universität wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„Geometry of Twisted D-branes in Group Manifolds"

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2018 / Vienna 2018

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 066 876
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Master's degree programme Physics
Betreut von / Supervisor:	Univ.-Prof. Dipl.-Phys. Dr. Stefan Fredenhagen
Mitbetreut von / Co-Supervisor:	

Abstract

The main topic addressed in this work is the geometry of D-branes in the WZW model on a compact, simple, simply connected Lie group. At first we recall some main ideas related to open strings in group manifolds and their gluing conditions. The D-branes given in terms of some special gluing conditions correspond to twisted conjugacy classes with respect to an automorphism of the Lie group. After setting up some mathematical machinery to allow us to work with them, we parametrize the space of twisted conjugacy classes using a rather explicit computation and comparing it with the abstract method. We also point out some (what we believe to be) errors in [12] on which both of the methods we use agree. Finally, as stabilizer of the twisted conjugacy classes also fit into the picture of D-branes, we compute them explicitly and again more abstractly for SU(4).

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Acknowledgements

Firstly, I would like to thank my thesis supervisor Univ.-Prof. Stefan Fredenhagen, who introduced me to this topic and made working on it very enjoyable and educational. He was always ready to help me progress in the right direction and gave many suggestions that helped me bring the thesis into the form it is in now.

Further, my thanks goes to my colleagues and friends, who made me company and from whose side I have heard only positive and supporting comments throughout my master studies. Among these I would like to mention in particular Else Novac, Timon Gutleb, Peter Allmer and Jakob Möller. Moreover, I would like to thank the entire mathematical physics group of the University of Vienna, for allowing me to work comfortably on my thesis.

Additionally, I would like to show my appreciation to Dr. Richard Koschuch who convinced me to study physics in the first place and who was always willing to give a helping hand throughout the time I've spent at the university.

Last but by no means least, I would like to express my gratitude to my grandparents who made my entire stay at the university possible and who supported me whenever it was needed and still continue to do so.

1 Introduction

In this section we discuss the WZW model in compact Lie groups and observe that the D-branes of this model are certain geometric objects called twisted conjugacy classes. This motivates the work done in the later parts of the thesis which are ariented at describing these conjugacy classes and their stabilizers.

1.1 Open strings and boundary conditions in a flat background

We begin with a short summary of some results about strings in flat backgrounds (for more see [1]). A string in $\mathbb{R}^d$ with Minkowski metric is a smooth map:

$$X : \mathbb{R} \times [0, \pi] \to \mathbb{R}^d .$$

In the conformal gauge the action of such a string is given by

$$S[X] = -\frac{T}{2} \int \eta^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \, d\tau d\sigma , \tag{1}$$

where the integral goes over the world sheet $\mathbb{R} \times [0, \pi]$. One usually allows open and closed strings, however here we will be only considering the open ones.

An open string does not have the boundary condition $X(\tau, 0) = X(\tau, \pi)$ like in the case of a closed one (hence the name). Instead one usually chooses one of the following conditions:

$$\text{Neumann (N):} \qquad \partial_\sigma X(\tau, 0) = 0 \qquad \text{or} \qquad \partial_\sigma X(\tau, \pi) = 0. \tag{2}$$

$$\text{Dirichlet (D):} \qquad \partial_\tau X(\tau, 0) = 0 \qquad \text{or} \qquad \partial_\tau X(\tau, \pi) = 0. \tag{3}$$

In the case (D) the endpoints of the string at 0 or π are constant while in the case (N) they move with velocity 1. Moreover, one admits combinations of the (N) and (D) conditions in different directions of X. The Euler Lagrange equation takes the form:

$$(-\partial_\tau^2 + \partial_\sigma^2) X = 0 , \tag{4}$$

and the solution is

$$X^\mu = X_L^\mu(\xi^+) + X_R^\mu(\xi^-) \tag{5}$$

where $\xi^\pm = \tau \pm \sigma$, and

$$\partial_+ X_L^\mu = \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-in\xi^+} ,$$

$$\partial_- X_R^\mu = \sum_{n \in \mathbb{Z}} \tilde{\alpha}_n^\mu e^{-in\xi^-} .$$

After canonical quantization, the α_n^μ and $\tilde{\alpha}_n^\mu$ become ladder operators with the commutator relations:

$$\begin{aligned}
&[\alpha_n^\mu, \alpha_m^\nu] = n\delta_{n,-m}\eta^{\mu\nu}\,, \\
&[\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] = n\delta_{n,-m}\eta^{\mu\nu}\,, \\
&[\alpha_n^\mu, \tilde{\alpha}_m^\nu] = 0\,.
\end{aligned} \tag{6}$$

In the case that we impose (N) conditions in a given direction at $\sigma = 0$ we obtain the relation:

$$(N) \qquad \alpha_n^\mu - \tilde{\alpha}_n^\mu = 0\,. \tag{7}$$

For the (D) condition we have:

$$(D) \qquad \alpha_n^\mu + \tilde{\alpha}_n^\mu = 0\,. \tag{8}$$

These relations preserve the algebra (6). Moreover, writing $z = \exp(i(\tau+\sigma))$ they translate respectively into

$$\begin{aligned}
&(N) \quad \partial_z X(\tau, 0) - \partial_{\bar{z}} X(\tau, 0) = 0\,, \\
&(D) \quad \partial_z X(\tau, 0) + \partial_{\bar{z}} X(\tau, 0) = 0\,.
\end{aligned} \tag{9}$$

To implement the different possible combinations of (N) and (D) conditions one generalizes the above equation by introducting a linear map $R \in SO(d-1) \subset SO(d-1, 1)$, that is

$$R = \begin{pmatrix} 1 & 0 \\ 0 & O_{d-1} \end{pmatrix}\,,$$

where O_{d-1} is an element of the orthogonal group $\mathrm{O}(d-1)$. and the equation (see [2]) :

$$\partial_z X(\tau, 0) = R\partial_{\bar{z}} X(\tau, 0)\,. \tag{10}$$

The eigenvectors corresponding to the eigenvalue -1 give a direction with (D) condition being required, while the vectors orthogonal to these have (N) boundary conditions. Notice that the (D) conditions are only in $\mathbb{R}^{d-1} \times \{0\} \subset \mathbb{R}^d$.

D-Branes are closely related to the boundary conditions and are dynamical objects necessary for the understanding of string theory. The following is a description of a geometric realization of a D-brane of an open string in a flat background (which easily generalizes to curved backgrounds).

Description 1. *If X is an open string with the boundary condition given by* (10)*, then the D-brane of this string at the boundary $\sigma = 0$ is a hyperplane orthogonal to the eigenspace of R with the eigenvalue -1.*

Notice that a D-brane covers all the directions with (N) boundary conditions while it is orthogonal to the directions with the (D) condition. A (D) condition fixes the coordinate of an endpoint as mentioned above such that it is constant. An (N) condition gives a direction in which the endpoint $\sigma = 0$ can travel as a function of τ. Thus the geometric interpretation of D-branes is that they specify the hyperplane in which the endpoint of a free string can move. For a more precise and general discussion of D-branes see [3].

1.2 Open strings and boundary conditions in Lie groups

We recall here the treatment of this topic presented in [2]. For simplicity we assume that the Lie group is compact, semi-simple. This implies especially that its Killing form κ is negative definite and thus by left translation of $-\kappa$ one obtains a Riemannian metric on G which we denote by $(\cdot,\cdot)$. Generalizing the action from (1), we can write the kinetic term of the action for the string

$$\gamma : \Sigma \to G$$

as

$$S_{\text{kin}}[\gamma] = \int_\Sigma (\partial_\mu \gamma, \partial^\mu \gamma)\,. \tag{11}$$

Here Σ is an orientable surface with flat Minkowski metric, parametrized by τ and σ as before (or by $\xi^\pm$). One can construct from the orientable surface a 3 dimensional object by attaching a 3-cell to it (as it is its own 2-skeleton, for more on CW complexes see [4]). Moreover as G is semi-simple and compact, its second homotopy group $\pi_2(G)$ vanishes by [5, Theorem 1.45]. This implies that we can construct a smooth map:

$$\tilde{\gamma} : \tilde{\Sigma} \to G$$

where $\tilde{\Sigma}$ is obtained from Σ by gluing the 3 cell, and $\tilde{\gamma}|_\Sigma = \gamma$. Thus we can write the Wess-Zumio action term:

$$S_{WZ}[\gamma] = \frac{1}{6}\int_{\tilde{\Sigma}} \varepsilon^{ijk}(\partial_i\tilde{\gamma}, [\partial_j\tilde{\gamma}, \partial_k\tilde{\gamma}])\,. \tag{12}$$

Here ∂_i means a partial derivative in one of the 3 dimensions of $\tilde{\Sigma}$ and ε^{ijk} is completely antisymmetric in its indices. The WZW action on a string in G is a sum of these two terms

$$S_{WZW}[\gamma] = S_{\text{kin}}[\gamma] + S_{WZ}[\gamma]\,. \tag{13}$$

This action has an infinite dimensional symmetry group. Let us write γ as a function of $z = \exp(i(\tau + \sigma))$, then the following is a symmetry of the model as mentioned in [6]:

$$\gamma(z, \bar{z}) \to \Omega(z)\gamma(z, \bar{z})\bar{\Omega}^{-1}(\bar{z}), \tag{14}$$

where Ω and $\bar{\Omega}$ are smooth maps from Σ to G depending only on z and $\bar{Z}$ respectively. This symmetry allows us to introduce the Lie algebra valued conserved currents:

$$J(z) = -\rho_{\gamma^{-1}}(\partial_z \gamma), \qquad \bar{J}(\bar{z}) = \lambda_{\gamma^{-1}}(\partial_{\bar{z}} \gamma), \tag{15}$$

where ρ_g and λ_g are the derivatives of right, resp. left, translation diffeomorphisms by g on G. Let T_a be a basis of the Lie algebra $\mathfrak{g}$ of G, then we can write $J = J^a T_a$ and $\bar{J}^a = \bar{J}^a T_a$. Expanding J^a and $\bar{J}^a$ into their modes:

$$J^a(z) = \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1}, \qquad \bar{J}_a(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{J}^a_n \bar{z}^{-n-1}, \tag{16}$$

and quantizing the coefficients J^a_n and $\bar{J}^a_n$, we obtain the following commutator relations [2]:

$$\begin{aligned}
[J^a_n, J^b_m] &= f^{ab}{}_c J^c_{n+m} + n\delta_{n+m,0}(T_a, T_b)_e, \\
[\bar{J}^a_n, \bar{J}^b_m] &= f^{ab}{}_c \bar{J}^c_{n+m} + n\delta_{n+m,0}(T_a, T_b)_e, \\
[J^a_n, \bar{J}^b_m] &= 0 .,
\end{aligned}$$

where $\sum_c f^{ab}{}_c T_c = [T_a, T_b]$. One can compare it with the affine Lie algebra constructed as a central extension of the loop algebra of $\mathfrak{g}$, see [13, 7.7]. The commutators given there coincide with the above, thus we have obtained the affine Kac Moody algebra for $\mathfrak{g}$.

In [2] one considered the WZW model as a generalization of the one in flat background discussed in the previous subsection. The conserved currents $J(z)$ and $\bar{J}(\bar{z})$ correspond then to the quantities $-\partial_z X(z)$ and $\partial_{\bar{z}} X(\bar{z})$ respectively. The gluing conditions from (9) induce then the conditions:

$$\begin{aligned}
(N) \qquad & J(z) = -\bar{J}(\bar{z}), \\
(D) \qquad & J(z) = \bar{J}(\bar{z}),
\end{aligned}$$

which are given for $z = \bar{z}$. The more general condition (10) gives the gluing condition at $z = \bar{z}$:

$$J(z) = R\bar{J}(\bar{z}), \tag{17}$$

where R is a constant linear map on the Lie algebra $\mathfrak{g}$ which is an isometry for the metric $(\cdot\,,\cdot)_e$ (this requirement is enforced by preserving the conformal symmetry).

The previous equation is slightly misleading, as in (10) the relation was given on the tangent space of the point $(\tau, 0)$, while here the equation relates only quantities in the tangent space at the identity corresponding to the Lie algebra. To remedy this, one can rewrite it using the defining equation of the currents (15) and

$$\mathcal{R}(\gamma) = -\rho_\gamma \circ R \circ \lambda_{\gamma^{-1}} \tag{18}$$

to obtain

$$\partial_z \gamma = \mathcal{R}(\gamma) \partial_{\bar{z}} \gamma \,. \tag{19}$$

This is now the actual boundary condition at $z = \bar{z}$ obtained from the gluing condition (17).

At any endpoint on the boundary $z = \bar{z}$, we can now distinguish the Neumann directions and Dirichlet directions as it was done in the flat case. Consider such a point $\gamma(z)$ and the corresponding $\mathcal{R}(\gamma(z))$. The eigenspace in $T_{\gamma(z)}G$ for the eingevalue -1 gives the directions with Dirichlet conditions and the eigenspaces orthogonal to it (w.r.t. $(\cdot\,,\cdot)$) with all the other possible eigenvalues (which can be complex) span the directions with (N) conditions. Now we can generalize Description 1 to give a description of the D-brane in a group manifold. We do it here in terms of distributions of the tangent bundle TG of the Lie group (see for example [7] to recall the definition).

Generalizing flat D-branes: Let γ be an open string in the Lie group G with the gluing condition at $z = \bar{z}$ given by (17) and the corresponding boundary condition (19). Then the generalization of D-branes in flat space is given by the distribution of TG consisting at each $\gamma(z)$, where $z = \bar{z}$, of the subspace orthogonal to the -1 eigenspace of $R(\gamma(z))$ in $T_{\gamma(z)}G$. Notice that again a D-brane specifies at each point $\gamma(z)$ the possible directions in which the string's endpoint can propagate.

