_{n_k}, which converges almost surely to f. Now the uniform integrability implies that $\mathbb{E}[|f_n|] < \infty$ for every n and thus by using Fatous lemma we get

$$\mathbb{E}[|f|] \leq \limsup_{k \to \infty} \mathbb{E}[|f_{n_k}|] < \infty.$$

To prove L^1 convergence let $\varepsilon > 0$ and consider the set $A_\varepsilon^n = \{|f_n - f| > \varepsilon\}$, then

$$\begin{aligned} \|f_n - f\|_1 &\leq \int_{\Omega \setminus A_\varepsilon^n} |f_n - f| d\mathbb{P} + \int_{A_\varepsilon^n} |f_n - f| d\mathbb{P} \\ &\leq \varepsilon + \int_{A_\varepsilon^n} |f_n| d\mathbb{P} + \int_{A_\varepsilon^n} |f_n| d\mathbb{P}. \end{aligned}$$

However since $\mathbb{P}(A_\varepsilon^n) \to 0$ as $n \to \infty$ both integrals are bounded by ε by Proposition 4 and thus $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

Now for the second direction of the statement assume (ii). Convergence in L^1 already implies convergence in probability, so the only part left to be shown is the uniform integrability. Consider a measurable set A, then

$$\int_A |f_n| d\mathbb{P} \leq \int_A |f| d\mathbb{P} + \|f_n - f\|_1 < \infty \quad \text{for all} \quad n \geq 1 \tag{7}$$

and thus in particular $\sup_n \int_\Omega |f_n| d\mathbb{P} < \infty$. Let $\varepsilon > 0$ and take n_0 such that $\|f_n - f\|_1 < \varepsilon/2$ for $n \geq n_0$. Now consider the finite sequence $F = \{f_1, f_2, ..., f_{n_0}, f\}$. This F is uniformly integrable by Remark 5, so by Proposition 4 there exists a $\delta > 0$ such that

$$\int_A |g| d\mathbb{P} < \frac{\varepsilon}{2}$$

for $g \in F$ and $\mathbb{P}(A) < \delta$. However using (7) we get

$$\int_A |f_n| d\mathbb{P} < \varepsilon$$

for all $n \geq 1$ if $\mathbb{P}(A) < \delta$ and thus by Proposition 4 again that the whole sequence f_n is uniformly integrable. $\square$

2.5 The Hellinger Distance

Before we start with the main chapter, there is one subject left that we should discuss, namely the Hellinger Distance between two measures. Details can be found in [7].

Definition 11. *Consider two measures P and Q with densities p and q with respect to a dominating measure λ. The* Hellinger distance between P and Q *is defined as the L^2-distance between the square roots of their densities multiplied by the factor $\frac{1}{2}$. That is*

$$\begin{aligned} d(P,Q)^2 &= \frac{1}{2}\int_{\mathbb{X}} (\sqrt{p} - \sqrt{q})^2 d\lambda = \frac{1}{2}\int_{\mathbb{X}} (p + q + 2\sqrt{pq}) d\lambda \\ &= \frac{1}{2}(P(\mathbb{X}) + Q(\mathbb{X})) - \int_{\mathbb{X}} \sqrt{pq}) d\lambda. \end{aligned}$$

Moreover we call the term $\int_{\mathbb{X}} \sqrt{pq}\, d\lambda$, the Hellinger affinity*, which is denoted by $\alpha_2(P,Q)$.*

Remark 6. Note that if P and Q give finite measure to the whole space $\mathbb{X}$, the Hellinger distance is finite, as can easily seen by the last expression in the definition above, when considering the fact that the square roots of p and q are square integrable.

Furthermore the Hellinger distance does not depend on the choice of λ, as the densities p and q are chosen with respect to λ.

The factor $\frac{1}{2}$ is chosen so that in the case, where P and Q are probability measures, the Hellinger distance satisfies $0 \leq d(P,Q) \leq 1$.

Since $\sqrt{pq} \geq p \wedge q$, we also have

$$\int_{\mathbb{X}} \sqrt{pq}\, d\lambda \geq \int_{\mathbb{X}} p \wedge q \, d\lambda$$

and thus $d(P,Q)^2 \leq ||P-Q||_1$, where $||.||_1$ denotes the total variation norm. On the other hand for $P(\mathbb{X}), Q(\mathbb{X}) < \infty$ we get by the Cauchy-Schwarz inequality

$$\begin{aligned}
||P-Q||_1^2 &= \Big(\int_{\mathbb{X}} |\sqrt{p} - \sqrt{q}|\, |\sqrt{p} + \sqrt{q}| d\lambda \Big)^2 \\
&\leq \int_{\mathbb{X}} |\sqrt{p} - \sqrt{q}|^2 d\lambda \int_{\mathbb{X}} |\sqrt{p} + \sqrt{q}|^2 d\lambda \\
&= 2d(P,Q)^2 (P(\mathbb{X}) + Q(\mathbb{X}) + 2\alpha_2) \\
&\leq c \cdot d(P,Q)
\end{aligned}$$

for some positive constant $c \in \mathbb{R}$. Putting this together yields

$$d(P,Q) \leq ||P-Q||_1 \leq c \cdot d(P,Q).$$

In other words convergence of the Hellinger distance is equivalent to convergence in the total variation norm for finite measures P and Q.

3 Absolute Continuity of Poisson processes

In this section we want to analyse the connection between Poisson processes and their intensity measures. For a given measure λ on $\mathbb{X}$, we denote the Poisson process with intensity measure λ by η_λ. Details for this section can be found in [2].

3.1 Preparation

Definition 12. *Let η_λ be a Poisson process with intensity measure λ. We denote the distribution of this process by π_λ and call it the* Poisson measure, *since it is a probability measure on the space of counting measures $(\mathbf{N}, \mathcal{N})$.*

Furthermore we call two Poisson processes η_λ and η_ρ mutually absolutely continuous, *if their distributions π_λ and π_ρ are mutually absolutely continuous in the usual measure theoretic sense, i.e. they have the same null sets.*

Remark 7. Note that with this notation we can actually write expectations, for example in the Laplace functional or the Mecke equation, as integrals over the space of counting measures ξ and integrate with respect to $\pi_\lambda(d\xi)$.

