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Independent and almost disjoint families

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Abstract (english)

This thesis deals with two sorts of collections of infinite subsets of the naturals: almost disjoint and independent families. However, the main focus clearly lies on independent families and the cardinal characteristic associated to them - the independence number $\mathfrak{i}$.

The thesis is composed as follows:

Section 1 provides a thorough introduction to both independent and almost disjoint families and a number of examples of such families. It is concluded with a proof of the relative consistency of $\mathfrak{a} < \mathfrak{i}$. While carrying out this proof we have our expository first encounter with the concept of indestructibility with respect to a forcing notion.

In Section 2 we gain more insight into the structure of independent families by studying two ideals associated to them. The *independence diagonalization ideal* gives rise to a forcing notion enabling us, for example, to give consistency proofs about the value of $\mathfrak{i}$ compared to $\mathfrak{c}$ as well as possible structures of the spectrum of independence. On the other hand, the *independence density ideal* can be associated to another poset $\mathbb{P}$ which adds a generic maximal independent family with some very special properties. We also find interesting connections between the two discussed ideals in this section.

Finally, in Section 3 we introduce Sacks forcing in order to show that the independent family added by $\mathbb{P}$ is indestructible with respect to a countable support iteration of this forcing notion. We give a very detailed proof of this theorem, which is perhaps the most demanding part of this thesis.

In comparison to other cardinal characteristics, few people have studied the independence number and we know little about independent families thus far. One important contribution was made in [14], which essentially laid the ground work for the study of independent families as seen today. This basis was recently built upon in [4] and [5], two papers which in return heavily influenced this thesis.

One of the most interesting open problems in this context is the question whether the inequality $\mathfrak{i} < \mathfrak{a}$ is consistent with ZFC. Nevertheless, there are many other open problems regarding independent and almost disjoint families, a selected few of them can be found at the end of this thesis.

Abstract (deutsch)

Diese Arbeit handelt von unabhängigen und fast disjunkten Familien gemeinsam mit den dazugehörigen Kardinalzahlen $\mathfrak{i}$ und $\mathfrak{a}$. Der Hauptfokus liegt klar auf unabhängigen Familien.

Die Arbeit richtet sich nach dem folgenden Schema:

Abschnitt 1 behandelt die grundlegenden Definitionen unabhängiger und fast disjunkter Familien gemeinsam mit einigen Beispielen davon. Wir beenden den Abschnitt mit einem Beweis der relativen Konsistenz von $\mathfrak{a} < \mathfrak{i}$, in dem das Konzept Unzerstörbarkeit bezüglich eines Forcings das erste Mal auftaucht.

Abschnitt 2 gewährt einen tieferen Einblick in die Struktur unabhängiger Familien im Rahmen zweier Ideale auf ω und diesen jeweils korrespondierenden Focrings. Mit dem *Unabhängigkeits-Diagonalisierungs-Ideal* kann unter anderem die relative Konsistenz von $\mathfrak{i} < \mathfrak{c}$ bewiesen werden. Das *Unabhängigkeits-Dichtheits-Ideal* kann mit einem Poset $\mathbb{P}$ assoziiert werden das eine unabhängige Familien mit speziellen Eigenschaften hinzufügt. Zuletzt werden Beziehungen zwischen diesen beiden Idealen aufgedeckt.

Abschnitt 3 widmet sich dem Sacks Forcing und dem Beweis der Tatsache dass die generische maximale unabhängige Familie die $\mathbb{P}$ hinzufügt nach einer ω_2-Iteration von Sacks Forcing mit abzählbarem Träger maximal bleibt.

Verglichen mit anderen kardinalen Charakteristiken des Kontinuums wurde die Unabhängigkeitsnummer $\mathfrak{i}$ bis jetzt erst wenig erforscht. Ein wichtiger Beitrag dazu wurde in [14] geleistet. Darauf bauen die Arbeiten [4] und [5] auf, welche wiederum diese Masterarbeit stark beeinflusst haben.

Eines der interessantesten ungelösten Probleme in diesem Kontext ist die Frage nach der Konsistenz von $\mathfrak{i} < \mathfrak{a}$. Es gibt aber noch viele andere offene Fragen, ein paar ausgewählte davon finden sich am Ende der Arbeit.

Contents

1 Preliminaries.

***Overview*:**

Section 1.1 starts with the basic definitions concerning independent families and the independence number $\mathfrak{i}$. We show the relations between $\mathfrak{i}$ and the cardinals $\mathfrak{r}$, $\mathfrak{d}$ and $\mathfrak{c}$ provable in ZFC. We conclude the section with some examples of continuum-sized independent families and apply the existence of such a family to determine the number of ultrafilters on ω.

Section 1.2 is structured similarly to the previous one, dealing with almost disjoint families and the almost disjoint number $\mathfrak{a}$. Afterwards we present a prototypical argument using isomorphisms of names to describe the behaviour of mad families in Cohen-extensions.

In Section 1.3 the behaviour of mad and independent families in Cohen-extensions is further explored, resulting in the proof of the consistency of $\mathfrak{a} < \mathfrak{i}$. To achieve this we construct a Cohen-indestructible mad family.

***References*:**

An overview about cardinal characteristics and their ZFC-relations can be found in Chapter 9 of Lorenz Halbeisen's book [9]. The short paper [8] by Stefan Geschke provides a nice list of concrete independent and almost disjoint families of size continuum. They inspired both Section 1.1 (aside from the result on ultrafilters, which is found in Thomas Jech's book [10]) and Section 1.2.
Kenneth Kunen was the first to describe a Cohen-indestructible mad family as presented in Section 1.3, this material can be found in his book [12], while one source for the fact that there are no mad families of intermediate size is the paper [2] by Jörg Brendle.

1.1 Independent Families.

Independent families are the main focus of this thesis. They are collections of infinite subsets of the natural numbers characacterized by a closure property concerning their finite boolean combinations. The study of maximal independent families gives rise to an interesting cardinal number - the independence number $\mathfrak{i}$. It is an uncountable cardinal below the continuum $\mathfrak{c}$, and thus subject to various questions of consistency.

We start with the basic definitions. The notation we use throughout this text essentially identifies finite boolean combinations with finite partial functions.

Definition 1.1. Let $\mathcal{A} \subseteq [\omega]^\omega$ be a family of infinite subsets of ω. Then $FF(\mathcal{A})$ denotes the collection of all partial functions $\mathcal{A} \to 2$ with finite domain. Given such a function $h \in FF(\mathcal{A})$ we define the corresponding *boolean combination* by

$$\mathcal{A}^h = \bigcap\{A \in dom(h) \mid A \in h(A) = 0\} \setminus \bigcup\{A \in dom(h) \mid h(A) = 1\},$$

or equivalently

$$\mathcal{A}^h = \bigcap\{A^{h(A)} \mid A \in dom(h)\},$$

where we stipulate $A^0 = A$ and $A^1 = \omega \setminus A$. The set of all finite boolean combinations of $\mathcal{A}$ is given by $BC(A) = \{\mathcal{A}^h \mid h \in FF(\mathcal{A})\}$.

Definition 1.2. A family $\mathcal{A} \subseteq [\omega]^\omega$ is called *independent* if $BC(\mathcal{A}) \subseteq [\omega]^\omega$, that is if every finite boolean combination of $\mathcal{A}$ remains infinite. The independent family $\mathcal{A}$ is *maximal* if there is no other independent family $\mathcal{A}' \neq \mathcal{A}$ which contains $\mathcal{A}$.

Definition 1.3. The *spectrum of independence* is the set of possible sizes of maximal independent families and will be denoted by $Spec(mif)$. The *independence number* $\mathfrak{i}$ is the minimal such size $\mathfrak{i} = \min(Spec(mif))$.

Uncountable cardinals $\kappa \leq 2^{\aleph_0}$ are often referred to as *cardinal characteristics* (or *cardinal invariants*) of the continuum. While there are some relations between such cardinals provable within ZFC, others remain independent. A meaningful discussion of cardinal characteristics obviously requires the absence of CH.

Our first goal is to prove that $\mathfrak{i}$ is indeed a valid cardinal characteristic, meaning that we always have the inequality $\omega_1 \leq \mathfrak{i} \leq \mathfrak{c}$. Here ω_1 denotes the first uncountable cardinal and $\mathfrak{c} = 2^{\aleph_0}$ is the size of the continuum.
Before we describe the somewhat better lower bounds for $\mathfrak{i}$ given by other cardinal characteristics, as a warm-up we first show directly that $\omega_1 \leq \mathfrak{i}$ holds:

Proposition 1.4. *Let $\mathcal{A}$ be a countable independent family. Then there exists a set $X \in [\omega]^\omega \setminus \mathcal{A}$ such that $\mathcal{A} \cup \{X\}$ is still independent, hence $\mathcal{A}$ is not maximal.*

Proof. Since $\mathcal{A}$ is countable, the set $FF(\mathcal{A})$ is countable as well, so we enumerate is as $\{h_n \mid n \in \omega\} = FF(\mathcal{A})$. We can inductively find a strictly increasing sequence $(a_n)_{n\in\omega}$ such that we have

$$a_{2n}, a_{2n+1} \in \mathcal{A}^{h_n} \text{ for all } n \in \omega.$$

This is possible since $\mathcal{A}^{h_n}$ is infinite for every $n \in \omega$ due to the independence of $\mathcal{A}$. Now the set $X = \{a_{2n} \mid n \in \omega\} \in [\omega]^\omega$ has the desired properties:
Notice that X can not already be in $\mathcal{A}$, since any $A \in \mathcal{A}$ is by itself a boolean combination, meaning that $A = \mathcal{A}^{h_n}$ for some $n \in \omega$, which implies $a_{2n+1} \in A \setminus X$. Furthermore, for any given $h \in FF(\mathcal{A})$ we have that $\mathcal{A}^h \cap X$ as well as $\mathcal{A}^h \cap (\omega \setminus X)$

is infinite because there are infinitely many extensions $h_m \supseteq h$ contributing an element a_{2m} to $\mathcal{A}^h \cap X$ and an element a_{2m+1} to $\mathcal{A}^h \cap (\omega \setminus X)$. $\square$

The reaping number $\mathfrak{r}$ is one of the cardinal characteristics which bounds $\mathfrak{i}$ from below. It is defined as follows:

Definition 1.5. A family $\mathcal{R} \subseteq [\omega]^\omega$ is called *reaping* (or *unsplittable*) if for every $X \in [\omega]^\omega$ there is a $R \in \mathcal{R}$ such that either $Y \cap R$ or $Y \setminus R$ is finite. The *reaping number* $\mathfrak{r}$ is the least possible size of a reaping family.

Remark 1.6. For $X, Y \in [\omega]^\omega$ we say that X *splits* Y if both $X \cap Y$ and $Y \setminus X$ are infinite. This explains why reaping families are also called unsplittable:
$\mathcal{R}$ is reaping if and only if there is no single $X \subseteq \omega$ splitting all members of $\mathcal{R}$.

Proposition 1.7. $\mathfrak{r} \leq \mathfrak{i}$.

Proof. Let $\mathcal{A}$ be a maximal independent family of size $\mathfrak{i}$ and consider the collection of boolean combinations $\mathcal{R} = BC(\mathcal{A})$. Then $|\mathcal{R}| = \mathfrak{i}$ and $\mathcal{R}$ is reaping:
For any $X \in [\omega]^\omega$, due to the maximality of $\mathcal{A}$ there is a $R \in BC(\mathcal{A})$ such that either $X \cap R$ or $(\omega \setminus X) \cap R = R \setminus X$ is finite. Thus we found a reaping family of size $\mathfrak{i}$, which finishes the proof. $\square$

The second cardinal characteristic serving as a lower bound for the independence number is the dominating number $\mathfrak{d}$:

Definition 1.8. For $f, g \in \omega^\omega$ we say that g *dominates* f if $f(n) < g(n)$ holds for almost all $n \in \omega$. In this case we write $f <^* g$. A family $\mathcal{D} \subseteq \omega^\omega$ is *dominating* if any $f \in \omega^\omega$ is dominated by a member of $\mathcal{D}$. The *dominating number* $\mathfrak{d}$ is the least possible size of a dominating family.

It is easily shown that $\omega_1 \leq \mathfrak{d}$ holds, since for a countable family of functions $\{f_n \mid n \in \omega\} \subseteq \omega^\omega$ we can define the function $f(n) = max\{f_k(n) + 1 \mid k \leq n\}$, which is not dominated by any of them. On the other hand $\mathfrak{d} \leq \mathfrak{c}$ trivially follows from the fact that ω^ω is itself a dominating family.

For the next proof we will need one more definition:

Definition 1.9. Given $X, Y \subseteq \omega$ we say that X is *almost contained* in Y if $X \setminus Y$ is finite. We denote this by $X \subseteq^* Y$. A set $P \in [\omega]^\omega$ is called a *pseudo-intersection* of the collection $\mathcal{C} \subseteq [\omega]^\omega$ if $P \subseteq^* C$ holds for every $C \in \mathcal{C}$.

We now want to show that $\mathfrak{d}$ is indeed below $\mathfrak{i}$, which will require a longer argument which is much more complicated than in the case of $\mathfrak{r}$. The central idea of the proof is to show that an independent family of size $< \mathfrak{d}$ can't be maximal. We do this by constructing a collection of $< \mathfrak{d}$ functions associated to this family in order to make use of the fact that there must be a function which is not dominated by any of them.

Theorem 1.10. $\mathfrak{d} \leq \mathfrak{i}$.

Proof. Let $\mathcal{A} = \{A_\alpha \mid \alpha \in \kappa\} \subseteq [\omega]^\omega$ be an independent family of size $\kappa < \mathfrak{d}$. We let $\mathcal{A}_\omega = \{A_n \mid n \in \omega\}$ be the first ω-many members of $\mathcal{A}$ and have $\mathcal{A}' = \mathcal{A} \setminus \mathcal{A}_\omega$ consist of the remaining ones. Given $g \in 2^\omega$ and $n \in \omega$ we consider

$$C_{n,g} = \mathcal{A}^{g \restriction n} = \bigcap \{A_k^{g(k)} \mid k \in n\}.$$

We first show how to deduce the theorem under the following assumption, which we will prove afterwards:

(∗) There exists a pseudo-intersection $Y_g \in [\omega]^\omega$ of the family $\{C_{n,g} \mid n \in \omega\})$ for which $Y_g \cap B$ is infinite whenever $B \in BC(\mathcal{A}')$.

If this is the case, take the sets of functions which eventually become constant

$$Q_i = \{g \in 2^\omega \mid \exists n_0 \forall k > n_0 : g(k) = 0\} \text{ for } i \in 2.$$

We enumerate $Q_0 \cup Q_1 = \{g_n \mid n \in \omega\}$, which is countable since it stands in bijection with $2^{<\omega}$. Now we inductively define $Y'_{g_n} = Y_{g_n} \setminus \bigcup \{Y_{g_k} \mid k < n\}$ for all $n \in \omega$ to get countably many pairwise disjoint infinite subsets of ω. We set

$$Z_i = \bigcup \{Y'_g \mid g \in Q_i\} \text{ for } i \in 2$$

and show that both Z_0 and Z_1 have infinite intersection with every boolean combination of $\mathcal{A}$. Because of $Z_1 \subseteq \omega \setminus Z_0$ this will imply that $\mathcal{A} \cup \{Z_0\}$ remains independent, and thus that $\mathcal{A}$ is not maximal, concluding the proof.
Given $h \in FF(\mathcal{A})$ let h_ω and h' be the respective restrictions to $\mathcal{A}_\omega$ and $\mathcal{A}'$. We choose $m \in \omega$ such that $\operatorname{dom}(h_\omega) \subseteq \{A_k \mid k \in m\}$ and $g \in Q_0$ such that $h_\omega(A_k) = 0 \leftrightarrow g(k) = 0$ whenever $A_k \in \operatorname{dom}(h_\omega)$. Now altogether we get

$$\mathcal{A}^h = \mathcal{A}^{h'} \cap \mathcal{A}^{h_\omega} \supseteq \mathcal{A}^{h'} \cap C_{m,g} \overset{*}{\supseteq} \mathcal{A}^{h'} \cap Y_g.$$

The set $\mathcal{A}^{h'} \cap Y_g$ is infinite due to (∗) and is a subset of Z_0 because $g \in Q_0$. Therefore it is almost contained in $\mathcal{A}^h \cap Z_0$ as well, which implies that this set has to be infinite. The case of $\mathcal{A}^h \cap Z_1$ is done in the same way.

To finish the proof we still have to show that (∗) holds. Note that we haven't made use of the fact that $\kappa < \mathfrak{d}$ yet, thus the following can be somehow regarded as the 'heart of the proof':

Proof of (∗). Given $s \in \omega^\omega$ we let $Y_g^s = \bigcup \{C_{n,g} \cap s(n) \mid n \in \omega\}$, which is almost contained in each $C_{n,g}$ be our candidate for the pseudo-intersection. However, in general this set is not infinite. For $B \in BC(\mathcal{A}')$, $n \in \omega$ we have that $B \cap C_{n,g} \in BC(\mathcal{A})$ is infinite, so $f_B(n) =$"the n-th element of $B \cap C_{n,g}$

in increasing order" is an element of ω^ω. The family $\{f_B \mid B \in BC(\mathcal{A}')\}$ has size $\kappa < \mathfrak{d}$, so we can find some $s_0 \in \omega^\omega$ such that for any B the set $\{n \in \omega \mid s_0(n) > f_B(n)\}$ is infinite, which implies that $B \cap Y_g^{s_0}$ is infinite as well, so $Y_g^{s_0}$ has the desired properties. □

Let us now consider the inequality $\mathfrak{i} \leq \mathfrak{c}$. We will give various examples of independent families of size $\mathfrak{c}$, the first of which is most commonly used to prove the above inequality and which is generalizable to higher cardinals:

Proposition 1.11. *There exists an independent family of size* $\mathfrak{c}$.

Proof. Instead of constructing the family on ω we will use the countable set

$$\mathcal{C} = \{(s, A) \mid s \in [\omega]^{<\omega}, A \subseteq \mathcal{P}(s)\}.$$

For every $X \in [\omega]^\omega$ we let $P_X = \{(s, A) \in \mathcal{C} \mid X \cap s \in A\}$. For $X, Y \in [\omega]^\omega$ distinct we can choose some finite $s \subset \omega$ such that $s \cap X \neq s \cap Y$ and find that $(s, \{s \cap X\}) \in P_X \setminus P_Y$, which implies that $\mathcal{A} = \{P_X \mid X \in [\omega]^\omega\}$ is of size $\mathfrak{c}$.
It remains to show that $\mathcal{A}$ is an independent family as well. Given $h \in FF(\mathcal{A})$, let $h^{-1}(\{0\}) = \{P_{X_1}, \dots, P_{X_m}\}$ and $h^{-1}(\{1\}) = \{P_{X_{m+1}}, \dots, P_{X_{m+n}}\}$. First, choose $s \in [\omega]^\omega$ sufficiently large to assure that for all distinct $i, j \leq m+n$ we get $s \cap X_i \neq s \cap X_j$ and set $A = \{s \cap X_i \mid i \leq m\}$. For $k \in \omega \setminus s$ define $s_k = s \cup \{k\}$ and $A_k = A \cup \{t \cup \{k\} \mid t \in A\}$. Then the infinite set $\{(s_k, A_k) \mid k \in \omega \setminus s\}$ is contained in $\mathcal{A}^h$, because for $i \leq m + n$ the following holds:

$$(s_k, A_k) \in P_{X_i} \Leftrightarrow s_k \cap X_i \in A_k \Leftrightarrow (X_i \cap s) \cup (X_i \cap \{k\}) \in A_k \Leftrightarrow i \leq m.$$

Thus $\mathcal{A}^h$ is infinite and therefore $\mathcal{A}$ is indeed an independent family. □

A straightforward application of Zorn's lemma now implies the existence of a *maximal* independent family of size $\mathfrak{c}$ as well:

Corollary 1.12. $\mathfrak{i} \leq \mathfrak{c}$.

Proof. Let $\mathcal{F} = \{\mathcal{A} \subseteq [\omega]^\omega \mid \mathcal{A}$ is independent and of size $\mathfrak{c}\}$. By the above Proposition $\mathcal{F}$ is nonempty, and given an increasing chain $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots$ in $\mathcal{F}$ it is easy to see that $\mathcal{A} = \bigcup\{\mathcal{A}_n \mid n \in \omega\}$ lies in $\mathcal{F}$ as well, as for any $h \in FF(\mathcal{A})$ there exists an $n \in \omega$ such that $\mathrm{dom}(h) \subseteq \mathcal{A}_n$. Thus by Zorn's Lemma we get that $\mathcal{F}$ contains a maximal element, which is a maximal independent family of size $\mathfrak{c}$ witnessing that $\mathfrak{i} \leq \mathfrak{c}$. □

As demonstrated in Proposition 1.11. it is often easier to find independent families on a countable set that somehow has more structure than ω. In the following two examples we look at other interesting constructions of maximal independent families of size continuum. A third example related to almost disjoint families will be presented in Section 1.2.

Example 1.13. (***Polynomials***)
Consider the countable set $\mathbb{Q}[X]$ of polynomials with rational coefficients. For $r \in \mathbb{R}$ define $A_r = \{p \in \mathbb{Q}[X] \mid p(r) > 0\}$. We show that $\mathcal{A} = \{A_r \mid r \in \mathbb{R}\}$ is independent. Given $r_1, \ldots, r_m, r_{m+1}, \ldots, r_{m+n}$ distinct real numbers, we can find a polynomial $p \in \mathbb{Q}[X]$ such that $p(r_i) > 0$ for $i \leq m$ and $p(r_{m+i}) < 0$ for $i \leq n$. The boolean combination $\bigcap\{A_{r_i} \mid i \leq m\} \setminus \bigcup\{A_{r_{m+i}} \mid i \leq n\}$ thus contains the infinite set of all positive rational multiples of the polynomial p.

