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„Nonabelian Extensions of the Standard Model with Classical Scale Invariance"

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Abstract

This master's thesis discusses a nonabelian, classically scale invariant extension of the Standard Model (SM), where the SM gauge group is enlarged by an additional SU(2) gauge symmetry. This new gauge group, under which the SM particles transform trivially, comes along with the appearance of three new gauge bosons. Although classical scale invariance forbids a bilinear Higgs mass term with operator dimension 2 in the Lagrangian density, spontaneous symmetry breaking is induced by radiative corrections at one-loop level (Coleman-Weinberg mechanism). Furthermore, the well-known masses of the top-quark ($\sim$ 170 GeV) and of the Higgs boson (125 GeV) make it necessary to introduce additional scalar degrees of freedom to stabilize the effective potential. As a first step, a new scalar particle is added to the theory, which is a doublet with respect to the new SU(2) gauge symmetry and a singlet under the SM gauge group. Afterwards, a real scalar singlet and three right-handed neutrinos are introduced, which enable the implementation of the seesaw mechanism. After a summary of the theoretical background and the derivation of the effective potential for a very general theory, the Gildener-Weinberg method is used to perturbatively analyse both of these models up to one-loop level in great detail.

Zusammenfassung

Diese Masterarbeit behandelt eine nicht-abelsche, klassisch skaleninvariante Erweiterung des Standardmodells (SM), wobei die Eichgruppe des SM um eine zusätzliche SU(2) Eichsymmetrie erweitert wird. Diese neue Eichgruppe, unter welcher die Teilchen des SM trivial transformieren, geht einher mit dem Auftreten von drei neuen Eichbosonen. Obwohl klassische Skaleninvarianz einen bilinearen Higgs-Massenterm mit Operatordimension 2 in der Lagrange-Dichte verbietet, wird spontane Symmetriebrechung auf 1-Schleifenniveau durch Strahlungskorrekturen induziert (Coleman-Weinberg Mechanismus). Des Weiteren machen es die bekannten Massen des Top-Quarks ($\sim$ 170 GeV) und des Higgs-Bosons (125 GeV) notwendig, zusätzliche skalare Freiheitsgrade einzuführen, um das effektive Potential zu stabilisieren. Als ersten Schritt wird der Theorie ein neues skalares Teilchen hinzugefügt, welches ein Dublett bezüglich der neuen SU(2) Eichsymmetrie und ein Singlett unter der Eichgruppe des SM ist. Anschließend werden ein reelles skalares Singlett und drei rechtshändige Neutrinos eingeführt, wodurch die Implementierung des Seesaw-Mechanismus ermöglicht wird. Nach einer Zusammenfassung der theoretischen Grundlagen und der Herleitung des effektiven Potentials für eine allgemeine Theorie wird die Gildener-Weinberg Methode zur detaillierten, störungstheoretischen Analyse dieser beiden Modelle auf 1-Schleifenniveau herangezogen.

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Contents

1. Introduction

In the Standard Model (SM), the masses of the gauge bosons $W^\pm$ and Z, as well as the fermion masses, are generated via the famous Higgs mechanism [1, 2], where the vacuum expectation value (VEV) of the Higgs field H is generated via spontaneous symmetry breaking (SSB) due to the special form of the tree-level Higgs potential:

$$V = -\mu^2 H^\dagger H + \frac{\lambda}{4} \left(H^\dagger H \right)^2 \ . \tag{1.0.1}$$

Despite its huge success explaining most of the experimental data, it is clear that the SM needs to be extended in some way to be able to address some open questions like the appearance and tininess of neutrino masses[1] [3, 4], the nature of dark matter, the stability of the Higgs potential [5, 6] or the Hierarchy problem of the Higgs mass.

One promising class of such extensions of the SM are so-called classically scale-invariant (CSI) extensions of the SM, where only terms possessing operator dimension 4 are allowed in the Lagrangian density (the term $-\mu^2 H^\dagger H$ is therefore forbidden) and where the VEV of a scalar field is not generated at zero-loop order but with the help of radiative corrections to the tree-level potential (Coleman-Weinberg mechanism [7]). Such theories have to come along with an extension of the SM gauge group and/or the introduction of additional scalar fields to explain the experimentally observed Higgs mass of 125 GeV.

In this thesis, we will focus on a (nonabelian) extension of the Standard Model gauge group [8–15], where we introduce an extra SU(2) gauge symmetry, under which the SM particle content is invariant, and a corresponding second Higgs doublet, which transforms nontrivially under this new SU(2) gauge group only. We investigate this model with the help of the Gildener-Weinberg analysis [16] and try to find constraints on the coupling parameters as well as on the masses of the new particles. In a second step, we introduce an additional real scalar singlet and three right-handed neutrino fields to be able to implement the seesaw mechanism [17–21] in our model and thus, to explain small neutrino masses.

This thesis is structured in the following way: Chapter 2 gives a short overview over conventions and the notation of this work, as well as a brief introduction into classical scale invariance. We introduce the Coleman-Weinberg mechanism in chapter 3 and discuss the effective potential and its properties in detail. An explicit calculation of the effective potential for a simple toy model with the help of Feynman diagrams can also be found in this chapter. This is followed by the derivation of the effective potential for a very general theory in chapter 4, where the path integrals are calculated explicitly, and a discussion of the Gildener-Weinberg approach in chapter 5. Chapter 6 finally deals with nonabelian classically scale invariant extensions of the Standard Model and discusses its properties in great detail.

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[1] A consequence of the observation of neutrino oscillation.

2. Preliminaries

2.1. Notation and Constants

In this thesis, we work with the following convention for the Minkowski metric (*particle physics* or *west coast* convention):

$$
(g_{\mu\nu}) = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} .
\tag{2.1.1}
$$

If we are dealing with four-vectors, we indicate this by the use of Greek indices, where we distinguish between upper and lower ones[1]:

$$
\sum_{\mu=1}^{4} q_\mu \, p^\mu = \sum_{\mu=1}^{4} \sum_{\nu=1}^{4} g_{\mu\nu} \, q^\mu p^\nu .
\tag{2.1.2}
$$

Unless not stated otherwise, we sum over repeating indices, if they appear twice in a single term (Einstein summation):

$$
\sum_{i=1}^{n} x_i \, y_i = x_1 y_1 + x_2 y_2 + \ldots x_n y_n = x_i \, y_i \ .
\tag{2.1.3}
$$

$$
\sum_{\mu=1}^{4} q_\mu \, p^\mu = q_0 \, p_0 - q_1 \, p_1 - q_2 \, p_2 - q_3 \, p_3 = q_\mu \, p^\mu \ .
\tag{2.1.4}
$$

We set the speed of light c equal to one, but still work with

$$
\hbar = 1.055 \cdot 10^{-34} \, \text{Js}
\tag{2.1.5}
$$

in the first few chapters [22]. In Table 2.1 we list important Standard Model particle masses.

Particles	Pole masses
M_Z	91.19 GeV
M_W	80.38 GeV
m_t	173.0 GeV
$M_{h_{\text{SM}}}$	125.18 GeV

Table 2.1.: Pole masses of Standard Model particles [22].

---

[1]Otherwise, we use Latin letters and don't distinguish between upper and lower indices.

2.2. Kronecker-Product

To be able to deal in a compact and clear way with the four-dimensional Dirac substructure of a set ψ of m Dirac bispinor fields, we briefly introduce the Kronecker product and list its most important and useful properties. A more detailed discussion can be found in [23] and [24].

Whenever in this thesis a Kronecker product of the form $A \otimes B$ appears, the substructure B is described by a 4×4 matrix and the dimension of A will be clear from the context or explicitly displayed otherwise.

2.2.1. Definition

For two matrices $A = (a_{ij}) \in \mathbb{K}^{m \times n}$ and $B \in \mathbb{K}^{r \times s}$, where $\mathbb{K}^{m \times n}$ denotes the space of real or complex $m \times n$ matrices, the Kronecker product is defined as:

$$A \otimes B = \begin{bmatrix} a_{11} \cdot B & \dots & a_{1n} \cdot B \\ \vdots & \ddots & \vdots \\ a_{m1} \cdot B & \dots & a_{mn} \cdot B \end{bmatrix} \in K^{mr \times ns} \quad . \tag{2.2.1}$$

2.2.2. Properties

Let $A_{1,2} \in \mathbb{K}^{m \times n}$, $B_{1,2} \in \mathbb{K}^{r \times s}$, $C \in \mathbb{K}^{n \times o}$, $D \in \mathbb{K}^{s \times t}$ and $\alpha \in \mathbb{K}$. Then

- $A \otimes B \neq B \otimes A$

- $(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha (A \otimes B)$

- $(A_1 + A_2) \otimes B = (A_1 \otimes B) + (A_2 \otimes B)$

- $A \otimes (B_1 + B_2) = (A \otimes B_1) + (A \otimes B_2)$

- $(A \otimes B)(C \otimes D) = AC \otimes BD$

- $(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$

- $(A \otimes B)^{\dagger} = A^{\dagger} \otimes B^{\dagger}$.

Let $A \in \mathbb{K}^{n \times n}$ and $B \in \mathbb{K}^{m \times m}$. Then

- $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A) \cdot \mathrm{Tr}(B)$

- $\mathrm{Det}(A \otimes B) = \mathrm{Det}(A)^m \cdot \mathrm{Det}(B)^n$.

2.3. Classical Scale Invariance

A theory in d dimensions is said to be classically scale invariant, if its action is invariant under a scale transformation of the form:

$$\begin{aligned} x &\to \rho\, x & H &\to \rho^{-\frac{d-2}{2}}\, H \\ A^{\mu} &\to \rho^{-\frac{d-2}{2}}\, A^{\mu} & \psi &\to \rho^{-\frac{d-1}{2}}\, \psi \, . \end{aligned} \tag{2.3.1}$$

Therefore, only terms with operator-dimension d are allowed in the Lagrangian density, or, with other words, only terms with dimensionless coupling constants.

To obtain a classically scale invariant version of the Standard Model (SM), we have to get rid of the imaginary mass term $-\mu^2 H^\dagger H$, which breaks scale invariance, since:

$$\int \mathrm{d}^d x \; \mu^2 H^\dagger(x) H(x) \; \longrightarrow \; \rho^2 \cdot \int \mathrm{d}^d x \; \mu^2 H^\dagger(x) H(x) \; . \tag{2.3.2}$$

As a consequence, the potential of the Lagrangian density has to be convex, which seems to lead to vanishing vacuum expectation values of the scalar fields and hence, no masses would be generated through the famous Higgs-mechanism. However, as it turns out, this remains true only at tree level [7] and elevation of classical scale invariance to a fundamental property of nature can serve as a promising guiding principle for theories beyond the SM (BSM) with great predictive power.

3. The Coleman-Weinberg Mechanism

Spontaneous symmetry breaking (SSB) is usually studied within a tree-level approximation, which simply means that one searches for minima of the potential of the Lagrangian density (i.e. all the non-derivative terms). From this point of view, SSB should not take place in a basic φ^4-theory or in any classically scale invariant theory, due to the absence of nontrivial minima. However, higher-order effects can change this simple structure and S.Coleman and E.Weinberg demonstrated in their famous paper [7] that radiative corrections can induce spontaneous symmetry breaking even in theories without nontrivial vacuum expectation values (VEV) at zero-loop order. They studied a classically scale-invariant electrodynamic theory with a charged scalar and showed that the scalar boson gets a nontrivial VEV and becomes, as well as the vector boson, massive due to quantum effects [7]. Hereby, the appearance of the massive scalar at one-loop level is associated with the breaking of the scale invariance by higher-order corrections.

An appropriate tool to investigate this described feature of such theories is the so-called effective potential, a function whose minima give the true vacuum expectation values up to all orders and coincides at lowest order with the tree-level potential (see section 3.4 and 4.1).

In this section we will follow Coleman's and Weinberg's approach [7], which is based on the work of Jona-Lasinio [25], to get an intuitive understanding of SSB in classically scale invariant theories and use functional methods and a diagrammatic expansion to obtain the effective potential and to explore its properties. Many of the following definitions and derivations can also be found in [26].

3.1. Generating Functional of Correlation Functions

We start with defining the generating functional of correlation functions[1] $Z[J]$ as

$$
\begin{aligned}
Z[J] &:= \lim_{T \to \infty(1-\mathrm{i}0^+)} \frac{1}{N} \int [\mathrm{d}\varphi]\, e^{\frac{\mathrm{i}}{\hbar} \int_{-T}^{T} \mathrm{d}^4 x \left(\mathcal{L}(\varphi) + J(x)\varphi(x) \right)} \\
&\mathrel{\hat{=}} \frac{1}{N} \int [\mathrm{d}\varphi]\, e^{\frac{\mathrm{i}}{\hbar} \left(S(\varphi) + \int \mathrm{d}^4 x\, J(x)\varphi(x) \right)} \,,
\end{aligned}
\tag{3.1.1}
$$

where, in order to keep the notation as simple as possible, $S[\varphi]$ describes the classical action of a theory with only one real scalar field. All of the later arguments are of course still valid for a more complex theory. The limit $T \to \infty(1 - \mathrm{i}0^+)$ is rarely displayed explicitly in the literature, but will become of importance when we calculate path integrals in appendix D. The extra field $J(x)$ denotes an external source with

$$
J(x) \overset{|x| \to \infty}{\longrightarrow} 0 \,,
\tag{3.1.2}
$$

---

[1] The nomenclature will become clear in a few lines.

and the normalization constant N is chosen as

$$N = \int [\mathrm{d}\varphi] \, e^{\frac{\mathrm{i}}{\hbar} \int_{-T}^{T} \mathrm{d}^4 x \, \mathcal{L}(\varphi)} \, . \tag{3.1.3}$$

With the help of the functional derivative, which satisfies (see e.g. [26])

$$\frac{\delta}{\delta f(y)} f(x) = \delta^{(4)}(x - y) \, , \tag{3.1.4}$$

and the definition of the n-point correlation function[2] [27],

$$\langle 0 | T(\hat{\varphi}(x_1)...\hat{\varphi}(x_n)) | 0 \rangle = \lim_{T \to \infty(1 - \mathrm{i}0^+)} \frac{\int [\mathrm{d}\varphi] \, e^{\frac{\mathrm{i}}{\hbar} \int_{-T}^{T} \mathrm{d}^4 x \, \mathcal{L}(\varphi)} \varphi(x_1)...\varphi(x_1)}{\int [\mathrm{d}\varphi] \, e^{\frac{\mathrm{i}}{\hbar} \int_{-T}^{T} \mathrm{d}^4 x \, \mathcal{L}(\varphi)}} \, , \tag{3.1.5}$$

we find

$$\frac{\hbar}{\mathrm{i}} \frac{\delta Z[J]}{\delta J(x_1)} \bigg|_{J=0} = \lim_{T \to \infty(1 - \mathrm{i}0^+)} \frac{\int [\mathrm{d}\varphi] \, e^{\frac{\mathrm{i}}{\hbar} \int_{-T}^{T} \mathrm{d}^4 x \, \mathcal{L}(\varphi)} \varphi(x_1)}{\int [\mathrm{d}\varphi] \, e^{\frac{\mathrm{i}}{\hbar} \int_{-T}^{T} \mathrm{d}^4 x \, \mathcal{L}(\varphi)}} = \langle 0 | \hat{\varphi}(x_1) | 0 \rangle \, , \tag{3.1.6}$$

or more generally:

$$\boxed{Z^{(n)}(x_1, \ldots, x_n) := \left(\frac{\hbar}{\mathrm{i}} \right)^n \frac{\delta^n Z[J]}{\delta J(x_1)...\delta J(x_n)} \bigg|_{J=0} = \langle 0 | T(\hat{\varphi}(x_1)...\hat{\varphi}(x_n)) | 0 \rangle \, .} \tag{3.1.7}$$

The above relation makes clear why we call $Z[J]$ the generating functional of correlation functions and with the help of this very equation (3.1.7) it is straightforward to write down the generating functional as an expansion of n-point correlation functions $Z^{(n)}$:

$$\boxed{Z[J] = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{\hbar^n n!} \int \mathrm{d}^4 x_1...\mathrm{d}^4 x_n \, Z^{(n)}(x_1, \ldots, x_n) \, J(x_1)...J(x_n) \, .} \tag{3.1.8}$$

Furthermore, we realize that the vacuum state is reasonably normalized[3] through the definition (3.1.3) :

$$Z[0] = \langle 0 | 0 \rangle = 1 \, . \tag{3.1.9}$$

Following equation (3.1.7), we can also define the n-point correlation function in presence of an external source J_0:

$$Z_J^{(n)}(x_1, \ldots, x_n) := \left(\frac{\hbar}{\mathrm{i}} \right)^n \frac{\delta^n Z[J]}{\delta J(x_1)...\delta J(x_n)} = \langle 0 | T(\hat{\varphi}(x_1)...\hat{\varphi}(x_n)) | 0 \rangle_J \, . \tag{3.1.10}$$

3.2. Generating Functional of Connected Correlation Functions

As a next step, we introduce the generating functional of connected correlation function $W[J]$:

$$\boxed{W[J] := -\mathrm{i}\hbar \cdot \ln Z[J] \, .} \tag{3.2.1}$$

---

[2]While $\varphi(x)$ is a real valued function, $\hat{\varphi}(x)$ denotes an operator.
[3]The normalization constant is chosen such that it cancels all the vacuum graphs.

Again, the name is revealing and it can be shown (see [26]) that the function

$$W^{(n)}(x_1, \ldots, x_n) := \left(\frac{\hbar}{i} \right)^{n-1} \frac{\delta^n W[J]}{\delta J(x_1) \ldots \delta J(x_n)} \bigg|_{J=0} \qquad (3.2.2)$$

is the connected n-point correlation function $\langle 0 | T(\hat{\varphi}(x_1) \ldots \hat{\varphi}(x_n)) | 0 \rangle_c$. Not as a strict proof, but as an illustration of this statement, it is enlightening to have a look at the first few relations between $Z^{(n)}$ and $W^{(n)}$

$$\begin{aligned}
Z^{(1)}(x_1) =& W^{(1)}(x_1) \\
Z^{(2)}(x_1, x_2) =& W^{(2)}(x_1, x_2) + W^{(1)}(x_1) \cdot W^{(1)}(x_2) \\
Z^{(3)}(x_1, x_2, x_3) =& W^{(3)}(x_1, x_2, x_3) + W^{(2)}(x_1, x_2) \cdot W^{(1)}(x_3) \\
& + W^{(2)}(x_2, x_3) \cdot W^{(1)}(x_1) + W^{(2)}(x_3, x_1) \cdot W^{(1)}(x_2) \\
& + W^{(1)}(x_1) \cdot W^{(1)}(x_2) \cdot W^{(1)}(x_3) .
\end{aligned} \qquad (3.2.3)$$

These equations arise from the definitions (3.2.1),(3.1.7) and (3.2.2) and underline the fact that any n-point correlation function can be written as the sum of all possible combinations of connected m_i-point functions with $\sum m_i = n$.

As before, we can write down the generating functional as a Taylor series:[4]

$$W[J] = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar} \right)^{n-1} \int d^4 x_1 \ldots d^4 x_n \, W^{(n)}(x_1, \ldots, x_n) \, J(x_1) \ldots J(x_n) . \qquad (3.2.4)$$

By analogy with the definition (3.1.10), we also mention the connected n-point correlation function in presence of the source J_0:

$$W_J^{(n)}(x_1, \ldots, x_n) := \left(\frac{\hbar}{i} \right)^{n-1} \frac{\delta^n W[J]}{\delta J(x_1) \ldots \delta J(x_n)} . \qquad (3.2.5)$$

3.3. The Effective Action

The effective action $\Gamma[\phi]$ is defined by the functional Legendre-transformation of $W[J]$,

$$\Gamma[\phi] = W[J] - \int d^4 x \, J(x) \phi(x) , \qquad (3.3.1)$$

with

$$\phi(x) = \frac{\delta W[J]}{\delta J(x)} , \qquad (3.3.2)$$

where $J = J_\phi$ is actually an implicit functional of $\phi(x)$ and given by the solution of

$$\phi(x) = \frac{\delta W[J]}{\delta J(x)} \bigg|_{J=J_\phi} . \qquad (3.3.3)$$

---

[4]Be aware that $W^{(0)} = 0$, since $Z[0] = 1$

Furthermore, $\Gamma[\phi]$ fulfils the following relation:

$$\boxed{\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = -J(x)} + \underbrace{\int \mathrm{d}^4 y \frac{\delta J(y)}{\delta\phi(x)} \left[\frac{\delta}{\delta J(y)} W[J] - \phi(y) \right]}_{\overset{(3.3.2)}{=} 0} . \qquad (3.3.4)$$

Vice versa, with the help of relation (3.3.4), the field $\phi = \phi_J(x)$ can also be seen as an implicit functional of $J(x)$,

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_J} = -J(x) , \qquad (3.3.5)$$

and therefore, the inverse Legendre-Transformation of (3.3.1) is given by:

$$W[J] = \Gamma[\phi] + \int \mathrm{d}^4 x J(x)\phi(x) . \qquad (3.3.6)$$

We find that $\phi(x)$ is just the vacuum expectation value of the scalar field $\varphi(x)$ in presence of the external source $J(x)$,

$$\frac{\delta W[J]}{\delta J(x)} = \phi(x) \overset{(3.2.5)}{=} W_J^{(1)}(x) \overset{(3.2.3)}{=} Z_J^{(1)}(x) \overset{(3.1.10)}{=} \langle 0|\hat\varphi(x)|0\rangle_J , \qquad (3.3.7)$$

and in case we send the external field $J(x)$ to zero ($\phi(x) = \langle 0|\hat\varphi(x)|0\rangle_{J=0} = \langle 0|\hat\varphi|0\rangle$[5]) the effective action is extremized (compare equation (3.3.5)):

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{\phi=\langle 0|\hat\varphi|0\rangle} = 0 . \qquad (3.3.8)$$

We now introduce the shifted field

$$\chi(x) = \phi(x) - W^{(1)}(x) , \qquad (3.3.9)$$

and write the effective action as an expansion around $W^{(1)}(x) = \langle\hat\varphi\rangle$,

$$\boxed{\Gamma[\phi] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}^4 x_1 .. \mathrm{d}^4 x_n \frac{\hbar}{\mathrm{i}} \Gamma^{(n)}(x_1, .., x_n) \, \chi(x_1)...\chi(x_n) ,} \qquad (3.3.10)$$

with

$$\Gamma^{(n)}(x_1, .., x_n) = \frac{\mathrm{i}}{\hbar} \left.\frac{\delta\Gamma[\phi]}{\delta\phi(x_1)..\delta\phi(x_n)}\right|_{\phi=\langle\hat\varphi\rangle} . \qquad (3.3.11)$$

We want to show that these functions $\Gamma^{(n)}(x_1, .., x_n)$ are the one particle irreducible (1PI) n-point correlation functions, which also go under the catchier name of proper vertices[6]. To be precise, this statement is only true for $n > 2$, because for $n = 1$ we get zero and for $n = 2$ we get the inverse of the full propagator, as we will see in a moment.

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[5]Since we are always dealing with translation invariant theories the VEV has to be a constant quantity.
[6]For that reason, the effective action is sometimes also referred to as the generating functional of proper vertices.

For that purpose, we follow the methods of [26] and expand $\phi(x)$, using equation (3.2.4) and (3.3.3):

$$\chi(x) = \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \left(\frac{i}{\hbar}\right)^{n-1} \int d^4x_1...d^4x_{n-1} \, W^{(n)}(x, x_1, \ldots, x_{n-1}) J(x_1)...J(x_{n-1})$$

$$= \frac{i}{\hbar} \int d^4x_1 \, W^{(2)}(x, x_1) J(x_1)$$

$$+ \frac{1}{2} \left(\frac{i}{\hbar}\right)^2 \int d^4x_1 d^4x_2 W^{(3)}(x, x_1, x_2) J(x_1) J(x_2)$$

$$+ \frac{1}{3!} \left(\frac{i}{\hbar}\right)^3 \int d^4x_1 d^4x_2 d^4x_3 W^{(4)}(x, x_1, x_2, x_3) J(x_1) J(x_2) J(x_3) + \ldots \, .$$

$$(3.3.12)$$

We define the inverse of the connected 2-point correlation function $W^{(2)}(y, z)$ as

$$\int d^4y \, S(x, y) W^{(2)}(y, z) = \delta^{(4)}(x - z) \, , \qquad (3.3.13)$$

and apply it to equation (3.3.12) to express $J(x)$:

$$J(x) = \frac{\hbar}{i} \int d^4y \, S(x, y) \left(\phi(y) - W^{(1)}(y)\right)$$

$$- \frac{1}{2} \frac{i}{\hbar} \int d^4x_1 d^4x_2 d^4y \, S(x, y) W^{(3)}(y, x_1, x_2) J(x_1) J(x_2)$$

$$- \frac{1}{3!} \left(\frac{i}{\hbar}\right)^2 \int d^4x_1 d^4x_2 d^4x_3 d^4y \, S(x, y) W^{(4)}(y, x_1, x_2, x_3) J(x_1) J(x_2) J(x_3)$$

$$+ \ldots \, .$$

$$(3.3.14)$$

The external field $J(x)$ can now be written as a series in powers of $\phi(x)$

$$J(x) = \frac{\hbar}{i} \Bigg[\int d^4y_1 \, S(x, y_1) \left(\phi(y_1) - W^{(1)}(y_1)\right)$$

$$- \frac{1}{2} \int d^4y_1 d^4y_2 \, W^{(3)}_{\text{amp}}(x, y_1, y_2) \left(\phi(y_1) - W^{(1)}(y_1)\right) \left(\phi(y_2) - W^{(1)}(y_2)\right)$$

$$- \frac{1}{3!} \int d^4y_1 d^4y_2 d^4y_3 \Bigg(W^{(4)}_{\text{amp}}(x, y_1, y_2, y_3)$$

$$- \int d^4v \, d^4w \, \Big\{ W^{(3)}_{\text{amp}}(x, y_1, v) \, W^{(2)}(v, w) \, W^{(3)}_{\text{amp}}(w, y_2, y_3)$$

$$+ W^{(3)}_{\text{amp}}(x, y_2, v) \, W^{(2)}(v, w) \, W^{(3)}_{\text{amp}}(w, y_1, y_3)$$

$$+ W^{(3)}_{\text{amp}}(x, y_3, v) \, W^{(2)}(v, w) \, W^{(3)}_{\text{amp}}(w, y_1, y_2) \Big\} \Bigg)$$

$$\cdot \left(\phi(y_1) - W^{(1)}(y_1)\right) \left(\phi(y_2) - W^{(1)}(y_2)\right) \left(\phi(y_3) - W^{(1)}(y_3)\right) + \ldots \Bigg] \, ,$$

$$(3.3.15)$$

where $W^{(n)}_{\text{amp}}(x_1, \ldots, x_n)$ denotes the amputated, connected n-point correlation function:

$$W^{(n)}_{\text{amp}}(x_1, \ldots, x_n) = \int d^4y_1...d^4y_n \, S(x_1, y_1)...S(x_n, y_n) W^{(n)}(y_1, \ldots, y_n) \, . \qquad (3.3.16)$$

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Comparison of the expansion (3.3.10) and the result we get from equation (3.3.4) and (3.3.15) leads to the desired relations:

$$
\begin{aligned}
\Gamma^{(1)}(x_1) &= 0 \\
\Gamma^{(2)}(x_1, x_2) &= -S(x_1, x_2) \\
\Gamma^{(3)}(x_1, x_2, x_3) &= W_{\text{amp}}^{(3)}(x_1, x_2, x_3) \\
\Gamma^{(4)}(x_1, x_2, x_3, x_4) &= W_{\text{amp}}^{(4)}(x_1, x_2, x_3, x_4) \\
&\quad - \int \mathrm{d}^4 v\, \mathrm{d}^4 w \left[W_{\text{amp}}^{(3)}(x_1, x_2, v) W_{\text{amp}}^{(3)}(w, x_3, x_4) \right. \\
&\qquad\qquad\qquad + W_{\text{amp}}^{(3)}(x_1, x_3, v) W_{\text{amp}}^{(3)}(w, x_2, x_4) \\
&\qquad\qquad\qquad \left. + W_{\text{amp}}^{(3)}(x_1, x_4, v)\, W_{\text{amp}}^{(3)}(w, x_2, x_3) \right] W^{(2)}(v, w) \\
&\;\;\vdots
\end{aligned}
\tag{3.3.17}
$$

As stated above, we find that $\Gamma^{(1)}(x_1)$ is equal to zero and that $-\Gamma^{(2)}(x_1, x_2)$ is the inverse of the full propagator. The amputated, connected three-point correlation function $W_{\text{amp}}^{(3)}(x_1, x_2, x_3)$ is already 1PI and to make $W_{\text{amp}}^{(4)}(x_1, x_2, x_3, x_4)$ 1PI we have to subtract all the one-particle reducible parts.

3.4. The Effective Potential

In contrast to equation (3.3.10), it is also possible to expand Γ in powers of $\partial_\mu \phi$ [7]:

$$
\Gamma[\phi] = \int \mathrm{d}^4 x \left(-V(\phi) + \frac{1}{2} Z(\phi)(\partial_\mu \phi)^2 + Y(\phi)(\partial_\mu \phi)^4 + \dots \right).
\tag{3.4.1}
$$

Since the VEV of the scalar field is a constant quantity, it is justified to evaluate $\Gamma[\phi]$ also for a constant field ϕ_c only. Therefore, we obtain the simpler expression

$$
\Gamma[\phi_c] = -\int \mathrm{d}^4 x\, V(\phi_c),
\tag{3.4.2}
$$

where we call the function $V(\phi_c)$ the effective potential and we find from equation (3.3.8) that the effective potential is extremized at $\phi_c = \langle 0|\hat\varphi|0\rangle$:

$$
\left. \frac{\delta\Gamma[\phi_c]}{\delta\phi_c} \right|_{\phi_c = \langle 0|\hat\varphi|0\rangle} = -\left. \frac{\partial V(\phi_c)}{\partial\phi_c} \right|_{\phi_c = \langle 0|\hat\varphi|0\rangle} = 0.
\tag{3.4.3}
$$

Hereby, we achieved our goal of finding a function, the effective potential $V(\phi_c)$, whose minima give the true vacuum expectation values without any approximations.

Since a perturbative expansion of the effective potential leads in general to a non-convex function [28, 29], the extremum is ambiguous. For a classically scale-invariant theory with radiative induced SSB for example, one of the extrema is still the trivial tree-level VEV. To get the other one, we had to introduce the external source $J(x)$ as a perturbation,

Figure 3.1.: Perturbation of the unstable, trivial vacuum state (grey dot) leads to a stable vacuum state away from zero (black dot). Afterwards the perturbation can be turned off.
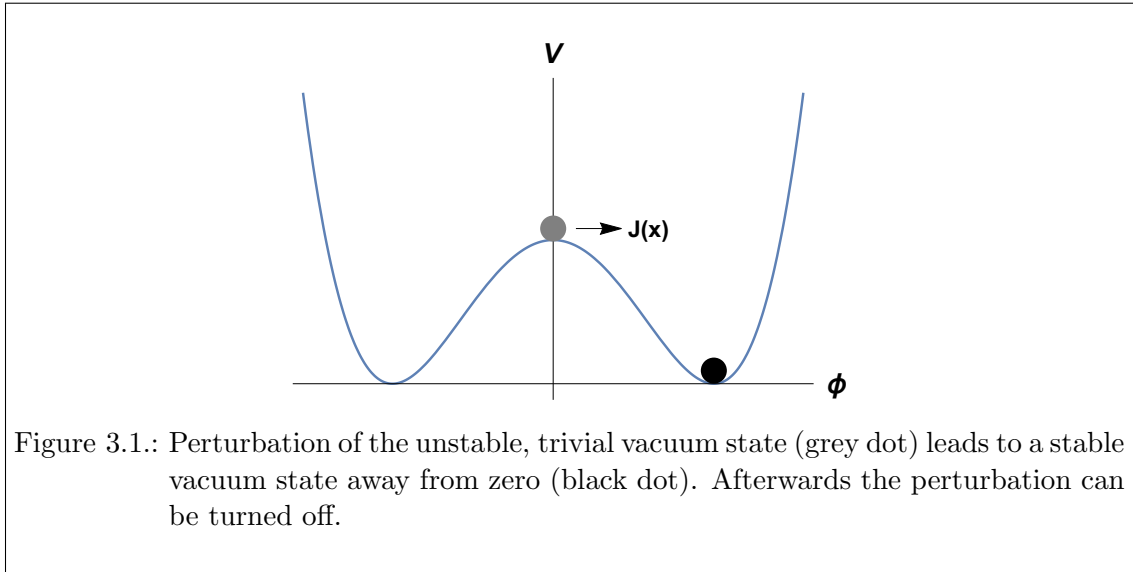
although in the end, we set it equal to zero anyway. Otherwise we only would have been able to find the tree-level VEV $W^{(1)}(x) = 0$. This feature is illustrated in Fig. 3.1.

To find a useful expression for the effective potential, we write down the Fourier transform $\tilde{\Gamma}^{(n)}(k_1, \ldots, k_n)$ of the vertex function $\Gamma^{(n)}(x_1, \ldots, x_n)$

$$(2\pi)^4 \delta^4\left(\sum_{i=1}^{n} k_i\right) \tilde{\Gamma}^{(n)}(k_1, \ldots, k_n) = \int \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_n \, e^{-\mathrm{i}(k_1 x_1 + \ldots + k_n x_n)} \, \Gamma^{(n)}(x_1, \ldots, x_n) \,,$$

(3.4.4)

where the total momentum conservation corresponds to the translation invariance of the vertex function $\Gamma^{(n)}(x_1, \ldots, x_n)$ [26] and k denotes the wavevector.

From now on we will concentrate on classically scale-invariant theories, where the one-point function $W^{(1)}(x_1)$ is equal to zero and therefore, after setting all external momenta to zero, we can use equation (3.4.4) to write the effective action (eq. (3.3.10)) as

$$\Gamma[\phi_c] = (2\pi)^4 \delta^4(0) \sum_{n=2}^{\infty} \frac{1}{n!} \frac{\hbar}{\mathrm{i}} \tilde{\Gamma}^{(n)}(0, \ldots, 0) \, \phi_c^n \,.$$

(3.4.5)

If we realize that $\delta^4(k)$ is the Fourier transform of 1,

$$\delta^4(k) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 x \, e^{-\mathrm{i}kx}$$

$$(2\pi)^4 \delta^4(0) = \int \mathrm{d}^4 x \, 1 \,,$$

(3.4.6)

we find from equation (3.4.2) and (3.4.5) for the effective potential

$$-V(\phi_c) = \sum_{n=2}^{\infty} \frac{1}{n!} \frac{\hbar}{\mathrm{i}} \tilde{\Gamma}^{(n)}(0, \ldots, 0) \, \phi_c^n \,,$$

(3.4.7)

where $\tilde{\Gamma}^{(n)}(0, \ldots, 0)$ are the 1PI n-point correlation functions in momentum space with vanishing external momenta.[7]

---

[7]For a classically scale invariant theory this is even true for $n = 2$, as we will see in subsection 3.4.2.

3.4.1. The Loop Expansion

Since it is clear that it is impossible to write down the exact effective potential for a non-trivial model, we have to come up with an appropriate approximation method. Coleman and Weinberg demonstrated in [7] that an expansion in powers of an overall factor a of the form

$$\mathcal{L}(\varphi, \partial_\mu \varphi, a) = a^{-1} \mathcal{L}(\varphi, \partial_\mu \varphi) \tag{3.4.8}$$

corresponds to a loopwise expansion of the effective potential. The reason for this correlation is that every propagator, as the inverse of the quadratic terms in the Lagrangian, carries a factor of a and every vertex a factor of a^{-1}. For an arbitrary 1PI diagram the power of a, P, is then given by

$$P = I - V \, , \tag{3.4.9}$$

where I denotes the number of internal lines and V the number of vertices. The number of loops, L, corresponds to the number of momenta as variables of integration and is the difference between the number of momenta, I, and the number of energy-momentum δ-functions, V, without counting one δ-function for the total energy-momentum conservation:

$$L = I - V + 1 \, . \tag{3.4.10}$$

If we combine the equations (3.4.9) and (3.4.10), we finally find the connection between the power of a and the number of loops:

$$P = L - 1 \, . \tag{3.4.11}$$

As Coleman and Weinberg stated [7], the important point is not that a has to be a small parameter (actually a can be equal to 1), but that we found a suitable organization scheme for an infinite sum of Feynman-diagrams where each higher order of the expansion contributes less than the one before and that the parameter a, as an overall factor of the Lagrangian density, is not affected by shifts of the fields. Furthermore, as discussed in [30] and [31], $\hbar$ is an appropriate choice for the overall factor a (compare the prefactor of the Lagrangian density in equation (3.1.1)) and therefore, a loop-expansion corresponds to an expansion in powers of $\hbar$. That is the reason why we made the effort to set $\hbar$ not equal to 1 from the beginning.

3.4.2. A Simple Example of the Derivation of the Effective Potential via Feynman Diagrams
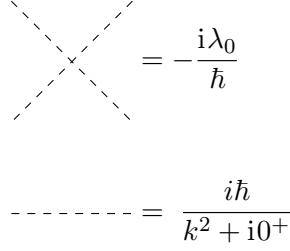
Just as Coleman and Weinberg in [7], we compute the one-loop effective potential for the simplest case of a massless, quartically self-interacting scalar field with the Lagrangian density[8]:

$$\mathcal{L} = \frac{1}{2} \left(\partial \varphi_0 \right)^2 - \frac{\lambda_0}{4!} \varphi_0^4 \, . \tag{3.4.12}$$

If we explicitly display $\hbar$, the Feynman rules for the propagator and the vertex are scaled with $\hbar$ and $\frac{1}{\hbar}$ respectively and are given in Fig. 3.2 [9]. With the help of equation (3.4.7) and knowledge of the loop expansion we are now in the position to write down the effective

---

[8]The subscript 0 identifies the parameters as bare quantities.
[9]All the Feynman diagrams in this work have been generated with the help of TikZ-Feynman [32].

Figure 3.2.: Feynman rules for an overall factor $\hbar^{-1}$ of the Lagrangian density.

potential up to one-loop order. Therefore, we define the vertex functions with vanishing external momenta at l-loop order $\tilde{\Gamma}_l^{(n)}$ and write the effective potential as
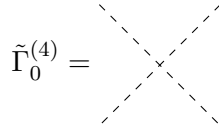
$$V(\phi_c) = i\hbar \sum_{l=0}^{\infty} \sum_{n=2}^{\infty} \frac{1}{n!} \tilde{\Gamma}_l^{(n)} \phi_{0c}^n = \sum_{l=0}^{\infty} \hbar^l V_l(\phi_{0c}) \,, \qquad (3.4.13)$$

where $V_l(\phi_{0c})$ denotes the l-loop contribution to the effective potential.

The only possible tree-level diagram for the theory under consideration is shown in Figure 3.3 and contributes to the effective potential in the form

$$V_0 = i\hbar \sum_{n=2}^{\infty} \frac{1}{n!} \tilde{\Gamma}_0^{(n)} \phi_{0c}^n = i\hbar \frac{1}{4!} \tilde{\Gamma}_0^{(4)} \phi_{0c}^4 = \frac{\lambda}{4!} \phi_{0c}^4 \,, \qquad (3.4.14)$$

which is, as it should be, equal to the potential of the Lagrangian density (i.e. all the non-derivative terms).

Figure 3.3.: Tree-level contribution to the effective potential.

