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Abstract

In this thesis we discuss the topic topology optimization for an elastoplastic object which is influenced by external forces and show that there exists an optimal configuration for the sharp-interface model, as well as for the phase-field approach, where the material density is continuous.

First, we introduce our setting and model the motion of the medium. Here, the elastoplastic behaviour is described by the minimization of a specific energy functional. Following this, we prove the existence of such a minimizer, using the *Direct Method of the Calculus of Variations*. Subsequently, we prove that both the sharp-interface problem, as well as the phase-field problem admit a solution. Eventually, we show that under suitable conditions the phase-field model converges to the sharp-interface model with respect to Γ-convergence.

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Contents

Chapter 1

Introduction

Topology optimization describes the method of finding the optimal shape of a certain object, placed within the container Ω. Of course the word 'optimal' is too vague and can include many different aspects, like production costs or specific geometrical properties. From a mathematical standpoint this is usually expressed by minimizing a certain target functional, subjected to the distribution of the material within this object. In other words, the material density $z : \Omega \to [0, 1]$ at each point is the control parameter that needs to be optimized.

There are many applications of topology optimization, especially in the area of airplane construction. Finding the best shape of airfoils and wings can lead to major advantages. Hence, there are many papers written on this topic, see for example [NS11] or [SLS14]. Recently, a growing interest in the field of 3D-printing has arisen. Here we refer to [BAH11] and [CAS16]. Applications of topology optimization will not be discussed in this thesis, as the focus lies on the theoretical analysis. More precisely, we want to show existence of such an optimal structure and its approximation.

There have been some contributions on this matter, like [BC03]. In these theoretical settings the material is usually assumed to behave in an elastic way. However, properties such as permanent deformation and damage, that are of the utmost importance in engineering, are not captured in these models. This thesis is trying to examine such inelastic effects. To be specific, the focus lies on incremental finite plasticity. In other words, the whole body is behaving in an elastoplastic way.

This chapter describes the problem of topology optimization for an elastoplastic object in a mathematical way and introduces the general notation.

1.1 Modelling plasticity

We are given a container $\Omega \subset \mathbb{R}^d$ for $d = \{2, 3\}$ and we assume to be able to adjust the density of the material within this object. This is described by the order parameter

$$z : \Omega \to [0, 1].$$

If at some point $x \in \Omega$ there is no material then $z(x) = 0$ and vice versa $z(x) = 1$ if there is material. Here we will make a distinction between the case where we allow intermediate states, like $z(x) = \frac{1}{2}$, and the case where this is not possible. Moreover, we assume the order parameter z to be in the space $L^1(\Omega)$ of Lebesgue integrable functions in Ω.

The deformation is described by the mapping

$$\varphi : \Omega \to \mathbb{R}^d.$$

Here, we assume that the whole container Ω gets deformed, both in its solid point $\{z = 1\}$ and its void point $\{z = 0\}$. As a matter of fact, we model the void $\{z = 0\}$ as a very soft elastoplastic medium. One can think of a very soft polymeric matrix, for instance. As such, we often refer to Ω as *body* in the following. Furthermore, we assume that φ is smooth enough such that we can define the deformation gradient $\nabla\varphi$. The body is assumed to behave elastoplastically. This can be described by a multiplicative decomposition of the deformation gradient, the so called *Lee-Liu* decomposition [LL67] which is given by

$$\nabla\varphi = F_{el}P.$$

Here, F_{el} represents the elastic part of the deformation gradient, the one directly related to stresses, while P is the plastic strain, recording the plastic state of the body instead. A good motivation for this model can be found in [Has20, ch. 8, p. 255]. However, note that the quality of this model is still debated, see for example [DF15]. Another parameter that is needed in this configuration is the so-called geometric dislocation tensor $\mathcal{G}(P)$. It represents the relation between the plastic deformation P and the surface elements. This notion was first introduced by Nye in 1953 [Nye53].

The internal energy H of the elastoplastic body depends on the configuration (z, φ, P) and can be characterized via the energy density F as

$$H(z, \varphi, P) = \int_\Omega F(x, z, \nabla\varphi P^{-1}, P, \mathcal{G}(P)) \, \mathrm{d}x.$$

The object will deform in such a way that it minimizes its stored energy H. By assuming F to be defined relative to some previous plastic state, the above-mentioned minimization specifies the increment in plastic strain. As such the problem is called incremental. More details can be found in [MM06].

However, the motion of the body is not only influenced by the type of the material but also by external forces and boundary conditions. The boundary $\Gamma := \partial\Omega$ is being clamped in $\Gamma_D \subset \Gamma$ and we impose a traction force

$$g : \Gamma_N \to \mathbb{R}^d$$

on $\Gamma_N \subset \Gamma \setminus \Gamma_D$. Here, Γ_D and Γ_N are assumed to be relatively open in the topology of $\partial\Omega$ such that $\Gamma_D \cap \Gamma_N = \emptyset$ and $\mathcal{H}^{d-1}(\Gamma_D) \neq 0$ where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. Moreover, a force density acting on the whole body is applied

$$f : \Omega \to \mathbb{R}^d.$$

In order to specify the force, one has to multiply f by the density z. The actual force that is acting on the body is therefore given by

$$zf : \Omega \to \mathbb{R}^d.$$

The two forces f and g are conservative in a sense that if a particle moves along a closed trajectory then the work done by the force is zero. This can be described by the potential energy

$$U(z, \varphi, P) = -\left(\int_\Omega zf \cdot \varphi \, \mathrm{d}x + \int_{\Gamma_N} g \cdot \varphi \, \mathrm{d}x\right)$$

Note that the negative sign comes from the fact that one has to put work into the system in order to move it against the force. The total energy can be written as

$$\mathcal{E}(z, \varphi, P) = H(z, \varphi, P) + U(z, \varphi, P)$$

This is the functional that describes the motion of Ω and that we are aiming to minimize in Chapter 2.

1.2 Modelling the optimal shape

Knowing how φ behaves, we can define the objective functional for the sharp-interface case where z is either 0 or 1, as

$$J_0(z, \varphi, P) = \int_\Omega z f \cdot \varphi \,\mathrm{d}x + \int_{\Gamma_N} g \cdot \varphi \,\mathrm{d}\mathcal{H}^{d-1} + c_\psi \mathrm{Per}(\{z = 1\}, \Omega).$$

Here $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure and $c_\psi > 0$ is a given constant. The first two terms in J_0 are called *compliance*. They model the mechanical forces that we want to minimize and represent the ability to withstand the external forces f and g. The last term models the surface area of the solid $\{z = 1\}$. Thus, the body avoids having a large boundary.

Furthermore, we approximate the objective functional by a phase field approach, allowing z to take values between 0 and 1. The boundary of the solid is then given by a diffuse interface. The thickness of this boundary is described by a small parameter $\varepsilon > 0$ and the perimeter is replaced by the Ginzburg-Landau energy

$$J_\varepsilon(z, \varphi, P) = \int_\Omega z f \cdot \varphi \,\mathrm{d}x + \int_{\Gamma_N} g \cdot \varphi \,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\,\mathrm{d}x.$$

The function ψ is a so called double obstacle potential and has zeros at 0 and 1. Hence, configurations where the density is either 0 or 1 are preferred. The Ginzburg-Landau energy functional approximates the perimeter and in fact converges to $c_\psi \mathrm{Per}(\{z = 1\}, \Omega)$ as $\varepsilon \to 0$ with respect to the so called Γ-convergence. This was shown by Modica and Mortola in 1977 [MM77].

We will be analyzing the existence of minimizers of J_0 and J_ε under the condition that φ and P minimize $\mathcal{E}$ in Chapter 3.

Ultimately, in Chapter 4, we show Γ-convergence of the phase-field model to the sharp-interface model, under the assumption that the deformation φ is unique.

1.3 Notation and assumptions

Throughout this thesis we only look at the 2- and 3-dimensional case, i.e. $d \in \{2,3\}$. The body $\Omega \subset \mathbb{R}^d$ is assumed to be an open and bounded Lipschitz domain. This means that Ω is a bounded set, the boundary $\partial\Omega$ is not part of Ω, and it can be described by a Lipschitz function. If we denote by λ^d the Lebesgue measure in $\mathbb{R}^d$, the boundedness of Ω implies that $\lambda^d(\Omega) < \infty$.

Additionally, we can define the Lebesgue integral over Ω of a function $f : \Omega \to \mathbb{R}$ and write

$$\int_\Omega f \, \mathrm{d}x := \int \mathbb{1}_\Omega f \, \mathrm{d}\lambda$$

where

$$\mathbb{1}_\Omega(x) := \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

is the characteristic function of Ω. Recall the definition of the L^p-norm of a function f for $p \in [1,\infty)$

$$\|f\|_p := \left(\int_\Omega |f|^p \, \mathrm{d}x \right)^{1/p}$$

and the L^∞-norm

$$\|f\|_\infty := \inf\{C \geq 0 : |f(x)| \leq C \text{ for a.e. } x \in \Omega\}$$

Note that the space

$$L^p(\Omega) := \{f : \Omega \to \mathbb{R} \ \text{ measurable} : \|f\|_p < \infty\}$$

is a Banach space for all $p \in [1,\infty]$ and is reflexive for all $p \in (1,\infty)$.

Another important concept is the notion of weak derivatives.

Definition 1.3.1. Let $f \in L^1(\Omega)$. Then, $f^{(\alpha)} \in L^1(\Omega)$ is called the α-th weak derivative of f if

$$\int_\Omega f D^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f^{(\alpha)} \varphi \, \mathrm{d}x$$

for all $\varphi \in C_0^\infty(\Omega)$ where $\alpha = (\alpha_1, ..., \alpha_d)$ is a multi-index and D^α the multi-derivative.

Based on this definition we can introduce the Sobolev spaces

$$W^{k,p} := \{f \in L^p(\Omega) : f^{(\alpha)} \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}$$

These are again Banach spaces when endowed with the norm

$$\|f\|_{k,p} := \left(\sum_{|\alpha| \leq k} \left\| f^{(\alpha)} \right\|_p^p \right)^{1/p}$$

for $p \in [1, \infty)$ and

$$\|f\|_{k,\infty} := \max_{|\alpha| \leq k} \left\| f^{(\alpha)} \right\|_\infty$$

for the limit case. The case $p = 2$ is special since $W^{k,2}(\Omega) =: H^k(\Omega)$ is a Hilbert space. Moreover, for all $p \in (1, \infty)$ these spaces are reflexive.

Functions that only take the values 0 and 1 in Ω do not have a weak derivative which is the motivation for the following definition.

Definition 1.3.2. Let $u \in L^1(\Omega)$. Define the variation $V(u, \Omega)$ of u in Ω by

$$V(u, \Omega) := \int_\Omega |Du| = \sup \left\{ \int_\Omega u \operatorname{div} v \, \mathrm{d}x : v \in C_0^1(\Omega; \mathbb{R}^d), \|v\|_\infty \leq 1 \right\}$$

We say that u has bounded variation and write $u \in BV(\Omega)$, if $\int_\Omega |Du| < \infty$.

Functions with this property are more general than functions in $W^{1,1}(\Omega)$. In other words if $u \in W^{1,1}(\Omega)$ then $\int_\Omega |Du| = \int_\Omega |\nabla u| \, \mathrm{d}x$ since

$$\int_\Omega u \operatorname{div} v \, \mathrm{d}x = -\int_\Omega \nabla u \cdot v \, \mathrm{d}x$$

for all $v \in C_0^1(\Omega;\mathbb{R}^d)$ with $\|v\|_\infty \leq 1$ and we can approximate $\operatorname{sgn}(\nabla u)$ by such functions. The space $BV(\Omega)$ together with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_1 + \int_\Omega |Du|$$

is a Banach space as well. Through this notation we can give a mathematical desription of the perimeter of a set.

Definition 1.3.3. Let $A \subset \mathbb{R}^d$ be a Borel set. Then we define the perimeter of A in Ω to be

$$\operatorname{Per}(A,\Omega) := V(\mathbb{1}_A,\Omega) = \sup\left\{\int_\Omega \mathbb{1}_A \operatorname{div} v \,\mathrm{d}x : v \in C_0^1(\Omega;\mathbb{R}^d), \|v\|_\infty \leq 1\right\}$$

Chapter 2

Existence result for minimizers of the energy functional

In this chapter we are going to show existence of minimizers (z, φ, P) for the total energy functional $\mathcal{E}$. This is done using the *Direct Method of the Calculus of Variations*, which is a tool proposed by David Hilbert around the year 1900 to prove existence of solutions of minimization problems.

First, we will state the problem in a rigorous way. Then, we will show coercivity in order to get weak convergence of subsequences. In Sections 2.3 and 2.4 we will prove (lower semi-)continuity results which will be the last step to obtain the existence of a minimizer.

This chapter mainly follows [MM06] as well as [KR19] and [Dac08].

2.1 Stating the problem

The plasticity functional is given by

$$H(z, \varphi, P) = \int_\Omega F(x, z, \nabla\varphi P^{-1}, P, \mathcal{G}(P))\,\mathrm{d}x \tag{2.1}$$

where $F : \Omega \times [0,1] \times \mathbb{R}^{d\times d} \times \mathrm{SL(d)} \times \mathbb{R}^{\overbrace{d \times \ldots \times d}^{(d-1)-times}} \to \mathbb{R} \cup \{\infty\}$. SL(d) describes the special linear group of matrices of order d, i.e. $d \times d$ matrices with determinant equal to 1. This means that throughout this thesis we will assume $\det P = 1$. In other words we say that the plastic deformation is volume preserving.

By $\mathcal{G}(P)$ we denote the geometric dislocation tensor which for $d = 2$ is given by

$$\mathcal{G}(P) = \mathrm{curl}_2 P = \begin{pmatrix} \partial_1 P_{12} - \partial_2 P_{11} \\ \partial_1 P_{22} - \partial_2 P_{21} \end{pmatrix} \in \mathbb{R}^2.$$

And for $d = 3$ by

$$\mathcal{G}(P) = (\mathrm{curl}_3 P)P^T \in \mathbb{R}^{3\times 3}$$

where the curl_3 of a matrix is given by the vectorial curl applied to each row separately, hence generating a 3×3 matrix, i.e.,

$$(\mathrm{curl} P)_{ij} = \partial_{j+1} P_{ij+2} - \partial_{j+2} P_{ij+1}.$$

This term is sometimes included in models of crystal plasticity, since an argument can be made that the surface where two differently sheared subdomains meet admits a density of geometrically necessary dislocations. A more detailed description can be found in [CG01]. From now on we will omit the subscript. Sometimes we will also use the notation $F_{el} = \nabla\varphi P^{-1}$ and $G = \mathcal{G}(P)$ for simplicity.

2.2 Coercivity

In order to get convergence of sequences we want to apply the Banach-Alaoglu Theorem A.0.1 and hence we require uniform boundedness first. This means that for a sequence $(f^{(k)})$ there exists a constant $C > 0$ such that $\left\| f^{(k)} \right\| \leq C$ for all $k \in \mathbb{N}$. Therefore we have to study coercivity of the different parameters.