Let us now consider the distribution V_R which consists of all the subspaces orthogonal to the -1 eigenspace of $\mathcal{R}(g)$ in T_gG going over all $g \in G$. Assume that V_R is involutive. That is, let X and Y be sections of V_R as the tangent fields on G, then we have that their Lie bracket is again a section of V_R:

$$[X, Y] \in \Gamma(G, V_R) \,. \tag{20}$$

Frobenius theorem [7] tells us that there exists a maximal submanifold of G going through a point $\gamma(z)$ whose tangent bundle consists of the spaces of the distribution V_R. One calls it the maximal integral submanifold for V_R in G. Thus in the case of V_R being involutive there is an equivalence between the generalization of D-branes described above and the corresponding maximal integral submanifolds of G going through the endpoints of the string. These manifolds replace the hyperplanes for the flat case and describe the space in which the endpoints of an open string can move.

One can now give a restriction on the map R on the Lie algebra $\mathfrak{g}$ which will secure that the corresponding distribution V_R is involutive.

Statement 2. *Let R be a Lie algebra automorphism (thus an element of the Lie group $Aut(\mathfrak{g})$), that is it solves:*

$$R([X,Y]) = [R(X), R(Y)]\,. \tag{21}$$

Then the distribution V_R constructed above with respect to $\mathcal{R}(g)$ is an involutive distribution.

Proof. Consider the section of TG

$$\xi_X : G \to TG\,, \qquad \xi_X(g) = \rho_g(X) - \lambda_g R^{-1}(X) \tag{22}$$

given for every $X \in \mathfrak{g}$. The map

$$X \mapsto \xi_X(g) \tag{23}$$

is surjective onto $V_R|_g$. This follows because the -1 eigenspace of $\mathcal{R}(g)$ is the kernel of the map

$$\mathrm{Id} - \rho_g \circ R \circ \lambda_{g^{-1}}\,,$$

thus $V_R|_g$ is the image of its adjoint with respect to the product $(\cdot,\cdot)_g$. Using that $\rho_g \circ R \circ \lambda_{g^{-1}}$ is unitary with respect to the inner product, we obtain that the adjoint is given by

$$\mathrm{Id} - \lambda_g \circ R^{-1} \circ \rho_{g^{-1}}\,.$$

Composing this map with the derivative of the right translation ρ_g, gives us the map (23) which shows its surjectivity. To show that the sections of V_R are closed under the Lie bracket, it is enought to show that acting on any sections forming a local frame of V_R with Lie bracket gives again sections of V_R. For that, notice that if X_i are a basis of $\mathfrak{g}$, then ξ_{X_i} will span $V_R|_g$ at every point $g \in G$. A simple calculation shows that

$$[\xi_{X_i}, \xi_{X_j}] = \xi_{[X_i,X_j]}\,.$$

This completes the proof. □

The proof has additionally shown that every vector in $V_R|_g$ is of the form $\rho_g(X) - \lambda_g R^{-1}(X)$ for some $X \in \mathfrak{g}$. Let us now assume that $R^{-1} \in \mathrm{Aut}(\mathfrak{g})$ can be lifted to an automorphism $\tau_R \in \mathrm{Aut}(G)$ of the Lie group. Consider the action of G on itself called τ_R-twisted conjugation:

$$G \times G \to G \qquad (h, g) \mapsto hg\tau_R(h^{-1})\,. \tag{24}$$

The orbit of this action going through the point $g \in G$ is called the twisted conjugacy class $C_{\tau_R}(g)$. Taking a derivative of the map (24) for a given $g \in G$, one obtains the map

$$X \mapsto \xi_X(g)$$

which has $V_R|_g$ as its image.

Conclusion 3. *The D-branes for the gluing condition given by $R \in Aut(\mathfrak{g})$, where R can be lifted to an automorphism on the Lie group, and thus R^{-1} lifts to $\tau_R \in Aut(G)$, are given by the τ_R twisted conjugacy classes.*

The stabilizers of g with respect to the twisted conjugation also get a special meaning. Consider a -1 eigenvector U in $T_g G$ of $\mathcal{R}(g)$ and write it as $U = \lambda_g(R^{-1}(X))$ for some $X \in \mathfrak{g}$. We know that it solves

$$\rho_g R(\lambda_g^{-1}(U)) = U\,,$$

and we obtain the following equation for X:

$$\mathrm{ad}_g(R^{-1}(X)) = X\,.$$

This describes precisely the Lie algebra of the stabilizer subgroup of g with respect to the action described in (24). Thus the right translation by g of the Lie algebra of the twisted stabilizer of g gives the space of directions with Dirichlet conditions. This also implies that the dimension of every D-brane given by a twisted conjugacy class $C_\tau(g)$ is given by:

$$\dim\big(C_\tau(g)\big) = \dim(G) - \dim\big(\mathrm{Stab}_\tau(g)\big)\,. \tag{25}$$

1.3 Summary of contents

The main goal of what follows will be to observe the behavior of twisted conjugacy classes in the Lie groups $\mathrm{SU}(n)$. In the section 2 we introduce the necessary mathematical background and compute an outer automorphism τ on $\mathrm{SU}(n)$ with respect to which we will be taking the twisted conjugacy classes. In section 3 we discuss a method for working with twisted conjugation via Cartan subgroups in non-connected Lie groups. This then gives us the result that the space of twisted conjugacy classes is parametrized by the quotient:

$$T_0^\tau // W(G, T, \tau)\,,$$

where T_0^τ is the maximal torus of τ invariants and $W(G, T, \tau)$ the twisted Weyl group on G. We apply this statement directly to the cases SU(n), n=4,5,6,7 in section 4, computing the spaces of twisted conjugation classes and their twisted stabilizers directly. Comparing our explicitly computed results with [12], we notice that there are some inconsistencies. In the last section, we simply recall the method used in [12] and check for SU(4) that our explicit results are also obtained by this abstract method. Most of the necessary mathematical notions are discussed in the text, however we suggest the reader to have some basic knowledge of semi-simple Lie algebras [9], affine Lie algebras [13] and compact Lie groups [14].

2 Compact Lie groups, compact forms and automorphisms on Dynkin diagram

In what follows, we will recall some results about compact Lie groups, their compact forms, and Dynkin diagrams. This will allow us to construct outer automorphisms on compact Lie groups, and we will obtain a particular automorphism on SU(n) coming from the non-trivial Dynkin automorphism of A_{n-1}. The twisted conjugacy classes considered in 4, will be taken with respect to this automorphism. We also discuss the folding of structures related to the Dynkin diagrams. The resulting folded root systems will be later used for classifying all twisted conjugacy classes and determining their stabilizers via a method from [12].

2.1 Some standard results about compact Lie groups

Let G be a compact Lie group with its Lie algebra $\mathfrak{g}$. A maximal torus $T \subset G$ is a maximal abelian connected subgroup of G. By [10, Theorem 3.6], it is itself a compact Lie group isomorphic to a torus (a finite product of $U(1)$). Its Lie algebra $\mathfrak{t} \subset \mathfrak{g}$ is a maximal abelian Lie sub-algebra. If G is additionally semisimple, then $\mathfrak{t}$ is a Cartan sub-algebra of $\mathfrak{g}$. In the following, we present the standard result for maximal tori.

Lemma 4. *[10, Theorem 3.8 and Corollary 3.9]*
Let T be a maximal torus of a connected compact Lie group G, then any

element of G is conjugate in G to an element of T. The exponential map $exp : g \to G$ is surjective.

If $N_G(T)$ is the normalizer of the maximal torus T, then $W(G,T) = N_G(T)/T$ defines the *Weyl group* of T in G. As T is abelian, there is a well-defined action of $W(G,T)$ on T induced by conjugation. Another standard result states that two elements in T are conjugate in G if and only if they are in the same orbit of $W(G,T)$. Thus the set of conjugacy classes in G can be identified with the set of orbits

$$T//W(G,T)\,. \tag{26}$$

2.2 Compact form

Let $\mathfrak{g}$ be a complex Lie algebra. A real subalgebra $\mathfrak{g}_0$, such that its complexification $(\mathfrak{g}_0)_{\mathbb{C}}$ gives again $\mathfrak{g}$, is called a *real form* of $\mathfrak{g}$. We will be interested in working with the compact forms of simple complex Lie algebras which get their name from the corresponding Lie groups being compact.

Definition 5. *Let $\mathfrak{g}$ be a Lie algebra with a Cartan subalgebra $\mathfrak{h}$ and positive roots Δ^+. Its compact form $\mathfrak{g}_0$ is a real form with a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{h}$, such that $(\mathfrak{h}_0)_{\mathbb{C}} = \mathfrak{h}$, and the the simple roots of $\mathfrak{g}$ take only imaginary values on $\mathfrak{h}_0$.*

By [11, 26.1], the compact form of $\mathfrak{g}$ is unique, and it has the form:

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{l}_\alpha\,,$$

where $\mathfrak{l}_\alpha$ is a two dimensional real vector space which is the intersection of $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ with $\mathfrak{g}_0$. The elements of $\mathrm{ad}(H)$ act on $\mathfrak{l}_\alpha$ by a rotation. More precisely, let ω_0 be a complex antilinear map on $\mathfrak{g}$ which leaves the Lie bracket invariant, defined uniquely by:

$$E_i \mapsto F_i\,, \quad F_i \mapsto E_i\,, \quad H_i \mapsto -H_i\,,$$

where E_i, F_i and H_i are the Chevalley's generators for all the simple roots α_i of $\mathfrak{g}$. The compact form $\mathfrak{g}_0$ is equal to $\mathfrak{g}^{\omega_0}$ - the set of ω_0 fixed points in $\mathfrak{g}$. Recall also that the subalgebra generated by $\mathfrak{l}_\alpha$ is isomorphic to the compact form of $\mathfrak{sl}(2,\mathbb{C})$ which is $\mathfrak{su}(2)$. This means that we can choose elements S_α, T_α and P_α in this subalgebra such that:

$$[P_\alpha, S_\alpha] = 2T_\alpha\,, \quad [P_\alpha, T_\alpha] = -2S_\alpha\,, \text{ and } [S_\alpha, T_\alpha] = 2P_\alpha\,.$$

In fact, if E_α, F_α, and H_α in the complex subalgebra generated by $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ are such that

$$[H_\alpha, E_\alpha] = 2E_\alpha \,, \quad [H_\alpha, F_\alpha] = -2F_\alpha \,, \text{ and } [E_\alpha, F_\alpha] = H_\alpha \,,$$

then they can be given by:

$$P_\alpha = \frac{1}{2} i H_\alpha \,, \quad S_\alpha = \frac{1}{2}(E_\alpha + F_\alpha) \,, \quad T_\alpha = \frac{i}{2}(E_\alpha - F_\alpha) \,. \tag{27}$$

2.3 Automorphisms on Dynking diagrams

A *finite graph* Q is a pair (V, E) where V is a finite set of its *vertices* and E consists of pairs of V such that the pairs can repeat themselves (thus they have some integer multiplicity). The elements of E are called the *edges* of Q. Graphs form a category with the morphism between $Q_1 = (V_1, E_1)$ and $Q_2 = (V_2, E_2)$ being a map from V_1 to V_2 such that it maps every pair from E_1 to a pair of E_2 with the same multiplicity. One can construct graphs from matrices of a special form.

Definition 6. *A generalized Cartan matrix* $(a_{ij}) = A \in Mat(n, n, \mathbb{Z})$ *is an integer valued matrix with the following properties:*

1. *The diagonal entries are equal to 2.*
2. *Whenever* $i \neq j$, *then* $a_{ij} \leq 0$.
3. *If* $a_{ij} = 0$, *then* $a_{ji} = 0$.

For a given generalized Cartan $n \times n$ matrix $(a_{ij}) = A$, one can construct a graph in the following way:

1. The set of vertices V consists of the V_i corresponding to the i'th index of the matrix A for each $1 \leq i \leq n$.
2. The multiplicity of the edge between the vertices V_i and V_j is given by $a_{ij}a_{ji}$.

Additionally, one gives some edges with a multiplicity higher than one an orientation in the following way: If $a_{ij}a_{ji} \leq 4$, and $|a_{ij}| \geq |a_{ji}|$, then the arrow goes from V_j to V_i. A morphism of these graphs (diagrams) has to preserve the orientation too. If A is a generalized Cartan matrix, we will denote its corresponding graph by $Q(A)$.

Here, we will be interested mainly in the *finite Dynkin diagrams* and their automorphisms. Finite Dynkin digram is the diagram given by the above method for a generalized Cartan matrix A, such that all of its eigenvalues are positive. Such a matrix A is simply called a *Cartan matrix.*

Theorem 7. *Let A be a Cartan matrix and $Q(A)$ its Dynking diagram. Then $Q(A)$ is given up to an isomorphisms by one of the diagrams in Figure 1.*

Proof. See Theorem 4.8 in [13] for example. □

Looking at the figure we especially notice the following statement about the automorphisms of Dynkin diagrams.