We ultimately want to prove the following theorem.

Theorem 9. *Let λ and ρ be two σ-finite, nonatomic measures on $\mathbb{X}$. Then the Poisson processes η_λ and η_ρ are mutually absolutely continuous if and only if*

(i) λ and ρ are mutually absolutely continuous and

(ii) the Hellinger distance $d(\rho, \lambda)$ between ρ and λ is finite, that is

$$d(\rho, \lambda)^2 := \frac{1}{2} \int_{\mathbb{X}} |\sqrt{d\rho} - \sqrt{d\lambda}|^2 < \infty.$$

Furthermore for the Hellinger Distance $D(\pi_\rho, \pi_\lambda)$ between the Poisson measures π_ρ and π_λ the following formula holds in the case of absolute continuity

$$D(\pi_\rho, \pi_\lambda)^2 := \frac{1}{2} \int_{\mathbb{X}} |\sqrt{d\pi_\rho} - \sqrt{d\pi_\lambda}|^2 = 1 - exp\{-d(\rho, \lambda)^2\}.$$

Note that since the Hellinger distance is symmetric, λ and ρ are interchangeable.

This theorem was established in [2] in a slightly less general setup, where $\mathbb{X}$ was assumed to be a locally compact Hausdorff space with countable basis and the intensity measures λ and ρ were nonatomic infinite nonnegative Radon measures.

One reason why this theorem is interesting, is because one could start with some Poisson process η_λ with intensity measure λ and use some measurable transformation T on this process. By the Mapping theorem this is again a Poisson process. To see if the processes η_λ and $T(\eta_\lambda)$ are basically the same, i.e. are absolutely continuous, one only needs to check mutual absolute continuity for the intensity measures λ and $T(\lambda)$, as well as the finiteness of the Hellinger distance, if one has access to the theorem above.

Remark 8. Statement (i) in the theorem implies that the Radon-Nikodym derivative $\phi := \frac{d\rho}{d\lambda}$ is positive and finite λ-almost everywhere.

Furthermore statement (ii) can be rewritten as

$$\sqrt{\phi} - 1 \in \mathbf{L}^2(\mathbb{X}, \lambda). \tag{8}$$

This can be easily checked by plugging in ϕ into (ii)

$$\int_{\mathbb{X}} |\sqrt{d\rho} - \sqrt{d\lambda}|^2 = \int_{\mathbb{X}} |\sqrt{\phi} - 1|^2 d\lambda.$$

Let us also recall the formula for the Laplace functional for Poisson processes

$$\int_{\mathbf{N}} \pi_\lambda(d\xi) exp\{-\langle \xi, u\rangle\} = exp\Big\{ - \int_{\mathbb{X}} (1 - e^{-u}) d\lambda \Big\}.$$

For easier notation we write the integral on the left hand side as a functional

$$\int u(x)\xi(dx) = \langle \xi, u\rangle.$$

Also we write the expectation as integral, so that the Poisson measure π_λ can be seen in the formula.

3.2 Finite Measure Case

Let us first consider the case where the whole space $\mathbb{X}$ has finite measure under λ and ρ respectively.

As we have seen in Remark 6, the Hellinger distance between two measures λ and ρ is finite, if the measures give finite measure to the whole space. This gives us a much simpler version of our main theorem.

Theorem 10. *Let λ and ρ be two finite, nonatomic, measures on $\mathbb{X}$. Then the Poisson processes η_λ and η_ρ are mutually absolutely continuous if and only if λ and ρ are mutually absolutely continuous.*

We can prove the "only if" direction using the Laplace functional. To see this let $B \in \mathcal{X}$ be a measurable set and let

$$u(x) := \begin{cases} \infty, & \text{if } x \in B \\ 0, & \text{else} \end{cases}$$

Now if we plug this $u(x)$ into the formula for the Laplace functional, we get

$$\begin{aligned} \int_{\mathbf{N}} \pi_\lambda(d\xi) exp\{-\langle \xi, u \rangle\} &= exp\Big\{ - \int_{\mathbb{X}} (1 - e^{-u})\, d\lambda \Big\} \\ &= exp\Big\{ - \int_{B} (1 - 0)\, d\lambda \Big\} \\ &= exp\{-\lambda(B)\} \end{aligned}$$

and on the other hand

$$\begin{aligned} \int_{\mathbf{N}} \pi_\lambda(d\xi) exp\{-\langle \xi, u \rangle\} &= \int_{\{\xi(B)=0\}} \pi_\lambda(d\xi) \cdot 1 \\ &= \pi_\lambda\{\xi : \xi(B) = 0\}. \end{aligned}$$

So in summary this yields

$$\pi_\lambda\{\xi : \xi(B) = 0\} = exp\{-\lambda(B)\}$$

and thus of course also

$$\pi_\lambda\{\xi : \xi(B) > 0\} = 1 - exp\{-\lambda(B)\}.$$

However this already proves the "only if" direction. We formulate it as a Lemma.

Lemma 1. *If π_λ and π_ρ are mutually absolutely continuous, then for each measurable set $B \in \mathcal{X}$*

$$0 < \lambda(B) \quad \textit{if and only if} \quad 0 < \rho(B).$$

In other words, the measures λ and ρ are mutually absolutely continuous.

For the "if" direction note that in the finite case the total variation of the signed measure $\rho - \lambda$ is finite, that is

$$\int_{\mathbb{X}} |d\rho - d\lambda| = \int_{\mathbb{X}} |\phi - 1| d\lambda < \infty.$$

This is because, as we have seen in section 2.5., for finite measures convergence of the Hellinger distance is equivalent to convergence in total variation norm.