For the order variable z we only get this bound in Chapter 3. In this chapter we will assume convergence, i.e. for a sequence $(z^{(k)}) \subset L^1(\Omega; [0, 1])$ there exists an element $z \in L^1(\Omega; [0, 1])$ such that

$$z^{(k)} \to z \in L^1(\Omega).$$

Note that since $\|z\|_\infty < \infty$ and $\lambda^d(\Omega) < \infty$ by the Riesz-Thorin interpolation theorem [Tho48] we get

$$z^{(k)} \to z \in L^p(\Omega)$$

for $p \in [1, \infty)$.

Growth properties of F play an important role. The condition that we we will assume is the following

$$F(x, z, F_{el}, P, G) \geq c(|F_{el}|^{q_F} + |P|^{q_P} + |P^{-1}|^{q_P} + |G|^{q_G}) - g(x) \tag{2.2}$$

for some $c > 0$ and $g \in L^1(\Omega)$. Before we can prove coercivity we recall some basic notions of Linear Algebra.

Let $F \in \mathbb{R}^{d\times d}$ then we define the Euclidean norm of the matrix F via

$$|F|^2 := \sum_{i,j=1}^{d} F_{ij}^2.$$

Recall the following properties:

- $|Fa| \leq |F||a|$ for all $a \in \mathbb{R}^d$,
- $|FH| \leq |F||H|$ for all $H \in \mathbb{R}^{d\times d}$,
- $|FP^{-1}| \leq |F|/|P|$ for all $P \in \mathbb{R}^{d\times d}$ where $P \in \mathrm{GL}(\mathbb{R}^d)$.

Furthermore, we need the following three Lemmas.

Lemma 2.2.1. *Let $P \in SL(d)$ and $\mathcal{G}(P)$ as in Chapter 2.1. Then, we have the following inequalities*

- $|\mathcal{G}(P)| \geq |\mathrm{curl}P|$ *for* $d = 2$,
- $|\mathcal{G}(P)| \geq \frac{|\mathrm{curl}P|}{|P^{-1}|}$ *for* $d = 3$.

Proof. We will prove the case $d = 3$ as the 2-dimensional case is trivial. Recall that $\mathcal{G}(P) = (\mathrm{curl}_3 P)P^T$. Using the properties of the Euclidean norm of matrices results in

$$|\mathcal{G}(P)||P^{-1}| = |\mathcal{G}(P)||P^{-T}| \geq |\mathcal{G}(P)P^{-T}| = |(\mathrm{curl}_3 P)P^T P^{-T}| = |\mathrm{curl}_3 P|.$$

$\square$

Lemma 2.2.2. *Let q_1, q_2, q be such that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ and let $f : \Omega \to \mathbb{R}^{d\times d}$ and $g : \Omega \to \mathrm{GL}(\mathbb{R}^d)$ with $g \in L^{q_2}(\Omega; \mathbb{R}^{d\times d})$ and $fg^{-1} \in L^{q_1}(\Omega; \mathbb{R}^{d\times d})$. Then we have $f \in L^q(\Omega; \mathbb{R}^{d\times d})$ and*

$$\left\|fg^{-1}\right\|_{q_1} \geq \frac{\|f\|_q}{\|g\|_{q_2}}.$$

Proof. Taking $h = fg^{-1}$ and using Hölder's inequality for h and g we get

$$\|hg\|_q \leq \|h\|_{q_1} \|g\|_{q_2}.$$

Plugging $f = hg$ in we get our result. □

Lemma 2.2.3. *Let $a, b > 0$, $\varepsilon > 0$ and $r > 1$ then we have*

$$\frac{a}{b} \geq r\varepsilon^{(r-1)/r} a^{1/r} - (r-1)\varepsilon b^{1/(r-1)}. \tag{2.3}$$

Proof. Recalling Young's inequality $AB \leq \frac{A^p}{p} + \frac{B^q}{q}$ for $A, B > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ and setting $r = p$, $A = (\varepsilon^{r-1}a)^{1/r}$ and $B = \varepsilon^{r-1}b$ we get

$$(\varepsilon^{r-1}a)^{1/r}\varepsilon^{r-1}b \leq \frac{\varepsilon^{r-1}a}{r} + \frac{(\varepsilon^{r-1}b)^{r/(r-1)}}{r/(r-1)}.$$

Subtracting the last term on both sides and multiplying by $\frac{r}{\varepsilon^{r-1}b}$ we get

$$r\varepsilon^{(r-1)/r} a^{1/r} - (r-1)\varepsilon b^{1/(r-1)} \leq \frac{a}{b}$$

which is the desired result. □

Using the Growth Condition 2.2 and applying the Lemmas above we are able to prove coercivity.

Proposition 2.2.4. *Assume that*

$$c(|F_{el}|^{q_F} + |P|^{q_P} + |P^{-1}|^{q_P} + |G|^{q_G}) - g(x) \leq F(x, z, F_{el}, P, G)$$

holds for some $c > 0$ and $g \in L^1(\Omega)$ and let

- $\frac{1}{q_\varphi} := \frac{1}{q_F} + \frac{1}{q_P} \leq 1$,

- $\frac{1}{q_C} := \begin{cases} \frac{1}{q_G} \leq 1 & \textit{for} \quad d = 2, \\ \frac{1}{q_G} + \frac{1}{q_P} \leq 1 & \textit{for} \quad d = 3. \end{cases}$

If $H(z, \varphi, P) \leq C_H$ for some $C_H \in \mathbb{R}$ then there exists $C > 0$ which depends on C_H, q_F, q_P, q_G, and g such that

$$\|\nabla\varphi\|_{q_\varphi} + \|P\|_{q_P} + \left\|P^{-1}\right\|_{q_P} + \|\mathrm{curl}P\|_{q_C} \leq C.$$

Proof. For simplicity, we will only consider the case $d = 3$ as the case $d = 2$ follows by similar arguments with different constants.

From the assumptions it follows that

$$\int_\Omega F(x, z, F_{el}, P, G)\mathrm{d}x = H(z, \varphi, P) \leq C_H$$

$$c(\|F_{el}\|_{q_F}^{q_F} + \|P\|_{q_P}^{q_P} + \left\|P^{-1}\right\|_{q_P}^{q_P} + \|G\|_{q_G}^{q_G}) - \int_\Omega g\mathrm{d}x \leq C_H$$

Adding the integral of g on both sides as well as applying Lemma 2.2.1 on G and Lemma 2.2.2 on F_{el} and G we get

$$c(\|\nabla\varphi\|_{q_\varphi}^{q_F} / \|P\|_{q_P}^{q_F} + \|P\|_{q_P}^{q_P} + \left\|P^{-1}\right\|_{q_P}^{q_P} + \|\mathrm{curl}P\|_{q_C}^{q_G} / \left\|P^{-1}\right\|_{q_P}^{q_G}) \leq \widetilde{C_H}$$

Now we are going to use Lemma 2.2.3 on the first and fourth term of the left hand side with $r = \frac{q_F}{q_\varphi}$, $\tilde{r} = \frac{q_G}{q_C}$ and $\varepsilon, \tilde{\varepsilon}$ to be chosen. One can see that $r - 1 = \frac{q_F}{q_P}$ and $\tilde{r} - 1 = \frac{q_G}{q_P}$ holds. Hence, we get the following

$$\begin{aligned} c(&\frac{q_F}{q_\varphi}\varepsilon^{\frac{q_\varphi}{q_P}} \|\nabla\varphi\|_{q_\varphi}^{q_\varphi} - \frac{q_F}{q_P}\varepsilon \|P\|_{q_P}^{q_P} + \|P\|_{q_P}^{q_P} \\ &+ \left\|P^{-1}\right\|_{q_P}^{q_P} + \frac{q_G}{q_C}\tilde{\varepsilon}^{\frac{q_C}{q_P}} \|\mathrm{curl}P\|_{q_C}^{q_C} - \frac{q_G}{q_P}\tilde{\varepsilon} \left\|P^{-1}\right\|_{q_P}^{q_P}) \leq \widetilde{C_H}. \end{aligned}$$

Choosing ε and $\tilde{\varepsilon}$ such that $\frac{q_F}{q_P}\varepsilon = \frac{1}{2}$ and $\frac{q_G}{q_P}\tilde{\varepsilon} = \frac{1}{2}$ we get

$$c(c_1 \|\nabla\varphi\|_{q_\varphi}^{q_\varphi} + \frac{1}{2} \|P\|_{q_P}^{q_P} + \frac{1}{2} \left\|P^{-1}\right\|_{q_P}^{q_P} + c_2 \|\mathrm{curl}P\|_{q_C}^{q_C}) \leq \widetilde{C_H}.$$

for some $c_1, c_2 > 0$. Finally taking $\widetilde{C} = \frac{\widetilde{C_H}}{c \cdot \min\{c_1, c_2, 1/2\}} > 0$ we obtain our result with $C = \widetilde{C}^{1/q_\varphi} + \widetilde{C}^{1/q_P} + \widetilde{C}^{1/q_C} > 0$. □

We have shown that under suitable conditions (in our case the growth condition) we get boundedness of $\nabla\varphi$, P and G. This will help us extract weakly convergent subsequences which are precisely the topic of the next chapter.

2.3 Lower semi-continuity of H

Starting from weakly convergent sequences our goal now is to show lower semi-continuity of H. Here, F is assumed to be polyconvex in the matrix-valued components, i.e. F is a convex function in the minors of the matrix-valued components. Hence, we will need some results on weak convergence of minors.

The following theorem is called the div-curl lemma and describes convergence of the product of two weakly converging functions.

Theorem 2.3.1 (div-curl lemma). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Furthermore let $(f_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ be sequences such that*

$$\begin{aligned} f_k &\rightharpoonup f \text{ in } L^p(\Omega;\mathbb{R}^d), \\ g_k &\rightharpoonup g \text{ in } L^q(\Omega;\mathbb{R}^d), \end{aligned}$$

for $\frac{1}{p}+\frac{1}{q}=\frac{1}{\sigma}<1$. If

$$\{\mathrm{curl} f_k : k\in\mathbb{N}\} \text{ is bounded in } \begin{cases} L^p(\Omega) & for\ d=2, \\ L^p(\Omega;\mathbb{R}^3) & for\ d=3, \end{cases}$$
$$\{\mathrm{div} g_k : k\in\mathbb{N}\} \text{ is bounded in } L^q(\Omega),$$

then,

$$f_k\cdot g_k \rightharpoonup f\cdot g \text{ in } L^\sigma(\Omega).$$

Murat proved this statement in [Mur78, p. 490] in the sense of distributions, i.e. for functions $\phi\in C_0^\infty(\Omega)$. But since $C_0^\infty(\Omega)$ is dense in $L^{\sigma'}(\Omega)$ for $\frac{1}{\sigma}+\frac{1}{\sigma'}=1$ and $\|f_k\cdot g_k\|_\sigma \le \|f_k\|_p\|g_k\|_q<\infty$ for all $k\in\mathbb{N}$ and $\|f\cdot g\|_\sigma \le \|f\|_p\|g\|_q<\infty$ we get weak convergence in $L^\sigma(\Omega)$ by approximating a function $h\in L^{\sigma'}(\Omega)$ with test functions $\phi_k\in C_0^\infty(\Omega)$.

For a given matrix $A\in\mathbb{R}^{d\times d}$ we denote by $M(A)$ the vector of all minors of the matrix A. In the case $d=2$ this means $M(A)=(A,\mathrm{det}A)$ and in the case $d=3$ this means $M(A)=(A,\mathrm{Cof}A,\mathrm{det}A)$. Here $\mathrm{Cof}A$ is the cofactor matrix which is defined by $(\mathrm{Cof}A)_{ij}:=(-1)^{i+j}\mathrm{det}C_{ij}$ where $C_{ij}\in\mathbb{R}^{2\times 2}$ is a submatrix obtained by removing the i-th row and j-th column of A. If

A is invertible then we have $\mathrm{Cof}A = (\det A)A^{-T}$. Furthermore, if we count indices modulo 3, i.e., $4 \mapsto 1, 5 \mapsto 2$, then we get

$$(\mathrm{Cof}A)_{ij} := A_{i+1,j+1}A_{i+2,j+2} - A_{i+1,j+2}A_{i+2,j+1}.$$

The number of components of $M(A)$ is $\binom{2d}{d} - 1$. For $d = 2$ these are 5 components and for $d = 3$ we get 19 components. By $M_s(A)$ we denote the s-th entry of the vector $M(A)$. Each minor $M_s(A)$ for $s = 1, ..., d$ has $\binom{d}{s}$ entries. The following theorem shows the importance of minors.

Theorem 2.3.2 (Weak convergence of minors)**.** *Let $1 \leq s \leq d$, $\frac{1}{\sigma} = \frac{1}{p_1} + ... + \frac{1}{p_s} < 1$ and let $f_i^{(k)} : \Omega \to \mathbb{R}^d$ for $i \in \{1, ..., s\}$ satisfy*

(1) $f_i^{(k)} \rightharpoonup f_i^\star \quad in\ L^{p_i}(\Omega; \mathbb{R}^d) \quad for\ k \to \infty,$

(2) $\{\mathrm{curl} f_i^{(k)} : k \in \mathbb{N}\}$ is bounded in $\begin{cases} L^{q_i}(\Omega; \mathbb{R})\ for\ d = 2, \\ L^{q_i}(\Omega; \mathbb{R}^3)\ for\ d = 3, \end{cases}$

for $\frac{1}{q_i} \leq \min\{\frac{1}{p_i} + \frac{1}{d}, 1\}$.

Let $F^{(k)} \in \mathbb{R}^{s\times d}$ be the matrix with rows $(f_i^{(k)})_{i=1}^s$ and $F^\star \in \mathbb{R}^{s\times d}$ with rows $(f_i^\star)_{i=1}^s$. Then, $M_s(F^{(k)}) \in \mathbb{R}^{1\times\binom{d}{s}}$ and for all entries of the s-th minor we have

$$(M_s(F^{(k)}))_{1j} \rightharpoonup (M_s(F^\star))_{1j} \quad in\ L^\sigma(\Omega) \quad for\ k \to \infty$$

for all $1 \leq j \leq \binom{d}{s}$.