Example 1.14. (***Finite approximation***) This example is built on the following fact : For all $n \in \omega$ we can find a family $(X_k)_{k<n}$ such that $X_k \subseteq \mathcal{P}(n)$ and for any disjoint $S, T \subseteq n$ we get that

$$\bigcap_{k \in S} X_k \cap (\mathcal{P}(n) \setminus \bigcup_{k \in T} X_k) \neq \emptyset.$$

For example, we may set $X_k = \{A \subseteq n \mid k \in A\}$ and find that $S \setminus T$ will surely be contained in the above set.
Now for all $n \in \omega$ choose a family $\{X^n_s \mid s \in 2^n\}$ of subsets of some finite set Y_n with $|Y_n| \geq 2^{2^n}$, such that for disjoint $S, T \subseteq 2^n$ we have that

$$\bigcap_{s \in S} X^n_s \cap (Y_n \setminus \bigcup_{s \in T} X^n_s) \neq \emptyset.$$

Furthermore we choose the Y_n to be pairwise disjoint, then $Y = \bigcup\{Y_n \mid n \in \omega\}$ is of size ω. For each $\sigma \in 2^\omega$ we now set $X_\sigma = \bigcup_{n \in \omega} X^n_{\sigma \restriction n}$. Then the family $\mathcal{A} = \{X_\sigma \mid \sigma \in 2^\omega\}$ is independent on Y:
Given pairwise distinct $\sigma_1, \ldots \sigma_k, \sigma_{k+1}, \ldots, \sigma_{k+l}$ in 2^ω we can find some $n_0 \in \omega$ such that for all $n > n_0$ all the $(\sigma_i \restriction n)_{i \leq k+l}$ are pairwise distinct. Therefore for all $n > n_0$ the set $\bigcap_{i \leq k} X^n_{\sigma_i \restriction n} \cap (Y_n \setminus \bigcup_{j \leq l} X^n_{\sigma_{k+j} \restriction n})$ is nonempty subset of Y_n. Therefore

$$\bigcap_{i \leq k} X_{\sigma_i} \cap (Y \setminus \bigcup_{j \leq l} X_{\sigma_{k+j}}) = \bigcup_{n \in \omega} \Big(\bigcap_{i \leq k} X^n_{\sigma_i \restriction n} \cap (Y_n \setminus \bigcup_{j \leq l} X^n_{\sigma_{k+j} \restriction n})\Big)$$

is infinite.

A nice application of the fact that there exists such an independent family lies in determining how many ultrafilters there are on ω. Since every ultrafilter is an element of $\mathcal{P}(\mathcal{P}(\omega))$, clearly there are at most $2^{\mathfrak{c}}$ many ultrafilters. We show that we can indeed find $2^{\mathfrak{c}}$ distinct ultrafilters on ω using independent families.

Proposition 1.15. *There are $2^{\mathfrak{c}}$ many distinct ultrafilters on ω.*

Proof. Let $\mathcal{A}$ be an independent family of size $\mathfrak{c}$. For every total function $f : \mathcal{A} \to 2$ we consider the set $G_f = FR(\omega) \cup \{A^{f(A)} \mid A \in \mathcal{A}\}$ where $FR(\omega)$ denotes the Frechét-filter $\{X \subseteq \omega \mid \omega \setminus X \text{ is finite}\}$ of cofinite sets. Every G_f has the (strong) finite intersection property and can therefore be extended to an ultrafilter $\mathcal{U}_f \supseteq G_f$. Furthermore for $f \neq g$ the ultrafilters $\mathcal{U}_f$ and $\mathcal{U}_g$ are distinct, because for $A \in \mathcal{A}$ with $f(A) \neq g(A)$ we have that $A \in \mathcal{U}_f \setminus \mathcal{U}_g$ or $A \in \mathcal{U}_g \setminus \mathcal{U}_f$. □

In fact, this proposition is generalizable to higher cardinals κ.

Remark 1.16. For every infinite cardinal κ there are $2^{2^{\kappa}}$ many ultrafilters on κ. To prove this, the above may be generalized as follows:
Similarly to Proposition 1.11. one shows that there exists a family $\mathcal{A} \subseteq [\kappa]^{\kappa}$ of size 2^{κ} such that all finite boolean combinations of $\mathcal{A}$ are of size κ, that is $BC(\mathcal{A}) \subseteq [\kappa]^{\kappa}$. Then repeat the procedure of the proof of Proposition 1.15. using this family (cf. [10]).

1.2 Almost disjoint Families.

We will now introduce almost disjoint families and the almost disjoint number $\mathfrak{a}$. The results here will make apparent that there are similarities between almost disjoint and independent families. This, in return, will give us the opportunity to introduce the concept of indestructibility through a comparatively easy example in the framework of almost disjoint families.

Definition 1.17. An infinite family $\mathfrak{A} \subseteq [\omega]^{\omega}$ is called *almost disjoint* if any two distinct elements of $\mathfrak{A}$ have finite intersection, formally

$$\forall A, B \in \mathfrak{A} : A \neq B \Rightarrow |A \cap B| < \omega.$$

Furthermore $\mathfrak{A}$ is *maximal almost disjoint* (often abbreviated *mad*) if it is not properly contained in any other almost disjoint family.

Definition 1.18. By $Spec(mad)$ we denote the *spectrum of almost disjointness*, which is the set of possible sizes of maximal almost disjoint families. The minimal such size is the *almost disjoint number* $\mathfrak{a} = \min(Spec(mad))$.

We quickly verify that $\omega_1 \leq \mathfrak{a} \leq \mathfrak{c}$ holds, starting with the leftmost inequality using a straightforward diagonalization argument.

Lemma 1.19. *Let $\mathfrak{A} \subseteq [\omega]^{\omega}$ be a countable almost disjoint family. Then there is some $A \in [\omega]^{\omega} \setminus \mathfrak{A}$ such that $\mathfrak{A} \cup \{A\}$ remains almost disjoint.*

Proof. Enumerate $\mathfrak{A} = \{A_n \mid n \in \omega\}$. We inductively construct the infinite set $A = \{a_n \mid n \in \omega\}$ starting with an arbitrary $a_0 \in A_0$ and such that $a_{n+1} \in A_{n+1} \setminus \bigcup\{A_k \mid k \leq n\}$ holds for all $n \in \omega$. It is always possible to pick such an element due to the fact that

$$A_{n+1} \setminus \bigcup\{A_k \mid k \leq n\} = A_{n+1} \setminus \bigcup\{A_{n+1} \cap A_k \mid k \leq n\}$$

is infinite, since the almost disjointness of $\mathfrak{A}$ implies that $\bigcup\{A_{n+1} \cap A_k \mid k \leq n\}$ must be finite. $A \notin \mathfrak{A}$ follows immediately from $a_{n+1} \in A \setminus A_n$ for all $n \in \omega$, and since by construction

$$A \cap A_n \subseteq \{a_0, \dots, a_n\}$$

holds for each $n \in \omega$ we get that $\mathfrak{A} \cup \{A\}$ is still almost disjoint. $\square$

The existence of an almost disjoint familiy of size $\mathfrak{c}$ is, for example, witnessed by the *full binary tree* 2^ω consisting of all infinite $0-1-$ *sequences* with the partial ordering given by end-extensions.

Proposition 1.20. *There exists an almost disjoint family of size* $\mathfrak{c}$.

Proof. We construct the family on the countable set $2^{<\omega}$. For a branch $t \in 2^\omega$ of the full binary tree let $A_t = \{t \restriction n \mid n \in \omega\}$ consist of all its finite initial segments. Our almost disjoint family is now given by $\mathfrak{A} = \{A_t \mid t \in 2^\omega\}$.
The fact that $|\mathfrak{A}| = |2^\omega| = \mathfrak{c}$ is clear, and to show almost disjointness we take distinct $s, t \in 2^{<\omega}$ and note that the intersection $A_s \cap A_t = \{s \restriction n \mid n < n_0\}$ is finite where $n_0 = min_k\{s(k) \neq t(k)\} \in \omega$, the first place where s and t differ is well-defined as long as $s \neq t$. $\square$

Now a straightforward application of Zorn's Lemma guarantees the existence of a continuum-sized *mad* family as well. Since this has already been demonstrated in Corollary 1.12. we omit the proof of the following:

Corollary 1.21. $\mathfrak{a} \leq \mathfrak{c}$. $\square$

Example 1.22. We look at another easy construction of an almost disjoint family, this time constructed on the set $\mathbb{Q}$. Given $r \in \mathbb{R}$ let $(r_n)_{n\in\omega} \subseteq \mathbb{Q}$ be a sequence of rationals converging towards it and set $A_r = \{r_n \mid n \in \omega\}$. Then $\mathfrak{A} = \{A_r \mid r \in \mathbb{R}\}$ is almost disjoint:
For distinct reals $r, s \in \mathbb{R}$ there is some $\epsilon > 0$ such that the two intervals $B_\epsilon(r) = (r-\epsilon, r+\epsilon)$ and $B_\epsilon(s) = (s-\epsilon, s+\epsilon)$ are disjoint. But by the definition of convergence only finitely many elements of A_x are outside of $B_\epsilon(x)$ for every $x \in \mathbb{R}$. Therefore $A_r \cap A_s$ must be finite.

We can apply the existence of an almost disjoint family of size $\mathfrak{c}$ to construct another example of an independent family of the same size as well.

Example 1.23. Given an almost disjoint family $\mathfrak{A} \subseteq [\omega]^\omega$ of size continuum and one of its members $A \in \mathfrak{A}$, we set

$$X_A = \{x \in [\omega]^{<\omega} \mid x \cap A \neq \emptyset\},$$

in other words X_A consists of all finite subsets of ω intersecting A. Clearly $A \in [\omega]^\omega$ and $|[\omega]^{<\omega}| = \omega$ imply $|X_A| = \omega$. Now consider the family given by $\mathcal{A} = \{X_A \mid A \in \mathfrak{A}\}$, which is of size $\mathfrak{c}$ as well since for $A \neq B$ and find some $a \in A \setminus B$ and notice that $\{a\} \in X_A \setminus X_B$. Furthermore $\mathcal{A}$ is independent:
Given pairwise distinct $A_1, \ldots, A_m, B_1, \ldots, B_n \in \mathfrak{A}$, the set $A_i \setminus \bigcup_{j \leq n} B_j$ is infinite for every $1 \leq i \leq n$. Thus we can find infinitely many finite sets intersecting each A_i and disjoint from each B_j, meaning that the boolean combination $\bigcap_{i \leq m} X_{A_i} \setminus \bigcup_{j \leq n} X_{B_j}$ is infinite.

We end this section with a demonstration of an argument using *isomorphisms of names* to get a grasp of the spectrum of almost disjointness in the extension $V^{\mathbb{C}_\lambda}$ adding λ-many Cohen-reals to a ground model V satisfying CH.

Theorem 1.24. *Let $V \models CH$ and $\mathbb{C}_\lambda = (Fn(\omega \times \lambda, 2), \supseteq)$ be the Cohen-poset for a regular uncountable cardinal λ. Then there are no mad families of size κ with $\aleph_1 < \kappa < \lambda$ in the generic extension $V^{\mathbb{C}_\lambda}$.*

Proof. Let $\{\dot{A}_\alpha\}_{\alpha \in \kappa}$ be names for subsets of ω. Using nice names we identify each of those $\dot{A}_\alpha$ with $\{(p^\alpha_{n,i}, k^\alpha_{n,i})\}_{n,i \in \omega}$ where the $\{p^\alpha_{n,i}\}_{i \in \omega}$ are maximal antichains and $k^\alpha_{n,i} \in 2$ are such that

$$k^\alpha_{n,i} = 0 \Leftrightarrow p^\alpha_{n,i} \Vdash \check{n} \in \dot{A}_\alpha \text{ and } k^\alpha_{n,i} = 1 \Leftrightarrow p^\alpha_{n,i} \Vdash \check{n} \notin \dot{A}_\alpha.$$

Now let $B_\alpha = \bigcup\{dom(p^\alpha_{n,i}) \mid i, n \in \omega\}$. Then the union $\bigcup\{B_\alpha \mid \alpha \in \kappa\}$ is of size $\leq \kappa$ since all B_α are countable. By the Δ-system lemma we may assume without loss of generality that $\{B_\alpha\}_{\alpha < \omega_2}$ forms a Δ-system with root R. Furthermore, for $\alpha, \beta < \omega_2$ choose bijections $\varphi_{\alpha,\beta} : B_\alpha \to B_\beta$ fixing R. One can find such bijections by extending id_R by a bijection witnessing $|B_\alpha \setminus R| = |B_\beta \setminus R|$. Each bijection $\phi_{\alpha,\beta}$ induces an order-isomorphism $\psi_{\alpha,\beta} : \mathbb{C}_{B_\alpha} \to \mathbb{C}_{B_\beta}$ assigning to $p \in Fn(B_\alpha, 2)$ the finite partial function $\psi(p) \in Fn(B_\beta, 2)$ with $dom(\psi(p)) = \phi_{\alpha,\beta}[dom(p)]$ and $\psi(p)(x) = p(\phi^{-1}_{\alpha,\beta}(x))$.
Note that by CH there are only ω_1 many isomorphism types of names, since every name essentially corresponds to an 'ω^2-sized binary matrix' $(k_{n,i})$. Therefore, by the infinite pigeonhole-principle, amongst the names $(\dot{A}_\alpha)_{\omega_2}$ there must be ω_2 isomorphic ones, allowing us to assume without loss of generality that $\psi_{\alpha,\beta}$

sends $\dot{A}_\alpha$ to $\dot{A}_\beta$:

$$\forall n, i \in \omega : k_{n,i} = k^\alpha_{n,i} = k^\beta_{n,i} \text{ and } \psi_{\alpha,\beta}(p^\alpha_{n,i}) = p^\beta_{n,i}.$$

We now define a new name $\dot{A}_\kappa$, choosing B_κ such that $B_\kappa \cap \bigcup_{\alpha<\kappa} B_\alpha = R$ (this is possible since $\kappa < \lambda$, thus there is "enough space left") and such that there is a bijection $\phi_{\alpha,\kappa} : B_\alpha \to B_\kappa$ fixing R for every $\alpha < \omega_2$. As before let $\psi_{\alpha,\kappa}$ denote the indused isomorphisms, then we finish the construction of $\dot{A}_\kappa$ by stipulating

$$p^\kappa_{n,i} = \psi_{\alpha,\kappa}(p^\alpha_{n,i}) \text{ and } k^\kappa_{n,i} = k_{n,i} \text{ for an arbitrary } \alpha < \omega_2.$$

We now show that $\{\dot{A}_\alpha\}_{\alpha<\kappa}$ can't correspond to a maximal almost disjoint family in $V^{\mathbb{C}_\lambda}$. Let $\beta < \kappa$ be arbitrary. Since there are only $\aleph_1$-many subsets of B_β, there are $\alpha, \gamma < \omega_2$ such that $B_\alpha \cap B_\beta = B_\gamma \cap B_\beta$. Now we have that

$$B_\alpha \cap B_\beta = (B_\alpha \cap B_\beta) \cap (B_\gamma \cap B_\beta) = (B_\alpha \cap B_\gamma) \cap B_\beta = R \cap B_\beta \subseteq R,$$

and therefore $B_\alpha \cap B_\beta = B_\kappa \cap B_\beta$. This means that there is a valid bijective extension of $\phi_{\alpha,\kappa}$ mapping $B_\alpha \cup B_\beta$ to $B_\kappa \cup B_\beta$. This induces an isomorphism $\mathbb{C}_{B_\alpha \cup B_\beta} \cong \mathbb{C}_{B_\kappa \cup B_\beta}$. So we conclude that

$$(\Vdash_{\mathbb{C}_\lambda} \text{"}\dot{A}_\alpha \cap \dot{A}_\beta \text{ is finite"}) \Leftrightarrow (\Vdash_{\mathbb{C}_\lambda} \text{"}\dot{A}_\kappa \cap \dot{A}_\beta \text{ is finite"}).$$

Now if $\mathfrak{A} = \{\dot{A}_\alpha[G] \mid \alpha < \kappa\}$ ends up being an almost disjoint family for a $\mathbb{C}_\lambda$-generic filter G the above implies that the set $\dot{A}_\kappa[G]$ witnesses that this family is not maximal. This finishes the proof. □

1.3 Cohen-Indestructibility and Con($\mathfrak{a} < \mathfrak{i}$).

We now further explore the behaviour of both independent and almost disjoint families in Cohen -extensions of a ground model satisfying CH. The main result here is the consistency of $\mathfrak{a} < \mathfrak{i}$ and the concept of *indestructibility* will play a crucial part in showing this. While towards the end of the thesis we consider the more complicated case of Sacks-indestructible maximal independent families, in this section we will shed some light on the concept by describing a Cohen-indestructible *mad* family.

Theorem 1.25. *Let $V \models CH$ and $\kappa \in V$ be uncountable with $\kappa \geq \omega_2$. There exists a Cohen-indestructible mad family $\mathfrak{A}$ in V, meaning that $\mathfrak{A}$ remains mad in every $\mathbb{C}_\kappa$-generic extension $V^{\mathbb{C}_\kappa}$.*

Proof. The idea is to construct a family $\mathfrak{A}$ in the ground model in such a way that it remains mad after adding a single Cohen real through $\mathbb{C} = (Fn(\omega, 2), \supseteq)$.

Afterwards we verify that this family also remains mad after adding κ many Cohen reals through $\mathbb{C}_\kappa$.
We construct $\mathfrak{A} = \{A_\xi \mid \xi \in \omega_1\} \subseteq [\omega]^\omega$ in V as follows: For $\{A_n \mid n \in \omega\}$ we choose any collection of pairwise disjoint infinite subsets of ω.
Next, let $\{(p_\xi, \tilde{x}_\xi) \mid \omega \leq \xi < \omega_1\}$ enumerate all pairs $(p, \tilde{x})$ where $p \in Fn(\omega, 2)$ is a condition of $\mathbb{C}$ and $\tilde{x}$ is a nice name for a subset of ω. Due to CH, there are ω_1-many nice names in V and therefore it is possible to find such an enumeration.
Now let $\omega \leq \xi < \omega_1$ and inductively assume that A_η is already defined for $\eta < \xi$. The set $A_\xi \in [\omega]^\omega$ should then be chosen such that the following conditions hold:

(1) For every $\eta < \xi$ the intersection $A_\eta \cap A_\xi$ is finite.

(2) If
$$p_\xi \Vdash |\tilde{x}_\xi| = \omega \text{ and } \forall \eta < \xi : p_\xi \Vdash |\tilde{x}_\xi \cap A_\eta| < \omega, \tag{$*$}$$
then the set of conditions $\{q \leq p_\xi \mid q \Vdash |A_\xi \cap \tilde{x}_\xi| = \omega\}$ is dense above p_ξ.

We verify that it is always possible to choose A_ξ in this manner. If $(*)$ doesn't hold we don't have to concern ourselves with satisfying condition (2) and thus can find a A_ξ almost disjoint from all the A_η already constructed since $\{A_\eta \mid \eta < \xi\}$ is countable, and thus not maximal. Now suppose that $(*)$ holds and see what happens in a $\mathbb{C}$-generic extension $V[G]$ with $p_\xi \in G$:
Due to (2), we have that
$$V[G] \models \tilde{x}_\xi[G] \in [\omega]^\omega \text{ and } \forall \eta < \xi : |\tilde{x}_\xi[G] \cap A_\eta| < \omega.$$
This essentially means that after adding $\tilde{x}_\xi[G]$ to $\{A_\eta \mid \eta < \xi\}$ the family remains almost disjoint. We will construct A_ξ making sure that this potential candidate is eliminated, that is we assure that $V[G] \models |\tilde{x}_\xi[G] \cap A_\xi| = \omega$. Here is how we construct the set A_ξ appropriately:
Since $\eta < \omega_1$ is countable we 're-enumerate' our family constructed thus far as $\{B_n \mid n \in \omega\} = \{A_\eta \mid \eta < \xi\}$. Furthermore we let $\{(n_i, q_i) \mid i \in \omega\}$ be an enumeration of the countable set $\omega \times \{q \in Fn(\omega, 2) \mid q \leq p_\xi\}$. Since we are assuming that $(*)$ holds, for each $i \in \omega$ we have that
$$q_i \Vdash |\tilde{x}_\xi \setminus (B_0 \cup \cdots \cup B_i)| = \omega.$$
Therefore we may find an extension $r_i \leq q_i$ together with an integer $m_i \geq n_i$ such that $m_i \notin B_0 \cup \cdots \cup B_i$ and $r_i \Vdash m_i \in \tilde{x}_\xi$. Now we set $A_\xi = \{m_i \mid i \in \omega\}$.
For any $q \leq p_\xi$, $n \in \omega$ and finite collection $\{B_{i_0}, \ldots, B_{i_N}\}$ we now have a condition $q' \leq q$ and $m \geq n$ with
$$q' \Vdash m \in \tilde{x}_\xi \text{ and } m \notin B_{i_0} \cup \cdots \cup B_{i_N}.$$

But we constructed A_ξ in such a way that $\tilde{x}_\xi[G]$ can't be added to $\{A_\eta \mid \eta \leq \xi\}$, and thus the mad family $\mathfrak{A} = \{A_\xi \mid \xi \in \omega_1\}$ remains mad in $V[G]$. Finally we show that this $\mathfrak{A}$ remains *mad* in $V[G_\kappa]$ as well whenever G_κ is a $\mathbb{C}_\kappa$-generic filter:
Suppose towards a contradiction that $\mathfrak{A}$ is not mad in $V[G_\kappa]$, meaning that

$$V[G_\kappa] \models \exists x \in [\omega]^\omega : \forall A_\xi \in \mathfrak{A} : |x \cap A_\xi| < \omega.$$

This means that there is a $\mathbb{C}_\kappa$-name $\tilde{x}$ and a $\mathbb{C}_\kappa$-condition p forcing this for every $\xi < \omega_1$:

$$p \Vdash |\tilde{x}| = \omega \text{ and } |\tilde{x} \cap A_\xi| < \omega.$$

However, since $\mathbb{C}_\kappa$ satisfies *ccc* there is a countable set $I_0 \subseteq \kappa$ such that there is a nice $\mathbb{C}_{I_0}$-name $\tilde{x}_0$ for a subset of ω together with a $\mathbb{C}_{I_0}$ condition p_0 such that again for all $\xi < \omega_1$ we have

$$p_0 \Vdash |\tilde{x}_0| = \omega \text{ and } |\tilde{x}_0 \cap A_\xi| < \omega.$$

But since there is an isomorphism $\mathbb{C}_\omega \simeq \mathbb{C}$ we can replace $\mathbb{C}_{I_0}$ by $\mathbb{C}$ and make use of our construction of $\mathfrak{A}$. There is a pair $(p_{\xi_0}, \tilde{x}_{\xi_0})$ such that for all $\xi < \omega_1$

$$p_{\xi_0} \Vdash |\tilde{x}_{\xi_0}| = \omega \text{ and } |\tilde{x}_{\xi_0} \cap A_\xi| < \omega.$$

which in particular leads to $p_{\xi_0} \Vdash |\tilde{x}_{\xi_0} \cap A_{\xi_0}| < \omega$, contradicting condition (2) in our construction of A_{ξ_0}. $\square$

Combining this result with Theorem 1.24., we now have a full description of the spectrum of almost disjointness in the Cohen model:

Corollary 1.26. *Let κ be regular uncountable and $V \models CH$. Then*

$$V^{\mathbb{C}_\kappa} \models \ \textit{Spec}(mad) = \{\aleph_1, \mathfrak{c}\}.$$

$\square$

We now turn our interest towards the behaviour of independent families in Cohen extensions. Contrary to *mad* families Cohen forcing can't preserve the maximality of any independent family. The ground work for the proof of the relative consistency of $\mathfrak{a} < \mathfrak{i}$ is done in the next proposition:

Proposition 1.27. *Let $\mathcal{A} \subseteq [\omega]^\omega$, $\mathcal{A} \in V$ be an independent family and let $I \in V$ be infinite. If G is a $\mathbb{C}_I$-generic filter over V then $\mathcal{A}$ is not a maximal independent family in $V[G]$.*

Proof. It suffices to consider $I = \omega$, since in any other case we may pick $I_0 \subseteq I$ with $(|I_0| = \omega)^V$ and consider the $\mathbb{C}$-generic model $V[G \cap Fn(I_0, 2)] \subseteq V[G]$. So we are left with showing that for the set A corresponding to generic real $\bigcup G$ we have that $\mathcal{A} \cup \{A\}$ is still independent in $V[G]$. In fact we will show that whenever $B \subseteq \omega$, $B \in V$ is infinite, both $A \cap B$ and $B \setminus A = B \cap (\omega \setminus A)$ are infinite.
For any $m \in \omega$ we have that the set

$$D_m = \{p \in Fn(\omega, 2) \mid \exists n > m : p(n) = 1 \wedge n \in B\}$$

is dense in $\mathbb{C}$, since for any condition p we can choose an element of the infinite set $\{n \in B \setminus dom(p) \mid n > m\}$ and have $p \cup \{(n, 1)\} \supseteq p$. This implies that in $V[G]$ we have that $A \cap B$ is infinite. To show that $B \setminus A$ is infinite as well we use the dense sets

$$E_m = \{p \in Fn(\omega, 2) \mid \exists n > m : p(n) = 0 \wedge n \in B\}$$

for all $m \in \omega$ in the exact same way. $\square$

Now it will be fairly easy to prove the main result of this section.

