universität wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„Phase Transitions in the Coupled Vicsek-Stokes Model "

verfasst von / submitted by

Judith Staudner, BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2020 / Vienna, 2020

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Mathematik
Betreut von / Supervisor:	Ass.-Prof. Sara Merino Aceituno, PhD

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1 Abstract

In this thesis the phase transition that occurs in the non-normalized Vicsek-BGK Model for self propelled particles is analyzed. The Vicsek-BGK model is a partial differential equation that describes the time-evolution of distribution of agents undergoing collective dynamics. The equilibria of the kinetic Vicsek-BGK equation are described as well as their stability, in different regions (high or low density). Then the hydrodynamic equations for self alignment are derived for the high density regions, as well as a diffusion equation for the density in the low density regions. In the next step, the Vicsek Model for self-propelled particles is coupled with the Stokes equations for fluids to investigate the interplay of particles and fluid. We look at a coupled Vicsek-Stokes system [2]. In the coupled model we derive the self-organized Hydrodynamics Stokes system for high density regimes as well as the diffusion equation for low density regimes. The overall results do not change in the coupled model, however the resulting model does not only treat the particles mean velocity and orientation but also the fluid velocity and pressure.

2 Introduction

In this master thesis collective behavior of particles, described by their position, orientation and velocity is studied and in particular the phase transition that occurs when a certain density threshold is crossed. In the context of self propelled agents, phase transition", means a change in particle behavior emerges and particles do no longer move independently of one another. Once that certain density is exceeded, particles start to move in unison, collective behavior emerges.
Collective and self-organized motion can be frequently found in nature. Collective motion is an emergent phenomenon that appears when a lot of interaction between individual agents without a leader occurs. Examples that immediately come to mind are flocking birds or schools of fish, however self-organization also happens on a microscopic level, such as in bacterial suspensions or sperm motion in viscous fluids. In the present work, both collective motion in terms of the Bhatnagar-Gross-Krook (BGK) Vicsek Model is being studied as well as collective motion within a fluid, where the Vicsek Model is coupled with Stokes equation for fluids. In the context of the BGK Vicsek model, the underlying system consists of a large number of self-propelled particles or agents, trying to align their orientation with the average of their neighbours' orientations.

The field of collective behavior has been rigorously studied in mathematical literature, and there are various models to describe collective motion. The Vicsek model distinguishes itself by considering self-propelled particles (e.g.birds) and by a geometrical constraint: the velocities of the particles have constant norm and therefore the direction of their velocities ω (their orientation) lies on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$.
To study collective behavior in a mathematical context, we adopt the methodology used in mathematical kinetic theory of gases, since gases can also be viewed as many particle systems. Such a particle system can be examined on three different levels:

- The microscopic scale: here, the motion of each individual particle is detailed with ODEs for each particle. These Individual based models (IBM) are either derived from Newton's laws or from stochastic processes.

- The mesosopic scale: sometimes, it can be more reasonable to look at the system of particles as a whole, especially when we consider a large number of agents. In that case, kinetic differential equations like the Boltzmann, Fokker Planck or, like in this thesis, BGK equation can be used, which model the evolution of the distribution function of the system.
  The BGK Equation is a simplification of the Boltzmann Equation, where the BGK operator is used instead of the classical collision operator. The BGK operator is a relaxation Operator toward a Maxwellian (in the case of this thesis a von Mises Distribution). [3]

- The macroscopic scale: large scale dynamics, obtained by averaging, for example described by Euler or Navier-Stokes equations.

The challenge with these different levels of examination is the derivation of one from the other (e.g. deriving the kinetic equation from the IBM or the macroscopic equations from the kinetic model). This issue has been extensively studied over the past century, especially since the question is included in Hilbert's sixth problem, which asks, whether it is possible to expand the axiomatic method outside the existing mathematical disciplines, to physics and beyond.

Usually, conservation laws are a necessity to transition from the kinetic model to macroscopic equations. However, in the case of self-propelled particles, the conservation laws typically valid in the classic kinetic theory of gasses, do not hold anymore. As a sort of "workaroundthe Generalized Collision Invariant (GCI) has been introduced. The GCI is a great advance to accurately perform the passage between meso- and macroscopic models when concerned with collective behavior of self-propelled particles [4]. This GCI is extremely useful to derive the macroscopic equations or the Self-Organised Hydrodynamics (SOH), in sections 3.4.2 and 4.1.2. Unlike on the micro- and mesoscopic levels, where we have a single equation describing the system, the SOH are a system of partial differential equations, each describing one physical quantity.

In this thesis, we start out with the BGK equation, which is a variation of the Boltzmann equation [3]:

$$\partial_t f + \omega \cdot \nabla_x f = \rho_f M_{J_f} - f. \tag{1}$$

Here $f(t, x, \omega)$ is a probability density function describing the distribution of particles having position $x \in \mathbb{R}^3$ and orientation $\omega \in \mathbb{S}^2$ at time $t \in \mathbb{R}_+$. The particle density ρ_f, the flux (amount of a given quantity flowing per unit time, through a unit area [5]) J_f; and the von Mises Distribution, a continuous probability distribution on the circle [6], M_{J_f} are given by, respectively:

$$\begin{aligned} \rho_f(t, x) &= \int_{\mathbb{S}^2} f(t, x, \omega) d\omega, \\ J_f &= \int_{\mathbb{S}^2} \omega f(t, x, \omega) d\omega, \\ M_{J_f}(\omega) &= \frac{e^{\omega \cdot J_f}}{\int_{\mathbb{S}^{d-1}} e^{\omega \cdot J_f} d\omega}. \end{aligned}$$

As in the Boltzmann equation, the left hand side of the BGK equation (1) models the particle movement, or transport, describing a particle moving with orientation ω. On the right hand side of the equation is the BGK operator, substituting the collision term. This operator models the interaction between the particles since they do not collide, but adopt the average direction of motion of their neighbors [7]. It is based on the assumption that f relaxes over time toward a von Mises distribution.

