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Zhangwen Guo, BSc

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Abstract

As a contact structure can be equivalently viewed as a filtered manifold whose symbol algebra is isomorphic to the Heisenberg algebra in each point, there is a natural frame bundle associated to a contact structure, and a Lagrangean contact structure can be viewed as a reduction of such a frame bundle. We encode the latter as a filtered G-structure, thus obtain an equivalent description of Lagrangean contact structures.

Moreover, we will extend such a filtered G-structure to a canonical Cartan geometry, which is parallel to the construction of a canonical Cartan connection associated to a CR structure due to Tanaka [6] and Chern-Moser [3]. In particular, we obtain an upper bound of the dimension of the automorphism group of a Lagrangean contact structure.

The thesis also includes as an easier analogy to the construction on Lagrangean contact structures an equivalent description of a Riemannian manifold as a G-structure, coming from the orthonormal frame bundle, and as a canonical Cartan geometry, coming from the G-structure and the Levi-Civita connection.

Abstract

Eine Kontaktstruktur kann als fitrierte Mannigfaltigkeit interpretiert werden, deren Symbolalgebra in jedem Punkt isomorph zur Heisenberg-Algebra ist. Dadurch kann man für eine Kontaktsruktur ein natürliches Rahmenbündel konstruieren. Eine Lagrange-Kontaktstrukture kann dann äquivalent als Reduktion dieses Rahmenbündels beschrieben werden. Diese Beschreibung von Lagrange-Kontaktstrukturen ist ein filtriertes Analogon zum klassischen Konzept einer G-Struktur.

Als nächsten Schritt erweitern wir diese filtrierte G-Struktur zu einer kanonischen Cartan Geometrie, was analog zu den Resultaten von Tanaka [6] und Chern-Moser [3] über die Existenz von kanonischen Cartan Konnexionen für CR-strukturen ist. Insbesondere liefert das eine obere Schranke and die Dimension der Automorphismengruppe einer Lagrange-Kontaktstruktur.

Als Motivation für den Fall von Lagrange-Kontaktstrukturen werden in der Masterabeit auch die (viel einfacheren) analogen Konstruktionen für Riemann Mannigfaltigkeiten besprochen. Über das orthonormale Rahmenbündel kann man eine Riemann Metrik äquivalent als G-Struktur beschreiben. Die Levi-Civita Konnexion macht diese G-Strukture zu einer kanonischen Cartan Geometrie, die eine äquivalente Beschreibung der Riemann Metrik liefert.

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Contents

Chapter 1

Introduction

Recall that a symplectic manifold is an even dimensional manifold M^{2n} equipped with a closed nondegenerate differential 2-form $\omega \in \Omega^2(M)$. Thus $(T_xM, \omega(x))$ at each $x \in M$ is a symplectic vector space, and so they are all isomorphic. Moreover, a theorem by Darboux implies that all symplectic manifolds of the same dimension are locally isomorphic, hence there are no local invariants on symplectic manifolds.

As an odd dimensional analogue, a contact structure on M^{2n+1} is a subbundle $H \subseteq TM$ of codimension 1 satisfying an additional condition. In particular, there is a natural partial 2-form $\mathcal{L} \in \Gamma(L(\Lambda^2 H, TM/H))$, called the Levi-bracket, such that at each $x \in M$, $(H_x, \mathcal{L}(x))$ is a symplectic vector space. Thus $(T_xM/H_x \oplus H_x, \mathcal{L}(x))$ at each $x \in M$ are all isomorphic in an obvious sense. Analogous to the Darboux's theorem, a theorem by Pfaff implies that all contact structures of the same dimension are locally isomorphic, hence there are no local invariants on contact structures either.

Observe that given a $2n$-dimensional symplectic vector space $(V, [\ ,\ ])$, there is a (non-unique) decomposition of V into two Lagrangean subspaces $V = V_1 \oplus V_2$. Recall that $W \subseteq V$ is a Lagrangean subspace if and only if $W^\perp = W$, or equivalently, W is an n-dimensional subspace such that $[\ ,\ ]$ vanishes on $W \times W$. Similarly a Lagrangean contact structure is defined as a contact structure $(M^{2n+1}, H \subseteq TM)$ together with a decomposition $H = E \oplus F$ to n-dimensional subbundles such that $\mathcal{L}(x)$ is trivial on $E_x \times E_x$ and on $F_x \times F_x$ for all $x \in M$.

It turns out that Lagrangean contact structures do have local invariants, similar to the curvature of a Riemannian manifold. In fact, the analogy to Riemannian manifolds goes much further and our aim is to obtain a similarly nice description of Lagrangean contact structures.

On an n-dimensional Riemannian manifold M, we refer to the full frame bundle of M as the bundle of all linear isomorphisms $\mathbb{R}^n \xrightarrow{\cong} T_xM$, $x \in M$, and refer to the orthonormal frame bundle of M as the bundle of all isometries $\mathbb{R}^n \xrightarrow{\cong} T_xM$, $x \in M$. We will see that the reduction of the full frame bundle to the orthonormal frame bundle is an equivalent encoding of a Riemannian metric. On the other hand, given a contact structure (M, H), $(H_x, \mathcal{L}(x))$ at each $x \in M$ is a symplectic vector space, hence there is a frame bundle associated to a contact structure. It turns out that a Lagrangean contact structure can be equivalently encoded as a reduction of structure group of that frame bundle. This yields an equivalence of the category of Riemannian manifolds (resp. Lagrangean contact structures) and the category of G-structures (resp. filtered G-structures) with a certain structure group.

As a fundamental result on Riemannian geometry, each Riemannian manifold has a unique Levi-Civita connection, which leads to the curvature of a Riemannian manifold. We will see that the Levi-Civita connection is induced by a principal connection on the orthonormal frame bundle. This yields the description of a Riemannian n-manifold as a normal Cartan geometry, which has the advantage of being formally very similar to the description of Euclidean space as a homogeneous space of the Euclidean group. Hence this Cartan geometry is of type $(Euc(n), O(n))$. Again, this yields a categorial equivalence. As a direct consequence, the automorphism group of a connected Riemannian n-manifold is a Lie group of dimension at most the dimension of $Euc(n)$. Similarly there is a description of a Lagrangean contact structure on a $(2n+1)$-dimensional manifold as a regular normal Cartan geometry, which is formally similar to the description of the canonical Lagrangean contact structure on the flag manifold $F_{1,n+1}(\mathbb{R}^{n+2})$ as a homogeneous space of $PGL(n+2, \mathbb{R})$. The construction of this canonical Cartan geometry and the resulting categorical equivalence is the main result of this thesis. This result is parallel to the famous construction of a canonical Cartan connection associated to a CR structure due to Tanaka [6] and Chern-Moser [3]. As a direct consequence, the automorphism group of a connected Lagrangean contact structure is a Lie group of dimension at most the dimension of $PGL(n + 2, \mathbb{R})$.

Overview of the text

In our parallel study of Riemannian geometry and of Lagrangean contact structures, chapter 2 corresponds to chapter 3, and chapter 4 corresponds to chapter 5.

In chapter 2, we describe orthonormal frame bundles of Riemannian n-manifolds as $O(n)$-structures of type $\mathbb{R}^n$ and establish the equivalence between the two categories.

In chapter 3, after introducing Lagrangean contact structures, we define the frame bundle of a Lagrangean contact structure on a $(2n+1)$-dimensional manifold, which is parallel to the orthonormal frame bundle of a Riemannian n-manifold. We describe these frame bundles as regular filtered G_0-structures of type $\mathfrak{g}_-$, which is parallel to $O(n)$-structures of type $\mathbb{R}^n$. Similarly we establish the equivalence between the category of Lagrangean contact structures on $(2n + 1)$-dimensional manifolds and the category of regular filtered G_0-structures of type $\mathfrak{g}_-$.

In chapter 4, we find that a Riemannian n-manifold together with the Levi-Civita connection can be described as an $O(n)$-structure of type $\mathbb{R}^n$ together with a certain principal connection on it. Generalizing from the homogeneous model, we describe $O(n)$-structures of type $\mathbb{R}^n$ together with canonical principal connections as a normal Cartan geometry of type $(Euc(n), O(n))$ and establish the equivalence between the category of $O(n)$-structures of type $\mathbb{R}^n$ and the category of normal Cartan geometries of type $(Euc(n), O(n))$.

In chapter 5, we generalize from the homogeneous model a functor from regular Cartan geometries of type (G, P) to regular filtered G_0-structures of type $\mathfrak{g}_-$. We define the normalising condition on regular Cartan geometries of type (G, P) parallel to the normalising condition on Cartan geometries of type $(Euc(n), O(n))$, thus establish the equivalence between the category of normal regular Cartan geometries of type (G, P) and the category of regular filtered G_0-structures of type $\mathfrak{g}_-$.

Throughout the text we assume that all manifolds are smooth and admits partitions of unity. We also assume that all representations are finite dimensional.

Chapter 2

Riemannian manifolds

As a motivation for parallel developments on Lagrangean contact structures, this chapter recalls the description of Riemannian manifolds as G-structures with structure group $O(n)$.

2.1 The orthonormal frame bundle

Let M be an n-dimensional manifold. We associate to M a natural frame bundle $GL(\mathbb{R}^n, TM)$ whose fiber over each $x \in M$ is the space $GL(\mathbb{R}^n, T_xM)$ of all linear isomorphisms $\mathbb{R}^n \xrightarrow{\cong} T_xM$. This is a principal bundle on M with structure group $GL(n, \mathbb{R})$. Its local sections $U \to GL(\mathbb{R}^n, TM|_U)$ on any open subset $U \subseteq M$ are exactly given by the local trivialisations $U \times \mathbb{R}^n \xrightarrow{\cong} TM|_U$, or equivalently, by local frames for TM defined on U.

Let g be a Riemannian metric on M. Then for each $x \in M$, $(T_xM, g(x))$ is isomorphic to the standard inner product space $(\mathbb{R}^n, \langle\ ,\ \rangle)$, thus we associate to (M, g) a natural frame bundle $O(\mathbb{R}^n, TM)$, whose fiber over each $x \in M$ is the space $O(\mathbb{R}^n, T_xM)$ of isometries $(\mathbb{R}^n, \langle\ ,\ \rangle) \xrightarrow{\cong} (T_xM, g(x))$. This is a principal subbundle of $GL(\mathbb{R}^n, TM)$ with structure group $O(n)$. Its local sections $U \to O(\mathbb{R}^n, TM|_U)$ on any open subset $U \subseteq M$ are exactly given by those local trivialisations $U \times \mathbb{R}^n \xrightarrow{\cong} TM|_U$ such that $\langle\ ,\ \rangle$ corresponds to g, which can be interpreted as the ordered orthonormal local frames on U of TM. We call $O(\mathbb{R}^n, TM)$ the *orthonormal frame bundle* of (M, g).

Proposition 2.1.1. *Let $\mathcal{G} \subseteq GL(\mathbb{R}^n, TM)$ be a principal subbundle with structure group $O(n)$. Then there is a unique Riemannian metric g on M such that $\mathcal{G}$ is the orthonormal frame bundle of (M, g).*

Proof. Given $u \in \mathcal{G}$ lying above $x \in M$, the inner product $g(x) : T_xM \times T_xM \to \mathbb{R}$ is uniquely determined by demanding $u : \mathbb{R}^n \to T_xM$ to be an isometry, thus for any tangent vectors $\xi, \eta \in T_xM$, $g(x)(\xi, \eta) = \langle u^{-1}(\xi), u^{-1}(\eta)\rangle$. The construction does not depend on the choice of $u \in \mathcal{G}_x$, because any element in $\mathcal{G}_x$ is of the form $u \circ A$ for some $A \in O(n)$, but $\langle\ ,\ \rangle$ is $O(n)$-invariant.

It remains to show that $g(x)$ puts together a smooth section $g \in \Gamma(S^2T^*M)$. Let $\sigma : U \to \mathcal{G}|_U$ be a local section of $\mathcal{G}$ define on an open subset $U \subseteq M$. Denote by $e_1, \cdots, e_n$ the standard basis of $\mathbb{R}^n$, then $(\sigma(e_1), \cdots, \sigma(e_n))$ is a local orthonormal frame of TM. In particular, g is smooth. $\square$

In particular, a Riemannian metric on M is equivalently encoded as a principal $O(n)$-subbundle of $GL(\mathbb{R}^n, TM)$, i.e. by a reduction to the structure group $O(n)$.

Now we ask that how local isometries of Riemannian n-manifolds relate to maps between orthonormal frame bundles. As an immediate observation, each local isometry $f : M \to M'$ between Riemannian n-manifolds has a natural lift

$$F : O(\mathbb{R}^n, TM) \to O(\mathbb{R}^n, TM'), u \mapsto Tf \circ u$$

with base map f. F is $O(n)$-equivariant, hence is a principal bundle map. Note that there are other lifts of f to principal bundle maps $O(\mathbb{R}^n, TM) \to O(\mathbb{R}^n, TM')$, namely those $u \mapsto F(u) \circ \varphi(u)$, where $\varphi : O(\mathbb{R}^n, TM) \to O(n)$ is a smooth map such that $\varphi(u \circ A) = A^{-1} \circ \varphi(u) \circ A$ for all $u \in O(\mathbb{R}^n, TM), A \in O(n)$. As we will see in the next section, F becomes the only admitted lift of f by demanding that the lift should respect the soldering form.

2.2 $O(n)$-structures of type $\mathbb{R}^n$

Let M be an n-dimensional manifold. Recall that the *soldering form* $\theta \in \Omega^1(GL(\mathbb{R}^n, TM), \mathbb{R}^n)$ sends a tangent vector $\tilde{\xi} \in T_u GL(\mathbb{R}^n, TM)$ (with $u \in GL(\mathbb{R}^n, T_xM)$) which lies above $\xi \in T_xM$ to

$$\theta(u)(\tilde{\xi}) = u^{-1}(\xi).$$

It is a smooth one-form because it is given by

$$TGL(\mathbb{R}^n, TM) \to GL(\mathbb{R}^n, TM) \times_M TM \to \mathbb{R}^n$$

where $TGL(\mathbb{R}^n, TM) \to GL(\mathbb{R}^n, TM)$ is the projection to base point, $TGL(\mathbb{R}^n, TM) \to TM$ is the tangent map of the natural map $GL(\mathbb{R}^n, TM) \to M$, and

$$GL(\mathbb{R}^n, TM) \times_M TM \to \mathbb{R}^n$$

is the natural pairing, which at the fiber above each $x \in M$ is given by

$$GL(\mathbb{R}^n, T_xM) \times T_xM \to \mathbb{R}^n, (u, \xi) \mapsto u^{-1}(\xi).$$

The last map is smooth because a local section of $GL(\mathbb{R}^n, TM) \times_M TM$ defined on an open subset $U \subseteq M$ can be expressed as a local trivialisation of TM together with a local vector field of M, both defined on U. Thus pairing them produces a smooth map $U \to \mathbb{R}^n$.

The restriction of the soldering form to any principal subbundle $\mathcal{G} \subseteq GL(\mathbb{R}^n, TM)$ is also called the *soldering form* on $\mathcal{G}$. For a discussion on Riemannian n-manifolds, we are only concerned about soldering forms on principal $O(n)$-subbundles of $GL(\mathbb{R}^n, TM)$.

Definition 2.2.1. *Let $p : \mathcal{P} \to M$ be a principal bundle on an arbitrary manifold M, then $\mathcal{VP} := ker(Tp) \subseteq T\mathcal{P}$ is called the vertical bundle of $\mathcal{P}$. Let V be a vector space. A differential form $\omega \in \Omega^k(\mathcal{P}, V)$ is said to be horizontal and is denoted by $\omega \in \Omega^k_{hor}(\mathcal{P}, V)$ if whenever at least one of $\xi_1, \dots, \xi_k$ lies in $\mathcal{VP}$, then $\omega(\xi_1, \dots, \xi_k) = 0$. In the case $\omega \in \Omega^1(\mathcal{P}, V)$, ω is said to be strictly horizontal if $ker(\omega) = \mathcal{VP}$.*

Denote by H the structure group of $\mathcal{P}$ and denote by $r^h : \mathcal{P} \to \mathcal{P}$ the principal right action by any $h \in H$. Suppose V is an H-representation, then a smooth map $\Phi : \mathcal{P} \to V$ is said to be H-equivariant if $\Phi \circ r^h = h^{-1} \circ \Phi$ for all $h \in H$, and we write $\Phi \in C^\infty(\mathcal{P}, V)^H$; similarly $\omega \in \Omega^k(\mathcal{P}, V)$ is said to be H-equivariant if $(r^h)^*\omega = h^{-1} \circ \omega$ for all $h \in H$, and we write $\omega \in \Omega^k(\mathcal{P}, V)^H$.

Lemma 2.2.1. *Let M be an n-dimensional manifold and $\mathcal{G} \subseteq GL(\mathbb{R}^n, TM)$ a principal $O(n)$-subbundle. Then the soldering form $\theta \in \Omega^1(\mathcal{G}, \mathbb{R}^n)$ on $\mathcal{G}$ is $O(n)$-equivariant and strictly horizontal, i.e. $ker(\theta) = \mathcal{VG}$.*

Proof. Denote by $p : \mathcal{G} \to M$ the bundle projection. At each $u \in \mathcal{G}$ with base point $x \in M$, u is a linear isomorphism $\mathbb{R}^n \xrightarrow{\cong} T_xM$. Thus $\theta(u)$ is the map

$$T_u\mathcal{G} \xrightarrow{T_up} T_xM \xrightarrow{u^{-1}} \mathbb{R}^n$$

from which we see that θ is strictly horizontal and $O(n)$-equivariant. □

In particular, a principal $O(n)$-subbundle $\mathcal{G} \subseteq GL(\mathbb{R}^n, TM)$ together with its soldering form θ is an $O(n)$-structure of type $\mathbb{R}^n$, which we define below. We call $(\mathcal{G}, \theta)$ the *canonical $O(n)$-structure of type $\mathbb{R}^n$* on $\mathcal{G}$.

Definition 2.2.2. *An $O(n)$-structure (also called a G-structure with structure group $O(n)$) of type $\mathbb{R}^n$ is a principal $O(n)$-bundle $\mathcal{G}$ together with a strictly horizontal, $O(n)$-equivariant one-form $\theta \in \Omega^1(\mathcal{G}, \mathbb{R}^n)$.*

A morphism of $O(n)$-structures of type $\mathbb{R}^n$ is a principal bundle map $\Phi : (\mathcal{G}, \theta) \to (\mathcal{G}', \theta')$ such that $\Phi^*\theta' = \theta$.

Proposition 2.2.1. *Any $O(n)$-structure $(\mathcal{G} \to M, \theta)$ of type $\mathbb{R}^n$ has an n-dimensional base manifold. Moreover, there is a unique reduction $\iota : \mathcal{G} \hookrightarrow GL(\mathbb{R}^n, TM)$ such that θ is the pullback of the soldering form on $\mathcal{G}$.*

Proof. The dimension of the base manifold M equals the rank of $T\mathcal{G}/\mathcal{VG}$, and the latter equals n as the trictly horizontal one-form θ induces a trivialisation $T\mathcal{G}/\mathcal{VG} \cong \mathcal{G} \times \mathbb{R}^n$.

Since θ is strictly horizontal, at each $u \in \mathcal{G}$ with base point $x \in M$, $\theta(u) : T_u\mathcal{G} \to \mathbb{R}^n$ descends to a linear isomorphism $T_xM \xrightarrow{\cong} \mathbb{R}^n$. Denote its inverse by $\iota(u) \in GL(\mathbb{R}^n, T_xM)$. Thus the map $\iota : \mathcal{G} \to GL(\mathbb{R}^n, TM)$ is the only possible reduction such that θ is the pullback of the soldering form on $\mathcal{G}$. We just need to show that ι is indeed a reduction.

First we show that ι is $O(n)$-equivariant: let $v \in \mathbb{R}^n$ and $\iota(u)(v) =: \xi \in T_xM$. Then $\theta(u)(\tilde{\xi}) = v$ for any tangent vector $\tilde{\xi} \in T_u\mathcal{G}$ lifting ξ. Since θ is $O(n)$-equivariant, $\theta(uA)(T_ur^A(\tilde{\xi})) = A^{-1}v$ for any $A \in O(n)$. Since $T_ur^A(\tilde{\xi}) \in T_{uA}\mathcal{G}$ is a lift of ξ, $\iota(uA)(A^{-1}v) = \xi$. Hence $\iota(uA) = \iota(u)A$.

Next, ι is smooth if and only if the map $F : \mathcal{G} \times \mathbb{R}^n \to TM, (u, v) \mapsto \iota(u)(v)$ is smooth. As θ is strictly horizontal, it induces a global trivialisation $T\mathcal{G}/\mathcal{V}\mathcal{G} \cong \mathcal{G} \times \mathbb{R}^n$. Now F is the composition of $T\mathcal{G}/\mathcal{V}\mathcal{G} \to TM$, which descends from the tangent map $T\mathcal{G} \to TM$, and $\mathcal{G} \times \mathbb{R}^n \xrightarrow{\cong} T\mathcal{G}/\mathcal{V}\mathcal{G}$, the inverse of the trivialisation. Hence F is smooth.

It's clear that ι covers the identity on M, hence ι is a reduction. $\square$

Hence each $O(n)$-structure $(\mathcal{G} \to M, \theta)$ of type $\mathbb{R}^n$ admits a unique isomorphism covering id_M to the canonical $O(n)$-structure of type $\mathbb{R}^n$ on a principal $O(n)$-subbundle of $GL(\mathbb{R}^n, TM)$. As a principal $O(n)$-subbundle of $GL(\mathbb{R}^n, TM)$ induces a Riemannian metric on M by requesting it to be the orthonormal frame bundle on M, each $O(n)$-structure of type $\mathbb{R}^n$ has an underlying Riemannian metric on the base manifold. This can be explicitly described as follows.

Corollary 2.2.1. *Let $(p : \mathcal{G} \to M, \theta)$ be an $O(n)$-structure of type $\mathbb{R}^n$ and let g be its induced Riemannian metric on M. Then whenever $\tilde{\xi}, \tilde{\eta} \in \mathfrak{X}(\mathcal{G})$ are p-related to $\xi, \eta \in \mathfrak{X}(M)$, respectively, then $\langle\theta(\tilde{\xi}), \theta(\tilde{\eta})\rangle \in C^\infty(\mathcal{G})$ is p-related to $g(\xi, \eta) \in C^\infty(M)$.*

Proof. Denote by $\iota : \mathcal{G} \xrightarrow{\cong} \iota(\mathcal{G}) \subseteq GL(\mathbb{R}^n, TM)$ the unique isomorphism covering id_M from $(\mathcal{G}, \theta)$ to the canonical $O(n)$-structure of type $\mathbb{R}^n$ on some $O(n)$-subbundle of $GL(\mathbb{R}^n, TM)$. Then for any $u \in \mathcal{G}$ above $x \in M$, $\iota(u)^{-1} : T_xM \xrightarrow{\cong} \mathbb{R}^n$ is supposed to be an isometry. By the definition of soldering form, $\iota(u)^{-1}$ can be given by first taking any lift $T_xM \to T_{\iota(u)}\iota(\mathcal{G})$ then applying the soldering form, which gives the same result as first taking any lift $T_xM \to T_u\mathcal{G}$ and then applying θ. Hence g is characterised as claimed.

$\square$

Let $(\mathcal{G}, \theta)$ be an $O(n)$-structure of type $\mathbb{R}^n$ with underlying Riemannian n-manifold (M, g), then $(\mathcal{G}, \theta)$ is isomorphic over id_M to the canonical $O(n)$-structure of type $\mathbb{R}^n$ on $O(\mathbb{R}^n, TM)$. With such an identification there is a very simple interpretation for morphisms of $O(n)$-structures of type $\mathbb{R}^n$:

Lemma 2.2.2. *Let (M, g) and (M', g') be Riemannian n-manifolds.*

(i) Any morphism

$$\Phi : O(\mathbb{R}^n, TM) \to O(\mathbb{R}^n, TM')$$

between the canonical $O(n)$-structures of type $\mathbb{R}^n$ on the orthonormal frame bundles with base map $f : M \to M'$ is of the form $u \mapsto Tf \circ u$. In particular, f is a local isometry.

(ii) Conversely, a local isometry $f : M \to M'$ admits a unique lift to a morphism $O(\mathbb{R}^n, TM) \to O(\mathbb{R}^n, TM')$ between the canonical $O(n)$-structures of type $\mathbb{R}^n$ on the orthonormal frame bundles.

Proof. Denote the orthonormal frame bundles by $p : \mathcal{G} := O(\mathbb{R}^n, TM) \to M$ and $p' : \mathcal{G}' := O(\mathbb{R}^n, TM') \to M'$, and denote by θ resp. θ' the soldering form on $\mathcal{G}$ resp. on $\mathcal{G}'$.

(i) Let $u \in \mathcal{G}$ be above $x \in M$. We show that $\Phi(u) \in O(\mathbb{R}^n, T_{f(x)}M')$ equals $T_x f \circ u$. Indeed, for any tangent vector $\xi \in T_x M$ with any lift $\tilde{\xi} \in T_u\mathcal{G}$ we have $\theta(u)(\tilde{\xi}) = u^{-1}(\xi)$. Since the tangent vector $T_u\Phi(\tilde{\xi}) \in T_{\Phi(u)}\mathcal{G}'$ lifts the tangent vector $T_x f(\xi) \in T_{f(x)}M'$, we have

$$u^{-1}(\xi) = \theta(u)(\tilde{\xi}) = \theta'(\Phi(u))(T_u\Phi(\tilde{\xi})) = \Phi(u)^{-1}(T_x f(\xi)).$$

Hence $\Phi(u) = T_x f \circ u$. In particular, Tf restricts to an isometry $T_x M \xrightarrow{\cong} T_{f(x)}M'$ at each $x \in M$, hence f is a local isometry.

(ii) Let $f : M \to M'$ be a local isometry. We show that the principal bundle map $\Phi : \mathcal{G} \to \mathcal{G}', u \mapsto Tf \circ u$ satisfies $\Phi^*\theta' = \theta$, thus by (i), Φ is the unique lift to a morphism $(\mathcal{G}, \theta) \to (\mathcal{G}', \theta')$. Indeed, let $\tilde{\xi} \in T_u\mathcal{G}$ be a tangent vector and $p(u) =: x \in M$. Then by the definition of soldering form, $\theta(u)(\tilde{\xi}) = u^{-1}(T_u p(\tilde{\xi}))$ and since $p' \circ \Phi = f \circ p$, the soldering form θ' applied to the Φ-related tangent vector $T_u\Phi(\tilde{\xi})$ gives

$$\theta'(\Phi(u))(T_u\Phi(\tilde{\xi})) = \Phi(u)^{-1}(T_{\Phi(u)}p'(T_u\Phi(\tilde{\xi}))) = u^{-1} \circ (T_x f)^{-1} \circ (T_x f(T_u p(\tilde{\xi}))) = \theta(u)(\tilde{\xi}).$$

Hence $\Phi^*\theta' = \theta$. □

Hence given two $O(n)$-structures $(\mathcal{G}, \theta)$ resp. $(\mathcal{G}', \theta')$ with underlying Riemannian n-manifolds (M, g), (M', g'), descending to the base map yields a bijection from the space of morphisms $(\mathcal{G}, \theta) \to (\mathcal{G}', \theta')$ to the space of local isometries $(M, g) \to (M', g')$.

Corollary 2.2.2. *The category of $O(n)$-structures of type $\mathbb{R}^n$ is equivalent to the category of Riemannian n-manifolds, whose morphisms are the local isometries.*

Proof. There is a functor from $O(n)$-structures of type $\mathbb{R}^n$ to Riemannian n-manifolds, which is given by taking the underlying Riemannian n-manifold and taking the base map. The functor is full and faithful as descending to the base map yields a bijection from the space of morphisms between two $O(n)$-structures of type $\mathbb{R}^n$ to the space of local isometries between the underlying Riemannian n-manifolds.

It remains to show that the functor is essentially surjective. But given an Riemannian n-manifold (M, g), the underlying Riemannian manifold of the canonical $O(n)$-structure of type $\mathbb{R}^n$ demands $O(\mathbb{R}^n, TM)$ be the orthonormal frame bundle, hence it is just (M, g).

We conclude that the functor yields a categorical equivalence.

□

Chapter 3

Lagrangean contact structures

3.1 Motivation: symplectic manifolds

We briefly review some basics on symplectic manifolds in order to develop its contact analogue.

Definition 3.1.1. *A symplectic form $\omega \in \Omega^2(M)$ on a manifold M is a closed (i.e. $d\omega = 0$) two-form such that $\omega(x) : T_xM \times T_xM \to \mathbb{R}^n$ is a skew-symmetric nondegenerate bilinear form for each $x \in M$. In this case (M, ω) is called a symplectic manifold.*

A symplectomorphism $f : (M, \omega) \to (M', \omega')$ is a local diffeomorphism $f : M \to M'$ such that $f^*\omega' = \omega$.