Theorem 1.28. *Let $V \models CH$ and let $\kappa \in V$ be regular with $\kappa \geq \omega_2$. Furthermore let G be $\mathbb{C}_\kappa$-generic over V. Then*

$$V[G] \models \mathit{Spec}(mad) = \{\aleph_1, \mathfrak{c}\} \wedge \mathit{Spec}(mif) = \{\mathfrak{c}\}.$$

In particular $V[G] \models \mathfrak{a} < \mathfrak{i}$.

Proof. We only have to show that in $V[G]$ every maximal independent family has size $\kappa = \mathfrak{c}$. Suppose that $\mathcal{A}$ is an independent family in $V[G]$ of size $\lambda < \kappa$. We show that $\mathcal{A}$ is not maximal:
Let $\mathcal{A} = \{A_\xi \mid \xi \in \lambda\}$ and consider the set of pairs

$$X = \{(\xi, n) \mid n \in A_\xi\} \subseteq \lambda \times \omega.$$

There exists a $I_0 \subset \kappa$ with $(|I_0| \leq \lambda)^V$ and $X \in V[G_0]$ where $G_0 = G \cap Fn(I_0, 2)$. Let $I_1 = \kappa \setminus I_0$, then we can split up $V[G] = V[G_0][G_1]$ for a $\mathbb{C}_{I_1}$-generic filter G_1 over $V[G_0]$. Now $\mathcal{A} \in V[G_0]$ and we may apply the previous proposition to conclude that $\mathcal{A}$ is not maximal in $V[G]$. $\square$

2 Two Ideals of Independence.

***Overview*:**
We explore two ideals associated to an independent families as well as two forcing notions corresponding to them:

In Section 2.1 we introduce independence diagonalization ideals which can be used to adjoin independent reals, thereby enabling us to gain some control over the value of $\mathfrak{i}$ compared to $\mathfrak{c}$ as well as the composition of $Spec(mif)$ in extensions obtained by iterated forcing.

In Section 2.2 we have a look at the independence density ideal together whith a poset $\mathbb{P}$ which adjoins a generic maximal independent family. We give a thorough combinatorial description of this poset. Lastly we show that the filter dual to the density ideal of the generic maximal independent family is Ramsey.

In Section 2.3 we compare the two ideals of independence and give a sufficient condition for them being equal. Densely maximal independent families are introduced and we show that their corresponding diagonalization ideals add dominating reals.

***References*:**
Most of the material covered in this Section is motivated by the papers [4] by Vera Fischer & Diana Montoya and [5] by Vera Fischer & Saharon Shelah. The independence density ideal introduced in Section 2.2 was also studied by Michael Perron in his doctoral thesis [13].

2.1 The independence diagonalization ideal.

Recall that an *ideal* on ω is by definition a collection of subsets $\mathcal{I} \subseteq \mathcal{P}(\omega)$ which contains $\emptyset$ and is closed under subsets and finite unions. Usually it is helpful to think of the elements of an ideal as being small in some sense.

Starting with an independent family $\mathcal{A}$ we will find some associated ideal $\mathcal{I}_\mathcal{A}$ which will then give rise to a forcing notion $\mathbb{B}(\mathcal{I}_\mathcal{A})$ adding a real which is independent over $\mathcal{A}$ in the generic extension. This enables us to prove, for example, the consistency of $\mathfrak{i} < \mathfrak{c}$. We start with the following construction using transfinite induction:

Lemma 2.1. *Let $\mathcal{A}$ be an independent family. There exists an ideal $\mathcal{I}_\mathcal{A}$ on ω which satisifes the following two properties:*

(i). $\mathcal{I}_\mathcal{A} \cap BC(\mathcal{A}) = \emptyset$.

(ii). For all infinite subsets $X \subseteq \omega$ there exists an $h \in FF(\mathcal{A})$ such that either $X \cap \mathcal{A}^h \in \mathcal{I}_\mathcal{A}$ or $\mathcal{A}^h \setminus X \in \mathcal{I}_\mathcal{A}$.

Proof. Let $\{X_\alpha \mid \alpha < \mathfrak{c}\}$ be some enumeration of $[\omega]^\omega$. We will inductively construct an increasing sequence of ideals $(\mathcal{I}_\alpha)_{\alpha<\mathfrak{c}}$ on ω and obtain the desired ideal $\mathcal{I}_\mathcal{A}$ as their union. In the successor step at $\alpha+1<\mathfrak{c}$ we will make sure that property (ii) holds for the set $X_\alpha \in [\omega]^\omega$.
Start with $\mathcal{I}_0 = [\omega]^{<\omega}$, the finite subsets of ω, and for limit ordinals $\lambda < \mathfrak{c}$ we simply set $\mathcal{I}_\lambda = \bigcup\{\mathcal{I}_\alpha \mid \alpha < \lambda\}$. For the successor case assume $\mathcal{I}_\alpha$ is already constructed and distinguish the following two cases:

- If there is some $h \in FF(\mathcal{A})$ and $Y \in \mathcal{I}_\alpha$ such that $\mathcal{A}^h \subseteq X_\alpha \cup Y$, then we do nothing and set $\mathcal{I}_{\alpha+1} = \mathcal{I}_\alpha$.
- If there is no such boolean combination we let $\mathcal{I}_{\alpha+1}$ be the ideal generated by $\mathcal{I}_\alpha \cup \{X_\alpha\}$.

Now we show that the ideal $\mathcal{I}_\mathcal{A} = \bigcup\{\mathcal{I}_\alpha \mid \alpha < \mathfrak{c}\}$ does indeed have the desired properties.
(i): Assume towards a contradiction that $\mathcal{I}_\mathcal{A}$ contains a boolean combination $\mathcal{A}^{h_0}$ for some $h_0 \in FF(\mathcal{A})$. Since this is infinite it is not contained in $\mathcal{I}_0$, thus there is a least $\alpha < \mathfrak{c}$ such that $\mathcal{A}^{h_0} \in \mathcal{I}_{\alpha+1} \setminus \mathcal{I}_\alpha$. Since $\mathcal{I}_\alpha \neq \mathcal{I}_{\alpha+1}$ holds, the second case in the construction applied and thus we have that for all $Y \in \mathcal{I}_\alpha$ the set $X_\alpha \cup Y$ does not contain a boolean combination of $\mathcal{A}$. However, because $\mathcal{I}_{\alpha+1}$ is the ideal generated by $\mathcal{I}_\alpha \cup \{X_\alpha\}$ there must be some $Y_0 \in \mathcal{I}_\alpha$ such that $\mathcal{A}^{h_0} \subseteq X_\alpha \cup Y_0$, a contradiction.
(ii): Every $X \in [\omega]^\omega$ appears in the enumeration and we can choose $\alpha < \mathfrak{c}$ minimal with $X_\alpha = X$. If the first case in the construction applied then $\mathcal{A}^h \subseteq X_\alpha \cup Y$ for some $h \in FF(\mathcal{A})$ and $Y \in \mathcal{I}_\alpha$. Then $\mathcal{A}^h \setminus X_\alpha \subseteq Y$ is an element of $\mathcal{I}_\alpha \subseteq \mathcal{I}_\mathcal{A}$ because $\mathcal{I}_\mathcal{A}$ is an ideal, thus closed under subsets. Otherwise $X_\alpha \in \mathcal{I}_{\alpha+1}$ and thus for every $h \in FF(\mathcal{A})$ the set $X_\alpha \cap \mathcal{A}^h \subseteq X_\alpha$ belongs to $\mathcal{I}_\mathcal{A}$. $\square$

Note that if $\mathcal{A}$ is a maximal independent family, then $\mathcal{I}_\mathcal{A} = [\omega]^{<\omega}$ also has those two properties: The first property clearly holds since $[\omega]^{<\omega} \cap BC(\mathcal{A}) = \emptyset$ follows from $BC(\mathcal{A}) \subseteq [\omega]^\omega$, and property (ii) simply turns into the definition of maximality in this case. Also note that $\mathcal{A}$ is maximal but the ideal $\mathcal{I}_\mathcal{A}$ is far away from being maximal.

However, since this set is countable, it is not very useful for the forcing used later in this section. This motivates the following definition:

Definition 2.2. We refer to an ideal $\mathcal{I}_\mathcal{A}$ corresponding to an independent family $\mathcal{A}$ obtained by the construction in the above proof as *independence diagonalization ideal.*

While we now know that independence diagonalization ideals exist, we can't say anything about uniqueness. In fact the above construction does depend on the enumeration of $[\omega]^\omega$ and can result in different ideals for different enumerations.

We now define the poset $\mathbb{B}(\mathcal{I}_\mathcal{A})$, which essentially is a variation of Mathias forcing:

Definition 2.3. For an independent family $\mathcal{A}$ and a corresponding independence diagonalization ideal $\mathcal{I}_\mathcal{A}$ let $\mathbb{B}(\mathcal{I}_\mathcal{A}) = \{(s, E) \mid s \in [\omega]^{<\omega}, E \in [\mathcal{I}_\mathcal{A}]^{<\omega}\}$ be the poset endowed with the folloing extension relation:

$$(s_1, E_1) \leq (s_2, E_2) \Leftrightarrow s_2 \subseteq s_1, E_2 \subseteq E_1 \text{ and } (s_1 \setminus s_2) \cap \bigcup E_2 = \emptyset.$$

Remark 2.4. (i). This is indeed a partial order: Reflexivity and antisymmetry are obvious, and to show transitivity assume $(s_1, E_1) \leq (s_2, E_2) \leq (s_3, E_3)$ and notice that this implies

$$(s_1 \setminus s_3) \cap \bigcup E_3 \subseteq \Big((s_1 \setminus s_2) \cap \bigcup E_2\Big) \cup \Big((s_2 \setminus s_3) \cap \bigcup E_3\Big) = \emptyset.$$

Therefore we have $(s_1, E_1) \leq (s_3, E_3)$ as well.

(ii). Two conditions (s, E) and (s, F) with the same stem are always compatible since $(s, E \cup F) \in \mathbb{B}(\mathcal{I}_\mathcal{A})$ extends both of them. Therefore, in any uncountable collection of conditions $A \subseteq \mathbb{B}(\mathcal{I}_\mathcal{A})$ there are two compatible elements, since there are only countably many different stems $s \in [\omega]^{<\omega}$. This means the poset $\mathbb{B}(\mathcal{I}_\mathcal{A})$ satisfies the countable chain condition.

The following proposition will prove extremely useful as the primary tool in proving the main theorems of this section. From now on V will always denote the ground model we work in.

Proposition 2.5. *Let G be a $\mathbb{B}(\mathcal{I}_\mathcal{A})$-generic filter over V and let*

$$x_G = \bigcup\{s \in [\omega]^{<\omega} \mid \exists E \in [\mathcal{I}_\mathcal{A}]^{<\omega} : (s, E) \in G\}$$

denote the corresponding $\mathbb{B}(\mathcal{I}_\mathcal{A})$-generic real over V. Then $x_G \in [\omega]^\omega$ has the following properties in $V[G]$:

(i). $\mathcal{A} \cup \{x_G\}$ is an independent family.

(ii). For every $Y \in ([\omega]^\omega \setminus \mathcal{A}) \cap V$ the family $\mathcal{A} \cup \{x_G, Y\}$ is not independent.

Proof. The set x_G is indeed infinite because for all $n \in \omega$ the set of conditions $\{(s, E) \mid |s| > n\}$ is dense: Given an arbitrary condition (s, E) we have that $\omega \setminus \bigcup E$ is infinite, thus we can find a finite $s' \subset \omega \setminus \bigcup E$ of cardinality greater than n and find $(s \cup s', E) \leq (s, E)$. Now we show that the two properties hold:

(i): Given $h \in FF(\mathcal{A})$ we want to show that for every $n \in \omega$ the sets of conditions

$$A_{n,h} = \{(s, E) \in \mathbb{B}(\mathcal{I}_\mathcal{A}) : |s \cap \mathcal{A}^h| > n\},$$
$$B_{n,h} = \{(s, E) \in \mathbb{B}(\mathcal{I}_\mathcal{A}) : |\bigcup E \cap \mathcal{A}^h| > n\}$$

are dense. Then the density of the $A_{n,h}$ clearly implies that $|x_G \cap \mathcal{A}^h| = \omega$ and $|(\omega \setminus x_G) \cap \mathcal{A}^h|$ will follow from the density of the $B_{n,h}$ and the fact that $\bigcup E \subseteq \omega \setminus x_G$ holds for all $(s, E) \in G$.
So let (t, F) be an arbitrary condition. Notice that $\mathcal{A}^h \setminus \bigcup F$ must be infinite, since otherwise $\mathcal{A}^h = (\mathcal{A}^h \setminus \bigcup F) \cup \bigcup F \in \mathcal{I}_{\mathcal{A}}$ would contradict $BC(\mathcal{A}) \cap \mathcal{I}_{\mathcal{A}} = \emptyset$. So we may choose a finite subset $s \subset \mathcal{A}^h \setminus \bigcup F$ of size greater than n and have $(t \cup s, F) \in A_{n,h}$ extending (t, F). Similarly, since $\mathcal{A}^h$ is infinite we can pick a finite subset $F' \in [\mathcal{A}^h]^{<\omega} \subset \mathcal{I}_{\mathcal{A}}$ of size greater than n, and $(t, F \cup \{F'\})$ is an element of $B_{n,h}$ extending (t, F).
(ii): Let $Y \in ([\omega]^\omega \setminus \mathcal{A}) \cap V$. By property (ii) in Lemma 2.1. we get that there exists $h \in FF(\mathcal{A})$ such that either $Y \cap \mathcal{A}^h \in \mathcal{I}_{\mathcal{A}}$ or $\mathcal{A}^h \setminus Y \in \mathcal{I}_{\mathcal{A}}$ holds. In the first case the set

$$C_Y = \{(s, E) \mid Y \cap \mathcal{A}^h \in E\}$$

is clearly dense, since in this case we always have $(s, E \cap \{Y \cap \mathcal{A}^h\})$ as condition extending an arbitrary $(s, E) \in \mathbb{B}(\mathcal{I}_{\mathcal{A}})$. So, due to the genericity of G there is some $(s, E) \in C_Y \cap G$, and so in $V[G]$ we have $x_G \cap Y \cap \mathcal{A}^h \subseteq s$.
In the other case, where $\mathcal{A}^h \setminus Y \in \mathcal{I}_{\mathcal{A}}$ holds, similarly the set of conditions

$$D_Y = \{(s, E) \mid \mathcal{A}^h \setminus Y \in E\}$$

is dense and we get that $x_G \cap \mathcal{A}^h \setminus Y \subseteq s$ for some $(s, E) \in D_Y \cap G$. Therefore, in both cases there will be a boolean combination of $\mathcal{A} \cup \{x_G, Y\}$ contained in some finite set s, hence this family can't be independent in $V[G]$. □

Now we want to make use of this proposition to keep the size of $\mathfrak{i}$ comparatively small in the theorems that follow. For this we want to add more than one generic independent real to a family. Therefore we will first recall some basic facts about iterated forcing:

Given a poset $\mathbb{P} \in V$ in the ground model and another poset $\mathbb{Q} \in V^{\mathbb{P}}$ in its generic extension we can define the two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ in V, where $\dot{Q}$ is a $\mathbb{P}$-name for $\mathbb{Q}$. The conditions of $\mathbb{P} * \dot{\mathbb{Q}}$ are of the form $(p, \dot{q})$ with $p \in \mathbb{P}$ such that $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$. The extension relation is given by

$$(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2) \Leftrightarrow p_1 \leq p_2 \wedge p_1 \Vdash (\dot{q}_1 \leq \dot{q}_2).$$

Now a general iteration of length $\gamma \in ORD$ is given by a sequence of the form $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma)$ where $\mathbb{P}_\alpha$ is an iteration of length $\alpha < \gamma$ and $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a poset $\mathbb{Q}_\alpha \in V^{\mathbb{P}_\alpha}$. Here we always have $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.

At limit ordinals $\lambda < \gamma$ it is tempting to simply let $\mathbb{P}_\lambda$ consist of all sequences of names $p = (\dot{q}_\alpha \mid \alpha < \gamma)$ for conditions $q_\alpha \in \mathbb{Q}_\alpha$. However, we do have some freedom here and in order to make sure important preservation properties are kept we impose restrictions upon the *support* of those sequences, which is given by $supp(p) = \{\alpha < \lambda : \dot{q}_\alpha \neq 1_\alpha\}$, where 1_α denotes the largest element of $\mathbb{P}_\alpha$. The most important iterations are the *finite support iteration* where $|supp(p)| < \omega$ is required and the *countable support iteration* with $|supp(p)| \leq \omega$.

In the following theorems we will rely on the following crucial property of finite support iterations (a proof can be found in [9] or [11]):

Proposition 2.6. *If $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma)$ is a finite support iteration of forcing notions satisfying ccc, that is*

$$\Vdash_{\mathbb{P}_\alpha} \text{"}\dot{\mathbb{Q}}_\alpha \textit{ satisfies ccc" holds for every } \alpha < \gamma,$$

then $\mathbb{P}_\gamma$ satisfies ccc itself. Furthermore, if $\lambda \leq \gamma$ is an infinite limit ordinal of uncountable cofinality, we have

$$[\omega]^\omega \cap V^{\mathbb{P}_\lambda} = \bigcup \{[\omega]^\omega \cap V^{\mathbb{P}_\alpha} \mid \alpha < \lambda\},$$

in other words, no new reals are added at stage λ of the iteration. ☐

With these tools at hand we are ready to prove the main theorems of this section.