Proof. We start with the case $d = 2$. Here we distinguish between the cases $s = 1$ and $s = 2$. The former one is rather trivial since

$$M_1(F^{(k)}) = F^{(k)} = f_1^{(k)} \rightharpoonup f_1^\star = F^\star = M_1(F^\star)$$

in $L^{p_1}(\Omega; \mathbb{R}^2)$ by assumption *(1)*.
For the case $s = 2$ we set $f_i^{(k)} = (f_{i1}^{(k)}, f_{i2}^{(k)})$ for $i = 1, 2$. We can write

$$F^{(k)} = \begin{pmatrix} f_{11}^{(k)} & f_{12}^{(k)} \\ f_{21}^{(k)} & f_{22}^{(k)} \end{pmatrix}.$$

Now let $g_2^{(k)} = (g_{21}^{(k)}, g_{22}^{(k)}) = (f_{22}^{(k)}, -f_{21}^{(k)})$ then we get

$$\begin{aligned} M_2(F^{(k)}) = \det F^{(k)} &= f_{11}^{(k)} f_{22}^{(k)} - f_{12}^{(k)} f_{21}^{(k)} \\ &= (f_{11}^{(k)}, f_{12}^{(k)}) \cdot (f_{22}^{(k)}, -f_{21}^{(k)}) = f_1^{(k)} \cdot g_2^{(k)}. \end{aligned}$$

The same holds for the matrix $F^\star$. By assumption *(1)* we know that

$$\begin{aligned} f_1^{(k)} &\rightharpoonup f_1^\star \quad \text{in } L^{p_1}(\Omega; \mathbb{R}^2), \\ g_2^{(k)} &\rightharpoonup g_2^\star \quad \text{in } L^{p_2}(\Omega; \mathbb{R}^2). \end{aligned}$$

Since $\operatorname{div} g_2^{(k)} = \operatorname{curl} f_2^{(k)}$ we can deduce by assumption *(2)* that

$$\begin{aligned} &\{\operatorname{curl} f_1^{(k)} : k \in \mathbb{N}\} \text{ is bounded in } L^{p_1}(\Omega; \mathbb{R}), \\ &\{\operatorname{div} g_2^{(k)} : k \in \mathbb{N}\} \text{ is bounded in } L^{p_2}(\Omega). \end{aligned}$$

Hence we can apply the div-curl lemma 2.3.1 to obtain

$$M_2(F^{(k)}) = f_1^{(k)} \cdot g_2^{(k)} \rightharpoonup f_1^\star \cdot g_2^\star = M_2(F^\star)$$

in $L^\sigma(\Omega)$ where $\frac{1}{\sigma} = \frac{1}{p_1} + \frac{1}{p_2} < 1$.
The case $d = 3$ is similar. In fact for $s = 1$ and $s = 2$ we can use the same argument again. For $s = 3$ we apply the Laplace expansion to calculate the determinant of $F^{(k)}$

$$\det F^{(k)} = f_1^{(k)} \cdot (\operatorname{Cof} F^{(k)})_1$$

where $(\operatorname{Cof} F^{(k)})_1$ denotes the first row of $\operatorname{Cof} F^{(k)}$. We already know from the case $s = 2$ that these cofactors converge in $L^\sigma(\Omega)$ for $\frac{1}{\sigma} = \frac{1}{p_2} + \frac{1}{p_3} < 1$ which is why we want to use the div-curl lemma once more. Setting $f_i^{(k)} = (f_{i1}^{(k)}, f_{i2}^{(k)}, f_{i3}^{(k)})$ for $i = 1, 2, 3$ we get

$$\begin{aligned} (\operatorname{Cof} F^{(k)})_1 &= (f_{22}^{(k)} f_{33}^{(k)} - f_{23}^{(k)} f_{32}^{(k)}, f_{23}^{(k)} f_{31}^{(k)} - f_{21}^{(k)} f_{33}^{(k)}, f_{21}^{(k)} f_{32}^{(k)} - f_{22}^{(k)} f_{31}^{(k)}) \\ &= (f_2^{(k)} \times f_3^{(k)}). \end{aligned}$$

One can check that the following vector calculus identity holds

$$\operatorname{div}(A \times B) = (\operatorname{curl} A) \cdot B - (\operatorname{curl} B) \cdot A.$$

Using this identity it follows that

$$\operatorname{div}(\operatorname{Cof}F^{(k)})_1 = \operatorname{div}(f_2^{(k)} \times f_3^{(k)}) = (\operatorname{curl}f_2^{(k)}) \cdot f_3^{(k)} - (\operatorname{curl}f_3^{(k)}) \cdot f_2^{(k)}.$$

We know that $f_2^{(k)}$ and $f_3^{(k)}$ are weakly convergent and therefore bounded in $L^{p_2}(\Omega;\mathbb{R}^3)$ and $L^{p_3}(\Omega;\mathbb{R}^3)$. Furthermore $\operatorname{curl}f_2^{(k)}$ and $\operatorname{curl}f_3^{(k)}$ are bounded in $L^{p_2}(\Omega;\mathbb{R}^3)$ and $L^{p_3}(\Omega;\mathbb{R}^3)$ as well and hence by the Hölder inequality it follows that

$$\{\operatorname{div}(\operatorname{Cof}F^{(k)})_1 : k \in \mathbb{N}\} \text{ is bounded in } L^{\sigma}(\Omega)$$

for $\frac{1}{q} = \frac{1}{p_2} + \frac{1}{p_3}$. Applying the div-curl lemma one last time yields the desired result. $\square$

This theorem helps us show weak convergence of the matrix-valued components of F. The following lemma is more specific.

Lemma 2.3.3. *Let $q_P > d$, $\frac{1}{q_C} < \frac{1}{d}+\frac{1}{q_P}$ and $\frac{1}{q^\star} := \frac{1}{q_C}+\frac{d-2}{q_P} < 1$. Furthermore, let $P^{(k)} : \Omega \to \mathrm{SL(d)}$ be a sequence such that*

$$\begin{aligned} P^{(k)} &\rightharpoonup P \quad \text{in } L^{q_P}(\Omega;\mathbb{R}^{d\times d}),\\ \operatorname{curl}P^{(k)} &\rightharpoonup A \quad \text{in } \begin{cases} L^{q_C}(\Omega;\mathbb{R}^2) \ \ for \ \ d=2,\\ L^{q_C}(\Omega;\mathbb{R}^{3\times 3}) \ \ for \ \ d=3. \end{cases} \end{aligned}$$

Then, we have the following:

(a) $A = \operatorname{curl}P$,

(b) $M_s(P^{(k)}) \rightharpoonup M_s(P) \quad$ *in* $L^{q_P/s}(\Omega;\mathbb{R}^{\binom{d}{s}\times\binom{d}{s}})$,

(c) $\mathcal{G}(P^{(k)}) \rightharpoonup \mathcal{G}(P) \quad$ *in* $\begin{cases} L^{q^\star}(\Omega;\mathbb{R}^2) \ \ for \ \ d=2,\\ L^{q^\star}(\Omega;\mathbb{R}^{3\times 3}) \ \ for \ \ d=3. \end{cases}$

Proof.
(a) We will prove this statement for the case $d=3$. Take $P \in C^1(\bar{\Omega};\mathbb{R}^{3\times 3})$ and note that if we again count indices modulo 3 we get

$$(\operatorname{curl}P)_{ij} = \partial_{j+1}P_{ij+2} - \partial_{j+2}P_{ij+1}.$$

Multiplying by a test function $\phi \in C_0^\infty(\Omega)$, integrating over Ω, and integrating by parts yields

$$\int_\Omega (\mathrm{curl}P)_{ij}\phi \;\mathrm{d}x = -\int_\Omega P_{ij+2}\partial_{j+1}\phi \;\mathrm{d}x + \int_\Omega P_{ij+1}\partial_{j+2}\phi \;\mathrm{d}x. \tag{2.4}$$

This is continuous in $C^1(\bar{\Omega};\mathbb{R}^{3\times 3})$ equipped with the $L^{q_P}(\Omega;\mathbb{R}^{d\times d})$-norm if ϕ is fixed, since

$$\left|\int_\Omega (\mathrm{curl}P)_{ij}\phi \;\mathrm{d}x\right| \le \|\mathrm{curl}P\|_1 \|\phi\|_\infty \le C(\phi)\,\|P\|_{q_P}$$

for a constant $C(\phi) > 0$. Since $C^1(\bar{\Omega};\mathbb{R}^{3\times 3})$ is dense in $L^{q_P}(\Omega;\mathbb{R}^{d\times d})$ the equation (2.4) holds true for all $P \in L^{q_P}(\Omega;\mathbb{R}^{d\times d})$. Using the weak convergence of $P^{(k)}$ we get

$$\int_\Omega P^{(k)}_{ij+2}\partial_{j+1}\phi \;\mathrm{d}x \to \int_\Omega P_{ij+2}\partial_{j+1}\phi \;\mathrm{d}x,$$
$$\int_\Omega P^{(k)}_{ij+1}\partial_{j+2}\phi \;\mathrm{d}x \to \int_\Omega P_{ij+1}\partial_{j+2}\phi \;\mathrm{d}x,$$

as $k \to \infty$. And hence

$$\lim_{k\to\infty}\int_\Omega [(\mathrm{curl}P^{(k)})_{ij} - (\mathrm{curl}P)_{ij}]\phi \;\mathrm{d}x = 0$$

for all $\phi \in C_0^\infty(\Omega)$. Applying the classical density argument and the uniqueness of the weak limit we get the desired result.

The case $d = 2$ is exactly the same except that in this case we have $(\mathrm{curl}P)_i = \partial_1 P_{i2} - \partial_2 P_{i1}$.

(b) This follows from Theorem 2.3.2 by setting $p_j = q_P$ and $f_j^{(k)} = e_j^T P^{(k)}$.

(c) The case $d = 2$ is trivial since $\mathcal{G}(P) = \mathrm{curl}P$.
For the case $d = 3$ we have $\mathcal{G}(P) = (\mathrm{curl}P)P^T$. It is clear that we want to apply the div-curl Lemma 2.3.1. Hence, we have to verify its assumptions. Setting $f_j^{(k)} = e_j^T P^{(k)}$ and $g_j^{(k)} = e_j^T \mathrm{curl}P^{(k)}$ we can conclude that

$$f_j^{(k)} \rightharpoonup f_j = e_j^T P \quad \text{in } L^{q_P}(\Omega;\mathbb{R}^3),$$
$$g_j^{(k)} \rightharpoonup g_j = e_j^T \mathrm{curl}P \quad \text{in } L^{q_C}(\Omega;\mathbb{R}^3).$$

Furthermore we know that $\{\mathrm{curl} f_j^{(k)} : k \in \mathbb{N}\}$ is bounded since the weak convergence of each row of $\mathrm{curl} P^{(k)}$ implies boundedness. Note also that

$$\mathrm{div} g_j^{(k)} = \mathrm{div}(e_j^T \mathrm{curl} P^{(k)}) = \mathrm{div}\ \mathrm{curl}(e_j^T P^{(k)}) = 0$$

and therefore we can apply the div-curl lemma for $j = 1, 2, 3$ and obtain

$$(\mathrm{curl} P^{(k)}) P^{(k)T} \rightharpoonup (\mathrm{curl} P) P^T \quad \text{in } L^{q^\star}(\Omega; \mathbb{R}^{3\times 3})$$

which completes our proof. $\square$

In order to get weak convergence for the product $\nabla\varphi P^{-1}$ we need the following Lemma.

Lemma 2.3.4. *Let $F, P \in \mathbb{R}^{d\times d}$ with $\det P = 1$. Then for all $1 \leq s \leq d$ and $1 \leq i, j \leq \binom{d}{s}$ there exists a matrix $K(s, i, j) \in \mathbb{R}^{d\times d}$ such that*

$$M_s(FP^{-1})_{ij} = \det K(s, i, j)$$

where $K(s, i, j)$ consists of s rows of F and $d - s$ rows of P.

Proof. For the case $d = 2$ we can use the explicit formula for the inverse which is given by

$$P^{-1} = \begin{pmatrix} P_{22} & -P_{12} \\ -P_{21} & P_{11} \end{pmatrix}.$$

Via standard matrix computation we can identify all entries of the matrix FP^{-1} with determinants of matrices K containing one row of F and one row of P

$$(FP^{-1})_{11} = F_{11}P_{22} - F_{12}P_{21} = \det \begin{pmatrix} F_{11} & F_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

$$(FP^{-1})_{12} = -F_{11}P_{12} + F_{12}P_{11} = \det \begin{pmatrix} P_{11} & P_{12} \\ F_{11} & F_{12} \end{pmatrix},$$

$$(FP^{-1})_{21} = F_{21}P_{22} - F_{22}P_{21} = \det \begin{pmatrix} F_{21} & F_{22} \\ P_{21} & P_{22} \end{pmatrix},$$

$$(FP^{-1})_{22} = -F_{21}P_{11} - F_{22}P_{11} = \det \begin{pmatrix} P_{11} & P_{12} \\ F_{21} & F_{22} \end{pmatrix}.$$

Furthermore, by using the calculation rules for the determinant we get

$$\det(FP^{-1}) = \frac{\det F}{\det P} = \det F$$

which finishes the proof for the case $d = 2$.

The same argument can be made for the determinant in the case $d = 3$. These calculation rules also hold for the cofactor matrix. Additionally we have the identity

$$\mathrm{Cof} A = (\det A)A^{-T}.$$

Let $i, j \in \{1, 2, 3\}$ and define H as the matrix P where the j-th row is replaced by the i-th row of F. This gives

$$\det H = \sum_k H_{jk}(\mathrm{Cof} H)_{jk} = \sum_k F_{ik}(\mathrm{Cof} P)_{jk}$$

and therefore

$$(FP^{-1})_{ij} = (F(\mathrm{Cof} P)^T)_{ij} = \sum_k F_{ik}(\mathrm{Cof} P)_{jk} = \det H$$

which leaves us with proving the result for the entries of the cofactor matrix. Note that

$$\mathrm{Cof}(FP^{-1}) = (\mathrm{Cof} F)(\mathrm{Cof} P^{-1}) = (\mathrm{Cof} F)(\mathrm{Cof} P)^{-1} = (\mathrm{Cof} F)P^T$$

Again let $i, j \in \{1, 2, 3\}$ and this time define H as the matrix P where we fix the j-th row and replace the other ones by the rows $\{1, 2, 3\} \setminus \{i\}$ of F. We have

$$\det H = \sum_k H_{jk}(\mathrm{Cof} H)_{jk} = \sum_k (\mathrm{Cof} F)_{ik} P_{jk}.$$

Combining this with the equality from above we get

$$\mathrm{Cof}(FP^{-1})_{ij} = \sum_k (\mathrm{Cof} F)_{ik} P_{jk} = \det H,$$

which finishes the proof. $\square$

Now we finally can show weak lower semi-continuity. First, we will recall the definition.

Definition 2.3.5. Let $\Omega \subset \mathbb{R}^d$ be open, $z : \Omega \to \mathbb{R}^s$, $u : \Omega \to \mathbb{R}^t$ and $F : \Omega \times \mathbb{R}^s \times \mathbb{R}^t \to \mathbb{R} \cup \{\infty\}$ with the functional $I(z,u) := \int_\Omega F(x,z,u)\,\mathrm{d}x$. We say that I is weakly lower semi-continuous if for all sequences

$$\begin{aligned} z_n &\rightharpoonup z \text{ in } W^{1,p}(\Omega;\mathbb{R}^s) \\ u_n &\rightharpoonup u \text{ in } L^q(\Omega;\mathbb{R}^t) \end{aligned}$$

we have

$$I(z,u) \le \liminf_{n\to\infty} I(z_n,u_n)$$

We will usually assume the following condition on F.

Definition 2.3.6. F is said to be a Carathéodory function if

i) $F(\cdot,z,u) : \Omega \to \mathbb{R} \cup \{\infty\}$ is measurable for all $(z,u) \in \mathbb{R}^s \times \mathbb{R}^t$

ii) $F(x,\cdot,\cdot) : \mathbb{R}^s \times \mathbb{R}^t \to \mathbb{R} \cup \{\infty\}$ is continuous for a.e. $x \in \Omega$

In order to fully state and prove the theorem on lower semi-continuity we still need some preparation. Particularly several results from measure theory will be of importance.