Recall that a *symplectic vector space* (V, ω) is a vector space V together with a skew-symmetric nondegenerate bilinear form $\omega : \wedge^2 V \to \mathbb{R}$, and a symplectic map $f : (V, \omega) \to (V', \omega')$ is a linear map $f : V \to W$ such that $f^*\omega' = \omega$. By linear algebra, a symplectic vector space must have an even dimension, and all symplectic vector spaces of the same dimension are isomorphic. Denote the standard $2n$-dimensional symplectic vector space by $(\mathbb{R}^{2n}, [\ ,\ ])$ with

$$\mathbb{R}^{2n} = span\{e_1, ..., e_n, f_1, ..., f_n\}$$

and

$$[\ ,\ ] : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$$

defined by $[f_i, e_j] = -[e_j, f_i] = \delta_{ij}$, $[e_i, e_j] = 0$, $[f_i, f_j] = 0$ for all $i, j = 1, ..., n$. We denote by $Sp(2n, \mathbb{R})$ the group of symplectic automorphisms on $(\mathbb{R}^{2n}, [\ ,\ ])$.

Hence a symplectic manifold (M, ω) must have some even dimension $2n$, and its tangent bundle TM is a locally trivial bundle over $(\mathbb{R}^{2n}, [\ ,\ ])$. We associate to (M, ω) a natural frame bundle whose fiber over each $x \in M$ is the space of symplectic isomorphisms

$$(\mathbb{R}^{2n}, [\ ,\ ]) \xrightarrow{\cong} (T_xM, \omega(x)).$$

Then its structure group is $Sp(2n, \mathbb{R})$. Conversely, however, a reduction of $GL(\mathbb{R}^{2n}, TM)$ to structure group $Sp(2n, \mathbb{R})$ does not induce a symplectic form on M in general. In fact, such a reduction induces a two-form on M whose value in each point is nondegenerate, but this form is not closed in general.

We cite a result in linear algebra, thus obtain an alternative definition of symplectic forms:

Lemma 3.1.1. *A two-form $\omega \in \wedge^2 V^*$ on a $2n$-dimensional vector space V is nondegenerate if and only if $\wedge^n \omega$ is a volumn form on V ([5]: 31.3).*

Hence a two-form $\omega \in \Omega^2(M)$ on a $2n$-dimensional manifold M is a symplectic form if and only if $d\omega = 0$ and $\wedge^n \omega \in \Omega^{2n}(M)$ is a volumn form.

Example 3.1.1. *(The canonical symplectic form on the cotangent bundle) Let N be an n-dimensional manifold and $T^*N \to N$ be its cotangent bundle. The tautological one-form $\alpha \in \Omega^1(T^*N)$ on T^*N maps any tangent vector $\tilde{\xi} \in T_u T^*N$ (with $u \in T_x^*N$) lying above $\xi \in T_x N$ to*

$$\alpha(u)(\tilde{\xi}) = u(\xi).$$

α is smooth because it is given by

$$\alpha : TT^*N \to T^*N \times_N TN \to \mathbb{R}$$

where $TT^*N \to T^*N$ is the projection to the base point, $TT^*N \to TN$ is the tangent map of the natural map $T^*N \to N$, and $T^*N \times_N TN \to \mathbb{R}$ is the natural pairing. We claim that $d\alpha \in \Omega^2(T^*N)$ is a symplectic form on T^*N.

Indeed, any local chart

$$(q^1, \ldots, q^n) : U \xrightarrow{\cong} U' \subseteq \mathbb{R}^n$$

defined on an open subset $U \subseteq N$ induces a local chart

$$(q^1, \ldots, q^n, p_1, \ldots, p_n) : T^*N|_U \xrightarrow{\cong} U' \times \mathbb{R}^n$$

characterized by the fact that for $x \in U$ and $u \in T_x^*N$, one gets $u = \Sigma_i p_i(u) dq^i(x)$. Thus

$$\alpha = \Sigma_i p_i dq^i$$

and so

$$d\alpha = \Sigma_i dp_i \wedge dq^i$$

with respect to the local chart. In particular we see that $d\alpha$ is a nondegenerate two form on T^*N. As $dd\alpha = 0$, $d\alpha$ is a symplectic form on T^*N.

Moreover, we claim that any diffeomorphism $f : N \to N$ lifts to a symplectomorphism $\Phi : (T^*N, d\alpha) \to (T^*N, d\alpha)$. Indeed, let Φ send each $u \in T_x^*N$ to $u \circ (T_x f)^{-1} \in T_{f(x)}^*N$. Clearly Φ covers f and restricts to a linear isomorphism on each fiber. To see that Φ is *smooth, notice that $Tf : TN \to TN$ is an automorphism of vector bundle, and for any smooth sections $s \in \Gamma(T^*N) = \Omega^1(N)$ and $\xi \in \mathfrak{X}(N)$, we have $\Phi(s)(\xi) = s \circ (Tf)^{-1} \circ \xi \in C^\infty(N)$, which means that Φ is smooth. Therefore Φ is an automorphism on T^*N. It remains to show that $\Phi^*\alpha = \alpha$, thus $\Phi^*(d\alpha) = d(\Phi^*\alpha) = d\alpha$ and Φ is a symplectomorphism lifting f. Indeed, for a tangent vector $\tilde{\xi} \in T_uT^*N$ (with $u \in T_x^*N$) lying above the tangent vector $\xi \in T_xN$, $T_u\Phi(\tilde{\xi})$ is a tangent vector with base point $\Phi(u) = u \circ (T_xf)^{-1} \in T^*_{f(x)}N$ and it is a lift of $T_xf(\xi) \in T_{f(x)}N$. Hence*

$$\alpha(\Phi(u))(T_u\Phi(\tilde{\xi})) = (u \circ (T_xf)^{-1})(T_xf(\xi)) = u(\xi) = \alpha(u)(\tilde{\xi})$$

and so $\Phi^*\alpha = \alpha$.

In particular, the group of automorphisms on $(T^*N, d\alpha)$ contains the group of diffeomorphisms $N \to N$, hence is infinite dimensional. Since all $2n$-dimensional symplectic manifolds are locally isomorphic, this locally extends to all $2n$-dimensional symplectic manifolds.

Moreover, the Darboux theorem ([5]: 31.15) implies that any $2n$-dimensional symplectic manifold (M, ω) admits a symplectic atlas, i.e. an atlas with charts that have local coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ such that $\omega = \Sigma_i dp_i \wedge dq^i$. In particular, all $2n$-dimensional symplectic manifolds are locally isomorphic. Therefore there are no local invariants on symplectic manifolds.

3.2 Contact manifolds and contact structures

Now we define contact forms in analogy to the alternative definition (Lemma 3.1.1) of symplectic forms.

Definition 3.2.1. *A contact form $\alpha \in \Omega^1(M)$ on a manifold M of odd dimension $2n+1$ is a one-form such that $\alpha \wedge (d\alpha)^n \in \Omega^{2n+1}(M)$ is a volumn form.*

A contact form $\alpha \in \Omega^1(M)$ on a $(2n+1)$-dimensional manifold M is non-vanishing as $\alpha \wedge (d\alpha)^n$ is non-vanishing. Hence $H := ker(\alpha) \subseteq TM$ is a vector subbundle of corank 1, which is called the *contact subbundle* of (M, α).

By linear algebra (Lemma 3.1.1), there is an alternative definition of contact forms:

Lemma 3.2.1. *A non-vanishing one-form $\alpha \in \Omega^1(M)$ on a manifold M is a contact form if and only if $d\alpha$ is a skew-symmetric nondegenerate bilinear form on each fiber of $H := ker(\alpha)$.*

Proof. At each $x \in M$ we may choose a basis for H_x and extend it to a basis for T_xM. With this we see that $\alpha(x) \wedge (d\alpha)^n(x)$ is a volumn form on T_xM if and only if $(d\alpha)^n(x)$ is a volumn form on H_x, which is equivalent to that $d\alpha(x)$ is a skew-symmetric nondegenerate two-form on H_x. Hence α a contact form if and only if $d\alpha$ is nondegenerate on H. $\square$

Notice that a contact subbundle determines the contact form up to a smooth scalar:

Lemma 3.2.2. *Let (M, α) be a contact manifold with contact subbundle $H \subseteq TM$. Then any non-vanishing one-form $TM \to \mathbb{R}$ with kernel H is a contact form on M with contact subbundle H.*

Proof. Any one-form with kernel H is of the form $f\alpha$ for any non-vanishing smooth map $f \in C^\infty(M)$. For $\xi, \eta \in \Gamma(H)$,

$$d\alpha(\xi, \eta) = \xi.\alpha(\eta) - \eta.\alpha(\xi) - \alpha([\xi, \eta]) = -\alpha([\xi, \eta])$$

$$d(f\alpha)(\xi, \eta) = \xi.(f\alpha)(\eta) - \eta.(f\alpha)(\xi) - f\alpha([\xi, \eta]) = -f\alpha([\xi, \eta])$$

hence $d(f\alpha)$ restricted to H is also nondegenerate, in particular, $f\alpha$ is a contact form with contact subbundle H. $\square$

Hence we want to focus on the contact subbundle rather than on a contact form. This yields the definition of contact structures.

Definition 3.2.2. *Let M be a manifold. A contact structure (M, H) on M is a subbundle $H \subseteq TM$ of corank 1 such that each $x \in M$ has an open neighborhood $U \subseteq M$, on which there is a contact form whose contact subbundle is $H|_U$.*

A contactomorphism $f : (M, H) \to (M', H')$ of contact structures is a local diffeomorphism $f : M \to M'$ such that $Tf(H) \subseteq H'$.

From the last lemma, the local contact forms on a contact structure (M, H) are exactly those local one-forms with kernel H, which is equivalent to a local trivialisation on TM/H. In particular, H is the contact subbundle of a globally defined contact form if and only if TM/H is globally trivial.

On the other hand, if α is such a local one-form, then there is a locally defined bilinear form $H \times H \to TM/H$ induced from $d\alpha$ and the induced local trivialisation on TM/H. We will show that this TM/H-valued bilinear form is just the negative of the Levi bracket when M is viewed as a filtered manifold with filtration $TM = T^{-2}M \supseteq T^{-1}M = H$. This yields an alternative definition of contact structures.

Definition 3.2.3. *A filtered manifold is a manifold M together with a filtration*

$$TM = T^{-k}M \supseteq T^{-k+1}M \supseteq ... \supseteq T^{-1}M, k > 0$$

of TM by vector subbundles, such that for each $\xi \in \Gamma(T^iM), \eta \in \Gamma(T^jM)$, we have $[\xi, \eta] \in T^{i+j}M$. We follow the convention that $T^iM = TM$ for $i < -k$ and $T^iM = 0$ for $i \geq 0$.

The *associated graded bundle* of a filtered manifold M is given by $gr(TM) = \oplus_i gr_i(TM)$, a direct sum of quotient bundles $gr_i(TM) := T^iM/T^{i+1}M$. For each i denote by $q_i : T^iM \twoheadrightarrow gr_i(TM)$ the natural quotient map. Consider the operator

$$\Gamma(T^iM) \times \Gamma(T^jM) \to gr_{i+j}(TM), (\xi, \eta) \mapsto q_{i+j}([\xi, \eta]).$$

We claim that the operator is $C^\infty(M)$-bilinear, hence is a vector bundle map. Indeed, for $f \in C^\infty(M)$, $\xi \in \Gamma(T^iM)$, $\eta \in \Gamma(T^jM)$, we have

$$[f\xi, \eta] - f[\xi, \eta] = -(\eta.f)\xi \in T^iM \subseteq T^{i+j+1}M$$

as $j \leq -1$. Hence $q_{i+j}([f\xi, \eta]) = q_{i+j}(f[\xi, \eta]) = fq_{i+j}([\xi, \eta])$ as q_{i+j} is $C^\infty(M)$-linear. Similarly we have $q_{i+j}([\xi, f\eta]) = fq_{i+j}([\xi, \eta])$. Moreover, we observe that if $\xi \in \Gamma(T^{i+1}M)$ or $\eta \in \Gamma(T^{j+1}M)$, then $[\xi, \eta] \in \Gamma(T^{i+j+1}M)$ by the property of the filtered manifold, hence $q_{i+j}([\xi, \eta]) = 0$. In particular, the operator descends to a tensorial map $gr_i(TM) \times gr_j(TM) \to gr_{i+j}(TM)$. Taking these maps together, we obtain a grading-preserving tensorial map

$$\mathcal{L} : gr(TM) \times gr(TM) \to gr(TM)$$

which is called the *Levi bracket* on $gr(TM)$.

Lemma 3.2.3. *Let M be a manifold. A vector subbundle $H \subseteq TM$ of corank 1 is a contact structure on M if and only if on the filtered manifold M with filtration*

$$TM = T^{-2}M \supseteq T^{-1}M = H,$$

the Levi-bracket $\mathcal{L} : H \times H \to TM/H$ is nondegenerate at each fiber.

Proof. Choose a local contact form $\alpha \in \Omega^1_{loc}(M)$ defined on an open subset $U \subseteq M$, such that $ker(\alpha) = H|_U$, and let α descend to a map $\underline{\alpha} : (TM/H)|_U \to \mathbb{R}$ which is a linear isomorphism at each fiber. Then (M, H) is a contact structure if and only if $d\alpha$ is nondegenerate on H_U for all such α. But for any $\xi, \eta \in \Gamma_U(H)$,

$$d\alpha(\xi, \eta) = -\alpha([\xi, \eta]) = -\underline{\alpha} \circ \mathcal{L}(\xi, \eta)$$

hence $d\alpha$ is nondegenerate on H_U if and only if $\mathcal{L}$ is nondegenerate on $H|_U$, and so H is a contact structure if and only if $\mathcal{L}$ is nondegenerate on each fiber. $\square$

Definition 3.2.4. *A graded Lie algebra is a Lie algebra $\mathfrak{g}$ with a decomposition*

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \mathfrak{g}_{-k+1} \oplus ... \oplus \mathfrak{g}_l, k, l \geq 0$$

into vector subspaces such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all $i, j \in \mathbb{Z}$. We take the convention that $\mathfrak{g}_i = 0$ for $i < -k$ and for $i > l$.

A graded Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism such that $\varphi(\mathfrak{g}_i) \subseteq \mathfrak{h}_i$ for all i.

We define the $(2n + 1)$-dimensional *Heisenberg algebra* to be the graded algebra $\mathfrak{g}_- := \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ where $\mathfrak{g}_{-1} := \mathbb{R}^{2n} = span\{e_1, ..., e_n, f_1, ..., f_n\}$ is the standard symplectic vector space, $\mathfrak{g}_{-2} := \mathbb{R}$ and $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is the standard symplectic form, i.e. $[\ ,\ ]$ is generated by $[f_i, e_j] = \delta_{ij}$, $i, j = 1, ..., n$. Denote by $Aut_{gr}(\mathfrak{g}_-)$ the group of automorphisms of graded Lie algebra on $\mathfrak{g}_-$, which consists of all pairs $(\varphi_{-2}, \varphi_{-1})$ of linear isomorphisms

$$\varphi_{-2} : \mathfrak{g}_{-2} \xrightarrow{\cong} \mathfrak{g}_{-2}$$

$$\varphi_{-1} : \mathfrak{g}_{-1} \xrightarrow{\cong} \mathfrak{g}_{-1}$$

such that

$$[\varphi_{-1}(X), \varphi_{-1}(Y)] = \varphi_{-2}([X, Y])$$

for all $X, Y \in \mathfrak{g}_{-1}$.

Corollary 3.2.1. *A vector subbundle $H \subseteq TM$ is a contact structure on M if and only if $(T_xM/H_x \oplus H_x, \mathcal{L}(x))$ is isomorphic to the Heisenberg algebra $\mathfrak{g}_-$ for each $x \in M$.*

Thus for any contact structure H on a $(2n+1)$-dimensional manifold M, we associate to $gr(TM) = TM/H \oplus H$ a frame bundle $\mathcal{P}(gr(TM))$, whose fiber above each $x \in M$ is the space of isomorphisms $\mathfrak{g}_- \xrightarrow{\cong} (gr(T_xM), \mathcal{L}(x))$ of graded Lie algebras, that is, all pairs (u_{-2}, u_{-1}) of linear isomorphisms

$$u_{-2} : \mathfrak{g}_{-2} \xrightarrow{\cong} T_xM/H_x$$

$$u_{-1} : \mathfrak{g}_{-1} \xrightarrow{\cong} H_x$$

such that

$$\mathcal{L}(x)(u_{-1}(X), u_{-1}(Y)) = u_{-2}([X, Y])$$

for all $X, Y \in \mathfrak{g}_{-1}$. The structure group of $\mathcal{P}(gr(TM))$ is then $Aut_{gr}(\mathfrak{g}_-)$.

Observe that for each $(\varphi_{-2}, \varphi_{-1}) \in Aut_{gr}(\mathfrak{g}_-)$, φ_{-2} is completely determined by φ_{-1} as $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is surjective. Hence $Aut_{gr}(\mathfrak{g}_-) \subseteq GL(\mathfrak{g}_{-1}) = GL(2n, \mathbb{R})$, In particular, let $GL(\mathfrak{g}_{-1}, H)$ denote the full frame bundle of H whose fiber above each $x \in M$ is the space of linear isomorphisms $\mathbb{R}^{2n} \xrightarrow{\cong} H_x$, then $(u_{-2}, u_{-1}) \mapsto u_{-1}$ is a reduction $\mathcal{P}(gr(TM)) \hookrightarrow GL(\mathbb{R}^{2n}, H)$ to structure group $Aut_{gr}(\mathfrak{g}_-) \subseteq GL(2n, \mathbb{R})$. Conversely, however, a reduction of $GL(\mathbb{R}^{2n}, H)$ to structure group $Aut_{gr}(\mathfrak{g}_-)$ cannot be viewed as the frame bundle of $gr(TM)$ in general as an arbitrary isomorphism $\varphi_{-1} : \mathfrak{g}_{-1} \xrightarrow{\cong} \mathfrak{g}_{-1}$ need not extend to an element $(\varphi_{-2}, \varphi_{-1}) \in Aut_{gr}(\mathfrak{g}_-)$.

Let $i : Sp(2n, \mathbb{R}) \hookrightarrow Aut_{gr}(\mathfrak{g}_-)$ send a symplectic automorphism φ_{-1} on $\mathbb{R}^{2n} = \mathfrak{g}_{-1}$ to $(1, \varphi_{-1}) \in Aut_{gr}(\mathfrak{g}_-)$, and let $p : Aut_{gr}(\mathfrak{g}_-) \to \mathbb{R}^*, (\varphi_{-2}, \varphi_{-1}) \mapsto \varphi_{-2}$ denote the canonical projection. Then

Lemma 3.2.4.

$$1 \to Sp(2n, \mathbb{R}) \xrightarrow{i} Aut_{gr}(\mathfrak{g}_-) \xrightarrow{p} \mathbb{R}^* \to 1$$

is an exact sequence of group homomorphisms. Hence $Aut_{gr}(\mathfrak{g}_-) \cong Sp(2n, \mathbb{R}) \rtimes \mathbb{R}^*$.

Proof. p is surjective because given $\varphi_{-2} =: c \in \mathbb{R}^*$, define

$$\varphi_{-1} : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}, \Sigma_i a^i e_i + b^i f_i \mapsto \Sigma_i c a^i e_i + b^i f_i$$

then p sends $(\varphi_{-2}, \varphi_{-1}) \in Aut_{gr}(\mathfrak{g}_-)$ to φ_{-2}.

The kernel of p is the set of all $(1, \varphi_{-1}) \in Aut_{gr}(\mathfrak{g}_-)$, thus we must have $\varphi_{-1} \in Sp(2n, \mathbb{R})$. As i is injective, we obtain the claimed exact sequence. □

Analogous to the Darboux theorem, the Pfaff theorem ([4]: 1.9.0.56) implies that for any contact structure (M, H) on a $(2n+1)$-dimensional manifold, M admits locall coordinates $(t, q^i, p_i : 1 \leq i \leq n)$ around each point, such that $\alpha = dt + \Sigma_i p_i dq^i$ is a local contact form with kernel H. In particular, all contact structures on manifolds of the same dimension are locally isomorphic. Therefore there are no local invariants on contact structures either.

Example 3.2.1. *(Canonical contact structure on the projectivised cotangent bundle) Let N be a manifold of dimension $n \geq 2$. Denote by $PT^*N \to N$ the projectivised cotangent bundle of N, whose fiber above each $x \in N$ are the lines in T_x^*N, hence this is a fiber bundle with standard fiber $\mathbb{R}\mathcal{P}^{n-1}$. Moreover, the natural projection $(T^*N \setminus 0_N) \to PT^*N$ is a principal bundle with structure group $\mathbb{R}^* = \mathbb{R} - \{0\}$. Recall from Example 3.1.1 the tautological one-forn $\alpha \in \Omega^1(T^*N)$, we claim that the kernel of α restricted to $(T^*N \setminus 0_N)$ descends to a contact structure $H \subseteq TPT^*N$ on PT^*N. At each $\ell \in PT^*N$, where ℓ is a line in T_x^*N, H_ℓ is just the preimage by $T_l PT^*N \to T_x N$ of the hyperplane in $T_x N$ annihilated by ℓ.*

We now show that H is smooth. Indeed, $ker(\alpha) \subseteq TT^*N$ has corank 1 and contains the vertical bundle of $T^*N \to N$, hence $ker(\alpha)|_{(T^*N - 0_N)}$ also has corank 1 and contains the vertical bundle of $(T^*N - 0_N) \to PT^*N$. Moreover, $ker(\alpha)|_{(T^*N-0_N)}$ is $\mathbb{R}^*$ invariant because each $c \in \mathbb{R}^*$ acting on a tangent vector $\tilde{\xi} \in T_u(T^*N - 0_N)$ $(0 \neq u \in T_x^*N)$ lying above $\xi \in T_x N$ gives $T_u r^c(\tilde{\xi}) \in T_{cu}(T^*N - 0_N)$ lying above $c\xi \in T_x N$. By

$$\alpha(cu)(T_u r^c(\tilde{\xi})) = cu(x) = c\alpha(u)(\tilde{\xi}),$$

$ker(\alpha)|_{(T^*N-0_N)}$ is $\mathbb{R}^*$ invariant. In particular, $H \subseteq TPT^*N$ is smooth subbundle of corank 1 because for each local section σ of $(T^*N \setminus 0_N) \to PT^*N$, H equals the kernel of

$$TPT^*N \xrightarrow{T\sigma} TT^*N \twoheadrightarrow TT^*N/ker(\alpha).$$

It remains to show that H is a contact structure. Recall that any local chart $(q^i) : U \xrightarrow{\cong} U' \subseteq \mathbb{R}^n$ defined on an open subset $U \subseteq N$ induces a local chart $(q^i, p_i) : T^*N|_U \xrightarrow{\cong} U' \times \mathbb{R}^n$, thus $\alpha = \Sigma_i p_i dq^i$. The chart restricts to a chart

$$(q^i, p_i) : (T^*N - 0_N)|_U \xrightarrow{\cong} U' \times (\mathbb{R}^n - \{0\}).$$

For $k = 1, ..., n$ choose local sections

$$\sigma_k : PT^*N|_U \supseteq \tilde{U}_k \to (T^*N - 0_N)|_U$$

of $(T^*N - 0_N) \to PT^*N$ such that

$$(q^i, p_i) \circ \sigma_k(\tilde{U}_k) = U' \times (\mathbb{R} \times ... \times \{1\} \times ... \times \mathbb{R})$$

where $\{1\}$ is the k-th entry of $\mathbb{R}^n$. Then $PT^*N|_U = \cup_i \tilde{U}_i$ and $(q^i, p_i) \circ \sigma_k$ induces a local chart of PT^*N. By abuse of notation we denote the chart by

$$(q^1, ..., q^n, p_1, ..., p_{k-1}, p_{k+1}, ...p_n) : PT^*N|_U \supseteq \tilde{U}_k \xrightarrow{\cong} U' \times \mathbb{R}^{n-1}.$$

We show that H is a contact structure by showing that $\theta := \alpha \circ \sigma_k$ for any $1 \leq k \leq n$ is a local contact form, as $ker(\theta) = H|_{\tilde{U}_k}$. Indeed, we have

$$\theta = dq^k + \Sigma_{i \neq k} p_i dq^i$$

with respect to the local chart above. In particular $(dp_i, dq^i)_{i \neq k} : ker(\theta) \to \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ is a linear isomorphism at each fiber, hence $d\theta = \Sigma_{i \neq k} dp_i \wedge dq^i$ is nondegenerate on $ker(\theta)$ and θ is a contact form on $\tilde{U}_k$.

Now we claim that any diffeomorphism $f : N \to N$ lifts to a contactomorphism $\underline{\Phi} : (PT^*N, H) \to (PT^*N, H)$. Recall that f lifts to a diffeomorphism $\Phi : T^*N \to T^*N$ sending $u \in T^*_x N$ to $u \circ (T_x f)^{-1}$ such that $\Phi^*\alpha = \alpha$. Since a line $\mathbb{R}u \in T^*_x N$ is mapped by Φ to a line $\mathbb{R}u \circ (T_x f)^{-1} \in T^*_{f(x)} N$, Φ descends to a diffeomorphism $\underline{\Phi} : PT^*N \to PT^*N$. $\Phi^*\alpha = \alpha$ implies $T\Phi(ker(\alpha)) = ker(\alpha)$, hence $T\underline{\Phi}(H) = H$ and $\underline{\Phi}$ is a contactomorphism.

In particular, the automorphism group on (PT^*N, H) also contains the diffeomorphisms $N \to N$, hence is infinite dimensional. Since all contact structures on manifolds of the same dimension are locally isomorphic, this locally extends to all contact structures on $(2n-1)$-dimensional manifolds.

3.3 Lagrangean contact structures

Recall that on a $2n$-dimensional symplectic vector space (V, ω), a linear subspace $W \subseteq V$ is said to be *isotropic* if $\omega|_{W \times W} = 0$, i.e. $W \subseteq W^\perp$; W is said to be *Lagrangean* if $W = W^\perp$. Note that an isotropic subspace $W \subseteq V$ is Lagrangean if and only if W is n-dimensional because by the nondegeneracy of ω, any r-dimensional isotropic subspace W extends to a $2r$-dimensional nondegenerate subspace $\tilde{W}$. Thus $V = \tilde{W} \oplus \tilde{W}^\perp$, and $\tilde{W}^\perp$ is a $(2n-2r)$-dimensional subspace of $W^\perp$.

Let (M, H) be a contact structure and $\mathcal{L} : H \times H \to TM/H$ be the Levi bracket of the filtered manifold $TM = T^{-2}M \supseteq T^{-1}M = H$. Then $(H_x, \mathcal{L}(x))$ is a symplectic vector space for each $x \in M$ (Lemma 3.2.3). We call a vector subbundle $F \subseteq H$ a *Lagrangean subbundle* if $F_x \subseteq H_x$ is a Lagrangean subspace for each $x \in M$.

Example 3.3.1. *Let N be a manifold of dimension $n \geq 2$. Recall the canonical contact structure (PT^*N, H) given in Example 3.2.1. We have shown that the vertical bundle F of $PT^*N \to N$ lies in H. As a vertical bundle, the Lie bracket on vector fields of PT^*N sends $\Gamma(F) \times \Gamma(F)$ to $\Gamma(F)$, and since F has rank $(n-1)$ and H has rank $(2n-2)$, F is a Lagrangean subbundle of H. Clearly, the contactomorphism $\underline{\Phi} : PT^*N \to PT^*N$ lifted from any diffeomorphism $f : N \to N$ as given in Example 3.2.1 preserves the vertical bundle F. Hence the group of contactomorphisms $PT^*N \to PT^*N$ preserving F is infinite dimensional, and we conclude that the automorphism group of a contact structure with one distinguished Lagrangean subbundle may have infinite dimension. In fact, all contact structures with one distinguished involutive Lagrangean subbundle on manifolds of the same dimension are also locally isomorphic ([1]), therefore there again are no local invariants.*

Definition 3.3.1. *Let (M, H) be a contact structure. Then a Lagrangean contact structure $(M, E \oplus F)$ on (M, H) is a decomposition $H = E \oplus F$ into two Lagrangean subbundles.*

A morphism $f : (M, E \oplus F) \to (M', E' \oplus F')$ of Lagrangean contact structures is a local diffeomorphism $f : M \to M'$ such that $Tf(E) \subseteq E'$ and $Tf(F) \subseteq F'$ (which implies that f is a contactomorphism).