Proof of the "if"-direction (finite case). Let $u \in \mathbb{R}_+(\mathbb{X})$. Then by using the Laplace functional we get

$$\begin{aligned}\int_{\mathbf{N}} \pi_\rho(d\xi) exp\{-\langle \xi, u\rangle\} &= exp\Big\{ - \int_{\mathbb{X}} (1 - e^{-u}) d\rho \Big\} \\ &= exp\Big\{ - \int_{\mathbb{X}} (1 - e^{-u}) \phi \, d\lambda \Big\}.\end{aligned}$$

Now since

$$(1 - \phi) - (1 - e^{-u + log\,\phi}) = (1 - \phi) - (1 - e^{-u}\phi) = -(1 - e^{-u})\phi$$

we can also write the expression above as

$$
\begin{aligned}
&= exp\Big\{ \int_{\mathbb{X}} (1-\phi) d\lambda \Big\} exp\Big\{ - \int_{\mathbb{X}} (1 - e^{-u + log\,\phi}) d\lambda \Big\} \\
&= exp\Big\{ \int_{\mathbb{X}} (1-\phi) d\lambda \Big\} \int_{\mathbf{N}} \pi_\lambda(d\xi) exp\{-\langle \xi, u - log\,\phi \rangle\} \\
&= \int_{\mathbf{N}} \pi_\lambda(d\xi) exp\Big\{ \int_{\mathbb{X}} (1-\phi) d\lambda + \langle \xi, log\,\phi \rangle \Big\} exp\{-\langle \xi, u \rangle\}
\end{aligned}
$$

and thus

$$
\frac{d\pi_\rho}{d\pi_\lambda} = exp\Big\{ \langle \xi, log\,\phi \rangle + \int_{\mathbb{X}} (1-\phi) d\lambda \Big\}.
$$

□

3.3 Proof of "Only If" direction

Let us from here on assume that λ and ρ give infinite measure to the whole space $\mathbb{X}$. As in the finite case, we get the first part of this direction rather quickly using the Laplace functional. To see this let $B \in \mathcal{X}$ be a measurable set and let

$$
u(x) := \begin{cases} \infty, & \text{if } x \in B \\ 0, & \text{else} \end{cases}
$$

If we plug this $u(x)$ into the formula for the Laplace functional we get analogously to the finite case

$$
\pi_\lambda\{\xi : \xi(B) = 0\} = exp\{-\lambda(B)\} \tag{9}
$$

$$
\pi_\lambda\{\xi : \xi(B) > 0\} = 1 - exp\{-\lambda(B)\}. \tag{10}
$$

This gives us the following Lemma, which already covers the first part of the "only if" direction of our theorem.

Lemma 2. *If π_λ and π_ρ are mutually absolutely continuous, then for each measurable set $B \in \mathcal{X}$*

$$
0 < \lambda(B) < \infty \quad \textit{if and only if} \quad 0 < \rho(B) < \infty.
$$

In particular, the measures λ and ρ are mutually absolutely continuous.

From here on let us assume mutual absolute continuity between π_λ and π_ρ for the rest of this subsection.

The next Lemma due to Kakutani can be found in [6].

Lemma 3. *(Kakutani) Let X_n be a sequence of random variables and consider the σ-algebras*

$$\mathcal{F} = \sigma(X_k, k \in \mathbb{N}), \quad \mathcal{F}_n = \sigma(X_k, 1 \leq k \leq n).$$

Furthermore let f_n and g_n be two everywhere positive density functions on our probability space and let P and Q be the according probability densities, under which X_n are independent. If Q is absolutely continuous with respect to P, then the Radon-Nikodym derivative $\frac{dQ}{dP}$ is given by

$$\frac{dQ}{dP} = \prod_{i=1}^{\infty} Y_i \in (0, \infty) \quad P\text{-almost surely,}$$

where $Y_i = \frac{g_i(X_i)}{f_i(X_i)}$.

Proof. Can be found in [6] on pages 150 - 151. □

Lemma 4. *For every positive constant M we have*

(i) $\lambda\{x \in \mathbb{X} : \phi(x) > 1 + M\} < \infty$

(ii) $\int_{\phi > 1+M} \phi \, d\lambda < \infty$.

Proof. Statement *(ii)* follows from *(i)* with Lemma 2 since

$$\rho\{x \in \mathbb{X} : \phi(x) > 1 + M\} = \int_{\phi > 1+M} \phi \, d\lambda.$$

We want to prove *(i)* by contradiction, so let us assume the contrary. Then there exists an infinite partition $\{B_i\}$ of the set

$$\{x \in \mathbb{X} : \phi(x) > 1 + M\},$$

such that for each i

$$\lambda(B_i) = 1.$$

Note that this can be done since λ is assumed σ-finite and nonatomic and the range of nonatomic measures are compact intervals, see for example [5] on page 39. Now consider

$$X_i := sgn\,\xi(B_i) = \begin{cases} 1 & \text{if } \xi(B_i) > 0 \\ 0 & \text{if } \xi(B_i) = 0 \end{cases}$$

where ξ are the counting measures one gets from the Poisson processes η_ρ and η_λ. Let P and Q be the laws of $(X_i)_{i\geq 1}$ under η_λ and η_ρ respectively. Then by using (9) and (10)

$$\begin{aligned} &Q(X_i = 0) = e^{-\rho(B_i)}, \quad Q(X_i = 1) = 1 - e^{-\rho(B_i)} \\ &P(X_i = 0) = e^{-1} \quad \text{and} \quad P(X_i = 1) = 1 - e^{-1}. \end{aligned}$$

Since η_ρ and η_λ are mutually absolutely continuous, so are the probability measures P and Q and by Lemma 3

$$\frac{dQ}{dP} = \prod_{i=1}^{\infty} \Big(1_{[X_i=0]} \frac{e^{-\rho(B_i)}}{e^{-1}} + 1_{[X_i=1]} \frac{1 - e^{-\rho(B_i)}}{1 - e^{-1}} \Big).$$