Theorem 2.3.7 (Lusin)**.** *Let $\Omega \subset \mathbb{R}^d$ be bounded and $F : \Omega \to \mathbb{R}$ be measurable. Then for all $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $F : K \to \mathbb{R}$ is continuous and $\lambda^d(\Omega \setminus K) < \varepsilon$.*

A proof to this statement can be optained via standard measure theory arguments and can be found in [Fel81]. Another well-known result is the following.

Theorem 2.3.8 (Egorov's theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be bounded and measurable and let f_n be measurable functions in Ω converging almost everywhere to a function f. Then for all $\varepsilon > 0$ there exists a closed and measurable set $M \subset \Omega$ such that f_n converges uniformly to f in M and $\lambda^d(\Omega \setminus M) < \varepsilon$.*

Here, we refer to [Bog07, thm. 2.2.1, p. 110].

A direct consequence from the above theorems which applies to our case is the so called Scorza-Dragoni theorem (see [Giu03, lemma 4.6, p. 128]).

Theorem 2.3.9 (Scorza-Dragoni theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be bounded and measurable, $S \subset \mathbb{R}^s \times \mathbb{R}^t$ compact and $F : \Omega \times \mathbb{R}^s \times \mathbb{R}^t \to \mathbb{R} \cup \{\infty\}$ a Carathéodory function. Then for all $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $F : K \times S \to \mathbb{R} \cup \{\infty\}$ is continuous and $\lambda^d(\Omega \setminus K) < \varepsilon$.*

Proof. For $n \in \mathbb{N}$ define

$m_n := \sup\{|F(x,z,u) - F(x,y,v)| : (z,u),(y,v) \in S,\ |(z,u)-(y,v)| < 1/n\}.$

Since F is a Carathéodory function, $m_n \to 0$ a.e. in Ω as $n \to \infty$. Applying Egorov's theorem and observing that Ω is bounded, for all $\varepsilon > 0$ we can find a compact and measurable set $M \subset \Omega$ where $m_n \to 0$ uniformly in M and $\lambda^d(\Omega \setminus M) < \varepsilon/2$.

This means that for every $\eta > 0$ and $(z,u) \in S$ there exists a $\delta_1 > 0$ such that for all $x \in M$ and $(y,v) \in S$ we have

$$|(z,u) - (y,v)| < \delta_1 \implies |F(x,z,u) - F(x,y,v)| < \eta/4. \tag{2.5}$$

Since S is compact, we can find a countable dense subset $\{(z_n,u_n)\}_{n=1}^{\infty}$. Now we can apply Lusin's theorem for fixed (z_n,u_n). Hence, for every $n \in \mathbb{N}$ there exists a compact set $K_n \subset \Omega$ such that $F(\cdot, z_n, u_n) : K_n \to \mathbb{R}$ is continuous and $\lambda^d(\Omega \setminus K_n) < \varepsilon/2^{n+1}$.

Let $N = \bigcap_{n\in\mathbb{N}} K_n$. Then for all $n \in \mathbb{N}$ we know that $F(\cdot, z_n, u_n) : N \to \mathbb{R}$ is continuous and $\lambda^d(\Omega \setminus N) < \varepsilon/2$.

This means that for every $\eta > 0$, $x \in N$ and $(z,u) \in S$ there exists a $\delta_2 > 0$ such that for all $\bar{x} \in N$ we have

$$|x - \bar{x}| < \delta_2 \implies |F(x,z_n,u_n) - F(\bar{x},z_n,u_n)| < \eta/4. \tag{2.6}$$

Now set $K = M \cap N$. We need to show that F restricted to $K \times S$ is continuous. Let $\eta > 0$, $x \in N$, $(z,u) \in S$ and let $\delta_1 > 0$ such that (2.5) is satisfied. Furthermore, choose (z_n,u_n) such that $|(z_n,u_n) - (z,u)| < \delta_1$.

This implies that if $x, \bar{x} \in K$ and $(z,u),(y,v) \in S$ with $|(z,u) - (y,v)| < \delta_1$ then

$$\begin{aligned}
|F(x,z,u) - F(x,z_n,u_n)| &< \eta/4,\\
|F(\bar{x},z,u) - F(\bar{x},z_n,u_n)| &< \eta/4,\\
|F(\bar{x},z,u) - F(\bar{x},y,v)| &< \eta/4.
\end{aligned}$$

Furthermore, we can choose $\delta_2 > 0$ such that (2.6) holds for all $\bar{x} \in K$. Hence, if we set $\delta := \min\{\delta_1, \delta_2\}$ then for every $\bar{x} \in K$ and $(y, v) \in S$ such that $|x - \bar{x}| + |(z, u) - (y, v)| < \delta$ we get

$$
\begin{aligned}
|F(x,z,u) - &F(\bar{x},y,v)| \leq \\
&|F(x,z,u) - F(x,z_n,u_n)| + |F(x,z_n,u_n) - F(\bar{x},z_n,u_n)| \\
&+|F(\bar{x},z,u) - F(\bar{x},z_n,u_n)| + |F(\bar{x},z,u) - F(\bar{x},y,v)| \\
&<\eta/4 + \eta/4 + \eta/4 + \eta/4 = \eta
\end{aligned}
$$

which shows that F is continuous on $K \times S$.
Moreover we know that $\lambda^d(\Omega \setminus M) < \varepsilon/2$ and $\lambda^d(\Omega \setminus N) < \varepsilon/2$ and hence it follows that $\lambda^d(\Omega \setminus K) < \varepsilon$. □

Another important result that connects weak and strong convergence is the following (see [Rud91, Thm. 3.13, p. 67]).

Lemma 2.3.10 (Mazur's lemma). *Let V be a Banach space and $\{u_n\} \subset V$ be a weakly convergent sequence with $u_n \rightharpoonup u$ for $n \to \infty$. Then there exists a function $m : \mathbb{N} \to \mathbb{N}$ and $\lambda_j^n \geq 0$ with $\sum_{j=n}^{m(n)} \lambda_j^n = 1$ such that for*

$$v_n := \sum_{j=n}^{m(n)} \lambda_j^n u_j$$

it follows that

$$v_n \to u \text{ in } V \text{ as } n \to \infty.$$

In order to prove weak lower semi-continuity we will first need a slightly simpler theorem where the dependence is only in the matrix-valued components, i.e. $F : \Omega \times \mathbb{R}^t \to \mathbb{R} \cup \{\infty\}$ (see [Dac08, Thm. 3.20, p. 94]).

Theorem 2.3.11. *Let $\Omega \subset \mathbb{R}^d$ be open and $F : \Omega \times \mathbb{R}^t \to \mathbb{R} \cup \{\infty\}$ be a Carathéodory function such that for all $u \in \mathbb{R}^t$ we have $F(\cdot, u) \geq \psi$ for some $\psi \in L^1(\Omega)$. Furthermore assume that $F(x, \cdot) : \mathbb{R}^t \to \mathbb{R} \cup \{\infty\}$ is convex and that*

$$u_n \rightharpoonup u \text{ in } L^q(\Omega; \mathbb{R}^t).$$

Then for $I(u) := \int_\Omega F(x, u) \, \mathrm{d}x$ we have

$$I(u) \leq \liminf_{n \to \infty} I(u_n).$$

Proof. Without loss of generality we can assume $F \geq 0$ for otherwise we can work with $F - \psi$. First we want to show strong lower semi-continuity, hence let

$$u_n \to u \text{ in } L^q(\Omega; \mathbb{R}^t)$$

and extract a (non-relabeled) subsequence such that

$$u_n \to u \quad \text{a.e. in } \Omega.$$

Since F is continuous in u and $F \geq 0$, it follows by Fatou's lemma that

$$\int_\Omega F(x, u) \, \mathrm{d}x = \int_\Omega \liminf_{n\to\infty} F(x, u_n) \, \mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega F(x, u_n) \, \mathrm{d}x. \tag{2.7}$$

Our goal is to go from strong to weak lower semi-continuity. Take a sequence $\{u_n\}_{n=1}^\infty$ with

$$u_n \rightharpoonup u \text{ in } L^q(\Omega; \mathbb{R}^t).$$

Define $L := \liminf_{n\to\infty} I(u_n)$ and extract a (non-relabeled) subsequence such that $L = \lim_{n\to\infty} F(x, u_n)$. Note that $L \geq 0$ and assume that $L < \infty$, otherwise there is nothing to show. From the definition of limits we get that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$I(u_n) \leq I(u) + \varepsilon \quad \text{for all } n \geq N. \tag{2.8}$$

Fixing $\varepsilon > 0$ and applying Mazur's lemma to the sequence $\{u_n\}_{n=N}^\infty$ we obtain a sequence $\{v_n\}_{n=N}^\infty$ with

$$v_n \to u \text{ in } L^q(\Omega; \mathbb{R}^t)$$

such that there exists a function $m : (N, N+1, ...) \to (N, N+1, ...)$ and $\lambda_j^n \geq 0$ with $\sum_{j=n}^{m(n)} \lambda_j^n = 1$ and

$$v_n := \sum_{j=n}^{m(n)} \lambda_j^n u_j.$$

Using the convexity of F in the second component and inequality (2.8) it follows that

$$\int_\Omega F(x, v_n)\,\mathrm{d}x = \int_\Omega F(x, \sum_{j=n}^{m(n)} \lambda_j^n u_j)\,\mathrm{d}x \le \sum_{j=n}^{m(n)} \lambda_j^n \int_\Omega F(x, u_j)\,\mathrm{d}x$$
$$\le \sum_{j=n}^{m(n)} \lambda_j^n (L+\varepsilon) = L + \varepsilon. \tag{2.9}$$

Since we know that v_n converges strongly to u we also get from (2.7) that

$$\int_\Omega F(x, u)\,\mathrm{d}x = \int_\Omega \liminf_{n\to\infty} F(x, v_n)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega F(x, v_n)\,\mathrm{d}x. \tag{2.10}$$

Combining (2.9) and (2.10) we obtain

$$I(u) \le L + \varepsilon.$$

This finishes our proof since $\varepsilon > 0$ was fixed but arbitrary. $\square$

Now we can state and prove our main lower semi-continuity result (see [Dac08, thm. 3.23, p. 96]).

Theorem 2.3.12. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $F : \Omega \times \mathbb{R}^s \times \mathbb{R}^t \to \mathbb{R} \cup \{\infty\}$ be a Carathéodory function such that for all $(z, u) \in \mathbb{R}^s \times \mathbb{R}^t$ we have $F(\cdot, z, u) \ge \psi$ for some $\psi \in L^1(\Omega)$. Furthermore, assume that $F(x, z, \cdot) : \mathbb{R}^t \to \mathbb{R} \cup \{\infty\}$ is convex and that*

$$z_n \to z^\star \text{ in } L^p(\Omega;\mathbb{R}^s) \quad \text{and} \quad u_n \rightharpoonup u^\star \text{ in } L^q(\Omega;\mathbb{R}^t).$$

Then for $I(z, u) := \int_\Omega F(x, z, u)\,\mathrm{d}x$ we have

$$I(z^\star, u^\star) \le \liminf_{n\to\infty} I(z_n, u_n).$$

Proof. Again without loss of generality we can assume $F \ge 0$. As before we define $L := \liminf_{n\to\infty} I(z_n, u_n)$ and extract a (non-relabeled) subsequence such that $L = \lim_{n\to\infty} F(x, z_n, u_n)$ with $0 \le L < \infty$.

Step 1: Our first goal is to show that for all $\varepsilon > 0$ there exists a measurable set $\Omega_\varepsilon \subset \Omega$ with $\lambda^d(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and a subsequence n_k with $n_k \to \infty$, such that

$$\int_{\Omega_\varepsilon} |F(x, z_{n_k}, u_{n_k}) - F(x, z^\star, u_{n_k})| \,\mathrm{d}x < \varepsilon \lambda^d(\Omega). \tag{2.11}$$

Now choose $\varepsilon > 0$ and let

$$\begin{aligned} K_{1,\varepsilon} &:= \{x \in \Omega : |z^\star(x)| \leq M_\varepsilon \text{ or } |z_n(x)| \leq M_\varepsilon\}, \\ K_{2,\varepsilon} &:= \{x \in \Omega : |u_n(x)| \leq M_\varepsilon\}, \end{aligned}$$

where M_ε is such that $\lambda^d(K_{1,\varepsilon}) < \varepsilon/6$ and $\lambda^d(K_{2,\varepsilon}) < \varepsilon/6$ for every $n \in \mathbb{N}$. We can find such a M_ε since $z, z_n \in L^p(\Omega; \mathbb{R}^s)$ and $u_n \in L^q(\Omega; \mathbb{R}^t)$.
Set $\Omega_{1,n,\varepsilon} := \Omega \setminus (K_{1,\varepsilon} \cup K_2, \varepsilon)$ and it follows that $\lambda^d(\Omega \setminus \Omega_{1,n,\varepsilon}) < \varepsilon/3$.
Applying the Scorza-Dragoni theorem with

$$S := \{(z, u) :\in \mathbb{R}^s \times \mathbb{R}^t : |z| < M_\varepsilon \text{ and } |u| < M_\varepsilon\}$$

we can conclude that there exists a compact set $\Omega_{2,n,\varepsilon} \subset \Omega_{1,n,\varepsilon}$ with $\lambda^d(\Omega_{1,n,\varepsilon} \setminus \Omega_{2,n,\varepsilon}) < \varepsilon/3$ and such that F is continuous on $\Omega_{2,n,\varepsilon} \times S$.
Hence there exists $\delta > 0$ such that if $|z - y| < \delta$ then

$$|F(x, z, u) - F(x, y, u)| < \varepsilon \tag{2.12}$$

for all $x \in \Omega_{2,n,\varepsilon}$, $z, y \in \mathbb{R}^s$ with $|z| < M_\varepsilon$ and $|y| < M_\varepsilon$ and $u \in \mathbb{R}^t$ with $|u| < M_\varepsilon$.
Choosing such a $\delta > 0$ we know that since $z_n \to z^\star$ in $L^p(\Omega; \mathbb{R}^s)$ there exists $N_\varepsilon \in \mathbb{N}$ such that for

$$\Omega_{3,n,\varepsilon} := \{x \in \Omega : |z_n(x) - z^\star(x)| < \delta\}$$

holds that $\lambda^d(\Omega \setminus \Omega_{3,n,\varepsilon}) < \varepsilon/3$ for all $n \geq N_\varepsilon$.
Now let $\Omega_{n,\varepsilon} := \Omega_{2,n,\varepsilon} \cap \Omega_{3,n,\varepsilon}$ then $\lambda^d(\Omega \setminus \Omega_{n,\varepsilon}) < \varepsilon$ and from (2.12) it follows that

$$\int_{\Omega_{n,\varepsilon}} |F(x, z^\star, u_n) - F(x, z_n, u_n)| \,\mathrm{d}x < \varepsilon \lambda^d(\Omega)$$

for every $n \geq N_\varepsilon$. The same holds for $\varepsilon_k := \varepsilon/2^k$ for $k \in \mathbb{N}$. Hence, by choosing a sequence $n_k \geq N_{\varepsilon_k}$ with $n_k \to \infty$ and setting $\Omega_\varepsilon := \bigcap_{k=1}^\infty \Omega_{n_k,\varepsilon_k}$ the bound (2.11) follows immediately.