The objective of the first part of the present thesis is showing the emergence of collective motion and self organization resulting in patterns such as clusters or traveling bands on a macroscopic scale. It is crucial to understand that such behavior is not a characteristic of the individual agents described in the IBM, but only emerges once we successfully derive the macroscopic model corresponding to the IBM. The Self Organized Hydrodynamics (SOH), a continuum version of the Vicsek model, are a macroscopic model capable of describing this emergence of collective motion. It is known that the agents follow a von Mises distribution when in equilibrium and with the right scaling, and are no longer uniformly distributed.

The major results in the first part of this thesis can be summarized with two theorems:

Theorem 1. *Considering the spatially homogeneous BGK equation, for* $f = f(x,t)$, $x \in \mathbb{R}^n$*:*

$$\partial_t f = \rho M_{J_f} - f$$

with $\rho > 0$ *constant, a given particle density, we obtain the following classification of equilibria:*

- *If* $\rho \leq n$, $\kappa = 0$ *is the only solution to the compatibility condition* $\kappa = \rho c(\kappa)$. *In that case the only equilibrium is the isotropic or uniform equilibria* $f = \rho$ *for arbitrary* $0 \leq \rho \leq n$. *This equilibrium corresponds to no alignment at all, that is* $|J| = 0$. *[1]*
- *If* $\rho > n$, $\kappa = \rho c(\kappa)$ *has two solutions:* $\kappa = 0$ *as in the previous case, and an additional unique and strictly positive root* $\kappa(\rho)$. *For the zero solution, this gives again the isotropic equilibrium* $f = \rho$. *The equilibrium associated with the second root are of the form* $\rho M_{\kappa(\rho)\Omega}$ *with arbitrary* $\rho > n$ *and arbitrary* $\Omega \in \mathbb{S}$. *This set of equilibria forms a manifold of dimension n. (See Figure 3 in [3])*

The detailed theorem with proof can be found in Section 3.2.2, Theorem 4.
The second important result concerns the macroscopic equations and the emergence of collective behavior.

Theorem 2. *We consider the rescaled, spatially inhomogeneous equation in dimension* $n = 2$.

$$\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon}\left(\rho_{f_\varepsilon} M_{J_f^\varepsilon}(\omega) - f^\varepsilon\right)$$

with $\rho_{f_\varepsilon} = \int_{\mathbb{S}} f^\varepsilon d\omega$ *and* $J_f^\varepsilon = \int_{\mathbb{S}} \omega f(t,x,\omega)\, d\omega$

1. *Supposing that in an ordered region,* f_ε *converges strong enough as* $\varepsilon \to 0$. *Then* f_ε *converges to a stable equilibrium of the form* $\rho M_{\kappa,\Omega}$ *where* $\rho = \rho(t,x)$ *describes*

the density of the particles and $\Omega = \Omega(t,x) \in \mathbb{S}$ *gives the mean direction of the particles. These two quantities satisfy the following system of equations:*

$$\begin{cases} \partial_t \rho + \nabla_x \left(c(\kappa) \rho \Omega \right) = 0, \\ \rho \partial_t \Omega_0 + \frac{1}{\kappa(\rho)} P_{\Omega_0^\perp} \nabla_x \rho + \frac{c_1}{c_0} \rho (\Omega \cdot \nabla_x) \Omega = 0, \end{cases}$$

with $c_0, c_1, c(\kappa)$ *being defined later in the full theorem.*

2. *In a disordered region, where* f_ε *converges to an isotropic equilibrium, where the orientation is uniformly distributed. The density* ρ^ε *satisfies formally at first order the following diffusion equation:*

$$\partial_t \rho^\varepsilon = \frac{\varepsilon}{2} \nabla_x \cdot \left(\frac{\nabla_x \rho^\varepsilon}{\left(1 - \frac{\rho^\varepsilon}{2}\right)} \right).$$

These results can be found with proof in Section 3.4.2, Theorem 7 and in Section 3.4.4, Theorem 8.

The second part of the present thesis deals with the coupling of the Vicsek-BGK model with the Stokes equation.
Here we consider self-propelled particles 'swimming' in a viscous fluid. Now we also have to consider the interplay between these 'swimmers' and the fluid. Swimmers create perturbations within the surrounding fluid through their motion, these perturbations in turn have some effect on neighboring swimmers. This leads to highly non-linear interaction between neighboring swimmers, in particular, when swimmer-density is high.
Swimmer-swimmer interaction is modelled with the Vicsek-BGK equation, since it is assumed that these interactions create spontaneous alignment of the direction of motion of the swimmers. To model the fluid, we use the Stokes equation. These two models are coupled to include the interplay between swimmers and the fluid.
Since the BGK equation is of mesoscopic scale and the Stokes equation is a macroscopic equation, the coupled BGK-Stokes dynamics presented here are hybrid mesoscopic/macroscopic dynamics.

As before, the objective is the derivation of the macroscopic equations for the coupled system. The aim is to present a coarse-grained description of the hybrid system through a purely macroscopic description for the swimmers as well as the fluid. We have already performed the transition to macroscopic dynamics, the 'self organized hydrodynamics' (SOH) for the BGK equation in the first part of this thesis, obtaining equations for the density and orientation. However, in the coupled case the SOH-Stokes Model is a fully coupled system of equations for the fluid velocity and pressure and, again, as before, one equation describing the swimmer density and orientation respectively.
Again, we make use of the Generalized Collision Invariant concept to obtain these equations.
The major results of this section of the thesis are again summarized in a theorem:

Theorem 3. *We consider the rescaled, couples BGK-Stokes System. (see (34a)-(34d))*

1. *Supposing that in an ordered region, f_ε converges strong enough as $\varepsilon \to 0$. Then $f_\varepsilon \to \rho M_{\kappa,\Omega}$, where $\rho = \rho(t,x)$ describes the mass of the particles, $\Omega = \Omega(t,x)$ gives the mean direction of the particles, $v = v(t,x)$ describes the velocity of the fluid and $p = p(t,x)$ is the fluid pressure. These four quantities satisfy the following system of equations:*

$$\begin{cases} \partial_t \rho + \nabla_x \cdot \rho \left(v + \Omega_0 c(\kappa)\right) = 0, \\ \rho(c_0\kappa(\rho)\partial_t\Omega_0 + \kappa(\rho)c_1(\Omega_0 \cdot \nabla_x)\Omega_0) + c_0 P_{\Omega_0^\perp}\nabla_x\rho + \rho c_0 \nabla_x \Omega_0 \cdot v \\ \quad + \rho\left[\lambda P_{\Omega_0^\perp} S(v)\Omega_0 \left(\kappa c_0 - 6c_1 - \kappa c_2\right) - \kappa c_0 P_{\Omega_0^\perp} A(v)\Omega_0\right] \\ \Delta_x v + \nabla_x p = -b\ c_4 \nabla_x \cdot \left[\rho\left(\Omega_0 \otimes \Omega_0 - \frac{1}{2}Id\right)\right] \\ \nabla_x \cdot v = 0 \end{cases}$$

   with c_0, c_1, c_2, c_4 being defined later in the full theorem.