Or maybe more elegantly

$$\frac{dQ}{dP} = exp\Big\{ \sum_{i=1}^{\infty} Z_i \Big\} \in (0, \infty) \quad P\text{-almost surely,} \tag{11}$$

where

$$Z_i := 1_{[X_i=0]}(1 - \rho(B_i)) + 1_{[X_i=1]} log \frac{1 - e^{-\rho(B_i)}}{1 - e^{-1}}.$$

Since every Z_i only depends on the corresponding X_i and the X_i are clearly independent, so are the random variables Z_i. Now (11) implies that $\sum_i Z_i$ converges almost surely, which means in particular that

$$P(Z_i < -M \text{ infinitely often}) = 0.$$

By using the second Borel-Cantelli lemma we get

$$\sum_i P(Z_i < -M) < \infty.$$

On the other hand however if $X_i = 0$

$$\begin{aligned} Z_i &= 1 - \rho(B_i) = 1 - \int_{B_i} d\rho = 1 - \int_{B_i} \phi \, d\lambda < 1 - \int_{B_i} 1 + M \, d\lambda \\ &= 1 - \int_{B_i} d\lambda - \int_{B_i} M \, d\lambda = 1 - \lambda(B_i) - M\lambda(B_i) = -M \end{aligned}$$

and thus

$$\sum_i P(Z_i < -M) \geq \sum_i P(X_i = 0) = \sum_i e^{-1} = \infty,$$

which is clearly a contradiction and we are finished with the proof. □

Lemma 5. *For every positive constant M we have*

(i) $\lambda\{x \in \mathbb{X} : \phi(x) < 1 - M\} < \infty$

(ii) $\rho\{x \in \mathbb{X} : \phi(x) < 1 - M\} = \int_{\phi<1-M} \phi \, d\lambda < \infty$.

Proof. The first statement can be proved similarly to the previous lemma by interchanging the roles of λ and ρ and the second statement follows immediately from *(i)*. □

We are now ready to prove the "only if" direction of our main Theorem.

Proof of the "only if" direction. Let M be an arbitrary positive constant. Using the two previous lemmas we get

$$\int_{\phi>e^M \text{ or } \phi<e^{-M}} |\phi - 1| d\lambda < \infty.$$

Where we replaced ϕ by $|\phi - 1|$, which has no influence on the finiteness of the integral and instead of $1 + M$ and $1 - M$ we wrote e^M and e^{-M}, which is nothing but a change in notation. Also since for nonnegative x

$$|\sqrt{x}-1|^2 \leq |(\sqrt{x}-1)^2| \leq |x-2\sqrt{x}+1| \leq |x+1|$$

we get

$$\int_{\phi>e^M \text{ or } \phi<e^{-M}} |\sqrt{\phi}-1|^2 d\lambda < \infty.$$

So the only part left to show is

$$\int_{1<\phi\leq e^M} |\sqrt{\phi}-1|^2 d\lambda < \infty \quad \text{and} \tag{12}$$

$$\int_{1>\phi\geq e^{-M}} |\sqrt{\phi}-1|^2 d\lambda < \infty. \tag{13}$$

We will only look at (12), since (13) follows immediately by interchanging ρ and λ. Now if the set $\{x \in \mathbb{X} : 1 < \phi(x) \leq e^M\}$ is finite under λ, the statement is trivial. Therefore consider the opposite, then we can take a partition of measurable sets B_i of this set with $\lambda(B_i) = 1$ and

$$B_i \text{ is contained in } \{x \in \mathbb{X} : b_i \geq \sqrt{\phi}(x) - 1 \geq b_{i+1}\}$$

for some nonincreasing sequence $b_1 \geq b_2 \geq ... \geq b_n \to 0$. Let us use the same notation as in the proof of lemma (4) for the bounded random variables Z_i. We have already seen, that if we have a partition of measurable sets, then $\sum_i Z_i < \infty$ almost surely, so by using the three series theorem we get for the expectation with respect to P that

$$\sum_i \mathbb{E}[Z_i^2] < \infty.$$

Note that by the boundedness of Z_i we do not need to write $\mathbb{E}[(Z_i^{[a]})^2]$ as in the statement of the three series theorem. We can now use the expectation restricted to the event $X_i = 0$ to find an upper bound

$$\begin{aligned}\mathbb{E}[Z_i^2] \geq \mathbb{E}[Z_i^2 1_{[X_i=0]}] &= P(X_i = 0)(\rho(B_i) - \lambda(B_i))^2 \\ &= e^{-1}(\rho(B_i) - 1) = e^{-1}\Big\{\int_{B_i} (\phi - 1) d\lambda\Big\}^2\end{aligned}$$

and therefore

$$\sum_i \Big\{ \int_{B_i} (\phi - 1) d\lambda \Big\}^2 < \infty.$$

Now on B_i we have

$$\phi - 1 = (\sqrt{\phi} - 1)(\sqrt{\phi} + 1) = (\sqrt{\phi} - 1)((\sqrt{\phi} - 1) + 2) \geq b_{i+1}^2 + 2b_{i+1}.$$

Next we want to use this and the fact that $\lambda(B_i) = 1$

$$\begin{aligned}
\infty > \sum_i \Big(\int_{B_i} (\phi - 1) d\lambda \Big)^2 &\geq \sum_i \Big(\int_{B_i} (b_i^2 + 2b_i) d\lambda \Big)^2 \\
= \sum_i ((b_i^2 + 2b_i)\lambda(B_i))^2 &\geq \sum_i b_i^2.
\end{aligned}$$

Using convergence of this sum and $\lambda(B_i)) = 1$ again we get

$$\begin{aligned}
\int_{1 < \phi \leq e^M} |\sqrt{\phi} - 1|^2 d\lambda &= \sum_i \int_{B_i} |\sqrt{\phi} - 1|^2 d\lambda \\
&\leq \sum_i \lambda(B_i) b_i^2 \\
&= \sum_i b_i^2 < \infty
\end{aligned}$$

and therefore (12) and by interchanging ρ and λ also (13), which were the only two parts left to be shown and the result is proven. $\square$

3.4 Proof of "If" direction

In this section let us assume finiteness of the Hellinger distance $d(\rho, \lambda)$ and mutual absolute continuity between ρ and λ.