Theorem 2.7. *There exists a model $V[G]$ in which $\omega_1 = \mathfrak{i} < \mathfrak{c}$ holds.*

Proof. Let V be a ground model in which CH fails, that is $V \models \omega_1 < \mathfrak{c}$. We will inductively define a finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \omega_1, \beta < \omega_1)$ together with independent families $(\mathcal{A}_\alpha \mid \alpha \leq \omega_1)$, where $\mathcal{A}_\alpha \in V^{\mathbb{P}_\alpha}$, as follows:
Start with $\mathcal{A}_0 = \emptyset$. If, for $\alpha < \omega_1$, we already constructed $\mathcal{A}_\alpha$, choose some independence diagonalization ideal $\mathcal{I}_{\mathcal{A}_\alpha}$ for it and let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the poset $\mathbb{B}(\mathcal{I}_{\mathcal{A}_\alpha})$. We then define $\mathcal{A}_{\alpha+1} = \mathcal{A} \cup \{x_\alpha\}$, where x_α denotes the generic real added by $\mathbb{B}(\mathcal{I}_{\mathcal{A}_\alpha})$ as described in Proposition 2.5. For limit ordinals $\lambda < \omega_1$ we simply take unions $\mathcal{A}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{A}_\alpha$.
Now $\mathcal{A} = \mathcal{A}_{\omega_1} = \{x_\alpha \mid \alpha < \omega_1\} \subseteq [\omega]^\omega \cap V^{\mathbb{P}_{\omega_1}}$, being an increasing union of independent families, is an independent family itself and clearly has size ω_1. We show that $\mathcal{A}$ is maximal as well:
Let $Y \in ([\omega]^\omega \setminus \mathcal{A}) \cap V^{\mathbb{P}_{\omega_1}}$ be an infinite set in the generic extension. Since no new reals are added at limit steps we can find an index $\alpha < \omega_1$ such that $Y \in ([\omega]^\omega \setminus \mathcal{A}_\alpha) \cap V^{\mathbb{B}(\mathcal{I}_{\mathcal{A}_\alpha})}$, and due to Proposition 2.5. we know that $\mathcal{A}_\alpha \cup \{x_\alpha, Y\}$ is not independent. Hence $\mathcal{A} \cup \{Y\}$ is not independent.
Since $\mathbb{P}_{\omega_1}$, being a finite support iteration of *ccc*-posets satisfies *ccc* itself, cardinals are preserved throughout the iteration. Since in our ground model $V \models \omega_1 < \mathfrak{c}$ is true the same holds in the extension, and $\mathcal{A}$ is a witness for $V^{\mathbb{P}_{\omega_1}} \models \omega_1 = \mathfrak{i} < c$ as desired. ☐

In fact we can modify the proof a bit to arrive at the following generalization of this result:

Theorem 2.8. *For regular uncountable cardinals $\kappa < \lambda$ there exists a model in which $\kappa = \mathfrak{d} = \mathfrak{i} < \mathfrak{c} = \lambda$ holds.*

Proof. Once again we will use an iteration, the length of which shall be the ordinal product $\gamma = \lambda \cdot \kappa$, which is of cardinality λ and of cofinality κ. Choose a cofinal subset $E \subseteq \gamma$ consisting of successor ordinals with $|E| = \kappa$.
We inductively define the finite support iteration $\mathbb{P}_\gamma = (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma)$ together with independent families $(\mathcal{A}_\alpha \mid \alpha < \gamma)$ where $\mathcal{A}_\alpha \in V^{\mathbb{P}_\alpha}$. We start with $\mathcal{A}_0 = \emptyset$ and at limit steps $\mu < \gamma$ we set $\mathcal{A}_\mu = \bigcup_{\alpha<\mu} \mathcal{A}_\alpha$. In the definition at successor steps we distinguish two cases:
Suppose we already constructed $\mathcal{A}_\alpha$. If $\alpha + 1 \in E$ we proceed as in the last theorem, setting $\mathbb{Q}_{\alpha+1} = \mathbb{B}(\mathcal{I}_{\mathcal{A}_\alpha})$ and $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha \cup \{x_\alpha\}$ where x_α again denotes the $\mathbb{B}(\mathcal{I}_{\mathcal{A}_\alpha})$-generic real. If, on the other hand, $\alpha + 1 \notin E$ we add a Cohen real, that is we set $\mathbb{Q}_{\alpha+1} = \mathbb{C}$ and leave $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha$ intact. Now we show that $V^{\mathbb{P}_\gamma}$ has the desired properties. As before, $\mathbb{P}_\gamma$ satisfies *ccc* and thus cardinals are preserved.
$V^{\mathbb{P}_\gamma} \models \kappa \leq \mathfrak{d}$ follows from the fact that Cohen forcing adds unbounded reals and $cf(\gamma) = \kappa$, as this implies that, for cofinal sequences in γ, that the set of all Cohen reals added along this sequence is unbounded.
$V^{\mathbb{P}_\gamma} \models \mathfrak{i} \leq \kappa$ is witnessed by the independent family $\mathcal{A} = \bigcup_{\alpha<\gamma} \mathcal{A}_\alpha$. The reason for this family being maximal is similar to Theorem 2.7. - If $\mathcal{A} \cup \{Y\}$ is still independent for some $Y \in ([\omega]^\omega \setminus \mathcal{A}) \cap V^{\mathbb{P}_\gamma}$, then there is some $\alpha' < \gamma$ such that $Y \in ([\omega]^\omega \setminus \mathcal{A}) \cap V^{\mathbb{P}_{\alpha'}}$. But since E is unbounded there is some $\alpha \geq \alpha'$ such that $\alpha + 1 \in E$, and Proposition 2.5. yields that $\mathcal{A}_\alpha \cup \{Y, x_\alpha\} \subseteq \mathcal{A} \cup \{Y\}$ is not independent, a contradiction.
Finally, since throughout the iteration λ many Cohen reals are added it is clear that $V^{\mathbb{P}_\gamma} \models \mathfrak{c} = \lambda$. Altogether, using the fact that $\kappa \leq \mathfrak{d} \leq \mathfrak{i} \leq \kappa$ we have the desired property $V^{\mathbb{P}_\gamma} \models \kappa = \mathfrak{d} = \mathfrak{i} < \mathfrak{c} = \lambda$. □

Our next theorem makes a statement about the spectrum of independence. In fact, we can assure that a chosen finite amount of regular uncountable cardinals will show up in this spectrum by simultaneously constructing maximal independend families of those sizes as before.

Theorem 2.9. *Let $n \in \omega$ and $\kappa_1 < \kappa_2 < \cdots < \kappa_n$ be regular uncountable cardinals. Then there exists a model in which $\{\kappa_i \mid i \in n\} \subseteq Spec(mif)$.*

Proof. The length of our iteration will now be given by the ordinal product $\gamma = \kappa_n \cdot \kappa_{n-1} \cdot \dots \cdot \kappa_1$. We choose pairwise disjoint unbounded subsets

$E_1, \dots E_n \subset \gamma$ such that for $i \in \{1, \dots, n\}$ we have $|E_i| = \kappa_i$. Furthermore these subsets should only contain successor ordinals. Our finite support iteration $\mathbb{P}_\gamma = (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma)$ will be defined inductively together with independent families $(\mathcal{A}^1_\alpha, \dots, \mathcal{A}^n_\alpha \mid \alpha < \gamma)$ in the following way:
Start with $\mathcal{A}^1_0 = \dots = \mathcal{A}^n_0 = \emptyset$. At limits $\lambda < \gamma$ for $i \in \{1, \dots, n\}$ we set $\mathcal{A}^i_\lambda = \bigcup_{\alpha<\lambda} \mathcal{A}^i_\alpha$. Now for successor ordinals $\alpha + 1$ we proceed like this:
If there exists an $i \in \{1, \dots, n\}$ with $\alpha \in E_i$, we pick an independence diagonalization ideal $\mathcal{I}_{\mathcal{A}^i_\alpha}$ for $\mathcal{A}^i_\alpha$ and let $\mathbb{Q}_{\alpha+1} = \mathbb{B}(\mathcal{I}_{\mathcal{A}^i_\alpha})$. Let x_α then denote the generic real added by this poset and define $\mathcal{A}^i_{\alpha+1} = \mathcal{A}^i_\alpha \cup \{x_\alpha\}$. For $i \neq j \in \{1, \dots, n\}$ we leave the independent family unchanged, that is $\mathcal{A}^j_{\alpha+1} = \mathcal{A}^j_\alpha$. If $\alpha \notin \bigcup_{i=1}^{n} E_i$ then $\mathbb{Q}_{\alpha+1} = \mathbb{C}$ is a Cohen poset and none of the $\mathcal{A}^i_{\alpha+1}$ will differ from $\mathcal{A}^i_\alpha$. This finishes the definition. Now we look at the extension $V^{\mathbb{P}_\gamma}$.
A maximal independent family of size κ_i is now given by $\mathcal{A}^i = \bigcup_{\alpha<\gamma} \mathcal{A}^i_\alpha$. The reason for its maximality is again exactly the same as in Theorem 2.8. Thus $V^{\mathbb{P}_\gamma} \models \kappa_i \in Spec(mif)$ for all $i \in \{1, \dots, n\}$ as desired. □

Under some large cardinal assumptions it is even possible to alter the construction further to exclude all the values in-between the κ_i. In other words, the spectrum of independence may not be convex. This can be proved if there are measurable cardinals:

Definition 2.10. An uncountable cardinal κ is called *measurable* if there exists a nonprincipal ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\kappa)$ which is κ-complete, meaning that the intersection of fewer than κ many members of $\mathcal{U}$ is always a member of $\mathcal{U}$ as well.

Remark 2.11. Every measurable cardinal is inaccessible, thus ZFC does not prove the existence of measurable cardinals.

For a poset $\mathbb{P}$ we can define the ultrapower $\mathbb{P}^\kappa/\mathcal{U}$ similar to model theory, consisting of equivalence classes

$$[f] = \{g : \kappa \to \mathbb{P} \mid \{\alpha \mid f(\alpha) = g(\alpha)\} \in \mathcal{U}\}.$$

This is itself a poset with the natural partial order defined by

$$[f] \leq [q] \Leftrightarrow \{\alpha \in \kappa \mid f(\alpha) \leq g(\alpha)\} \in \mathcal{U}.$$

By identifying $p \in \mathbb{P}$ with the equivalence class of the constant function $f_p(\alpha) = p$ for all $\alpha \in \kappa$ we can always assume $\mathbb{P} \subseteq \mathbb{P}^\kappa/\mathcal{U}$. We will make use of the following result, which we will not prove here:

Fact 2.12. If $\mathbb{P}$ has the countable chain condition, then the ultrapower $\mathbb{P}^\kappa/\mathcal{U}$ has it as well.

A very interesting result of taking such an ultrapower is the following, which shows that it is able to destroy maximality of some independent families. We will only sketch the proof of the following:

Proposition 2.13. *Let $\mathbb{P}$ be a poset satisfying ccc and let $\dot{\mathcal{A}}$ be a $\mathbb{P}$-name with $\Vdash_{\mathbb{P}}$ "$\dot{\mathcal{A}}$ is independent". Then we have that $\Vdash_{\mathbb{P}^\kappa/\mathcal{U}}$ "$\dot{\mathcal{A}}$ is not maximal".*

Proof Sketch. Let $\lambda \geq \kappa$ and $\{A_\alpha \mid \alpha \in \lambda\} = \mathcal{A}$. Similar to Theorem 1.24. we can represent every $\dot{A}_\alpha$ as $\{(p^\alpha_{n,i}), k^\alpha_{n,i}\}_{n,i\in\omega}$ where $\{p^\alpha_{n,i}\}_{i\in\omega}$ are maximal antichains and $k^\alpha_{n,i} \in 2$ satisfy

$$k^\alpha_{n,i} = 0 \Leftrightarrow p^\alpha_{n,i} \Vdash \check{n} \in \dot{A}_\alpha \text{ and } k^\alpha_{n,i} = 1 \Leftrightarrow p^\alpha_{n,i} \Vdash \check{n} \notin \dot{A}_\alpha.$$

The idea is now to take the 'average' $\dot{A}$ of the first κ names $\{\dot{A}_\alpha \mid \alpha < \kappa\}$, which is defined by

$$[p_{n,i}] = (p^\alpha_{n,i} \mid \alpha < \kappa)/\mathcal{U} \text{ and } k_{n,i} = (k^\alpha_{n,i} \mid \alpha < \kappa)/\mathcal{U}$$

for every $n, i \in \omega$. Using Łoś's theorem one now shows that

$$\Vdash_{\mathbb{P}^\kappa/\mathcal{U}} \text{"}\dot{\mathcal{A}} \cup \{\dot{A}\} \text{ is independent"}$$

holds, which is due to the fact that for $\dot{B}$ a $\mathbb{P}$-name for a boolean combination $B \in BC(\mathcal{A})$ we have that $\Vdash_{\mathbb{P}}$ "$\dot{B} \cap \dot{A}_\alpha$ is infinite" holds for all except finitely many α. $\square$

We can now use this to prove a slightly altered version of Theorem 2.9., where we not only construct independent families $\mathcal{A}_i$ witnessing $\kappa_i \in Spec(mif)$ but also make sure that there are no maximal independent families of size in-between those κ_i.

Theorem 2.14. *Let $n \in \omega$ and $\kappa_1 < \cdots < \kappa_n$ be measurable cardinals. Then there is a ccc generic extension in which the following hold:*

$$\{\kappa_j\}_{j=1}^n \subseteq Spec(mif) \text{ and for every } 1 \leq j < n : (\kappa_j, \kappa_{j+1}) \cap Spec(mif) = \emptyset.$$

Proof. Let $EVEN$ denote the class of all ordinals $\lambda + 2k$ for λ limit ordinal and $k \in \omega$ and ODD analogously be the class of ordinals of the form $\lambda + 2k + 1$. As in Theorem 2.7. we want to define a finite support iteration of lenght $\gamma = \kappa_n \cdot \kappa_{n-1} \cdot \cdots \cdot \kappa_1$. Once again we fix pairwise disjoint disjoint unbounded subsets $E_1, \cdots, E_n \subset \gamma$ with $|E_i| = \kappa_i$. For every $i \in \{1, \cdots, n\}$ let E_i consist of

successor ordinals and satisfy $\gamma = sup(E_i \cap EVEN) = sup(E_i \cap ODD)$. We now inductively define our finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma)$, together with independent families $(\mathcal{A}^i_\alpha \mid i \in \{1, \cdots, n\}, \alpha \leq \gamma)$ where $\mathcal{A}^1_\alpha, \cdots, \mathcal{A}^n_\alpha \in V^{\mathbb{P}_\alpha}$. Here is the inductive construction. Let $\alpha < \gamma$ be a successor ordinal, say $\alpha = \beta + 1$ and distinguish cases.
If there is some i such that $\alpha \in E_i \cap EVEN$ pick an independence diagonalization ideal $\mathcal{I}_{\mathcal{A}^i_\beta}$ and let $\dot{\mathbb{Q}}_\alpha$be a $\mathbb{P}_\alpha-$name for $\mathbb{B}(\mathcal{I}_{\mathcal{A}^i_\beta})$. Set $\mathcal{A}^i_\alpha = \mathcal{A}^i_\beta \cup \{x_\alpha\}$ where x_α denotes the generic real adjoined by this poset and leave all the other $\mathcal{A}^j_\alpha$ unchanged. In short, in this case we proceed exactly like we did in the proof of Theorem 2.9.
In case there is some i such that $\alpha \in E_i \cap ODD$ let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\beta$-name for the poset $\mathbb{P}^{\kappa_i}/\mathcal{U}_i$ where $\mathcal{U}_i$ is a κ_i-complete filter witnessing the measurability of κ_i. In this case we don't add any elements to the independent families constructed thus far.
Finally, if none of the above two hold add a Cohen real as in Theorem 2.9.
Now the extension satisfies $V^{\mathbb{P}_\gamma} \models \{\kappa_i\}_{i=1}^n \subseteq Spec(mif)$ for the exact same reason as in the previous theorems of this form. So we are only left with proving the second statement: Assume towards contradiction that there is some $\kappa_j < \lambda < \kappa_{j+1}$ such that there exists a maximal independent family $\mathcal{A}$ of size (λ) in $V^{\mathbb{P}_\gamma}$. The poset $\mathbb{P}_\gamma$ satisfies *ccc*, therefore there is some $\alpha_0 < \gamma$ such that $\mathcal{A} \in V^{\mathbb{P}_{\alpha_0}}$. But by our choice of E_j we can find an $\alpha \in E_j \cap ODD$ with $\alpha = \beta + 1$ and $\alpha_0 < \beta$. Due to the previous proposition we can now conclude that $\mathcal{A}$ is not maximal in $V^{\mathbb{P}_\alpha}$ and therefore also not maximal in $V^{\mathbb{P}_\gamma}$.

□

2.2 The independence density ideal.

In this section we introduce another ideal $id(\mathcal{A})$ associated to an independent family $\mathcal{A}$ together with a poset $\mathbb{P}$ allowing us to adjoin a generic maximal independent family with interesting properties.

Definition 2.15. For an independent family $\mathcal{A}$ the *independence density ideal associated to* $\mathcal{A}$ is defined as

$$id(\mathcal{A}) = \{X \subseteq \omega \mid \forall h \in FF(\mathcal{A}) \exists h' \in FF(\mathcal{A}) : (h \subseteq h' \wedge |\mathcal{A}^{h'} \cap X| < \omega\}.$$

Equivalently, if we set $\mathcal{D}(X) = \{h \in FF(\mathcal{A}) \mid |\mathcal{A}^h \cap X| < \omega\}$ we get that $id(\mathcal{A}) = \{X \subseteq \omega \mid \mathcal{D}(X)$ is dense in $FF(\mathcal{A})\}$.

One might interpret a subset X in $id(\mathcal{A})$ as a very bad candidate for adding to the family $\mathcal{A}$ - any boolean combination h of $\mathcal{A}$ can be extended to $h' \supseteq h$ witnessing that $\mathcal{A} \cup \{X\}$ is not independent anymore.

We quickly verify some easy properties the ideal satisfies:

Remark 2.16. (i): Let us first show that this is indeed an ideal: $\emptyset \in id(\mathcal{A})$ is clear by definition and if $X \cap \mathcal{A}^{h'}$ is finite then for any $Y \subseteq X$ the set $Y \cap \mathcal{A}^{h'} \subseteq X \cap \mathcal{A}^{h'}$ also needs to be finite. For two elements $X, Y \in id(\mathcal{A})$ and $h \in FF(\mathcal{A})$ first let $h' \supseteq h$ be such that $X \cap \mathcal{A}^{h'}$ is finite, afterwards choose another extension $h'' \supseteq h'$ to be such that $Y \cap \mathcal{A}^{h''}$ is finite. Then

$$(X \cup Y) \cap \mathcal{A}^{h''} = (X \cap \mathcal{A}^{h''}) \cup (Y \cap \mathcal{A}^{h''})$$

is finite as well, thus $X \cup Y \in id(\mathcal{A})$ holds.

(ii): If $\mathcal{A}_0 \subseteq \mathcal{A}_1$ are independent families, it follows that $id(\mathcal{A}_0) \subseteq id(\mathcal{A}_1)$: Given $X \in id(\mathcal{A}_0)$ and $h \in FF(\mathcal{A}_1)$ let $h_0 = h \restriction \mathcal{A}_0$ and choose $h_0 \subseteq h_0'$ such that $\mathcal{A}_0^{h_0'} \cap X$ is finite. Then $h' = h \cup h_0' \in FF(\mathcal{A}_1)$ has the property that $\mathcal{A}_1^{h'} \cap X$ is finite as well.

(iii): If $\mathcal{A}$ is an infinite independent family, and $\mathcal{A}^h \cap X$ is finite, then there is $h \subseteq h'$ such that $\mathcal{A}^{h'} \cap X = \emptyset$: If $n \in \mathcal{A}^h \cap X$ consider some $A \in \mathcal{A} \setminus dom(h)$. Now let $dom(h') = dom(h) \cup \{A\}$ and set $h'(A) = 0$ iff $n \notin A$. Then $n \notin \mathcal{A}^{h'} \cap X$, and we can repeat this procedure finitely many times until finally arriving at the empty set.

Definition 2.17. We define the following forcing poset:

$$\begin{aligned} \mathbb{P} = \{(\mathcal{A}, A) \mid & \ \mathcal{A} \text{ countable independent family}; \\ & A \in [\omega]^\omega : \forall h \in FF(\mathcal{A}) : |A \cap \mathcal{A}^h| = \omega\}, \end{aligned}$$

where the extension relation is given by

$$(\mathcal{B}, B) \leq (\mathcal{A}, A) \Leftrightarrow \mathcal{B} \supseteq \mathcal{A} \text{ and } B \subseteq^* A.$$

It is immediately clear that $\mathbb{P}$ is reflexive and transitive (since $\subseteq^*$ is transitive). However, it is easy to find a counter-example for antisymmetry - given any countable independent family $\mathcal{A}$ we get $(\mathcal{A}, \omega) \leq (\mathcal{A}, \omega \setminus \{0\}) \leq (\mathcal{A}, \omega)$.

The following lemma shows a connection between this poset $\mathbb{P}$ and the independence density ideal $id(\mathcal{A})$. Modifications of a condition $(\mathcal{A}, A)$ with elements of $id(\mathcal{A})$ are negligible in the following sense:

Lemma 2.18. *If $(\mathcal{A}, A) \in \mathbb{P}$ and $X \in id(\mathcal{A})$, then for all boolean combinations $h \in FF(\mathcal{A})$ the set $\mathcal{A}^h \cap (A \setminus X)$ is infinite, therefore $(\mathcal{A}, A \setminus X)$ is still a condition in $\mathbb{P}$.*

Proof. Since $X \in id(\mathcal{A})$ we can, by part (iii) of the above remark, find $h' \supseteq h$ such that $\mathcal{A}^{h'} \cap X = \emptyset$, which implies that $\mathcal{A}^{h'} \cap (A \setminus X) = \mathcal{A}^{h'} \cap A$ is infinite.

Furthermore $\mathcal{A}^{h'} \cap (A \setminus X)$ is a subset of $\mathcal{A}^h \cap (A \setminus X)$, which therefore needs to be infinite as well. $\square$

Remark 2.19. For a given countable independent family $\mathcal{A}$ and $X \in id(\mathcal{A})$ the lemma also implies that the set of conditions $\{(\mathcal{B}, B) \mid B \cap X = \emptyset\}$ is dense below $(\mathcal{A}, A) \in \mathbb{P}$: given $(\mathcal{A}', A') \leq (\mathcal{A}, A)$, since $X \in id(\mathcal{A}) \subseteq id(\mathcal{A}')$ holds, by the lemma we find that $(\mathcal{A}', A' \setminus X) \in \mathbb{P}$, and clearly $(\mathcal{A}', A' \setminus X) \leq (\mathcal{A}', A')$ holds.

Next we look at some important combinatorial properties of this poset $\mathbb{P}$.