2. *In a disordered region, where f_ε converges to an isotropic equilibrium, where the orientation is uniformly distributed. The density ρ^ε satisfies formally at first order the following diffusion equation:*

$$\partial_t \rho^\varepsilon = \nabla_x \cdot \left(\frac{\varepsilon \nabla_x \rho^\varepsilon}{2 - \rho^\varepsilon} - v\rho\right)$$

These results are rigorously proven and stated in more detail in Section 4 Theorem 9 and Theorem 10.

The graphic below, Figure 1, shows what has been done so far in the world of collective behavior and what the this present thesis contributes to the field.

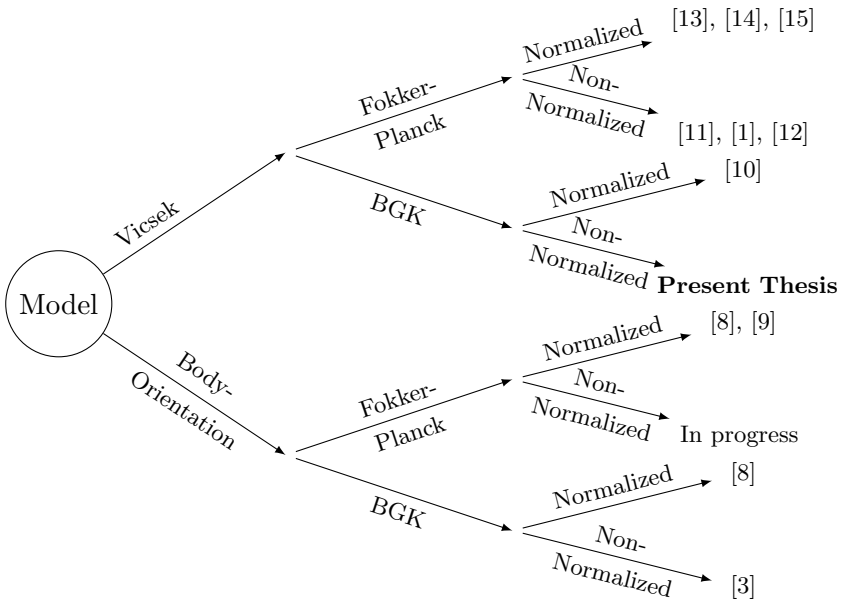


Abbildung 1: Overview of the different models and their respective status.
See [3] FIGURE 1 on p.10.

The outline of this thesis is as follows. In Section 3 the Vicsek-BGK model is studied. It is shown that, depending on the density ρ, there exist two equilibria that are stable under certain conditions. Also, the hydrodynamic model for the ordered regime, above the density threshold, is derived as well as the diffusion equation in the disordered regime, below the density threshold. In Section 4 the Vicsek model is coupled with the Stokes equation for fluids and the self ordered hydrodynamic model for the coupled system is derived, as well as the diffusion equation for the density in the Vicsek-Stokes model. The results of the numerical Simulations presented in section 5. The conclusions are presented in Section 6. Three appendices are added. In Appendix A a useful result is proven rigorously, as well as numerically. In Appendix B the code for the numerical simulations of the model can be found, and Appendix C contains the abstract in German.

3 Phase transitions for the Vicsek-BGK equation

In this section we analyse the Vicsek-BGK equation, find its equilibria and perform the rescaling which eventually leads to the phase transition. Throughout this section, we look at the non-normalized equation, since phase transitions only occur in non-normalized systems. In a normalized system the flux $J_f(t,x)$ is replaced by its normalization $\Omega_f(t,x) = \frac{J_f(t,x)}{|J_f(t,x)|}$. The normalized Vicsek model has been studied in [14] and [15]. The insight that phase transitions in fact only occur in the non-normalized Vicsek model, as studied in this thesis, has been shown in [1] and [11].

3.1 Review of the Vicsek-BGK model

The Vicsek-BGK model is a kinetic or mesoscopic model, describing collective motion in terms of the distribution function for the system. It has been derived from a particle model by looking at a large number of particles ($N \to \infty$). This is called the Mean-Field limit. For a detailed derivation see [1], [3] and [5]. The underlying particle model can be described by a Markov process:

The N agents are described at time $t \in \mathbb{R}_+$ by their positions x and orientation ω. With $Z_t = \left\{\left(x_t^i, \omega_t^i\right)\right\}_{i\in\{1,\dots,N\}} \in \left(\mathbb{R}^2 \times \mathbb{S}\right)^N$ describing the whole system. The set T_n with $n \in \mathbb{N}$ describes the jumping times of the process, they are independent, identically distributed random variables which are exponentially distributed with parameter equal to 1. Interactions between agents can be described by a PDMP (piece-wise deterministic Markov process), see [8] and [10]:

1. The system evolves in a deterministic way when in between two jump times (T_n, T_{n+1}):
$$dx_t^i = \omega_t^i dt$$
This means, when no interaction that triggers a change in orientation happens, the individual particles moves in direction of its corresponding orientation.

2. When the next jump time T_{n+1} arrives, a particle $i \in \{1, ..., N\}$ is chosen uniformly, with probability $\frac{1}{N}$. Then, at T_{n+1}^+, a new orientation is chosen according to the probability density function $M_{J^i(Z_{T_{n+1}^-})}$, where
$$J^i(Z^i) := \frac{1}{N}\sum_{j=1}^{N} K\left(|x^i - x^j|\right)\omega^j,$$
describes the flux and with K being a smooth observation kernel. This simply gives an average of the neighboring particles' orientation, weighed by their distance to the particle considered.