Lemma 6. *Assume that the total variation of the signed measure $\rho - \lambda$ is finite, that is*

$$\int_{\mathbb{X}} |d\rho - d\lambda| = \int_{\mathbb{X}} |\phi - 1|\, d\lambda < \infty.$$

Then π_ρ and π_λ are mutually absolutely continuous and

$$\frac{d\pi_\rho}{d\pi_\lambda}(\xi) = exp\Big\{\langle \xi, log\, \phi \rangle + \int_{\mathbb{X}} (1-\phi) d\lambda \Big\}.$$

Proof. This is basically just the proof of the "if" direction for the finite case, since we never actually used the finiteness of the space $\mathbb{X}$ in the computation for said proof. $\square$

Lemma 7. *For every positive constant M, the following holds*

$$\int_{\phi > e^M} (\phi + 1) d\lambda < \infty \quad \textit{and} \tag{14}$$

$$\int_{\phi < e^{-M}} (\phi + 1) d\lambda < \infty. \tag{15}$$

Proof. By symmetry in ρ and λ it suffices to prove (14). This however follows from the fact that

$$(\sqrt{x} - 1)^2 = x + 1 - 2\sqrt{x} > C(x+1) \quad \text{if } x > e^M$$

for some constant C depending on M. Now the finiteness of the integral follows immediately from (8). $\square$

Corollary 3. *Set*

$$E := \Big\{ x : \phi(x) \geq 2 \ or\ \phi(x) \leq \frac{1}{2} \Big\},$$

then

$$\int_E |\phi - 1| d\lambda < \infty.$$

Proof. Choose $M = log(2)$ in the Lemma above and use the fact that $|\phi - 1| \leq (\phi + 1)$ on E. $\square$

The following Lemma can be found in [1] chapter 12.

Lemma 8. *Let $f, g \in L^1(\lambda) \cap L^2(\lambda)$ and let us use the notation*

$$I(f) = \int_{\mathbf{N}} f \, \pi_\lambda(d\xi) - \int_{\mathbb{X}} f d\lambda.$$

Then

$$\mathbb{E}[I(f)I(g)] = \int_{\mathbb{X}} f g \, d\lambda.$$

Proof. Can be found in [1] on page 112. $\square$

Let us consider a sequence C_i of measurable sets with finite measure under λ such that $\bigcup_i C_i = \mathbb{X}$. Now the sets

$$K_n := \bigcup_{i=1}^{n} C_i$$

form an exhausting, increasing sequence of measurable sets of finite measure.

Lemma 9. *Let $(K_i)_{i \geq 0}$ be an exhausting, increasing sequence of measurable sets of finite measure as constructed above and set*

$$B_0 := E \quad and \quad B_n := E \cup K_n \ (n \geq 1).$$

Furthermore let ξ_B be the restriction of the measure ξ to the measurable set B and set

$$Y_n(\xi) := \langle \xi_{B_n}, log \, \phi \rangle + \int_{B_n} (1 - \phi) d\lambda.$$

Then the following limit exists π_λ-almost everywhere:

$$Y(\xi) = \lim_{n \to \infty} Y_n(\xi).$$

Moreover the convergence also holds in $L^2(\pi_\lambda)$.

Proof. Set $B'_n := B_n \setminus B_{n-1}$. Since the B'_n are disjoint, the random variables $\xi_{B'_n}$ are pairwise independent by the Restriction Theorem. Thus the random variables

$$Z_n := Y_n - Y_{n-1} \quad \text{and} \quad Z_0 := Y_0$$

are also independent. The case where $\phi(x) \geq 2$ or $\phi(x) \leq \frac{1}{2}$ it is clear that the integral is finite, so let $\frac{1}{2} \leq \phi(x) \leq 2$. Since in this case $\phi(x)$ is finite we can always find a constant C such that

$$|log\, \phi + 1 - \phi| \leq C(\sqrt{\phi} - 1)^2 \quad \text{and} \quad |log\, \phi|^2 \leq C(\sqrt{\phi} - 1)^2. \tag{16}$$

To see why this is useful we need to do a little computation

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{E}[Z_n] &= \sum_{n=1}^{\infty} \int_{\mathbf{N}} \pi_\lambda(d\xi)\Big(\langle \xi_{B'_n}, log\, \phi\rangle + \int_{B'_n} (1-\phi) d\lambda\Big) \\
&= \sum_{n=1}^{\infty} \int_{B'_n} (1-\phi) d\lambda + \int_{\mathbf{N}} \pi_\lambda(d\xi) \langle \xi_{B'_n}, log\, \phi\rangle \\
&= \sum_{n=1}^{\infty} \int_{B'_n} (1-\phi) d\lambda + \int_{B'_n} log\, \phi \, d\lambda \\
&= \sum_{n=1}^{\infty} \int_{B'_n} (log\, \phi + 1 - \phi) d\lambda,
\end{aligned}$$

where we used the Mecke equation in the third step. Similarly, although a bit more complicated, we can compute $\sum_n \mathbb{E}[Z_n^2]$. For this let us first recall Lemma 8 for $f = g = log\, \phi \in L^1(\lambda) \cap L^2(\lambda)$, that is

$$\mathbb{E}[I(log\, \phi)^2] = \int_{\mathbb{X}} (log\, \phi)^2 \, d\lambda. \tag{17}$$