Lemma 8. *The group of automorphisms of the Dynkin diagrams A_n ($n \geq 2$), D_n ($n \geq 5$), E_6 is isomorphic to $\mathbb{Z}_2$. The group of automorphisms of D_4 is isomorphic to $\mathbb{Z}_3$. For other Dynkin diagrams these groups are trivial (notice that this especially holds for all non-simply laced Dynkin diagrams).*

Let $\mathfrak{g}$ be a complex simple Lie algebra with its Cartan subalgebra $\mathfrak{h}$, its root system Δ, simple root system Π, and its Dynkin diagram Q with n vertices. Every automorphism from $\mathrm{Aut}(Q)$ can be extended to an automorphism of the Lie algebra $\mathfrak{g}$: Let τ be such an automorphism and let V_i be the vertices corresponding to the simple roots α_i. If E_i, H_i, F_i are the Chevalley generators associated to the simple root α_i, then we define the automorphism τ' on $\mathfrak{g}$ by extending the following relations to an automorphism of the Lie algebra $\mathfrak{g}$:

$$\begin{aligned} &\tau'(H_i) = H_j\,, \quad \tau'(E_i) = E_j\,, \quad \tau'(F_i) = F_j \\ &\text{if} \quad \tau(V_i) = V_j\,. \end{aligned} \tag{28}$$

This is always possible, as the Serre relations are preserved under this map, because the Dynkin diagram and thus the Cartan matrix giving the Serre relations are preserved. Notice that constructing this from a composition of automorphisms on Q gives us a composition of the corresponding morphisms on $\mathfrak{g}$. This implies that the mapping $\tau \mapsto \tau'$ is a group homomorphism into $\mathrm{Aut}(\mathfrak{g})$. Thus we can view the group $\mathrm{Aut}(Q)$ as a subgroup of $\mathrm{Aut}(\mathfrak{g})$.

One can restrict this newly constructed automorphism to $\mathfrak{h}$ where it becomes simply a permutation of H_i. Let $(-,-)$ denote the Killing form on $\mathfrak{g}$, and

$$B : \mathfrak{h} \to \mathfrak{h}^*$$

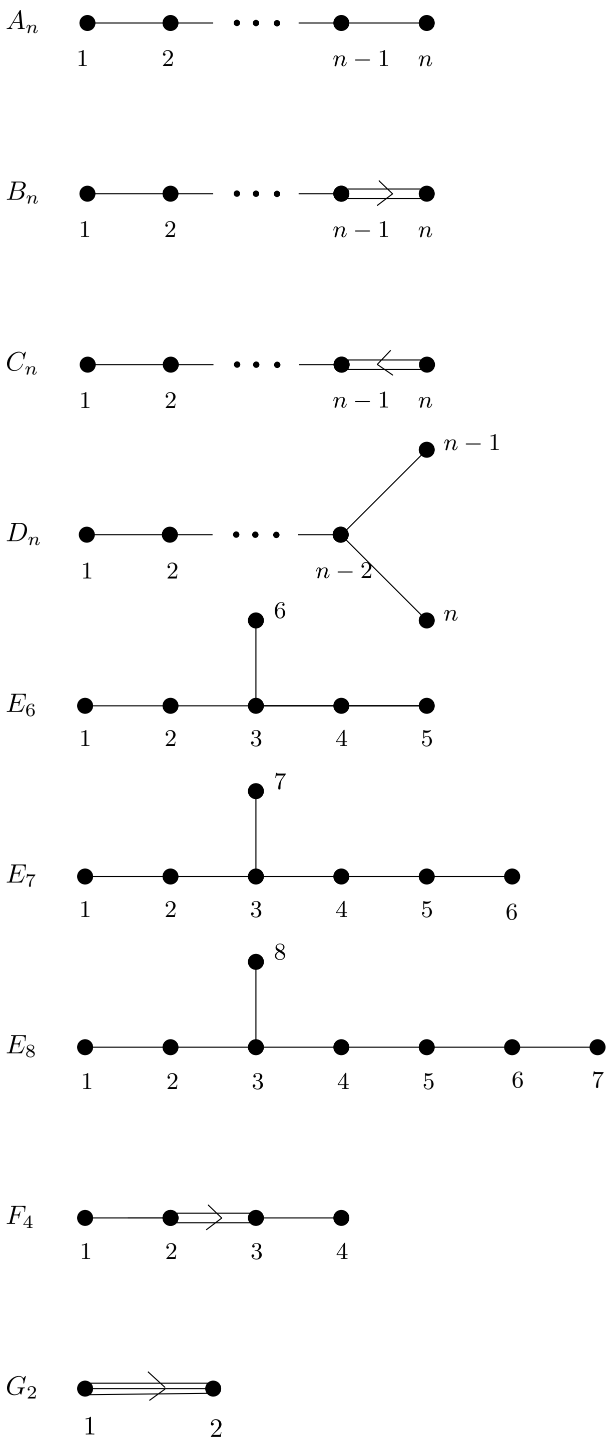


Figure 1: Dynking diagrams

the corresponding isomorphism which identifies $\mathfrak{h}$ and $\mathfrak{h}^*$ via the equation:

$$B(X)(Y) = (X, Y)\,.$$

We can now view τ' as acting on $\mathfrak{h}^*$ such that B is τ' equivariant. Then we have that for each $\alpha \in \mathfrak{h}^*$

$$\tau'(\alpha) = \alpha \circ \tau'\,.$$

One sees immediately that τ' permutes the simple roots the way that it permutes the corresponding vertices.

We want to consider a special example which will be useful later. We set $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{h}$ consists of the diagonal matrices in $\mathfrak{g}$. We choose the simple roots to be $\Pi = \{\alpha_i = e_i - e_{i+1}\}_{i=1}^{n-1}$ where $e_i\big(\mathrm{diag}(x_1, \ldots, x_n)\big) = x_i$. The Chevalley generators are now:

$$E_i = E_{i,i+1}, \quad F_i = E_{i+1,i}, \quad H_i = E_{ii} - E_{i+1,i+1}\,. \tag{29}$$

Here E_{ij} denotes an $n \times n$ matrix that has all of its entries equal to zero apart from the entry at the position (i, j) which is equal to 1. The unique non-trivial automorphism of the Dynkin diagram A_{n-1} maps the vertex V_i to the vertex V_{n-i}. We label this automorphism τ and the corresponding automorphism of $\mathfrak{g}$ is called τ'.

Lemma 9. *Let $\mathfrak{g}$, τ, and τ' be defined as above. If $A = (a_{ij})$ is an element of $\mathfrak{g}$, then:*

$$\tau'(A) = \big((-1)^{i+j+1} a_{n+1-j,n+1-i}\big)\,. \tag{30}$$

Proof. Notice that $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$. Thus we get for $i = 1, \ldots, n$ and $0 < a \leq n - i$ that

$$E_{i,i+a} = [\ldots[[E_{i,i+1}, E_{i+1,i+2}], E_{i+2,i+3}] \ldots, E_{i+a-1,i+a}]\,.$$

Applying τ' to both sides of the equation we obtain

$$\tau'(E_{i,i+a}) = (-1)^{a-1}[E_{n-i-a+1,n-i-a+2}, \ldots [E_{n-i-1,n-i}, E_{n-i,n-i+1}] \ldots]\,.$$

The right hand side of the last equation is equal to $(-1)^{a-1} E_{n-i-a+1,n-i+1}$. A similar statement can be shown for $-i + 1 \leq a < 0$. We also know that $E_{ii} - E_{i+1,i+1} = [E_{i+1,i}, E_{i,i+1}]$ is mapped to $[E_{n-i+1,n-i}, E_{n-i,n-i+1}] = E_{n-i,n-i} - E_{n-i+1,n-i+1}$. By linearity and combining the previous results, we conclude the statement of the lemma. □

Consider the compact form $\mathfrak{su}(n)$ of $\mathfrak{sl}(n, \mathbb{C})$ which consists of traceless anti-hermitian matrices. One can easily check that $\mathfrak{su}(n)$ is τ' invariant, thus we can restrict it to an automorphisms on this real subalgebra (which we now label τ for convenience).

Corollary 10. *If τ is defined on $\mathfrak{su}(n)$ as given above and A is a matrix in $\mathfrak{su}(n)$ then*

$$\tau(A) = J_n \bar{A} J_n^{-1}, \tag{31}$$

where J_n is an anti-diagonal matrix with its $(i, n-i+1)$ entry being $(-1)^i$.

Proof. We apply $a_{ij} = -\bar{a}_{ji}$ to the equation (30) to get:

$$\tau(A) = \left((-1)^{i+j} \bar{a}_{n+1-i,n+1-j}\right).$$

The right hand side corresponds to first taking the complex conjugate of A, and then conjugating by J_n. □

Comparing it to the automorphisms used in [12, Example 3.2], we see that it is slightly different. When restricted to the Cartan subalgebra of $\mathfrak{su}(n)$ of diagonal antihermitian matrices it still acts the same way, so it doesn't change the results in 4. However, it affects the form of the Lie algebras of the stabilizers in 5.1.

2.4 Outer automorphisms

Let G be a simple compact Lie group. The group $\mathrm{Aut}(\mathfrak{g})$ of Lie algebra automorphisms on $\mathfrak{g}$ is a matrix group in $\mathrm{GL}(\mathfrak{g})$ and thus a Lie group. It's Lie algebra is denoted by $\mathfrak{Der}(\mathfrak{g})$ and consists of the elements of the form $\mathrm{ad}(X)$ where $X \in \mathfrak{g}$. We denote the group of inner automorphisms on $\mathfrak{g}$ by $\mathrm{Inn}(\mathfrak{g}) = \{\mathrm{Ad}(g) \mid g \in G\}$. By the compactness of G, this subgroup coincides with the image of $\mathfrak{Der}(\mathfrak{g})$ under the exponential map. As such it is the connected component of $\mathrm{Aut}(\mathfrak{g})$. It was shown in [8] that the component group $\mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ is finite.

Let us now consider the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. The Proposition 26.4 in [11] states that because G is compact then $\mathfrak{g}$ must be the unique compact form of $\mathfrak{g}_{\mathbb{C}}$. We have already shown that if Q the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$, then there is an embedding of $\mathrm{Aut}(Q)$ into $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. We want to show that any automorphisms in this subgroup of $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ commutes with the compact involution ω_0. This follows because if $\tau \in \mathrm{Aut}(Q)$, then both $\omega_0 \circ \tau$ and

$\tau \circ \omega_0$ are antilinear. And if E_i, F_i, and H_i are the Chevalley's generators, then the compositions both act on them in the same way:

$$E_i \mapsto F_j\,, \quad F_i \mapsto E_j\,, \quad H_i \mapsto -H_j\,,$$

whenever $\tau(V_i) = V_j$ on the corresponding vertices of the graph Q. This implies that we can restrict the automorphisms from the subgroup $\mathrm{Aut}(Q)$ to act on $\mathfrak{g}$. This restriction gives an injective homomorphism from $\mathrm{Aut}(Q)$ into $\mathrm{Aut}(\mathfrak{g})$, because the restriction of τ permutes the $\mathfrak{su}(2)$ subalgebras of $\mathfrak{g}$ generated by $\mathfrak{l}_{\alpha_i}$ and thus acts non-trivially unless τ was the identity.

We can also extend uniquely every automorphism of $\mathfrak{g}$ to an automorphism on its complexification. This gives us an embedding

$$\mathrm{Aut}(\mathfrak{g}) \hookrightarrow \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})\,.$$

Which takes $\mathrm{Aut}(Q)$ to itself and $\mathrm{Inn}(\mathfrak{g})$ to a subgroup of $\mathrm{Inn}(\mathfrak{g}_{\mathbb{C}})$. Moreover, one has the following statement about the group of automorphisms on $\mathfrak{g}_{\mathbb{C}}$.

Lemma 11. *Let $\mathfrak{g}_{\mathbb{C}}$ be a simple complex Lie algebra. Then its automorphism group is given by a semi-direct product of its subgroups* Aut(Q) *and* $\mathrm{Inn}(\mathfrak{g}_{\mathbb{C}})$*:*

$$\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}) = \mathrm{Aut(Q)} \rtimes \mathrm{Inn}(\mathfrak{g}_{\mathbb{C}})\,.$$

Proof. See the proof of Proposition D.40 in [11]. □

This together with our previous results implies the same result for the compact forms:

Proposition 12. *Let $\mathfrak{g}$ be a Lie algebra of a compact simple Lie group G. The group of automorphisms on $\mathfrak{g}$ is a semidirect product:*

$$\mathrm{Aut}(\mathfrak{g}) = \mathrm{Aut(Q)} \rtimes \mathrm{Inn}(\mathfrak{g})\,. \tag{32}$$

The group of outer automorphisms $\mathrm{Out}(\mathfrak{g})$ *is isomorphic to* Aut(Q).

Remark 13. *Additionally, if G is simply connected, then the statement of Proposition 12 extends to it. That is,* Out(G) *is isomorphic to* Aut(Q)*, where each τ in* $\mathrm{Aut(Q)} \subset \mathrm{Aut}(\mathfrak{g})$ *is lifted to be the corresponding unique automorphism of G with derivative τ.*

This together with Lemma 8 shows us that the automorphism τ on $\mathfrak{su}(n)$ given by (31) is a representative of the unique non-trivial element in $\mathrm{Out}(\mathfrak{su}(\mathrm{n}))$. It lifts to an automorphism of $SU(n)$ given by the same equation $\tau(A) = J_n \bar{A} J_n^{-1}$.

2.5 Folding of Lie groups, Lie algebras and root systems

We describe the concept of folding of a simple, compact Lie group, Lie algebra and the root system, to give a better idea of the tools used in 5.2. Let Q be a Dynkin diagram from Figure 1, and denote by $*Q$ any structure associated with it, e.g. complex simple Lie algebra, its compact form, compact Lie group, Cartan subalgebra, root system. We have shown in 2.3 and 2.4 that the automorphism τ of Q can be extended to $*Q$. We denote the resulting automorphism there by $*\tau$. As mentioned in Lemma 8, we can restrict the consideration to simply laced Dynkin diagrams and their non-trivial automorphisms.

We will first address the cases where $Q \neq A_{2m}$.