From this PDMP describing the individual particle model, the following kinetic description of the system can be derived:

$$\partial_t f + \omega \cdot \nabla_x f = \left(\int_{\mathbb{S}} f d\omega \right) M_J(\omega) - f \tag{2}$$

with (x_i, ω_i), $x_i \in \mathbb{R}^2$, $\omega \in \mathbb{S}$, and $f = f(t, x, \omega)$ a probability density function for the system.

$$M_{\Omega_f}(\omega) = \frac{e^{\kappa \omega \cdot \Omega}}{z}, \quad z = \int_{\mathbb{S}^{d-1}} e^{\kappa \omega \cdot \Omega} d\omega, \quad \kappa > 0, \quad \omega \in \mathbb{S}$$

is the von Mises distribution from which the changes in orientation are sampled. And

$$\Omega_f = \frac{J_f}{|J_f|}, \quad J_f(t, x) = \int \int K(|x - y|) \omega f(t, y, \omega) \; d\omega \; dy$$

is the normalized system, where Ω_f is the is the mean normalized orientation, J_f is the total flux, and ω is an orientation vector .

However, from now on forth, we shall look at non-normalized Vicsek Model, since we aim to observe phase transition phenomena.
That means we consider J_f rather than the normalized Ω_f. We also neglect the sensing kernel K, meaning the influence particles have on each other's orientation is not weighed by distance anymore i.e.:

$$\partial_t f + \omega \cdot \nabla_x f = \rho_f M_{J_f}(\omega) - f \tag{3}$$

where

$$J_f = \int_{\mathbb{S}} \omega f(t, x, \omega) \; d\omega,$$
$$M_{J_f} = \frac{e^{\omega \cdot J_f}}{z} = M_{\kappa \Omega_f}, \; \kappa = |J_f|, \; \kappa > 0$$

and $\rho_f = \int_{\mathbb{S}} f d\omega$.

3.2 Equilibria of the Vicsek-BGK model

We shall determine the equilibria of the kinetic equation. To make things a little easier, we consider the spatially homogeneous BGK equation, where we neglect the transport term $\omega \cdot \nabla_x f$ describing the change of position of the single particles. This leaves us with the following space homogeneous, non-normalized kinetic equation:

$$\partial_t f = \rho_f M_{J_f} - f = Q_{BGK}(f). \tag{4}$$

$Q_{BGK}(f)$ is called the BGK operator and is a relaxation operator towards the von Mises distribution M_{J_f}. As before, $J_f = \int \omega f d\omega$ and $\rho_f = \int f d\omega$ is the particle density.
The objective now is to find/ characterize equilibria of the spatially homogeneous Vicsek-BGK equation. This means, we want to find the functions f such that $Q_{BGK}(f) = 0$. For equilibria to exist, compatibility conditions depending on the density of particles ρ need to be fulfilled [3].

3.2.1 Compatibility conditions

The compatibility condition can be easily derived by assuming f is already an equilibrium of the BGK operator. A few transformations give the properties that an equilibrium has to fulfill, which leads precisely to the compatibility condition.
We proceed as follows, for f to be an equilibrium it must hold that:

$$\rho_f M_{J_f} - f = 0 \tag{5}$$

multiplying by ω and integrating with respect to ω, it must hold that

$$J_f = \rho_f \ \langle \omega \rangle_{M_J}$$

where

$$\langle \omega \rangle_{M_J} = \int M_{J_f} \ \omega \ d\omega.$$

Denoting $\kappa = |J_f|$ we can write $J_f = \kappa\Omega$ Substituting this in equation (5) we get

$$\kappa\Omega = \rho \int \omega \frac{e^{\kappa\omega\Omega}}{z} d\omega = I. \tag{6}$$

To check under which conditions this equation is fulfilled, we compute first the integral I. To solve I, we do the orthogonal decomposition of ω :
$\omega = P_\Omega(\omega) + P_{\Omega_\perp}(\omega)$, where

$$\Omega = (\cos(\alpha), \sin(\alpha)) \ \text{ and } \Omega_\perp = (-\sin(\alpha), \cos(\alpha))$$

give an orthogonal basis in $\mathbb{R}^2$.

Using this basis, we can decompose ω as $\omega = (\omega \cdot \Omega)\Omega + (\omega \cdot \Omega_\perp)\Omega_\perp$. By making the change of variables $\omega \to \theta$, we get $\omega = \cos(\theta)\Omega + \sin(\theta)\Omega_\perp$ and further:

$$\begin{aligned} I &= \int_{-\pi}^{\pi} (\cos(\theta)\Omega + \sin(\theta)\Omega_{\perp}) \frac{e^{\kappa \ \cos(\theta)}}{z} d\theta \\ &= \Omega \int_{-\pi}^{\pi} \cos(\theta) \frac{e^{\kappa \ \cos(\theta)}}{z} d\theta \\ &=: c(\kappa)\Omega \end{aligned} \tag{7}$$

Where in the second line the integral on $\sin(\theta)$ vanishes since the integrand is odd.

Substituting in (6) the value of I in equation(7), we conclude that the compatibility condition can be simplified into:

$$\kappa = \rho c(\kappa). \tag{8}$$

3.2.2 Equilibria of the BGK-operator depending on the density ρ

Particularly, going back to equation (5) this implies the relation

$$J_{M_{\kappa\Omega}} = \langle \omega \rangle_{M_{\kappa\Omega}} = c(\kappa)\Omega, \tag{9}$$

where $c(\kappa)$ measures how concentrated the von Mises distribution is around Ω and we have $0 \leq c(\kappa) < 1$ as stated in [1] on p.8.