Proposition 2.20. *The poset $\mathbb{P}$ is σ-closed, meaning that every countable decreasing chain of conditions $(\mathcal{A}_0, A_0) \geq (\mathcal{A}_1, A_1) \geq \dots$ has a lower bound $(\mathcal{A}, A) \in \mathbb{P}$. Moreover, if CH and $2^{\aleph_1} = \aleph_2$ hold then $\mathbb{P}$ is $\aleph_2$-cc, meaning that every antichain is of size strictly less than $\aleph_2$.*

Proof. To show σ-closedness take a countable decreasing chain of conditions $\{(\mathcal{A}_n, A_n) \mid n \in \omega\}$. Due to Lemma 2.18. and the fact that clearly all finite sets are contained in the independence density ideal we may assume without loss of generality that $A_{n+1} \subseteq A_n$ holds for all $n \in \omega$. We pick enumerations $FF(\mathcal{A}_n) = \{h_{n,l} \mid l \in \omega\}$ and observe that for $m \leq n$ we have that $\mathcal{A}_m \subseteq \mathcal{A}_n$ and furthermore that $FF(\mathcal{A}_m) \subseteq FF(\mathcal{A}_n)$. This implies that in this case the set $A_n \cap \mathcal{A}_n^{h_{m,l}}$ is always infinite. We now choose, for every $n \in \omega$ a set of pairwise distinct elements $\{k_{n,m,l} \mid m \leq n, l \leq n\}$ such that $k_{n,m,l} \in A_n \cap \mathcal{A}_n^{h_{m,l}}$. The lower bound is now given by $(\mathcal{A}, A)$ where

$$\mathcal{A} = \bigcup \{\mathcal{A}_n \mid n \in \omega\} \text{ and } A = \{k_{n,m,l} \mid n \in \omega, m \leq n, l \leq n\}.$$

To see that $(\mathcal{A}, A) \in \mathbb{P}$ let $h \in FF(\mathcal{A})$ be arbitrary. Then there is some $m \in \omega$ such that $h \in FF(\mathcal{A}_m)$, thus it appears in the enumeration as some $h = h_{m,l}$. Then the infinite subset of A given by $\{k_{n,m,l} \mid n \in \omega\}$ is contained in $\mathcal{A}^h$.
To confirm that $(\mathcal{A}, A)$ is a lower bound observe that A_n is almost contained in A since $A \setminus A_n = \{k_{i,m,l} \mid i < n, m \leq i, l \leq i\}$ is finite.
Now for the second part of the Proposition assume that CH and $2^{\aleph_1} = \aleph_2$ hold. Given a set of $\aleph_2$ many conditions $\{(\mathcal{A}_\alpha, A_\alpha) \mid \alpha \in \aleph_2\}$, first notice that because of CH and $\{A_\alpha \mid \alpha \in \aleph_2\} \subseteq [\omega]^\omega$ we can assume that $A_\alpha = A_\beta$ for all $\alpha, \beta < \aleph_2$. Similarly $2^{\aleph_1} = \aleph_2$ implies that there are only $\aleph_1$ many countable subsets of $\aleph_1 \simeq [\omega]^\omega$. Therefore there must be an uncountable set of compatible conditions amongst $\{(\mathcal{A}_\alpha, A_\alpha) \mid \alpha \in \aleph_2\}$, so it is certainly not an antichain. $\square$

Our next goal will be to show that the poset $\mathbb{P}$ adjoins a maximal independent family. Preceeding this we prove the following lemma which is simple but gives a useful insight into the poset $\mathbb{P}$ and will also be used in the proofs of various other results later on.

Lemma 2.21. *For a condition $(\mathcal{A}, A) \in \mathbb{P}$ there is $B \notin \mathcal{A}$ with $B \subseteq A$ such that $(\mathcal{A} \cup \{B\}, A) \leq (\mathcal{A}, A)$.*

Proof. Fix an enumeration $\{h_n \mid n \in \omega\}$ of $FF(\mathcal{A})$. Choose distinct elements $k_{0,0}, k_{0,1} \in \mathcal{A}^{h_0} \cap A$ and proceed inductively: If we have defined $(k_{i,0}, k_{i,1})_{i<n}$ then pick distinct $k_{n,0}, k_{n,1} \in \mathcal{A}^{h_n} \cap A \setminus \{k_{i,j} \mid i < n, j < 2\}$. This is possible because the set $\mathcal{A}^{h_n} \cap A$ is infinite for every $n \in \omega$. Now let $B = \{k_{n,0} \mid n \in \omega\} \subseteq A$.
We confirm that $\mathcal{A} \cup \{B\}$ is independent to finish the proof. So consider an arbitrary $h \in FF(\mathcal{A})$. Of course there are infinitely many $m \in \omega$ such that $h \subseteq h_m$, which implies that for infinitely many m we have that

$$k_{m,0} \in \mathcal{A}^{h_m} \cap B \subseteq \mathcal{A}^h \cap B \text{ and } k_{m,1} \in \mathcal{A}^{h_m} \cap (\omega \setminus B) \subseteq \mathcal{A}^h \cap (\omega \setminus B),$$

so both of these boolean combinations $\mathcal{A}^h \cap B$ and $\mathcal{A}^h \cap (\omega \setminus B)$ are infinite. $\square$

This tells us that $(\mathcal{A}, A) \in \mathbb{P}$ means that A contains an infinite subset B such that $\mathcal{A} \cup \{B\}$ is still independent, that is a subset of A witnesses that $\mathcal{A}$ is not maximal.

We now make use of this to prove one of the main theorems of this section:

Theorem 2.22. *Let G be $\mathbb{P}$-generic over the ground model $V \models GCH$ and set*

$$\mathcal{A}_G = \bigcup \{\mathcal{A} \mid \exists A \in [\omega]^\omega : (\mathcal{A}, A) \in G\}.$$

Then $\mathcal{A}_G$ is a maximal independent family in the generic extension $V^{\mathbb{P}}$.

Proof. Since $\mathcal{A}_G$ is a directed union of independent families (since the filter G is directed), it is itself independent. So it remains to show that $\mathcal{A}_G$ is maximal in the extension $V^{\mathbb{P}}$.
Suppose otherwise, that is suppose there is some $X \in [\omega]^\omega \setminus \mathcal{A}_G$ such that $\mathcal{A}_G \cup \{X\}$ is still independent. Note that since $\mathbb{P}$ is σ-closed we assume that X is a ground-model real. There is some condition $(\mathcal{A}, A) \in G$ forcing that this is the case:

$$(\mathcal{A}, A) \Vdash \text{"}\mathcal{A}_G \cup \{X\} \text{ is independent and } X \notin \mathcal{A}_G\text{"}.$$

Now if for all $h \in FF(\mathcal{A})$ we have that $|\mathcal{A}^h \cap A \cap X| = |\mathcal{A}^h \cap A \cap (\omega \setminus X)| = \omega$ then $(\mathcal{A} \cup \{X\}, A)$ extends $(\mathcal{A}, A)$, but $(\mathcal{A} \cup \{X\}, A) \Vdash X \in \mathcal{A}_G$, a contradiction. On the other hand, if there exists $h \in FF(\mathcal{A})$ such that either $|\mathcal{A}^h \cap A \cap X|$ or $|\mathcal{A}^h \cap A \cap (\omega \setminus X)|$ is finite, apply the previous lemma to find $B \notin \mathcal{A}$ with $B \subseteq A$ and $(\mathcal{A} \cup \{B\}, A) \leq (\mathcal{A}, A)$.

But now this condition forces the following:

$$(\mathcal{A} \cup \{B\}, A) \Vdash \text{"}\exists h \in FF(\mathcal{A}_G) : \mathcal{A}_G^h \cap X \text{ or } \mathcal{A}_G^h \setminus X \text{ is finite."}$$

This is equivalent to $(\mathcal{A} \cup \{B\}, A) \Vdash$ "$\mathcal{A}_G \cup \{X\}$ is not independent.", so we arrived at another contradiction and thus we have shown that no X with the assumed properties can exists in the first place, which means that $\mathcal{A}_G$ is indeed a maximal independent family. $\square$

We will now further explore the poset $\mathbb{P}$, in particular its behaviour towards partitions of ω. First we introduce some more terminology.

Definition 2.23. Let $\mathcal{E}$ be a partition of ω and $A \in [\omega]^\omega$. We say that $\chi(\mathcal{E}, A)$ holds if there is a single $E \in \mathcal{E}$ containing all of A or if for each $E \in \mathcal{E}$ we have that $|E \cap A| \leq 1$. In this latter case we call A a *semiselector* for $\mathcal{E}$.

In particular, if $\mathcal{E}$ partitions ω into finite pieces, that is $|E| < \omega$ for all $E \in \mathcal{E}$, then $\chi(\mathcal{E}, A)$ always means that A is a semiselector for $\mathcal{E}$. As the name suggests this means that from every $E \in \mathcal{E}$ the set A picks at most (but not necessarily) one element. A *selector* then is a semiselector for which $|E \cap A| = 1$ holds for every $E \in \mathcal{E}$.

Lemma 2.24. *Let $\mathcal{E}$ be a partition of ω, $(\mathcal{A}, A) \in \mathbb{P}$ and $h^0 \in FF(\mathcal{A})$. Then there are $h^1 \supseteq h^0$ and $B \subseteq A$ such that $(\mathcal{A}, B) \leq (\mathcal{A}, A)$ and $\chi(\mathcal{E}, \mathcal{A}^{h^1} \cap B)$. In particular, if $\mathcal{E}$ partitions ω into finite sets then $\mathcal{A}^{h^1} \cap B$ is a semiselector for $\mathcal{E}$.*

Proof. Pick an enumeration of $\{h \in FF(\mathcal{A}) \mid h^0 \subseteq h\}$ given by $\{h_n \mid n \in \omega\}$, and furthermore let it be such that $h_0 = h^0$. Given an element $k \in \omega$ let $\mathcal{E}(k)$ denote the unique $E \in \mathcal{E}$ with $k \in E$. Now we distinguish two cases:
First assume it is possible to inductivly construct a sequence $\{k_n \mid n \in \omega\}$ such that for every n we have

$$k_n \in (\mathcal{A}^{h_n} \cap A) \setminus \bigcup \{\mathcal{E}(k_l) \mid l < n\}.$$

Then set $B = \{k_n \mid n \in \omega\} \cup A \setminus \mathcal{A}^{h^0} \subseteq A$. Now we show that $(\mathcal{A}, B) \in \mathbb{P}$, so let $h \in FF(\mathcal{A})$ be given. If h and h^0 are compatible, look at $h' = h \cup h^0$. There are infinitely many $m \in \omega$ such that $h' \subseteq h_m$, and therefore the set $\mathcal{A}^{h'} \cap B \subseteq \mathcal{A}^h \cap B$ contains infinitely many of the k_m. If, on the other hand, h and h^0 are incompatible, then there is some $C \in \mathcal{A}$ with $h(C) \neq h^0(C)$, wlog assume that $h(C) = 0$. Then $h^0(C) = 1$ implies that $\mathcal{A}^{h^0} \subseteq \omega \setminus C$ which then implies that $A \cap C \subseteq A \setminus \mathcal{A}^{h^0} \subseteq B$. If we set $h' = h \setminus \{(C, 0)\}$ we get the following:

$$\mathcal{A}^h \cap B \supseteq (\mathcal{A}^{h'} \cap C) \cap A \cap C = (\mathcal{A}^{h'} \cap C) \cap A = \mathcal{A}^h \cap A.$$

Therefore, since $B \subseteq A$ and $(\mathcal{A}, A) \in \mathbb{P}$ we have that $\mathcal{A}^h \cap B = \mathcal{A}^h \cap A$ is infinite and the lemma holds with $h^1 = h^0$.
Secondly, in the case that it is not possible to construct an infinite sequence as above, the construction terminates at some step $n \in \omega$ and we are left with a finite set $\{k_l \mid l < n\}$ such that

$$k_l \in (\mathcal{A}^{h_l} \cap A) \setminus \bigcup\{\mathcal{E}(k_j) \mid j < l\}$$

and

$$\mathcal{A}^{h_n} \cap A \subseteq \bigcup\{\mathcal{E}(k_l) \mid l < n\}.$$

For $l \leq n$, inductively try to construct $h_{n,l} \in FF(\mathcal{A})$ such that $h_{n,0} = h_n, h_{n,l} \subseteq h_{n,l'}$ for $l \leq l'$ and $\mathcal{A}^{h_{n,l+1}} \cap A \cap \mathcal{E}(k_l)$ finite. If this was possible, then the infinite set $\mathcal{A}^{h_{n,n}} \cap A$ would be covered by the finite set $\bigcup\{\mathcal{A}^{h_{n,l+1}} \cap A \cap \mathcal{E}(k_l) \mid l < n\}$, a contradiction. So there must be some $l < n$ such that for all $h \supset h_{n,l}$ we have that $\mathcal{A}^h \cap A \cap \mathcal{E}(k_l)$ is infinite. Now the lemma holds with

$$h^1 = h_{n,l} \text{ and } B = (\mathcal{A}^{h^1} \cap A \cap \mathcal{E}(k_l)) \cup A \setminus \mathcal{A}^{h^1}$$

for the following reason:
Since $\mathcal{A}^{h^1} \cap B \subseteq \mathcal{E}(k_l)$ by definition $\chi(\mathcal{E}, \mathcal{A}^{h^1} \cap B)$ holds and we only have to show that $(\mathcal{A}, B)$ is a condition of $\mathbb{P}$. Let $h \in FF(\mathcal{A})$ be arbitrary. If h and h^1 are compatible then $h' = h \cup h^1$ extends h^1 and we find that $\mathcal{A}^h \cap B$ contains $\mathcal{A}^{h'} \cap B = \mathcal{A}^{h'} \cap A \cap \mathcal{E}(k_l)$, an infinite set. Otherwise h and h^1 are incompatible and again wlog let $h(C) = 0$ while $h^1(C) = 1$ for some $C \in \mathit{dom}(h) \cap \mathit{dom}(h^1)$. Then $A \cap C \subseteq A \setminus \mathcal{A}^{h^1} \subseteq B$ implies that $\mathcal{A}^h \cap B$ contains the infinite set $A \cap C$ and we are done.
For the last part of the lemma notice that, if $\mathcal{E}$ consists only of finite sets, the first case - producing a semiselector - will always apply. □

Corollary 2.25. *Let $\mathcal{E}$ be a partition of ω.*

(i). The set of conditions $(\mathcal{A}, A)$ in $\mathbb{P}$ with the property that

$$\forall h \in FF(\mathcal{A}) \exists h' \in FF(\mathcal{A}) : h' \supseteq h \textit{ and } \chi(\mathcal{E}, \mathcal{A}^{h'})$$

is dense in $\mathbb{P}$.

(ii). If $\mathcal{E}$ consists only of finite subsets of ω then the set of conditions $(\mathcal{A}, A)$ such that A is a semiselector for $\mathcal{E}$ is dense in $\mathbb{P}$.

Proof. (i). Let $(\mathcal{A}, A) \in \mathbb{P}$ be arbitrary and let $h^0 \in FF(\mathcal{A})$. Through the above lemma we find $h^1 \supseteq h^0$ and $B \subseteq A$ with $(\mathcal{A}, B) \leq (\mathcal{A}, A)$ and $\chi(\mathcal{E}, \mathcal{A}^{h^1} \cap B)$. By

Lemma 2.21. we can now find $B' \subseteq B$ such that $(\mathcal{A} \cup \{B'\}, B) < (\mathcal{A}, B)$ and thus $h^2 = h^1 \cup \{(B', 0)\}$ extends h^0. If we set $\mathcal{A}_1 = \mathcal{A} \cup \{B'\}$ we have that $\chi(\mathcal{E}, \mathcal{A}^{h^2})$ holds. We repeat this procedure countably many times and get a decreasing secuence of conditions $\{(\mathcal{A}_n, A_n) \mid n \in \omega\}$. Now set $\mathcal{A}_\omega = \bigcup\{\mathcal{A}_n \mid n \in \omega\}$ and let A_ω be a pseudo-intersection of the A_n. Then $(\mathcal{A}_\omega, A_\omega)$ has the desired properties:
To see that $(\mathcal{A}_\omega, A_\omega) \in \mathbb{P}$ let $h \in FF(\mathcal{A})$ be arbitrary. There is some $n \in \omega$ such that $dom(h) \subseteq \mathcal{A}_n$ and $\mathcal{A}_n^h \cap A_n = \mathcal{A}_\omega^h \cap A_n$ is infinite. Therefore, since $A_\omega \subseteq^* A_n$, it follows that $\mathcal{A}_\omega^h \cap A_\omega$ is infinite as well and $(\mathcal{A}_\omega, A_\omega) \leq (\mathcal{A}, A)$.

(ii). Let $(\mathcal{A}, A) \in \mathbb{P}$ be arbitrary and enumerate $FF(\mathcal{A}) = \{h_n \mid n \in \omega\}$. Inductively construct a sequence $(k_n)_{n \in \omega} \subseteq \omega$ such that

$$k_n \in (\mathcal{A}^{h_n} \cap A) \setminus \bigcup\{\mathcal{E}(k_l) \mid l < n\}$$

holds for all $n \in \omega$. Note that if $\mathcal{E}$ consists of finite sets this is possible since in this case the set $\bigcup\{\mathcal{E}(k_l) \mid l < n\}$ is finite while $\mathcal{A}^{h_n} \cap A$ is infinite by definition. Now set $B = \{k_n \mid n \in \omega\}$, which by construction is a semiselector for $\mathcal{E}$. Furthermore $(\mathcal{A}, B) \in \mathbb{P}$ holds since for every $h \in FF(\mathcal{A})$ there are infinitely many extensions $h_m \supseteq h$ appearing in the enumeration, so $\mathcal{A}^h \cap B$ is always infinite. Lastly we clearly have $(\mathcal{A}, B) \leq (\mathcal{A}, A)$ because of $B \subseteq A$. $\square$

Next we want to find out more about the structure of $id(\mathcal{A}_G)$ for the generic maximal independent family added by the poset $\mathbb{P}$. As expected we can correlate it to the density ideals corresponding to the independent families occuring in the $\mathbb{P}$-generic filter G.

Lemma 2.26. *For a $\mathbb{P}$-generic filter G and the corresponding maximal independent family $\mathcal{A}_G$ we have that $id(\mathcal{A}_G) = \bigcup\{id(\mathcal{A}) \mid \exists A : (\mathcal{A}, A) \in G\}$.*

Proof. Suppose the lemma does not hold, then there is some condition $(\mathcal{A}, A) \in \mathbb{P}$ and $X \in [\omega]^\omega \cap V$ such that

$$(\mathcal{A}, A) \Vdash X \in id(\mathcal{A}_G) \setminus \bigcup\{id(\mathcal{A}) \mid \exists A : (\mathcal{A}, A) \in G\}.$$

For given $h \in FF(\mathcal{A})$, since $(\mathcal{A}, A) \Vdash X \in id(\mathcal{A}_G)$, we find that the following holds as well:

$$(\mathcal{A}, A) \Vdash \exists h' \in FF(\mathcal{A}_G) : h' \supseteq h \wedge \mathcal{A}_G^{h'} \cap X = \emptyset.$$

Therefore we can find an element $(\mathcal{A}', A') \in G$ such that $(\mathcal{A}', A') \leq (\mathcal{A}, A)$ and $h' \in FF(\mathcal{A}')$. Now inductively construct a decreasing sequence of conditions $(\mathcal{A}_n, A_n)_{n \in \omega}$ below $(\mathcal{A}, A)$ such that $\mathcal{A}_\omega = \bigcup\{\mathcal{A}_n \mid n \in \omega\}$ is closed under this property, that is for all $h \in FF(\mathcal{A}_\omega)$ there is $h' \in FF(\mathcal{A}_\omega)$ such that $h' \supseteq h$ and $\mathcal{A}_\omega^{h'} \cap X = \emptyset$, and thus $X \in id(\mathcal{A}_\omega)$. If we choose A_ω to be a pseudo-intersection of the A_n we get a condition $(\mathcal{A}_\omega, A_\omega) \leq (\mathcal{A}, A)$ with

$$(\mathcal{A}_\omega, A_\omega) \Vdash X \in \bigcup \{id(\mathcal{A}) \mid \exists A : (\mathcal{A}, A) \in G\},$$

a contradiction. □

Now this indicates that the structure of $id(\mathcal{A}_G)$ is rather comprehensible. We specify this in the next proposition:

Proposition 2.27. *For any $\mathbb{P}$-generic filter G, in the extension $V[G]$ the density ideal $id(\mathcal{A}_G)$ is generated by the set $\{\omega \setminus A \mid \exists \mathcal{A} : (\mathcal{A}, A) \in G\}$.*

Proof. Using the previous lemma we know that $id(\mathcal{A}_G)$ is given by the union $\bigcup\{id(\mathcal{A}) \mid \exists A : (\mathcal{A}, A) \in G\}$. Let $\mathcal{I}_G = \langle\{\omega \setminus A \mid \exists \mathcal{A} : (\mathcal{A}, A) \in G\}\rangle$ be the ideal generated in $V[G]$. We want to show that $\mathrm{id}(\mathcal{A}_G) = \mathcal{I}_G$ holds:
Given $X \in id(\mathcal{A}_G)$, by the above there is a condition $(\mathcal{A}, A) \in G$ such that $X \in id(\mathcal{A})$. Since $\{(\mathcal{B}, B) \mid X \cap B = \emptyset\}$ is dense below $(\mathcal{A}, A)$ there is $(\mathcal{B}, B) \in G$ such that $X \cap B = \emptyset$, or equivalently $X \subseteq \omega \setminus B$ and therefore $X \in \mathcal{I}_G$. So the inclusion $\mathrm{id}(\mathcal{A}_G) \subseteq \mathcal{I}_G$ holds.
On the other hand, for $X \in \mathcal{I}_G$ by definition there is a finite set of conditions $(\mathcal{A}_i, A_i)_{i \in n} \subseteq G$ such that

$$X \subseteq \bigcup_{i \in n} (\omega \setminus A_i) = \omega \setminus \bigcap_{i \in n} A_i.$$

Now in fact we can condense this information into one condition

$$(\mathcal{B}, B) = (\bigcup_{i \in n} \mathcal{A}_i, \bigcap_{i \in n} A_i),$$

which lies in G as well. Fix an $h \in FF(\mathcal{A}_G)$ and find $(\mathcal{C}, C) \in G$ such that $(\mathcal{C}, C) \leq (\mathcal{B}, B)$ with $h \in FF(\mathcal{C})$. Now observe that the set of conditions

$$D = \{(\mathcal{C}', C') \mid \exists Y \in \mathcal{C}' : Y \subseteq B\}$$

is dense below $(\mathcal{B}, B)$, so there is some $(\mathcal{C}', C') \in G \cap D$ for which by definition there is $Y \in \mathcal{C}'$ such that $Y \subseteq B$. So we can choose $h' = h \cup \{(Y, 0)\}$ extending h to witness $\mathcal{A}_G^{h'} \cap X = \emptyset$ and thus $X \in id(\mathcal{A}_G)$. Thus we showed that $\mathcal{I}_G \subseteq id(\mathcal{A})$. □

We conclude this section with an examination of the dual filter of $id(\mathcal{A}_G)$, which will be denoted by $fil(\mathcal{A}_G)$. To begin with we introduce P-sets, Q-sets and Ramsey filters in general.

Definition 2.28. Let $\mathcal{F} \subseteq [\omega]^\omega$.

(i). $\mathcal{F}$ is a *Q-set* if for every partition $\mathcal{E}$ of ω into finite sets there exists $A \in \mathcal{F}$ such that $\chi(\mathcal{E}, A)$ holds, meaning that A is a semiselector for $\mathcal{E}$.

(ii). $\mathcal{F}$ is a *P-set* if it is a filter on ω such that every countable subfamily of $\mathcal{F}$ has a pseudointersection in $\mathcal{F}$.