Consider the $(2n+1)$-dimensional Heisenberg algebra $\mathfrak{g}_-$. The $2n$-dimensional symplectic vector space $(\mathfrak{g}_{-1}, [\ ,\ ])$ with

$$\mathfrak{g}_{-1} = span\{e_1, ..., e_n, f_1, ..., f_n\}$$

$$[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$$

generated by $[f_i, e_j] = \delta_{ij}$ has a natural decomposition $\mathfrak{g}_{-1} = \mathfrak{g}^E_{-1} \oplus \mathfrak{g}^F_{-1}$ into two Lagrangean subspaces

$$\mathfrak{g}^E_{-1} := span\{e_1, ..., e_n\}$$

and

$$\mathfrak{g}^F_{-1} := span\{f_1, ..., f_n\}.$$

Let $G_0 \subseteq Aut_{gr}(\mathfrak{g}_-)$ be the subgroup of automorphisms preserving the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}^E_{-1} \oplus \mathfrak{g}^F_{-1}$, i.e. G_0 consists of all triples $(\varphi_{-2}, \varphi^E_{-1}, \varphi^F_{-1})$ of linear isomorphisms

$$\varphi_{-2} : \mathfrak{g}_{-2} \xrightarrow{\cong} \mathfrak{g}_{-2}$$

$$\varphi^E_{-1} : \mathfrak{g}^E_{-1} \xrightarrow{\cong} \mathfrak{g}^E_{-1}$$

$$\varphi^F_{-1} : \mathfrak{g}^F_{-1} \xrightarrow{\cong} \mathfrak{g}^F_{-1}$$

such that

$$[\varphi^E_{-1}(X), \varphi^F_{-1}(Y)] = \varphi_{-2}([X, Y])$$

for all $X \in \mathfrak{g}^E_{-1}$ and $Y \in \mathfrak{g}^F_{-1}$. We claim that any pair of linear isomorphisms $\varphi_{-2} : \mathfrak{g}_{-2} \xrightarrow{\cong} \mathfrak{g}_{-2}$ and $\varphi^E_{-1} : \mathfrak{g}^E_{-1} \xrightarrow{\cong} \mathfrak{g}^E_{-1}$ determines a unique $(\varphi_{-2}, \varphi^E_{-1}, \varphi^F_{-1}) \in G_0$. Indeed,

$[\ ,\ ]: \mathfrak{g}^F_{-1} \times \mathfrak{g}^E_{-1} \to \mathfrak{g}_{-2}$ is just the standard inner product on $\mathbb{R}^n$, thus for each $i = 1, ..., n$ there is a unique $\varphi^F_{-1}(f_i) \in \mathfrak{g}^F_{-1}$ such that $[\varphi^F_{-1}(f_i), \varphi^E_{-1}(e_j)] = \varphi_{-2}(\delta_{ij})$ for all $j = 1, ..., n$. In particular, $G_0 \cong GL(\mathfrak{g}_{-2}) \times GL(\mathfrak{g}^E_{-1}) \cong \mathbb{R}^* \times GL(n, \mathbb{R})$.

Let H be a contact structure on a $(2n+1)$-dimensional manifold M. Recall that there is a frame bundle $\mathcal{P}(gr(TM))$ with structure group $Aut_{gr}(\mathfrak{g}_-)$ modelling $(gr(TM), \mathcal{L})$ over the $(2n+1)$-dimensional Heisenberg algebra $\mathfrak{g}_-$. Then a Lagrangean contact structure $H = E \oplus F$ induces a frame bundle $\mathcal{G}_0$, which is a subbundle of $\mathcal{P}(gr(TM))$ modelling E over g^E_{-1} and F over g^F_{-1}. Then the structure group of $\mathcal{G}_0$ is G_0.

Proposition 3.3.1. *Let H be a contact structure on a $(2n+1)$-dimensional manifold M, and let $\mathcal{P}(gr(TM))$ denote its frame bundle with structure group $Aut_{gr}(\mathfrak{g}_-)$. Then for any principal subbundle $\mathcal{G}_0 \subseteq \mathcal{P}(gr(TM))$ with structure group $G_0 \subseteq Aut_{gr}(\mathfrak{g}_-)$, there is a unique Lagrangean contact structure $H = E \oplus F$ on (M, H) such that $\mathcal{G}_0$ is the frame bundle associated to it.*

Proof. For any $(u_{-2}, u_{-1}) \in \mathcal{G}_0 \subseteq \mathcal{P}(gr(TM))$ with base point $x \in M$, we must have $E_x = u_{-1}(\mathfrak{g}^E_{-1})$ and $F_x = u_{-1}(\mathfrak{g}^F_{-1})$, then $H_x = E_x \oplus F_x$ is clearly a decomposition into Lagrangean subspaces. Since G_0 preserves the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}^E_{-1} \oplus \mathfrak{g}^F_{-1}$, different elements in $\mathcal{G}_0$ in the fiber above x induce the same subspaces E_x and F_x, thus we obtain a Lagrangean contact structure $H = E \oplus F$. The local frames of E resp. F can be given by the local sections of $\mathcal{G}_0$ evaluated at any basis of $\mathfrak{g}^E_{-1}$ resp. $\mathfrak{g}^F_{-1}$, and so E and F are smooth vector subbundles. $\square$

In particular, just like given an n-dimensional manifold M with frame bundle $GL(n, \mathbb{R})$, then principal $O(n)$-subbundles of $GL(n, \mathbb{R})$ correspond to Riemannian metrics on M, the contact analogue is that when (M, H) is a contact structure on $(2n+1)$-dimensional manifold with frame bundle $\mathcal{P}(gr(TM))$, then principal G_0-subbundles of $\mathcal{P}(gr(TM))$ correspond to Lagrangean contact structures on (M, H).

Example 3.3.2. *(Canonical Lagrangean contact structure on the flag manifold) The points of the flag manifold $F_{1,n+1}(\mathbb{R}^{n+2})$ are the pairs (V_1, V_{n+1}) such that $V_{n+1} \subseteq \mathbb{R}^{n+2}$ is a hyperplane and $V_1 \subseteq V_{n+1}$ is a line. Then there is a natural projection*

$$\begin{aligned} F_{1,n+1}(\mathbb{R}^{n+2}) &\to \mathbb{R}P^{n+1} =: N \\ (V_1, V_{n+1}) &\mapsto V_1 \end{aligned}$$

which we may interpret as the projective cotangent bundle $PT^*N \to N$ because for each $\ell = \mathbb{R}x \in N$, where $0 \neq x \in \mathbb{R}^{n+2}$, we may realise $T_\ell N$ as $(T_x\mathbb{R}^{n+2})/\ell = \mathbb{R}^{n+2}/\ell$, hence realise $PT^*_\ell N$ as lines in $(\mathbb{R}^{n+2}/\ell)^*$, i.e. lines in $\mathbb{R}^{(n+2)*}$ annihilating ℓ, which is the same as hyperplanes in $\mathbb{R}^{n+2}$ containing ℓ. It's clear that such realisation does not depend on the choice of representative $0 \neq x \in \ell$.

Consider the canonical contact structure H on PT^*N as given in Example 3.2.1 and the Lagrangean subbundle $F \subseteq H$ given by the vertical bundle of $PT^*N \to N$ as in Example 3.3.1. Let E be the vertical bundle of the natural fiber bundle

$$F_{1,n+1}(\mathbb{R}^{n+2}) = PT^*N \to \mathbb{R}P^{(n+1)*}$$
$$(V_1, V_{n+1}) \mapsto V_{n+1}$$

with standard fiber $\mathbb{R}P^n$. We see that $E \subseteq H$ because at each $(V_1, V_{n+1}) \in PT^*N$, express V_{n+1} as a line $\mathbb{R}\varphi$ in $\mathbb{R}^{(n+2)*}$ annihilating V_1, then the image of $E_{(V_1,V_{n+1})}$ by the natural projection $TPT^*N \to TN$ is a subspace of $T_{V_1}N$ annihilated by φ.

Since $N = \mathbb{R}P^{n+1}$ is $(n+1)$-dimensional, $F_{1,n+1}(\mathbb{R}^{n+2}) = PT^*N$ is $(2n+1)$-dimensional, and E and F both has rank n. As vertical subbundles, E and F are involutive, hence are Lagrangean subbundles. Clearly $E \cap F = \{0\}$, therefore $H = E \oplus F$ is a Lagrangean contact structure.

Later in Section 5.1 we will see that the Lagrangean contact structure on $F_{1,n+1}(\mathbb{R}^{n+2})$ is homogeneous under $GL(n+2,\mathbb{R})$, thus is homogeneous under the projective linear group $PGL(n+2,\mathbb{R})$ by an effective and transitive action. In particular, each element in $PGL(n+2,\mathbb{R})$ yields a distinct automorphism on the Lagrangean contact structure on $F_{1,n+1}(\mathbb{R}^{n+2})$. After obtaining the description of a Lagrangean contact structure as a Cartan geometry, we will see that the automorphism group on the Lagrangean contact structure on $F_{1,n+1}(\mathbb{R}^{n+2})$ is exactly $PGL(n+2,\mathbb{R})$, a group of finite dimension. This will provide an upper bound on the dimension of the automorphism group of any Lagrangean contact structure.

3.4 Regular filtered G_0-structures of type $\mathfrak{g}_-$

Recall from Section 2.2 the soldering form

$$TGL(\mathbb{R}^n, TM) \to GL(\mathbb{R}^n, TM) \times_M TM \to \mathbb{R}^n$$

on the frame bundle $GL(\mathbb{R}^n, TM)$ of an n-dimensional manifold M. We will construct an analogue on the frame bundles of contact structures.

Now let (M, H) be a contact structure on a $(2n+1)$-dimensional manifold M and let $\mathcal{P}(gr(TM))$ be its frame bundle. Recall that the fiber of $\mathcal{P}(gr(TM))$ above $x \in M$ consists of all pairs (u_{-2}, u_{-1}) of linear isomorphisms $u_{-2} : \mathfrak{g}_{-2} \xrightarrow{\cong} T_xM/H_x$, $u_{-1} : \mathfrak{g}_{-1} \xrightarrow{\cong} H_x$ such that $\mathcal{L}(x)(u_{-1}(X), u_{-1}(Y)) = u_{-2}([X,Y])$ for all $X, Y \in \mathfrak{g}_{-1}$. Moreover, $(u_{-2}, u_{-1}) \mapsto u_{-1}$ yields a reduction $\mathcal{P}(gr(TM)) \hookrightarrow GL(\mathfrak{g}_{-1}, H)$ of the full frame bundle of H.

On $GL(\mathfrak{g}_{-1}, H)$ there is an obvious partial soldering form θ_{-1}: denote by $T^{-1}GL(\mathfrak{g}_{-1}, H) \subseteq TGL(\mathfrak{g}_{-1}, H)$ the preimage of $H = T^{-1}M$ under the tangent map of $GL(\mathfrak{g}_{-1}, H) \to M$. $T^{-1}GL(\mathfrak{g}_{-1}, H)$ is smooth because it is the kernel of

$$TGL(\mathfrak{g}_{-1}, H) \to TM \twoheadrightarrow TM/H.$$

Let $\theta_{-1} \in \Gamma(L(T^{-1}GL(\mathfrak{g}_{-1}, H), \mathfrak{g}_{-1}))$ map each tangent vector $\tilde{\xi} \in T_u^{-1}GL(\mathfrak{g}_{-1}, H)$ (with $u : \mathfrak{g}_{-1} \xrightarrow{\cong} H_x$) lying above $\xi \in H_x$ to $\theta_{-1}(u)(\tilde{\xi}) = u^{-1}(\xi)$. θ_{-1} is smooth because it is given by the natural maps

$$T^{-1}GL(\mathfrak{g}_{-1}, H) \to GL(\mathfrak{g}_{-1}, H) \times_M H \to \mathfrak{g}_{-1}.$$

Denote by $T^{-1}\mathcal{P}(gr(TM)) \subseteq T\mathcal{P}(gr(TM))$ the preimage of $H = T^{-1}M$ by the tangent map of $\mathcal{P}(gr(TM)) \to M$. Then the reduction $\mathcal{P}(gr(TM)) \hookrightarrow GL(\mathfrak{g}_{-1}, H)$ pulls back θ_{-1} to a partial soldering form

$$T^{-1}\mathcal{P}(gr(TM)) \to \mathcal{P}(gr(TM)) \times_M H \to \mathfrak{g}_{-1}$$

which we still denote by θ_{-1}. That is, for each tangent vector $\tilde{\xi} \in T_u^{-1}\mathcal{P}(gr(TM))$ with base point $u = (u_{-2}, u_{-1}) \in \mathcal{P}(gr(TM))_x$ such that $\tilde{\xi}$ descends to $\xi \in H_x$, we have

$$\theta_{-1}(u)(\tilde{\xi}) = (u_{-1})^{-1}(\xi).$$

On the other hand, we define a one-form $\theta_{-2} \in \Omega^1(\mathcal{P}(gr(TM)), \mathfrak{g}_{-2})$: for each $\tilde{\xi} \in T_u\mathcal{P}(gr(TM))$ with base point $u = (u_{-2}, u_{-1}) \in \mathcal{P}(gr(TM))_x$ such that $\tilde{\xi}$ descends to $\xi \in T_xM$ and further descends to $\underline{\xi} \in T_xM/H_x$, let

$$\theta_{-2}(u)(\tilde{\xi}) = (u_{-2})^{-1}(\underline{\xi}).$$

θ_{-2} is smooth because it is given by

$$T\mathcal{P}(gr(TM)) \to \mathcal{P}(gr(TM)) \times_M TM/H \to \mathfrak{g}_{-2}$$

where $T\mathcal{P}(gr(TM)) \to \mathcal{P}(gr(TM))$ is the projection to base point, $T\mathcal{P}(gr(TM)) \to TM/H$ is the composition of the tangent map of $\mathcal{P}(gr(TM)) \to M$ to the natural projection $TM \twoheadrightarrow TM/H$, and $\mathcal{P}(gr(TM)) \times_M TM/H \to \mathfrak{g}_{-2}$ is the natural pairing sending $(u_{-2}, u_{-1}) \in \mathcal{P}(gr(TM))_x$, $\underline{\xi} \in T_xM/H_x$ to $(u_{-2})^{-1}(\underline{\xi}) \in \mathfrak{g}_{-2}$.

Thus we define the *soldering form* on $\mathcal{P}(gr(TM))$ as $\theta = (\theta_{-2}, \theta_{-1})$, with $\theta_{-2} \in \Omega^1(\mathcal{P}(gr(TM)), \mathfrak{g}_{-2})$ and $\theta_{-1} \in \Gamma(L(T^{-1}\mathcal{P}(gr(TM)), \mathfrak{g}_{-1}))$ as given above. Moreover, we also call the restriction of θ to any principal subbundle $\mathcal{G}_0 \subseteq \mathcal{P}(gr(TM))$ the *soldering form* on $\mathcal{G}_0$, which is a $\mathfrak{g}_{-2}$-valued one-form together with a $\mathfrak{g}_{-1}$-valued partial one-form defined on the preimage $T^{-1}\mathcal{G}_0$ of H by the tangent map of $\mathcal{G}_0 \to M$. We only discuss the case when $\mathcal{G}_0$ is a reduction to the structure group G_0 as it correspond to a Lagrangean contact structure. We think of $T^{-1}\mathcal{G}_0$ as the subbundle in the G_0-invariant filtration

$$T\mathcal{G}_0 = T^{-2}\mathcal{G}_0 \supseteq T^{-1}\mathcal{G}_0 \supseteq T^0\mathcal{G}_0 = \mathcal{V}\mathcal{G}_0$$

lifted from the filtration $TM = T^{-2}M \supseteq T^{-1}M = H$.

We describe the properties of the soldering form on $\mathcal{G}_0 \subseteq \mathcal{P}(gr(TM))$. Note that G_0 comes with a distinguished representation on $\mathfrak{g}_{-2}$ and on $\mathfrak{g}_{-1}$ via the canonical inclusions $\mathfrak{g}_{-2} \hookrightarrow \mathfrak{g}_-$ and $\mathfrak{g}_{-1} \hookrightarrow \mathfrak{g}_-$.

Lemma 3.4.1. *Let (M,H) be a contact structure on a $(2n+1)$-dimensional manifold with frame bundle $\mathcal{P}(gr(TM))$, and let $\mathcal{G}_0 \subseteq \mathcal{P}(gr(TM))$ be a principal subbundle with structure group G_0. Denote by $\theta = (\theta_{-2}, \theta_{-1})$ the soldering form on $\mathcal{G}_0$. Then*

(i) $\theta_{-2} \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-2})$ is G_0-equivariant and its kernel is $T^{-1}\mathcal{G}_0$, and

(ii) $\theta_{-1} \in \Gamma(L(T^{-1}\mathcal{G}_0, \mathfrak{g}_{-1}))$ is G_0-equivariant and its kernel is $T^0\mathcal{G}_0$.

(iii)

$$d\theta_{-2}(\tilde{\xi}, \tilde{\eta}) = -[\theta_{-1}(\tilde{\xi}), \theta_{-1}(\tilde{\eta})]$$

for all $\tilde{\xi}, \tilde{\eta} \in \Gamma(T^{-1}\mathcal{G}_0)$.

Proof. Denote by $p : \mathcal{G}_0 \to M$ the bundle projection and let $\varphi = (\varphi_{-2}, \varphi_{-1}) \in G_0$ be any element in the structure group. Let $u = (u_{-2}, u_{-1}) \in \mathcal{G}_0$ be any point with base point $x \in M$.

(i) $\theta_{-2}(u)$ is given by

$$T_u\mathcal{G}_0 \xrightarrow{Tp} T_xM \twoheadrightarrow T_xM/H_x \xrightarrow[\cong]{(u_{-2})^{-1}} \mathfrak{g}_{-2},$$

from which we see that the kernel of θ_{-2} is exactly $(Tp)^{-1}(H) = T^{-1}\mathcal{G}_0$.

At $u\varphi = (u_{-2} \circ \varphi_{-2}, u_{-1} \circ \varphi_{-1})$, $\theta_{-2}(u\varphi)$ is given by

$$T_{u\varphi}\mathcal{G}_0 \xrightarrow{Tp} T_xM \twoheadrightarrow T_xM/H_x \xrightarrow[\cong]{(u_{-2})^{-1}} \mathfrak{g}_{-2} \xrightarrow[\cong]{(\varphi_{-2})^{-1}} \mathfrak{g}_{-2}.$$

Since Tp is G_0-invariant, θ_{-2} is G_0-equivariant.

(ii) Similarly $\theta_{-1}(u)$ is given by

$$T_u^{-1}\mathcal{G}_0 \xrightarrow{Tp|_{T^{-1}\mathcal{G}_0}} H_x \xrightarrow[\cong]{(u_{-1})^{-1}} \mathfrak{g}_{-1},$$

from which we see that the kernel of θ_{-1} is just the vertical bundle $T^0\mathcal{G}_0$, and $\theta_{-1}(u\varphi)$ is given by

$$T_{u\varphi}^{-1}\mathcal{G}_0 \xrightarrow{Tp|_{T^{-1}\mathcal{G}_0}} H_x \xrightarrow[\cong]{(u_{-1})^{-1}} \mathfrak{g}_{-1} \xrightarrow[\cong]{(\varphi_{-1})^{-1}} \mathfrak{g}_{-1},$$

from which we see that θ_{-1} is G_0-equivariant.

(iii) Since both sides of the claimed equation are tensorial maps $T^{-1}\mathcal{G}_0 \times T^{-1}\mathcal{G}_0 \to \mathfrak{g}_{-2}$, it suffices to prove the equality at a point $u \in \mathcal{G}_0$. Indeed, for any $\tilde{\xi}, \tilde{\eta} \in \Gamma(T^{-1}\mathcal{G}_0)$,

$$d\theta_{-2}(\tilde{\xi}, \tilde{\eta}) = \tilde{\xi}.\theta_{-2}(\tilde{\eta}) - \tilde{\eta}.\theta_{-2}(\tilde{\xi}) - \theta_{-2}([\tilde{\xi}, \tilde{\eta}]) = -\theta_{-2}([\tilde{\xi}, \tilde{\eta}])$$

as $ker(\theta_{-2}) = T^{-1}\mathcal{G}_0$. Without loss of generality, we may assume that $\tilde{\xi}, \tilde{\eta}$ are G_0-invariant vector fields, hence descend to $\xi, \eta \in \Gamma(H)$, respectively. In particular, $[\tilde{\xi}, \tilde{\eta}] \in \mathfrak{X}(\mathcal{G}_0)$ is a G_0-invariant vector field and it descends to $[\xi, \eta] \in \mathfrak{X}(M)$, which further descends to $\mathcal{L}(\xi, \eta) \in \Gamma(TM/H)$. Therefore

$$\theta_{-2}(u)([\tilde{\xi}, \tilde{\eta}]) = (u_{-2})^{-1}(\mathcal{L}(x)(\xi, \eta))$$

$$[\theta_{-1}(u)(\tilde{\xi}), \theta_{-1}(u)(\tilde{\eta})] = [(u_{-1})^{-1}(\xi), (u_{-1})^{-1}(\eta)]$$

which are equal as $u = (u_{-2}, u_{-1})$ is an isomorphism $\mathfrak{g}_- \xrightarrow{\cong} (gr(T_xM), \mathcal{L}(x))$ of graded Lie algebras. Hence

$$d\theta_{-2}(u)(\tilde{\xi}, \tilde{\eta}) = -\theta_{-2}(u)([\tilde{\xi}, \tilde{\eta}]) = -[\theta_{-1}(u)(\tilde{\xi}), \theta_{-1}(u)(\tilde{\eta})].$$

□

In particular, a principal G_0-subbundle $\mathcal{G}_0 \subseteq \mathcal{P}(gr(TM))$ of the frame bundle of a contact structure (M, H) together with the lifted filtration

$$T\mathcal{G}_0 = T^{-2}\mathcal{G}_0 \supseteq T^{-1}\mathcal{G}_0 \supseteq T^0\mathcal{G}_0 = \mathcal{V}\mathcal{G}_0$$

from $TM = T^{-2}M \supseteq T^{-1}M = H$ and the soldering form θ yields a regular filtered G_0-structure of type $\mathfrak{g}_-$, which we define below. We call $(\mathcal{G}, \theta)$ the *canonical regular filtered G_0-structure of type $\mathfrak{g}_-$* on $\mathcal{G}_0$.

Definition 3.4.1. *A regular filtered G_0-structure (also called a regular filtered G-structure with structure group G_0) of type $\mathfrak{g}_-$ consists of*

(i) a principal G_0-bundle $\mathcal{G}_0 \to M$;

(ii) a G_0-invariant filtration $T\mathcal{G}_0 = T^{-2}\mathcal{G}_0 \supseteq T^{-1}\mathcal{G}_0 \supseteq T^0\mathcal{G}_0 = \mathcal{V}\mathcal{G}_0$;

(iii) a pair $\theta = (\theta_{-2}, \theta_{-1})$, such that $\theta_{-2} \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-2})$ is a G_0-equivariant one-form with kernel $T^{-1}\mathcal{G}_0$, $\theta_{-1} \in \Gamma(L(T^{-1}\mathcal{G}_0, \mathfrak{g}_{-1}))$ is a G_0-equivariant partial one-form with kernel $T^0\mathcal{G}_0$, which satisfies the regularity condition, i.e.

$$d\theta_{-2}(\tilde{\xi}, \tilde{\eta}) = -[\theta_{-1}(\tilde{\xi}), \theta_{-1}(\tilde{\eta})]$$

for all $\tilde{\xi}, \tilde{\eta} \in \Gamma(T^{-1}\mathcal{G}_0)$.

A morphism $\Phi : (\mathcal{G}_0, \theta) \to (\mathcal{G}'_0, \theta')$ of regular filtered G_0-structures of type $\mathfrak{g}_-$ is a principal bundle map $\Phi : \mathcal{G}_0 \to \mathcal{G}'_0$ such that $T\Phi$ is filtration preserving and satisfies $\Phi^*\theta'_i = \theta_i$ for $i = -2, -1$.

Proposition 3.4.1. *The base manifold M of a regular filtered G_0-structure $(\mathcal{G}_0 \to M, \theta)$ of type $\mathfrak{g}_-$ has dimension $(2n+1)$. The subbundle $T^{-1}\mathcal{G}_0$ in the filtration $T\mathcal{G}_0 = T^{-2}\mathcal{G}_0 \supseteq T^{-1}\mathcal{G}_0 \supseteq T^0\mathcal{G}_0 = \mathcal{V}\mathcal{G}_0$ descends to a contact structure H on M.*

Moreover, there is a unique reduction $\iota : \mathcal{G}_0 \hookrightarrow \mathcal{P}(gr(TM))$ of the frame bundle of the contact structure (M, H), such that ι pulls back the soldering form to θ.

Proof. Since for $i = -2, -1$, $ker(\theta_i) = T^{i+1}\mathcal{G}_0$, we have

$$rank(T^i\mathcal{G}_0) - rank(T^{i+1}\mathcal{G}_0) = dim(\mathfrak{g}_i).$$

Hence

$$\begin{aligned} dim(M) =& rank(T^{-2}\mathcal{G}_0) - rank(T^0\mathcal{G}_0) \\ =& (rank(T^{-2}\mathcal{G}_0) - rank(T^{-1}\mathcal{G}_0)) + (rank(T^{-1}\mathcal{G}_0) - rank(T^0\mathcal{G}_0)) \\ =& dim(\mathfrak{g}_{-2}) + dim(\mathfrak{g}_{-1}) \\ =& 2n+1. \end{aligned}$$

Since $T^{-1}\mathcal{G}_0$ is G_0 invariant and contains the vertical bundle, it descends to a subbundle $H \subseteq TM$ which also has corank $dim(\mathfrak{g}_{-2}) = 1$. H is smooth because whenever $\sigma : M \supseteq U \to \mathcal{G}_0$ is a local section of $\mathcal{G}_0$ defined on an open subset $U \subseteq M$, $H|_U$ is the kernel of

$$TM|_U \xrightarrow{T\sigma} T\mathcal{G}_0|_U \twoheadrightarrow (T^{-2}\mathcal{G}_0/T^{-1}\mathcal{G}_0)|_U.$$

In particular, $T\sigma$ is filtration-preserving. Define

$$F_{-1} : H|_U \xrightarrow{T\sigma} T^{-1}\mathcal{G}_0|_U \xrightarrow{\theta_{-1}} \mathfrak{g}_{-1}$$

which is a linear isomorphism in each fiber. Similarly, the composition

$$\overline{F_{-2}} : TM|_U \xrightarrow{T\sigma} T\mathcal{G}_0|_U \xrightarrow{\theta_{-2}} \mathfrak{g}_{-2}$$

has kernel $H|_U$, hence descends to

$$F_{-2} : (TM/H)|_U \to \mathfrak{g}_{-2}$$

which is a linear isomorphism at each fiber. We claim that for the filtered manifold given by $TM = T^{-2}M \supseteq T^{-1}M = H$, $F = (F_{-2}, F_{-1})$ is a local trivialisation of the associated graded bundle $(gr(TM), \mathcal{L})$ over the graded Lie algebra $\mathfrak{g}_-$. This means that H is a contact structure on M.

Indeed, for any open subset $U \subseteq M$, any local sections $\xi, \eta \in \Gamma_{loc}(H)$ defined on U can be lifted to some local sections $\tilde{\xi}, \tilde{\eta} \in \Gamma_{loc}(T^{-1}\mathcal{G}_0)$ defined on U, respectively. Then the local section $[\tilde{\xi}, \tilde{\eta}] \in \mathfrak{X}_{loc}(\mathcal{G}_0)$ is a lift of the local section $[\xi, \eta] \in \mathfrak{X}_{loc}(M)$. For each $x \in U$, let $u := \sigma(x) \in \mathcal{G}_0$, then

$$\tilde{\xi}(u) - T_x\sigma(\xi(x)) \in T^0_u\mathcal{G}_0$$

$$\tilde{\eta}(u) - T_x\sigma(\eta(x)) \in T^0_u\mathcal{G}_0$$

$$[\tilde{\xi}, \tilde{\eta}](u) - T_x\sigma([\xi, \eta](x)) \in T^0_u\mathcal{G}_0.$$

Since $T^0\mathcal{G}_0$ lies in the kernels of both θ_{-1} and θ_{-2},

$$F_{-1}(x)(\xi) = \theta_{-1}(u)(\tilde{\xi})$$

$$F_{-1}(x)(\eta) = \theta_{-1}(u)(\tilde{\eta})$$

$$F_{-2}(x)(\mathcal{L}(\xi,\eta)) = \overline{F_{-2}}(x)([\xi,\eta]) = \theta_{-2}(u)([\tilde{\xi},\tilde{\eta}]).$$

Hence

$$\begin{aligned}[F_{-1}(x)(\xi), F_{-1}(x)(\eta)] =&[\theta_{-1}(u)(\tilde{\xi}), \theta_{-1}(u)(\tilde{\eta})]\\ =&-d\theta_{-2}(u)(\tilde{\xi},\tilde{\eta})\\ =&\theta_{-2}(u)([\tilde{\xi},\tilde{\eta}])\\ =&F_{-2}(x)(\mathcal{L}(\xi,\eta))\end{aligned}$$

so F is indeed a local trivialisation of $(gr(TM),\mathcal{L})$ with fiber $\mathfrak{g}_-$.