Proposition 14. *Let $*Q$ be a structure associated with Q as described above and $*\tau$ the associated automorphism to the Dynkin diagram automorphism τ. Define $*Q * \tau$ to be a new structure of the same type which has for the following cases these meanings:*

1. *If $*Q = G$ is the complex algebraic group/compact Lie group. Then $*Q * \tau = G_0^\tau$ is the connected component of the $*\tau$ fixed points of G.*

2. *If $*Q = \mathfrak{g}$ or $\mathfrak{h}$ is the complex Lie algebra/its compact form or a Cartan subalgebra, then $*Q * \tau = \mathfrak{g}^\tau$ or $\mathfrak{h}^\tau$ is a set of its $*\tau$ fixed points.*

3. *If $*Q = (\Delta, \mathfrak{h}^*)$ is the root-system with the roots δ then $*Q * \tau$ is the root system $(\Delta^\tau, (\mathfrak{h}^*)^\tau)$ where*

$$\Delta^\tau = \left\{ \sum_{i=1}^{ord(\tau)} \tau^i(\alpha) : \alpha \in \Delta \right\}. \tag{33}$$

Then $*Q * \tau \cong *(Q^\tau)$ where Q^τ is a "folded" Dynkin diagram given by the following table:

Q	A_{2m-1}	$D_n, n \geq 5$	D_4	E_6
Q^τ	C_m	B_n	G_2	F_4

Proof. The statement that this holds for the root system was shown in Proposition 13.2.2 [16]. For a complex simple Lie algebra $\mathfrak{g}$ with τ of finite order, we can write it as a finite direct sum of $\mathfrak{g}_i$, where $\mathfrak{g}_i$ is the eigenspace corresponding to the eigenvalue $\exp(\frac{2\pi i}{\mathrm{ord}\tau})$ of τ. The set of fixed points can then be labeled as $\mathfrak{g}_0$. The same can be done for the Cartan subalgebra $\mathfrak{h}$. From [13]

Proposition 7.9 and Proposition 8.3, we know that $\mathfrak{g}_0$ is a simple Lie algebra with Cartan subalgebra $\mathfrak{h}_0$ and then the root system of $\mathfrak{g}_0$ is given by

$$\Delta_0 = \{\alpha|_{\mathfrak{h}_0} : \alpha \in \Delta\}\,. \tag{34}$$

However the map

$$\mathfrak{h}^* \to (\mathfrak{h}_0)^* : \alpha \mapsto \alpha|_{\mathfrak{h}_0} \tag{35}$$

induces an isomorphism of the root systems $(\Delta^\tau, (\mathfrak{h}^*)^\tau)$ and $(\Delta_0, \mathfrak{h}_0^*)$. This concludes that $\mathfrak{g}^\tau$ is a simple Lie algebra with the Dynkin diagram Q^τ.

We have already shown that τ and the compact involution commutes on $\mathfrak{g}$, which implies that the set of fixed points of the compact form of $\mathfrak{g}$ is the compact form of $\mathfrak{g}^\tau$, because the compact involution on $\mathfrak{g}^\tau$ is the restriction of the compact involution on $\mathfrak{g}$.

Finally the statement about Lie groups and algebraic groups follows because the connected component of the fixed points G_0^τ has as its Lie algebra the subalgebra of fixed points $\mathfrak{g}^\tau$. □

For $Q = A_{2m}$ the statement becomes different because the folding of the root system is no longer a reduced root system but a non-reduced one.

Proposition 15. *Let $Q = A_{2m}$, τ its non-trivial automorphism and $(\Delta, \mathfrak{h}^*)$ its root system, then the folded root system $(\Delta^\tau, (\mathfrak{h}^*)^\tau)$ where Δ^τ is given by* (33) *is the non-reduced root system BC_m. The root system $(\Delta_0, \mathfrak{h}_0^*)$ of the Lie algebra $\mathfrak{g}^\tau$ of fixed points is the root system B_m contained in BC_m which is the image of $(\Delta^\tau, (\mathfrak{h}^*)^\tau)$ under* (35).

The Lie algebra $\mathfrak{g}^\tau$ and its compact form is simple of type B_m and the connected component G_0^τ of the fixed points in the corresponding compact Lie group G is a simple Lie group of type B_m.

Proof. It was shown in Proposition 13.2.2 [16] that the folded root system is BC_n. Further, from Proposition 8.3 in [13], we see the statement about $(\Delta_0, \mathfrak{h}_0^*)$. This implies the rest of the proposition by the same argument as in the previous proposition. □

Remark 16. *Because the map τ acts on the simple roots of the simple root system $\Pi \subset \Delta$ by interchanging them, we see that acting with the map* (35) *on Π gives us a simple root system Π_0 of the folded root system. Moreover, for the case A_{2m} the root system Δ_0 contains Π_0. Thus in all cases Π_0 is a simple root system in Δ_0.*

Remark 17. *One can show that the highest weight of $\mathfrak{g}_1$ as a representation of $\mathfrak{g}_0$ with respect to $\mathfrak{h}_0$ is equal to the highest short root of Δ_0 in the case*

$Q \neq A_{2m}$. If $Q = A_{2m}$, then this highest weight is equal to the highest root of the non-reduced folded root system $\Delta^\tau = BC_m$. We denote this root by θ_τ in both cases. Notice also that from Proposition 8.3 in [13] it follows that θ_τ is the root corresponding to the additional point of the twisted affine Dynkin diagram constructed from the Dynkin diagram Q for the automorphism τ. More precisely, we can consider $(\Delta_0, \mathfrak{h}_0^*)$ as a subsystem of a root system $(\tilde{\Delta}, \tilde{\mathfrak{h}_0^*})$ of the twisted affine algebra constructed from $\mathfrak{g}$ as described for example in 8.3 [13]. We have then :

$$\tilde{\mathfrak{h}_0^*} = \mathfrak{h}_0^* \oplus \mathbb{C}\delta \text{ and } \delta \text{ orthogonal to } \mathfrak{h}_0^*\,, \tag{36}$$

and for the simple root system $\tilde{\Pi}$ of $\tilde{\Delta}$

$$\tilde{\Pi} = \Pi_0 \cup \{\alpha_0 = \delta - \theta_\tau\}\,, \tag{37}$$

where Π_0 is described in Remark 16.

3 Twisted conjugation

We can concentrate now on recovering the methods for working with twisted conjugacy classes which will be applied to $\mathrm{SU}(4)$ in 4. Essentially, we combine the results from [12], [15], and [14]. We connect statements about a more general topic of Cartan subgroups to our problem and explain how one can apply the general results to the case that we are working with. At the end, we will be able to express the space of twisted conjugacy classes in a similar way as it was done for the non-twisted case in (26).

From now on, we will only consider simple, simply connected, compact Lie groups. Let G be such a Lie group. Let us first recall the notion of twisted conjugation that we obtained in 1: Choosing an automorphism $\tau \in \mathrm{Aut}(G)$ allows us to define a left action of G on itself given by:

$$(g, x) \mapsto g x \tau(g)^{-1}\,. \tag{38}$$

This action is called *twisted conjugation* with respect to $\tau \in \mathrm{Aut}(G)$. The orbits of the twisted conjugation are called *twisted conjugacy classes.*

If two automorphisms τ and σ of the Lie group G are related by a conjugation i_g by an element g (that is, $\tau = \sigma \circ i_g$), then the map $x \mapsto gx$ restricted to a τ-twisted conjugacy class containing the element $h \in G$ maps onto the σ-twisted conjugacy class containing the element gh as a diffeomorphism. Further, this induces a bijection between the τ-twisted and σ-twisted conjugacy classes of G. This implies that we can content ourselves to work with a single representative of a class of outer automorphisms $\mathrm{Out}(G)$.

3.1 An equivalent way to view twisted conjugation

To be able to consider the τ-twisted conjugacy classes for a non-trivial $[\tau] \in \mathrm{Out}(G)$, one can approach this by defining a new group

$$\hat{G} = G \rtimes \Gamma$$

where Γ is the finite subgroup of $\mathrm{Out}(G)$ generated by τ. From now on τ will be an element of $\mathrm{Aut}(Q)$ lifted to act as an automorphisms on the Lie group as described in the subsection 2.4. The group-homomorphisms $\langle\tau\rangle = \Gamma \to \mathrm{Aut}(G)$ defining the semidirect product $G \rtimes \Gamma$ is simply the inclusion. We will use the notation $G \times \{\tau\} = G\tau$.

Notice that acting with the standard conjugation by the subgroup $G \subset \hat{G}$ on $G\tau$ gives an action of G on $G\tau$ of the following form:

$$\begin{aligned} &G \times G\tau \to G\tau \\ &(g, (h, \tau)) \mapsto (gh\tau(g)^{-1}, \tau)\,. \end{aligned} \tag{39}$$

One uses the identification

$$\begin{aligned} &G\tau \to G \\ &(g, \tau) \mapsto g \end{aligned}$$

to identify the τ-twisted conjugacy classes in G with the orbits of the above action for a given element (h, τ).

The general ideas about the space of conjugacy classes in a compact connected Lie group recalled in subsection 2.1 can be generalized to Lie groups that are not necessarily connected by introducing *Cartan subgroups*.

3.2 Cartan subgroups

Definition 18. *Let G be a compact Lie group. A Cartan subgroup $C < G$ of G is a topologically cyclic subgroup such that the quotient $N(C)/C$ is finite ($N(A)$ denotes the normalizer of a subgroup A).*

We will now repeat the statement of Proposition 4.2 in [14] applied to our specific case of a Lie group $\hat{G} = G \rtimes \Gamma$ which will tell us how to construct a Cartan subgroup containing (id, τ):

Proposition 19. *Let $\hat{G}$ be given as above, G^τ the subgroup of τ invariants in G and G_0^τ its connected component. If T_0^τ is a maximal torus of G_0^τ, then the subgroup C generated by T_0^τ and (id, τ) is a Cartan subgroup of $\hat{G}$.*

Proof. Follows from the proof of Proposition 4.2 in [14] for the Lie group $G \rtimes \Gamma$ and the element $g = (\mathrm{id}, \tau)$. □

Let C be a Cartan subgroup constructed in such a way and $N(C)$ its normalizer in $\hat{G}$. Notice that because τ acts as an identity on T_0^τ, then the semidirect product $C = T_0^\tau \rtimes \Gamma$ is in fact the direct product. We also see that $T_0^\tau \times \{\tau\} = A$ generates this subgroup. Take the element (g, τ^n) with its inverse

$$(\tau^{-n}(g^{-1}), \tau^{-n})\,.$$

The conjugation of (h, τ) where $h \in T_0^\tau$ by (g, τ^n) is given by

$$(gh\tau(g^{-1}), \tau)\,. \tag{40}$$

Thus we see that

$$(g, \tau^n)A(g, \tau^n)^{-1} = A\,,$$

if an only if g normalizes T_0^τ with respect to the τ-twisted conjugation, that is:

$$gT_0^\tau\tau(g^{-1}) = T_0^\tau\,.$$

As A generates C as a subgroup we have for every $\hat{g} \in \hat{G}$ that

$$\hat{g}A\hat{g}^{-1} = A \text{ implies } \hat{g}C\hat{g}^{-1} = C\,. \tag{41}$$

We also see from (40) that conjugation by any element in $N(C)$ maps $A = T_0^\tau \times \{\tau\}$ into itself, so we can replace implication in (41) by equivalence. Conclusively, we have shown that the normalizer of C in $\hat{G}$ has the following form:

$$N(C) = N_{(G,\tau)}(T_0^\tau) \rtimes \Gamma\,, \tag{42}$$

where $N_{(G,\tau)}(T_0^\tau)$ is the normalizer of T_0^τ in G with respect to the τ-twisted conjugation:

$$N_{(G,\tau)}(T_0^\tau) := \left\{ g \in G \mid gT_0^\tau\tau(g^{-1}) = T_0^\tau \right\}. \tag{43}$$

3.3 Twisted Weyl group

Before we summarize the results from the previous subsection, let us introduce some new notation.

Definition 20. *Let T be a maximal torus of G such that the connected component of $T \cap G^\tau$ is T_0^τ. We define the twisted Weyl group*

$$W(G, T, \tau) := N_{(G,\tau)}(T_0^\tau)/T_0^\tau \,. \tag{44}$$

Corollary 21. *Let everything be given as above. The twisted Weyl group $W(G, T, \tau)$ can be equivalently computed as:*

$$W(G, T, \tau) = N(C)/C \,. \tag{45}$$

Proof. This follows immediately from the equation (42), because

$$N_{(G,\tau)}(T_0^\tau)/T_0^\tau = (N_{(G,\tau)}(T_0^\tau)/T_0^\tau \rtimes \Gamma)/(T_0^\tau \rtimes \Gamma) \,.$$

□

We can also give a different approach to constructing the Cartan subgroup. The following reverses the statement in Definition 20 which assumes the existence of a maximal torus such that the connected component of its τ invariants is T_0^τ (which always holds, as T_0^τ is a torus in G).

Proposition 22. *Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ and $\tau \in Aut(Q)$ an automorphism of $\mathfrak{g}$ (or the lifted automorphism on G) constructed from a Dynkin diagram automorphism as in 2.4 with respect to $\mathfrak{h}$. Take the maximal torus $T = exp(\mathfrak{h})$ and the connected component T_0^τ of $T \cap G^\tau$. Then T_0^τ is a maximal torus in G^τ.*

Proof. This is equivalent to showing that $\mathfrak{h}^\tau$ (the set of τ-invariants in $\mathfrak{h}$) is a maximal abelian subalgebra of $\mathfrak{g}^\tau$. From lemma 8 we know that we are only considering the simply laced Dynkin diagrams. However, for these we know that the set of fixed points $\mathfrak{g}^\tau$ is the corresponding folded non-simply laced Lie algebra with its Cartan subalgebra $\mathfrak{h}^\tau$. This is shown for example in proposition 7.9 in [13]. □

Notice in particular that the torus T as given here is invariant under the action of τ which follows from τ leaving the Cartan algebra $\mathfrak{h}$ invariant.