(iii). $\mathcal{F}$ is a *Ramsey* if it is a filter such that for every partition $\mathcal{E}$ of ω for which $\{\omega \setminus E \mid E \in \mathcal{E}\} \subseteq \mathcal{F}$ holds there is a $F \in \mathcal{F}$ such that $|F \cap E| \leq 1$ holds for every $E \in \mathcal{E}$.

These three concepts are connected in the following way:

Proposition 2.29. *Let $\mathcal{F} \subseteq [\omega]^\omega$ be a nonprincipal filter. Then $\mathcal{F}$ is Ramsey if and only if $\mathcal{F}$ is both a P-set and a Q-set.*

Proof. $(\Rightarrow)$: Assume $\mathcal{F}$ is Ramsey. Then clearly $\mathcal{F}$ is a Q-set by definition since $\{\omega \setminus E \mid E \in \mathcal{E}\} \subseteq \mathcal{F}$ holds for every partition of ω into finite sets.
To see that $\mathcal{F}$ is a P-set let $\{A_n \mid n \in \omega\} \in \mathcal{F}$ be a countable subfamily and suppose that $\bigcap\{A_n \mid n \in \omega\} \notin \mathcal{F}$, since otherwise there is nothing to show. Now apply the Ramsey property to the partition

$$\mathcal{E} = \{\omega \setminus A_n \mid n \in \omega\} \cup \{\bigcap_{n\in\omega} A_n\}$$

to find the pseudointersection $F \in \mathcal{F}$. Indeed we have that $F \subseteq^* A_n$ holds since we have $|F \cap (\omega \setminus A_n)| \leq 1$ for all $n \in \omega$.
$(\Leftarrow)$: Let $\mathcal{E}$ be a partition with $\{\omega \setminus E \mid E \in \mathcal{E}\} \subseteq \mathcal{F}$. Since $\mathcal{F}$ is a P-set we can find a pseudointersection $P \subseteq^* \omega \setminus E$, thus all the sets $P \cap E$ for $E \in \mathcal{E}$ are finite. We get a partition of ω into finite pieces consisting of those $P \cap E$ together with singletons $\{x\}$ for all x not contained within some $P \cap E$. Now, since $\mathcal{F}$ is also a Q-set we find a semiselector $Q \in \mathcal{F}$. Now $F = P \cap Q$, which lies in $\mathcal{F}$ since P and Q do, has the desired property.

□

Lemma 2.30. *For a $\mathbb{P}$-generic filter G, consider*

$$\mathcal{F}_G^0 = \{A \in [\omega]^\omega \mid \exists \mathcal{A} : (\mathcal{A}, A) \in G\}.$$

This is a Q-set and the filter $\mathcal{F}_G$ generated by $\mathcal{F}_G^0 \cup FR(\omega)$ is a P-set, where $FR(\omega)$ denotes the Frechét-filter consisting of all cofinite sets.

Proof. The fact that $\mathcal{F}_G^0$ is a Q-set follows from part (ii) of Corollary 2.25. - the set of conditions $(\mathcal{A}, A)$ such that A is a semiselector for $\mathcal{E}$ is dense, thus there

is such a condition $(\mathcal{A}, A) \in G$, and thus the semiselector A is in $\mathcal{F}_G^0$.
Next we show that $\mathcal{F}_G$ is closed under finite intersections: Let $A_1, A_2 \in \mathcal{F}_G$ be given from $(\mathcal{A}_1, A_1), (\mathcal{A}_2, A_2) \in G$, find a common extension $(\mathcal{C}, C) \in G$. By definition this means that $\mathcal{C} \supseteq \mathcal{A}_1 \cup \mathcal{A}_2$ and $C \subseteq^* A_1 \cap A_2$. The latter implies that there is a finite set K such that $C \setminus K \subseteq A_1 \cap A_2$. Now $C \setminus K$ is an element of $\mathcal{F}_G$ since $(\mathcal{C}, C) \in G$ and $C \setminus K = C \cap \omega \setminus K$. This means that the superset $A_1 \cap A_2$ is an element of the filter $\mathcal{F}_G$ as well.
Finally we show that $\mathcal{F}_G$ is a P-set. In fact it is sufficient to show that $\mathbb{P}$ forces that $\mathcal{F}_G^0$ is a P-set. Suppose otherwise: Then there is $p \in \mathbb{P}$ forcing that some countable subfamily has no pseudo-intersection in $\mathcal{F}_G^0$, more formally

$$p \Vdash \exists \mathcal{H} \in [\mathcal{F}_G^0]^\omega : \forall F \in \mathcal{F}_G^0 : \exists H \in \mathcal{H} : (F \not\subseteq^* H).$$

Since $\mathbb{P}$ is countably closed there is $\mathcal{H} \in V$ witnessing this property, say it is enumerated by $\mathcal{H} = \{A_n \mid n \in \omega\}$ with corresponding conditions $(\mathcal{A}_n, A_n)$ in G. Without loss of generality we may assume $(\mathcal{A}_0, A_0) \leq p$ and that the sequence $((\mathcal{A}_n, A_n))_{n \in \omega}$ is decreasing. Let $(\mathcal{A}_\omega, A_\omega)$ be a lower bound of the sequence, then $(\mathcal{A}_\omega, A_\omega) \leq p$ forces that $A_\omega \in \mathcal{F}_G^0$ is a pseudointersection of $\mathcal{H}$, a contradiction. □

Corollary 2.31. *Let G be $\mathbb{P}$-generic and denote by $fil(\mathcal{A}_G)$ the dual filter of the density ideal $id(\mathcal{A}_G)$ associated to the generic maximal independent family $\mathcal{A}_G$. Then, with the above notation, $fil(\mathcal{A}_G) = \mathcal{F}_G$ is a Ramsey filter.*

Proof. We already know that $id(\mathcal{A}_G)$ is generated by $\{\omega \setminus A \mid \exists \mathcal{A} : (\mathcal{A}, A) \in G\}$ as shown in Proposition 2.29., and therefore the dual filter is generated by the respective complements $\{A \mid \exists \mathcal{A} : (\mathcal{A}, A) \in G\}$. The above lemma therefore suggests that $fil(\mathcal{A}_G) = \mathcal{F}_G$. Since $fil(\mathcal{A}_G)$ is both a P-set and a Q-set it follows from Proposition 2.29. that $fil(\mathcal{A}_G)$ is Ramsey. □

2.3 Correlations between the two ideals of independence.

Next we want to point out some interesting and surprising connections between the density ideal $id(\mathcal{A})$ and some independence diagonalization ideal $\mathcal{I}_\mathcal{A}$ for a given independent family $\mathcal{A}$.

Recall that the diagonalization ideal as constructed in Lemma 2.1. is not unique and depends on the chosen enumeration of $[\omega]^\omega$. Still, in any case the resulting ideal will at least contain $id(\mathcal{A})$:

Lemma 2.32. *Let $\mathcal{I}_\mathcal{A}$ be an independene diagonalization ideal corresponding to an independent family $\mathcal{A}$. Then $id(\mathcal{A}) \subseteq \mathcal{I}_\mathcal{A}$.*

Proof. As in the construction of $\mathcal{I}_{\mathcal{A}}$ in Lemma 2.1. let $\{X_\alpha \mid \alpha \in \mathfrak{c}\}$ be an enumeration of $[\omega]^\omega$ and let $\{\mathcal{I}_\alpha \mid \alpha \in \mathfrak{c}\}$ be the increasing sequence of ideals with $\mathcal{I}_{\mathcal{A}} = \bigcup_{\alpha \in \mathfrak{c}} \mathcal{I}_\alpha$. Both $\mathcal{I}_{\mathcal{A}}$ and $id(\mathcal{A})$ contain all of $[\omega]^{<\omega}$, now suppose towards contradiction that there was some $X \in (id(\mathcal{A}) \cap [\omega]^\omega) \setminus \mathcal{I}_{\mathcal{A}}$.
Let $\alpha \in \mathfrak{c}$ be minimal such that $X = X_\alpha$, then $X \notin \mathcal{I}_{\alpha+1}$ and the way the construction in Lemma 2.1. goes imply that there is some $h \in FF(\mathcal{A})$ and $Y \in \mathcal{I}_\alpha$ such that $\mathcal{A}^h \subseteq X_\alpha \cup Y$ holds. This is equivalent to $\mathcal{A}^h \setminus X \subseteq Y$, and since $Y \in \mathcal{I}_\alpha \subseteq \mathcal{I}_{\mathcal{A}}$ it follows that $\mathcal{A}^h \setminus X \in \mathcal{I}_{\mathcal{A}}$.
Furthermore, since we assumed that $X \in id(\mathcal{A})$ by definition there is some $h' \supseteq h$ in $FF(\mathcal{A})$ such that $\mathcal{A}^{h'} \cap X$ is finite. Now $\mathcal{A}^{h'} \subseteq \mathcal{A}^h$ implies that $\mathcal{A}^{h'} \setminus X \subseteq \mathcal{A}^h \setminus X$ is in $\mathcal{I}_{\mathcal{A}}$ as well. But then we can conclude that

$$\mathcal{A}^{h'} = (\mathcal{A}^{h'} \setminus X) \cup (\mathcal{A}^{h'} \cap X) \in \mathcal{I}_{\mathcal{A}},$$

which is a contradiction to the property $\mathcal{I}_{\mathcal{A}} \cap BC(\mathcal{A}) = \emptyset$ which is required of independence diagonalization ideals. $\square$

This inclusion can be proper. In fact an easy argument shows that this always happens for independent families which are not maximal.

Lemma 2.33. *Let $\mathcal{A}$ be an independent family which is not maximal. Then $id(\mathcal{A})$ is a proper subset of any diagonalization ideal $\mathcal{I}_{\mathcal{A}}$.*

Proof. Let $X \notin \mathcal{A}$ be such that $\mathcal{A} \cup \{X\}$ is still independent. By definition there is some $h \in FF(\mathcal{A})$ such that either $X \cap \mathcal{A}^h$ or $\mathcal{A}^h \setminus X$ belongs to $\mathcal{I}_{\mathcal{A}}$. However, neither of those two sets can be in $id(\mathcal{A})$, since for any given $h' \in FF(\mathcal{A})$ with $h' \supseteq h$ the independence of $\mathcal{A} \cup \{X\}$ assures that both

$$X \cap \mathcal{A}^h \cap \mathcal{A}^{h'} = X \cap \mathcal{A}^{h'} \text{ and } (\mathcal{A}^h \setminus X) \cap \mathcal{A}^{h'} = \mathcal{A}^{h'} \setminus X$$

are infinite. $\square$

It is natural to ask about conditions for the two ideals to coincide. One sufficient condition for $id(\mathcal{A}) = \mathcal{I}_{\mathcal{A}}$ is given by $\mathcal{A}$ being a special kind of maximal independent family of the following kind:

Definition 2.34. An independent family $\mathcal{A}$ is called *densely maximal* if for every $X \in [\omega]^\omega \setminus \mathcal{A}$ and $h \in FF(\mathcal{A})$ there exists $h' \in FF(\mathcal{A})$ with $h' \supseteq h$ such that either $X \cap \mathcal{A}^{h'}$ or $\mathcal{A}^{h'} \setminus X$ is finite.

Looking at the definitions we can already assume that $id(\mathcal{A})$ will be particularly nice for a densely maximal independent family $\mathcal{A}$. We will show that in this case any other ideal living outside of $BC(\mathcal{A})$ must be contained in $id(\mathcal{A})$. From this it easily follows that densely maximal independent families only have one independence diagonalization ideal, namely $\mathcal{I}_{\mathcal{A}} = id(\mathcal{A})$.

Lemma 2.35. *Let $\mathcal{A}$ be a densely maximal independent family and $\mathcal{I}$ be an ideal on ω such that $\mathcal{I} \cap BC(\mathcal{A}) = \emptyset$. Then $\mathcal{I} \subseteq id(\mathcal{A})$.*

Proof. Suppose towards contradiction that there is some $X \in \mathcal{I} \setminus id(\mathcal{A})$. Since $X \notin id(\mathcal{A})$ there must be some $h \in FF(\mathcal{A})$ such that $\mathcal{A}^{h'} \cap X$ is infinite for every extension $h' \supseteq h$. However, since $\mathcal{I} \cap BC(\mathcal{A}) = \emptyset$ and $X \in \mathcal{I}$ hold, it follows that $X \notin \mathcal{A}$, thus we can apply dense maximality to X and h to find an extension $h' \supseteq h$ such that either $X \cap \mathcal{A}^{h'}$ or $\mathcal{A}^{h'} \setminus X$ is finite.
In fact only the latter case can hold by the above, so $\mathcal{A}^{h'} \setminus X$ needs to be finite. We may choose a further extension $h'' \supseteq h'$ such that $\mathcal{A}^{h''} \setminus X = \emptyset$, which is equivalent to $\mathcal{A}^{h''} \subseteq X$. But we assumed that X is an element of the ideal $\mathcal{I}$, so $\mathcal{A}^{h''} \subseteq X$ implies that $\mathcal{A}^{h''} \in \mathcal{I}$, a contradiction to the assumption that $\mathcal{I}$ does not contain any boolean combinations of $\mathcal{A}$. $\square$

Corollary 2.36. *If $\mathcal{A}$ is a densely maximal independent family then there is precisely one independence diagonalization ideal $\mathcal{I}_{\mathcal{A}}$ for $\mathcal{A}$, namely $\mathcal{I}_{\mathcal{A}} = id(\mathcal{A})$.*

Proof. Let $\mathcal{I}_{\mathcal{A}}$ be some independence diagonalizazion ideal. By Lemma 2.32. we know that $id(\mathcal{A}) \subseteq \mathcal{I}_{\mathcal{A}}$ holds in any case. The other inclusion $\mathcal{I}_{\mathcal{A}} \subseteq id(\mathcal{A})$ is a consequence of the previous lemma since $\mathcal{I}_{\mathcal{A}} \cap BC(\mathcal{A}) = \emptyset$ is a prerequisite for every independence diagonalization ideal. $\square$

An equivalent but perhaps more technical characterization of densely maximal independent families is given in the following:

Lemma 2.37. *An independent family $\mathcal{A}$ is densely maximal if and only if it satisfies the following:*

(DM) For all $h \in FF(\mathcal{A})$ and $X \subseteq \mathcal{A}^h$ one of the following holds: Either there is $B \in id(\mathcal{A})$ such that $\mathcal{A}^h \setminus X \subseteq B$, or there is $h' \in FF(\mathcal{A})$ such that $h' \supseteq h$ and $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \setminus X$.

Proof. We first show that the condition (DM) is sufficient for dense maximality: Given $X \in [\omega]^\omega \setminus \mathcal{A}$ and $h \in FF(\mathcal{A})$ we consider the set $Y = X \cap \mathcal{A}^h$ and distinguish the two cases in (DM):
Case 1: There is $B \in id(\mathcal{A})$ such that $\mathcal{A}^h \setminus Y \subseteq B$. This implies that

$$\mathcal{A}^h \setminus X = \mathcal{A}^h \setminus Y \subseteq B$$

is itself an element of $id(\mathcal{A})$, so by definititon there is $h' \supseteq h$ such that the set $\mathcal{A}^{h'} \cap (\mathcal{A}^h \setminus X) = \mathcal{A}^{h'} \setminus X$ is finite.

Case 2: There is some $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \setminus Y = \mathcal{A}^h \setminus X$, and thus in particular $\mathcal{A}^{h'} \cap X = \emptyset$. In any case the definition of dense maximality is fulfilled, and thus (DM) implies that $\mathcal{A}$ is densely maximal.
Now we show that the condition (DM) is also necessary. Let $\mathcal{A}$ be densely maximal and let $h \in FF(\mathcal{A})$ as well as $X \subseteq \mathcal{A}^h$ be given. We will suppose there is no $B \in id(\mathcal{A})$ containing $\mathcal{A}^h \setminus X$ and show that then there is some $h' \supseteq h$ with $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \setminus X$. By definition of $id(\mathcal{A})$ and since by assumption $\mathcal{A}^h \setminus X \notin id(\mathcal{A})$ we find some $h' \in FF(\mathcal{A})$ such that for all extensions $h'' \supseteq h'$ we get that $\mathcal{A}^{h''} \cap (\mathcal{A}^h \setminus X)$ is infinite. The functions h and h' must be compatible, since otherwise it follows that $\mathcal{A}^{h'} \cap (\mathcal{A}^h \setminus X) = \emptyset$, contradicting this. Thus we may assume without loss of generality that $h' \supseteq h$ and the above property simplifies to the fact that

$$\text{for all extensions } h'' \supseteq h' : \mathcal{A}^{h''} \cap (\mathcal{A}^h \setminus X) = \mathcal{A}^{h''} \setminus X \text{ is infinite.}$$

Now we may apply dense maximality of $\mathcal{A}$ to $\mathcal{A}^{h'} \setminus X$ and h'. This yields an extension $h'' \supseteq h'$ such that either $\mathcal{A}^{h''} \setminus X$ or $\mathcal{A}^{h''} \cap X$ is finite. In fact by the above only the latter case can hold, and we can again further extend $h''' \supseteq h''$ to find $\mathcal{A}^{h'''} \cap X = \emptyset$. So we found an extension h''' of h such that $\mathcal{A}^{h'''} \subseteq \mathcal{A}^h \setminus X$ holds, and thus (DM) is satisfied. ☐

As part of our main proof in Section 3.2 we use this criterion (there denoted $(*)_0$) to show that the family $\mathcal{A}_G$ adjoined by the forcing poset $\mathbb{P}$ is in fact densely maximal.

One might assume that the discussed ideals are maximal if the independent family $\mathcal{A}$ is maximal. However, as shown in the following result, this turns out to be wrong - an independence diagonalization ideal can't be maximal, and therefore the same is true for the subideal $id(\mathcal{A})$.

Proposition 2.38. *Let $\mathcal{I}_\mathcal{A}$ be an independence diagonalization ideal associated to some independent family $\mathcal{A}$. Then there is a set $X \in [\omega]^\omega \setminus BC(\mathcal{A})$ such that $X \notin \mathcal{I}_\mathcal{A}$ as well as $\omega \setminus X \notin \mathcal{I}_\mathcal{A}$.*

Proof. We start by introducing some notation: For $h \in FF(\mathcal{A})$ let $h^\perp$ denote the 'opposite boolean combination', meaning that $dom(h) = dom(h^\perp)$ and $h^\perp(A) = 1 - h(A)$ for all $A \in dom(h)$. So whenever h chooses the complement of a set $A \in \mathcal{A}$ we have that $h^\perp$ chooses the set itself and vice versa. Now for the construction of X: Choose any $g \in FF(\mathcal{A})$ and a finite nonempty subset $X_0 \subseteq \omega \setminus \mathcal{A}^g$. Now set $X = \mathcal{A}^g \cup X_0$.
First we show that $X \notin BC(\mathcal{A})$. Suppose otherwise and let $h \in FF(\mathcal{A})$ be such that $X = \mathcal{A}^g \cup X_0 = \mathcal{A}^h$. From this subset relation it follows that g properly

extends h, that is $h \subsetneq g$. Set $g_0 = g \restriction (dom(g) \setminus dom(h))$ and consider the boolean combination $h' = h \cup g_0^\perp$. Then

$$\mathcal{A}^{h'} = \mathcal{A}^h \cap \mathcal{A}^{g_0^\perp} = (\mathcal{A}^g \cup X_0) \cap \mathcal{A}^{g_0^\perp} = X_0 \cap \mathcal{A}^{g_0^\perp}$$

yields a contradiction, since on the left hand side there is a supposedly infinite set while on the right hand side we have a subset of the finite set X_0.
It remains to show that neither X itself nor its complement $\omega \setminus X$ is an element of $\mathcal{I}_\mathcal{A}$. Since $\mathcal{I}_\mathcal{A} \cap BC(\mathcal{A}) = \emptyset$ we have that $X \notin \mathcal{I}_\mathcal{A}$, because otherwise $\mathcal{A}^g \subseteq X$ would also be in $\mathcal{I}_\mathcal{A}$. Furthermore $\mathcal{A}^{g^\perp} \setminus X_0 \subseteq \omega \setminus X$, so assuming that $\omega \setminus X \in \mathcal{I}_\mathcal{A}$ leads to

$$\mathcal{A}^{g^\perp} \subseteq (\omega \setminus X) \cup X_0 \in \mathcal{I}_\mathcal{A},$$

once again contradicting the fact that independence diagonalizazion ideals do not contain any boolean combinations. $\square$

We finish this section with a continued discussion of the poset $\mathbb{B}(\mathcal{I}_\mathcal{A})$. This poset adjoint an independent real and the question arises whether it is unbounded or even dominating over the ground-model reals. While the first question always has a positive answer, the second one is more complicated and a partial answer is again related to dense maximality.

Proposition 2.39. *Let $\mathcal{I}_\mathcal{A}$ be an independence diagonalization ideal for a given independent family $\mathcal{A}$ and let $f_G \in V[G]$ be the increasing enumeration of the set x_G added by the $\mathbb{B}(\mathcal{I}_\mathcal{A})$-generic filter G. Then for every ground model function $g \in V \cap \omega^\omega$ we have that*

$$V[G] \Vdash \forall n \exists m > n : g(m) < f_G(m).$$

In other words, $\mathbb{B}(\mathcal{I}_\mathcal{A})$ adds unbounded reals.

Proof. We fix $g \in V \cap \omega^\omega$ and show that for all $n \in \omega$ the set

$$D_n = \{q \in \mathbb{B}(\mathcal{I}_\mathcal{A}) \mid \exists m > n : q \Vdash g(m) < f_G(m)\}$$

is dense. So choose an arbitrary condition $(s, E) \in \mathbb{B}(\mathcal{I}_\mathcal{A})$ and recall that this means that $s \in [\omega]^{<\omega}$ and $E \in [\mathcal{I}_\mathcal{A}]^{<\omega}$. Now consider the set $C = \omega \setminus (\bigcup E \cup s)$ which is infinite since $(\bigcup E \cup s) \in \mathcal{I}_\mathcal{A}$ and otherwise it follows that

$$\omega = (\bigcup E \cup s) \cup C \in \mathcal{I}_\mathcal{A},$$

a contradiction. Take an initial segment $s \cup t \subset s \cup C$ of size $m > n$. The condition $(s \cup t, E)$ now forces that the increasing enumerating function of $(s \cup t)$

coincides with $f_G \restriction m$.