Now the left hand side of this can be rewritten as

$$\mathbb{E}\Big[\Big(\int_{\mathbf{N}}\pi_\lambda(d\xi)log\,\phi\Big)^2-2\Big(\int_{\mathbb{X}}log\,\phi\,d\lambda\Big)\Big(\int_{\mathbf{N}}\pi_\lambda(d\xi)log\,\phi\Big)+\Big(\int_{\mathbb{X}}log\,\phi\,d\lambda\Big)^2\Big]$$
$$=\Big(\int_{\mathbb{X}}log\,\phi\,d\lambda\Big)^2-2\Big(\int_{\mathbb{X}}log\,\phi\,d\lambda\Big)\mathbb{E}\Big[\Big(\int_{\mathbf{N}}\pi_\lambda(d\xi)log\,\phi\Big)\Big]+\mathbb{E}\Big[\Big(\int_{\mathbf{N}}\pi_\lambda(d\xi)log\,\phi\Big)^2\Big]$$
$$=\Big(\int_{\mathbb{X}}log\,\phi\,d\lambda\Big)^2-2\Big(\int_{\mathbb{X}}log\,\phi\,d\lambda\Big)^2+\mathbb{E}\Big[\Big(\int_{\mathbf{N}}\pi_\lambda(d\xi)log\,\phi\Big)^2\Big].$$

Putting this and (17) together yields

$$\mathbb{E}\Big[\Big(\int_{\mathbf{N}}\pi_\lambda(d\xi)log\,\phi\Big)^2\Big]=\int_{\mathbb{X}}(log\,\phi)^2d\lambda+\Big(\int_{\mathbb{X}}log\,\phi\,d\lambda\Big)^2.$$

Thus for fixed n we get

$$\begin{aligned}\mathbb{E}[Z_n^2]&=\Big(\int_{B_n'}(1-\phi)d\lambda\Big)^2+2\int_{B_n'}(1-\phi)d\lambda\,\mathbb{E}\Big[\int_{\mathbf{N}}\pi_\lambda(d\xi)log\,\phi\Big]\\&\qquad+\int_{B_n'}(log\,\phi)^2d\lambda+\Big(\int_{B_n'}log\,\phi\,d\lambda\Big)^2\\&=\Big(\int_{B_n'}(1-\phi)d\lambda\Big)^2+2\int_{B_n'}(1-\phi)d\lambda\int_{B_n'}log\,\phi\,d\lambda\\&\qquad+\int_{B_n'}(log\,\phi)^2d\lambda+\Big(\int_{B_n'}log\,\phi\,d\lambda\Big)^2\\&=\Big(\int_{B_n'}(log\,\phi+1-\phi)\,d\lambda\Big)^2+\int_{B_n'}(log\,\phi)^2d\lambda.\end{aligned}$$

Since the Z_n are disjoint, we can simply sum over n to finally get

$$\sum_{n=1}^{\infty}\mathbb{E}[Z_n^2]=\sum_{n=1}^{\infty}\Big(\int_{B_n'}(\log\,\phi+1-\phi)\,d\lambda\Big)^2+\sum_{n=1}^{\infty}\int_{B_n'}(log\,\phi)^2d\lambda.$$

Now by combining (16) and (8) we get that the series

$$\sum_{n=1}^{\infty} \mathbb{E}[Z_n] = \sum_{n=1}^{\infty} \int_{B'_n} (log\,\phi + 1 - \phi) d\lambda \quad \text{and} \tag{18}$$

$$\sum_{n=1}^{\infty} \mathbb{E}[Z_n^2] = \sum_{n=1}^{\infty} \Big(\int_{B'_n} (log\,\phi + 1 - \phi) d\lambda \Big)^2 + \sum_{n=1}^{\infty} \int_{B'_n} (log\,\phi)^2 d\lambda \tag{19}$$

are absolutely convergent. Thus by the three-series theorem $\sum_n Z_n$ converges almost surely.

Note that it is sufficient to only check those two sums, since the random variables Z_n are bounded by the boundedness of ϕ and $\lambda(B'_n)$. Also by (19) the sum converges in $L^2(\pi_\lambda)$. In summary we have shown that the limit of Y_n exists almost surely and in $L^2(\pi_\lambda)$ and we are finished with the proof. □

Lemma 10. *The limit $Y(\xi)$ defined in the lemma above satisfies*

$$\int_{\mathbf{N}} \pi_\lambda(d\xi) exp Y(\xi) = 1.$$

Proof. On one hand

$$\begin{aligned}
\int_{\mathbf{N}} \pi_\lambda(d\xi) exp Y_n &= \int_{\mathbf{N}} \pi_\lambda(d\xi) exp(\langle \xi, log\,\phi \rangle - \langle \lambda, \phi - 1 \rangle) \\
&= exp\Big\{ \int_{\mathbb{X}} \phi - 1\, d\lambda \Big\} \int_{\mathbf{N}} \pi_\lambda(d\xi) exp\{\langle \xi, log\,\phi \rangle\} \\
&= exp\Big\{ \int_{\mathbb{X}} \phi - 1\, d\lambda \Big\} exp\Big\{ \int_{\mathbb{X}} 1 - e^{log\,\phi} d\lambda \Big\} \\
&= exp\Big\{ \int_{\mathbb{X}} \phi - 1\, d\lambda \Big\} exp\Big\{ \int_{\mathbb{X}} 1 - \phi\, d\lambda \Big\} \\
&= e^0 = 1
\end{aligned}$$

Where we used the Laplace functional to compute the integral over $\mathbf{N}$. Now we get with the Fatou lemma

$$\begin{aligned}
\int_{\mathbf{N}} \pi_\lambda(d\xi) exp Y &= \int_{\mathbf{N}} \pi_\lambda(d\xi) \liminf_{n\to\infty} exp Y_n \\
&\le \liminf_{n\to\infty} \int_{\mathbf{N}} \pi_\lambda(d\xi) exp Y_n = 1.
\end{aligned}$$

On the other hand we can use the computation above and the Jensen inequality to get

$$\begin{aligned}
\int_{\mathbf{N}} \pi_\lambda(d\xi) exp Y &= \int_{\mathbf{N}} \pi_\lambda(d\xi) exp\{Y - Y_n\} \\
&\geq exp\Big\{ \int_{\mathbf{N}} \pi_\lambda(d\xi)(Y - Y_n) \Big\} \\
&\geq exp\Big\{ \int_{\mathbf{N}} \pi_\lambda(d\xi) |Y - Y_n|^2 \Big\}
\end{aligned}$$

as $n \to \infty$ this expression converges to 1 by lemma 9 and the proof is finished. □