We state here now the most important properties of the twisted Weyl group and the twisted conjugacy classes which generalize the ones for standard conjugation. Firstly, notice that $W(G, T, \tau)$ acts canonically on T_0^τ:

$$\begin{aligned} W(G, T, \tau) \times T_0^\tau &\to T_0^\tau \\ ([g], h) &\mapsto g h \tau(g^{-1}) \,. \end{aligned} \tag{46}$$

Proposition 23. *The torus T_0^τ intersects each τ-twisted conjugacy class of G in a non-empty subset. Additionally, the elements g and h in T_0^τ are τ-twisted conjugated to each other in G if and only if they lie in the same orbit of the $W(G,T,\tau)$ action on T_0^τ.*

Proof. The first statement follows from applying Proposition 4.3 in [14] to $G \rtimes \Gamma$ with its Cartan subgroup $T_0^\tau \rtimes \Gamma$ and the generating component $T_0^\tau \rtimes \{\tau\}$. The action of G on $G\tau$ is described in (39) and it corresponds to the twisted conjugation. The second part of the statement is again a direct application of [14, Proposition 4.7] to our case. □

We have thus shown that the space of τ-twisted conjugacy classes is given by:

$$T_0^\tau \,//\, W(G,T,\tau)\,. \tag{47}$$

3.4 Twisted Weyl group as a semidirect product

One can further specify the general form of the twisted Weyl group. For that, we need the following two groups ($W(G,T)$ is the Weyl group of the torus T in G):

$$W(G,T)^\tau := \left\{ nT \in W(G,T) \,\middle|\, \tau(n) = ns_n \text{ for some } s_n \in T \right\}, \tag{48}$$

$$(T/T_0^\tau)^\tau = \left\{ nT_0^\tau \in T/T_0^\tau \,\middle|\, \tau(n) = ns_n \text{ for some } s_n \in T_0^\tau \right\}. \tag{49}$$

Remark 24. *Because τ leaves T invariant and thus is an automorphism on it, and $W(G,T)$ can be viewed as a group of automorphism on T acting by conjugation by the representants n of nT, we can define the group W^τ of all the element of $W(G,T)$ commuting on T with τ. It holds that*

$$W^\tau = W(G,T)^\tau\,.$$

Thus for any $nT \in W(G,T)^\tau$ the element n normalizes T_0^τ, and we get another description of this group:

$$W(G,T)^\tau := \left\{ nT \,\middle|\, n \in N(T_0^\tau)\,, \tau(n) = ns_n \textit{ for some } s_n \in T \right\}. \tag{50}$$

Proof. If i_n denotes the conjugation by the element n, then the above requirement on $nT \in W(G,T)$ translates into the equation

$$\tau(n)t\tau(n^{-1}) = \tau \circ i_n \circ \tau^{-1}(t) = i_n(t) = ntn^{-1}$$

which has to hold for all $t \in T$. Obviously this is true if $\tau(n) = ns_n$ for some $s_n \in T$. Conversely, assume that this equation holds. We can write $\tau(n) = ns_n$ for some s_n in G. But then

$$ns_nts_n^{-1}n^{-1} = ntn^{-1}$$

implies that $s_n \in Z(T) = T$. □

Remark 25. *The group* $(T/T_0^\tau)^\tau$ *can be equivalently expressed as the subgroup of* T/T_0^τ *consisting of all cosets* nT_0^τ *such that* $\tau(n)T_0^\tau = nT_0^\tau$ *.*

Remark 26. *We can give a description of* $W(G,T,\tau)$ *similar to the ones in* (48)*. In fact:*

$$W(G,T,\tau) = \left\{ nT_0^\tau \,\middle|\, n \in N(T_0^\tau)\,, \tau(n) = ns_n \text{ for some } s_n \in T_0^\tau \right\}. \tag{51}$$

Notice that $N(T_0^\tau)$ *is the standard (non-twisted) normalizer of* T_0^τ*.*

Proof. The definition of the τ-twisted normilazer in (43) tells us especially that if n is its element, then applying twisted conjugation to the identity element gives

$$n\tau(n)^{-1} = \tilde{s}_n \in T_0^\tau \,.$$

This especially shows that there exists an $s_n \in T_0^\tau$, such that $\tau(n) = ns_n$. Using this, we can show that for any $t \in T_0^\tau$, we have:

$$ntn^{-1} = nts_n\tau(n^{-1}) \in T_0^\tau$$

which shows that n also normalizes T_0^τ. Conversely, if n satisfies the conditions in (51) then it is obviously an element of the twisted normalizer. □

Let us now consider the following diagram:

$$1 \longrightarrow (T/T_0^\tau)^\tau \xrightarrow{\phi_1} W(G,T,\tau) \xrightarrow{\phi_2} W^\tau \longrightarrow 1\,, \tag{52}$$

where ϕ_1 is the inclusion, and

$$\phi_2(nT_0^\tau) = nT$$

which gives a well defined map by comparing the equations (50) and (51) for $W(G,T)^\tau$ and $W(G,T,\tau)$ respectively.

We can now state the final idea of this section.

Theorem 27. *The group* $W(G,T,\tau)$ *has the following form:*

$$W(G,T,\tau) = W^\tau \ltimes (T/T_0^\tau)^\tau\,, \tag{53}$$

where the semidirect product is given by the short exact sequence (52) *that splits.*

Proof. Firstly, notice that

$$\ker(\phi_2) = \{ nT_0^\tau \,|\, n \in T, \tau(n) = ns_n : s_n \in T_0^\tau \}$$

which is exactly the image of ϕ_1. Further, let $nT \in W^\tau$, then $\tau(n) = ns_n$ for some $s_n \in T$. The rest was shown for algebraic groups over algebraicly closed fields in [15, Lemma 2.6] in the parts 1.b and 2. of the proof. The same proof applies to compact Lie groups. □

If we identify $W^\tau \rtimes (T/T_0^\tau)^\tau$ with $W(G,T,\tau)$ then the elements of the former will act on T_0^τ. From the form of ϕ_1 in (52) we see that the elements of $(T/T_0^\tau)^\tau$ act by twisted conjugation by their representatives. The split homomorphism from W^τ to $W(G,T,\tau)$ described in the proof of [15, Lemma 2.6] maps every element nT to an element of $W(G,T,\tau)$ which acts on $(T/T_0^\tau)^\tau$ by conjugation by n. Thus we know now how to replace $W(G,T,\tau)$ with $W^\tau \rtimes (T/T_0^\tau)^\tau$ in (47).

Now that we have the general theory, we would like to consider the twisted conjugation for $SU(n)$.

4 Example on SU(n)

The authors of [12] have addressed this example. However, there seem to be some discrepancies regarding the results. We want to correct these by doing a direct computation using just Theorem 27 without computing the fundamental alcoves of the action of the twisted Weyl group. The results we obtain from the method suggested by [12]n will be compared to the explicit computation at the end of the section.

We have already found the automorphism of $\mathfrak{su}(n)$ which results from lifting the Dynkin diagram automorphism. If τ is given by (31), then the corresponding automorphisms, of which τ is a derivative, on the Lie group $SU(n)$ is given by the same equation. The maximal torus $T = \exp(\mathfrak{h})$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{su}(n)$ with respect to which we have constructed τ, is given by:

$$T = \left\{ \mathrm{diag}(e^{i\phi_1}, \ldots, e^{i\phi_n}) \in SU(n) \right\}.$$

This is diffeomorphic to

$$\left\{ (\phi_1, \ldots \phi_n) : -\pi \leq \phi_i \leq \pi \;\forall\, i = 1, 2, \ldots, n \text{ and } \sum_{i=1}^{n} \phi_i = 2\pi k \text{ for some } k \in \mathbb{Z} \right\} / \sim,$$

where $x \sim y$ if and only if $(x - y) \in 2\pi\mathbb{Z}^n$.

4.1 The case $n = 2m$

Let us examine the case for an even $n = 2m$. The set of τ-invariants in T is the set

$$T^\tau = \left\{ \mathrm{diag}(e^{i\phi_1}, \ldots, e^{i\phi_m}, e^{-i\phi_m}, \ldots, e^{-i\phi_1}) \in SU(n) \right\}.$$

This is diffeomorphic to

$$A/\sim, \text{ where } A = \{ (\phi_1, \ldots, \phi_m) : \phi_i \in [-\pi, \pi] \} \tag{54}$$

with the equivalence relation from before. The mapping is given by:

$$(\phi_1, \ldots, \phi_m) \mapsto \mathrm{diag}(e^{\phi_1}, \ldots, e^{\phi_m}, e^{-\phi_m}, \ldots, e^{-\phi_1}).$$

We see that this group is connected and so is equal to T_0^τ. To find the quotient group T/T_0^τ, notice that the elements

$$\mathrm{diag}(1, \ldots, 1, e^{i\phi_1}, \ldots, e^{i\phi_m}) \text{ where } \sum_{i=1}^{m} \phi_i = 0$$

give unique representatives of the cosets: Every element

$$\mathrm{diag}(e^{i\phi_1}, \ldots, e^{i\phi_m}, e^{i\phi_{m+1}}, \ldots e^{i\phi_n}) \in SU(n)$$

can be written as a product of

$$(e^{i\phi_1}, \ldots, e^{i\phi_m}, e^{-i\phi_m}, \ldots, e^{-i\phi_1}) \text{ and } (1, \ldots, 1, e^{i(\phi_m+\phi_{m+1})}, \ldots, e^{i(\phi_1+\phi_{2m})}).$$

One observes that this factorization is unique in terms of elements of this form. Thus the quotient group T/T_0^τ is isomorphic to the torus:

$$\left\{ (e^{i\phi_1}, \ldots, e^{i\phi_m}) : \sum_{i=1}^{m} \phi_i = 0 \right\} \tag{55}$$

with component-wise multiplication. Acting with τ on

$$\mathrm{diag}(1, \ldots, 1, e^{i\phi_1}, \ldots, e^{i\phi_m}),$$

we obtain

$$\mathrm{diag}(e^{-i\phi_m}, \ldots, e^{-i\phi_1}, 1, \ldots, 1)$$

which represents the same coset as $\mathrm{diag}(1,\dots,1,e^{-i\phi_1},\dots,e^{-i\phi_m})$. From this, we conclude that the $(T/T_0^\tau)^\tau$ is isomorphic to the group:

$$\left\{(\epsilon_1,\dots,\epsilon_m):\epsilon_i=\pm 1,\prod_{i=1}^{m}\epsilon_i=1\right\} \tag{56}$$

which is clearly finite. We can describe this group also as the m-tuples with entries in $\{\pm 1\}$, such that the number of negative signs is even.

The Weyl group of the chosen torus T is the group of permutations of $n=2m$ elements which acts by permuting the diagonal entries. The subgroup W^τ then coincides with the subgroup of those permutations that commute with the permutation

$$\nu=\begin{pmatrix}1 & 2 & \dots & 2m-1 & 2m\\ 2m & 2m-1 & \dots & 2 & 1\end{pmatrix}.$$

From $\nu\sigma\nu=\sigma$, one gets that the elements of this subgroup have to solve the requirement:

$$\sigma(2m+1-i)=2m+1-\sigma(i)\,.$$

Thus σ is already specified by $\sigma(i)$ where i goes from 1 to m. Choosing a value for $\sigma(i)$ determines $\sigma(2m+1-i)$, so the choice of $\sigma(i+1)$ has 2 less possible values. The cardinality of W^τ is $(2m)!!=2m\cdot(2m-2)\cdot\ldots\cdot 4\cdot 2$. As such, we have found the twisted Weyl group $W(G,T,\tau)=(T/T_0^\tau)^\tau\rtimes W^\tau$.

In the following, we will find the fundamental domain α_τ for the action of the twisted Weyl group on A which corresponds to the space of twisted conjugacy classes.

4.2 SU(4) and SU(6)

Taking the description of T_0^τ to be given by (54), take the fundamental set with respect to the action of

$$(T/T_0^\tau)^\tau=\{\,\mathrm{id},\mathrm{diag}(1,1,-1,-1)\,\}\,.$$

The action corresponds to adding $(\pm\pi,\pm\pi)$ in the set A. So the fundamental domain is

$$[0,\pi]\times[-\pi,\pi]\,.$$

The cardinality of W^τ is 8 and it acts on T_0^τ by permuting the order of the 4 elements and thus permuting the order of ϕ_1 and ϕ_2 in A and multiplying them with minuses. The fundamental domain α_τ as a subset of A is one 16'th of it and is shown in Figure 2. It can be described as

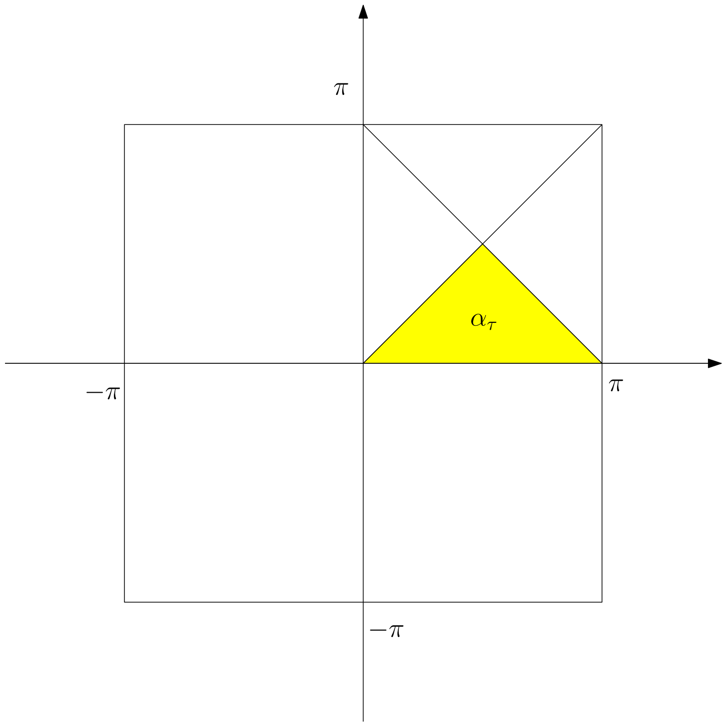


Figure 2: The fundamental alcove $\boldsymbol{\alpha_\tau}$ of SU(4) parameterizing twisted conjugacy classes is represented by the yellow area.