At one-loop order we have to deal with the special case of $\tilde{\Gamma}^{(2)}(0,0)$ as the negative inverse full propagator. But since the full propagator is given by

$$
\begin{aligned}
\left(-\tilde{\Gamma}^{(2)}(-k,k) \right)^{-1} &= \text{---} \bigcirc \text{---} \\
&= \text{-----} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} \bigcirc \text{---} + ... \\
&= \frac{i\hbar}{k^2 - \left(\frac{m}{\hbar}\right)^2 + i0^+} \sum_{n=0}^{\infty} \left(i\Pi \frac{i\hbar}{k^2 - \left(\frac{m}{\hbar}\right)^2 + i0^+} \right)^n \\
&= \frac{i\hbar}{k^2 - \left(\frac{m}{\hbar}\right)^2 + \hbar\Pi + i0^+} \,,
\end{aligned}
\qquad (3.4.15)
$$

15

where the self-energy $\Pi(k)$ is related to the 1PI 2-point diagrams,

$$i\Pi(k) = \text{---}\ \varhexagonblack\ \text{---}\ , \tag{3.4.16}$$

the vertex-function $\tilde{\Gamma}^{(2)}(0,0)$ for the given massless theory is just:

$$\tilde{\Gamma}^{(2)}(0,0) = i\Pi(0) = \ \text{(diagram)}\ + \ \text{(diagram)}\ + \dots \ . \tag{3.4.17}$$

Therefore, we have to consider the infinite sum of all 1PI 2n-point correlation functions with vanishing external momenta at one-loop level

$$\sum_{n=1}^{\infty} \tilde{\Gamma}_1^{(2n)} = S_1 \cdot \text{(diagram)} + S_2 \cdot \text{(diagram)} + S_3 \cdot \text{(diagram)} + S_4 \cdot \text{(diagram)} + \dots \ , \tag{3.4.18}$$

and for the next-order contribution to the effective potential we find:

$$V_1 = i \sum_{n=1}^{\infty} \frac{1}{(2n)!} \underbrace{\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, S_n \left(\frac{\lambda_0}{k^2 + i0^+} \right)^n}_{\tilde{\Gamma}_1^{(2n)}} \phi_{0c}^{2n} \ . \tag{3.4.19}$$

To determine the symmetry factor S_n we have to count the number of different graphs we can get by reordering of n vertices and $2n$ external lines:

There are $(n-1)!$ different ways to connect n vertices and the irrelevance of the direction of the internal lines is accounted by an extra factor of $\frac{1}{2}$. The $2n$ external lines can be interchanged in $(2n!)$ different ways, but since the pairwise exchange of direct connected external lines is equivalent to the exchange of vertices, these $n!$ different graphs are already covered by the factor $(n-1)!$ (Fig. 3.4). For that reason, we have to add an additional factor of $\frac{1}{n!}$. Furthermore, due to the fact that two scalars at the same vertex are indistinguishable, the swap of two external lines at the same vertex does not lead to a new graph, which gives an extra factor of $\frac{1}{2^n}$.

Hence, the symmetry factor S_n has the form

$$S_n = \frac{(n-1)!}{2} \cdot \frac{(2n!)}{2^n n!} \ , \tag{3.4.20}$$

and we obtain for the effective potential up to one-loop order

$$V = \frac{\lambda_0}{4!} \phi_{0c}^4 + i\hbar \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{2n} \left(\frac{\frac{1}{2}\lambda_0 \phi_{0c}^2}{k^2 + i0^+} \right)^n \ , \tag{3.4.21}$$

which is in perfect agreement with the result of [7]. The computation of this integral can be found in appendix A.1 and leads to the renormalized effective potential:

$$V = \frac{\lambda}{4!} \phi_c^4 + \hbar \frac{\lambda^2 \phi_c^4}{256\pi^2} \left(\ln \frac{\frac{1}{2}\lambda \phi_c^2}{\Lambda^2} - \frac{3}{2} \right) \ . \tag{3.4.22}$$

Figure 3.4.: Pairwise exchange of direct connected external lines (a) leads to the same diagram as the interchange of vertices (b).

This result confirms that a loopwise expansion of the effective potential corresponds to an expansion in powers of $\hbar$ and the discrepancy between the final expression of the effective potential in [7] and equation (3.4.22) arises from a different choice of renormalization schemes (cutoff instead of $\overline{\text{MS}}$).

Furthermore, we find that scale invariance is broken by the loop contributions to the tree-level potential, since

$$\int \mathrm{d}^4 x \; \phi^4 \ln\phi^2 \;\longrightarrow\; \int \mathrm{d}^4 x \; \phi^4 \left(\ln\phi^2 - \ln\rho^2\right) \; . \tag{3.4.23}$$

This simple example illustrates the drastic change of the potential caused by radiative corrections and the supposed appearance of a new, nontrivial minimum (Fig.3.5), which meaning is further discussed in [7].

Since it is much more complicated to find the minimum of the effective potential for a theory with multiple scalar particles, Gildener and Weinberg (GW) came up with a new method to study the effective potential of arbitrary classically scale invariant gauge theories [16]. But before we discuss in detail the GW analysis of the effective potential in chapter 5, we want to derive the one-loop effective potential for a general gauge-invariant and renormalizable Lagrangian density. For this purpose, we choose a pure functional approach and explicitly evaluate path integrals instead of performing a diagrammatic expansion. The lack of visualizability is compensated by the ability to apply this approach to a much more general theory.
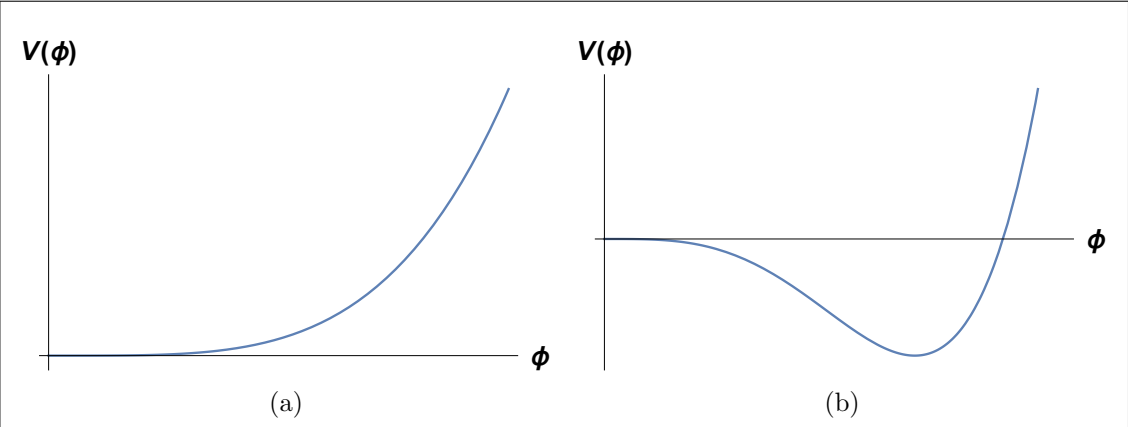
Figure 3.5.: Illustration of the difference between the tree-level potential (a) and effective potential at one-loop level (b).

4. Derivation of the Effective Potential

In section 3.4, we calculated the effective potential for a simple model with the help of Feynman diagrams up to one-loop order and discovered that an expansion of the effective potential in powers of $\hbar$ corresponds to a loopwise expansion. In this chapter, we choose a slightly different approach, in which we perform a semiclassical expansion and compute path integrals explicitly.

4.1. The Semiclassical Expansion

By analogy with chapter 3, we start with considering a simple theory with only one real scalar field φ and stress that all later results are still valid for a more complex theory, only more attention would have to be paid to the computation of the path integrals (see appendix D). The following discussion is based on [26] and [33].

According to equation (3.1.1), the generating functional of correlation functions reads

$$Z[J] = \frac{1}{N} \int [\mathrm{d}\varphi] e^{\frac{\mathrm{i}}{\hbar}\left(S[\varphi] + \int \mathrm{d}^4 x J(x)\varphi(x)\right)} . \tag{4.1.1}$$

In the classical limit, $S \gg \hbar$, the path integral is dominated by the classical field $\varphi_{\mathrm{cl}}(J)$, given by the solution of the saddle-point equation

$$\left.\frac{\delta S[\varphi]}{\delta \varphi}\right|_{\varphi_{\mathrm{cl}}(J)} = -J , \tag{4.1.2}$$

since an expansion of the field φ around this saddle-point solution,

$$\varphi = \varphi_{\mathrm{cl}} + \sqrt{\hbar}\,\bar{\varphi} , \tag{4.1.3}$$

leads to

$$Z[J] = \frac{\bar{Z}[J]}{\bar{Z}[0]} , \tag{4.1.4}$$

with

$$\bar{Z}[J] = e^{\frac{i}{\hbar}\left(S[\varphi_{\mathrm{cl}}] + \int \mathrm{d}^4 x J(x)\varphi_{\mathrm{cl}}(x)\right)} \int [\mathrm{d}\bar{\varphi}]\, e^{\frac{\mathrm{i}}{2}\int \mathrm{d}x_1 \mathrm{d}x_2 \left.\frac{\delta S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\varphi_{\mathrm{cl}}} \bar{\varphi}(x_1)\bar{\varphi}(x_2) + \mathcal{O}(\sqrt{\hbar})} . \tag{4.1.5}$$

The linear terms vanish due to equation (4.1.2) and the contributions proportional to $\sqrt{\hbar}$ are suppressed due to cancellations caused by rapidly varying phases (Method of stationary phase). Therefore, the semiclassical solution of the path integral is given by:

$$\bar{Z}[J] \approx e^{\frac{i}{\hbar}\left(S[\varphi_{\mathrm{cl}}] + \int \mathrm{d}^4 x J(x)\varphi_{\mathrm{cl}}(x)\right)} \int [\mathrm{d}\bar{\varphi}] e^{\frac{\mathrm{i}}{2}\int \mathrm{d}x_1 \mathrm{d}x_2 \left.\frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\varphi_{\mathrm{cl}}} \bar{\varphi}(x_1)\bar{\varphi}(x_2)}$$

$$= e^{\frac{i}{\hbar}\left(S[\varphi_{\mathrm{cl}}] + \int \mathrm{d}^4 x J(x)\varphi_{\mathrm{cl}}(x)\right)} \hat{C} \left[\mathrm{Det}\left.\frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\varphi_{\mathrm{cl}}}\right]^{-\frac{1}{2}} . \tag{4.1.6}$$

The calculation of the path integral can be found in appendix D.1.2 and the exact form of the constant $\hat{C}$ shall not bother us here, since it gets canceled anyway (compare equation (4.1.4)).

For the generating functional of connected correlation functions we find from equation (3.2.1) and (4.1.6)

$$
\begin{aligned}
W[J] = -\mathrm{i}\hbar \ln(Z) &= \left(S[\varphi_{\mathrm{cl}}] + \int \mathrm{d}^4 x J(x)\varphi_{\mathrm{cl}}(x) - S[\varphi_{\mathrm{cl}}^0] \right) \\
&+ \frac{\mathrm{i}\hbar}{2} \left(\ln \mathrm{Det}\, \frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\bigg|_{\varphi_{\mathrm{cl}}} - \ln \mathrm{Det}\, \frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\bigg|_{\varphi_{\mathrm{cl}}^0} \right) \quad (4.1.7) \\
&+ \mathcal{O}(\hbar^2) \\
&= W_0(J) + \hbar W_1(J) + \mathcal{O}(\hbar^2) \,,
\end{aligned}
$$

where the superscript 0 indicates that the external source J is equal to zero ($\varphi_{\mathrm{cl}}^0 = \varphi_{\mathrm{cl}}(0)$). Therefore, at tree level, the classical saddle-point solution φ_{cl} is indistinguishable from the vacuum expectation value of the field in presence of the external source J:

$$
\langle \hat{\varphi}(x) \rangle_J = \phi(x) \overset{(3.3.7)}{\approx} \frac{\delta W_0[J]}{\delta J(x)} = \int \mathrm{d}^4 y \left(\underbrace{\frac{\delta S[\varphi_{\mathrm{cl}}]}{\delta\varphi_{\mathrm{cl}}(y)} \frac{\delta\varphi_{\mathrm{cl}}(y)}{\delta J(x)} + J(y)\frac{\delta\varphi_{\mathrm{cl}}(y)}{J(x)}}_{\overset{(4.1.2)}{=}0} \right) + \varphi_{\mathrm{cl}}(x) \,.
$$

$$(4.1.8)$$

As a consequence, the classical field φ_{cl}^0 is constant for a translation-invariant theory.

With the help of the Legendre transformation and equation (3.3.1) we can now relate $W_0[J]$ with the tree-level effective action $\Gamma_0[\phi]$,

$$
\begin{aligned}
W_0[J] &= \Gamma_0[\phi] + \int \mathrm{d}^4 x J(x)\phi(x) \\
&= S[\varphi_{\mathrm{cl}}] + \int \mathrm{d}^4 x J(x)\varphi_{\mathrm{cl}}(x) - S[\varphi_{\mathrm{cl}}^0] \,,
\end{aligned}
$$

$$(4.1.9)$$

and find that the effective action differs from the classical action at tree level at most by an insignificant constant term:

$$
\boxed{\Gamma_0[\phi] = S[\phi] - S[\phi^0] \,.}
$$

$$(4.1.10)$$

At one-loop order, we discover from an expansion of the field $\phi(x)$,

$$
\phi(x) = \frac{\delta(W_0(J) + \hbar W_1(J))}{\delta J(x)} = \varphi_{\mathrm{cl}}(x) + \hbar\phi_1(x) \,,
$$

$$(4.1.11)$$

that we still can identify the classical field $\varphi_{\mathrm{cl}}(x)$ with the vacuum expectation value $\phi(x)$

in the course of the calculation of the effective action:

$$\Gamma_0[\phi] + \hbar\Gamma_1[\phi] = W_0[J] + \hbar W_1[J] - \int \mathrm{d}^4x\, J(x)\phi(x)$$

$$= S[\varphi_{\mathrm{cl}}] + \int \mathrm{d}^4x\, \cancel{J(x)\varphi_{\mathrm{cl}}} - S[\varphi_{\mathrm{cl}}^0] - \int \mathrm{d}^4 J(x)\left(\cancel{\varphi_{\mathrm{cl}}(x)} + \hbar\phi_1(x)\right)$$

$$+ \frac{\mathrm{i}\hbar}{2}\left(\ln\mathrm{Det}\left.\frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\varphi_{\mathrm{cl}}} - \ln\mathrm{Det}\left.\frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\varphi_{\mathrm{cl}}^0}\right)$$

$$= S[\phi] - \hbar\int \mathrm{d}^4x\left(\left.\frac{\delta S[\varphi]}{\delta\varphi(x)}\right|_{\varphi_{\mathrm{cl}}} + J(x)\right)\phi_1(x)$$

$$- S[\phi^0] + \hbar\int \mathrm{d}^4x\left.\frac{\delta S[\varphi]}{\delta\varphi_x}\right|_{\varphi_{\mathrm{cl}}^0}\phi_1^0(x)$$

$$+ \frac{\mathrm{i}\hbar}{2}\left(\ln\mathrm{Det}\left.\frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\phi-\hbar\phi_1} - \ln\mathrm{Det}\left.\frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\phi^0-\hbar\phi_1^0}\right)$$

$$\overset{(4.1.2)}{=} S[\phi] - S[\phi^0] + \frac{\mathrm{i}\hbar}{2}\left(\ln\mathrm{Det}\left.\frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\phi} - \ln\mathrm{Det}\left.\frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\phi^0}\right)$$

$$+ \mathcal{O}(\hbar^2)\,. \tag{4.1.12}$$

The effective action at one-loop order is then given by

$$\boxed{\Gamma_1[\phi] = \frac{\mathrm{i}}{2}\left(\ln\mathrm{Det}\left.\frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\phi} - \ln\mathrm{Det}\left.\frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)}\right|_{\phi^0}\right)\,,} \tag{4.1.13}$$

and, as we know from section 3.4, the relation between the effective action and the effective potential for a constant field ϕ_{c} reads:

$$\boxed{\Gamma_0[\phi_{\mathrm{c}}] + \hbar\,\Gamma_1[\phi_{\mathrm{c}}] = -\int \mathrm{d}^4x\left(V_0(\phi_{\mathrm{c}}) + \hbar\,V_1(\phi_{\mathrm{c}})\right)\,.} \tag{4.1.14}$$

4.1.1. A Simple Example of the Derivation of the Effective Potential via Semiclassical Expansion

Once again, we consider the Lagrangian density of a massless, quartically self-interacting scalar field[1],

$$\mathcal{L} = \frac{1}{2}(\partial\varphi_0)^2 - \frac{\lambda_0}{4!}\varphi_0^4\,, \tag{4.1.15}$$

---

[1]Compare subsection 3.4.2 and remember that the subscript 0 indicates that we are dealing with bare quantities.

and find from equation (4.1.13) for $\Gamma_1[\phi_{0c}]^2$:

$$\Gamma_1[\phi_{0c}] = \frac{\mathrm{i}}{2}\left(\ln\mathrm{Det}\left(\frac{-\partial_\mu\partial^\mu - \frac{\lambda_0}{2}\phi_{0c}^2}{-\partial_\mu\partial^\mu}\,\delta^4(x-y)\right)\right)$$
$$= \frac{\mathrm{i}}{2}\left(\ln\mathrm{Det}\left(\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{k^2 - \frac{\lambda_0}{2}\phi_{0c}^2}{k^2}\,e^{\mathrm{i}\,k(x-y)}\right)\right). \tag{4.1.16}$$

Since we know from appendix D.1.2 that

$$\ln\mathrm{Det}\big(A(x-y)\big) = \int\mathrm{d}^4x\int\frac{\mathrm{d}^4k}{(2\pi)^4}\ln\det\big(\tilde{A}(k)\big) = \mathrm{Tr}\int\mathrm{d}^4x\int\frac{\mathrm{d}^4k}{(2\pi)^4}\ln\big(\tilde{A}(k)\big)\,, \tag{4.1.17}$$

where

$$A(x-y) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\tilde{A}(k)e^{\mathrm{i}\,k(x-y)}\,, \tag{4.1.18}$$

we obtain for $V_1(\phi_{0c})$:

$$V_1(\phi_{0c}) = -\frac{\mathrm{i}}{2}\,\mathrm{Tr}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\ln\left(\frac{k^2 - \frac{\lambda_0}{2}\phi_{0c}^2}{k^2}\right)\,. \tag{4.1.19}$$

This integral is after a Wick rotation in agreement with the potential from (A.1.1) and therefore, we reproduce the result from (3.4.22),

$$\boxed{V = \frac{\lambda}{4!}\phi_c^4 + \hbar\,\frac{\lambda^2\phi_c^4}{256\pi^2}\left(\ln\frac{\frac{1}{2}\lambda\phi_c^2}{\Lambda^2} - \frac{3}{2}\right)\,,} \tag{4.1.20}$$

which demonstrates that both the diagrammatic and the semiclassical approach lead to the same effective potential.

4.2. Derivation of the Effective Potential for a General Lagrangian Density

As discussed in detail in appendix B, a general Lorentz-invariant, gauge-invariant[3] and renormalizable Lagrangian density can be written as[4] (see e.g. [34–37])

$$\boxed{\begin{aligned}\mathcal{L}_{\mathrm{YM}} = {}&-\frac{1}{4}\,F_a^{\mu\nu}F_{\mu\nu,a} + \frac{1}{2}\left(D_\mu\varphi\right)_i\left(D^\mu\varphi\right)_i - V_0(\varphi)\\ &+ \overline{\psi}\,\mathrm{i}\slashed{D}\,\psi - \frac{1}{2}\left(\overline{\omega_{\mathrm{L}}}\,M(\omega_{\mathrm{L}})^{\mathrm{c}} + \overline{(\omega_{\mathrm{L}})^{\mathrm{c}}}\,M^\dagger\,\omega_{\mathrm{L}}\right)\,,\end{aligned}} \tag{4.2.1}$$

where φ denotes a set of n real spinless boson fields and ψ a set of m Dirac bispinor fields. The generalized field strength tensor $F_\alpha^{\mu\nu}$ has the form

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - \tilde{\Lambda}^{\frac{4-d}{2}}\,f_{ajk}A_j^\mu A_k^\nu\,, \tag{4.2.2}$$

---

[2] $\phi_{0c}^0 = \phi_{0c}(J=0) = 0$.
[3] With respect to a direct product of SU(N) and U(1) gauge-groups.
[4] For the sake of simplicity we choose to work from now on with $\hbar = 1$.

with the totally antisymmetric structure constant f_{ajk} and s real gauge fields A_a. The covariant derivative, acting on bosonic and fermionic fields, is defined as

$$(D^\mu \varphi)_i = \partial^\mu \varphi_i + \mathrm{i}\, \tilde{\Lambda}^{\frac{4-d}{2}} \,(\theta_a)_{ik}\, \varphi_k A_a^\mu \tag{4.2.3}$$

$$(D^\mu \psi)_i = \partial^\mu \psi_i + \mathrm{i}\, \tilde{\Lambda}^{\frac{4-d}{2}} \,(t_a)_{ik}\, \psi_k A_a^\mu \,, \tag{4.2.4}$$

and for the potential $V(\varphi)$ we choose the most general 4th-order polynomial

$$V_0(\varphi) = \tilde{\Lambda}^{\frac{d-4}{2}} \kappa_i \varphi_i + \mu_{ik}\varphi_i\varphi_k + \tilde{\Lambda}^{\frac{4-d}{2}} \rho_{ikm}\varphi_i\varphi_k\varphi_m + \tilde{\Lambda}^{4-d}\lambda_{ikmn}\varphi_i\varphi_k\varphi_m\varphi_n \,. \tag{4.2.5}$$

The vector ω_L is built up by the left-handed fields[5] ψ_L and $(\psi_\mathrm{R})^c$,

$$\omega_\mathrm{L} = \begin{pmatrix} \psi_\mathrm{L} \\ (\psi_\mathrm{R})^c \end{pmatrix} \,, \tag{4.2.6}$$

and the symmetric matrix M is given by

$$M = \begin{bmatrix} M_\mathrm{L} + \tilde{\Lambda}^{\frac{4-d}{2}}\Gamma_{\mathrm{L},i}\varphi_i & M_D + \tilde{\Lambda}^{\frac{4-d}{2}}\Gamma_{\mathrm{D},i}\varphi_i \\ M_D^T + \tilde{\Lambda}^{\frac{4-d}{2}}\Gamma_{\mathrm{D},i}^\mathrm{T}\varphi_i & M_\mathrm{R} + \tilde{\Lambda}^{\frac{4-d}{2}}\Gamma_{\mathrm{R},i}\varphi_i \end{bmatrix} \otimes \mathbb{1}_{4\times 4} \,, \tag{4.2.7}$$

where $\Gamma_{\{\mathrm{L,R,D}\},i}$ labels the $m \times m$ Yukawa coupling matrices and $M_{\{\mathrm{L,R,D}\}}$ refers to possible explicit Dirac or Majorana mass terms in the Lagrangian density. Furthermore, the factor $\tilde{\Lambda}$, with $[\tilde{\Lambda}] = E$, keeps the dimension of the coupling constants unchanged with respect to $d = 4$ and enables us to work in d dimensions and to perform dimensional regularization in subsection 4.2.2.

To avoid integrating over physically equivalent gauge fields in the path integral[6], we have to add a gauge-fixing term to the Lagrangian density, where we choose to work with the t'Hooft-R_ξ-Gauge [34]

$$\mathcal{L}_\mathrm{GF} = -\frac{\xi}{2}\left(\partial_\mu A_a^\mu + \mathrm{i}\tilde{\Lambda}^{\frac{4-d}{2}}\xi^{-1}\phi_i\,(\theta_a)_{ik}\,(\varphi_k - \phi_k)\right)^2 \,, \tag{4.2.8}$$

with the gauge parameter ξ. The corresponding Faddeev-Popov-Ghost term is then given by [27, 38]

$$\mathcal{L}_\mathrm{FP} = +\tilde{\Lambda}^{\frac{4-d}{2}}(\partial_\mu \chi_a^*)f_{abc}\chi_b A_c^\mu + \partial_\mu \chi_a^* \partial^\mu \chi_a - \tilde{\Lambda}^{4-d}\frac{1}{\xi}\chi_a^*\phi_i\,(\theta_a\theta_b)_{ik}\,\varphi_k\chi_b \,, \tag{4.2.9}$$

where χ denotes a vector of s anticommuting complex scalar ghost-fields and our whole Lagrangian density finally reads:

$$\mathcal{L} = \mathcal{L}_\mathrm{YM} + \mathcal{L}_\mathrm{GF} + \mathcal{L}_\mathrm{FP} \,. \tag{4.2.10}$$

---

[5]Therefore $(\omega_\mathrm{L})^c$ is right-handed.

[6]Due to the general gauge transformation $A_a^\mu(x) \longrightarrow A_a^\mu(x) - \partial^\mu\alpha_a(x) + f_{abc}A_b^\mu(x)\alpha_c(x)$ (see (B.1.3)), all gauge fields of the form $A_a^\mu(x) = \partial^\mu\alpha_a(x) - f_{abc}A_b^\mu(x)\alpha_c(x)$ are equivalent to $A_a^\mu(x) = 0$.

4.2.1. Semiclassical Expansion of the General Lagrangian Density

We want to apply the semiclassical approach to the general Lagrangian density (4.2.1) and perform an expansion of the fields around their saddle-point solutions (see (4.1.2)):

$$
\begin{aligned}
\varphi(x) &= \varphi_{\text{cl}}(x) + \varphi'(x) \\
A(x) &= A_{\text{cl}}(x) + A'(x) \\
\chi(x) &= \chi_{\text{cl}}(x) + \chi'(x) \\
\psi(x) &= \psi_{\text{cl}}(x) + \psi'(x) \ .
\end{aligned}
\tag{4.2.11}
$$

As we just demonstrated, up to one-loop level, the saddle-point solutions can be identified with the vaccum expectation values in presence of an external source and for translation invariant theories, it is justified to assume them to be constant in the progress of the derivation of the effective potential. Furthermore, to avoid breaking of Lorentz symmetry, we claim that only scalar fields can acquire a nonzero VEV and hence, we get

$$
\varphi_{\text{cl}}(x) = \phi(x) = \phi_{\text{c}} \ ,
\tag{4.2.12}
$$

as well as

$$
\partial_\mu \phi_{\text{c}} = A_{\text{cl}} = \chi_{\text{cl}} = \psi_{\text{cl}} = 0 \ .
\tag{4.2.13}
$$

Therefore, we can write the classical action in the presence of external fields[7] up to order one as

$$
\begin{aligned}
S^J[\varphi, A, \psi, \chi] &:= \int \mathrm{d}^d x \left(\mathcal{L}_{\text{YM}} + J_i \varphi_i + f_{\mu a} A_a^\mu + \chi_a^* \epsilon_a + \chi_a \epsilon_a^* + \overline{\eta} \psi + \overline{\psi} \eta + \overline{\eta^{\text{c}}} \psi^{\text{c}} + \overline{\psi^{\text{c}}} \eta^{\text{c}} \right) \\
&= \underbrace{\int \mathrm{d}^d x \left(-V(\phi_c) + J_i \phi_{\text{c},i} \right)}_{S_{\text{cl}}^J} + \left(S_{\varphi'} + S_{A'} + S_{\psi'} + S_{\chi'} \right) \ ,
\end{aligned}
\tag{4.2.14}
$$

where the fields decouple[8] and the various parts read:

- Scalar term $S_{\varphi'}$:

$$
S_{\varphi'} = -\int \mathrm{d}^d x \, \frac{1}{2} \, \varphi'^{\text{T}} \underbrace{\left(\mathbb{1}_{n \times n} \partial^2 + \frac{1}{\xi} \tilde{\Lambda}^{4-d} \theta_a \phi_{\text{c}} \phi_{\text{c}}^{\text{T}} \theta_a + M_\varphi^2 \right)}_{K_\varphi = K_\varphi^{\text{T}}} \varphi'
\tag{4.2.15}
$$

with

$$
(M_\varphi^2)_{ik} = \left. \frac{\partial^2 V_0(\varphi)}{\partial \varphi_i \partial \varphi_k} \right|_{\varphi = \phi_{\text{c}}}
\tag{4.2.16}
$$

---

[7]Since we are only interested in the derivation of the effective potential and the calculation of the scalar VEV, we could have set all external fields, except for $J(x)$, equal to zero right from the beginning. They are only of importance for the computation of correlation functions.

[8]The mixture term between scalar and gauge fields cancel due to our choice of the gauge fixing condition.

- Gauge term $S_{A'}$:

$$S_{A'} = \int \mathrm{d}^d x \, \frac{1}{2} \, A'_{\mu,a} \left[\delta_{ab} \, g^{\mu\nu} \partial^2 - \delta_{ab}(1-\xi)\partial^\mu \partial^\nu + g^{\mu\nu}(M_A^2)_{ab} \right] A'_{\nu,b}$$

$$= -\frac{1}{2} \int \mathrm{d}^d x \, A'^{\mathrm{T}} \underbrace{\left(-\mathbb{1} \otimes g \, \partial^2 + \mathbb{1} \otimes (g \, \partial \partial^{\mathrm{T}} g) \cdot (1-\xi) - M_A^2 \otimes g \right)}_{K_A = K_A^{\mathrm{T}}} A' \qquad (4.2.17)$$

with

$$(M_A^2)_{ab} = \tilde{\Lambda}^{4-d} \phi_{\mathrm{c},i}(\theta_a \theta_b)_{ij} \phi_{\mathrm{c},j} \qquad (4.2.18)$$

- Ghost term $S_{\chi'}$:

$$S_{\chi'} = \int \mathrm{d}^d x \, \chi'^*_a \left[-\delta_{ab}\partial^2 - \frac{1}{\xi}(M_A^2)_{ab} \right] \chi'_b$$

$$= \int \mathrm{d}^d x \, \chi'^\dagger \underbrace{\left[-\mathbb{1}_{s \times s}\partial^2 - \frac{1}{\xi} M_A^2 \right]}_{K_\chi = K_\chi^\dagger} \chi' \qquad (4.2.19)$$

- Fermionic term $S_{\psi'}$:

$$S_{\psi'} = -\frac{1}{2} \int \mathrm{d}^d x \, \omega'^{\mathrm{T}} \underbrace{(\mathbb{1} \otimes C^{-1}) \left[I \otimes i\slashed{\partial} - M_\psi \otimes P_{\mathrm{L}} - \left(I M_\psi^\dagger I \right) \otimes P_{\mathrm{R}} \right]}_{K_\psi = -K_\psi^{\mathrm{T}}} \omega' \qquad (4.2.20)$$

with

$$\omega' = \begin{pmatrix} \psi' \\ \psi'^{\mathrm{c}} \end{pmatrix} \quad , \quad I = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} , \qquad (4.2.21)$$

and

$$M_\psi = \begin{bmatrix} M_{\mathrm{L}} + \tilde{\Lambda}^{\frac{4-d}{2}}\Gamma_{\mathrm{L},i}\phi_{\mathrm{c},i} & M_D + \tilde{\Lambda}^{\frac{4-d}{2}}\Gamma_{\mathrm{D},i}\phi_{\mathrm{c},i} \\ M_D^{\mathrm{T}} + \tilde{\Lambda}^{\frac{4-d}{2}}\Gamma_{\mathrm{D},i}^{\mathrm{T}}\phi_{\mathrm{c},i} & M_{\mathrm{R}} + \tilde{\Lambda}^{\frac{4-d}{2}}\Gamma_{\mathrm{R},i}\phi_{\mathrm{c},i} \end{bmatrix} . \qquad (4.2.22)$$

We can now easily write down the semiclassical expansion of the generating functional

$$Z[J] = \frac{\int [\mathrm{d}\varphi'][\mathrm{d}A'][\mathrm{d}\omega'][\mathrm{d}\chi'][\mathrm{d}\chi'^*] \, \exp\left[\mathrm{i}S_{\mathrm{cl}}^J + \mathrm{i}\left(S_{\varphi'} + S_{A'} + S_{\psi'} + S_{\chi'} \right) \right]}{\int [\mathrm{d}\varphi'][\mathrm{d}A'][\mathrm{d}\omega'][\mathrm{d}\chi'][\mathrm{d}\chi'^*] \, \exp\left[\mathrm{i}S_{\mathrm{cl}}^{J=0} + \mathrm{i}\left(S_{\varphi'}^0 + S_{A'}^0 + S_{\psi'}^0 + S_{\chi'}^0 \right) \right]} , \qquad (4.2.23)$$

where the superscript zero in the denominator reveals that the terms are evaluated at $\phi_{\mathrm{c}}^0 = \phi_{\mathrm{c}}(J = 0)$. Solving the above Gaussian integrals (see appendix D and especially (D.1.23),(D.2.24) and (D.2.26)) leads to

$$Z = e^{i(S_{\mathrm{cl}}^J - S_{\mathrm{cl}}^{J=0})}\mathrm{Det}\left(\frac{K_A^0}{K_A} \right)^{\frac{1}{2}} \cdot \mathrm{Det}\left(\frac{K_\varphi^0}{K_\varphi} \right)^{\frac{1}{2}} \cdot \mathrm{Det}\left(\frac{K_\psi}{K_\psi^0} \right)^{\frac{1}{2}} \cdot \mathrm{Det}\left(\frac{K_\chi}{K_\chi^0} \right) . \qquad (4.2.24)$$

Again, we can express the generating functional Z through the connected generating functional W and a loop expansion,

$$Z[J] = e^{\mathrm{i}W[J]} = e^{\mathrm{i}W_0[J]}e^{\mathrm{i}W_1[J]}\dots , \qquad (4.2.25)$$

helps us to find (compare (D.1.25), (D.2.25) and (D.2.26)):

$$
\begin{aligned}
W_0[J] &= S_{\text{cl}}^J - S_{\text{cl}}^{J=0} \\
W_1[J] &= \mathrm{i} \int \mathrm{d}^d x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[\frac{1}{2} \left(\ln \det \hat{K}_A(k) - \ln \det \hat{K}_A^0(k) \right) \right. \\
&\qquad\qquad + \frac{1}{2} \left(\ln \det \hat{K}_\varphi(k) - \ln \det \hat{K}_\varphi^0(k) \right) \\
&\qquad\qquad - \frac{1}{2} \left(\ln \det \hat{K}_\psi(k) - \ln \det \hat{K}_\psi^0(k) \right) \\
&\qquad\qquad \left. - \left(\ln \det \hat{K}_\chi(k) - \ln \det \hat{K}_\chi^0(k) \right) \right] .
\end{aligned}
\tag{4.2.26}
$$

4.2.2. Explicit Calculation of the Effective Potential

We want to solve these decoupled integrals separately and start with the

I: Scalar field term

$$
W_\varphi := \frac{\mathrm{i}}{2} \int \mathrm{d}^d x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\ln \det \hat{K}_\varphi(k) - \ln \det \hat{K}_\varphi^0(k) \right) .
\tag{4.2.27}
$$

Since the determinant can be rewritten as[9]

$$
\begin{aligned}
\det \hat{K}_\varphi(k) &= \det \left(- \mathbb{1} \cdot (k^2 + \mathrm{i}0^+) + D_\varphi \right) \\
&= (-k^2 - \mathrm{i}0^+)^n \cdot \det \left(\mathbb{1} - \frac{D_\varphi}{k^2 + \mathrm{i}0^+} \right) ,
\end{aligned}
\tag{4.2.28}
$$

where the diagonal matrix D is given by

$$
D_\varphi = P_\varphi^{-1} \left(\frac{1}{\xi} \tilde{\Lambda}^{4-d} \theta_a \phi_{\mathrm{c}} \phi_{\mathrm{c}}^{\mathrm{T}} \theta_a + M_\varphi^2 \right) P_\varphi ,
\tag{4.2.29}
$$

we get

$$
W_\varphi = \frac{\mathrm{i}}{2} \int \mathrm{d}^d x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\ln \det \left(\mathbb{1} - \frac{D_\varphi}{k^2 + \mathrm{i}0^+} \right) - \ln \det \left(\mathbb{1} - \frac{D_\varphi^0}{k^2 + \mathrm{i}0^+} \right) \right) .
\tag{4.2.30}
$$

If we further perform a Wick rotation and make use of the identity

$$
\ln \det A = \operatorname{Tr} \ln A ,
\tag{4.2.31}
$$

---

[9]The appearance of $+\mathrm{i}0^+$ is explained in appendix D.1.2 and enables us to perform the Wick rotation. Afterwards it is safe to set it to zero, $\mathrm{i}0^+ \to 0$.

the integral reads:

$$
\begin{aligned}
W_\varphi &= -\frac{1}{2} \int \mathrm{d}^d x \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \left(\ln \det \left(\mathbb{1} + \frac{D_\varphi}{k_E^2 - \mathrm{i}0^+} \right) - \ln \det \left(\mathbb{1} + \frac{D_\varphi^0}{k_E^2 - \mathrm{i}0^+} \right) \right) \\
&= -\frac{1}{2} \operatorname{Tr} \int \mathrm{d}^d x \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \left(\ln \left(\mathbb{1} + \frac{D_\varphi}{k_E^2} \right) - \ln \left(\mathbb{1} + \frac{D_\varphi^0}{k_E^2} \right) \right) \\
&= -\frac{1}{2} \sum_{i=1}^n \int \mathrm{d}^d x \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \left(\ln \left(1 + \frac{(D_\varphi)_{ii}}{k_E^2} \right) - \ln \left(1 + \frac{(D_\varphi^0)_{ii}}{k_E^2} \right) \right) \\
&\overset{(\mathrm{A.1.5})}{=} -\frac{1}{2} \sum_{i=1}^n \int \mathrm{d}^d x \, \frac{2}{d} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left((D_\varphi)_{ii}^{\frac{d}{2}} - (D_\varphi^0)_{ii}^{\frac{d}{2}} \right) \\
&= - \int \mathrm{d}^d x \, \operatorname{Tr} \left[\frac{1}{d} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left(\left(\frac{1}{\xi} \tilde{\Lambda}^{4-d} \theta_a \phi_\mathrm{c} \phi_\mathrm{c}^\mathrm{T} \theta_a + M_\varphi^2 \right)^{\frac{d}{2}} \right. \right. \\
&\qquad\qquad\qquad\qquad \left. \left. - \left(\frac{1}{\xi} \tilde{\Lambda}^{4-d} \theta_a \phi_\mathrm{c}^0 \phi_\mathrm{c}^{0\mathrm{T}} \theta_a + M_\varphi^{02} \right)^{\frac{d}{2}} \right) \right] .
\end{aligned}
$$
(4.2.32)

For $d = 4 - 2\epsilon$ dimensions we can expand the above expression in powers of ϵ,

$$
\begin{aligned}
&\frac{1}{4 - 2\epsilon} \frac{1}{(4\pi)^{2-\epsilon}} \Gamma(-1 + \epsilon) \, A^{2-\epsilon} = \\
&= \frac{1}{64\pi^2} \left(1 + \frac{\epsilon}{2} \right) (1 + \epsilon \ln 4\pi) \left(-\frac{1}{\epsilon} + \gamma_E - 1 + \mathcal{O}(\epsilon) \right) \tilde{\Lambda}^{-2\epsilon} \underbrace{\tilde{\Lambda}^{2\epsilon}}_{1 + \epsilon \ln \tilde{\Lambda}^2} A^2 \left(1 - \epsilon \ln A \right) + \mathcal{O}(\epsilon) \\
&= \frac{1}{64\pi^2} \left[\left(-\frac{1}{\epsilon} - \frac{3}{2} \right) \tilde{\Lambda}^{-2\epsilon} A^2 + A^2 \ln \frac{A}{\Lambda^2} \right] + \mathcal{O}(\epsilon) ,
\end{aligned}
$$
(4.2.33)

where γ_E denotes the Euler–Mascheroni constant and with $\tilde{\Lambda}^2 = \Lambda^2 \cdot e^{\gamma_E - \ln(4\pi)}$. Therefore, in the $\overline{\mathrm{MS}}$-scheme the counter term reads

$$
\boxed{\mathrm{CT}_\varphi = \tilde{\Lambda}^{-2\epsilon} \left(\frac{1}{\epsilon} \right) \left[\left(\frac{1}{\xi} \tilde{\Lambda}^{2\epsilon} \theta_a \phi_\mathrm{c} \phi_\mathrm{c}^\mathrm{T} \theta_a + M_\varphi^2 \right)^2 - \left(\frac{1}{\xi} \tilde{\Lambda}^{2\epsilon} \theta_a \phi_\mathrm{c}^0 \phi_\mathrm{c}^{0\mathrm{T}} \theta_a + M_\varphi^{02} \right)^2 \right] ,}
$$
(4.2.34)

and the limit $\epsilon \to 0$ yields:

$$
\boxed{\begin{aligned}
W_\varphi = &-\frac{1}{64\pi^2} \int \mathrm{d}^4 x \, \operatorname{Tr} \left[\left(\frac{1}{\xi} \theta_a \phi_\mathrm{c} \phi_\mathrm{c}^\mathrm{T} \theta_a + M_\varphi^2 \right)^2 \cdot \left(\ln \left(\frac{\frac{1}{\xi} \theta_a \phi_\mathrm{c} \phi_\mathrm{c}^\mathrm{T} \theta_a + M_\varphi^2}{\Lambda^2} \right) - \frac{3}{2} \right) \right. \\
&\left. - \left(\frac{1}{\xi} \theta_a \phi_\mathrm{c}^0 \phi_\mathrm{c}^{0\mathrm{T}} \theta_a + M_\varphi^{02} \right)^2 \cdot \left(\ln \left(\frac{\frac{1}{\xi} \theta_a \phi_\mathrm{c}^0 \phi_\mathrm{c}^{0\mathrm{T}} \theta_a + M_\varphi^{02}}{\Lambda^2} \right) - \frac{3}{2} \right) - \mathrm{CT}_\varphi \right] .
\end{aligned}}
$$
(4.2.35)