Finally we construct the reduction ι. For each $u \in \mathcal{G}_0$ with base point $x \in M$,

$$\theta_{-1}(u) : T_u^{-1}\mathcal{G}_0 \to \mathfrak{g}_{-1}$$

descends to a linear isomorphism $H_x \xrightarrow{\cong} \mathfrak{g}_{-1}$, and we denote its inverse by

$$\iota_{-1}(u) : \mathfrak{g}_{-1} \xrightarrow{\cong} H_x.$$

Similarly

$$\theta_{-2}(u) : T_u\mathcal{G}_0 \to \mathfrak{g}_{-2}$$

descends to a linear map $T_xM \to \mathfrak{g}_{-2}$ with kernel H_x, hence it further descends to a linear isomorphism $T_xM/H_x \xrightarrow{\cong} \mathfrak{g}_{-2}$. We denote its inverse by

$$\iota_{-2}(u) : \mathfrak{g}_{-2} \xrightarrow{\cong} T_xM/H_x.$$

Observe that if we let $\sigma : M \supseteq U \to \mathcal{G}_0$ be a local section of $\mathcal{G}_0$ defined on an open neighborhood U of x such that $\sigma(x) = u$, then

$$\iota(u) = (\iota_{-2}(u), \iota_{-1}(u)) : \mathfrak{g}_- \xrightarrow{\cong} gr(T_xM)$$

is the inverse of $F(x) = (F_{-2}(x), F_{-1}(x))$, which is an isomorphism $(gr(T_xM), \mathcal{L}(x)) \xrightarrow{\cong} \mathfrak{g}_-$ of graded Lie algebras. Hence $\iota(u) \in \mathcal{P}(gr(TM))$.

Denote by $p : \mathcal{G}_0 \to M$ the principal bundle. Observe that for $i = -2, -1$, $\theta_i(u)$ is of the form

$$\theta_{-2}(u) : T_u\mathcal{G}_0 \xrightarrow{Tp} T_xM \twoheadrightarrow T_xM/H_x \xrightarrow[\cong]{(\iota_{-2}(u))^{-1}} \mathfrak{g}_{-2}$$

$$\theta_{-1}(u) : T_u^{-1}\mathcal{G}_0 \xrightarrow{Tp} H_x \xrightarrow[\cong]{(\iota_{-1}(u))^{-1}} \mathfrak{g}_{-1}.$$

Since Tp is G_0-invariant and θ is G_0 equivariant, we must have $\iota_i(u\varphi) = \iota_i(u) \circ \varphi_i$ for $i = -2, -1$ and for any $\varphi = (\varphi_{-2}, \varphi_{-1}) \in G_0$, i.e.

$$\iota : \mathcal{G}_0 \to \mathcal{P}(gr(TM)), u \mapsto \iota(u)$$

is G_0-equivariant.

Moreover, ι is smooth if any only if for any local section $\sigma : M \supseteq U \to \mathcal{G}_0$ of $\mathcal{G}_0$, the map

$$\tilde{\iota} : U \times \mathfrak{g}_- \to gr(TM), (x, X) \mapsto \iota(\sigma(x))(X)$$

is smooth. But use σ to define F as above, then F induces an isomorphism $gr(TM)|_U \cong U \times \mathfrak{g}_-$, whose inverse is exactly $\tilde{\iota}$.

Since ι covers id_M, it is a reduction. By construction, it is the unique reduction such that θ equals the pullback of the soldering form on $\iota(\mathcal{G}_0)$. □

Corollary 3.4.1. *Each regular filtered G_0-structure of type $\mathfrak{g}_-$ induces a Lagrangean contact structure on its base manifold.*

Proof. Let $(\mathcal{G}_0, \theta)$ be a regular filtered G_0-structure of type $\mathfrak{g}_-$. By the last proposition, it has an underlying contact structure (M, H) on its base manifold, and there is a unique reduction $\iota : \mathcal{G}_0 \hookrightarrow \mathcal{P}(gr(TM))$ of the frame bundle of (M, H) such that ι is an isomorphism from $(\mathcal{G}_0, \theta)$ to the canonical regular filtered G_0-structure of type $\mathfrak{g}_-$ on $\iota(\mathcal{G}_0)$. In particular, we obtain a Lagrangean contact structure $H = E \oplus F$ by requesting the principal G_0-subbundle $\iota(\mathcal{G}_0) \subseteq \mathcal{P}(gr(TM))$ to be its frame bundle.

□

Lemma 3.4.2. *Let $(M, E \oplus F)$, $(M', E' \oplus F')$ be two Lagrangean contact structures on $(2n+1)$-dimensional manifolds, and let $(\mathcal{G}_0, \theta)$ resp. $(\mathcal{G}_0', \theta')$ their frame bundles together with the soldering forms.*

(i) For any morphism

$$\Phi : (\mathcal{G}_0, \theta) \to (\mathcal{G}_0', \theta')$$

of regular filtered G_0-structures of type $\mathfrak{g}_-$, the base map $f : M \to M'$ is a morphism between the induced Lagrangean contact structures. Moreover, Φ is of the form $u \mapsto gr(Tf) \circ u$.

(ii) Conversely, a morphism $f : (M, E \oplus F) \to (M', E' \oplus F')$ of Lagrangean contact structures admits a unique lift to a morphism of regular filtered G_0-structures of type $\mathfrak{g}_-$ from $(\mathcal{G}_0, \theta)$ to $(\mathcal{G}_0', \theta')$.

Proof. (i) Put $T^{-1}M = H := E \oplus F$ and $T^{-1}M' = H' := E' \oplus F'$. Since Φ is filtration-preserving, so is f, i.e. f is a contactomorphism. In particular, Tf restricts to a map

$$gr_{-1}(Tf) = (Tf)|_H : H \to H'$$

and Tf descends to a map

$$gr_{-2}(Tf) = \underline{Tf} : TM/H \to TM'/H'.$$

Thus we obtain

$$gr(Tf) = (gr_{-2}(Tf), gr_{-1}(Tf)) : gr(TM) \to gr(TM').$$

Let $u \in \mathcal{G}_0$ have base point $x \in M$, then u is an isomorphism $\mathfrak{g}_- \xrightarrow{\cong} gr(T_xM)$ of graded Lie algebras.

We first check that the maps $\Phi(u), gr(T_xf) \circ u : \mathfrak{g}_- \xrightarrow{\cong} gr(T_{f(x)}M)$ agree on $\mathfrak{g}_{-1}$, i.e., for any $X \in \mathfrak{g}_{-1}$, let $u(X) =: \xi \in H_x$, then we have $\Phi(u)(X) = T_xf(\xi)$. Indeed, for any lift $\tilde{\xi} \in T_u^{-1}\mathcal{G}_0$ of ξ, we have $\theta_{-1}(u)(\tilde{\xi}) = u^{-1}(\xi)$. Since the tangent vector $T_u\Phi(\tilde{\xi}) \in T_{\Phi(u)}\mathcal{G}'_0$ is a lift of $T_xf(\xi) \in T_{f(x)}M'$, we have

$$X = u^{-1}(\xi) = \theta_{-1}(u)(\tilde{\xi}) = \theta'_{-1}(\Phi(u))(T_u\Phi(\tilde{\xi})) = \Phi(u)^{-1}(T_xf(\xi))$$

hence $\Phi(u)(X) = T_xf(\xi)$.

Next we check that $\Phi(u), gr(T_xf) \circ u : \mathfrak{g}_- \xrightarrow{\cong} gr(T_{f(x)}M)$ agree on $\mathfrak{g}_{-2}$, i.e., for any $Y \in \mathfrak{g}_{-2}$, let $u(Y) = \underline{\eta} \in T_xM/H_x$ for a tangent vector $\eta \in T_xM$, then we have $\Phi(u)(Y) = gr_{-2}(T_xf)(\underline{\eta})$. Indeed, recall that $gr_{-2}(T_xf)(\underline{\eta}) = \underline{T_xf(\eta)} \in T_{f(x)}M'/H'_{f(x)}$ descends from $T_xf(\eta) \in T_{f(x)}M'$. For any lift $\tilde{\eta} \in T_u\mathcal{G}_0$ of η, we have $\theta_{-2}(u)(\tilde{\eta}) = u^{-1}(\underline{\eta})$. Since $T_u\Phi(\tilde{\eta}) \in T_{\Phi(u)}\mathcal{G}'_0$ is a lift of $T_xf(\eta)$, we have

$$Y = u^{-2}(\underline{\eta}) = \theta_{-2}(u)(\tilde{\eta}) = \theta'_{-2}(\Phi(u))(T\Phi(\tilde{\eta})) = \Phi(u)^{-1}(\underline{T_xf(\eta)})$$

hence $\Phi(u)(Y) = \underline{Tf(\eta)} = gr_{-2}(T_xf)(\underline{\eta})$.

By $\Phi(u) = gr(T_xf) \circ u$, we have

$$E'_{f(x)} = \Phi(u)(\mathfrak{g}^E_{-1}) = gr(T_xf) \circ u(\mathfrak{g}^E_{-1}) = gr(T_xf)(E_x) = T_xf(E_x).$$

Similarly $T_xf(F_x) = F'_{f(x)}$. Hence f is a morphism of Lagrangean contact structures.

(ii) Let $f : (M, E \oplus F) \to (M', E' \oplus F')$ be a morphism of Lagrangean contact structures and let $u \in \mathcal{G}_0$ be above $x \in M$, i.e.

$$u : \mathfrak{g}_- \xrightarrow{\cong} (gr(T_xM), \mathcal{L}(x))$$

is an isomorphism of graded Lie algebras such that u restricts to linear isomorphisms $\mathfrak{g}^E_{-1} \xrightarrow{\cong} E_x$ and $\mathfrak{g}^F_{-1} \xrightarrow{\cong} F_x$. Since Tf is filtration-preserving, we define $gr(Tf)$ as in (i). Then

$$gr(T_xf) : gr(T_xM) \to gr(T_{f(x)}M')$$

restricts to linear isomorphisms $E_x \xrightarrow{\cong} E'_{f(x)}$, $F_x \xrightarrow{\cong} F'_{f(x)}$ and $T_xM/H_x \xrightarrow{\cong} T_{f(x)}M'/H'_{f(x)}$. Moreover, we claim that

$$(gr(T_xf))^*\mathcal{L}(f(x)) = \mathcal{L}(x).$$

Indeed, let $\xi, \eta \in \Gamma_{loc}(H)$ be local vector fields defined around x, $\xi', \eta' \in \Gamma_{loc}(H')$ be local vector fields defined around $f(x)$, such that $f^*(\xi') = \xi$ and $f^*(\eta') = \eta$ (i.e. $(gr_{-1}(Tf))^*(\xi') = \xi$ and $(gr_{-1}(Tf))^*(\eta') = \eta$). Then $f^*([\xi', \eta']) = [\xi, \eta]$, and so $(gr_{-2}(Tf))^*(\mathcal{L}(f(x))(\xi', \eta')) = \mathcal{L}(x)(\xi, \eta)$. In particular,

$$gr(T_xf) : (gr(T_xM), \mathcal{L}(x)) \xrightarrow{\cong} (gr(T_{f(x)}M'), \mathcal{L}(f(x)))$$

is an isomorphism of graded Lie algebras. Hence $gr(Tf) \circ u \in \mathcal{G}_0'$ and we may define a map

$$\Phi : (\mathcal{G}_0, \theta) \to (\mathcal{G}_0', \theta'), u \mapsto gr(Tf) \circ u.$$

Clearly, Φ is G_0-equivariant and smooth. We have $\Phi^*\theta_i' = \theta_i$ for $i = -2, -1$ because for each tangent vector $\tilde{\xi} \in T_u^{-1}\mathcal{G}_0$ lying above $\xi \in H_x$ and for each tangent vector $\tilde{\eta} \in T_u\mathcal{G}_0$ lying above $\eta \in T_xM$ and descends to $\underline{\eta} \in T_xM/H_x$, we have $T_u\Phi(\tilde{\xi})$ lying above $T_xf(\xi) = gr(T_xf)(\xi)$ and $T_u\Phi(\tilde{\eta})$ lying above $T_xf(\eta)$, which descends to $\underline{T_xf(\eta)} = gr(T_xf)(\underline{\eta}) \in T_{f(x)}M'/H'_{f(x)}$. We compute the soldering forms:

$$\theta'_{-1}(\Phi(u))(T_u\Phi(\tilde{\xi})) = \Phi(u)^{-1}(gr(T_xf)(\xi)) = u^{-1}(\xi) = \theta_{-1}(u)(\tilde{\xi})$$

$$\theta'_{-2}(\Phi(u))(T_u\Phi(\tilde{\eta})) = \Phi(u)^{-1}(gr(T_xf)(\underline{\eta})) = u^{-1}(\underline{\eta}) = \theta_{-2}(u)(\tilde{\eta}).$$

Hence Φ is a morphism lifting f. By (i), this is the unique lift. □

Corollary 3.4.2. *Descending to the underlying Lagrangean contact structure yields an equivalence of the category of regular filtered G_0-structures of type $\mathfrak{g}_-$ and the category of Lagrangean contact structures on $(2n+1)$-dimensional manifolds.*

Proof. By Corollary 3.4.1 and Lemma 3.4.2, taking the underlying Lagrangean contact structure and taking the base map is clearly a functor from the category of regular filtered G_0-structures of type $\mathfrak{g}_-$ to the category of Lagrangean contact structures on $(2n+1)$-dimensional manifolds. The functor is essentially surjective because for each Lagrangean contact structure $(M, E \oplus F)$, the underlying Lagrangean contact structure of the frame bundle of $(M, E \oplus F)$ together with the soldering form is again $(M, E \oplus F)$. The functor is full and faithful because descending to base map yields a bijective correspondence between the space of morphisms of regular filtered G_0-structures of type $\mathfrak{g}_-$ and the space of morphisms between the underlying Lagrangean contact structures. Hence the fuctor yields an equivalence of categories. □

Chapter 4

Cartan geometries of type $(Euc(n), O(n))$

4.1 Motivation: the Levi-Civita connection

Let $(\mathcal{G} \to M, \theta)$ be an $O(n)$-structure of type $\mathbb{R}^n$ and let $\mathcal{G} \times_{O(n)} \mathbb{R}^n$ be the associated bundle with respect to the standard representation of $O(n)$ on $\mathbb{R}^n$. We denote the natural projection $\mathcal{G} \times \mathbb{R}^n \to \mathcal{G} \times_{O(n)} \mathbb{R}^n$ by $(u, v) \mapsto [(u, v)]$. Recall that the space of sections of $\mathcal{G} \times_{O(n)} \mathbb{R}^n$ is identified with the space of $O(n)$-equivariant smooth maps $\mathcal{G} \to \mathbb{R}^n$, such that $\sigma \in \Gamma(\mathcal{G} \times_{O(n)} \mathbb{R}^n)$ corresponds to $f \in C^\infty(\mathcal{G}, \mathbb{R}^n)^{O(n)}$ if and only if $\sigma(x) = [(u, f(u))]$ for all $u \in \mathcal{G}$ with base point $x \in M$.

Proposition 4.1.1. *There is a natural isomorphism of vector bundles*

$$TM \xrightarrow{\cong} \mathcal{G} \times_{O(n)} \mathbb{R}^n.$$

In the resulting identification, $\xi \in \mathfrak{X}(M)$ corresponds to the $O(n)$-equivariant function $\theta(\tilde{\xi}) \in C^\infty(\mathcal{G}, \mathbb{R}^n)^{O(n)} = \Gamma(\mathcal{G} \times_{O(n)} \mathbb{R}^n)$, where $\tilde{\xi} \in \mathfrak{X}(\mathcal{G})$ is any lift of ξ.

Proof. We may view $\mathcal{G}$ as a principal $O(n)$-subbundle of $GL(\mathbb{R}^n, TM)$ such that θ is the soldering form (Proposition 2.2.1). Thus there is a natural smooth map

$$\mathcal{G} \times \mathbb{R} \to TM, (u, v) \mapsto u(v)$$

which descends to a vector bundle map

$$\mathcal{G} \times_{O(n)} \mathbb{R} \to TM$$

over id_M, which restricts to a linear isomorphism at each fiber. Hence it is an isomorphism of vector bundles.

Let $\xi \in \mathfrak{X}(M)$ and let $\tilde{\xi} \in \mathfrak{X}(\mathcal{G})$ be any lift of ξ. Then for each $u \in \mathcal{G}$ with base point $x \in M$, $u^{-1}(\xi(x)) = \theta(u)(\tilde{\xi}(u))$ by the definition of soldering form. Therefore $\xi(x) \in T_xM$ corresponds to $[(u, \theta(u)(\tilde{\xi}))] \in \mathcal{G} \times_{O(n)} \mathbb{R}$ by the resulting isomorphism. Hence ξ corresponds to $\theta(\tilde{\xi}) : \mathcal{G} \to \mathbb{R}^n$.

□

Since $TM \cong \mathcal{G} \times_{O(n)} \mathbb{R}^n$, any principal connection $\gamma \in \Omega^1(\mathcal{G}, \mathfrak{o}(n))$ on $\mathcal{G}$ induces an affine connection

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

on M as follows: denote by

$$\mathfrak{X}(M) \to \mathfrak{X}(\mathcal{G}), \xi \mapsto \xi^h$$

the horizontal lift corresponding to γ, then for $\xi, \eta \in \mathfrak{X}(M)$ let $\Phi \in C^\infty(\mathcal{G}, \mathbb{R}^n)^{O(n)} \cong \Gamma(\mathcal{G} \times_{O(n)} \mathbb{R}^n)$ correspond to η. Then $\nabla_\xi \eta \in \mathfrak{X}(M)$ corresponds to $\xi^h.\Phi \in C^\infty(\mathcal{G}, \mathbb{R}^n)^{O(n)} \cong \Gamma(\mathcal{G} \times_{O(n)} \mathbb{R}^n)$.

Proposition 4.1.2. ∇ *is compatible with the underlying Riemannian metric g on M.*

Proof. For $\xi, \eta_i \in \mathfrak{X}(M)$, $i = 1, 2$, η_i corresponds to $\theta(\eta_i^h) \in C^\infty(\mathcal{G}, \mathbb{R}^n)^{O(n)}$ by Proposition 4.1.1, and $\nabla_\xi \eta_i \in \mathfrak{X}(M)$ corresponds to $\xi^h.\theta(\eta_i^h) \in C^\infty(\mathcal{G}, \mathbb{R}^n)^{O(n)}$. By Corollary 2.2.1,

$$g(\nabla_\xi \eta_1, \eta_2) + g(\eta_1, \nabla_\xi \eta_2) \in C^\infty(M)$$

is related by $\mathcal{G} \to M$ to

$$\langle \xi^h.\theta(\eta_1^h), \theta(\eta_2^h) \rangle + \langle \theta(\eta_1^h), \xi^h.\theta(\eta_2^h) \rangle = \xi^h.\langle \theta(\eta_1^h), \theta(\eta_2^h) \rangle \in C^\infty(\mathcal{G}),$$

and the right hand side is related to $\xi.g(\eta_1, \eta_2) \in C^\infty(M)$, hence $g(\nabla_\xi \eta_1, \eta_2) + g(\eta_1, \nabla_\xi \eta_2) = \xi.g(\eta_1, \eta_2)$. □

It turns out that exactly one of the principal connections on $\mathcal{G}$ induces a torsion-free affine connection on M, i.e. induces the Levi-Civita connection on the underlying Riemannian manifold M, which we will show in the next section. By the end of this section, we give the example of homogeneous affine n-space, in which there is a canonical principal connection on a canonical $O(n)$-structure of type $\mathbb{R}^n$, which induces the Levi-Civita connection on the affine n-space.

First we observe that with the notations above, for $\xi, \eta \in \mathfrak{X}(M)$, the torsion

$$\tau(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta] \in \mathfrak{X}(M)$$

corresponds to the $O(n)$-equivariant map $\mathcal{G} \to \mathbb{R}^n$ given by

$$\xi^h.\theta(\eta^h) - \eta^h.\theta(\xi^h) - \theta([\xi^h, \eta^h])$$

as $[\xi^h, \eta^h]$ is a lift of $[\xi, \eta]$.

Example 4.1.1. *(The Levi-Civita connection on Euclidean n-space) There is a homogeneous model on the affine n-space*

$$E^n = \begin{pmatrix} 1 \\ \mathbb{R}^n \end{pmatrix} \subseteq \mathbb{R}^{n+1}$$

by the natural action of

$$Euc(n) = \begin{pmatrix} 1 & 0 \\ \mathbb{R}^n & O(n) \end{pmatrix} \subseteq GL(n+1, \mathbb{R})$$

on it: if we fix a base point $x_0 := (1, 0, ..., 0)^t \in E^n$, we obtain a principal bundle

$$Euc(n) \to E^n, g \mapsto gx_0$$

whose structure group is given by the isotropy group

$$\begin{pmatrix} 1 & 0 \\ 0 & O(n) \end{pmatrix} =: O(n)$$

of x_0. Hence it descends to $Euc(n)/O(n) \cong E^n$. We denote by $\mathfrak{euc}(n)$ the Lie algebra of $Euc(n)$ and denote by

$$\omega \in \Omega^1(Euc(n), \mathfrak{euc}(n))$$

the left Maurer-Cartan form on $Euc(n)$. Thus ω is an $O(n)$-equivariant one-form with respect to the adjoint action of the isotropy group on $\mathfrak{euc}(n)$.

Notice that the representation on $\mathfrak{euc}(n) = \begin{pmatrix} 0 & 0 \\ \mathbb{R}^n & \mathfrak{o}(n) \end{pmatrix}$ has an $O(n)$-invariant decomposition into the standard action of $O(n)$ on $\mathbb{R}^n$ and the usual adjoint action of $O(n)$ on $\mathfrak{o}(n)$. In particular, ω decompose to its $\mathbb{R}^n$ component

$$\theta \in \Omega^1(Euc(n), \mathfrak{o}(n))^{O(n)}$$

and its $\mathfrak{o}(n)$-component

$$\gamma \in \Omega^1(Euc(n), \mathfrak{euc}(n))^{O(n)},$$

and both components are $O(n)$-equivariant. Clearly $(Euc(n), \theta)$ is an $O(n)$-structure of type $\mathbb{R}^n$ on E^n, and its underlying Riemannian metric is just the standard Riemannian metric on E^n, and γ is a principal connection on this $O(n)$-structure of type $\mathbb{R}^n$. We claim that γ induces an affine connection on E^n, whose torsion τ vanished identically. That means, γ induces the Levi-Civita connection on E^n.

Indeed, as a property of the Maurer-Cartan form, the curvature two-form $K \in \Omega^2(Euc(n), \mathfrak{euc}(n))$ defined by

$$\begin{aligned} K(\tilde{\xi}, \tilde{\eta}) :=& d\omega(\tilde{\xi}, \tilde{\eta}) + [\omega(\tilde{\xi}), \omega(\tilde{\eta})] \\ =& \tilde{\xi}.\omega(\tilde{\eta}) - \tilde{\eta}.\omega(\tilde{\xi}) - \omega([\tilde{\xi}, \tilde{\eta}]) + [\omega(\tilde{\xi}), \omega(\tilde{\eta})] \end{aligned}$$

for all $\tilde{\xi}, \tilde{\eta} \in \mathfrak{X}(Euc(n))$ vanishes identically. This can be checked by inserting in the left invariant vector fields of $Euc(n)$.

Now let $h : \mathfrak{X}(E^n) \to \mathfrak{X}(Euc(n))$ denote the horizontal lift corresponding to γ. Then $\omega \circ h$ always takes values in $\mathbb{R}^n$. Hence $[\omega(\xi^h), \omega(\eta^h)] \subseteq [\mathbb{R}^n, \mathbb{R}^n] = 0$ for all $\xi, \eta \in \mathfrak{X}(E^n)$. In paticular, the $\mathbb{R}^n$-component of $K(\xi^h, \eta^h)$ equals

$$\xi^h.\theta(\eta^h) - \eta^h.\theta(\xi^h) - \theta([\xi^h, \eta^h])$$

which corresponds to $\tau(\xi, \eta)$. As K vanishes identically, so does its $\mathbb{R}^n$-component. Hence $\tau(\xi, \eta) = 0$.

4.2 Normal Cartan geometries of type $(Euc(n), O(n))$

We continue with the task of finding a principal connection on an $O(n)$-structure of type $\mathbb{R}^n$ which induces the Levi-Civita connection on the underlying Riemannian manifold, this is easier in the language of Cartan geometries. Recall that in Example 4.1.1, the left Maurer-Cartan form ω on $Euc(n)$ includes the information of an $O(n)$-structure (via the form θ) and a principal connection γ which induces the Levi-Civita connection on E^n. ω can be phrased as a Cartan connection of type $(Euc(n), O(n))$ on the principal $O(n)$-bundle $Euc(n) \to E^n$, in the sense of the following definition.

Definition 4.2.1. *Let $H \subseteq G$ be a Lie subgroup, and denote by $\mathfrak{h} \subseteq \mathfrak{g}$ their Lie algebras. Consider the representation of H on $\mathfrak{g}$ obtained from restricting the adjoint representation of G.*

A Cartan geometry $(\mathcal{G}, \omega)$ of type (G, H) is a principal H-bundle $\mathcal{G}$ together with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})^H$, which is an H-equivariant $\mathfrak{g}$-valued one-form, such that ω is a linear isomorphism at each fiber, and ω reproduces the generators of fundamental vector fields, i.e. for each fundamental vector field ζ_X generated by $X \in \mathfrak{h}$, we have $\omega(u)(\zeta_X) = X$ at all points $u \in \mathcal{G}$.

Moreover, the curvature two-form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of ω is given by

$$\begin{aligned} K(\tilde{\xi}, \tilde{\eta}) :=& d\omega(\tilde{\xi}, \tilde{\eta}) + [\omega(\tilde{\xi}), \omega(\tilde{\eta})] \\ =& \tilde{\xi}.\omega(\tilde{\eta}) - \tilde{\eta}.\omega(\tilde{\xi}) - \omega([\tilde{\xi}, \tilde{\eta}]) + [\omega(\tilde{\xi}), \omega(\tilde{\eta})]. \end{aligned}$$

for all $\tilde{\xi}, \tilde{\eta} \in \mathfrak{X}(\mathcal{G})$. Equivalently, the curvature two-form can be encoded as the map

$$\kappa : \mathcal{G} \to L(\Lambda^2\mathfrak{g}, \mathfrak{g})$$

$$\kappa(X, Y) := K(\omega^{-1}(X), \omega^{-1}(Y))$$

for all $X, Y \in \mathfrak{g}$, where $\omega^{-1}(X), \omega^{-1}(Y) \in \mathfrak{X}(\mathcal{G})$ refer to the vector fields corresponding to X resp. Y in the trivialisation $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$ induced by ω. We call κ the curvature function of ω.

A morphism $\Phi : (\mathcal{G}, \omega) \to (\mathcal{G}', \omega')$ between two Cartan geometries of type (G, H) is a principal bundle map $\Phi : \mathcal{G} \to \mathcal{G}'$ such that $\Phi^*\omega' = \omega$.

Lemma 4.2.1. *The curvature two-form K of a Cartan geometry $(\mathcal{G}, \omega)$ of type (G, H) is (i) horizontal and (ii) H-equivariant.*

Hence κ is an H-equivariant map $\mathcal{G} \to L(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$, where the action of H on $\mathfrak{g}/\mathfrak{h}$ is induced by the action on $\mathfrak{g}$ coming from the adjoint action of G.

Proof. (i) Let ζ_X be the fundamental vector field generated by X in the Lie algebra $\mathfrak{h}$ of H, since

$$d\omega(\zeta_X, \cdot) = i_{\zeta_X} d\omega = \mathcal{L}_{\zeta_X}\omega - di_{\zeta_X}\omega = \mathcal{L}_{\zeta_X}\omega = \frac{d}{dt}\Big|_{t=0}(Fl_t^{\zeta_X})^*\omega$$

$$= \frac{d}{dt}\Big|_{t=0}(r^{exp(tX)})^*\omega = \frac{d}{dt}\Big|_{t=0}(-Ad \circ exp(tX))\omega = ad(-X) \circ \omega,$$

we have $d\omega(\zeta_X(u), \eta(u)) = [-X, \omega(\eta(u))]$, thus $K(\zeta_X(u), \eta(u)) = 0$.

In particular, for $X \in \mathfrak{h}, Y \in \mathfrak{g}$ we have $\kappa(X, Y) = K(\zeta_X, \omega^{-1}(Y)) = 0$. Hence κ is a map $\mathcal{G} \to L(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$.

(ii) Follows from the H-equivariancy of ω and the fact that d commutes with the pull-back of the principal right action by h for all $h \in H$.

In particular, let $X, Y \in \mathfrak{g}, u \in \mathcal{G}, h \in H$, and define $\xi := \omega^{-1}(X)(u) \in T_u\mathcal{G}$ and $\eta := \omega^{-1}(Y)(u) \in T_u\mathcal{G}$. Since ω is H-equivariant, $\omega(uh)(T_u r^h(\xi)) = Ad(h^{-1})(X)$. Hence $\omega^{-1}(Ad(g^{-1})(X))(uh) = T_u r^h(\xi)$. Similarly $\omega^{-1}(Ad(g^{-1})(Y))(uh) = T_u r^h(\eta)$. We have

$$\begin{aligned}
\kappa(ug)(Ad(g^{-1})(X), Ad(g^{-1})(Y)) =& K(ug)(\omega^{-1}(Ad(g^{-1})(X)), \omega^{-1}(Ad(g^{-1})(Y))) \\
=& K(ug)(T_u r^h(\xi), T_u r^h(\eta)) \\
=& K(u)(\xi, \eta) \\
=& \kappa(u)(X, Y)
\end{aligned}$$

hence κ is H-equivariant.

□

In Example 4.1.1, we also see that the fact that γ induces a torsion free connection follows from the fact that K vanishes identically, more precisely that the $\mathbb{R}^n$-component of K vanishes identically.