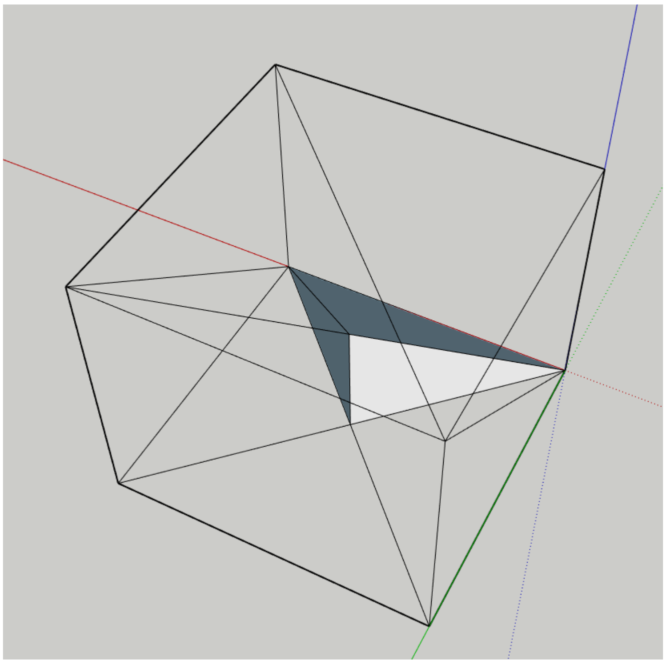


Figure 3: The fundamental alcove α_τ of SU(6) parameterizing twisted conjugacy classes is represented by the set surrounded by white walls contained in the cube $[0,\pi]\times[0,\pi]\times[0,\pi]$.

$$\alpha_\tau = \{(\phi_1,\phi_2) : \phi_1 \geq \phi_2 \geq 0, \phi_1+\phi_2 \leq \pi\}\,. \tag{57}$$

Let us now consider $SU(6)$. The group $(T/T_0^\tau)^\tau$ consists of the elements

$$\mathrm{diag}(1,1,1,-1,-1,1)\,,\quad \mathrm{diag}(1,1,1,-1,1,-1)\,,\quad \mathrm{diag}(1,1,1,1,-1,-1)\,,$$

and the identity. Working with the set A the action of this group corresponds to adding $(\pm\pi,\pm\pi,0)$, $(\pm\pi,0,\pm\pi)$, or $(0,\pm\pi,\pm\pi)$. So we can restrict our considerations to the set

$$[-\pi,\pi]\times[0,\pi]\times[0,\pi]\,.$$

Because of the form of W^τ (consisting of permutations and all combinations of minus signs), we get the following fundamental domain α_τ shown in Figure 3. The set in this picture as a subset of A can be described as

$$\alpha_\tau = \{(\phi_1,\phi_2,\phi_3)\in\mathbb{R}^3\,|\,\phi_1\geq\phi_2\geq\phi_3\geq 0,\ \phi_1+\phi_2\leq\pi\}\,. \tag{58}$$

This especially shows that the result in [12, Example 3.2] is incorrect as the set described there as α_τ is strictly larger than ours.

4.3 The case $n = 2m + 1$

For $n = 2m + 1$, we have:

$$\begin{aligned} T^\tau &= \left\{ \mathrm{diag}(e^{i\phi_1}, \dots, e^{i\phi_m}, 1, e^{-i\phi_m}, \dots, e^{-i\phi_1}) \in SU(n) \right\} \\ &\cong A/\sim\,, \\ A &= \left\{ (\phi_1, \dots \phi_m) \,:\, -\pi \le \phi_i \le \pi \;\forall\, i = 1, 2, \dots, m \right\} \end{aligned}$$

which is connected. The cosets of the the quotient group T/T_0^τ are represented by the elements

$$\mathrm{diag}(1, \dots, 1, e^{i\phi_1}, \dots, e^{i\phi_m}, e^{i\phi_{m+1}}), \text{ where } \sum_{i=1}^{m+1} \phi_i = 0\,,$$

and

$$\left\{(e^{i\phi_1}, \dots, e^{i\phi_{m+1}}) | \sum_{i=1}^{m+1} \phi_i = 0\right\} \cong T/T_0^\tau\,. \tag{59}$$

For the subgroup of τ invariants we have again:

$$(T/T_0^\tau)^\tau = \left\{(\epsilon_1, \dots, \epsilon_m, \epsilon_{m+1}) : \epsilon_i = \pm 1, \prod_{i=1}^{m+1} \epsilon_i = 1\right\}. \tag{60}$$

The Weyl group is also a permutation of $n = 2m + 1$ elements, and its subgroup W^τ consists of σ that solve

$$\sigma(2m + 1 - i) = 2m + 1 - \sigma(i)\,.$$

Thus again these permutations are given uniquely by their values

$$\sigma(1), \sigma(2), \dots, \sigma(m) \in [n]/\{m + 1\}$$

and $\sigma(m + 1) = m + 1$. The cardinality of this group is $(2m)!!$.

4.4 SU(5) and SU(7)

Similarly as we did with SU(4) and SU(6), we ask again about the subset of A parameterizing the conjugacy classes in SU(5). The only difference here is that the group $(T/T_0^\tau)^\tau$ consists of the elements

$$\mathrm{diag}(1, 1, -1, 1, -1)\,,\ \mathrm{diag}(1, 1, -1, -1, 1)\,,\ \mathrm{diag}(1, 1, 1, -1, -1)$$

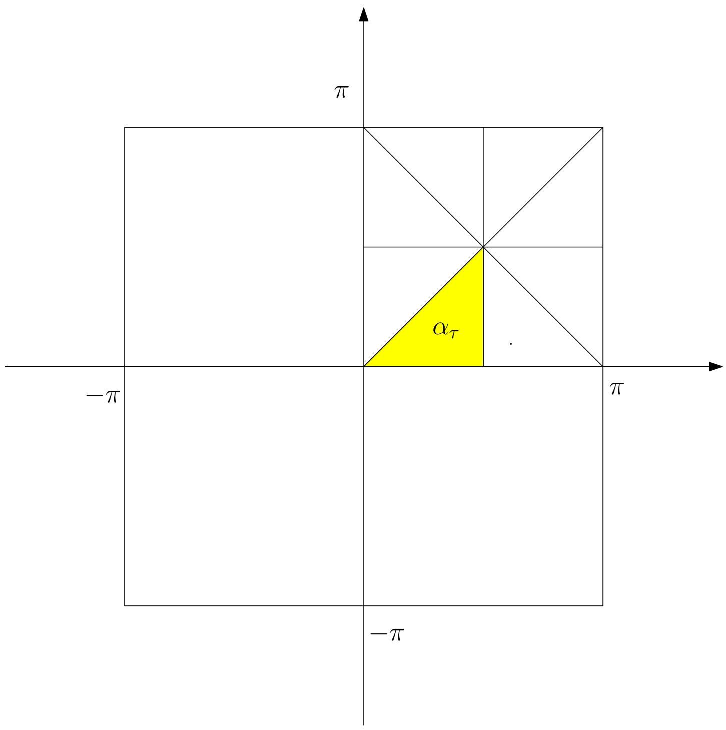


Figure 4: The fundamental alcove α_τ of SU(5) parameterizing twisted conjugacy classes is represented by the yellow area.

and the identity, which acts on A by adding $(0, \pm\pi)$, $(\pm\pi, 0)$, and $(\pm\pi, \pm\pi)$. As such one gets one 32'th of A as the fundamental alcove α_τ shown in the Figure 4 with the equation:

$$\alpha_\tau = \{(\phi_1, \phi_2) : \phi_1 \geq \phi_2 \geq 0, \phi_1 \leq \frac{\pi}{2}\}\,. \tag{61}$$

The group $SU(n)$ of highest dimension we can consider here is for $n = 7$. Under the action of the group $(T/T_0^\tau)^\tau$, we can restrict our considerations to the set $[0, \pi] \times [0, \pi] \times [0, \pi]$. The set α_τ, as a subset of this set which is the fundamental domain for W^τ is given by

$$\alpha_\tau = \{(\phi_1, \phi_2, \phi_3)\,|\,\phi_1 \geq \phi_2 \geq \phi_3 \geq 0,\ \phi_1 \leq \frac{\pi}{2}\}\,, \tag{62}$$

which looks as shown in the Figure 5.

4.5 Abstract approach

Notice that the fundamental domains α_τ as described here are all subsets of $\mathfrak{h}^\tau$. It is worth stating that by following the abstract approach laid out in

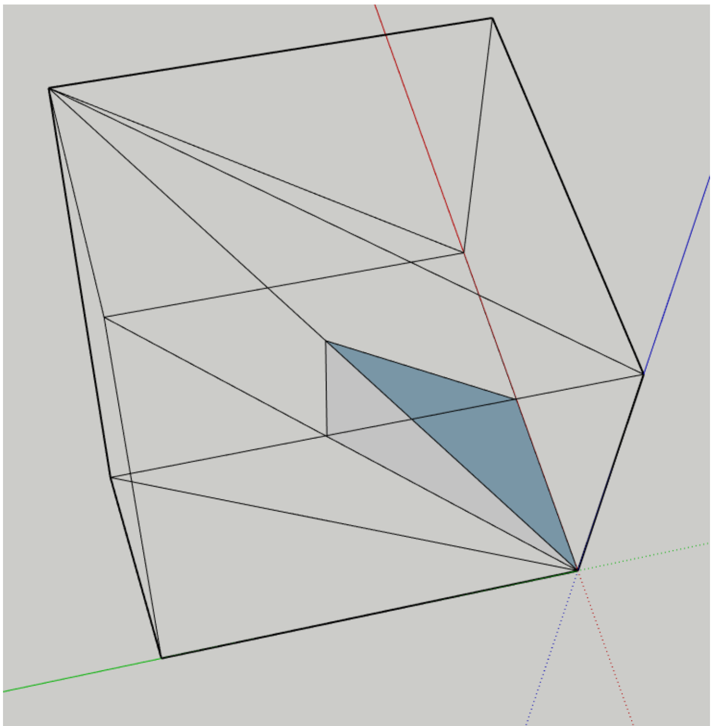


Figure 5: The fundamental alcove α_τ of SU(7) parameterizing twisted conjugacy classes is represented by the set surrounded by white walls.

[12, p. 5], one acquires equal results to these, while also generalizing them to higher dimensions. We outline the method here.

Let $\mathfrak{g}$ be the Lie algebra of G with the Cartan subalgebra $\mathfrak{h}$. Take the root system Δ of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ with its simple roots Π. Then the fundamental alcove which under the exponential map from $\mathfrak{h}^{\tau}$ parametrizes the space of twisted conjugacy classes is given by:

$$\alpha_{\tau} = \left\{ X \in \mathfrak{h}^{\tau} : \frac{1}{2\pi i}\alpha(X) \geq 0 \,\forall\, \alpha \in \Pi_0 \text{ and } \frac{1}{2\pi i}\theta_{\tau}(X) \leq \frac{1}{\operatorname{ord}(\tau)} \right\}. \quad (63)$$

Here Π_0 and θ_{τ} are as defined in remarks 16 and 17. This is equivalent to finding the fundamental domain for the action of the Weyl group of the twisted affine Lie algebra constructed from $\mathfrak{g}$ with respect to τ.

Let us apply it now to $SU(n)$. We have already observed that

$$\begin{aligned} &\text{for } n = 2m: \quad \mathfrak{h}^{\tau} = \{\operatorname{diag}(i\phi_1, \ldots, i\phi_m, -i\phi_m, \ldots, -i\phi_1)\}, \\ &\text{for } n = 2m+1: \quad \mathfrak{h}^{\tau} = \{\operatorname{diag}(i\phi_1, \ldots, i\phi_m, 0, -i\phi_m, \ldots, -i\phi_1)\}. \end{aligned}$$

Further we know that for $\mathfrak{su}(n)_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$ the roots are given by:

$$\begin{aligned} \Delta &= \{e_i - e_j : i \neq j, 1 \leq i, j \leq n\}, \\ \Pi &= \{e_i - e_{i+1} : 1 \leq i \leq n-1\}, \end{aligned}$$

where e_i acts on a diagonal matrix by taking its i'th element on the diagonal and up to a normalization we have $(e_i, e_j) = \delta_{ij}$ for the bilinear form induced by the Killing form (see for example 3.8 in [9]). We compute the root systems Δ_0 by restricting to $\mathfrak{h}^{\tau}$ and taking the subsystem B_m in the case $n = 2m+1$ (see propositions 14 and 15)):

$$\begin{aligned} n = 2m : \Delta_0 &= \{e_i - e_j : i \neq j, 1 \leq i, j \leq m\} \cup \pm\{e_i + e_j : 1 \leq i, j \leq m\} \\ \Pi_0 &= \{e_i - e_{i+1} : 1 \leq i \leq m-1\} \cup \{2e_m\} \\ n = 2m+1 : \Delta_0 &= \{e_i - e_j : i \neq j, 1 \leq i, j \leq m\} \cup \pm\{e_i + e_j : 1 \leq i, j \leq m\} \\ &\quad \cup \pm\{e_i : 1 \leq i \leq m\} \\ \Pi_0 &= \{e_i - e_{i+1} : 1 \leq i \leq m-1\} \cup \{e_m\}. \end{aligned}$$

To get the full non-reduced system in the odd case one simply adds

$$\pm\{2e_i : 1 \leq i \leq m\}.$$

The bilinear form on $\mathfrak{h}_0^*$ induced by the killing form on $\mathfrak{g}_0$ is given up to a multiplication by a constant by restricting the form from $\mathfrak{h}^*$. This allows us to determine the relative length of the roots, and we see that θ_τ is given by

$$n = 2m : \quad \theta_\tau = e_1 + e_2 \,, \qquad n = 2m+1 : \quad \theta_\tau = 2e_1 \,.$$

Applying the equation (63), we obtain:

$$\begin{aligned} n = 2m : \quad & \alpha_\tau = \{ (\phi_1, \ldots, \phi_m) \in A | \phi_1 \geq \ldots \geq \phi_m \geq 0, \, \phi_1 + \phi_2 \leq \pi \} \,, \\ n = 2m+1 : \quad & \alpha_\tau = \{ (\phi_1, \ldots, \phi_m) \in A | \phi_1 \geq \ldots \geq \phi_m \geq 0, \, \phi_1 \leq \frac{\pi}{2} \} \,. \end{aligned} \tag{64}$$

Notice that these results coincide with the ones from before for $n = 4, 5, 6$ and 7 and not with the ones from [12]. Further also notice that this proves two of the cases of folding discussed in propositions 14 and 15.

