universität
wien

DISSERTATION/DOCTORAL THESIS

Titel der Dissertation / Title of the Doctoral Thesis

"Generalized notions of recurrence: Bases and the existence of invariant probability measures"

verfasst von / submitted by

Manuel Jürgen Inselmann

angestrebter akademischer Grad / in partial fulfillment of the requirements for the degree of

Doktor der Naturwissenschaften (Dr. rer. nat.)

Wien, 2019 / Vienna, 2019

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 796 605 405
Dissertationsgebiet lt. Studienblatt: field of study as it appears on the student record sheet:	Mathematik
Betreut von / Supervisor:	Univ.-Prof. Benjamin Miller, PhD

Abstract

We establish basis and anti-basis theorems for a broad collection of recurrence notions appearing in descriptive, measurable, and topological dynamics, and show that such notions cannot characterize the existence of invariant probability measures in the descriptive milieu.

Zusammenfassung

Wir zeigen Basis- und Antibasisresultate für eine Vielzahl von Rekurrenzarten, die in deskriptiver, maßtheoretischer und topologischer Dynamik auftreten und zeigen, dass solche Rekurrenzbedingungen nicht die Existenz von invarianten Wahrscheinlichkeitsmaßen im deskriptiven Kontext charakterisieren können.

Acknowledgments

First of all I would like to thank Benjamin Miller for the supervision of my thesis. When I started my PhD I did not expect to end up in Vienna, but I am very glad I did. The past years have been very inspiring mathematically and beyond and I am grateful to have been given this opportunity.

I want to thank my fellow PhD-Colleagues (and all the other members of the KGRC) for the joint coffee brakes and simply making the KGRC a nice place to do mathematics at (in the order I met them): Stefan, Marlene, Anda, Diana, Jonathan, Ömer.
I thank Fabio for the numerous meetings and discussions we had.

I would like to thank Alexander Kechris for making it possible to visit him at the beginning of my PhD.

It would take to long to thank all the nice people I had the chance to meet at conferences, so I will only mention Cristina and also Chris here.

I would like to thank Virginia for nothing in particular but just for being there and for proofreading the acknowledgments. You did an amazing job.

Auch Danke an Petra und Richard für eure Hilfe im Alltag des KGRCs.

Ein spezieller Dank geht an die Studienstiftung nicht nur für die finanzielle Förderung, sondern auch für die unzähligen Momente der ideellen Förderung, an denen ich teilhaben durfte.

Zu guter Letzt möchte ich meiner Familie danken.

Table of Contents

Chapter 1

Introduction

1.1 Descriptive Set Theory and group actions

Descriptive set theory is the study of definable subsets of Polish spaces, i.e., separable completely metrizable spaces. Typical examples are the Cantor space ($2^{\mathbb{N}}$), the Baire space ($\mathbb{N}^{\mathbb{N}}$), and the set of real numbers ($\mathbb{R}$). In these spaces, *Borel* sets can be classified in hierarchies according to the complexity of their definitions. Starting with the open subsets of a Polish space X, we obtain the Borel hierarchy by closing under countable unions and complements:

$$\Sigma^0_1 = \{U : U \text{ is open}\},$$

$$\Pi^0_\alpha = \{A \mid \sim A \in \Sigma^0_\alpha\},$$

$$\Sigma^0_\alpha = \{\bigcup_{n\in\mathbb{N}} A_{\alpha_n} : A_{\alpha_n} \in \Pi_{\alpha_n}, \alpha_n < \alpha\}.$$

It is well known that $(\Sigma^0_n \setminus \Sigma^0_n)_\sigma$, $(\Pi^0_n \setminus \Pi^0_n)_\sigma$, and Σ^0_{n+1} coincide on metric spaces (see, for example, [Kec95, §11.B]). Σ^0_2-sets are also referred to as *F_σ-sets* and Π^0_2-sets are also referred to as *G_δ-sets*. We say that a set $A \subseteq X$ is *Borel* if $A \in \Sigma^0_\alpha$ for some $\alpha < \omega_1$. A Borel space whose σ-algebra arises as above from the open sets of a Polish space is called a *standard Borel space*. It is well known that all uncountable standard Borel spaces are Borel isomorphic. Above the Borel sets, *analytic* sets are continuous images of Borel sets, and *co-analytic* sets are complements of analytic sets.
An *equivalence relation* on a standard Borel space X is a transitive, symmetric and reflexive relation E on X. It is *Borel* if E is a Borel subset of $X \times X$.

An *action* of a group G on a set X is a map $G \times X \to X$, denoted by $(g, x) \mapsto g \cdot x$ such that the following hold:

- $\forall g, h \in G\, x \in X\, g \cdot (h \cdot x) = (gh) \cdot x$,
- $\forall x \in X\, 1_G \cdot x = x$.

Given an action of G on a set X (in short $G \curvearrowright X$), let E_G^X denote its orbit equivalence relation given by $x \; E_G^X \; y$ if there exists a $g \in G$ such that $g \cdot x = y$. The set Gx is called the *orbit* of x. A set $C \subseteq X$ is *invariant* if $GC = C$ and a Borel probability measure on X is *invariant* if $g_* \mu = \mu$ for every $g \in G$. When X is a standard Borel space, a Borel equivalence relation F on X is *smooth* if there is a standard Borel space Z for which there is a Borel function $\pi \colon X \to Z$ such that $x \; F \; y \iff \pi(x) = \pi(y)$ for all $x, y \in X$. An example of a non-smooth Borel equivalence relation is $\mathbb{E}_0$ defined on $2^{\mathbb{N}}$ by $x \; \mathbb{E}_0 \; y$ if there exists $n \in \mathbb{N}$ such that $x(m) = y(m)$ for all $m \geq n$.

We say that a set $Y \subseteq X$ is *complete* if $X = GY$, and *σ-complete* if there is a countable set $H \subseteq G$ for which $X = HY$. When $G \curvearrowright X$ is continuous and Y is open, these notions are equivalent.

1.2 The scope

The main part of this thesis consists of a joining of the articles [IM17] and [IM19], where the first one is slightly generalized to a setting that the second paper suggests. Both papers are joint work with Benjamin Miller.

It is well-known that if ν is a Borel probability measure on X and T is a Borel automorphism on X, then the inexistence of a weakly-wandering ν-positive set yields a T-invariant Borel probability measure $\mu \gg \nu$ (see [Zak93] for the generalization to groups of Borel automorphisms). We introduce a generalized notion of recurrence and show that recurrence conditions do not yield invariant Borel probability measures in the descriptive set-theoretic milieu, in the sense that if a Borel action of a locally compact Polish group on a standard Borel space satisfies such a condition but does not have an orbit supporting an invariant Borel probability measure, then there is an invariant Borel set on which the action satisfies the condition but does not have an invariant Borel probability measure.

Given an ordered family $(\mathcal{O}, \preceq)$ of mathematical structures and an upward-closed property Φ of structures in $\mathcal{O}$, a *basis* for the family $\mathcal{O}_\Phi = \{O \in \mathcal{O} \mid \Phi(O)\}$ under $\preceq$ is a set $\mathcal{B} \subseteq \mathcal{O}_\Phi$ with the property that $\forall O \in \mathcal{O}_\Phi \, \exists B \in \mathcal{B} \; B \preceq O$. Singleton bases are particularly useful, as their existence ensures that satisfying Φ is equivalent to containing a copy of a canonical structure. Even when there are no small bases, the existence of a basis consisting solely of particularly simple structures nevertheless yields substantial insight into the nature of Φ. Here we show that this is the case for myriad properties of actions of locally-compact Polish groups, including non-smoothness, the inexistence of suitably-large weakly-wandering Borel sets, and weak mixing.

In Chapter 2 we consider a generalized notion of recurrence. Suppose that $d \in \mathbb{Z}^+$ and X is a set, and $G \curvearrowright X$ is an action. For all sets $R \subseteq X^{\{0,\ldots,d\}}$, define $\Delta_G^X(R) = \{\mathbf{g} \in G^{\{1,\ldots,d\}} \mid \exists x \in X \, \overline{\mathbf{g}}x \in R\}$, where $\overline{\mathbf{g}} \in G^{\{0,\ldots,d\}}$ is the extension of $\mathbf{g}$ given by $\overline{\mathbf{g}}_0 = 1_G$. When $\boldsymbol{\mathcal{F}} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ is a *family of upward closed families*, i.e., for all $d \in \mathbb{Z}^+$ and $\mathcal{F} \in \boldsymbol{\mathcal{F}} \cap \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ the family $\mathcal{F}$ is upward closed in $\mathcal{P}(G^{\{1,\ldots,d\}})$, and $\emptyset \in \mathcal{H}$ is a subset of $\bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(X^{\{0,\ldots,d\}})$, we say that a set $Y \subseteq X$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$*-transient* if there exists $d \in \mathbb{Z}^+$, $H \in \mathcal{H} \cap \mathcal{P}(X^{\{0,\ldots,d\}})$, and $\mathcal{F} \in \boldsymbol{\mathcal{F}} \cap \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ for which $\Delta_G^X(Y^{\{0,\ldots,d\}} \setminus H) \notin \mathcal{F}$. When C is an invariant subset of X and Γ is a subset of $\mathcal{P}(C)$, we say that $G \curvearrowright C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$*-recurrent on* Γ if no set $A \in \Gamma$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-transient, if $\mathcal{H} = \{\emptyset\}$ we just say that the action is $\boldsymbol{\mathcal{F}}$-recurrent and if $\boldsymbol{\mathcal{F}} = \{\mathcal{F}\}$ is single family, we omit the brackets in the definitions of recurrence and transience. This generalizes the usual notion of recurrence in topological dynamics, where one says that a continuous-in-X action of a group on a topological space X is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$*-recurrent* if it is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent on non-empty open sets.

When G is a group and X is a Polish space, the *decomposition into minimal components* of a continuous-in-X action $G \curvearrowright X$ is the equivalence relation on X given by $x \; F_G^X \; y \Leftrightarrow \overline{Gx} = \overline{Gy}$. Let $(U_n)_{n \in \mathbb{N}}$ be a countable basis for the topology of X and define the map $\varphi : X \to 2^{\mathbb{N}}$ by $\varphi(x)(n) = 1$ if and only if $x \in GU_n$. Then $x \; F_G^X \; y \Leftrightarrow \varphi(x) = \varphi(y)$ and since the preimage of any open set under φ is F_σ, it follows that F_G^X is G_δ and smooth. Furthermore, for each F_G^X-class C, the action $G \curvearrowright C$ is *minimal*, in the sense that every orbit is dense. When G is a topological group and X is Borel, the *recurrence spectrum* of a Borel action $G \curvearrowright X$ is the collection of all pairs $(\boldsymbol{\mathcal{F}}, \mathcal{H})$, where $\boldsymbol{\mathcal{F}} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ is a family of upward closed families and $\emptyset \in \mathcal{H}$ is a countable subset of $\bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(X^{\{0,\ldots,d\}})$ such that every $H \in \mathcal{H}$ is closed whenever τ is a topology on X generating its Borel sets such that $G \curvearrowright X$ is continuous, such that every smooth Borel superequivalence relation F of E_G^X has an equivalence class C for which $G \curvearrowright C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent on σ-complete Borel sets. We establish the basic properties of the recurrence spectrum of a Borel action of a Polish group on a standard Borel space, which codifies the suitably robust forms of recurrence that it satisfies.

We show that locally-compact non-compact Polish groups have free Borel actions on Polish spaces with maximal recurrence spectra.

We generalize the generic compressibility theorem of Kechris-Miller (see [KM04, Theorem 13.1]) to Borel actions of locally compact Polish groups on standard Borel spaces. We simultaneously replace comeagerness with a stronger notion under which the recurrence spectrum is invariant, thereby ensuring that no condition on the latter yields an invariant Borel probability measure.

In Chapter 3, we introduce the actions in our bases. In the special case of $\mathbb{Z}$-actions, these are made up of actions induced by transformations obtained via cutting and stacking with a sufficiently quickly growing number of insertions at each stage. In order to endow our actions with appropriate topologies and handle groups other than $\mathbb{Z}$, we use quotients associated with cocycles to generalize the cutting and stacking construction to produce continuous minimal actions of non-compact locally-compact Polish groups G on locally-compact Polish spaces. We refer to these actions as being obtained through *expansive cutting and stacking*. More generally, we define *continuous disjoint unions* of such actions.

In Chapter 4, we consider a refined notion of recurrence. Given $d \in \mathbb{Z}^+$ and a binary relation R on a set X, we say that a sequence $x \in X^{\{0,\dots,d\}}$ is *R-discrete* if there do not exist distinct $i, j \leq d$ for which $x_i \mathrel{R} x_j$. The *orbit relation* on X associated with an action $G \curvearrowright X$ and a set $K \subseteq G$ is given by $x \mathrel{R_K^X} y \iff x \in Ky$. Note that the set $F_K^d = \{\mathbf{x} \in X^{\{0,\dots,d\}} \mid \mathbf{x} \text{ is not } R_K^X\text{-discrete}\}$ is closed for every $d \in \mathbb{Z}^+$, compact set $K \subseteq G$, and continuous action $G \curvearrowright X$. Given $d \in \mathbb{Z}^+$ and a set $S \in \mathcal{P}(G^{\{1,\dots,d\}})$ we define the family $\mathcal{F}_S = \{F \subseteq G^{\{1,\dots,d\}} \mid F \cap S \neq \emptyset\}$. Note that any upward closed family $\mathcal{F} \subseteq G^{\{1,\dots,d\}}$ is an intersection of families of the form $\mathcal{F}_S$ for a suitable family of $S \in \mathcal{P}(G^{\{1,\dots,d\}})$. Given an *exhaustive increasing sequence* $(K_n)_{n\in\mathbb{N}}$ *of compact subsets of* G, i.e., an increasing sequence of compact subsets of G such that each compact set $K \subseteq G$ is contained in K_n for some $n \in \mathbb{N}$ and a family $\mathcal{S} \subseteq \bigcup_{d\in\mathbb{Z}^+} \mathcal{P}(G^{\{1,\dots,d\}})$, we say that a set $Y \subseteq X$ is *expansively $\mathcal{S}$-transient* if Y is $(\{\mathcal{F}_S \mid S \in \mathcal{S}\}, \{F_{K_n}^d \mid n \in \mathbb{N}, d \in \mathbb{Z}^+\} \cup \{\emptyset\})$-transient, i.e., if there exist $d \in \mathbb{Z}^+$, a compact set $K \subseteq G$, and $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\dots,d\}})$ for which $\Delta_G^X(\{y \in Y^{\{0,\dots,d\}} \mid y \text{ is } R_K^X\text{-discrete}\}) \cap S = \emptyset$. We say that a G-action by homeomorphisms of a topological space is *expansively $\mathcal{S}$-recurrent* if no non-empty open set is expansively $\mathcal{S}$-transient, and a Borel G-action on a standard Borel space X is *σ-expansively $\mathcal{S}$-transient* if X is a union of countably-many expansively-$\mathcal{S}$-transient Borel sets. We note that every minimal continuous G-action on a Polish space is either expansively $\mathcal{S}$-recurrent or σ-expansively $(\bigcup_{g\in G} g\mathcal{S}g^{-1})$-transient. Given a family $\boldsymbol{\mathcal{S}}$ of subsets of $\bigcup_{d\in\mathbb{Z}^+} \mathcal{P}(G^{\{1,\dots,d\}})$, we say that a Borel G-action on a standard Borel space is *σ-expansively $\boldsymbol{\mathcal{S}}$-transient* if it is σ-expansively $\mathcal{S}$-transient for some $\mathcal{S} \in \boldsymbol{\mathcal{S}}$. A *homomorphism* from $G \curvearrowright X$ to $G \curvearrowright Y$ is a function $\varphi\colon X \to Y$ with the property that $\varphi(g \cdot x) = g \cdot \varphi(x)$ for all $g \in G$ and $x \in X$, a *stabilizer-preserving homomorphism* is a homomorphism whose restriction to each orbit is injective, and an *embedding* is an injective homomorphism.

Building on arguments of [Wei84], we show that if $\mathcal{S}$ is a non-empty countable family, then among all non-σ-expansively-$(\bigcup_{g\in G} g\mathcal{S}g^{-1})$-transient Borel G-actions on Polish spaces, those obtained via expansive cutting and stacking form a basis under continuous embeddability. Similarly, we show that if $\boldsymbol{\mathcal{S}}$ is a family of non-empty countable families, then among all non-σ-expansively-$\{\bigcup_{g\in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}\}$-transient Borel G-actions on Polish spaces, those that are continuous disjoint unions of actions obtained via expansive cutting and stacking form a basis under continuous stabilizer-preserving homomorphism.

Building on arguments of [EHW98], we show that if $\mathcal{S}$ is a non-empty countable family and $G \curvearrowright X$ is a non-σ-expansively-$(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient Borel action on a Polish space, then there is a family $\mathcal{O}$ of $2^{\aleph_0}$-many non-σ-expansively-$(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient Borel actions on Polish spaces such that every action in $\mathcal{O}$ admits a continuous embedding into $G \curvearrowright X$, but every Borel G-action on a standard Borel space admitting a Borel stabilizer-preserving homomorphism to at least two actions in $\mathcal{O}$ is σ-expansively $\{G\}$-transient. Building on this, we show that if $\boldsymbol{\mathcal{S}}$ is a family of non-empty countable families and $G \curvearrowright X$ is a non-σ-expansively-$\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}\}$-transient Borel action on a Polish space, then there is no countable basis, under Borel stabilizer-preserving homomorphism, for the family of non-σ-expansively-$\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}\}$-transient Borel G-actions on Polish spaces that admit a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$.

In Chapter 5, we turn our attention to actions that are particularly simple from the descriptive-set-theoretic point of view. A *reduction* of an equivalence relation E on X to an equivalence relation F on Y is a function $\pi \colon X \to Y$ such that $w \mathrel{E} x \iff \pi(w) \mathrel{F} \pi(x)$ for all $w, x \in X$, so, in particular, a Borel equivalence relation on a standard Borel space is smooth if it admits a Borel reduction to equality on a standard Borel space, and a Borel action $G \curvearrowright X$ on a standard Borel space is *smooth* if E_G^X is smooth. It is easy to see that the latter notion is equivalent to σ-expansive $\{G\}$-transience, from which it follows that the family of actions obtained via expansive cutting and stacking is a basis, under continuous embeddability, for the family of all non-smooth Borel G-actions on Polish spaces. This generalizes and strengthens the original Glimm-Effros dichotomy [Gli61; Eff65], as well as the subsequent results of [SW82; Wei84] (and strengthens the corresponding special case of [HKL90]). It also follows that if $G \curvearrowright X$ is a non-smooth Borel action on a Polish space, then there is no basis of cardinality strictly less than $2^{\aleph_0}$, under Borel stabilizer-preserving homomorphism, for the family of non-smooth Borel G-actions on Polish spaces that admit a continuous embedding into $G \curvearrowright X$. This negatively answers Louveau's question as to whether there is a singleton basis, under Borel embeddability, for the family of all non-smooth Borel $\mathbb{Z}$-actions on standard Borel spaces.

In an attempt to salvage the hope underlying Louveau's question, we also consider Borel free G-actions on standard Borel spaces that *contain a basis*, in the sense that their non-smooth G-invariant Borel restrictions form a basis, under Borel embeddability, for the family of all non-smooth Borel free G-actions on standard Borel spaces. We show that this notion is robust, in the sense that it remains unchanged if Borel embedding is replaced with Borel stabilizer-preserving homomorphism. Recalling that the *diagonal product* of $G \curvearrowright X$ and $G \curvearrowright Y$ is the action $G \curvearrowright X \times Y$ given by $g \cdot (x, y) = (g \cdot x, g \cdot y)$, we also show that a Borel free action $G \curvearrowright X$ on a standard Borel space contains a basis if and only if $G \curvearrowright X \times Y$ is non-smooth for every non-smooth Borel free action $G \curvearrowright Y$ on a standard Borel space. Examples of such actions include all continuous free G-actions on compact Polish spaces, as well as all Borel free G-actions on standard Borel spaces that are invariant with respect to some Borel probability measure μ on X. Let $\mathbb{s}$ denote the *shift* on the class of $\mathbb{N}$-sequences given by $\mathbb{s}_n(\mathbf{g}) = \mathbf{g}_{n+1}$, and define $\mathrm{IP}(\mathbf{g}) = \{\mathbf{g}^s \mid s \in 2^{<\mathbb{N}}\}$ for all $\mathbf{g} \in G^{\mathbb{N}}$, where $\mathbf{g}^s = \prod_{n<|s|} \mathbf{g}_n^{s(n)}$ for all $s \in 2^{<\mathbb{N}}$. Letting $\boldsymbol{\mathcal{S}}_{cb}$ denote the family of sets of the form $\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}$, where $\mathbf{g} \in G^{\mathbb{N}}$ is an injective sequence for which $\mathrm{IP}(\mathbf{g})\mathrm{IP}(\mathbf{g})^{-1}$ is closed and discrete, we show that if G is abelian, then a Borel free G-action on a standard Borel space contains a basis if and only if it is not σ-expansively $\boldsymbol{\mathcal{S}}_{cb}$-transient. It follows that among all Borel free G-actions on Polish spaces that contain a basis, those that are continuous disjoint unions of actions obtained via expansive cutting and stacking form a basis under Borel stabilizer-preserving homomorphism. It also follows that if $G \curvearrowright X$ is a Borel free action on a Polish space containing a basis, then there is no countable basis, under Borel stabilizer-preserving homomorphism, for the family of Borel free G-actions on Polish spaces that contain a basis and admit a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$.

We also consider sets $Y \subseteq X$ that are *weakly wandering*, in the sense that there is an infinite set $S \subseteq G$ such that Y is S*-wandering*, i.e., such that $g^{-1}Y \cap h^{-1}Y = \emptyset$ for all distinct $g, h \in S$ and sets $Y \subseteq X$ that are *very weakly wandering*, in the sense that there are arbitrarily large finite sets $S \subseteq G$ such that Y is S-wandering. We show that the existence of weakly-wandering and very-weakly-wandering suitably-complete Borel sets, as well as suitably-complete Borel sets satisfying the minimal non-trivial notion of transience corresponding to the failure of the strongest notion of recurrence, can be characterized in terms of the recurrence spectrum. Our arguments also yield complexity bounds leading to implications between many of these notions. For instance, it follows that if X is a standard Borel space, $T \colon X \to X$ is a Borel automorphism, and there is no smooth Borel superequivalence relation F of E_T^X with the property that there is a weakly-wandering $(T \upharpoonright C)$-complete Borel set for every F-class C, then there is no locally-weakly-wandering T-complete Borel set, where a set is *locally-weakly-wandering* if its intersection with each E_T^X-class is weakly-wandering.

Letting $\boldsymbol{\mathcal{S}}_{ww\sigma}$ denote the family of sets consisting of a single closed discrete infinite subset of G of the form SS^{-1}, and $\boldsymbol{\mathcal{S}}_{\sigma ww}$ denote the family of countable sets of closed discrete infinite subsets of G of the form SS^{-1}, we note that a Borel free G-action on a standard Borel space admits a weakly-wandering σ-complete Borel set if and only if it is σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{ww\sigma}\}$-transient, whereas the underlying space is a union of countably-many weakly-wandering Borel sets if and only if it is σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{\sigma ww}\}$-transient. These notions are the same for minimal continuous free actions, but the latter is strictly weaker outside of the minimal case. Strengthening the earlier measure-theoretic result, we show that the failure of either of these properties ensures that the action in question contains a basis. We also show that if G admits a compatible two-sided-invariant metric, then the failure of either of these properties is strictly stronger than containing a basis. It also follows that among all Borel free G-actions on Polish spaces that do not have one of these properties, those that are continuous disjoint unions of actions obtained via expansive cutting and stacking form a basis under Borel stabilizer-preserving homomorphism. In addition, we show that if $G \curvearrowright X$ is a Borel free action on a Polish space that does not have one of these properties, then there is no countable basis, under Borel stabilizer-preserving homomorphism, for the family of Borel G-actions on Polish spaces that do not have the property and admit a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$. This answers [EHN93, Question 1] concerning the circumstances under which a Borel $\mathbb{Z}$-action on a standard Borel space admits a weakly-wandering σ-complete Borel set.

The main result of [EHN93] is the existence of a Borel $\mathbb{Z}$-action on a standard Borel space that admits neither an invariant Borel probability measure nor a weakly-wandering σ-complete Borel set. Their example is a disjoint union of $2^{\aleph_0}$-many $\mathbb{Z}$-actions obtained via expansive cutting and stacking. We show that there is an example that is itself obtained via expansive cutting and stacking, and retains the advantages of the more recent examples appearing in [Mil04; Tse15], in that the same straightforward argument not only rules out weakly-wandering σ-complete Borel sets, but also σ-complete Borel sets satisfying still weaker wandering conditions, yielding a structurally simpler negative answer to [EHN93, Question 2].

In Chapter 6, we turn our attention towards mixing conditions. An action $G \curvearrowright X$ by homeomorphisms of a topological space is *topologically transitive* if $\Delta_G^X(U \times V) \neq \emptyset$ for all non-empty open sets $U, V \subseteq X$. More generally, such an action is *topologically d-transitive* if $G \curvearrowright X^d$ is topologically transitive. In the special case that $d = 2$, we also say that $G \curvearrowright X$ is *weakly mixing*. Fix a countable dense subset H of G. Setting $\mathcal{S}_{tdt} = H^{\{1,\dots,2d-1\}}\{g \in G^{\{1,\dots,2d-1\}} \mid \forall 0 < i < d\ g_{2i+1} = g_1 g_{2i}\}$, we note that a topologically-transitive continuous G-action on a Polish space with no open orbits is topologically d-transitive if and only if it is not σ-expansively $(\bigcup_{g \in G} g\mathcal{S}_{tdt}g^{-1})$-transient. It follows that among all topologically-d-transitive continuous G-actions on Polish spaces with no open orbits, those obtained via expansive cutting and stacking form a basis under continuous embeddability. It also follows that if $G \curvearrowright X$ is a topologically-d-transitive continuous G-action on a Polish space with no open orbits, then there is no basis of cardinality strictly less than $2^{\aleph_0}$, under Borel stabilizer-preserving homomorphism, for the family of topologically-d-transitive continuous G-actions on Polish spaces with no open orbits that admit a continuous embedding into $G \curvearrowright X$.

A Borel action $G \curvearrowright X$ on a standard Borel space is *ergodic* with respect to a Borel measure μ on X if every G-invariant Borel set is μ-conull or μ-null, and *weakly mixing* with respect to μ if $G \curvearrowright X \times X$ is $(\mu \times \mu)$-ergodic. In the spirit of [SW82; Wei84], we show that if G is abelian, then a Borel action $G \curvearrowright X$ on a standard Borel space is weakly mixing with respect to a Polish topology compatible with the Borel structure of X on a G-invariant closed set if and only if it is weakly mixing with respect to a G-invariant σ-finite Borel measure on X.

We also note if G has a compatible two-sided-invariant metric and $G \curvearrowright X$ is a continuous action on a Polish space with no open orbits satisfying any mixing condition at least as strong as weak mixing, then there is no basis of cardinality strictly less than the additivity of the meager ideal on $\mathbb{R}$, under continuous stabilizer-preserving homomorphism, for the family of continuous G-actions on Polish spaces with no open orbits satisfying the mixing condition and admitting a continuous embedding into $G \curvearrowright X$.

We say that a continuous action $G \curvearrowright X$ on a Polish space with no open orbits is *mildly mixing* if $G \curvearrowright X \times Y$ is topologically transitive for every topologically-transitive continuous action $G \curvearrowright Y$ on a Polish space with no open orbits. Letting $\boldsymbol{\mathcal{S}}_{mm}$ denote the family of sets consisting of a single closed discrete subset of G of the form $g\mathrm{IP}(\mathbf{g})\mathrm{IP}(\mathbf{g})^{-1}$, where $g \in G$ and $\mathbf{g} \in G^{\mathbb{N}}$ is injective, we note that a topologically-transitive continuous G-action on a Polish space with no open orbits is mildly mixing if and only if it is not σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{mm}\}$-transient if and only if there is a non-σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{mm}\}$-transient continuous disjoint union of G-actions obtained via expansive cutting and stacking that admits a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$.

A continuous action $G \curvearrowright X$ on a Polish space is *strongly mixing* if $\Delta_G^X(U \times V)$ is co-compact for all non-empty open sets $U, V \subseteq X$. Letting $\boldsymbol{\mathcal{S}}_{sm}$ denote the family of sets consisting of a single closed discrete infinite subset of G, we note that a topologically-transitive continuous G-action on a Polish space with no open orbits is strongly mixing if and only if it is not σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{sm}\}$-transient if and only if there is a non-σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{sm}\}$-transient continuous disjoint union of G-actions obtained via expansive cutting and stacking that admits a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$.

In Chapter 7 we gather some results that do not fit into the context of the previous chapters but are nevertheless of interest in their own right. The conjugation action of F_2 on the space of its subgroups is universal among countable Borel equivalence relations under Borel reducibility [TV99]. We give a different proof of this fact.

Given a continuous action $G \curvearrowright X$ and $d \in \mathbb{Z}^+$ we say that $G \curvearrowright X$ is *strongly d-mixing* if $\Delta_G^X(\Pi_{k \le d} U_k)$ contains the complement of $\{\mathbf{g} \in G^{\{1,\ldots,d\}} \mid \exists i \neq j \in \{0,...,d\}\, \overline{\mathbf{g}}_i \overline{\mathbf{g}}_j^{-1} \in K\}$ for some compact set $K \subset G$ for all non-empty open sets U_k for $k \in \{0,...,d\}$. For countable groups we construct minimal actions of G which are strongly d-mixing but not strongly $(d+1)$-mixing for every $d \in \mathbb{Z}^+$.

At last we give an example of a countably infinite family of actions of the free group in at least two generators whose induced Borel equivalence relations are universal among countable Borel equivalence relations under Borel embeddability such that the diagonal product action for any two distinct actions from this family is smooth.

Chapter 2

Recurrence and measures

2.1 The recurrence spectrum

Suppose that $G \curvearrowright X$ is a group action.We start with the following observation which ensures that the notions of completeness and σ-completeness coincide under mild hypothesis on open sets.

Proposition 2.1.1. Suppose that G is a topological group, $H \subseteq G$ is dense, X is a topological space, $G \curvearrowright X$ is an action, and $U \subseteq X$ has the property that for every $x \in U$ the set $\{g \mid gx \in U\}$ contains a non-empty open subset of G. Then $GU = HU$. In particular, if $G \curvearrowright X$ is continuous-in-G and $U \subseteq X$ is open, then $GU = HU$.

Proof. If $V \subseteq G$ is non-empty and open and $g \in G$, then there exists $h \in gV^{-1} \cap H$, thus $g \in HV$, hence $HV = G$. Now, suppose that $x \in U$. If $V \subseteq \{g \mid gx \in U\}$ is non-empty and open, then $Gx = HVx \subseteq HU$ thus $GU = HU$. If $G \curvearrowright X$ is continuous-in-G and U is open, then for every $x \in U$ there is an open neighborhood $V \subseteq G$ of 1_G for which $Vx \subseteq U$, thus by the previous argument $GU = HU$. ⊠

We will abuse language by saying that a subset of X is $\aleph_0$-*universally Baire* if its preimage under every Borel function from a Polish space to X has the Baire property, and an $\aleph_0$-universally Baire equivalence relation E on X *smooth* if there is no Borel function $\pi : 2^{\mathbb{N}} \to X$ such that $x \mathbb{E}_0 y \Leftrightarrow \pi(x) \; E \; \pi(y)$. The Harrington-Kechris-Loveau generalization of the Glimm-Effros dichotomy (see [HKL90, Theorem 1.1]) ensures that this is compatible with the usual notion of smoothness for Borel equivalence relations on standard Borel spaces.

Proposition 2.1.2. Suppose that G is a group, X is a Polish space, $G \curvearrowright X$ is continuous-in-X, $B \subseteq X$ is E_G^X-invariant, $C \subseteq X$ is an E_G^X-invariant G_δ set for which $G \curvearrowright C$ is topologically transitive and in which B is comeager, and F is a smooth $\aleph_0$-universally-Baire superequivalence relation of E_G^B. Then there is an F-class that is comeager in C.

Proof. Fix a dense G_δ set $C' \subseteq C$ contained in B, and note that F has the Baire property in $C' \times C'$, thus in $C \times C$. The straightforward generalization of the Becker-Kechris criterion for continuously embedding $\mathbb{E}_0$ from orbit equivalence relations induced by groups of homeomorphisms (see [BK96, Theorem 3.4.5]) to superequivalence relations of such orbit equivalence relations (see, for example, [KMS14, Theorem 2.1]) ensures that the union of F and $(C \setminus B) \times (C \setminus B)$ is non-meager in $C \times C$, so the Kuratowski-Ulam theorem (see, for example, [Kec95, Theorem 8.41]) yields an F-class that is non-meager and has the Baire property in C, thus comeager in C by topological transitivity. ⊠

The following fact is the obvious generalization of Pettis's Lemma (see, for example, [Kec95, Theorem 9.9]) to group actions.

Proposition 2.1.3. Suppose that G is a group, X is a Baire space, $G \curvearrowright X$ is continuous-in-X, $d \in \mathbb{Z}^+$, $R \subseteq X^{\{0,\ldots,d\}}$ is closed, $(V_k)_{k \leq d}$ is a sequence of non-empty open subsets of X, and B_k is comeager in V_k for all $k \leq d$. Then $\Delta_G^X((\Pi_{k \leq d} V_k) \setminus R) \subseteq \Delta_G^X((\Pi_{k \leq d} B_k) \setminus R)$.

Proof. If $g \in \Delta_G^X((\Pi_{k \leq d} V_k) \setminus R)$, then there exists $x \in \bigcap_{k \leq d} (\overline{g}_k)^{-1} V_k$ such that $\overline{g} x \notin R$. Fix an open neighborhood of $V \subseteq \bigcap_{k \leq d} (\overline{g}_k)^{-1} V_k$ of x with the property that $(\Pi_{k \leq d} \overline{g}_k V) \cap R = \emptyset$. As $(\overline{g}_k)^{-1} B_k$ is comeager in $(\overline{g}_k)^{-1} V_k$ for all $k \leq d$, it follows that $\bigcap_{k \leq d} (\overline{g}_k)^{-1} B_k$ is comeager in $\bigcap_{k \leq d} (\overline{g}_k)^{-1} V_k$ and therefore intersects V, from which it follows that $g \in \Delta_G^X((\Pi_{k \leq d} B_k) \setminus R)$. ⊠

We next note that, under mild hypotheses, $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrence propagates to $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrence on non-meager sets with the Baire property. When Y is a topological space, we use $\mathcal{F}(Y)$ to denote the family of all closed subsets of Y.

Proposition 2.1.4. Suppose that G is a group, X is a Baire space, $\boldsymbol{\mathcal{F}} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ is a family of upward closed families, $\emptyset \in \mathcal{H} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{F}(X^{\{0,\ldots,d\}})$ and $G \curvearrowright X$ is continuous-in-X and $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent. Then $G \curvearrowright X$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent on non-meager sets with the Baire property.

Proof. If $B \subseteq X$ is a non-meager set with the Baire property, then there is a non-empty open set $U \subseteq X$ in which B is comeager, and Proposition 2.1.3 ensures that $\Delta_G^X((\Pi_{k \leq d} U) \setminus R) \subseteq \Delta_G^X((\Pi_{k \leq d} B) \setminus R)$, i.e., if B is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-transient, then U is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-transient. ⊠

As a theorem of Becker-Kechris ensures that every Borel action of a Polish group on a standard Borel space is Borel isomorphic to a continuous action on a Polish space (see [BK96, Theorem 5.2.1]), the following observation ensures that, under mild hypotheses, the notion of recurrence spectrum is robust, in the sense that it does not depend on the particular notion of definability, and in the sense that it is invariant under passage to sufficiently large E_G^X-invariant subsets.

Proposition 2.1.5. Suppose, that G is a separable group, X is a Polish space, $G \curvearrowright X$ in continuous, $\boldsymbol{\mathcal{F}} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ is a family of upward closed families, $\emptyset \in \mathcal{H} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{F}(X^{\{0,\ldots,d\}})$, and $B \subseteq X$ is E_G^X-invariant and comeager in every F_G^X-class. Then the following are equivalent:

(1) Every smooth $\aleph_0$-universally-Baire superequivalence relation F of E_G^B has a class C for which $G \curvearrowright C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent on σ-complete $\aleph_0$-universally-Baire sets.

(2) There is an F_G^X-class C for which $G \curvearrowright C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent.

Proof. To see (1) $\Rightarrow$ (2), note that $F_G^X | B$ is a smooth $\aleph_0$-universally-Baire superequivalence relation of E_G^B and fix an F_G^X class C for which $G \curvearrowright B \cap C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent on σ-complete $\aleph_0$-universally-Baire sets. To see that $G \curvearrowright C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent, suppose that $U \subseteq C$ is a non-empty open set, and note that the minimality of $G \curvearrowright C$ ensures that U is complete and therefore σ-complete by Proposition 2.1.1, thus $\Delta_G^X(U^{\{0,\ldots,d\}} \setminus H) \in \mathcal{F}$ for all $d \in \mathbb{Z}^+$, $\mathcal{F} \in \boldsymbol{\mathcal{F}} \cap \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ and $H \in \mathcal{H} \cap \mathcal{F}(X^{\{0,\ldots,d\}})$. To see (2) $\Rightarrow$ (1), fix an F_G^X-class C for which $G \curvearrowright C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent, and suppose that F is a smooth $\aleph_0$-universally-Baire superequivalence relation of E_G^B. Proposition 2.1.2 then yields an F-class D that is comeager in C. To see that $G \curvearrowright D$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent on σ-complete $\aleph_0$-universally-Baire sets, suppose that $A \subseteq D$ is such a set, and note that σ-completeness ensures that A is non-meager in C. Fix a dense G_δ set $C' \subseteq C$ contained in D, and note that $A \cap C'$ has the Baire property in C', thus Proposition 2.1.4 ensures that $\Delta_G^X(A^{\{0,\ldots,d\}} \setminus H) \in \mathcal{F}$ for all $d \in \mathbb{Z}^+$, $\mathcal{F} \in \boldsymbol{\mathcal{F}} \cap \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ and $H \in \mathcal{H} \cap \mathcal{F}(X^{\{0,\ldots,d\}})$. ⊠

Let $\check{\Gamma}$ denote the family of sets whose complements are in Γ, let $\Gamma \setminus \Gamma$ denote the family of differences of sets in Γ, and let Γ_σ denote the family of countable unions of sets in Γ. The *horizontal sections* of a set $R \subseteq X \times Y$ are the sets of the form $R^y = \{x \in X \mid x \mathrel{R} y\}$, whereas the *vertical sections* of a set $R \subseteq X \times Y$ are the sets of the form $R_x = \{y \in X \mid x \mathrel{R} y\}$. Given a family of upward closed families $\boldsymbol{\mathcal{F}} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ and $\emptyset \in \mathcal{H} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{F}(X^{\{0,\ldots,d\}})$, we say that $\boldsymbol{\mathcal{F}}$ is Γ*-on-open* if for all $d \in \mathbb{Z}^+$, $\mathcal{F} \in \boldsymbol{\mathcal{F}} \cap \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$, and all open sets $U \subseteq G^{\{0,\ldots,d\}} \times X$ the set $\{x \in X \mid U^x \in \mathcal{F}\}$ is in Γ. Given a superequivalence relation E of E_G^X, we say that an action $G \curvearrowright X$ is E*-locally* $(\boldsymbol{\mathcal{F}}, \mathcal{H})$*-recurrent on* Γ if for all $B \in \Gamma$, there is an E-class C such that $\Delta_G^X((B \cap C)^{\{0,\ldots,d\}} \setminus H) \in \mathcal{F}$ for all $d \in \mathbb{Z}^+$, $\mathcal{F} \in \boldsymbol{\mathcal{F}} \cap \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ and $H \in \mathcal{H} \cap \mathcal{F}(X^{\{0,\ldots,d\}})$. We next note that, under mild hypotheses, the recurrence spectrum can also be characterized in terms of local recurrence of $G \curvearrowright X$ itself.

Proposition 2.1.6. Suppose that G is a separable group, X is a Polish space, $G \curvearrowright X$ is continuous, $\Gamma \subseteq \mathcal{P}(X)$ is a family of $\aleph_0$-universally-Baire sets containing the open sets and closed under finite intersections and finite unions, $\boldsymbol{\mathcal{F}} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ is a countable family of upward closed families, $\emptyset \in \mathcal{H} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{F}(X^{\{0,\ldots,d\}})$ is countable, and $\boldsymbol{\mathcal{F}}$ is $\check{\Gamma}$-on-open. Then the following are equivalent:

1. (1) There is an F_G^X-class C for which $G \curvearrowright C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent.
2. (2) The action $G \curvearrowright X$ is F_G^X-locally $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent on σ-complete $(\Gamma \setminus \Gamma)_\sigma$ sets.

Proof. To see $\neg(2) \Rightarrow \neg(1)$, observe that if $B \subseteq X$ is a σ-complete $(\Gamma \setminus \Gamma)_\sigma$ set, then it is non-meager and has the Baire property in every F_G^X-class and if $B \cap C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-transient for every such class, then there is no F_G^X-class for which $G \curvearrowright C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent by Proposition 2.1.4.
To see $\neg(1) \Rightarrow \neg(2)$, fix a basis $(U_n)_{n \in \mathbb{N}}$ for the open subsets of X. For all $n \in \mathbb{N}$, $d \in \mathbb{Z}^+$, and $H \in \mathcal{H} \cap \mathcal{F}(X^{\{0,\ldots,d\}})$, define $V_n^H = \{(\mathbf{g}, x) \in G^{\{1,\ldots,d\}} \times X \mid \mathbf{g} \in \Delta_G^X((U_n \cap [x]_{F_G^X})^{\{0,\ldots,d\}} \setminus H)\}$. Observe that if $(\mathbf{g}, x) \in V_n^H$, then the minimality of $G \curvearrowright [x]_{F_G^X}$ yields $h \in G$ for which $\overline{\mathbf{g}}(hx) \in U_n^{\{0,\ldots,d\}} \setminus H$, so the continuity of $G^{\{0,\ldots,d\}} \curvearrowright X^{\{0,\ldots,d\}}$ yields open neighborhoods of $U_{\mathbf{g}}$ of $\mathbf{g}$ and U_x of x such that $\overline{\mathbf{g}'} h U_x \subseteq U_n^{\{0,\ldots,d\}} \setminus H$ for all $\mathbf{g}' \in U_{\mathbf{g}}$, thus $U_{\mathbf{g}} \times U_x \subseteq V_n^H$, hence V_n^H is open. It follows that the F_G^X-invariant sets $A_n^{H,\mathcal{F}} = \{x \in GU_n \mid (V_n^H)^x \notin \mathcal{F}\}$ are in Γ for all $\mathcal{F} \in \boldsymbol{\mathcal{F}} \cap \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$. Let $\pi : \mathbb{N} \to \mathbb{N} \times \{(H, \mathcal{F}) \in \mathcal{H} \times \boldsymbol{\mathcal{F}} \mid \exists d \in \mathbb{Z}^+ H \subseteq X^{\{0,\ldots,d\}} \; \mathcal{F} \subseteq \mathcal{P}(G^{\{1,\ldots,d\}})\}$ be a bijection and define $A_n = A_{\mathrm{proj}_{\mathbb{N}}(\pi(n))}^{\mathrm{proj}_{\mathcal{H} \times \boldsymbol{\mathcal{F}}}(\pi(n))}$. Then the sets $B_n = A_n \setminus \bigcup_{m<n} A_m$ are in $\Gamma \setminus \Gamma$, thus the set $B = \bigcup_{n \in \mathbb{N}} B_n \cap U_{\mathrm{proj}_{\mathbb{N}}(\pi(n))}$ is in $(\Gamma \setminus \Gamma)_\sigma$. But if there is no F_G^X-class C for which $G \curvearrowright C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent, then B is complete, and therefore σ-complete by Proposition 2.1.1, thus $G \curvearrowright X$ is not F_G^X-locally $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent on σ-complete $(\Gamma \setminus \Gamma)_\sigma$ sets.

⊠

We next show that, under an additional mild hypothesis on G, Propositions 2.1.3, 2.1.4, and 2.1.5 can be strengthened so as to show that the recurrence spectrum is also robust in the sense that it does not depend on whether the underlying notion of recurrence is local.
The following fact is a somewhat more intricate generalization of the special case of Pettis's Lemma for second-countable groups to topologically-transitive actions of such groups.

Proposition 2.1.7. Suppose that G is a second-countable Baire group, X is a second-countable Baire space, $G \curvearrowright X$ is continuous, $U \subseteq X$ is non-empty and open, $B \subseteq U$ is comeager, $d \in \mathbb{Z}^+$, and $H \subseteq X^{\{0,\ldots,d\}}$ is closed. Then $\Delta_G^X((U \cap Gx)^{\{0,\ldots,d\}} \setminus H) \subseteq \Delta_G^X((B \cap Gx)^{\{0,\ldots,d\}} \setminus H)$ for comeagerly many $x \in X$.