Similarly we may also describe an $O(n)$-structure of type $\mathbb{R}^n$ together with a principal connection as a Cartan geometry, and observe a relation between the torsion of the induced connection and the curvature two-form.

Proposition 4.2.1. *Let $(\mathcal{G} \to M, \theta)$ be an $O(n)$-structure of type $\mathbb{R}^n$, and let γ be a principal connection on $\mathcal{G}$. Then $\omega := \theta + \gamma \in \Omega^1(\mathcal{G}, \mathfrak{euc}(n))$ is a Cartan connection of type $(Euc(n), O(n))$ on $\mathcal{G}$. Let K be the curvature of ω, then the $\mathfrak{o}(n)$-component of K is the principal curvature of γ.*

Moreover, γ induces a torsion-free, hence the Levi-Civita connection on M if and only if the $\mathbb{R}^n$-component of K vanishes identically.

Proof. Since θ is strictly horizontal and γ reproduces the generators of fundamental vector fields, ω is a linear isomorphism at each fiber, and it reproduces the generators of fundamental vector fields. Since θ and γ are both $O(n)$-equivariant, so is ω. Therefore ω is a Cartan connection of type $(Euc(n), O(n))$ on $\mathcal{G}$.

Recall that the principal curvature $\Omega \in \Omega^2(\mathcal{G}, \mathfrak{o}(n))$ of γ is given by

$$\begin{aligned}\Omega(\tilde{\xi}, \tilde{\eta}) =& d\gamma(\tilde{\xi}, \tilde{\eta}) + [\gamma(\tilde{\xi}), \gamma(\tilde{\eta})] \\ =& \tilde{\xi}.\gamma(\tilde{\eta}) - \tilde{\eta}.\gamma(\tilde{\xi}) - \gamma([\tilde{\xi}, \tilde{\eta}]) + [\gamma(\tilde{\xi}), \gamma(\tilde{\eta})]\end{aligned}$$

for all $\tilde{\xi}, \tilde{\eta} \in \mathfrak{X}(\mathcal{G})$. On the otherhand, we have

$$K(\tilde{\xi}, \tilde{\eta}) = \tilde{\xi}.\omega(\tilde{\eta}) - \tilde{\eta}.\omega(\tilde{\xi}) - \omega([\tilde{\xi}, \tilde{\eta}]) + [\omega(\tilde{\xi}), \omega(\tilde{\eta})]$$

where $[\omega(\tilde{\xi}), \omega(\tilde{\eta})]$ equals

$$[\gamma(\tilde{\xi}), \gamma(\tilde{\eta})] + [\gamma(\tilde{\xi}), \theta(\tilde{\eta})] + [\theta(\tilde{\xi}), \gamma(\tilde{\eta})] + [\theta(\tilde{\xi}), \theta(\tilde{\eta})],$$

whose first summand is in $\mathfrak{o}(n)$, second and third summands in $\mathbb{R}^n$, and last summand is zero. Therefore the $\mathfrak{o}(n)$-component of $K(\tilde{\xi}, \tilde{\eta})$ is exactly $\Omega(\tilde{\xi}, \tilde{\eta})$.

Let $h : \mathfrak{X}(M) \to \mathfrak{X}(\mathcal{G})$ be the horizontal lift induced by γ, and τ be the torsion of the affine connection on M induced by γ. Then $\omega \circ h$ takes values in $\mathbb{R}^n$, hence for all $\xi, \eta \in \mathfrak{X}(M)$, the $\mathbb{R}^n$-component of $K(\xi^h, \eta^h)$ equals

$$\xi^h.\theta(\eta^h) - \eta^h.\theta(\xi^h) - \theta([\xi^h, \eta^h]) + 0$$

which corresponds to $\tau(\xi, \eta)$. Since K is horizontal, τ vanishes identically if and only if the $\mathbb{R}^n$-component of K vanishes identically. □

In this case we describe $(\mathcal{G}, \omega)$ as a normal Cartan geometry of type $(Euc(n), O(n))$.

Definition 4.2.2. *A Cartan geometry $(\mathcal{G}, \omega)$ of type $(Euc(n), O(n))$ is said to be normal if and only if its curvature two-form $K \in \Omega^2_{hor}(\mathcal{G}, \mathfrak{euc}(n))^{O(n)}$ takes values in $\mathfrak{o}(n)$, or equivalently, κ is an $O(n)$-equivariant map $\mathcal{G} \to L(\Lambda^2(\mathfrak{euc}(n)/\mathfrak{o}(n)), \mathfrak{euc}(n)) = L(\Lambda^2\mathbb{R}^n, \mathfrak{euc}(n))$ which takes values in $\mathfrak{o}(n)$.*

Thus we rephrase our task as finding a principal connection γ on an $O(n)$-structure $(\mathcal{G}, \theta)$ of type $\mathbb{R}^n$ such that $\omega := \theta + \gamma$ is a normal Cartan connection of type $(Euc(n), O(n))$. Recall that a principal connection γ on $\mathcal{G}$ always exists. By choosing such γ, we fix a trivialisation

$$T\mathcal{G} \cong \mathcal{G} \times \mathfrak{euc}(n)$$

induced by $\omega = \theta + \gamma$. We also recall that the space of all principal connections on $\mathcal{G}$ is an affine space modelled over the space $\Omega^1_{hor}(\mathcal{G}, \mathfrak{o}(n))^{O(n)}$ of $\mathfrak{o}(n)$-valued horizontal equivariant one-forms on $\mathcal{G}$. That is, a one-form $\tilde{\gamma} \in \Omega^1(\mathcal{G}, \mathfrak{o}(n))$ is a principal connection on $\mathcal{G}$ if and only if

$$\tilde{\gamma} - \gamma : T\mathcal{G} \to \mathfrak{o}(n)$$

is an $O(n)$-equivariant one-form whose kernel contains $\mathcal{V}\mathcal{G}$. We use the trivialisation $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{euc}(n)$ to identify $\Omega^1_{hor}(\mathcal{G}, \mathfrak{o}(n))^{O(n)}$ with the space of $O(n)$-equivariant smooth maps

$$\mathcal{G} \to L(\mathfrak{euc}(n)/\mathfrak{o}(n), \mathfrak{o}(n)) = L(\mathbb{R}^n, \mathfrak{o}(n)).$$

In particular, a principal connection $\tilde{\gamma} \in \Omega^1(\mathcal{G}, \mathfrak{o}(n))$ corresponds to an $O(n)$-equivariant map

$$\Phi : \mathcal{G} \to L(\mathbb{R}^n, \mathfrak{o}(n)),$$
$$\Phi(u)(X) = (\tilde{\gamma} - \gamma)(\omega(u)^{-1}(X))$$

for all $u \in \mathcal{G}$, $X \in \mathbb{R}^n$. Conversely, an $O(n)$-equivariant map $\Phi : \mathcal{G} \to L(\mathbb{R}^n, \mathfrak{o}(n)) = L(\mathfrak{euc}(n)/\mathfrak{o}(n), \mathfrak{o}(n))$ corresponds to a principal connection $\tilde{\gamma} \in \Omega^1(\mathcal{G}, \mathfrak{o}(n))$ such that for each $\tilde{\xi} \in T_u\mathcal{G}$,

$$\tilde{\gamma}(u)(\tilde{\xi}) = \gamma(u)(\tilde{\xi}) + \Phi(u)(\omega(u)(\tilde{\xi})).$$

Let $\tilde{\gamma}$ be a principal connection on $\mathcal{G}$ with corresponding $O(n)$-equivariant map $\Phi : \mathcal{G} \to L(\mathbb{R}^n, \mathfrak{o}(n))$. Then $\tilde{\omega} := \theta + \tilde{\gamma}$ is a Cartan connection of type $(Euc(n), O(n))$. Let $\tilde{K} \in \Omega^2(\mathcal{G}, \mathfrak{euc}(n))$ and $\tilde{\kappa} : \mathcal{G} \to L(\Lambda^2\mathbb{R}^n, \mathfrak{euc}(n))$ be the curvature two-forms of $\tilde{\omega}$, and let $K \in \Omega^2(\mathcal{G}, \mathfrak{euc}(n))$ and $\kappa : \mathcal{G} \to L(\Lambda^2\mathbb{R}^n, \mathfrak{euc}(n))$ be the curvature two-forms of ω.

Lemma 4.2.2. *Changing from γ to $\tilde{\gamma}$ with corresponding function $\Phi : \mathcal{G} \to L(\mathbb{R}^n, \mathfrak{o}(n))$, the $\mathbb{R}^n$-component of $\tilde{\kappa} - \kappa : \mathcal{G} \to L(\Lambda^2\mathbb{R}^n, \mathfrak{euc}(n))$ equals $\partial \circ \Phi$. Here*

$$\partial : L(\mathbb{R}^n, \mathfrak{o}(n)) \to L(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$$

is the linear map defined by $\partial\varphi(X, Y) = \varphi(X)Y - \varphi(Y)X$ for all $\varphi \in L(\mathbb{R}^n, \mathfrak{o}(n))$, $X, Y \in \mathbb{R}^n$.

Proof. For each $u \in \mathcal{G}$, $X, Y \in \mathbb{R}^n$,

$$\begin{aligned}\tilde{\kappa}(u)(X, Y) =& \tilde{K}(u)(\tilde{\omega}(u)^{-1}(X), \tilde{\omega}(u)^{-1}(Y)) \\ =& \tilde{K}(u)(\omega(u)^{-1}(X), \omega(u)^{-1}(Y))\end{aligned}$$

because $\theta(u)(\tilde{\omega}(u)^{-1}(X)) = X = \theta(u)(\omega(u)^{-1}(X))$ implies that

$$\tilde{\omega}(u)^{-1}(X) - \omega(u)^{-1}(X) \in \mathcal{V}_u\mathcal{G},$$

and similarly

$$\tilde{\omega}(u)^{-1}(Y) - \omega(u)^{-1}(Y) \in \mathcal{V}_u\mathcal{G}.$$

Then

$$
\begin{aligned}
(\tilde{\kappa}-\kappa)(u)(X,Y) =&(\tilde{K}-K)(u)(\omega^{-1}(X),\omega^{-1}(Y))\\
=&\omega^{-1}(X)(u).\Phi(Y)-\omega^{-1}(Y)(u).\Phi(X)-\Phi(u)(\omega([\omega^{-1}(X),\omega^{-1}(Y)]))\\
&+[\Phi(u)(X),\Phi(u)(Y)]+[X,\Phi(u)(Y)]+[\Phi(u)(X),Y].
\end{aligned}
$$

Since the Lie bracket in $\mathfrak{euc}(n)$ restricted to $\mathbb{R}^n\times\mathbb{R}^n$ is zero, and restricted to $\mathfrak{o}(n)\times\mathbb{R}^n$ is the usual matrix multiplication $\mathfrak{o}(n)\times\mathbb{R}^n\to\mathbb{R}^n$, the $\mathbb{R}^n$-component of $(\tilde{\kappa}-\kappa)(u)(X,Y)$ equals

$$-[\Phi(u)(Y),X]+[\Phi(u)(X),Y]=-\Phi(u)(Y)(X)+\Phi(u)(X)(Y)=(\partial\circ\Phi(u))(X,Y).$$

□

Lemma 4.2.3. $\partial : L(\mathbb{R}^n,\mathfrak{o}(n))\to L(\Lambda^2\mathbb{R}^n,\mathbb{R}^n)$ *is an* $O(n)$*-equivariant linear isomorphism.*

Proof. We check that ∂ is $O(n)$-equivariant. Let $A\in O(n)$, and let $A.$ denote the $O(n)$-representations on $L(\mathbb{R}^n,\mathfrak{o}(n))$ and $L(\Lambda^2\mathbb{R}^n,\mathbb{R}^n)$ by A. Then

$$(A.\varphi)(X)=Ad(A)(\varphi(A^{-1}X))=A\varphi(A^{-1}X)A^{-1}$$

where the right hand side denotes the multiplication of three matrices, thus

$$
\begin{aligned}
&(\partial(A.\varphi))(X,Y)\\
=&A\varphi(A^{-1}X)A^{-1}Y-A\varphi(A^{-1}Y)A^{-1}X\\
=&A(\partial\varphi(A^{-1}X,A^{-1}Y))=(A.(\partial\varphi))(X,Y).
\end{aligned}
$$

Now we show that ∂ is injective, hence is a linear isomorphism as φ maps between two vector spaces of the same dimension. Recall that a matrix $A\in\mathfrak{gl}(n,\mathbb{R})$ is in $\mathfrak{o}(n)$ if and only if $\langle Av,w\rangle+\langle v,Aw\rangle=0$ for all $v,w\in\mathbb{R}^n$, where $\langle\ ,\ \rangle$ is the standard inner product on $\mathbb{R}^n$. Thus $\varphi(u)\in\mathfrak{o}(n)$ for all $u\in\mathbb{R}^n$ means

$$\langle\varphi(u)(v),w\rangle=-\langle\varphi(u)(w),v\rangle \tag{4.1}$$

for all $u,v,w\in\mathbb{R}$. Suppose $\partial\varphi=0$, i.e. $\varphi(u)(v)-\varphi(v)(u)=0$ for all $u,v\in\mathbb{R}^n$, thus

$$\langle\varphi(u)(v),w\rangle=\langle\varphi(v)(u),w\rangle \tag{4.2}$$

for all $u,v,w\in\mathbb{R}$. Applying Eq. (4.1) and Eq. (4.2) alternatingly, we have

$$\langle\varphi(u)(v),w\rangle=-\langle\varphi(u)(w),v\rangle=-\langle\varphi(w)(u),v\rangle=\langle\varphi(w)(v),u\rangle$$

$$=\langle\varphi(v)(w),u\rangle=-\langle\varphi(v)(u),w\rangle=-\langle\varphi(u)(v),w\rangle$$

for all $u,v,w\in\mathbb{R}$, which implies that φ is trivial. □

In particular, $\tilde{\omega}$ is a normal Cartan connection if and only if the $\mathbb{R}^n$-component $(\tilde{\kappa}-\kappa)_{\mathbb{R}^n}$ of $\tilde{\kappa}-\kappa$, which equals $\partial \circ \Phi$, coincides with the $\mathbb{R}^n$-component $(-\kappa)_{\mathbb{R}^n}$ of $-\kappa$. In this case $\Phi = \partial^{-1} \circ (-\kappa)_{\mathbb{R}^n}$. Note that we have $\Phi \in C^\infty(\mathcal{G}, L(\mathbb{R}^n, \mathfrak{o}(n)))^{O(n)}$ because both $\partial^{-1} : L(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) \xrightarrow{\cong} L(\mathbb{R}^n, \mathfrak{o}(n))$ and $(-\kappa)_{\mathbb{R}^n} : \mathcal{G} \to L(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$ are $O(n)$-equivariant. Thus the corresponding principal connection $\tilde{\gamma}$ is the unique principal connection such that $\tilde{\omega} = \theta + \tilde{\gamma}$ is a normal Cartan connection.

Moreover, recall (Corollary 2.2.2) that a morphism of $O(n)$-structures of type $\mathbb{R}^n$ is an equivalent description of its underlying local isometry on Riemannian n-manifolds. From Riemannian geometry we know that a local isometry pulls back the Levi-Civita connection. In the language of Cartan geometry, this is described as that each morphism of $O(n)$-structure of type $\mathbb{R}^n$ is a morphism of normal Cartan geometries of type $(Euc(n), O(n))$.

Proposition 4.2.2. *Let $\Phi : (\mathcal{G}, \theta) \to (\mathcal{G}', \theta')$ be a morphism of $O(n)$-structures of type $\mathbb{R}^n$, and let γ resp. γ' be the principal connections on $\mathcal{G}$ resp. $\mathcal{G}'$ inducing the Levi-Civita connections on the underlying Riemannian manifolds. Then Φ is a morphism of Cartan geometries of type $(Euc(n), O(n))$ from $(\mathcal{G}, \theta + \gamma)$ to $(\mathcal{G}', \theta' + \gamma')$.*

Proof. $\Phi^*\gamma'$ is a principal connection on $\mathcal{G}$ because Φ is $O(n)$ equivariant, thus sends a fundamental vector field of $\mathcal{G}$ to fundamental vector field of $\mathcal{G}'$ with the same generator. Since $\Phi^*\theta' = \theta$, $\theta + \Phi^*\gamma' = \Phi^*(\theta' + \gamma')$ is a Cartan connection on $\mathcal{G}$. It curvature equals the pullback of the curvature of $\theta' + \gamma'$, hence $\theta + \Phi^*\gamma'$ is a normal Cartan connection on $\mathcal{G}$, and we must have $\Phi^*\gamma' = \gamma$, so Φ is also a morphism of Cartan geometries. □

Corollary 4.2.1. *The category of normal Cartan geometries of type $(Euc(n), O(n))$ is equivalent to the category of $O(n)$-structures of type $\mathbb{R}^n$.*

Proof. There is a functor from the category of normal Cartan geometries of type $(Euc(n), O(n))$ to the category of $O(n)$-structures of type $\mathbb{R}^n$: if $(\mathcal{G}, \omega)$ is a normal Cartan geometry of type $(Euc(n), O(n))$, we may decompose ω to its $\mathbb{R}^n$-component $\theta \in \Omega^1(\mathcal{G}, \mathbb{R}^n)^{O(n)}$ and its $\mathfrak{o}(n)$-component $\gamma \in \Omega^1(\mathcal{G}, \mathfrak{o}(n))^{O(n)}$. Since ω reproduces the generators, so does γ, hence γ is a principal connection on $\mathcal{G}$; for the same reason, θ is horizontal, since ω is a linear isomorphism at each fiber, θ must be strictly horizontal, hence $(\mathcal{G}, \theta)$ is an $O(n)$-structure of type $\mathbb{R}^n$. Moreover, a morphism of normal Cartan geometries of type $(Euc(n), O(n))$ preserves the Cartan connections, hence preserves the $\mathbb{R}^n$-component of the Cartan connections, i.e. preserves the $O(n)$-structure forms.

Conversely, we know that for each $O(n)$-structure $(\mathcal{G}, \theta)$ of type $\mathbb{R}^n$, there is a unique principal connection γ on $\mathcal{G}$ such that $(\mathcal{G}, \theta + \gamma)$ is a normal Cartan geometry of type $(Euc(n), O(n))$, and we also know that a morphism between two $O(n)$-structures of type $\mathbb{R}^n$ is itself a morphism between the induced normal Cartan connections. Thus we obtain a functor from the category of $O(n)$-structures of type $\mathbb{R}^n$ to the category of normal Cartan geometries of type $(Euc(n), O(n))$.

Clearly the composition of the two functors in either order yields the identity functor. Hence the two categories are equivalent. □

Remark 4.2.1. *We know that the group of isometries on E^n equals $Euc(n)$. This can also be concluded from the theory of Cartan geometries: we cite that*

- *If $G \to G/H$ is a homogeneous structure, G/H is connected and ω is the left Maurer-Cartan form on G, then the automorphisms on the Cartan geometry (G, ω) of type (G, H) are exactly the left multiplications by G ([2]: Proposition 1.5.2(2)).*

Hence the automorphism group on $(Euc(n) \to E^n, \omega)$ is $Euc(n)$. By the categorial equivalence, this is isomorphic to the group of isometries on E^n. Moreover, from Riemannian geometry we have that geodesics are preserved by isometries. In particular an isometry $f : M \to M$ on a connected Riemannian n-manifold is completely determined by $f(x)$ and $T_x f : T_x M \to T_{f(x)} M$. If M is connected, we see that the isometry group has dimension at most $n + dim(O(n)) = dim(Euc(n))$. In the theory of Cartan geometries, we have that

- *The automorphism group of a Cartan geometry of type (G, H) over a connected manifold is a Lie group of dimension at most $dim(G)$ ([2]: Theorem 1.5.11).*

Hence the automorphism group of a normal Cartan geometry of type $(Euc(n), O(n))$ over a connected manifold is a Lie group of dimension $\leq dim(Euc(n))$. By the categorial equivalence, the isometry group on the underlying Riemannian manifold is also a Lie group of dimension at most $dim(Euc(n))$.

Chapter 5

Cartan geometry description of Lagrangean contact structures

5.1 The homogeneous model

Recall from Example 3.3.2 the canonical Lagrangean contact structure $H = E \oplus F$ on the flag manifold $F_{1,n+1}(\mathbb{R}^{n+2})$, with E being the vertical bundle of

$$F_{1,n+1}(\mathbb{R}^{n+2}) \to \mathbb{R}P^{(n+1)*}, (V_1, V_{n+1}) \mapsto V_{n+1}$$

and F being the vertical bundle of

$$F_{1,n+1}(\mathbb{R}^{n+2}) \to \mathbb{R}P^{(n+1)}, (V_1, V_{n+1}) \mapsto V_1.$$

We claim that this Lagrangean contact structure is homogeneous under $PGL(n+2,\mathbb{R})$.

Indeed, the standard action of $GL(n+2,\mathbb{R})$ on $\mathbb{R}^{n+2}$ maps subspaces to subspaces, hence $GL(n+2,\mathbb{R})$ acts on $F_{1,n+1}(\mathbb{R}^{n+2})$. We see that the action is transitive and the subbundles E and F are invariant under this action. An element in $GL(n+2,\mathbb{R})$ acts trivially on $F_{1,n+1}(\mathbb{R}^{n+2})$ if and only if it preserves all lines in $\mathbb{R}^{n+2}$, i.e. it is a multiple of the identity. In particular, identifying each matrix in $GL(n+2,\mathbb{R})$ with its nonzero multiples we obtain a group

$$G := PGL(n+2,\mathbb{R}),$$

which acts transitively and effectively on $F_{1,n+1}(\mathbb{R}^{n+2})$. If n is odd, we realise G as $SL(n+2,\mathbb{R})$; if n is even, we realise G as elements in $GL(n+2,\mathbb{R})$ with determinant ± 1 and identify each matrix with its negative. By the realisation in either parity, G around the identity is locally isomorphic to $SL(n+2,\mathbb{R})$ around $\mathbb{I}_{n+2}$, thus the Lie algebra $\mathfrak{g}$ of G is $\mathfrak{sl}(n+2,\mathbb{R})$.

Let's fix a base point

$$x_0 := (\mathbb{R} \times \{0\}^{n+1}, \mathbb{R}^{n+1} \times \{0\}) \in F_{1,n+1}(\mathbb{R}^{n+2})$$

and fix the block size $(1, n, 1)$ on all $(n+2)\times(n+2)$ matrices. Then the isotropy subgroup $P \subseteq G$ fixing the base point is the image of block-upper triangular matrices under the quotient map $GL(n+2,\mathbb{R}) \twoheadrightarrow PGL(n+2,\mathbb{R})$. We obtain a principal P-bundle

$$p : G \to F_{1,n+1}(\mathbb{R}^{n+2}), g \mapsto g.x_0$$

which descends to an isomorphism $G/P \cong F_{1,n+1}(\mathbb{R}^{n+2})$.

We decompose the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+2,\mathbb{R})$ as

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^E & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^F & \mathfrak{g}_0 \end{pmatrix}$$

indicated as block matrices. Then the Lie algebra $\mathfrak{p}$ of P equals $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and we have

$$TF_{1,n+1}(\mathbb{R}^{n+2}) \cong G \times_P \mathfrak{g}/\mathfrak{p}. \tag{$*$}$$

At the base point we see that $T_e p(\mathfrak{g}_{-1}^E) = E_{x_0}$ and $T_e p(\mathfrak{g}_{-1}^F) = F_{x_0}$. Hence for all $g \in G$, $x := p(g) \in F_{1,n+1}(\mathbb{R}^{n+2})$ we have

$$T_g p(\{L_X(g) : X \in \mathfrak{g}_{-1}^E\}) = E_x$$

$$T_g p(\{L_X(g) : X \in \mathfrak{g}_{-1}^F\}) = F_x.$$

In other words, denote by $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{p}, X \mapsto [X]$ the natural projection, then $(*)$ restricts to

$$E \cong G \times_P [\mathfrak{g}_{-1}^E] \text{ and } F \cong G \times_P [\mathfrak{g}_{-1}^F].$$

We conclude that $(F_{1,n+1}(\mathbb{R}^{n+2}), E \oplus F)$ is homogeneous under G. In particular, the left multiplication by each element of G is a distinct automorphism on $(F_{1,n+1}(\mathbb{R}^{n+2}), E \oplus F)$.

Observe that the subalgebra $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$ is exactly the Heisenberg algebra together with a decomposition $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$ into Lagrangean subspaces as defined in Section 3.3. Moreover, we claim that the group $G_0 \subseteq Aut_{gr}(\mathfrak{g}_-)$ of isomorphisms on the graded Lie algebra $\mathfrak{g}_-$ preserving the subspaces $\mathfrak{g}_{-1}^E$ and $\mathfrak{g}_{-1}^F$ as defined in Section 3.3 is also isomorphic to a subgroup of G.

Indeed, with $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$, $\mathfrak{g}$ becomes a semisimple graded Lie algebra, with respect to which P is the corresponding parabolic subgroup of G, meaning that

$$P = \{g \in G : Ad(g)(\mathfrak{g}^i) = \mathfrak{g}^i \ \forall i\}$$

where $\mathfrak{g}^i := \oplus_{j \geq i} \mathfrak{g}_j$ We denote by G_0 the Levi-subgroup of P, meaning that

$$G_0 := \{g \in P : Ad(g)(\mathfrak{g}_i) = \mathfrak{g}_i \ \forall i\}.$$

Then G_0 equals the image of block-diagonal matrices under the quotient projection $GL(n+2,\mathbb{R}) \twoheadrightarrow PGL(n+2,\mathbb{R})$. When n is odd, with the realisation $G = SL(n+2,\mathbb{R})$ we see that the adjoint action of G_0 on $\mathfrak{g}_-$ is given by

$$Ad\begin{pmatrix} a & & \\ & A & \\ & & b \end{pmatrix}\begin{pmatrix} & & \\ X & & \\ \beta & Y & \end{pmatrix} = \begin{pmatrix} & & \\ a^{-1}AX & & \\ a^{-1}b\beta & Y(bA^{-1}) & \end{pmatrix} \qquad (**)$$

for any $diag(a, A, b) \in G_0 \subseteq G = SL(n+2,\mathbb{R})$. When n is even, let's always realise a short curve though the identity of G as a short curve in $SL(n+2,\mathbb{R})$ through $\mathbb{I}_{n+2}$, thus the conjugate action on such a curve by elements in G is realised as the conjutate action on $SL(n+2,\mathbb{R})$ by $GL(n+2,\mathbb{R})$. In particular, the adjoint action of G_0 on $\mathfrak{g}_-$ is computed by the same equation $(**)$ for any representative $diag(a, A, b) \in GL(n+2,\mathbb{R})$ of G_0. Clearly $Ad(G_0)$ on $\mathfrak{g}_-$ is faithful. Moreover, we may request $(a^{-1}b, a^{-1}A)$ to take any value in $GL(\mathfrak{g}_{-2}) \times GL(\mathfrak{g}^E_{-1})$, hence Ad is a group isomorphism from G_0 to all graded Lie algebra isomorphisms on $\mathfrak{g}_-$ preserving $\mathfrak{g}^E_{-1}$ and $\mathfrak{g}^F_{-1}$, i.e. G_0 also agrees with the definition of G_0 in Section 3.3.

Let P_+ be the image of strictly block-upper triangular matrices under the quotient projection $GL(n+2,\mathbb{R}) \twoheadrightarrow PGL(n+2,\mathbb{R})$, then the Lie algebra $\mathfrak{p}_+$ of P_+ equals $\mathfrak{g}^1$ and we have $G_0 \cong P/P_+$.

We claim that there is a regular filtered G_0-structure of type $\mathfrak{g}_-$ on the space G/P_+ of cosets, whose underlying Lagrangean contact structure is $(F_{1,n+1}(\mathbb{R}^{n+2}), E \oplus F)$. Note that G/P_+ does not carry a group structure.

Indeed, we see that $G/P_+ \to F_{1,n+1}(\mathbb{R}^{n+2})$ is a principal G_0-bundle and $G \to G/P_+$ is a principal P_+-bundle. The subbundle $E \oplus F = H \subseteq TF_{1,n+1}(\mathbb{R}^{n+2})$ lifts to a G_0-invariant subbundle $T^{-1}(G/P_+) \subseteq T(G/P_+)$. Thus we obtain a G_0-invariant filtration

$$T(G/P_+) = T^{-2}(G/P_+) \supseteq T^{-1}(G/P_+) \supseteq T^0(G/P_+) = \mathcal{V}(G/P_+ \to F_{1,n+1}(\mathbb{R}^{n+2})).$$

Consider the filtration on TG defined by $T^iG := \omega^{-1}(\mathfrak{g}^i)$, where $\omega \in \Omega^1(G, \mathfrak{g})$ is the left Maurer-Cartan form on G. Then $T^{-1}(G/P_+)$ lifts to $T^{-1}G$, $T^0(G/P_+)$ lifts to T^0G, and T^1G is the vertical bundle of $G \to G/P_+$.