Since the integral of the

II: <u>Ghost field term</u>

$$W_\chi := -\mathrm{i} \int \mathrm{d}^d x \int \frac{d^d k}{(4\pi)^d} \left[\ln\det\left(\mathbb{1} - \frac{\xi^{-1} M_A{}^2}{k^2 + \mathrm{i}0^+} \right) - \ln\det\left(\mathbb{1} - \frac{\xi^{-1} M_A^0{}^2}{k^2 + \mathrm{i}0^+} \right) \right],$$

(4.2.36)

has the exact same structure as the scalar part (4.2.27) we can immediately write down the result

$$\boxed{W_\chi = \frac{2}{64\pi^2} \int \mathrm{d}^4 x \, \mathrm{Tr}\left[\frac{1}{\xi^2} M_A^4 \ln\left(\frac{M_A{}^2}{\xi\,\Lambda^2} - \frac{3}{2} \right) - \frac{1}{\xi^2} M_A^0{}^4 \ln\left(\frac{M_A^0{}^2}{\xi\,\Lambda^2} - \frac{3}{2} \right) - \mathrm{CT}_\chi \right],}$$

(4.2.37)

with

$$\boxed{\mathrm{CT}_\chi = \tilde{\Lambda}^{-2\epsilon} \frac{1}{\epsilon} \frac{1}{\xi^2} \left[M_A^4 - M_A^0{}^4 \right].}$$

(4.2.38)

Next, we consider the

III: <u>Gauge field term</u>

$$W_A := \frac{\mathrm{i}}{2} \int d^d x \int \frac{d^d k}{(2\pi)^d} \left(\ln\det\hat{K}_A(k) - \ln\det\hat{K}_A^0(k) \right),$$

(4.2.39)

with

$$\hat{K}_A(k) = \left(\mathbb{1}_{s\times s} \otimes g \right)\left(\mathbb{1}_{s\times s} \otimes k^2\, \mathbb{1}_{d\times d} - (1-\xi)\cdot \mathbb{1}_{s\times s} \otimes kk^\mathrm{T} g - M_A^2 \otimes \mathbb{1}_{d\times d} \right).$$

(4.2.40)

The $d \times d$ dimensional matrix $kk^\mathrm{T}g$ has one eigenvector $\hat{k} \parallel k$ with eigenvalue k^2 and $d-1$ eigenvectors $\tilde{k} \perp k$ with eigenvalue 0. Hence, we can find a matrix P_A,

$$P_A = \left(P^M_{s\times s} \otimes P^k_{d\times d} \right),$$

(4.2.41)

with

$$P^M_{s\times s} D_A \left(P^M_{s\times s} \right)^{-1} = M_A^2 \quad \text{and} \quad P^k_{d\times d} \, \mathrm{diag}\left(k^2, 0, \ldots, 0 \right) \left(P^k_{d\times d} \right)^{-1} = kk^\mathrm{T} g$$

(4.2.42)

such that

$$\hat{K}_A(k) = \left(\mathbb{1}_{s\times s} \otimes g \right) P_A \left(\mathbb{1}_{s\times s} \otimes (k^2 + \mathrm{i}0^+) \begin{bmatrix} \xi & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - D_A \otimes \mathbb{1}_{d\times d} \right) P_A^{-1}$$

$$= \left(\mathbb{1}_{s\times s} \otimes g \right) P_A \left(\mathbb{1}_{s\times s} \otimes (k^2 + \mathrm{i}0^+) \begin{bmatrix} \xi & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \right)$$

$$\cdot \left(\mathbb{1}_{s\times s} \otimes \mathbb{1}_{d\times d} - \frac{1}{k^2 + \mathrm{i}0^+} D_A \otimes \begin{bmatrix} \frac{1}{\xi} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \right) P_A^{-1},$$

(4.2.43)

where we again have displayed the $+\mathrm{i}0^+$ term in the last equation. The diagonalization matrix P_A does not play any role in the course of the derivation of the determinant and since $\hat{K}_A(k)$ and $\hat{K}_A^0(k)$ both exhibit the same factor

$$\left(\mathbb{1}_{s\times s}\otimes g\right)\cdot\left(\mathbb{1}_{s\times s}\otimes(k^2+\mathrm{i}0^+)\begin{bmatrix}\xi&&&\\&1&&\\&&\ddots&\\&&&1\end{bmatrix}\right), \tag{4.2.44}$$

they cancel each other out. Therefore, after a Wick rotation and applying the identity (4.2.31) we are left with:

$$W_A = -\frac{1}{2}\,\mathrm{Tr}\int\mathrm{d}^d x\int\frac{\mathrm{d}^d k_E}{(2\pi)^d}\left[\ln\left(\mathbb{1}_{s\times s}\otimes\mathbb{1}_{d\times d}+\frac{1}{k_E^2}D_A\otimes\begin{bmatrix}\frac{1}{\xi}&&&\\&1&&\\&&\ddots&\\&&&1\end{bmatrix}\right)\right.$$
$$\left.-\ln\left(\mathbb{1}_{s\times s}\otimes\mathbb{1}_{d\times d}+\frac{1}{k_E^2}D_A^0\otimes\begin{bmatrix}\frac{1}{\xi}&&&\\&1&&\\&&\ddots&\\&&&1\end{bmatrix}\right)\right]$$
$$= -\frac{1}{2}\sum_{i=1}^s\int\mathrm{d}^d x\int\frac{\mathrm{d}^d k_E}{(2\pi)^d}\left[\ln\left(1+\frac{\frac{1}{\xi}(D_A)_{ii}}{k_E^2}\right)+(d-1)\cdot\ln\left(1+\frac{(D_A)_{ii}}{k_E^2}\right)\right.$$
$$\left.-\ln\left(1+\frac{\frac{1}{\xi}(D_A^0)_{ii}}{k_E^2}\right)+(d-1)\cdot\ln\left(1+\frac{(D_A^0)_{ii}}{k_E^2}\right)\right]. \tag{4.2.45}$$

By analogy with the above calculations we find in the $\overline{\mathrm{MS}}$-scheme

$$\boxed{\mathrm{CT}_A = \tilde{\Lambda}^{-2\epsilon}\frac{1}{\epsilon}\left[\left(3+\frac{1}{\xi^2}\right)\left(M_A^4-M_A^{0\,4}\right)\right],} \tag{4.2.46}$$

and for $\epsilon\to 0$:

$$\boxed{\begin{aligned}W_A = &-\frac{1}{64\pi^2}\int\mathrm{d}^4 x\,\mathrm{Tr}\left[M_A^4\left(3\cdot\left(\ln\frac{M_A^2}{\Lambda^2}-\frac{5}{6}\right)+\frac{1}{\xi^2}\left(\ln\frac{\xi^{-1}M_A^2}{\Lambda^2}-\frac{3}{2}\right)\right)\right.\\&\left.-M_A^{0\,4}\left(3\cdot\left(\ln\frac{M_A^{0\,2}}{\Lambda^2}-\frac{5}{6}\right)-\frac{1}{\xi^2}\left(\ln\frac{\xi^{-1}M_A^{0\,2}}{\Lambda^2}+\frac{3}{2}\right)\right)-\mathrm{CT}_A\right].\end{aligned}} \tag{4.2.47}$$

The last missing part is the

IV: <u>Fermion field term</u>

$$W_\psi := -\frac{\mathrm{i}}{2}\int\mathrm{d}^d x\int\frac{\mathrm{d}^d k}{(4\pi)^d}\left[\mathrm{Tr}\ln\left(\mathbb{1}_{2n\times 2n}\otimes\mathbb{1}_{4\times 4}+A\right)-\mathrm{Tr}\ln\left(\mathbb{1}_{2n\times 2n}\otimes\mathbb{1}_{4\times 4}+A^0\right)\right], \tag{4.2.48}$$

with

$$A = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} M_\psi \otimes \frac{\slashed{k}}{k^2 + \mathrm{i}0^+} P_{\mathrm{L}} + M_\psi^\dagger \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} \otimes \frac{\slashed{k}}{k^2 + \mathrm{i}0^+} P_{\mathrm{R}} \,, \qquad (4.2.49)$$

where we already have canceled the factor

$$\left(\mathbb{1}_{2n \times 2n} \otimes C^{-1} \right) \left(\begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} \otimes -\slashed{k} \right) \,, \qquad (4.2.50)$$

and made use of equation (4.2.31). Since the trace of an odd number of gamma matrices vanishes,

$$\begin{aligned} \mathrm{Tr}\left(\gamma_1 ... \gamma_{2n+1} \right) &= \mathrm{Tr}\left(\gamma_1 ... \gamma_{2n+1} \gamma_5 \gamma_5 \right) \overset{(\mathrm{C}.1.4)}{=} -\mathrm{Tr}\left(\gamma_5 \gamma_1 ... \gamma_{2n+1} \gamma_5 \right) \\ &= -\mathrm{Tr}\left(\gamma_1 ... \gamma_{2n+1} \right) = 0 \,, \end{aligned} \qquad (4.2.51)$$

and we know that (see appendix C.3)

$$\begin{aligned} P_{\mathrm{L}} P_{\mathrm{R}} &= P_{\mathrm{R}} P_{\mathrm{L}} = 0 \\ \mathrm{Tr} P_{\mathrm{L}} &= \mathrm{Tr} P_{\mathrm{R}} = 2 \\ P_{\mathrm{L}}^2 &= P_{\mathrm{L}} \\ P_{\mathrm{R}}^2 &= P_{\mathrm{R}} \,, \end{aligned} \qquad (4.2.52)$$

we find for the trace of the logarithm

$$\begin{aligned} \mathrm{Tr} \ln\left(\mathbb{1} + A \right) &= \mathrm{Tr} \sum_{l=1}^\infty (-1)^{l+1} \frac{A^l}{l} = \mathrm{Tr} \sum_{l=1}^\infty (-1)^{2l+1} \frac{(A^2)^l}{2l} \\ &= \mathrm{Tr} \sum_{l=1}^\infty \frac{(-1)^{2l+1}}{2l} \left(\frac{1}{k^2 + \mathrm{i}0^+} \left(\begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} M_\psi M_\psi^\dagger \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} \otimes P_{\mathrm{R}} + M_\psi^\dagger M_\psi \otimes P_{\mathrm{L}} \right) \right)^l \\ &= 2 \cdot \mathrm{Tr} \sum_{l=1}^\infty \frac{(-1)^{l+1}}{l} \left(-\frac{1}{k^2 + \mathrm{i}0^+} M_\psi M_\psi^\dagger \right)^l = 2 \cdot \mathrm{Tr} \ln\left(\mathbb{1} - \frac{M_\psi M_\psi^\dagger}{k^2 + \mathrm{i}0^+} \right) \,. \end{aligned} \qquad (4.2.53)$$

Therefore, we have reduced our integral to the meanwhile well known form

$$W_\psi = -\mathrm{i} \int \mathrm{d}^d x \int \frac{\mathrm{d}^d k}{(4\pi)^d} \left[\ln \det\left(\mathbb{1} - \frac{M_\psi M_\psi^\dagger}{k^2 + \mathrm{i}0^+} \right) - \ln \det\left(\mathbb{1} - \frac{M_\psi^0 M_\psi^{0\dagger}}{k^2 + \mathrm{i}0^+} \right) \right] \,, \qquad (4.2.54)$$

and thus we get

$$\boxed{W_\psi = \frac{2}{64\pi^2} \int \mathrm{d}^4 x \, \mathrm{Tr}\left[\left(M_\psi M_\psi^\dagger \right)^2 \ln\left(\frac{M_\psi M_\psi^\dagger}{\Lambda^2} - \frac{3}{2} \right) \right. \\ \left. - \left(M_\psi^0 M_\psi^{0\dagger} \right)^2 \ln\left(\frac{M_\psi^0 M_\psi^{0\dagger}}{\Lambda^2} - \frac{3}{2} \right) - \mathrm{CT}_\psi \right] \,,} \qquad (4.2.55)$$

with

$$\mathrm{CT}_\psi = \tilde{\Lambda}^{-2\epsilon} \frac{1}{\epsilon} \left[\left(M_\psi\, M_\psi^\dagger \right)^2 - \left(M_\psi^0 M_\psi^{0\,\dagger} \right)^2 \right] .$$
(4.2.56)

With the help of equation (3.4.2) we can now finally write down the effective potential at one-loop order as[10]

$$V(\phi_c) = V_0(\phi_c) + \frac{1}{64\pi^2} \operatorname{Tr}\left[\left(\frac{1}{\xi}\theta_a\phi_c\phi_c^{\mathrm{T}}\theta_a + M_\varphi^2 \right)^2 \cdot \left(\ln\left(\frac{\frac{1}{\xi}\theta_a\phi_c\phi_c^{\mathrm{T}}\theta_a + M_\varphi^2}{\Lambda^2} \right) - \frac{3}{2} \right) \right.$$
$$+ M_A^4 \left(3 \cdot \left(\ln\frac{M_A^2}{\Lambda^2} - \frac{5}{6} \right) - \frac{1}{\xi^2}\left(\ln\frac{\xi^{-1}M_A^2}{\Lambda^2} - \frac{3}{2} \right) \right)$$
$$\left. - 2 \cdot \left(M_\psi\, M_\psi^\dagger \right)^2 \ln\left(\frac{M_\psi\, M_\psi^\dagger}{\Lambda^2} - \frac{3}{2} \right) \right] + \mathrm{const.} + \mathrm{CT} ,$$
(4.2.57)

where the counterterm CT is given by:

$$\mathrm{CT} = \frac{1}{64\pi^2}\operatorname{Tr}\left[-\mathrm{CT}_\varphi - \mathrm{CT}_A + 2\mathrm{CT}_\chi + 2\mathrm{CT}_\psi \right] .$$
(4.2.58)

This effective potential is for Dirac fermions[11] and in Landau gauge ($\xi \to \infty$), up to a different choice of renormalization schemes, in perfect agreement with the result found in [16].

---

[10]The derivation of this general effective potential for a general R_ξ gauge was done in cooperation with M. Fink and thus can also be found in his master's thesis [39].

[11]For Dirac fermions ($M_{\mathrm{L}} = M_{\mathrm{R}} = 0$) the eigenvalues of the $2m \times 2m$ mass matrix $M_\psi M_\psi^\dagger$ degenerate and the trace delivers an extra factor of 2.

5. The Gildener-Weinberg Approach

To be able to determine the minimum of the effective potential for theories with an arbitrary number of massless scalar particles, E.Gildener and S.Weinberg came up with a new efficient method to explore SSB in such classically scale invariant theories. Since we want to apply this Gildener-Weinberg approach to a SU(2)-extension of the Standard Model in the next section, we give a short review of their work, based on their paper [16].

We consider a general, classically scale invariant and renormalizable gauge theory with n weakly-coupled real scalar fields φ_i. The tree-level potential is then given by

$$V_0(\varphi) = \frac{1}{24}\lambda_{iklm}\varphi_i\varphi_k\varphi_l\varphi_m \ . \tag{5.0.1}$$

Since the coupling constant λ_{iklm}, which we assume to be totally symmetric, depends on the renormalization scale Λ, $\lambda_{iklm} = \lambda_{iklm}(\Lambda)$, we can choose $\Lambda = \Lambda_{\mathrm{GW}}$ such that the potential V_0 possesses a nontrivial minimum along some ray $\varphi_i^{\mathrm{flat}} = n_i\sigma$ (Figure 5.1).

(a) Tree-level potential for arbitrary Λ (b) Tree-level potential for $\Lambda = \Lambda_{\mathrm{GW}}$

Figure 5.1.: Illustration of the appearance of a flat direction for the right choice of the renormalization scale - here for a theory with 2 real scalar fields.
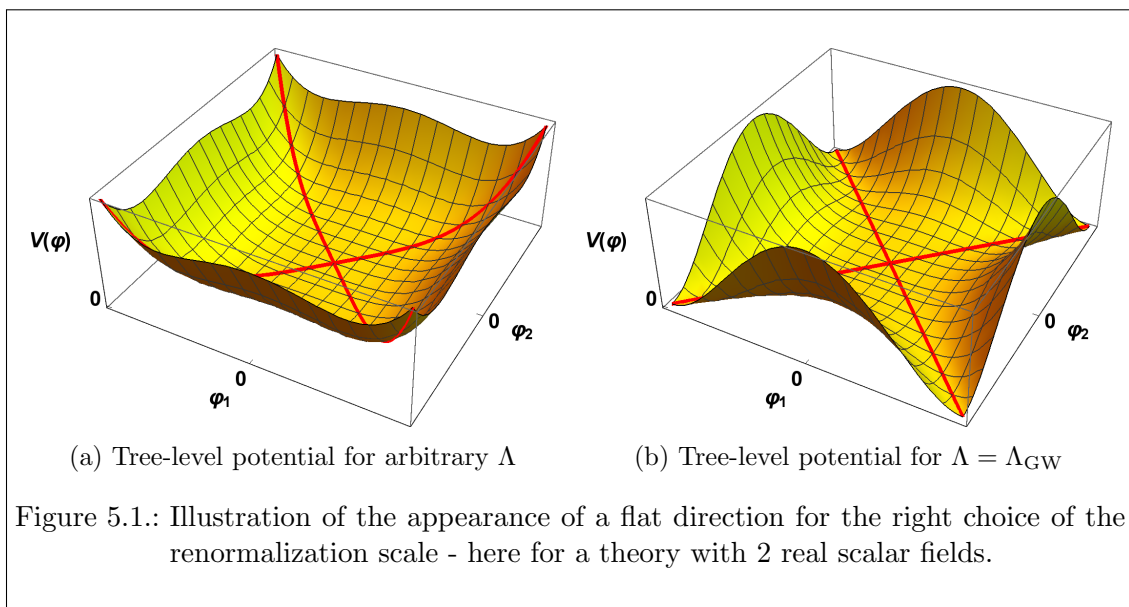
To find this special direction we claim that the effective potential has to fulfill the following condition for its minimum on the unit sphere:

$$\min_{N_jN_j=1}\Big(\lambda_{iklm}(\Lambda_{\mathrm{GW}})N_iN_kN_lN_m\Big) = 0 \ . \tag{5.0.2}$$

Therefore, if the potential $V_0(N)$ is equal to zero for a unit vector $N_i = n_i$, the potential is also zero along the ray $\varphi_i^{\mathrm{flat}} = n_i\sigma$, which gives the flat direction of Figure 5.1. The Gildener-Weinberg condition (5.0.2) puts one single constraint on the coupling constants

and enables us to trade a dimensionless coupling constant for the dimensionful Gildener-Weinberg scale Λ_{GW}. This phenomenon is known as dimensional transmutation and is, as described in [7], a general feature of SSB in classically scale invariant theories .

To ensure that the flat direction defines a minimum of the tree-level potential, we have to claim that the Hessian matrix P is positive semidefinite:

$$P_{ik} u_i u_k = \left. \frac{\partial^2 V_0(\varphi)}{\partial \varphi_i \partial \varphi_k} \right|_{\varphi = n} u_i u_k = \frac{1}{2} \lambda_{iklm}(\Lambda_{\text{GW}}) u_i u_k n_k n_l \geq 0 \quad , \quad \forall u \ . \tag{5.0.3}$$

Furthermore, higher order contributions δV to the zero-loop potential V_0 lead to a small curvature in the flat direction, which produces a distinct minimum, and to a small shift $\delta \varphi_i$ of this minimum in the direction of φ_i. Therefore, the extremum is defined by

$$0 = \left[\frac{\partial}{\partial \varphi_i} \Big(V_0(\varphi) + \delta V(\varphi) \Big) \right]_{n \langle \sigma \rangle + \delta \varphi} , \tag{5.0.4}$$

which reads at one-loop order[1]

$$0 = P_{ik} \delta \varphi_k \langle \sigma \rangle^2 + \left. \frac{\partial V_1(\varphi)}{\partial \varphi_i} \right|_{n \langle \sigma \rangle} . \tag{5.0.5}$$

In the last line we made use of the fact that n is an eigenvector of P_{ik} with eigenvalue 0,

$$P_{ik} n_k = 0 \ , \tag{5.0.6}$$

since a necessary condition for the minimum in the direction of n is just:

$$\lambda_{iklm}(\Lambda_{\text{GW}}) \, n_k n_l n_m = 0 \ . \tag{5.0.7}$$

For a gauge symmetry with the infinitesimal transformation law

$$\varphi \to \varphi + i \epsilon \, \alpha_a \Theta_a \, \varphi \ , \tag{5.0.8}$$

each broken generator, $\Theta_a n \neq 0$, corresponds to a massless Goldstone boson and delivers another eigenvector of P with eigenvalue zero (Goldstone's theorem [40]):

$$P_{ik} (\Theta n)_k = 0 \ . \tag{5.0.9}$$

We can now make use of equation (5.0.6) and contract equation (5.0.5) with n_i to find a condition, which determines $\langle \sigma \rangle$:

$$0 = n_i \left. \frac{\partial V_1(\varphi)}{\partial \varphi_i} \right|_{n \langle \sigma \rangle} = \left. \frac{\partial V_1(n\sigma)}{\partial \sigma} \right|_{\langle \sigma \rangle} . \tag{5.0.10}$$

The one-loop effective potential (in Landau gauge) along the ray $\varphi^{\text{flat}} = n\sigma$ can be written in a very compact way as (compare equation (4.2.57))[2]

$$\boxed{ V_1(n\sigma) = A \, \sigma^4 + B \, \sigma^4 \ln \left(\frac{\sigma^2}{\Lambda_{\text{GW}}^2} \right) \ , } \tag{5.0.11}$$

---

[1] $\delta V = V_1 + V_2 + \ldots$

[2] We ignore possible constant terms and postpone the discussion of the counterterms and renormalization to the next chapter.

with the dimensionless constants

$$A = \frac{1}{64\pi^2 \langle\sigma\rangle^4} \, \mathrm{Tr}\left[m_\varphi^4 \left(\ln\left(\frac{m_\varphi^2}{\langle\sigma\rangle^2} - \frac{3}{2} \right) \right) + 3 m_A^4 \left(\ln\left(\frac{m_A^2}{\langle\sigma\rangle^2} - \frac{5}{6} \right) \right) \right.$$
$$\left. - 2 \left(m_\psi m_\psi^\dagger \right)^2 \left(\ln\left(\frac{m_\psi m_\psi^\dagger}{\langle\sigma\rangle^2} - \frac{3}{2} \right) \right) \right] , \tag{5.0.12}$$

and

$$B = \frac{1}{64\pi^2 \langle\sigma\rangle^4} \, \mathrm{Tr}\left[m_\varphi^4 + 3 m_A^4 - 2 \left(m_\psi m_\psi^\dagger \right)^2 \right] . \tag{5.0.13}$$

The matrices m_φ, m_A and m_ψ are the tree-level mass matrices of the scalar bosons, gauge bosons and fermions[3],

$$\left(m_\varphi^2 \right)_{ik} = \left(M_\varphi^2 \right)_{ik} \Big|_{n\langle\sigma\rangle} \overset{(4.2.16)}{=} \frac{\partial^2 V_0(\varphi)}{\partial\varphi_i \partial\varphi_k} \Big|_{n\langle\sigma\rangle}$$

$$\left(m_A^2 \right)_{ab} = \left(M_A^2 \right)_{ab} \Big|_{n\langle\sigma\rangle} \overset{(4.2.18)}{=} n_i \left(\Theta_a \Theta_b \right)_{ik} n_k \langle\sigma\rangle^2 \tag{5.0.14}$$

$$m_\psi = M_\psi \Big|_{n\langle\sigma\rangle} \overset{(4.2.22)}{=} \begin{bmatrix} \Gamma_{\mathrm{L},i} n_i & \Gamma_{\mathrm{D},i}^{\mathrm{T}} n_i \\ \Gamma_{\mathrm{D},i} n_i & \Gamma_{\mathrm{R},i} n_i \end{bmatrix} \cdot \langle\sigma\rangle ,$$

and hence, we can write the constant B as a sum over all tree-level masses $\hat{m}_h$, $\hat{m}_g$ and $\hat{m}_f$ of scalar bosons, gauge bosons and fermions respectively:

$$\boxed{ B = \frac{1}{64\pi^2 \langle\sigma\rangle^4} \left[\sum_h \hat{m}_h^4 + 3 \sum_g \hat{m}_g^4 - \sum_f c_f \hat{m}_f^4 \right] . } \tag{5.0.15}$$

For Dirac fermions ($\Gamma_{\mathrm{L}} = \Gamma_{\mathrm{R}} = 0$) the eigenvalues of the mass matrix degenerate and therefore the factor c_f is 4 for Dirac fermions and 2 for Majorana fermions.

A stationary point of the potential, see equation (5.0.10), is given by the condition

$$\ln\left(\frac{\langle\sigma\rangle^2}{\Lambda_{\mathrm{GW}}^2} \right) = -\frac{1}{2} - \frac{A}{B} , \tag{5.0.16}$$

and as long as $\ln\left(\frac{\langle\sigma\rangle^2}{\Lambda_{\mathrm{GW}}^2} \right)$ is of order unity, perturbation theory should be valid [16]. Since the one-loop potential is not bounded from below for $B < 0$ and purely quartic for B=0, we have to require that

$$\boxed{ B > 0 . } \tag{5.0.17}$$

Furthermore, it is not hard to show that the value of the potential at this nontrivial stationary point is smaller than its value at the origin $V(0) = 0$:

$$V(n\langle\sigma\rangle) = V_1(n\langle\sigma\rangle) = A \langle\sigma\rangle^4 + B \langle\sigma\rangle^4 \ln\left(\frac{\langle\sigma\rangle^2}{\Lambda_{\mathrm{GW}}^2} \right) \overset{(5.0.16)}{=} -\frac{1}{2} B \langle\sigma\rangle^4 < 0 . \tag{5.0.18}$$

---

[3]We are dealing with a classically scale invariant theory and therefore, explicit fermion mass terms are forbidden.

Since the squared masses of the scalar bosons at tree level are just the eigenvalues of the matrix

$$(m_\varphi^2)_{ik} = \left.\frac{\partial^2 V_0(\varphi)}{\partial \varphi_i \varphi_k}\right|_{n\langle\sigma\rangle} = P_{ik} \langle\sigma\rangle^2 \ , \tag{5.0.19}$$

we immediately see from equation (5.0.6) and (5.0.9) that we find at zero-loop order - apart from a set of massive bosons corresponding to positive eigenvalues of m_φ^2 and a set of massless Goldstone bosons - one extra massless scalar boson, the scalon [16].

If we consider the one-loop contributions to the mass matrix

$$\left(m_\varphi^2 + \delta m_\varphi^2\right)_{ik} = \left.\frac{\partial^2 \left(V_0(\varphi) + V_1(\varphi)\right)}{\partial \varphi_i \partial \varphi_k}\right|_{n\langle\sigma\rangle + \delta\varphi} \ , \tag{5.0.20}$$

we find at one-loop order:

$$\left(\delta m_\varphi^2\right)_{ik} = \lambda_{iklm} \, n_l \, \langle\sigma\rangle \, \delta\varphi_m + \left.\frac{\partial^2 V_1(\varphi)}{\partial \varphi_i \partial \varphi_k}\right|_{n\langle\sigma\rangle} \ . \tag{5.0.21}$$

As long as δm_φ^2 is only a small perturbation compared to m_φ^2, the positive eigenvectors of m_φ^2 remain positive at higher orders and if we assume that the effective potential is still invariant under Θ at higher loop orders, the Goldstone bosons stay massless.

The squared mass of the scalon at one-loop order is just the eigenvalue of the mass matrix $m_\varphi^2 + \delta m_\varphi^2$ with respect to the eigenvector $n_i + \delta\varphi_i$ and given by:

$$m_s^2 = (n_i + \delta\varphi_i) \left(P_{ik} \langle\sigma\rangle^2 + \lambda_{iklm} \, n_l \, \langle\sigma\rangle \, \delta\varphi_m + \left.\frac{\partial^2 V_1(\varphi)}{\partial \varphi_i \partial \varphi_k}\right|_{n\langle\sigma\rangle} \right) (n_k + \delta\varphi_k)$$

$$\stackrel{(5.0.6)}{=} n_i n_k \left.\frac{\partial^2 V_1(\varphi)}{\partial \varphi_i \varphi_k}\right|_{n\langle\sigma\rangle} = \left.\frac{\partial^2 V_1(n\sigma)}{\partial^2 \sigma}\right|_{\langle\sigma\rangle} \stackrel{(5.0.16)}{=} 8B \langle\sigma\rangle^2 \ . \tag{5.0.22}$$

Therefore, we found that the scalon becomes massive at one-loop order[4], where its mass is determined by the tree-level masses of the other particles contained in the theory under consideration:

$$\boxed{m_s^2 = \frac{1}{8\pi^2 \langle\sigma\rangle^2} \left[\sum_h \hat{m}_h^4 + 3 \sum_g \hat{m}_g^4 - 2 \sum_f c_f \hat{m}_f^4 \right] \ .} \tag{5.0.23}$$

As long as the negative fermion contributions are compensated by the other particle masses (or simpler, as long as $B > 0$), all eigenvalues are positive-definite and the zero-loop result, that $n \langle\sigma\rangle + \delta\varphi$ is a local minimum, is still valid at one-loop order.

---

[4]For that reason we call the scalon a pseudo-Goldstone boson.

6. Classically Scale Invariant Extensions of the Standard Model

If we trust in classical scale invariance as a guiding principle of model building, it might be tempting to apply the GW approach to the classically scale invariant version of the well-investigated Standard Model ($\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$), which is simply achieved by getting rid of the imaginary scalar mass term $-\mu^2 H^\dagger H$ in the Lagrangian density.

6.1. The Classically Scale Invariant Standard Model

The classically scale invariant Lagrangian density reads[1] (see e.g. [36, 37])

$$
\begin{aligned}
\mathcal{L}_{\mathrm{SM}} = -\,& \frac{1}{4}\, W^a_{\mu\nu} W^{a,\mu\nu} - \frac{1}{4}\, G^A_{\mu\nu} G^{A,\mu\nu} - \frac{1}{4}\, B_{\mu\nu} B^{\mu\nu} \\
+\,& \mathrm{i}\, \overline{Q}_\mathrm{L} \slashed{D} Q_\mathrm{L} + \mathrm{i}\, \overline{u}_\mathrm{R} \slashed{D} u_\mathrm{R} + \mathrm{i}\, \overline{d}_\mathrm{R} \slashed{D} d_\mathrm{R} \\
+\,& \mathrm{i}\, \overline{L} \slashed{D} L + \mathrm{i}\, \overline{l}_\mathrm{R} \slashed{D} l_\mathrm{R} + (D^\mu H)^\dagger (D_\mu H) - V_{\mathrm{SM}} \\
-\,& \overline{Q}_\mathrm{L} \Gamma^{(d)}_\mathrm{D} D_\mathrm{R} H - \overline{Q}_\mathrm{L} \Gamma^{(u)}_\mathrm{D} U_\mathrm{R} \, (\mathrm{i}\tau_2 H^*) + \mathrm{h.c.} \\
-\,& \overline{L}\, \Gamma^{(l)}_\mathrm{D} l_\mathrm{R} H + \mathrm{h.c.}\,,
\end{aligned}
\tag{6.1.1}
$$

with the CSI tree-level potential:

$$
V_{\mathrm{SM}} = \frac{\lambda}{4} \left(H^\dagger H \right)^2 . \tag{6.1.2}
$$

The complex scalar field H denotes the Higgs doublet

$$
H = \frac{1}{\sqrt{2}} \begin{pmatrix} h_1 + \mathrm{i}h_2 \\ h_3 + \mathrm{i}h_4 \end{pmatrix} , \tag{6.1.3}
$$

and the quark and lepton fields are defined as

$$
Q_\mathrm{L} = \begin{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}_\mathrm{L} \\ \begin{pmatrix} c \\ s \end{pmatrix}_\mathrm{L} \\ \begin{pmatrix} t \\ b \end{pmatrix}_\mathrm{L} \end{pmatrix} \;,\;\; U_\mathrm{R} = \begin{pmatrix} u_\mathrm{R} \\ c_\mathrm{R} \\ t_\mathrm{R} \end{pmatrix} \;,\;\; D_\mathrm{R} = \begin{pmatrix} d_\mathrm{R} \\ s_\mathrm{R} \\ b_\mathrm{R} \end{pmatrix} \;,\;\; L = \begin{pmatrix} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_\mathrm{L} \\ \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_\mathrm{L} \\ \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_\mathrm{L} \end{pmatrix} \;,\;\; l_\mathrm{R} = \begin{pmatrix} e^-_\mathrm{R} \\ \mu^-_\mathrm{R} \\ \tau^-_\mathrm{R} \end{pmatrix} \;,
\tag{6.1.4}
$$

where the quark fields transform as triplets with respect to $\mathrm{SU}(3)_c$[2]. The weak isospins and weak hypercharges (T, Y) of these fields can be found in Table 6.1 and the electric

---

[1] We simplify our notation a little bit and do not explicitly display the Dirac structure of the fermions anymore.

[2] e.g. $u := (u_r \;\; u_g \;\; u_b)$.

Fields		(T, Y)
$Q_{\mathrm{L}i}$	$\sim$	$\left(\frac{1}{2}, \frac{1}{3}\right)$
$U_{\mathrm{R}i}$	$\sim$	$\left(0, \frac{4}{3}\right)$
$D_{\mathrm{R}i}$	$\sim$	$\left(0, -\frac{2}{3}\right)$
L_i	$\sim$	$\left(\frac{1}{2}, -1\right)$
$l_{\mathrm{R}i}$	$\sim$	$(0, -2)$
H	$\sim$	$\left(\frac{1}{2}, 1\right)$

Table 6.1.: Weak isospin and weak hypercharge.

charge is then given by

$$Q = T_3 + \frac{Y}{2} \; . \tag{6.1.5}$$

The covariant derivative can be written as

$$D^\mu = \partial^\mu + \mathrm{i} g_s \frac{\mathcal{T}_A}{2} G_A^\mu + \mathrm{i} g \frac{T_a}{2} W_a^\mu + \mathrm{i} g' \frac{Y}{2} B^\mu \; , \tag{6.1.6}$$

where G_A^μ labels the eight gluon gauge fields and W_a^μ and B^μ denote the four electroweak gauge bosons. The generators $\mathcal{T}_A$ are the Gell-Mann matrices λ_A when applied to $\mathrm{SU}(3)_\mathrm{c}$ triplets and zero otherwise, the generators T_a are the Pauli matrices τ_a for $\mathrm{SU}(2)_\mathrm{L}$ doublets and equal to zero for $\mathrm{SU}(2)_\mathrm{L}$ singlets. Furthermore, the field strength tensors are defined by

$$\begin{aligned}
G_{\mu\nu}^A &= \partial_\mu G_\nu^A - \partial_\nu G_\mu^A - \mathrm{i} g_s f^{ABC} G_\mu^B G_\nu^C \; , \\
W_{\mu\nu}^a &= \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - \mathrm{i} g \, \varepsilon^{abc} A_\mu^b A_\nu^c \; , \\
B_{\mu\nu} &= \partial_\mu B_\nu \; - \partial_\nu B_\mu \; ,
\end{aligned} \tag{6.1.7}$$

and $\Gamma_\mathrm{D}^{(d)}$, $\Gamma_\mathrm{D}^{(u)}$ and $\Gamma_\mathrm{D}^{(l)}$ label the complex 3×3 Yukawa matrices.

6.1.1. Tree-Level Masses

If we assume that the Higgs field acquires a vacuum expectation value of the form

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \langle h_3 \rangle \end{pmatrix} = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; , \tag{6.1.8}$$

we obtain the following gauge boson mass term from the kinetic scalar term:

$$\mathcal{L}_m^{(g)} = \frac{v^2}{8} \left[g^2 \left(A_1^\mu + \mathrm{i} A_2^\mu \right) \left(A_1^\mu - \mathrm{i} A_2^\mu \right) + \left(g' B^\mu - g A_2^\mu \right)^2 \right] \; . \tag{6.1.9}$$

The mass eigenstates are then given by

$$W_\mu^\pm = \frac{W_{1\mu} \mp \mathrm{i} W_{2\mu}}{\sqrt{2}}$$

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{bmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{bmatrix} \begin{pmatrix} B_\mu \\ W_{3\mu} \end{pmatrix} \; , \tag{6.1.10}$$

where θ_W labels the Weinberg angle,

$$\cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}} \; , \tag{6.1.11}$$

and the corresponding masses are

$$M_W^2 = \frac{1}{4} g^2 v^2 \quad \text{and} \quad M_Z^2 = \frac{1}{4} \left(g^2 + g'^2 \right) v^2 \ . \tag{6.1.12}$$

The Yukawa generated Dirac mass term for the leptons read

$$\mathcal{L}_D^{(l)} = -\bar{l}_{\mathrm{L}} \frac{v}{\sqrt{2}} \Gamma_{\mathrm{D}}^{(l)} l_{\mathrm{R}} + \text{h.c.} \ , \tag{6.1.13}$$

and after a suitable unitary transformation,

$$\begin{aligned} U_{\mathrm{R}}^{(l)} l_{\mathrm{R}} &= l_{\mathrm{R}}' \\ U_{\mathrm{L}}^{(l)} L &= L' \ , \end{aligned} \tag{6.1.14}$$

where l_{R}' and L' denote the mass eigenfields, the diagonalized leptonic mass matrix is given by:

$$M_{\mathrm{D}}^{(l)} = U_{\mathrm{L}}^{(l)} \frac{v}{\sqrt{2}} \Gamma_{\mathrm{D}}^{(l)} U_{\mathrm{R}}^{(l)\dagger} = \mathrm{diag}(m_e, m_\mu, m_\tau) \ . \tag{6.1.15}$$

If we rewrite the Lagrangian density in terms of these leptonic mass eigenstates, all the unitary transformation matrices cancel and thus no interactions between different generations of leptons appear.