Step 2: We want to apply Theorem 2.3.11. Hence, we denote by $\mathbb{1}_{\Omega_\varepsilon} : \Omega \to \{0,1\}$ the characteristic function of Ω_ε in Ω, i.e.,

$$\mathbb{1}_{\Omega_\varepsilon}(x) = \begin{cases} 1 & \text{if } x \in \Omega_\varepsilon \\ 0 & \text{if } x \notin \Omega_\varepsilon \end{cases}$$

and let

$$G(x,u) := \mathbb{1}_{\Omega_\varepsilon}(x) F(x, z^\star, u).$$

It follows that $G : \Omega \times \mathbb{R}^t \to \mathbb{R} \cup \{\infty\}$ is a Carathéodroy function such that $G(\cdot, u) \geq \psi$ for all $u \in \mathbb{R}^t$ and $G(x,\cdot) : \mathbb{R}^t \to \mathbb{R} \cup \{\infty\}$ is convex. Hence, we can apply Theorem 2.3.11 and obtain

$$\int_\Omega \mathbb{1}_{\Omega_\varepsilon}(x) F(x, z^\star, u^\star) \, \mathrm{d}x \leq \liminf_{n_k \to \infty} \int_\Omega \mathbb{1}_{\Omega_\varepsilon}(x) F(x, z^\star, u_{n_k}) \, \mathrm{d}x.$$

Using (2.11) and the fact that $F \geq 0$ we have that

$$\begin{aligned}
&\int_\Omega \mathbb{1}_{\Omega_\varepsilon}(x) F(x, z^\star, u_{n_k}) \, \mathrm{d}x - \varepsilon \lambda^d(\Omega) = \int_{\Omega_\varepsilon} F(x, z^\star, u_{n_k}) \, \mathrm{d}x - \varepsilon \lambda^d(\Omega) \\
&\leq \int_{\Omega_\varepsilon} F(x, z^\star, u_{n_k}) \, \mathrm{d}x - \int_{\Omega_\varepsilon} |F(x, z_{n_k}, u_{n_k}) - F(x, z^\star, u_{n_k})| \, \mathrm{d}x \\
&\leq \int_{\Omega_\varepsilon} F(x, z_{n_k}, u_{n_k}) \, \mathrm{d}x \leq \int_\Omega F(x, z_{n_k}, u_{n_k}) \, \mathrm{d}x.
\end{aligned}$$

In particular, we have

$$\int_\Omega \mathbb{1}_{\Omega_\varepsilon}(x) F(x, z^\star, u^\star) \, \mathrm{d}x - \varepsilon \lambda^d(\Omega) \leq \liminf_{n_k \to \infty} \int_\Omega F(x, z_{n_k}, u_{n_k}) \, \mathrm{d}x.$$

Since $F \geq 0$ we can apply the Monotone Convergence Theorem (see A.0.4) on the left hand side and let $\varepsilon \to 0$ to conclude that

$$\int_\Omega F(x, z^\star, u^\star) \, \mathrm{d}x \leq \liminf_{n_k \to \infty} \int_\Omega F(x, z_{n_k}, u_{n_k}) \, \mathrm{d}x.$$

$\square$

As mentioned before, we will not assume convexity on the function F but only polyconvexity. Hence, we will need an even stronger result on weak lower semi-continuity which can be applied to our problem (2.1).

Theorem 2.3.13. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let H and F be given as in (2.1) with $F(\cdot, z, F_{el}, P, G) \geq \psi$ for some $\psi \in L^1(\Omega)$. Furthermore, assume that there exists a Carathéodory function*

$$\widetilde{F} : \Omega \times [0,1] \times \mathbb{R}^{\binom{2d}{d}-1} \times \mathbb{R}^{\binom{2d}{d}-1} \times \mathbb{R}^{\overbrace{d \times ... \times d}^{(d-1)-times}} \to \mathbb{R} \cup \{\infty\}$$

such that $\widetilde{F}(x, z, \cdot, \cdot, \cdot) : \mathbb{R}^{\binom{2d}{d}-1} \times \mathbb{R}^{\binom{2d}{d}-1} \times \mathbb{R}^{\overbrace{d \times ... \times d}^{(d-1)-times}} \to \mathbb{R} \cup \{\infty\}$ *is convex and*

$$F(x, z, F_{el}, P, G) = \widetilde{F}(x, z, M(F_{el}), M(P), G)$$

Let $q_\varphi > d$, $q_P > d$, and q_C satisfy

$$\frac{d-2}{q_P} + \frac{1}{q_C} < 1,$$
$$\frac{1}{q_C} < \frac{1}{d} + \frac{1}{q_P},$$

and let

$$z^{(k)} \to z^\star \text{ in } L^p(\Omega; [0,1]),$$
$$\varphi^{(k)} \rightharpoonup \varphi^\star \text{ in } W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d),$$
$$P^{(k)} \rightharpoonup P^\star \text{ in } A_{\det}^{q_P, q_C}(\Omega),$$

where $W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d) := \{\varphi \in W^{1,q_\varphi}(\Omega; \mathbb{R}^d) : \varphi|_{\Gamma_D} = 0\}$ for some $\Gamma_D \subset \partial\Omega$ and $A_{\det}^{q_P, q_C}(\Omega)$ consists of all elements $P \in L^{q_P}(\Omega; \mathbb{R}^{d \times d})$ such that

$$\mathrm{curl} P \in \begin{cases} L^{q_C}(\Omega; \mathbb{R}^2) \text{ for } d = 2, \\ L^{q_C}(\Omega; \mathbb{R}^{3\times 3}) \text{ for } d = 3 \end{cases},$$
$$\det P = 1 \text{ a.e. in } \Omega.$$

Then,

$$H(z^\star, \varphi^\star, P^\star) \leq \liminf_{k\to\infty} H(z^{(k)}, \varphi^{(k)}, P^{(k)}).$$

Proof. Since $\mathrm{curl}P^{(k)}$ is bounded in $L^{q_C}(\Omega;\mathbb{R}^d)$ for $d=2$ and $L^{q_C}(\Omega;\mathbb{R}^{d\times d})$ for $d=3$ we know that there exists a (non-relabeled) converging subsequence and can apply Lemma 2.3.3 to obtain

$$\begin{aligned}
\mathrm{curl}P^{(k)} &\rightharpoonup \mathrm{curl}P^\star \quad \text{in } \begin{cases} L^{q_C}(\Omega;\mathbb{R}^2) \ for\ d=2, \\ L^{q_C}(\Omega;\mathbb{R}^{3\times 3}) \ for\ d=3, \end{cases} \\
M(P^{(k)}) &\rightharpoonup M(P^\star) \quad \text{in } L^{q_P/d}(\Omega;\mathbb{R}^{\binom{2d}{d}-1}), \\
\mathcal{G}(P^{(k)}) &\rightharpoonup \mathcal{G}(P^\star) \quad \text{in } \begin{cases} L^{q^\star}(\Omega;\mathbb{R}^2) \ for\ d=2, \\ L^{q^\star}(\Omega;\mathbb{R}^{3\times 3}) \ for\ d=3, \end{cases} \qquad \text{for some } q^\star > 1.
\end{aligned}$$

To get weak convergence of $M(F_{el})$ we note that by Lemma 2.3.4 each component $M_s(\nabla\varphi^{(k)}(P^{(k)})^{-1})_{ij}$ is the determinant of a $d\times d$ matrix $K(s,i,j)$ where $K(s,i,j)$ consists of s rows of $\nabla\varphi^{(k)}$ and $d-s$ rows of $P^{(k)}$. This means that the rows of $K(s,i,j)$ are either curl-free or the curl is bounded in $L^{q_C}(\Omega)$. Hence, we can use Theorem 2.3.2 to get

$$M(\nabla\varphi^{(k)}(P^{(k)})^{-1}) \rightharpoonup M(\nabla\varphi^\star(P^\star)^{-1}) \quad \text{in } L^\sigma(\Omega;\mathbb{R}^{\binom{2d}{d}-1}) \text{ for some } \sigma > 1$$

Now we apply Theorem 2.3.12 with

$$\begin{aligned}
z_n &:= z^{(k)}, \\
z^\star &:= z^\star, \\
u_n &:= (M(F_{el}^{(k)}), M(P^{(k)}), G^{(k)}), \\
u^\star &:= (M(F_{el}^\star), M(P^\star), G^\star),
\end{aligned}$$

and $q = \min\{q_P/d, q^\star, \sigma\} > 1$ to get our desired result. □

2.4 Continuity of U

Recall that the functional U describes the work done by conservative forces and is defined as

$$U(z,\varphi,P) = -\left(\int_\Omega zf\cdot\varphi \,\mathrm{d}x + \int_{\Gamma_N} g\cdot\varphi \,\mathrm{d}x\right). \tag{2.13}$$

In this section, we want to show that U is continuous on $L^1(\Omega;[0,1]) \times W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$.

Theorem 2.4.1. *Let $\Omega \subset \mathbb{R}^d$ be an open Lipschitz domain and $q_\varphi > d$. Furthermore, assume that $f \in L^{q_f}(\Omega)$ for some $q_f > \frac{q_\varphi}{q_\varphi - 1} =: (q_\varphi)'$ and $g \in L^{q_g}(\Gamma_N)$ for some $q_g = (q_\varphi)'$. If*

$$\begin{aligned}
z^{(k)} &\to z \quad \text{in } L^{q_z}(\Omega) \quad \text{where} \quad \frac{1}{q_z} + \frac{1}{q_\varphi} + \frac{1}{q_f} = 1,\\
\varphi^{(k)} &\rightharpoonup \varphi \quad \text{in } W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d),\\
P^{(k)} &\rightharpoonup P \quad \text{in } A_{\det}^{q_P,q_C}(\Omega),
\end{aligned}$$

then,

$$\lim_{k\to\infty} U(z^{(k)}, \varphi^{(k)}, P^{(k)}) \to U(z,\varphi,P).$$

Proof. Note that for the exponents it holds that

$$\frac{1}{q_f} + \frac{1}{q_\varphi} = \frac{1}{q_z'} \quad \text{and} \quad \frac{1}{q_f} + \frac{1}{q_z} = \frac{1}{q_\varphi'}.$$

Looking at the first term of U we obtain

$$\int_\Omega z^{(k)} f \cdot \varphi^{(k)} \,\mathrm{d}x = \int_\Omega (z^{(k)} - z) f \cdot \varphi^{(k)} \,\mathrm{d}x + \int_\Omega z f \cdot \varphi^{(k)} \,\mathrm{d}x.$$

Now we can use Hölder's inequality and see that

$$\begin{aligned}
\int_\Omega (z^{(k)} - z) f \cdot \varphi^{(k)} \,\mathrm{d}x &\le \left\| z^{(k)} - z \right\|_{q_z} \left\| f \cdot \varphi^{(k)} \right\|_{q_z'}\\
&\le \left\| z^{(k)} - z \right\|_{q_z} \left\| f \right\|_{q_f} \left\| \varphi^{(k)} \right\|_{q_\varphi} \to 0
\end{aligned}$$

as $k \to \infty$. Additionally we know that $zf \in L^{q_\varphi'}(\Omega)$ since

$$\|zf\|_{q_\varphi'} \le \|z\|_{q_z} \|b\|_{q_f} < \infty.$$

Hence, we can conclude that

$$\int_\Omega z^{(k)} f \cdot \varphi^{(k)} \,\mathrm{d}x \to \int_\Omega z f \cdot \varphi \,\mathrm{d}x \quad \text{for } k \to \infty.$$

For the second term of U we can use Theorem B.0.3 and from the continuity of the trace operator T (here, we will omit T in the notation) it follows that

$$\varphi^{(k)} \rightharpoonup \varphi \quad \text{in } L^{q_\varphi}(\partial\Omega).$$

Now since $\frac{1}{q_\varphi} + \frac{1}{q_g} = 1$ and $\Gamma_N \subset \partial\Omega$ we obtain

$$\int_{\Gamma_N} g \cdot \varphi^{(k)} \, \mathrm{d}\mathcal{H}^{d-1} \to \int_{\Gamma_N} g \cdot \varphi \, \mathrm{d}\mathcal{H}^{d-1} \quad \text{for } k \to \infty.$$

All in all we have shown that

$$-U(z^{(k)}, \varphi^{(k)}, P^{(k)}) \to -U(z, \varphi, P) \quad \text{for } k \to \infty$$

which finishes the proof. □

Remark 2.4.2. Using some analogous arguments one can see that for $z \in L^{q_z}(\Omega)$ there exists a constant $C > 0$ such that

$$|U(z, \varphi, P)| \leq C \, \|\varphi\|_{q_\varphi} .$$

2.5 Existence of minimizers

Combining the coercivity result of Chapter 2.2, the lower semi-continuity result of Chapter 2.3 and the continuity result of Chapter 2.4, we now show existence of at least one global minimizer of the plasticity functional H given by (2.1).

Theorem 2.5.1. *Let H and F be given as in (2.1) and U as in (2.13). Moreover, we assume the following properties:*

1. *there exists $(z, \varphi, P) \in L^1(\Omega; [0,1]) \times W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d) \times A_{\det}^{q_P, q_C}(\Omega)$ such that $\mathcal{E}(z, \varphi, P) < \infty$,*
2. *there exists $c > 0$ and $g \in L^1(\Omega)$ such that*

$$c(|F_{el}|^{q_F} + |P|^{q_P} + |P^{-1}|^{q_P} + |G|^{q_G}) - g(x) \leq F(x, z, F_{el}, P, G)$$

holds with

- $\frac{1}{q_\varphi} := \frac{1}{q_F} + \frac{1}{q_P} < \frac{1}{d}$,
- $\frac{1}{q_C} := \begin{cases} \frac{1}{q_G} & \textit{for} \quad d = 2, \\ \frac{1}{q_G} + \frac{1}{q_P} & \textit{for} \quad d = 3, \end{cases}$
- $\frac{1}{q_G} + \frac{\min\{2d-4,d\}}{q_P} < 1$,
- $\frac{1}{q_G} + \frac{\min\{d-3,1\}}{q_P} < \frac{1}{d}$.

3. *The order variable z is uniformly bounded in $H^1(\Omega;[0,1])$ or $BV(\{0,1\})$.*

4. *There exists a Carathéodory function*

$$\widetilde{F} : \Omega \times [0,1] \times \mathbb{R}^{\binom{2d}{d}-1} \times \mathbb{R}^{\binom{2d}{d}-1} \times \mathbb{R}^{\overbrace{d \times \ldots \times d}^{(d-1)-times}} \to \mathbb{R} \cup \{\infty\}$$

such that $\widetilde{F}(x, z, \cdot, \cdot, \cdot) : \mathbb{R}^{\binom{2d}{d}-1} \times \mathbb{R}^{\binom{2d}{d}-1} \times \mathbb{R}^{\overbrace{d \times \ldots \times d}^{(d-1)-times}} \to \mathbb{R} \cup \{\infty\}$ *is convex and*

$$F(x, z, F_{el}, P, G) = \widetilde{F}(x, z, M(F_{el}), M(P), G).$$

Then, the infimum of $\mathcal{E}(z, \varphi, P)$ is attained on $H^1(\Omega;[0,1]) \times W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$ or $BV(\Omega;\{0,1\}) \times W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$.