The following Lemma is crucial for the classification of the equilibria

Lemma 1. *$\kappa \mapsto \frac{c(\kappa)}{\kappa}$ is decreasing, since $c(\kappa)$ is bounded and tends to $\frac{1}{n}$ as $\kappa \to 0$. We shall prove this Lemma for the case $n = 2$ (see [11] p.434):*

Beweis. To compute the limit $\lim\limits_{\kappa \to 0} \frac{c(\kappa)}{\kappa}$, we apply L'Hôpital's rule, since

$$c(0) = \int_{-\pi}^{\pi} \cos(\theta) d\theta = 0.$$

First we compute

$$c'(\kappa) = \frac{\int_{-\pi}^{\pi} e^{\kappa \cos(\theta)} d\theta \int_{-\pi}^{\pi} \cos^2(\theta) e^{\kappa \cos(\theta)} d\theta - \int_{-\pi}^{\pi} \cos(\theta) e^{\kappa \cos(\theta)} d\theta \int_{-\pi}^{\pi} \cos(\theta) e^{\kappa \cos(\theta)} d\theta}{\left(\int_{-\pi}^{\pi} e^{\kappa \cos(\theta)} d\theta \right)^2},$$

which at $\kappa = 0$ gives

$$\frac{2\pi \int_{-\pi}^{\pi} \cos^2(\theta) d\theta - \left(\int_{-\pi}^{\pi} \cos(\theta) d\theta\right)^2}{4\pi^2} = \frac{2\pi^2 - 0}{4\pi^2} = \frac{1}{2}.$$

We apply L'Hôpital's rule here and since the denominator is one, it suffices to look at $c'(\kappa)$. We know $\frac{c(\kappa)}{\kappa} \overset{k\to 0}{\longrightarrow} \frac{1}{2}$ and $\frac{c(\kappa)}{\kappa}$ is decreasing and therefore $\left(\frac{c(\kappa)}{\kappa}\right)' < 0$ for $\kappa > 0$ (for proof see [16] Proposition 3.3 and appendix A.1.), clearly $\frac{c(\kappa)}{\kappa}$ has its maximum at $\frac{1}{2}$, in dimension 2 and at $\frac{1}{n}$ in the general case, however this is not shown in this thesis. See [16] for general results. From now on, the general case is assumed. □

Transforming the compatibility condition (8) to obtain an expression for ρ,

$$\frac{1}{\rho} = \frac{c(\kappa)}{\kappa} < \frac{1}{n} \quad \forall \kappa > 0,$$

we see that ρ has to be bigger than n for positive κ.

In conclusion, it follows that if $\rho \leq n$, $\kappa = 0$ is the only solution. However, if $\rho > n$, another solution exists and that solution is unique and strictly positive. This leads to the following classification of the two equilibria, similarly to Proposition 2.2 in [11] and Theorem 5 in [3]:

Theorem 4. *Classification of equilibria*

- *If $\rho \leq n$, $\kappa = 0$ is the only solution to the compatibility condition $\kappa = \rho c(\kappa)$. In that case the only equilibria of the kinetic equation (4) are the isotropic or uniform equilibria $f = \rho$ for arbitrary $0 \leq \rho \leq n$. These equilibria correspond to no alignment at all, that is $|J| = 0$. [1]*
- *If $\rho > n$, $\kappa = \rho c(\kappa)$ has two solutions: $\kappa = 0$ as in the previous case, and an additional unique and strictly positive root $\kappa(\rho)$. For the zero solution, this gives again the isotropic equilibria $f = \rho$. The equilibria associated with the second root are of the form $\rho M_{\kappa(\rho)\Omega}$ with arbitrary $\rho > n$ and arbitrary $\Omega \in \mathbb{S}$. This set of equilibria forms a manifold of dimension n. (See Figure 3 in [3])*

Beweis. First, we show that zero is, in fact, a solution to the compatibility condition (8):

It is easy to check that $c(0) = 0$:

$$c(0) = \int_{-\pi}^{\pi} \cos(\theta) d\theta = 0.$$

Now we simply plug into the equation for the compatibility equation:

$$\begin{aligned} \kappa &= \rho c(\kappa)|_{\kappa=0} \\ 0 &= \rho c(0) = 0. \end{aligned} \tag{10}$$

Clearly, since the compatibility condition (8) is just a simplification of the initial compatibility condition (5), and $\kappa = 0$ implies $J = 0$, zero is a solution to that initial equation as well:

$$J_f = \rho_f \ \langle \omega \rangle_{M_J} = \rho_f \int_{\mathbb{S}} \omega M_J d\omega. \tag{11}$$

Now we use $J = 0$:

$$\rho_f \int_{\mathbb{S}} \omega M_0 d\omega = \rho_f \int_{\mathbb{S}} \omega d\omega = 0,$$

since M_0 is the uniform distribution and equal to 1 and ω is an odd function.

This also proves that in the case $\rho \leq n$ we have no alignment.
From this solution we can now check under which circumstances $Q_{BGK}(f) = 0$ when $J = 0$ and derive the corresponding equilibrium f [11]:

$$0 = \rho_f M_0 - f = \rho_f - f.$$

This gives $\rho_f = f$ when $J = 0$.

Here we used the fact that the von Mises distribution M_0 with concentration parameter $J = 0$ corresponds to the uniform distribution and therefore M_0 is constant and equal to one.
The compatibility condition, $\rho c(\kappa) = \kappa$ can be rewritten as $\frac{1}{\rho} = \frac{c(\kappa)}{\kappa} := h(\kappa)$. $h(\kappa)$ is decreasing, and as κ approaches zero, it converges toward $\frac{1}{2}$ in the two dimensional case and $\frac{1}{n}$ in the general case. This means:

$$max\ h(\kappa) = max\ \frac{c(\kappa)}{\kappa} = \frac{1}{n},\ for\ \kappa \geq 0.$$

So when $\rho \leq n$ that in return means $\frac{1}{\rho} \geq \frac{1}{n}$ which combined with the fact that $\frac{1}{n}$ is a maximum of $h(\kappa)$ for $\kappa \geq 0$ gives that κ must be equal to zero in that case, since the

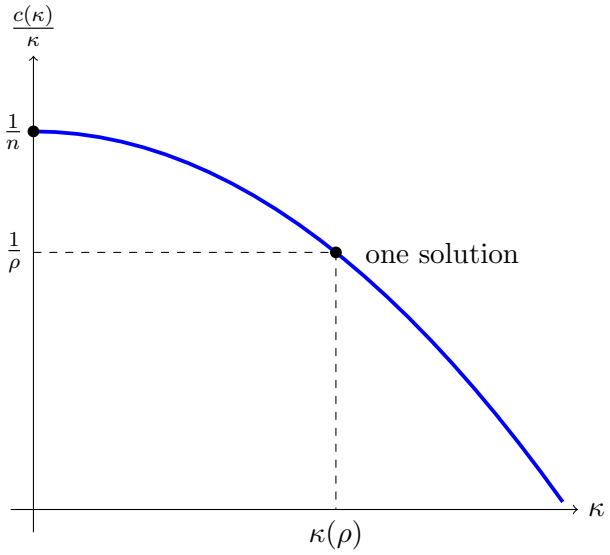


Abbildung 2: Unique, non-zero solution $\kappa(\rho)$ of the compatibility condition

zero solution exists independently of ρ.