Furthermore, we find by analogy with the leptonic case for quark fields

$$\mathcal{L}_D^{(q)} = -\overline{D}_{\mathrm{L}} \frac{v}{\sqrt{2}} \Gamma_{\mathrm{D}}^{(d)} D_{\mathrm{R}} - \overline{U}_{\mathrm{L}} \frac{v}{\sqrt{2}} \Gamma_{\mathrm{D}}^{(u)} U_{\mathrm{R}} + \text{h.c.} \ , \tag{6.1.16}$$

and introducing the mass eigenstates

$$\begin{aligned} V_{\mathrm{R}}^{(d)} D_{\mathrm{R}} &= D_{\mathrm{R}}' \\ V_{\mathrm{L}}^{(d)} D_{\mathrm{L}} &= D_{\mathrm{L}}' \\ V_{\mathrm{R}}^{(u)} U_{\mathrm{R}} &= U_{\mathrm{R}}' \\ V_{\mathrm{L}}^{(u)} U_{\mathrm{L}} &= U_{\mathrm{L}}' \ , \end{aligned} \tag{6.1.17}$$

yields the diagonalized quark mass matrices:

$$\begin{aligned} M_{\mathrm{D}}^{(u)} &= V_{\mathrm{L}}^{(u)} \frac{v}{\sqrt{2}} \Gamma_{\mathrm{D}}^{(u)} V_{\mathrm{R}}^{(l)\dagger} = \mathrm{diag}(m_u, m_c, m_t) \\ M_{\mathrm{D}}^{(d)} &= V_{\mathrm{L}}^{(d)} \frac{v}{\sqrt{2}} \Gamma_{\mathrm{D}}^{(d)} V_{\mathrm{R}}^{(d)\dagger} = \mathrm{diag}(m_d, m_s, m_b) \ . \end{aligned} \tag{6.1.18}$$

This time, rewriting the Lagrangian density with the help of the quark mass eigenfields leads to the CP-violationg term

$$\frac{g}{\sqrt{2}} \gamma^\mu W_\mu^+ \overline{U'}_{\mathrm{L}} \underbrace{V_{\mathrm{L}}^{(u)\dagger} V_{\mathrm{L}}^{(d)}}_{V_{\mathrm{CKM}}} D_{\mathrm{L}}' + \text{h.c.} \ , \tag{6.1.19}$$

where V_{CKM} labels the unitary Cabibbo-Kobayashi-Maskawa matrix [41, 42], and thus transitions between different generations of quark fields can occur.

6.1.2. Scalon Mass

Since the SM Higgs boson, $h_3 := h_{\mathrm{SM}}$, is the only massive scalar particle of such a theory, it is clear that we have to identify it with the scalon. Therefore, at tree level massless, it should acquire its mass (125.09 GeV) by radiative corrections at one-loop order[3]:

$$(125.09\mathrm{GeV})^2 \overset{!}{=} m_s^2 = 8B\left\langle\sigma\right\rangle^2 = \frac{1}{8\pi^2\left\langle\sigma\right\rangle^2}\left[\underbrace{3\cdot\left(2\cdot M_W^4 + M_Z^4\right) - 4\cdot 3\cdot m_t^4}_{-(318\,\mathrm{GeV})^4}\right] < 0\,.$$

(6.1.20)

Obviously, this result is in contradiction to the requirement $B > 0$ for finding a minimum of the potential and we are not able to reproduce the experimentally measured Higgs mass. Hence, a classically scale invariant version of the SM is clearly excluded by mass measurements.

The failure of this reduced Standard Model is caused by the dominating top quark mass. Nevertheless, this negative fermionic contribution could be compensated by additional scalars (extended Higgs sector) and/or gauge bosons (extended gauge group).

Out of the vast amount of possible classically scale invariant extentions of the SM, we choose to concentrate on a nonabelian extension (see for example [8–14]) for the following discussion. There, we first introduce an extra SU(2) gauge group, as well as a corresponding scalar doublet, and add a real scalar singlet in a second step to be able to implement the seesaw mechanism and to explain nonzero neutrino masses.

6.2. A Classically Scale Invariant SU(2) Extension of the Standard Model

We consider the classically scale invariant version of the SM from above and extend the SM gauge group $\mathcal{G}_{\mathrm{SM}}$ by an additional $\mathrm{SU}(2)_{\mathrm{X}}$ symmetry,

$$\mathrm{SU}(3)_{\mathrm{c}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}} \longrightarrow \mathrm{SU}(3)_{\mathrm{c}} \times \mathrm{SU}(2)_{\mathrm{X}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}\,,$$

(6.2.1)

under which all the SM particles act like singlets. Furthermore, we introduce a new scalar boson Φ as a doublet with respect to $\mathrm{SU}(2)_{\mathrm{X}}$, which transforms trivially under $\mathcal{G}_{\mathrm{SM}}$. Hence, the whole Lagrangian density is given by[4]

$$\mathcal{L} = \mathcal{L}'_{\mathrm{SM}} - \frac{1}{4}X_{\mu\nu}X^{\mu\nu} + \left(\tilde{D}_\mu\Phi\right)^\dagger\left(\tilde{D}^\mu\Phi\right) - V_0(\Phi, H)\,,$$

(6.2.2)

with

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + \mathrm{i}\phi_2 \\ \phi_3 + \mathrm{i}\phi_4\end{pmatrix} \quad\text{and}\quad H = \frac{1}{\sqrt{2}}\begin{pmatrix}h_1 + \mathrm{i}h_2 \\ h_3 + \mathrm{i}h_4\end{pmatrix}\,.$$

(6.2.3)

The covariant derivative has the form

$$\tilde{D}^\mu\Phi_i = \partial^\mu\Phi_i + \mathrm{i}g_{\mathrm{x}}X_a^\mu\frac{(\tau_a)_{ik}}{2}\Phi_k\,,$$

(6.2.4)

---

[3]The factor 3 in front of the top quark mass arises from the three different color charges and all the other fermion masses are neglected due to their smallness. The particle masses can be found in Table 2.1.

[4]We define $\mathcal{L}_{\mathrm{SM}} = \mathcal{L}'_{\mathrm{SM}} - V_{\mathrm{SM}}$.

with the new coupling constant g_{x} and the $\mathrm{SU}(2)_{\mathrm{x}}$ related gauge fields[5] X_a, and the corresponding field strength tensor $X^{\mu\nu}$ is defined by

$$X^{\mu\nu} = \partial^\mu X_a^\nu - \partial^\nu X_a^\mu - g_{\mathrm{x}}\varepsilon_{abc}X_b^\mu X_c^\nu \ . \tag{6.2.5}$$

The classically scale invariant potential reads

$$V_0(\Phi, H) = \lambda_\phi \left(\Phi^\dagger \Phi\right)^2 + \lambda_H \left(H^\dagger H\right)^2 - 2\lambda_p \left(\Phi^\dagger \Phi\right)\left(H^\dagger H\right) \ , \tag{6.2.6}$$

where λ_ϕ, λ_H and λ_p denote dimensionless and real[6] coupling constants and since the field Φ interacts with the Standard Model only via the mixing term $-2\lambda_p \left(\Phi^\dagger \Phi\right)\left(H^\dagger H\right)$, this interaction is sometimes referred to as Higgs portal.

We can decompose the two complex scalar doublets into their real components,

$$\tilde{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \quad \text{and} \quad \tilde{H} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} \ , \tag{6.2.7}$$

or

$$\varphi = \begin{pmatrix} \tilde{\Phi} \\ \tilde{H} \end{pmatrix} \ , \tag{6.2.8}$$

and rewrite the potential as

$$\boxed{V_0(\tilde{\Phi}, \tilde{H}) = \frac{1}{4}\lambda_\phi \left(\tilde{\Phi}_i\tilde{\Phi}_i\right)^2 + \frac{1}{4}\lambda_H \left(\tilde{H}_i\tilde{H}_i\right)^2 - \frac{1}{2}\lambda_p \left(\tilde{\Phi}_i\tilde{\Phi}_i\right)\left(\tilde{H}_k\tilde{H}_k\right) \ ,} \tag{6.2.9}$$

which enables us to apply the Gildener-Weinberg approach, as described in the last chapter.

Firstly, we realize that we can use the $\mathrm{SU}(2)$ and $\mathrm{SU}(2)_{\mathrm{X}}$ symmetry to restrict the flat direction of the tree-level potential to the plane

$$\varphi^{\mathrm{flat}} = n\sigma = \begin{pmatrix} 0 \\ 0 \\ n_1 \\ 0 \\ 0 \\ 0 \\ n_2 \\ 0 \end{pmatrix} \cdot \sigma \,\hat{=}\, \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \cdot \sigma \ , \tag{6.2.10}$$

so that we are able to simplify the above potential along this flat direction to

$$V_0(\varphi^{\mathrm{flat}}) \propto \frac{1}{4}\lambda_\phi n_1^4 + \frac{1}{4}\lambda_H n_2^4 - \frac{1}{2}\lambda_p n_1^2 n_2^2 \ . \tag{6.2.11}$$

From equation (5.0.2) follows that the conditions

$$\begin{aligned} \mathrm{I}: & \quad \lambda_\phi \left(\Lambda_{\mathrm{GW}}\right) n_1^2 - \lambda_p \left(\Lambda_{\mathrm{GW}}\right) n_2^2 = 0 \\ \mathrm{II}: & \quad \lambda_H \left(\Lambda_{\mathrm{GW}}\right) n_2^2 - \lambda_p \left(\Lambda_{\mathrm{GW}}\right) n_1^2 = 0 \end{aligned} \tag{6.2.12}$$

---

[5]As the gauge boson X_a does not interact with the SM, we sometimes refer to it as dark boson.
[6]The action has to be a real quantity.

have to be fulfilled to obtain $V_0(\varphi^{\text{flat}}) = 0$. Therefore, the GW condition at the GW scale reads

$$\lambda_p^2 \left(\Lambda_{\text{GW}}\right) = \lambda_\phi \left(\Lambda_{\text{GW}}\right) \lambda_H \left(\Lambda_{\text{GW}}\right) , \tag{6.2.13}$$

and we find[7]

$$\varphi^{\text{flat}} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \cdot \sigma = \begin{pmatrix} \sqrt{\frac{\lambda_p}{\lambda_p + \lambda_\phi}} \\ \sqrt{\frac{\lambda_\phi}{\lambda_p + \lambda_\phi}} \end{pmatrix} \cdot \sigma := \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} \cdot \sigma \quad, \sigma > 0 \,, \, 0 \leq \alpha \leq \frac{\pi}{2} \,, \tag{6.2.14}$$

where we have dropped the explicit scale dependence of the coupling constants in the last equation for brevity and from now keep in mind that $\lambda \equiv \lambda \left(\Lambda_{\text{GW}}\right)$. Furthermore, the conditions (6.2.12) can only be nontrivially satisfied if

$$\text{sgn}(\lambda_p) = \text{sgn}(\lambda_H) = \text{sgn}(\lambda_\phi) . \tag{6.2.15}$$

The vacuum expectation values of the scalar fields, defined by the minimum of the potential at one-loop level, can be expressed as[8]

$$\langle \varphi \rangle = \begin{pmatrix} \langle \phi_3 \rangle \\ \langle h_3 \rangle \end{pmatrix} := \begin{pmatrix} \langle \phi \rangle \\ \langle h \rangle \end{pmatrix} = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} \cdot \langle \sigma \rangle . \tag{6.2.16}$$

Since only the scalar field h interacts with the particles of the Standard Model and possesses a nontrivial VEV, the mass of the W-boson is, after symmetry breaking, given by

$$m_{\text{W}}^2 = \frac{1}{4} g^2 \langle h \rangle^2 , \tag{6.2.17}$$

and with the definition of the Fermi coupling constant [43],

$$G_{\text{F}} = \frac{\sqrt{2} g^2}{8 m_{\text{W}}^2} = 1.166 \cdot 10^{-5} \, (\text{GeV})^{-2} , \tag{6.2.18}$$

we find

$$\langle h \rangle = \sin(\alpha) \langle \sigma \rangle = 246 \, \text{GeV} , \tag{6.2.19}$$

and hence:

$$\langle \phi \rangle = \frac{246 \, \text{GeV}}{\tan(\alpha)} \quad , \quad \langle \sigma \rangle = \frac{246 \, \text{GeV}}{\sin(\alpha)} . \tag{6.2.20}$$

Furthermore, the mass term of the dark gauge bosons X_a^μ reads (compare (6.1.9))

$$\mathcal{L}_m^{(g_x)} = \frac{\langle \phi \rangle^2}{8} g_x^2 \left(X_1^2 + X_2^2 + X_3^2\right) , \tag{6.2.21}$$

and thus the masses of the dark gauge bosons are degenerated:

$$M_X^2 = \frac{1}{4} g_x^2 \langle \phi \rangle^2 . \tag{6.2.22}$$

---

[7]Actually, one finds four different flat directions (compare figure 5.1), but all of them are physically identical and we choose to work with $n_i \geq 0$ and $\sigma > 0$.

[8]We assume $\delta\varphi$ to be small enough to safely ignore it.

The second derivatives of the tree-level potential are given by

$$\left(\frac{\partial^2 V_0(\tilde{\Phi}, \tilde{H})}{\partial \varphi_i \partial \varphi_k}\right) = \begin{bmatrix} \lambda_\phi \left((\tilde{\Phi}^{\mathrm{T}}\tilde{\Phi} - \frac{\lambda_p}{\lambda_\phi}\tilde{H}^{\mathrm{T}}\tilde{H})\, \mathbb{1} + 2\tilde{\Phi}\tilde{\Phi}^{\mathrm{T}}\right) & -2\lambda_p\tilde{H}\tilde{\Phi}^{\mathrm{T}} \\ -2\lambda_p\tilde{\Phi}\tilde{H}^{\mathrm{T}} & \lambda_H \left((\tilde{H}^{\mathrm{T}}\tilde{H} - \frac{\lambda_p}{\lambda_H}\tilde{\Phi}^{\mathrm{T}}\tilde{\Phi})\, \mathbb{1} + 2\tilde{H}\tilde{H}^{\mathrm{T}}\right) \end{bmatrix},$$

$$(6.2.23)$$

which, when evaluated at the vacuum expectation value $n\langle\sigma\rangle$, yields the squared scalar tree-level mass matrix (see (5.0.19)):

$$m_\varphi^2 = \left(\frac{\partial^2 V_0}{\partial \varphi_i \partial \varphi_k}\right)\bigg|_{n\langle\sigma\rangle}. \qquad (6.2.24)$$

Considering the conditions from (6.2.12), this mass matrix reduces to the simpler form

$$m_\varphi^2 = \begin{bmatrix} 3\lambda_\phi n_1^2 - \lambda_p n_2^4 & -2\lambda_p n_1 n_2 \\ -2\lambda_p n_1 n_2 & 3\lambda_H n_2^2 - \lambda_p n_1^2 \end{bmatrix} \cdot \langle\sigma\rangle^2, \qquad (6.2.25)$$

where all other entries of the 8×8 matrix vanish and n_1 and n_2 are defined by equation (6.2.14). Therefore, we find 6 massless Goldstone bosons and the two eigenvalues

$$\boxed{m_s^2 = 0 \quad \text{and} \quad m_h^2 = 2\lambda_p\langle\sigma\rangle^2} \qquad (6.2.26)$$

of the remaining matrix correspond to the squared tree-level masses of the scalon field h_s and a second Higgs field h_m respectively, where the requirement of positive definiteness restricts λ_p to be positive. The mass of the scalon is then generated at one-loop order and reads for this model

$$\boxed{m_s^2 = 8B\langle\sigma\rangle^2 = \frac{1}{8\pi^2\langle\sigma\rangle^2}\left[3 \cdot \left(2 \cdot M_W^4 + M_Z^4 + 3 \cdot M_X^4\right) + m_h^4 - 4 \cdot 3 \cdot m_t^4\right] \overset{!}{>} 0.}$$

$$(6.2.27)$$

Furthermore, the scalar flavour eigenstates ϕ and h can be written as a mixture of the mass eigenstates h_s and h_m

$$\boxed{\begin{aligned}\begin{pmatrix} \phi \\ h \end{pmatrix} &= \begin{pmatrix} \langle\phi\rangle + \phi' \\ \langle h\rangle + h' \end{pmatrix} = n\langle\sigma\rangle + nh_s + n_\perp h_m \\ &= \begin{pmatrix} \langle\phi\rangle + \cos(\alpha)\, h_s - \sin(\alpha)\, h_m \\ \langle h\rangle + \sin(\alpha)\, h_s + \cos(\alpha)\, h_m \end{pmatrix}.\end{aligned}} \qquad (6.2.28)$$

Therefore, the two scalar mass eigenstates are both candidates for the Standard Model Higgs field h_{SM}, where it is far from clear which of them has to be linked with the SM Higgs mass of 125 GeV.

Before we investigate these two possible scenarios, we first want to derive the beta functions for all the coupling parameters of this model.

6.2.1. Running Couplings

We will catch up on the discussion about renormalization of the effective potential, which we skipped in section 4.2 and chapter 5 and concentrate on the pure scalar part of the

Lagrangian density,

$$\mathcal{L}_{\text{scalar}} = \left(\partial_\mu \begin{pmatrix} \phi_0 \\ h_0 \end{pmatrix} \right)^{\mathrm{T}} \left(\partial^\mu \begin{pmatrix} \phi_0 \\ h_0 \end{pmatrix} \right) \underbrace{- \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \lambda_\phi^0 \, \phi_0^4 - \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \lambda_H^0 \, h_0^4 + \frac{1}{2} \tilde{\Lambda}^{2\epsilon} \lambda_p^0 \, \phi_0^2 \, h_0^2}_{-V_0(\varphi)} \, , \quad (6.2.29)$$

where we have explicitly displayed the bar quantities and work in $d = 4 - 2\epsilon$ dimensions. Introducing the scale-dependent renormalized scalar fields and coupling constants up to one-loop order,

$$\begin{pmatrix} \phi_0 \\ h_0 \end{pmatrix} = Z_\varphi^{\frac{1}{2}} \begin{pmatrix} \phi \\ h \end{pmatrix} = \begin{bmatrix} 1 + \frac{1}{2} \delta Z_{\phi\phi} & \delta Z_{\phi h}^{\frac{1}{2}} \\ \delta Z_{h\phi}^{\frac{1}{2}} & 1 + \frac{1}{2} \delta Z_{hh} \end{bmatrix} \begin{pmatrix} \phi \\ h \end{pmatrix} \, , \qquad (6.2.30a)$$

$$\begin{aligned}
\lambda_\phi^0 &= Z_{\lambda_\phi} \lambda_\phi &&= \left(1 + \delta Z_{\lambda_\phi} \right) \lambda_\phi \, , \\
\lambda_H^0 &= Z_{\lambda_H} \lambda_H &&= \left(1 + \delta Z_{\lambda_H} \right) \lambda_H \, , \\
\lambda_p^0 &= Z_{\lambda_p} \lambda_p &&= \left(1 + \delta Z_{\lambda_p} \right) \lambda_p \, ,
\end{aligned} \qquad (6.2.30b)$$

leads to

$$\begin{aligned}
\mathcal{L}_{\text{scalar}} = & \left(\partial_\mu \begin{pmatrix} \phi \\ h \end{pmatrix} \right)^{\mathrm{T}} \left(\partial^\mu \begin{pmatrix} \phi \\ h \end{pmatrix} \right) - \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \lambda_\phi \, \phi^4 - \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \lambda_H \, h^4 + \frac{1}{2} \tilde{\Lambda}^{2\epsilon} \lambda_p \, \phi^2 \, h^2 \\
& + \left(\partial_\mu \begin{pmatrix} \phi \\ h \end{pmatrix} \right)^{\mathrm{T}} \begin{bmatrix} \delta Z_{\phi\phi} & \delta Z_{\phi h}^{\frac{1}{2}} + \delta Z_{h\phi}^{\frac{1}{2}} \\ \delta Z_{\phi h}^{\frac{1}{2}} + \delta Z_{h\phi}^{\frac{1}{2}} & \delta Z_{hh} \end{bmatrix} \left(\partial^\mu \begin{pmatrix} \phi \\ h \end{pmatrix} \right) - \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \left(\delta Z_{\lambda_H} + 2\delta Z_{hh} \right) \lambda_H h^4 \\
& - \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \left(\delta Z_{\lambda_\phi} + 2\delta Z_{\phi\phi} \right) \lambda_\phi \phi^4 + \frac{1}{2} \tilde{\Lambda}^{2\epsilon} \left(\delta Z_{\lambda_p} + \delta Z_{hh} + \delta Z_{\phi\phi} \right) \lambda_p \, \phi^2 \, h^2 \\
& - \tilde{\Lambda}^{2\epsilon} \left(\lambda_\phi - \lambda_p \right) \delta Z_{h\phi}^{\frac{1}{2}} \phi^3 h - \tilde{\Lambda}^{2\epsilon} \left(\lambda_H - \lambda_p \right) \delta Z_{\phi h}^{\frac{1}{2}} \phi h^3 \, .
\end{aligned} \qquad (6.2.31)$$

As demonstrated in section 4.2, the effective potential comes at one-loop level along with the counterterm (compare equation (4.2.58))

$$\mathrm{CT} = \frac{1}{64\pi^2} \mathrm{Tr} \left[- \mathrm{CT}_\varphi - \mathrm{CT}_A + 2\mathrm{CT}_\psi \right] \, , \qquad (6.2.32)$$

where we find for our model[9]

$$\begin{aligned}
\mathrm{Tr} \, \mathrm{CT}_\varphi &\overset{(4.2.34)}{=} \tilde{\Lambda}^{-2\epsilon} \frac{1}{\epsilon} \, \mathrm{Tr} \left(\frac{\partial^2 V_0}{\partial \varphi_i \partial \varphi_k} \right)^2 \overset{(6.2.23)}{=} \tilde{\Lambda}^{2\epsilon} \frac{1}{\epsilon} \Bigg[\left(12\lambda_\phi^2 + 4\lambda_p^2 \right) \phi^4 + \left(12\lambda_H^2 + 4\lambda_p^2 \right) h^4 \\
& \hspace{6cm} + \left(8\lambda_p^2 - 12\lambda_\phi \lambda_p - 12\lambda_H \lambda_p \right) \phi^2 h^2 \Bigg] \, , \\[4pt]
\mathrm{Tr} \, \mathrm{CT}_A &\overset{(4.2.46)}{=} \tilde{\Lambda}^{-2\epsilon} \frac{3}{\epsilon} \, \mathrm{Tr} \, M_A^4 \overset{(4.2.18)}{=} \tilde{\Lambda}^{2\epsilon} \frac{3}{\epsilon} \left[\frac{1}{16} \left(3g^4 + 2g^2 g'^2 + g'^4 \right) h^4 + \frac{3}{16} g_x^4 \phi^4 \right] \, , \\[4pt]
\mathrm{Tr} \, \mathrm{CT}_\psi &\overset{(4.2.56)}{=} \tilde{\Lambda}^{-2\epsilon} \frac{1}{\epsilon} \mathrm{Tr} \left(M_\psi M_\psi^\dagger \right)^2 \overset{(4.2.22)}{=} \tilde{\Lambda}^{2\epsilon} \frac{6}{\epsilon} \frac{y_t^4}{4} h^4 \, .
\end{aligned} \qquad (6.2.33)$$

---

[9]Since the counterterms appear only at one-loop order, we don't have to distinguish between bare and renormalized quantities and can work with the renormalized ones.

Since we want to obtain a finite effective potential

$$V = V_0 + V_1 + \text{CT} \stackrel{!}{=} \text{finite} \tag{6.2.34}$$

the following relations have to hold in the $\overline{\text{MS}}$ scheme[10]:

$$\left(\delta Z_{\lambda_\phi} + 2\delta Z_{\phi\phi}\right)\lambda_\phi = \frac{1}{\epsilon}\frac{1}{16\pi^2}\left[12\lambda_\phi^2 + 4\lambda_p^2 + \frac{9}{16}g_x^4\right]. \tag{6.2.35a}$$

$$\left(\delta Z_{\lambda_H} + 2\delta Z_{\phi\phi}\right)\lambda_H = \frac{1}{\epsilon}\frac{1}{16\pi^2}\left[12\lambda_H^2 + 4\lambda_p^2 + \frac{3}{16}\left(3g^4 + 2g^2g'^2 + g'^4\right) - 3y_t^4\right]. \tag{6.2.35b}$$

$$\left(\delta Z_{\lambda_p} + \delta Z_{hh} + \delta Z_{\phi\phi}\right)\lambda_p = \frac{1}{\epsilon}\frac{1}{32\pi^2}\left[-8\lambda_p^2 + 12\lambda_\phi\lambda_p + 12\lambda_H\lambda_p\right]. \tag{6.2.35c}$$

The field strength renormalization counterterms $\delta Z_{\phi\phi}$ and δZ_{hh} can be calculated with the help of the scalar self energy $\Pi(k^2)$ (see appendix A.2) and read

$$\delta Z_{\phi\phi} = \frac{1}{\epsilon}\frac{9g_x^2}{64\pi^2}$$

$$\delta Z_{hh} = \frac{1}{\epsilon}\frac{9g^2 + 3g'^2 - 12y_t^2}{64\pi^2}. \tag{6.2.36}$$

Since $\tilde{\Lambda}^{2\epsilon}\lambda_0$ has to be scale-invariant, we find

$$0 = \frac{\mathrm{d}}{\mathrm{d}\ln\Lambda}\left(\tilde{\Lambda}^{2\epsilon}\lambda_0\right) = \frac{\mathrm{d}}{\mathrm{d}\ln\Lambda}\left(\tilde{\Lambda}^{2\epsilon}Z_\lambda\lambda\right)$$

$$= 2\epsilon\,\tilde{\Lambda}^{2\epsilon}Z_\lambda\lambda + \tilde{\Lambda}^{2\epsilon}\frac{\mathrm{d}\delta Z_\lambda}{\mathrm{d}\ln\Lambda}\lambda + \tilde{\Lambda}^{2\epsilon}\left(1 + \delta Z_\lambda\right)\frac{\mathrm{d}\lambda}{\mathrm{d}\ln\Lambda}, \tag{6.2.37}$$

and hence, we get up to one-loop order

$$\beta(\lambda) := \frac{\mathrm{d}\lambda}{\mathrm{d}\ln\Lambda} = -2\epsilon\lambda - \frac{\mathrm{d}\delta Z_\lambda}{\mathrm{d}\ln\Lambda}\lambda. \tag{6.2.38}$$

In addition to that, this relation reads for gauge and Yukawa couplings at tree level

$$\beta(g) = \frac{\mathrm{d}g}{\mathrm{d}\ln\Lambda} = -\epsilon\,g \quad \text{and} \quad \beta(y_t) = \frac{\mathrm{d}y_t}{\ln\Lambda} = -\epsilon\,y_t. \tag{6.2.39}$$

Therefore, the derivation of equation (6.2.35a) with respect to $\ln\Lambda$ gives

$$-\beta(\lambda_\phi) - 2\epsilon\lambda_\phi + 2\underbrace{\frac{\mathrm{d}\,\delta Z_{\phi\phi}}{\mathrm{d}\ln\Lambda}}_{-2\cdot\frac{9g_x^2}{64\pi^2}}\lambda_\phi - 2\epsilon\underbrace{\left(\delta Z_{\lambda_\phi} + 2\,\delta Z_{\phi\phi}\right)\lambda_\phi}_{\frac{1}{\epsilon}\frac{1}{16\pi^2}\left(12\lambda_\phi^2 + 4\lambda_p^2 + \frac{9}{16}g_x^4\right)} = \frac{1}{\epsilon}\frac{1}{16\pi^2}4\epsilon\left(-12\lambda_\phi^2 - 4\lambda_p^2 - \frac{9}{16}g_x^4\right),$$

$$\tag{6.2.40}$$

which yields in the limit $\epsilon \to 0$:

$$\boxed{\beta(\lambda_\phi) = \frac{1}{(4\pi)^2}\left[-9\lambda_\phi g_x^2 + 24\lambda_\phi^2 + 8\lambda_p^2 + \frac{9}{8}g_x^4\right].} \tag{6.2.41}$$

---

[10]In addition to these relations we also find: $\delta Z_{\phi h} = \delta Z_{h\phi} = 0$

In exactly the same way as above we find

$$\beta(\lambda_H) = \frac{1}{(4\pi)^2}\left[\left(12y_t^2 - 9g^2 - 3g'^2\right)\lambda_H + 24\lambda_H^2 + 8\lambda_p^2 + \frac{3}{8}\left(3g^4 + 2g'^2g^2 + g'^4\right) - 6y_t^4\right],$$
(6.2.42)

and

$$\beta(\lambda_p) = \frac{1}{(4\pi)^2}\lambda_p\left[-\frac{9}{2}g_x^2 - \frac{9}{2}g^2 - \frac{3}{2}g'^2 + 6y_t^2 + 12\lambda_\phi + 12\lambda_H - 8\lambda_p\right].$$
(6.2.43)

These one-loop renormalization group equations (RGEs) are, despite of a slightly different definition of the coupling constants, in perfect agreement with the results found in [9] and [10]. The initial conditions for these scalar couplings will be defined at the Gildener-Weinberg scale $\Lambda = \Lambda_{\mathrm{GW}}$.

The beta function of the gauge coupling g_x can be calculated with the help of the general result for a SU(N) gauge group ([44], [45])

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}N - n_f\frac{4}{3}T(R) - n_s\frac{1}{6}T(R)\right],$$
(6.2.44)

where n_f and n_s are the number of fermions and real scalars (without Goldstone bosons), which transform under this gauge group, and $T(R)$ is defined by the trace of the generators $\mathcal{T}_a$ in the corresponding representation:

$$\mathrm{Tr}\left[\mathcal{T}_a\mathcal{T}_b\right] = T(R)\delta_{ab}.$$
(6.2.45)

For our SU(2)$_X$ gauge group[11] we therefore find $N = 2$, $n_f = 0$, $n_s = 1$, $T(R) = 1$ and thus

$$\beta(g_x) = -\frac{1}{(4\pi)^2}\frac{43}{6}g_x^3,$$
(6.2.46)

where we define:

$$g_x(\Lambda = M_X) = \frac{2M_X}{\langle\phi\rangle}.$$
(6.2.47)

The Standard Model couplings are not affected by the introduction of the new gauge group and the corresponding beta functions are hence given by (see [46])

$$\beta(g') = \frac{1}{(4\pi)^2}\frac{41}{6}g'^3$$

$$\beta(g) = \frac{1}{(4\pi)^2}\left(-\frac{19}{6}\right)g^3$$

$$\beta(g_s) = \frac{1}{(4\pi)^2}(-7)g_s^3,$$
(6.2.48)

---

[11]For real scalars the generators read: $\mathcal{T}_1 = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}$, $\mathcal{T}_2 = \frac{1}{2}\begin{pmatrix} 0 & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & 0 \end{pmatrix}$ and

$\mathcal{T}_3 = \frac{1}{2}\begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}$.

and

$$\beta(y_t) = \frac{1}{(4\pi)^2}\left(\frac{9y_t^2}{2} - \frac{17g'^2}{12} - \frac{9g^2}{4} - 8g_s^2\right)y_t \ . \tag{6.2.49}$$

The initial conditions for the Standard Model coupling constants at $\Lambda = M_t$ are taken from [46]:

$$\begin{aligned} g'(\Lambda = M_t) &= 0.358 \\ g(\Lambda = M_t) &= 0.648 \\ g_s(\Lambda = M_t) &= 1.167 \\ y_t(\Lambda = M_t) &= 0.937 \ . \end{aligned} \tag{6.2.50}$$

6.2.2. General Constraints

As we know from equation (6.2.28), the scalar field h' can be written as a linear combination of the mass eigenstates h_s and h_m,

$$h' = \sin(\alpha)\,h_s + \cos(\alpha)\,h_m \ , \tag{6.2.51}$$

and interacts, unlike the second scalar field ϕ', with the SM particles. Thus, depending on whether h_s or h_m is related to the SM Higgs boson, the couplings of the Higgs boson to the other SM particles get rescaled by $\sin(\alpha)$ or $\cos(\alpha)$. Therefore, we have to claim that the respective scaling factor is of order one to avoid coming into conflict with the experimentally verified predictions of the Standard Model and more precisely, we can use

$$\sin(\alpha) > 0.87 \iff h_s \hat{=} h_{\text{SM}} \tag{6.2.52a}$$

$$\cos(\alpha) > 0.87 \iff h_m \hat{=} h_{\text{SM}} \tag{6.2.52b}$$

as a lower bound for our model [12].

Furthermore, from $B > 0$ follows the condition

$$9M_X^4 + m_h^4 > (318\,\text{GeV})^4 \ , \tag{6.2.53}$$

and the validity of perturbation theory is guaranteed as long as[12]

$$\lambda_H(\Lambda) \ll \frac{8\pi^2}{3}, \ \ \lambda_\phi(\Lambda) \ll \frac{8\pi^2}{3} \ \ \text{and} \ \ \lambda_p(\Lambda) \ll \frac{4\pi^2}{3} \ . \tag{6.2.54}$$

Our tree-level potential (6.2.9) can be rewritten as

$$V_0(\tilde{\phi}, \tilde{H}) = \frac{1}{4}\left(\tilde{H}^2 - \frac{\lambda_p}{\lambda_H}\tilde{\phi}^2\right)^2 + \frac{1}{4}\left(\lambda_\phi - \frac{\lambda_p^2}{\lambda_H}\right)\tilde{\phi}^4 \ , \tag{6.2.55}$$

and thus we can read off the tree-level conditions for vacuum stability:

$$\lambda_H \geq 0 \quad \text{and} \quad \lambda_p^2 \leq \lambda_\phi \lambda_H \ . \tag{6.2.56}$$

---

[12]For a theory with the potential $\frac{\lambda}{4!}\phi^4$ the condition $\lambda \ll (4\pi)^2$ has to be fulfilled.

As argued in [47], these conditions have to hold for large scales even at one-loop order and we already found out that $\lambda_H(\Lambda)$ is positive at the GW scale as well. However, the GW condition for symmetry breaking reads

$$\lambda_p^2(\Lambda_{\text{GW}}) = \lambda_\phi(\Lambda_{\text{GW}})\lambda_H(\Lambda_{\text{GW}}) , \tag{6.2.57}$$

and thus, we have to assume that at least the second vacuum stability condition is violated for scales less than the GW scale and that the potential in this region is stabilized by radiative corrections.

Our extended Lagrangian density (6.2.2) contains four free parameters: λ_ϕ, λ_H, λ_p and g_x. This number can be halved by the condition (6.2.19) for the vacuum expectation value and by the identification of one of the mass eigenstates with the SM Higgs particle h_{SM}. In addition to that, it would also be possible to trade one of the coupling constants for the Gildener-Weinberg scale Λ_{GW} (Dimensional transmutation). Therefore, we are now going to investigate this 2-dimensional parameter space for our two different possible choices of interpreting the physically observed Higgs boson.

6.2.3. Identification of the Mass Eigenstate h_m with the Physically Observed Higgs Boson h_{SM}

In this case, we can identify the experimentally measured Higgs mass $M_{h_{\text{SM}}}$ with m_h and find from equation (6.2.20) and (6.2.26)

$$m_h^2 = 2\,\lambda_p \left\langle\sigma\right\rangle^2 = 2\,\lambda_p \frac{\left\langle h\right\rangle^2}{\sin^2(\alpha)} \overset{!}{=} M_{h_{\text{SM}}}^2 , \tag{6.2.58}$$

and thus:

$$\boxed{\lambda_p = \frac{M_{h_{\text{SM}}}^2}{2\left\langle h\right\rangle^2} \cdot \sin^2(\alpha) .} \tag{6.2.59}$$

Using the definition of $\sin(\alpha)$ (see equation (6.2.14)) and the Gildener-Weinberg condition (6.2.13) immediately leads to

$$\boxed{\lambda_\phi = \frac{2\left\langle h\right\rangle^2 \lambda_p^2}{M_{h_{\text{SM}}}^2 - 2\left\langle h\right\rangle^2 \lambda_p} ,} \tag{6.2.60}$$

and

$$\boxed{\lambda_H = \frac{M_{h_{\text{SM}}}^2}{2\left\langle h\right\rangle^2} - \lambda_p .} \tag{6.2.61}$$

Considering (6.2.52b) and the experimentally measured values of $M_{h_{\text{SM}}}^2$ and $\left\langle h\right\rangle$ strongly restrict the allowed parameter space[13]:

$$0 < \lambda_\phi < \lambda_p < 0.031 \tag{6.2.62a}$$

$$0.098 < \lambda_H < 0.129 \tag{6.2.62b}$$

---

[13]Be once more reminded that all these scalar coupling parameters are defined at the Gildener-Weinberg scale Λ_{GW}.

In addition to that, we get from (6.2.53) a lower bound for the mass of the dark boson:

$$M_X = \frac{1}{2} g_x \langle \phi \rangle \geq 183 \text{ GeV} . \tag{6.2.63}$$

We choose λ_p and g_x as our free parameters, which are restricted by condition (6.2.62a) and (6.2.63). The resulting allowed parameter space is displayed in Figure 6.1, where g_x is assumed to be of the order of the other gauge couplings[14]. Further restrictions could be

Figure 6.1.: The dark area of the parameter space is excluded and the blue lines and the dashed red lines represent fixed values of M_X and m_s respectively.
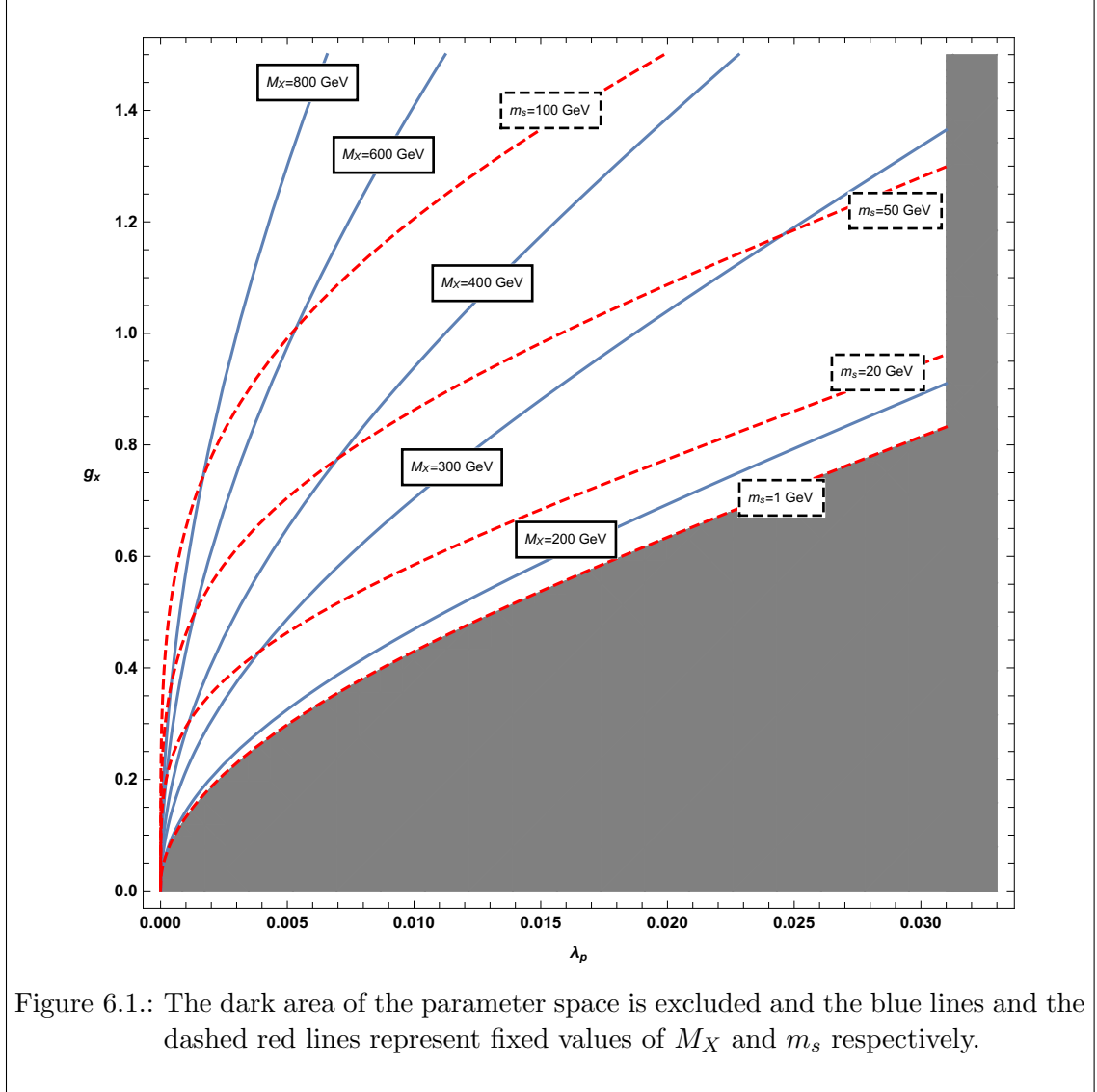
found by considering stability requirements, where one would have to scan over the whole allowed parameter space of Figure 6.1 and perform an analysis of the running of the scalar couplings. Since this would go well beyond the scope of this master's thesis, we refer to [8] and discuss stability only for two exemplary points of the parameter space:

---

[14]A too large gauge coupling would spoil our perturbative analysis.