Proof. We write $\forall^* x \in X\, \varphi(x)$ to indicate that $\{x \in X \mid \varphi(x)\}$ is comeager. As the fact that $G \curvearrowright X$ is continuous-in-X ensures that it is open, it follows that $\{(g,x) \in G \times X \mid g \cdot x \notin U \setminus B\}$ is comeager, so the set $C = \{x \in X \mid \forall^* g \in G\, g \cdot x \notin U \setminus B\}$ is comeager by the Kuratowski-Ulam theorem. Observe that if $h' \in G$ and $x \in X$, then $\{g \in G \mid g \cdot x \notin h'(U \setminus B)\} = h'\{g \in G \mid g \cdot x \notin U \setminus B\}$, so the fact that $G \curvearrowright X$ is continuous-in-X also ensures that if $x \in C$ and $\mathbf{h} \in G^{\{1,\ldots,d\}}$, then $\forall^* g \in G\, g \cdot x \notin \bigcup_{k \leq d} \overline{\mathbf{h}}_k(U \setminus B)$, in which case the fact that $\bigcap_{k \leq d} \overline{\mathbf{h}}_k U \setminus \bigcap_{k \leq d} \overline{\mathbf{h}}_k B \subseteq \bigcup_{k \leq d} \overline{\mathbf{h}}_k(U \setminus B)$ therefore implies that $\forall^* g \in G\, g \cdot x \notin \bigcap_{k \leq d} \overline{\mathbf{h}}_k U \setminus \bigcap_{k \leq d} \overline{\mathbf{h}}_k B$. Note that for all $\mathbf{h} \in G^{\{1,\ldots,d\}}$, the fact that $G \curvearrowright X$ is continuous-in-X ensures that the set $\{y \mid \overline{\mathbf{h}} y \in U^{\{0,\ldots,d\}} \setminus H\}$ is open, so the fact that $G \curvearrowright X$ is continuous-in-G implies that if $x \in C$ and $\{y \mid \overline{\mathbf{h}} y \in U^{\{0,\ldots,d\}} \setminus H\} \cap Gx$ is non-empty, then there are non-meagerly many $g \in G$ for which $g \cdot x \in \{y \mid \overline{\mathbf{h}} y \in U^{\{0,\ldots,d\}} \setminus H\}$. In particular, since $\{y \mid \overline{\mathbf{h}} y \in U^{\{0,\ldots,d\}} \setminus H\} \setminus \{y \mid \overline{\mathbf{h}} y \in B^{\{0,\ldots,d\}} \setminus H\} \subseteq \bigcap_{k \leq d} \overline{\mathbf{h}}_k U \setminus \bigcap_{k \leq d} \overline{\mathbf{h}}_k B$, it follows that if $x \in C$ and $\{y \mid \overline{\mathbf{h}} y \in U^{\{0,\ldots,d\}} \setminus H\} \cap Gx$ is non-empty, then so too is $\{y \mid \overline{\mathbf{h}} y \in B^{\{0,\ldots,d\}} \setminus H\} \cap Gx$, hence $\Delta_G^X((U \cap Gx)^{\{0,\ldots,d\}} \setminus H) \subseteq \Delta_G^X((B \cap Gx)^{\{0,\ldots,d\}} \setminus H)$ for all $x \in C$. ⊠

We next note that, under mild hypotheses, $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrence of topologically transitive actions not only propagates to $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrence on non-meager sets with the Baire property, but to its E_G^X-local strengthening.

Proposition 2.1.8. Suppose that G is a second-countable Baire group, X is a second-countable Baire space, $\boldsymbol{\mathcal{F}} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ is a family of upward closed families, $\emptyset \in \mathcal{H} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{F}(X^{\{0,\ldots,d\}})$ is countable, and $G \curvearrowright X$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent, and topologically transitive. Then $G \curvearrowright X$ is E_G^X-locally $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent on non-meager sets with the Baire property.

Proof. Suppose that $B \subseteq X$ is a non-meager set with the Baire property, and fix a non-empty open set $U \subseteq X$ in which B is comeager. The topological transitivity of $G \curvearrowright X$ ensures that the set $C = \{x \in X \mid Gx \,\text{is dense}\}$ is comeager, and Proposition 2.1.7 implies that the set $D_d^H = \{x \in X \mid \Delta_G^X((U \cap Gx)^{\{0,\ldots,d\}} \setminus H) \subseteq \Delta_G^X((B \cap Gx)^{\{0,\ldots,d\}} \setminus H)$, comeager for all $d \in \mathbb{Z}^+$ and $H \in \mathcal{H} \cap \mathcal{F}(X^{\{0,\ldots,d\}})$. So it only remains to observe that if $x \in C \cap \bigcap_{d \in \mathbb{Z}^+, H \in \mathcal{H} \cap \mathcal{F}(X^{\{0,\ldots,d\}})} D_d^H$, then $\Delta_G^X(U^{\{0,\ldots,d\}} \setminus H) \subseteq \Delta_G^X((U \cap Gx)^{\{0,\ldots,d\}} \setminus H) \subseteq \Delta_G^X((B \cap Gx)^{\{0,\ldots,d\}} \setminus H)$ for all $d \in \mathbb{Z}^+$ and $H \in \mathcal{H} \cap \mathcal{F}(X^{\{0,\ldots,d\}})$, thus $\Delta_G^X((B \cap Gx)^{\{0,\ldots,d\}} \setminus H) \in \mathcal{F}$ for all $d \in \mathbb{Z}^+$, $\mathcal{F} \in \boldsymbol{\mathcal{F}} \cap \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ and $H \in \mathcal{H} \cap \mathcal{F}(X^{\{0,\ldots,d\}})$. ⊠

We can now establish the promised robustness result.

Proposition 2.1.9. Suppose that G is a second countable Baire group, X is a Polish space, $G \curvearrowright X$ is continuous, $\boldsymbol{\mathcal{F}} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ is a family of upward closed families, $\emptyset \in \mathcal{H} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{F}(X^{\{0,\ldots,d\}})$ is countable, and $B \subseteq X$ is E_G^X-invariant and comeager in every F_G^X-class. Then the following are equivalent.

(1) Every smooth $\aleph_0$-universally-Baire superequivalence relation F of E_G^B has a class C for which $G \curvearrowright C$ is E_G^C-locally $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent on σ-complete $\aleph_0$-universally-Baire sets.

(2) There is an F_G^X-class C for which $G \curvearrowright C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent.

Proof. To see (1) $\Rightarrow$ (2), note that $F_G^X|B$ is a smooth $\aleph_0$-universally-Baire superequivalence relation of E_G^B and fix an F_G^X-class C for which $G \curvearrowright B \cap C$ is E_G^C-locally $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent on σ-complete $\aleph_0$-universally-Baire sets. To see that $G \curvearrowright C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent, suppose that $U \subseteq C$ is a non-empty open set, and note that the minimality of $G \curvearrowright C$ ensures that U is complete and therefore σ-complete by Proposition 2.1.1, thus there exists $x \in C$ such that $\Delta_G^X((U \cap Gx)^{\{0,\ldots,d\}} \setminus H) \in \mathcal{F}$ for all $d \in \mathbb{Z}^+$, $\mathcal{F} \in \boldsymbol{\mathcal{F}} \cap \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ and $H \in \mathcal{H} \cap \mathcal{F}(X^{\{0,\ldots,d\}})$, thus $G \curvearrowright C$ is E_G^C-locally $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent on non-empty open sets, thus $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent. To see (2) $\Rightarrow$ (1), fix an F_G^X-class C for which $G \curvearrowright C$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent, and suppose that F is a smooth $\aleph_0$-universally-Baire superequivalence relation of E_G^B. Proposition 2.1.2 then yields an F-class D that is comeager in C. To see that $G \curvearrowright D$ is E_G^D-locally $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent on σ-complete $\aleph_0$-universally-Baire sets, suppose that $A \subseteq D$ is such a set, and note that σ-completeness ensures that A is non-meager in C. Fix a dense G_δ set $C' \subseteq C$ contained in D, and note that $A \cap C'$ has the Baire property and is non-meager in C', thus Proposition 2.1.8 ensures that there exists $x \in C'$ such that $\Delta_G^X((A \cap Gx)^{\{0,\ldots,d\}} \setminus H) \in \mathcal{F}$ for all $d \in \mathbb{Z}^+$, $\mathcal{F} \in \boldsymbol{\mathcal{F}} \cap \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ and $H \in \mathcal{H} \cap \mathcal{F}(X^{\{0,\ldots,d\}})$. ⊠

2.2 The strongest notion of recurrence

Recall that for a subset $S \subseteq G^{\{1,\ldots,d\}}$ the set $\mathcal{F}_S$ denotes the family of sets $T \subseteq G^{\{1,\ldots,d\}}$ such that $T \cap S \neq \emptyset$. Note that a set $Y \subseteq G$ is $\mathcal{F}_S$-transient if $\Delta_G^X(Y^{\{0,\ldots,d\}}) \cap S = \emptyset$, i.e., if $SY \cap Y^{\{1,\ldots,d\}} = \emptyset$ in which case we just say that Y is *S-transient.* When $K \subseteq G$ is compact, we define the set $C_K^d = \{\mathbf{g} \in G^{\{1,\ldots,d\}} \mid \exists i \neq j \in \{0,\ldots,d\}\, \overline{\mathbf{g}}_i \overline{\mathbf{g}_j}^{-1} \in K\}$ and the set D_K^d to be the complement of C_K^d.

Proposition 2.2.1. Suppose that G is a topological group, X is a Hausdorff space, $G \curvearrowright X$ is continuous, $K \subseteq G$ is compact, $x \in X$ is not fixed by any element of K. Then there is a C_K^d-transient open neighborhood of x.

Proof. For each $g \in K$, the fact that X is Hausdorff yields open neighborhoods $V_g \subseteq X$ of x and $W_g \subseteq X$ of gx such that V_g and W_g are disjoint. The continuity of $G \curvearrowright X$ yields open neighborhoods $U_g \subseteq G$ of g and $V'_g \subseteq V_g$ of x for which $U_g V'_g \subseteq W_g$, thus $(U_g V'_g) \cap V'_g = \emptyset$. The compactness of K then yields a finite set $F \subseteq K$ for which $K \subseteq \bigcup_{g \in F} U_g$, in which case $V = \bigcap_{g \in F} V'_g$ is a K-transient open neighborhood of x. If $\mathbf{g} \in \Delta_G^X(V^{\{0,\ldots,d\}})$ and $i \neq j \in \{0, \ldots, d\}$ then $(\overline{\mathbf{g}_i}\overline{\mathbf{g}_j}^{-1})(\overline{\mathbf{g}_j}x) = \overline{\mathbf{g}_i}x$ thus $\overline{\mathbf{g}_i}\overline{\mathbf{g}_j}^{-1} \notin K$ hence $\mathbf{g} \notin C_K^d$. ⊠

It follows that upward closed families $\mathcal{F} \subseteq \mathcal{P}(G^{\{1,\ldots,d\}})$ whose corresponding notions of recurrence are realizable by suitable free actions necessarily contain D_K^d, for all compact sets $K \subseteq G$ not containing 1_G.

Proposition 2.2.2. Suppose that G is a topological group, X is a Hausdorff space, $\boldsymbol{\mathcal{F}} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ is a family of upward closed families, $\emptyset \in \mathcal{H} \subseteq \bigcup_{d \in \mathbb{Z}^+} F(X^{\{0,\ldots,d\}})$, and $G \curvearrowright X$ is $(\boldsymbol{\mathcal{F}}, \mathcal{H})$-recurrent, continuous, and free. Then for all $d \in \mathbb{Z}^+$, every $\mathcal{F} \in \boldsymbol{\mathcal{F}} \cap \mathcal{P}(\mathcal{P}(G^{\{1,\ldots,d\}}))$ contains D_K^d for all compact $K \subseteq G$ not containing 1_G.

Proof. This is a direct consequence of Proposition 2.2.1. ⊠

When X is a topological space and $d \in \mathbb{Z}^+$, recall that a continuous-in-X action $G \curvearrowright X$ is strongly d-mixing if $\Delta_G^X(\Pi_{k \leq d} U_k)$ contains D_K^d for some compact set $K \subseteq G$ for all sequences $(U_k)_{k \leq d}$ of non-empty open subsets of X, $G \curvearrowright X$ is strongly mixing if $G \curvearrowright X$ is strongly 1-mixing, and strongly $(< \omega)$-mixing if $G \curvearrowright X$ is strongly d-mixing for all $d \in \mathbb{Z}^+$.

Proposition 2.2.3. Suppose that G is a topological group, X is a topological space, $\mathcal{F} \subseteq \mathcal{P}(G^{\{1,\ldots,d\}})$ is the family of subsets of $G^{\{1,\ldots,d\}}$ containing D_K^d for some compact $1_G \notin K \subseteq G$, and $G \curvearrowright X$ is continuous-in-X, $\mathcal{F}$-recurrent, and topologically transitive. Then $G \curvearrowright X$ is strongly d-mixing.

Proof. Given a sequence $(U_k)_{k \leq d}$ of non-empty open subsets of X, the topological transitivity of $G \curvearrowright X$ recursively yields $\mathbf{g} \in G^{\{1,\ldots,d\}}$ and non-empty opens sets $V_k \subseteq U_0$ for $k \leq d$ for which $\mathbf{g}_k V_k \subseteq U_k$ and $V_{k+1} \subseteq V_k$ for $k < d$, so the fact that $G \curvearrowright X$ is $\mathcal{F}$-recurrent ensures that $\Delta_G^X(V_d^{\{0,\ldots,d\}})$ contains D_K^d for some compact $1_G \notin K \subseteq G$. As $\mathbf{h} \in \Delta_G^X(V_d^{\{0,\ldots,d\}}) \Leftrightarrow \mathbf{h}(V_d) \cap V_d^{\{1,\ldots,d\}} \neq \emptyset \Leftrightarrow (\mathbf{g}_k \mathbf{h}_k)_{k \in \{1,\ldots,d\}} \in \Delta_G^X(\Pi_{k \in \{0,\ldots,d\}} \overline{\mathbf{g}}_k V_d)$, it follows that $\mathbf{g}\Delta_G^X(V_d^{\{0,\ldots,d\}}) = \Delta_G^X(\Pi_{k \in \{0,\ldots,d\}} \overline{\mathbf{g}}_k V_d)$, so $\Delta_G^X(\Pi_{k<d} U_d)$ contains $\mathbf{g}\Delta_G^X(V_d^{\{0,\ldots,d\}})$ and thus $D^d_{\bigcup_{i,j \leq d} \overline{\mathbf{g}_i} K \overline{\mathbf{g}_j}^{-1}}$ since $D^d_{\bigcup_{i,j \leq d} \overline{\mathbf{g}_i} K \overline{\mathbf{g}_j}^{-1}} \subseteq \mathbf{g} D_K^d$. ⊠

It follows that the existence of a suitable free Borel action of G whose recurrence spectrum contains $\mathcal{F}$, where $\mathcal{F} \subseteq \mathcal{P}(G^{\{1,\dots,d\}})$ is the family of subsets of $G^{\{1,\dots,d\}}$ containing D_K^d for some compact $1_G \notin K \subseteq G$, is equivalent to the existence of a suitable continuous strongly-d-mixing free action of G.

Proposition 2.2.4. Suppose that G is a separable group, X is a Polish space, $G \curvearrowright X$ is continuous, $d \in \mathbb{Z}^+$, and $\mathcal{F} \subseteq \mathcal{P}(G^{\{1,\dots,d\}})$ is the family of subsets of $G^{\{1,\dots,d\}}$ containing D_K^d for some compact $1_G \notin K \subseteq G$. Then $\mathcal{F}$ is in the recurrence spectrum of $G \curvearrowright X$ if and only if there is an equivalence class C of F_G^X for which $G \curvearrowright C$ is strongly d-mixing.

Proof. If $\mathcal{F}$ is in the recurrence spectrum of $G \curvearrowright X$, then Propositions 2.1.5 and 2.2.3 imply that there is an equivalence class C of F_G^X for which $G \curvearrowright C$ is strongly d-mixing. It remains to show that if there is an equivalence class C of F_G^X for which $G \curvearrowright C$ is strongly d-mixing, then $G \curvearrowright C$ is $\mathcal{F}$-recurrent. Suppose that $U \subseteq C$ is non-empty and open. Then we can find an open and symmetric neighborhood $W \subseteq G$ of 1_G and a non-empty open subset $V \subseteq U$ such that $W^d V \subseteq U$. Then find a compact set $K \subseteq G$ such that $D_K^d \subseteq \Delta_G^C(V^{\{0,\dots,d\}})$. Suppose that $\mathbf{g} \notin C_{K\setminus W}^d$. If $\mathbf{g} \in D_K^d$, then $\mathbf{g} \in \Delta_G^C(V^{\{0,\dots,d\}})$ and if $\mathbf{g} \in C_K^d \setminus C_{K\setminus W}^d$ then there exists $0 \leq d' < d$, $\mathbf{g}' \in D_K^{d'}$ and an injective map $f : d' \to d$ such that $\mathbf{g}_{f(i)} = \mathbf{g}'_i$ for all $i \in \{1, \dots, d'\}$ and $\mathbf{g}_j \in W^d\{\overline{\mathbf{g}'_1}, \dots, \overline{\mathbf{g}'_{d'}}\})$ for all $j \in \{1, \dots, d\}$. As there exists $x \in V$ such that $\mathbf{g}' x \in V^{\{1,\dots,d'\}}$ we obtain that $\mathbf{g}_j x \in W^d V \subseteq U$ for all $j \in \{1, \dots, d\}$ thus $\mathbf{g} \in \Delta_G^C(V^{\{0,\dots,d\}})$, hence $D_{K\setminus W}^d \subseteq \Delta_G^C(U^{\{0,\dots,d\}})$. ⊠

To our surprise, we were unable to find a proof in the literature of the fact that locally-compact non-compact Polish groups have free strongly mixing continuous free actions on Polish spaces. In a pair of private emails, Glasner-Weiss suggested that the strengthening in which the underlying space is compact should be a consequence of the generalizations of the results of [Wei84] to locally compact groups, and that a substantially simpler construction should yield the aforementioned result. However, we give an elementary proof by checking that the action of G by left multiplication on the space $\mathcal{F}(G)$ of closed subsets of G is strongly $(< \omega)$-mixing, where $\mathcal{F}(G)$ is equipped with the *Fell topology* generated by the sets $V_K = \{F \in \mathcal{F}(G) \mid F \cap K = \emptyset\}$ and $W_U = \{F \in \mathcal{F}(G) \mid F \cap U \neq \emptyset\}$, where $K \subseteq G$ is compact and $U \subseteq G$ is open. It is well known that $\mathcal{F}(G)$ is a compact Polish space (see, for example, [Kec95, Exercise 12.7]).

Proposition 2.2.5. Suppose that G is a locally compact non-compact Polish group. Then there is a Polish space X for which there is a free continuous action $G \curvearrowright X$ which is strongly $(< \omega)$-mixing.

Proof. While it is well-known that $G \curvearrowright \mathcal{F}(G)$ is continuous, we will provide a proof for the reader's convenience. Towards this end, it is sufficient to show that if $g \in G, F \in \mathcal{F}(G)$, and U_{gF} is an open neighborhood of gF, then there are open neighborhoods $U_g \subseteq G$ of g and $U_F \subseteq \mathcal{F}(G)$ of F for which $U_g U_F \subseteq U_{gF}$. Clearly we can assume that $U_{gF} = V_K$ for some compact $K \subseteq K$ or $U_{gF} = W_U$ for some open set $U \subseteq G$. In the former case, it follows that $F \cap g^{-1}K = \emptyset$, so the local compactness of G ensures that for all $h \in K$ there are a pre-compact open neighborhood $U_{g,h}$ of g and an open neighborhood $V_{g,h}$ of h such that $F \cap \overline{U_{g,h}}^{-1} V_{g,h} = \emptyset$, and the compactness of K yields a finite set $L \subseteq K$ such that $K \subseteq \bigcup_{h \in L} V_{g,h}$, in which case the sets $U_g = \bigcap_{h \in L} U_{g,h}$ and $U_F = V_{\overline{U_g^{-1}}K}$ are as desired. In the latter case, there exists $h \in F$ for which $gh \in U$, so there are open neighborhoods $U_g \subseteq G$ of g and U_h of h such that $U_g U_h \subseteq U$, thus the sets U_g and W_{U_h} are as desired.

Given $d \in \mathbb{Z}^+$ and non-empty open subsets U_i for $i \leq d$ we need to show that $\Delta_G^X(\Pi_{i \leq d} U_i)$ contains D_L^d for some compact set $L \subseteq G$. Assume that $U_i = V_{K_i} \cap \bigcap_{j < n_i} W_{U_{i,j}}$ for $n_i \in \mathbb{Z}^+$ and $i \leq d$ where $K_i \subseteq G$ are compact and $U_{i,j} \subseteq \sim K_i$ are open and pre-compact for all $j < n_i$ and $i \leq d$. Set $L = \bigcup_{i,i' \leq d, j < n_i} K_{i'} U_{i,j}^{-1}$ and suppose that $\mathbf{g} \in D_L^d$. Choose $g_{i,j} \in U_{i,j}$ for $j < n_i$ and $i \leq d$ and set $F = \{\mathbf{g}_i^{-1} g_{i,j} \mid j < n_i, i \leq d\}$. To see that $\mathbf{g}F \in \Pi_{i \leq d} U_i$ note that $\mathbf{g}_i F \cap U_{i,j} \neq \emptyset$ for all $j < n_i, i \leq d$ and the L-discreteness of $\mathbf{g}$ and the fact that K_i is disjoint from $U_{i,j}$ for all $j < n_i$ ensures that $\mathbf{g}_i F \cap K_i = \emptyset$ for all $i \leq d$.

The *free part* of the action $G \curvearrowright \mathcal{F}(G)$ is the set X of $F \in \mathcal{F}(G)$ that are not fixed by any non-identity element of G. The local-compactness and separability of G ensure that X is the intersection of countable many sets of the form $X_K = \{F \in \mathcal{F}(G) \mid \forall g \in K \, gF \neq F\}$, where $K \subseteq G \setminus \{1_G\}$ is compact. As Proposition 2.2.1 ensures that each X_K is open, it follows that X is G_δ and therefore Polish. To see that $G \curvearrowright X$ is the desired action, it only remains to establish that X is comeager. And for this, it is sufficient to show that if $K \subseteq G \setminus \{1_G\}$ is compact, then X_K is dense. To this end, suppose that $U = V_L \cap \bigcap_{i<n} W_{U_i}$ is non-empty, where $L \subseteq G$ is compact and $U_i \subseteq \sim L$ is open for all $i < n$, and fix $g_i \in U_i$ for all $i < n$. As G is locally compact, by passing to open neighborhoods of g_i contained in U_i, we can assume that each of the sets U_i is pre-compact. As G is not compact, there exists $g \in \sim (L \cup \bigcup_{i<n} K^{-1} U_i)$. Then the set $F = \{g\} \cup \{g_i \mid i < n\}$ is in U, and the fact that $F \cap Kg = \emptyset$ ensures that $F \in X_K$. ⊠

2.3 Generic compressibility

Suppose that E is a Borel equivalence relation on X that is *countable*, in the sense that all of its equivalence classes are countable. We say that a function

$\rho\colon E \to (0,\infty)$ is a *cocycle* if $\rho(x,z) = \rho(x,y)\rho(y,z)$ whenever $x \mathrel{E} y \mathrel{E} z$. When $\rho\colon E \to (0,\infty)$ is a Borel cocycle, we say that a Borel measure μ on X is *ρ-invariant* if $\mu(T(B)) = \int_B \rho(T(x),x)\, d\mu(x)$ for all Borel sets $B \subseteq X$ and Borel automorphisms $T\colon X \to X$ such that $\text{graph}T \subseteq E$. We say that ρ is *aperiodic* if $\sum_{y\in[x]_E} \rho(y,x) = \infty$ for all $x \in X$. Here we generalize the following fact to orbit equivalence relations induced by Borel actions of locally compact Polish groups, while simultaneously strengthening comeagerness to a notion under which the recurrence spectrum is invariant.

Theorem 2.3.1 (Kechris-Miller)**.** *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho\colon E \to (0,\infty)$ is an aperiodic Borel cocycle. Then there is an E-invariant comeager Borel set $C \subseteq X$ that is null with respect to every ρ-invariant Borel probability measure.*

A function $\varphi\colon X \to Z$ is *E-invariant* if $\varphi(x) = \varphi(y)$ whenever $x \mathrel{E} y$. The *E-saturation* of a set $Y \subseteq X$ is the set of $x \in X$ for which there exists $y \in Y$ such that $x \mathrel{E} y$. We say that a Borel probability measure μ on X is *E-quasi-invariant* if the E-saturation of every μ-null set $N \subseteq X$ is μ-null. Let $P(X)$ denote the standard Borel space of Borel probability measures on X (see, for example, [Kec95, §17.E]). The *push-forward* of a Borel measure μ on X through a Borel function $\varphi\colon X \to Y$ is the Borel measure $\varphi_*\mu$ on Y given by $(\varphi_*\mu)(B) = \mu(\varphi^{-1}(B))$ for all Borel sets $B \subseteq Y$.

Proposition 2.3.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\varphi\colon X \to P(X)$ is an E-invariant Borel function such that μ is E-quasi-invariant and $\varphi^{-1}(\mu)$ is μ-conull for all $\mu \in \varphi(X)$. Then there is a Borel cocycle $\rho\colon E \to (0,\infty)$ such that μ is ρ-invariant for all $\mu \in \varphi(X)$.*

Proof. By standard change of topology results (see, for example, [Kec95, §13]), we can assume that X is a zero-dimensional Polish space. Fix a compatible complete ultrametric on X. By [FM77, Theorem 1], there is a countable group Γ of Borel automorphisms of X whose induced orbit equivalence relation is E. For all $\gamma \in \Gamma$, define $\rho_\gamma\colon X_\gamma \to (0,\infty)$ by $\rho_\gamma(x) = \lim_{\varepsilon\to 0}((\gamma^{-1})_*\varphi(x))(\mathcal{B}(x,\varepsilon))/\varphi(x)(\mathcal{B}(x,\varepsilon))$, where X_γ is the set of $x \in X$ for which this limit exists and lies in $(0,\infty)$.

Note that if $\gamma \in \Gamma$, $\mu \in \varphi(X)$, and $\psi\colon X \to (0,\infty)$ is a Radon-Nikodým derivative of $(\gamma^{-1})_*\mu$ with respect to μ (see, for example, [Kec95, §17.A]), then the straightforward generalization of the Lebesgue density theorem for Polish ultrametric spaces (see, for example, [Mil08a, Proposition 2.10]) to integrable functions ensures that $\psi(x) = \lim_{\varepsilon\to 0} \int_{\mathcal{B}(x,\varepsilon)} \psi\, d\mu/\mu(\mathcal{B}(x,\varepsilon)) = \rho_\gamma(x)$ for μ-almost all $x \in X$.

It immediately follows that for all $\gamma \in \Gamma$, the complement of X_γ is null with respect to every $\mu \in \varphi(X)$. Moreover, if $B \subseteq X$ is Borel, $\gamma, \delta \in \Gamma$, and $\mu \in \varphi(X)$, then

$$\begin{aligned}(\gamma\delta)_*^{-1}\mu(B) &= \int_{\delta(B)} \rho_\gamma(x)\ d\mu(x) \\ &= \int_B \rho_\gamma(\delta \cdot x)\ d((\delta^{-1})_*\mu)(x) \\ &= \int_B \rho_\gamma(\delta \cdot x)\rho_\delta(x)\ d\mu(x),\end{aligned}$$

so the almost-everywhere uniqueness of Radon-Nikodým derivatives ensures that the set of $x \in X$ for which there exist $\gamma, \delta \in \Gamma$ such that $\rho_{\gamma\delta}(x) \neq \rho_\gamma(\delta \cdot x)\rho_\delta(x)$ is null with respect to every $\mu \in \varphi(X)$.

Let N denote the E-saturation of the union of these sets, and let $\rho\colon E \to (0, \infty)$ be the extension of the constant cocycle on $E \upharpoonright N$ given by $\rho(\gamma \cdot x, x) = \rho_\gamma(x)$ for all $\gamma \in \Gamma$ and $x \in X$. ⊠

As a consequence, we obtain the following.

Theorem 2.3.3. *Suppose that X is a Polish space, E is a Borel equivalence relation on X admitting a Borel complete set $B \subseteq X$ on which E is countable, F is a superequivalence relation of E for which every F-class is G_δ and the F-saturation of every open set is Borel, and $\varphi\colon X \to P(X)$ is an E-invariant Borel function for which every measure $\mu \in \varphi(X)$ has μ-conull φ-preimage and concentrates off of Borel sets on which E is smooth. Then there is an E-invariant Borel set $C \subseteq X$ that is comeager in every F-class, but null with respect to every measure in $\varphi(X)$.*

Proof. By the Lusin-Novikov uniformization theorem (see, for example, [Kec95, Theorem 18.10]), there is a Borel extension $\pi\colon X \to B$ of the identity function on B whose graph is contained in E. Fix a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers whose sum is 1, in addition to a countable group $\{\gamma_n \mid n \in \mathbb{N}\}$ of Borel automorphisms of B whose induced orbit equivalence relation is $E \upharpoonright B$, and define $\psi\colon B \to P(B)$ by $\psi(x) = \sum_{n\in\mathbb{N}}(\gamma_n \circ \pi)_*\varphi(x)/\varepsilon_n$. As each $\nu \in \psi(B)$ is $(E \upharpoonright B)$-quasi-invariant, Proposition 2.3.2 yields a Borel cocycle $\rho\colon E \upharpoonright B \to (0, \infty)$ such that every $\nu \in \psi(B)$ is ρ-invariant.

Given $\nu \in \psi(B)$, fix $x \in B$ for which $\nu = \psi(x)$, set $\mu = \varphi(x)$, and observe that $\nu(\psi^{-1}(\nu)) \geq \mu(\varphi^{-1}(\mu)) = 1$. Moreover, as E is smooth on the periodic part $P = \{x \in B \mid \sum_{y\in[x]_{E\upharpoonright B}} \rho(y, x) < \infty\}$ of ρ (see, for example, [Mil08b, Proposition 2.1.1]), and therefore on its E-saturation, it follows that $[P]_E$ is null with respect to every measure in $\varphi(X)$, thus P is null with respect to every measure in $\psi(B)$.

By the proof of Theorem 2.3.1 (see [KM04, Theorem 13.1]), there is a Borel set $R \subseteq \mathbb{N}^\mathbb{N} \times B$, whose vertical sections are $(E \upharpoonright B)$-invariant and null with respect to every ρ-invariant Borel probability measure, such that every $x \in B$ is contained in comeagerly-many vertical sections of R. It follows that the vertical sections of the set $S = (\mathrm{id} \times \pi)^{-1}(R)$ are E-invariant and null with respect to every measure in $\varphi(X)$, and every $x \in X$ is contained in comeagerly-many vertical sections of S. The Kuratowski-Ulam theorem therefore ensures that for all $x \in X$, comeagerly-many vertical sections of R are comeager in $[x]_F$.

By [Sri79], there is a Borel set $D \subseteq X$ intersecting every F-class in a single point. As the F-saturation of every open set is Borel, the usual proof of the Montgomery-Novikov theorem that the pointclass of Borel sets is closed under category quantifiers (see, for example, [Kec95, Theorem 16.1]) shows that $\{(b, x) \in \mathbb{N}^{\mathbb{N}} \times X \mid R_b \text{ is comeager in } [x]_F\}$ and $\{(b, x) \in \mathbb{N}^{\mathbb{N}} \times X \mid R_b \text{ is non-meager in } [x]_F\}$ are Borel, so [Kec95, Theorem 18.6] yields a Borel function $\beta \colon D \to \mathbb{N}^{\mathbb{N}}$ such that $R_{\beta(x)}$ is comeager in $[x]_F$ for all $x \in D$. Then the set $C = \bigcup_{x \in D} R_{\beta(x)} \cap [x]_F$ is as desired. ⊠

We say that a function $\rho \colon G \times X \to (0, \infty)$ is a *cocycle* if $\rho(gh, x) = \rho(g, h \cdot x)\rho(h, x)$ for all $g, h \in G$ and $x \in X$. When $\rho \colon G \times X \to (0, \infty)$ is a Borel cocycle, we say that a Borel measure μ on X is *ρ-invariant* if $\mu(gB) = \int_B \rho(g, x) \, d\mu(x)$ for all Borel sets $B \subseteq X$ and group elements $g \in G$. The following fact is the desired generalization of Theorem 2.3.1.

Theorem 2.3.4. *Suppose that G is a locally compact Polish group, X is a Polish space, $G \curvearrowright X$ is a continuous action, F is a superequivalence relation of E_G^X for which every F-class is G_δ and the F-saturation of every open set is Borel, and $\rho \colon G \times X \to (0, \infty)$ is a Borel cocycle with the property that every G-orbit is null with respect to every ρ-invariant Borel probability measure. Then there is a G-invariant Borel set $C \subseteq X$ that is comeager in every F-class, but null with respect to every ρ-invariant Borel probability measure.*

Proof. By [Kec92, Theorem 1.1], there is a complete Borel set $B \subseteq X$ on which E_G^X is countable. Fix a ρ-invariant uniform ergodic decomposition $\varphi \colon X \to P(X)$ of $G \curvearrowright X$ (see [GS00, Theorem 5.2]), and appeal to Theorem 2.3.3. ⊠

We next check that the special case of Theorem 2.3.4 for $F = F_G^X$ provides a proper strengthening of Theorem 2.3.1. While this can be seen as a consequence of the Kuratowski-Ulam theorem, we will show that the usual proof of the latter easily adapts to yield a generalization to a natural class of equivalence relations containing F_G^X.

Theorem 2.3.5. *Suppose that X is a second-countable Baire space, E is an equivalence relation on X such that every E-class is a Baire space and the E-saturation of every open subset of X is open, and $B \subseteq X$ has the Baire property. Then:*

(1) $\forall^* x \in X$ *B has the Baire property in* $[x]_E$.

(2) *B is comeager* $\iff \forall^* x \in X$ *B is comeager in* $[x]_E$.

Proof. We begin with a simple observation.

Lemma 2.3.6. Suppose that $U \subseteq X$ is a non-empty open set and $V \subseteq U$ is a dense open set. Then $[V]_E$ is dense in $[U]_E$.

Proof. If $W \subseteq X$ is open and $[V]_E \cap W = \emptyset$, then $V \cap [W]_E = \emptyset$, so the openness of $[W]_E$ ensures that $\overline{V} \cap [W]_E = \emptyset$, thus the density of V implies that $U \cap [W]_E = \emptyset$, hence $[U]_E \cap W = \emptyset$. ⊠

To see the special case of ($\Longrightarrow$) of (2) when $B \subseteq X$ is open, note that if $U \subseteq X$ is non-empty and open, then Lemma 2.3.6 yields that $[B \cap U]_E$ is dense in $[U]_E$, and therefore $\forall^* x \in X$ $(x \in [U]_E \Longrightarrow x \in [B \cap U]_E)$, or equivalently, $\forall^* x \in X$ $(U \cap [x]_E \neq \emptyset \Longrightarrow B \cap U \cap [x]_E \neq \emptyset)$. As X is second countable, it follows that $\forall^* x \in X$ B is dense in $[x]_E$.

To see ($\Longrightarrow$) of (2), suppose that $B \subseteq X$ is comeager, fix dense open sets $B_n \subseteq X$ for which $\bigcap_{n \in \mathbb{N}} B_n \subseteq B$, and appeal to the special case for open sets to obtain that $\forall^* x \in X$ $\bigcap_{n \in \mathbb{N}} B_n$ is comeager in $[x]_E$.

To see (1), fix an open set $U \subseteq X$ for which $B \triangle U$ is meager, and note that $\forall^* x \in X$ $B \triangle U$ is meager in $[x]_E$, by ($\Longrightarrow$) of (2).

To see ($\Longleftarrow$) of (2), suppose that B is not comeager, fix a non-empty open set $V \subseteq X$ in which B is meager, note that $\forall x \in V$ $V \cap [x]_E \neq \emptyset$, and appeal to ($\Longrightarrow$) of (2) to obtain that $\forall^* x \in X$ $B \cap V$ is meager in $[x]_E$, thus $\forall^* x \in V$ B is not comeager in $[x]_E$. ⊠

Finally, we check that no condition on the recurrence spectrum can yield the existence of an invariant Borel probability measure. When B is a subset of X and $\mathcal{H} \subseteq \bigcup \mathcal{P}(X^{\{0,\ldots,d\}})$, we denote $\{H \cap B^{\{0,\ldots,d\}} \mid d \in \mathbb{Z}^+, H \in \mathcal{H} \cap \mathcal{P}(X^{\{0,\ldots,d\}})\}$ by $\mathcal{H} \upharpoonright B$.

Theorem 2.3.7. *Suppose that G is a locally compact Polish group, X is a standard Borel space, $G \curvearrowright X$ is Borel, and $\rho \colon G \times X \to (0, \infty)$ is a Borel cocycle for which every G-orbit is null with respect to every ρ-invariant Borel probability measure. Then there is a G-invariant Borel set $B \subseteq X$ that is null with respect to every ρ-invariant Borel probability measure but for which $(\boldsymbol{\mathcal{F}}, \mathcal{H})$ is in the recurrence spectrum of $G \curvearrowright X$ if and only if $(\boldsymbol{\mathcal{F}}, \mathcal{H} \upharpoonright B)$ is in the recurrence spectrum of $G \curvearrowright B$.*

Proof. We can assume that X is a Polish space and $G \curvearrowright X$ is continuous. By Theorem 2.3.4, there is an E_G^X-invariant Borel set $B \subseteq X$ that is comeager in every F_G^X-class, but null with respect to every ρ-invariant Borel probability measure. Proposition 2.1.5 then ensures that $(\boldsymbol{\mathcal{F}}, \mathcal{H})$ is in the recurrence spectrum of $G \curvearrowright X$ if and only if $(\boldsymbol{\mathcal{F}}, \mathcal{H} \upharpoonright B)$ is in the recurrence spectrum of $G \curvearrowright B$. ⊠

Chapter 3

A generalization of cutting and stacking

3.1 Quotients

Given a topological space X and an equivalence relation E on X, we endow X/E with the topology consisting of all sets $U \subseteq X/E$ for which $\bigcup U$ is an open subset of X. We begin by noting a sufficient condition under which such quotients are Polish spaces.

Proposition 3.1.1. *Suppose that X is a Polish space and E is an equivalence relation on X for which every E-class is closed, E-saturations of open sets are open, and there is a basis of open sets $U \subseteq X$ such that $\overline{[U]_E} \subseteq [\overline{U}]_E$. Then X/E is a Polish space.*

Proof. The fact that every E-class is closed ensures that X/E is T_1, and the fact that X is second countable implies that so too is X/E, for if $(U_n)_{n \in \mathbb{N}}$ is a basis for X, then $([U_n]_E/E)_{n \in \mathbb{N}}$ is a basis for X/E. To see that X/E is regular, note that if $V \subseteq X/E$ is an open neighborhood of $[x]_E$, then there is an open neighborhood $U \subseteq \bigcup V$ of x such that $\overline{U} \subseteq \bigcup V$ and $\overline{[U]_E} \subseteq [\overline{U}]_E$, in which case $\overline{[U]_E/E} = \overline{[U]_E}/E \subseteq [\overline{U}]_E/E \subseteq V$. The Urysohn metrization theorem (see, for example, [Kec95, Theorem 1.1]) therefore ensures that X is metrizable. As the surjection $\pi \colon X \to X/E$ given by $\pi(x) = [x]_E$ is continuous and open, it follows that X/E is Polish (see, for example, [Kec95, Theorem 8.19]). ⊠

In the special case that X is locally compact, so too is the quotient.

Proposition 3.1.2. *Suppose that X is a locally-compact Polish space and E is a closed equivalence relation on X for which E-saturations of open sets are open. Then X/E is a locally-compact Polish space.*

Proof. To see that X/E is Hausdorff, note that if $[x]_E$ and $[y]_E$ are distinct elements of X/E, then there are open neighborhoods $U \subseteq X$ of x and $V \subseteq X$ of y whose product is disjoint from E, in which case $[U]_E/E$ and $[V]_E/E$ are disjoint open neighborhoods of x and y. As the function $\pi\colon X \to X/E$ given by $\pi(x) = [x]_E$ is continuous, it follows that if $U \subseteq X$ is an open set with compact closure, then the set $\pi(\overline{U}) = [\overline{U}]_E/E$ is compact, so $[\overline{U}]_E$ is closed, thus $[U]_E/E$ is an open set with compact closure and $\overline{[U]_E} \subseteq [\overline{U}]_E$, hence X/E is locally compact and Proposition 3.1.1 ensures that it is Polish. ⊠

Suppose that R and S are binary relations on X and Y. A *homomorphism* from R to S is a function $\varphi\colon X \to Y$ for which $(\varphi \times \varphi)(R) \subseteq S$, a *reduction* of R to S is a homomorphism from R to S that is also a homomorphism from $\sim R$ to $\sim S$, an *embedding* of R into S is an injective reduction of R to S, and an *isomorphism* of R with S is a surjective embedding of R into S. Note that if G is a group and $G \curvearrowright X$ is an action by homomorphisms from E to E, then it is an action by isomorphisms of E with E, and we obtain an action $G \curvearrowright X/E$ by setting $g \cdot [x]_E = [g \cdot x]_E$ for all $g \in G$ and $x \in X$.

Proposition 3.1.3. *Suppose that G is a topological group, X is a topological space, E is an equivalence relation on X for which the E-saturation of every open set is open, and $G \curvearrowright X$ is a continuous action by homomorphisms from E to E. Then $G \curvearrowright X/E$ is continuous.*

Proof. Suppose that $g \in G$, $x \in X$, and $W \subseteq X/E$ is an open neighborhood of $g \cdot [x]_E$. Then there are open neighborhoods $U \subseteq G$ of g and $V \subseteq X$ of x such that $UV \subseteq \bigcup W$, in which case U and $[V]_E/E$ are open neighborhoods of g and $[x]_E$ such that $U([V]_E/E) \subseteq W$. ⊠

Suppose that G is a group, X is a set, and E is an equivalence relation on X. A function $\rho\colon E \to G$ is a *cocycle* if $\rho(x,z) = \rho(x,y)\rho(y,z)$ for all $x \mathrel{E} y \mathrel{E} z$. This trivially implies that $\rho(x,x) = 1_G$ for all $x \in X$, thus $\rho(x,y) = \rho(y,x)^{-1}$ for all $x \mathrel{E} y$.

More generally, we say that a function $P\colon E \to \mathcal{P}(G) \setminus \{\emptyset\}$ is a *cocycle* if $P(x,z) = P(x,y)g$ for all $x \mathrel{E} y \mathrel{E} z$ and $g \in P(y,z)$. This trivially implies that $1_G \in P(x,x)$ for all $x \in X$, so $P(x,y) = P(y,x)^{-1}$ for all $x \mathrel{E} y$, thus $P(x,z) = gP(y,z)$ for all $x \mathrel{E} y \mathrel{E} z$ and $g \in P(x,y)$.

Let $\mathcal{S}(G)$ denote the set of all subgroups of G. We say that a function $\mathbf{G}\colon X \to \mathcal{S}(G)$ is *compatible* with a cocycle $\rho\colon E \to G$ if $\mathbf{G}_x\rho(x,y) = \rho(x,y)\mathbf{G}_y$ for all $x \mathrel{E} y$, in which case we define $P\colon E \to \mathcal{P}(G) \setminus \{\emptyset\}$ by setting $P(x,y) = \rho(x,y)\mathbf{G}_y$. Observe that if $x \mathrel{E} y \mathrel{E} z$ and $g \in P(y,z)$, then there exists $h \in \mathbf{G}_z$ for which

$g = \rho(y,z)h$, and it follows that $P(x,z) = \rho(x,z)\mathbf{G}_z = \rho(x,y)\rho(y,z)\mathbf{G}_z h = \rho(x,y)\mathbf{G}_y\rho(y,z)h = P(x,y)g$, thus P is a cocycle.

The *orbit cocycle* on E_G^X associated with an action $G \curvearrowright X$ is given by $P_G^X(x,y) = \{g \in G \mid x = g \cdot y\}$. For each cocycle $P\colon E \to \mathcal{P}(G) \setminus \{\emptyset\}$, define $E_P \subseteq E$ by $x \mathrel{E_P} y \iff 1_G \in P(x,y)$. Suppose now that $E_G^X \subseteq E$ and $P_G^X(x,y) \subseteq P(x,y)$ for all $x \mathrel{E_G^X} y$. If $g \in G$ and $x \mathrel{E} y$, then the facts that $g \in P(g \cdot x, x)$ and $g^{-1} \in P(y, g \cdot y)$ ensure that $P(g \cdot x, g \cdot y) = gP(x,y)g^{-1}$, so $x \mathrel{E_P} y \Longrightarrow g \cdot x \mathrel{E_P} g \cdot y$, thus $G \curvearrowright X$ is an action by homomorphisms from E_P to E_P. The fact that $g^{-1} \in P(y, g \cdot y)$ also implies that $P(x, g \cdot y) = P(x,y)g^{-1}$, so $[x]_{E_P} = g \cdot [y]_{E_P} \iff 1_G \in P(x, g \cdot y) \iff g \in P(x,y)$, thus P factors over E_P to the orbit cocycle of $G \curvearrowright X/E_P$.

Let $G \curvearrowright G \times X$ denote the action given by $g \cdot (h,x) = (gh, x)$, set $I(G) = G \times G$, identify the product of equivalence relations E on X and F on Y with the equivalence relation on $X \times Y$ given by $(x_1, y_1) \ (E \times F) \ (x_2, y_2) \iff (x_1 \ E \ x_2$ and $y_1 \ F \ y_2)$, and let $\overline{P}$ denote the cocycle on $I(G) \times E$ given by $\overline{P}((g,x),(h,y)) = gP(x,y)h^{-1}$. Clearly $E_G^{G \times X} \subseteq I(G) \times E$. Moreover, if $g \in G$ and $(h,x) \in G \times X$, then $\overline{P}(g \cdot (h,x),(h,x)) = ghP(x,x)h^{-1}$, so $g \in \overline{P}(g \cdot (h,x),(h,x))$, thus $P_G^{G \times X}(g \cdot (h,x),(h,x)) \subseteq \overline{P}(g \cdot (h,x),(h,x))$.

Recall that an equivalence relation on a topological space is minimal if its equivalence classes are dense.