When $\rho > n$ there exists exactly one $\kappa > 0$ such that $\frac{1}{\rho} = \frac{c(\kappa)}{\kappa}$. This is due to the fact that $\frac{c(\kappa)}{\kappa}$ is strictly decreasing for $\kappa > 0$. So depending on $\rho > n$ there always exists exactly one solution $\kappa(\rho)$. This solution can be seen in Figure 2 below.
This concludes the Theorem. □

In Theorem 4, we have seen that the spatially homogeneous Vicsek- BGK model has two types of equilibria. The uniform equilibrium where the solution to (5) is $J = 0$ and therefore the equilibrium itself is $f = \rho M_0$ which of course is $f = \rho$ since M_0 corresponds to the uniform distribution which is constant and equal to one. This equilibrium always exists since the corresponding solution to the compatibility condition ($\kappa = 0$) does not depend on ρ.
The existence of the second equilibrium, that is, when the solution of (5) is $\kappa(\rho) > 0$, on the other hand clearly does depend on ρ.
This change in the number of equilibria as the density crosses the threshold corresponding to the dimension of the problem, called critical density, leads to a phase transition or bifurcation [1].

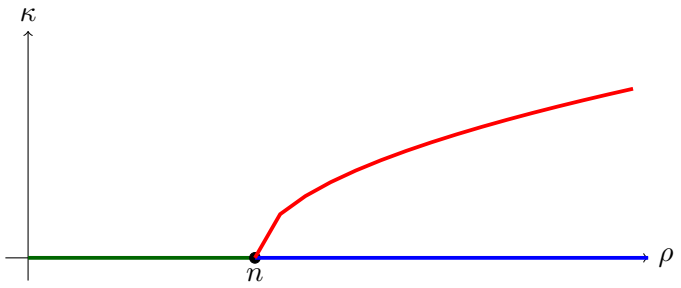


Abbildung 3: Bifurcation diagram showing the existence of different equilibria

3.3 Stability of the equilibria

Theorem 4 leads to the following bifurcation diagram, which shows that there exists one equilibrium while $\rho \leq n$, this is the zero solution, and two equilibria as soon as $\rho > n$, the zero solution as well as the solution $\kappa(\rho)$ which is unique for each ρ.

Now that we know about the different equilibria for the Vicsek-BGK model, it is natural, to wonder about the asymptotic behavior of $f(t, x, \omega)$ and how the function converges toward these equilibria as $t \to \infty$. That is, we want to analyze the stability of the different equilibria. However, it is much simpler, to look at the asymptotic behavior of J_f instead. This is admissible, since if $J_f \to J_\infty$ also $f(t) \to \rho M_{J_\infty}$ as $t \to \infty$. This can be seen when writing Duhamel's formula for the spatially homogeneous BGK equation equation $\partial_t f = \rho_f M_{J_f} - f$:
See [3] p.24.

$$f(t) = e^{-t} f_0 + \rho \int_0^t e^{-(t-s)} M_{J_{f(s)}(\omega)} \, ds.$$

Since $e^{-t} f_0 \to 0$, we only look at the integral

$$\int_0^t e^{-(t-s)} M_{J_{f(s)}(\omega)} \, ds.$$

We perform a substitution to solve the integral and set $r = (t - s)$.

$$\int_t^0 e^{-r} M_{J_{f(t-r)}(\omega)} \, (-dr) = \int_0^t e^{-r} M_{J_{f(t-r)}(\omega)} \, dr \; = \int_0^\infty e^{-r} M_{J_{f(t-r)}(\omega)} \mathbb{1}_{r \in [0,t]} \, dr.$$

Now passing to the limit $t \to \infty$, we finally see that it in fact is admissible to only look at the asymptotic behavior of J_f :

$$\int_0^\infty e^{-r} M_{J_\infty(\omega)}\, dr = M_{J_\infty(\omega)} \int_0^\infty e^{-r}\, dr = -M_{J_\infty} \left[e^{-r}\right]_0^\infty = -\, M_{J_\infty}[0-1] = M_{J_\infty}.$$

We have shown that in fact, $f(t)$ converges towards ρM_{J_∞} as $t \to \infty$. This makes the analysis of asymptotic behavior simpler since J_f has simpler asymptotic behavior, because we already know:

$$\frac{d}{dt} J_f = \rho_f\ \langle\omega\rangle_{M_J} - J_f. \tag{12}$$

We obtained this equality in J_f by multiplying the homogeneous BGK equation (4) with ω and integrating with respect to ω.

It is clear to see that solutions of the compatibility conditions in J_f (5) are exactly the equilibria of the dynamical system (12).

Proposition 1. *A distribution $f = \rho M_J$ is a root of the BGK operator $\rho_f M_{J_f} - f$, if and only if J is an equilibrium of (12).*
[3] p.24.