Next we want to find an extension which ensures that at place m the generic real will be above g. We define $u = (C \setminus t) \cap (g(m)+1)$ and choose $k = min(C \setminus (t \cup u))$. Finally we have

$$q := (s \cup t \cup \{k\}, E \cup \{u\}) \leq (s \cup t, E) \leq (s, E).$$

Note that $(s \cup t, E) \leq (s, E)$ is valid since $t \subset C$ and $C \cap \bigcup E = \emptyset$, also $q \leq (s \cup t, E)$ holds because of $k \notin \bigcup E$. Furthermore we have that $q \Vdash g(m) < f_G(m)$, because q forces that $f_G(m) = k > g(m)$, which finishes the proof. $\square$

Proposition 2.40. *Let $\mathcal{I}_\mathcal{A}$ be an independence diagonalization ideal for an independent family $\mathcal{A} \in V$. If there is a family $\{X_g \mid g \in \omega^\omega$ increasing $\}$ in $\mathcal{P}(\omega) \cap V$ with the properties*

(i). $X_g \subseteq [g(0), \omega)$

(ii). $\omega \setminus X_g \in \mathcal{I}_\mathcal{A}$

(iii). $|X_g \cap [g(n), g(n+1))| \leq 1$ *for almost all* $n \in \omega$,

then $\mathbb{B}(\mathcal{I}_\mathcal{A})$ adds a dominating real f_G, meaning that $V[G] \models \exists m \forall n > m : g(n) < f_G(n)$ for each $g \in V \cap \omega^\omega$. In particular this is the case if $\mathcal{A}$ is densely maximal.

Proof. We show that for an arbitrary $g \in \omega^\omega \cap V$ the set of conditions

$$D = \{q \in \mathbb{B}(\mathcal{I}_\mathcal{A}) \mid \exists m : q \Vdash \forall n > m : g(n) < f_G(n)\}$$

is dense. Let a condition $(s, E) \in \mathbb{B}(\mathcal{I}_\mathcal{A})$ be given and let m be the least number such that $m > max(s)$ and for all $n \geq m$ we have

$$|X_g \cap [g(n), g(n+1))| \leq 1.$$

Such a m exists due to property (iii). Now again let $C = \omega \setminus (\bigcup E \cup s)$ and let $s \cup t$ be an initial segment of $s \cup C$ with $|s \cup t| = m$. Now set $F = E \cup \{\omega \setminus X_g, [0, g(m))\}$ and note that $F \in [\mathcal{I}_\mathcal{A}]^{<\omega}$ follows from property (ii). Now consider the condition

$$q := (s \cup t, F) \leq (s \cup t, E) \leq (s, E).$$

Note that

$$q \Vdash \forall n > m : f_G(n) \notin (\omega \setminus X_g) \cup [0, g(m)),$$

which implies

$$q \Vdash \forall n > m : f_G(n) \in X_g \cap [0, g(m)).$$

With this and properties (i) and (iii) we conclude that $q \Vdash \forall n > m : g(n) \leq f_G(n)$, so we found an extension $(s, E) \geq q \in D$ as desired.
If $\mathcal{A}$ is a densely maximal independent family, by Corollary 2.31. we know that the dual filter $fil(\mathcal{A})$ of $id(\mathcal{A}) = \mathcal{I}_{\mathcal{A}}$ is Ramsey. We use this property to construct X_g for a given increasing g accordingly:
Consider the partition $\mathcal{E}$ of ω into finite sets given by $[0, g(0))$ and $[g(n), g(n+1))$ for every $n \in \omega$. Now clearly $fil(\mathcal{A})$ contains all the cofinite sets $\omega \setminus E$ for $E \in \mathcal{E}$ and thus, since it is Ramsey, we get some $X'_g \in fil(\mathcal{A})$ with $|X'_g \cap E| \leq 1$, so X'_g satisfies condition (iii) already. Since $X'_g \in fil(\mathcal{A})$ we also have that $\omega \setminus X'_g \in id(\mathcal{A})$, which is condition (ii). To satisfy (i) as well we restrict to the set $X_g = X'_g \cap [g(0), \omega)$, which still lies in $fil(\mathcal{A})$ since $X'_g \in fil(\mathcal{A})$ and $[g(0), \omega)$, being cofinite, also is an element of $fil(\mathcal{A})$. $\square$

3 Sacks Indestructibility.

Overwiew:

In Section 3.1. we introduce Sacks forcing in terms of perfect subtrees and show that it satisfies Axiom A, the Sacks property and is ω^ω-bounding. Many of the proofs in this section rely on an important technique when it comes to Sacks forcing - the construction of fusion sequences.

In Section 3.2. we show that the independent family $\mathcal{A}_G$ adjoined by the poset $\mathbb{P}$ is Sacks-indestructible. The whole section is dedicated to the proof of this fact.

References:

Stefan Geschke and Sandra Quickert give a valuable exposition of Sacks forcing and the Sacks property in their paper [7]. Together with Chapter 23 of Halbeisen's book [9] it provided a basis for Section 3.1.

The full proof of the Sacks indestructibility of $\mathcal{A}_G$ as presented in Section 3.2 was given in the (as of yet unpublished) paper [6] by Vera Fischer.

3.1 Sacks Forcing and the Sacks Property

This section provides an overview of Sacks forcing and its properties. Our basic definitions concern trees and perfect subtrees. Recall that a *tree* $(T, \sqsubseteq)$ is a partially ordered set such that for every $t \in T$ the set of predecessors $t{\downarrow}= \{s \in T \mid s \sqsubset t\}$ is well-ordered by $\sqsubset$. We call $s, t \in T$ *compatible* (denoted $s \not\perp t$) if one extends the other, that is either $s \sqsubseteq t$ or $t \sqsubseteq s$ holds. Otherwise we call them *incompatible*, denoted $t \perp s$. We call $S \subseteq T$ a *subtree* of T if $(S, \sqsubseteq)$ is downwards closed, meaning that for all $s \in S$ we have that $s{\downarrow}\subseteq S$. The subtree is called *perfect* if every $s \in S$ has two incompatible extensions, that is there are $t_1, t_2 \in S$ with $s \sqsubseteq t_1, s \sqsubseteq t_2$ and $t_1 \perp t_2$.

Applying these concepts to the binary tree, we will now define the notion of Sacks forcing.

Definition 3.1. *Sacks forcing* $\mathbb{S}$ is the set of all perfect subtrees of the binary tree $2^{<\omega}$ with the extension relation given by inclusion:

$$S_1 \leq S_2 \Leftrightarrow S_1 \subseteq S_2.$$

This means that stronger conditions are the smaller subtrees.

For a given $\mathbb{S}$-generic filter G, the corresponding *Sacks real* is given by $\bigcap G \in 2^\omega$.

Remark 3.2. To see that $\bigcap G$ is a well-defined element of 2^ω first notice that all branches of elements of $\mathbb{S}$ are infinite since a leaf doesn't have two incompatible extensions. Therefore the set $D_n = \{T \in \mathbb{S} \mid |stem(T)| \geq n\}$ is dense in $\mathbb{S}$ for every $n \in \omega$.

Sacks forcing does not satisfy *ccc*. Our proof of this fact is yet another nice application of the existence of almost disjoint families of size $\mathfrak{c}$:

Lemma 3.3. *The poset $\mathbb{S}$ contains an antichain of size $\mathfrak{c}$, and thus is not ccc.*

Proof. Let $\mathfrak{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\} \subseteq [\omega]^\omega$ be an almost disjoint family. For each $\alpha < \mathfrak{c}$ we set $T_\alpha = \{s \in 2^{<\omega} \mid \forall n < height(s) : n \notin A_\alpha \to s(n) = 0\}$. Each T_α is a perfect tree which splits exactly at the levels $n \in A_\alpha$. Since $\mathfrak{A}$ is almost disjoint, for $\alpha < \beta < \mathfrak{c}$ we have that $T_\alpha \cap T_\beta$ can only have finitely many splitting levels and thus does not contain a perfect subtree of $2^{<\omega}$. This means exactly that T_α and T_β are incompatible, thus $\{T_\alpha \mid \alpha \in \mathfrak{c}\}$ is an antichain of size $\mathfrak{c}$. □

However, there is a weaker property similar to the countable chain condition which still assures that $\aleph_1$ is preserved, commonly referred to as Baumgartner's *Axiom A*:

Proposition 3.4. *Sacks forcing $(\mathbb{S}, \leq)$ satisfies (Axiom A), meaning that there is a decreasing chain of partial orders $(\leq_n)_{n\in\omega}$ on $\mathbb{S}$ with $\leq_0 = \leq$ satisfying the following two conditions:*

(i). Every fusion sequence $(S_n)_{n\in\omega} \subseteq \mathbb{S}$*, that is $S_{n+1} \leq_n S_n$ for all $n \in \omega$, has a* fusion $S \in \mathbb{S}$*, meaning that $S \leq_n S_n$ for all $n \in \omega$.*

(ii). For every antichain A, condition T in $\mathbb{S}$ and $n \in \omega$ there is an extension $S \leq_n T$ such that $\{T' \in \mathbb{S} \mid T' \not\perp S\} \cap A$ is countable - in other words only countably many conditions in A are compatible with S.

Proof. Given a perfect subtree $T \in \mathbb{S}$, for $n \in \omega$ let $split_n(T)$ denote the set of all nodes $t \in T$ minimal with respect to the following: t has exactly n proper initial segments $s \sqsubset t$ such that both $s^\frown 0$ and $s^\frown 1$ are contained in T. Note that there is a natural bijection $split_n(T) \simeq 2^n$. Now define the relations $\leq_n$ as follows for all $S_1, S_2 \in \mathbb{S}$:

$$S_1 \leq_n S_2 \Leftrightarrow S_1 \leq S_2 \wedge split_n(S_1) = split_n(S_2).$$

Clearly $\leq_0 = \leq$ follows by definition and the fact that $\leq_n \supseteq \leq_{n+1}$ for all $n \in \omega$ is shown by a straightforward induction. We proceed to verify the two properties:
(i): Given a fusion sequence $(S_n)_{n\in\omega}$, notice that $S = \bigcap_{n\in\omega} S_n$ is still a perfect tree, and thus a member of $\mathbb{S}$. Furthermore $S \leq_n S_n$ holds for all $n \in \omega$.
(ii): Let $A \subseteq \mathbb{S}$ be an antichain and $T \in \mathbb{S}$ be some condition. Using the natural correspondence $split_n(T) \simeq 2^n$ we find that every $\sigma \in 2^n$ corresponds to some unique $t_\sigma \in split_n(T)$. This defines an extension $T * \sigma \leq T$ given by the set $T * \sigma = \{s \in T \mid s \sqsubseteq t_\sigma \vee t_\sigma \sqsubset s\}$. For every $\sigma \in 2^n$ we can pick an extension $S_\sigma \leq T * \sigma$ compatible with at most one element of A. Finally

define $S = \bigcup_{\sigma \in 2^n} S_\sigma$, which as a union of perfect trees is itself a perfect tree and furthermore satisfies $S \leq_n T$. It also has the property that $S_\sigma = S * \sigma$ for all $\sigma \in 2^n$. Now if $S \not\perp R$ holds for some $R \in A$ then in particular $S * \sigma \not\perp R$ holds for some $\sigma \in 2^n$. But we made sure that $S * \sigma = S_\sigma$ is compatible with at most one element of A, and therefore S is compatible with at most 2^n distinct elements of A - at most one for every $\sigma \in 2^n$. □

Next the Sacks property will be defined in general before we show that Sacks forcing actually satisfies it.

Definition 3.5. Let $V_0 \subseteq V_1$ be models of ZFC. We say that V_1 has the *Sacks property* over V_0 if for every real $r \in \omega^\omega \cap V_1$ in the larger model there is a map $C : \omega \to [\omega]^{<\omega}$ in V_0 with $|C(n)| \leq 2^n$ and $r(n) \in C(n)$ for every $n \in \omega$.
A forcing notion $\mathbb{P}$ has the Sacks property if $V[G]$ has the Sacks property over the ground model V for every $\mathbb{P}$-generic filter G.

We will in fact show that Sacks forcing has a property which is even stronger. For this purpose we use the following notion: a subtree $\mathcal{T} \subseteq \omega^{<\omega}$ is *binary* if every $t \in \mathcal{T}$ has at most two immediate successors. We say that a real $r \in \omega^{<\omega}$ is *covered* by $\mathcal{T}$ if $r \in [\mathcal{T}]$ where $[\mathcal{T}] = \{x \in \omega^\omega \mid \forall n \in \omega : x \restriction n \in \mathcal{T}\}$ denotes the set of infinite branches through T.

Definition 3.6. Let $V_0 \subseteq V_1$ be models of ZFC. Then V_1 has the *2-localization property* over V_0 if every real $r \in \omega^\omega \cap V_1$ in the larger model V_1 is covered by a binary tree in V_0.
As before, we say that a forcing notion $\mathbb{P}$ has this property if every $\mathbb{P}$-generic extension has this property over the ground model.

Clearly the 2-localization property implies the Sacks property - simply set $C(n) = level_n(\mathcal{T})$ where $\mathcal{T}$ denotes the binary tree covering r.

Proposition 3.7. *Sacks forcing $\mathbb{S}$ has the 2-localization property, and therefore it has the Sacks property as well.*

Proof. Let $\dot{r}$ be a $\mathbb{S}$-name and $T \in \mathbb{S}$ a condition with $T \Vdash \dot{r} \in \omega^\omega$. We want to find a binary tree $\mathcal{T} \subseteq \omega^{<\omega}$ and a stronger condition $S \leq T$ with $S \Vdash \dot{r} \in [\mathcal{T}]$. Note that any condition $S \leq T$ forces that the tree defined by the equation

$$\mathcal{T}_S = \{x \in \omega^{<\omega} \mid \exists S' \leq S : S' \Vdash x \in \dot{r}\}$$

will have $\dot{r}$ as a branch. We are done if we can choose the condition S such that the corresponding tree $\mathcal{T}_S$ is binary. If there is some condition $S \leq T$ deciding all of $\dot{r}$ in the first place, then $\mathcal{T}_S$ does not split at all and we are done. So assume that there is no condition below T deciding all of $\dot{r}$.

Following the notation of the proof of Proposition 3.4. we inductively define a fusion sequence of conditions $(T_n)_{n\in\omega} \subseteq \mathbb{S}$ with $T_{n+1} \leq_n T_n$ and in the end will set $S = \bigcap_{n\in\omega} T_n$. Start with $T_0 = T$ and inductively assume that T_n is already defined.
For $\sigma \in 2^n$ we consider the conditions $T_n * (\sigma^\frown 0)$ and $T_n * (\sigma^\frown 1)$. By assumption neither of them decides all of $\dot{r}$, so we can find incompatible conditions $S_{\sigma^\frown 0} \leq T_n * (\sigma^\frown 0)$ and $S_{\sigma^\frown 1} \leq T_n * (\sigma^\frown 1)$ forcing incomparable initial segments of $\dot{r}$. More specifically, having $\dot{r}_S$ denote the longest initial segment of $\dot{r}$ decided by $S \in \mathbb{S}$, we can assure that $\dot{r}_{S_{\sigma^\frown 0}}$ and $\dot{r}_{S_{\sigma^\frown 1}}$ are incomparable. Now we may set $T_{n+1} = \bigcup\{S_{\sigma^\frown i} \mid i \in 2, \sigma \in 2^n\}$.
Now for the condition $S = \bigcap_{n\in\omega} T_n$ and some $n \in \omega$ we can look at the induced finite tree $\mathcal{T}_n$ consisting of all initial segments of $\{\dot{r}_{S*\sigma} \mid \sigma \in 2^n\}$. This is a finite binary tree of height at least n with $\mathcal{T}_n \subseteq \mathcal{T}_S$ containing all elements of $\mathcal{T}_S$ having length $\leq n$. Therefore $\mathcal{T}_S = \bigcup_{n\in\omega} \mathcal{T}_n$ is a binary tree as desired. $\square$

Later on we shall also use the fact that Sacks forcing has another important property:

Definition 3.8. A forcing notion $\mathbb{Q}$ is called *ω^ω-bounding* if every function in any $\mathbb{Q}$-generic extension $V^{\mathbb{Q}}$ is dominated by a ground model function, that is for every $f \in \omega^\omega \cap V^{\mathbb{Q}}$ there exists $g \in \omega^\omega \cap V$ such that

$$V^{\mathbb{P}} \models \exists n \in \omega : \forall m > n : f(m) < g(m).$$

Proposition 3.9. *Sacks forcing is ω^ω-bounding.*

Before we prove this propostition we introduce some notation. Given $T \in \mathbb{S}$ and $t \in T$ we set

$$T[t] = \{s \in T \mid s \sqsubseteq t \vee t \sqsubseteq s\}.$$

Note that since $T \in \mathbb{S}$ we always have that $T[t]$ is still a perfect subtree, thus $T[t] \in \mathbb{S}$. Furthermore clearly $T[t] \leq T$ holds by definition. Now for the proof of the proposition:

Proof. Let $f \in \omega^\omega \cap V^{\mathbb{S}}$ be given and let $\dot{f}$ be a $\mathbb{S}$-name for it. We show that for every condition $S \in \mathbb{S}$ there is an extension $T \leq S$ forcing that a ground model function $g \in \omega^\omega \cap V$ will dominate f:

$$T \Vdash \exists n \in \omega : \forall m > n : \dot{f}(m) < g(m).$$

In other words, we will show that the set of conditions forcing this is dense, thus proving the proposition.
We will now define a decreasing sequence of conditions $(T_i \mid i \in \omega)$ below S

together with a sequence of integers $(k_i \mid i \in \omega)$ such that with $T_i \Vdash \dot{f}(i) < k_i$. The condition $T = \bigcap_{i\in\omega} T_i$ and the ground model function defined by $g(n) = k_n$ will then fulfill our requirements. The construction goes as follows:
Choose $T_0 \leq S$ with $T_0 \Vdash \dot{f}(0) < k_0$, where $k_0 \in \omega$ is the smallest integer for which such a condition exists. Now let $t \in T_0$ be such that both $t_0 := t^\frown 0$ and $t_1 := t^\frown 1$ are in T_0. Let $k_1 \in \omega$ be minimal such that there exist extensions $T_{0,0} \leq T_0[t_0]$ and $T_{0,1} \leq T_0[t_1]$ such that for both $i \in 2$ we have that $T_{0,i} \Vdash \dot{f}(1) < k_1$. Now $T_1 = T_{0,0} \cup T_{0,1}$ works, as shown in the following:
Assume towards contradiction that $T_1 \nVdash \dot{f}(1) < k_1$. Then there is $T' \leq T_1$ forcing the contrary $T' \Vdash \dot{f}(1)$. But then we also have that every extension of both T' and either $T_{0,0}$ or $T_{0,1}$ also forces this, a contradiction.
To clarify the argument we also show the next step to obtain $T_2 \leq T_1$. For $i \in 2$ we choose $t_{0,i}, t_{1,i} \sqsupseteq t_i$ dinstinct but of the same length and inside $T_{0,i}$. Pick $k_2 \in \omega$ minimal such that for all $i, j \in 2$ we can find conditions $T_{0,i,j} \leq T_{0,i}[t_{i,j}]$ with $T_{0,i,j} \Vdash \dot{f}(2) < k_2$. Now $T_2 = \bigcup_{i,j\in 2} T_{0,i,j}$ works for the same reason as before. Proceeding with this construction we obtain our sequence of conditions as desired, finishing the proof. $\square$

Finally, since in the next section we look at the countable support iteration of Sacks forcing we also cite some important preservation theorems found in [15] at this point:

Theorem 3.10. *Let $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma)$ be a countable support iteration.*

(i). If $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is proper and ω^ω-bounding" for every $\alpha < \gamma$ then $\mathbb{P}_\gamma$ is proper and ω^ω-bounding itself.

(ii). If $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is proper and has the Sacks property" for every $\alpha < \gamma$ then $\mathbb{P}_\gamma$ is proper and has the Sacks property itself.

$\square$

A direct proof of the fact that Sacks forcing is proper can, for example, be found in [9]. However, in [1] Baumgartner proved the useful result that every forcing notion satisfying Axiom A is proper.

3.2 A Sacks indestructible maximal independent family.

In this section we show that the maximal independent family adjoined by the poset $\mathbb{P}$ remains maximal after a length ω_2 countable support iteration of Sacks forcing. We start with a ground model V in which both CH and $2^{\aleph_1} = \aleph_2$ hold. Furthermore we set $V_0 = V^{\mathbb{P}}$ and denote the maximal independent family which $\mathbb{P}$ adjoins by $\mathcal{A}_G$.

We first describe a correspondence between the independence density ideal $id(\mathcal{A}_G)$ in V_0 and in its Sacks extension $V_0^{\mathbb{S}}$ - we show that these two have the same generating set.

Lemma 3.11. *For each $X \in id(\mathcal{A}_G)^{V_0^{\mathbb{S}}}$ there exists a $X_0 \in id(\mathcal{A}_G)^{V_0}$ with $X \subseteq X_0$.*

Proof. Let $G * H$ be $\mathbb{P} * \mathbb{S}$-generic over V and let $\dot{X}$ be a $\mathbb{P} * \mathbb{S}$-name for X. Then there is a condition $((\mathcal{A}, A), S) \in G * H$ such that $((\mathcal{A}, A), S) \Vdash \dot{X} \in id(\mathcal{A}_G)$. Since $\mathbb{P}$ is countably closed we may assume without loss of generality that $((\mathcal{A}, A), S) \Vdash \dot{X} \in id(\mathcal{A})$. Since $(\mathcal{A}, A) \in G$ we have that $\mathcal{A}$ is countable and therefore there is a natural correspondence $FF(\mathcal{A}) \simeq \omega^{<\omega}$.
Recall that $id(\mathcal{A}) = \{X \subseteq \omega \mid \mathcal{D}(X) \text{ is dense in } FF(\mathcal{A})\}$, where $\mathcal{D}(X)$ is given by $\mathcal{D}(X) = \{h \in FF(\mathcal{A}) \mid |\mathcal{A}^h \cap X| = \emptyset\}$. Let $\dot{\mathcal{D}}$ be a $\mathbb{P} * \mathbb{S}$-name for $\mathcal{D}(X)$, which by the above correspondence can be associated with an open dense subset of ω^ω. In particular we have that $((\mathcal{A}, A), S) \Vdash$ " $\dot{\mathcal{D}}$ is open dense in ω^ω ".
We now want to show the following:

($*$): In V_0 the set of conditions $T \in \mathbb{S}$ for which there exists a dense open subset $D_0 \subseteq \omega^\omega \cap V_0$ such that $T \Vdash \check{D} \subseteq \dot{\mathcal{D}}$ is dense below S.