Proof. From assumption 1. we know that $\mathcal{E} \not\equiv \infty$ and from assumption 2. we can can conclude that

$$H(z,\varphi,P) \geq c(\|F_{el}\|_{q_F}^{q_F} + \|P\|_{q_P}^{q_P} + \left\|P^{-1}\right\|_{q_P}^{q_P} + \|G\|_{q_G}^{q_G}) - \|g\|_1 \geq C$$

for some $C \in \mathbb{R}$ which implies that H is bounded from below. Additionally, U is bounded from above for $\varphi \in W_D^{1,q_\varphi}$. Hence, we can pick a minimizing sequence $(z^{(k)}, \varphi^{(k)}, P^{(k)}) \in L^1(\Omega;[0,1]) \times W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$ which means that

$$m := \inf \mathcal{E}(z,\varphi,P) = \lim_{k\to\infty} \mathcal{E}(z^{(k)}, \varphi^{(k)}, P^{(k)}).$$

We know that $\mathcal{E}(z^{(k)}, \varphi^{(k)}, P^{(k)}) < \infty$ for all $k \in \mathbb{N}$ which implies that (using assumption 2.) we can apply Proposition 2.2.4. Hence, the sequence $(\varphi^{(k)}, P^{(k)})$ is bounded in the reflexive Banach space $W^{1,q_\varphi}(\Omega; \mathbb{R}^d) \times L^{q_P}(\Omega; \mathbb{R}^{d\times d})$ and, by using the Banach-Alaoglu Theorem A.0.1, there exists a (non-relabelled) weakly convergent subsequence. By Lemma 2.3.3 and the continuity of traces B.0.3 we know that

$$\begin{aligned} \varphi^{(k)} &\rightharpoonup \varphi^\star \quad \text{in } W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d), \\ P^{(k)} &\rightharpoonup P^\star \quad \text{in } A_{\det}^{q_P,q_C}(\Omega). \end{aligned}$$

Assumption 3. tells us that the sequence $(z^{(k)})$ is uniformly bounded in $H^1(\Omega; [0,1])$ or $BV(\{0,1\})$. Using the Rellich-Kondrachov Theorem B.0.1 for $z^{(k)} \in H^1(\Omega; [0,1])$ or the compactness theorem for BV functions (see Thm. B.0.2) we get

$$\begin{aligned} &z^{(k)} \to z^\star \text{ in } L^1(\Omega; [0,1]), \\ &\varphi^{(k)} \rightharpoonup \varphi^\star \text{ in } W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d), \\ &P^{(k)} \rightharpoonup P^\star \text{ in } A_{\det}^{q_P,q_C}(\Omega). \end{aligned}$$

This and assumption 4. are exactly the conditions that we need to be able to apply Theorem 2.3.13. Thus, we know that H is weakly lower semicontinuous. Additionally, U is continuous which implies that $\mathcal{E}$ is weakly lower semicontinuous

$$m \le \mathcal{E}(z^\star, \varphi^\star, P^\star) \le \liminf_{k\to\infty} \mathcal{E}(z^{(k)}, \varphi^{(k)}, P^{(k)}) = m.$$

And consequently $(z^\star, \varphi^\star, P^\star) \in H^1(\Omega; [0,1]) \times W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$ (or $(z^\star, \varphi^\star, P^\star) \in BV(\Omega; \{0,1\}) \times W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$) is a minimizer of $\mathcal{E}$. $\square$

Chapter 3

Existence results for minimizers of the target functionals

In this chapter we are going to show the existence of minimizers for the phase-field problem where $z \in H^1(\Omega;[0,1])$ and the sharp-interface problem where $z \in BV(\Omega;\{0,1\})$. Here, we are again applying the *Direct Method of the Calculus of Variations*. Hence, we have to check boundedness in order to extract convergent subsequences, as well as proving (lower semi-) continuity results.
This chapter follows [BGHR16].

3.1 Stating the problem

Throughout this chapter, we will assume $\Omega \subset \mathbb{R}^d$ to be a bounded Lipschitz domain. Moreover, we split the boundary into two parts $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \emptyset$ and $\mathcal{H}^{d-1}(\Gamma_D) > 0$ where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. We fix Ω on Γ_D, i.e. $\varphi|_{\Gamma_D} = 0$ and on Γ_N the surface load $g \in L^{q_g}(\Gamma_N)$ should be minimized. In Ω we have the body force $f \in L^{q_f}(\Omega)$ which we want to minimize. We can then define the phase field functional as

$$J_\varepsilon(z,\varphi,P) = \int_\Omega zf \cdot \varphi \,\mathrm{d}x + \int_{\Gamma_N} g \cdot \varphi \,\mathrm{d}\mathcal{H}^{d-1} + \int_\Omega \frac{\varepsilon}{2}|\nabla z|^2 + \frac{1}{\varepsilon}\psi(z)\,\mathrm{d}x \quad (3.1)$$

with a continuous double-well potential $\psi : \mathbb{R} \to [0, \infty)$ that vanishes only at 0 and 1. We define the sharp interface functional as

$$J_0(z, \varphi, P) = \int_\Omega z f \cdot \varphi \,\mathrm{d}x + \int_{\Gamma_N} g \cdot \varphi \,\mathrm{d}\mathcal{H}^{d-1} + c_\psi \mathrm{Per}(\{z = 1\}, \Omega)$$

where c_ψ is a given positive function depending solely on the structure of ψ and $\mathrm{Per}(\{z = 1\}, \Omega)$ is the perimeter of the set $\{z = 1\}$ in Ω. For the sake of notational simplicity we write these functionals as

$$\begin{aligned} J_\varepsilon(z, \varphi, P) &= K(z, \varphi, P) + E_\varepsilon(z), \\ J_0(z, \varphi, P) &= K(z, \varphi, P) + E_0(z), \end{aligned}$$

with

$$\begin{aligned} K(z, \varphi, P) &= \int_\Omega z f \cdot \varphi \,\mathrm{d}x + \int_{\Gamma_N} g \cdot \varphi \,\mathrm{d}\mathcal{H}^{d-1}, \\ E_\varepsilon(z) &= \int_\Omega \frac{\varepsilon}{2} |\nabla z|^2 + \frac{1}{\varepsilon} \psi(z) \,\mathrm{d}x, \\ E_0(z) &= c_\psi \mathrm{Per}(\{z = 1\}, \Omega). \end{aligned}$$

Our goal is to minimize the phase field functional for $z \in H^1(\Omega; [0, 1])$ and the sharp interface functional for $z \in BV(\Omega; \{0, 1\})$ under the condition that (φ, P) minimizes $\mathcal{E}(z, \cdot, \cdot)$. Or in other terms we want to show that the following exist

$$\min_{z \in H^1(\Omega; [0,1])} \{J_\varepsilon(z, \varphi, P) : (\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z, \cdot, \cdot)\}, \tag{3.2}$$

$$\min_{z \in BV(\Omega; \{0,1\})} \{J_0(z, \varphi, P) : (\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z, \cdot, \cdot)\}. \tag{3.3}$$

We start by showing the existence of minimizing sequences.

3.2 Existence of a proper minimizing sequence

In the last chapter we showed that there exists at least one minimizer $(z^*, \varphi^*, P^*) \in L^1(\Omega) \times W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d) \times A_{\det}^{q_P, q_C}(\Omega)$ for $\mathcal{E}$ if we consider the design variable z to be uniformly bounded in $H^1(\Omega;[0,1])$ or $BV(\Omega;\{0,1\})$. Taking the constant sequence $z^{(k)} = z^*$ for all $k \in \mathbb{N}$ and for $z^* \in H^1(\Omega;[0,1])$ or $z^* \in BV(\Omega;\{0,1\})$ yields exactly this condition. Hence, for fixed $z \in H^1(\Omega;[0,1])$ or $z \in BV(\Omega;\{0,1\})$ there exists $(\varphi, P) \in W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$ such that $(\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z,\cdot,\cdot)$.

Now we want to minimize the target functionals over z. In order to extract weakly convergent subsequences we need to show boundedness. First, we know that K is bounded from below, since $\varphi \in W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d)$. Additionally, there exists

$$(z,\varphi,P) \in H^1(\Omega;[0,1]) \times W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$$

with $(\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z,\cdot,\cdot)$ such that

$$J_\varepsilon(z,\varphi,P) < \infty$$

and there exists

$$(z,\varphi,P) \in BV(\Omega;\{0,1\}) \times W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$$

with $(\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z,\cdot,\cdot)$ such that

$$J_0(z,\varphi,P) < \infty.$$

Since $E_\varepsilon \geq 0$ and $E_0 \geq 0$ we also know that J_ε and J_0 are bounded from below. This means that we can choose minimizing sequences $z_\varepsilon^{(k)} \in H^1(\Omega;[0,1])$ and $z_0^{(k)} \in BV(\Omega;\{0,1\})$. For these sequences we can find $(\varphi_\varepsilon^{(k)}, P_\varepsilon^{(k)}) \in \operatorname{Arg\,min} \mathcal{E}(z_\varepsilon^{(k)},\cdot,\cdot)$ and $(\varphi_0^{(k)}, P_0^{(k)}) \in \operatorname{Arg\,min} \mathcal{E}(z_0^{(k)},\cdot,\cdot)$ such that

- $J_\varepsilon(z_\varepsilon^{(k)}, \varphi_\varepsilon^{(k)}, P_\varepsilon^{(k)})$
- $J_0(z_0^{(k)}, \varphi_0^{(k)}, P_0^{(k)})$
- $H(z_\varepsilon^{(k)}, \varphi_\varepsilon^{(k)}, P_\varepsilon^{(k)})$
- $H(z_0^{(k)}, \varphi_0^{(k)}, P_0^{(k)})$

are uniformly bounded sequences. This implies that $(z_\varepsilon^{(k)})$ is uniformly bounded in $H^1(\Omega)$, $(z_0^{(k)})$ is uniformly bounded in $BV(\Omega)$, and $(\varphi_\varepsilon^{(k)}, P_\varepsilon^{(k)})$, $(\varphi_0^{(k)}, P_0^{(k)})$ are uniformly bounded in $W^{1,q_\varphi}(\Omega) \times L^{q_P}(\Omega; \mathbb{R}^{d\times d})$. By the same arguments as in Chapter 2 we can extract (non-relabelled) convergent subsequences

$$\begin{aligned} z_\varepsilon^{(k)} &\to \hat{z}_\varepsilon \quad \text{in } L^1(\Omega), \\ \varphi_\varepsilon^{(k)} &\rightharpoonup \hat{\varphi}_\varepsilon \quad \text{in } W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d), \\ P_\varepsilon^{(k)} &\rightharpoonup \hat{P}_\varepsilon \quad \text{in } A_{\det}^{q_P,q_C}(\Omega), \\ z_0^{(k)} &\to \hat{z}_0 \quad \text{in } L^1(\Omega), \\ \varphi_0^{(k)} &\rightharpoonup \hat{\varphi}_0 \quad \text{in } W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d), \\ P_0^{(k)} &\rightharpoonup \hat{P}_0 \quad \text{in } A_{\det}^{q_P,q_C}(\Omega). \end{aligned}$$

We found suitable minimizing sequences but we still have to check that $(\hat{\varphi}_\varepsilon, \hat{P}_\varepsilon) \in \operatorname{Arg\,min} \mathcal{E}(z,\cdot,\cdot)$ and $(\hat{\varphi}_0, \hat{P}_0) \in \operatorname{Arg\,min} \mathcal{E}(z,\cdot,\cdot)$. First, we need to show pointwise convergence of F.

Lemma 3.2.1. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain and let F be given as in (2.1) such that there exists a Carathéodory function*

$$\widetilde{F} : \Omega \times [0,1] \times \mathbb{R}^{\binom{2d}{d}-1} \times \mathbb{R}^{\binom{2d}{d}-1} \times \mathbb{R}^{\overbrace{d \times \ldots \times d}^{(d-1)-times}} \to \mathbb{R} \cup \{\infty\}$$

with

$$F(x, z, F_{el}, P, G) = \widetilde{F}(x, z, M(F_{el}), M(P), G).$$

If $(z^{(k)})$ is uniformly bounded in $H^1(\Omega; [0,1])$ or $BV(\Omega; \{0,1\})$ then there exists a convergent subsequence $(z^{(k_n)}) \subset (z^{(k)}) \to \hat{z}$ in $L^1(\Omega)$ such that

$$\lim_{n\to\infty} F(x, z^{(k_n)}, F_{el}, P, G) = F(x, \hat{z}, F_{el}, P, G)$$

for a.e. $x \in \Omega$ and for all $(F_{el}, P, G) \in \mathbb{R}^{d\times d} \times SL(d) \times \mathbb{R}^{\overbrace{d \times \ldots \times d}^{(d-1)-times}}$.

Proof. For $z^{(k)} \in H^1(\Omega;[0,1])$ we obtain boundedness of $z^{(k)}$ in $W^{1,1}(\Omega;[0,1])$ using the Hölder inequality. Now since $p < d$ we can apply Rellich-Kondrachov (see B.0.1) to get strong convergence of $z^{(k)}$ in $L^1(\Omega;[0,1])$. For $z^{(k)} \in BV(\Omega;\{0,1\})$ we can immediately use compactness of BV functions (see B.0.2). Extracting a (non-relabelled) subsequence we find that $z^{(k)} \to \hat{z}$ pointwise almost everywhere. But $\widetilde{F}$ is a Carathéodory function and therefore continuous in z which means that

$$\widetilde{F}(x, z^{(k)}, M(F_{el}), M(P), G) \to \widetilde{F}(x, \hat{z}, M(F_{el}), M(P), G)$$

for almost every $x \in \Omega$. Consequently the same holds for F. $\square$

Using this lemma together with the Dominated Convergence Theorem we are able to show our claim but need to require an additional condition on F.

Proposition 3.2.2. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain and F and $\widetilde{F}$ as before such that there exist $c, C > 0$ and $g, h \in L^1(\Omega)$ with*

$$F \geq c(|F_{el}|^{q_F} + |P|^{q_P} + |P^{-1}|^{q_P} + |G|^{q_G}) - g(x),$$
$$F \leq C(|F_{el}|^{q_F} + |P|^{q_P} + |P^{-1}|^{q_P} + |G|^{q_G}) + h(x).$$

Then, for any uniformly bounded sequence $(z^{(k)}) \subset H^1(\Omega;[0,1])$ or $(z^{(k)}) \subset BV(\Omega;\{0,1\})$ there exists a sequence $(\varphi^{(k)}, P^{(k)}) \in \operatorname{Arg\,min} \mathcal{E}(z^{(k)}, \cdot, \cdot)$ with

$$\varphi^{(k)} \rightharpoonup \hat{\varphi} \quad \text{in } W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d)$$
$$P^{(k)} \rightharpoonup \hat{P} \quad \text{in } A_{\det}^{q_P,q_C}(\Omega)$$

such that $(\hat{\varphi}, \hat{P}) \in \operatorname{Arg\,min} \mathcal{E}(z, \cdot, \cdot)$.