Example 1: $\lambda_p(\Lambda_{\mathrm{GW}}) = 0.005$ **and** $g_x(M_x) = 1.5$

For these values we find

$$M_X = 923 \text{ GeV} \quad , \quad m_s = 229 \text{ GeV} \quad \text{and} \quad \Lambda_{\mathrm{GW}} = 783 \text{ GeV} , \tag{6.2.64}$$

and the RG evolution is shown in Figure 6.2, where we used the beta functions and initial conditions from subsection 6.2.1. Furthermore, we also display the running of the

Figure 6.2.: Running couplings for $\lambda_p(\Lambda_{\mathrm{GW}}) = 0.005$ and $g_x(M_x) = 1.5$.
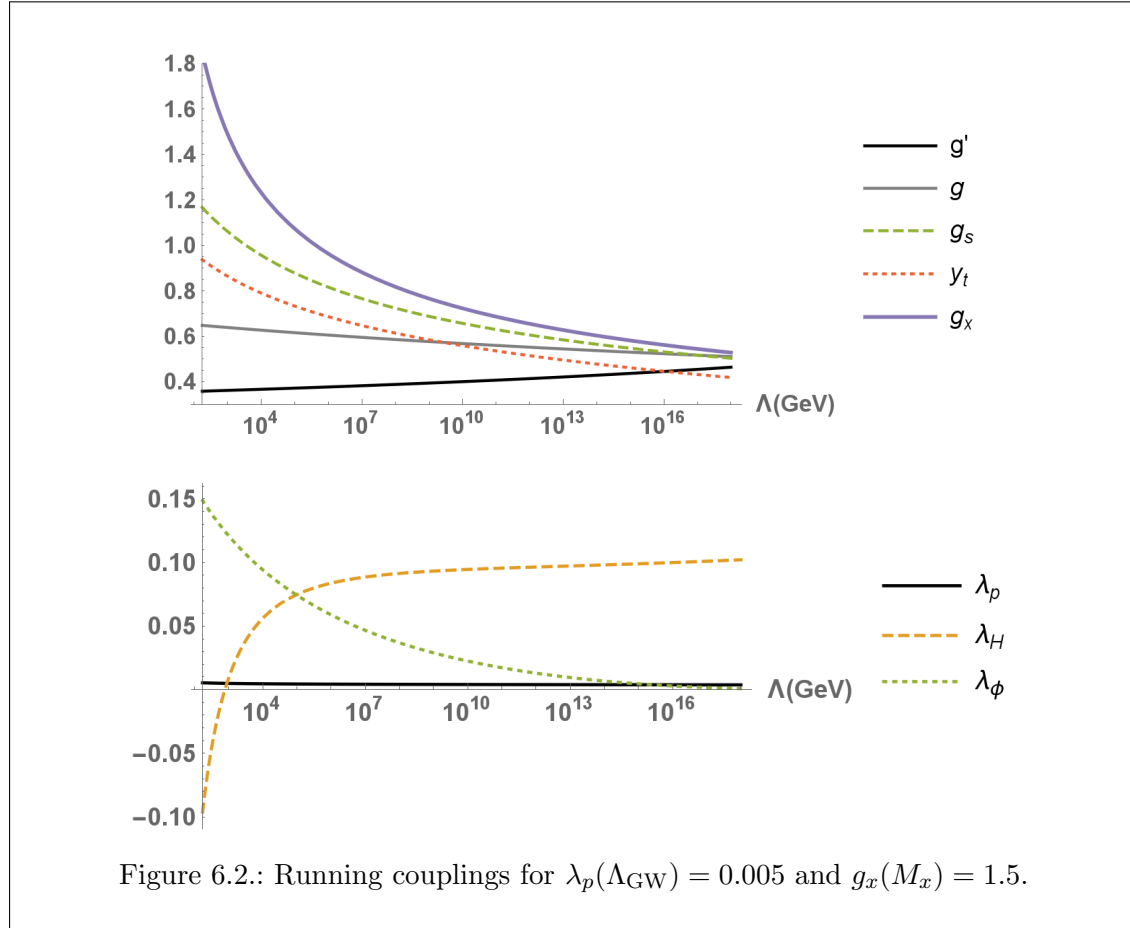
Gildener-Weinberg condition in Figure 6.3 and together with the results from Figure 6.2 this demonstrates that the tree-level conditions for stability (compare (6.2.56)) are fulfilled at one-loop order for large scales, but necessarily violated for $\Lambda < \Lambda_{\mathrm{GW}}$.

Example 2: $\lambda_p(\Lambda_{\mathrm{GW}}) = 0.025$ **and** $g_x(M_x) = 0.9$

We perform our analysis in the same way as before and get

$$M_X = 227 \text{ GeV} \quad , \quad m_s = 24 \text{ GeV} \quad \text{and} \quad \Lambda_{\mathrm{GW}} = 294 \text{ GeV} . \tag{6.2.65}$$

The scale dependence of the couplings and the GW condition is shown in Figure 6.4 and Figure 6.5. In contrast to the first example we find that both tree-level stability conditions are obviously violated for large scales and hence assume the potential to be instable.

Figure 6.3.: Scale dependence of the GW condition for $\lambda_p(\Lambda_{\mathrm{GW}}) = 0.005$ and $g_x(M_x) = 1.5$.

Figure 6.4.: Scale dependence of the GW condition for $\lambda_p(\Lambda_{\mathrm{GW}}) = 0.025$ and $g_x(M_x) = 0.9$.
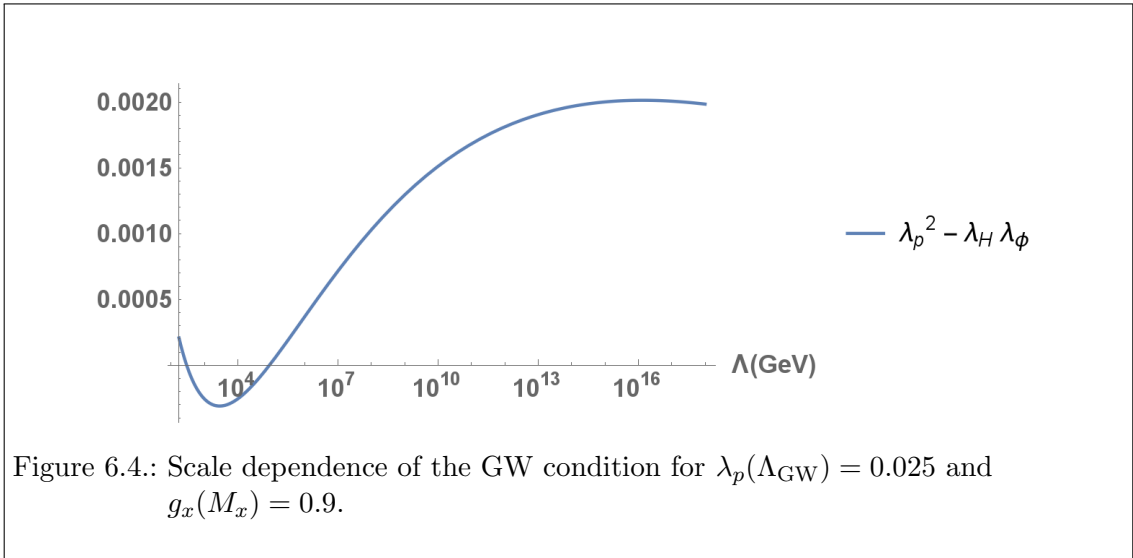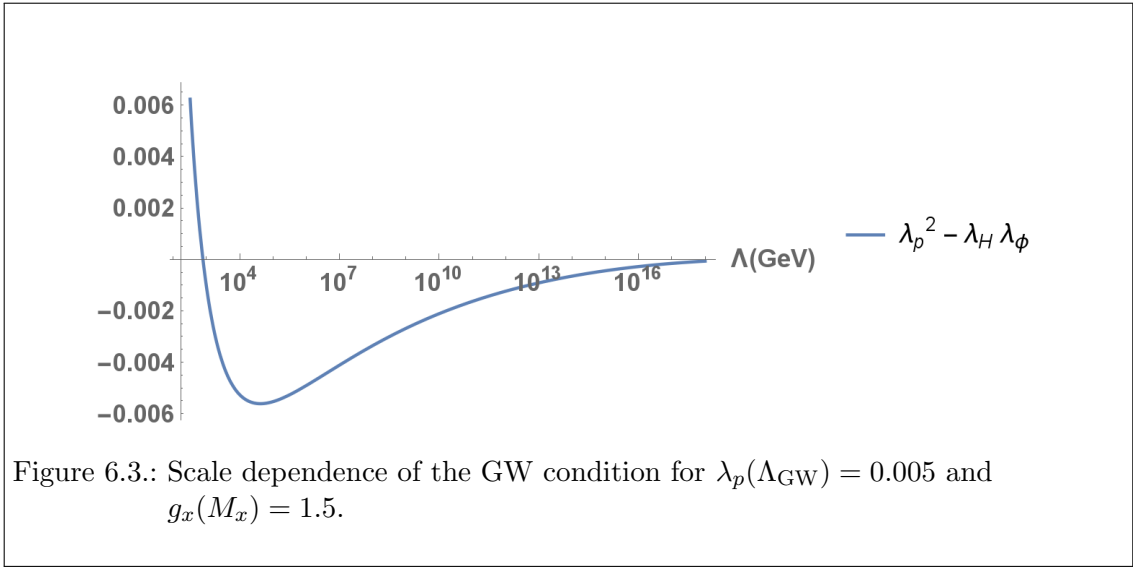
In addition to that, we notice the appearance of a second Gildener-Weinberg scale at $8.9 \cdot 10^4$ GeV, which corresponds to a second flat direction and thus to a second extremum of the potential. Knowing all coupling parameters at this new scale enables us to calculate A and B (A and B depend on the flat direction n) and with the help of equation (5.0.16) we get

$$B > 0 \quad , \quad \frac{A}{B} \approx 1 \quad \text{and} \quad \langle \sigma' \rangle \approx 3 \cdot 10^4 \text{ GeV} . \tag{6.2.66}$$

Thus, perturbation theory should be valid (compare chapter 5) and we find a second minimum, which is smaller than the minimum corresponding to $\Lambda_{\mathrm{GW}} = 294 \, \mathrm{GeV}$ and $\langle h \rangle = 246$ GeV:

$$\frac{V(n \langle \sigma \rangle)}{V(n' \langle \sigma' \rangle)} \approx 10^{-6} . \tag{6.2.67}$$

Hence, this is another evidence for the instability of this potential. Nevertheless, a more precise investigation of this new minimum, and in general of the appearance of a second

Figure 6.5.: Running couplings for $\lambda_p(\Lambda_{\text{GW}}) = 0.025$ and $g_x(M_x) = 0.9$.
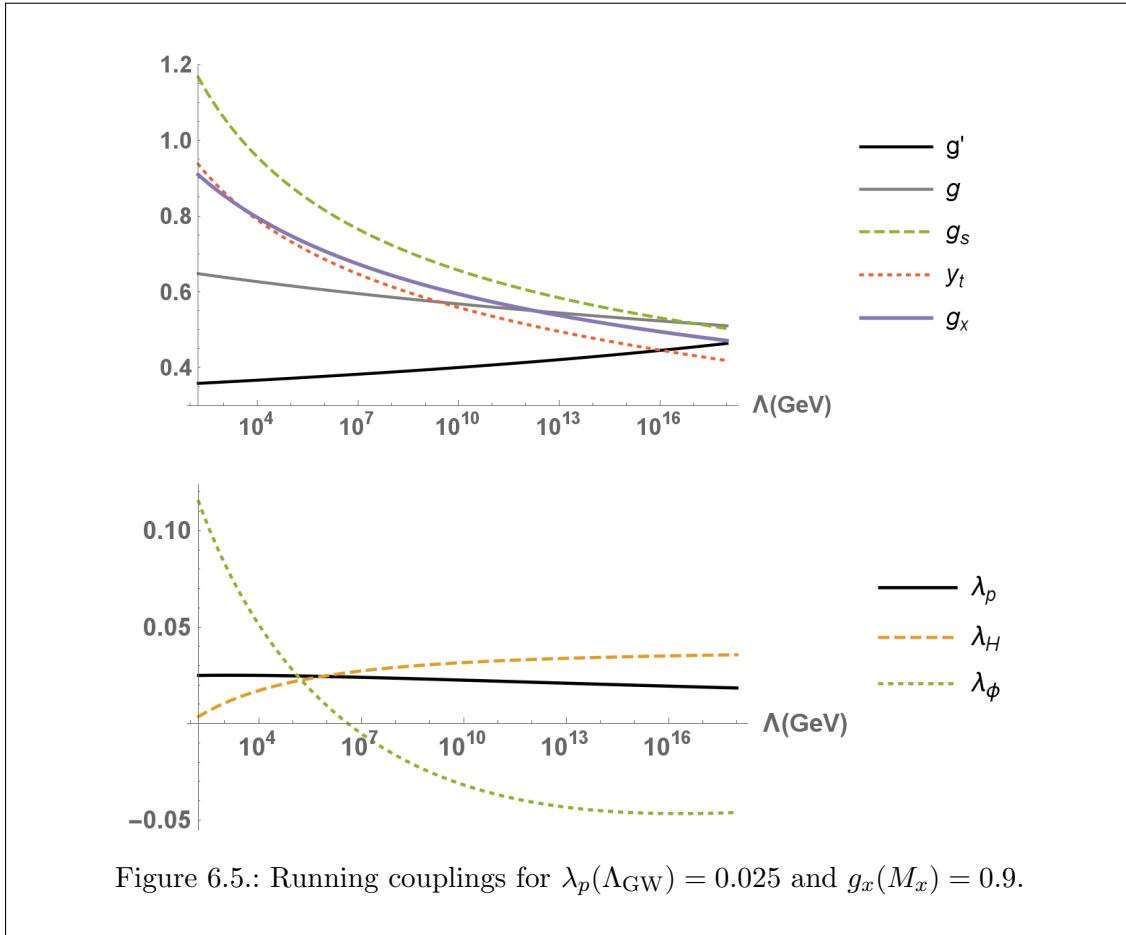
Gildener-Weinberg scale, is rescheduled to later research.

6.2.4. Identification of the Mass Eigenstate h_s with the Physically Observed Higgs Boson h_{SM}

We set the scalon mass equal to the Standard Model Higgs mass,

$$m_s^2 = \frac{\sin^2(\alpha)}{8\pi^2 \langle h \rangle^2} \left[3 \cdot \left(2 \cdot M_W^4 + M_Z^4 + 3 \cdot M_X^4 \right) + m_h^4 - 12 \cdot m_t^4 \right] \stackrel{!}{=} M_{h_{\text{SM}}}^2 \,, \qquad (6.2.68)$$

and find from the experimental restriction (6.2.52a):

$$\boxed{(540 \text{ GeV})^4 \leq 9 \cdot M_X^4 + m_h^4 \leq (575 \text{ GeV})^4 \,.} \qquad (6.2.69)$$

Hence, the upper mass bounds read

$$\boxed{m_h \leq 575 \text{ GeV} \qquad \text{and} \qquad M_X \leq 332 \text{ GeV} \,.} \qquad (6.2.70)$$

With the help of the definitions of these two masses,

$$M_X^2 \stackrel{(6.2.17)}{=} \frac{1}{4} g_x^2 \langle \phi \rangle^2 = \frac{1}{4} g_x^2 \frac{\langle h \rangle^2}{\tan^2(\alpha)} \,, \qquad (6.2.71)$$

and

$$m_h^2 \stackrel{(6.2.26)}{=} 2\,\lambda_p\,\langle\sigma\rangle^2 = 2\,\lambda_p\,\frac{\langle h\rangle^2}{\sin^2(\alpha)}\ ,\tag{6.2.72}$$

we can rewrite $\sin(\alpha)$ as a function of g_x and λ_p, which leads together with (6.2.52a) to the restricted parameter space in Figure 6.6. Thus, if we assume g_x to be smaller than
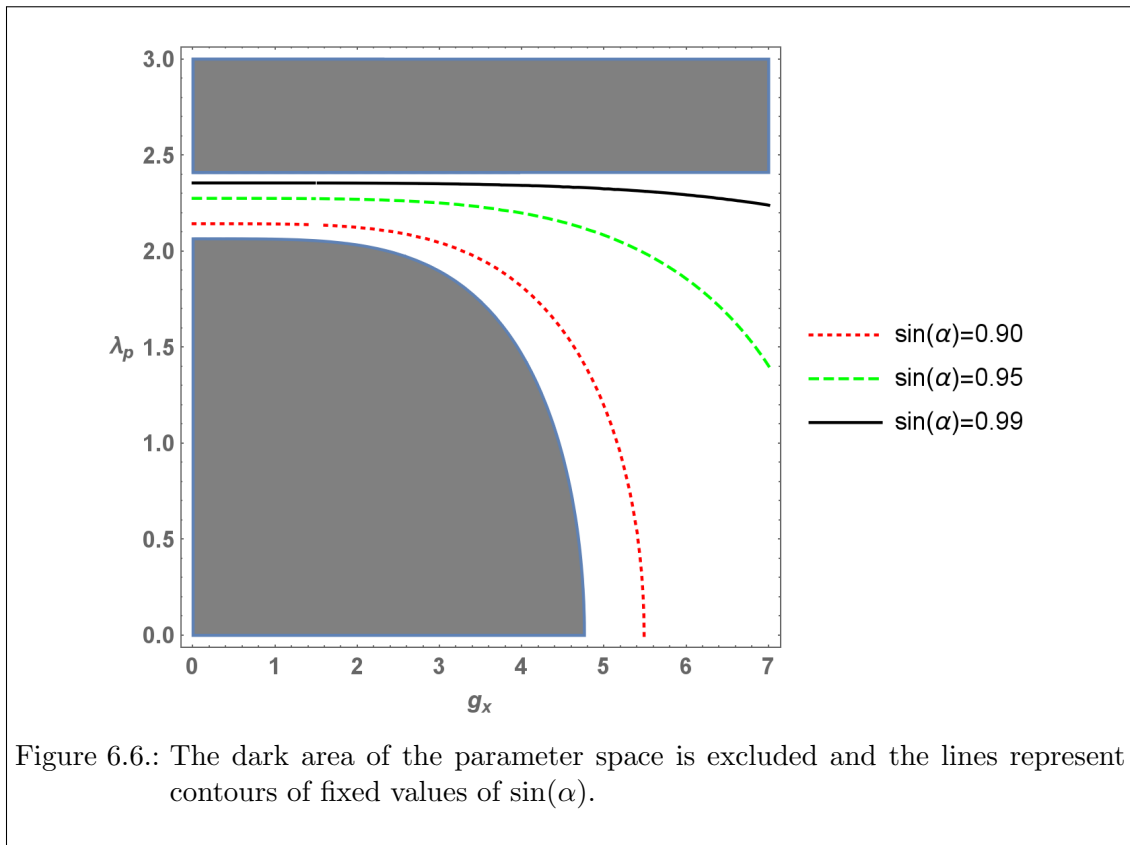
Figure 6.6.: The dark area of the parameter space is excluded and the lines represent contours of fixed values of $\sin(\alpha)$.

4.5, λ_p has to be at least greater than 1.5, which leads in any case to the appearance of a Landau pole (Figure 6.7a). Larger values of the gauge coupling would allow smaller scalar couplings, but spoil perturbation theory for small scales (Figure 6.7b) and therefore the identification of the scalon with the physically observed Higgs Boson is rather excluded.

6.2.5. Conclusion

We demonstrated that we have to identify the Standard Model Higgs boson with our mass eigenstate h_m to be able to find a parameter space (g_x, λ_p), which does not lead to any inconsistencies. All experimental observations that are predicted by the SM can then also be explained by our SU(2)$_X$ extended model. In addition to the particle content of the SM, one obtains a second scalar particle and three additional dark gauge bosons, which are possible candidates for dark matter [8–10]. We also showed that the the running of the Gildener-Weinberg condition can be used to find possible other minima of the effective potential (in addition to the one reproducing $\langle h\rangle = 246$ GeV), where a deeper investigation of this feature is required.

Nevertheless, the experimentally observed neutrino oscillations and the resulting appearance of light, but massive, left-handed neutrinos cannot be explained in a satisfactory

(a) Illustration of the appearance of a Landau Pole for $\lambda_p = 2.1$ and $g_x = 1$.

(b) Behaviour of the dark gauge coupling for small scales for $\lambda_p = 0.3$ and $g_x = 5$.

Figure 6.7.: Exemplary behaviour of λ_p and g_x in case of $h_s \hat{=} h_{\mathrm{SM}}$.
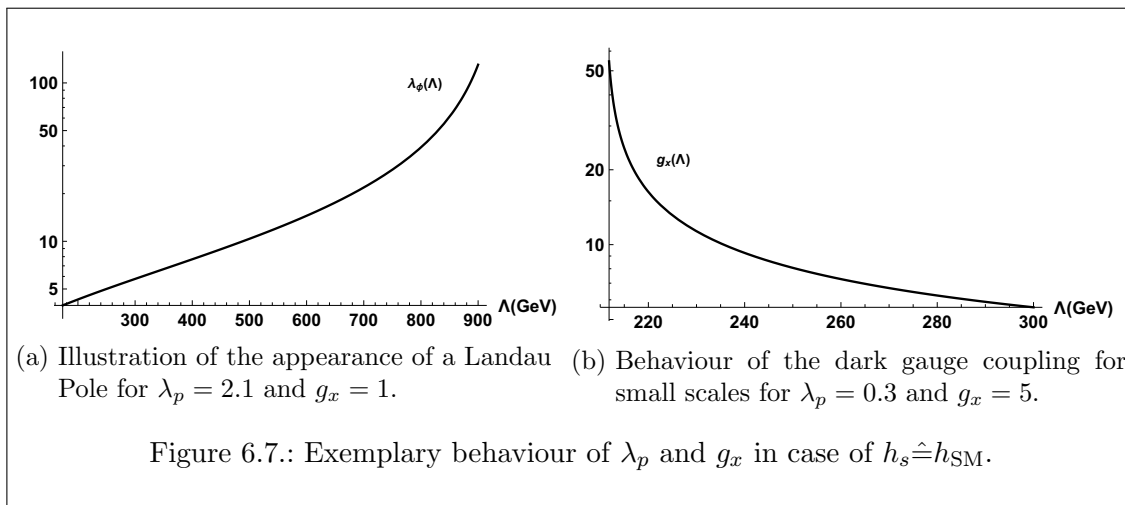
manner within this theory. Classical scale invariance prohibits explicit right-handed neutrino Majorana mass terms, which would enables us to explain the lightness of the observed neutrinos (with respect to the masses of the other Leptons) with help of the seesaw mechanism[15]. This problem can be solved by the introduction of an additional real scalar singlet, which acquires a nontrivial vacuum expectation value. Such a model was also discussed in detail in [14].

6.3. A Classically Scale Invariant SU(2) Extension of the Standard Model with Implemented Seesaw Mechanism

The Lagrangian density of our extended model reads

$$
\begin{aligned}
\mathcal{L} = \mathcal{L}'_{\mathrm{SM}} &- \frac{1}{4} X_{\mu\nu} X^{\mu\nu} + \left(\tilde{D}_\mu \Phi \right)^\dagger \left(\tilde{D}^\mu \Phi \right) + \frac{1}{2} \left(\partial_\mu s \right) \left(\partial^\mu s \right) + \frac{1}{2} \overline{\nu_{\mathrm{R}}}\, \mathrm{i} \slashed{\partial}\, \nu_{\mathrm{R}} \\
&- V_0(\Phi, H, s) - \left(\overline{L}\, \Gamma_{\mathrm{D}}^{(\nu)} \left(\mathrm{i}\tau_2 H^* \right) + \frac{1}{2} \overline{(\nu_{\mathrm{R}})^{\mathrm{c}}} \Gamma_{\mathrm{R}}^{(\nu)} s \right) \nu_{\mathrm{R}} + \text{h.c.} \,,
\end{aligned}
\tag{6.3.1}
$$

where s denotes the new real scalar boson, which is a singlet under the gauge group (6.2.1), and ν_{R} labels a vector of three right-handed neutrinos[16]. In contrast to the simpler model before, we added a Dirac-type and a Majorana-type neutrino Yukawa interaction term with the corresponding complex 3×3 Yukawa matrices $\Gamma_{\mathrm{D}}^{(\nu)}$ and $\Gamma_{\mathrm{R}}^{(\nu)}$ to the Lagrangian density and enlarged the tree-level potential to

$$
\begin{aligned}
V_0(\Phi, H, s) = \lambda_\phi \left(\Phi^\dagger \Phi \right)^2 &+ \lambda_H \left(H^\dagger H \right)^2 + \frac{1}{4} \lambda_s s^4 \\
&- 2\lambda_p \left(\Phi^\dagger \Phi \right) \left(H^\dagger H \right) - \lambda_{\phi s}\, \Phi^\dagger \Phi\, s^2 - \lambda_{Hs}\, H^\dagger H s^2 \,,
\end{aligned}
\tag{6.3.2}
$$

---

[15]Of course, we could add right-handed neutrinos to our theory and generate a Dirac mass via a Yukawa interaction term in the same way we did for the other leptons. However, this would leave open the question of the tininess of neutrino masses.

[16]As discussed in [48], at least two SU(2) Higgs doublets are necessary to generate neutrino masses at one-loop order. Since we are dealing with one SU(2) and one SU(2)$_X$ Higgs doublet, we thus have to introduce three right-handed neutrinos to describe three massive left-handed neutrinos.

with real and dimensionless coupling constants.

We perform the analysis of the Gildener-Weinberg mechanism in the same way as before and decompose the two complex doublets into their real components

$$
\tilde{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \quad \text{and} \quad \tilde{H} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} , \tag{6.3.3}
$$

and define the real vector

$$
\varphi = \begin{pmatrix} \tilde{\Phi} \\ \tilde{H} \\ s \end{pmatrix} . \tag{6.3.4}
$$

The potential can then be written as

$$
\begin{aligned}
V(\Phi, H, s) = {} & \frac{1}{4}\, \lambda_\phi \left(\tilde{\Phi}_i \tilde{\Phi}_i \right)^2 + \frac{1}{4}\, \lambda_H \left(\tilde{H}_i \tilde{H}_i \right)^2 + \frac{1}{4}\, \lambda_s s^4 \\
& - \frac{1}{2}\, \lambda_p \left(\tilde{\Phi}_i \tilde{\Phi}_i \right) \left(\tilde{H}_i \tilde{H}_i \right) - \frac{1}{2}\lambda_{\phi s}\, \tilde{\Phi}_i \tilde{\Phi}_i\, s^2 - \frac{1}{2}\, \lambda_{Hs}\, \tilde{H}_i \tilde{H}_i s^2 ,
\end{aligned} \tag{6.3.5}
$$

and with the help of the SU(2) and the SU(2)$_{\mathrm{X}}$ gauge symmetry we restrict the flat direction to

$$
\varphi^{\text{flat}} = n\sigma = \begin{pmatrix} 0 \\ 0 \\ n_1 \\ 0 \\ 0 \\ 0 \\ n_2 \\ 0 \\ n_3 \end{pmatrix} \cdot \sigma \,\hat{=}\, \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \sigma . \tag{6.3.6}
$$

Hence, the above potential along this direction is given by

$$
\begin{aligned}
V_0(\varphi^{\text{flat}}) \propto {} & \frac{1}{4}\lambda_\phi n_1^4 + \frac{1}{4}\lambda_H n_2^4 + \frac{1}{4}\lambda_s n_3^4 \\
& - \frac{1}{2}\lambda_p n_1^2 n_2^2 - \frac{1}{2}\lambda_{\phi s} n_1^2 n_3^2 - \frac{1}{2}\lambda_{Hs} n_2^2 n_3^2 ,
\end{aligned} \tag{6.3.7}
$$

and since the flat direction has to fulfill equation (5.0.2), the conditions

$$
\begin{aligned}
\text{I}: \quad & \lambda_\phi n_1^2 - \lambda_p n_2^2 - \lambda_{\phi s} n_3^2 = 0 \\
\text{II}: \quad & \lambda_H n_2^2 - \lambda_p n_1^2 - \lambda_{Hs} n_3^2 = 0 \\
\text{III}: \quad & \lambda_s n_3^2 - \lambda_{\phi s} n_1^2 - \lambda_{Hs} n_2^2 = 0
\end{aligned} \tag{6.3.8}
$$

have to hold. Again, all scalar couplings are evaluated at the Gildener-Weinberg scale Λ_{GW}. The Gildener-Weinberg relation reads

$$
\boxed{\lambda_s = \frac{\lambda_{\phi s}^2 \lambda_H + 2\lambda_{\phi s}\lambda_p \lambda_{Hs} + \lambda_{Hs}^2 \lambda_\phi}{\lambda_\phi \lambda_H - \lambda_p^2}} , \tag{6.3.9}
$$

and the flat direction can be written as

$$\varphi^{\text{flat}} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \sigma = \begin{pmatrix} \cos(\alpha)\sin(\beta) \\ \sin(\alpha)\sin(\beta) \\ \cos(\beta) \end{pmatrix} \cdot \sigma \quad , \sigma > 0 \, , \; 0 \leq \alpha, \beta \leq \frac{\pi}{2} \, , \tag{6.3.10}$$

with

$$\sin(\alpha) = \sqrt{\frac{\lambda_{\phi s}\lambda_p + \lambda_{Hs}\lambda_\phi}{\lambda_{\phi s}(\lambda_H + \lambda_p) + \lambda_{Hs}(\lambda_p + \lambda_\phi)}} \, , \tag{6.3.11}$$

and

$$\sin(\beta) = \sqrt{\frac{\lambda_{\phi s}(\lambda_H + \lambda_p) + \lambda_{Hs}(\lambda_p + \lambda_\phi)}{\lambda_\phi \lambda_H - \lambda_p^2 + \lambda_{\phi s}(\lambda_H + \lambda_p) + \lambda_{Hs}(\lambda_p + \lambda_\phi)}} \, . \tag{6.3.12}$$

For

$$\text{sgn}(\lambda_\phi) = \text{sgn}(\lambda_H) = \text{sgn}(\lambda_s) = \text{sgn}(\lambda_p) = \text{sgn}(\lambda_{\phi s}) = \text{sgn}(\lambda_{Hs}) \tag{6.3.13}$$

we can find a nontrivial solution of (6.3.8), where none of the components n_1, n_2 or n_3 are allowed to be equal to zero. While we cannot follow the argumentation of [14] that this would only be true if one of the mixing scalar couplings (i.e. λ_p, $\lambda_{\phi s}$, λ_{Hs}) is negative[17], all the other results are in perfect agreement with [14].

The vacuum expectation values take the form

$$\langle \varphi \rangle = \begin{pmatrix} \langle \phi_3 \rangle \\ \langle h_3 \rangle \\ \langle s \rangle \end{pmatrix} := \begin{pmatrix} \langle \phi \rangle \\ \langle h \rangle \\ \langle s \rangle \end{pmatrix} = \begin{pmatrix} \cos(\alpha)\sin(\beta) \\ \sin(\alpha)\sin(\beta) \\ \cos(\beta) \end{pmatrix} \cdot \langle \sigma \rangle \, , \tag{6.3.14}$$

and for the same reason as in section 6.2 we find

$$\boxed{\langle h \rangle = 246 \text{ GeV} \, ,} \tag{6.3.15}$$

and thus:

$$\boxed{\begin{aligned} \langle \phi \rangle &= \frac{246 \text{ GeV}}{\tan(\alpha)} \, . \\ \langle s \rangle &= \frac{246 \text{ GeV}}{\sin(\alpha)\tan(\beta)} \, . \\ \langle \sigma \rangle &= \frac{246 \text{ GeV}}{\sin(\alpha)\sin(\beta)} \, . \end{aligned}} \tag{6.3.16}$$

Furthermore, the angles α and β, which define the flat direction, are given by the simple relations:

$$\tan^2(\alpha) = \frac{\langle h \rangle^2}{\langle \phi \rangle^2} \quad \text{and} \quad \tan^2(\beta) = \frac{\langle h \rangle^2 + \langle \phi^2 \rangle}{\langle s \rangle^2} \, . \tag{6.3.17}$$

---

[17]Especially in that case it would be possible to find a nontrivial solution with one vanishing component.

In a next step, we aim to find expressions for the scalar masses in terms of the scalar couplings. Therefore, we have to calculate the second derivatives of the tree-level potential (see (5.0.19)):

$$\left(\frac{\partial^2 V_0(\tilde{\Phi}, \tilde{H}, s)}{\partial\varphi_i\partial\varphi_k}\right) = \tag{6.3.18}$$

$$\begin{bmatrix} \lambda_\phi\left(\tilde{\Phi}^{\mathrm{T}}\tilde{\Phi} - \frac{\lambda_p}{\lambda_\phi}\tilde{H}^{\mathrm{T}}\tilde{H} - \frac{\lambda_{\phi s}}{\lambda_\phi}s^2\right)\mathbb{1} \\ +2\lambda_\phi\tilde{\Phi}\tilde{\Phi}^{\mathrm{T}} & -2\lambda_p\tilde{H}\tilde{\Phi}^{\mathrm{T}} & -2\lambda_{\phi s}\,s\,\tilde{\phi} \\[2ex] \hline -2\lambda_p\tilde{\Phi}\tilde{H}^{\mathrm{T}} & \begin{array}{c}\lambda_H\left(\tilde{H}^{\mathrm{T}}\tilde{H} - \frac{\lambda_p}{\lambda_H}\tilde{\Phi}^{\mathrm{T}}\tilde{\Phi} - \frac{\lambda_{Hs}}{\lambda_H}s^2\right)\mathbb{1} \\ +2\lambda_H\tilde{H}\tilde{H}^{\mathrm{T}}\end{array} & -2\lambda_{Hs}\,s\,\tilde{h} \\[2ex] \hline -2\lambda_{\phi s}\,s\,\tilde{\phi}^{\mathrm{T}} & -2\lambda_{Hs}\,s\,\tilde{h}^{\mathrm{T}} & \begin{array}{c}3\,\lambda_s\,s^2 - \lambda_{\phi s}\,\tilde{\phi}^{\mathrm{T}}\tilde{\phi} \\ -\lambda_{Hs}\,\tilde{h}^{\mathrm{T}}\tilde{h}\end{array}\end{bmatrix}$$

Considering the conditions (6.3.8) reduces the resulting mass matrix to

$$m_\varphi^2 = \left(\frac{\partial^2 V_0}{\partial\varphi_i\partial\varphi_k}\right)\Bigg|_{n\langle\sigma\rangle} = 2\cdot\begin{bmatrix} \lambda_\phi n_1^2 & -\lambda_p n_1 n_2 & -\lambda_{\phi s}n_1 n_3 \\ -\lambda_p n_1 n_2 & \lambda_H n_2^2 & -\lambda_{Hs}n_2 n_3 \\ -\lambda_{\phi s}n_1 n_3 & -\lambda_{Hs}n_2 n_3 & \lambda_{\phi s}n_1^2 + \lambda_{Hs}n_2^2 \end{bmatrix}\cdot\langle\sigma\rangle^2$$

$$= 2\cdot\begin{bmatrix} \lambda_\phi\cos^2(\alpha)\sin^2(\beta) & -\lambda_p\cos(\alpha)\sin(\alpha)\sin^2(\beta) \\ -\lambda_p\cos(\alpha)\sin(\alpha)\sin^2(\beta) & \lambda_H\sin^2(\alpha)\sin^2(\beta) \\ -\lambda_{\phi s}\cos(\alpha)\sin(\beta)\cos(\beta) & -\lambda_{Hs}\sin(\alpha)\sin(\beta)\cos(\beta) \end{bmatrix} \tag{6.3.19}$$

$$\left.\begin{array}{c} -\lambda_{\phi s}\cos(\alpha)\sin(\beta)\cos(\beta) \\ -\lambda_{Hs}\sin(\alpha)\sin(\beta)\cos(\beta) \\ \left(\lambda_{\phi s}\cos^2(\alpha) + \lambda_{Hs}\sin^2(\alpha)\right)\sin^2(\beta) \end{array}\right]\cdot\langle\sigma\rangle^2\ ,$$

where all other entries of the 9×9 matrix vanish. Since we already know the eigenvector

$$v_1 = \begin{pmatrix}\cos(\alpha)\sin(\beta) \\ \sin(\alpha)\sin(\beta) \\ \cos(\beta)\end{pmatrix} \tag{6.3.20}$$

with eigenvalue 0, we can rotate our squared mass matrix with the help of the orthogonal matrix

$$D = \begin{bmatrix}\cos(\alpha)\sin(\beta) & -\sin(\alpha) & -\cos(\alpha)\cos(\beta) \\ \sin(\alpha)\sin(\beta) & \cos(\alpha) & -\sin(\alpha)\cos(\beta) \\ \cos(\beta) & 0 & \sin(\beta)\end{bmatrix}\ , \tag{6.3.21}$$

which obeys

$$D\begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix} = v_1 \quad\text{and}\quad \det D = 1\ , \tag{6.3.22}$$

and find[18]:

$$D^T m_\varphi^2 D = \frac{2}{\langle\phi\rangle^2 + \langle h\rangle^2}$$

$$\cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_{Hs} \langle\phi\rangle^2 \langle h\rangle^2 + \lambda_p \left(\langle\phi\rangle^2 + \langle h\rangle^2\right)^2 + \lambda_{\phi s} \langle h\rangle^2 \langle s\rangle^2 & (\lambda_{\phi s} - \lambda_{Hs}) \langle\phi\rangle \langle h\rangle \langle s\rangle \langle\sigma\rangle \\ 0 & (\lambda_{\phi s} - \lambda_{Hs}) \langle\phi\rangle \langle h\rangle \langle s\rangle \langle\sigma\rangle & \lambda_{\phi s} \langle\phi\rangle^2 \langle\sigma\rangle^2 + \lambda_{Hs} \langle h\rangle^2 \langle\sigma\rangle^2 \end{bmatrix}$$

$$= 2 \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & w_{11} & w_{12} \\ 0 & w_{21} & w_{22} \end{bmatrix} . \tag{6.3.23}$$

Therefore, the diagonalized squared scalar mass matrix reads

$$A^T D^T m_\varphi^2 D A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & \sin(\gamma) \\ 0 & -\sin(\gamma) & \cos(\gamma) \end{bmatrix} D^T m_\varphi^2 D \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix} \tag{6.3.24}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & m_1^2 & 0 \\ 0 & 0 & m_2^2 \end{bmatrix} ,$$

with

$$\tan(2\gamma) = \frac{2w_{12}}{w_{11} - w_{22}} , \tag{6.3.25}$$

and the tree-level mass eigenvalues are given by

$$\boxed{m_s^2 = 0} \tag{6.3.26}$$

and

$$\boxed{\begin{aligned} m_{1,2}^2 &= w_{11} + w_{22} \mp \sqrt{(w_{11} - w_{22})^2 + 4w_{12}^2} \\ &= (\lambda_{Hs} + \lambda_p) \langle h\rangle^2 + (\lambda_{\phi s} + \lambda_p) \langle\phi\rangle^2 + (\lambda_{Hs} + \lambda_{\phi s}) \langle s\rangle^2 \\ &\quad \mp \sqrt{\begin{aligned} &(\lambda_{Hs} - \lambda_p)^2 \langle h\rangle^4 + \left((\lambda_{Hs} - \lambda_{\phi s}) \langle s\rangle^2 + (\lambda_p - \lambda_{\phi s}) \langle\phi\rangle^2\right)^2 \\ &+ 2(\lambda_{Hs} - \lambda_p) \langle h\rangle^2 \left((\lambda_{Hs} - \lambda_{\phi s}) \langle s\rangle^2 + (\lambda_{\phi s} - \lambda_p) \langle\phi\rangle^2\right) \end{aligned}} . \end{aligned}} \tag{6.3.27}$$

The mass of the scalon is again generated at one-loop order and given by

$$\boxed{m_s^2 = \frac{1}{8\pi^2 \langle\sigma\rangle^2} \left[3 \cdot \left(2 \cdot M_W^4 + M_Z^4 + 3 \cdot M_X^4\right) + m_1^4 + m_2^4 - 4 \cdot 3 \cdot m_t^4 - 2\sum_{i=1}^{3} m_{\nu_i}^4 \right] \overset{!}{>} 0 ,} \tag{6.3.28}$$

---

[18]Be aware that $\langle\sigma\rangle^2 = \langle\phi\rangle^2 + \langle h\rangle^2 + \langle s\rangle^2$.

where m_{ν_i} denotes the masses of the heavy right-handed Majorana neutrinos. Therefore, the mass inequality

$$9 \cdot M_X^4 + m_1^4 + m_2^4 - 2 \cdot \sum_{i=1}^{3} m_{\nu_i}^4 > (318\text{GeV})^4 \qquad (6.3.29)$$

has to be fulfilled.