Proposition 3.1.4. *Suppose that G is a topological group, X is a topological space, E is a minimal equivalence relation on X for which the E-saturation of every open set is open, and $P\colon E \to \mathcal{P}(G) \setminus \{\emptyset\}$ is a cocycle. Then $G \curvearrowright (G \times X)/E_{\overline{P}}$ is minimal.*

Proof. Suppose that $W \subseteq (G \times X)/E_{\overline{P}}$ is a non-empty G-invariant open set. Then there are non-empty open sets $U \subseteq G$ and $V \subseteq X$ with the property that $U \times V \subseteq \bigcup W$. The fact that $\bigcup W$ is G-invariant then ensures that $G \times V \subseteq \bigcup W$. To see that $G \times X \subseteq \bigcup W$, suppose that $g \in G$ and $x \in X$, fix $y \in V$ such that $x \mathrel{E} y$, fix $h \in gP(x,y)$, and observe that $1_G \in gP(x,y)h^{-1} = \overline{P}((g,x),(h,y))$, so the $E_{\overline{P}}$-invariance of $\bigcup W$ ensures that it contains (g,x). ⊠

Recall that, when Y is a topological space, we use $\mathcal{F}(Y)$ to denote the family of all closed subsets of Y and we equip $\mathcal{F}(Y)$ with the *Fell topology* generated by the sets of the form $\{F \mid F \cap K = \emptyset\}$ and $\{F \mid F \cap U \neq \emptyset\}$, where $K \subseteq Y$ is compact and $U \subseteq Y$ is open. We say that a function $\varphi\colon X \to \mathcal{F}(Y)$ is *upper semi-continuous* if it is continuous with respect to the topology generated by the sets of the former type, and *lower semi-continuous* if it is continuous with respect to the topology generated by the sets of the latter type.

We say that a sequence $(E_n)_{n\in\mathbb{N}}$ of subequivalence relations of E is *exhaustive* if $E = \bigcup_{n\in\mathbb{N}} E_n$.

Proposition 3.1.5. *Suppose that G is a topological group, X is a topological space, E is an equivalence relation on X, and $P\colon E \to \mathcal{F}(G)$ is a cocycle for which there is an exhaustive increasing sequence $(E_n)_{n\in\mathbb{N}}$ of subequivalence relations of E such that E_n-saturations of open sets are open and $P \restriction E_n$ is lower semi-continuous for all $n \in \mathbb{N}$. Then $E_{\overline{P}}$-saturations of open sets are open.*

Proof. Suppose that $U \times V \subseteq G \times X$ is an open rectangle. Given $(g, x) \in [U \times V]_{E_{\overline{P}}}$, fix $(h, y) \in U \times V$ for which $(g, x)\ E_{\overline{P}}\ (h, y)$, as well as $n \in \mathbb{N}$ for which $x\ E_n\ y$, and open neighborhoods $U_g, U_h \subseteq G$ of g and h for which $U_g g^{-1} U_h \subseteq U$. As $g^{-1}h \in P(x, y)$, there is an open neighborhood $V_x \times V_y \subseteq X \times V$ of (x, y) with the property that $g^{-1}U_h \cap P(x', y') \neq \emptyset$ for all $(x', y') \in E_n \cap (V_x \times V_y)$. Define $V'_x = V_x \cap [V_y]_{E_n}$, and note that if $(g', x') \in U_g \times V'_x$, then there exists $y' \in V_y$ for which $x'\ E_n\ y'$, and since $g^{-1}U_h \cap P(x', y') \neq \emptyset$, there exists $h' \in g'P(x', y') \cap U$, so $(g', x')\ E_{\overline{P}}\ (h', y')$, thus $U_g \times V'_x \subseteq [U \times V]_{E_{\overline{P}}}$. ⊠

Recall that an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G is exhaustive if every compact subset of G is contained in some K_n.

Proposition 3.1.6. *Suppose that G is a locally-compact separable group. Then there is an exhaustive increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G.*

Proof. Fix a countable dense set $D \subseteq G$ and a non-empty open set $U \subseteq G$ with compact closure. As $D^{-1}g$ is dense and therefore intersects U for all $g \in G$, it follows that $G = DU$. Fix an enumeration $(g_n)_{n\in\mathbb{N}}$ of D, set $F_n = \{g_m \mid m \leq n\}$ and $K_n = F_n\overline{U}$ for all $n \in \mathbb{N}$, and observe that if $K \subseteq G$ is compact, then the fact that $K \subseteq DU$ yields $n \in \mathbb{N}$ for which $K \subseteq F_n U \subseteq K_n$. ⊠

For each set $K \subseteq G$ and cocycle $P\colon E \to \mathcal{F}(G)$, define $R^X_K = P^{-1}(\{H \subseteq G \mid H \cap K \neq \emptyset\})$. Note that the relations R^X_K associated with an action and its orbit cocycle coincide. We say that P is $(E_n, K_n)_{n\in\mathbb{N}}$*-expansive* if $R^X_{K_n} \subseteq E_n$ for all $n \in \mathbb{N}$.

Proposition 3.1.7. *Suppose that G is a locally-compact group, X is a Polish space, E is an equivalence relation on X, $P\colon E \to \mathcal{F}(G)$ is a cocycle, $(K_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, and there is an exhaustive increasing sequence $(E_n)_{n\in\mathbb{N}}$ of closed subequivalence relations of E such that P is $(E_n, K_n)_{n\in\mathbb{N}}$-expansive and $P \restriction E_n$ is upper semi-continuous for all $n \in \mathbb{N}$. Then $E_{\overline{P}}$ is closed.*

Proof. If $((g, x), (h, y)) \in \sim E_{\overline{P}}$, then $x \mathrel{E} y \implies g^{-1}h \notin P(x, y)$. The fact that every topological group is regular yields an open neighborhood $U_g \times U_h \subseteq G \times G$ of (g, h) for which $\overline{U_g^{-1}U_h}$ is compact and $x \mathrel{E} y \implies P(x, y) \cap \overline{U_g^{-1}U_h} = \emptyset$. Fix $n \in \mathbb{N}$ sufficiently large that $U_g^{-1}U_h \subseteq K_n$, as well as an open neighborhood $V_x \times V_y \subseteq X \times Y$ of (x, y) with the property that $P(x', y') \cap \overline{U_g^{-1}U_h} = \emptyset$ for all $(x', y') \in E_n \cap (V_x \times V_y)$, and observe that $(U_g \times V_x) \times (U_h \times V_y)$ is disjoint from $E_{\overline{P}}$. ⊠

We say that an equivalence relation E on a metric space X is *locally generated by continuous actions of compact Polish groups* if X is the union of E-invariant open sets $U \subseteq X$ for which there are compact Polish groups G and continuous actions $G \curvearrowright U$ such that $E \restriction U = E_G^U$. Note that every such equivalence relation is necessarily closed, for if $(x, y) \in \overline{E}$, then it is the limit of a sequence $(x_n, y_n)_{n \in \mathbb{N}}$ of elements of E, and if $U \subseteq X$ is an E-invariant open neighborhood of x for which there is a compact Polish group G and a continuous action $G \curvearrowright U$ such that $E \restriction U = E_G^U$, then by passing to a terminal subsequence, we can assume that $x_n \in U$ for all $n \in \mathbb{N}$, in which case there is a sequence $(g_n)_{n \in \mathbb{N}}$ of elements of G such that $g_n \cdot x_n = y_n$ for all $n \in \mathbb{N}$, and by passing to infinite subsequences, we can assume that $(g_n)_{n \in \mathbb{N}}$ converges to some $g \in G$, so $g \cdot x = y$, thus $x \mathrel{E} y$.

Proposition 3.1.8. *Suppose that G is a locally-compact Polish group, X is a metric space, E is an equivalence relation on X, $P \colon E \to \mathcal{F}(G)$ is a cocycle, $(K_n)_{n \in \mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, and there is an exhaustive increasing sequence $(E_n)_{n \in \mathbb{N}}$ of subequivalence relations of E such that E_n is locally generated by continuous actions of compact Polish groups, P is $(E_n, K_n)_{n \in \mathbb{N}}$-expansive, and $P \restriction E_n$ is upper semi-continuous for all $n \in \mathbb{N}$. Then $\overline{[R]_{E_{\overline{P}}}} \subseteq [\overline{R}]_{E_{\overline{P}}}$ for all sets $R \subseteq G \times X$ with the property that $\overline{\mathrm{proj}_G(R)}$ is compact.*

Proof. Suppose that $(g, x) \in \overline{[R]_{E_{\overline{P}}}}$, and fix a sequence $(g_n, x_n)_{n \in \mathbb{N}}$ of elements of $[R]_{E_{\overline{P}}}$ for which $(g_n, x_n) \to (g, x)$, as well as a sequence $(h_n, y_n)_{n \in \mathbb{N}}$ of elements of R such that $(g_n, x_n) \mathrel{E_{\overline{P}}} (h_n, y_n)$ for all $n \in \mathbb{N}$. By passing to infinite subsequences, we can assume that $(h_n)_{n \in \mathbb{N}}$ converges to some $h \in G$. As the closure of $\{g_n \mid n \in \mathbb{N}\} \cup \{h_n \mid n \in \mathbb{N}\}$ is compact, so too is the closure of $\{g_n^{-1}h_n \mid n \in \mathbb{N}\}$. Fix $m \in \mathbb{N}$ for which the latter set is contained in K_m. As $g_n^{-1}h_n \in P(x_n, y_n)$ for all $n \in \mathbb{N}$, it follows that $x_n \mathrel{E_m} y_n$ for all $n \in \mathbb{N}$. Fix an E_m-invariant open neighborhood $V \subseteq X$ of x, a compact Polish group K, and a continuous action $K \curvearrowright V$ such that $E_K^V = E_m \restriction V$. By passing to terminal subsequences, we can assume that $x_n \in V$ for all $n \in \mathbb{N}$, so there is a sequence $(k_n)_{n \in \mathbb{N}}$ of elements of K such that $y_n = k_n \cdot x_n$ for all $n \in \mathbb{N}$. By passing to infinite subsequences, we can assume that $(k_n)_{n \in \mathbb{N}}$ converges to some $k \in K$, in which case $(y_n)_{n \in \mathbb{N}}$ converges to the point $y = k \cdot x$, so $x \mathrel{E_m} y$ and $(h_n, y_n) \to (h, y)$, thus $(h, y) \in \overline{R}$. To see that $(g, x)\ E_{\overline{P}}\ (h, y)$, note that if $U \subseteq G$ is an open neighborhood of $g^{-1}h$, then $g_n^{-1}h_n \in U$ for all but finitely many $n \in \mathbb{N}$, so $P(x_n, y_n) \cap U \neq \emptyset$ for all but finitely many $n \in \mathbb{N}$, so the local compactness of G and upper semi-continuity of $P \restriction E_m$ ensure that $g^{-1}h \in P(x, y)$. ⊠

Suppose that $P\colon E \to \mathcal{P}(G)$ and $\Sigma\colon F \to \mathcal{P}(G)$. A *homomorphism* from P to Σ is a homomorphism φ from E to F such that $P(x, y) \subseteq \Sigma(\varphi(x), \varphi(y))$ for all $x\ E\ y$, a *reduction* of P to Σ is a reduction φ of E to F such that $P(x, y) = \Sigma(\varphi(x), \varphi(y))$ for all $x\ E\ y$, and an *embedding* of P into Σ is an injective reduction of P to Σ. Given an action $G \curvearrowright Y$ and a function $\varphi\colon X \to Y$, define $\varphi_G\colon G \times X \to Y$ by $\varphi_G(g, x) = g \cdot \varphi(x)$.

Proposition 3.1.9. *Suppose that G is a group, X and Y are sets, E is an equivalence relation on X, $P\colon E \to \mathcal{P}(G) \setminus \{\emptyset\}$ is a cocycle, and $G \curvearrowright Y$ is an action.*

1. *(1) If $\varphi\colon X \to Y$ is a homomorphism from P to P_G^Y, then $\varphi_G / E_{\overline{P}}$ is a homomorphism from $G \curvearrowright (G \times X)/E_{\overline{P}}$ to $G \curvearrowright Y$.*
2. *(2) If $\varphi\colon X \to Y$ is a reduction of P to P_G^Y, then $\varphi_G / E_{\overline{P}}$ is an embedding of $G \curvearrowright (G \times X)/E_{\overline{P}}$ into $G \curvearrowright Y$.*

Proof. If $\varphi\colon X \to Y$ is a homomorphism from P to P_G^Y, $g, h \in G$, and $w\ E\ x$, then $\overline{P}((g, w), (h, x)) = gP(w, x)h^{-1} \subseteq gP_G^Y(\varphi(w), \varphi(x))h^{-1} = P_G^Y(\varphi_G(g, w), \varphi_G(h, x))$, so φ_G is a homomorphism from $\overline{P}$ to P_G^Y, and therefore factors over $E_{\overline{P}}$ to a homomorphism from $\overline{P}/E_{\overline{P}}$ to P_G^Y, thus to a homomorphism from $G \curvearrowright (G \times X)/E_{\overline{P}}$ to $G \curvearrowright Y$.

Similarly, if $\varphi\colon X \to Y$ is a reduction of P to P_G^Y, $g, h \in G$, and $w\ E\ x$, then $\overline{P}((g, w), (h, x)) = gP(w, x)h^{-1} = gP_G^Y(\varphi(w), \varphi(x))h^{-1} = P_G^Y(\varphi_G(g, w), \varphi_G(h, x))$, so φ_G is a reduction of $\overline{P}$ to P_G^Y, and therefore factors over $E_{\overline{P}}$ to an embedding of $\overline{P}/E_{\overline{P}}$ into P_G^Y, thus to an embedding of $G \curvearrowright (G \times X)/E_{\overline{P}}$ into $G \curvearrowright Y$. ⊠

3.2 Cutting and stacking

For all $n \in \mathbb{N}$, let $\mathbb{E}_{0,n}(\mathbb{N})$ denote the equivalence relation on $\mathbb{N}^{\mathbb{N}}$ given by $a\ \mathbb{E}_{0,n}(\mathbb{N})\ b \iff \forall m \geq n\ a_m = b_m$, and define $\mathbb{E}_0(\mathbb{N}) = \bigcup_{n \in \mathbb{N}} \mathbb{E}_{0,n}(\mathbb{N})$. For all $s \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{<\mathbb{N}}$, define $X_s = \prod_{n < |s|} \{0, \ldots, |s_n|\}$, and for all $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, set $T_{\mathbf{g}} = \bigcup_{n \in \mathbb{N}} X_{\mathbf{g} \restriction n}$ and $X_{\mathbf{g}} = \prod_{n \in \mathbb{N}} \{0, \ldots, |\mathbf{g}_n|\}$, and let $\wp_{\mathbf{g}}$ be the cocycle on $\mathbb{E}_0(\mathbb{N}) \restriction X_{\mathbf{g}}$ given by $\wp_{\mathbf{g}}((0)^n \frown (k) \frown c, (0)^n \frown (0) \frown c) = (\mathbf{g}_n)_k$ for all $n \in \mathbb{N}$, $c \in X_{\mathfrak{s}^{n+1}(\mathbf{g})}$, and $1 \leq k \leq |\mathbf{g}_n|$. We say that a function $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{S}(G)$ is *compatible* with $\mathbf{g}$ if it is compatible with $\wp_{\mathbf{g}}$. For every such $\mathbf{G}$, define $\mathbb{P}_{\mathbf{g},\mathbf{G}} \colon \mathbb{E}_0(\mathbb{N}) \upharpoonright X_{\mathbf{g}} \to G$ by $\mathbb{P}_{\mathbf{g},\mathbf{G}}(c,d) = \wp_{\mathbf{g}}(c,d)\mathbf{G}_d$, and set $E_{\mathbf{g},\mathbf{G}} = E_{\overline{\mathbb{P}_{\mathbf{g},\mathbf{G}}}}$ and $\mathbb{X}_{\mathbf{g},\mathbf{G}} = (G \times X_{\mathbf{g}})/E_{\mathbf{g},\mathbf{G}}$.

In the special case that $\mathbf{G}$ is the function $\mathbf{1_G}$ with constant value $\{1_G\}$, we use $\mathbb{P}_{\mathbf{g}}$, $E_{\mathbf{g}}$, and $\mathbb{X}_{\mathbf{g}}$ to denote $\mathbb{P}_{\mathbf{g},\mathbf{G}}$, $E_{\mathbf{g},\mathbf{G}}$, and $\mathbb{X}_{\mathbf{g},\mathbf{G}}$. When $G = \mathbb{Z}$ and $\forall n \in \mathbb{N} \forall k < |\mathbf{g}_n|\ (\mathbf{g}_n)_{k+1} > (\overline{\mathbf{g}_n})_k + \sum_{m<n} (\mathbf{g}_m)_{|\mathbf{g}_m|}$, it is not difficult to see that $G \curvearrowright \mathbb{X}_{\mathbf{g}}$ is essentially generated by the automorphism obtained via cutting and stacking with stacks of height $|\mathbf{g}_n| + 1$ and $(\mathbf{g}_n)_{k+1} - 1 - (\overline{\mathbf{g}_n})_k - \sum_{m<n} (\mathbf{g}_m)_{|\mathbf{g}_m|}$ insertions between the k^{th} and $(k+1)^{\text{st}}$ levels of the n^{th} stack.

For all $s \in T_{\mathbf{g}}$, define $\mathbf{g}^s = \prod_{n<|s|} (\overline{\mathbf{g}_n})_{s(n)}$.

Proposition 3.2.1. *Suppose that G is a group, $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{S}(G)$ is compatible with $\mathbf{g}$, $n \in \mathbb{N}$, $c \in X_{\$^n(\mathbf{g})}$, and $s,t \in X_{\mathbf{g} \upharpoonright n}$. Then $\mathbb{P}_{\mathbf{g},\mathbf{G}}(s \frown c, t \frown c) = \mathbf{g}^s \mathbf{G}_{(0)^n \frown c} (\mathbf{g}^t)^{-1}$.*

Proof. As

$$\begin{aligned}
\mathbb{P}_{\mathbf{g},\mathbf{G}}(s \frown c, t \frown c) &= \wp_{\mathbf{g}}(s \frown c, t \frown c)\mathbf{G}_{t \frown c} \\
&= \wp_{\mathbf{g}}(s \frown c, (0)^n \frown c)\wp_{\mathbf{g}}((0)^n \frown c, t \frown c)\mathbf{G}_{t \frown c} \\
&= \wp_{\mathbf{g}}(s \frown c, (0)^n \frown c)\mathbf{G}_{(0)^n \frown c}\wp_{\mathbf{g}}((0)^n \frown c, t \frown c),
\end{aligned}$$

it is sufficient to show that $\wp_{\mathbf{g}}(s \frown c, (0)^n \frown c) = \mathbf{g}^s$ for all $n \in \mathbb{N}$, $c \in X_{\$^n(\mathbf{g})}$, and $s \in X_{\mathbf{g} \upharpoonright n}$. But if this holds at n and $c \in X_{\$^{n+1}(\mathbf{g})}$, $k \leq |\mathbf{g}_n|$, and $s \in X_{\mathbf{g} \upharpoonright n}$, then

$$\begin{aligned}
&\wp_{\mathbf{g}}(s \frown (k) \frown c, (0)^n \frown (0) \frown c) \\
&\quad = \wp_{\mathbf{g}}(s \frown (k) \frown c, (0)^n \frown (k) \frown c)\wp_{\mathbf{g}}((0)^n \frown (k) \frown c, (0)^n \frown (0) \frown c) \\
&\quad = \mathbf{g}^s (\overline{\mathbf{g}_n})_k \\
&\quad = \mathbf{g}^{s \frown (k)},
\end{aligned}$$

so it holds at $n+1$. ⊠

Given a binary relation R on X, we say that a sequence $(X_i)_{i \in I}$ of subsets of X is *R-discrete* if every element of $\prod_{i \in I} X_i$ is R-discrete. For all $n \in \mathbb{N}$, define $\mathrm{IP}(\mathbf{g} \upharpoonright n) = \{\mathbf{g}^s \mid s \in X_{\mathbf{g} \upharpoonright n}\}$. We say that $(\mathbf{g}, \mathbf{G})$ is *$(K_n)_{n \in \mathbb{N}}$-expansive* if $\overline{\mathbf{g}_n}\mathbf{G}_{(0)^{n+1} \frown c}$ is $R^G_{\mathrm{IP}(\mathbf{g} \upharpoonright n)^{-1} K_n \mathrm{IP}(\mathbf{g} \upharpoonright n)}$-discrete for all $n \in \mathbb{N}$ and $c \in X_{\$^{n+1}(\mathbf{g})}$. In the special case that $\mathbf{G} = \mathbf{1_G}$, we say that $\mathbf{g}$ is *$(K_n)_{n \in \mathbb{N}}$-expansive.*

Proposition 3.2.2. *Suppose that G is a topological group, $(K_n)_{n \in \mathbb{N}}$ is an increasing sequence of compact subsets of G, $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{S}(G)$ is compatible with $\mathbf{g}$, and $(\mathbf{g}, \mathbf{G})$ is $(K_n)_{n \in \mathbb{N}}$-expansive. Then $\mathbb{P}_{\mathbf{g},\mathbf{G}}$ is $(\mathbb{E}_{0,n}(\mathbb{N}) \upharpoonright X_{\mathbf{g}}, K_n)_{n \in \mathbb{N}}$-expansive.*

Proof. Simply observe that if $n \in \mathbb{N}$, $c \in X_{\mathfrak{s}^{n+1}(\mathbf{g})}$, $j, k \leq |\mathbf{g}_n|$ are distinct, and $s, t \in X_{\mathbf{g} \upharpoonright n}$, then $(\overline{\mathbf{g}_n})_j \mathbf{G}_{(0)^{n+1} \frown c} \cap (\mathbf{g}^s)^{-1} K_n \mathbf{g}^t (\overline{\mathbf{g}_n})_k = \emptyset$, so $\mathbf{g}^s (\overline{\mathbf{g}_n})_j \mathbf{G}_{(0)^{n+1} \frown c} (\overline{\mathbf{g}_n})_k^{-1} (\mathbf{g}^t)^{-1} \cap K_n = \emptyset$, thus Proposition 3.2.1 ensures that $\mathbb{P}_{\mathbf{g},\mathbf{G}}(s \frown (j) \frown c, t \frown (k) \frown c) \cap K_n = \emptyset$. ⊠

We say that an action of a locally compact Polish group is obtained via *expansive cutting and stacking* if it is of the form $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$, where $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ is compatible with $\mathbf{g}$ and continuous, and $(\mathbf{g}, \mathbf{G})$ is $(K_n)_{n \in \mathbb{N}}$-expansive for some exhaustive increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of G.

Proposition 3.2.3. *Suppose that G is a locally-compact Polish group and $G \curvearrowright X$ is obtained via expansive cutting and stacking. Then X is a locally-compact Polish space and $G \curvearrowright X$ is minimal and continuous.*

Proof. Fix an exhaustive increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of G, $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, as well as a continuous function $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ compatible with $\mathbf{g}$ for which $(\mathbf{g}, \mathbf{G})$ is $(K_n)_{n \in \mathbb{N}}$-expansive and $G \curvearrowright X$ is $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$. As $\mathbb{E}_0(\mathbb{N}) \upharpoonright X_{\mathbf{g}}$ is minimal, Proposition 3.1.4 implies that $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is minimal. Proposition 3.2.1 ensures that $\mathbb{P}_{\mathbf{g},\mathbf{G}} \upharpoonright (\mathbb{E}_{0,n}(\mathbb{N}) \upharpoonright X_{\mathbf{g}})$ is continuous for all $n \in \mathbb{N}$. As $(\mathbb{E}_{0,n}(\mathbb{N}) \upharpoonright X_{\mathbf{g}})$-saturations of open sets are open for all $n \in \mathbb{N}$, Proposition 3.1.5 implies that $E_{\mathbf{g},\mathbf{G}}$-saturations of open sets are open, so Proposition 3.1.3 ensures that $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is continuous. Proposition 3.2.2 ensures that $\mathbb{P}_{\mathbf{g},\mathbf{G}}$ is $(\mathbb{E}_{0,n}(\mathbb{N}) \upharpoonright X_{\mathbf{g}}, K_n)_{n \in \mathbb{N}}$-expansive. As $\mathbb{E}_{0,n}(\mathbb{N}) \upharpoonright X_{\mathbf{g}}$ is closed for all $n \in \mathbb{N}$, Proposition 3.1.7 implies that $E_{\mathbf{g},\mathbf{G}}$ is closed. As $G \times X_{\mathbf{g}}$ is a locally-compact Polish space, Proposition 3.1.2 ensures that so too is $\mathbb{X}_{\mathbf{g},\mathbf{G}}$. ⊠

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is given by $RS = \{(x, z) \in X \times Z \mid \exists y \in Y \ x \mathrel{R} y \mathrel{S} z\}$.

Proposition 3.2.4. *Suppose that $G \curvearrowright X$ is a continuous action of a topological group on a topological space, $K, L \subseteq G$ are compact, $R \subseteq X \times X$ is closed, and $(x, y) \in {\sim} R^X_{K^{-1}} R R^X_L$. Then there are open sets $U_K \supseteq K$ and $U_L \supseteq L$ and open neighborhoods V_x and V_y of x and y such that $(V_x \times V_y) \cap R^X_{U_K^{-1}} R R^X_{U_L} = \emptyset$.*

Proof. The fact that R is closed ensures that for all $(g, h) \in K \times L$, there are open neighborhoods $W_{g,h,x}, W_{g,h,y} \subseteq X$ of $g \cdot x$ and $h \cdot y$ such that $R \cap (W_{g,h,x} \times W_{g,h,y}) = \emptyset$. As $G \curvearrowright X$ is continuous, there are open neighborhoods $U_{g,h,x}, U_{g,h,y} \subseteq G$ of g and h and $V_{g,h,x}, V_{g,h,y} \subseteq X$ of x and y such that $U_{g,h,x} V_{g,h,x} \subseteq W_{g,h,x}$ and $U_{g,h,y} V_{g,h,y} \subseteq W_{g,h,y}$. As $K \times L$ is compact, there is a finite set $F \subseteq K \times L$ such that $K \times L \subseteq \bigcup_{(g,h) \in F} U_{g,h,x} \times U_{g,h,y}$. Define $\mathcal{F}' =$

$\{F' \subseteq F \mid L \subseteq \bigcup_{(g,h)\in F'} U_{g,h,y}\}$, and observe that the sets $U_K = \bigcup_{(g,h)\in F} U_{g,h,x}$, $U_L = \bigcap_{F'\in\mathcal{F}'} \bigcup_{(g,h)\in F'} U_{g,h,y}$, $V_x = \bigcap_{(g,h)\in F} V_{g,h,x}$, and $V_y = \bigcap_{(g,h)\in F} V_{g,h,y}$ are as desired. ⊠

A *homomorphism* from $\rho\colon E \to G$ to $\Sigma\colon F \to \mathcal{P}(G)$ is a homomorphism from the function $P\colon E \to \mathcal{P}(G)$ given by $P(w,x) = \{\rho(w,x)\}$ to Σ. Given an equivalence relation E on X and a binary relation R on X, we say that a function $\varphi\colon X_{\mathbf{g}} \to X$ is *doubly* $(R, (K_n)_{n\in\mathbb{N}})$*-expansive* with respect to a cocycle $P\colon E \to \mathcal{P}(G) \setminus \{\emptyset\}$ if it is a homomorphism from $\sim\mathbb{E}_{0,n}(\mathbb{N}) \restriction X_{\mathbf{g}}$ to $\sim R^X_{K_n \mathrm{IP}(\mathbf{g}\restriction n)} R R^X_{\mathrm{IP}(\mathbf{g}\restriction n)^{-1} K_n}$ for all $n \in \mathbb{N}$. When R is equality on X, we say that φ is *doubly* $(K_n)_{n\in\mathbb{N}}$*-expansive.* As before, we say that $(\mathbf{g}, \mathbf{G})$ is *doubly* $(K_n)_{n\in\mathbb{N}}$*-expansive* if $\overline{\mathbf{g}_n}\mathbf{G}_{(0)^{n+1}\frown c}$ is $R^G_{(\mathrm{IP}(\mathbf{g}\restriction n)^{-1} K_n \mathrm{IP}(\mathbf{g}\restriction n))^2}$-discrete for all $n \in \mathbb{N}$ and $c \in X_{\mathbb{s}^{n+1}(\mathbf{g})}$.

Proposition 3.2.5. *Suppose that G is a locally-compact separable group, $(K_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of compact subsets, $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, E is an equivalence relation on a set X, $P\colon E \to \mathcal{P}(G) \setminus \{\emptyset\}$ is a cocycle, and $\varphi\colon X_{\mathbf{g}} \to X$ is a doubly-$(K_n)_{n\in\mathbb{N}}$-expansive homomorphism from $\mathbb{p}_{\mathbf{g}}$ to P. Then the function $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{S}(G)$ given by $\mathbf{G}_c = P(\varphi(c), \varphi(c))$ is compatible with $\mathbf{g}$, $(\mathbf{g}, \mathbf{G})$ is doubly $(K_n)_{n\in\mathbb{N}}$-expansive, and φ is a reduction of $\mathbb{P}_{\mathbf{g},\mathbf{G}}$ to P.*

Proof. To see that $\mathbf{G}$ is compatible with $\mathbf{g}$, note that

$$\begin{aligned} \mathbb{p}_{\mathbf{g}}(c,d)\mathbf{G}_d &= \mathbb{p}_{\mathbf{g}}(c,d) P(\varphi(d), \varphi(d)) \\ &= P(\varphi(c), \varphi(d)) \\ &= P(\varphi(c), \varphi(c)) \mathbb{p}_{\mathbf{g}}(c,d) \\ &= \mathbf{G}_c \mathbb{p}_{\mathbf{g}}(c,d). \end{aligned}$$

To see that $(\mathbf{g}, \mathbf{G})$ is doubly $(K_n)_{n\in\mathbb{N}}$-expansive, suppose that $n \in \mathbb{N}$, $c \in X_{\mathbb{s}^{n+1}(\mathbf{g})}$, $j, k \leq |\mathbf{g}_n|$ are distinct, and $s, t \in X_{\mathbf{g}\restriction n}$. The fact that $P(\varphi(s \frown (j) \frown c), \varphi(t \frown (k) \frown c))$ and $K_n \mathrm{IP}(\mathbf{g} \restriction n) \mathrm{IP}(\mathbf{g} \restriction n)^{-1} K_n$ are disjoint ensures that so too are $P(\varphi(s \frown (j) \frown c), \varphi((0)^{n+1} \frown c))$ and $K_n \mathrm{IP}(\mathbf{g} \restriction n) \mathrm{IP}(\mathbf{g} \restriction n)^{-1} K_n P(\varphi(t \frown (k) \frown c), \varphi((0)^{n+1} \frown c))$. As $\mathbf{g}^r(\overline{\mathbf{g}_n})_i \mathbf{G}_{(0)^{n+1}\frown c} = P(\varphi(r \frown (i) \frown c), \varphi((0)^{n+1} \frown c))$ for all $(r,i) \in \{(s,j),(t,k)\}$, it follows that $\overline{\mathbf{g}_n}\mathbf{G}_{(0)^{n+1}\frown c}$ is $R^G_{(\mathrm{IP}(\mathbf{g}\restriction n)^{-1} K_n \mathrm{IP}(\mathbf{g}\restriction n))^2}$-discrete.

To see that φ is a homomorphism from $\mathbb{P}_{\mathbf{g},\mathbf{G}}$ to P, simply observe that if $n \in \mathbb{N}$, $c \in X_{\mathbb{s}^n(\mathbf{g})}$, and $s, t \in X_{\mathbf{g}\restriction n}$, then $\mathbb{P}_{\mathbf{g},\mathbf{G}}(s \frown c, t \frown c) = \mathbb{p}_{\mathbf{g}}(s \frown c, t \frown c) P(\varphi(t \frown c), \varphi(t \frown c)) = P(\varphi(s \frown c), \varphi(t \frown c))$.

To see that φ is a homomorphism from $\sim\mathbb{E}_0(\mathbb{N}) \restriction X_{\mathbf{g}}$ to $\sim E$, note that if $c, d \in X_{\mathbf{g}}$ are $\mathbb{E}_0(\mathbb{N})$-inequivalent but $\varphi(c) \mathrel{E} \varphi(d)$, then $K_n \cap P(\varphi(c), \varphi(d)) = \emptyset$ for all $n \in \mathbb{N}$, so $\varphi(c)$ and $\varphi(d)$ are E-inequivalent. ⊠

A *homomorphism parameter* for an action $G \curvearrowright X$ of a group by homeomorphisms of a Polish space is a sequence of the form $P = (d_X^P, (\varepsilon_n^P)_{n\in\mathbb{N}}, \mathbf{g}^P, \mathcal{V}^P)$, where d_X^P is a compatible complete metric on X, $(\varepsilon_n^P)_{n\in\mathbb{N}}$ is a sequence of positive real numbers converging to zero, $\mathbf{g}^P \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, and $\mathcal{V}^P$ is a countable basis for X.

A *P-code* is a sequence $\mathbf{V} \in (\mathcal{V}^P)^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, the following hold:

(1) $\forall k \leq |\mathbf{g}_n^P|\ (\overline{\mathbf{g}_n^P})_k \overline{\mathbf{V}_{n+1}} \subseteq \mathbf{V}_n$.

(2) $\forall s \in X_{\mathbf{g}^P \restriction \{0,\ldots,n\}}\ \mathrm{diam}_{d_X^P}((\mathbf{g}^P)^s \mathbf{V}_{n+1}) \leq \varepsilon_n^P$.

Condition (1) yields that if $c \in X_{\mathbf{g}^P}$ and $n \in \mathbb{N}$, then $(\mathbf{g}^P)^{c\restriction(n+1)}\overline{\mathbf{V}_{n+1}} = (\mathbf{g}^P)^{c\restriction n}(\overline{\mathbf{g}_n})_{c(n)}\overline{\mathbf{V}_{n+1}} \subseteq (\mathbf{g}^P)^{c\restriction n}\mathbf{V}_n$, so condition (2) implies that we obtain a continuous function $\varphi^{P,\mathbf{V}} \colon X_{\mathbf{g}^P} \to X$ by letting $\varphi^{P,\mathbf{V}}(c)$ be the unique element of $\bigcap_{n\in\mathbb{N}}(\mathbf{g}^P)^{c\restriction n}\mathbf{V}_n$.

Proposition 3.2.6. *Suppose that $G \curvearrowright X$ is an action of a group by homeomorphisms of a Polish space, P is a homomorphism parameter, and $\mathbf{V} \in (\mathcal{V}^P)^{\mathbb{N}}$ is a P-code. Then $\varphi^{P,\mathbf{V}}$ is a homomorphism from $\mathbb{p}_{\mathbf{g}^P}$ to P_G^X.*

Proof. Simply observe that

$$\begin{aligned}
&\{\mathbb{p}_{\mathbf{g}^P}((0)^n \frown (k) \frown c, (0)^n \frown (0) \frown c) \cdot \varphi^{P,\mathbf{V}}((0)^n \frown (0) \frown c)\} \\
&\quad = \{(\overline{\mathbf{g}_n^P})_k \cdot \varphi^{P,\mathbf{V}}((0)^n \frown (0) \frown c)\} \\
&\quad = \textstyle\bigcap_{m\in\mathbb{N}}(\overline{\mathbf{g}_n^P})_k(\mathbf{g}^P)^{(0)^n\frown(0)\frown c\restriction m}\mathbf{V}_{n+1+m} \\
&\quad = \textstyle\bigcap_{m\in\mathbb{N}}(\mathbf{g}^P)^{(0)^n\frown(k)\frown c\restriction m}\mathbf{V}_{n+1+m} \\
&\quad = \{\varphi^{P,\mathbf{V}}((0)^n \frown (k) \frown c)\}
\end{aligned}$$

for all $n \in \mathbb{N}$, $c \in X_{\mathbb{s}^{n+1}(\mathbf{g}^P)}$, and $k \leq |\mathbf{g}_n^P|$. ⊠

An *embedding parameter* for an action $G \curvearrowright X$ of a σ-compact group by homeomorphisms of a Polish space is a sequence of the form $P = (d_X^P, (\varepsilon_n^P)_{n\in\mathbb{N}}, \mathbf{g}^P, (K_n^P)_{n\in\mathbb{N}}, R^P, \mathcal{V}^P)$ with the property that the sequence $P' = (d_X^P, (\varepsilon_n^P)_{n\in\mathbb{N}}, \mathbf{g}^P, \mathcal{V}^P)$ is a homomorphism parameter, $(K_n^P)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, and R^P is a closed binary relation on X.

A *P-code* is a P'-code $\mathbf{V} \in (\mathcal{V}^P)^{\mathbb{N}}$ such that $\overline{\mathbf{g}_n^P}\mathbf{V}_{n+1}$ is $R_{L_n^P}^X R^P R_{L_n^P}^X$-discrete, where $L_n^P = \mathrm{IP}(\mathbf{g}^P \restriction n)^{-1} K_n^P \mathrm{IP}(\mathbf{g}^P \restriction n)$, for all $n \in \mathbb{N}$.

Proposition 3.2.7. *Suppose that $G \curvearrowright X$ is an action of a σ-compact group by homeomorphisms of a Polish space, P is an embedding parameter, and $\mathbf{V} \in (\mathcal{V}^P)^{\mathbb{N}}$ is a P-code. Then $\varphi^{P,\mathbf{V}}$ is doubly $(R^P, (K_n^P)_{n\in\mathbb{N}})$-expansive.*

Proof. Note that if $n \in \mathbb{N}$, $c, d \in X_{\mathfrak{s}^{n+1}(\mathbf{g}^P)}$, $s, t \in X_{\mathbf{g}^P \upharpoonright n}$, and $j, k \leq |\mathbf{g}_n^P|$ are distinct, then $\varphi^{P,\mathbf{V}}(r \frown (i) \frown c) \in \mathrm{IP}(\mathbf{g}^P \upharpoonright n)(\overline{\mathbf{g}_n^P})_i \mathbf{V}_{n+1}$ for all $i \in \{j, k\}$ and $r \in \{s, t\}$ by Propositions 3.2.1 and 3.2.6, in which case $(\varphi^{P,\mathbf{V}}(s \frown (j) \frown c), \varphi^{P,\mathbf{V}}(t \frown (k) \frown d)) \notin R^X_{K_n^P \mathrm{IP}(\mathbf{g}^P \upharpoonright n)} R^P R^X_{\mathrm{IP}(\mathbf{g}^P \upharpoonright n)^{-1} K_n^P}$, as $\overline{\mathbf{g}_n^P} \mathbf{V}_{n+1}$ is $R^X_{L_n^P} R^P R^X_{L_n^P}$-discrete. ⊠

3.3 Continuous disjoint unions

We associate with each function $\mathbf{g} \colon I \to (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ the set $X_{\mathbf{g}} = \{(i, c) \in I \times \mathbb{N}^{\mathbb{N}} \mid c \in X_{\mathbf{g}(i)}\}$ and the cocycle $\wp_{\mathbf{g}} \colon (= \times \mathbb{E}_0(\mathbb{N})) \upharpoonright X_{\mathbf{g}} \to G$ given by $\wp_{\mathbf{g}}((i, c), (i, d)) = \wp_{\mathbf{g}(i)}(c, d)$. We say that a function $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{S}(G)$ is *compatible* with $\mathbf{g}$ if $\mathbf{G}(i)$ is compatible with $\mathbf{g}(i)$ for all $i \in I$, in which case we define $\mathbb{P}_{\mathbf{g},\mathbf{G}} \colon (= \times \mathbb{E}_0(\mathbb{N})) \upharpoonright X_{\mathbf{g}} \to \mathcal{S}(G)$ by $\mathbb{P}_{\mathbf{g},\mathbf{G}}((i, c), (i, d)) = \mathbb{P}_{\mathbf{g}(i),\mathbf{G}(i)}(c, d)$, and set $E_{\mathbf{g},\mathbf{G}} = E_{\overline{\mathbb{P}_{\mathbf{g},\mathbf{G}}}}$ and $\mathbb{X}_{\mathbf{g},\mathbf{G}} = (G \times X_{\mathbf{g}}) / E_{\mathbf{g},\mathbf{G}}$. We say that $(\mathbf{g}, \mathbf{G})$ is $(K_n)_{n \in \mathbb{N}}$-*expansive* if $(\mathbf{g}(i), \mathbf{G}(i))$ is $(K_n)_{n \in \mathbb{N}}$-expansive for all $i \in I$. We say that an action of a locally compact Polish group is a *continuous disjoint union* of actions obtained via expansive cutting and stacking if it is of the form $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$, where I is a Polish space, $\mathbf{g} \colon I \to (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ is continuous, $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ is both compatible with $\mathbf{G}$ and continuous, and $(\mathbf{g}, \mathbf{G})$ is $(K_n)_{n \in \mathbb{N}}$-expansive for some exhaustive increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of G.

Proposition 3.3.1. *Suppose that G is a locally-compact Polish group and $G \curvearrowright X$ is a continuous disjoint union of actions obtained via expansive cutting and stacking. Then X is Polish and $G \curvearrowright X$ is continuous.*

Proof. Fix a exhaustive increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of G, a Polish space I, a continuous function $\mathbf{g} \colon I \to (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, and a continuous function $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ compatible with $\mathbf{g}$ for which $(\mathbf{g}, \mathbf{G})$ is $(K_n)_{n \in \mathbb{N}}$-expansive and $G \curvearrowright X$ is $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$. Note that $(= \times \mathbb{E}_{0,n}(\mathbb{N})) \upharpoonright X_{\mathbf{g}}$ is locally generated by continuous actions of compact groups, $((= \times \mathbb{E}_{0,n}(\mathbb{N})) \upharpoonright X_{\mathbf{g}})$-saturations of open sets are open, and Proposition 3.2.1 ensures that $\mathbb{P}_{\mathbf{g},\mathbf{G}} \upharpoonright ((= \times \mathbb{E}_{0,n}(\mathbb{N})) \upharpoonright X_{\mathbf{g}})$ is continuous for all $n \in \mathbb{N}$. Proposition 3.1.5 ensures that $E_{\mathbf{g},\mathbf{G}}$-saturations of open sets are open. As Proposition 3.2.2 implies that $\mathbb{P}_{\mathbf{g},\mathbf{G}}$ is $((= \times \mathbb{E}_{0,n}(\mathbb{N})) \upharpoonright X_{\mathbf{g}}, K_n)_{n \in \mathbb{N}}$-expansive, Proposition 3.1.7 yields that $E_{\mathbf{g},\mathbf{G}}$ is closed, thus $\mathbb{X}_{\mathbf{g},\mathbf{G}}$ is Polish by Propositions 3.1.1 and 3.1.8. Proposition 3.1.3 ensures that $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is continuous. ⊠

The *stabilizer function* associated with an action $G \curvearrowright X$ is given by $\mathrm{Stab}(x) = \{g \in G \mid g \cdot x = x\}$ for all $x \in X$.

Proposition 3.3.2. *Suppose that $G \curvearrowright X$ is a continuous action of a topological group on a Hausdorff space. Then the corresponding stabilizer function is upper semicontinuous.*

Proof. If $K \subseteq G$ is compact, then $\mathrm{Stab}^{-1}(\{F \subseteq G \mid F \cap K = \emptyset\}) = \{x \in X \mid \neg x \ R_K^X \ x\}$, and the latter set is open by Proposition 3.2.4. ⊠

A function $\varphi \colon X \to Y$ between topological spaces is *Baire class one* if the preimage of every open subset of Y is F_σ. In the special case that Y is second countable, this is equivalent to the existence of a sequence $(F_n)_{n\in\mathbb{N}}$ of closed subsets of X for which the preimage of every open subset of Y is a union of sets along $(F_n)_{n\in\mathbb{N}}$.

Proposition 3.3.3. *Suppose that X is a topological space, Y is a locally-compact regular second-countable space, and $\varphi \colon X \to \mathcal{F}(Y)$ is upper semicontinuous. Then φ is Baire class one.*

Proof. If $U \subseteq Y$ is open, then there are compact sets $K_n \subseteq Y$ with the property that $U = \bigcup_{n\in\mathbb{N}} K_n$, so $\varphi^{-1}(\{F \subseteq Y \mid F \cap U \neq \emptyset\}) = \bigcup_{n\in\mathbb{N}} \varphi^{-1}(\{F \subseteq Y \mid F \cap K_n \neq \emptyset\}$ $)$, and the latter set is F_σ. ⊠

A *universal embedding parameter* for a Borel action $G \curvearrowright X$ of a locally-compact Polish group on a Polish space is a sequence of the form $P = (d_G^P, d_X^P, (\varepsilon_n^P)_{n\in\mathbb{N}}, (F_n^P)_{n\in\mathbb{N}}, (K_n^P)_{n\in\mathbb{N}}, R^P, \mathcal{U}^P, \mathcal{V}^P, (W_n^P)_{n\in\mathbb{N}})$ for which there is a Polish topology τ on X such that X and (X, τ) have the same Borel sets, $G \curvearrowright (X, \tau)$ is continuous, d_G^P is a compatible complete metric on G, d_X^P is a compatible complete metric on (X, τ), $(\varepsilon_n^P)_{n\in\mathbb{N}}$ is a sequence of positive real numbers converging to zero, $(F_n^P)_{n\in\mathbb{N}}$ is a sequence of closed subsets of (X, τ) such that the preimage of every open subset of $\mathcal{F}(G)$ under the stabilizer function is a union of sets along $(F_n^P)_{n\in\mathbb{N}}$, $(K_n^P)_{n\in\mathbb{N}}$ is a exhaustive increasing sequence of compact subsets of G containing 1_G, R^P is a closed binary relation on (X, τ), $\mathcal{U}^P$ is a countable basis for G, $\mathcal{V}^P$ is a countable basis for (X, τ), and $(W_n^P)_{n\in\mathbb{N}}$ is a sequence of dense open subsets of (X, τ) for which the topology $\bigcap_{n\in\mathbb{N}} W_n^P$ inherits from (X, τ) is finer than that it inherits from X.

For all $n \in \mathbb{N}$, $d \in (\mathbb{Z}^+)^n$, and $U \in \prod_{m<n} \mathcal{P}(G)^{\{0,\ldots,d_m\}}$, define $\mathrm{IP}(U) = \{U^s \mid s \in \prod_{m<n}\{0, \ldots, d_m\}\}$, where $U^s = \prod_{m<n}(U_m)_{s_m}$ for all $s \in \prod_{m<n}\{0, \ldots, d_m\}$.

A *P-code* is a pair $(\mathbf{U}, \mathbf{V}) \in (\prod_{n\in\mathbb{N}} \prod_{m<n} (\mathcal{U}^P)^{\{0,\ldots,d_m\}}) \times (\mathcal{V}^P)^{\mathbb{N}}$, where $d \in (\mathbb{Z}^+)^{\mathbb{N}}$, such that for all $n \in \mathbb{N}$, the following hold:

(1) $\forall m < n \forall k \leq d_m \ \overline{((\mathbf{U}_{n+1})_m)_k} \subseteq ((\mathbf{U}_n)_m)_k$.