We can further simplify the ODE we want to investigate using $J_f(t) = \kappa(t)\Omega(t)$ and (9). This gives:

$$\frac{d}{dt} J_f = \frac{d}{dt} \left[\kappa(t)\Omega(t)\right].$$

But we already know that

$$\frac{d}{dt} J_f = \rho_f\ \langle\omega\rangle_{M_J} - J_f = \rho_f c(\kappa)\Omega - \kappa\Omega.$$

We derived the first equality here by taking the right hand side of (12) and substituting J_f by $\kappa\Omega$ and by substituting $\langle\omega\rangle_{M_J}$ by $c(\kappa)\Omega$ (see (9)).
Now we can set the two expressions for $\frac{d}{dt} J_f$ equal and apply the product rule to obtain:

$$\rho_f c(\kappa)\Omega - \kappa\Omega = \kappa'(t)\Omega(t) + \kappa(t)\frac{d\Omega(t)}{dt}. \tag{13}$$

For the next step it is helpful to note that $\Omega \cdot \Omega = 1$ and therefore

$$0 = \frac{d|\Omega|}{dt} = \frac{d(\Omega \cdot \Omega)}{dt} = 2\Omega \frac{d\Omega}{dt}.$$

Which implies $\frac{d\Omega}{dt} = 0$.
Eventually, by dividing (13) by Ω, this leads to an ODE describing the behavior of κ and it follows:

$$\kappa'(t) = \rho_f c(\kappa) - \kappa.$$

Let us now call the right hand side of this last equation $Q_\rho(\kappa)$. This gives:

$$\frac{d}{dt}\kappa(t) = Q_\rho(\kappa). \tag{14}$$

This non-linear ODE can now be linearized around an equilibrium to investigate the asymptotic behavior. A result from Dynamical Systems and Nonlinear Ordinary Differential Equations [17] is very useful for further analysis:

Theorem 5. *Let $n = 1$ and let $\bar{\kappa}$ be a steady state of (14). If $Q'_\rho(\bar{\kappa}) < 0$, then $\bar{\kappa}$ is locally asymptotically stable. If $Q'_\rho(\bar{\kappa}) > 0$, then $\bar{\kappa}$ is unstable.*
There is no conclusion for $Q'_\rho(\bar{\kappa}) = 0$,because in this case the local behavior of Q_ρ around $\bar{\kappa}$, and therefore also its stability properties, depend on higher order terms in the Taylor expansion.

Now, we can classify equilibria of the Vicsek-BGK equation as stable/unstable, depending on whether $\bar{\kappa}$ is a stable or unstable equilibrium of the ODE (14). The kernel of (14) is:

$$Ker\ Q_\rho(\kappa) = \{\kappa = 0,\ \kappa(\rho) > 0\}.$$

We start with the $\kappa = 0$ equilibrium, which corresponds to the equilibrium $f = \rho$ in the original model.

$$Q'_\rho(\kappa) = \rho c'(\kappa) - 1,$$
$$Q'_\rho(0) = \rho\frac{1}{2} - 1 < 0,$$

which is equivalent to $\rho < 2$.

We can conclude that the uniform distribution, which is the equilibrium associated with the solution $\kappa = 0$ of the compatibility condition, is stable as long as $\rho < 2$ (or $\rho < n$ in the general case).

Now we look at the positive equilibrium of the ODE, i.e. $\kappa(\rho) > 0$, which corresponds to a van Mises distribution in the original model.

$$Q'_\rho(\kappa(\rho)) = \rho c'(\kappa(\rho)) - 1 < 0,$$

which is equivalent to

$$\rho < \frac{1}{c'(\kappa(\rho))}.$$

The other equilibrium, a van-Mises distribution $\rho M_{\kappa(\rho)\Omega}$, which corresponds to the positive solution $\kappa(\rho)$ of the compatibility condition, exists and is stable when $2 < \rho < \frac{1}{c'(\kappa(\rho))}$ or simply $Q'_\rho(\kappa) < 0$
We use the compatibility condition $\rho c(\kappa) = \kappa$ (8) as starting point. We already know:

- $0 \leq c(\kappa) \leq 1$
- $\kappa \mapsto \frac{c(\kappa)}{\kappa}$ is a decreasing function, since $c(\kappa)$ is limited and κ is not.
- $\kappa \mapsto \frac{c(\kappa)}{\kappa}$ is strictly decreasing for $\kappa > 0$.
- Therefore: $\frac{\kappa}{c(\kappa)}$ is increasing with minimum $\frac{\kappa}{c(\kappa)} = 2$ as κ approaches 0.

This leads to:

$$\rho_0 c(\kappa) = \kappa, \tag{15}$$

and it follows: $\rho_0 \geq 2$.

Now we shall check the stability of $\kappa(\rho)$:

$$Q_\rho(\kappa) = \rho c(\kappa) - \kappa,$$
$$Q'_\rho(\kappa) = \rho c'(\kappa) - 1.$$

The equilibrium is stable, which is equivalent to $Q'_\rho(\kappa) < 0$ which means $\rho c'(\kappa) < 1$.

We know that for $\kappa(\rho) \neq 0,\ \rho \geq 2$ (see (15)) and that an equilibrium is stable, exactly when:

$$\rho c'(\kappa(\rho)) < 1,$$

and since $\rho = \frac{\kappa(\rho)}{c(\kappa(\rho))}$:

$$\frac{\kappa(\rho)}{c(\kappa(\rho))} c'(\kappa(\rho)) < 1.$$

The following Lemma gives a result that is very helpful in proving stability:

Lemma 2. *$c'(\kappa) < \frac{c(\kappa)}{\kappa}$ for positive κ.*

Beweis. First we recall that $h(\kappa) = \frac{c(\kappa)}{\kappa}$ is a strictly decreasing function for positive κ and therefore $h'(\kappa) < 0$ (see Appendix A.1.):

$$0 > h'(\kappa) = \frac{c'(\kappa)\kappa - c(\kappa)}{\kappa^2} = \frac{c'(\kappa)}{\kappa} - h(\kappa)\frac{1}{\kappa}.$$

Since $\kappa > 0$, it is admissible to multiply by κ, which gives $c'(\kappa) - h(\kappa) < 0$ and eventually $c'(\kappa) < h(\kappa)$.