The eigenvectors, which correspond to the mass eigenvalues, take the form

$$v_s = \begin{pmatrix} \cos(\alpha)\sin(\beta) \\ \sin(\alpha)\sin(\beta) \\ \cos(\beta) \end{pmatrix}$$

$$v_1 = \begin{pmatrix} -\sin(\alpha)\cos(\gamma) - \cos(\alpha)\cos(\beta)\sin(\gamma) \\ \cos(\alpha)\cos(\gamma) - \sin(\alpha)\cos(\beta)\sin(\gamma) \\ \sin(\beta)\sin(\gamma) \end{pmatrix} \qquad (6.3.30)$$

$$v_2 = \begin{pmatrix} \sin(\alpha)\sin(\gamma) - \cos(\alpha)\cos(\beta)\cos(\gamma) \\ -\cos(\alpha)\sin(\gamma) - \sin(\alpha)\cos(\beta)\cos(\gamma) \\ \sin(\beta)\cos(\gamma) \end{pmatrix} \;,$$

and hence, the scalar fields ϕ, h and s can be written as a mixture of mass eigenstates:

$$\begin{pmatrix} \phi \\ h \\ s \end{pmatrix} = \begin{pmatrix} \langle\phi\rangle + \phi' \\ \langle h\rangle + h' \\ \langle s\rangle + s' \end{pmatrix} = \begin{pmatrix} \langle\phi\rangle \\ \langle h\rangle \\ \langle s\rangle \end{pmatrix} + v_s h_s + v_1 h_{m_1} + v_2 h_{m_2} \;. \qquad (6.3.31)$$

In particular, we find:

$$h = \langle h\rangle + \sin(\alpha)\sin(\beta)\, h_s + \Big(\cos(\alpha)\cos(\gamma) - \sin(\alpha)\cos(\beta)\sin(\gamma) \Big) h_{m_1}$$
$$- \Big(\cos(\alpha)\sin(\gamma) + \sin(\alpha)\cos(\beta)\cos(\gamma) \Big) h_{m_2} \;. \qquad (6.3.32)$$

All these results are again in agreement with [14].

6.3.1. Seesaw mechanism

We want to review the seesaw mechanism [17–21] and apply it to our model to explain small neutrino masses.

After spontaneous symmetry breaking, the Lagrangian density (6.3.1) delivers a Dirac neutrino mass term,

$$\mathcal{L}_{\text{D},\nu} = -\overline{\nu_\text{L}} \underbrace{\frac{\langle h\rangle}{\sqrt{2}} \Gamma_\text{D}^{(\nu)}}_{M_\text{D}} \nu_\text{R} + \text{h.c.} \;, \qquad (6.3.33)$$

as well as a Majorana neutrino mass term,

$$\mathcal{L}_{\text{M},\nu} = -\frac{1}{2} \overline{(\nu_\text{R})^\text{c}} \underbrace{\langle s\rangle\, \Gamma_\text{R}^{(\nu)}}_{M_\text{R}} \nu_\text{R} + \text{h.c.} \;, \qquad (6.3.34)$$

with $M_{\mathrm{R}} = M_{\mathrm{R}}^{\mathrm{T}}$ (see subsection B.3.3). Introducing the left-handed vector

$$\omega_{\mathrm{L}} = \begin{pmatrix} \nu_{\mathrm{L}} \\ (\nu_{\mathrm{R}})^{\mathrm{c}} \end{pmatrix} \tag{6.3.35}$$

enables us to combine these two mass terms to

$$\mathcal{L}_{\mathrm{D+M},\nu} = -\frac{1}{2} \overline{\omega_{\mathrm{L}}} \underbrace{\begin{bmatrix} 0 & M_{\mathrm{D}} \\ M_{\mathrm{D}}^{\mathrm{T}} & M_R \end{bmatrix}}_{M_\nu = M_\nu^{\mathrm{T}}} (\omega_{\mathrm{L}})^{\mathrm{c}} + \mathrm{h.c.} \ . \tag{6.3.36}$$

Since M_ν is a complex and symmetric matrix, there exists a unitary matrix U such that

$$\hat{M}_\nu = U^\dagger M_\nu U^* = \mathrm{diag}\,(\hat{m}_1, \hat{m}_2, \ldots, \hat{m}_6) \tag{6.3.37}$$

is diagonal and non-negative [49, 50], where the diagonal elements $\hat{m}_i$ are the positive square roots of the eigenvalues of $M_\nu M_\nu^\dagger$ [24, 51]. Defining

$$\omega_{\mathrm{L}} = U \omega_{\mathrm{L}}' \tag{6.3.38}$$

leads to

$$\mathcal{L}_{\mathrm{D+M},\nu} = -\frac{1}{2} \overline{N} \hat{M}_\nu N = -\frac{1}{2} \sum_{i=1}^{6} \overline{\nu_i}\, \hat{m}_i\, \nu_i \ , \tag{6.3.39}$$

where

$$N = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_6 \end{pmatrix} = \omega_{\mathrm{L}}' + (\omega_{\mathrm{L}}')^{\mathrm{c}} \tag{6.3.40}$$

satisfies the Majorana condition

$$N^{\mathrm{c}} = N \ . \tag{6.3.41}$$

Therefore, introducing a Dirac-Majorana mass term yields mass eigenstates, which are Majorana fields. Furthermore, with the help of the decomposed matrix U [48],

$$U = \begin{pmatrix} U_{\mathrm{L}} \\ U_{\mathrm{R}}^* \end{pmatrix} \ , \tag{6.3.42}$$

the neutrino weak eigenfields can be expressed as linear combinations of these mass eigenstates:

$$\nu_{\mathrm{L}} = U_{\mathrm{L}} \omega_{\mathrm{L}}' = U_{\mathrm{L}} P_{\mathrm{L}} N \quad , \quad \nu_{\mathrm{R}} = U_{\mathrm{R}} \left(\omega_{\mathrm{L}}'\right)^{\mathrm{c}} = U_{\mathrm{R}} P_{\mathrm{R}} N \ . \tag{6.3.43}$$

If we assume that the eigenvalues of M_{R} are much bigger than the elements of M_{D} ($\langle s \rangle \gg \langle h \rangle$), we can block-diagonalize the matrix M_ν (up to corrections $\propto M_{\mathrm{R}}^{-1} M_{\mathrm{D}}$) by performing a unitary transformation [51–53],

$$W^\dagger \begin{bmatrix} 0 & M_{\mathrm{D}} \\ M_{\mathrm{D}}^{\mathrm{T}} & M_R \end{bmatrix} W^* = \begin{bmatrix} M_{\mathrm{light}} & 0 \\ 0 & M_{\mathrm{heavy}} \end{bmatrix} \ , \tag{6.3.44}$$

with

$$W \approx \begin{pmatrix} 1 - \frac{1}{2} J J^\dagger & J \\ -J^\dagger & 1 - \frac{1}{2} J^\dagger J \end{pmatrix} \quad , \quad J = M_{\mathrm{D}} M_{\mathrm{R}}^{-1} \ , \tag{6.3.45}$$

and where M_{light} and M_{heavy} are symmetric 3×3 matrices:

$$\boxed{M_{\text{light}} \approx -M_{\text{D}} M_{\text{R}}^{-1} M_{\text{D}}^{\text{T}} \quad , \quad M_{\text{heavy}} \approx M_{\text{R}} \, .}$$
(6.3.46)

Hence, we obtain three light and three heavy neutrinos, where the tininess of the mass of the observed neutrinos is naturally explained by the existence of heavy neutrinos.

6.3.2. Running couplings

For the sake of completeness, we also derive the RGE's of the scalar couplings for this model (compare subsection 6.2.1). The pure scalar part of the Lagrangian density reads

$$\mathcal{L}_{\text{scalar}} = \left(\partial_\mu \begin{pmatrix} \phi_0 \\ h_0 \\ s_0 \end{pmatrix} \right)^{\text{T}} \left(\partial^\mu \begin{pmatrix} \phi_0 \\ h_0 \\ s_0 \end{pmatrix} \right) - \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \lambda_\phi^0 \phi_0^4 - \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \lambda_H^0 h_0^4 - \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \lambda_s^0 s_0^4$$
$$+ \frac{1}{2} \tilde{\Lambda}^{2\epsilon} \lambda_{\phi s}^0 \phi_0^2 s_0^2 + \frac{1}{2} \tilde{\Lambda}^{2\epsilon} \lambda_{Hs}^0 h_0^2 s_0^2 + \frac{1}{2} \tilde{\Lambda}^{2\epsilon} \lambda_p^0 \phi_0^2 h_0^2 \, ,$$
(6.3.47)

and introducing the scale-dependent, renormalized scalar fields and coupling constants[19],

$$\begin{pmatrix} \phi_0 \\ h_0 \\ s_0 \end{pmatrix} = Z_\varphi^{\frac{1}{2}} \begin{pmatrix} \phi \\ h \\ s \end{pmatrix} = \begin{bmatrix} 1 + \frac{1}{2}\delta Z_{\phi\phi} & \delta Z_{\phi h}^{\frac{1}{2}} & \delta Z_{\phi s}^{\frac{1}{2}} \\ \delta Z_{h\phi}^{\frac{1}{2}} & 1 + \frac{1}{2}\delta Z_{hh} & \delta Z_{hs}^{\frac{1}{2}} \\ \delta Z_{s\phi}^{\frac{1}{2}} & \delta Z_{sh}^{\frac{1}{2}} & 1 + \frac{1}{2}\delta Z_{ss} \end{bmatrix} \begin{pmatrix} \phi \\ h \\ s \end{pmatrix} \, ,$$
(6.3.48a)

$$\lambda_i^0 = Z_{\lambda_i} \lambda_i = (1 + \delta Z_{\lambda_i}) \lambda_i \, ,$$
(6.3.48b)

leads to

$$\mathcal{L}_{\text{scalar}} = \left(\partial_\mu \begin{pmatrix} \phi \\ h \\ s \end{pmatrix} \right)^{\text{T}} \left(\partial^\mu \begin{pmatrix} \phi \\ h \\ s \end{pmatrix} \right) - \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \lambda_\phi \phi^4 - \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \lambda_H h^4 - \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \lambda_s s_4$$

$$+ \frac{1}{2} \tilde{\Lambda}^{2\epsilon} \lambda_{\phi s} \phi^2 s^2 + \frac{1}{2} \tilde{\Lambda}^{2\epsilon} \lambda_{Hs} h^2 s^2 + \frac{1}{2} \tilde{\Lambda}^{2\epsilon} \lambda_p \phi^2 h^2$$

$$+ \left(\partial_\mu \begin{pmatrix} \phi \\ h \\ s \end{pmatrix} \right)^{\text{T}} \left(\left(Z_\varphi^{\frac{1}{2}} \right)^{\text{T}} Z_\varphi^{\frac{1}{2}} - \mathbb{1} \right) \left(\partial^\mu \begin{pmatrix} \phi \\ h \\ s \end{pmatrix} \right) - \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \left(\delta Z_{\lambda_H} + 2\delta Z_{hh} \right) \lambda_H h^4$$

$$- \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \left(\delta Z_{\lambda_\phi} + 2\delta Z_{\phi\phi} \right) \lambda_\phi \phi^4 - \frac{1}{4} \tilde{\Lambda}^{2\epsilon} \left(\delta Z_{\lambda_s} + 2\delta Z_{ss} \right) \lambda_s s^4$$

$$+ \frac{1}{2} \tilde{\Lambda}^{2\epsilon} \left(\delta Z_{\lambda_p} + \delta Z_{hh} + \delta Z_{\phi\phi} \right) \lambda_p \phi^2 h^2 + \frac{1}{2} \tilde{\Lambda}^{2\epsilon} \left(\delta Z_{\lambda_{\phi s}} + \delta Z_{\phi\phi} + \delta Z_{ss} \right) \lambda_{\phi s} \phi^2 s^2$$

$$+ \frac{1}{2} \tilde{\Lambda}^{2\epsilon} \left(\delta Z_{\lambda_{Hs}} + \delta Z_{hh} + \delta Z_{ss} \right) \lambda_{Hs} h^2 s^2 + \text{Terms} \left(\propto \delta Z_{\phi h}, \propto \delta Z_{\phi s}, \propto \delta Z_{hs} \right)$$

$$+ \text{ higher orders} \, .$$
(6.3.49)

---

[19] No summation over i.

The counterterms for our model are given by[20]

$$
\begin{aligned}
\operatorname{Tr} \mathrm{CT}_\varphi = \tilde\Lambda^{2\epsilon} \frac{1}{\epsilon} \Bigg[& \left(12\lambda_\phi^2 + 4\lambda_p^2 + \lambda_{\phi s}^2\right) \phi^4 + \left(12\lambda_H^2 + 4\lambda_p^2 + \lambda_{Hs}^2\right) h^4 + \left(9\lambda_s^2 + 4\lambda_{\phi s}^2 + 4\lambda_{Hs}^2\right) s^4 \\
& + \left(8\lambda_p^2 - 12\lambda_\phi\lambda_p - 12\lambda_H\lambda_p + 2\lambda_{\phi s}\lambda_{Hs}\right) \phi^2 h^2 \\
& + \left(8\lambda_{\phi s}^2 - 12\lambda_\phi\lambda_{\phi s} - 6\lambda_s\lambda_{\phi s} + 8\lambda_p\lambda_{Hs}\right) \phi^2 s^2 \\
& + \left(8\lambda_{Hs}^2 - 12\lambda_H\lambda_{Hs} - 6\lambda_s\lambda_{Hs} + 8\lambda_p\lambda_{\phi s}\right) h^2 s^2 \Bigg] ,
\end{aligned}
$$

$$
\operatorname{Tr} \mathrm{CT}_A = \tilde\Lambda^{2\epsilon} \frac{3}{\epsilon} \left[\frac{1}{16}\left(3g^4 + 2g^2 g'^2 + g'^4\right) h^4 + \frac{3}{16} g_x^4 \phi^4 \right] ,
$$

$$
\operatorname{Tr} \mathrm{CT}_\psi = \tilde\Lambda^{2\epsilon} \frac{1}{\epsilon} \left(6\frac{y_t^4}{4} h^4 + \operatorname{Tr}\left(\Gamma_\mathrm{R}\Gamma_\mathrm{R}^*\Gamma_\mathrm{R}\Gamma_\mathrm{R}^*\right) s^4 \right) ,
$$

$$\tag{6.3.50}$$

and from equation (6.2.34) follows that the relations

$$
\left(\delta Z_{\lambda_\phi} + 2\delta Z_{\phi\phi}\right)\lambda_\phi = \frac{1}{\epsilon}\frac{1}{16\pi^2}\left[12\lambda_\phi^2 + 4\lambda_p^2 + \lambda_{\phi s}^2 + \frac{9}{16}g_x^4\right] , \tag{6.3.51a}
$$

$$
\left(\delta Z_{\lambda_H} + 2\delta Z_{\phi\phi}\right)\lambda_H = \frac{1}{\epsilon}\frac{1}{16\pi^2}\left[12\lambda_H^2 + 4\lambda_p^2 + \lambda_{Hs}^2 + \frac{3}{16}\left(3g^4 + 2g^2 g'^2 + g'^4\right) - 3y_t^4\right] , \tag{6.3.51b}
$$

$$
\left(\delta Z_{\lambda_s} + 2\delta Z_{ss}\right)\lambda_s = \frac{1}{\epsilon}\frac{1}{16\pi^2}\left[9\lambda_s^2 + 4\lambda_{\phi s} + 4\lambda_{Hs} - 2\operatorname{Tr}\left(\Gamma_\mathrm{R}\Gamma_\mathrm{R}^*\Gamma_\mathrm{R}\Gamma_\mathrm{R}^*\right)\right] , \tag{6.3.51c}
$$

$$
\left(\delta Z_{\lambda_p} + \delta Z_{hh} + \delta Z_{\phi\phi}\right)\lambda_p = \frac{1}{\epsilon}\frac{1}{32\pi^2}\left[-8\lambda_p^2 + 12\lambda_\phi\lambda_p + 12\lambda_H\lambda_p - 2\lambda_{\phi s}\lambda_{Hs}\right] , \tag{6.3.51d}
$$

$$
\left(\delta Z_{\lambda_{\phi s}} + \delta Z_{\phi\phi} + \delta Z_{ss}\right)\lambda_{\phi s} = \frac{1}{\epsilon}\frac{1}{32\pi^2}\left[-8\lambda_{\phi s}^2 + 12\lambda_\phi\lambda_{\phi s} + 6\lambda_s\lambda_{\phi s} - 8\lambda_p\lambda_{Hs}\right] , \tag{6.3.51e}
$$

$$
\left(\delta Z_{\lambda_{Hs}} + \delta Z_{hh} + \delta Z_{ss}\right)\lambda_{Hs} = \frac{1}{\epsilon}\frac{1}{32\pi^2}\left[-8\lambda_{Hs}^2 + 12\lambda_H\lambda_{Hs} + 6\lambda_s\lambda_{Hs} - 8\lambda_p\lambda_{\phi s}\right] , \tag{6.3.51f}
$$

have to hold. With the help of the field strength renormalization counterterms from appendix A.2,

$$
\begin{aligned}
\delta Z_{\phi\phi} &= \frac{1}{\epsilon}\frac{9g_x^2}{64\pi^2} \\
\delta Z_{hh} &= \frac{1}{\epsilon}\frac{9g^2 + 3g'^2 - 12y_t^2}{64\pi^2} \\
\delta Z_{ss} &= \frac{1}{\epsilon}\left(-\frac{4}{64\pi^2}\operatorname{Tr}\left(\Gamma_\mathrm{R}\Gamma_\mathrm{R}^*\right)\right) ,
\end{aligned}
\tag{6.3.52}
$$

the RGE's of the scalar couplings are calculated by analogy with subsection 6.2.1 and take the form:

---

[20]We only consider contributions from heavy neutrinos and neglect light neutrinos.

$$\beta(\lambda_\phi) = \frac{1}{(4\pi)^2}\left[-9\lambda_\phi g_x^2 + 24\lambda_\phi^2 + 8\lambda_p^2 + 2\lambda_{\phi s}^2 + \frac{9}{8}g_x^4\right].$$

$$\beta(\lambda_H) = \frac{1}{(4\pi)^2}\left[\left(12y_t^2 - 9g^2 - 3g'^2\right)\lambda_H + 24\lambda_H^2 + 8\lambda_p^2 + 2\lambda_{Hs}^2\right.$$
$$\left. + \frac{3}{8}\left(3g^4 + 2g'^2g^2 + g'^4\right) - 6y_t^4\right].$$

$$\beta(\lambda_s) = \frac{1}{16\pi^2}\left[18\lambda_s^2 + 8\lambda_{\phi s}^2 + 8\lambda_{Hs}^2 - 4\,\mathrm{Tr}\left(\Gamma_R\Gamma_R^*\Gamma_R\Gamma_R^*\right) + 4\,\mathrm{Tr}\left(\Gamma_R\Gamma_R^*\right)\right].$$

$$\beta(\lambda_p) = \frac{1}{(4\pi)^2}\left[\lambda_p\left(-\frac{9}{2}g_x^2 - \frac{9}{2}g^2 - \frac{3}{2}g'^2 + 6y_t^2 + 12\lambda_\phi\right.\right. \tag{6.3.53}$$
$$\left.\left. + 12\lambda_H - 8\lambda_p\right) - 2\lambda_{\phi s}\lambda_{Hs}\right].$$

$$\beta(\lambda_{\phi s}) = \frac{1}{(4\pi)^2}\left[\lambda_{\phi s}\left(2\,\mathrm{Tr}\left(\Gamma_R\Gamma_R^*\right) - \frac{9}{2}g_x^2 + 12\lambda_\phi + 6\lambda_s - 8\lambda_{\phi s}\right) - 8\lambda_p\lambda_{Hs}\right].$$

$$\beta(\lambda_{Hs}) = \frac{1}{(4\pi)^2}\left[\lambda_{Hs}\left(2\,\mathrm{Tr}\left(\Gamma_R\Gamma_R^*\right) - \frac{9}{2}g^2 - \frac{3}{2}g'^2 + 6y_t^2 + 12\lambda_H\right.\right.$$
$$\left.\left. + 6\lambda_s - 8\lambda_{Hs}\right) - 8\lambda_p\lambda_{\phi s}\right].$$

Once more, these results are in perfect agreement with [14].

6.3.3. Interpretation of the Physically Observed Higgs Boson

As discussed in subsection 6.3.1, the existence of heavy neutrinos is mandatory to explain the observed small neutrino masses (with respect to the masses of the other Leptons) via the seesaw mechanism. If we assume the heavy neutrino masses to lie in the TeV region (or above), we get

$$\boxed{\frac{\langle h\rangle}{\langle s\rangle} \overset{(6.3.15)}{=} \sin(\alpha)\tan(\beta) \ll 1\,,} \tag{6.3.54}$$

because we know that

$$M_{\mathrm{heavy}} \approx M_R \propto \langle s\rangle\,. \tag{6.3.55}$$

Accordingly, the scalon as the physically observed Higgs boson is excluded, since this identification would require

$$\sin(\alpha)\sin(\beta) \approx 1\,. \tag{6.3.56}$$

Otherwise, we would come into conflict with the experimentally observed values for the coupling constants[21]. Thus, we find

$$\boxed{m_1 = 125\text{ GeV}\quad\text{or}\quad m_2 = 125\text{ GeV}\,.} \tag{6.3.57}$$

---

[21]See equation (6.3.32) and compare with the argumentation in subsection 6.2.2

6.3.4. Conclusion

We demonstrated that the introduction of an additional real scalar field, which acquires a nontrivial vacuum expectation value and is a singlet with respect to our extended gauge group (6.2.1), enables us to implement the seesaw mechanism in our model and thus to explain small neutrino masses in a satisfactory manner. Furthermore, we argued that the scalon, which gets massive at one-loop order, cannot be identified with the physically observed Higgs particle at 125 GeV. This is a consequence of the requirement of heavy neutrino masses.

In addition to the particle content of the Standard Model, we obtain two new scalar particles, three dark gauge bosons, which are again possible candidates for dark matter [14] and three heavy neutrinos. The mass inequality (6.3.29) demands large scalar and/or dark gauge boson masses (comparable to the heavy neutrino masses), in order to obtain a positive squared scalon mass.

Our extended version of the SM contains many free dimensionless coupling parameters: 6 scalar couplings (λ_ϕ, λ_H, λ_s, λ_p, $\lambda_{\phi s}$, λ_{Hs}), the dark gauge coupling g_x and the Yukawa couplings $\Gamma_D^{(\nu)}$ and $\Gamma_R^{(\nu)}$. This number gets reduced by two after identifying one scalar mass with 125 GeV and the vacuum expectation value $\langle h \rangle$ with 246 GeV. In spite of this reduction, a scan over the remaining parameter space is still quite challenging, but it was shown in [14] that it is possible to choose the free parameters such, that no inconsistencies with experimental observation occur and all requirements for stability and perturbativity are met.

Altogether, this model is a promising classically scale invariant extension of the Standard Model, since it is not in contradiction to experimental data and explains small neutrino masses via the seesaw mechanism. Furthermore, it predicts additional scalar particles as well as vector dark matter, which may be probed and discovered in future experiments.

Appendices

A. Auxiliary Calculations

A.1. Explicit Calculation of the Effective Potential for a Simple Toy Model

To compute the integral of equation (3.4.21), we start by performing a Wick rotation and write the infinite sum as a logarithm:

$$V_1 = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_{E0}}{(2\pi)^4} \int \mathrm{d}\vec{k}_E \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{\frac{1}{2}\lambda_0 \phi_{0c}^2}{k_{E0}^2 + \vec{k}_E^2} \right)^n}_{\ln\left(1 + \frac{\frac{1}{2}\lambda_0 \phi_{0c}^2}{k_E^2}\right)}. \tag{A.1.1}$$

To calculate this integral with the help of dimensional regularization, our theory is continued to a d-dimensional space-time,

$$V_1 = \frac{1}{2} \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \ln\left(1 + \frac{\frac{1}{2}\tilde{\Lambda}^{4-d} \lambda_0 \, \phi_{0c}^2}{k_E^2}\right) \,, \tag{A.1.2}$$

where the factor $\tilde{\Lambda}^{4-d}$, with $[\tilde{\Lambda}] = \frac{1}{L}$, arises from a redefinition of the coupling constant, $\lambda_0 \to \tilde{\Lambda}^{4-d}\lambda_0$, to keep the dimension of λ_0, $[\lambda_0] = \frac{1}{EL}$, unchanged in a d-dimensional space-time.

An integral of the form of equation (A.1.2) can easily be rewritten as

$$\int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \ln\left(\frac{k_E^2 + a}{k_E^2 + b}\right) = \int_b^a \mathrm{d}\Delta \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{1}{k_E^2 + \Delta} \,, \tag{A.1.3}$$

and is linked to the well-known and common integral ([27, 54]):

$$\int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{1}{k_E^2 + \Delta} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1 - \frac{d}{2})}{\Gamma(1)} \Delta^{\frac{d}{2}-1} \,. \tag{A.1.4}$$

Therefore, we can solve equation (A.1.3),

$$\int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \ln\left(\frac{k_E^2 + a}{k_E^2 + b}\right) = \frac{2}{d} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left(a^{\frac{d}{2}} - b^{\frac{d}{2}}\right) \,, \tag{A.1.5}$$

and find for the special case, $a = \frac{1}{2}\tilde{\Lambda}^{4-d}\lambda_0 \phi_{0c}^2$ and $b = 0$, in $d = 4 - 2\epsilon$ dimensions:

$$V_1 = \frac{1}{64\pi^2} \left(\frac{1}{2}\lambda_0 \phi_{0c}^2\right)^2 \left(-\frac{1}{\epsilon}\Lambda^{2\epsilon} + \ln\left(\frac{1}{2}\lambda_0 \phi_{0c}^2\right) + \gamma_e - \ln(4\pi) - \ln(\tilde{\Lambda}^2) - \frac{3}{2}\right) + \mathcal{O}(\epsilon) \,. \tag{A.1.6}$$

To get rid of the divergency in the limit $\epsilon \to 0$, we introduce the $\overline{\text{MS}}$-renormalized, Λ-dependent parameters $\lambda(\Lambda)$ and $\phi_c(\Lambda)$,

$$
\begin{aligned}
\lambda_0 &= Z_\lambda\, \lambda(\Lambda) \ = (1 + \delta Z_\lambda)\, \lambda(\Lambda) \\
\phi_{0c} &= Z_\varphi^{\frac{1}{2}}\, \phi_c(\Lambda) = (1 + \frac{1}{2}\delta Z_\varphi)\, \phi_c(\Lambda)\ ,
\end{aligned}
\tag{A.1.7}
$$

with the one-loop renormalization constants δZ_λ and δZ_φ and find at one-loop order:

$$
V = \frac{\lambda}{4!}\phi_c^4 + \hbar\, \frac{\lambda^2 \phi_c^4}{256\pi^2} \left(\ln \frac{\frac{1}{2}\lambda \phi_c^2}{\Lambda^2} - \frac{3}{2} \right) + \left(2\delta Z_\varphi + \delta Z_\lambda - \frac{1}{\epsilon}\frac{3\hbar\lambda}{32\pi^2} \right) \Lambda^{2\epsilon} \frac{\lambda}{4!}\phi_c^4\ .
\tag{A.1.8}
$$

In the $\overline{\text{MS}}$-scheme, with $\tilde{\Lambda}^2 = \Lambda^2 \cdot e^{\gamma_E - \ln(4\pi)}$ and for

$$
2\delta Z_\varphi + \delta Z_\lambda = \frac{1}{\epsilon}\frac{3\hbar\lambda}{32\pi^2}\ ,
\tag{A.1.9}
$$

the effective potential reads:

$$
\boxed{V = \frac{\lambda}{4!}\phi_c^4 + \hbar\, \frac{\lambda^2 \phi_c^4}{256\pi^2} \left(\ln \frac{\frac{1}{2}\lambda \phi_c^2}{\Lambda^2} - \frac{3}{2} \right) .}
\tag{A.1.10}
$$

A.2. Computation of the Field Strength Renormalization Matrix

Since the full scalar propagator (see equation (3.4.15)) reads for our classically scale invariant theory (compare equation (6.2.31))

$$
\varphi_i \, \text{---}\!\!\bigcirc\!\!\text{---} \, \varphi_k \;\; = \frac{\mathrm{i}}{p^2 + \left((Z_\varphi^{\frac{1}{2}})^\mathrm{T} Z_\varphi^{\frac{1}{2}} - \mathbb{1} \right)_{ik} p^2 + \Pi_{ik}(p^2)} \;, \tag{A.2.1}
$$

the scalar self energy $\Pi(p^2)$, which is just the sum of all 1PI 2-point diagrams (equation (3.4.16)),

$$
\mathrm{i}\Pi(p^2) = \text{---}\!\!\bigcirc\!\!\text{---} \;, \tag{A.2.2}
$$

is related to the renormalized one by ([55, 56])

$$
\begin{bmatrix} \Pi_{\phi\phi}(p^2) & \Pi_{\phi h}(p^2) \\ \Pi_{h\phi}(p^2) & \Pi_{hh}(p^2) \end{bmatrix} = \begin{bmatrix} \Pi_{\phi\phi}^{(\mathrm{r})}(p^2) & \Pi_{\phi h}^{(\mathrm{r})}(p^2) \\ \Pi_{h\phi}^{(\mathrm{r})}(p^2) & \Pi_{hh}^{(\mathrm{r})}(p^2) \end{bmatrix} - \begin{bmatrix} \delta Z_{\phi\phi} & \delta Z_{\phi h}^{\frac{1}{2}} + \delta Z_{h\phi}^{\frac{1}{2}} \\ \delta Z_{\phi h}^{\frac{1}{2}} + \delta Z_{h\phi}^{\frac{1}{2}} & \delta Z_{hh} \end{bmatrix} p^2 \;. \tag{A.2.3}
$$

Hence, we have to calculate all infinite, momentum-dependent contributions to the scalar 1PI 2-point correlation functions to obtain the field strength renormalization matrix.

In the course of the following calculations, we make use of the Feynman parametrization [27]

$$
\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{(xA + (1-x)B)^2} \;, \tag{A.2.4}
$$

and that hence

$$
\frac{1}{\left((p-k)^2 - m^2 + \mathrm{i}0^+\right)\left(k^2 - m^2 + \mathrm{i}0^+\right)} =
$$
$$
= \int_0^1 \mathrm{d}x \frac{1}{\left((k - (p(1-x))^2 + p^2 x(1-x) - m^2 + \mathrm{i}0^+\right)^2} \;, \tag{A.2.5}
$$

where afterwards one is able to perform a shift, $k^\mu \to p^\mu(1-x)$, in momentum space. Furthermore, we are aware of the well-known d-dimensional integral in Minkowski space [27]

$$
\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta + \mathrm{i}0^+)^n} = \mathrm{i} \frac{(-1)^n}{(4\pi)^d} \frac{\Gamma(n - \frac{d}{2})}{\Gamma n} \left(\Delta - \mathrm{i}0^+\right)^{\frac{d}{2} - n} \;, \tag{A.2.6}
$$

and [57] serves as a helpful source for Feynman rules. Therefore, the relevant Feynman diagrams are given by:

$$i\Pi_{hh}^{f} = \text{h} \dashrightarrow \bigcirc \dashrightarrow \text{h}$$

$$= -3\,\tilde{\Lambda}^{4-d}\,\frac{y_t^2}{2}\int\frac{d^dk}{(2\pi)^d}\,\frac{\text{Tr}\left[(\slashed{p}+\slashed{k}+m_t)(\slashed{k}+m_t)\right]}{\left((p+k)^2-m_t^2+\text{i}0^+\right)\left(k^2-m_t^2+\text{i}0^+\right)}$$

$$= -3\,\tilde{\Lambda}^{4-d}\,\frac{y_t^2}{2}\int\frac{d^dk}{(2\pi)^d}\,\frac{4\left(p\cdot k+k^2+m_t^2\right)}{\left((p+k)^2-m_t^2+\text{i}0^+\right)\left(k^2-m_t^2+\text{i}0^+\right)}$$

$$= -3\,\tilde{\Lambda}^{4-d}\,y_t^2\int\frac{d^dk}{(2\pi)^d}\,\frac{\left((p+k)^2-m_t^2\right)+\left(k^2-m_t^2\right)+4m_t^2-p^2}{\left((p+k)^2-m_t^2+\text{i}0^+\right)\left(k^2-m_t^2+\text{i}0^+\right)}$$

$$= -3\,\tilde{\Lambda}^{4-d}\,y_t^2\int\frac{d^dk}{(2\pi)^d}\,\frac{4m_t^2-p^2}{\left((p+k)^2-m_t^2+\text{i}0^+\right)\left(k^2-m_t^2+\text{i}0^+\right)}+\ldots \tag{A.2.7}$$

$$= -3\,\tilde{\Lambda}^{4-d}\,y_t^2\int\frac{d^dk}{(2\pi)^d}\int_0^1\text{d}x\,\frac{4m_t^2-p^2}{\left(k^2+\underbrace{p^2x(1-x)-m_t^2}_{-\Delta}+\text{i}0^+\right)^2}+\ldots$$

$$= -3\,\tilde{\Lambda}^{4-d}\,y_t^2\int_0^1\text{d}x\,\left(4m_t^2-p^2\right)\frac{\text{i}}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\left(\Delta-\text{i}0^+\right)^{\frac{d}{2}-2}+\ldots$$

$$\overset{d=4-2\epsilon}{=} -3\,y_t^2\int_0^1\text{d}x\,\left(4m_t^2-p^2\right)\frac{\text{i}}{(4\pi)^2}\left(1+\epsilon\ln4\pi\right)$$
$$\cdot\left(\frac{1}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\right)\left(1-\epsilon\ln\Delta\right)\left(1+\epsilon\ln\tilde{\Lambda}^2\right)+\ldots$$

$$= \text{i}\,\frac{1}{\epsilon}\frac{3}{(4\pi)^2}\,y_t^2\,p^2+\ldots$$

$$
\mathrm{i}\Pi^{W^\pm}_{hh} = \quad \text{h} \cdots \cdots \underset{-p-k}{\overset{\overset{-k}{\underset{\substack{W^\pm \\ G^\pm_h}}{}}}{\cdots}} \cdots \text{h}
$$

$$
= -2\,\tilde\Lambda^{4-d}\,\frac{g^2}{4}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{(2p+k)_\mu (2p+k)_\nu}{k^2 - M_W^2 + \mathrm{i}0^+}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{(k^2+\mathrm{i}0^+)}\right)\left(\frac{1}{(k+p)^2+\mathrm{i}0^+}\right)
$$

$$
= -\tilde\Lambda^{4-d}\,\frac{g^2}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{\left(k^2 - M_W^2 + \mathrm{i}0^+\right)\left((k+p)^2+\mathrm{i}0^+\right)}\left((2p+k)^2 - \frac{((2p+k)\cdot k)^2}{k^2+\mathrm{i}0^+}\right)
$$

$$
= -\tilde\Lambda^{4-d}\,\frac{g^2}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{4p^2 k^2 - 4(pk)^2}{\left(k^2 - M_W^2 + \mathrm{i}0^+\right)\left((k+p)^2+\mathrm{i}0^+\right)\left(k^2+\mathrm{i}0^+\right)}
$$

$$
= -\tilde\Lambda^{4-d}\,\frac{g^2}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{4\left[k^2 p^2 - \left(\frac{1}{2}(p+k)^2 - \frac{1}{2}p^2 - \frac{1}{2}k^2\right)^2\right]}{\left(k^2 - M_W^2 + \mathrm{i}0^+\right)\left((k+p)^2+\mathrm{i}0^+\right)\left(k^2+\mathrm{i}0^+\right)}
$$

$$
= -\tilde\Lambda^{4-d}\,\frac{4g^2}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{\frac{1}{2}k^2 p^2 - (p+k)^2\cdot\left[\frac{1}{4}(p+k)^2 - \frac{1}{4}(p^2+k^2)\right] + \frac{1}{2}pk\left[p^2+k^2\right]}{\left(k^2 - M_W^2 + \mathrm{i}0^+\right)\left((k+p)^2+\mathrm{i}0^+\right)\left(k^2+\mathrm{i}0^+\right)}
$$

$$
= -\tilde\Lambda^{4-d}\,\frac{4g^2}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\left[\frac{\frac{1}{2}p^2}{\left(k^2 - M_W^2 + \mathrm{i}0^+\right)\left((k+p)^2+\mathrm{i}0^+\right)}\right.
$$
$$
\left.+\frac{\frac{1}{4}p^2}{\left(k^2 - M_W^2 + \mathrm{i}0^+\right)\left(k^2+\mathrm{i}0^+\right)} + \dots\right]
$$

$$
= -\tilde\Lambda^{4-d}\,\frac{4g^2}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\int_0^1 \mathrm{d}x\left[\frac{\frac{1}{2}p^2}{\left(k^2 - \bar\Delta^2 + \mathrm{i}0^+\right)^2} + \frac{\frac{1}{4}p^2}{\left(k^2 - \Delta^2 + \mathrm{i}0^+\right)^2}\right] + \dots
$$

$$
\overset{d=4-2\epsilon}{=} -\mathrm{i}\,\frac{1}{\epsilon}\,\frac{6}{4(4\pi)^2}\,g^2\,p^2 + \dots
$$

$$
\text{(A.2.8)}
$$

$$
\mathrm{i}\Pi^Z_{hh} = \quad \text{h} \cdots \cdots \underset{-p-k}{\overset{\overset{-k}{\underset{\substack{Z \\ G^0_h}}{}}}{\cdots}} \cdots \text{h}
$$

$$
= -\tilde\Lambda^{4-d}\,\frac{g'^2 + g^2}{4}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{(2p+k)_\mu (2p+k)_\nu}{k^2 - M_Z^2 + \mathrm{i}0^+}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{(k^2+\mathrm{i}0^+)}\right)\left(\frac{1}{(k+p)^2+\mathrm{i}0^+}\right)
$$

$$
= \dots
$$

$$
\overset{d=4-2\epsilon}{=} -\mathrm{i}\,\frac{1}{\epsilon}\,\frac{3}{4(4\pi)^2}\left(g'^2 + g^2\right)p^2 + \dots
$$

$$
\text{(A.2.9)}
$$

$$i\Pi_{\phi\phi}^X = \phi \cdots\cdots \overset{\overset{\displaystyle -k}{\frown}}{\underset{\underset{\displaystyle -p-k}{\leftarrow}}{\overset{X_i}{\underset{G_\phi^i}{}}}} \cdots\cdots \phi$$
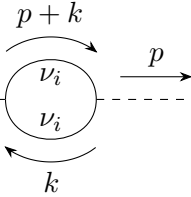
$$= -3\,\tilde{\Lambda}^{4-d}\,\frac{g_x^2}{4}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{(2p+k)_\mu (2p+k)_\nu}{k^2 - M_X^2 + \mathrm{i}0^+}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{(k^2 + \mathrm{i}0^+)}\right)\left(\frac{1}{(k+p)^2 + \mathrm{i}0^+}\right)$$

$$= \ldots$$

$$\overset{d=4-2\epsilon}{=} -\mathrm{i}\,\frac{1}{\epsilon}\,\frac{9}{4(4\pi)^2}\,g_x^2\,p^2 + \ldots \tag{A.2.10}$$

Since none of the other Feynman diagrams possess infinite and p-dependent contributions, we finally obtain the following results for the scalar field strength renormalization matrix:

$$\boxed{\delta Z_{\phi\phi} = \frac{1}{\epsilon}\,\frac{9g_x^2}{64\pi^2} \quad \text{and} \quad \delta Z_{hh} = \frac{1}{\epsilon}\,\frac{9g^2 + 3g'^2 - 12y_t^2}{64\pi^2}} \ . \tag{A.2.11}$$

In case of a theory with an additional real scalar field and right-handed neutrinos (see section 6.3 and especially (6.3.1)), we also have to calculate δZ_{ss} (compare (6.3.48a)) and hence have to consider the diagram

$$i\Pi_{ss}^N = \mathrm{s} \cdots\cdots \overset{\overset{\displaystyle p+k}{\frown}}{\underset{\underset{\displaystyle k}{\leftarrow}}{\bigcirc\begin{matrix}\nu_i\\\nu_i\end{matrix}}} \cdots\cdots \mathrm{s} \tag{A.2.12}$$

with a Majorana fermion loop. Using the Feynman rules for Majorana fields from [58] and neglecting light neutrino masses[1] leads by analogy with (A.2.7) to

$$i\Pi_{ss}^N = \mathrm{i}\frac{1}{\epsilon}\frac{1}{(4\pi)^2}\mathrm{Tr}\left(\Gamma_\mathrm{R}\Gamma_\mathrm{R}^*\right)p^2 + \ldots \ , \tag{A.2.13}$$

and thus:

$$\boxed{\delta Z_{ss} = \frac{1}{\epsilon}\left(-\frac{4}{64\pi^2}\mathrm{Tr}\left(\Gamma_\mathrm{R}\Gamma_\mathrm{R}^*\right)\right)} \ . \tag{A.2.14}$$

---

[1] $\sum \hat{m}_i^2 \approx \mathrm{Tr}\left(\Gamma_\mathrm{R}\Gamma_R^*\right)$ - see subsection 6.3.1.