(2) $\forall m \leq n \forall k \leq d_m \ \mathrm{diam}_{d_G^P}(((\mathbf{U}_{n+1})_m)_k) \leq \varepsilon_n^P$.

(3) $\forall s \in \prod_{m \leq n}\{0, \ldots, d_m\}$ $\overline{\mathbf{U}^s_{n+1}\mathbf{V}_{n+1}} \subseteq \mathbf{U}^{s \restriction n}_n \mathbf{V}_n$.

(4) $\forall s \in \prod_{m \leq n}\{0, \ldots, d_m\}$ $\operatorname{diam}_{d^P_X}(\mathbf{U}^s_{n+1}\mathbf{V}_{n+1}) \leq \varepsilon^P_n$.

(5) $\forall s \in \prod_{m \leq n}\{0, \ldots, d_m\}$ $\mathbf{U}^s_{n+1}\mathbf{V}_{n+1} \subseteq W^P_n$.

(6) $\forall s \in \prod_{m \leq n}\{0, \ldots, d_m\} \exists F \in \{F^P_n, \sim F^P_n\}$ $\mathbf{U}^s_{n+1}\mathbf{V}_{n+1} \subseteq F$.

(7) $(((\mathbf{U}_{n+1})_n)_k \mathbf{V}_{n+1})_{k \leq d_n}$ is $R^X_{L^P_n} R^P R^X_{L^P_n}$-discrete, where

$$L^P_n = \mathrm{IP}(\mathbf{U}_{n+1} \restriction n)^{-1} K^P_n \mathrm{IP}(\mathbf{U}_{n+1} \restriction n).$$

(8) $\forall m \leq n$ $1_G \in ((\mathbf{U}_{n+1})_m)_0$.

Let I_P denote the set of all P-codes. Conditions (1) and (2) ensure that we obtain a continuous function $\mathbf{g}^P \colon I_P \to (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ by letting $(\mathbf{g}^P_n(\mathbf{U}, \mathbf{V}))_k$ be the unique element of $\bigcap_{n>m}((\mathbf{U}_n)_m)_k$. Conditions (3) and (4) imply that we obtain a continuous function $\varphi^P \colon X_{\mathbf{g}^P} \to (X, d^P_X)$ by letting $\varphi^P((\mathbf{U}, \mathbf{V}), c)$ be the unique element of $\bigcap_{n \geq m} \mathbf{U}^s_n V_n$. Define $\mathbf{G}^P \colon X_{\mathbf{g}^P} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ by $\mathbf{G}^P = \mathrm{Stab} \circ \varphi^P$.

Proposition 3.3.4. *Suppose that $G \curvearrowright X$ is a Borel action of a locally-compact Polish group on a Polish space, P is a universal embedding parameter, and $(\mathbf{U}, \mathbf{V})$ is a P-code. Then $\varphi^P \colon X_{\mathbf{g}^P} \to X$ and $\mathbf{G}^P$ are continuous.*

Proof. As condition (5) ensures that $\varphi^P(X_{\mathbf{g}^P}) \subseteq \bigcap_{n \in \mathbb{N}} W^P_n$, it follows that $\varphi^P \colon X_{\mathbf{g}^P} \to X$ is continuous.

To see that $\mathbf{G}^P$ is continuous, note that if $((\mathbf{U}, \mathbf{V}), c) \in X_{\mathbf{g}^P}$ and $U \subseteq \mathcal{F}(G)$ is an open neighborhood of the stabilizer of $\varphi^P((\mathbf{U}, \mathbf{V}), c)$, then there exists $n \in \mathbb{N}$ with the property that $\varphi^P((\mathbf{U}, \mathbf{V}), c) \in F^P_n$ and $\mathrm{Stab}(F^P_n) \subseteq U$, so condition (6) ensures that $\mathbf{U}^{c \restriction (n+1)}_{n+1} \mathbf{V}_{n+1} \subseteq F^P_n$, thus $\mathbf{G}^P((\mathcal{N}_{\mathbf{U} \restriction (n+2)} \times \mathcal{N}_{\mathbf{V} \restriction (n+2)}) \times \mathcal{N}_{c \restriction (n+1)}) \subseteq U$. ⊠

Chapter 4

Transience

4.1 Basis theorems

Given a sequence $d \in \mathbb{N}^\mathbb{N}$, we say that a set $T \subseteq \bigcup_{n\in\mathbb{N}} \prod_{m<n}\{0,\ldots,d_m\}$ is *dense* if for all $s \in \bigcup_{n\in\mathbb{N}} \prod_{m<n}\{0,\ldots,d_m\}$, there exists $t \in T$ such that $s \sqsubseteq t$. Given a set $\mathcal{S} \subseteq \bigcup_{d\in\mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$, we say that a sequence $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^\mathbb{N}$ is $\mathcal{S}$*-dense* if for all $S \in \mathcal{S}$, there are densely-many $g \in G$ such that there are $\sqsubseteq$-densely-many $t \in T_\mathbf{g}$ for which $g\mathbf{g}^t\mathbf{g}_{|t|}(g\mathbf{g}^t)^{-1} \in S$.

Proposition 4.1.1. *Suppose that G is a topological group, $(K_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, $\mathcal{S} \subseteq \bigcup_{d\in\mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$, $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^\mathbb{N}$ is $\mathcal{S}$-dense, $\mathbf{G}\colon X_\mathbf{g} \to \mathcal{S}(G)$ is compatible with $\mathbf{g}$, and $(\mathbf{g},\mathbf{G})$ is $(K_n)_{n\in\mathbb{N}}$-expansive. Then $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is expansively $\mathcal{S}$-recurrent.*

Proof. Suppose that $d \in \mathbb{Z}^+$, $K \subseteq G$ is compact, $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots d\}})$, and $V \subseteq \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is a non-empty open set. Fix $s \in T_\mathbf{g}$ and a non-empty open set $U \subseteq G$ for which $U \times \mathcal{N}_s \subseteq \bigcup V$. As $\mathbf{g}$ is $\mathcal{S}$-dense, there exist $g \in U$, $n \in \mathbb{N}$, and $t \in X_{\mathbf{g}\restriction n}$ for which $g^{-1}Kg \subseteq K_n$, $s \sqsubseteq t$, and $g\mathbf{g}^t\mathbf{g}_n(g\mathbf{g}^t)^{-1} \in S$. Fix $c \in X_{\mathfrak{s}^{n+1}(\mathbf{g})}$. Proposition 3.2.1 yields that $\mathbb{P}_{\mathbf{g},\mathbf{G}}((g, t \frown (j) \frown c), (g, t \frown (k) \frown c)) = g\mathbf{g}^t(\overline{\mathbf{g}_n})_j \mathbf{G}_{(0)^{n+1}\frown c}(\overline{\mathbf{g}_n})_k^{-1}(g\mathbf{g}^t)^{-1}$ for all $j, k \leq d$. As $(\mathbf{g},\mathbf{G})$ is $(K_n)_{n\in\mathbb{N}}$-expansive, the latter set is disjoint from K whenever $j \neq k$, so the sequence $x \in V^{\{0,\ldots,d\}}$ given by $x_k = [(g, t \frown (k) \frown c)]_{E_{\mathbf{g},\mathbf{G}}}$ is $R_K^{\mathbb{X}_{\mathbf{g},\mathbf{G}}}$-discrete. But $x_k = g\mathbf{g}^t(\mathbf{g}_n)_k(g\mathbf{g}^t)^{-1} \cdot x_0$ for all $1 \leq k \leq d$, so $\Delta_G^{\mathbb{X}_{\mathbf{g},\mathbf{G}}}(\{y \in V^{\{0,\ldots,d\}} \mid y \text{ is } R_K^{\mathbb{X}_{\mathbf{g},\mathbf{G}}}\text{-discrete}\}) \cap S \neq \emptyset$. ⊠

If $\mathcal{S}$ is conjugation invariant and a continuous action is not σ-expansively $\mathcal{S}$-transient, then it is somewhere expansively $\mathcal{S}$-recurrent:

Proposition 4.1.2. *Suppose that $G \curvearrowright X$ is an action of a group by homeomorphisms of a second-countable topological space whose open subsets are F_σ,*

$\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$*, and* X *is not a union of countably-many expansively-*$(\bigcup_{g \in G} g\mathcal{S}g^{-1})$*-transient closed sets. Then there is a* G*-invariant non-empty closed set* $C \subseteq X$ *such that* $G \curvearrowright C$ *is expansively* $\mathcal{S}$*-recurrent.*

Proof. As X is second countable, there is a maximal open set $V \subseteq X$ contained in a union of countably-many expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient closed sets. To see that the G-invariant non-empty closed set $C = {\sim}V$ is as desired, suppose that $W \subseteq C$ is an expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient open set, and fix an open set $W' \subseteq X$ such that $W = C \cap W'$, as well as closed sets $C_n \subseteq X$ for which $W' = \bigcup_{n \in \mathbb{N}} C_n$. As the sets $C \cap C_n$ are expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient, the maximality of V ensures that it contains W', thus $W = \emptyset$. ⊠

Given a binary relation R on X, we say that a point $x \in X$ is R*-expansively* $\mathcal{S}$*-recurrent* if for all open neighborhoods $V \subseteq X$ of x, $d \in \mathbb{Z}^+$, compact sets $K \subseteq G$, and $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}})$, there exists $g \in S$ such that $x \in \bigcap_{k \le d} (\overline{g}_k)^{-1} V$ and $\overline{g} \cdot x$ is $R^X_{K^{-1}} R R^X_K$-discrete. In the special case that R is equality, we say that x is *expansively* $\mathcal{S}$*-recurrent.*

Proposition 4.1.3. *Suppose that* G *is a locally-compact separable group,* $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$ *is countable, and* $G \curvearrowright X$ *is an expansively* $\mathcal{S}$*-recurrent continuous action on a second-countable topological space. Then there are comeagerly-many expansively* $\mathcal{S}$*-recurrent points.*

Proof. By Proposition 3.1.6, we need only show that if $V \subseteq X$ is a non-empty open set, $d \in \mathbb{Z}^+$, $K \subseteq G$ is compact, and $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}})$, then there exist $g \in S$ and a non-empty open set $W \subseteq \bigcap_{k \le d} (\overline{g}_k)^{-1} V$ for which $\overline{g} W$ is R^X_K-discrete. But this is a straightforward consequence of Proposition 3.2.4. ⊠

Let $\le_{\mathrm{lex}}$ denote the linear ordering of $\mathbb{N}^{<\mathbb{N}}$ given by $s <_{\mathrm{lex}} t \iff (|s| < |t| \text{ or } (|s| = |t| \text{ and } s_{\delta(s,t)} < t_{\delta(s,t)}))$, where $\delta(s,t)$ is the least natural number for which $s_{\delta(s,t)} \neq t_{\delta(s,t)}$, and let $\langle \cdot \rangle \colon 2^{<\mathbb{N}} \to \mathbb{N}$ denote the isomorphism of $\le_{\mathrm{lex}} \restriction 2^{<\mathbb{N}}$ with $\le$. For all $d \in 2^{\mathbb{N}}$, $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, and $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$, define both $\mathbf{g} * d \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ and $\mathbf{G} * d \colon X_{\mathbf{g} * d \restriction n} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ by $(\mathbf{g} * d)_n = \mathbf{g}_{\langle d \restriction n \rangle}$ and $(\mathbf{G} * d)_c = \mathbf{G}_{\varphi_d(c)}$, where $\varphi_d \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is given by $\varphi_d(b) = \bigoplus_{n \in \mathbb{N}} b_n \frown (0)^{\langle b \restriction (n+1) \rangle - \langle b \restriction n \rangle - 1}$.

Proposition 4.1.4. *Suppose that* $G \curvearrowright X$ *is a Borel action of a locally-compact Polish group on a Polish space,* P *is a universal embedding parameter,* $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$ *is countable, and* $G \curvearrowright (X, d^P_X)$ *has comeagerly-many* R^P*-expansively* $\mathcal{S}$*-recurrent points. Then there is a* P*-code* $(\mathbf{U}, \mathbf{V})$ *such that* $\mathbf{g}^P(\mathbf{U}, \mathbf{V}) * d$ *is* $\mathcal{S}$*-dense for all* $d \in 2^{\mathbb{N}}$*.*

Proof. Fix a countable dense set $H \subseteq G$, and define $\mathcal{T} = \bigcup_{h \in H} h^{-1}\mathcal{S}h$.

Lemma 4.1.5. There exists a sequence $\mathbf{T} \in \mathcal{T}^{\mathbb{N}}$ with the property that $\forall d \in 2^{\mathbb{N}} \forall T \in \mathcal{T} \exists^\infty n \in \mathbb{N}\ T = \mathbf{T}_{\langle d \upharpoonright n \rangle}$.

Proof. Fix an enumeration $(T_n)_{n \in \mathbb{N}}$ of $\mathcal{T}$, as well as a sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers such that $\forall k \in \mathbb{N} \exists^\infty n \in \mathbb{N}\ k_n = k$, and define $\mathbf{T}_{\langle s \rangle} = T_{k_{|s|}}$ for all $s \in 2^{<\mathbb{N}}$. ⊠

Lemma 4.1.6. There exists a sequence $\mathbf{s} \in \prod_{n \in \mathbb{N}} \prod_{m<n} \{0, \ldots, d_m\}$ such that $\operatorname{supp}(\mathbf{s}_{\langle t \rangle}) \subseteq \{\langle t \upharpoonright n \rangle \mid n < |t|\}$ for all $t \in 2^{<\mathbb{N}}$, and $\operatorname{supp}(s) \subseteq \{\langle d \upharpoonright n \rangle \mid n \in \mathbb{N}\} \implies \exists n \in \mathbb{N}\ (s \sqsubseteq \mathbf{s}_{\langle d \upharpoonright n \rangle}$ and $T = \mathbf{T}_{\langle d \upharpoonright n \rangle})$ for all $d \in 2^{\mathbb{N}}$, $s \in \bigcup_{n \in \mathbb{N}} \prod_{m<n} \{0, \ldots, d_m\}$, and $T \in \mathcal{T}$.

Proof. Fix natural numbers $d_n > 0$ such that $\mathbf{T}_n \subseteq G^{\{1,\ldots,d_n\}}$ for all $n \in \mathbb{N}$, and recursively define $\mathbf{s}_{\langle t \rangle} = s \frown (0)^{\langle t \rangle - |s|}$, where $t \in 2^{<\mathbb{N}}$ and s is the $\leq_{\text{lex}}$-least element of $\bigcup_{n \leq |t|} \prod_{m<n} \{0, \ldots, d_m\}$ such that $\operatorname{supp}(s) \subseteq \{\langle t \upharpoonright \ell \rangle \mid \ell < |t|\}$ but there does not exist $\ell < |t|$ for which $s \sqsubseteq \mathbf{s}_{\langle t \upharpoonright \ell \rangle}$ and $\mathbf{T}_{\langle t \upharpoonright \ell \rangle} = \mathbf{T}_{\langle t \rangle}$. ⊠

Set $\mathbf{U}_0 = \emptyset$ and fix a non-empty set $\mathbf{V}_0 \in \mathcal{V}^P$. We will recursively find $\mathbf{g}_n \in G^{\{1,\ldots,d_n\}}$, sequences $(((\mathbf{U}_{n+1})_m)_k)_{k \leq d_m, m \leq n}$ of non-empty sets in $\mathcal{U}^P$, and non-empty sets $\mathbf{V}_{n+1} \in \mathcal{V}^P$ such that:

(1) $\forall m < n \forall k \leq d_m\ \overline{((\mathbf{U}_{n+1})_m)_k} \subseteq ((\mathbf{U}_n)_m)_k$.

(2) $\forall m \leq n \forall k \leq d_m\ \operatorname{diam}_{d_G^P}(((\mathbf{U}_{n+1})_m)_k) \leq \varepsilon_n^P$.

(3) $\forall s \in X_{\mathbf{g} \upharpoonright \{0,\ldots,n\}}\ \overline{\mathbf{U}_{n+1}^s \mathbf{V}_{n+1}} \subseteq \mathbf{g}^{s \upharpoonright n} \mathbf{V}_n$.

(4) $\forall s \in X_{\mathbf{g} \upharpoonright \{0,\ldots,n\}}\ \operatorname{diam}_{d_X^P}(\mathbf{U}_{n+1}^s \mathbf{V}_{n+1}) \leq \varepsilon_n^P$.

(5) $\forall s \in X_{\mathbf{g} \upharpoonright \{0,\ldots,n\}}\ \mathbf{U}_{n+1}^s \mathbf{V}_{n+1} \subseteq W_n^P$.

(6) $\forall s \in X_{\mathbf{g} \upharpoonright \{0,\ldots,n\}} \exists F \in \{F_n^P, {\sim} F_n^P\}\ \mathbf{U}_{n+1}^s \mathbf{V}_{n+1} \subseteq F$.

(7) $(((\mathbf{U}_{n+1})_n)_k \mathbf{V}_{n+1})_{k \leq d_n}$ is $R^X_{L_n^P} R^P R^X_{L_n^P}$-discrete.

(8) $\forall m \leq n \forall k \leq d_m\ (\overline{\mathbf{g}_m})_k \in ((\mathbf{U}_{n+1})_m)_k$.

Suppose that $n \in \mathbb{N}$ and we have already found $\mathbf{g} \upharpoonright n$, $\mathbf{U}_n$, and $\mathbf{V}_n$. Fix an R^P-expansively $\mathcal{T}$-recurrent point $y_n \in \mathbf{g}^{\mathbf{s}_n} \mathbf{V}_n$, and define $L'_n = \operatorname{IP}(\mathbf{g} \upharpoonright n)^{-1} K_n^P \operatorname{IP}(\mathbf{g} \upharpoonright n)$. Then there exists $g_n \in \mathbf{T}_n$ for which $y_n \in \bigcap_{k \leq d_n} (\overline{g_n})_k^{-1} \mathbf{g}^{\mathbf{s}_n} \mathbf{V}_n$ and $\overline{g_n} \cdot y_n$ is $R^X_{\mathbf{g}^{\mathbf{s}_n} L'_n} R^P R^X_{L'_n (\mathbf{g}^{\mathbf{s}_n})^{-1}}$-discrete. Set $\mathbf{g}_n = (\mathbf{g}^{\mathbf{s}_n})^{-1} g_n \mathbf{g}^{\mathbf{s}_n}$. Then the point $x_n = (\mathbf{g}^{\mathbf{s}_n})^{-1} \cdot y_n$ is in $\bigcap_{k \leq d_n} (\overline{\mathbf{g}_n})_k^{-1} \mathbf{V}_n$ and $\overline{\mathbf{g}_n} \cdot x_n$ is $R^X_{L'_n} R^P R^X_{L'_n}$-discrete. For all $s \in \prod_{m \leq n} \{0, \ldots, d_m\}$, the regularity of X and the fact that $\mathbf{g}^s \cdot x_n = \mathbf{g}^{s \upharpoonright n} (\overline{\mathbf{g}_n})_{s(n)} \cdot x_n \in \mathbf{g}^{s \upharpoonright n} \mathbf{V}_n$ yield an open neighborhood $W_s \subseteq X$ of $\mathbf{g}^s \cdot x_n$ whose closure is contained in $\mathbf{g}^{s \upharpoonright n} \mathbf{V}_n$ and whose d_X^P-diameter is at most ε_n^P, and the continuity of $G \curvearrowright (X, d_X^P)$

yields open neighborhoods $U_{m,s} \subseteq G$ of $(\overline{\mathbf{g}_m})_{s(m)}$ and an open neighborhood $V_s \subseteq X$ of x_n for which $(\prod_{m \leq n} U_{m,s})V_s \subseteq W_s$, in which case the intersections $((\mathbf{U}_{n+1})_m)_k$ of the sets $U_{m,s}$ where $k = s(m)$ and the intersection $\mathbf{V}_{n+1}$ of the sets V_s satisfy conditions (3) and (4). The regularity of G ensures that we can thin down the sets $((\mathbf{U}_{n+1})_m)_k$ to neighborhoods of $(\overline{\mathbf{g}_m})_k$ satisfying conditions (1) and (2). For all $s, t \in \prod_{m<n}\{0, \dots, d_m\}$ and $s', t' \in \prod_{m \leq n}\{0, \dots, d_m\}$ such that $s'(n) \neq t'(n)$, Proposition 3.2.4 yields open neighborhoods $(U_{s,s',t,t'})_m \subseteq G$ of $(\overline{\mathbf{g}_m})_{s(m)}$ and $(V_{s,s',t,t'})_m \subseteq G$ of $(\overline{\mathbf{g}_m})_{t(m)}$ for all $m < n$, $(U'_{s,s',t,t'})_m \subseteq G$ of $(\overline{\mathbf{g}_m})_{s'(m)}$ and $(V'_{s,s',t,t'})_m \subseteq G$ of $(\overline{\mathbf{g}_m})_{t'(m)}$ for all $m \leq n$, and $W_{s,s',t,t'} \subseteq X$ of x_n with the property that the product of $(\prod_{m<n}(U_{s,s',t,t'})_m)^{-1} K_n^P (\prod_{m<n}(U'_{s,s',t,t'})_m) W_{s,s',t,t'}$ with $(\prod_{m<n}(V_{s,s',t,t'})_m)^{-1} K_n^P (\prod_{m<n}(V'_{s,s',t,t'})_m) W_{s,s',t,t'}$ is disjoint from R^P, so we obtain sets satisfying condition (7) by replacing $((\mathbf{U}_{n+1})_m)_k$ with its intersection with the sets $(U_{s,s',t,t'})_m$ where $k = s(m)$, $(U'_{s,s',t,t'})_m$ where $k = s'(m)$, $(V_{s,s',t,t'})_m$ where $k = t(m)$, and $(V'_{s,s',t,t'})_m$ where $k = t'(m)$, and $\mathbf{V}_{n+1}$ with its intersection with the sets $W_{s,s',t,t'}$. As the intersection W_n of the sets $(\mathbf{g}^s)^{-1} W_n^P$ for $s \in \prod_{m \leq n}\{0, \dots, \mathbf{d}_m\}$ is dense, there exists $x'_n \in \mathbf{V}_{n+1} \cap W_n$. For all $s \in \prod_{m \leq n}\{0, \dots, d_m\}$, the continuity of $G \curvearrowright X$ yields open neighborhoods $U'_{m,s} \subseteq G$ of $(\overline{\mathbf{g}_m})_{s(m)}$ and $V'_s \subseteq X$ of x'_n for which $(\prod_{m \leq n} U'_{m,s})V'_s \subseteq W_n^P$, in which case we obtain sets satisfying condition (5) by replacing each $((\mathbf{U}_{n+1})_m)_k$ with its intersection with the sets $U'_{m,s}$ where $k = s(m)$ and $\mathbf{V}_{n+1}$ with its intersection with the sets V'_s. Note that if $s \in X_{\mathbf{g} \upharpoonright \{0,\dots,n\}}$, then there is a non-empty open set $W'_s \subseteq \mathbf{g}^s \mathbf{V}_{n+1}$ contained in F_n^P or $\sim F_n^P$, and the continuity of $G \curvearrowright X$ yields neighborhoods $U''_{m,s} \subseteq ((\mathbf{U}_{n+1})_m)_{s(m)}$ of $(\overline{\mathbf{g}_m})_{s(m)}$ and a non-empty open set $V''_s \subseteq \mathbf{V}_{n+1}$ for which $(\prod_{m \leq n} U''_{m,s})V''_s \subseteq W'_s$, so by replacing $((\mathbf{U}_{n+1})_m)_{s(m)}$ with $U''_{m,s}$ and $\mathbf{V}_{n+1}$ with V''_s, we obtain sets satisfying the instance of condition (6) at s. By recursively applying this observation to each $s \in \prod_{m \leq n}\{0, \dots, d_m\}$, we obtain sets satisfying condition (6). Replacing each of the sets $((\mathbf{U}_{n+1})_m)_k$ with non-empty subsets in $\mathcal{U}^P$ and $\mathbf{V}_{n+1}$ with a non-empty subset in $\mathcal{V}^P$, this completes the construction.

To complete the proof, it only remains to note that $(\mathbf{U}, \mathbf{V})$ is a P-code, $\mathbf{g}^P(\mathbf{U}, \mathbf{V}) * d$ is $\mathcal{S}$-dense for all $d \in 2^{\mathbb{N}}$, and $\mathbf{g} = \mathbf{g}^P(\mathbf{U}, \mathbf{V})$. ⊠

We next characterize σ-expansive $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transience:

Theorem 4.1.7. *Suppose that G is a locally-compact Polish group, I is a finite set, $(X_i)_{i \in I}$ is a sequence of Polish spaces, and $(G \curvearrowright X_i)_{i \in I}$ is a sequence of Borel actions such that* $\mathrm{Stab}(x_i) = \mathrm{Stab}(x_j)$ *for all distinct $i, j \in I$, $x_i \in X_i$, and $x_j \in X_j$, and $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\dots,d\}})$ is a countable non-empty set. Then the following are equivalent:*

(1) *The action $G \curvearrowright \prod_{i \in I} X_i$ is not σ-expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient.*

(2) *There is an expansively $\mathcal{S}$-recurrent action, obtained via expansive cutting and stacking, that admits a Baire-measurable stabilizer-preserving homomorphism to each $G \curvearrowright X_i$.*

(3) *There is an expansively $\mathcal{S}$-recurrent action, obtained via expansive cutting and stacking, that admits a continuous embedding into each $G \curvearrowright X_i$.*

Proof. Clearly (3) $\Longrightarrow$ (2).

To see $\neg(1) \Longrightarrow \neg(2)$, observe that if $G \curvearrowright X$ is a continuous action on a Polish space that admits a Baire-measurable stabilizer-preserving homomorphism to each $G \curvearrowright X_i$, then it admits a Baire-measurable stabilizer-preserving homomorphism to $G \curvearrowright \prod_{i \in I} X_i$, and since pullbacks of expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient sets through stabilizer-preserving homomorphisms are expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient, it follows that if $G \curvearrowright \prod_{i \in I} X_i$ is σ-expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient, then $G \curvearrowright X$ admits an expansively $\mathcal{S}$-transient non-meager Baire-measurable set, in which case Propositions 3.2.4 and 2.1.3 ensure that $G \curvearrowright X$ is not expansively $\mathcal{S}$-recurrent.

To see (1) $\Longrightarrow$ (3), appeal to [BK96, Theorem 5.2.1] to obtain a Polish topology τ_i on each X_i for which X_i and (X_i, τ_i) have the same Borel sets and $G \curvearrowright (X_i, \tau_i)$ is continuous, and set $X = \prod_{i \in I} X_i$ and $\tau = \prod_{i \in I} \tau_i$. By Proposition 4.1.2, there is a G-invariant non-empty closed set $C \subseteq (X, \tau)$ such that $G \curvearrowright (C, \tau)$ is expansively $\mathcal{S}$-recurrent, so Proposition 4.1.3 ensures that $G \curvearrowright (C, \tau)$ has comeagerly-many expansively $\mathcal{S}$-recurrent points. Set $R = \bigcup_{i \in I} \{(x, y) \in X \times X \mid x_i = y_i\}$. As $\mathrm{Stab}(x_i) = \mathrm{Stab}(x_j)$ for all distinct $i, j \in I$, $x_i \in X_i$, and $x_j \in X_j$, it follows that every expansively $\mathcal{S}$-recurrent point is R-expansively $\mathcal{S}$-recurrent. As the "identity" function from (C, τ) to C is Borel, and therefore Baire measurable, there is a comeager subset of (C, τ) on which it is continuous, in which case the topology that the comeager subset inherits from τ is finer than that it inherits from X. In particular, it follows that there is a universal embedding parameter P for $G \curvearrowright C$ such that d_C^P is compatible with (C, τ) and $R^P = R$, in which case Proposition 4.1.4 yields a P-code $(\mathbf{U}, \mathbf{V})$ for which $\mathbf{g}^P(\mathbf{U}, \mathbf{V})$ is $\mathcal{S}$-dense. Proposition 3.3.4 ensures that $\mathbf{G}^P(\mathbf{U}, \mathbf{V})$ is continuous, Proposition 3.2.7 implies that each of the functions $\varphi_i = \mathrm{proj}_{X_i} \circ \varphi^P((\mathbf{U}, \mathbf{V}), \cdot)$ is a doubly-$(K_n^P)_{n \in \mathbb{N}}$-expansive homomorphism from $\mathbb{p}_{\mathbf{g}^P(\mathbf{U}, \mathbf{V})}$ to $P_G^{X_i}$, and Proposition 3.2.5 yields that $\mathbf{G}^P(\mathbf{U}, \mathbf{V})$ is compatible with $\mathbf{g}^P(\mathbf{U}, \mathbf{V})$, $(\mathbf{g}^P(\mathbf{U}, \mathbf{V}), \mathbf{G}^P(\mathbf{U}, \mathbf{V}))$ is $(K_n^P)_{n \in \mathbb{N}}$-expansive, and each φ_i is a reduction of $\mathbb{P}_{\mathbf{g}^P(\mathbf{U}, \mathbf{V}), \mathbf{G}^P(\mathbf{U}, \mathbf{V})}$ to $P_G^{X_i}$. Then $G \curvearrowright \mathbb{X}_{\mathbf{g}^P(\mathbf{U}, \mathbf{V}), \mathbf{G}^P(\mathbf{U}, \mathbf{V})}$ is obtained via expansive cutting and stacking. Proposition 3.1.9 yields that each $(\varphi_i)_G / E_{\mathbf{g}^P(\mathbf{U}, \mathbf{V}), \mathbf{G}^P(\mathbf{U}, \mathbf{V})}$ is an embedding of $G \curvearrowright \mathbb{X}_{\mathbf{g}^P(\mathbf{U}, \mathbf{V}), \mathbf{G}^P(\mathbf{U}, \mathbf{V})}$ into $G \curvearrowright X_i$, and Proposition 4.1.1 implies that $G \curvearrowright \mathbb{X}_{\mathbf{g}^P(\mathbf{U}, \mathbf{V}), \mathbf{G}^P(\mathbf{U}, \mathbf{V})}$ is expansively $\mathcal{S}$-recurrent. ⊠

The *σ-expansive-transience spectrum* of $G \curvearrowright X$ is the family of all countable non-empty sets $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$ for which $G \curvearrowright X$ is σ-expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient.

Theorem 4.1.8. *Suppose that $G \curvearrowright X$ is a Borel (continuous) action of a locally-compact Polish group on a Polish space. Then there is a continuous disjoint union of actions obtained via expansive cutting and stacking that has the same σ-expansive-transience spectrum as $G \curvearrowright X$ and admits a Borel (continuous) stabilizer-preserving homomorphism to $G \curvearrowright X$.*

Proof. By [BK96, Theorem 5.2.1], it is sufficient to establish the parenthetical (continuous) version of the theorem. Towards this end, fix a universal embedding parameter P for $G \curvearrowright X$ such that d_X^P is compatible with X and R^P is equality on X. Proposition 3.3.4 ensures that φ^P and $\mathbf{G}^P$ are continuous, Propositions 3.2.6 and 3.2.7 imply that $\varphi^P((\mathbf{U}, \mathbf{V}), \cdot)$ is a doubly-$(K_n^P)_{n \in \mathbb{N}}$-expansive homomorphism from $\mathbb{p}_{\mathbf{g}^P(\mathbf{U},\mathbf{V})}$ to P_G^X for all P-codes $(\mathbf{U}, \mathbf{V})$, and Proposition 3.2.5 yields that $\mathbf{G}^P$ is compatible with $\mathbf{g}^P$, $(\mathbf{g}^P, \mathbf{G}^P)$ is $(K_n^P)_{n \in \mathbb{N}}$-expansive, and $\varphi^P((\mathbf{U}, \mathbf{V}), \cdot)$ is a reduction of $\mathbb{P}_{\mathbf{g}^P(\mathbf{U},\mathbf{V}),\mathbf{G}^P(\mathbf{U},\mathbf{V})}$ to P_G^X for all P-codes $(\mathbf{U}, \mathbf{V})$. It follows that $G \curvearrowright \mathbb{X}_{\mathbf{g}^P,\mathbf{G}^P}$ is a continuous disjoint union of actions obtained via expansive cutting and stacking, and Proposition 3.1.9 implies that $(\varphi_i)_G / E_{\mathbf{g}^P,\mathbf{G}^P}$ is a stabilizer-preserving homomorphism from $G \curvearrowright \mathbb{X}_{\mathbf{g}^P,\mathbf{G}^P}$ to $G \curvearrowright X$. To see that the σ-expansive-transience spectrum of $G \curvearrowright X$ is contained in that of $G \curvearrowright \mathbb{X}_{\mathbf{g}^P,\mathbf{G}^P}$, observe that if $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$ is a countable non-empty set for which $G \curvearrowright X$ is σ-expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient, then the fact that pullbacks of expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient sets through stabilizer-preserving homomorphisms are themselves expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient ensures that $G \curvearrowright \mathbb{X}_{\mathbf{g}^P,\mathbf{G}^P}$ is σ-expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient. To see that the two spectra actually coincide, note that if $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$ is a countable non-empty set for which $G \curvearrowright X$ is not σ-expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient, then Proposition 4.1.4 yields a P-code $(\mathbf{U}, \mathbf{V})$ for which $\mathbf{g}^P(\mathbf{U}, \mathbf{V})$ is $\mathcal{S}$-dense, in which case Propositions 3.2.4, 4.1.1, and 2.1.3 ensure that $G \curvearrowright \mathbb{X}_{\mathbf{g}^P,\mathbf{G}^P}$ is not σ-expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient. ⊠

4.2 Anti-basis theorems

We begin with the following observation:

Proposition 4.2.1. *Suppose that $G \curvearrowright X$ is a continuous action of a locally-compact separable group on a Polish space, $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$ is a countable non-empty set, and $\emptyset \in \mathcal{H} \subseteq \bigcup_{d \in \mathbb{Z}^+} F(X^{\{0,\ldots,d\}})$ is countable. Then the following are equivalent:*

(1) The pair $(\{\mathcal{F}_S \mid S \in \mathcal{S}\}, \mathcal{H})$ *is not in the recurrence spectrum of* $G \curvearrowright X$*.*

(2) There exists $g_n \in G$*,* $S_n \in \mathcal{S}$*, and* $(\mathcal{F}_{g_n S_n g_n^{-1}}, g_n \mathcal{H})$*-transient* Σ^0_2 *sets* B_n *for which* $X = \bigcup_{n \in \mathbb{N}} B_n$*.*

Proof. Note that if $U \subseteq G^{\{1,\dots,d\}} \times X$, $d \in \mathbb{Z}^+$, $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\dots,d\}})$, and $x \in X$, then $U^x \in \mathcal{F}_S \Leftrightarrow x \in \bigcup_{g \in S} U_g$, so $\{\mathcal{F}_S \mid S \in \mathcal{S}\}$ is Σ^0_1-on-open and the claim immediately follows from the proof of Proposition 2.1.6. ⊠

Recall that we use $\forall^* x \in X\ \varphi(x)$ to indicate that $\{x \in X \mid \varphi(x)\}$ is comeager, and $\exists^* x \in X\ \varphi(x)$ to indicate that $\{x \in X \mid \varphi(x)\}$ is non-meager. An *almost stabilizer-preserving-homomorphism* from a continuous action $G \curvearrowright X$ to a Borel action $G \curvearrowright Y$ is a function $\varphi \colon X \to Y$ such that $\mathrm{Stab}(\varphi(x)) \subseteq \mathrm{Stab}(x)$ and $\forall^* g \in G\ g \cdot \varphi(x) = \varphi(g \cdot x)$ for comeagerly many $x \in X$.

Proposition 4.2.2. *Suppose that* G *is a locally-compact Polish group,* X *is a Polish space,* Y *and* Z *are standard Borel spaces,* $G \curvearrowright X$ *is a continuous action,* $G \curvearrowright Y$ *is a Borel action,* $\varphi \colon Y \to Z$ *is a* G*-invariant Borel function, and* R *is the set of* $z \in Z$ *for which there is a Borel almost stabilizer-preserving-homomorphism from* $G \curvearrowright X$ *to* $G \curvearrowright \varphi^{-1}(\{z\})$*. Then* R *is analytic.*

Proof. Fix a compact zero-dimensional Polish topology τ on X whose Borel sets coincide with those of X, and recall that every Borel function $\varphi \colon (X, \tau) \to Y$ is continuous on a comeager set (see, for example, [Kec95, Theorem 8.38]), every continuous function $\varphi \colon (B, \tau) \to Y$ on a G_δ set $B \subseteq (X, \tau)$ is the restriction of a Baire-class-one function on (X, τ) (see, for example, [Kur58, §3.31.6]), and every Baire class one function $\varphi \colon (X, \tau) \to Y$ is a pointwise limit of continuous functions (see, for example, [Kec95, p. 24.10]). It follows that R is the set of $z \in Z$ for which there are continuous functions $\varphi_n \colon (X, \tau) \to Y$ such that:

(1) $\forall^* x \in X\ \varphi(\lim_{n \to \infty} \varphi_n(x)) = z$.

(2) $\forall^* x \in X\ \mathrm{Stab}(\lim_{n \to \infty} \varphi_n(x)) \subseteq \mathrm{Stab}(x)$.

(3) $\forall^* x \in X \forall^* g \in G\ g \cdot \lim_{n \to \infty} \varphi_n(x) = \lim_{n \to \infty} \varphi_n(g \cdot x)$.

As there is a Polish topology on the set of continuous functions from (X, τ) to Y with respect to which the evaluation function $(f, x) \mapsto f(x)$ is Borel (see, for example, [Kec95, Theorem 4.19]) and the pointclass of Borel sets is closed under category quantification (see, for example, [Kec95, Theorem 16.1]), it follows that R is analytic. ⊠

The following observation yields our primary means of producing incompatible actions:

Proposition 4.2.3. *Suppose that G is a locally-compact Polish group, $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ is compatible with $\mathbf{g}$ and continuous, $(K_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G with the property that $(\mathbf{g}, \mathbf{G})$ is doubly $(K_n)_{n\in\mathbb{N}}$-expansive, and $d_0, d_1 \in 2^{\mathbb{N}}$ are distinct. Then no expansively-$\{G\}$-recurrent continuous action $G \curvearrowright X$ on a Polish space admits a Borel almost stabililzer-preserving-homomorphism φ_i to $G \curvearrowright \mathbb{X}_{\mathbf{g}*d_i,\mathbf{G}*d_i}$ for all $i < 2$.*

Proof. Suppose, towards a contradiction, that there are such almost stabilizer-preserving-homomorphisms. Then there is a compact set $L \subseteq G$ with the property that the set $B = \bigcap_{i<2} \varphi_i^{-1}(L_i)$ is non-meager, where $L_i = (L \times X_{\mathbf{g}*d_i}) / E_{\mathbf{g}*d_i,\mathbf{G}*d_i}$ for all $i < 2$. Fix $m \in \mathbb{N}$ sufficiently large that $L \cup L^{-1}L \subseteq K_m$ and $d_0 \restriction \{0,\ldots,m\} \neq d_1 \restriction \{0,\ldots,m\}$, set $K = K_m \mathrm{IP}(\mathbf{g} \restriction \{0,\ldots,\langle(1)^m\rangle\})$, and fix a non-empty open set $V \subseteq X$ in which B is comeager. As $G \curvearrowright X$ is expansively $\{G\}$-recurrent, there exist $g \in G$ and $x \in V \cap g^{-1}V$ for which $\neg x \ R^X_{KK^{-1}} \ g \cdot x$. By Proposition 3.2.4, there are open neighborhoods $U \subseteq G$ of g and $W \subseteq V$ of x for which $UW \subseteq V$ and $R^X_{KK^{-1}} \cap (W \times UW) = \emptyset$. As $\forall h \in U \forall^* y \in W \ h \cdot y \in B$, the Kuratowski-Ulam theorem (see, for example, [Kec95, Theorem 8.41]) ensures that $\forall^* y \in W \forall^* h \in U \ h \cdot y \in B$, so the definition of almost stabilizer-preserving-homomorphism yields $h \in U$ and $y \in B \cap h^{-1}B$ with the property that $\neg y \ R^X_{KK^{-1}} \ h \cdot y$, $\mathrm{Stab}(\varphi_i(y)) \subseteq \mathrm{Stab}(y)$ for all $i < 2$, and $\varphi_i(h \cdot y) = h \cdot \varphi_i(y)$ for all $i < 2$, so $P_G^{\mathbb{X}_{\mathbf{g}*d_i,\mathbf{G}*d_i}}(\varphi_i(h \cdot y), \varphi_i(y)) = h\mathrm{Stab}(\varphi_i(y)) \subseteq h\mathrm{Stab}(y) = P^X_G(h \cdot y, y)$ for all $i < 2$.

For all $i, j < 2$, fix $g_{i,j} \in L$ and $c_{i,j} \in X_{\mathbf{g}*d_i}$ such that $\psi_i(h^j \cdot y)$ is the $E_{\mathbf{g}*d_i,\mathbf{G}*d_i}$-class of $(g_{i,j}, c_{i,j})$. Note that for all $i < 2$, there exists $m_i \geq m$ for which $c_{i,0}(m_i) \neq c_{i,1}(m_i)$, since otherwise Proposition 3.2.1 ensures that $L\mathrm{IP}(\mathbf{g} \restriction \{0,\ldots,\langle(1)^m\rangle\})\mathrm{IP}(\mathbf{g} \restriction \{0,\ldots,\langle(1)^m\rangle\})^{-1}L^{-1} \cap P^X_G(y, h \cdot y) \neq \emptyset$, contradicting the fact that $\neg y \ R^X_{KK^{-1}} \ h \cdot y$.

For all $i < 2$, let m_i be the maximal natural number with the property that $c_{i,0}(m_i) \neq c_{i,1}(m_i)$, set $c_i = \mathbb{s}^{m_i+1}(c_{i,0}) = \mathbb{s}^{m_i+1}(c_{i,1})$ and $n_i = \langle d_i \restriction \{0,\ldots,m_i\}\rangle$, and fix $i < 2$ with the property that $n_i > n_{1-i}$. As $L\mathrm{IP}(\mathbf{g} \restriction \{0,\ldots,n_{1-i}\})\mathrm{IP}(\mathbf{g} \restriction \{0,\ldots,n_{1-i}\})^{-1}L^{-1} \cap P^X_G(y, h \cdot y) \neq \emptyset$ and $P^X_G(y, h \cdot y) \subseteq L\mathrm{IP}(\mathbf{g} \restriction \{0,\ldots,n_i - 1\})(\overline{\mathbf{g}_{n_i}})_{(c_{i,0})_{n_i}}(\mathbf{G} * d_i)_{(0)^{m_i+1} \frown c_i}(\overline{\mathbf{g}_{n_i}})^{-1}_{(c_{i,1})_{n_i}}\mathrm{IP}(\mathbf{g} \restriction \{0,\ldots,n_i - 1\})^{-1}L^{-1}$ by Proposition 3.2.1, the fact that $n_i \geq \langle d \restriction \{0,\ldots,m\}\rangle \geq m$ contradicts the double $(K_n)_{n\in\mathbb{N}}$-expansivity of $(\mathbf{g}, \mathbf{G})$. ⊠

We now establish our primary anti-basis results:

Theorem 4.2.4. *Suppose that G is a locally-compact Polish group, $\mathcal{S} \subseteq \bigcup_{d\in\mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$ is a non-empty countable set, and $G \curvearrowright X$ is a non-σ-expansively-$\mathcal{S}$-transient Borel action on a standard Borel space. Then there is a family $\mathcal{B}$*

of continuum-many G-invariant Borel subsets of X on which $G \curvearrowright X$ is not σ-expansively-$\mathcal{S}$-transient such that every non-σ-expansively-$\{G\}$-transient Borel action on a standard Borel space admits a Borel stabilizer-preserving homomorphism to at most one action of the form $G \curvearrowright B$, where $B \in \mathcal{B}$.

Proof. Fix an exhaustive increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G. By Proposition 4.1.4 and Theorem 4.1.7, we can assume that $G \curvearrowright X$ is of the form $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$, where $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ and $\mathbf{g} * d$ is $\mathcal{S}$-dense for all $d \in 2^{\mathbb{N}}$, $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ is compatible with $\mathbf{g}$ and continuous, and $(\mathbf{g}, \mathbf{G})$ is doubly $(K_n)_{n\in\mathbb{N}}$-expansive. Proposition 4.2.3 then ensures that the family $\mathcal{B} = \{(G \times \varphi_d(X_{\mathbf{g}*d}))/E_{\mathbf{g},\mathbf{G}} \mid d \in 2^{\mathbb{N}}\}$ is as desired. ⊠

Theorem 4.2.5. *Suppose that G is a locally-compact Polish group, $G \curvearrowright X$ is a Borel action on a standard Borel space, and $\mathcal{O}$ is a countable family of non-σ-expansively-$\{G\}$-transient Borel actions on standard Borel spaces. Then there is a Borel G-action on a standard Borel space that admits a Borel stabilizer-preserving homomorphism to $G \curvearrowright X$ and has the same σ-expansive-transience spectrum as $G \curvearrowright X$, but to which no action in $\mathcal{O}$ admits a Borel almost stabilizer-preserving-homomorphism.*

Proof. By Proposition 4.2.1, we can assume that each action in $\mathcal{O}$ is continuous and minimal. Fix a universal embedding parameter P, and let R be the set of pairs $((\mathbf{U}, \mathbf{V}), d) \in I_P \times 2^{\mathbb{N}}$ with the property that no action in $\mathcal{O}$ admits a Borel almost stabilizer-preserving-homomorphism to $G \curvearrowright \mathbb{X}_{\mathbf{g}^P(\mathbf{U},\mathbf{V})*d,\mathbf{G}^P(\mathbf{U},\mathbf{V})*d}$.

Proposition 4.2.2 ensures that R is co-analytic, whereas Proposition 4.2.3 implies that every vertical section of R is co-countable. The usual uniformization results for co-analytic sets with large vertical sections (see, for example, [Kec95, Corollary 36.24]) therefore yield a Borel uniformization $\delta\colon I_P \to 2^{\mathbb{N}}$ of R. Define $\mathbf{g}\colon I_P \to (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ and $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ by $\mathbf{g}(\mathbf{U}, \mathbf{V}) = \mathbf{g}^P(\mathbf{U}, \mathbf{V}) * \delta(\mathbf{U}, \mathbf{V})$ and $\mathbf{G}(\mathbf{U}, \mathbf{V}) = \mathbf{G}^P(\mathbf{U}, \mathbf{V}) * \delta(\mathbf{U}, \mathbf{V})$.