5 Twisted stabilizer subgroups

In [12], an abstract approach was given for computing the stabilizer groups of each of the conjugacy classes parametrized by α_τ. Instead, we want to compute these explicitly together with their Lie algebras and compare it to the results in [12] and the ones that we obtain by using the abstract method later. We will use the following remark.

Remark 28. *With G compact and an automorphism τ on it, the Lie algebra of the stabilizer of $g \in G$ with respect to the τ-twisted conjugation denoted by $Stab_\tau(g)$ is given by:*

$$Lie\big(Stab_\tau(g)\big) = \{ X \in \mathfrak{g} : Ad_g \circ \tau(X) = X \} \,. \tag{65}$$

This follows because each element $h \in Stab_\tau(g)$ has to solve

$$g\tau(h)g^{-1} = h$$

from which we get (65) *by deriving at the identity.*

In 1.2, we have shown that D-branes of the WZW model correspond to the twisted conjugacy classes. We are particularly interested in the geometry of D-branes which give the lowest number of degrees of freedom for the trajectory of the endpoints of a string. These correspond to the twisted conjugacy classes with lowest dimension and thus highest dimension of their stabilizers. Such stabilizers can be found in the vertices of α_τ as will be obvious from Proposition 29. Thus we consider for now only the vertices of the fundamental alcove α_τ of SU(4) given in Figure 2.

5.1 Explicit computation of stabilizer groups in SU(4)

A simple calculation shows that the Lie algebras of twisted stabilizers at the corresponding vertices of α_τ are given by matrices in $\mathfrak{su}(n)$ of the following forms:

At $(0,0) \in \alpha_\tau$, *stabilizing the identity:*

$$\begin{pmatrix} i\phi_1 & z_{12} & z_{13} & z_{14} \\ -\bar{z}_{12} & i\phi_2 & z_{23} & -z_{13} \\ -\bar{z}_{13} & -\bar{z}_{23} & -i\phi_2 & z_{12} \\ -\bar{z}_{14} & \bar{z}_{13} & -\bar{z}_{12} & -i\phi_1 \end{pmatrix} \tag{66}$$

At $(\pi,0) \in \alpha_\tau$, *stabilizing diag*$(-1,1,1,-1)$*:*

$$\begin{pmatrix} i\phi_1 & z_{12} & z_{13} & z_{14} \\ -\bar{z}_{12} & i\phi_2 & z_{23} & z_{13} \\ -\bar{z}_{13} & -\bar{z}_{23} & -i\phi_2 & -z_{12} \\ -\bar{z}_{14} & -\bar{z}_{13} & \bar{z}_{12} & -i\phi_1 \end{pmatrix} \tag{67}$$

At $\left(\frac{\pi}{2},\frac{\pi}{2}\right) \in \alpha_\tau$, *stabilizing diag*$(i,i,-i,-i)$*:*

$$\begin{pmatrix} i\phi_1 & z_{12} & z_{13} & 0 \\ -\bar{z}_{12} & i\phi_2 & 0 & -z_{13} \\ -\bar{z}_{13} & 0 & -i\phi_2 & z_{12} \\ 0 & -\bar{z}_{13} & -\bar{z}_{12} & -i\phi_1 \end{pmatrix} \tag{68}$$

Let us address the first two stabilizers given here. The Lie algebra $\mathfrak{sp}(4)$ of SP(4) which is the compact form of $\mathfrak{sp}(4,\mathbb{C})$ consists of matrices of the following form:

$$\begin{pmatrix} i\phi_1 & z_{12} & z_{13} & z_{14} \\ -\bar{z}_{12} & i\phi_2 & z_{14} & z_{24} \\ -\bar{z}_{13} & -\bar{z}_{14} & -i\phi_1 & \bar{z}_{12} \\ -\bar{z}_{14} & -\bar{z}_{24} & -z_{12} & i\phi_2 \end{pmatrix} \tag{69}$$

Notice that conjugation of (69) by

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \text{ resp. } B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

gives matrices of the form (66), resp. (67). All three of these Lie algebras have dimension 10. Thus we have shown that they are isomorphic. But this implies that conjugating $\mathrm{SP}(4)$ by B_1, resp. by B_2, gives a matrix group with its Lie algebra corresponding to the Lie algebra of the twisted stabilizer at $(0,0)$, resp. at $(\pi, 0)$. As all of these groups are connected they must coincide.

The Lie algebra $\mathfrak{f}$ of matrices described in 68 has dimension 6. Let us introduce the standard notation for Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Notice that $\mathfrak{su}(2)$ is spanned by

$$i\sigma_1,\, i\sigma_2,\, i\sigma_3\,.$$

One easily computes now that the Lie algebra $\mathfrak{f}$ is spanned by the following generators which in their respective triples span ideals isomorphic to $\mathfrak{su}(2)$:

$$\begin{pmatrix} i\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad \begin{pmatrix} i\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad \begin{pmatrix} i\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \tag{70}$$

and

$$\begin{pmatrix} i\,\mathrm{id} & 0 \\ 0 & -i\,\mathrm{id} \end{pmatrix}, \qquad \begin{pmatrix} 0 & i\,\mathrm{id} \\ i\,\mathrm{id} & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \mathrm{id} \\ \text{-id} & 0 \end{pmatrix}. \tag{71}$$

Thus we see that $\mathfrak{f} \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and the form in which both of the $\mathfrak{su}(2)$ are contained in it suggests the following group homomorphism $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SU}(4)$:

$$(g, g') \mapsto \begin{pmatrix} g_{11}g' & g_{12}g' \\ g_{21}g' & g_{22}g' \end{pmatrix}. \tag{72}$$

It can be shown to be a Lie group homomorphism and its kernel is given by:

$$\big\{(\mathrm{id}, \mathrm{id}), (-\mathrm{id}, -\mathrm{id})\big\}\,.$$

One can also easily check that this induces an isomorphism $\mathfrak{su}(2) \times \mathfrak{su}(2) \to \mathfrak{f}$ where the generators $(i\sigma_k, 0)$ are mapped to the generators in (70) and the generators $(0, \sigma_k)$ to the ones in (71). We conclude that the Lie algebra of the image of the (72) coincides with the Lie algebra of the stabilizer. As both groups are connected, they must be equal and we have the statement that:

$$\mathrm{Stab}_\tau\big(\mathrm{diag}(i, i, -i, -i)\big) \cong SU(2) \times SU(2) \Big/ \big\{(\mathrm{id}, \mathrm{id}), (-\mathrm{id}, -\mathrm{id})\big\}\,. \tag{73}$$

Additionally, comparison of the results that we get later in Figure 6 with the ones from Figure 1 in [12] suggests that we should examine the stabilizer at $(\pi/2, 0)$ of the element $\mathrm{diag}(i, 1, 1, -i)$. We obtain that its Lie algebra consists of all matrices of the form:

$$\begin{pmatrix} i\phi_1 & 0 & 0 & 0 \\ 0 & i\phi_2 & z_{23} & 0 \\ 0 & -\bar{z}_{23} & -i\phi_2 & 0 \\ 0 & 0 & 0 & -i\phi_1 \end{pmatrix}. \tag{74}$$

One immediately observes that the corresponding compact connected Lie group with this Lie algebra is isomorphic to $SU(2) \times U(1)$, and it is the group of the following diagonal block matrices:

$$\left\{ \begin{pmatrix} e^{i\phi} & & \\ & g & \\ & & e^{-i\phi} \end{pmatrix} : g \in SU(2), \phi \in \mathbb{R} \right\}. \tag{75}$$

5.2 Abstract approach to the computation of stabilizers

Here we choose to work with the method suggested in [12]. Let G be a compact, simple, simply connected Lie group, $\mathfrak{g}$ its Lie algebra with its Cartan subalgebra $\mathfrak{h}$. Take the complexification $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{h}_\mathbb{C}$ with its root system Δ and choose a system of simple roots Π. Let τ be an the automorphism of G constructed from the Dynkin diagram, Π_0 and θ_τ as defined in remarks 16 and 17, we have then the following statement about the τ-twisted stabilizers:

Proposition 29. *Let $H \in \alpha_\tau$ and $\exp(H)$ its corresponding element in T_0^τ. One sets*

$$\Pi_0^H := \left\{ \alpha_i \in \Pi_0 : \frac{\alpha_i(H)}{2\pi i} \in \mathbb{Z} \right\},$$

and

$$\tilde{\Pi}^H = \begin{cases} \Pi_0^H \cup \{\theta_\tau\} \ \textit{if} \ \frac{\theta_\tau(H)}{2\pi i} \in \frac{1}{ord(\tau)} + \mathbb{Z} \\ \Pi_0^H \ \textit{otherwise.} \end{cases}$$

The Lie algebra

$$\mathcal{L}(\mathrm{Stab}(\exp(\mathrm{H})))$$

of the τ-twisted stabilizer of $\exp(H)$ *is a reductive Lie algebra as a direct sum of a semisimple Lie algebra with a simple root system $\tilde{\Pi}^H$ and its centrum of center*

$$\dim(\mathfrak{h}_0) - |\tilde{\Pi}^H|\,.$$

Proof. In the proof of Proposition 4.1 [12], it was shown the the complexification of $\mathcal{L}(\mathrm{Stab}(\exp(H)))$ is a subalgebra of the complexification of $\mathfrak{g}_{\mathbb{C}}$ generated by:

$$\bigoplus_{\alpha \in \pm\Pi_0^H} (\mathfrak{g}_{\mathbb{C}})_{0,\alpha} \oplus (\mathfrak{h}_{\mathbb{C}})_0 \oplus \mathfrak{g}^H_{\mathbb{C},\theta_\tau}\,, \tag{76}$$

where $(\mathfrak{g}_{\mathbb{C}})_{0,\alpha}$ is the root space for the root α of $\mathfrak{g}^\tau_{\mathbb{C}}$ and

$$\mathfrak{g}^H_{\mathbb{C},\theta_\tau} = (\mathfrak{g}_{\mathbb{C}})_{1,-\theta_\tau} \oplus (\mathfrak{g}_{\mathbb{C}})_{-1,\theta_\tau}$$

if $\theta_\tau(H) = 1/\mathrm{ord}(\tau)$ and trivial otherwise. The space $(\mathfrak{g}_{\mathbb{C}})_{1,-\theta_\tau}$ is the $-\theta_\tau$ weightspace of the $(\mathfrak{g}_{\mathbb{C}})_0$-module $(\mathfrak{g}_{\mathbb{C}})_1$. We know that the resulting Lie algebra is reductive, and it can be written as a direct sum of a semi-simple Lie algebra and its center. The rootsystem of the semi-simple Lie algebra is spanned by the elements of $\tilde{\Pi}^H$ which are linearly independent for all H. The center of the stabilizer Lie algebra is a subalgebra of $(\mathfrak{h}_{\mathbb{C}})_0$ which commutes with all rootspaces and thus lies in the kernel of all roots from $\tilde{\Pi}^H$. This specifies its dimension. □

Remark 30. *We have already mentioned in 17 that $\Pi_0 \cup \{\theta_\tau\}$ is related to the simple root system of the twisted affine algebra. From the above proposition, we conclude that the Dynkin diagrams of the semisimple part of a twisted stabilizer are subdiagrams of the twisted affine diagram corresponding to the roots in $\tilde{\Pi}^H$ which form its simple rootsystem.*

Remark 31. *Proposition 29 suggests that for H in the interior of α_τ its corresponding stabilizer is isomorphic to a torus. The stabilizers will have the highest dimension for H being a vertex of α_τ, because all hyperplanes forming the boundary of the fundamental alcove and corresponding to the roots of the twisted affine algebra except one intersect there.*

Because we are now able to specify the Lie algebra of the stabilizer as a direct sum of a certain semisimple Lie algebra and an abelian Lie algebra, we know that the stabilizer group itself will be a direct product of a semisimple connected Lie group and a torus. The only missing piece to determining the stabilizers is the fundamental group of the semisimple Lie group. For that we will be relying on the following result.