B. General Lagrangian Density

Since we do not want to restrict our derivation of the effective potential in section 4.2 only to some special cases, the aim of this section is to construct a general Lorentz-invariant, gauge-invariant and renormalizable[1] Lagrangian density for a theory with n real spinless fields, m Dirac bispinor fields and s real gauge fields. In the following discussion we draw some inspiration from Steven Weinberg ([34–37]) and adapt his formalism here and there.

B.1. Gauge Field Lagrangian Density

We assume that the Lagrangian density is gauge-invariant with respect to a gauge group $\mathcal{G}$, which can be written as a direct product of subgroups SU(n) and U(1). Hence, the s various generators T_a of the gauge group $\mathcal{G}$ are hermitian and obey the commutation relation

$$[T_a, T_b] = i f_{abc} T_c \ , \tag{B.1.1}$$

with the real and totally antisymmetric structure constant f_{abc}. The generalized field strength tensor is defined as

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - f_{abc} A_b^\mu A_c^\nu \ , \tag{B.1.2}$$

where the s real gauge fields A_a transform as:

$$A_a^\mu(x) \longrightarrow A_a^\mu(x) - \partial^\mu \alpha_a(x) + f_{abc} A_b^\mu(x) \alpha_c(x). \tag{B.1.3}$$

Out of the field strength tensor $F_a^{\mu\nu}$ we can build up a general Lorentz- and gauge-invariant Lagrangian density of the simple form

$$\boxed{\mathcal{L}_A = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu,a} \ ,} \tag{B.1.4}$$

since the second possible term $c \cdot \varepsilon_{\rho\sigma\mu\nu} F_a^{\rho\sigma} F_a^{\mu\nu}$ can be written as a total derivative .

B.1.1. Parity-violating Gauge Term

We want to show as a little exercise that the term $\varepsilon^{\rho\sigma\mu\nu} F_{\rho\sigma}^a F_{\mu\nu}^a$ can indeed be rewritten as a total derivative. Using the fact that the Levi-Civita-Symbol is total antisymmetric, it is easy to expand the above expression in the following way:

$$\varepsilon^{\rho\sigma\mu\nu} F_{\rho\sigma}^a F_{\mu\nu}^a = \varepsilon^{\rho\sigma\mu\nu} \left(2 \cdot \partial_\rho A_\sigma^a + f^{abc} A_\rho^b A_\sigma^c \right) \left(2 \cdot \partial_\mu A_\nu^a + f^{ade} A_\mu^d A_\nu^e \right) =$$

$$= \varepsilon^{\rho\sigma\mu\nu} \left[\underbrace{4 \cdot (\partial_\rho A_\sigma^a)(\partial_\mu A_\nu^a)}_{\text{I}} + \underbrace{4 \cdot f^{abc} A_\mu^b A_\nu^c (\partial_\rho A_\sigma^a)}_{\text{II}} + \underbrace{f^{abc} f^{ade} A_\rho^b A_\sigma^c A_\mu^d A_\nu^e}_{\text{III}} \right] \ . \tag{B.1.5}$$

---

[1] For $\hbar = 1$, only coupling constants with mass dimension ≥ 0 are allowed.

I: The first part is straight forward,

$$
\begin{aligned}
4 \cdot \varepsilon^{\rho\sigma\mu\nu} (\partial_\rho A_\sigma^a)(\partial_\mu A_\nu^a) &= 4 \cdot \varepsilon^{\rho\sigma\mu\nu} [\partial_\rho (A_\sigma^a \, \partial_\mu A_\nu^a) - A_\sigma^a \, \partial_\rho \partial_\mu A_\nu^a] \\
&= 4 \cdot \varepsilon^{\rho\sigma\mu\nu} \partial_\rho (A_\sigma^a \, \partial_\mu A_\nu^a) \ ,
\end{aligned}
\tag{B.1.6}
$$

since $\varepsilon^{\rho\sigma\mu\nu}$ is antisymmetric in ρ and μ, while $\partial_\rho \partial_\mu$ is symmetric.

II: If we realize that f^{abc} is also totally antisymmetric, we can write the second term as:

$$
4 \cdot \epsilon^{\rho\sigma\mu\nu} f^{abc} A_\mu^b A_\nu^c (\partial_\rho A_\sigma^a) = \frac{4}{3} \cdot \varepsilon^{\rho\sigma\mu\nu} f^{abc} \, \partial_\rho \left(A_\sigma^a A_\mu^b A_\nu^c \right) \ .
\tag{B.1.7}
$$

III: We can relabel the Latin indices and write the third part in the form

$$
\begin{aligned}
\varepsilon^{\rho\sigma\mu\nu} f^{abc} f^{ade} A_\rho^b A_\sigma^c A_\mu^d A_\nu^e = \frac{1}{3} \cdot \epsilon^{\rho\sigma\mu\nu} [\, & f^{abc} f^{ade} A_\rho^b A_\sigma^c A_\mu^d A_\nu^e \\
+ & f^{acd} f^{abe} A_\rho^c A_\sigma^d A_\mu^b A_\nu^e \\
+ & f^{adb} f^{ace} A_\rho^d A_\sigma^b A_\mu^c A_\nu^e \,] = \ ,
\end{aligned}
\tag{B.1.8}
$$

and if we relabel the Greek indices as well and have the antisymmetry of $\varepsilon^{\rho\sigma\mu\nu}$ in mind, we find

$$
= \frac{1}{3} \cdot \varepsilon^{\rho\sigma\mu\nu} \left[f^{abc} f^{ade} + f^{acd} f^{abe} + f^{adb} f^{ace} \right] A_\rho^b A_\sigma^c A_\mu^d A_\nu^e \ .
\tag{B.1.9}
$$

Since the structure constants satisfy the Jacoby identity

$$
f^{abc} f^{ade} + f^{acd} f^{abe} + f^{adb} f^{ace} = 0 \ ,
\tag{B.1.10}
$$

the third term is equal to zero.

Therefore, we are finally able to write down the above expression as a total derivative:

$$
\boxed{\varepsilon^{\rho\sigma\mu\nu} F_{\rho\sigma}^a F_{\mu\nu}^a = 4 \cdot \varepsilon^{\rho\sigma\mu\nu} \partial_\rho \left(A_\sigma^a \, \partial_\mu A_\nu^a + \frac{1}{3} \cdot A_\sigma^a A_\mu^b A_\nu^c \right) \ .}
\tag{B.1.11}
$$

B.2. Scalar Lagrangian Density

Seeing that a complex scalar field $\varphi(x)$ can always be rewritten with the help of two real spinless fields $\varphi_{1,2}$,

$$
\varphi(x) = \mathrm{Re}[\varphi(x)] + \mathrm{i}\, \mathrm{Im}[\varphi(x)] = \varphi_1(x) + \mathrm{i}\, \varphi_2(x) \ ,
\tag{B.2.1}
$$

it is sufficient to consider a theory with only real scalar fields and hence, we define φ to be a set of n such real fields.

B.2.1. Kinetic Term

A general kinetic term,

$$
\mathcal{L}_{\mathrm{Kin}} = \frac{1}{2} Z_{ik} \left(\partial_\mu \varphi_i' \right) \left(\partial^\mu \varphi_k' \right) \ ,
\tag{B.2.2}
$$

with the real and symmetric[2] mixing matrix Z, can easily be diagonalized with the help of an orthogonal matrix O through

$$Z = O^{\mathrm{T}} \hat{Z} O , \tag{B.2.3}$$

and after an appropriate field redefinition,

$$\varphi = \hat{Z}^{\frac{1}{2}} O \varphi' , \tag{B.2.4}$$

the Lagrangian density reads:

$$\mathcal{L}_{Kin} = \frac{1}{2} \left(\partial_\mu \varphi \right)_i \left(\partial^\mu \varphi \right)_i . \tag{B.2.5}$$

Since this Lagrangian density is clearly not invariant under an arbitrary gauge transformation of the form

$$\varphi \rightarrow \mathrm{e}^{\mathrm{i} \alpha_a(x)\, \theta_a} \varphi , \tag{B.2.6}$$

we have to introduce the covariant derivative

$$(D^\mu \varphi)_i = \partial^\mu \varphi_i + \mathrm{i} (\theta_a)_{ik} \varphi_k A_a^\mu , \tag{B.2.7}$$

where θ_a denotes the matrix representation of the generator T_a in the representation of $\mathcal{G}$, under which the scalar fields φ_i transform. These s antisymmetric and hermitian $n \times n$ gauge coupling matrices (group generators) are proportional to the gauge coupling constants and satisfy the commutation relation

$$[\theta_a, \theta_b] = \mathrm{i} f_{abc} \theta_c . \tag{B.2.8}$$

From the transformation behaviour of gauge fields (B.1.3) follows that $D^\mu \varphi$ changes under a gauge transformation in the same way as φ and we finally obtain a gauge invariant kinetic Lagrangian density:

$$\boxed{\mathcal{L}_{\mathrm{Kin}} = \frac{1}{2} \left(D_\mu \varphi \right)_i \left(D^\mu \varphi \right)_i .} \tag{B.2.9}$$

For a more detailed discussion of (non-abelian) gauge invariance the interested reader is, amongst others, referred to [27, 36, 37, 59].

B.2.2. Interaction Term

For the potential $V_0(\varphi)$ we choose the most general 4th-order polynomial

$$\boxed{V_0(\varphi) = \kappa_i \varphi_i + \mu_{ik} \varphi_i \varphi_k + \rho_{ikm} \varphi_i \varphi_k \varphi_m + \lambda_{ikmn} \varphi_i \varphi_k \varphi_m \varphi_n ,} \tag{B.2.10}$$

with real coefficients κ_i, μ_{ik}, ρ_{ikm} and λ_{ikmn}. Gauge invariance requires that the condition

$$\frac{\partial V_0(\varphi)}{\partial \varphi_i} (\theta_a)_{ik} \varphi_k = 0 \tag{B.2.11}$$

is fulfilled, since an infinitesimal gauge transformation yields

$$V_0(\varphi) \overset{!}{=} V_0(e^{\mathrm{i}\alpha_a \theta_a} \varphi) = V_0(\varphi) + \mathrm{i}\alpha_a \frac{\partial V_0(\varphi)}{\varphi_i} (\theta_a)_{ik} \varphi_k . \tag{B.2.12}$$

---

[2] A consequence of the fact that the Lagrangian has to be a real quantity and that the antisymmetric parts cancel each other out.

B.3. Fermionic Lagrangian Density

B.3.1. Kinetic Term

If we consider a set ψ of m Dirac bispinor fields and want to be able to describe the dynamics of the fermions (and antifermions) of the theory, we have to introduce a kinetic term of the general form[3]

$$
\begin{aligned}
\mathcal{L}_{\text{Kin}} = \frac{1}{2}\Big[& \overline{\psi}\left(\mathbb{1}\otimes \mathrm{i}\gamma_\mu\partial^\mu\right)\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5\right)\psi \\
& + \overline{\psi^{\mathrm{c}}}\left(\mathbb{1}\otimes \mathrm{i}\gamma_\mu\partial^\mu\right)\left(A^{\mathrm{T}}\otimes\mathbb{1}+\mathrm{i}\cdot B^{\mathrm{T}}\otimes\gamma_5\right)\psi^{\mathrm{c}} \\
& + \overline{\psi}\left(\mathbb{1}\otimes \mathrm{i}\gamma_\mu\partial^\mu\right)\left(E\otimes\mathbb{1}+\mathrm{i}\cdot F\otimes\gamma_5\right)\psi^{\mathrm{c}} \\
& + \overline{\psi^{\mathrm{c}}}\left(\mathbb{1}\otimes \mathrm{i}\gamma_\mu\partial^\mu\right)\left(E\otimes\mathbb{1}+\mathrm{i}\cdot F\otimes\gamma_5\right)^{\dagger}\psi \Big]\,,
\end{aligned}
\tag{B.3.1}
$$

with the adjoint spinor

$$
\overline{\psi}_i = \psi_i^{\dagger}\gamma_0\,,
\tag{B.3.2}
$$

and the charge-conjugated field (see appendix C.2)

$$
\psi_i^{\mathrm{c}} = C\gamma_0^{\mathrm{T}}\psi_i^{*}\,.
\tag{B.3.3}
$$

We already made use of the identities (see (C.4.4))

$$
\begin{aligned}
\overline{\psi^{\mathrm{c}}}\left(\mathbb{1}\otimes \mathrm{i}\gamma_\mu\partial^\mu\right)\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5\right)\psi^{\mathrm{c}} = \\
= \overline{\psi}\left(\mathbb{1}\otimes \mathrm{i}\gamma_\mu\partial^\mu\right)\left(A^{\mathrm{T}}\otimes\mathbb{1}+\mathrm{i}\cdot B^{\mathrm{T}}\otimes\gamma_5\right)\psi\,,
\end{aligned}
\tag{B.3.4}
$$

and (see (C.4.5))

$$
\begin{aligned}
\left[\overline{\psi}\left(\mathbb{1}\otimes \mathrm{i}\gamma_\mu\partial^\mu\right)\left(E\otimes\mathbb{1}+\mathrm{i}\cdot F\otimes\gamma_5\right)\psi^{\mathrm{c}}\right]^{\dagger} = \\
= \overline{\psi^{\mathrm{c}}}\left(\mathbb{1}\otimes \mathrm{i}\gamma_\mu\partial^\mu\right)\left(E\otimes\mathbb{1}+\mathrm{i}\cdot F\otimes\gamma_5\right)^{\dagger}\psi\,,
\end{aligned}
\tag{B.3.5}
$$

and introduced the Kronecker product, denoted by $\otimes$, to deal with the 4-dimensional substructure of the m Dirac bispinor fields ψ_i. Furthermore, one can show that (see (C.4.6))

$$
\begin{aligned}
\left[\overline{\psi}\left(\mathbb{1}\otimes \mathrm{i}\gamma_\mu\partial^\mu\right)\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5\right)\psi\right]^{\dagger} = \\
= \overline{\psi}\left(\mathbb{1}\otimes \mathrm{i}\gamma_\mu\partial^\mu\right)\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5\right)^{\dagger}\psi\,,
\end{aligned}
\tag{B.3.6}
$$

and hence, A and $\mathrm{i}B$ have to be hermitian matrices. The kinetic part of the Lagrangian density can then be written in a very compact way:

$$
\mathcal{L}_{\text{Kin}} = \frac{1}{2}\underbrace{\left(\overline{\psi}\ \ \overline{\psi^{\mathrm{c}}}\right)}_{\overline{\omega}}\left(\mathbb{1}_{2m}\otimes \mathrm{i}\gamma_\mu\partial^\mu\right)\left(\underbrace{\begin{bmatrix}A & E\\ E^{\dagger} & A^{\mathrm{T}}\end{bmatrix}\otimes\mathbb{1}+\mathrm{i}\cdot\begin{bmatrix}B & F\\ -F^{\dagger} & B^{\mathrm{T}}\end{bmatrix}\otimes\gamma_5}_{Y}\right)\underbrace{\begin{pmatrix}\psi\\ \psi^{\mathrm{c}}\end{pmatrix}}_{\omega}\,.
\tag{B.3.7}
$$

Since the Matrix Y is hermitian, it can be diagonalized with the help of a unitary matrix U,

$$
Y = U\hat{Y}U^{\dagger}\,,
\tag{B.3.8}
$$

---

[3] $\overline{\psi}_i\gamma_\mu\psi_j$ and $\overline{\psi}_i\gamma_\mu\gamma_5\psi_j$ transform as four-vectors and therefore, Lorentz-invariance is guaranteed.

and after a suitable redefinition of the fermionic fields,

$$\omega' = \hat{Y}^{\frac{1}{2}} U^\dagger \omega \,, \tag{B.3.9}$$

we find

$$\mathcal{L}_{\text{Kin}} = \frac{1}{2} \overline{\omega} \left(\mathbb{1}_{2m} \otimes \mathrm{i}\gamma_\mu \partial^\mu \right) \omega = \overline{\psi} \left(\mathbb{1}_m \otimes \mathrm{i}\gamma_\mu \partial^\mu \right) \psi \,, \tag{B.3.10}$$

where we immediately have dropped the apostrophe for the sake of simplicity.

Again, we have to introduce the covariant derivative,

$$\left(D^\mu \psi \right)_i = \partial^\mu \psi_i + i \left(t_a \right)_{ik} \psi_k A_a^\mu \,, \tag{B.3.11}$$

to make the Lagrangian density invariant under an arbitrary gauge transformation of the form:

$$\psi \to \mathrm{e}^{\mathrm{i}\alpha_a(x) \, t_a} \psi \,. \tag{B.3.12}$$

The m hermitian gauge coupling matrices t_a, which are the matrix representations of the generators T_a in the representation of $\mathcal{G}$, under which the fermion fields ψ transform, are proportional to the gauge coupling constants and satisfy the commutation relation

$$[t_a, t_b] = \mathrm{i} f_{abc} t_c \,. \tag{B.3.13}$$

For the same reason as for the scalar Lagrangian density we obtain a gauge invariant kinetic fermionic Lagrangian density:

$$\boxed{\mathcal{L}_{\text{Kin}} = \frac{1}{2} \overline{\omega} \left(\mathbb{1}_{2m} \otimes \mathrm{i}\slashed{D} \right) \omega = \overline{\psi} \left(\mathbb{1}_m \otimes \mathrm{i}\slashed{D} \right) \psi \,.} \tag{B.3.14}$$

To be able to deal with chiral theories (e.g. the weak interaction), we rewrite the Lagrangian density with the help of left-handed and right-handed fermion-fields (appendix C.3)

$$\psi_{\text{L},i} = \frac{1}{2} \left(\mathbb{1} - \gamma_5 \right) \psi_i \quad \text{and} \quad \psi_{\text{R},i} = \frac{1}{2} \left(\mathbb{1} + \gamma_5 \right) \psi_i \,, \tag{B.3.15}$$

and find (compare (C.4.10))

$$\boxed{\mathcal{L}_{\text{Kin}} = \overline{\psi_{\text{L}}} \left(\mathbb{1} \otimes \mathrm{i}\slashed{D} \right) \psi_{\text{L}} + \overline{\psi_{\text{R}}} \left(\mathbb{1} \otimes \mathrm{i}\slashed{D} \right) \psi_{\text{R}} \,.} \tag{B.3.16}$$

B.3.2. Dirac-Mass-Term

An explicit Dirac mass term for a set ψ of m Dirac bispinor fields has the form[4]

$$\mathcal{L}_{\text{D}} = -\frac{1}{2} \left[\overline{\psi} \left(A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5 \right) \psi + \overline{\psi^{\text{c}}} \left(A^{\text{T}} \otimes \mathbb{1} + \mathrm{i} \cdot B^{\text{T}} \otimes \gamma_5 \right) \psi^{\text{c}} \right] \,, \tag{B.3.17}$$

where A and B must be hermitian matrices, due to the fact that the Lagrangian has to be real and (see (C.4.7))

$$\left[\overline{\psi} \left(A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5 \right) \psi \right]^\dagger = \overline{\psi} \left(A^\dagger \otimes \mathbb{1} + \mathrm{i} \cdot B^\dagger \otimes \gamma_5 \right) \psi \,. \tag{B.3.18}$$

---

[4] $\overline{\psi}_i \psi_j$ and $\overline{\psi}_i \gamma_5 \psi_j$ transform as scalars and pseudoscalars respectively.

Furthermore, we already used the relation (see (C.4.8)):

$$\overline{\psi^{\mathrm{c}}}\left(A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5\right)\psi^{\mathrm{c}} = \overline{\psi}\left(A^{\mathrm{T}} \otimes \mathbb{1} + \mathrm{i} \cdot B^{\mathrm{T}} \otimes \gamma_5\right)\psi \, , \tag{B.3.19}$$

to simplify (B.3.17). With the help of the chiral fields ψ_{L} and ψ_{R} we rewrite the above expression (see (C.4.11)),

$$
\begin{aligned}
\overline{\psi}\left[A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5\right]\psi &= \\
&= \overline{\psi_{\mathrm{L}}}\left[A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5\right]\psi_{\mathrm{R}} + \overline{\psi_{\mathrm{R}}}\left[A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5\right]\psi_{\mathrm{L}} = \\
&= \overline{\psi_{\mathrm{L}}}\left[\underbrace{(A + \mathrm{i} \cdot B)}_{M_{\mathrm{D}}} \otimes \mathbb{1}\right]\psi_{\mathrm{R}} + \overline{\psi_{\mathrm{R}}}\left[\underbrace{(A - \mathrm{i} \cdot B)}_{M_{\mathrm{D}}^{\dagger}} \otimes \mathbb{1}\right]\psi_{\mathrm{L}} \, ,
\end{aligned}
\tag{B.3.20}
$$

and we finally get:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{D}} = -\frac{1}{2}\bigg[&\overline{\psi_{\mathrm{L}}}\left(M_{\mathrm{D}} \otimes \mathbb{1}\right)\psi_{\mathrm{R}} + \overline{\psi_{\mathrm{R}}}\left(M_{\mathrm{D}}^{\dagger} \otimes \mathbb{1}\right)\psi_{\mathrm{L}} \\
&+ \overline{(\psi_{\mathrm{L}})^{\mathrm{c}}}\left(M_{\mathrm{D}}^{*} \otimes \mathbb{1}\right)(\psi_{\mathrm{R}})^{\mathrm{c}} + \overline{(\psi_{\mathrm{R}})^{\mathrm{c}}}\left(M_{\mathrm{D}}^{\mathrm{T}} \otimes \mathbb{1}\right)(\psi_{\mathrm{L}})^{\mathrm{c}}\bigg] \\
\overset{(\mathrm{B.3.19})}{=}& -\overline{\psi_{\mathrm{L}}}\left(M_{\mathrm{D}} \otimes \mathbb{1}\right)\psi_{\mathrm{R}} - \overline{\psi_{\mathrm{R}}}\left(M_{\mathrm{D}}^{\dagger} \otimes \mathbb{1}\right)\psi_{\mathrm{L}} \, .
\end{aligned}
\tag{B.3.21}
$$

We have to emphasize that only those matrix elements are allowed to be nonzero, which lead to gauge invariant mass terms. Since chiral fermion fields transform under an arbitrary gauge transformation as (appendix (C.3))

$$
\begin{aligned}
\psi_{\mathrm{R}} &\to \mathrm{e}^{\mathrm{i}\alpha_{\mathrm{i}}(x)t_{\mathrm{R},i}}\psi_{\mathrm{R}} \\
\psi_{\mathrm{L}} &\to \mathrm{e}^{\mathrm{i}\alpha_{\mathrm{i}}(x)t_{\mathrm{L},i}}\psi_{\mathrm{L}} \, ,
\end{aligned}
\tag{B.3.22}
$$

the following relation has to be fulfilled:

$$t_{\mathrm{L},i}\left(M_{\mathrm{D}} \otimes \mathbb{1}\right) - \left(M_{\mathrm{D}} \otimes \mathbb{1}\right)t_{\mathrm{R},i} = 0 \, . \tag{B.3.23}$$

For the chiral Standard Model this condition is only true for $M_{\mathrm{D}} = 0$, due to the fact that right-handed fermions are singlets with respect to the weak interaction ($t_{\mathrm{R},SU(2)} = 0$, but $t_{\mathrm{L},SU(2)} \neq 0$).

Nevertheless, even forbidden explicit Dirac mass terms can still appear in the Lagrangian density after spontaneous symmetry breaking. This can be achieved through the famous Higgs mechanism, where two fermions couple to a real scalar field φ_i (Yukawa interaction) with a nonzero vacuum expectation value:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{D}}^{\mathrm{Yuk}} = -\frac{1}{2}\bigg[&\overline{\psi_{\mathrm{L}}}\left(\Gamma_{\mathrm{D},i}\varphi_i \otimes \mathbb{1}\right)\psi_{\mathrm{R}} + \overline{\psi_{\mathrm{R}}}\left(\Gamma_{\mathrm{D},i}^{\dagger}\varphi_i \otimes \mathbb{1}\right)\psi_{\mathrm{L}} \\
&+ \overline{(\psi_{\mathrm{L}})^{\mathrm{c}}}\left(\Gamma_{\mathrm{D},i}^{*}\varphi_i \otimes \mathbb{1}\right)(\psi_{\mathrm{R}})^{\mathrm{c}} + \overline{(\psi_{\mathrm{R}})^{\mathrm{c}}}\left(\Gamma_{\mathrm{D},i}^{\mathrm{T}}\varphi_i \otimes \mathbb{1}\right)(\psi_{\mathrm{L}})^{\mathrm{c}}\bigg] \\
\overset{(\mathrm{B.3.19})}{=}& -\overline{\psi_{\mathrm{L}}}\left(\Gamma_{\mathrm{D},i}\varphi_i \otimes \mathbb{1}\right)\psi_{\mathrm{R}} - \overline{\psi_{\mathrm{R}}}\left(\Gamma_{\mathrm{D},i}^{\dagger}\varphi_i \otimes \mathbb{1}\right)\psi_{\mathrm{L}} \, .
\end{aligned}
\tag{B.3.24}
$$

Again, only those matrix elements of the Yukawa coupling matrix $\Gamma_{\mathrm{D},i}$ are nonzero that lead to gauge invariant terms. From the equations (B.2.6) and (B.3.22) we find the following condition for gauge invariance:

$$t_{\mathrm{L},i}(\Gamma_{\mathrm{D},m} \otimes \mathbb{1}) - (\Gamma_{\mathrm{D},m} \otimes \mathbb{1})t_{\mathrm{R},i} = \Gamma_{\mathrm{D},n}(\theta_i)_{nm} \otimes \mathbb{1} \, . \tag{B.3.25}$$

B.3.3. Majorana-Mass-Term

We are able to obtain a second type of mass terms, the so called Majorana mass term, if we connect fermions with antifermions and from the identity (see (C.4.9))

$$\left[\overline{\psi^c}\left(A_2 \otimes \mathbb{1} + \mathrm{i} \cdot B_2 \otimes \gamma_5\right)\psi\right]^\dagger = \overline{\psi}\left(A_2^\dagger \otimes \mathbb{1} + \mathrm{i} \cdot B_2^\dagger \otimes \gamma_5\right)\psi^c\ , \tag{B.3.26}$$

we find:

$$\mathcal{L}_{\mathrm{Maj}} = -\frac{1}{2}\left[\overline{\psi}\left(A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5\right)\psi^c + \overline{\psi^c}\left(A^\dagger \otimes \mathbb{1} + \mathrm{i} \cdot B^\dagger \otimes \gamma_5\right)\psi\right]\ . \tag{B.3.27}$$

After defining

$$\begin{aligned} A &= \frac{1}{2}\left(M_{\mathrm{R}}^\dagger + M_{\mathrm{L}}\right) \\ B &= \frac{\mathrm{i}}{2}\left(M_{\mathrm{R}}^\dagger - M_{\mathrm{L}}\right)\ , \end{aligned} \tag{B.3.28}$$

we can rewrite the above Lagrangian density as

$$\begin{aligned} \mathcal{L}_{\mathrm{Maj}} = -\frac{1}{2}\Bigg[&\overline{\psi}\left(M_{\mathrm{L}} \otimes \frac{1}{2}\left(\mathbb{1} + \gamma_5\right) + M_{\mathrm{R}}^\dagger \otimes \frac{1}{2}\left(\mathbb{1} - \gamma_5\right)\right)\psi^c \\ &+ \overline{\psi^c}\left(M_{\mathrm{L}}^\dagger \otimes \frac{1}{2}\left(\mathbb{1} - \gamma_5\right) + M_{\mathrm{R}} \otimes \frac{1}{2}\left(\mathbb{1} + \gamma_5\right)\right)\psi\Bigg]\ , \end{aligned} \tag{B.3.29}$$

and if we again introduce the chiral fields ψ_{L} and ψ_{R}, we finally get:

$$\boxed{\begin{aligned} \mathcal{L}_{\mathrm{Maj}} = -\frac{1}{2}\Bigg[&\overline{\psi_{\mathrm{L}}}\left(M_{\mathrm{L}} \otimes \mathbb{1}\right)\psi_{\mathrm{L}}^c + \overline{(\psi_{\mathrm{L}})^c}\left(M_{\mathrm{L}}^\dagger \otimes \mathbb{1}\right)\psi_{\mathrm{L}} \\ &+ \overline{\psi_{\mathrm{R}}}\left(M_{\mathrm{R}}^\dagger \otimes \mathbb{1}\right)\left(\psi_{\mathrm{R}}\right)^c + \overline{(\psi_{\mathrm{R}})^c}\left(M_{\mathrm{R}} \otimes \mathbb{1}\right)\psi_{\mathrm{R}}\Bigg]\ . \end{aligned}} \tag{B.3.30}$$

Since chiral antifermions transform, analog to equation (B.3.22), as

$$\begin{aligned} \left(\psi_{\mathrm{R}}\right)^c &\to \mathrm{e}^{-\mathrm{i}\alpha_i(x)t_{\mathrm{R},i}^*}\left(\psi_{\mathrm{R}}\right)^c \\ \left(\psi_{\mathrm{L}}\right)^c &\to \mathrm{e}^{-\mathrm{i}\alpha_i(x)t_{\mathrm{L},i}^*}\left(\psi_{\mathrm{L}}\right)^c\ , \end{aligned} \tag{B.3.31}$$

the condition for gauge-invariance is given by:

$$\begin{aligned} t_{\mathrm{L},i}M_{\mathrm{L}} + M_{\mathrm{L}}t_{\mathrm{L},i}^* &= 0 \\ t_{\mathrm{R},i}M_{\mathrm{R}}^\dagger + M_{\mathrm{R}}^\dagger t_{\mathrm{R},i}^* &= 0\ . \end{aligned} \tag{B.3.32}$$

In the Standard Model, this is only nontrivially true for sterile fermions, e.g. right-handed neutrinos, which only interact via gravity.

Furthermore, we realize that M_{L} and M_{R} are symmetric matrices

$$M_{\mathrm{L}} = M_{\mathrm{L}}^{\mathrm{T}} \quad\text{and}\quad M_{\mathrm{R}} = M_{\mathrm{R}}^{\mathrm{T}}\ , \tag{B.3.33}$$

since the following identity holds (see (C.4.12)):

$$\overline{\psi_{\mathrm{L}}}\left(M_{\mathrm{L}} \otimes \mathbb{1}\right)\left(\psi_{\mathrm{L}}\right)^c = \overline{\psi_{\mathrm{L}}}\left(M_{\mathrm{L}}^{\mathrm{T}} \otimes \mathbb{1}\right)\left(\psi_{\mathrm{L}}\right)^c\ . \tag{B.3.34}$$

Again, primary forbidden terms can be generated via Yukawa interactions and spontaneous symmetry breaking:

$$\mathcal{L}_{\mathrm{Maj}}^{\mathrm{Yuk}} = -\frac{1}{2}\left[\overline{\psi_{\mathrm{L}}}\big(\Gamma_{\mathrm{L},i}\varphi_i \otimes \mathbb{1}\big)(\psi_{\mathrm{L}})^{\mathrm{c}} + \overline{(\psi_{\mathrm{L}})^{\mathrm{c}}}\big(\Gamma_{\mathrm{L},i}^{\dagger}\varphi_i \otimes \mathbb{1}\big)\psi_{\mathrm{L}} \right.$$
$$\left. + \overline{(\psi_{\mathrm{R}})^{\mathrm{c}}}\big(\Gamma_{\mathrm{R},i}\varphi_i \otimes \mathbb{1}\big)\psi_{\mathrm{R}} + \overline{\psi_{\mathrm{R}}}\big(\Gamma_{\mathrm{R},i}^{\dagger}\varphi_i \otimes \mathbb{1}\big)(\psi_{\mathrm{R}})^{\mathrm{c}} \right]. \tag{B.3.35}$$

Gauge invariance is guaranteed as long as

$$t_{\mathrm{L},i}\left(\Gamma_{\mathrm{L},m} \otimes \mathbb{1}\right) + \left(\Gamma_{\mathrm{L},m} \otimes \mathbb{1}\right)t_{\mathrm{L},i}^{*} = \Gamma_{\mathrm{L},n}(\theta_i)_{nm} \otimes \mathbb{1}$$
$$t_{\mathrm{R},i}(\Gamma_{\mathrm{R},m}^{\dagger} \otimes \mathbb{1}) + (\Gamma_{\mathrm{R},m}^{\dagger} \otimes \mathbb{1})t_{\mathrm{R},i}^{*} = \Gamma_{\mathrm{R},n}^{\dagger}(\theta_i)_{nm} \otimes \mathbb{1} \tag{B.3.36}$$

is fulfilled and for the same reason as above, $\Gamma_{\mathrm{L},i}$ and $\Gamma_{\mathrm{R},i}$ are symmetric $m \times m$ matrices.

C. Fun with Gamma Matrices

We want to prove some of the relations used in appendix B and start with a short summary about gamma matrices, charge-conjugation and chiral fermion fields. For more details see [51, 60] .