The usual change-of-topology results (see, for example, [Kec95, §13]) and Proposition 3.3.1 ensure that $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is a Borel action on a standard Borel space. Note that if $\varphi\colon X_{\mathbf{g}*d} \to X$ is given by $\varphi((\mathbf{U}, \mathbf{V}), c) = (\varphi_P \circ \varphi_{\delta(\mathbf{U},\mathbf{V})})(c)$, then $\overline{\varphi}/E_{\mathbf{g},\mathbf{G}}$ is a stabilizer-preserving Borel homomorphism from $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ to $G \curvearrowright X$.

To see that the σ-expansive-transience spectrum of $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is contained in that of $G \curvearrowright X$, note that if $\mathcal{S} \subseteq \bigcup_{d\in\mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$ is a countable non-empty set for which $G \curvearrowright X$ is not σ-expansively $\mathcal{S}$-transient, then Proposition 4.1.4 yields $(\mathbf{U}, \mathbf{V}) \in I_P$ for which $\mathbf{g}(\mathbf{U}, \mathbf{V})$ is $\mathcal{S}$-dense, so Proposition 4.1.1 ensures that $G \curvearrowright \mathbb{X}_{\mathbf{g}(\mathbf{U},\mathbf{V}),\mathbf{G}(\mathbf{U},\mathbf{V})}$ is expansively $\mathcal{S}$-recurrent, thus Proposition 2.1.3 implies that $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is not σ-expansively $\mathcal{S}$-transient.

To see that none of the actions $G \curvearrowright Y$ in $\mathcal{O}$ admit a Borel almost-stabilizer-preserving-homomorphism φ to $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$, note that the minimality of $G \curvearrowright Y$ would otherwise yield $(\mathbf{U}, \mathbf{V}) \in I_P$ with the property that $\varphi^{-1}(\mathbb{X}_{\mathbf{g}(\mathbf{U},\mathbf{V}),\mathbf{G}(\mathbf{U},\mathbf{V})})$ is comeager, contradicting the fact that $((\mathbf{U}, \mathbf{V}), \delta(\mathbf{U}, \mathbf{V})) \in R$. ⊠

Chapter 5

Wandering

5.1 Smoothness

A *transversal* of an action $G \curvearrowright X$ is a set $Y \subseteq X$ containing exactly one point of every orbit. Burgess has shown that a Borel action of a Polish group on a standard Borel space is smooth if and only if it has a Borel transversal [Bur79].

Proposition 5.1.1. *A Borel action $G \curvearrowright X$ of a locally-compact Polish group on a standard Borel space is smooth if and only if it is σ-expansively $\{G\}$-transient.*

Proof. By [BK96, Theorem 5.2.1], we can assume that X is Polish and $G \curvearrowright X$ is continuous. Fix a compatible complete metric d on X.

To see ($\Longrightarrow$), fix a Borel transversal $B \subseteq X$ of $G \curvearrowright X$, and let s be the unique function from X to B whose graph is contained in E_G^X. As the graph of s is Borel, so too is s (see, for example, [Kec95, Theorem 14.12]). It follows that if $K \subseteq G$ is compact, then KB is Borel, for if H is a countable dense subset of K, then $x \in KB \iff x \in Ks(x) \iff \forall \varepsilon > 0 \exists h \in H \ d(x, h \cdot s(x)) < \varepsilon$ for all $x \in X$. But if $(K_n)_{n \in \mathbb{N}}$ is a sequence of compact subsets of G whose union is G, then $(K_n B)_{n \in \mathbb{N}}$ is a sequence of expansively $\{G\}$-transient Borel sets whose union is X.

To see ($\Longleftarrow$), suppose that $(B_n)_{n \in \mathbb{N}}$ is a sequence of expansively $\{G\}$-transient Borel sets whose union is X, and fix compact sets $K_n \subseteq G$ such that $E_G^X \upharpoonright B_n \subseteq R_{K_n}^X$ for all $n \in \mathbb{N}$. Then the uniformization theorem for Borel subsets of the plane with non-meager vertical sections (see, for example, [Kec95, Corollary 18.7]) ensures that the corresponding sets $C_n = \{x \in X \mid \exists^* g \in G \ g \cdot x \in B_n\}$ are Borel and there are Borel functions $\varphi_n \colon C_n \to B_n$ whose graphs are contained in E_G^X. For all $n \in \mathbb{N}$, Proposition 3.2.4 ensures that $E_G^X \upharpoonright B_n$ is closed, which easily implies that $E_G^X \upharpoonright B_n$ is smooth (see, for example, [Kec95, Exercise 18.20]), thus so too is $G \curvearrowright C_n$. As the sets C_n are G-invariant and $X = \bigcup_{n \in \mathbb{N}} C_n$, it follows that $G \curvearrowright X$ is smooth. ⊠

We now establish our strengthening of the Glimm-Effros dichotomy for Borel actions of locally-compact Polish groups on Polish spaces:

Theorem 5.1.2. *Suppose that $G \curvearrowright X$ is a Borel action of a locally-compact Polish group on a Polish space. Then the following are equivalent:*

(1) The action $G \curvearrowright X$ is not smooth.

(2) There is a Baire-measurable stabilizer-preserving homomorphism from a G-action obtained via expansive cutting and stacking to $G \curvearrowright X$.

(3) There is a continuous embedding of a G-action obtained via expansive cutting and stacking into $G \curvearrowright X$.

Proof. As the proof of Proposition 4.1.1 shows that every G-action obtained via expansive cutting and stacking is expansively $\{G\}$-recurrent, the desired result follows from Theorem 4.1.7 and Proposition 5.1.1. ⊠

We now establish our anti-basis theorem for non-smooth Borel actions of locally-compact Polish groups on standard Borel spaces:

Theorem 5.1.3. *Suppose that $G \curvearrowright X$ a non-smooth Borel action of a locally-compact Polish group on a standard Borel space. Then there is a family $\mathcal{B}$ of continuum-many G-invariant Borel subsets of X on which $G \curvearrowright X$ is non-smooth such that every non-smooth Borel G-action on a standard Borel space admits a Borel stabilizer-preserving homomorphism to at most one action of the form $G \curvearrowright B$, where $B \in \mathcal{B}$.*

Proof. Again appealing to the proof of Proposition 4.1.1 to see that every G-action obtained via expansive cutting and stacking is expansively $\{G\}$-recurrent, the desired result follows from Theorem 4.2.4 and Proposition 5.1.1. ⊠

5.2 Containing bases

The following fact is a local refinement of our promised results on the robustness of the property of containing bases and its characterization via diagonal products:

Theorem 5.2.1. *Suppose that $G \curvearrowright X$ and $G \curvearrowright Y$ are Borel free actions of a locally-compact Polish group on Polish spaces. Then the following are equivalent:*

(1) The action $G \curvearrowright X \times Y$ is not smooth.

(2) There is a Baire-measurable stabilizer-preserving homomorphism from a G-action obtained via expansive cutting and stacking to $G \curvearrowright X$ and $G \curvearrowright Y$.

(3) There is a continuous embedding of a G-action obtained via expansive cutting and stacking into $G \curvearrowright X$ and $G \curvearrowright Y$.

Proof. Once more appealing to the proof of Proposition 4.1.1 to see that every action obtained via expansive cutting and stacking is expansively $\{G\}$-recurrent, the desired result follows from Theorem 4.1.7 and Proposition 5.1.1. ⊠

When $\mathbf{g} \in G^{\mathbb{N}}$, we use $X_{\mathbf{g}}$, $E_{\mathbf{g}}$, and $\mathbb{X}_{\mathbf{g}}$ to denote $X_{\mathbf{h}}$, $E_{\mathbf{h}}$, and $\mathbb{X}_{\mathbf{h}}$, where $\mathbf{h} \in (G^{\{1\}})^{\mathbb{N}}$ is given by $(\mathbf{h}_n)_1 = \mathbf{g}_n$ for all $n \in \mathbb{N}$. In light of Theorems 5.1.2 and 5.2.1, the fact that every homomorphism between free actions is stabilizer preserving, and the fact that there is a continuous embedding of $G \curvearrowright \mathbb{X}_{(\mathbf{g}^{s_n})_{n\in\mathbb{N}}}$ into $G \curvearrowright \mathbb{X}_{\mathbf{g}}$ whenever $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, $(k_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, and $s_n \in T_{\mathbf{g}}$ is supported on $[k_n, k_{n+1})$ for all $n \in \mathbb{N}$, the following fact ensures that continuous free actions of locally-compact Polish groups on compact Polish spaces contain bases:

Proposition 5.2.2. *Suppose that G is a locally-compact Polish group, $(K_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, $G \curvearrowright X$ is a continuous action on a compact Polish space, and $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ is $(K_n)_{n\in\mathbb{N}}$-expansive. Then there exist a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers, sequences $s_n \in T_{\mathbf{g}}$ with non-trivial support contained in $[k_n, k_{n+1})$ for all $n \in \mathbb{N}$, and a continuous homomorphism from $G \curvearrowright \mathbb{X}_{(\mathbf{g}^{s_n})_{n\in\mathbb{N}}}$ to $G \curvearrowright X$.*

Proof. The following fact will allow us to mimic the proof of the existence of G-invariant non-empty closed sets on which $G \curvearrowright X$ is minimal.

Lemma 5.2.3. If $x \in X$ and $y \in \bigcap_{n\in\mathbb{N}} \overline{(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x}$, then $\bigcap_{n\in\mathbb{N}} \overline{(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot y} \subseteq \bigcap_{n\in\mathbb{N}} \overline{(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x}$.

Proof. It is sufficient to show that if $z \in \bigcap_{n\in\mathbb{N}} \overline{(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot y}$, $n \in \mathbb{N}$, and $W \subseteq X$ is an open neighborhood of z, then W intersects $(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x$. Fix a sequence $s \in T_{\mathbb{s}^n(\mathbf{g})}$ with non-trivial support for which $\mathbb{s}^n(\mathbf{g})^s \cdot y \in W$. As $G \curvearrowright X$ is an action by homeomorphisms, there is an open neighborhood $V \subseteq X$ of y such that $\mathbb{s}^n(\mathbf{g})^s V \subseteq W$. Fix $t \in X_{\mathbb{s}^{n+|s|}(\mathbf{g})}$ with the property that $\mathbb{s}^{n+|s|}(\mathbf{g})^t \cdot x \in V$, and observe that $\mathbb{s}^n(\mathbf{g})^{s\frown t} \cdot x = \mathbb{s}^n(\mathbf{g})^s \mathbb{s}^{n+|s|}(\mathbf{g})^t \cdot x \in \mathbb{s}^n(\mathbf{g})^s V \subseteq W$ and the $(K_n)_{n\in\mathbb{N}}$-expansivity of $\mathbf{g}$ ensures that $\mathbb{s}^n(\mathbf{g})^{s\frown t} \neq 1_G$. ⊠

By Lemma 5.2.3, there is an ordinal λ for which there is a maximal sequence $(x_\alpha)_{\alpha<\lambda}$ such that $x_\alpha \in \bigcap_{\beta<\alpha} \overline{(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x_\beta}$ but $\overline{(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x_\alpha} \neq \bigcap_{\beta<\alpha} \overline{(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x_\beta}$ for all $\alpha < \lambda$. Fix any point $x \in \bigcap_{\alpha<\lambda} \overline{(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x_\alpha}$, and observe that x is $\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\} \mid n \in \mathbb{N}\}$-recurrent.

Fix a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers converging to zero, as well as a compatible complete metric on X, and set $k_0 = 0$ and $V_0 = X$. We will recursively construct k_{n+1}, s_n, and open neighborhoods V_{n+1} of x. Given $n \in \mathbb{N}$ for which we have already found k_n and V_n, fix a sequence $s_n \in T_{\mathbf{g}}$, whose support is non-empty and contained in $[k_n, \infty)$, for which $\mathbf{g}^{s_n} \cdot x \in V_n$, set $k_{n+1} = |s_n|$, and fix an open neighborhood $V_{n+1} \subseteq X$ of x such that $\overline{V_{n+1}} \subseteq V_n \cap (\mathbf{g}^{s_n})^{-1} V_n$ and $\operatorname{diam}(\mathbf{g}^s V_{n+1}) \leq \varepsilon_n$ for all $s \in T_{\mathbf{g}}$ of length k_{n+1}.

Define a continuous function $\varphi \colon X_{(\mathbf{g}^{s_n})_{n\in\mathbb{N}}} \to X$ by $\varphi(c) =$ the unique element of $\bigcap_{n\in\mathbb{N}} (\mathbf{g}^{s_n})^{c \restriction n}_{n\in\mathbb{N}} V_n$. Then $\varphi_G / E_{(\mathbf{g}^{s_n})_{n\in\mathbb{N}}}$ is a homomorphism from $G \curvearrowright \mathbb{X}_{(\mathbf{g}^{s_n})_{n\in\mathbb{N}}}$ to $G \curvearrowright X$ by the proof of Proposition 3.2.6. ⊠

In light of Theorem 5.2.1, the following fact ensures that Borel-probability-measure-preserving Borel free actions of locally-compact Polish groups on standard Borel spaces contain bases:

Proposition 5.2.4. *Suppose that G is a locally-compact Polish group, X and Y are standard Borel spaces, $G \curvearrowright X$ is a Borel action that is invariant with respect to a Borel probability measure μ on X, and $G \curvearrowright Y$ is a Borel action for which $G \curvearrowright X \times Y$ is free and smooth. Then $G \curvearrowright Y$ is smooth.*

Proof. Fix a Borel transversal $B \subseteq X \times Y$ of $G \curvearrowright X \times Y$, and define $\varphi \colon X \times Y \to G$ by letting $\varphi(x, y)$ be the unique $g \in G$ for which $g \cdot (x, y) \in B$. Let $P(Y)$ denote the standard Borel space of Borel probability measures on Y (see, for example, [Kec95, §17.E]), and define $\nu \colon Y \to P(G)$ by $\nu(y) = \varphi(\cdot, y)_* \mu$. If $H \subseteq G$ and $y \in Y$, then

$$\begin{aligned}\nu(y)(H) &= \mu(\{x \in X \mid \exists h \in H \ h \cdot (x, y) \in B\}) \\ &= \mu(\{x \in X \mid (x, y) \in H^{-1} B\}),\end{aligned}$$

so the G-invariance of μ ensures that if $g \in G$, then

$$\begin{aligned}\nu(g \cdot y)(H) &= \mu(g^{-1} \cdot \{x \in X \mid (x, g \cdot y) \in H^{-1} B\}) \\ &= \mu(\{x \in X \mid (g \cdot x, g \cdot y) \in H^{-1} B\}) \\ &= \mu(\{x \in X \mid (x, y) \in (Hg)^{-1} B\}) \\ &= \nu(y)(Hg).\end{aligned}$$

But if $K \subseteq G$ is compact and $g \notin K^{-1} K$, then $K \cap Kg = \emptyset$, in which case $\{y \in Y \mid \nu(y)(K) > 1/2\}$ is σ-expansively $\{G\}$-transient, thus Theorem 5.1.1 ensures that $G \curvearrowright Y$ is smooth. ⊠

We next characterize expansive $\{G\}$-recurrence of products with free actions obtained via expansive cutting and stacking:

Proposition 5.2.5. *Suppose that G is a locally-compact Polish group, X is a Polish space, $G \curvearrowright X$ is a continuous free action, and $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$. Then $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is expansively $\{G\}$-recurrent $\iff$ $G \curvearrowright X$ is expansively $\{IP(\mathbb{s}^n(\mathbf{g}))IP(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}$-recurrent.*

Proof. To see ($\Longrightarrow$), suppose that $K \subseteq G$ is compact, $n \in \mathbb{N}$, and $V \subseteq X$ is a non-empty open set, and fix an open neighborhood $U \subseteq G$ of 1_G with compact closure and a non-empty open set $V' \subseteq X$ for which $UV' \subseteq V$. As $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is expansively $\{G\}$-recurrent, it follows that $\Delta_G^X(V' \times V') \cap \Delta_G^{\mathbb{X}_{\mathbf{g}}}((U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}} \times (U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}}) \nsubseteq U^{-1}KU$. But $U\Delta_G^X(V' \times V')U^{-1} = \Delta_G^X(UV' \times UV')$ and Proposition 3.2.1 ensures that $\Delta_G^{\mathbb{X}_{\mathbf{g}}}((U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}} \times (U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}}) = U^{-1}\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1}U$, so $\Delta_G^X(V \times V) \cap \mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \nsubseteq K$.

To see ($\Longleftarrow$), suppose that $K \subseteq G$ is compact, $s \in T_{\mathbf{g}}$, $U \subseteq G$ is a non-empty open set with compact closure, and $V \subseteq X$ is a non-empty open set. Then $\Delta_G^X((U\mathbf{g}^s)^{-1}V \times (U\mathbf{g}^s)^{-1}V) \cap \mathrm{IP}(\mathbb{s}^{|s|}(\mathbf{g}))\mathrm{IP}(\mathbb{s}^{|s|}(\mathbf{g}))^{-1} \nsubseteq (U\mathbf{g}^s)^{-1}KU\mathbf{g}^s$ by expansive $\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}$-recurrence. But $\Delta_G^X((U\mathbf{g}^s)^{-1}V \times (U\mathbf{g}^s)^{-1}V) = (U\mathbf{g}^s)^{-1}\Delta_G^X(V \times V)U\mathbf{g}^s$ and $U\mathbf{g}^s\mathrm{IP}(\mathbb{s}^{|s|}(\mathbf{g}))\mathrm{IP}(\mathbb{s}^{|s|}(\mathbf{g}))^{-1}(U\mathbf{g}^s)^{-1} = \Delta_G^{\mathbb{X}_{\mathbf{g}}}((U \times \mathcal{N}_s)/E_{\mathbf{g}} \times (U \times \mathcal{N}_s)/E_{\mathbf{g}})$ by Proposition 3.2.1, so $\Delta_G^X(V \times V) \cap \Delta_G^{\mathbb{X}_{\mathbf{g}}}((U \times \mathcal{N}_s)/E_{\mathbf{g}} \times (U \times \mathcal{N}_s)/E_{\mathbf{g}}) \nsubseteq K$. ⊠

We now establish a local version of the promised characterization of free actions containing bases in the abelian case:

Theorem 5.2.6. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space and $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ is expansive.*

1. *If $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is smooth, then $G \curvearrowright X$ is σ-expansively $(\bigcup_{g \in G} g\{IP(\mathbb{s}^n(\mathbf{g}))IP(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}g^{-1})$-transient.*
2. *If G is abelian, then the converse holds.*

Proof. By [BK96, Theorem 5.2.1], we can assume that X is Polish and $G \curvearrowright X$ is continuous. Proposition 4.2.1 ensures that $G \curvearrowright X$ is σ-expansively $(\bigcup_{g \in G} g\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}g^{-1})$-transient if and only if there does not exist $x \in X$ for which $G \curvearrowright [x]_{F_G^X}$ is expansively $\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}$-recurrent, and Proposition 5.2.5 implies that the latter condition holds if and only if there does not exist $x \in X$ for which $G \curvearrowright [x]_{F_G^X} \times \mathbb{X}_{\mathbf{g}}$ is expansively $\{G\}$-recurrent. So it is enough to prove the analog of the theorem in which the σ-expansive $(\bigcup_{g \in G} g\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}g^{-1})$-transience of $G \curvearrowright X$ is replaced with the condition that there is no $x \in X$ for which $G \curvearrowright [x]_{F_G^X} \times \mathbb{X}_{\mathbf{g}}$ is expansively $\{G\}$-recurrent.

To see the analog of (1), appeal to Proposition 5.1.1 to see that $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is σ-expansively $\{G\}$-transient, in which case Proposition 2.1.3 ensures that there does not exist $x \in X$ for which $G \curvearrowright [x]_{F_G^X} \times \mathbb{X}_{\mathbf{g}}$ is expansively $\{G\}$-recurrent.

To see the analog of (2), note that if $K \subseteq G$ is compact, $V \times W \subseteq X \times \mathbb{X}_{\mathbf{g}}$ is open, and $x \in X$, then the minimality of $G \curvearrowright [x]_{F_G^X}$ ensures that $V \cap [x]_{F_G^X} \neq \emptyset \iff V \cap Gx \neq \emptyset \iff x \in GV$, and the freeness of $G \curvearrowright X$ implies that $E_G^{X \times \mathbb{X}_{\mathbf{g}}} \restriction ((V \cap [x]_{F_G^X}) \times W) \subseteq R_K^{X \times \mathbb{X}_{\mathbf{g}}} \iff V \cap (\Delta_G^{\mathbb{X}_{\mathbf{g}}}(W \times W) \setminus K)^{-1} V \cap [x]_{F_G^X} = \emptyset \iff V \cap (\Delta_G^{\mathbb{X}_{\mathbf{g}}}(W \times W) \setminus K)^{-1} V \cap Gx = \emptyset \iff x \notin G(V \cap (\Delta_G^{\mathbb{X}_{\mathbf{g}}}(W \times W) \setminus K)^{-1} V)$, so the set of $x \in X$ for which $V \cap [x]_{F_G^X}$ is non-empty but $E_G^{X \times \mathbb{X}_{\mathbf{g}}} \restriction ((V \cap [x]_{F_G^X}) \times W) \subseteq R_K^{X \times \mathbb{X}_{\mathbf{g}}}$ is a difference of two G-invariant open sets. Appeal to Proposition 3.1.6 to obtain an exhaustive increasing sequence $(K_m)_{m \in \mathbb{N}}$ of compact subsets of G, fix an enumeration $(V_n \times W_n)_{n \in \mathbb{N}}$ of a basis for $X \times \mathbb{X}_{\mathbf{g}}$, and for all $(m, n) \in \mathbb{N} \times \mathbb{N}$, let $U_{m,n}$ be the set of $x \in X$ for which $V_n \cap [x]_{F_G^X}$ is non-empty but $E_G^{X \times \mathbb{X}_{\mathbf{g}}} \restriction ((V_n \cap [x]_{F_G^X}) \times W_n) \subseteq R_{K_m}^{X \times \mathbb{X}_{\mathbf{g}}}$. Fix a countable dense set $H \subseteq G$. Then the sets of the form $U_{m,n} \cap (gV_n \times hW_n)$, where $g, h \in H$ and $m, n \in \mathbb{N}$, cover $X \times \mathbb{X}_{\mathbf{g}}$, and the fact that G is abelian ensures that they are expansively $\{G\}$-transient, so Proposition 5.1.1 implies that $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is smooth. ⊠

The promised basis theorem easily follows:

Theorem 5.2.7. *Suppose that $G \curvearrowright X$ is a Borel (continuous) free action of an abelian locally-compact Polish group on a Polish space that contains a basis. Then there is a continuous disjoint union of actions obtained via expansive cutting and stacking that contains a basis and admits a Borel (continuous) stabilizer-preserving homomorphism to $G \curvearrowright X$.*

Proof. By Theorems 4.1.8, 5.1.2, and 5.2.6. ⊠

We similarly obtain the promised anti-basis theorem:

Theorem 5.2.8. *Suppose that $G \curvearrowright X$ is a Borel free action of an abelian locally-compact Polish group on a standard Borel space containing a basis, and $\mathcal{O}$ is a countable family of non-smooth Borel actions on standard Borel spaces. Then there is a Borel G-action on a standard Borel space that admits a Borel stabilizer-preserving homomorphism to $G \curvearrowright X$ and contains a basis, but to which no action in $\mathcal{O}$ admits a Borel almost stabilizer-preserving-homomorphism.*

Proof. By Theorems 4.2.5, 5.1.2, and 5.2.6. ⊠

5.3 Complete transient sets

When $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$, we say that a set $Y \subseteq X$ is *$\mathcal{S}$-transient* if there exist $d \in \mathbb{Z}^+$ and $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}})$ with the property that $\Delta_G^X(Y^{\{0,\ldots,d\}}) \cap S = \emptyset$. Note that if $S \subseteq G$, then a set $Y \subseteq X$ is S-wandering if and only if it is $\{SS^{-1} \setminus \{1_G\}\}$-transient. We say that a G-action by homeomorphisms of a topological space is *$\mathcal{S}$-recurrent* if it is $\mathcal{F}_{\mathcal{S}}$-recurrent, and a Borel G-action on a standard Borel space X is *σ-$\mathcal{S}$-transient* if X is a union of countably-many $\mathcal{S}$-transient Borel sets.

Proposition 5.3.1. *Suppose that G is a separable group, X is a Polish space, $G \curvearrowright X$ is continuous, $d \in \mathbb{Z}^+$, and $S \subseteq G^{\{1,\ldots,d\}}$. Then the following are equivalent:*

(1) The family $\mathcal{F}_S$ is not in the recurrence spectrum of $G \curvearrowright X$.

(2) There is a smooth $\aleph_0$-universally Baire superequivalence relation F of E_G^X for which each action $G \curvearrowright [x]_F$ has an S-transient σ-complete $\aleph_0$-universally-Baire set.

(3) The action $G \curvearrowright X$ has an S-transient σ-complete Σ_2^0 set.

(4) $G \curvearrowright X$ is σ-$(\bigcup_{g \in G} g\{S\}g^{-1})$-transient.

Proof. As a set $Y \subseteq X$ is S-transient if and only if $\Delta_G^X(Y^{\{0,\ldots,d\}}) \notin \mathcal{F}_S$ Proposition 2.1.5 yields (1) $\Leftrightarrow$ (2). Note that if $U \subseteq G^{\{1,\ldots,d\}} \times X$ and $x \in X$, then $U^x \in \mathcal{F}_S \Leftrightarrow x \in \bigcup_{g \in S} U_g$, so $\mathcal{F}_S$ is Σ_1^0-on-open. As a set $Y \subseteq X$ is S-transient if and only if $\Delta_G^X((Y \cap C)^{\{0,\ldots,d\}}) \notin \mathcal{F}_S$ for all equivalence classes C of F_G^X, Proposition 2.1.6 yields (1) $\Leftrightarrow$ (3). Obviously (3) $\Rightarrow$ (4) and Proposition 2.1.3 implies that (4) $\Rightarrow$ (1). ⊠

For a set $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$, we say that a set $Y \subseteq X$ is *$\mathcal{S}$-transient* if there is a set $S \in \mathcal{S}$ for which Y is S-transient.

Proposition 5.3.2. *Suppose that G is a separable group, X is a Polish space, $G \curvearrowright X$ is continuous, and $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$. Then the following are equivalent:*

(1) There exists $S \in \mathcal{S}$ for which the family $\mathcal{F}_S$ is not in the recurrence spectrum of $G \curvearrowright X$.

(2) There exist $S \in \mathcal{S}$ and a smooth $\aleph_0$-universally Baire superequivalence relation F of E_G^X for which each action $G \curvearrowright [x]_F$ has an S-transient σ-complete $\aleph_0$-universally-Baire set.

(3) The action $G \curvearrowright X$ has an $\mathcal{S}$-transient σ-complete Σ_2^0-set.

Proof. This is a direct consequence of Proposition 5.3.1. ⊠

We say that a set $Y \subseteq X$ is *non-trivially d-transient* if there is a set $S \subseteq G^{\{1,\ldots,d\}}$ intersecting every set of the form D^d_K for $K \subseteq G$ compact for which Y is S-transient.

Proposition 5.3.3. *Suppose that G is a separable group, X is a Polish space, $G \curvearrowright X$ is continuous, $d \in \mathbb{Z}^+$, and $\mathcal{S}_d$ is the family of subsets of $G^{\{1,\ldots,d\}}$ which intersect every set of the form D^d_K where $K \subseteq G$ is compact. Then the following are equivalent:*

1. *There exists $S \in \mathcal{S}_d$ for which the family $\mathcal{F}_S$ is not in the recurrence spectrum of $G \curvearrowright X$.*
2. *There exist $S \in \mathcal{S}_d$ and a smooth $\aleph_0$-universally Baire superequivalence relation F of E^X_G for which each action $G \curvearrowright [x]_F$ has an S-transient σ-complete $\aleph_0$-universally-Baire set.*
3. *The action $G \curvearrowright X$ has a non-trivially-d-transient σ-complete $\mathbf{\Sigma}^0_2$ set.*

Proof. Observe that a set $Y \subseteq X$ is non-trivially d-transient if and only if it is $\mathcal{S}_d$-transient, and appeal to Proposition 5.3.2. ⊠

Proposition 5.3.4. *Suppose that G is a topological group, X is a topological space, $G \curvearrowright X$ is continuous, and $U \subseteq X$ is a non-empty open set. Then there exist a non-empty open set $V \subseteq U$ and an open neighborhood $W \subseteq G$ of 1_G for which $W^{\{1,\ldots,d\}}\Delta^X_G(V^{\{0,\ldots,d\}})W^{-1} \subseteq \Delta^X_G(U^{\{0,\ldots,d\}})$.*

Proof. The continuity of $G \curvearrowright X$ yields a non-empty open set $V \subseteq U$ and an open neighborhood $W \subseteq G$ of 1_G for which $WV \subseteq U$. To see that $W^{\{1,\ldots,d\}}\Delta^X_G(V^{\{0,\ldots,d\}})W^{-1} \subseteq \Delta^X_G(U^{\{0,\ldots,d\}})$, note that if $\mathbf{g} \in \Delta^X_G(V^{\{0,\ldots,d\}})$, $w_0 \in W$ and $\mathbf{w}_1 \in W^{\{1,\ldots,d\}}$, then there exists $x \in V$ for which $\mathbf{g}x \in V^{\{1,\ldots,d\}}$, in which case $\mathbf{w}_1\mathbf{g}w_0^{-1}(w_0x) \in U^{\{1,\ldots,d\}}$, thus $\mathbf{w}_1\mathbf{g}w_0^{-1} \in \Delta^X_G(U^{\{0,\ldots,d\}})$. ⊠

Proposition 5.3.5. *Suppose that G is a locally compact group, X is a topological space, $G \curvearrowright X$ is continuous, $d \in \mathbb{Z}^+$, and $S \subseteq G^{\{1,\ldots,d\}}$ intersects every set of the form D^d_K where $K \subseteq G$ is compact. Then every S-transient non-empty open set $U \subseteq X$ has a non-empty open subset $V \subseteq U$ such that for all dense sets $H \subseteq G$, there is a function $\varphi : S \to H$ with the property that $\varphi(S)$ intersects every set of the form D^d_K where $K \subseteq G$ is compact and V is $\varphi(S)$-transient.*

Proof. By Proposition 5.3.4 there exist a non-empty open set $V \subseteq U$ and an open neighborhood of $W \subseteq G$ of 1_G for which $W^{\{1,\ldots,d\}}\Delta^X_G(V^{\{0,\ldots,d\}}) \subseteq \Delta^X_G(U^{\{0,\ldots,d\}})$. As G is locally compact we can assume that W is pre-compact. As $H \subseteq G$ is dense, there is a function $\varphi : S \to H^{\{1,\ldots,d\}}$ with the property that $\varphi(\mathbf{g}) \in (W^{-1})^{\{1,\ldots,d\}}\mathbf{g}$ for all $\mathbf{g} \in S$. Then $S \subseteq W^{\{1,\ldots,d\}}\varphi(S)$, so $\varphi(S)$ intersects every set of the form D^d_K where $K \subseteq G$ is compact and $W^{\{1,\ldots,d\}}\Delta^X_G(V^{\{0,\ldots,d\}}) \cap S \subseteq \Delta^X_G(U^{\{0,\ldots,d\}}) \cap S = \emptyset$, so $\Delta^X_G(V^{\{0,\ldots,d\}}) \cap \varphi(S) \subseteq \Delta^X_G(V^{\{0,\ldots,d\}}) \cap (W^{-1})^{\{1,\ldots,d\}}S = \emptyset$, thus V is $\varphi(S)$-transient. ⊠

Given a superequivalence relation E of E^X_G , we say that a set $Y \subseteq X$ is E-*locally non-trivially-d-transient* if its intersection with each E-class is non-trivially d-transient. Given $\mathcal{S} \subseteq \mathcal{P}(G^{\{1,\ldots,d\}})$ define the family $\mathcal{F}_\mathcal{S} = \bigcap_{S\in\mathcal{S}} \mathcal{F}_S$.

Proposition 5.3.6. *Suppose that G is a locally compact Polish group, X is a Polish space, $G \curvearrowright X$ is continuous, $d \in \mathbb{Z}^+$, $F_d \subseteq \mathcal{P}(G^{\{1,\ldots,d\}})$ is the family of subsets of $G^{\{1,\ldots,d\}}$ containing D^d_K for some compact $1_G \notin K \subseteq G$, and $\mathcal{S}_d$ is the family of subsets of $G^{\{1,\ldots,d\}}$ which intersect every set of the form D^d_K where $K \subseteq G$ is compact. Then the following are equivalent:*

(1) The family $\mathcal{F}_d$ is not in the recurrence spectrum of $G \curvearrowright X$.

(2) The family $\mathcal{F}_{\mathcal{S}_d}$ is not in the recurrence spectrum of $G \curvearrowright X$.

(3) There is a smooth $\aleph_0$-universally Baire superequivalence relation F of E^X_G for which each action $G \curvearrowright [x]_F$ has an E^X_G-locally non-trivially-d-transient σ-complete $\aleph_0$-universally-Baire set.

(4) The action $G \curvearrowright X$ has an F^X_G-locally-non-trivially-d-transient σ-complete $\boldsymbol{\Sigma}^0_4$ set.

Proof. As $\mathcal{F}_{\mathcal{S}_d}$ is the family of subsets of $G^{\{1,\ldots,d\}}$ containing a set of the form D^d_K for some compact $K \subseteq G$, Proposition 2.2.4 yields (1) ⇔ (2). As a set $Y \subseteq X$ is E^X_G-locally non-trivially-d-transient if and only if $\Delta^X_G((C \cap Y)^{\{0,\ldots,d\}}) \notin \mathcal{F}_{\mathcal{S}_d}$ for all equivalence classes C of E^X_G , Proposition 2.1.9 yields (2) ⇔ (3). The fact that every F^X_G-locally non-trivially-d-transient set $Y \subseteq X$ is E^X_G-locally non-trivially-d-transient yields (4) ⇒ (2). To see (2) ⇒ (4), fix a countable dense set $H \subseteq G$, and let $\mathcal{T}_d$ denote the family of sets $T \subseteq H^{\{1,\ldots,d\}}$ which intersect every set of the form D^d_K where $K \subseteq G$ is compact. Observe that if condition (2) holds, then Proposition 2.1.5 ensures that there is no equivalence class C of F^X_G for which $G \curvearrowright C$ is $\mathcal{F}_{\mathcal{S}_d}$-recurrent, so Proposition 5.3.5 implies that there is no equivalence class C of F^X_G for which $G \curvearrowright C$ is $\mathcal{F}_{\mathcal{T}_d}$-recurrent, thus $\mathcal{F}_{\mathcal{T}_d}$ is not in the recurrence spectrum of $G \curvearrowright X$. Fix an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G that is exhausting in the sense that every compact set $K \subseteq G$ is contained in some K_n, and note that if $U \subseteq G^{\{1,\ldots,d\}} \times X$ and $x \in X$, then $U^x \in \mathcal{F}_{\mathcal{T}_d} \Leftrightarrow \exists n \in \mathbb{N}\, H^{\{1,\ldots,d\}} \subset C^d_{K_n} \cup U^x$, so $\mathcal{F}_{\mathcal{T}_d}$ is Σ^0_3-on-open. As every set $Y \subseteq X$ with the property that $\Delta((C \cap Y)^{\{0,\ldots,d\}}) \notin \mathcal{F}_{\mathcal{T}_d}$ for all equivalence classes C of F^X_G is F^X_G-locally non-trivially-d-transient, Proposition 2.1.6 yields condition (4). ⊠

5.4 Weak wandering

Given a set $S \subseteq G^{\{1,\ldots,d\}}$, we say that a set $Y \subseteq X$ is *S-wandering* if Y is $SS^{-1} \setminus \{(1_G)_{i \in \{1,\ldots,d\}}\}$-transient. We say that Y *d-weakly wandering* if there exists an infinite set $S \subseteq G^{\{1,\ldots,d\}}$ for which Y is S-wandering. Note that Y is weakly wandering when it is 1-weakly wandering.

Proposition 5.4.1. *Suppose that G is a separable group, X is a Polish space, $G \curvearrowright X$ is continuous, $d \in \mathbb{Z}^+$, and $\mathcal{S}$ is the family of sets of the form $SS^{-1} \setminus \{(1_G)_{i \in \{1,\ldots,d\}}\}$, where $S \subseteq G^{\{1,\ldots,d\}}$ is infinite. Then the following are equivalent:*

1. *There exists $S \in \mathcal{S}$ for which the family $\mathcal{F}_S$ is not in the recurrence spectrum of $G \curvearrowright X$.*
2. *There exist an infinite set $S \subseteq G^{\{1,\ldots,d\}}$ and a smooth $\aleph_0$-universally Baire superequivalence relation F of E_G^X for which each action $G \curvearrowright [x]_F$ has an S-wandering σ-complete $\aleph_0$-universally-Baire set.*
3. *The action $G \curvearrowright X$ has a d-weakly-wandering σ-complete Σ_2^0 set.*

Proof. It follows immediately from Proposition 5.3.2. ⊠

Give a superequivalence relation E of E_G^X, we say that a set $Y \subseteq X$ is *E-locally-d-weakly-wandering* if its intersection with each E-class is d-weakly-wandering.

Proposition 5.4.2. *Suppose that G is a Polish group, X is a Polish space, $G \curvearrowright X$ is continuous, $d \in \mathbb{Z}^+$, and $\mathcal{S}_d$ is the family of sets of the form $SS^{-1} \setminus \{(1_G)_{i \in \{1,\ldots,d\}}\}$, where $S \subseteq G^{\{1,\ldots,d\}}$ is infinite. Then the following are equivalent:*

1. *The family $\mathcal{F}_{\mathcal{S}_d}$ is not in the recurrence spectrum of $G \curvearrowright X$.*
2. *There is a smooth $\aleph_0$-universally Baire superequivalence relation F of E_G^X for which each action $G \curvearrowright [x]_F$ has an E_G^X-locally-d-weakly-wandering σ-complete $\aleph_0$-universally-Baire set.*
3. *The action $G \curvearrowright X$ has an F_G^X-locally-d-weakly-wandering σ-complete $(\Sigma_1^1 \setminus \Sigma_1^1)_\sigma$ set.*

Proof. As a set $Y \subseteq X$ is E_G^X-locally-d-weakly wandering if and only if $\Delta_G^X((Y \cap C)^{\{0,\ldots,d\}}) \notin \mathcal{F}_{\mathcal{S}_d}$ for all equivalence classes C of E_G^X, Proposition 2.1.9 yields (1) $\Leftrightarrow$ (2). Note that if $U \subseteq G^{\{1,\ldots,d\}} \times X$, then $U^x \in \mathcal{F}_{\mathcal{S}_d} \Leftrightarrow \forall (g_i)_{i \in \mathbb{N}} \in (G^{\{1,\ldots,d\}})^{\mathbb{N}} \exists i \neq j \, (g_i = g_j) \text{ or } g_i g_j^{-1} \in U^x$, so $\mathcal{F}_{\mathcal{S}_d}$ is Π_1^1-on-open. As a set $Y \subseteq X$ is F_G^X-locally-d-weakly wandering if and only if $\Delta_G^X((Y \cap C)^{\{0,\ldots,d\}}) \notin \mathcal{F}_{\mathcal{S}_d}$ for all equivalence classes C of F_G^X, Proposition 2.1.6 yields (1) $\Leftrightarrow$ (3). ⊠

We say that a set $Y \subseteq X$ is *very weakly d-wandering* if there are arbitrarily large finite sets $S \subseteq G$ for which Y is S-wandering.

Proposition 5.4.3. *Suppose that G is a separable group, X is a Polish space, $G \curvearrowright X$ is continuous, $d \in \mathbb{Z}^+$, and $\mathcal{S}_d$ is the family of sets of the form $\bigcup_{n \in \mathbb{N}} S_n S_n^{-1} \setminus \{(1_G)_{i \in \{1,\ldots,d\}}\}$, where $S_n \subseteq G^{\{1,\ldots,d\}}$ has cardinality n for all $n \in \mathbb{N}$. Then the following are equivalent:*

(1) There exists $S \in \mathcal{S}_d$ for which the family $\mathcal{F}_S$ is not in the recurrence spectrum of $G \curvearrowright X$.

(2) There exist sets $S_n \subseteq G^{\{1,\ldots,d\}}$ of cardinality n and a smooth $\aleph_0$-universally Baire superequivalence relation F of E_G^X for which each action $G \curvearrowright [x]_F$ has a σ-complete $\aleph_0$-universally-Baire set that is S_n-wandering for all $n \in \mathbb{N}$.

(3) The action $G \curvearrowright X$ has very-weakly-d-wandering σ-complete $\mathbf{\Sigma}_2^0$ set.

Proof. Observe that if $S_n \subseteq G^{\{1,\ldots,d\}}$ for all $n \in \mathbb{N}$, then a set $Y \subseteq X$ is S_n-wandering for all $n \in \mathbb{N}$ if and only if it is $\bigcup_{n \in \mathbb{N}} S_n S_n^{-1} \setminus \{(1_G)_{i \in \{1,\ldots,d\}}\}$-transient, and appeal to Proposition 5.3.2. ⊠

Although we are already in position to establish the analog of Proposition 5.4.2 for very weak d-wandering, the following observation will allow us to obtain a substantially stronger complexity bound.

Proposition 5.4.4. *Suppose that G is a topological group, X is a topological space, $G \curvearrowright X$ is continuous, and $S \subseteq G$. Then every S-wandering non-empty open set $U \subseteq X$ has a non-empty open subset $V \subseteq U$ such that for all dense sets $H \subseteq G$, there is an injection $\varphi : S \to H$ with the property that V is $\varphi(S)$-wandering.*

Proof. By Proposition 5.3.4, there exist a non-empty open set $V \subseteq U$ and an open neighborhood $W \subseteq G$ of 1_G for which $W^{-1}\Delta_G^X(V^{\{0,1\}})W \subseteq \Delta_G^X(U^{\{0,1\}})$. Note that if $g, h \in S$ and $(Wg)(Wh)^{-1} \cap \Delta_G^X(V^{\{0,1\}}) \neq \emptyset$, then the fact that $(Wg)(Wh)^{-1} = Wgh^{-1}W^{-1}$ yields that $gh^{-1} \in W^{-1}\Delta_G^X(V^{\{0,1\}})W \subseteq \Delta_G^X(U^{\{0,1\}})$, thus $g = h$. But if $H \subseteq G$ is dense, then there is a function $\varphi : S \to H$ with the property that $\varphi(g) \in Wg$ for all $g \in S$, and it follows that is injective and V is $\varphi(S)$-wandering. ⊠

Given a superequivalence relation E of E_G^X, we say that a set $Y \subseteq X$ is E-*locally very-weakly-wandering* if its intersection with each E-class is very weakly wandering.

Proposition 5.4.5. *Suppose that G is a Polish group, X is a Polish space, $G \curvearrowright X$ is continuous, and $\mathcal{S}$ is the family of sets of the form $\bigcup_{n \in \mathbb{N}} S_n S_n^{-1} \setminus \{1_G\}$, where $S_n \subseteq G$ has cardinality n for all $n \in \mathbb{N}$. Then the following are equivalent:*

(1) The family $\mathcal{F}_\mathcal{S}$ is not in the recurrence spectrum of $G \curvearrowright X$.

(2) There is a smooth $\aleph_0$-universally Baire superequivalence relation F of E^X_G for which each action $G \curvearrowright [x]_F$ has an E^X_G-locally very weakly wandering σ-complete $\aleph_0$-universally-Baire set.

(3) The action $G \curvearrowright X$ has an F^X_G-locally-very-weakly-wandering σ-complete Σ^0_4 set.

Proof. As a set $Y \subseteq X$ is E^X_G-locally very-weakly-wandering if and only if $\Delta^X_G((C \cap Y)^{\{0,1\}}) \notin \mathcal{F}_\mathcal{S}$ for all equivalence classes C of E^X_G , Proposition 2.1.9 yields (1) $\Leftrightarrow$ (2). The fact that every F^X_G-locally very-weakly-wandering set $Y \subseteq X$ is E^X_G-locally very-weakly-wandering yields (3) $\Rightarrow$ (2). To see (1) $\Rightarrow$ (3), fix a countable dense set $H \subseteq G$, and let $\mathcal{T}$ denote the family of sets of the form $\bigcup_{n \in \mathbb{N}} T_n T_n^{-1} \setminus \{1_G\}$, where $T_n \subseteq H$ has cardinality n for all $n \in \mathbb{N}$. Now observe that if condition (1) holds, then Proposition 2.1.5 ensures that there is no equivalence class C of F^X_G for which $G \curvearrowright C$ is $\mathcal{F}_\mathcal{S}$-recurrent, so Proposition 5.4.4 implies that there is no equivalence class C of F^X_G for which $G \curvearrowright C$ is $\mathcal{F}_\mathcal{T}$-recurrent, thus $\mathcal{F}_\mathcal{T}$ is not in the recurrence spectrum of $G \curvearrowright X$. Note that if $U \subseteq G \times X$ and $x \in X$, then $U^x \in \mathcal{F}_\mathcal{T} \Leftrightarrow \exists n \in \mathbb{N} \forall (h_i)_{i<n} \in H^n \exists i \neq j \, (h_i = h_j \text{ or } h_i h_j^{-1} \in U^x)$, so $\mathcal{F}_\mathcal{T}$ is Σ^0_3-on-open. As every set $Y \subseteq X$ with the property that $\Delta^X_G((Y \cap C)^{\{0,1\}}) \notin \mathcal{F}_\mathcal{T}$ for all equivalence classes C of F^X_G is F^X_G-locally very-weakly-wandering, Proposition 2.1.6 yields condition (3). ⊠

The following fact ensures that if $G \curvearrowright X$ is a minimal continuous action, then the existence of a weakly-wandering σ-complete Borel set is equivalent to the existence of a cover by countably-many weakly-wandering Borel sets:

Proposition 5.4.6. *Suppose that G is a separable group, $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$, and $G \curvearrowright X$ is a σ-$(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient minimal continuous action on a Baire space. Then there exists $S \in \mathcal{S}$ for which there is an $\{S\}$-transient complete open set.*

Proof. Fix an $\mathcal{S}$-transient non-meager Borel set $B \subseteq X$, as well as $S \in \mathcal{S}$ for which B is $\{S\}$-transient, and a non-empty open set $V \subseteq X$ in which B is comeager. Then Proposition 2.1.3 ensures that V is $\{S\}$-transient, and the minimality of $G \curvearrowright X$ implies that it is complete. ⊠

The following fact ensures that the above assumption of minimality is necessary:

Proposition 5.4.7. *Suppose that G is a locally-compact Polish group, $(K_n)_{n \in \mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, and $\mathbf{g} \in G^\mathbb{N}$ is* *doubly $(K_n)_{n\in\mathbb{N}}$-expansive. Then there is a continuous disjoint union $G \curvearrowright X$ of free actions obtained via expansive cutting and stacking, a continuous surjective homomorphism $\varphi\colon X \to 2^{\mathbb{N}}$ from E_G^X to equality, and a complete open set $V \subseteq X$ such that $V \cap \varphi^{-1}(\{d\})$ is IP$(\mathbf{g} * d)$-wandering for all $d \in 2^{\mathbb{N}}$, but for all sets $S \subseteq G$ with non-compact closure, there is at most one $d \in 2^{\mathbb{N}}$ with the property that $G \curvearrowright \varphi^{-1}(\{d\})$ is σ-expansively $\{S\}$-transient.*

Proof. We first note a pair of lemmas:

Lemma 5.4.8. Suppose that $d, e \in 2^{\mathbb{N}}$ are distinct, $K, L \subseteq G$ are compact, and $S \subseteq K\mathrm{IP}(\mathbf{g} * d)\mathrm{IP}(\mathbf{g} * d)^{-1}K^{-1}$. Then the closure of $L\mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1}L^{-1} \cap S$ is compact.