Proof of the "if" direction. The proof is rather similar to the proof of lemma 6 with some extra steps that arise when no finiteness of the total variation is assumed. First consider a function $u \in \mathbb{R}_+(\mathbb{X})$ for which there exist an integer n such that the support of u is containted in B_n except for a null set, where the B_n are as defined in Lemma 9. Then

$$\begin{aligned}
\int_{\mathbf{N}} \pi_\rho(d\xi) exp\{-\langle \xi, u\rangle\} &= \int_{\mathbf{N}} \pi_\rho(d\xi) exp\{-\langle \xi_{B_n}, u\rangle\} \\
&= exp\Big\{ - \int_{B_n} (1 - e^{-u}) d\rho \Big\} \\
&= exp\Big\{ - \int_{B_n} (1 - e^{-u}) \phi \, d\lambda \Big\} \\
&= exp\Big\{ \int_{B_n} (1 - \phi) d\lambda \Big\} exp\Big\{ - \int_{B_n} (1 - e^{-u}\phi) d\lambda \Big\} \\
&= \prod_{k=0}^{n} exp\Big\{ \int_{B'_k} (1 - \phi) d\lambda \Big\} exp\Big\{ - \int_{B'_k} (1 - e^{-u}\phi) d\lambda \Big\} \\
&= \prod_{k=0}^{n} \int_{\mathbf{N}} \pi_\lambda(d\xi) exp\{-\langle \xi_{B'_k}, u\rangle + Y_k(\xi) - Y_{k-1}(\xi)\},
\end{aligned}$$

where we used the Laplace functional in the second and last step. Now since Poisson processes on disjoint sets are independent, we can interchange the product and the integral to get

$$\int_{\mathbf{N}} \pi_\lambda(d\xi) exp\{-\langle \xi_{B_n}, u\rangle + Y_n(\xi)\} = \int_{\mathbf{N}} \pi_\lambda(d\xi) exp\{Y_n(\xi)\} exp\{-\langle \xi, u\rangle\}.$$

By Lemma 9 we can let $n \to \infty$ to get

$$\int_{\mathbf{N}} \pi_\rho(d\xi) exp\{-\langle \xi, u\rangle\} = \int_{\mathbf{N}} \pi_\lambda(d\xi) exp\{Y(\xi)\} exp\{-\langle \xi, u\rangle\}.$$

Now for general $u \in \mathbb{R}_+(\mathbb{X})$, let us consider a sequence of measurable functions $u_n := 1_{B_n} u$. Of course these u_n converge to u as n tends to ∞ and since

$$exp\{-\langle \xi, u_n\rangle\} \searrow exp\{-\langle \xi, u\rangle\} \geq 0$$

we can use the monotone convergence theorem twice to see that

$$\begin{aligned}
\int_{\mathbf{N}} \pi_\rho(d\xi) exp\{-\langle \xi, u\rangle\} &= \int_{\mathbf{N}} \lim_{n\to\infty} \pi_\rho(d\xi) exp\{-\langle \xi, u_n\rangle\} \\
&= \lim_{n\to\infty} \int_{\mathbf{N}} \pi_\rho(d\xi) exp\{-\langle \xi, u_n\rangle\} \\
&= \lim_{n\to\infty} \int_{\mathbf{N}} \pi_\lambda(d\xi) exp\{Y(\xi)\} exp\{-\langle \xi, u_n\rangle\} \\
&= \int_{\mathbf{N}} \lim_{n\to\infty} \pi_\lambda(d\xi) exp\{Y(\xi)\} exp\{-\langle \xi, u_n\rangle\} \\
&= \int_{\mathbf{N}} \pi_\lambda(d\xi) exp\{Y(\xi)\} exp\{-\langle \xi, u\rangle\}.
\end{aligned}$$

Therefore the measures π_ρ and π_λ are mutually absolutely continuous and the Radon-Nikodym derivative is given by

$$\frac{d\pi_\rho}{d\pi_\lambda}(\xi) = exp\, Y(\xi),$$

which completes the proof. $\square$

3.5 Computation of the Hellinger distance

The last part of the theorem that remains to be shown is the explicit expression for the Hellinger distance in case of mutual absolute continuity.

Lemma 11. *If the total variation of the signed measure $\rho - \lambda$ is finite, that is*

$$\int_{\mathbb{X}} |d\rho - d\lambda| = \int_{\mathbb{X}} |\phi - 1| \, d\lambda < \infty,$$

then the Hellinger distance $D(\pi_\rho, \pi_\lambda)$ can be written as

$$D(\pi_\rho, \pi_\lambda) = 1 - exp\{-d(\rho, \lambda)\},$$

where $d(\rho, \lambda)$ is the Hellinger distance between ρ and λ.

Proof. To prove the lemma let us first recall that in case of mutual absolute continuity and finiteness of the total variation, we can write the Radon-Nikodym derivative of the Poisson measures explicitly:

$$d\pi_\rho / d\pi_\lambda = exp\{\langle \xi, log\, \phi \rangle - \langle \lambda, \phi - 1 \rangle\}.$$

We therefore find

$$\begin{aligned} D(\pi_\rho, \pi_\lambda) &= \frac{1}{2} \int_{\mathbf{N}} |\sqrt{d\pi_\lambda} - \sqrt{d\pi_\rho}|^2 \\ &= \frac{1}{2} \int_{\mathbf{N}} d\pi_\lambda \, |1 - \sqrt{\frac{d\pi_\rho}{d\pi_\lambda}}|^2 \\ &= \frac{1}{2} \int_{\mathbf{N}} d\pi_\lambda \, |1 - \sqrt{exp\{\langle \xi, log\, \phi \rangle - \langle \lambda, \phi - 1 \rangle\}}|^2 \\ &= \frac{1}{2} \int_{\mathbf{N}} d\pi_\lambda \, |1 - 2\sqrt{exp\{\langle \xi, log\, \phi \rangle - \langle \lambda, \phi - 1 \rangle\}} + exp\{\langle \xi, log\, \phi \rangle - \langle \lambda, \phi - 1 \rangle\}|. \end{aligned}$$