Once we have shown this, we can find find a dense open $D \subseteq \omega^\omega \cap V_0$ such that $D \subseteq \mathcal{D}(X)$ holds in $V_0^{\mathbb{S}}$. For this D we find some $D_0 \subseteq \omega^{<\omega}$ such that $D = \bigcup_{t \in D_0}[t]$ holds, since D is open. Then we can define

$$X_0 = \bigcap \{\omega \setminus \mathcal{A}^h \mid h \in D_0\},$$

which has the desired properties - indeed we have $X_0 \in id(\mathcal{A}) \subseteq id(\mathcal{A}_G)$ since $D_0 \subseteq \mathcal{D}(X_0)$ is dense in $FF(\mathcal{A})$ and $X \subseteq X_0$ holds because $X \subseteq \omega \setminus \mathcal{A}^h$ is true for all $h \in \mathcal{D}(X)$, so in particular it holds for the elements of $D_0 \subseteq \mathcal{D}(X)$. So we are left with proving ($*$):
First we pick an enumeration $\{s_n \mid n \in \omega\}$ of $\omega^{<\omega}$. We construct a fusion sequence $(T_n)_{n \in \omega}$ and elements $(t_n \mid n \in \omega) \subseteq \omega^{<\omega}$ with $s_n \sqsubseteq t_n$ such that $T_n \Vdash [t_n] \subseteq \dot{\mathcal{D}}$. We start with $T_0 = S$ and proceed inductively. If T_n is already constructed, enumerate $split_n(T_n) = \{u_i \mid i \in 2^n\}$. We use the natural lexicographic ordering of $\{u_i^\frown j \mid i \in 2^n, j \in 2\} \simeq 2^n \times 2$ and construct $t_{i,j} \in \omega^{<\omega}$ for every $(i, j) \in 2^n \times 2$ according to the following rules:
Since $T_n[u_0^\frown 0] \leq T_n \leq S$ we have $T_n[u_0^\frown 0] \Vdash$ " $\dot{\mathcal{D}}$ is open dense in ω^ω ". Therefore there is some $U_{0,0} \leq T_n[u_0^\frown 0]$ and $t_{0,0} \in \omega^{<\omega}$ with $s_{n+1} \sqsubseteq t_{0,0}$ and $U_{0,0} \Vdash [t_{0,0}] \subseteq \dot{\mathcal{D}}$.
Since $T_n[u_0^\frown 1] \Vdash$ " $\dot{\mathcal{D}}$ is open dense in ω^ω " holds as well, we can now find $U_{0,1} \leq T_n[u_0^\frown 1]$ and $t_{0,1} \sqsupseteq t_{0,0}$ such that $U_{0,1} \Vdash [t_{0,1}] \subseteq \dot{\mathcal{D}}$. Proceed like this

and finally set

$$T_{n+1} = \bigcup\{U_{i,j} \mid (i,j) \in 2^n \times 2\} \text{ and } t_{n+1} = t_{2^n,1}.$$

For the fusion T of $(T_n)_{n\in\omega}$ and $D_0 = \bigcup_{n\in\omega}[t_n]$ we now have $T \Vdash \check{D}_0 \subseteq \dot{D}$ because $T \leq T_n$ implies $T \Vdash [t_n] \subseteq \dot{\mathcal{D}}$ for all $n \in \omega$. Furthermore D_0 is dense open in ω^ω since we made sure in our construction that it meets every basic open set $[s_n]$. $\square$

Now we state and prove the main theorem of this section. The proof goes via induction and furthermore shows that the maximal independent family adjoined by $\mathbb{P}$ is densely maximal, and remains so throughout every step of the iteration of Sacks forcing.

Theorem 3.12. *The generic maximal independent family $\mathcal{A}_G$ adjoined by $\mathbb{P}$ remains maximal in the extension $V_0^{\mathbb{P}_{\omega_2}}$ where $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \omega_2, \beta < \omega_2)$ is a countable support iteration of $\mathbb{S}$ over V_0.*

Proof. Consider the following statement for $\alpha \leq \omega_2$:

$(*)_\alpha$: In $V_0^{\mathbb{P}_\alpha}$ the following holds: For each $h \in FF(\mathcal{A}_G)$ and $X \subseteq \mathcal{A}_G^h$ either there is some $B \in id(\mathcal{A}_G)^{V_0}$ with $\mathcal{A}_G^h \setminus X \subseteq B$, or there is an extension $h' \supseteq h$ such that $\mathcal{A}_G^{h'} \subseteq \mathcal{A}_G^h \setminus X$.

The statement $(*)_{\omega_2}$ actually implies the Theorem as follows:
Suppose otherwise that $\mathcal{A}_G$ is not maximal in $V_0^{\mathbb{P}_{\omega_2}}$. Then there is $X \in [\omega]^\omega \cap V_0^{\mathbb{P}_{\omega_2}}$ such that both $\mathcal{A}_G^h \cap X$ and $\mathcal{A}_G^h \setminus X$ are infinite for every $h \in FF(\mathcal{A}_G)$. Take some fixed $h \in FF(\mathcal{A}_G)$ and set $X^h = \mathcal{A}_G^h \cap X$. Since clearly $X^h \subseteq \mathcal{A}_G^h$ holds, we can apply $(*)_{\omega_2}$ to this set.
In the case that there is some extension $h' \supseteq h$ such that $\mathcal{A}_G^{h'} \subseteq \mathcal{A}_G^h \setminus X^h$ we find that $\mathcal{A}_G^{h'} \cap X^h = \mathcal{A}_G^{h'} \cap X = \emptyset$ contradicting the fact that $\mathcal{A}_G^{h'} \cap X$ should remain infinite.
In the other case there is $B \in id(\mathcal{A}_G)^{V_0}$ such that $\mathcal{A}_G^h \setminus X^h \subseteq B$. This means there is an extension $h' \supseteq h$ with $\mathcal{A}_G^{h'} \cap B = \emptyset$, or equivalently $\mathcal{A}_G^{h'} \subseteq \omega \setminus B$. But

$$\mathcal{A}_G^h \setminus X = \mathcal{A}_G^h \setminus X^h \subseteq B$$

implies

$$\mathcal{A}_G^{h'} \setminus X \subseteq (\mathcal{A}_G^h \setminus X) \cap (\omega \setminus B) = \emptyset,$$

which also yields a contradiction.

So to finish the proof we inductively show that $(*)_\alpha$ holds for all $\alpha \leq \omega_2$, starting with the base case $(*)_0$: Let $h \in FF(\mathcal{A}_G)$ and $X \subseteq \mathcal{A}_G^h$ be given. If

$\mathcal{A}_G^h \setminus X \notin id(\mathcal{A}_G)$, then by definition of $id(\mathcal{A}_G)$ there is some $h_0 \in FF(\mathcal{A})_G$ such that for all extensions $h_1 \supseteq h_0$ the set $\mathcal{A}_G^{h_1} \cap (\mathcal{A}_G^h \setminus X)$ is infinite. We apply Corollary 2.23.(i) to the partition $\mathcal{E} = \{\mathcal{A}_G^h \setminus X, (\mathcal{A}_G^h \setminus X)^c\}$ and h_0 to find $h_1 \supseteq h_0$ such that $\chi(\mathcal{E}, \mathcal{A}_G^{h_1})$ holds. Since $\mathcal{A}_G^{h_1} \subseteq (\mathcal{A}_G^h \setminus X)^c$ yields the contradiction $\mathcal{A}_G^{h_1} \cap (\mathcal{A}_G^h \setminus X) = \emptyset$ we conclude that $\mathcal{A}_G^{h_1} \subseteq \mathcal{A}_G^h \setminus X$ holds, which confirms $(*)_0$.

Successor case: We assume that $V_0^{\mathbb{P}_\alpha} \models (*)_\alpha$ and want to show that this implies $V_0^{\mathbb{P}_{\alpha+1}} \models (*)_{\alpha+1}$ as well. Suppose towards a contradiction that this is not the case, meaning that the following holds:

$\neg(*)_{\alpha+1}$: In $V^{\mathbb{P}_{\alpha+1}}$ there exists $h \in FF(\mathcal{A}_G)$ and $X \subseteq \mathcal{A}_G^h$ such that $\mathcal{A}_G^h \setminus X \nsubseteq B$ for all $B \in id(\mathcal{A}_G)^{V_0}$, and such that $\mathcal{A}_G^{h'} \cap X \neq \emptyset$ for every extension $h' \supseteq h$.

Thus there is $(\mathcal{A}, A) \in G$ with $h \in FF(\mathcal{A})$, a $\mathbb{P}_{\alpha+1}$-name $\dot{x}$ and $(\bar{p}, \dot{S}) \in \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ for which the following hold for $\tilde{p} = ((\mathcal{A}, A), \bar{p}, \dot{S})$ in V:

(i'). $\tilde{p} \Vdash \dot{x} \subseteq \mathcal{A}^h$,

(ii'). $\tilde{p} \Vdash \forall B \in id(\mathcal{A}_G)^{V_0} : \mathcal{A}^{\langle} \setminus \dot{x} \nsubseteq B$,

(iii'). $\tilde{p} \Vdash \mathcal{A}_G^{h'} \cap \dot{x} \neq \emptyset$ for every $h' \in FF(\mathcal{A}_G)$ which extends h.

However, we would like to work in $V_0^{\mathbb{P}_\alpha}$ in order to use the inductive assumption. This can be achieved by passing to a quotient name. For a $\mathbb{P}_\alpha$-generic filter G_α over V_0 with $\bar{p} \in G_\alpha$ set $S = \dot{S}[G_\alpha]$. Now there is a $\mathbb{S}$-name $\dot{X}$ such that the following holds in $V_0[G_\alpha] = V_0^{\mathbb{P}_\alpha}$:

(i). $S \Vdash \dot{X} \subseteq \mathcal{A}^h$,

(ii). $S \Vdash \forall B \in id(\mathcal{A}_G)^{V_0} : \mathcal{A}^h \setminus \dot{X} \nsubseteq B$,

(iii). $S \Vdash \mathcal{A}_G^{h'} \cap \dot{X} \neq \emptyset$ for every $h' \in FF(\mathcal{A}_G)$ which extends h.

Now for every $l \in \omega$ consider the following set in $V_0^{\mathbb{P}_\alpha}$:

$$Y_l = \{n \in \omega \mid \exists S' \leq_{l+1} S : S' \nVdash \check{n} \notin \dot{X}\}.$$

We claim that the following holds for every $l \in \omega$:

$$S \Vdash \dot{X} \subseteq \check{Y}_l \subseteq \check{\mathcal{A}}^h.$$

To see that $S \Vdash \dot{X} \subseteq \check{Y}_l$ let $n \in \omega$ be such that $S \Vdash \check{n} \in \dot{X}$. If $S' \leq_{l+1} S$ then we also have that $S' \Vdash \check{n} \in \dot{X}$. Therefore $S' \nVdash \check{n} \notin \dot{X}$ and therefore $S \Vdash \check{n} \in \check{Y}_l$ as well.
For $S \Vdash \check{Y}_l \subseteq \check{\mathcal{A}}^h$ we show the contrapositive. Let $n \notin \mathcal{A}^h$. Due to (i) we have

that $S \Vdash \check{n} \notin \dot{X}$, and therefore the same holds for every $S' \leq_{l+1} S$, implying that $n \notin Y_l$ holds as well.
For each $l \in \omega$ we can now apply the inductive hypothesis $(*)_\alpha$ to the set $Y_l \subseteq \mathcal{A}^h$. This means either there is some $B_l \in id(\mathcal{A}_G)^{V_0}$ such that $\mathcal{A}^h \setminus Y_l \subseteq B_l$ or there is an extension $h' \supseteq h$ with $\mathcal{A}_G^{h'} \cap Y_l = \emptyset$. However, assuming the latter leads to a contradiction:
If there was some $h' \supseteq h$ with $\mathcal{A}_G^{h'} \cap Y_l = \emptyset$, then since $S \Vdash \dot{X} \subseteq Y_l$ we also have $S \Vdash \dot{X} \cap \mathcal{A}_G^{h'} = \emptyset$, which contradicts property (iii).
Therefore for each $l \in \omega$ we find some $B_l \in id(\mathcal{A}_G)^{V_0}$ for which $\mathcal{A}^h \setminus Y_l \subseteq B_l$ holds. Therefore the $\mathcal{A}^h \setminus Y_l$ lie in $id(\mathcal{A}_G)^{V_0}$ as well, or equivalently their complements lie in the dual filter:

$$\omega \setminus (\mathcal{A}^h \setminus Y_l) = (\omega \setminus \mathcal{A}^h) \cup Y_l \in fil(\mathcal{A}_G)^{V_0},$$

where the equality follows from $Y_l \subseteq \mathcal{A}^h$. Now, since $fil(\mathcal{A}_G)^{V_0}$ is Ramsey, thus in particular a P-set, we can find a pseudo-intersection $B \in fil(\mathcal{A}_G)^{V_0}$, so we have

$$\forall l \in \omega : B \subseteq^* (\omega \setminus \mathcal{A}^h) \cup Y_l.$$

Since B is an element of $fil(\mathcal{A}_G)^{V_0}$, by Proposition 2.27. we know that there is a countable independent family $\mathcal{B} \in V$ for which $(\mathcal{B}, B) \in G$. We can assume that $\mathcal{A} \subseteq \mathcal{B}$, which implies that $(\mathcal{A}, B) \in G$ due to $(\mathcal{B}, B) \leq (\mathcal{A}, B)$.
We have that the set $(\mathcal{A}^h \cap B) \setminus Y_l$ is bounded for every $l \in \omega$, therefore we can find a strictly increasing function $f \in \omega^\omega \cap V_0^{\mathbb{P}_\alpha}$ with $(\mathcal{A}^h \cap B) \setminus Y_l \subseteq f(l)$. Since $\mathbb{P}_\alpha$ is ω^ω-bounding we can assume that f already lies in V_0.
Now, for all $l \in \omega$ we let

$$k_l = min((\mathcal{A}^h \cap B) \setminus (f(l) + 1)),$$

which always exists since $(\mathcal{A}, B) \in G$ means that $\mathcal{A}^h \cap B$ is infinite. With this we get a strictly increasing sequence $(k_l)_{l \in \omega}$ with the property

$$n \in (\mathcal{A}^h \cap B) \setminus (k_l + 1) \Rightarrow n \in Y_l.$$

Furthermore we can identify this sequence with an interval partition $\mathcal{E}$ of $\omega \simeq \mathcal{A}^h \cap B$. Therefore we can apply part (ii) of Corollary 2.25. to find a semi-selector $C \subseteq B$ in $fil(\mathcal{A}_G)^{V_0}$ for which $(\mathcal{A}, C) \in G$. Enumerate $C = \{c_l \mid l \in \omega\}$ in increasing order and note that $k_l < c_l$ holds for every $l \in \omega$.
We are done if we can find a condition $T \leq S$ such that $T \Vdash \check{C} \subseteq \dot{X}$, since this

together with property (i.) of S implies that

$$T \Vdash \check{C} \subseteq \dot{X} \subseteq \check{\mathcal{A}}^h,$$

from which we can conclude

$$T \Vdash \check{\mathcal{A}}^h \setminus \dot{X} \subseteq \check{\mathcal{A}}^h \setminus \check{C} \subseteq \omega \setminus \check{C}.$$

This, however, is a contradiction to property $(ii.)$ of S, since $\omega \setminus C \in id(\mathcal{A})_G^{V_0}$.
So we construct such a $T \leq S$ as a fusion of a sequence of conditions $(T_n \mid n \in \omega)$, where we will make sure that $T_{n+1} \Vdash c_n \in \dot{X}$ holds for all $n \in \omega$. Start with $T_0 = S$ and inductively assume that T_n is already constructed. Since $c_n \in Y_n$ we have that there is some condition $T'_{n+1} \leq_{n+1} S$ forcing $\check{c}_n \in \dot{X}$. This can be refined to our desired condition $T_{n+1} \leq_{n+1} T_n$. This concludes the successor step of our transfinite induction.

The limit case $(*)_\lambda$ for limit ordinals λ is a consequence of the following lemma, which Shelah proved in [14] and which we will first state in all its generality.

Lemma 3.13. *Let $\mathcal{F}, \mathcal{H} \subseteq \mathcal{P}(\omega)$ be such that:*

(i). $\mathcal{F}$ contains all cofinite sets, $\emptyset \notin \mathcal{F}$, $\mathcal{F}$ is closed under finite intersections and is Ramsey.

(ii). $\mathcal{H}$ does not contain any elements of the filter $\langle \mathcal{F} \rangle$ generated by $\mathcal{F}$ and furthermore we have

$$\mathcal{P}(\omega) \setminus \mathcal{F} \subseteq \{Y \subseteq \omega \mid \exists X \in \mathcal{H} : Y \subseteq^* X\}.$$

Then for any countable support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \lambda)$ of ω^ω-bounding proper forcings: If

$$\Vdash_{\mathbb{P}_\alpha} \mathcal{P}(\omega) \setminus \mathcal{F} \subseteq \{Y \subseteq \omega \mid \exists X \in \mathcal{H} : Y \subseteq^* X\}.$$

holds for all $\alpha < \lambda$, then it also holds for λ itself.

First note that this lemma is applicable here since we already showed that $\mathbb{S}$ is ω^ω-bounding, and furthermore it is proper since every forcing satisfying Axiom A is proper (shown in [1]). In our specific case we apply this lemma to $\mathcal{F} = \mathcal{F}_G^0 \cup FR(\omega)$ and $\mathcal{H} = \{\omega \setminus \mathcal{A}_G^h \mid h \in FF(\mathcal{A}_G)\}$. Condition (i) is easily verified since we already know that $\mathcal{F}$ is Ramsey. Verifying condition (ii) requires a bit more work:
Fix $\alpha < \lambda$ and let $Y \in \mathcal{P}(\omega) \setminus fil(\mathcal{A}_G)$, which implies that $\omega \setminus Y \notin id(\mathcal{A}_G)$.

Therefore we find some $h \in FF(\mathcal{A}_G)$ such that

$$\forall h' \supseteq h : |\mathcal{A}_G^{h'} \cap (\omega \setminus Y)| = |\mathcal{A}_G^{h'} \setminus Y| = \omega.$$

Now consider $Y' = \mathcal{A}_G^h \setminus Y$ for which we clearly have $Y' \subseteq \mathcal{A}_G^h$, which means we may again apply $(*)_\alpha$. If there is some $h' \supseteq h$ with $\mathcal{A}_G^{h'} \subseteq \mathcal{A}_G^h \setminus Y'$ we find

$$\mathcal{A}_G^{h'} \subseteq \mathcal{A}_G^h \setminus Y' = \mathcal{A}_G^h \cap Y,$$

and thus $\mathcal{A}_G^{h'} \setminus Y = \emptyset$, contradicting that, as established above, this set should be infinite.
Therefore there is some $B \in id(\mathcal{A}_G)^{V_0}$ with $\mathcal{A}_G^h \setminus Y' \subseteq B$, implying that $\mathcal{A}_G^h \setminus Y' \in id(\mathcal{A}_G)$. This means that there is some $h' \supseteq h$ with $\mathcal{A}_G^{h'} \cap Y$ finite, and therefore $Y \subseteq^* \omega \setminus \mathcal{A}_G^{h'}$. So condition (ii) of the lemma is satisfied if we stipulate $X = \omega \setminus \mathcal{A}_G^{h'}$. □

Therefore the generic maximal independent family $\mathcal{A}_G$ witnesses the following:

Corollary 3.14. *Let $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_2)$ be the countable support iteration of Sacks forcing and $V_0 = V^{\mathbb{P}}$ for a ground model $V \models CH$. Then*

$$V_0^{\mathbb{P}_{\omega_2}} \models \mathfrak{i} = \aleph_1 < \mathfrak{c} = \aleph_2.$$

□

Remark 3.15. (i): Since the condition $(*)_\alpha$ is an equivalent condition for dense maximality in $V_0^{\mathbb{P}_\alpha}$ - as shown in Lemma 2.37. - this proof actually shows that $\mathcal{A}_G$ is densely maximal and remains so throughout the iteration. In particular we showed that the existence of a densely maximal independent family is consistent with the failure of CH, as witnessed by $\mathcal{A}_G$ in the extension $V^{\mathbb{P}_{\omega_2}}$

(ii): A similar argument shows that $\mathcal{A}_G$ also remains maximal after forcing with a countable support *product* of Sacks forcing of lenght λ with $cof(\lambda) > \omega$. In this case we find a model of $Spec(mif) = \{\aleph_1, \lambda\}$ (cf. [4]).

Open Questions

At this time independent families remain rather unexplored and there are still many open questions concerning them. The following list contains a selected few of these questions related to the content of this thesis.

Question 1. *Is it consistent that* $\mathfrak{i} < \mathfrak{a}$ *?*
While it is rather easy to prove $Con(\mathfrak{a} < \mathfrak{i})$ as we did in Section 1.3. through Cohen forcing, the opposite inequality remains one of the most important open questions concerning cardinal characteristics.

Question 2. *Is it consistent that* $cof(\mathfrak{i}) = \omega$ *?*
We know that consistently $\mathfrak{a}$ may be of countable cofinality and thus not regular (see, for example, [3]). For $\mathfrak{i}$, on the other hand, no analogous result has been found so far.

In the following, let $\mathfrak{i}_d$ denote the minimal size of a densely maximal independent family, assuming that such a family exists.

Question 3. *Is it consistent that* $\mathfrak{i} < \mathfrak{i}_d$ *?*
Clearly $\mathfrak{i} \leq \mathfrak{i}_d$ holds since every densely maximal independent family is a maximal independent family in the standard sense. Whether this inequality might be strict is unknown.

Question 4. *Is there a Sacks indestructible maximal independent family which is not densely maximal?*
As we've seen in Section 3.2., dense maximality played an essential role in showing that the independent family adjoined by $\mathbb{P}$ is Sacks indestructible. But does a Sacks indestructible maximal independent family necessarily have to be densely maximal?

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