Proof. Let s be the maximal common initial segment of d and e. As $\mathbf{g}$ is doubly $(K_n)_{n\in\mathbb{N}}$-expansive, there is a natural number $n > \langle s \rangle$ such that $\mathbf{g}_m \notin (\mathrm{IP}(\mathbf{g} \upharpoonright m)^{-1}(K^{-1}L)^{\pm 1}\mathrm{IP}(\mathbf{g} \upharpoonright m))^2$ for all $m \geq n$, in which case a straightforward calculation reveals that

$$\begin{aligned} &L\mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1}L^{-1} \cap S \\ &\quad \subseteq K\mathrm{IP}(\mathbf{g} * d)\mathrm{IP}(\mathbf{g} * d)^{-1}K^{-1} \cap L\mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1}L^{-1} \\ &\quad \subseteq K\mathrm{IP}(\mathbf{g} \upharpoonright n)\mathrm{IP}(\mathbf{g} \upharpoonright n)^{-1}K^{-1} \cap L\mathrm{IP}(\mathbf{g} \upharpoonright n)\mathrm{IP}(\mathbf{g} \upharpoonright n)^{-1}L^{-1}, \end{aligned}$$

so it only remains to note that the latter set is compact. ⊠

Lemma 5.4.9. Suppose that $K \subseteq G$ is compact, but the closure of $S \subseteq G$ is not compact. Then there exists $d \in 2^{\mathbb{N}}$ such that for all $e \in \sim\{d\}$, there is a $(K_n)_{n\in\mathbb{N}}$-expansive $\{S\}$-dense sequence $\mathbf{g}_e \in G^{\mathbb{N}}$ for which $K\mathrm{IP}(\mathbf{g}_e)\mathrm{IP}(\mathbf{g}_e)^{-1}K^{-1} \cap \mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1} = \{1_G\}$.

Proof. Fix a countable dense set $H \subseteq G$, as well as a sequence $\mathbf{h} \in H^{\mathbb{N}}$ such that $\forall h \in H \exists^\infty n \in \mathbb{N}\ h = \mathbf{h}_n$, and a sequence $\mathbf{s} \in \prod_{n\in\mathbb{N}} 2^n$ such that $\{\mathbf{s}_n \mid n \in \mathbb{N} \text{ and } h = \mathbf{h}_n\}$ is $\sqsubseteq$-dense for all $h \in H$. As the closure of S is not compact, Lemma 5.4.8 yields $d \in 2^{\mathbb{N}}$ such that $S \nsubseteq L\mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1}L^{-1}$ for all $e \in \sim\{d\}$ and compact sets $L \subseteq G$, in which case a simple recursive construction yields $\mathbf{g}_e \in G^{\mathbb{N}}$ such that:

(1) $\forall n \in \mathbb{N}\ (\mathbf{g}_e)_n \notin \mathrm{IP}(\mathbf{g}_e \upharpoonright n)K_n\mathrm{IP}(\mathbf{g}_e \upharpoonright n)^{-1}$.

(2) $\forall n \in \mathbb{N}\ (\mathbf{g}_e)_n \in (\mathbf{g}_e^{\mathbf{s}_n})^{-1}\mathbf{h}_n^{-1}S\mathbf{h}_n\mathbf{g}_e^{\mathbf{s}_n}$.

(3) $\forall n \in \mathbb{N}\ (\mathbf{g}_e)_n \notin \mathrm{IP}(\mathbf{g}_e \upharpoonright n)^{-1}K^{-1}\mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1}K\mathrm{IP}(\mathbf{g}_e \upharpoonright n)$.

The first condition ensures that $\mathbf{g}_e$ is $(K_n)_{n\in\mathbb{N}}$-expansive, the second condition implies that $\mathbf{g}_e$ is $\{S\}$-dense, and the third condition yields that $K\mathrm{IP}(\mathbf{g}_e)\mathrm{IP}(\mathbf{g}_e)^{-1}K^{-1}$ $\cap\mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1} = \{1_G\}$. ⊠

Fix an open neighborhood $U \subseteq G$ of 1_G with compact closure, let I be the set of $(d, \mathbf{g}_d) \in 2^{\mathbb{N}} \times G^{\mathbb{N}}$ for which $\mathbf{g}_d$ is $(K_n)_{n \in \mathbb{N}}$-expansive and $U\mathrm{IP}(\mathbf{g}_d)\mathrm{IP}(\mathbf{g}_d)^{-1}U^{-1} \cap \mathrm{IP}(\mathbf{g} * d)\mathrm{IP}(\mathbf{g} * d)^{-1} = \{1_G\}$, and define $X = \mathbb{X}_{\mathrm{proj}_{G^{\mathbb{N}}} \upharpoonright I}$. Then $G \curvearrowright X$ is a continuous disjoint union of actions obtained via expansive cutting and stacking, the function $\varphi \colon X \to 2^{\mathbb{N}}$ given by $\varphi([(g, ((d, \mathbf{g}_d), c))]_{E_{\mathrm{proj}_{G^{\mathbb{N}}} \upharpoonright I}}) = d$ is a homomorphism from E_G^X to equality, and $V = (U \times X_{\mathrm{proj}_{G^{\mathbb{N}}} \upharpoonright I}) / E_{\mathrm{proj}_{G^{\mathbb{N}}} \upharpoonright I}$ is a complete open set.

Proposition 3.2.1 ensures that $V \cap \varphi^{-1}(\{d\})$ is $\mathrm{IP}(\mathbf{g} * d)$-wandering for all $d \in 2^{\mathbb{N}}$. If the closure of $S \subseteq G$ is not compact, then Lemma 5.4.9 yields $d \in 2^{\mathbb{N}}$ such that for all $e \in \sim\{d\}$, there is an $\{S\}$-dense sequence $\mathbf{g}_e \in I_e$, thus $G \curvearrowright \varphi^{-1}(\{e\})$ is not σ-expansively $\{S\}$-transient by Propositions 4.1.1 and 2.1.3. If $d \in 2^{\mathbb{N}}$, $e \in \sim\{d\}$, and $\mathbf{g}_e \in I_e$, then $G \curvearrowright \mathbb{X}_{\mathbf{g}_e}$ is not $\{\mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1} \setminus \{1_G\}\}$-recurrent, and therefore not expansively $\{(\mathbf{g} * e)(\mathbb{N})\}$-recurrent, so Proposition 4.1.1 ensures that $\mathbf{g}_e$ is not $\{(\mathbf{g} * e)(\mathbb{N})\}$-dense, thus Lemma 5.4.9 implies that there is a $\{(\mathbf{g} * e)(\mathbb{N})\}$-dense sequence $\mathbf{g}_d \in I_d$, hence φ is surjective. ⊠

We next note a restriction on the sets $S \subseteq G$ appearing in the definition of weak wandering in the topological setting:

Proposition 5.4.10. *Suppose that $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a Polish space, $S \subseteq G$, and there is an S-wandering non-empty open set $U \subseteq X$. Then S is closed and discrete.*

Proof. Otherwise, there is an injective sequence $(g_n)_{n \in \mathbb{N}}$ of elements of S that converges to some $g \in G$, so $g_n g^{-1} \to 1_G$. But if $x \in U$, then $g_n g^{-1} \cdot x \to x$, so there exists $n \in \mathbb{N}$ such that $g_m g^{-1} \cdot x \in U$ for all $m \geq n$, thus $g^{-1} \cdot x \in \bigcap_{m \geq n} g_m^{-1} U$, a contradiction. ⊠

Proposition 5.4.11. *Suppose that G is a locally-compact Polish group and the closure of $S \subseteq G$ is not compact. Then there is an infinite set $T \subseteq S$ for which TT^{-1} is closed and discrete.*

Proof. Fix an increasing sequence $(U_n)_{n \in \mathbb{N}}$ of open subsets of G with compact closures whose union is G, and recursively construct $g_n \in S \setminus (U_n^{\pm 1}\{g_i \mid i < n\})$ for all $n \in \mathbb{N}$. To see that the set $T = \{g_n \mid n \in \mathbb{N}\}$ is as desired, note that for all $g \in G$, there exists $n \in \mathbb{N}$ such that $g \in U_n$, but $TT^{-1} \cap U_n \subseteq \{g_i g_j^{-1} \mid i, j < n\}$. ⊠

Clearly $\{S \setminus \{1_G\}\}$-transience implies expansive $\{S\}$-transience. When S is closed and discrete, a natural weakening of the converse also holds:

Proposition 5.4.12. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space, $S \subseteq G$ is closed and discrete, and $B \subseteq X$ is an expansively $\{S\}$-transient Borel set. Then B is a union of finitely-many $\{S \setminus \{1_G\}\}$-transient Borel sets.*

Proof. Fix a compact set $K \subseteq G$ for which $R^B_S \subseteq R^B_K$. As $G \curvearrowright X$ is free, it follows that $R^B_S \subseteq R^B_{K \cap S}$. As S is closed and discrete, it follows that $K \cap S$ is finite. Set $F = (K \cap S)^{\pm 1} \setminus \{1_G\}$, and note that R^X_F is a Borel graph of vertex degree $|F|$, and therefore has a Borel $(|F|+1)$-coloring (see [KST99, Proposition 4.6]), so B is the union of $(|F|+1)$-many $\{S \setminus \{1_G\}\}$-transient Borel sets. ⊠

In light of Proposition 5.3.1, the following fact characterizes both the existence of a weakly-wandering σ-complete Borel set and the existence of a cover by weakly-wandering Borel sets:

Proposition 5.4.13. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space and $\mathcal{S} \subseteq \mathcal{P}(G)$. Then the following are equivalent:*

(1) There are infinite sets $S_n \in \bigcup_{S \in \mathcal{S}} \mathcal{P}(S)$ and S_n-wandering Borel sets $B_n \subseteq X$ for which $X = \bigcup_{n \in \mathbb{N}} B_n$.

(2) There are infinite sets $T_n \in \bigcup_{S \in \mathcal{S}} \mathcal{P}(S)$ for which $T_n T_n^{-1}$ is closed and discrete with the property that $G \curvearrowright X$ is σ-expansively $(\bigcup_{g \in G} g\{T_n T_n^{-1} \mid n \in \mathbb{N}\}g^{-1})$-transient.

Proof. To see (1) $\implies$ (2), note first that we can assume that X is Polish and $G \curvearrowright X$ is continuous by [BK96, Theorem 5.2.1]. Proposition 2.1.3 then ensures that for all $x \in X$, there exists $n \in \mathbb{N}$ for which $G \curvearrowright [x]_{F^X_G}$ is not $\{S_n S_n^{-1} \setminus \{1_G\}\}$-recurrent, in which case Proposition 5.4.10 implies that S_n is closed and discrete. Define $N = \{n \in \mathbb{N} \mid S_n \text{ is closed and discrete}\}$, and for all $n \in N$, appeal to Proposition 5.4.11 to obtain an infinite set $T_n \subseteq S_n$ for which $T_n T_n^{-1}$ is closed and discrete. Then Proposition 4.2.1 ensures that $G \curvearrowright X$ is σ-$(\bigcup_{g \in G} g\{T_n T_n^{-1} \setminus \{1_G\} \mid n \in N\}g^{-1})$-transient, and therefore σ-expansively $(\bigcup_{g \in G} g\{T_n T_n^{-1} \mid n \in N\}g^{-1})$-transient.

To see (2) $\implies$ (1), appeal to Proposition 5.4.12 to see that $G \curvearrowright X$ is σ-$(\bigcup_{g \in G} g\{T_n T_n^{-1} \setminus \{1_G\} \mid n \in \mathbb{N}\}g^{-1})$-transient. ⊠

We next note that finite changes to S have little influence on the existence of large S-wandering Borel sets:

Proposition 5.4.14. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space, $g \in G$, $S \subseteq G$ is countable, and $B \subseteq X$ is an S-wandering Borel set. Then B is a union of countably-many $(\{g\} \cup S)$-wandering Borel sets.*

Proof. We can assume that $g \notin S$. Note that for all $x \in B$, there is at most one pair $(h, y) \in S \times B$ for which $g^{-1} \cdot x = h^{-1} \cdot y$. Let $\varphi \colon B \rightharpoonup B$ be the partial function sending x to y. The freeness of $G \curvearrowright X$ ensures that φ is fixed-point free, in which case graph$\varphi^{\pm 1}$ is a graph generated by a Borel function, and therefore has a Borel $\aleph_0$-coloring (see [KST99, Proposition 4.5]), thus B is a union of countably-many $(\{g\} \cup S)$-wandering Borel sets. ⊠

In light of Theorem 5.1.2, the following fact ensures that if a free Borel action does not contain a basis, then it admits a weakly-wandering σ-complete Borel set:

Proposition 5.4.15. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space, $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ is expansive, and $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is smooth. Then $G \curvearrowright X$ admits a $\mathbf{g}(\mathbb{N})$-wandering σ-complete Borel set.*

Proof. Appeal first to Theorem 5.2.6 to see that $G \curvearrowright X$ is σ-expansively $(\bigcup_{g \in G} g\{\mathbf{g}(\mathbb{N} \setminus n)\mathbf{g}(\mathbb{N} \setminus n)^{-1} \mid n \in \mathbb{N}\}g^{-1})$-transient. The expansivity of $\mathbf{g}$ yields that $\mathbf{g}(\mathbb{N})\mathbf{g}(\mathbb{N})^{-1}$ is closed and discrete, so $G \curvearrowright X$ is σ-$(\bigcup_{g \in G} g \ \{\mathbf{g}(\mathbb{N} \setminus n)\mathbf{g}(\mathbb{N} \setminus n)^{-1} \ \setminus \{1_G\} \mid n \in \mathbb{N}\}g^{-1})$-transient by Proposition 5.4.12, thus σ-$(\bigcup_{g \in G} g\{\mathbf{g}(\mathbb{N})\mathbf{g}(\mathbb{N})^{-1} \setminus \{1_G\}\}g^{-1})$-transient by Proposition 5.4.14, in which case Proposition 5.3.1 yields a $\mathbf{g}(\mathbb{N})$-wandering σ-complete Borel set. ⊠

The following fact yields a sufficient condition for the existence of a non-smooth restriction with a suitably transient complete Borel set:

Proposition 5.4.16. *Suppose that G is a locally-compact Polish group, $(K_n)_{n \in \mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, $\mathbf{g} \in G^{\mathbb{N}}$ is $(K_n)_{n \in \mathbb{N}}$-expansive, $S \subseteq G$ is disjoint from a neighborhood of 1_G, and there is no compact set $K \subseteq G$ with the property that $IP(\mathbf{g})IP(\mathbf{g})^{-1} \subseteq K^{-1}SK$. Then there is a G-action obtained via expansive cutting and stacking that admits a continuous embedding into $G \curvearrowright \mathbb{X}_{\mathbf{g}}$ and an $\{S\}$-transient non-empty open set.*

Proof. Note that for all compact sets $K \subseteq G$ and $n \in \mathbb{N}$, there exist $s_0, s_1 \in 2^{<\mathbb{N}}$ for which $\mathbb{s}^n(\mathbf{g})^{s_1}(\mathbb{s}^n(\mathbf{g})^{s_0})^{-1} \notin K \cup K^{-1}SK$. Fix an open neighborhood $U \subseteq G$ of 1_G with the property that $\overline{U}$ is compact and $S \cap UU^{-1} = \{1_G\}$, recursively find $\ell_n \in \mathbb{N}$ and $s_{0,n}, s_{1,n} \in 2^{\ell_n}$ such that $\mathbf{h}_n \notin \mathrm{IP}(\mathbf{h} \restriction n)^{-1}(K_n \cup U^{-1}SU)\mathrm{IP}(\mathbf{h} \restriction n)$ for all $n \in \mathbb{N}$, where $\mathbf{h}_n = \mathbf{g}^{\bigoplus_{m<n} s_{0,m}} \mathbb{s}^{\sum_{m<n} \ell_m}(\mathbf{g})^{s_{1,n}}(\mathbb{s}^{\sum_{m<n} \ell_m}(\mathbf{g})^{s_{0,n}})^{-1}(\mathbf{g}^{\bigoplus_{m<n} s_{0,m}})^{-1}$ for all $n \in \mathbb{N}$, and define $\varphi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $\varphi(c) = \bigoplus_{n \in \mathbb{N}} s_{c(n),n}$. Then $\mathbf{h}$ is $(K_n)_{n \in \mathbb{N}}$-expansive, so $G \curvearrowright \mathbb{X}_{\mathbf{h}}$ is obtained via expansive cutting and stacking, φ_G factors over $E_{\mathbf{h}}$ and $E_{\mathbf{g}}$ to a continuous embedding of $G \curvearrowright \mathbb{X}_{\mathbf{h}}$ into $G \curvearrowright \mathbb{X}_{\mathbf{g}}$, and Proposition 3.2.1 ensures that $(U \times 2^{\mathbb{N}})/E_{\mathbf{h}}$ is an $\{S\}$-transient non-empty open set. ⊠

For each set N, let $[N]^{\aleph_0}$ denote the family of countably-infinite subsets of N, and for each sequence of sets $(X_n)_{n \in N}$, define $\limsup_{n \in N} X_n = \{x \mid \exists^\infty n \in N\ x \in X_n\}$. We say that a sequence $\mathbf{h} \in G^{\mathbb{N}}$ is *sufficiently* $(K_n)_{n \in \mathbb{N}}$*-expansive* if the following hold, where $H_n = \{\mathbf{h}_m \mid m < n\}$:

(1) $\forall n \in \mathbb{N}\ \mathbf{h}_n \notin (K_n H_n H_n^{-1})^3 K_n H_n$.

(2) $\forall n \in \mathbb{N} \forall m > n$
$$\mathbf{h}_n \notin K_n \mathbf{h}_m H_n^{-1} K_n H_n \mathbf{h}_m^{-1} K_n H_n \cup K_n H_n \mathbf{h}_m^{-1} K_n \mathbf{h}_m H_n^{-1} K_n H_n \cup K_n^{-1} H_n \mathbf{h}_m^{-1} K_n^{-1} H_n H_n^{-1} K_n^{-1} \mathbf{h}_m \cup K_n H_n H_n^{-1} K_n H_n \mathbf{h}_m^{-1} K_n \mathbf{h}_m.$$

(3) $\forall K \subseteq G$ compact$\forall N \in [\mathbb{N} \times \mathbb{N}]^{\aleph_0} \exists M \in [N]^{\aleph_0}$
$\overline{\limsup_{(m,n) \in M} K \mathbf{h}_m \mathbf{h}_n^{-1} K \mathbf{h}_n \mathbf{h}_m^{-1} K}$ is compact.

Proposition 5.4.17. *Suppose that G is a non-compact locally-compact Polish group that admits a compatible two-sided-invariant metric, and $(K_n)_{n \in \mathbb{N}}$ is an increasing sequence of compact subsets of G. Then there is a sufficiently-$(K_n)_{n \in \mathbb{N}}$-expansive sequence $\mathbf{h} \in G^{\mathbb{N}}$.*

Proof. The primary observation is as follows:

Lemma 5.4.18. Suppose that $K \subseteq G$ is compact and $H \in [G]^{\aleph_0}$. Then there exists $H' \in [H]^{\aleph_0}$ such that $\overline{\limsup_{g \in H'} KgKg^{-1}K}$ is compact.

Proof. By [Kle52, p. 1.5], there is a conjugation-invariant open neighborhood $U \subseteq G$ of 1_G with compact closure. Fix a finite set $F \subseteq G$ for which $K \subseteq FU$. By a straightforward induction, it is sufficient to show that for all $f \in F$ and $H \in [G]^{\aleph_0}$, there exists $H' \in [H]^{\aleph_0}$ for which $\overline{\limsup_{g \in H'} Kgfg^{-1}KU}$ is compact. Towards this end, we can assume that there is a set $H' \in [H]^{\aleph_0}$ for which $\bigcap_{g \in H'} Kgfg^{-1}KU \neq \emptyset$. Fix $h \in H'$, and note that $\forall g \in H'\ gfg^{-1} \in K^{-1}Khfh^{-1}KK^{-1}UU^{-1}$, so $\bigcup_{g \in H'} Kgfg^{-1}KU \subseteq KK^{-1}Khfh^{-1}KK^{-1}KUU^{-1}U$. As the latter set has compact closure, so too does $\limsup_{g \in H'} Kgfg^{-1}KU$. ⊠

As G is not compact, there is a discrete set $G_0 \in [G]^{\aleph_0}$. Given $n \in \mathbb{N}$, $G_n \in [G_0]^{\aleph_0}$, and $\mathbf{h} \restriction n$, set $H_n = \{\mathbf{h}_m \mid m < n\}$ and define

$$L_{g,n} = K_n g H_n^{-1} K_n H_n g^{-1} K_n H_n \cup K_n H_n g^{-1} K_n g H_n^{-1} K_n H_n \cup K_n^{-1} H_n g^{-1} K_n^{-1} H_n H_n^{-1} K_n^{-1} g \cup K_n H_n H_n^{-1} K_n H_n g^{-1} K_n g$$

for all $g \in G_n$, and observe that four successive applications of Lemma 5.4.18 yield a set $G_n' \in [G_n]^{\aleph_0}$ with the property that the closure of $\limsup_{g \in G_n'} L_{g,n}$ is compact. As G_n is discrete and infinite, there exists $\mathbf{h}_n \in G_n \setminus ((K_n H_n H_n^{-1})^3 K_n H_n \cup \limsup_{g \in G_n'} L_{g,n})$, in which case the set $G_{n+1} = \{g \in G_n' \mid \mathbf{h}_n \notin L_{g,n}\}$ is infinite. Clearly $\mathbf{h}$ is as desired. ⊠

The following observation ensures that one can obtain a Borel free action $G \curvearrowright X$ that contains a basis and admits a weakly-wandering σ-complete Borel set by fixing an exhaustive increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G and a sufficiently-$(K_n)_{n\in\mathbb{N}}$-expansive sequence $\mathbf{h} \in G^{\mathbb{N}}$, and taking a continuous disjoint union of the actions and weakly-wandering sets obtained by applying Proposition 5.4.16 to every $(K_n)_{n\in\mathbb{N}}$-expansive sequence $\mathbf{g} \in G^{\mathbb{N}}$ with $S = \mathbf{h}(\mathbb{N})\mathbf{h}(\mathbb{N})^{-1} \setminus \{1_G\}$:

Proposition 5.4.19. *Suppose that G is a locally-compact Polish group, $(K_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, $\mathbf{g} \in G^{\mathbb{N}}$ is $(K_n)_{n\in\mathbb{N}}$-expansive, and $\mathbf{h} \in G^{\mathbb{N}}$ is sufficiently $(K_n)_{n\in\mathbb{N}}$-expansive. Then there is no compact set $K \subseteq G$ with the property that $IP(\mathbf{g})IP(\mathbf{g})^{-1} \subseteq K^{-1}\mathbf{h}(\mathbb{N})\mathbf{h}(\mathbb{N})^{-1}K$.*

Proof. Suppose, towards a contradiction, that there is such a K, and set $H_n = \{\mathbf{h}_m \mid m < n\}$ for all $n \in \mathbb{N}$. The $(K_n)_{n\in\mathbb{N}}$-expansivity of $\mathbf{g}$ ensures that $\mathbf{g}(\mathbb{N})$ is closed, discrete, and infinite, so by passing to a subsequence of $\mathbf{g}$, we can assume that there is a strictly increasing sequence $k \in \mathbb{N}^{\mathbb{N}}$ such that $\mathbf{g}_n \in K^{-1}(\mathbf{h}_{k_n}H_{k_n}^{-1})^{\pm 1}K$ for all $n \in \mathbb{N}$. By passing to a terminal segment of $\mathbf{g}$, we can assume that $KK^{-1} \subseteq K_{k_0}$.

Lemma 5.4.20. For all $n \in \mathbb{N}$, the set $\mathrm{IP}(\mathbf{g} \restriction n)\mathbf{g}_n(\mathrm{IP}(\mathbf{g} \restriction n))^{-1}$ is contained in $K^{-1}(\mathbf{h}_{k_n}H_{k_n}^{-1})^{\pm 1}K$.

Proof. Granting that we have established the lemma below n, suppose that $s, t \in 2^n$, fix $k \in \mathbb{N}$ for which $\mathbf{g}^s\mathbf{g}_n(\mathbf{g}^t)^{-1} \in K^{-1}(\mathbf{h}_k H_k^{-1})^{\pm 1}K$, and note that $\mathbf{g}^s\mathbf{g}_n(\mathbf{g}^t)^{-1} \in K^{-1}H_{k_n}H_{k_n}^{-1}KK^{-1}(\mathbf{h}_{k_n}H_{k_n})^{\pm 1}KK^{-1}H_{k_n}H_{k_n}^{-1}K$. A simple calculation then reveals that if $k \neq k_n$ and $\ell = \max(k, k_n)$, then $\mathbf{h}_\ell \in (K_\ell H_\ell H_\ell^{-1})^3 K_\ell H_\ell$, contradicting the sufficient $(K_n)_{n\in\mathbb{N}}$-expansivity of $\mathbf{h}$. ⊠

Lemma 5.4.21. Suppose that $k, m \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ and $t \in 2^n$ such that $\forall s \in 2^m\ \mathbf{g}^{s \frown t \frown (1)} \in K^{-1}(\mathbf{h}_{k_{m+n}}(H_{k_{m+n}} \setminus H_k)^{-1})^{\pm 1}K$.

Proof. Suppose that the lemma fails, and fix $n \in \mathbb{N}$ for which $k_{m+n} \geq k$. Then there exist $i \in \{\pm 1\}$, $s_0, s_1 \in 2^m$, and distinct $t_0, t_1 \in 2^2$ such that $\forall j < 2\ \mathbf{g}^{s_j \frown (0)^n \frown t_j \frown (1)} \in K^{-1}(\mathbf{h}_{k_{m+n+2}}H_k^{-1})^i K$, and $\ell \in \{m+n, m+n+1\}$ for which $\mathbf{g}^{s_0 \frown (0)^n \frown t_0}(\mathbf{g}^{s_1 \frown (0)^n \frown t_1})^{-1} \in K^{-1}(\mathbf{h}_{k_\ell}H_{k_\ell}^{-1})^{\pm 1}K$. A simple calculation then yields that $\mathbf{h}_{k_\ell} \in K_{k_\ell}\mathbf{h}_{k_{m+n+2}}H_{k_\ell}^{-1}K_{k_\ell}H_{k_\ell}\mathbf{h}_{k_{m+n+2}}^{-1}K_{k_\ell}H_{k_\ell} \cup K_{k_\ell}H_{k_\ell}\mathbf{h}_{k_{m+n+2}}^{-1}K_{k_\ell}\mathbf{h}_{k_{m+n+2}} H_{k_\ell}^{-1}K_{k_\ell}H_{k_\ell}$, which contradicts the sufficient $(K_n)_{n\in\mathbb{N}}$-expansivity of $\mathbf{h}$. ⊠

In particular, there exist sequences $s_n \in 2^{<\mathbb{N}}$ such that $\mathbf{g}^{\varphi(t \frown (1))} \in K^{-1} (\mathbf{h}_{k_{n+\sum_{m\leq n}|s_m|}}(H_{k_{n+\sum_{m\leq n}|s_m|}} \setminus H_{k_{n+\sum_{m<n}|s_m|}})^{-1})^{\pm 1}K$ for all $n \in \mathbb{N}$ and $t \in 2^n$, where $\varphi\colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ is given by $\varphi(t) = \bigoplus_{n<|t|} s_n \frown t_n$.

Lemma 5.4.22. Suppose that $i \in \{\pm 1\}$, $n \in \mathbb{N}$, $\ell_0, \ell_1 \in [k_{n+\sum_{m<n}|s_m|}, k_{n+\sum_{m\leq n}|s_m|})$, $t_0, t_1 \in 2^n$, and $\mathbf{g}^{\varphi(t_j \frown (1))} \in K^{-1}(\mathbf{h}_{k_{n+\sum_{m\leq n}|s_m|}} \mathbf{h}_{\ell_j}^{-1})^i K$ for all $j < 2$. Then $\ell_0 = \ell_1$.

Proof. Observe that if $\ell_0 \neq \ell_1$, $k = k_{n+\sum_{m\leq n}|s_m|}$, and $\ell = \max(\ell_0, \ell_1)$, then $K^{-1} H_\ell H_\ell^{-1} K K^{-1}(\mathbf{h}_k H_\ell^{-1})^i K \cap K^{-1}(\mathbf{h}_k \mathbf{h}_\ell^{-1})^i K \neq \emptyset$, so a straightforward calculation reveals that $\mathbf{h}_\ell \in K_\ell^{-1} H_\ell \mathbf{h}_k^{-1} K_\ell^{-1} H_\ell H_\ell^{-1} K_\ell^{-1} \mathbf{h}_k \cup K_\ell H_\ell H_\ell^{-1} K_\ell H_\ell \mathbf{h}_k^{-1} K_\ell \mathbf{h}_k$, contradicting the sufficient $(K_n)_{n\in\mathbb{N}}$-expansivity of $\mathbf{h}$. ⊠

In particular, there are integers $\ell_{i,n} \in [k_{n+\sum_{m<n}|s_m|}, k_{n+\sum_{m\leq n}|s_m|})$ such that $\mathbf{g}^{\varphi(t\frown(1))} \in \bigcup_{i\in\{\pm 1\}} K^{-1}(\mathbf{h}_{k_{n+\sum_{m\leq n}|s_m|}} \mathbf{h}_{\ell_{i,n}}^{-1})^i K$ for all $n \in \mathbb{N}$ and $t \in 2^n$. Fix $N \in [\mathbb{N}]^{\aleph_0}$ with the property that the closure L_i of $\limsup_{n\in N} K^{-1}(\mathbf{h}_{k_{n+\sum_{m\leq n}|s_m|}} \mathbf{h}_{\ell_{i,n}}^{-1})^i$ $KK^{-1}(\mathbf{h}_{k_{n+\sum_{m\leq n}|s_m|}} \mathbf{h}_{\ell_{i,n}}^{-1})^{-i} K$ is compact for all $i \in \{\pm 1\}$, as well as $n \in \mathbb{N}$ such that $L_{-1} \cup L_1 \subseteq K_n$, and $i < 2$, $N' \in [N \setminus (n+2)]^{\aleph_0}$, and distinct $t_0, t_1 \in 2^2$ with the property that $\mathbf{g}^{\varphi((0)^n \frown t_j \frown (0)^{n'-n-2} \frown (1))} \in K^{-1}(\mathbf{h}_{k_{n'+\sum_{m\leq n'}|s_m|}} \mathbf{h}_{\ell_{i,n'}}^{-1})^i K$ for all $j < 2$ and $n' \in N'$. Then $\mathbf{g}^{\varphi((0)^n \frown t_0)}(\mathbf{g}^{\varphi((0)^n \frown t_1)})^{-1} \in L_i$, contradicting the $(K_n)_{n\in\mathbb{N}}$-expansivity of $\mathbf{g}$. ⊠

We now establish our basis and anti-basis theorems for our two notions of admitting large weakly-wandering Borel sets:

Theorem 5.4.23. *Suppose that $G \curvearrowright X$ is a Borel (continuous) free action of an locally-compact Polish group on a Polish space that does not admit a weakly-wandering σ-complete Borel set. Then there is a continuous disjoint union of actions obtained via expansive cutting and stacking that does not admit a weakly-wandering σ-complete Borel set but does admit a Borel (continuous) stabilizer-preserving homomorphism to $G \curvearrowright X$.*

Proof. By Theorem 4.1.8, Proposition 5.3.1, and Theorem 5.4.13. ⊠

Theorem 5.4.24. *Suppose that $G \curvearrowright X$ is a Borel (continuous) free action of an locally-compact Polish group on a Polish space that does not admit a cover by countably-many weakly-wandering Borel set. Then there is a continuous disjoint union of actions obtained via expansive cutting and stacking that does not admit a cover by countably-many weakly-wandering Borel sets but does admit a Borel (continuous) stabilizer-preserving homomorphism to $G \curvearrowright X$.*

Proof. By Theorems 4.1.8 and 5.4.13. ⊠

Theorem 5.4.25. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space that does not admit a weakly-wandering σ-complete Borel set, and $\mathcal{O}$ is a countable family of non-smooth Borel actions on standard Borel spaces. Then there is a Borel G-action on a standard Borel space that admits a Borel stabilizer-preserving homomorphism to $G \curvearrowright X$ and does not admit a weakly-wandering σ-complete Borel set, but to which no action in $\mathcal{O}$ admits a Borel almost stabilizer-preserving-homomorphism.*

Proof. By Theorems 4.2.5 and 5.2.1, Proposition 5.3.1, and Theorem 5.4.13. ⊠

Theorem 5.4.26. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space that does not admit a cover by countably-many weakly-wandering Borel sets, and $\mathcal{O}$ is a countable family of non-smooth Borel actions on standard Borel spaces. Then there is a Borel G-action on a standard Borel space that admits a Borel stabilizer-preserving homomorphism to $G \curvearrowright X$ and does not admit a cover by countably-many weakly-wandering Borel sets, but to which no action in $\mathcal{O}$ admits a Borel almost stabilizer-preserving-homomorphism.*

Proof. By Theorems 4.2.5, 5.2.1, and 5.4.13. ⊠

Recall that a set $Y \subseteq X$ is E_G^X-locally very-weakly-wandering if for all $n \in \mathbb{N}$ and $x \in X$, there is a set $S \subseteq G$ of cardinality n such that $Gx \cap Y$ is S-wandering.

Proposition 5.4.27. *Suppose that $\mathbf{g}_n = 3^n$ for all $n \in \mathbb{N}$. Then there is neither a $\mathbb{Z}$-invariant Borel probability measure on $\mathbb{X}_\mathbf{g}$ nor a smooth Borel superequivalence relation F of $E_\mathbb{Z}^{\mathbb{X}_\mathbf{g}}$ such that $\mathbb{Z} \curvearrowright [x]_F$ admits a E_G^X-locally-very-weakly-wandering complete Borel set for all $x \in \mathbb{X}_\mathbf{g}$.*

Proof. A straightforward induction shows that for every $z \in \mathbb{Z}$ there exists a unique pair (F_0, F_1) of disjoint finite sets $F_0, F_1 \subseteq \mathbb{N}$ such that $z = \sum_{k \in F_0} 3^k - \sum_{k \in F_1} 3^k$. This implies that the sets $B_n^k = (\{k\} \times \mathcal{N}_{(0)^{n+1}})/E_\mathbf{g}$ for $k \in [0, 3^n)$ are pairwise disjoint for $n \in \mathbb{N}$. If μ is a $\mathbb{Z}$-invariant Borel measure on $\mathbb{X}_\mathbf{g}$, a straightforward calculation shows that $\mu(\mathbb{X}_\mathbf{g}) \geq \mu(\bigcup_{k<3^n} B_n^k) = 3^n/2^{n+1}\mu((\{0\} \times 2^\mathbb{N})/E_\mathbf{g})$ for all $n \in \mathbb{N}$. It follows that $\mu(\mathbb{X}_\mathbf{g}) \in \{0, \infty\}$.

As Proposition 3.2.3 ensures that $\mathbb{Z} \curvearrowright \mathbb{X}_\mathbf{g}$ is minimal, Proposition 5.4.5 ensures that it is enough to show that there exists no very-weakly-wandering non-empty open set. But if $U = (\{n\} \times \mathcal{N}_s)/E_\mathbf{g}$ for some $n \in \mathbb{N}$ and $s \in 2^{<\mathbb{N}}$, then $\Delta_G^X(U^{\{0,1\}}) = 3^{|s|}\mathbb{Z}$. Suppose that U is S-wandering for a set $S \subseteq \mathbb{Z}$ of cardinality strictly greater than $3^{|s|}$. Then there exist $i \in \mathbb{Z}$ and $j, k \in S$ such that $j - k = i \cdot 3^{|s|} \in 3^{|s|}\mathbb{Z} = \Delta_G^X(U^{\{0,1\}})$, contradicting the fact that $\Delta_G^X(U^{\{0,1\}}) \cap ((S - S) \setminus \{0\}) = \emptyset$. ⊠

Remark 5.4.28. The *odometer* on $3^{\mathbb{N}}$ is the isometry $\sigma\colon 3^{\mathbb{N}} \to 3^{\mathbb{N}}$ given by $\sigma((2)^n \frown (i) \frown c) = (0)^n \frown (i+1) \frown c$, where $c \in 3^{\mathbb{N}}$ and $i < 2$. It is easy to see that the above action $\mathbb{Z} \curvearrowright \mathbb{X}_{\mathbf{g}}$ is Borel isomorphic to that generated by the restriction of σ to the saturation of $2^{\mathbb{N}}$.

Chapter 6

Mixing

6.1 Weak mixing

Given a family $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$, we say that an action $G \curvearrowright X$ by homeomorphisms of a topological space is *$\mathcal{S}$-transitive* if $\Delta_G^X(\prod_{k \leq d} V_k) \cap \mathcal{S} \neq \emptyset$ for all $d \in \mathbb{Z}^+$, $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}})$, and sequences $(V_k)_{k \leq d}$ of non-empty open subsets of X.

Proposition 6.1.1. *Suppose that G is a group, $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$, and $G \curvearrowright X$ is an $\mathcal{S}$-transitive action by homeomorphisms of a topological space. Then $G \curvearrowright X$ is $\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}}(\mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}}))G$-transitive.*

Proof. Note that if $d \in \mathbb{Z}^+$, $g \in G^{\{0,\ldots,d\}}$, $h \in G^{\{1,\ldots,d\}}$, and $(X_k)_{k \leq d}$ is a sequence of subsets of X, then

$$\begin{aligned} h \in \Delta_G^X(\textstyle\prod_{k \leq d} g_k X_k) &\iff g_0 X_0 \cap \textstyle\bigcap_{1 \leq k \leq d} h_k^{-1} g_k X_k \neq \emptyset \\ &\iff X_0 \cap \textstyle\bigcap_{1 \leq k \leq d} (g_k^{-1} h_k g_0)^{-1} X_k \neq \emptyset \\ &\iff (g_k^{-1} h_k g_0)_{1 \leq k \leq d} \in \Delta_G^X(\textstyle\prod_{k \leq d} X_k) \\ &\iff h \in (g_k)_{1 \leq k \leq d} \Delta_G^X(\textstyle\prod_{k \leq d} X_k) g_0^{-1}. \end{aligned}$$

It follows that if $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}})$ and $(U_k)_{k \leq d}$ is a sequence of non-empty open subsets of X, then the fact that $\Delta_G^X(\prod_{k \leq d} g_k U_k) \cap S \neq \emptyset$ ensures that $\Delta_G^X(\prod_{k \leq d} U_k) \cap (g_k^{-1})_{1 \leq k \leq d} S g_0 \neq \emptyset$. ⊠

Proposition 6.1.2. *Suppose that G is a topological group, X is a topological space, $H \subseteq G$ is dense, $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$, and $G \curvearrowright X$ is continuous, $\bigcup_{d \in \mathbb{Z}^+} H^{\{1,\ldots,d\}}(\mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}}))$-recurrent, and topologically transitive. Then $G \curvearrowright X$ is $\mathcal{S}$-transitive.*

Proof. Suppose that $d \in \mathbb{Z}^+$, $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}})$, and $(U_k)_{k \leq d}$ is a sequence of non-empty open subsets of X. Set $V_0 = U_0$, and construct $h \in H^{\{1,\ldots,d\}}$

by recursively appealing to the topological transitivity of $G \curvearrowright X$ to obtain $h_{k+1} \in H$ such that the set $V_{k+1} = h_{k+1}U_{k+1} \cap V_i$ is non-empty for all $k < d$. As $\Delta_G^X(V_d^{\{0,\ldots,d\}}) \cap hS \neq \emptyset$, the same calculation as in the proof of Proposition 6.1.1 reveals that $\Delta_G^X((\overline{h})^{-1}V_d) = h^{-1}\Delta_G^X(V_d^{\{0,\ldots,d\}})$, so $\Delta_G^X((\overline{h})^{-1}V_d) \cap S \neq \emptyset$. As $(\overline{h}_k)^{-1}V_d \subseteq U_k$ for all $k \leq d$, it follows that $\Delta_G^X(\prod_{k \leq d} U_k) \cap S \neq \emptyset$. ⊠

Observe that if $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a Polish space, $x \in X$, and Gx is non-meager, then there is a compact set $K \subseteq G$ for which Kx is non-meager, and therefore comeager in some non-empty open set $U \subseteq X$, in which case the fact that Kx is closed ensures that $U \subseteq Kx$, thus $Gx = GU$ is an expansively-$\{G\}$-transitive open orbit.

Proposition 6.1.3. *Suppose that $G \curvearrowright X$ is a continuous action of a topological group on a Hausdorff space with no open orbits, $K \subseteq G$ is compact, $d \in \mathbb{Z}^+$, and $(U_k)_{k \leq d}$ is a sequence of non-empty open subsets of X. Then there are non-empty open sets $V_k \subseteq U_k$ for which $(V_k)_{k \leq d}$ is R_K^X-discrete.*

Proof. By the obvious induction, it is sufficient to show that for all distinct $j, k \leq d$, there are non-empty open sets $V_j \subseteq U_j$ and $V_k \subseteq U_k$ such that $V_j \cap KV_k = \emptyset$. Towards this end, fix $x_k \in U_k$, and note that $U_j \not\subseteq Gx_k$, since otherwise $GU_j = Gx_k$, contradicting the fact that Gx_k is not open. Fix $x_j \in U_j \setminus KU_k$, and observe that Proposition 3.2.4 yields open neighborhoods $V_j \subseteq U_j$ of x_j and $V_k \subseteq U_k$ of x_k such that $V_j \cap KV_k = \emptyset$. ⊠

Along similar lines, we say that $G \curvearrowright X$ is *expansively $\mathcal{S}$-transitive* if $\Delta_G^X(\{y \in \prod_{k \leq d} V_k \mid y \text{ is } R_K^X\text{-discrete}\}) \cap \mathcal{S} \neq \emptyset$ for all $d \in \mathbb{Z}^+$, compact sets $K \subseteq G$, $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}})$, and sequences $(V_k)_{k \leq d}$ of non-empty open subsets of X.

Proposition 6.1.4. *Suppose that $d \in \mathbb{Z}^+$, $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a Polish space, and $H \subseteq G$ is dense. Then $G \curvearrowright X$ is topologically d-transitive and has no open orbits if and only if it is topologically transitive and expansively $(\bigcup_{g \in G} gH^{\{1,\ldots,2d-1\}}\{g \in G^{\{1,\ldots,2d-1\}} \mid \forall 0 < i < d \; g_{2i+1} = g_1 g_{2i}\}g^{-1})$-recurrent.*

Proof. Clearly $G \curvearrowright X^d$ is topologically transitive if and only if $G \curvearrowright X$ is $\{\{g \in G^{\{1,\ldots,2d-1\}} \mid \forall 0 < i < d \; g_1 g_{2i} = g_{2i+1}\}\}$-transitive. By Proposition 6.1.1, the latter condition holds if and only if $G \curvearrowright X$ is $H^{\{1,\ldots,2d-1\}}\{g \in G^{\{1,\ldots,2d-1\}} \mid \forall 0 < i < d \; g_1 g_{2i} = g_{2i+1}\}$-transitive. By Proposition 6.1.3 and the comment immediately preceding it, the conjunction of this with the inexistence of open orbits is equivalent to the expansive $H^{\{1,\ldots,2d-1\}}\{g \in G^{\{1,\ldots,2d-1\}} \mid \forall 0 < i < d \; g_1 g_{2i} = g_{2i+1}\}$-transitivity of $G \curvearrowright X$. And this holds if and only if $G \curvearrowright X$ is expansively $H^{\{1,\ldots,2d-1\}}\{g \in G^{\{1,\ldots,2d-1\}} \mid \forall 0 < i < d \; g_1 g_{2i} = g_{2i+1}\}$-recurrent and topologically transitive, by Proposition 6.1.2. ⊠

We now establish our basis theorem for weakly-mixing continuous actions of Polish groups:

Theorem 6.1.5. *Suppose that $G \curvearrowright X$ is a topologically-transitive continuous action of a locally-compact Polish group on a Polish space with no open orbits. Then the following are equivalent:*

(1) The action $G \curvearrowright X$ is weakly mixing.

(2) There is a Baire-measurable stabilizer-preserving homomorphism from a weakly-mixing G-action obtained via expansive cutting and stacking to $G \curvearrowright X$.