Now we can pull out the first and the last term of the integral and use Lemma 10 to simplify this expression to

$$
\begin{aligned}
&= 1 - \int_{\mathbf{N}} d\pi_\lambda exp\Big\{\frac{1}{2}\langle \xi, log\,\phi\rangle - \langle \lambda, \phi - 1\rangle\Big\} \\
&= 1 - exp\Big\{-\frac{1}{2}\langle \lambda, \phi - 1\rangle\Big\} \int_{\mathbf{N}} d\pi_\lambda\, exp\Big\{\langle \xi, \frac{1}{2} log\,\phi\Big\}.
\end{aligned}
$$

Next we use the Laplace functional and get

$$
\begin{aligned}
&= 1 - exp\Big\{-\frac{1}{2}\langle \lambda, \phi - 1\rangle\Big\} exp\Big\{\int_{\mathbb{X}} 1 - e^{\frac{1}{2} log\,\phi} d\lambda\Big\} \\
&= 1 - exp\Big\{-\frac{1}{2}\langle \lambda, \phi - 1\rangle\Big\} exp\Big\{\int_{\mathbb{X}} 1 - \sqrt{\phi}\, d\lambda\Big\}.
\end{aligned}
$$

However since

$$
\begin{aligned}
-\frac{1}{2}(\phi - 1) + \sqrt{\phi} - 1 &= -\frac{1}{2}\phi + \frac{1}{2} + \sqrt{\phi} - 1 \\
&= -\frac{1}{2}\phi + \sqrt{\phi} - \frac{1}{2} = -\frac{1}{2}(\sqrt{\phi} - 1)^2
\end{aligned}
$$

we can simplify the expression for the Hellinger distance $D(\pi_\rho, \pi_\lambda)$ even further to

$$
\begin{aligned}
&= 1 - exp\Big\{-\frac{1}{2}\langle \lambda, (\sqrt{\phi} - 1)^2\rangle\Big\} \\
&= 1 - exp\{-d(\rho, \lambda)\}.
\end{aligned}
$$

and we have proven the lemma. □

It is now time to prove the final part of the theorem, the expression for the Hellinger distance in case of mutual absolute continuity.

Proof of the final Part of the Theorem. Similar to the proof above, let us recall, that in case of absolute continuity without the assertion of finiteness of the total variation, we can write the Radon-Nikodym derivative as

$$
d\pi_\rho / d\pi_\lambda = exp\, Y,
$$

where $Y = \lim_{n\to\infty} Y_n$ and

$$Y_n(\xi) = \langle \xi_{B_n}, log\, \phi \rangle + \int_{B_n} (1 - \phi) d\lambda.$$

Another way of writing Y_n however is the following

$$Y_n(\xi) = \langle \xi, log\, \phi_n \rangle + \langle \lambda, 1 - \phi_n \rangle,$$

where

$$\phi_n := \begin{cases} \phi, & \text{on } B_n \\ 1, & \text{else} \end{cases} .$$

Now

$$\begin{aligned} D(\pi_\rho, \pi_\lambda) &= \frac{1}{2} \int_{\mathbf{N}} |\sqrt{d\pi_\rho} - \sqrt{d\pi_\lambda}|^2 \\ &= \frac{1}{2} \int_{\mathbf{N}} |\frac{\sqrt{d\pi_\rho}}{\sqrt{d\pi_\lambda}} - 1|^2 \\ &= \frac{1}{2} \int_{\mathbf{N}} |exp\, Y - 2exp\sqrt{Y} + 1|^2 \\ &= \frac{1}{2} + \frac{1}{2} - \int_{\mathbf{N}} d\pi_\lambda exp\, \frac{1}{2} Y_n \\ &= 1 - \lim_{n\to\infty} \int_{\mathbf{N}} d\pi_\lambda exp \frac{1}{2} Y_n, \end{aligned}$$

where we used the fact that $exp\, \frac{1}{2} Y_n$ are uniformly integrable functions by corollary 2 and thus converge in L^1 by theorem 8, in the last step to pull out the limit. However we have already computed a similar integral in the previous proof, as the total variation is of course finite for each n. This leads us to

$$= 1 - \lim_{n\to\infty} exp\Big\{ -\frac{1}{2} \int_{\mathbb{X}} (\sqrt{\phi_n} - 1)^2 d\lambda \Big\}.$$

Finally since $|\sqrt{\phi_n} - 1| \leq |\sqrt{\phi} - 1|$ and $|\sqrt{\phi} - 1|$ is square integrable by (8), we can use the dominated convergence theorem to interchange integral and limit again to get

$$
\begin{aligned}
&= 1 - exp\Big\{ -\frac{1}{2}\int_{\mathbb{X}} (\sqrt{\phi} - 1)^2 d\lambda \Big\} \\
&= 1 - exp\{-d(\rho, \lambda)^2\},
\end{aligned}
$$

which completes the proof.

$\square$

References

[1] Guenter Last, Mathew Penrose (2017). Lectures on the Poisson Process. *Cambridge University Press*

[2] Yoichiro Takahashi (1990). Absolute Continuity of Poisson Random Fields. *Publ. RIMS Kyoto Univ. 26 (1990), 629-647*

[3] P.E. Kopp (1984) Martingales and stochastic integrals. *Cambridge University Press*

[4] J.F.C. Kingman (1993) Poisson Processes *Clarendon Press*

[5] Andrzej Fryszkowski (2005) Fixed Point Theory for Decomposable Sets (Topological Fixed Point Theory and Its Applications). *Springer*

[6] David Williams (1991) Probability with Martingales. *Cambridge University Press*

[7] D.Pollard (2002) A User's Guide to Measure Theoretic Probability. *Cambridge University Press*