Notice that for each i, the adjoint representation of P_+ on $\mathfrak{g}^i$ restricts to the identity on $\mathfrak{g}_i$; there is an adjoint representation of G_0 on $\mathfrak{g}_i$; and $G \to G/P_+$ is filtration-preserving and G_0-equivariant. Let $\omega_{-2} : TG \to \mathfrak{g}_{-2}$ be the $\mathfrak{g}_{-2}$-component of ω. Then ω_{-2} is G_0-equivariant, P_+-invariant, and $ker(\omega^{-2}) = T^{-1}G \supseteq T^1G$. Hence ω^{-2} descends along $G \to G/P_+$ to a G_0-equivariant one-form

$$\underline{\omega}_{-2} : T(G/P_+) \to \mathfrak{g}_{-2}$$

whose kernel is $T^{-1}(G/P_+)$. Similarly let $\omega_{-1}|_{T^{-1}G} : T^{-1}G \to \mathfrak{g}_{-1}$ be the $\mathfrak{g}_{-1}$-component of ω restricted to $T^{-1}G$. Then $\omega_{-1}|_{T^{-1}G}$ is G_0-equivariant, P_+-invariant, and $ker(\omega_{-1}|_{T^{-1}G}) =$

$T^0G \supseteq T^1G$. Hence $\omega_{-1}|_{T^{-1}G}$ descends along $G \to G/P_+$ to a G_0-equivariant one-form

$$\underline{\omega}_{-1} : T^{-1}(G/P_+) \to \mathfrak{g}_{-1}$$

whose kernel is $T^0(G/P_+)$. We check that $(\underline{\omega}_{-2}, \underline{\omega}_{-1})$ is regular. Let $\tilde{\xi}, \tilde{\eta} \in \Gamma(T^{-1}G)$ be lifts of $\xi, \eta \in \Gamma(T^{-1}(G/P_+))$, respectively. Then $[\tilde{\xi}, \tilde{\eta}] \in \mathfrak{X}(G)$ is a lift of $[\xi, \eta] \in \mathfrak{X}(G/P_+)$. Fix $g \in G$ which descends to $[g] \in G/P_+$, and let $X := \omega(g)(\tilde{\xi}), Y := \omega(g)(\tilde{\eta}) \in \mathfrak{g}^{-1}$. Then

$$\begin{aligned}
d\underline{\omega}_{-2}([g])(\xi, \eta) &= - \underline{\omega}_{-2}([g])([\xi, \eta]) \\
&= - \omega_{-2}(g)([\tilde{\xi}, \tilde{\eta}]) \\
&= d\omega_{-2}(g)(\tilde{\xi}, \tilde{\eta}) \\
&= d\omega_{-2}(g)(L_X, L_Y) \\
&= - \omega_{-2}(g)([L_X, L_Y]) \\
&= - [X, Y]_{\mathfrak{g}_{-2}}
\end{aligned}$$

where $-[X, Y]_{\mathfrak{g}_{-2}}$ denotes the $\mathfrak{g}_{-2}$-component of $[X, Y]$, and

$$\begin{aligned}
-[\underline{\omega}_{-1}([g])(\xi), \underline{\omega}_{-1}([g])(\eta)] &= - [\omega_{-1}(g)(\tilde{\xi}), \omega_{-1}(g)(\tilde{\eta})] \\
&= - [X_{\mathfrak{g}_{-1}}, Y_{\mathfrak{g}_{-1}}] \\
&= - [X, Y]_{\mathfrak{g}_{-2}}
\end{aligned}$$

where $X_{\mathfrak{g}_{-1}}, Y_{\mathfrak{g}_{-1}}$ denote the $\mathfrak{g}_{-1}$-component of X, Y, respectively. Hence G/P_+ together with $(\underline{\omega}_{-2}, \underline{\omega}_{-1})$ is indeed a regular filtered G_0-structure of type $\mathfrak{g}_-$. Since $(\underline{\omega}_{-1})^{-1}(\mathfrak{g}^E_{-1}) \subseteq T^{-1}(G/P_+)$ lies under $(\omega_{-1}|_{T^{-1}G})^{-1}(\mathfrak{g}^E_{-1}) = \omega^{-1}(\mathfrak{g}^E_{-1})$, it corresponds to E, and similar for F, hence the underlying Lagrangean contact structure is just $E \oplus F$.

We observe that the regularity of $(\underline{\omega}_{-2}, \underline{\omega}_{-1})$ is a consequence of the Maurer-Cartan equation, more precisely, of the fact that

$$d\omega_{-2}(\tilde{\xi}, \tilde{\eta}) + [\omega_{-1}(\tilde{\xi}), \omega_{-1}(\tilde{\eta})] = 0$$

for all $\tilde{\xi}, \tilde{\eta} \in \Gamma(T^{-1}G)$, which is the same as

$$K(\tilde{\xi}, \tilde{\eta}) \in \mathfrak{g}^{-1}$$

for all $\tilde{\xi}, \tilde{\eta} \in \Gamma(T^{-1}G)$, where K is the curvature form of ω. Since K is horizontal, this is the same as saying that K has homogeneity ≥ 1 in the sense that $K(T^iG, T^jG) \subseteq \mathfrak{g}^{i+j+1}$.

5.2 Underlying filtered G_0-structures

We can generalize the construction of the underlying filtered G_0-structure to Cartan geometries, provided that we impose the following condition.

Definition 5.2.1. *Let $(\mathcal{G}, \omega)$ be a Cartan geometry of type (G, P). Then there is a filtration on $T\mathcal{G}$ given by $T^i\mathcal{G} := \omega^{-1}(\mathfrak{g}^i)$. Thus $(\mathcal{G}, \omega)$ is said to be regular if its curvature two-form K has homogeneity ≥ 1 in the sense that $K(T^i\mathcal{G}, T^j\mathcal{G}) \subseteq \mathfrak{g}^{i+j+1}$.*

Equivalently, $(\mathcal{G}, \omega)$ is regular if its curvature function $\kappa : \mathcal{G} \to L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$ takes values in $L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})^1 = \{\varphi \in L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g}) : \varphi(\mathfrak{g}^i + \mathfrak{p}, \mathfrak{g}^j + \mathfrak{p}) \subseteq \mathfrak{g}^{i+j+1}\ \forall i, j\}$.

The generalization now reads as follows.

Proposition 5.2.1. *Every regular Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) has an underlying regular filtered G_0-structure. Moreover, this construction is functorial.*

Proof. Let $\mathcal{G}/P_+$ denote the space of orbits under the restriction to P_+ of the principal right action on $\mathcal{G}$. Then $\mathcal{G} \to \mathcal{G}/P_+$ is a principal P_+-bundle with vertical bundle $T^1\mathcal{G}$. Since $T^{-1}\mathcal{G}$ is P-invariant and contains $T^1\mathcal{G}$, it descends to a G_0-invariant subbundle of $T(\mathcal{G}/P_+)$, which we denote by $T^{-1}(\mathcal{G}/P_+)$.

We also see that $\mathcal{G}/P_+ \to M$ is a principal G_0-bundle. Since $T^0\mathcal{G}$ is the vertical bundle of $\mathcal{G} \to M$, it descends along $\mathcal{G} \to \mathcal{G}/P_+$ to the vertical bundle of $\mathcal{G}/P_+ \to M$, which we denote by $T^0(\mathcal{G}/P_+) \subseteq T(\mathcal{G}/P_+)$. Now we have a G_0-invariant filtration

$$T(\mathcal{G}/P_+) = T^{-2}(\mathcal{G}/P_+) \supseteq T^{-1}(\mathcal{G}/P_+) \supseteq T^0(\mathcal{G}/P_+) = \mathcal{V}(\mathcal{G}/P_+ \to M).$$

We have already noticed that for each i, the adjoint representation of P_+ on $\mathfrak{g}^i$ fixes the $\mathfrak{g}_i$-component; there is an adjoint representation of G_0 on $\mathfrak{g}_i$; and $\mathcal{G} \to \mathcal{G}/P_+$ is filtration-preserving and G_0-equivariant. For $i = -2, -1$, let ω_i denote the $\mathfrak{g}_i$-component of ω. Then $\omega_{-2} : T\mathcal{G} \to \mathfrak{g}_{-2}$ is G_0-equivariant, P_+-invariant and has kernel $T^{-1}\mathcal{G} \supseteq T^1\mathcal{G}$, hence descends to a G_0-equivariant one-form

$$\underline{\omega}_{-2} : T(\mathcal{G}/P_+) \to \mathfrak{g}_{-2}$$

whose kernel is $T^{-1}(\mathcal{G}/P_+)$; and $\omega_{-1}|_{T^{-1}\mathcal{G}} : T^{-1}\mathcal{G} \to \mathfrak{g}_{-1}$ is G_0-equivariant, P_+-invariant, and has kernel $T^0\mathcal{G} \supseteq T^1\mathcal{G}$, hence it descends to a G_0-equivariant partial one-form

$$\underline{\omega}_{-1} : T^{-1}(\mathcal{G}/P_+) \to \mathfrak{g}_{-1}$$

whose kernel is $T^0(G/P_+)$. We check that $(\underline{\omega}_{-2}, \underline{\omega}_{-1})$ is regular. Notice that K has homogeneity ≥ 1 if and only if

$$d\omega_{-2}(\tilde{\xi}, \tilde{\eta}) + [\omega_{-1}(\tilde{\xi}), \omega_{-1}(\tilde{\eta})] = 0$$

for all $\tilde{\xi}, \tilde{\eta} \in \Gamma(T^{-1}\mathcal{G})$. Hence for $\xi, \eta \in \Gamma(T^{-1}(G/P_+))$, let $\tilde{\xi}, \tilde{\eta} \in \Gamma(T^{-1}\mathcal{G})$ be lifts of ξ and η, respectively, then $[\tilde{\xi}, \tilde{\eta}] \in \mathfrak{X}(\mathcal{G})$ is a lift of $[\xi, \eta] \in \mathfrak{X}(\mathcal{G}/P_+)$. Let $u_0 \in \mathcal{G}/P_+$ be lifted to

$u \in \mathcal{G}$, then

$$\begin{aligned} d\underline{\omega}_{-2}(u_0)(\xi,\eta) &= -\, \underline{\omega}_{-2}(u_0)([\xi,\eta]) \\ &= -\, \omega_{-2}(u)([\tilde{\xi},\tilde{\eta}]) \\ &= d\omega_{-2}(u)(\tilde{\xi},\tilde{\eta}) \\ &= -\, [\omega_{-1}(u)(\tilde{\xi}), \omega_{-1}(u)(\tilde{\eta})] \\ &= -\, [\underline{\omega}_{-1}(u_0)(\xi), \underline{\omega}_{-1}(u_0)(\eta)] \end{aligned}$$

Hence $(\mathcal{G}/P_+, (\underline{\omega}_{-2}, \underline{\omega}_{-1}))$ is a regular filtered G_0-structure of type $\mathfrak{g}_-$.

Let $\Phi : (\mathcal{G}, \omega) \to (\mathcal{G}', \omega')$ be a morphism of regular Cartan geometries of type (G, P). By the P-equivariancy of Φ, it descends to a principal bundle map $\underline{\Phi} : \mathcal{G}/P_+ \to \mathcal{G}'/P_+$. Since Φ is filtration-preserving, so does $\underline{\Phi}$. Since $\Phi^*\omega'_{-2} = \omega_{-2}$ and $\Phi^*(\omega'_{-1}|_{T^{-1}\mathcal{G}'}) = \omega_{-1}|_{T^{-1}\mathcal{G}}$, we have $\underline{\Phi}^*(\underline{\omega}'_{-2}, \underline{\omega}'_{-1}) = (\underline{\omega}_{-2}, \underline{\omega}_{-1})$. Hence $\underline{\Phi}$ is a morphism between the underlying regular filtered G_0-structures of type $\mathfrak{g}_-$. $\square$

Proposition 5.2.2. *Let $(\mathcal{G}_0 \to M, \theta)$ be a regular filtered G_0-structure of type $\mathfrak{g}_-$. Then there is a regular Cartan geometry of type (G, P) inducing $(\mathcal{G}_0 \to M, \theta)$.*

Proof. Let $\mathcal{G} := \mathcal{G}_0 \times_{G_0} P$, then $\mathcal{G} \to M$ is a principal P-bundle. (This is because the cocycle information on $\mathcal{G}_0$ passes along $G_0 \hookrightarrow P$ to the cocycle information on $\mathcal{G}$, making the latter a principal P-bundle. More precisely, a trivialisation $\mathcal{G}_0|_U \cong U \times G_0$ above an open subset $U \subseteq M$ yields a trivialisation $(\mathcal{G}_0 \times P)|_U \cong U \times G_0 \times P$, which descends to a trivialisation $(\mathcal{G}_0 \times_{G_0} P)|_U \cong U \times P$ by the group multiplication restricted to $G_0 \times P \to P$. If there are two such trivialisations on $\mathcal{G}_0|_U$, they are related by $U \times G_0 \to U \times G_0, (x, g) \mapsto (x\varphi(x)g)$ for a smooth map $\varphi : U \to G_0$, hence their induced trivialisations on $\mathcal{G}|_U$ are related by $U \times P \to U \times P, (x, g) \mapsto (x, \varphi(g))$.)

Let $P^{op} \subseteq G$ denote the opposite parabolic subgroup, consisting of the image of block-lower diagonal matrices by $GL(n+2, \mathbb{R}) \twoheadrightarrow G$. Its Lie algebra is $\mathfrak{g}_- \oplus \mathfrak{g}_0$. We will first extend the filtered one-form θ on $\mathcal{G}_0$ to a Cartan connection $\tilde{\theta} \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_- \oplus \mathfrak{g}_0)$ of type (P^{op}, G_0), and then extend $\tilde{\theta}$ to a Cartan connection ω of type (G, P) on $\mathcal{G}$.

First we choose a principal connection on $\mathcal{G}_0 \to M$ and express it as an G_0-invariant decomposition $T\mathcal{G}_0 = T_-\mathcal{G}_0 \oplus T_0\mathcal{G}_0$, where $T_0\mathcal{G}_0 := T^0\mathcal{G}_0 = \mathcal{V}(\mathcal{G}_0 \to M)$. Then choose a projection $TM \to H$, where H is the underlying contact structure, and denote its kernel by Q. This yields a decomposition $TM = Q \oplus H$, which is lifted by the principal connection on $\mathcal{G}_0$ to a G_0-invariant decomposition $T_-\mathcal{G}_0 = T_{-2}\mathcal{G}_0 \oplus T_{-1}\mathcal{G}_0$. Since $T^{-1}\mathcal{G}_0$ is the preimage of H by $T\mathcal{G}_0 \to TM$, we have $T^{-1}\mathcal{G}_0 = T_{-1}\mathcal{G}_0 \oplus T_0\mathcal{G}_0$. Now $T\mathcal{G}_0 = T_{-2}\mathcal{G}_0 \oplus T_{-1}\mathcal{G}_0 \oplus T_0\mathcal{G}_0$. Let $\tilde{\theta}$ restricted to $T_i\mathcal{G}_0 \subseteq T^i\mathcal{G}_0$ be given by θ_i for $i = -2, -1$, and on $T_0\mathcal{G}_0$ be the reproduction of generators of fundamental vector fields. Then $\tilde{\theta}$ is a Cartan connection of type (P^T, G_0) such that the $\mathfrak{g}_{-2}$-component of $\tilde{\theta}$ agrees with θ_{-2} and the $\mathfrak{g}_{-1}$-component of $\tilde{\theta}|_{T^{-1}\mathcal{G}_0}$ agrees with θ_{-1}.

Let $\iota : \mathcal{G}_0 \hookrightarrow \mathcal{G}$ denote the canonical embedding. Then for each $u_0 \in \mathcal{G}_0$, $T_{\iota(u_0)}\mathcal{G} = T_{u_0}\iota(T_{u_0}\mathcal{G}_0) + \mathcal{V}_{u_0}\mathcal{G}$, and the intersection of the two subspaces consists of tangent vectors $T_{u_0}\iota(\zeta_X(u_0)) = \zeta_X(\iota(u_0))$ for all $X \in \mathfrak{g}_0$. Since the linear map $T_{u_0}\iota(T_{u_0}\mathcal{G}_0) \to \mathfrak{g}$ given by $T_{u_0}\iota(\xi(u_0)) := \tilde{\theta}(\xi(u_0))$ for each $\xi(u_0) \in T_{u_0}\mathcal{G}_0$ reproduces the generators of fundamental vector fields evaluated at u_0 when the generators are in $\mathfrak{g}_0$, it extends to a linear map $\omega(\iota(u_0)) : T_{\iota(u_0)}\mathcal{G} \to \mathfrak{g}$ reproducing all generators of fundamental vector fields evaluated at u_0. The map is surjective, hence is bijective. Since the collection $\{\omega(\iota(u_0)) : u_0 \in \mathcal{G}_0\}$ is G_0-equivariant, it extends to a P-equivariant one-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ which is again a linear isomorphism at each fiber. It reproduces the generators of fundamental vector fields because $Tr^g(\zeta_X(u)) = \zeta_{Ad(g^{-1})X}(ug)$ for all $X \in \mathfrak{p}, g \in P, u \in \mathcal{G}$. Thus ω is a Cartan connection of type (G, P) such that $\iota^*\omega = \tilde{\theta}$.

To prove that ω is regular, since the curvature K of ω is P-equivariant, it suffices to check that

$$K(\iota(u_0))(T^{-1}\mathcal{G}, T^{-1}\mathcal{G}) \subseteq \mathfrak{g}^{-1}$$

for all $u_0 \in \mathcal{G}_0$. Since K is horizontal and $T_{\iota(u_0)}\mathcal{G} = T_{u_0}\iota(T_{u_0}\mathcal{G}_0) + \mathcal{V}_{u_0}\mathcal{G}$, it suffices to check that

$$K(\iota(u_0))(T_{u_0}\iota(\xi), T_{u_0}\iota(\eta)) \in \mathfrak{g}^{-1}$$

for all $\xi, \eta \in \Gamma(T^{-1}\mathcal{G}_0)$. But

$$K(\iota(u_0))(T_{u_0}\iota(\xi), T_{u_0}\iota(\eta) = d\tilde{\theta}(u_0)(\xi, \eta) + [\tilde{\theta}(u_0)(\xi), \tilde{\theta}(u_0)(\eta)],$$

whose $\mathfrak{g}_{-2}$-component equals

$$d\theta_{-2}(u_0)(\xi, \eta) + [\theta_{-1}(u_0)(\xi), \theta_{-1}(u_0)(\eta)] = 0$$

hence $K(\iota(u_0))(T_{u_0}\iota(\xi), T_{u_0}\iota(\eta)) \in \mathfrak{g}^{-1}$ and ω is regular.

We check that the underlying regular filtered G_0-structure $(\mathcal{G}/P_+, (\underline{\omega}_{-2}, \underline{\omega}_{-1}))$ is isomorphic to $(\mathcal{G}_0, \theta)$. Notice that ι is a lift of $\mathcal{G} \twoheadrightarrow \mathcal{G}/P_+ \cong \mathcal{G}_0$. For $i = -2, -1$, any tangent vector $\xi \in T^i_{u_0}\mathcal{G}_0$ can be lifted to $T_{u_0}\iota(\xi) \in T^i_{\iota(u_0)}\mathcal{G}$, and so

$$\underline{\omega}_i(u_0)(\xi) = \omega_i(\iota(u_0))(T_{u_0}\iota(\xi)) = (\iota^*\omega)(u_0)(\xi) = \tilde{\theta}_i(u_0)(\xi) = \theta_i(u_0)(\xi)$$

where $\tilde{\theta}_i$ denotes the $\mathfrak{g}_i$-component of $\tilde{\theta}$. Hence the underlying regular filtered G_0-structure is indeed $(\mathcal{G}_0, \theta)$. □

5.3 Some algebraic background

In the end, we want to impose restrictions on the curvature of a Cartan geometry which uniquely characterize one of the Cartan geometries inducing a regular filtered G_0-structure of type $\mathfrak{g}_-$. In order to do this, we first have to prove some purely algebraic results on the Lie algebra $\mathfrak{g}$.

Definition 5.3.1. *Let $\mathfrak{g}$ be an arbitrary graded Lie algebra and $\mathfrak{g}_-$ be its negative part. Define a grading-preserving cochain complex*

$$\mathfrak{g} \xrightarrow{\partial^0} L(\mathfrak{g}_-, \mathfrak{g}) \xrightarrow{\partial^1} L(\Lambda^2\mathfrak{g}_-, \mathfrak{g})$$

by

$$\partial^0(Z) := -ad(Z)|_{\mathfrak{g}_-}$$

for all $Z \in \mathfrak{g}$ and

$$\partial^1\varphi(X, Y) := [X, \varphi(Y)] - [Y, \varphi(X)] - \varphi([X, Y])$$

for all $\varphi \in L(\mathfrak{g}_-, \mathfrak{g})$, $X, Y \in \mathfrak{g}_-$.

We see that $\partial^1 \circ \partial^0 = 0$ by the Jacobi identity. Moreover, on $L(\mathfrak{g}_-, \mathfrak{g})$ there is a grading such that $L(\mathfrak{g}_-, \mathfrak{g})_k$ consists of maps $\varphi \in L(\mathfrak{g}_-, \mathfrak{g})$ of homogeneity k, which means that $\varphi(\mathfrak{g}_i) \subseteq \mathfrak{g}_{i+k}$ for all $i < 0$. Similarly, there is a grading on $L(\Lambda^2\mathfrak{g}_-, \mathfrak{g})$ such that $L(\Lambda^2\mathfrak{g}_-, \mathfrak{g})_k$ consists of maps $\varphi \in L(\Lambda^2\mathfrak{g}_-, \mathfrak{g})$ of homogeneity k, which means that $\varphi(\mathfrak{g}_i \wedge \mathfrak{g}_j) \subseteq \mathfrak{g}_{i+j+k}$ for all $i, j < 0$. It is clear that both ∂^0 and ∂^1 are grading-preserving by the property of Lie bracket on a graded algebra.

We will denote by $L(\mathfrak{g}_-, \mathfrak{g})^k$, $L(\Lambda^2\mathfrak{g}_-, \mathfrak{g})^k$ the spaces of all elements of homogeneity $\geq k$, and recall that $\mathfrak{g}^k = \oplus_{i \geq k}\mathfrak{g}_i$.

We say that $\mathfrak{g}$ is a *full prolongation* of $(\mathfrak{g}_-, \mathfrak{g}_0)$ if the cochain complex is exact in homogeneity ≥ 1, i.e. $im(\partial^0|_{\mathfrak{g}^1}) = ker(\partial^1|_{L(\mathfrak{g}_-,\mathfrak{g})^1})$.

Example 5.3.1. *Endow $\mathfrak{euc}(n) = \mathbb{R}^n \oplus \mathfrak{o}(n)$ with a grading $\mathfrak{euc}(n)_{-1} := \mathbb{R}^n$, $\mathfrak{euc}(n)_0 := \mathfrak{o}(n)$. Then the cochain complex in homogeneity ≥ 1 reads*

$$0 \to L(\mathbb{R}^n, \mathfrak{o}(n)) \xrightarrow{\partial} L(\Lambda^2\mathbb{R}^n, \mathfrak{euc}(n))$$

for $\partial : L(\mathbb{R}^n, \mathfrak{o}(n)) \xrightarrow{\cong} L(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) \subseteq L(\Lambda^2\mathbb{R}^n, \mathfrak{euc}(n))$ as given in Lemma 4.2.2. Hence the maps in the complex are $O(n)$-equivariant and $\mathfrak{euc}(n)$ is a full prolongation of $(\mathbb{R}^n, \mathfrak{o}(n))$, which is usually phrased as the fact that $\mathfrak{o}(n)$ has trivial first prolongation.

The space $L(\Lambda^2\mathbb{R}^n, \mathfrak{euc}(n))$ decomposes into the $O(n)$-subrepresentations

$$im(\partial) = L(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$$

and

$$\mathcal{N} := L(\Lambda^2\mathbb{R}^n, \mathfrak{o}(n)).$$

Recall that a Cartan geometry $(\mathcal{G}, \omega)$ of type $(Euc(n), O(n))$ is said to be normal if its curvature takes values in $\mathfrak{o}(n)$, that is, $\kappa : \mathcal{G} \to L(\Lambda^2\mathbb{R}^n, \mathfrak{euc}(n))$ takes values in $\mathcal{N}$.

In fact,

$$L(\Lambda^2\mathbb{R}^n, \mathfrak{euc}(n)) = im(\partial) \oplus \mathcal{N}$$

is an orthogonal decomposition with respect to the natural inner product on $L(\Lambda^2\mathbb{R}^n, \mathfrak{euc}(n)) = \Lambda^2\mathbb{R}^{n*} \otimes \mathfrak{euc}(n)$ *induced by the natural inner product on* $\mathbb{R}^{n*}$ *and on* $\mathfrak{euc}(n)$. *As the latter is the restriction of the standard inner product* $(A, B) \mapsto tr(A^tB)$ *on* $Mat(n+1, \mathbb{R})$ *to* $\mathfrak{euc}(n)$, *clearly* $\mathfrak{euc}(n) = \mathbb{R}^n \oplus \mathfrak{o}(n)$ *is a orthogonal decomposition. Hence so does* $L(\Lambda^2\mathbb{R}^n, \mathfrak{euc}(n)) = L(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) \oplus L(\Lambda^2\mathbb{R}^n, \mathfrak{o}(n))$.

From now on we resume the notations from the last section. Notice that $\mathfrak{g}_-$ is not a P-invariant subspace in $\mathfrak{g}$, so one should actually deal with $\mathfrak{g}/\mathfrak{p}$ instead of $\mathfrak{g}_-$. But for the first step, one is only interested in the G_0-module structure, for which $\mathfrak{g}_-$ and $\mathfrak{g}/\mathfrak{p}$ can be identified. In particular, we obtain grading-preserving isomorphisms

$$L(\mathfrak{g}_-, \mathfrak{g}) \cong L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})$$

$$L(\Lambda^2\mathfrak{g}_-, \mathfrak{g}) \cong L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$$

of G_0-representations. We aim for an analogue to the example above.

Lemma 5.3.1. *The maps in the complex*

$$\mathfrak{g} \xrightarrow{\partial^0} L(\mathfrak{g}_-, \mathfrak{g}) \xrightarrow{\partial^1} L(\Lambda^2\mathfrak{g}_-, \mathfrak{g})$$

are G_0*-equivariant.*

Proof. For $g \in G_0, A \in \mathfrak{g}$, $X, Y \in \mathfrak{g}_-$, $\varphi \in L(\mathfrak{g}_-, \mathfrak{g})$, we have

$$\begin{aligned}\partial^0(Ad(g)(A))(X) &= -[Ad(g)(A), X] \\ &= -Ad(g)[A, Ad(g^{-1})(X)] \\ &= Ad(g)\partial^0(A)(Ad(g^{-1})(X))\end{aligned}$$

meaning that ∂^0 is P-equivariant and

$$\begin{aligned}\partial^1(g.\varphi)(X, Y) =& [X, (g.\varphi)(Y)] - [Y, (g.\varphi)(X)] - [X, Y] \\ =& [X, Ad(g)\varphi(Ad(g^{-1})(Y))] - [Y, Ad(g)\varphi(Ad(g^{-1})(X))] - [X, Y] \\ =& Ad(g)[Ad(g^{-1})(X), \varphi(Ad(g^{-1})(Y))] - Ad(g)[Ad(g^{-1})(Y), \varphi(Ad(g^{-1})(X))] \\ & - Ad(g)[Ad(g^{-1})(X), Ad(g^{-1})(Y)] \\ =& Ad(g)(\partial^1\varphi)(Ad(g^{-1})(X), Ad(g^{-1})(Y)) \\ =& (g.(\partial^1\varphi))(X, Y)\end{aligned}$$

meaning that ∂^1 is P-equivariant. □

Since $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{R})$ is $|2|$-graded, the positive homogeneities of $L(\mathfrak{g}_-, \mathfrak{g})$ are $1, 2, 3, 4$.

Lemma 5.3.2. $\mathfrak{g}$ *is a full prolongation of* $(\mathfrak{g}_-, \mathfrak{g}_0)$ *in the sense that* ∂^1 *is injective in homogeneities* $3, 4$, *and in homogeneities* $1, 2$, ∂^0 *is injective with* $im\partial^0 = ker\partial^1$.

Proof. One can easily check that ∂^0 is injective, so we just show that ∂^1 is injective in homogeneities $3, 4$ and $\ker \partial^1 = im\partial^0$ in homogeneities $1, 2$.

We write $\partial^1 =: \partial$ and think of all matrices in block size $(1, n, 1)$. Unless otherwise stated, we denote arbitrary elements in $\mathfrak{g}$ by

$$\beta_{\mathfrak{g}_{-2}} + X_{\mathfrak{g}^E_{-1}} + Y_{\mathfrak{g}^F_{-1}} + (a, A, b)_{\mathfrak{g}_0} + Z_{\mathfrak{g}^E_1} + W_{\mathfrak{g}^F_1} + \gamma_{\mathfrak{g}_2} := \begin{pmatrix} a & Z & \gamma \\ X & A & W \\ \beta & Y & b \end{pmatrix} \in \mathfrak{g}.$$

(i) $l = 1$: Let $\varphi \in L(\mathfrak{g}_-, \mathfrak{g})_1 = L(\mathfrak{g}_{-2}, \mathfrak{g}_{-1}) \oplus L(\mathfrak{g}_{-1}, \mathfrak{g}_0)$. We notice that there is an element $X \in \mathfrak{g}_1$ such that $\varphi - ad(X)$ vanishes on $\mathfrak{g}_{-2}$. Hence it suffices to check that if $\varphi \in L(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ and $\partial\varphi = 0$, then $\varphi = 0$.