Proof. We still have to check the last claim, i.e., $(\hat{\varphi}, \hat{P}) \in \operatorname{Arg\,min} \mathcal{E}(z, \cdot, \cdot)$. Since we chose $(\varphi^{(k)}, P^{(k)}) \in \operatorname{Arg\,min} \mathcal{E}(z^{(k)}, \cdot, \cdot)$ we get that for all $(\varphi, P) \in W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$ it holds that

$$\mathcal{E}(z^{(k)}, \varphi^{(k)}, P^{(k)}) \leq \mathcal{E}(z^{(k)}, \varphi, P)$$
$$\liminf_{k\to\infty} \mathcal{E}(z^{(k)}, \varphi^{(k)}, P^{(k)}) \leq \liminf_{k\to\infty} \mathcal{E}(z^{(k)}, \varphi, P).$$

We pick a suitable (non-relabelled) subsequence and apply Theorem 2.3.13 and Theorem 2.4.1 on the LHS to conclude that

$$\mathcal{E}(\hat{z}, \hat{\varphi}, \hat{P}) \leq \lim_{k\to\infty} \mathcal{E}(z^{(k)}, \varphi, P).$$

Now we can use the assumption that F is bounded by an integrable function. Furthermore, by Lemma 3.2.1, we know that $F(x, z^{(k)}, \nabla\varphi P^{-1}, P, G)$ converges pointwise a.e. to $F(x, \hat{z}, \nabla\varphi P^{-1}, P, G)$. By the Dominated Convergence Theorem A.0.5 we get

$$\begin{aligned}\lim_{k\to\infty} H(z^{(k)}, \varphi, P) &= \lim_{k\to\infty} \int_\Omega F(x, z^{(k)}, \nabla\varphi P^{-1}, P, G)\, \mathrm{d}x \\ &= \int_\Omega F(x, \hat{z}, \nabla\varphi P^{-1}, P, G)\, \mathrm{d}x = H(\hat{z}, \varphi, P).\end{aligned}$$

Since U is continuous in z, we obtain

$$\lim_{k\to\infty} \mathcal{E}(z^{(k)}, \varphi, P) = \mathcal{E}(\hat{z}, \varphi, P)$$

All in all, we have

$$\mathcal{E}(\hat{z}, \hat{\varphi}, \hat{P}) \leq \mathcal{E}(\hat{z}, \varphi, P)$$

for all $(\varphi, P) \in W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$, which finishes the proof. □

As in Chapter 2 we want to show lower semi-continuity of the target functionals J_ε and J_0. The continuity of K follows from the fact that $K = -U$ and we have already shown continuity of U. We are left with showing lower semi-continuity of E_ε and E_0

3.3 Lower semi-continuity of E_ε and E_0

First, we want to show weak lower semi-continuity for E_ε. Recall that

$$E_\varepsilon(z) = \int_\Omega \frac{\varepsilon}{2}|\nabla z|^2 + \frac{1}{\varepsilon}\psi(z)\, \mathrm{d}x.$$

Theorem 3.3.1. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain. Then E_ε is weakly lower semi-continuous on $H^1(\Omega; [0,1])$.*

Proof. Let

$$z^{(k)} \rightharpoonup \hat{z} \quad \text{in } H^1(\Omega; [0,1])$$

As in the proof of Lemma 3.2.1 this implies that $z^{(k)}$ converges to $\hat{z}$ pointwise a.e. in Ω. Now since ψ is continuous we also know that $\psi(z^{(k)}) \to \psi(\hat{z})$ pointwise a.e. in Ω. On the other hand,

$$\nabla z^{(k)} \rightharpoonup \nabla \hat{z} \quad \text{in } L^2(\Omega; \mathbb{R}^d)$$

We can apply Fatou's lemma (see A.0.3) and we get

$$\int_\Omega \frac{\varepsilon}{2}|\nabla \hat{z}|^2 + \frac{1}{\varepsilon}\psi(\hat{z})\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega \frac{\varepsilon}{2}|\nabla z^{(k)}|^2 + \frac{1}{\varepsilon}\psi(z^{(k)})\,\mathrm{d}x,$$

which finishes the proof. $\square$

Next, we want to prove lower semi-continuity for

$$E_0(z) = c_\psi \mathrm{Per}(\{z=1\}, \Omega),$$

i.e., we need lower semi-continuity of the perimeter

$$\mathrm{Per}(\{z=1\}, \Omega) = \sup\left\{ \int_\Omega \mathbb{1}_{\{z=1\}}(x)\ \nabla \cdot \phi\ \mathrm{d}x : \phi \in C_c^1(\Omega; R^d), \|\phi\|_\infty \leq 1 \right\}.$$

Theorem 3.3.2. *Let $\Omega \subset \mathbb{R}^d$ be open and let $z^{(k)} \in BV(\Omega; \{0,1\})$ and $\hat{z} \in BV(\Omega; \{0,1\})$ such that*

$$z^{(k)} \to \hat{z} \quad \textit{in } L^1(\Omega; \{0,1\}).$$

Then,

$$\mathrm{Per}(\{\hat{z}=1\}, \Omega) \leq \liminf_{k\to\infty} \mathrm{Per}(\{z^{(k)}=1\}, \Omega).$$

Proof. First of all, note that

$$z^{(k)}(x) = \mathbb{1}_{\{z^{(k)}=1\}}(x) \quad \text{and} \quad \hat{z}(x) = \mathbb{1}_{\{\hat{z}=1\}}(x)$$

and hence

$$\mathbb{1}_{\{z^{(k)}=1\}} \to \mathbb{1}_{\{\hat{z}=1\}} \quad \text{in } L^1(\Omega; \{0,1\}).$$

In other words this means that for all $\phi \in C_0^1(\Omega; \mathbb{R}^d)$ with $\|\phi\|_\infty \leq 1$ we have

$$\int_\Omega \mathbb{1}_{\{\hat{z}=1\}}(x)\ \nabla \cdot \phi(x)\ \mathrm{d}x = \lim_{k\to\infty} \int_\Omega \mathbb{1}_{\{z^{(k)}=1\}}(x)\ \nabla \cdot \phi(x)\ \mathrm{d}x.$$

Taking the supremum for the term on the right hand side we get

$$\int_\Omega \mathbb{1}_{\{\hat{z}=1\}}(x)\ \nabla \cdot \phi(x)\ \mathrm{d}x \leq \liminf_{k\to\infty} \mathrm{Per}(\{z^{(k)} = 1\}, \Omega).$$

And now we can take the supremum on the left hand side too and obtain

$$\mathrm{Per}(\{\hat{z} = 1\}, \Omega) \leq \liminf_{k\to\infty} \mathrm{Per}(\{z^{(k)} = 1\}, \Omega)$$

which shows lower semi-continuity. □

Finally we can analyze the existence of minimizers for J_ε and J_0.

3.4 Existence of minimizers

We start by showing the existence of

$$\min_{z\in H^1(\Omega;[0,1])} \{J_\varepsilon(z, \varphi, P) : (\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z, \cdot, \cdot)\}. \tag{3.4}$$

Theorem 3.4.1. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain and let H and F be given as in (2.1) and J_ε as in (3.1). Assume that the conditions 1.-4. from Theorem 2.5.1 hold with the additional condition that there exists $C > 0$ and $h \in L^1(\Omega)$ with*

$$F \leq C(|F_{el}|^{q_F} + |P|^{q_P} + |P^{-1}|^{q_P} + |G|^{q_G}) + h(x)$$

Furthermore, let K be bounded from below and assume that $f \in L^{q_f}(\Omega)$ for some $q_f > (q_\varphi)'$ and $g \in L^{q_g}(\partial\Omega)$ for some $q_g = (q_\varphi)'$. Moreover, assume that there exists $(z, \varphi, P) \in H^1(\Omega; [0,1]) \times W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$ with $(\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z, \cdot, \cdot)$ such that $J_\varepsilon(z, \varphi, P) < \infty$.

Then, (3.4) admits a solution.

Proof. First, note that all the assumptions of Theorem 2.5.1 are satisfied which means that for every $z \in H^1(\Omega; [0,1])$ we can find
$(\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z, \cdot, \cdot)$ with $\varphi \in W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d)$ and $P \in A_{\det}^{q_P,q_C}(\Omega)$ such that $J_\varepsilon(z, \varphi, P) < \infty$. Since K is bounded from below we also know that J_ε is bounded from below. Using the arguments from Chapter 3.2 can find a minimizing sequence $z^{(k)} \in H^1(\Omega; [0,1])$ that is uniformly bounded, because

$$\begin{aligned}\lim_{k\to\infty} J_\varepsilon(z^{(k)}, \varphi^{(k)}, P^{(k)}) &= \inf_{z\in H^1(\Omega;[0,1])} \{J_\varepsilon(z, \varphi, P) : (\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z, \cdot, \cdot)\} \\ &=: m_\varepsilon, \\ J_\varepsilon(z^{(k)}, \varphi^{(k)}, P^{(k)}) &\leq J_\varepsilon(z^{(1)}, \varphi^{(1)}, P^{(1)}) < \infty.\end{aligned}$$

Additionally, the sequences $\varphi^{(k)} \in W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d)$ and $P^{(k)} \in A_{\det}^{q_P,q_C}(\Omega)$ with $(\varphi^{(k)}, P^{(k)}) \in \operatorname{Arg\,min} H(z^{(k)}, \cdot, \cdot)$ are uniformly bounded, which implies that we can find a weakly convergent subsequence

$$\begin{aligned} z^{(k)} &\rightharpoonup \hat{z} \quad \text{in } H^1(\Omega), \\ \varphi^{(k)} &\rightharpoonup \hat{\varphi} \quad \text{in } W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d), \\ P^{(k)} &\rightharpoonup \hat{P} \quad \text{in } A_{\det}^{q_P,q_C}(\Omega). \end{aligned}$$

Since the assumptions of Proposition 3.2.2 are satisfied we know that $(\hat{\varphi}, \hat{P}) \in \operatorname{Arg\,min} H(z, \cdot, \cdot)$.

Note that $z^{(k)} \to \hat{z}$ in $L^{q_z}(\Omega, [0,1])$ for all $q_z < \infty$ because $|z^{(k)}| \leq 1$ almost everywhere. Hence, we know that K is continuous and E_ε is lower semi-continuous which implies that also J_ε is lower semi-continuous. We get

$$m_\varepsilon \leq J_\varepsilon(\hat{z}, \hat{\varphi}, \hat{P}) \leq \liminf_{k\to\infty} J_\varepsilon(z^{(k)}, \varphi^{(k)}, P^{(k)}) = m_\varepsilon.$$

So $(\hat{z}, \hat{\varphi}, \hat{P})$ is a minimizer. $\square$

Next we can also show the existence of

$$\min_{z\in BV(\Omega;\{0,1\})} \{J_0(z, \varphi, P) : (\varphi, P) \in \operatorname{Arg\,min} H(z, \cdot, \cdot)\}. \tag{3.5}$$

Theorem 3.4.2. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain and let H and F be given as in (2.1) and J_0 as in (3.2). Assume that the conditions 1.-4. from Theorem 2.5.1 hold with the additional condition that there exists $C > 0$ and $h \in L^1(\Omega)$ with*

$$F \leq C(|F_{el}|^{q_F} + |P|^{q_P} + |P^{-1}|^{q_P} + |G|^{q_G}) + h(x)$$

Furthermore, let K be bounded from below and assume that $f \in L^{q_f}(\Omega)$ for some $q_f > (q_\varphi)'$ and $g \in L^{q_g}(\partial\Omega)$ for some $q_g > (q_\varphi)'$. Moreover, assume that there exists $(z, \varphi, P) \in BV(\Omega; \{0,1\}) \times W_D^{1,q_\varphi}(\Omega; \mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$ with $(\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z, \cdot, \cdot)$ such that $J_0(z, \varphi, P) < \infty$.

Then, (3.5) admits a solution.

Proof. Using the same arguments as before we get a minimizing sequence $z^{(k)}$ that converges in $L^1(\Omega; \{0,1\})$ to some $\hat{z} \in BV(\Omega; \{0,1\})$. Here, we can again apply the lower semi-continuity result for the perimeter and get our desired result. $\square$

Thus we have shown that both the phase-field problem and the sharp-interface problem are solvable. Our final goal is to check that the former one actually converges to the latter one. This needs additional requirements and is the topic of the next chapter.

Chapter 4

Γ-Convergence

In this section we are going to prove that the phase-field problem converges to the sharp-interface problem as $\varepsilon \to 0$. However, this does not hold in full generality and we require a uniqueness assumption. Moreover, we need to define the kind of variational convergence we are interested in, the so-called Γ-convergence.
This chapter follows [PRW12], as well as [Alb00].

4.1 Γ-convergence

Γ-convergence plays an important role in the Calculus of Variations and was first introduced by De Giorgi in 1975 in [DG75].

Definition 4.1.1. Let (X, d) be a metric space and let $(\mathcal{F}_n)$ be a sequence of functionals $\mathcal{F}_n : X \to [-\infty, \infty]$. We say that $(\mathcal{F}_n)$ converges to the Γ-limit $\mathcal{F} : X \to [-\infty, \infty]$ if the following two properties are satisfied:

(LB) for every $x \in X$ and every sequence $(x_n) \subset X$ with $x_n \to x$ it holds that

$$\mathcal{F}(x) \leq \liminf_{n\to\infty} \mathcal{F}_n(x_n), \tag{4.1}$$

(UB) for every $x \in X$ there exists a sequence $(x_n) \subset X$ with $x_n \to x$ such that

$$\mathcal{F}(x) \geq \limsup_{n\to\infty} \mathcal{F}_n(x_n). \tag{4.2}$$

The lower bound inequality (LB) ensures that the Γ-limit is never larger than the limit along an approximating sequence, while the upper bound inequality (UB) provides the existence of a sequence that certainly approximates the Γ-limit. This sequence is sometimes called recovery sequence. As a matter of fact, under condition (LB), we could exchange property (UB) with the following condition:

(UB*) For every $x \in X$ there exists a sequence $(x_n) \subset X$ with $x_n \to x$ such that

$$\mathcal{F}(x) = \lim_{n\to\infty} \mathcal{F}_n(x_n). \tag{4.3}$$

To simplify the notation we will write $\mathcal{F}_n \xrightarrow{\Gamma} \mathcal{F}$, if $\mathcal{F}_n$ converges to the Γ-limit $\mathcal{F}$. Furthermore, we are going to use the parameter $\varepsilon > 0$ instead of $n \in \mathbb{N}$ and write $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}$ as $\varepsilon \to 0$. These notions are equivalent, as we can set $\varepsilon_n := \frac{1}{n}$.

The importance of Γ-convergence is clarified by the following proposition, pointing out two useful properties.

Proposition 4.1.2.