Lemma 32. *Let G be a compact semi-simple Lie group with its Cartan algebra $\mathfrak{h} \subset \mathfrak{g}$ of its Lie algebra. Let $\Omega^\vee \subset \mathfrak{h}$ be the dual root lattice of $\mathfrak{g}$. Then the fundamental group of G is given by*

$$\pi_1(G) = \ker(\exp|_{\mathfrak{h}})/\Omega^\vee. \tag{77}$$

Proof. This follows from the Theorem 7.1 in [14] by using that

$$\pi_1(\exp(\mathfrak{h})) \cong \ker(\exp|_{\mathfrak{h}})$$

is an isomorphism given by lifting each closed curve of the fundamental group to a curve in $\mathfrak{h}$ (which is possible since $\exp|_{\mathfrak{h}}$ is a covering map), and connecting to it the end point of the lifted curve which is an element of the kernel. □

Thus, if we have the Lie algebra of a stabilizer of the form described in Proposition 29, and we want to determine the corresponding Lie group, we only need to find the intersection of ker(exp) with the Cartan subgroup of the semisimple Lie algebra and its dual root lattice $\Omega^\vee$.

The dual root lattice $\Omega^\vee$ of a semisimple Lie algebra $\mathfrak{g}$ with its Cartan subalgebra $\mathfrak{h}$ and a simple root system Π is generated by the simple coroots in $\mathfrak{h}$. If $\alpha_i \in \Pi$ is a simple root, then a simple coroot is denoted by $\alpha_i^\vee$ and is uniquely defined by

$$\alpha_j(\alpha_i^\vee) = a_{ij} \text{ for all } \alpha_j \in \Pi$$
$$a_{ij} = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}.$$

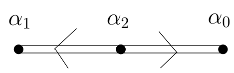
Under the identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ by the standard invariant form, $\alpha_i^\vee$ corresponds to $\frac{2\alpha_i}{(\alpha_i,\alpha_i)}$. We are now ready to apply this to our example SU(n).

5.3 All stabilizers of SU(4)

For SU(4) with τ given as before we have

$$\Pi_0 = \{\alpha_1 = e_1 - e_2, \alpha_2 = 2e_2\}, \qquad \theta_\tau = e_1 + e_2$$

as follows from 5.2. Thus we get the following diagram of the twisted affine Lie algebra (which is known as type $A_3^{(2)}$):



where the vertices are labeled by the roots and α_0 is given by Remark 17. The stabilizers obtained using the Proposition 29 together with Lemma 32 for different $H \in \alpha_\tau$ are summed up in the table below, where each row is numbered by a number corresponding to one in the Figure 6 and representing where H lies. We write $\mathrm{Stab}_\tau(\exp(H)) =: G_H^\tau$ and $\mathfrak{h}_H$ denotes the Cartan algebra of the semisimple subalgebra of $\mathcal{L}(G_H^\tau)$ and $Q(G_H^\tau)$ is its Dynkin diagram.

	$\tilde{\Pi}^G$	$Q(G_H^\tau)$	$\mathfrak{h}_H$	$\Omega^\vee$	$\pi_1(G_H^\tau)$	G_H^τ
1	$\{e_1 - e_2,\, 2e_2\}$	•⇒•	$\mathfrak{h}_0$	$\langle(2\pi, -2\pi)\,,\,(0, 2\pi)\rangle$	0	$Sp(4)$
2	$\{e_1 - e_2\}$	•	$\mathbb{R}\cdot(1,-1)$	$\langle(2\pi, -2\pi)\rangle$	0	$SU(2)\times U(1)$
3	$\{e_1 - e_2,\, e_1 + e_2\}$	• •	$\mathfrak{h}_0$	$\langle(2\pi, -2\pi)\,,\,(2\pi, 2\pi)\rangle$	$\mathbb{Z}_2$	$SU(2)\times SU(2)/\mathbb{Z}_2$
4	$\{e_1 + e_2\}$	•	$\mathbb{R}\cdot(1,1)$	$\langle(2\pi, 2\pi)\rangle$	0	$SU(2)\times U(1)$
5	$\{e_1 + e_2,\, 2e_2\}$	•⇐•	$\mathfrak{h}_0$	$\langle(2\pi, 2\pi)\,,\,(0, 2\pi)\rangle$	0	$Sp(4)$
6	$\{2e_2\}$	•	$\mathbb{R}\cdot(0,1)$	$\langle(0, 2\pi)\rangle$	0	$SU(2)\times U(1)$

Table 1: Table for Figure 6

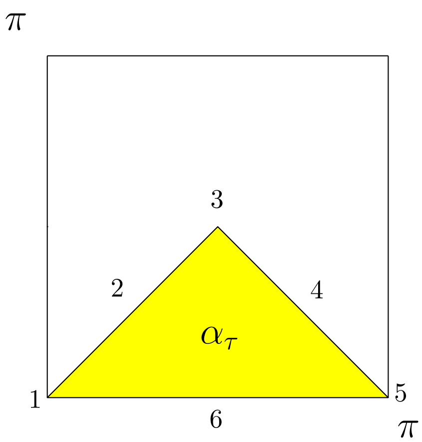


Figure 6: Each number in the picture corresponds to the twisted stabilizer group of the elements at the boundary of α_τ of $SU(4)$. These are given in Table 1

.

We are now able to compare our results through the abstract and explicit method with the ones given in [12, Figure 1]. The stabilizers 3 and 6 in

Table 1 coincide with the ones we computed in the subsection 5.1 (as do the stabilizers 1 and 5). However, this is a different outcome to the one in [12]. In the case of the stabilizer at 3 we find that one must additionally take a quotient by $\mathbb{Z}_2$ while for 6 the quotient by $\mathbb{Z}_2$ is excessive. We believe that our results are correct.

Using the known dimensions of the stabilizers, we can also determine the dimension of the twisted conjugacy classes $C_\tau\big(\exp(H)\big)$ for all H. This is summed up in Table 2.

	1	2	3	4	5	6
$\dim\big(C_\tau(\exp(H))\big)$	5	11	9	11	5	11

Table 2: The numbers labeling the columns correspond to the numbers representing the position of H in Figure 6. The dimensions of the twisted conjugacy classes containing $\exp(H)$ are given in the second row.

Appendices

A Summary

In this thesis, we have examined the question of geometry of D-branes of the WZW model in a simply connected, simple, compact Lie group G. In Section 1, we have recalled that for the gluing conditions

$$J(z) = R\bar{J}(\bar{z}),$$

where R is an automorphism on the Lie algebra of G, the corresponding D-branes are given by orbits of the action of G on itself called *τ-twisted conjugation.* Here τ is an automorphism on G the derivative of which is R^{-1}. This action has the following form:

$$G \times G \to G: \quad (h, g) \mapsto hg\tau(h)^{-1},$$

and its orbits are called *twisted conjugacy classes.*

In Section 2, we developed the necessary mathematical background and stated the well known result giving an isomorphism between the group of outer automorphisms $\mathrm{Out}(G)$ on G and the automorphisms of its Dynkin diagram $\text{Aut}(Q)$. We have also constructed the unique outer automorphism τ on $\text{SU}(n)$ for all n by lifting the corresponding Dynkin diagram automorphism.

Section 3 dealt with τ-twisted conjugation and an equivalent way to describe it on G. There the group G was replaced by

$$\hat{G} = G \rtimes \Gamma \,,$$

where Γ was the subgroup of $\text{Out}(G)$ generated by τ, and the tori in G were generalized to *Cartan subgroups* of $\hat{G}$. We combined the results from [12], [14] and [15] to describe the important properties of the Cartan subgroups. Additionally, we have connected these results about conjugation in $\hat{G}$ back to the twisted conjugation in G. Thus we were able to show that the space of τ-twisted conjugacy classes in G corresponds to the space of orbits

$$T_0^\tau // W(G, T, \tau) \,.$$

Here T_0^τ was the connected component of τ invariants in a maximal torus T and $W(G, T, \tau)$ was its twisted Weyl group.

Using this result, we computed in Section 4 explicitly the space of τ-twisted conjugacy classes in $\text{SU}(n)$, $n = 4, 5, 6$ and 7. We compared our results with ones that we obtained by applying the abstract method from [12]. Both of the methods applied by us gave the same results which didn't coincide with the ones in [12].

Finally, in Section 5, we determined the stabilizers of the twisted conjugacy classes and their Lie algebras using two different methods. While the Lie algebras were the same as the ones given in [12], the fundamental groups of the stabilizers didn't match in two cases.

B Zusammenfassung

In dieser Arbeit haben wir die Gometrie der D-branen in dem WZW Model in einer einfach zusammenhängenden , einfachen, kompakten Lie Gruppe betrachet. Im Kapitel 1 wurde daran erinnert, dass die D-Branen für die Randbedingung

$$J(z) = R\bar{J}(\bar{z}) \,,$$

wobei R ein Automorphismus auf der Lie algebra von G ist, den Bahnen einer Aktion von G auf sich selbst entsprechen. Diese Aktion wird τ-getwistete

Konjugation genannt und hat die folgende Form:

$$G \times G \to G : \quad (h, g) \mapsto hg\tau(h)^{-1} .$$

Als τ bezeichnen wir den Automorphismus auf G mit Ableitung R^{-1}. Die entsprechenden Bahnen werden *getwistete Konjugationsklassen* gennant.

Im Kapitel 2 haben wir den mathematischen Hintergrund entwickelt und den wohlbekannten Resultat gegeben, der sagt, dass die Gruppe der externen Automorphismen $\mathrm{Out}(G)$ auf G der Gruppe der automorphismen auf dem Dynkin Diagram $\mathrm{Aut}(Q)$ isomorph ist. Wir haben auch den eindeutigen externen Automorphismus von $\mathrm{SU}(n)$, der dem nicht-trivialen Automorphismus in $\mathrm{Aut}(A_{n-1})$ entsprich, hergeleitet.

Kapitel 3 hat die τ-getwistete Konjugation behandelt. Hier wurde eine equivalente Weise diese zu beschreiben besprochen. Die Gruppe G wurde durch

$$\hat{G} = G \rtimes \Gamma$$

ersetzt und die Toren in G durch *Cartan Untergruppen* von $\hat{G}$. Wir haben die Resultate aus [12], [14] und [15] zusammengefasst um wichtige Aussagen über die Cartan Untergruppen treffen zu können. Weiter haben wir diese Aussagen über Konjugation in $\hat{G}$ mit der getwisteten Konjugation auf G verbunden. Das hat uns erlaubt den Raum der τ-getwisteten Konjugations Klassen als

$$T_0^\tau // W(G, T, \tau)$$

zu beschreiben. Als T_0^τ bezeichnen wir die Zusammenhangskomponente von den τ Invarianten in einem maximalen Torus T und $W(G, T, \tau)$ ist die getwistete Weyl-Gruppe.

Durch Anwendung von diesem Resultat haben wir die Räume von den τ-getwisteten Konjugationsklassen in $\mathrm{SU}(n)$ mit $n = 4, 5, 6$ und 7 bestimmt. Weiter haben wir im Kapitel 4 diese Räume erneut durch eine abstrakte Methode in [12] berechnet. Wir haben bemerkt, dass unsere Ergebnisse, die für beide Methoden gleich sind, mit denen in [12] nicht übereinstimmen.

Letztendlich im Kapitel 5 haben wir auch die Stabilisatoren von den getwisteten Konjugationsklassen auf zwei unterschiedliche Wege bestimmt. Im Vergleich zu [12] gab es diesmal Diskrepanzen bei zwei Fundamentalgruppen, während die Lie Algebren übereinstimmten.

References

[1] J. Bedford, An Introduction to String Theory, arXiv:1107.3967v3 [hep-th] 7 Jun 2012

[2] Sonia Stanciu, D-branes in Group Manifolds, arXiv:hep-th/9909163v1 22 Sep 1999

[3] Joseph Polchinski, Tasi Lectures on D-branes, arXiv:hep-th/9611050v2 23 Apr 1997

[4] A. Kriegl, *Algebraic Topology*, https://www.mat.univie.ac.at/ kriegl/Skripten/alg-top.pdf

[5] Y. Félix, J. Oprea, D. Tanré, Algebraic Models in Geometry Volume 17 of Oxford graduate texts in mathematics Oxford mathematics, Oxford University Press, 2008

[6] D. Gepner and E. Witten, "String theory on group manifolds," Nuc. Phys. B278 (1986) 493–549.

[7] B. A. Sherwood, Frobenius' Theorem, http://www.math.uchicago.edu/∼may/vigre/vigre2011/ReuPapers/Sherwood.pdf

[8] S. Murakami, On the automorphisms of a real semisimple Lie algebra, Journal of the Mathematical Society of Japan Vol. 4 No. 2 (1952), 103-133

[9] A. Cap, *Lie Algebras and Representation Theory*, http://www.mat.univie.ac.at/ cap/files/LieAlgebras.pdf

[10] A. Cap, *Lie Groups*, http://www.mat.univie.ac.at/ cap/files/LieGroups.pdf

[11] W. Fulton, J. Harris, *Representation theory A First Course*, Springer-Verlag New York, 2004.

[12] S. Mohrdieck, R. Wendt, Twisted Conjugacy Classes, Coadjoint Orbits of Loop Groups, and D-Branes in the WZW Model, arXiv:math/0303118v4 [math.QA] 19 Jan 2004

[13] V.G. Kac, Infinite-dimensional Lie Algebras, 3rd edition, Cambridge University Press, Cambridge, 1990.

[14] T. Bröcker, T. tomDieck, Representations of Compact Lie Groups, Springer, Heidelberg, New York, 1985

[15] Mohrdieck, S. Transformation Groups (2003) 8: 377. https://doi.org/10.1007/s00031-003-0429-3

[16] R. W. Carter. Simple Groups of Lie Type. John Wiley and Sons, 1972.