Write $\varphi|_{\mathfrak{g}_{-1}} = (\varphi_1, \varphi_2, \varphi_3)_{\mathfrak{g}_0}$. Then $0 = \partial\varphi(1_{\mathfrak{g}_{-2}}, \cdot) = [1_{\mathfrak{g}_{-2}}, \varphi(\cdot)]$ on $\mathfrak{g}_{-1}$ implies $\varphi_1 = \varphi_3$. Fix any $X_{\mathfrak{g}^E_{-1}}$ and $Y_{\mathfrak{g}^F_{-1}}$, denote

$$\varphi(X_{\mathfrak{g}^E_{-1}}) = (a, A, a)_{\mathfrak{g}_0} \text{ and } \varphi(Y_{\mathfrak{g}^F_{-1}}) = (b, B, b)_{\mathfrak{g}_0}$$

with $a = -tr(A)/2$ and $b = -tr(B)/2$. Since $0 = \partial\varphi(X_{\mathfrak{g}^E_{-1}}, Y_{\mathfrak{g}^F_{-1}}) = [X_{\mathfrak{g}^E_{-1}}, \varphi(Y_{\mathfrak{g}^F_{-1}})] + [\varphi(X_{\mathfrak{g}^E_{-1}}), Y_{\mathfrak{g}^F_{-1}}]$, we have $[X_{\mathfrak{g}^E_{-1}}, \varphi(Y_{\mathfrak{g}^F_{-1}})] = [Y_{\mathfrak{g}^F_{-1}}, \varphi(X_{\mathfrak{g}^E_{-1}})]$, which gives

$$(b\mathbb{I} - B)X_{\mathfrak{g}^E_{-1}} = Y(A - a\mathbb{I})_{\mathfrak{g}^F_{-1}},$$

hence running $X_{\mathfrak{g}^E_{-1}}$ through $\mathfrak{g}^E_{-1}$ we get $B = b\mathbb{I}$, thus $tr(B) = bn = -tr(B)n/2$ and so $B = 0$; similarly running $Y_{\mathfrak{g}^F_{-1}}$ through $\mathfrak{g}^F_{-1}$ we get $A = 0$. Hence $\varphi = 0$.

(ii) $l = 2$: Similar to the case of $l = 1$, each linear map in $L(\mathfrak{g}_-, \mathfrak{g})_2$ can be written as $\varphi + ad(X)$ such that $X \in \mathfrak{g}_2$ and $\varphi \in L(\mathfrak{g}_-, \mathfrak{g})_2$, such that

$$\varphi(1_{\mathfrak{g}_{-2}}) = (0, A, -tr(A))_{\mathfrak{g}_0},$$

for some $A \in GL(n, \mathbb{R})$, and it suffices to show that if $\partial\varphi = 0$ then $\varphi = 0$.

But $\partial\varphi = 0$ together with the given $\varphi|_{\mathfrak{g}_{-2}}$ fix the value of φ on $\mathfrak{g}^E_{-1}$ and on $\mathfrak{g}^F_{-1}$, namely, by inserting $1_{\mathfrak{g}_{-2}} + X_{\mathfrak{g}^E_{-1}}$ and $1_{\mathfrak{g}_{-2}} + Y_{\mathfrak{g}^F_{-1}}$ in $0 = \partial\varphi$, we get

$$\varphi(X_{\mathfrak{g}^E_{-1}}) = (-AX)_{\mathfrak{g}^F_1} \text{ and } \varphi(Y_{\mathfrak{g}^F_{-1}}) = (Y(tr(A)\mathbb{I} + A))_{\mathfrak{g}^E_1}.$$

thus

$$0 = \partial\varphi(X_{\mathfrak{g}^E_{-1}} + Y_{\mathfrak{g}^F_{-1}}) = (-Y(tr(A)\mathbb{I} + A)X, *, *)_{\mathfrak{g}_0}.$$

Running through all possible X and Y forces $A = -tr(A)\mathbb{I}_n$, from which implies $A = 0$, hence $\varphi = 0$.

(iii) $l = 3$: we need to show that $\partial : L(\mathfrak{g}_-, \mathfrak{g})_3 \to L(\Lambda^2\mathfrak{g}_-, \mathfrak{g})_3$ is injective. Let $\varphi \in L(\mathfrak{g}_-, \mathfrak{g})_3$ such that $\partial\varphi = 0$, and define elements

$$\varphi(1_{\mathfrak{g}_{-2}}) =: Z_{\mathfrak{g}^E_1} + W_{\mathfrak{g}^F_1}, \varphi(X_{\mathfrak{g}^E_{-1}}) =: b_{\mathfrak{g}_2}, \varphi(Y_{\mathfrak{g}^F_{-1}}) =: c_{\mathfrak{g}_2}.$$

Then

$$0 = \partial\varphi(1_{\mathfrak{g}_{-2}}, X_{\mathfrak{g}^E_{-1}}) = (ZX - b, -XZ, b)_{\mathfrak{g}_0}.$$

Running through all possible $X_{\mathfrak{g}^E_{-1}}$ forces that $b = 0$ and $Z = 0$; similarly compute

$$0 = \partial\varphi(1_{\mathfrak{g}_{-2}}, Y_{\mathfrak{g}^F_{-1}}) = (-c, WY, c - YW)_{\mathfrak{g}_0},$$

running through all possible Y forces $c = 0$ and $W = 0$. Hence $\varphi = 0$.

(iv) $l = 4$: we need to show that $\partial : L(\mathfrak{g}_-, \mathfrak{g})_4 \to L(\Lambda^2\mathfrak{g}_-, \mathfrak{g})_4$ is injective. Let $\varphi \in L(\mathfrak{g}_-, \mathfrak{g})_4 = L(\mathfrak{g}_{-2}, \mathfrak{g}_2)$ such that $\partial\varphi = 0$. Then for all $X, Y \in \mathfrak{g}_{-1}$ we have $0 = \partial\varphi(X, Y) = -\varphi([X, Y])$. Since the Lie bracket restricted to $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is surjective, we must have $\varphi = 0$. □

For what follows we will have to describe a P-invariant subspace in $L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$. The first step towards this is a better description of the dual of the P-representation $\mathfrak{g}/\mathfrak{p}$.

Lemma 5.3.3. *There is a canonical isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ of P-representations, which is grading-preserving.*

In particular, this induces isomorphisms

$$L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \cong \mathfrak{p}_+ \otimes \mathfrak{g}$$

$$L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g}) \cong \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$$

of P-representations, which are grading-preserving.

Proof. The pairing $(X, Y) \mapsto tr(XY)$ by the trace map is clearly a bilinear form on $\mathfrak{g}$ that is easily seen to be non-degenerate. Since $Ad(g)(X) = gXg^{-1}$ for all $X \in \mathfrak{g}$, $g \in G$, it follows immediately that the pairing is invariant under the adjoint action of G. Now for $X \in \mathfrak{p}$ and $Y \in \mathfrak{p}_+$, XY is immediately seen to be trace free. This means that $\mathfrak{p}$ is contained in the annihilator of $\mathfrak{p}_+$ under the pairing, and since $dim(\mathfrak{g}/\mathfrak{p}) = dim(\mathfrak{p}_+)$, it follows that $\mathfrak{p}$ coincides with the annihilator of $\mathfrak{p}_+$. Thus the pairing factorizes to a non-degenerate bilinear form $\mathfrak{p}_+ \times (\mathfrak{g}/\mathfrak{p}) \to \mathbb{R}$, identifying $\mathfrak{p}_+$ with $(\mathfrak{g}/\mathfrak{p})^*$. Since $\mathfrak{p}_+$ and $\mathfrak{p}$ both are P-invariant subspaces of $\mathfrak{g}$, the invariance of the original pairing shows that the pairing $\mathfrak{p}_+ \times (\mathfrak{g}/\mathfrak{p}) \to \mathbb{R}$ is invariant for the natural P-actions on the two spaces. This implies that $\mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$ as P-representations. Clearly the isomorphism is grading-preserving.

□

The (positive definite) standard inner product $\langle A, B\rangle := tr(A^tB)$ on $Mat(n+2, \mathbb{R})$ restricts to inner products on $\mathfrak{g}$ and on $\mathfrak{p}_+$, inducing inner products on $\mathfrak{p}_+ \otimes \mathfrak{g}$ and on $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$. Recall that, if V, W are inner product spaces, then the induced inner product on $V \otimes W$ is generated by $\langle v \otimes w, v' \otimes w'\rangle = \langle v, v'\rangle\langle w, w'\rangle$ for $v, v' \in V, w, w' \in W$. One can check that it is positive-definite by fixing an orthonormal basis (e_i) on V, thus express any element in $V \otimes W$ by $\Sigma_i e_i \otimes W_i$ for $w_i \in W$. The induced inner product on $\Lambda^2 V$ is generated by $\langle v_1 \wedge v_2, w_1 \wedge w_2 \rangle = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle - \langle v_1, w_2 \rangle \langle v_2, w_1 \rangle$ for $v_1, v_2, w_1, w_2 \in V$. It is positive-definite because an orthonormal basis on V induces an orthonormal basis on $\Lambda^2 V$.

Lemma 5.3.4. *The inner product on $L(\mathfrak{g}_-, \mathfrak{g}) \cong L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \cong \mathfrak{p}_+ \otimes \mathfrak{g}$ can be computed as*

$$\langle Z \otimes A, \varphi \rangle = \langle A, \varphi(Z^t) \rangle$$

for all $\varphi \in L(\mathfrak{g}_-, \mathfrak{g}), Z \otimes A \in \mathfrak{p}_+ \otimes \mathfrak{g}$, and the inner product on $L(\Lambda^2 \mathfrak{g}_-, \mathfrak{g}) \cong L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g}) \cong \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$ can be computed as

$$\langle Z_1 \wedge Z_2 \otimes A, \varphi \rangle = \langle A, \varphi(Z_1, Z_2) \rangle$$

for all $\varphi \in L(\Lambda^2 \mathfrak{g}_-, \mathfrak{g}), Z_1 \wedge Z_2 \otimes A \in \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$.

Proof. For the first claim, it suffices to check in the case where φ corresponds to $W \otimes B \in \mathfrak{p}_+ \otimes \mathfrak{g}$, which means that $\varphi(X) = tr(WX)B$ for all $X \in \mathfrak{g}_-$. Then

$$\begin{aligned} \langle Z \otimes A, \varphi \rangle =& \langle Z \otimes A, W \otimes B \rangle = tr(Z^t W) tr(A^t B) \\ =& tr(A^t(tr(Z^t W)B)) = tr(A^t \varphi(Z^t)) = \langle A, \varphi(Z^t) \rangle. \end{aligned}$$

For the second claim, it suffices to check in the case where φ corresponds to $W_1 \wedge W_2 \otimes B \in \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$, which means that for $X, Y \in \mathfrak{g}_-$, $\varphi(X, Y) = (tr(W_1 X) tr(W_2 Y) - tr(W_2 X) tr(W_1 Y))B$. Then

$$\begin{aligned} \langle Z_1 \wedge Z_2 \otimes A, \varphi \rangle =& \langle Z_1 \wedge Z_2 \otimes A, W_1 \wedge W_2 \otimes B \rangle \\ =& (tr(Z_1^t W_1) tr(Z_2^t W_2) - tr(Z_1^t W_2) tr(Z_2^t W_1)) tr(A^t B) \\ =& tr(A^t \varphi(Z_1^t, Z_2^t)) \\ =& \langle A, \varphi(Z_1^t, Z_2^t) \rangle. \end{aligned}$$

□

Thus we may define $\mathcal{N} := im(\partial^1)^\perp \subseteq L(\Lambda^2 \mathfrak{g}_-, \mathfrak{g})$ and obtain an orthogonal decomposition

$$L(\Lambda^2 \mathfrak{g}_-, \mathfrak{g}) = im(\partial^1) \oplus \mathcal{N}.$$

For a better expression of $\mathcal{N}$, we define the *Kostant codifferential*, which is the linear map

$$\mathfrak{p}_+ \otimes \mathfrak{g} \xleftarrow{\partial^*} \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$$

generated by

$$\partial^*(Z \wedge W \otimes A) = -Z \otimes [W, A] + W \otimes [Z, A] + [Z, W] \otimes A \in \mathfrak{p}_+ \otimes \mathfrak{g}$$

for all $Z \wedge W \otimes A \in \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$. The explicit formula immediately implies that ∂^* is grading preserving and P-equivariant, which is very remarkable and crucial for what follows.

Notice that in the obvious sense ∂^* passes to a map $L(\Lambda^2 \mathfrak{g}_-, \mathfrak{g}) \to L(\mathfrak{g}_-, \mathfrak{g})$ which is grading-preserving and G_0-equivariant.

Lemma 5.3.5. *For any $\varphi \in L(\mathfrak{g}_-, \mathfrak{g}), \psi \in L(\Lambda^2\mathfrak{g}_-, \mathfrak{g})$ we have $\langle \partial^1\varphi, \psi\rangle = \langle \varphi, \partial^*\psi\rangle$.*

In particular, we have orthogonal decompositions

$$L(\mathfrak{g}_-, \mathfrak{g}) = im\partial^* \oplus ker\partial^1$$

and

$$L(\Lambda^2\mathfrak{g}_-, \mathfrak{g}) = im\partial^1 \oplus ker\partial^*.$$

The second decomposition implies that $\mathcal{N} = ker(\partial^*)$.

Proof. For the first claim, it suffice to check for $\psi = Z \wedge W \otimes A \in \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$. We have

$$\langle \partial^1\varphi, \psi\rangle = \langle \partial^1\varphi(Z^t, W^t), A\rangle = \langle [Z^t, \varphi(W^t)], A\rangle - \langle [W^t, \varphi(Z^t)], A\rangle - \langle \varphi([Z^t, W^t]), A\rangle$$

and

$$\begin{aligned}\langle \varphi, \partial^*\psi\rangle &= \langle \varphi, -Z \otimes [W, A] + W \otimes [Z, A] + [Z, W] \otimes A\rangle \\ &= -\langle \varphi(Z^t), [W, A]\rangle + \langle \varphi(W^t), [Z, A]\rangle + \langle \varphi([Z, W]^t), A\rangle.\end{aligned}$$

By $-[Z^t, W^t] = [Z, W]^t$, $-\langle \varphi([Z^t, W^t]), A\rangle = \langle \varphi([Z, W]^t), A\rangle$. Moreover, for $A, B, C \in \mathfrak{g}$,

$$\langle [A, B], C\rangle = tr(B^tA^tC - A^tB^tC) = tr(B^tA^tC - B^tCA^t) = tr(B^t(A^tC - CA^t)) = \langle B, [A^t, C]\rangle$$

hence $\langle [Z^t, \varphi(W^t)], A\rangle = \langle \varphi(W^t), [Z, A]\rangle$ and $\langle [W^t, \varphi(Z^t)], A\rangle = \langle \varphi(Z^t), [W, A]\rangle$. Hence the equality.

Now we prove the second claim. We have $(im\partial^*)^\perp = ker\partial^1$ because $\varphi \in (im\partial^*)^\perp \Leftrightarrow \langle \varphi, \partial^*\psi\rangle = 0$ for all $\psi \in L(\Lambda^2\mathfrak{g}_-, \mathfrak{g}) \Leftrightarrow \langle \partial^1\varphi, \psi\rangle = 0$ for all $\psi \in L(\Lambda^2\mathfrak{g}_-, \mathfrak{g}) \Leftrightarrow \partial^1\varphi = 0$. Hence $L(\mathfrak{g}_-, \mathfrak{g}) = im\partial^* \oplus (im\partial^*)^\perp = im\partial^* \oplus ker\partial^1$.

Similarly $(im\partial^1)^\perp = ker\partial^*$, and so $L(\Lambda^2\mathfrak{g}_-, \mathfrak{g}) = im\partial^1 \oplus (im\partial^1)^\perp = im\partial^1 \oplus ker\partial^*$. □

Finally we pass ∂^0 to a linear map $\mathfrak{g} \to L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})$, ∂^1 to a linear map $L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \to L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$, both being grading-preserving and G_0-equivariant, and we pass ∂^* to a linear map $L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g}) \to L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})$ which is grading-preserving and P-equivariant. Thus the orthogonal decompositions in the lemma above passes to orthogonal decompositions

$$L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g}) = im\partial^* \oplus ker\partial^1$$

and

$$L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g}) = im\partial^1 \oplus ker\partial^*.$$

Corollary 5.3.1. *We get that $\mathcal{N} = ker(\partial^*)$ is a P-invariant subspace of $L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$ which is complementary to $im(\partial^1)$. Moreover, $im(\partial^*) \subseteq L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})$ is a P-invariant subspace which is complementary to $ker(\partial^1)$. In particular, ∂^1 restricts to a linear isomorphism $im(\partial^*) \xrightarrow{\cong} im(\partial^1)$, and ∂^* restricts to a linear isomorphism $im(\partial^1) \xrightarrow{\cong} im(\partial^*)$.*

5.4 Normal Cartan geometries of type (G, P): existence

Similarly to the case of $Euc(n)$, we want to impose a normalization condition on the curvature of a Cartan geometry of type (G, P) to find a canonical geometry inducing an underlying structure. The algebraic considerations from Section 5.3 lead to such a normality condition:

Definition 5.4.1. *A regular Cartan geometry $(\mathcal{G}, \omega)$ of type (G, P) is said to be normal if its curvature function $\kappa : \mathcal{G} \to L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})^1$ takes values in $\mathcal{N}$, i.e. if $\partial^* \circ \kappa = 0$.*

For any regular filtered G_0-structure $(\mathcal{G}_0, \theta)$ of type $\mathfrak{g}_-$, by Proposition 5.2.2 there is a regular Cartan geometry $(\mathcal{G}, \omega)$ of type (G, P) inducing $(\mathcal{G}_0, \theta)$. Similar to what we did to obtain a normal Cartan connection of type $(Euc(n), O(n))$ which induces a certain $O(n)$-structure of type $\mathbb{R}^n$, we will modify ω to a normal regular Cartan connection $\tilde{\omega}$ of type (G, P) on $\mathcal{G}$ such that $(\mathcal{G}, \tilde{\omega})$ also induces $(\mathcal{G}_0, \theta)$.

We look at the space of all regular Cartan connections $\tilde{\omega} \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that $(\mathcal{G}, \tilde{\omega})$ induces $(\mathcal{G}_0, \theta)$. Each $(\mathcal{G}, \tilde{\omega})$ defines the same filtration on $T\mathcal{G}$, namely $T^i\mathcal{G}$ for $i = -2, -1, 0$ lifts $T^i\mathcal{G}_0$, and $T^i\mathcal{G}$ for $i = 0, 1, 2$ is determined by fundamental vector fields. Thus we may talk about the filtration on $T\mathcal{G}$ without specifying $\tilde{\omega}$.

Proposition 5.4.1. *The space of all regular Cartan connections $\tilde{\omega} \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that $(\mathcal{G}, \tilde{\omega})$ induces $(\mathcal{G}_0, \theta)$ is an affine space modelled over $\Omega^1_{hor}(\mathcal{G}, \mathfrak{g})^{1,P}$, i.e. all $\mathfrak{g}$-valued, P-equivariant horizontal one-forms on $\mathcal{G}$ with homogeneity ≥ 1.*

Using the trivialisation $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$ induced by ω, we may identify $\Omega^1_{hor}(\mathcal{G}, \mathfrak{g})^{1,P}$ with the space of all P-equivariant maps $\mathcal{G} \to L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})^1$.

Proof. We show that a one-form $\tilde{\omega} \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is a regular Cartan connection of type (G, P) such that $(\mathcal{G}, \tilde{\omega})$ induces $(\mathcal{G}_0, \theta)$ if and only if $\tilde{\omega} - \omega : T\mathcal{G} \to \mathfrak{g}$ is P-equivariant, horizontal, and has homogeneity ≥ 1.

Indeed, $\tilde{\omega}$ is P-equivariant if and only if $\tilde{\omega} - \omega$ is; $\tilde{\omega}$ reproduces the generators of the fundamental vector fields if and only if $\tilde{\omega} - \omega$ is horizontal; $\tilde{\omega}_{-2} = \omega_{-2}$ if and only if $(\tilde{\omega} - \omega)(T\mathcal{G}) \subseteq \mathfrak{g}^{-1}$; and $\tilde{\omega}_{-1}|_{T^{-1}\mathcal{G}} = \omega_{-1}|_{T^{-1}\mathcal{G}}$ if and only if $(\tilde{\omega} - \omega)(T^{-1}\mathcal{G}) \subseteq \mathfrak{g}^0$. Hence for any regular Cartan connection $\tilde{\omega}$ of type (G, P) on $\mathcal{G}$ such that $(\mathcal{G}, \tilde{\omega})$ induces $(\mathcal{G}_0, \theta)$, $\tilde{\omega} - \omega$ is P-equivariant, horizontal, and has homogeneity ≥ 1; conversely, if $\tilde{\omega} - \omega$ is P-equivariant, horizontal, and has homogeneity ≥ 1, then $\tilde{\omega}$ is a Cartan geometry of type (G, P) such that $\tilde{\omega}_{-2} = \omega_{-2}$ and $\tilde{\omega}_{-1}|_{T^{-1}\mathcal{G}} = \omega_{-1}|_{T^{-1}\mathcal{G}}$. We claim that $\tilde{\omega}$ is regular. Let K resp. $\tilde{K}$ denote the curvature of ω resp. $\tilde{\omega}$. For $\xi, \eta \in \Gamma(T^{-1}\mathcal{G})$,

$$
\begin{aligned}
&(\tilde{K} - K)(\xi, \eta) \\
=&\xi.(\tilde{\omega} - \omega)(\eta) - \tilde{\eta}.(\tilde{\omega} - \omega)(\xi) - (\tilde{\omega} - \omega)([\xi, \eta]) + [\tilde{\omega}(\xi), \tilde{\omega}(\eta)] - [\omega(\xi), \omega(\eta)] \\
=&\xi.(\tilde{\omega} - \omega)(\eta) - \tilde{\eta}.(\tilde{\omega} - \omega)(\xi) - (\tilde{\omega} - \omega)([\xi, \eta]) \\
&+ [(\tilde{\omega} - \omega)(\xi), (\tilde{\omega} - \omega)(\eta)] + [(\tilde{\omega} - \omega)(\xi), \omega(\eta)] + [\omega(\xi), (\tilde{\omega} - \omega)(\eta)]
\end{aligned}
$$

whose $\mathfrak{g}_{-2}$-component is zero as the six summands are in $\mathfrak{g}^0$, $\mathfrak{g}^0$, $\mathfrak{g}^{-1}$, $\mathfrak{g}^0$, $\mathfrak{g}^{-1}$, $\mathfrak{g}^{-1}$, respectively. By the regularity of ω, $K(\xi,\eta)$ takes values in $\mathfrak{g}^{-1}$, hence so does $\tilde{K}(\xi,\eta)$. Hence $\tilde{\omega}$ is regular. Also by $\tilde{\omega}_{-2} = \omega_{-2}$ and $\tilde{\omega}_{-1}|_{T^{-1}\mathcal{G}} = \omega_{-1}|_{T^{-1}\mathcal{G}}$, $(\mathcal{G},\tilde{\omega})$ induces $(\mathcal{G}_0,\theta)$.

Using the trivialisation $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$ induced by ω, a regular Cartan geometry $(\mathcal{G},\tilde{\omega})$ of type (G,P) inducing $(\mathcal{G}_0,\theta)$ corresponds to a P-equivariant map $\Phi : \mathcal{G} \to L(\mathfrak{g}/\mathfrak{p},\mathfrak{g})^1$ with

$$\Phi(u)(X) = (\tilde{\omega} - \omega)(u)(\omega^{-1}(X))$$

for all $u \in \mathcal{G}, X \in \mathfrak{g}$; conversely, a P-equivariant map $\Phi : \mathcal{G} \to L(\mathfrak{g}/\mathfrak{p},\mathfrak{g})^1$ corresponds to a Cartan connection $\tilde{\omega}$ of type (G,P) on $\mathcal{G}$ such that $(\mathcal{G},\tilde{\omega})$ induces $(\mathcal{G}_0,\theta)$ with

$$\tilde{\omega}(u)(\xi) = \omega(u)(\xi) + \Phi(u)(\omega(u)(\xi))$$

for all $\xi \in \mathfrak{X}(\mathcal{G}), u \in \mathcal{G}$. □

Let $\tilde{\omega}$ be a regular Cartan connection on $\mathcal{G}$ such that $(\mathcal{G},\tilde{\omega})$ induces $(\mathcal{G}_0,\theta)$, and let $\Phi : \mathcal{G} \to L(\mathfrak{g}/\mathfrak{p},\mathfrak{g})^1$ be the corresponding P-equivariant map. Let $\kappa,\tilde{\kappa} : \mathcal{G} \to L(\Lambda^2(\mathfrak{g}/\mathfrak{p}),\mathfrak{g})^1$ be the curvature functions of ω resp. $\tilde{\omega}$.

Lemma 5.4.1. *Suppose Φ takes values in $L(\mathfrak{g}/\mathfrak{p},\mathfrak{g})^l$ for some $l \in \{1,2,3,4\}$, then $\tilde{\kappa} - \kappa$ takes values in $L(\Lambda^2(\mathfrak{g}/\mathfrak{p}),\mathfrak{g})^l$. Moreover, we have*

$$(\tilde{\kappa} - \kappa)_l = \partial^1 \circ \Phi_l$$

where $(\tilde{\kappa} - \kappa)_l$ and Φ_l denote the homogeneity l component of $(\tilde{\kappa} - \kappa)$ and Φ, respectively.

Proof. Recall from Section 5.3 the grading preserving linear isomorphisms $L(\mathfrak{g}/\mathfrak{p},\mathfrak{g}) \cong L(\mathfrak{g}_-,\mathfrak{g})$ and $L(\Lambda^2(\mathfrak{g}/\mathfrak{p}),\mathfrak{g}) \cong L(\Lambda^2\mathfrak{g}_-,\mathfrak{g})$. Thus Φ passes to a smooth map $\mathcal{G} \to L(\mathfrak{g}_-,\mathfrak{g})^l$, Φ_l passes to $\mathcal{G} \to L(\mathfrak{g}_-,\mathfrak{g})_l$, and $\tilde{\kappa} - \kappa$ passes to $\mathcal{G} \to L(\Lambda^2\mathfrak{g}_-,\mathfrak{g})^1$.

For $u \in \mathcal{G}$, $X \in \mathfrak{g}_-$, define $\xi := \omega^{-1}(X)(u) \in T_u\mathcal{G}$, $\phi := \Phi(u) \in L(\mathfrak{g}_-,\mathfrak{g})^l$ and $\phi_l := \Phi_l(u) \in L(\mathfrak{g}_-,\mathfrak{g})_l$. Then $\tilde{\omega}(u)(\xi) = X + \phi(X)$. Applying $\tilde{\omega}(u)^{-1}$ we get

$$\omega^{-1}(X)(u) = \tilde{\omega}^{-1}(X)(u) + \tilde{\omega}^{-1}(\phi(X))(u).$$

Denote by K resp. $\tilde{K}$ the curvatures of ω resp. $\tilde{\omega}$. Then $(\tilde{\kappa} - \kappa)(u)$ maps $X \in \mathfrak{g}_i, Y \in \mathfrak{g}_j$ for $i,j \in \{-2,-1\}$ to

$$\begin{aligned}
&\tilde{\kappa}(u)(X,Y) - \kappa(u)(X,Y)\\
=&\tilde{K}(u)(\tilde{\omega}^{-1}(X),\tilde{\omega}^{-1}(Y)) - K(u)(\omega^{-1}(X),\omega^{-1}(Y))\\
=&(\tilde{K} - K)(u)(\omega^{-1}(X),\omega^{-1}(Y)) - \tilde{K}(u)(\omega^{-1}(X),\tilde{\omega}^{-1}(\phi(Y))\\
&- \tilde{K}(u)(\tilde{\omega}^{-1}(\phi(X)),\omega^{-1}(Y)) + \tilde{K}(u)(\tilde{\omega}^{-1}(\phi(X)),\tilde{\omega}^{-1}(\phi(Y)))
\end{aligned}$$

Since $\tilde{K}$ has homogeneity ≥ 1 and ϕ has homogeneity $\geq l$, the last three summands have homogeneities $\geq l+1$, and the first summand equals

$$\begin{aligned}
&(\tilde{K}-K)(u)(\omega^{-1}(X),\omega^{-1}(Y))\\
=&\omega^{-1}(X)(u).((\tilde{\omega}-\omega)(\omega^{-1}(Y))-\omega^{-1}(Y)(u).((\tilde{\omega}-\omega)(\omega^{-1}(X))\\
&-(\tilde{\omega}-\omega)(u)([\omega^{-1}(X),\omega^{-1}(Y)])\\
&+\left([\tilde{\omega}(u)(\omega^{-1}(X)),\tilde{\omega}(u)(\omega^{-1}(Y))]-[\omega(u)(\omega^{-1}(X)),\omega(u)(\omega^{-1}(Y))]\right)\\
=&\omega^{-1}(X)(u).\Phi(Y)-\omega^{-1}(Y)(u).\Phi(X)\\
&-\phi\left(\omega(u)([\omega^{-1}(X),\omega^{-1}(Y)])\right)\\
&+([X+\phi(X),Y+\phi(Y)]-[X,Y])\\
=&\omega^{-1}(X)(u).\Phi(Y)-\omega^{-1}(Y)(u).\Phi(X)\\
&-\phi\left(\omega(u)([\omega^{-1}(X),\omega^{-1}(Y)])\right)\\
&+([X,\phi(Y)]+[\phi(X),Y]+[\phi(X),\phi(Y)])\,.
\end{aligned}$$

The homogeneities of the six resulting summands are at least $i+l, j+l, l, l, l, 2l$, respectively. Hence we conclude that $(\tilde{\kappa}-\kappa)(u)$ has homogeneity $\geq l$.