C.1. Gamma Matrices

The 4×4 gamma matrices $(\gamma_\mu) = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ are defined by the commutation relation

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} \mathbb{1} \, , \tag{C.1.1}$$

and satisfy

$$\begin{aligned} \gamma_0^\dagger &= \gamma_0 \\ \gamma_i^\dagger &= -\gamma_i \, . \end{aligned} \tag{C.1.2}$$

Furthermore, the fifth gamma matrix, γ_5, is given by

$$\gamma_5 := \mathrm{i}\gamma_0 \gamma_1 \gamma_2 \gamma_3 \, , \tag{C.1.3}$$

and from (C.1.1) and (C.1.2) follow:

$$\begin{aligned} \{\gamma_5, \gamma_\mu\} &= 0 \\ \gamma_5^\dagger &= \gamma_5 \, . \end{aligned} \tag{C.1.4}$$

C.2. Charge-Conjugation

The charge-conjugation operator interchanges particles and antiparticles and hence flips all internal quantum numbers (charges). For a Dirac bispinor field ψ_i the charge conjugated field ψ_i^{c} is given by:

$$\boxed{\psi_i^{\mathrm{c}} = C\gamma_0^{\mathrm{T}} \psi_i^* \, .} \tag{C.2.1}$$

The charge conjugation matrix C is defined by the relation

$$C^{-1}\gamma_\mu C = -\gamma_\mu^{\mathrm{T}} \, , \tag{C.2.2}$$

and obeys [51]:

$$\begin{aligned} C^\dagger &= C^{-1} \\ C^{\mathrm{T}} &= -C \, . \end{aligned} \tag{C.2.3}$$

For the adjoint charge conjugated spinor we find

$$\overline{\psi_i^{\mathrm{c}}} = \left(C\gamma_0^{\mathrm{T}} \psi^*\right)^\dagger \gamma_0 \overset{(C.2.3)}{=} \psi^{\mathrm{T}} \gamma_0^* C^{-1} \gamma_0 \overset{(C.2.2)}{=} -\psi^{\mathrm{T}}(\gamma_0 \gamma_0^\dagger)^{\mathrm{T}} C^{-1} \overset{(C.1.1)}{=} -\psi^{\mathrm{T}} C^{-1} \, , \tag{C.2.4}$$

and from (C.1.1) and (C.2.2) we get a commutation relation for γ_5 and C:

$$C^{-1}\gamma_5 = \gamma_5^{\mathrm{T}} C^{-1} \ . \tag{C.2.5}$$

C.3. Chiral Fermion Fields

If we want to deal with chiral theories, we have to introduce the two projection operators

$$\boxed{P_{\mathrm{L}} = \frac{1}{2}(\mathbb{1} - \gamma_5) \quad \text{and} \quad P_{\mathrm{R}} = \frac{1}{2}(\mathbb{1} + \gamma_5) \ ,} \tag{C.3.1}$$

with the simple properties:

$$\begin{aligned} P_{\mathrm{L}} + P_{\mathrm{R}} &= \mathbb{1} \\ P_{\mathrm{L}} \cdot P_{\mathrm{R}} &= 0 \\ (P_{\mathrm{L/R}})^2 &= P_{\mathrm{L/R}} \ . \end{aligned} \tag{C.3.2}$$

Therefore, we can decompose a Dirac spinor ψ_i in its left-handed and right-handed components,

$$\psi_i = (P_{\mathrm{L}} + P_{\mathrm{R}})\psi_i = P_{\mathrm{L}}\psi_i + P_{\mathrm{R}}\psi_i = \psi_{i\mathrm{L}} + \psi_{i\mathrm{R}} \ , \tag{C.3.3}$$

which transform differently under a chiral gauge transformation and are eigenstates of γ_5:

$$\gamma_5\psi_{\mathrm{R}} = \psi_{i\mathrm{R}} \quad \text{and} \quad \gamma_5\psi_{i\mathrm{L}} = -\psi_{i\mathrm{L}} \ . \tag{C.3.4}$$

The behavior of the chiral fields under a general gauge transformation can be easily investigated for infinitesimal transformations,

$$\begin{aligned} \psi \to &\left(\mathbb{1} + \mathrm{i} t_i \alpha_i(x) \right) \psi = \\ = &\left(\mathbb{1} + \mathrm{i} \left(t_{1,i} \otimes \mathbb{1} + t_{2,i} \otimes \gamma_5 \right) \alpha_i(x) \right) (\psi_{\mathrm{R}} + \psi_{\mathrm{L}}) = \\ = &\left(\mathbb{1} + \mathrm{i} \underbrace{(t_{1,i} + t_{2,i}) \otimes \mathbb{1}}_{t_{\mathrm{R}}} \alpha_i(x) \right) \psi_{\mathrm{R}} \\ + &\left(\mathbb{1} + \mathrm{i} \underbrace{(t_{1,i} - t_{2,i}) \otimes \mathbb{1}}_{t_{\mathrm{L}}} \alpha_i(x) \right) \psi_{\mathrm{L}} \ , \end{aligned} \tag{C.3.5}$$

which leads, as required, to

$$\boxed{\begin{aligned} \psi_{\mathrm{R}} &\to \mathrm{e}^{\mathrm{i}\alpha_i(x) t_{\mathrm{R},i}} \psi_{\mathrm{R}} \\ \psi_{\mathrm{L}} &\to \mathrm{e}^{\mathrm{i}\alpha_i(x) t_{\mathrm{L},i}} \psi_{\mathrm{L}} \ . \end{aligned}} \tag{C.3.6}$$

Furthermore, a charge conjugated left-handed field is right-handed and vice versa:

$$\begin{aligned} (\psi_{\mathrm{L}})^{\mathrm{c}} &\overset{(\mathrm{C.2.1})}{=} C\gamma_0^{\mathrm{T}}\psi_{\mathrm{L}}^* = C\gamma_0^{\mathrm{T}}\frac{1}{2}\left(\mathbb{1} - \gamma_5^*\right)\psi^* \\ &\overset{(\mathrm{C.1.4})}{=} C\frac{1}{2}\left(\mathbb{1} + \gamma_5^*\right)\gamma_0^{\mathrm{T}}\psi^* \overset{(\mathrm{C.2.5})}{=} \frac{1}{2}\left(\mathbb{1} + \gamma_5^{\dagger}\right)C\gamma_0^{\mathrm{T}}\psi^* \\ &\overset{(\mathrm{C.1.4})}{=} P_{\mathrm{R}}\psi^{\mathrm{c}} = (\psi^{\mathrm{c}})_{\mathrm{R}} \ . \end{aligned} \tag{C.3.7}$$

C.4. Useful Relations

Due to the fact that fermion fields anticommute,

$$\psi_i \psi_j = -\psi_j \psi_i \ , \tag{C.4.1}$$

we find that the transpose of an expression of the following form comes along with a change of sign:

$$
\begin{aligned}
\psi^T \left(A_{m\times m} \otimes Z_{4\times 4} \right) \psi &= (\psi_i)_a A_{ij} Z_{ab} (\psi_j)_b \\
&= (\psi_i)_a (A^{\mathrm{T}})_{ji} (Z^{\mathrm{T}})_{ba} (\psi_j)_b \overset{(C.4.1)}{=} -(\psi_j)_b (A^{\mathrm{T}})_{ji} (Z^{\mathrm{T}})_{ba} (\psi_i)_a \\
&= -\psi^T \left(A_{m\times m} \otimes Z_{4\times 4} \right)^{\mathrm{T}} \psi \ .
\end{aligned}
\tag{C.4.2}
$$

With the help of the relations (C.1.1) - (C.4.2) we are now going to prove the identities used in appendix B and ignore total derivatives of the form:

$$
\begin{aligned}
\overline{\psi} \left(A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5 \right) & \left(\mathbb{1} \otimes \mathrm{i}\gamma_\mu \overset{\leftarrow}{\partial^\mu} \right) \psi = \\
&\overset{(C.1.4)}{=} \overline{\psi} \left(\mathbb{1} \otimes \mathrm{i}\gamma_\mu \overset{\leftarrow}{\partial^\mu} \right) \left(A \otimes \mathbb{1} - \mathrm{i} \cdot B \otimes \gamma_5 \right) \psi \\
&= \overline{\psi} \left(\mathbb{1} \otimes \mathrm{i}\gamma_\mu (-\overset{\rightarrow}{\partial^\mu}) \right) \left(A \otimes \mathbb{1} - \mathrm{i} \cdot B \otimes \gamma_5 \right) \psi \\
&+ \text{total derivative} \ .
\end{aligned}
\tag{C.4.3}
$$

- Relation (B.3.4)

$$
\boxed{\overline{\psi^{\mathrm{c}}} \left(\mathbb{1} \otimes \mathrm{i}\gamma_\mu \partial^\mu \right) \left(A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5 \right) \psi^{\mathrm{c}} = \overline{\psi} \left(\mathbb{1} \otimes \mathrm{i}\gamma_\mu \partial^\mu \right) \left(A^{\mathrm{T}} \otimes \mathbb{1} - \mathrm{i} \cdot B^{\mathrm{T}} \otimes \gamma_5 \right) \psi}
$$

$$
\begin{aligned}
\overline{\psi^{\mathrm{c}}} & \left(\mathbb{1} \otimes \mathrm{i}\gamma_\mu \partial^\mu \right) \left(A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5 \right) \psi^{\mathrm{c}} = \\
&\overset{(C.2.4)}{=} -\psi^{\mathrm{T}} \left(\mathbb{1} \otimes C^{-1} \right) \left(\mathbb{1} \otimes \mathrm{i}\gamma_\mu \partial^\mu \right) \left(A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5 \right) \left(\mathbb{1} \otimes C\gamma_0^{\mathrm{T}} \right) \psi^* \\
&\overset{(C.2.2)(C.2.5)}{=} \psi^{\mathrm{T}} \left(\mathbb{1} \otimes \mathrm{i}\gamma_\mu^{\mathrm{T}} \partial^\mu \right) \left(A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5^{\mathrm{T}} \right) \left(\mathbb{1} \otimes C^{-1} \right) \left(\mathbb{1} \otimes C\gamma_0^{\mathrm{T}} \right) \psi^* \\
&\overset{(C.4.2)}{=} -\psi^\dagger \left(\mathbb{1} \otimes \gamma_0 \right) \left(A^{\mathrm{T}} \otimes \mathbb{1} + \mathrm{i} \cdot B^{\mathrm{T}} \otimes \gamma_5 \right) \left(\mathbb{1} \otimes \mathrm{i}\gamma_\mu \overset{\leftarrow}{\partial^\mu} \right) \psi \\
&\overset{(C.1.4)}{=} \overline{\psi} \left(\mathbb{1} \otimes \mathrm{i}\gamma_\mu \overset{\rightarrow}{\partial^\mu} \right) \left(A^{\mathrm{T}} \otimes \mathbb{1} - \mathrm{i} \cdot B^{\mathrm{T}} \otimes \gamma_5 \right) \psi \qquad \Box
\end{aligned}
\tag{C.4.4}
$$

- Relation (B.3.5):

$$\left[\overline{\psi}\left(\mathbb{1}\otimes\mathrm{i}\gamma_\mu\partial^\mu\right)\left(E\otimes\mathbb{1}+\mathrm{i}\cdot F\otimes\gamma_5\right)\psi^{\mathrm{c}}\right]^\dagger = \overline{\psi^{\mathrm{c}}}\left(\mathbb{1}\otimes\mathrm{i}\gamma_\mu\partial^\mu\right)\left(E\otimes\mathbb{1}+\mathrm{i}\cdot F\otimes\gamma_5\right)^\dagger\psi$$

$$\left[\overline{\psi}\left(\mathbb{1}\otimes\mathrm{i}\gamma_\mu\partial^\mu\right)\left(E\otimes\mathbb{1}+iF\otimes\gamma_5\right)\psi^{\mathrm{c}}\right]^\dagger =$$

$$\overset{(\mathrm{C.1.2})(\mathrm{C.1.4})}{=} \psi^{\mathrm{T}}\left(\mathbb{1}\otimes\gamma_0^{\mathrm{T}}C^{-1}\right)\left(E^\dagger\otimes\mathbb{1}-\mathrm{i}\cdot F^\dagger\otimes\gamma_5\right)\left(\mathbb{1}\otimes(-\mathrm{i})\gamma_\mu^\dagger\overleftarrow{\partial^\mu}\right)(\mathbb{1}\otimes\gamma_0)\,\psi$$

$$\overset{(\mathrm{C.2.2})(\mathrm{C.4.4})}{=} \psi^{\mathrm{T}}\left(\mathbb{1}\otimes(-C^{-1}\gamma_0)\right)(\mathbb{1}\otimes\gamma_0)\left(\mathbb{1}\otimes\mathrm{i}\gamma_\mu\overrightarrow{\partial^\mu}\right)\left(E\otimes\mathbb{1}+\mathrm{i}\cdot F\otimes\gamma_5\right)^\dagger\psi$$

$$\overset{(\mathrm{C.1.1})(\mathrm{C.2.4})}{=} \overline{\psi^{\mathrm{c}}}\left(\mathbb{1}\otimes\mathrm{i}\gamma_\mu\partial^\mu\right)\left(E\otimes\mathbb{1}+\mathrm{i}\cdot F\otimes\gamma_5\right)^\dagger\psi \qquad \square$$

$$(\mathrm{C.4.5})$$

- Relation (B.3.6)

$$\left[\overline{\psi}\left(\mathbb{1}\otimes\mathrm{i}\gamma_\mu\partial^\mu\right)\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5\right)\psi\right]^\dagger = \overline{\psi}\left(\mathbb{1}\otimes\mathrm{i}\gamma_\mu\partial^\mu\right)\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5\right)^\dagger\psi$$

$$\left[\overline{\psi}\left(\mathbb{1}\otimes\mathrm{i}\gamma_\mu\partial^\mu\right)\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5\right)\psi\right]^\dagger =$$

$$\overset{(\mathrm{C.1.2})(\mathrm{C.1.4})}{=} \psi^\dagger\left(A^\dagger\otimes\mathbb{1}-\mathrm{i}\cdot B^\dagger\otimes\gamma_5\right)\left(\mathbb{1}\otimes(-\mathrm{i})\gamma_\mu^\dagger\overleftarrow{\partial^\mu}\right)(\mathbb{1}\otimes\gamma_0)\,\psi$$

$$\overset{(\mathrm{C.1.1})(\mathrm{C.1.4})}{=} \overline{\psi}\left(\mathbb{1}\otimes\mathrm{i}\gamma_\mu\overrightarrow{\partial^\mu}\right)\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5\right)^\dagger\psi \qquad \square$$

$$(\mathrm{C.4.6})$$

- Relation (B.3.18)

$$\left[\overline{\psi}\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5\right)\psi\right]^\dagger = \overline{\psi}\left(A^\dagger\otimes\mathbb{1}+\mathrm{i}\cdot B^\dagger\otimes\gamma_5\right)\psi$$

$$\left[\overline{\psi}\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5\right)\psi\right]^\dagger \overset{(\mathrm{C.1.2})(\mathrm{C.1.4})}{=} \psi^\dagger\left(A^\dagger\otimes\mathbb{1}-\mathrm{i}\cdot B^\dagger\otimes\gamma_5\right)(\mathbb{1}\otimes\gamma_0)\,\psi$$

$$\overset{(\mathrm{C.1.4})}{=} \overline{\psi}\left(A^\dagger\otimes\mathbb{1}+\mathrm{i}\cdot B^\dagger\otimes\gamma_5\right)\psi \qquad \square$$

$$(\mathrm{C.4.7})$$

- Relation (B.3.19)

$$\overline{\psi^{\mathrm{c}}}\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5\right)\psi^{\mathrm{c}} = \overline{\psi}\left(A^{\mathrm{T}}\otimes\mathbb{1}+\mathrm{i}\cdot B^{\mathrm{T}}\otimes\gamma_5\right)\psi$$

$$\overline{\psi^{\mathrm{c}}}\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5\right)\psi^{\mathrm{c}} \overset{(\mathrm{C.2.4})}{=} -\psi^{\mathrm{T}}\left(\mathbb{1}\otimes C^{-1}\right)\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5\right)\left(\mathbb{1}\otimes C\gamma_0^{\mathrm{T}}\right)\psi^*$$

$$\overset{(\mathrm{C.2.5})}{=} -\psi^{\mathrm{T}}\left(A\otimes\mathbb{1}+\mathrm{i}\cdot B\otimes\gamma_5^{\mathrm{T}}\right)\left(\mathbb{1}\otimes\gamma_0^{\mathrm{T}}\right)\psi^*$$

$$\overset{(\mathrm{C.4.2})}{=} \psi^\dagger\left(\mathbb{1}\otimes\gamma_0\right)\left(A^\dagger\otimes\mathbb{1}+\mathrm{i}\cdot B^{\mathrm{T}}\otimes\gamma_5\right)\psi \qquad \square$$

$$(\mathrm{C.4.8})$$

- Relation (B.3.26)

$$\left[\overline{\psi^{\mathrm{c}}} \left(A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5 \right) \psi \right]^{\dagger} = \overline{\psi} \left(A^{\dagger} \otimes \mathbb{1} + \mathrm{i} \cdot B^{\dagger} \otimes \gamma_5 \right) \psi^{\mathrm{c}}$$

$$\left[\overline{\psi^{\mathrm{c}}} \left(A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5 \right) \psi \right]^{\dagger} \overset{(\mathrm{C.1.2})}{=} \psi^{\dagger} \left(A^{\dagger} \otimes \mathbb{1} - \mathrm{i} \cdot B^{\dagger} \otimes \gamma_5 \right) \left(\mathbb{1} \otimes \gamma_0 \right) \psi^{\mathrm{c}}$$

$$\overset{(\mathrm{C.1.4})}{=} \psi^{\dagger} \left(\mathbb{1} \otimes \gamma_0 \right) \left(A^{\dagger} \otimes \mathbb{1} + \mathrm{i} \cdot B^{\dagger} \otimes \gamma_5 \right) \psi^{\mathrm{c}} \quad \square$$

(C.4.9)

- Relation (B.3.16)

$$\overline{\psi} \left(\mathbb{1} \otimes \mathrm{i} \not{D} \right) \psi = \overline{\psi_{\mathrm{L}}} \left(\mathbb{1} \otimes \mathrm{i} \not{D} \right) \psi_{\mathrm{L}} + \overline{\psi_{\mathrm{R}}} \left(\mathbb{1} \otimes \mathrm{i} \not{D} \right) \psi_{\mathrm{R}}$$

$$\overline{\psi_{\mathrm{L}}} \left(\mathbb{1} \otimes \mathrm{i} \not{D} \right) \psi_{\mathrm{R}} \overset{(\mathrm{C.1.4})}{=} \frac{1}{4} \psi^{\dagger} \left(\mathbb{1} \otimes \left(\mathbb{1} - \gamma_5 \right) \right) \left(\mathbb{1} \otimes \gamma_0 \right) \left(\mathbb{1} \otimes \mathrm{i} \gamma_{\mu} D^{\mu} \right) \left(\mathbb{1} \otimes \left(\mathbb{1} + \gamma_5 \right) \right) \psi$$

$$\overset{(\mathrm{C.1.4})}{=} \frac{1}{4} \overline{\psi} \left(\mathbb{1} \otimes \mathrm{i} \gamma_{\mu} D^{\mu} \right) \underbrace{\left(\mathbb{1} \otimes \left(\mathbb{1} - \gamma_5 \right) \right) \left(\mathbb{1} \otimes \left(\mathbb{1} + \gamma_5 \right) \right)}_{=0} \psi = 0 \quad \square$$

(C.4.10)

- Relation (B.3.20)

$$\overline{\psi} \left[A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5 \right] \psi = \overline{\psi_{\mathrm{L}}} \left[A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5 \right] \psi_{\mathrm{R}} + \overline{\psi_{\mathrm{R}}} \left[A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5 \right] \psi_{\mathrm{L}}$$

$$\overline{\psi_{\mathrm{R}}} (A \otimes \mathbb{1} + \mathrm{i} \cdot B \otimes \gamma_5) \psi_{\mathrm{R}} \overset{(\mathrm{C.1.4})}{=} \overline{\psi} (A \otimes \mathbb{1} + B \otimes \gamma_5) P_{\mathrm{L}} P_{\mathrm{R}} \psi$$

$$\overset{(\mathrm{C.3.2})}{=} 0 \quad \square$$

(C.4.11)

- Relation (B.3.35)

$$\overline{\psi_{\mathrm{L}}} \left(M_{\mathrm{L}} \otimes \mathbb{1} \right) (\psi_{\mathrm{L}})^{\mathrm{c}} = \overline{\psi_{\mathrm{L}}} \left(M_{\mathrm{L}}^{\mathrm{T}} \otimes \mathbb{1} \right) (\psi_{\mathrm{L}})^{\mathrm{c}}$$

$$\overline{\psi_{\mathrm{L}}} \left(M_{\mathrm{L}} \otimes \mathbb{1} \right) (\psi_{\mathrm{L}})^{\mathrm{c}} \overset{(\mathrm{C.2.1})}{=} \psi^{\dagger} \left(\mathbb{1} \otimes \gamma_0 \right) \left(M_{\mathrm{L}} \otimes \mathbb{1} \right) \left(\mathbb{1} \otimes C \gamma_0^{\mathrm{T}} \right) \psi_{\mathrm{L}}^{*}$$

$$\overset{(\mathrm{C.4.2})}{=} -\psi_{\mathrm{L}}^{\dagger} \left(\mathbb{1} \otimes \gamma_0 C^{\mathrm{T}} \right) \left(M_{\mathrm{L}}^{\mathrm{T}} \otimes \mathbb{1} \right) \left(\mathbb{1} \otimes \gamma_0^{\mathrm{T}} \right) \psi_{\mathrm{L}}^{*}$$

$$\overset{(\mathrm{C.2.3})}{=} \overline{\psi_{\mathrm{L}}} \left(M_{\mathrm{L}}^{\mathrm{T}} \otimes \mathbb{1} \right) (\psi_{\mathrm{L}})^{\mathrm{c}} \quad \square$$

(C.4.12)

D. Gaussian Integrals

We want to calculate various Gaussian Integrals, which are of great use in section 4.2 in the course of the derivation of the effective potential for a general Lagrangian density at one-loop order. This discussion is based on [27, 61, 62].

D.1. Bosonic Gaussian Integrals

We start with concentrating on Gaussian integrals over ordinary commuting quantities and consider integrals over real variables as well as over real fields.

D.1.1. Gaussian Integral over Real Coordinates

The one-dimensional Gaussian integral, with $a = |a|\, e^{\mathrm{i}\gamma}$, $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$,[1] is given by

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \ , \tag{D.1.1}$$

since it is not hard to show that

$$\left[\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{1}{2}ax^2} \right]^2 = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{d}y\, e^{-\frac{1}{2}a(x^2+y^2)} = 2\pi \int_0^{\infty} \mathrm{d}r\, r\, e^{-\frac{ar^2}{2}} = \frac{2\pi}{a} \ , \tag{D.1.2}$$

and where the complex square root is defined as:

$$\sqrt{a} = \sqrt{|a|} \cdot e^{\mathrm{i}\frac{\gamma}{2}} \ . \tag{D.1.3}$$

We can use this result to compute the n-dimensional real-valued integral of the form

$$\mathcal{I}_n = \int_{-\infty}^{\infty} \mathrm{d}^n x\, e^{-\frac{1}{2}x_i A_{ij} x_j} \ , \tag{D.1.4}$$

with the real, symmetric[2] and positive definite Matrix A_{ij}. Hence, there exists a real, orthogonal matrix Q such that

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = Q^{\mathrm{T}} A Q \ , \tag{D.1.5}$$

and if we define

$$\bar{x} = Qx \ , \tag{D.1.6}$$

---

[1] The integral converges only for $\mathrm{Re}\, a > 0$.
[2] The antisymmetric part cancels due to $x^{\mathrm{T}} A x = x^{\mathrm{T}} A^T x$.

we get

$$
\boxed{
\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}^n x \, e^{-\frac{1}{2} x^{\mathrm{T}} A x} &= \int_{-\infty}^{\infty} \mathrm{d}^n \bar{x} \, |\det Q^{\mathrm{T}}| \, e^{-\frac{1}{2} \bar{x}^{\mathrm{T}} D \bar{x}} = \prod_{i=1}^{n} \int_{-\infty}^{\infty} \mathrm{d}\bar{x}_i \, e^{-\frac{1}{2} \lambda_i \bar{x}_i^2} \\
&= \prod_{i=1}^{n} \sqrt{\frac{2\pi}{\lambda_i}} = \sqrt{\frac{(2\pi)^n}{\det A}} \, ,
\end{aligned}
}
\tag{D.1.7}
$$

where we used the fact that the determinant of an orthogonal matrix is ± 1.

We can also consider the related integral

$$
\begin{aligned}
\mathcal{I}_n &= \int_{-\infty}^{\infty} \mathrm{d}^n \mathrm{Re}(z) \, \mathrm{d}^n \mathrm{Im}(z) \, e^{-z^{\dagger} A z} \\
&= \int_{-\infty}^{\infty} \mathrm{d}^n \mathrm{Re}(z) \, \mathrm{d}^n \mathrm{Im}(z) \, e^{(\mathrm{Re}(z) - \mathrm{i}\,\mathrm{Im}(z))^{\mathrm{T}} A \, (\mathrm{Re}(z) + \mathrm{i}\,\mathrm{Im}(z))} \, ,
\end{aligned}
\tag{D.1.8}
$$

with the hermitian and positive definite matrix A. With the help of the unitary matrix U we can diagonalize the matrix A

$$
D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = U^{\dagger} A U \, ,
\tag{D.1.9}
$$

and after defining

$$
\bar{z} = U z \, ,
\tag{D.1.10}
$$

we find[3]

$$
\boxed{
\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}^n \mathrm{Re}(\bar{z}) \, \mathrm{d}^n \mathrm{Im}(\bar{z}) \, e^{-z^{\dagger} A z} &= \int_{-\infty}^{\infty} \mathrm{d}^n \mathrm{Re}(z) \, \mathrm{d}^n \mathrm{Im}(z) \, |\det U^{\dagger}| \, e^{-\bar{z}^{\dagger} D \bar{z}} \\
&= \prod_{i=1}^{n} \int_{-\infty}^{\infty} \mathrm{d}\mathrm{Re}(\bar{z}_i) \, \mathrm{d}\mathrm{Im}(\bar{z}_i) \, e^{-\lambda_i \mathrm{Re}(\bar{z}_i)^2 - \lambda_i \mathrm{Im}(\bar{z}_i)^2} \overset{(D.1.7)}{=} \frac{\pi^n}{\det A} \, .
\end{aligned}
}
\tag{D.1.11}
$$

D.1.2. Gaussian Integral over Real Fields

We want to calculate a path integral of the form

$$
\int [\mathrm{d}\phi] \, e^{\mathrm{i} S[\phi]} \, ,
\tag{D.1.12}
$$

with the classical action

$$
S[\phi] = -\frac{1}{2} \int \mathrm{d}^4 y \mathrm{d}^4 x \ \phi^{\mathrm{T}}(x) A(x - y) \phi(y) \, ,
\tag{D.1.13}
$$

---

[3] $|\det U^{\dagger}| = 1$ since U is a unitary matrix.

where ϕ denotes a vector of s real fields and A a real and symmetric $s \times s$ matrix. We limit our space-time volume to $V = L^4$ and write the fields and the matrix with the help of a Fourier series as

$$\phi(x) = \frac{1}{V} \sum_n e^{ik_n x} \hat{\phi}(k_n)$$

$$A(x-y) = \frac{1}{V} \sum_n e^{ik_n(x-y)} \hat{A}(k_n) \ . \tag{D.1.14}$$

Due to the finite volume and the choice of periodic boundary conditions the wave vector is discretized and given by

$$k_n^\mu = \frac{2\pi n^\mu}{L} \ , \tag{D.1.15}$$

where n^μ is an integer. Inserting these expansions back in (D.1.13) it is not hard to show that one gets:

$$S[\phi] = -\frac{1}{2V} \sum_n \hat{\phi}^{\mathrm{T}}(-k_n) \hat{A}(k_n) \hat{\phi}(k_n) \ . \tag{D.1.16}$$

Although $\phi(x)$ and $A(x-y)$ are real quantities, $\hat{\phi}(k_n)$ and $\hat{A}(k_n)$ are in general complex and have to obey

$$\hat{\phi}(-k_n) = \hat{\phi}^*(k_n)$$

$$\hat{A}(-k_n) = \hat{A}^*(k_n) \ . \tag{D.1.17}$$

Furthermore, from the symmetry of A, $A(x-y) = A^{\mathrm{T}}(y-x)$, follows

$$\hat{A}(k_n) = \hat{A}^{\mathrm{T}}(-k_n) = \hat{A}^{\mathrm{T}}(k_{-n}) \ , \tag{D.1.18}$$

and the relations from (D.1.17) enable us to find

$$S[\phi] = -\frac{1}{2V} \sum_n \hat{\phi}^\dagger(k_n) \hat{A}(k_n) \hat{\phi}(k_n) \overset{(S=S^*)}{=} -\frac{1}{2V} \sum_n \hat{\phi}^\dagger(-k_n) \hat{A}(-k_n) \hat{\phi}(-k_n)$$

$$= -\frac{1}{2V} \hat{\phi}^{\mathrm{T}}(0) \hat{A}(0) \hat{\phi}(0) - \frac{1}{V} \sum_{n>0} \hat{\phi}^\dagger(k_n) \hat{A}(k_n) \hat{\phi}(k_n) \ . \tag{D.1.19}$$

The measure of the path integral can then be written as

$$[\mathrm{d}\phi] = C \cdot \mathrm{d}^s \hat{\phi}(0) \prod_{n>0} \mathrm{d}^s \mathrm{Re}(\hat{\phi}(k_n)) \, \mathrm{d}^s \mathrm{Im}(\hat{\phi}(k_n)) \ , \tag{D.1.20}$$

where we are not interested in the exact form of the constant factor C, since we are always dealing with fractions of Gaussian integrals. Of course, the notation $n > 0$ for a vector is a misnomer and needs clarification. We define the vector n to be "positive" if

$$n^0 > 0 \quad \vee \quad n^0 = 0 \ , \ n^1 > 0 \quad \vee \quad n^0 = 0 \ , \ n^1 = 0 \ , \ n^2 > 0 \quad \vee \quad \ldots \ , \tag{D.1.21}$$

which is only one possible choice to systemize the sum over all n.

We can now reduce our path integral to simpler Gaussian integrals over real coordinates and we find from (D.1.7) and (D.1.11):

$$\int [\mathrm{d}\phi] e^{\mathrm{i}S[\phi]} = C \int \mathrm{d}^s \hat{\phi}(0) \prod_{n>0} \mathrm{d}^s \mathrm{Re}(\hat{\phi}(k_n))\, \mathrm{d}^s \mathrm{Im}(\hat{\phi}(k_n))$$
$$\cdot\, e^{-\frac{\mathrm{i}}{2V} \hat{\phi}^{\mathrm{T}}(0) \hat{A}(0) \hat{\phi}(0) - \frac{\mathrm{i}}{V} \sum_{n>0} \hat{\phi}^\dagger(k_n) \hat{A}(k_n) \hat{\phi}(k_n)} \qquad \text{(D.1.22)}$$
$$= C \cdot \sqrt{\frac{(\mathrm{i}V2\pi)^s}{\det \hat{A}(0)}} \prod_{n>0} \frac{(\mathrm{i}V\pi)^s}{\det \hat{A}(k_n)} \ .$$

If we realize that $\det A = \det A^{\mathrm{T}}$ and use relation (D.1.18), we can simply the above result to

$$\boxed{\int [\mathrm{d}\phi] e^{\mathrm{i}S[\phi]} = \bar{C} \cdot \prod_{\text{all } k_n} \sqrt{\frac{1}{\det \hat{A}(k_n)}} = \bar{C} \cdot \sqrt{\frac{1}{\mathrm{Det} A}}\ ,} \qquad \text{(D.1.23)}$$

with the new overall factor $\bar{C}$ and the functional determinant $\mathrm{Det} A$.

The alert reader might have cast doubt on the legitimacy of our calculation of the path integral in (D.1.22), since the Gaussian integrals seems not to converge at all, due to the purely imaginary phase. But, as defined in equation (3.1.1), we actually perform the integral in the limit $T \to \infty(1 - \mathrm{i}0^+)$, and therefore, we should have changed all k_n^0 to $k_n^0 + \mathrm{i}0^+$ [4] [27], where the term $\mathrm{i}0^+$ delivers a real contribution to the phase and the necessary convergence factor.

In a final step, we perform the limit $L \to \infty$, where the sum over the discretized momenta becomes again an integral,

$$\frac{(2\pi)^4}{V} \sum_{\text{all } k_n} \to \int \mathrm{d}^4 k\ , \qquad \text{(D.1.24)}$$

and in the end the path integral reads

$$\boxed{\int [\mathrm{d}\phi] e^{\mathrm{i}S[\phi]} = \hat{C}\, e^{-\frac{1}{2} \int \mathrm{d}^4 x \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \ln \det \hat{A}(k)}\ .} \qquad \text{(D.1.25)}$$

D.2. Fermionic Gaussian Integrals

We can also deal with Gaussian integrals over non-commuting quantities, so-called Grassmann numbers. Again we distinguish between integrals over variables and fields and start with a short introduction about these special numbers.

D.2.1. Grassmann Numbers

Grassmann numbers are anticommuting quantities (i.e. fermionic fields), which obey

$$\{\psi, \chi\} = 0\ , \qquad \text{(D.2.1)}$$

and

$$\psi\psi = \chi\chi = 0\ . \qquad \text{(D.2.2)}$$

---

[4] If we perform a Fourier transformation a shift of the T-axis in the direction $-\mathrm{i}0^+$ has to be compensated by a shift of the k^0-axis in the direction $+\mathrm{i}0^+$.

Therefore, a Taylor series expansion of a function $f(\psi)$ has the simple form:

$$f(\psi) = a + b \cdot \psi \ . \tag{D.2.3}$$

By analogy with an integral over ordinary numbers we can also introduce an integral over Grassmann numbers (Berezin integral [63]), which is defined as a linear functional:

$$\int \mathrm{d}\psi \ (a \cdot f(\psi) + b \cdot g(\psi)) = a \cdot \int \mathrm{d}\psi \ f(\psi) + b \cdot \int \mathrm{d}\psi \ g(\psi) \ . \tag{D.2.4}$$

The integral over a function $f(\psi)$ is then given by

$$\int \mathrm{d}\psi \ f(\psi) = a \int \mathrm{d}\psi + b \int \mathrm{d}\psi \ \psi \ , \tag{D.2.5}$$

and if we claim that such an integral should be invariant under a simple shift $\psi \to \psi + c$,

$$\int \mathrm{d}\psi \ f(\psi) = \int \mathrm{d}\psi \ f(\psi + c) = (a + bc) \int \mathrm{d}\psi + b \int \mathrm{d}\psi \ \psi \ , \tag{D.2.6}$$

we find:

$$\int \mathrm{d}\psi = 0 \ . \tag{D.2.7}$$

The second integral is not determined and is therefore set to be equal to 1:

$$\int \mathrm{d}\psi \ \psi = 1 \ . \tag{D.2.8}$$

As a consequence, integration and differentiation are identical for Grassmann variables:

$$\int \mathrm{d}\psi = \frac{\partial}{\partial \psi} \ . \tag{D.2.9}$$

Furthermore, it follows from equation (D.2.7) and (D.2.8) that[5]:

$$\int \mathrm{d}^n\psi \ \psi_1 \ldots \psi_m = \int \mathrm{d}\psi_n \ldots \mathrm{d}\psi_1 \ \psi_1 \ldots \psi_m = \begin{cases} 1, & \text{if } m = n. \\ 0, & \text{otherwise.} \end{cases} \tag{D.2.10}$$

D.2.2. Gaussian Integral over Grassmann Variables

With all this knowledge about Grassmann numbers we can now consider an integral of the form

$$\mathcal{I}_m = \int \mathrm{d}^m\psi \ e^{\frac{1}{2}\psi^{\mathrm{T}} A \psi} \ . \tag{D.2.11}$$

First of all, we notice from equation (D.2.10) that this integral is zero for an odd dimension m and that we can write:

$$\mathcal{I}_m = \mathcal{I}_{2n} = \int \mathrm{d}^{2n}\psi \ \frac{1}{n! \, 2^n} \left(\psi_i A_{ij} \psi_j \right)^n \ . \tag{D.2.12}$$

---

[5]The ordering of $\mathrm{d}\psi_i$ is conventional and different choices can result in a different sign. Since we always consider fractions of Gaussian integrals over Grassmann variables an overall factor does not bother us.

Moreover, due to the antisymmetry of $\psi_i\psi_j$, the symmetric part of A cancels and hence, we assume A to be antisymmetric from the beginning. Since $\psi_i\psi_i = 0$ (no summation over i), we can further simplify the integral and find

$$\mathcal{I}_{2n} = \int \mathrm{d}^{2n}\psi \, \frac{1}{n!\,2^n} \sum_{\sigma \in S_n} A_{\sigma(1)\sigma(2)}...A_{\sigma(2n-1)\sigma(2n)} \underbrace{\psi_{\sigma(1)}...\psi_{\sigma(2n)}}_{\varepsilon_{\sigma(1)..\sigma(2n)}\psi_1...\psi_{2n}}$$

$$\stackrel{\text{(D.2.10)}}{=} \frac{1}{n!\,2^n} \, \varepsilon_{i_1 i_2...i_{2n}} \, A_{i_1 i_2}...A_{i_{2n-1}i_{2n}} \, , \tag{D.2.13}$$

where S_n denotes the symmetric group. The result we found is just the definition of the pfaffian of the matrix A, which turns out to be the square root of the determinant for an antisymmetric complex matrix [64],[6]

$$\mathrm{pf}\, A = \frac{1}{n!\,2^n} \, \varepsilon_{i_1 i_2...i_{2n}} \, A_{i_1 i_2}...A_{i_{2n-1}i_{2n}} = \sqrt{\det A} \, , \tag{D.2.14}$$

and therefore, we finally get:

$$\boxed{\int \mathrm{d}^{2n}\psi \, e^{\frac{1}{2}\psi^{\mathrm{T}} A \psi} = \sqrt{\det A} \, .} \tag{D.2.15}$$

The second Gaussian integral over Grassmann variables we want to have a look at, is of the form:

$$\int \mathrm{d}^n\chi \mathrm{d}^n\chi^* \, e^{\chi^\dagger A \chi} \, . \tag{D.2.16}$$

If we realize that $\chi_i^*\chi_j$ just acts as a c-number,

$$\chi_i^*\chi_j \, \chi_m^*\chi_n = \chi_m^*\chi_n \, \chi_i^*\chi_j \, , \tag{D.2.17}$$

and that

$$\psi_1\chi_1\psi_2\chi_2...\psi_n\chi_n = (-1)^{1+2+...+(n-1)}\psi_1\psi_2...\psi_n\chi_1\chi_2...\chi_n$$

$$= (-1)^{\frac{n\cdot(n-1)}{2}}\psi_1\psi_2...\psi_n\chi_1\chi_2...\chi_n \, , \tag{D.2.18}$$

the above integral can be written as

$$\int \mathrm{d}^n\chi \mathrm{d}^n\chi^* \, e^{\chi^\dagger A \chi} \stackrel{\text{(D.2.10)}}{=} \frac{1}{n!} \int \mathrm{d}^n\chi \mathrm{d}^n\chi^* \, (\chi_i^* A_{ij} \chi_j)^n$$

$$\stackrel{\text{(D.2.17)}}{=} \frac{1}{n!} \int \mathrm{d}\chi_n...\mathrm{d}\chi_1\mathrm{d}\chi_n^*...\mathrm{d}\chi_1^* \, n! \cdot \sum_{\sigma \in S_n} \chi_1^* A_{1\sigma(1)}\chi_{\sigma(1)}\cdots\chi_n^* A_{n\sigma(n)}\chi_{\sigma(n)} \tag{D.2.19}$$

$$\stackrel{\text{(D.2.18)}}{=} (-1)^{\frac{n\cdot(n-1)}{2}} \int \mathrm{d}\chi_n...\mathrm{d}\chi_1\mathrm{d}\chi_n^*...\mathrm{d}\chi_1^* \, \varepsilon_{i_1...i_n} A_{1i_1}...A_{ni_n}\chi_1^*...\chi_n^*\chi_1...\chi_n \, ,$$

and we find:

$$\boxed{\int \mathrm{d}^n\chi \mathrm{d}^n\chi^* \, e^{\chi^\dagger A \chi} = (-1)^{\frac{n\cdot(n-1)}{2}} \det A \, .} \tag{D.2.20}$$

---

[6]Actually $\mathrm{pf}\, A = \pm\sqrt{\det A}$, but again, we don't care about an overall factor.

D.2.3. Gaussian Integral over Grassmann Fields

By analogy with the bosonic path integral we can also compute an integral of the form

$$\mathcal{I} = \int [\mathrm{d}\omega] \, e^{\mathrm{i}S[\omega]} = \int [\mathrm{d}\omega] \, e^{-\frac{\mathrm{i}}{2} \int \mathrm{d}^4 x \mathrm{d}^4 y \, \omega^{\mathrm{T}}(x) A(x-y)\omega(y)} \; , \tag{D.2.21}$$

with the antisymmetric $2n \times 2n$ matrix A and the $2n$ complex Grassmann fields ω_i.

Again, we limit our space-time volume to $V = L^4$ and after performing a Fourier series (compare equation (D.1.14)) it follows from the antisymmetry of the matrix A, $A(x-y) = -A^{\mathrm{T}}(y-x)$, that:

$$\hat{A}(k_n) = -\hat{A}^{\mathrm{T}}(-k_n) \quad . \tag{D.2.22}$$

This condition and the anticommutativity of Grassmann variables leads to

$$S[\omega] = -\frac{1}{2V}\hat{\omega}^{\mathrm{T}}(0)\hat{A}(0)\hat{\omega}(0) - \frac{1}{V}\sum_{n>0}\hat{\omega}^{\mathrm{T}}(-k_n)\hat{A}(k_n)\hat{\omega}(k_n) \; , \tag{D.2.23}$$

and the path integral then reads

$$\boxed{\begin{aligned}
\mathcal{I} &= C \int \mathrm{d}\hat{\omega}(0) \prod_{n>0} \mathrm{d}\hat{\omega}(k_n)\mathrm{d}\hat{\omega}(-k_n) \, e^{-\frac{\mathrm{i}}{2V}\hat{\omega}^{\mathrm{T}}(0)\hat{A}(0)\hat{\omega}(0) - \frac{\mathrm{i}}{V}\sum_{n>0}\hat{\omega}^{\mathrm{T}}(-k_n)\hat{A}(k_n)\hat{\omega}(k_n)} \\
&= \bar{C}\sqrt{\det\hat{A}(0)} \prod_{n>0} \det\hat{A}(k_n) = \hat{C} \prod_{\text{all } k_n} \sqrt{\det\hat{A}(k_n)} = \hat{C}\sqrt{\mathrm{Det}A} \; ,
\end{aligned}}$$
$$\tag{D.2.24}$$

where we applied equation (D.2.15) and (D.2.20) and used that $\det\hat{A}(k_n) = \det\hat{A}(-k_n)$[7]. Therefore, we finally get

$$\boxed{\int [\mathrm{d}\omega] \, e^{-\frac{\mathrm{i}}{2} \int \mathrm{d}^4 x \mathrm{d}^4 y \, \omega^{\mathrm{T}}(x) A(x-y)\omega(y)} = \hat{C} \, e^{\frac{1}{2} \int \mathrm{d}^4 x \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \ln\det\hat{A}(k)}} \; . \tag{D.2.25}$$

The calculation of the path integral version of the Grassmann integral (D.2.20) is now straightforward and leads to:

$$\boxed{\begin{aligned}
\int [\mathrm{d}\chi][\mathrm{d}\chi^*] \, e^{\mathrm{i}\int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \chi^{\dagger}(x) A(x-y)\chi(y)} &= \hat{C} \prod_{\text{all } k_n} \det\hat{A}(k_n) = \hat{C} \, \mathrm{Det}A \\
&\stackrel{L\to\infty}{=} \hat{C} \, e^{\int \mathrm{d}^4 x \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \ln\det\hat{A}(k)} \; .
\end{aligned}}$$
$$\tag{D.2.26}$$

---

[7] $\det\hat{A}(k_n) = \det\left(-\hat{A}^{\mathrm{T}}(-k_n)\right) = (-1)^{2n}\det\hat{A}(-k_n)$

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