(3) There is a continuous embedding of a weakly-mixing G-action obtained via expansive cutting and stacking into $G \curvearrowright X$.

Proof. By Theorem 4.1.7 and Proposition 6.1.4. ⊠

We now establish our anti-basis theorem for weakly-mixing continuous actions of Polish groups:

Theorem 6.1.6. *Suppose that $G \curvearrowright X$ is a weakly-mixing continuous action of a locally-compact Polish group on a Polish space. Then there is a family $\mathcal{A}$ of continuum-many weakly-mixing continuous G-actions on Polish spaces that admit continuous embeddings into $G \curvearrowright X$ such that every non-smooth Borel G-action on a standard Borel space admits a Borel stabilizer-preserving homomorphism to at most one action in $\mathcal{A}$.*

Proof. By Theorem 4.2.4 and Propositions 5.1.1 and 6.1.4. ⊠

We now establish the promised equivalence of the measure-theoretic and topological notions of weak mixing:

Theorem 6.1.7. *Suppose that $G \curvearrowright X$ is a continuous action of an abelian locally-compact Polish group on a Polish space. Then the following are equivalent:*

(1) There is a G-invariant σ-finite Borel measure μ on X with respect to which $G \curvearrowright X$ is weakly mixing.

(2) There is a G-invariant closed set $C \subseteq X$ for which $G \curvearrowright C$ is weakly mixing.

Proof. To see (1) $\implies$ (2), let C be the complement of the union of all μ-null non-empty open sets $U \subseteq X$, and observe that if $U, U', V, V' \subseteq C$ are non-empty open sets, then the G-saturations of $U \times V$ and $U' \times V'$ are $(\mu \times \mu)$-conull, thus $\Delta_G^{C \times C}((U \times V) \times (U' \times V')) \neq \emptyset$.

To see (2) $\implies$ (1), we first note the following:

Lemma 6.1.8. Suppose that $x \in C$ and Gx is an open subset of C. Then x is the unique element of Gx, and therefore of C.

Proof. Note that if $g \in G$ and $U \subseteq G$, then $R^{Gx}_{gU} = (g, 1_G)R^{Gx}_U$. It follows that if $H \subseteq G$ is a countable dense set, $H' = H \times \{1_G\}$, and $U \subseteq G$ is a non-empty open set, then $Gx \times Gx = \bigcup_{h \in H} R^{Gx}_{hU} = H' R^{Gx}_U$, so R^{Gx}_U is not meager.

Proposition 3.1.2 easily implies that $G/\mathrm{Stab}(x)$ is a Hausdorff space. It follows that if x is not the unique element of Gx, in which case $\mathrm{Stab}(x) \neq G$, then there are disjoint non-empty open sets $U, V \subseteq G/\mathrm{Stab}(x)$. As G is abelian, it follows that $R^{Gx}_{\bigcup U}$ and $R^{Gx}_{\bigcup V}$ are disjoint G-invariant non-meager sets with the Baire property, contradicting the fact that $G \curvearrowright C \times C$ is topologically transitive. ⊠

If C is a singleton, then any finite Borel measure concentrating on C is as desired. Otherwise, fix a countable dense subgroup H of G, and observe that by the proof of Theorem 6.1.5 and Lemma 6.1.8, we can assume that $G \curvearrowright X$ is of the form $G \curvearrowright \mathbb{X}_{\mathbf{h}}$, where $\mathbf{h} \in (H^{\{1,\ldots,3\}})^{\mathbb{N}}$ is expansive and $\forall h \in H \exists^\infty n \in \mathbb{N}\ h(\mathbf{h}_n)_1(\mathbf{h}_n)_2 = (\mathbf{h}_n)_3$.

For each $n \in \mathbb{N}$, let G_n denote the digraph on 2^n consisting of all pairs $(s, t) \in 2^n \times 2^n$ such that $\mathrm{supp}(s) \subseteq \mathrm{supp}(t)$ and $\mathrm{supp}(t) \setminus \mathrm{supp}(s)$ is a singleton.

Lemma 6.1.9. Suppose that $n \in \mathbb{N}$. Then there there is a partial injection $\varphi\colon 2^n \rightharpoonup 2^n$ whose graph is contained in G_n and whose domain has cardinality $2^n - \binom{n}{\lceil n/2 \rceil}$.

Proof. For all $m \leq n$, define $S_m = \{s \in 2^n \mid |\mathrm{supp}(s)| = m\}$.

If $m < n$, $A \subseteq S_m$, and $B = \{t \in S_{m+1} \mid \exists s \in A\ s\ G_n\ t\}$, then $|G_n \cap (A \times B)| = (n - m)|A|$ and $|G_n^{-1} \cap (B \times S_m)| = (m + 1)|B|$, so $(n - m)|A| \leq (m + 1)|B|$. It follows that if $m + 1 \leq n - m$, or equivalently, if $m \leq (n - 1)/2$, then $|A| \leq |B|$, in which case Hall's marriage theorem (see, for example, [HV50]) yields an injection $\varphi_m\colon S_m \to S_{m+1}$ whose graph is contained in G_m.

If $m < n$, $A \subseteq S_{m+1}$, and $B = \{s \in S_m \mid \exists t \in A\ s\ G_n\ t\}$, then $|G_n^{-1} \cap (A \times B)| = (m + 1)|A|$ and $|G_n \cap (B \times S_{m+1})| = (n - m)|B|$, so $(m + 1)|A| \leq (n - m)|B|$. It follows that if $n - m \leq m + 1$, or equivalently, if $m \geq (n - 1)/2$, then $|A| \leq |B|$, in which case one more application of Hall's marriage theorem yields an injection $\varphi_{m+1}\colon S_{m+1} \to S_m$ whose graph is contained in G_m^{-1}.

Finally, define $\varphi = \bigcup_{m \leq (n-1)/2} \varphi_m \cup \bigcup_{m > (n-1)/2} \varphi_{m+1}^{-1}$, and note that $|{\sim}\mathrm{dom}(\varphi)| = |S_n| + \sum_{\lceil n/2 \rceil \leq m < n} |S_m| - |S_{m+1}| = |S_{\lceil n/2 \rceil}| = \binom{n}{\lceil n/2 \rceil}$. ⊠

Let μ be the $\mathbb{N}$-fold power of the uniform probability measure on $\{0, 1, 2, 3\}$.

Lemma 6.1.10. Suppose that $\varepsilon > 0$, $n \in \mathbb{N}$, $h \in H$, and $s, t \in 4^n \times 4^n$. Then there exist a clopen set $C \subseteq \mathcal{N}_{s_0} \times \mathcal{N}_{s_1}$ and continuous functions $\varphi_i \colon C \to \mathcal{N}_{t_i}$ with the property that $\varphi_0 \times \varphi_1$ is injective, $(\mu \times \mu)(C) \geq (1-\varepsilon)(\mu \times \mu)(\mathcal{N}_{s_0} \times \mathcal{N}_{s_1})$, and $\mathbb{p}_{\mathbf{h}}(c_0, \varphi_0(c_0, c_1))h = \mathbb{p}_{\mathbf{h}}(c_1, \varphi_1(c_0, c_1))$ for all $(c_0, c_1) \in C$.

Proof. It is well-known that $\binom{k}{\lceil k/2 \rceil}/2^k$ converges to zero, so there exists $k \in \mathbb{N}$ for which $\binom{k}{\lceil k/2 \rceil}/2^k < \varepsilon$. For all $\ell \in \mathbb{N}$, appeal to Lemma 6.1.9 to obtain a partial injection $\varphi_\ell \colon 2^\ell \rightharpoonup 2^\ell$ whose graph is contained in G_ℓ and whose domain has cardinality $2^\ell - \binom{\ell}{\lceil \ell/2 \rceil}$. For all $(u_0, u_1) \in 4^{\leq \mathbb{N}} \times 4^{\leq \mathbb{N}}$, let $K_{(u_0,u_1)}$ be the set of $k \in \bigcap_{i<2} \operatorname{dom}(u_i)$ with the property that $h^{-1}(\mathbf{h}^{s_0})^{-1}\mathbf{h}^{t_0}(\mathbf{h}^{t_1})^{-1}\mathbf{h}^{s_1}(\mathbf{h}_{k+n})_1(\mathbf{h}_{k+n})_2 = (\mathbf{h}_{k+n})_3$ and $((u_0)_k, (u_1)_k) \in \{(0,2),(1,3)\}$. As $K_{(c_0,c_1)}$ is infinite for $(\mu \times \mu)$-almost every $(c_0, c_1) \in 4^{\mathbb{N}} \times 4^{\mathbb{N}}$, there exists $m \in \mathbb{N}$ such that $(\mu \times \mu)(\{(c_0, c_1) \in 4^{\mathbb{N}} \times 4^{\mathbb{N}} \mid |K_{(c_0 \restriction m, c_1 \restriction m)}| < k\}) + \binom{k}{\lceil k/2 \rceil}/2^k < \varepsilon$. For all $K \subseteq m$, let $(k_i^K)_{i<|K|}$ be the strictly increasing enumeration of K. For all $r_0, r_1 \in 4^{m \setminus K}$, set $U_{K,(r_0,r_1)} = \{(u_0, u_1) \in 4^m \times 4^m \mid K = K_{(u_0,u_1)} \text{ and } \forall i < 2 \ r_i \sqsubseteq u_i\}$, and define $\psi_{K,(r_0,r_1)} \colon U_{K,(r_0,r_1)} \to 2^{|K|}$ by $\psi_{K,(r_0,r_1)}(u_0, u_1)_i = (u_0)_{k_i^K}$. Define $\pi \colon 4^m \times 4^m \rightharpoonup 4^m \times 4^m$ by $\pi(u_0, u_1) = (\psi_{K,(r_0,r_1)}^{-1} \circ \varphi_{|K|} \circ \psi_{K,(r_0,r_1)})(u_0, u_1)$, where $K = K_{(u_0,u_1)}$ and $r_i = u_i \restriction (m \setminus K)$ for all $i < 2$, and observe that the partial function $(s_i \frown u_i \frown c_i)_{i<2} \mapsto (t_i \frown \pi_i(u_0, u_1) \frown c_i)_{i<2}$ is as desired, by Proposition 3.2.1. ⊠

Fix a Haar measure μ_G on G. Clearly $G \curvearrowright G \times 4^{\mathbb{N}}$ is invariant with respect to $\mu_G \times \mu$, and the latter is $E_{\mathbf{h}}$-invariant.

Lemma 6.1.11. Suppose that $B \subseteq (G \times 4^{\mathbb{N}}) \times (G \times 4^{\mathbb{N}})$ is G-invariant and $(E_{\mathbf{h}} \times E_{\mathbf{h}})$-invariant. Then B or $\sim B$ is $(\mu_G \times \mu) \times (\mu_G \times \mu)$-null.

Proof. Suppose that B is $(\mu_G \times \mu) \times (\mu_G \times \mu)$-positive. Then Fubini's theorem (see, for example, [Kec95, §17.A]) yields $g_0 \in G$ such that the set $B_{(g_0,g_1)} = \{(c_0, c_1) \in 4^{\mathbb{N}} \times 4^{\mathbb{N}} \mid ((g_0, c_0), (g_1, c_1)) \in B\}$ is $(\mu \times \mu)$-positive for a μ_G-positive set of $g_1 \in G$. Lemma 6.1.10 ensures that if $\varepsilon > 0$, $g_1 \in G$, $h \in H$, $s \in \bigcup_{n \in \mathbb{N}} 4^n \times 4^n$, and $B_{(g_0,g_1)}$ has density strictly greater than $1 - \varepsilon$ in $\mathcal{N}_{s_0} \times \mathcal{N}_{s_1}$, then $B_{(g_0,g_1h)}$ has density strictly greater than $1 - \varepsilon$ in $4^{\mathbb{N}} \times 4^{\mathbb{N}}$ for all $h \in H$. It follows that if $g_1 \in G$ and $(\mu \times \mu)(B_{(g_0,g_1)}) > 0$, then $(\mu \times \mu)(B_{(g_0,g_1h)}) = 1$ for all $h \in H$, so $(\mu \times \mu)(B_{(g_0,g_1)}) = 1$ for μ_G-almost all $g_1 \in G$, since the uniqueness of Haar measure up to a scaling factor ensures that $H \curvearrowright G$ is ergodic with respect to μ_G. As B is G-invariant, it follows that B is $(\mu_G \times \mu) \times (\mu_G \times \mu)$-conull. ⊠

It follows that the restriction of $\mu_G \times \mu$ to any Borel transversal of $E_{\mathbf{h}}$ induces the desired measure on $\mathbb{X}_{\mathbf{h}}$. ⊠

Remark 6.1.12. While the above arguments work just as well for topological d-transitive when $d > 2$, this does not yield any greater generality, as these notions coincide with weak mixing for abelian groups.

We next turn our attention to anti-basis theorems for strengthenings of weak mixing. The primary observation we will use to obtain such results is the following:

Proposition 6.1.13. *Suppose that G is a Polish group that admits a compatible two-sided-invariant metric, $G \curvearrowright X$ is a continuous action on a non-empty Polish space, $G \curvearrowright Y$ is a continuous action on a Polish space with at least two elements, and $G \curvearrowright X \times Y$ is topologically transitive. Then there exist $x \in X$ and a G-invariant dense G_δ set $C \subseteq Y$ for which there is no continuous homomorphism $\varphi\colon X \to Y$ from $G \curvearrowright X$ to $G \curvearrowright Y$ with the property that $\varphi(x) \in C$.*

Proof. Fix a compatible complete metric on X, positive real numbers $\varepsilon_n \to 0$, non-empty open sets $W_0, W_1 \subseteq Y$ with disjoint closures, and open neighborhoods $U \subseteq G$ of 1_G and non-empty open sets $W_0', W_1' \subseteq Y$ such that $UW_i' \subseteq W_i$ for all $i < 2$. By [Kle52, p. 1.5], we can assume that U is conjugation invariant. Fix natural numbers $i_n < 2$ and non-empty open sets $V_n \subseteq Y$ such that for all $i < 2$ and non-empty open sets $V \subseteq Y$, there are infinitely many $n \in \mathbb{N}$ for which $i_n = i$ and $V_n \subseteq V$.

Set $U_0 = X$. Given $n \in \mathbb{N}$ and a non-empty open set $U_n \subseteq X$, fix $g_n \in \Delta_G^{X \times Y}((U_n \times V_n) \times (U_n \times W_{i_n}'))$ and non-empty open sets $U_{n+1} \subseteq X$ and $V_n' \subseteq V_n$ such that $\operatorname{diam}(U_{n+1}) \leq \varepsilon_n$, $\overline{U_{n+1}} \cup g_n U_{n+1} \subseteq U_n$, and $g_n V_n' \subseteq W_{i_n}'$.

Let x be the unique point of $\bigcap_{n \in \mathbb{N}} U_n$. Note that for all $i < 2$ and $n \in \mathbb{N}$, the open set $V_{i,n} = \bigcup_{i = i_m, m \geq n} V_m'$ is dense, thus so too is the G_δ set $D = \bigcap_{i < 2, n \in \mathbb{N}} V_{i,n}$. Fix a countable dense set $H \subseteq G$, and observe that the G_δ set $D_H = \bigcap_{h \in H} h^{-1} D$ is also dense. Noting that $\forall g \in G \forall^* y \in Y \ g \cdot y \in D_H$, the Kuratowski-Ulam theorem ensures that the G-invariant set $C = \{y \in Y \mid \forall^* g \in G \ g \cdot y \in D_H\}$ is comeager. By [Vau75, Corollary 1.8], it is also G_δ.

Suppose now that $\varphi\colon X \to Y$ is a continuous homomorphism from $G \curvearrowright X$ to $G \curvearrowright Y$. To see that $\varphi(x) \notin C$, it is sufficient to show that if $y \in C$, then $g_n \cdot y \not\to y$, since $g_n \cdot x \to x$. Towards this end, fix $i < 2$ for which $y \notin \overline{W_i}$, as well as $g \in G$ for which $g \cdot y \in D_H$. As $G = UH$, there exists $h \in H$ for which $g^{-1} \in Uh$. As the set $N = \{n \in \mathbb{N} \mid hg \cdot y \in V_n' \text{ and } i = i_n\}$ is infinite, it only remains to note that if $n \in N$, then $g_n \cdot y \in g_n Uhg \cdot y = Ug_n hg \cdot y \subseteq W_i$. ⊠

In order to apply this result to obtain lower bounds on the cardinalities of bases consisting solely of weakly mixing actions, we will need the following straightforward observation:

Proposition 6.1.14. *Suppose that G is a group, $G \curvearrowright X$ is a weakly-mixing action by homeomorphisms of a topological space, $G \curvearrowright Y$ is a minimal action by homeomorphisms of a topological space, and there is a continuous homomorphism $\varphi \colon X \to Y$ from $G \curvearrowright X$ to $G \curvearrowright Y$. Then $G \curvearrowright X \times Y$ is topologically transitive.*

Proof. Suppose that $U \times V, U' \times V' \subseteq X \times Y$ are non-empty open sets. As $G \curvearrowright Y$ is minimal, the sets $\varphi^{-1}(V)$ and $\varphi^{-1}(V')$ are non-empty. As $G \curvearrowright X$ is weakly mixing, the set $\Delta_G^{X \times X}((U \times \varphi^{-1}(V)) \times (U' \times \varphi^{-1}(V')))$ is non-empty. But the fact that φ is a homomorphism ensures that this set is contained in $\Delta_G^{X \times Y}((U \times V) \times (U' \times V'))$. ⊠

As a corollary, we obtain the following:

Theorem 6.1.15. *Suppose that G is a Polish group that admits a compatible two-sided-invariant metric and $\mathcal{A}$ is a non-empty class of minimal continuous G-actions on Polish spaces of cardinality at least two that is closed under restrictions to G-invariant dense G_δ sets. Then any basis $\mathcal{B}$ for $\mathcal{A}$ under continuous homomorphism consisting solely of weakly-mixing actions has cardinality at least the additivity of the meager ideal.*

Proof. Fix an action $G \curvearrowright X$ in $\mathcal{A}$, and suppose, towards a contradiction, that there is an enumeration $(G \curvearrowright X_\alpha)_{\alpha < \kappa}$ of $\mathcal{B}$ of length strictly less than the additivity of the meager ideal. For all $\alpha < \kappa$, Proposition 6.1.14 ensures that $G \curvearrowright X \times X_\alpha$ is topologically transitive, so Proposition 6.1.13 yields a G-invariant dense G_δ set $C_\alpha \subseteq X$ for which there is no continuous homomorphism from $G \curvearrowright X_\alpha$ to $G \curvearrowright C_\alpha$. Fix a dense G_δ set $C \subseteq \bigcap_{\alpha < \kappa} C_\alpha$. Then $\forall g \in G \forall^* x \in X \; g \cdot x \in C$, so the Kuratowski-Ulam theorem ensures that $\forall^* x \in X \forall^* g \in G \; g \cdot x \in C$, in which case $B = \{x \in X \mid \forall^* g \in G \; g \cdot x \in C\}$ is a G-invariant dense G_δ set for which no action in $\mathcal{B}$ admits a continuous homomorphism to $G \curvearrowright B$, the desired contradiction. ⊠

6.2 Mild mixing

We begin this section with an alternative characterization of mild mixing:

Proposition 6.2.1. *Suppose that $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a Polish space with no open orbits and $(K_n)_{n \in \mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G. Then $G \curvearrowright X$ is mildly mixing if and only if $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is topologically transitive for all $(K_n)_{n \in \mathbb{N}}$-expansive sequences $\mathbf{g} \in G^{\mathbb{N}}$.*

Proof. By Proposition 3.2.3, it is sufficient to show ($\Longleftarrow$). Towards this end, suppose that $G \curvearrowright Y$ is a topologically-transitive continuous G-action with no open orbits, and fix $y \in Y$ for which $[y]_{F_G^Y}$ is comeager. The minimality of $G \curvearrowright [y]_{F_G^Y}$ ensures that it is topologically transitive. It also ensures that it has no open orbits, since otherwise $[y]_{F_G^Y}$ would itself be an orbit of $G \curvearrowright Y$, and since it is non-meager in Y, it would necessarily be open in Y.

Lemma 6.2.2. There exist a $(K_n)_{n \in \mathbb{N}}$-exhaustive sequence $\mathbf{g} \in G^{\mathbb{N}}$ and a continuous homomorphism $\varphi \colon \mathbb{X}_{\mathbf{g}} \to [y]_{F_G^Y}$ from $G \curvearrowright \mathbb{X}_{\mathbf{g}}$ to $G \curvearrowright Y$.

Proof. While it is easy enough to establish this directly, we will use the tools at hand: By Theorem 6.1.5, there exist a sequence $\mathbf{g} \in G^{\mathbb{N}}$, a continuous function $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ compatible with $\wp_{\mathbf{g}}$ for which $(\mathbf{g}, \mathbf{G})$ is $(K_n)_{n \in \mathbb{N}}$-expansive, and a continuous embedding $\psi \colon \mathbb{X}_{\mathbf{g},\mathbf{G}} \to [y]_{F_G^Y}$ from $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ to $G \curvearrowright Y$. As the function $\pi \colon \mathbb{X}_{\mathbf{g}} \to \mathbb{X}_{\mathbf{g},\mathbf{G}}$ given by $\pi([(g,x)]_{E_{\mathbf{g}}}) = [(g,x)]_{E_{\mathbf{g},\mathbf{G}}}$ is a homomorphism from $G \curvearrowright \mathbb{X}_{\mathbf{g}}$ to $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$, the function $\varphi = \psi \circ \pi$ is as desired. ⊠

Suppose now that $U_0, U_1 \subseteq X$ and $V_0, V_1 \subseteq Y$ are non-empty open sets. The fact that $[y]_{F_G^Y}$ is comeager ensures that it intersects each V_i, so the fact that $G \curvearrowright [y]_{F_G^Y}$ is minimal implies that the pullback of each V_i through φ is non-empty. The topological transitivity of $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ therefore implies that $\Delta_G^{X \times \mathbb{X}_{\mathbf{g}}}(\prod_{i<2} U_i \times \varphi^{-1}(V_i))$ is non-empty, and since φ is a homomorphism, this set is contained in $\Delta_G^{X \times Y}(\prod_{i<2} U_i \times V_i)$, so the latter set is non-empty as well. ⊠

In light of Proposition 6.2.1, the following facts can be viewed as local refinements of further alternative characterizations of mild mixing:

Proposition 6.2.3. *Suppose that $G \curvearrowright X$ is a continuous action of a topological group on a topological space and $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$. Then $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is topologically transitive if and only if $G \curvearrowright X$ is $\{IP(\mathfrak{s}^n(\mathbf{g}))IP(\mathfrak{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}$-transitive.*

Proof. Note that if $G \curvearrowright Y$ is topologically transitive, then $G \curvearrowright X \times Y$ is topologically transitive if and only if $\Delta_G^X(U \times V) \cap \Delta_G^Y(W \times W) \neq \emptyset$ for all non-empty open sets $U, V \subseteq X$ and $W \subseteq Y$, since $\Delta_G^X(U \times V) \cap \Delta_G^Y(W \times gW) = g(\Delta_G^X(U \times g^{-1}V) \cap \Delta_G^Y(W \times W))$ for all $g \in G$. In particular, this holds when $Y = \mathbb{X}_{\mathbf{g}}$, since Proposition 3.1.4 ensures that $G \curvearrowright \mathbb{X}_{\mathbf{g}}$ is minimal.

To see ($\Longrightarrow$), suppose that $n \in \mathbb{N}$ and $V, W \subseteq X$ are non-empty open sets, and fix an open neighborhood $U \subseteq G$ of 1_G and non-empty open sets $V', W' \subseteq X$ such that $UV' \subseteq V$ and $UW' \subseteq W$. Then $\Delta_G^X(V' \times W') \cap \Delta_G^{\mathbb{X}_{\mathbf{g}}}((U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}} \times$

$(U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}}) \neq \emptyset$. But $U\Delta_G^X(V' \times W')U^{-1} = \Delta_G^X(UV' \times UW')$, and it follows from Proposition 3.2.1 that $\Delta_G^{\mathbb{X}_{\mathbf{g}}}((U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}} \times (U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}}) = U^{-1}\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1}U$, so $\Delta_G^X(V \times W) \cap \mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \neq \emptyset$.

To see ($\Longleftarrow$), suppose that $s \in T_{\mathbf{g}}$, and $U \subseteq G$ and $V, W \subseteq X$ are non-empty open sets, and observe that $\Delta_G^X((U\mathbf{g}^s)^{-1}V \times (U\mathbf{g}^s)^{-1}W) \cap \mathrm{IP}(\mathbb{s}^{|s|}(\mathbf{g}))\mathrm{IP}(\mathbb{s}^{|s|}(\mathbf{g}))^{-1} \neq \emptyset$. Noting that $U\mathbf{g}^s\mathrm{IP}(\mathbb{s}^n\mathbf{g})\mathrm{IP}(\mathbb{s}^n\mathbf{g})^{-1}(U\mathbf{g}^s)^{-1} \subseteq \Delta_G^{\mathbb{X}_{\mathbf{g}}}((U \times \mathcal{N}_s)/E_{\mathbf{g}} \times (U \times \mathcal{N}_s)/E_{\mathbf{g}})$ by Proposition 3.2.1, the fact that $\Delta_G^X((U\mathbf{g}^s)^{-1}V \times (U\mathbf{g}^s)^{-1}W) = (U\mathbf{g}^s)^{-1}\Delta_G^X(V \times W)U\mathbf{g}^s$ ensures that $\Delta_G^X(V \times W) \cap \Delta_G^{\mathbb{X}_{\mathbf{g}}}((U \times \mathcal{N}_s)/E_{\mathbf{g}} \times (U \times \mathcal{N}_s)/E_{\mathbf{g}}) \neq \emptyset$. ⊠

Proposition 6.2.4. *Suppose that $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a Polish space and $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$. Then the following are equivalent:*

(1) The action $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is topologically transitive and the action $G \curvearrowright X$ has no open orbits.

(2) The action $G \curvearrowright X$ is topologically transitive and expansively $\{gIP(\mathbb{s}^n(\mathbf{g}))\, IP(\mathbb{s}^n(\mathbf{g}))^{-1} \mid g \in G$ and $n \in \mathbb{N}\}$-recurrent.

Proof. Note that $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is topologically transitive if and only if $G \curvearrowright X$ is $\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}$-transitive, by Proposition 6.2.3. The latter condition holds if and only if $G \curvearrowright X$ is $\{g\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid g \in G$ and $n \in \mathbb{N}\}$-transitive, by Proposition 6.1.1. The conjunction of this with the inexistence of open orbits is equivalent to the expansive $\{g\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid g \in G$ and $n \in \mathbb{N}\}$-transitivity of $G \curvearrowright X$, by Proposition 6.1.3 and the comment immediately preceding it. And the latter condition holds if and only if $G \curvearrowright X$ is expansively $\{g\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid g \in G$ and $n \in \mathbb{N}\}$-recurrent and topologically transitive, by Proposition 6.1.2. ⊠

As a consequence, we obtain a necessary and sufficient condition for an intransitive minimal continuous action to be mildly mixing:

Theorem 6.2.5. *Suppose that $G \curvearrowright X$ is an intransitive minimal continuous action of a locally-compact Polish group on a Polish space. Then the following are equivalent:*

(1) The action $G \curvearrowright X$ is mildly mixing.

(2) There is a continuous disjoint union of actions that is obtained via expansive cutting and stacking that is not σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{mm}\}$-transient but admits a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$.

Proof. Fix an exhaustive increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G.

To see (1) $\implies$ (2), note that if $S \in \bigcup \boldsymbol{\mathcal{S}}_{mm}$, then there exist $g \in G$ and a $(K_n)_{n\in\mathbb{N}}$-expansive sequence $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ for which $g\mathrm{IP}(\mathbf{g})\mathrm{IP}(\mathbf{g})^{-1} \subseteq S$. As the intransitivity and minimality of $G \curvearrowright X$ rule out the existence of open orbits, Proposition 6.2.4 ensures that $G \curvearrowright X$ is expansively $\{g\mathrm{IP}(\mathbf{g})\mathrm{IP}(\mathbf{g})^{-1}\}$-recurrent, so Lemma 2.1.3 implies that $G \curvearrowright X$ is not σ-expansively $(\bigcup_{g\in G} g\{S\}g^{-1})$-transient, thus Theorem 4.1.8 yields the desired disjoint union and embedding.

To see (2) $\implies$ (1), given a sequence $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ that is $(K_n)_{n\in\mathbb{N}}$-expansive, observe that if $g \in G$, $n \in \mathbb{N}$, and $S = g\mathrm{IP}(\mathfrak{s}^n(\mathbf{g}))\ \mathrm{IP}(\mathfrak{s}^n(\mathbf{g}))^{-1}$, then $G \curvearrowright X$ is not σ-expansively $\bigcup_{g\in G} g\{S\}g^{-1}$-transient, so the minimality of $G \curvearrowright X$ ensures that it is expansively $\{S\}$-recurrent. As $G \curvearrowright X$ is topologically transitive, Proposition 6.2.4 implies that $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is topologically transitive, so Proposition 6.2.1 yields that $G \curvearrowright X$ is mildly mixing. ⊠

We now establish the corresponding anti-basis theorem:

Theorem 6.2.6. *Suppose that G is a Polish group that admits a compatible two-sided-invariant metric and $\mathcal{A}$ is a non-empty class of mildly-mixing minimal continuous G-actions on Polish spaces of cardinality at least two that is closed under restrictions to G-invariant dense G_δ sets. Then any basis $\mathcal{B}$ for $\mathcal{A}$ under continuous homomorphism has cardinality at least the additivity of the meager ideal.*

Proof. Exactly as in the proof of Theorem 6.1.15, albeit without the need for Proposition 6.1.14. ⊠

6.3 Strong mixing

We begin this section with two local refinements of characterizations of strong mixing:

Proposition 6.3.1. *Suppose that $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a topological space. Then $G \curvearrowright X$ is strongly mixing if and only if it is $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-transitive.*

Proof. This is a straightforward consequence of the fact that a closed subset of G is compact if and only if it does not have a closed discrete infinite subset. ⊠

Proposition 6.3.2. *Suppose that $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a Polish space. Then $G \curvearrowright X$ is strongly mixing and has no open orbits if and only if it is topologically transitive and expansively $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-recurrent.*

Proof. Note that $G \curvearrowright X$ is strongly mixing if and only if it is $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-transitive, by Proposition 6.3.1. The conjunction of the latter condition with the inexistence of open orbits is equivalent to the expansive $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-transitivity of $G \curvearrowright X$, by Proposition 6.1.3 and the comment immediately preceding it. And the latter condition holds if and only if $G \curvearrowright X$ is expansively $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-recurrent and topologically transitive, by Proposition 6.1.2. ⊠

As a consequence, we obtain a necessary and sufficient condition for an intransitive minimal continuous action to be strongly mixing:

Theorem 6.3.3. *Suppose that $G \curvearrowright X$ is an intransitive minimal continuous action of a locally-compact Polish group on a Polish space. Then the following are equivalent:*

(1) The action $G \curvearrowright X$ is strongly mixing.

(2) There is a continuous disjoint union of actions obtained via expansive cutting and stacking that is not σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{sm}\}$-transient but admits a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$.

Proof. To see (1) $\Longrightarrow$ (2), note that the intransitivity and minimality of $G \curvearrowright X$ ensures that there are open orbits, in which case Proposition 6.3.2 implies that $G \curvearrowright X$ is expansively $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-recurrent, so Lemma 2.1.3 implies that $G \curvearrowright X$ is not σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{sm}\}$-transient, thus Theorem 4.1.8 yields the desired disjoint union and embedding.

To see (2) $\Longrightarrow$ (1), observe that $G \curvearrowright X$ is not σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{sm}\}$-transient, so the minimality of $G \curvearrowright X$ ensures that it is expansively $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-recurrent. As $G \curvearrowright X$ is topologically transitive, Proposition 6.3.2 implies that it is strongly mixing. ⊠

Chapter 7

Miscellaneous

In this chapter we gather some results that do not fit into the context of the previous chapters but are nevertheless of interest in their own right.

7.1 Conjugation action on subgroups

Suppose, G is a countable group. The space of subgroups of G can be realized as a compact subset $\mathcal{S}(G)$ of 2^G, by identifying a subgroup with its characteristic function. G acts continuously by conjugation on $\mathcal{S}(G)$.

Lemma 7.1.1. Suppose N is a normal subgroup of G, then the quotient map $\pi : G \to G/N$ induces a continuous injection of $\psi : \mathcal{S}(G/N) \to \mathcal{S}(G)$, defined by $\psi(H) = \pi^{-1}(H)$, which is an invariant embedding of $E_{G/N}^{\mathcal{S}(G/N)}$ to $E_G^{\mathcal{S}(G)}$. The image of $\mathcal{S}(G/N)$ under ψ consists of all subgroups of G containing N.

Proof. Note that $H_0, H_1 \in \mathcal{S}(G/N)$ are $E_{G/N}^{\mathcal{S}(G/N)}$ related, if and only if there is a $h \in G/N$ such that $hH_0h^{-1} = H_1$, if and only if there is a $g \in G$ such that $g\pi^{-1}(H_0)g^{-1} = \pi^{-1}(H_1)$. Thus ψ is an invariant embedding of $E_{G/N}^{\mathcal{S}(G/N)}$ to $E_G^{\mathcal{S}(G)}$. ⊠

If H is a subgroup of G, then $\mathcal{S}(H)$ is a compact subspace of $\mathcal{S}(G)$ by identifying $H_0 \in \mathcal{S}(H)$ with its characteristic function in $\mathcal{S}(G)$. Note that $E_H^{\mathcal{S}(H)}$ is a subequivalence relation of $E_G^{\mathcal{S}(G)}$ restricted to $\mathcal{S}(H)$. Assume that $H \in \mathcal{S}(G)$ is *malnormal* in G, i.e., whenever $g \in G \setminus H$, then $gHg^{-1} \cap H = \{1_G\}$. Then $E_H^{\mathcal{S}(H)}$ coincides with $E_G^{\mathcal{S}(G)}$ restricted to $\mathcal{S}(H)$, since if $gH_0g^{-1} = H_1$ for some $g \in G$ and $H_0, H_1 \in \mathcal{S}(H)$, then either H_0 and H_1 equal $\{1_G\}$, or g is actually in H, because of the malnormality of H in G. In either case H_0 and H_1 are $E_H^{\mathcal{S}(H)}$-related. There is a malnormal subgroup of F_2 which is isomorphic to F_ω, for example the group that is generated by the set $\{a^l b^l a^l \mid l > 0\}$ where a, b are free generators of F_2 (see [KS71] p. 950). This shows the following:

Proposition 7.1.2 (S. Thomas, B. Velickovic, [TV99])**.** *There exists a continuous embedding of* $E_{F_\omega}^{\mathcal{S}(F_\omega)}$ *into* $E_{F_2}^{\mathcal{S}(F_2)}$.

Suppose, that G acts on H by group-automorphisms. Then define the semidirect product of G and H to be the group $G \ltimes H$ with underlying set $G \times H$ and with the group operation given by $(g_0, h_0) \cdot (g_1, h_1) = (g_0 g_1, h_0(g_0(h_1)))$. The action of G on H propagates to an action of G on $\mathcal{S}(H)$ by $g \cdot H_0 = \{gh \mid h \in H_0\}$ for $g \in G$ and $H_0 \in \mathcal{S}(H)$. Let $i : \mathcal{S}(H) \to \mathcal{S}(G \ltimes H)$ be the map sending $H_0 \in \mathcal{S}(H)$ to $\{1_G\} \times H_0$. Then $G \ltimes H$ acts via conjugation on $i(\mathcal{S}(H))$ by $(g, h) \cdot H_0 = (g, h)H_0(g, h)^{-1} = (g, h)H_0(g^{-1}, g^{-1}h^{-1}) = h(g \cdot H_0)h^{-1}$.

Lemma 7.1.3. Suppose G is a countable group, H is an abelian group and G acts on H by group-automorphisms. Then i is an invariant embedding of $E_G^{\mathcal{S}(H)}$ to $E_{G \ltimes H}^{\mathcal{S}(G \ltimes H)}$. It is also an G-invariant embedding of the action of G on $\mathcal{S}(H)$ to the action of G viewed as a subgroup of $G \ltimes H$ by conjugation on $\mathcal{S}(G \ltimes H)$.

Proof. Just note that, in the special case in which H is abelian, $G \ltimes H$ acts on $i(\mathcal{S}(H))$ by $(g, h) \cdot i(H_0) = i(g \cdot H_0)$ for $(g, h) \in G \ltimes H$ and $H_0 \in \mathcal{S}(H)$. ⊠

For any countable group G and countable abelian group A, let $A[G]$ be the group with underlying set $\{(a_g)_{g \in G} \in A^G \mid |\{g \in G : a_g \neq 0_A\}| < \infty\}$ and pointwise addition. Let G act on $A[G]$ by $h(a_g)_{g \in G} = (a_{h^{-1}g})_{g \in G}$. This is an action by group-automorphisms. Let $G \ltimes A[G]$ be the corresponding semidirect product.

Lemma 7.1.4. The map $\varphi : \mathcal{S}(A)^G \to \mathcal{S}(A[G])$ sending $x \in \mathcal{S}(A)^G$ to the group $A[G] \cap \Pi_{g \in G} x(g)$ is an injective homomorphism of the shift-action of G on $\mathcal{S}(A)^G$ to $G \curvearrowright \mathcal{S}(A[G])$.

Proof. Note that for $h \in G$ and $x \in \mathcal{S}(A)^G$ the element $(a_g)_{g \in G} \in \varphi(hx)$ if and only if $a_g \in x(h^{-1}g)$ for all $g \in G$ if and only if $a_{hg} \in x(g)$ for all $g \in G$ if and only if $(a_g)_{g \in G} \in h\varphi(x)$. Note that φ is injective since, if $\pi_h : A[G] \to A$ is the projection defined by $\pi_h((a_g)_{g \in G}) = a_h$, then $x(h) = \pi_h(\varphi(x))$ for all $x \in \mathcal{S}(A)^G$ and $h \in G$. To see that φ is continuous just note that if $(a_g)_{g \in G} \in A[G]$, $F \subsetneq G$ is the finite set of $g \in G$ for which $a_g \neq 0_A$, for any $g \in G$ the set $U_g \subseteq \mathcal{S}(A)^G$ is the clopen set of all $x \in \mathcal{S}(A)^G$ with $\pi_g(x)$ containing a_g, and $U \subseteq A[G]$ is the clopen set of all subgroups of $A[G]$ containing $(a_g)_{g \in G}$, then $\varphi^{-1}(U) = \bigcap_{g \in F} U_g$ and thus $\varphi^{-1}(U)$ is also clopen. ⊠

Combing Lemmas 7.1.3 and 7.1.4 we immediately obtain:

Theorem 7.1.5. *Suppose G is a countable group and A is a countable abelian group. Then there is a continuous invariant-embedding $\varphi_G : \mathcal{S}(A)^G \to \mathcal{S}(G \ltimes A[G])$ of $E_G^{\mathcal{S}(A)^G}$ to $E_{G\ltimes A[G]}^{\mathcal{S}(G\ltimes A[G])}$.*

Remark 7.1.6. Suppose that G is a countable group, A is a countable abelian group, and μ is an invariant and ergodic probability measure for the shift action of G on $\mathcal{S}(A)^G$. Then the push-forward $(\varphi_G)_*(\mu)$ is invariant and ergodic with respect to the conjugation action of $G \ltimes A[G]$ on $\mathcal{S}(G \ltimes A[G])$ and also μ is weakly mixing with respect to the conjugation action of $G \ltimes A[G]$ if it is weakly mixing with respect to the conjugation action of G.

Proof. By Lemma 7.1.3 this is immediate from the observation that if $G \curvearrowright (X, \mu)$ is ergodic, then also the action $G \ltimes H \curvearrowright (X, \mu)$, given by $(g, h)x = gx$ is ergodic, for any countable group H. And this follows from the fact that a set $A \subseteq X$ is G-invariant if and only if it is $G \ltimes H$-invariant. ⊠

Note that there are countable abelian groups A such that $\mathcal{S}(A)$ is uncountable, for example $\mathbb{Q}$ or the infinite direct sum of $\mathbb{Z}/2\mathbb{Z}$. Let A be such a group. By Theorem 7.1.5 there is a continuous invariant-embedding $\varphi_{F_\omega} : \mathcal{S}(A)^{F_\omega} \to \mathcal{S}(F_\omega \ltimes A[F_\omega])$ of $E_{F_\omega}^{\mathcal{S}(A)^{F_\omega}}$ to $E_{F_\omega\ltimes A[F_\omega]}^{\mathcal{S}(F_\omega\ltimes A[F_\omega])}$. Now $(F_\omega \ltimes A[F_\omega])$ is a factor of F_ω and thus by Lemma 7.1.1 there is an invariant embedding of $E_{(F_\omega\ltimes A[F_\omega])}^{\mathcal{S}(F_\omega\ltimes A[F_\omega])}$ to $E_{F_\omega}^{\mathcal{S}(F_\omega)}$. Now, $\mathcal{S}(A)$ is Borel isomorphic to $\mathbb{R}$ and thus putting all together we obtain an invariant Borel embedding of the induced orbit equivalence relation of the shift action of F_ω on $\mathbb{R}^{F_\omega}$ to the induced orbit equivalence relation of the conjugation action of F_ω on its subgroups. The former is an invariantly universal countable Borel equivalence relation and it follows that $E_{F_\omega}^{\mathcal{S}(F_\omega)}$ is an invariantly universal countable Borel equivalence relation as well.

Combining Proposition 7.1.2 and the preceding observation we obtain:

Theorem 7.1.7 (S. Thomas, B. Velickovic, [TV99])**.** *$E_{F_2}^{\mathcal{S}(F_2)}$ is a universal countable Borel equivalence relation.*

Now, F_2 is a quotient of every free group F_n for $2 \leq n \leq \omega$, so by Lemma 7.1.1 we obtain:

Theorem 7.1.8 (S. Thomas, B. Velickovic, [TV99])**.** *$E_{F_n}^{\mathcal{S}(F_n)}$ is a universal countable Borel equivalence relation for every $2 \leq n \leq \omega$.*

For a group G and $H \in \mathcal{S}(H)$ let $N(H) = \{g \in G \mid gHg^{-1} = H\}$ denote its normalizer, i.e., the stabilizer of H under the conjugation action of G. For a group G let $\mathcal{SNS}(G)$ be the space of selfnormalizing subgroups of G, i.e., those subgroups for which $N(H) = H$. It is a G_δ subspace of $\mathcal{S}(G)$. To see this, note that its complement is the countable union of the closed sets $F_g = \{H \in \mathcal{S}(G) \mid g \notin H \wedge gHg^{-1} = H\}$ for $g \in G$. For any countable group G, let F_G be the free group with generating set $\{a_g \mid g \in G\}$. Then G acts on F_G by permuting the generators by left translation, i.e., $g \in G$ is the unique automorphism from F_G to F_G which sends a_h to a_{gh} for every $h \in G$. This action induces a continuous action of G on $\mathcal{S}(F_G)$. Define a map $\varphi : 2^G \to \mathcal{S}(F_G)$ by letting $\varphi(x)$ be the group generated by all a_g such that $x(g) = 1$. Then φ is continuous and injective. Note that $\varphi(2^G)$ is a partial transversal for the action of F_G by conjugation on $\mathcal{S}(F_G)$. To see this, suppose towards a contradiction, that there are distinct $x_0, x_1 \in 2^G$ and $h \in F_G$ such that $\varphi(x_0) = h\varphi(x_1)h^{-1}$. Then one can assume, by changing the roles of x_0 and x_1 if necessary, that there is a $g \in G$ such that $x_0(g) = 1$ and $x_1(g) = 0$. Let φ_g be the unique homomorphism of F_G to $\mathbb{Z}$ defined by $\varphi_g(a_g) = 1$ and $\varphi_g(a_h) = 0$ if $h \neq g$. Then $\varphi_g(H_0) = \mathbb{Z}$ and $hH_1h^{-1} \subseteq \ker(\varphi_g)$, a contradiction. Also note that φ is a G-embedding of the shift action of G on 2^G to the action of G on $\mathcal{S}(F_G)$. To see this, let $x \in 2^G$ and $g \in G$. Then $a_h \in \varphi(gx)$ if and only if $(gx)(h) = 1$, if and only if $x(g^{-1}h) = 1$, if and only if $a_{g^{-1}h} \in \varphi(x)$, thus $g\varphi(x) = \varphi(gx)$. Now, consider $\mathcal{S}(F_G)$ as a closed ($G \ltimes F_G$-invariant) subset of $\mathcal{S}(G \ltimes F_G)$ and suppose that $\varphi(x_0) = (g, h) \cdot \varphi(x_1)$ for some $x_0, x_1 \in 2^G$ and $(g, h) \in G \ltimes F_G$, i.e., $\varphi(x_0) = h(g \cdot \varphi(x_1))h^{-1}$. Since $\varphi(2^G)$ is a partial transversal for the action of F_G on $\mathcal{S}(F_G)$, this implies that $h(g \cdot \varphi(x_1))h^{-1} = g \cdot \varphi(x_1)$, so, since $g \cdot \varphi(x_1) = \varphi(g \cdot x_1)$ and φ is injective, φ is an embedding of the induced orbit equivalence relation of the shift action of G on 2^G to $E_{G \ltimes F_G}^{\mathcal{S}(G \ltimes F_G)}$. Note that if $x \in 2^G$ is in the free part $Fr(2^G)$ of the shift action, then if $(g, h)\varphi(x)(g, h)^{-1} = \varphi(x)$ we get that $h\varphi(gx)h^{-1} = \varphi(x)$. This implies that $\varphi(gx) = \varphi(x)$ since $\varphi(2^G)$ is a partial transversal for the action of F_G on $\mathcal{S}(F_G)$, but then $g = 1_G$, since x is in the free part of the shift action, thus $h\varphi(x)h^{-1} = \varphi(x)$. But since $\varphi(x)$ is malnormal in F_G this implies that $h \in \varphi(x)$. This shows that $\varphi(Fr(2^G)) \subseteq \mathcal{SNS}(G \ltimes F_G)$. Now, $G \ltimes F_G$ is a factor of F_ω and since the map ψ from Lemma 7.1.1 sends $\mathcal{SNS}(G \ltimes F_G)$ to $\mathcal{SNS}(F_\omega)$, we obtain the following:

Proposition 7.1.9. *For every countable group G there is a continuous embedding of $E_G^{Fr(2^G)}$ to $E_{F_\omega}^{\mathcal{SNS}(F_\omega)}$.*

Note that on the other hand, the induced equivalence relation of the action of G by conjugation on its malnormal subgroups is smooth, since the sets $A_g = \{H \in \mathcal{S}(G) \mid H \text{ is malnormal and } g \in H\}$ are partial transversals for all $g \in G \setminus \{1_G\}$.