Moreover, consider the third summand $-\phi\left(\omega(u)([\omega^{-1}(X),\omega^{-1}(Y)])\right)$. Its l-th homogeneity component equals $-\phi_l$ applied to the $\mathfrak{g}_{i+j}$-component of $\omega(u)([\omega^{-1}(X),\omega^{-1}(Y)])$, which is zero unless $i=j=-1$. In the case $i=j=-1$, by the regularity condition of ω, the $\mathfrak{g}_{-2}$-components of

$$\omega(u)([\omega^{-1}(X),\omega^{-1}(Y)])$$

coincides to the $\mathfrak{g}_{-2}$-component of

$$[\omega(u)(\omega^{-1}(X)),\omega(u)(\omega^{-1}(Y))]=[X,Y].$$

Since $[X,Y]=0$ unless $i=j=-1$, we conclude that the l-th homogeneity component of $-\phi\left(\omega(u)([\omega^{-1}(X),\omega^{-1}(Y)])\right)$ is $-\phi_l([X,Y])$. Thus

$$(\tilde{\kappa}-\kappa)_l(u)(X,Y)$$

equals

$$-\phi_l([X,Y])+[X,\phi_l(Y)]+[\phi_l(X),Y].$$

Recall from Definition 5.3.1, this is exactly $(\partial^1\circ\phi_l)(X,Y)$.

$\square$

Corollary 5.4.1. *There exists a normal regular Cartan connection $\tilde{\omega}$ of type (G,P) on $\mathcal{G}$ such that $(\mathcal{G},\tilde{\omega})$ induces $(\mathcal{G}_0,\theta)$.*

Proof. We start with $(\mathcal{G},\omega)$ which induces $(\mathcal{G}_0,\theta)$. Denote by $\kappa : \mathcal{G} \to L(\Lambda^2(\mathfrak{g}/\mathfrak{p}),\mathfrak{g})^1$ its curvature function. Suppose $\partial^* \circ \kappa$ has homogeneity $\geq l$ for some $l \in \{1,2,3,4\}$, so

$$\partial^* \circ \kappa : \mathcal{G} \to im(\partial^*) \cap L(\mathfrak{g}/\mathfrak{p},\mathfrak{g})^l.$$

We claim that there is a regular Cartan connection $\tilde{\omega}$ of type (G,P) on $\mathcal{G}$ with curvature function $\tilde{\kappa}$ such that $(\mathcal{G},\tilde{\omega})$ induces $(\mathcal{G}_0,\theta)$ and $\partial^* \circ \tilde{\kappa}$ takes values in $L(\mathfrak{g}/\mathfrak{p},\mathfrak{g})^{l+1}$. By Lemma 5.4.1, we just need to find a P-equivariant map

$$\Phi : \mathcal{G} \to L(\mathfrak{g}/\mathfrak{p},\mathfrak{g})^l$$

such that

$$\partial^* \circ \partial^1 \circ \Phi_l = (-\partial^* \circ \tilde{\kappa})_l$$

where Φ_l is the l-homogeneity component of Φ and $(-\partial^* \circ \tilde{\kappa})_l$ is the l-homogeneity component of $-\partial^* \circ \tilde{\kappa}$. Then we let $\tilde{\omega}$ correspond to Φ.

Indeed, let $p : \mathcal{G} \to M$ denote the bundle projection and let $\mathcal{U}$ be an open cover of M, such that for each $U \in \mathcal{U}$ there is a local section

$$\sigma : U \to \mathcal{G}.$$

We define

$$f : U \xrightarrow{\sigma} \mathcal{G} \xrightarrow{(-\partial^* \circ \kappa)_l} im(\partial^*) \cap L(\mathfrak{g}/\mathfrak{p},\mathfrak{g})_l.$$

Recall from Corollary 5.3.1 that ∂^1 and ∂^* restrict to grading-preserving linear isomorphisms $\partial^1 : im(\partial^*) \xrightarrow{\cong} im(\partial^1)$ and $\partial^* : im(\partial^1) \xrightarrow{\cong} im(\partial^*)$, which give rise to a grading preserving linear isomorphism

$$\psi := (\partial^* \circ \partial^1)^{-1} : im(\partial^*) \to im(\partial^*).$$

Let

$$\Phi^U : p^{-1}(U) \to im(\partial^*) \cap L(\mathfrak{g}/\mathfrak{p},\mathfrak{g})^l$$

be the unique P-equivariant map such that

$$\Phi^U \circ \sigma = \psi \circ f.$$

Denote by Φ^U_l the l-homogeneity component of Φ. Then for each $x \in U$, one has

$$\partial^* \circ \partial^1 \circ \Phi^U_l(\sigma(x)) = \partial^* \circ \partial^1 \circ \psi \circ f(\sigma(x)) = f(x) = (-\partial^* \circ \kappa)_l(\sigma(x)).$$

This means the following. Let $\tilde{\omega}_U \in \Omega^1(p^{-1}(U),\mathfrak{g})$ denote the locally defined Cartan connection corresponding to Φ^U. Then its curvature function $\tilde{\kappa}_U$ satisfies $\partial^* \circ \tilde{\kappa}_U(\sigma(x)) \in L(\mathfrak{g}/\mathfrak{p},\mathfrak{g})^{l+1}$ for all $x \in U$. Since $\tilde{\kappa}_U$ is P-equivariant, $\partial^* \circ \tilde{\kappa}_U$ always takes values in $L(\mathfrak{g}/\mathfrak{p},\mathfrak{g})^{l+1}$. This implies that

$$\partial^* \circ \partial^1 \circ \Phi^U_l = (-\partial^* \circ \kappa)_l.$$

In particular, using a partition of unity subordinate to $\mathcal{U}$, we patch together all Φ^U and obtain a P-equivariant map $\Phi : \mathcal{G} \to im(\partial^*) \cap L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})^l$ with homogeneity l component Φ_l, such that $\partial^* \circ \partial^1 \circ \Phi_l = (-\partial^* \circ \kappa)_l$.

We are able to iterate the process by describing the affine space over $\Omega^1_{hor}(\mathcal{G}, \mathfrak{g})^{1,P}$ of all regular Cartan connections on $\mathcal{G}$ of type (G, P) inducing $(\mathcal{G}_0, \theta)$ with the new center $\tilde{\omega}$. We also use the new trivialisation $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$ induced by $\tilde{\omega}$ to identify the space with the space of P-equivariant maps $\mathcal{G} \to L(\mathfrak{g}_-, \mathfrak{g})^1$. In particular, if $l + 1 = 5$, we conclude that $\partial^* \circ \tilde{\kappa} = 0$, and we get a normal regular Cartan connection of type (G, P) on $\mathcal{G}$ inducing $(\mathcal{G}_0, \theta)$. $\square$

Thus we proved that for any regular filtered G_0-structure of type $\mathfrak{g}_-$, there exists a normal regular Cartan geometry of type (G, P) inducing it.

5.5 Normal Cartan geometries of type (G, P): uniqueness

From last section we have that for a regular filtered G_0-structure $(\mathcal{G}_0, \theta)$ of type $\mathfrak{g}_-$, there exists a normal regular Cartan geometry $(\mathcal{G}, \omega)$ of type (G, P) inducing it.

We want to show that any normal regular Cartan geometry of type (G, P) inducing $(\mathcal{G}_0, \theta)$ is isomorphic to $(\mathcal{G}, \omega)$. For this we need an intermediate step of showing that given any normal regular Cartan connection $\tilde{\omega}$ of type (G, P) on $\mathcal{G}$ such that $(\mathcal{G}, \tilde{\omega})$ induces $(\mathcal{G}_0, \theta)$, then $(\mathcal{G}, \tilde{\omega})$ is isomorphic to $(\mathcal{G}, \omega)$ over $id_{\mathcal{G}_0}$.

Lemma 5.5.1. *A map $\Psi : \mathcal{G} \to \mathcal{G}$ is a principal bundle map lying above $id_{\mathcal{G}_0}$ if and only if it is of the form $u \mapsto uexp(f(u))$ for a P-equivariant smooth map $f : \mathcal{G} \to \mathfrak{p}_+$. In this case, $(\mathcal{G}, \Psi^*\omega)$ is a normal regular Cartan geometry of type (G, P) inducing $(\mathcal{G}_0, \theta)$.*

Proof. Let $\Psi : \mathcal{G} \to \mathcal{G}$ be a principal bundle map lying above $id_{\mathcal{G}_0}$. Since $exp : \mathfrak{p}_+ \to P_+$ is a diffeomorphism, we may write Ψ as $u \mapsto uexp(f(u))$ for a smooth map $f : \mathcal{G} \to \mathfrak{p}_+$. The P-equivariancy of Ψ exactly means that $exp(f(ug)) = conj(g^{-1})(exp(f(u))) = exp(Ad(g^{-1}) \circ f(u))$, that is, $f(ug) = Ad(g^{-1}) \circ f(u)$ for all $u \in \mathcal{G}, g \in P$. Conversely, a map $\Psi : \mathcal{G} \to \mathcal{G}, u \mapsto uexp(f(u))$ for a P-equivariant smooth map $f : \mathcal{G} \to \mathfrak{p}_+$ is smooth and P-equivariant, hence is a principal bundle map lying above $id_{\mathcal{G}_0}$.

Consider the one-form $\Psi^*\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Since Ψ is P-equivariant, $\Psi^*\omega$ is P-equivariant and reproduces the generators of the fundamental vector fields. Since $T\Psi$ is a linear isomorphism at each fiber, so does $\Psi^*\omega$. Hence $\Psi^*\omega$ is a Cartan connection of type (G, P). Moreover, since the curvature function of $\Psi^*\omega$ equals $\kappa \circ \Psi$, where κ is the curvature function of ω, $\Psi^*\omega$ is normal and regular as ω is.

It remains to show that $(\mathcal{G}, \Psi^*\omega)$ induces $(\mathcal{G}_0, \theta)$. As a morphism of Cartan geometries, Ψ is filtration preserving. Hence $(\mathcal{G}, \Psi^*\omega)$ induces the same filtration on $T\mathcal{G}_0$ as $(\mathcal{G}, \omega)$ does. For $i = -2, -1$ and $\xi \in T^i_{u_0}\mathcal{G}_0$, if $\tilde{\xi} \in T^i_u\mathcal{G}$ is a lift of ξ, then so does $T_u\Psi(\tilde{\xi}) \in T^i_{uexp(f(u))}\mathcal{G}$. Since $(\Psi^*\omega)(uexp(f(u)))(T_u\Psi(\tilde{\xi})) = \omega(u)(\tilde{\xi})$ we see that $(\mathcal{G}, \Psi^*\omega)$ and $(\mathcal{G}, \omega)$ induce the same regular filtered G_0-structure of type $\mathfrak{g}_-$, which is $(\mathcal{G}_0, \theta)$. $\square$

Let the affine space of all regular Cartan connections of type (G, P) on $\mathcal{G}$ inducing $(\mathcal{G}_0, \theta)$ centered at ω be identified with the space of P-equivariant smooth maps $\mathcal{G} \to L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})^1$ using the trivialisation $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$ induced by ω, and let $\Psi^*\omega$ correspond to $\Phi : \mathcal{G} \to L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})^1$ in the space.

We are able to describe the relation between $f : \mathcal{G} \to \mathfrak{g}^1$ and $(\Psi^*\omega - \omega) : \mathcal{G} \to L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})^1$ in the lowest homogeneity components.

Lemma 5.5.2. *If $f : \mathcal{G} \to \mathfrak{p}_+$ takes its images in $\mathfrak{g}^l$ for some $l \in \{1, 2\}$, then Φ takes its images in $L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})^l$. Moreover, we have*

$$\Phi_l = \partial^0 \circ f_l$$

where Φ_l and f_l denote the homogeneity l component of Φ and f, respectively.

Proof. Recall from Section 5.3 the grading preserving linear isomorphisms $L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \cong L(\mathfrak{g}_-, \mathfrak{g})$. Thus Φ passes to a smooth map $\mathcal{G} \to L(\mathfrak{g}_-, \mathfrak{g})^l$ and Φ_l passes to $\mathcal{G} \to L(\mathfrak{g}_-, \mathfrak{g})_l$.

For $u \in \mathcal{G}$, $X \in \mathfrak{g}_-$, define $\xi := \omega^{-1}(X)(u) \in T_u\mathcal{G}$. We denote by $\delta(exp \circ f) \in \Omega^1(\mathcal{G}, \mathfrak{p}_+)$ the left-logarithmic derivative of $exp \circ f$, by $\Psi(u) = uexp(f(u))$ we have

$$T_u\Psi(\xi) = T_u r^{exp(f(u))}(\xi) + \zeta_{\delta(exp\circ f)(u)(\xi)}(\Psi(u)).$$

Thus

$$\begin{aligned}
\Phi(u)(X) =&(\Psi^*\omega - \omega)(u)(\omega(u)^{-1}(X)) \\
=&(\Psi^*\omega)(u)(\xi) - X \\
=&\omega(u)(T_u\Psi(\xi)) - X \\
=&(Ad \circ exp(-f(u)))\omega(u)(\xi) + \delta(exp \circ f)(u)(\xi) - X
\end{aligned}$$

As

$$\begin{aligned}
(Ad \circ exp(-f(u)))\omega(u)(\xi) =&(Ad \circ exp(-f(u)))(X) \\
=&(e \circ ad(-f(u)))(X) \\
=&\Sigma_{i\geq 0}\frac{1}{i!}ad(-f(u))^i(X) \\
=&X + (ad(-f(u))(X) + \frac{1}{2!}ad(-f(u))^2(X) + \cdots),
\end{aligned}$$

$$\Phi(u)(X) = (ad(-f(u))(X) + \frac{1}{2!}ad(-f(u))^2(X) + \cdots) + \delta(exp \circ f)(u)(\xi).$$

Since $\delta(exp \circ f)(u)(\xi) \in \mathfrak{g}^l$ and any map $\mathfrak{g}_- \to \mathfrak{g}^l$ has homogeneity $\geq l + 1$,

$$\Phi_l(u)(X) = (ad(-f_l(u))(X).$$

Recall from Definition 5.3.1, this is exactly $(\partial^0 \circ f_l)(u)(X)$. □

Corollary 5.5.1. *(i) There is no nontrivial automorphism on $(\mathcal{G}, \omega)$ over $id_{\mathcal{G}_0}$.*

(ii) Let $(\mathcal{G}, \tilde{\omega})$ be another normal regular Cartan geometry of type (G, P) inducing $(\mathcal{G}_0, \theta)$. Then there exists a unique isomorphism from $(\mathcal{G}, \tilde{\omega})$ to $(\mathcal{G}, \omega)$ lying above $id_{\mathcal{G}_0}$.

Proof. (i) Let Ψ be an automorphism on $(\mathcal{G}, \omega)$ over $id_{\mathcal{G}_0}$, then Ψ is of the form $u \mapsto uexp(f(u))$ for a P-equivariant map $f : \mathcal{G} \to \mathfrak{p}_+$. Suppose f takes images in $\mathfrak{g}^l$ for $l \in \{1, 2\}$. Then its homogeneity l component f_l satisfies $\partial^0 \circ f_l = 0$ because $\Psi^* \omega = \omega$. By Lemma 5.3.2, $\partial^0|_{\mathfrak{p}_+}$ is injective, hence $f_l = 0$ and f takes values in $\mathfrak{g}^{l+1}$. Iterating the argument, we conclude that $f = 0$, hence $\Psi = id_{\mathcal{G}}$.

Observe that this also gives the uniqueness statement in (ii) by applying this to $\Psi_2 \circ \Psi_1^{-1}$ for two isomorphisms $\Psi_1, \Psi_2 : (\mathcal{G}, \tilde{\omega}) \to (\mathcal{G}, \omega)$.

(ii) Since $(\mathcal{G}, \omega)$ and $(\mathcal{G}, \tilde{\omega})$ both induce $(\mathcal{G}_0, \theta)$, ω and $\tilde{\omega}$ induce the same filtration on $T\mathcal{G}$, and $\tilde{\omega} - \omega : T\mathcal{G} \to \mathfrak{g}$ has homogeneity $\geq l$ for some $l \in \{1, 2, 3, 4\}$. In this case, $\tilde{\omega}$ corresponds to a P-equivariant map $\Phi : \mathcal{G} \to L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})^l$ in the sense of Proposition 5.4.1. Let's denote by κ resp. $\tilde{\kappa}$ the curvature functions of ω resp. $\tilde{\omega}$. By Lemma 5.4.1, $\tilde{\kappa} - \kappa$ has homogeneity l and

$$\partial^1 \circ \Phi_l = (\tilde{\kappa} - \kappa)_l$$

where Φ_l and $(\tilde{\kappa} - \kappa)_l$ are the homogeneity l components of Φ and $\tilde{\kappa} - \kappa$, respectively. Hence

$$\partial^* \circ \partial^1 \circ \Phi_l = \partial^* \circ (\tilde{\kappa} - \kappa)_l = 0.$$

By Corollary 5.3.1, this implies that for all $u \in \mathcal{G}$,

$$\partial^1 \circ \Phi_l(u) \in ker(\partial^*) \cap im(\partial^1) = \{0\},$$

and so $\Phi_l(u) \in ker(\partial^1)$. We discuss in two cases:

Case 1. If $l \in \{3, 4\}$, we have $\Phi_l(u) = 0$ because ∂^1 is injective in homogeneities $3, 4$ (Lemma 5.3.2). Hence Φ takes values in $L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})^{l+1}$. Iterating the argument, we get that Φ takes values in $L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})^5 = \{0\}$. Hence $\tilde{\omega} = \omega$.

Case 2. If $l \in \{1, 2\}$, let $p : \mathcal{G} \to M$ and $p_0 : \mathcal{G}_0 \to M$ denote the bundle projection. We claim that for any open subset $U \subseteq M$ with a local section

$$\sigma : U \to \mathcal{G},$$

then there is a unique isomorphism from $(p^{-1}(U), \tilde{\omega})$ to $(p^{-1}(U), \omega)$ over the identity on $p_0^{-1}(U)$.

Indeed, since for each $x \in U$, $\Phi_l(\sigma(x)) \in ker(\partial^1) = im(\partial^0)$ and since ∂^0 is injective on $\mathfrak{g}_l$ (Lemma 5.3.2), there is a (unique) smooth map $h : U \to \mathfrak{g}_l$ such that

$$\partial^0 \circ h = \Phi_l \circ \sigma.$$

Now let $f^U : p^{-1}(U) \to \mathfrak{g}^l$ be the unique P-equivariant map such that

$$f^U \circ \sigma = h.$$

Then

$$\partial^0 \circ f_l^U \circ \sigma = \Phi_l \circ \sigma$$

where f_l^U is the homogeneity l component of f^U. Define a principal bundle automorphism

$$\Psi_U : p^{-1}(U) \to p^{-1}(U), u \mapsto uexp(f^U(u)).$$

Then by Lemma 5.5.1 and Lemma 5.5.2, $(p^{-1}(U), \omega)$ and $(p^{-1}(U), \Psi_U^*\omega)$ has the same underlying structure, and $\Psi_U^*\omega$ corresponds to a P-equivariant map $\Phi' : p^{-1}(U) \to L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})^l$ such that

$$\Phi_l' = \partial^0 \circ f_l^U,$$

where Φ_l' is the homogeneity l component of Φ'. Hence

$$\Phi_l' \circ \sigma = \partial^0 \circ f_l^U \circ \sigma = \Phi_l \circ \sigma$$

meaning that $(\Phi' - \Phi)(\sigma(x)) \in L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})^{l+1}$ for all $x \in U$. Since $\Phi|_{p^{-1}(U)}$ and Φ' are both P-equivariant, we conclude that $\Phi - \Phi'$ takes values in $L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})^{l+1}$, hence $\tilde{\omega}|_{p^{-1}(U)} - \Psi_U^*\omega$ has homogeneity $\geq l+1$. If $l+1 = 3$, from the last part we have $\tilde{\omega}|_{p^{-1}(U)} = \Psi_U^*\omega$; if $l+1 = 2$, we describe $\tilde{\omega}|_{p^{-1}(U)}$ as a P-equivariant map $p^{-1}(U) \to L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})^{l+1}$ in the sense of Proposition 5.4.1, but use $\Psi_U^*\omega$ as the center instead of ω and use the trivialisation $(Tp)^{-1}(U) \xrightarrow{\cong} p^{-1}(U) \times \mathfrak{g}$ induced by $\Psi_U^*\omega$ instead of by ω. Repeating the argument, we obtain a principal bundle automorphism $\hat{\Psi}_U : p^{-1}(U) \to p^{-1}(U)$ such that $\tilde{\omega} - \hat{\Psi}_U^*\Psi_U^*\omega$ has homogeneity $l+2 = 3$. By the last part, this implies that $\tilde{\omega} = \hat{\Psi}_U^*\Psi_U^*\omega$, hence there is an isomorphism $\Psi_U \circ \hat{\Psi}_U : (p^{-1}(U), \tilde{\omega}) \to (p^{-1}(U), \omega)$ over the identity on $p_0^{-1}(U)$. By (i), such an isomorphism is unique.

In particular, all such isomorphisms piece together to a (unique) isomorphism from $(\mathcal{G}, \tilde{\omega})$ to $(\mathcal{G}, \omega)$ over $id_{\mathcal{G}_0}$. □

Now we can prove that the normal regular Cartan geometries of type (G, P) inducing $(\mathcal{G}_0, \theta)$ are all isomorphic.

Corollary 5.5.2. *If $(\mathcal{G}', \omega')$ is a normal regular Cartan geometry of type (G, P) inducing $(\mathcal{G}_0, \theta)$, then there is a unique isomorphism from $(\mathcal{G}, \omega)$ to $(\mathcal{G}', \omega')$ lying above $id_{\mathcal{G}_0}$.*

Proof. The proof is similar to that of Corollary 5.5.1. Let M denote the base manifold of $\mathcal{G}_0$, then it is also the base manifold of $\mathcal{G}$ and $\mathcal{G}'$. Choose an open cover $\{U_i\}$ of M such that for each i, the identity map on $\mathcal{G}_0|_{U_i}$ lifts to a principal bundle isomorphism $\Psi_i : \mathcal{G}|_{U_i} \to \mathcal{G}'|_{U_i}$. By Lemma 5.5.1, $(\mathcal{G}|_{U_i}, \Psi_i^*\omega')$ is a normal regular Cartan geometry inducing $(\mathcal{G}_0|_{U_i}, \theta)$. By Corollary 5.5.1(ii), there is a morphism $(\mathcal{G}|_{U_i}, \omega) \to (\mathcal{G}|_{U_i}, \Psi_i^*\omega')$ above the identity on $\mathcal{G}_0|_{U_i}$. Composing Ψ_i to this, we obtain a morphism $(\mathcal{G}|_{U_i}, \omega) \to (\mathcal{G}'|_{U_i}, \omega')$ above the identity on $\mathcal{G}_0|_{U_i}$, which we still denote by Ψ_i.

We claim that $\Psi_i : (\mathcal{G}|_{U_i}, \omega) \to (\mathcal{G}'|_{U_i}, \omega')$ is the unique morphism above the identity on $\mathcal{G}_0|_{U_i}$. Indeed, let Ψ_i' also be such a morphism. Being above $id_{\mathcal{G}_0|_{U_i}}$, Ψ_i and Ψ_i'

are isomorphisms and $\Psi_i^{-1} \circ \Psi_i'$ is an automorphism on $(\mathcal{G}|_{U_i}, \omega)$ above $id_{\mathcal{G}_0|_{U_i}}$. Then by Corollary 5.5.1(i), $\Psi_i^{-1} \circ \Psi_i'$ equals the identity on $\mathcal{G}|_{U_i}$.

Hence all the Ψ_i's piece together to define an isomorphism $\Psi : (\mathcal{G}, \omega) \to (\mathcal{G}', \omega')$ lying above $id_{\mathcal{G}_0}$. □

Moreover, let $(\mathcal{G}, \omega)$ resp. $(\mathcal{G}', \omega')$ be normal regular Cartan geometries of type (G, P), and denote their underlying regular filtered G_0-structures of type $\mathfrak{g}_-$ by $(\mathcal{G}_0, \theta)$ resp. $(\mathcal{G}_0', \theta')$. We claim that:

Proposition 5.5.1. *Any morphism $\Phi : (\mathcal{G}_0, \theta) \to (\mathcal{G}_0', \theta')$ admits a unique lift to a morphism $\Psi : (\mathcal{G}, \omega) \to (\mathcal{G}', \omega')$.*

Proof. Let $f : M \to M'$ denote the base map of Φ. Choose an open cover $\{U_i\}$ of M such that for each i, f restricts to an isomorphism $U_i \xrightarrow{\cong} f(U_i) =: U_i'$ and that $\Phi|_{\mathcal{G}_0|_{U_i}}$ lifts to a principal bundle map $\Psi_i : \mathcal{G}|_{U_i} \to \mathcal{G}'|_{U_i'}$. Similar to the proof of Lemma 5.5.1 we verify that $(\mathcal{G}|_{U_i}, \Psi_i^* \omega')$ is a normal regular Cartan geometry inducing $(\mathcal{G}_0|_{U_i}, \theta)$. By Corollary 5.5.2, there is a morphism $(\mathcal{G}|_{U_i}, \omega) \to (\mathcal{G}|_{U_i}, \Psi_i^* \omega')$ lying above $id_{\mathcal{G}_0|_{U_i}}$. Composing Ψ_i to this, we obtain a morphism $(\mathcal{G}|_{U_i}, \omega) \to (\mathcal{G}'|_{U_i}, \omega')$ above the identity on $\mathcal{G}_0|_{U_i}$, which we still denote by Ψ_i.

We claim that Ψ_i is the unique morphism $(\mathcal{G}|_{U_i}, \omega) \to (\mathcal{G}'|_{U_i}, \omega')$ lifting $\Phi|_{\mathcal{G}_0|_{U_i}}$. Indeed, let Ψ_i' also be such a morphism. Being with base map $f : U_i \xrightarrow{\cong} U_i'$, Ψ_i and Ψ_i' are isomorphisms and $\Psi_i^{-1} \circ \Psi_i'$ is an automorphism on $(\mathcal{G}|_{U_i}, \omega)$ above $id_{\mathcal{G}_0|_{U_i}}$. Then by Corollary 5.5.1(i), $\Psi_i^{-1} \circ \Psi_i'$ equals the identity on $\mathcal{G}|_{U_i}$.

Hence all the Ψ_i's piece together the unique morphism $(\mathcal{G}, \omega) \to (\mathcal{G}', \omega')$ lifting Φ. □

Corollary 5.5.3. *The category of normal regular Cartan geometries of type (G, P) is equivalent to the category of regular filtered G_0-structures of type $\mathfrak{g}_-$.*

Proof. Recall from Proposition 5.2.1 that there is a functor from regular Cartan geometries of type (G, P) to regular filtered G_0-structures of type $\mathfrak{g}_-$. We restrict it to the subcategory of normal regular Cartan geometries of type (G, P).

From the proposition above, the resulting functor is full and faithful. The functor is essentially surjective because for any regular filtered G_0-structures of type $\mathfrak{g}_-$, there exists a normal regular Cartan geometry of type (G, P) inducing it. Hence the functor yields an equivalence of categories. □

Remark 5.5.1. *From the theory of Cartan geometry as cited in Remark 4.2.1, we obtain the following consequences of this categorial equivalence:*

Since G together with its Maurer-Cartan form is a normal regular Cartan geometry of type (G, P) inducing the canonical Lagrangean contact structure on the flag manifold $F_{1,n+1}(\mathbb{R}^{n+2})$, and since $F_{1,n+1}(\mathbb{R}^{n+2})$ is a connected manifold as G is, from the theory of

Cartan geometries as cited in Remark 4.2.1, we conclude that the automorphism group of the Lagrangean contact structure on $F_{1,n+1}(\mathbb{R}^{n+2})$ *is* G.

Moreover, the automorphism group of a Lagrangean contact structure on a connected $(2n+1)$*-dimensional manifold is a Lie group of dimension at most* $dim(G)$.

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