1. *If $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}$ and $\mathcal{G}$ is continuous then $\mathcal{F}_\varepsilon + \mathcal{G} \xrightarrow{\Gamma} \mathcal{F} + \mathcal{G}$.*
2. *If $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}$ and x_ε are minimizers for $\mathcal{F}_\varepsilon$ over X, then every limit point of x_ε minimizes $\mathcal{F}$ over X.*

Proof.

1. One can easily check that $\mathcal{G}$ satisfies both conditions (LB) and (UB) for every sequence $x_\varepsilon \to x$. Hence by the properties of lim inf and lim sup we get

$$\liminf_{\varepsilon\to 0}(\mathcal{F}_\varepsilon + \mathcal{G})(x_\varepsilon) \geq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) + \liminf_{\varepsilon\to 0} \mathcal{G}(x_\varepsilon) \geq (\mathcal{F} + \mathcal{G})(x),$$
$$\limsup_{\varepsilon\to 0}(\mathcal{F}_\varepsilon + \mathcal{G})(y_\varepsilon) \leq \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(y_\varepsilon) + \limsup_{\varepsilon\to 0} \mathcal{G}(y_\varepsilon) \leq (\mathcal{F} + \mathcal{G})(y),$$

where (y_ε) is the recovery sequence for an arbitrary $y \in X$. That means $\mathcal{F} + \mathcal{G}$ is the Γ-limit of $\mathcal{F}_\varepsilon + \mathcal{G}$.

2. Suppose that $x \in X$ is a limit point of the minimizing sequence (x_ε), i.e. $x_\varepsilon \to x$ and $\min \mathcal{F}_\varepsilon = \mathcal{F}(x_\varepsilon)$. Take $y \in X$ and let (y_ε) be the recovery sequence of y. Then, we get

$$\mathcal{F}(y) = \lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(y_\varepsilon) \geq \liminf_{\varepsilon \to 0} \min \mathcal{F}_\varepsilon = \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(x_\varepsilon) \geq \mathcal{F}(x).$$

Since $y \in X$ was arbitrary we conclude that x minimizes $\mathcal{F}$ over X. $\square$

4.2 Modica-Mortola Theorem

Our next goal is to prove that J_0 is in fact the Γ-limit of J_ε. First, we only consider the order parameter z. In general, we choose $L^1(\Omega)$ as our metric space with its standard metric. Therefore, we have to adapt the functionals E_ε and E_0 in such a way that they are defined on $L^1(\Omega; [0,1])$. We have

$$E_\varepsilon(z) = \begin{cases} \int_\Omega (\frac{\varepsilon}{2} |\nabla z|^2 + \frac{1}{\varepsilon}\psi(z)) \, \mathrm{d}x & \text{if } z \in H^1(\Omega; [0,1]), \\ \infty & \text{if } z \in L^1(\Omega; [0,1]) \setminus H^1(\Omega; [0,1]), \end{cases} \tag{4.4}$$

$$E_0(z) = \begin{cases} c_\psi \mathrm{Per}(\{z = 1\}, \Omega) & \text{if } z \in BV(\Omega; \{0,1\}), \\ \infty & \text{if } z \in L^1(\Omega; [0,1]) \setminus BV(\Omega; \{0,1\}). \end{cases} \tag{4.5}$$

Moreover, the double well-potential ψ satisfies the following two conditions:

1. ψ is continuous and $\psi(z) = 0$ if and only if $z \in \{0,1\}$.

2. There exists $M > 0$ and $\bar{z} > 0$ such that

$$\psi(z) \geq M|z|,$$

for all $z \geq \bar{z}$.

With this notation one gets the following theorem, which was first proved in 1977 in [MM77].

Theorem 4.2.1 (Modica-Mortola). *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain. Then, E_0 is the Γ-limit of E_ε, i.e.,*

(LB) If $(z_\varepsilon) \subset H^1(\Omega; [0,1])$ and $z \in BV(\Omega; \{0,1\})$ with $z_\varepsilon \to z$ in $L^1(\Omega)$ then

$$E_0(z) \leq \liminf_{\varepsilon \to 0} E_\varepsilon(z_\varepsilon).$$

(UB) For every $z \in BV(\Omega; \{0,1\})$ there exists $(z_\varepsilon) \subset H^1(\Omega; [0,1])$ with $z_\varepsilon \to z$ in $L^1(\Omega)$ such that

$$E_0(z) \geq \limsup_{\varepsilon \to 0} E_\varepsilon(z_\varepsilon).$$

For a proof we refer to [Mod87]. Also note that, Braides showed this theorem in [Bra02, ch. 02] via a slicing argument.

We would like to apply the Modica-Mortola Theorem, but have to be careful, since we have the extra condition that $(\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z, \cdot, \cdot)$ and these configurations (φ, P) are not necessarily unique.

4.3 Convergence of the two models

Using the definitions from equations (4.4) and (4.5), we can write the phase-field problem as

$$\min_{z \in L^1(\Omega;[0,1])} \{J_\varepsilon(z, \varphi, P) : (\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z, \cdot, \cdot)\}, \tag{4.6}$$

and the sharp-interface problem as

$$\min_{z \in L^1(\Omega;\{0,1\})} \{J_0(z, \varphi, P) : (\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z, \cdot, \cdot)\}. \tag{4.7}$$

We get the following theorem.

Theorem 4.3.1. *If the minimizing deformation $\varphi \in W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d)$ is unique for any $z \in L^1(\Omega;[0,1])$, i.e., $\exists\ \varphi \in W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d)$ such that $\hat{\varphi} = \varphi$ for all $(\hat{\varphi}, P) \in \operatorname{Arg\,min}\ \mathcal{E}(z,\cdot,\cdot)$, then the sharp-interface problem is the Γ-limit of the phase-field problem. In other words:*

(LB) If $(z_\varepsilon) \subset H^1(\Omega;[0,1])$ and $z \in BV(\Omega;\{0,1\})$, such that $(\varphi_\varepsilon, P_\varepsilon) \in \operatorname{Arg\,min}\ \mathcal{E}(z_\varepsilon,\cdot,\cdot)$ and $(\varphi, P) \in \operatorname{Arg\,min}\ \mathcal{E}(z,\cdot,\cdot)$ with

$$\begin{aligned} z_\varepsilon &\to z \quad \text{in } L^1(\Omega),\\ \varphi_\varepsilon &\rightharpoonup \varphi \quad \text{in } W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d),\\ P_\varepsilon &\rightharpoonup P \quad \text{in } A_{\det}^{q_P,q_C}(\Omega). \end{aligned}$$

Then,

$$J_0(z,\varphi,P) \le \liminf_{\varepsilon\to 0} J_\varepsilon(z_\varepsilon,\varphi_\varepsilon,P_\varepsilon).$$

(UB) For every $z \in BV(\Omega;\{0,1\})$, such that $(\varphi, P) \in \operatorname{Arg\,min}\ \mathcal{E}(z,\cdot,\cdot)$, there exists a sequence $(z_\varepsilon) \subset H^1(\Omega;[0,1])$, such that $(\varphi_\varepsilon, P_\varepsilon) \in \operatorname{Arg\,min}\ \mathcal{E}(z_\varepsilon,\cdot,\cdot)$ with

$$\begin{aligned} z_\varepsilon &\to z \quad \text{in } L^1(\Omega),\\ \varphi_\varepsilon &\rightharpoonup \varphi \quad \text{in } W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d), \end{aligned}$$

and such that

$$J_0(z,\varphi,P) \ge \limsup_{\varepsilon\to 0} J_\varepsilon(z_\varepsilon,\varphi_\varepsilon,P_\varepsilon).$$

Proof. For the lower bound *(LB)* we can use the Modica-Mortola Theorem to obtain

$$E_0(z) \le \liminf_{\varepsilon\to 0} E_\varepsilon(z_\varepsilon).$$

Now, since K is continuous, we have

$$K(z,\varphi,P) = \lim_{\varepsilon\to 0} K(z_\varepsilon,\varphi_\varepsilon,P_\varepsilon).$$

It follows that

$$\begin{aligned} J_0(z,\varphi,P) = E_0(z) + K(z,\varphi,P) &\le \liminf_{\varepsilon\to 0} E_\varepsilon(z_\varepsilon) + \lim_{\varepsilon\to 0} K(z_\varepsilon,\varphi_\varepsilon,P_\varepsilon)\\ &\le \liminf_{\varepsilon\to 0} (E_\varepsilon(z_\varepsilon) + K(z_\varepsilon,\varphi_\varepsilon,P_\varepsilon)) = \liminf_{\varepsilon\to 0} J_\varepsilon(z_\varepsilon,\varphi_\varepsilon,P_\varepsilon). \end{aligned}$$

The upper bound *(UB)* makes use of the Modica-Mortola Theorem as well. In particular, for every $z \in BV(\Omega; \{0,1\})$ with $(\varphi, P) \in \operatorname{Arg\,min} \mathcal{E}(z,\cdot,\cdot)$ and $(\varphi, P) \in W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d) \times A_{\det}^{q_P,q_C}(\Omega)$, we can find a recovery sequence $(z_\varepsilon) \subset H^1(\Omega;[0,1])$ with $z_\varepsilon \to z$ in $L^1(\Omega)$ such that

$$E_0(z) = \lim_{\varepsilon\to 0} E_\varepsilon(z_\varepsilon).$$

By Proposition 3.2.2, there exists a sequence $(\varphi_\varepsilon, P_\varepsilon) \in \operatorname{Arg\,min} \mathcal{E}(z_\varepsilon,\cdot,\cdot)$ with

$$\begin{aligned} \varphi_\varepsilon &\rightharpoonup \hat{\varphi} \quad \text{in } W_D^{1,q_\varphi}(\Omega;\mathbb{R}^d), \\ P_\varepsilon &\rightharpoonup \hat{P} \quad \text{in } A_{\det}^{q_P,q_C}(\Omega), \end{aligned}$$

such that $(\hat{\varphi}, \hat{P}) \in \operatorname{Arg\,min} \mathcal{E}(z,\cdot,\cdot)$. Using the uniqueness of the minimizing deformation, we get $\varphi = \hat{\varphi}$. Additionally, since K is continuous, we have

$$\lim_{\varepsilon\to 0} K(z_\varepsilon, \varphi_\varepsilon, P_\varepsilon) = K(z, \hat{\varphi}, \hat{P}) = K(z, \varphi, P).$$

All in all, we obtain

$$\begin{aligned} J_0(z,\varphi,P) = E_0(z) + K(z,\varphi,P) &= \lim_{\varepsilon\to 0} E_\varepsilon(z_\varepsilon) + \lim_{\varepsilon\to 0} K(z_\varepsilon,\varphi_\varepsilon,P_\varepsilon) \\ &= \lim_{\varepsilon\to 0}(E_\varepsilon(z_\varepsilon) + K(z_\varepsilon,\varphi_\varepsilon,P_\varepsilon)) = \lim_{\varepsilon\to 0} J_\varepsilon(z_\varepsilon,\varphi_\varepsilon,P_\varepsilon), \end{aligned}$$

which is what we wanted to show. □

Remark 4.3.2. If the minimizing plastic part P is also unique, then $P = \hat{P}$ and we get weak convergence

$$P_\varepsilon \rightharpoonup P \quad \text{in } A_{\det}^{q_P,q_C}(\Omega).$$

Remark 4.3.3. Note that, in general, we cannot expect uniqueness of the minimizing deformation φ. Still, if one would identify a setting where uniqueness of the minimizing deformation can be guaranteed a priori, then this would lead to a complete Γ-convergence result.

Appendix A

Convergence Theorems

Theorem A.0.1 (Banach-Alaouglu theorem)**.** *Let X be a separable Banach space and X^* denote its dual, then the closed unit ball*

$$B := \{f \in X^* : \|f\|_{X^*} \leq 1\}$$

is weak-compact.*

For a proof see [Con85, thm. 3.1, p. 130]. A consequence of this theorem is the following lemma.

Lemma A.0.2. *If X is a reflexive Banach space, then every bounded sequence in X has a weakly convergent subsequence.*

Theorem A.0.3 (Fatou's Lemma)**.** *Let (Ω, Σ, μ) be a measure space and (f_n) a sequence of non-negative measurable functions on (Ω, Σ, μ).*
If f_n converges pointwise a.e. on Ω to some f then f is integrable over Ω and

$$\int_\Omega f \, \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu.$$

See [Roy88, 10., p. 86].

Theorem A.0.4 (Monotone Convergence Theorem)**.** *Let (Ω, Σ, μ) be a measure space and (f_n) an increasing sequence of non-negative measurable functions on (Ω, Σ, μ).*
If f_n converges pointwise a.e. on Ω to some f then f is integrable over Ω and

$$\lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu.$$

The theorem is shown in [Roy88, 10., p. 87].

Theorem A.0.5 (Dominated Convergence Theorem)**.** *Let (Ω, Σ, μ) be a measure space and (f_n) a sequence of measurable functions on (Ω, Σ, μ) such that there exists an integrable function g over Ω with $|f_n| \leq g$ for all $n \in \mathbb{N}$.*
If f_n converges pointwise a.e. on Ω to some f then f is integrable over Ω and

$$\lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu.$$

A proof is given in [Roy88, 16., p. 91].

Appendix B

Continuous Embeddings

Theorem B.0.1 (Rellich-Kondrachov)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain and let $d > p$. Then the embedding*

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact for $q < \frac{dp}{d-p}$, i.e. every bounded sequence in $W^{1,p}(\Omega)$ has a convergent subsequence in $L^q(\Omega)$.

See [DD12, thm. 2.80, p. 96] for more details.

Theorem B.0.2 (Compactness of BV functions)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain. Then every uniformly bounded sequence in $BV(\Omega)$ has a convergent subsequence in $L^1(\Omega)$ such that the limit lies in $BV(\Omega)$.*

We refer to [Giu84, thm. 1.19, p. 17] for a proof to this statement.

Theorem B.0.3 (Trace operator)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain and $1 < p < \infty$. Then, there exists a continuous linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

1. $T(u) = u|_{\partial\Omega}$ *for all* $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$,
2. $\|T(u)\|_{L^p(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$,

with $C > 0$ a constant depending on p and Ω.

See [Leo09, thm. 15.23, p. 473].

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Zusammenfassung

Wir behandeln das Thema Topologie Optimierung eines elastoplastischen Objekts, welches von äußeren Kräften beeinflusst wird und zeigen, dass es eine optimale Konfiguration des Modells mit eindeutiger Grenzschicht und des Modells mit Phasenübergang gibt, wo die Dichte des Materials stetig ist.

Zuerst präsentieren wir den Aufbau des Modells und modellieren die Bewegung des Objekts. Dabei wird das elastoplastische Verhalten durch die Minimierung eines Energie Funktionals beschrieben. Deshalb beweisen wir, mit Hilfe der *Direkten Methode der Variationsrechnung*, dass ein solcher Minimierer existiert. Anschließend beweisen wir, dass sowohl für das Problem mit eindeutiger Grenzschicht, als auch für das Problem mit Phasenübergang eine Lösung existiert. Schlussendlich zeigen wir, dass unter geeigneten Bedingungen das Modell mit Phasenübergangen, bezüglich Γ-Konvergenz, gegen das Modell mit eindeutiger Grenzschicht konvergiert.