Iterating $N : \mathcal{S}(G) \to \mathcal{S}(G)$ and taking unions at limit ordinals eventually yields a selfnormalizing group. The following observations shows that there exist countable groups for which the set of subgroups for which this process ends at the whole group is co-analytic hard.

For a group G and an action $G \curvearrowright X$ define a group $T(G,X)$ as follows. Its underlying set will be $T(G,X) = \{f \in G^{(X^{<\mathbb{N}})} \mid |\{s \in X^{<\mathbb{N}} : f(s) \neq 1_G\}| < \infty\}$. Define for every $f \in T(G,X)$ an map $\sigma_f : X^{<\mathbb{N}} \to X^{<\mathbb{N}}$ by recursively letting $\sigma_f(\emptyset) = \emptyset$ and $\sigma_f(s)(k) = f(\sigma_f(s|k)) \cdot s(k)$ for all $k \in \mathrm{dom}(s)$. Note that σ_f sends X^n bijectively to X^n. To see this, note that it sends X^0 to X^0. Assuming that σ_f sends X^n bijectively to X^n, note that if $s, t \in X^{n+1}$ and $\sigma_f(s) = \sigma_f(t)$, then $\sigma_f(s|n) = \sigma_f(t|n)$ and thus $s|n = t|n$ and $\sigma_f(s)(n) = f(\sigma_f(s|n)) \cdot s(n) = \sigma_f(t)(n) = f(\sigma_f(s|n)) \cdot t(n)$, thus $s(n) = t(n)$, so $s = t$. If $s \in X^{n+1}$, then take $t \in X^n$ with $\sigma_f(t) = s|n$ and choose $x \in X$ such that $f(\sigma_f(t|n)) \cdot x = s(n)$. Then $\sigma_f(t^\frown x) = s$. Now define a group operation on $T(G,X)$ by $(f \circ g)(s) = f(s)g(\sigma_f^{-1}(s))$ for $f, g \in T(G,X)$ and $s \in X^{<\mathbb{N}}$. $1_{T(G,X)}$ is given by $1_{T(G,X)}(s) = 1_G$ for all $s \in G^{<\mathbb{N}}$. Next we show that $\sigma_{f \circ g} = \sigma_f \circ \sigma_g$. Again we show this level by level, the case $n = 0$ being trivial. So suppose that $\sigma_{f\circ g}(s) = (\sigma_f \circ \sigma_g)(s)$ for all $s \in X^n$. If $s \in X^{n+1}$ then $\sigma_{f \circ g}(s)(n) = (f \circ g)(\sigma_{f\circ g}(s|n)) \cdot s(n) = (f \circ g)(\sigma_f(\sigma_g(s|n)) \cdot s(n) = f(\sigma_f(\sigma_g(s|n))g(\sigma_g(s|n)) \cdot s(n)$ and $(\sigma_f \circ \sigma_g)(s)(n) = (\sigma_f(\sigma_g(s))(n) = f(\sigma_f(\sigma_g(s))|n)) \cdot \sigma_g(s)(n) = f(\sigma_f(\sigma_g(s|n))) \cdot g(\sigma_g(s|n)) \cdot s(n)$. Next, we show that $\circ$ is associative: Let $f, g, h \in T(G,X), s \in X^{<N}$ and observe that $((f \circ g) \circ h)(s) = (f \circ g)(s)h(\sigma_{f\circ g}^{-1}(s)) = f(s)g(\sigma_f^{-1}(s))h(\sigma_g^{-1}\sigma_f^{-1}(s)) = f(s)(g \circ h)(\sigma_f^{-1}(s)) = (f \circ (g \circ h))(s)$. Finally note that the inverse of $f \in T(G,X)$ is given by $f^{-1}(s) = f^{-1}(\sigma_f(s))$ for $s \in X^{<\mathbb{N}}$. Note that $\sigma_f(s) = s$ if and only if $f(s|k) \cdot s(k) = s(k)$ for all $k \in \mathrm{dom}(s)$. For a tree $T \subseteq X^{<\mathbb{N}}$ define its pruning derivative by $T' = \{t \in T \mid \exists x \in X\, t^\frown x \in T\}$ and the group $H(T) \subseteq T(G,X)$ by $H(T) = \{f \in T(G,X) \mid \forall t \in T\, f(t) = 1_G\}$.

Lemma 7.1.10. *$H(T') \subseteq N(H(T))$.*

Proof. Observe that for $f \in H(T'), t \in T$, and $g \in H(T)$ we obtain $\sigma_f(t) = t$, thus $(fgf^{-1})(t) = f(t)(gf^{-1}(t)) = f(t)g(t)f^{-1}(t) = f(t)f^{-1}(t) = 1_G$. ⊠

Note that if $t \in T'$, $x \in X$ such that $t^\frown x \in T$, $g \in H(T)$, $f \in T(G,X)$ with $f(t) \neq 1_G$, and $\sigma_f(t) = t$, then $\sigma_f^{-1}(t^\frown x) = t^\frown(f^{-1}(t)x)$ and $\sigma_g^{-1}(t^\frown y) = t^\frown y$ for every $y \in X$, thus $f \circ g \circ f^{-1}(t^\frown x) = f(t^\frown x)g(t^\frown(f^{-1}(t)x))f^{-1}(t^\frown(f^{-1}(t)x))$. Now, if there exist $x \in X$ such that $t^\frown x \in T$ and $t^\frown(f^{-1}(t)x) \notin T$ then any $g \in H(T)$ with $g(t^\frown(f^{-1}(t)x)) \neq f^{-1}(t^\frown x)f(t^\frown(f^{-1}(t)x))$ witnesses that $f \notin N(H(T))$. In the special case where G acts on itself by left translation we abbreviate $T(G,G)$ by $T(G)$.

Proposition 7.1.11. *If G is a countable group such that each non-trivial element has infinite order, then there is a continuous injective map $\varphi : Tr(\mathbb{N}) \to \mathcal{S}(T(G))$ such that $N(\varphi(T)) = \varphi(T')$ for all $T \in Tr(\mathbb{N})$ and if $T_n \in Tr(\mathbb{N})$ for $n \in \mathbb{N}$ is decreasing, then $\varphi(\bigcap_{n\in\mathbb{N}} T_n) = \bigcup_{n\in\mathbb{N}} \varphi(T_n)$.*

Proof. By the previous paragraph and Lemma 7.1.10, all we have to show is that there is a sequence $(g_n)_{n\in\mathbb{N}}$ such that for all $g \in G$ and $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $g^m g_n$ is not in $\{g_n \mid n \in \mathbb{N}\}$. To see this, given such a sequence, define $\varphi_0 : \mathbb{N}^{<\mathbb{N}} \to G^{<\mathbb{N}}$ recursively by $\varphi_0(\emptyset) = \emptyset$ and having defined φ_0 on $\mathbb{N}^n$, define $\varphi_0(s^\frown m) = \varphi_0(s)^\frown g_m$ for all $s \in \mathbb{N}^n$ and $m \in \mathbb{N}$. Then define $\varphi_1(T) = \{\varphi_0(t) \mid t \in T\}$ and $\varphi(T) = H(\varphi_1(T))$ for all trees T on $\mathbb{N}$. Now, given a tree T on $\mathbb{N}$, $t \in T'$, and $g \in G$ there exists $n \in \mathbb{N}$ such that $\varphi_0(t^\frown n) \in \varphi_1(T)$ and $\varphi_0(t)^\frown(gg_n) \notin \varphi_1(T)$ implying the first part of the proposition. If $T_n \in Tr(\mathbb{N})$ for $n \in \mathbb{N}$ is decreasing, then $\varphi(\bigcap_{n\in\mathbb{N}} T_n) \supseteq \bigcup_{n\in\mathbb{N}} \varphi(T_n)$ and if $f \in \varphi(\bigcap_{n\in\mathbb{N}} T_n)$ then $f \in \varphi(T_N)$ for some $N \in \mathbb{N}$ since $|\{s \in G^{G^{<\mathbb{N}}} \mid f(s) \neq 1_G\}| < \infty$. It remains to show the existence of $(g_n)_{n\in\mathbb{N}}$. To do this, let $(h_n)_{n\in\mathbb{N}}$ be an enumeration of G and define $g_0 = h_0$. Having already defined $g_0, ..., g_n$, let g_{n+1} be any element in the complement of $\{h_l g_k \mid l, k \leq n\}$. To see that this works, let $n \in \mathbb{N}$ and $g \in G$ and let $N \in \mathbb{N}$ such that $g = h_N$. Let m be maximal such that $g^m g_n \in \{g_k \mid k \leq \max(n, N)\}$. Then for any $p > \max(n, N)$ we get that $g^{m+1} g_n \in \{h_l g_k \mid l, k \leq p\}$, thus $g^{m+1} g_n \notin \{g_n \mid n \in \mathbb{N}\}$. ⊠

7.2 Some remarks on strong mixing for countable groups

Let G be a countable group. When $K \subseteq G$, we say that $\mathbf{h} \in G^{\{1,...,d\}}$ is *K-discrete* if $\overline{\mathbf{h}}$ is R^G_K-discrete, where G acts on itself by left multiplication. When X is a topological space and $d \in \mathbb{Z}^+$, recall that a continuous-in-X action $G \curvearrowright X$ is strongly d-mixing if $\Delta^X_G(\Pi_{k\leq d} U_k)$ contains every K-discrete sequence $\mathbf{h} \in G^{\{1,...,d\}}$ for some compact set $K \subseteq G$ for all sequences $(U_k)_{k\leq d}$ of non-empty open subsets of X and $G \curvearrowright X$ is strongly $(< \omega)$-mixing if it is strongly d-mixing for every $d \in \mathbb{Z}^+$.

Proposition 7.2.1. *Suppose that $(F_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite subsets of G such that $\bigcup_{n\in\mathbb{N}} F_n = G$ and $1_G \in F_0$, $2 \leq D \leq \omega$, and $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,...,d\}})^{\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ are sequences such that $\mathbf{g}_n \in G^{\{1,...,d_n\}}$, $\mathbf{g}_n$ is F_n-discrete for every $n \in \mathbb{N}$, $S = \{\mathbf{g}_n \mid n \in \mathbb{N}\}$, and for every $d < D$ the set $\{n \in \mathbb{N} \mid d_n = d\}$ is finite. Then there exists a Polish space X_S, a continuous, minimal, and free action $G \curvearrowright X_S$ that is strongly d-mixing for all $d < D$, and an open set $U_S \subseteq X_S$ that is $\{\mathbf{g}_n\}$-transient for every $n \in \mathbb{N}$.*

Proof. Consider 2^G with the product topology and let G act on 2^G via the shift action defined by $(gx)(h) = x(g^{-1}h)$ for all $g, h \in G$ and $x \in 2^G$. Define the set $T_S = \{x \in 2^G \mid x(1_G) = 1 \land \forall \mathbf{g} \in S\, \exists 0 < i \leq |\mathbf{g}|\, x(\mathbf{g}_i^{-1}) = 0\}$. Then T_S is non-empty, closed, and $\{\mathbf{g}_n\}$-transient for all $n \in \mathbb{N}$. Let τ be the topology generated by the product topology on 2^G and the sets gT_S for $g \in G$. By Lemmas 13.2 and 13.3 of [Kec95] τ is a Polish topology. Let X_0 denote the τ-open and G-invariant set GT_S. Given $d < D$ and non-empty open sets $U_l \subseteq X_0$ for $l \leq d$ we show that there exists a finite set $K \subseteq G$ such that every K-discrete sequence in $G^{\{1,\dots,d\}}$ is in $\Delta_G^{X_0}(\Pi_{l \leq d} U_l)$. We can assume that U_l is of the form $\mathcal{N}_{s_l} \cap \bigcap_{k \leq K_l} e_k^l T_S \subseteq X_0$ for $K_l \in \mathbb{N}, e_k^l \in G$ for $k \leq K_l$, $s_l \in 2^{<\mathbb{N}}$ and $l \leq d$. We can further assume that $F = \mathrm{dom}(s_l)$ for all $l \leq d$, $e_k^l \in F$ for $k \leq K_l, l \leq d$, and F is symmetric for some finite set $F \subseteq G$. Let $x_l \in U_l$ for $l \leq d$. Choose $N \in \mathbb{N}$ such that $F^2 \cup F \bigcup_{\{(p,i) \in \mathbb{N}\times\mathbb{N} \mid d_p \leq d, i \in \{1,\dots,d_p\}\}} \{(\mathbf{g}_p)_i\} F \subseteq F_N$. Let $M = \{(\mathbf{g}_m)_i \mid i \in \{1,\dots,d_m\}, m < N\}$. If $\mathbf{h} \in G^{\{1,\dots,d\}}$ is $(F_N \cup FMF)$-discrete we define $x_{\mathbf{h}}$ by

(1) $x_{\mathbf{h}}(f) = x_l(\overline{\mathbf{h}}_l f)$ for $f \in \overline{\mathbf{h}}_l^{-1} F$ and $l \leq d$,

(2) $x_{\mathbf{h}}(f) = 0$ for $f \notin \bigcup_{l \leq d} \overline{\mathbf{h}}_l^{-1} F$.

This is well-defined since $\overline{\mathbf{h}}_i^{-1} F \cap \overline{\mathbf{h}}_j^{-1} F = \emptyset$ for $i \neq j \leq d$, since $\mathbf{h}$ is F^2-discrete. We now show that $\overline{\mathbf{h}}_l x_{\mathbf{h}} \in U_l$ for all $l \leq d$. Certainly $\overline{\mathbf{h}}_l x_{\mathbf{h}} \in \mathcal{N}_{s_l}$ for $l \leq d$. Also $\overline{\mathbf{h}}_l x_{\mathbf{h}}(e_k^l) = x_l(e_k^l) = 1$ for all $k \leq K_l$ and $l \leq d$. It remains to show that for every $m \in \mathbb{N}$ and $k \leq K_l$ there exists $0 < i \leq d_m$ such that $\mathbf{h}_l x_{\mathbf{h}}(e_k^l (\mathbf{g}_m)_i^{-1}) = 0$. Condition (2) implies that this is always the case when there exists $0 < i \leq d_m$ with $\overline{\mathbf{h}}_l^{-1} e_k^l (\mathbf{g}_m)_i^{-1} \notin \bigcup_{j \leq d} \overline{\mathbf{h}}_j^{-1} F$. So suppose that there exists $k \leq K_l$ and $m \in \mathbb{N}$ such that $\overline{\mathbf{h}}_l^{-1} e_k^l (\mathbf{g}_m)_i^{-1} \in \bigcup_{j \leq d} \overline{\mathbf{h}}_j^{-1} F$ for all $0 < i \leq d_m$. If $\overline{\mathbf{h}}_l^{-1} e_k^l (\mathbf{g}_m)_i^{-1} \in \overline{\mathbf{h}}_l^{-1} F$ for some $0 < i \leq d_m$, then $(\mathbf{g}_m)_i \in F^{-1} e_k^l \subseteq F^2 \subseteq F_N$. Since $\mathbf{g}_n$ is F_N-discrete for all $n \geq N$ this implies $m < N$. If then $\overline{\mathbf{h}}_l^{-1} e_k^l (\mathbf{g}_m)_p^{-1} \in \bigcup_{j \in \{0,\dots,d\} \setminus \{l\}} \overline{\mathbf{h}}_j^{-1} F$ for some $p \in \{1,\dots,d_m\} \setminus \{i\}$, then $\mathbf{h}_j \mathbf{h}_l^{-1} \in FMF$ for some $j \in \{0,\dots,d\} \setminus \{l\}$ contradicting the FMF-discreteness of $\mathbf{h}$. Thus $\overline{\mathbf{h}}_l^{-1} e_k^l (\mathbf{g}_m)_i^{-1} \in \overline{\mathbf{h}}_l^{-1} F$ for all $0 < i \leq d_m$ and since $x_l \in U_l$ there exists $0 < i \leq d_m$ such that $(\overline{\mathbf{h}}_l x_{\mathbf{h}})(e_k^l (\mathbf{g}_m)_i^{-1}) = x_l(e_k^l (\mathbf{g}_m)_i^{-1}) = 0$. If $\overline{\mathbf{h}}_l^{-1} e_k^l (\mathbf{g}_m)_i^{-1} \in \bigcup_{j \in \{0,\dots,d\} \setminus \{l\}} \overline{\mathbf{h}}_j^{-1} F$ for all $0 < i \leq d_m$, then necessarily $d_m > d$ since otherwise $\overline{\mathbf{h}}_j \overline{\mathbf{h}}_l^{-1} \in F \bigcup_{\{(p,i) \in \mathbb{N}\times\mathbb{N} \mid d_p \leq d, i \in \{1,\dots,d_p\}\}} \{(\mathbf{g}_p)_i\} F \subseteq F_N$ for some $j \in \{0,\dots,d\} \setminus \{l\}$. So there exists $j \in \{0,\dots,d\} \setminus \{l\}$ such that there are distinct $i_0, i_1 \in \{1,\dots,d_m\}$ with $\overline{\mathbf{h}}_l^{-1} e_k^l (\mathbf{g}_m)_{i_0}^{-1}, \overline{\mathbf{h}}_l^{-1} e_k^l (\mathbf{g}_m)_{i_1}^{-1} \in \overline{\mathbf{h}}_j^{-1} F$, thus $(\mathbf{g}_m)_{i_0} (\mathbf{g}_m)_{i_1}^{-1} \in F^2$. Thus again $m < N$, but then $\overline{\mathbf{h}}_j \overline{\mathbf{h}}_l^{-1} \in FMF$, contradicting the FMF-discreteness of $\mathbf{h}$. Thus we have shown that $\overline{\mathbf{h}}_l x_{\mathbf{h}} \in U_l$ for all $l \leq d$.

For all $g \in G \setminus \{1_G\}$ the set $\{x \in 2^G \mid gx = x\}$ has empty interior. To see this suppose that $U = \mathcal{N}_s \cap \bigcap_{k\leq K} e_k T_S$ is non-empty open for some $s \in 2^{<\mathbb{N}}, K \in \mathbb{N}$ and $e_k \in G$ for $k \leq K$. If $F = \text{dom}(s)$, we can assume that $e_k \in F$ for $k \leq K$, F is symmetric, and it contains 1_G and g. Choose $m \in \mathbb{N}$ such that $F^2 \subseteq F_m$ and $d_k \geq 2$ for all $k \geq m$, and $N > m$ such that $F(\mathbf{g}_l)_i^{-1} F \subseteq F_N$ for all $l \leq m, i \in \{1, ..., d_l\}$. Let $x \in U$. Define $x' \in 2^G$ by $x' \restriction F = x \restriction F$, $x'((\mathbf{g}_N)_1) = 1$ and $x'(h) = 0$ for all $h \notin (F \cup \{(\mathbf{g}_N)_1\})$. Then $g^{-1}(\mathbf{g}_N)_1 \notin F \cup \{(\mathbf{g}_N)_1\}$, thus $gx' \neq x'$ and if $e_k(\mathbf{g}_l)_i^{-1} = (\mathbf{g}_N)_1$ for some $l \in \mathbb{N}$, $i \in \{1, ..., d_l\}$, and $k \leq K$, then $l > m$ and thus $\mathbf{g}_l$ is F-discrete and $d_l \geq 2$ implying that $x' \in U$. Let X_S be the τ-dense, G-invariant G_δ set on which $G \curvearrowright X_0$ is free and every orbit is dense and set $U_S = T_S \cap X_S$. ⊠

As a corollary we obtain the following:

Proposition 7.2.2. *Let G be a countable group and $d \in \mathbb{Z}^+$. Then there exists a free continuous action on a Polish space which is strongly d-mixing but not strongly $(d+1)$-mixing.*

Proof. Let $(F_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of G such that $\bigcup_{n\in\mathbb{N}} F_n = G$ and $1_G \in F_0$. For any sequence $\mathbf{g} \in (G^{\{1,...,d+1\}})^{\mathbb{N}}$ such that $\mathbf{g}_n$ is F_n-discrete for all $n \in \mathbb{N}$ Proposition 7.2.1 yields a free continuous action $G \curvearrowright X_S$, where $S = \{\mathbf{g}_n \mid n \in \mathbb{N}\}$, which is strongly d-mixing and an open set U_S which is $\{\mathbf{g}_n\}$-transient for every $n \in \mathbb{N}$. Thus U_S is S-transient, thus $G \curvearrowright X_S$ is not strongly $(d+1)$-mixing. ⊠

Proposition 7.2.3. *Suppose $(F_n)_{n\in\mathbb{N}}$ is an increasing sequence of symmetric finite subsets of G such that $\bigcup_{n\in\mathbb{N}} F_n = G$, $(d_n)_{n\in\mathbb{N}}$ and $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,...,d\}})^{\mathbb{N}}$ are sequences such that $\mathbf{g}_n \in G^{\{1,...,d_n\}}$, the set $\{n \in \mathbb{N} \mid d_n = d\}$ is finite for every $d \in \mathbb{Z}^+$, $\mathbf{g}_n$ is $F_n\{(\overline{\mathbf{g}_m})_i(\overline{\mathbf{g}_m})_j^{-1} \mid m < n, i, j \leq d_m\}F_n$-discrete for all $n \in \mathbb{N}$, $A_i \subseteq \mathbb{N}$ are infinite for $i \in 2$, and $S_i = \{\mathbf{g}_n \mid n \in A_i\}$ for $i \leq 2$. Then the actions of G on X_{S_i} are strongly $(<\omega)$-mixing and $\mathbf{g}_n \in \Delta_G^{X_{S_0}}(U^{\{0,...,d_n\}})$ for every non-empty open set $U \subseteq X_{S_0}$ and all but finitely many $n \in A_1 \setminus A_0$.*

Proof. Let $\{a_n^i \mid n \in \mathbb{N}\}$ be the increasing enumeration of A_i and write $\mathbf{g}^i$ for $(\mathbf{g}_{a_n^i})_{n\in\mathbb{N}}$ and d_n^i for $d_{a_n^i}$ for $i \in \{0, 1\}$. By Proposition 7.2.1 the actions $G \curvearrowright X_{S_i}$ are strongly $(<\omega)$-mixing for all $i \in \{0, 1\}$. Now assume that $W \subseteq X_{S_0}$ is a non-empty open set of the form $W = \mathcal{N}_s \cap \bigcap_{k\leq K} e_k U_{S_0}$ for some $s \in 2^{<\mathbb{N}}, K \in \mathbb{N}$, and $e_k \in G$ for $k \leq K$. We can assume that $F = \text{dom}(s)$, $e_k \in F$ for $k \leq K$, and F is symmetric for some finite $F \subseteq G$. Let $N \in \mathbb{N}$ such that $F \cup F^2 \subseteq F_N$. Let $m \in \mathbb{N}$ be such that $a_m^1 \in A_1 \setminus A_0$ and $a_m^1 > N$. We show that $\bigcap_{i\leq d_m^1} (\overline{\mathbf{g}_m^1})_i^{-1} W$ is non-empty. With the notation of Proposition 7.2.1 it is sufficient to show that $\bigcap_{i\leq d_m^1} (\overline{\mathbf{g}_m^1})_i^{-1} U$ is non-empty where $U = \mathcal{N}_s \cap \bigcap_{k\leq K} e_k T_{S_0}$ since $\bigcap_{i\leq d_m^1} (\overline{\mathbf{g}_m^1})_i^{-1} W$ is comeager in $\bigcap_{i\leq d_m^1} (\overline{\mathbf{g}_m^1})_i^{-1} U$. Let $x \in U$ and define:

(1) $x_m(f) = x((\overline{\mathbf{g}^1_m})_i f)$ for $f \in (\overline{\mathbf{g}^1_m})_i^{-1}F$ and $i \leq d^1_m$,

(2) $x_m(f) = 0$ for $f \notin \bigcup_{i \leq d^1_m} (\overline{\mathbf{g}^1_m})_i^{-1}F$.

This is well-defined since $(\overline{\mathbf{g}^1_m})_i^{-1}F \cap (\overline{\mathbf{g}^1_m})_j^{-1}F = \emptyset$ for $i \neq j \leq d^1_m$, since $\mathbf{g}^1_m$ is F^2-discrete. As in Proposition 7.2.1 we now show that $(\overline{\mathbf{g}^1_m})_i x_m \in U$ for all $i \leq d^1_m$. Certainly $(\overline{\mathbf{g}^1_m})_i x_m \in \mathcal{N}_s$ for $i \leq d^1_m$ and thus $(\overline{\mathbf{g}^1_m})_i x_m(e_k) = x(e_k) = 1$ for all $i \leq d^1_m$ and $k \leq K$. It remains to show that for every $l \in \mathbb{N}$ and $k \leq K$ there exists $0 < j \leq d^0_l$ such that $((\mathbf{g}^1_m)_i x_m)(e_k(\mathbf{g}^0_l)_j^{-1}) = 0$. Condition (2) implies that this is always the case when there exists $0 < j \leq d^0_l$ with $(\overline{\mathbf{g}^1_m})_i^{-1} e_k(\mathbf{g}^0_l)_j^{-1} \notin \bigcup_{p \leq d^1_m} (\overline{\mathbf{g}^1_m})_p^{-1}F$. So suppose that there exist $k \leq K$ and $l \in \mathbb{N}$ such that $(\overline{\mathbf{g}^1_m})_i^{-1} e_k(\mathbf{g}^0_l)_j{}^{-1} \in \bigcup_{p \leq d^1_m} (\overline{\mathbf{g}^1_m})_p^{-1}F$ for all $0 < j \leq d^0_l$. If $(\overline{\mathbf{g}^1_m})_i^{-1} e_k(\mathbf{g}^0_l)_r{}^{-1} \in \bigcup_{p \in \{0,\ldots,d^1_m\} \setminus \{i\}} (\overline{\mathbf{g}^1_m})_p^{-1}F$ for some $0 < r \leq d^0_l$, then $(\overline{\mathbf{g}^1_m})_p (\overline{\mathbf{g}^1_m})_i^{-1} \in F_N\{(\overline{\mathbf{g}_q})_u (\overline{\mathbf{g}_q})_v^{-1} \mid q \leq a^0_l, u, v \leq d_q\}F_N$ for some $p \in \{0, \ldots, d^1_m\} \setminus \{i\}$ and $(\mathbf{g}^0_l)_r \in F_N\{(\overline{\mathbf{g}_q})_u (\overline{\mathbf{g}_q})_v^{-1} \mid q \leq a^1_m, u, v \leq d_q\}F_N$ contradicting the $F_{a^1_m}\{(\overline{\mathbf{g}_q})_u (\overline{\mathbf{g}_q})_v^{-1} \mid q < a^1_m, u, v \leq d_q\}F_{a^1_m}$-discreteness of $\mathbf{g}^1_m$ or the $F_{a^0_l}\{(\overline{\mathbf{g}_q})_u (\overline{\mathbf{g}_q})_v^{-1} \mid q < a^0_l, u, v \leq d_q\}F_{a^0_l}$-discreteness of $\mathbf{g}^0_l$ depending on whether $a^0_l < a^1_m$ or $a^1_m < a^0_l$. Thus $e_k(\mathbf{g}^0_l)_r^{-1} \in F$ for all $0 < r \leq d^0_l$ and since $x \in U$ there exists $0 < r \leq d^0_l$ such that $x_m((\overline{\mathbf{g}^1_m})_i^{-1} e_k(\mathbf{g}^0{}_l)_r^{-1}) = x(e_k(\mathbf{g}^0{}_l)_r^{-1}) = 0$. This shows that $x_m \in \bigcap_{0 \leq i \leq d^1_m} (\overline{\mathbf{g}^1_m})_i^{-1}U$.

⊠

Let $[\mathbb{N}]^{\mathbb{N}}$ denote the infinite subsets of $\mathbb{N}$. Define the quasi-order $\sqsubseteq^*$ of almost inclusion on $[\mathbb{N}]^{\mathbb{N}}$ by $A \sqsubseteq^* B$ if $A \setminus B$ is finite. When R is a relation on a set X and S is a relation on a set Y, a *cohomomorphism from (X, R) to (Y, S)* is a homomorphism from the complement of R to the complement of S.

Proposition 7.2.4. *Suppose that G is a countable group, $\mathcal{F} \subseteq \mathcal{P}(G)$ is the family of infinite subsets of G, X is a Polish space, and $G \curvearrowright X$ is continuous and $\mathcal{F}$-recurrent. Let $\mathcal{O}$ be the family of continuous and free G-actions on Polish spaces with σ-expansive-transience spectrum contained in that of $G \curvearrowright X$ for which there exists a continuous and surjective homomorphism to $G \curvearrowright X$ and let $\preccurlyeq$ denote the quasi order of Baire measurable homomorphisms on $\mathcal{O}$. Then there exists a cohomomorphism from $([\mathbb{N}]^{\mathbb{N}}, \sqsupseteq^*)$ to $(\mathcal{O}, \preccurlyeq)$. In case $G \curvearrowright X$ is free, the σ-expansive-transience spectra of all elements in $\mathcal{O}$ coincide with that of $G \curvearrowright X$.*

Proof. We begin with a simple lemma.

Lemma 7.2.5. Suppose X is a topological space, $G \curvearrowright X$ is continuous-in-X, and $G \curvearrowright X$ is $\mathcal{F}$-recurrent, $K \subseteq G$ is finite, $d \in \mathbb{Z}^+$, and $U \subseteq X$ is non-empty open. Then there exists $\mathbf{h} \in \Delta^X_G(U^{\{0,\ldots,d\}})$ such that $\mathbf{h}$ is K-discrete.

Proof. Recursively construct non-empty open subsets $V_{k-1} \subseteq U$ and $\mathbf{h}_k$ for $k \in \{1, ..., d\}$ such that $\mathbf{h}_k V_k \subseteq V_{k-1}$ and $(\mathbf{h}_k)_{k \in \{1,...,d\}}$ is K-discrete. Put $V_0 = U$. Having defined $V_0, ..., V_{k-1}$ and $\mathbf{h}_1, ..., \mathbf{h}_{k-1}$ there exists $\mathbf{h}_k \in \Delta_G^X(V_{k-1} \times V_{k-1}) \setminus (K \cup \bigcup_{i \in \{1,...,k-1\}} (K^{-1}\mathbf{h}_i \cup K\mathbf{h}_i))$, since $G \curvearrowright X$ is $\mathcal{F}$-recurrent. Then there exists $V_k \subseteq V_{k-1}$ such that $\mathbf{h}_k V_k \subseteq V_{k-1}$. This finishes the recursive construction and $\mathbf{h}$ is as desired, since $V_d \subseteq \bigcap_{i \in \{0,...,d\}} \overline{\mathbf{h}_i}^{-1} U$. ⊠

Let $(F_n)_{n \in \mathbb{N}}$ be an increasing sequence of finite subsets of G such that $\bigcup_{n \in \mathbb{N}} F_n = G$ and $1_G \in F_0$, let $(V_n)_{n \in \mathbb{N}}$ be a basis for the topology of X, $\beta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ a bijection, and $p = \text{proj}_0 \circ \beta^{-1}$. Recursively choose $\mathbf{g}_n \in \Delta_G^X(V_{p(n)}^{\{0,...,n\}})$ which is $F_n\{(\overline{\mathbf{g}_m})_i (\overline{\mathbf{g}_m})_j^{-1} \mid m < n, i, j \leq m\} F_n$-discrete. For any infinite subset $A \subseteq \mathbb{N}$ define the set $\varphi(A) = \{\beta(n,k) \mid n \in \mathbb{N}, k \in A\}$, let $(a_n^A)_{n \in \mathbb{N}}$ be the increasing enumeration of $\varphi(A)$, let $\mathbf{g}^A = (\mathbf{g}_{a_n^A})_{n \in \mathbb{N}}$, and define $X_A = X \times X_{\varphi(A)}$ and let $G \curvearrowright X_A$ be the diagonal product action. By Proposition 7.2.1 the action of G on $X_{\varphi(A)}$ is topological $(< \omega)$-strongly mixing. Given any non-empty family $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,...,d\}})$, a Polish space C, and an expansively $\mathcal{S}$-recurrent action $G \curvearrowright C$, the diagonal product action of G on $C \times X_{\varphi(A)}$ is again expansively $\mathcal{S}$-recurrent thus the σ-expansive-transience spectrum of $G \curvearrowright X_A$ is contained in that of $G \curvearrowright X$ by Proposition 4.1.2 and if $G \curvearrowright X$ is free they coincide. Note that the projection of X_A onto its first coordinate is a continuous surjective homomorphism. Now, suppose $A, B \subseteq \mathbb{N}$ are infinite, $A \not\subseteq^* B$, and there is a Baire-measurable homomorphism $\psi : X_B \to X_A$. Since $X_A = \bigcup_{g \in G} g(X \times U_{\varphi(A)})$, there is a $g \in G$ such that $\psi^{-1}(g(X \times U_{\varphi(A)}))$ is non-meager, thus since $\psi^{-1}(g(X \times U_{\varphi(A)})) = g\psi^{-1}(X \times U_{\varphi(A)})$ the set $\psi^{-1}(X \times U_{\varphi(A)})$ is non-meager. Then there exist non-empty open sets $V_m \subseteq X$ and $U \subseteq X_{\varphi(B)}$ such that $\psi^{-1}(X \times U_A)$ is comeager in $V_m \times U$, thus $\Delta_G^{X_B}((V_m \times U)^{\{0,...,d\}}) \subseteq \Delta_G^{X_B}((\psi^{-1}(X \times U_A))^{\{0,...,d\}})$ for all $d \in \mathbb{Z}^+$, thus $\psi^{-1}(X \times U_A)$ is $\mathbf{g}_n^A$-transient for all $n \in \mathbb{N}$ but by Proposition 7.2.3 there exist $n, k \in \mathbb{N}$ such that $\beta(m,k) = n$, $k \in A \setminus B$ and $(V_m \times U)^{\{0,...,d\}}$ is not $\mathbf{g}_n^A$-transient - a contradiction.

⊠

7.3 Universal actions with smooth products

The actions obtained by cutting and stacking all have the property that their induced Borel equivalence relations are hyperfinite. Here we give an example of a countably infinite family of actions of the free group in at least two generators whose induced Borel equivalence relations are universal among countable Borel equivalence relations under Borel embeddability such that the diagonal product action for any two distinct actions from this family is smooth.

Proposition 7.3.1. *Suppose that G is a countable group and G_1 and G_2 are subgroups of G for which $gG_1g^{-1} \cap G_2 = \{1_G\}$ for every $g \in G$. Then there are standard Borel spaces X_1 and X_2 and Borel actions of G on X_1 and X_2, such that $E_{G_i}^{2^{G_1} \setminus \{0\}^{G_i}}$ Borel embeds into $E_G^{X_i}$ for $i = 1, 2$ and the Borel equivalence relation induced by the diagonal product action of G on $X_1 \times X_2$ is smooth.*

Proof. For $i \in \{1, 2\}$ define $\varphi_i : 2^{G_i} \to 2^G$ by $\varphi_i(x)(g) = x(g)$ for all $g \in G_i$ and $\varphi_i(x)(g) = 0$ for all $g \notin G_i$ and set $Y_i = \varphi_i(2^{G_i} \setminus \{0\}^{G_i})$. Let G act on 2^G by the shift action. Set $X_i = G \cdot Y_i$ for $i \in \{1, 2\}$. Then φ_i is a Borel embedding of $E_{G_i}^{2^{G_i} \setminus \{0\}^{G_i}}$ into $E_G^{X_i}$ for $i = 1, 2$. Consider the diagonal product action of G on $X_1 \times X_2$ and let (x_1, x_2) be in $X_1 \times X_2$. Then there exists $h \in G$ such that $h \cdot x_1 \in Y_1$. So, $\bigcup_{g \in G} Y_1 \times gY_2$ is a complete section for $E_G^{X_1 \times X_2}$. It suffices to show that $Y_1 \times gY_2$ is a partial transversal for all $g \in G$. To this end assume that $h \cdot (x_1, x_2) \in Y_1 \times gY_2$ for some $(x_1, x_2) \in Y_1 \times gY_2$. Then necessarily $h \in G_1$ and $h \cdot x_2 = (hg) \cdot y' = g \cdot y''$, for some $y', y'' \in Y_2$. Thus $g^{-1}hg \in G_2$ and hence $h \in gG_2g^{-1} \cap G_1$, so $h = 1$. ⊠

Proposition 7.3.2. *Suppose that $N \in \mathbb{N} \setminus 2 \cup \{\omega\}$ and F_N is the free group in N generators. Then there exist standard Borel spaces X_n and actions $F_N \curvearrowright X_n$ for $n \in \mathbb{N}$ for which $E_{F_N}^{X_n}$ is a universal countable Borel equivalence relation and for any distinct $n, m \in \mathbb{N}$ the action $F_N \curvearrowright X_n \times X_m$ is smooth.*

Proof. By [KS71] p. 950 there exists a malnormal subgroup H of F_N which is isomorphic to F_ω. Let $\{x_n \mid n \in \mathbb{N}\}$ be a set of free generators for H. Suppose that $A_n \subset \mathbb{N}$ are pairwise disjoint for $n \in \mathbb{N}$ and let G_A be the group generated by $\{x_n \mid n \in A\}$ and $A \subseteq \mathbb{N}$. Note that F_N, G_{A_n}, and G_{A_m} fulfill the conditions of Proposition 7.3.1 for all distinct $n, m \in \mathbb{N}$. It remains to notice that $E_{F_\omega}^{2^{F_\omega} \setminus \{0\}^{F_\omega}}$ is a universal countable Borel equivalence relation. ⊠

Chapter 8

Bibliography

[BK96] Howard Becker and Alexander S. Kechris. *The descriptive set theory of Polish group actions.* Vol. 232. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1996, pp. xii+136. ISBN: 0-521-57605-9. DOI: `10.1017/CBO9780511735264`. URL: `http://dx.doi.org/10.1017/CBO9780511735264`.

[Bur79] John P. Burgess. "A selection theorem for group actions." In: *Pacific J. Math.* 80.2 (1979), pp. 333–336. ISSN: 0030-8730. URL: `http://projecteuclid.org/euclid.pjm/1102785707`.

[Eff65] Edward G. Effros. "Transformation groups and C^*-algebras." In: *Ann. of Math. (2)* 81 (1965), pp. 38–55. ISSN: 0003-486X.

[EHN93] S. Eigen, A. Hajian, and M. G. Nadkarni. "Weakly wandering sets and compressibility in descriptive setting." In: *Proc. Indian Acad. Sci. Math. Sci.* 103.3 (1993), pp. 321–327. ISSN: 0253-4142. DOI: `10.1007/BF02866994`. URL: `http://dx.doi.org/10.1007/BF02866994`.

[EHW98] S. Eigen, A. Hajian, and B. Weiss. "Borel automorphisms with no finite invariant measure." In: *Proc. Amer. Math. Soc.* 126.12 (1998), pp. 3619–3623. ISSN: 0002-9939. DOI: `10.1090/S0002-9939-98-04489-X`. URL: `https://doi.org/10.1090/S0002-9939-98-04489-X`.

[FM77] J. Feldman and C. C. Moore. "Ergodic equivalence relations, cohomology, and von Neumann algebras. I." In: *Trans. Amer. Math. Soc.* 234.2 (1977), pp. 289–324. ISSN: 0002-9947.

[Gli61] James Glimm. "Type I C^*-algebras." In: *Ann. of Math. (2)* 73 (1961), pp. 572–612. ISSN: 0003-486X.

[GS00] Gernot Greschonig and Klaus Schmidt. "Ergodic decomposition of quasi-invariant probability measures." In: *Colloq. Math.* 84/85.part 2 (2000). Dedicated to the memory of Anzelm Iwanik, pp. 495–514. ISSN: 0010-1354.

[HKL90] L. A. Harrington, A. S. Kechris, and A. Louveau. "A Glimm-Effros dichotomy for Borel equivalence relations." In: *J. Amer. Math. Soc.* 3.4 (1990), pp. 903–928. ISSN: 0894-0347. DOI: 10.2307/1990906. URL: http://dx.doi.org/10.2307/1990906.

[HV50] Paul R. Halmos and Herbert E. Vaughan. "The marriage problem." In: *Amer. J. Math.* 72 (1950), pp. 214–215. ISSN: 0002-9327. DOI: 10.2307/2372148. URL: https://doi.org/10.2307/2372148.

[IM17] M. Inselmann and B. D. Miller. "Recurrence and the existence of invariant measures." Sept. 2017.

[IM19] M. Inselmann and B. D. Miller. "Bases for notions of recurrence." Sept. 2019.

[Kec92] Alexander S. Kechris. "Countable sections for locally compact group actions." In: *Ergodic Theory Dynam. Systems* 12.2 (1992), pp. 283–295. ISSN: 0143-3857. DOI: 10.1017/S0143385700006751. URL: http://dx.doi.org/10.1017/S0143385700006751.

[Kec95] A.S. Kechris. *Classical descriptive set theory.* Vol. 156. Graduate Texts in Mathematics. New York: Springer-Verlag, 1995, pp. xviii+402. ISBN: 0-387-94374-9.

[Kle52] V. L. Klee Jr. "Invariant metrics in groups (solution of a problem of Banach)." In: *Proc. Amer. Math. Soc.* 3 (1952), pp. 484–487. ISSN: 0002-9939. DOI: 10.2307/2031907. URL: http://dx.doi.org/10.2307/2031907.

[KM04] A. S. Kechris and B. D. Miller. *Topics in orbit equivalence.* Vol. 1852. Lecture Notes in Mathematics. Berlin: Springer-Verlag, 2004, pp. x+134. ISBN: 3-540-22603-6.

[KMS14] Itay Kaplan, Benjamin D. Miller, and Pierre Simon. "The Borel cardinality of Lascar strong types." In: *J. Lond. Math. Soc. (2)* 90.2 (2014), pp. 609–630. ISSN: 0024-6107. DOI: 10.1112/jlms/jdu041. URL: http://dx.doi.org/10.1112/jlms/jdu041.

[KS71] A. Karrass and D. Solitar. "The Free Product of Two Groups with a Malnormal Amalgamated Subgroup." In: *Canadian Journal of Mathematics* 23.5 (1971), pp. 933–959. DOI: 10.4153/CJM-1971-102-8.

[KST99] A.S. Kechris, S. Solecki, and S. Todorcevic. "Borel chromatic numbers." In: *Adv. Math.* 141.1 (1999), pp. 1–44. ISSN: 0001-8708. DOI: 10.1006/aima.1998.1771. URL: http://dx.doi.org/10.1006/aima.1998.1771.

[Kur58] Casimir Kuratowski. *Topologie. Vol. I.* Monografie Matematyczne, Tom 20. 4ème éd. Państwowe Wydawnictwo Naukowe, Warsaw, 1958, pp. xiii+494.

[Mil04] B. D. Miller. "Full groups, classification, and equivalence relations." PhD thesis. University of California, Berkeley, 2004.

[Mil08a] Benjamin Miller. "The existence of measures of a given cocycle. I. Atomless, ergodic σ-finite measures." In: *Ergodic Theory Dynam. Systems* 28.5 (2008), pp. 1599–1613. ISSN: 0143-3857. DOI: 10.1017/S0143385707001113. URL: http://dx.doi.org/10.1017/S0143385707001113.

[Mil08b] Benjamin Miller. "The existence of measures of a given cocycle. II. Probability measures." In: *Ergodic Theory Dynam. Systems* 28.5 (2008), pp. 1615–1633. ISSN: 0143-3857. DOI: 10.1017/S0143385707001125. URL: http://dx.doi.org/10.1017/S0143385707001125.

[Sri79] S. M. Srivastava. "Selection theorems for G_δ-valued multifunctions." In: *Trans. Amer. Math. Soc.* 254 (1979), pp. 283–293. ISSN: 0002-9947. DOI: 10.2307/1998270. URL: http://dx.doi.org/10.2307/1998270.

[SW82] Saharon Shelah and Benjamin Weiss. "Measurable recurrence and quasi-invariant measures." In: *Israel J. Math.* 43.2 (1982), pp. 154–160. ISSN: 0021-2172. DOI: 10.1007/BF02761726. URL: https://doi.org/10.1007/BF02761726.

[Tse15] Anush Tserunyan. "Finite generators for countable group actions in the Borel and Baire category settings." In: *Adv. Math.* 269 (2015), pp. 585–646. ISSN: 0001-8708. DOI: 10.1016/j.aim.2014.10.013. URL: http://dx.doi.org/10.1016/j.aim.2014.10.013.

[TV99] Simon Thomas and Boban Velickovic. "On the Complexity of the Isomorphism Relation for Finitely Generated Groups." In: *Journal of Algebra* 217.1 (1999), pp. 352–373. ISSN: 0021-8693. DOI: https://doi.org/10.1006/jabr.1998.7825. URL: http://www.sciencedirect.com/science/article/pii/S0021869398978255.

[Vau75] Robert Vaught. "Invariant sets in topology and logic." In: *Fund. Math.* 82 (1974/75). Collection of articles dedicated to Andrzej Mostowski on his sixtieth birthday, VII, pp. 269–294. ISSN: 0016-2736. DOI: 10.4064/fm-82-3-269-294. URL: https://doi.org/10.4064/fm-82-3-269-294.

[Wei84] Benjamin Weiss. “Measurable dynamics.” In: *Conference in modern analysis and probability (New Haven, Conn., 1982)*. Vol. 26. Contemp. Math. Amer. Math. Soc., Providence, RI, 1984, pp. 395–421. DOI: 10.1090/conm/026/737417. URL: https://doi.org/10.1090/conm/026/737417.

[Zak93] Piotr Zakrzewski. “The existence of invariant probability measures for a group of transformations.” In: *Israel J. Math.* 83.3 (1993), pp. 343–352. ISSN: 0021-2172. DOI: 10.1007/BF02784061. URL: http://dx.doi.org/10.1007/BF02784061.