In particular it is also true that $c'(\kappa) < \sup h(\kappa) = \frac{1}{2}$. □

Using the result of the lemma, we get:

$$\frac{\kappa(\rho)}{c(\kappa(\rho))} c'(\kappa(\rho)) < \frac{\kappa(\rho)}{c(\kappa(\rho))} \frac{c(\kappa(\rho))}{\kappa(\rho)} = 1. \tag{16}$$

So now we have proven that once the positive equilibrium exists, which is as soon as $\rho > n$, it is stable.
However the critical case $\rho = n$ still has to be examined, which proves to be a little more challenging.
For $\rho = n = 2$ the derivative of ODE (12), which establishes the stability of the equilibria is:

$$Q_2'(\kappa) = 2c'(\kappa) - 1.$$

Since at $\rho = 2$ only the zero equilibrium exists and we have shown that $c'(0) = \frac{1}{2}$, we get:

$$Q_2'(0) = 0.$$

This makes stability analysis hard. Also looking at higher order terms is not very helpful, since $Q''(0) = 0$ as well. The expected behavior, however, is that the uniform distribution is stable at $\rho = 2$.
We have seen that both the first and second derivative of $Q_2(\kappa) = 2c(\kappa) - \kappa$ are zero at $\kappa = 0$. We now look at $Q_2'''(0) = 2c'''(\kappa)$.

According to [11] p.445 the Taylor expansion of $c(\kappa)$ at 0 is:

$$\frac{1}{2}\kappa - \frac{1}{2^2(2+2)}\kappa^3 + \mathcal{O}(\kappa^5).$$

We see the third derivative of $c(\kappa)$ is the first non-zero derivative and gives $-\frac{6}{16}$ at zero, which makes $Q'''(0) = -2\frac{6}{16} = -\frac{3}{4} < 0$, which proves the stability of the zero-equilibrium at $\rho = 2$.
Now we have analyzed the stability of all equilibria in dependence of the particle density ρ. The following theorem concludes our findings and corresponds to results in [1].

Theorem 6. *Stability of the equilibria of the ODE (12).*

- *In the subcritical case, $0 < \rho < n$, the only equilibrium is the uniform distribution $f = \rho$, this equilibrium is stable for arbitrary $0 < \rho < n$.*
- *In the supercritical case $n < \rho$, there exists another equilibrium $f = \rho M_{\kappa(\rho)\Omega}$, this equilibrium is stable for arbitrary $n < \rho$. The other equilibrium, the uniform distribution, becomes unstable for $n < \rho$.*
- *In the critical case where $\rho = n$ only the zero equilibrium exists and is stable.*

Definition 1. *Gradient Flow [18], [19], [5].*
Given a function $F : \mathbb{R}^n \to \mathbb{R}$ which is smooth enough, i.e. $C^{1,1}$ (differentiable with Lipschitz continuous derivative), and a point $x_0 \in \mathbb{R}^n$, a gradient flow is defined as a trajectory $x = x(t) \in \mathbb{R}^n$, with starting point at $t = 0$ given by x_0, which moves by choosing at each instant of time the direction which decreases the function F the most. More precisely the gradient flow is a smooth curve $x : \mathbb{R}^n \to X$ such that:

$$\frac{d}{dt}x = -\nabla F(x(t)).$$

The following Lemma explores the underlying gradient flow structure in the present model, similarly to Lemma 5.1 in [3].

Lemma 3. *Gradient flow structure of J.*

The equation

$$\frac{d}{dt}J_f = \rho_f \ \langle\omega\rangle_{M_J} - J_f,$$

can be written in gradient flow structure, i.e.:

$$\frac{d}{dt}J_f = -\nabla V(J). \tag{17}$$

Where $V(J) = \frac{1}{2}||J||^2 - \rho\log(Z(J))$, $Z(J) = \int_{\mathbb{S}} e^{J\omega}d\omega$ *is a potential.*

Beweis. Let $V(J) = \frac{1}{2}||J||^2 - \rho\log(Z(J))$ with $Z(J) = \int_{\mathbb{S}} e^{J\omega}d\omega$

$$\nabla_J V(J) = J - \rho\frac{1}{Z(J)}\int_{\mathbb{S}} e^{J\omega}\omega d\omega = J - \rho\int_{\mathbb{S}}\omega M_J d\omega = J - \rho\langle\omega\rangle_{M_J},$$

because of the definitions of M_J and $\langle\omega\rangle_{M_J}$.
It follows:

$$-\nabla_J V(J) = \rho\langle\omega\rangle_{M_J} - J_f = \frac{d}{dt}J_f,$$

as desired. □

Since $J_f = \kappa\Omega$ and $\kappa = |J_f|$ we can translate this gradient flow structure to a gradient flow structure in κ:

$$V(\kappa\Omega) = \frac{1}{2}\kappa^2 - \rho\log(Z(\kappa\Omega)),$$
$$Z(\kappa\Omega) = \int_{\mathbb{S}} e^{\kappa\Omega\omega}d\omega = \int_{\mathbb{S}} e^{\kappa\cos(\theta)}d\omega.$$

Lemma 4. *Gradient flow structure of* κ.
This now means, we have a gradient flow structure for the system

$$\frac{d}{dt}\kappa(t) = \rho c(\kappa) - \kappa,$$

with

$$V(\kappa) = \frac{1}{2}\kappa^2 - \rho\log\left(\int_{\mathbb{S}} e^{\kappa\cos(\theta)}d\theta\right) \tag{18}$$

being a potential.

Beweis. Let $V(\kappa) = \frac{1}{2}\kappa^2 - \rho\log\left(\int_{\mathbb{S}} e^{\kappa\cos(\theta)}d\theta\right)$

$$\nabla_\kappa V(\kappa) = \kappa - \rho\frac{1}{Z(\kappa)}\int_{\mathbb{S}}\cos(\theta)e^{\kappa\cos(\theta)}d\theta = \kappa - \rho c(\kappa).$$

It follows:

$$-\nabla_\kappa V(\kappa) = \rho c(\kappa) - \kappa = \frac{d}{dt}\kappa$$

as desired. □

A gradient flow structure implies that the corresponding dynamical system converges to an equilibrium, since we can now show that all trajectories of (12) are bounded along the lines of [3]

The potential $V(J)$ for (17) clearly is decreasing. Therefore we can write $\forall t > 0$:

$$\frac{1}{2} \|J(t)\|^2 \leq V(J_0) + \rho \log Z(J(t)).$$

Using the mean-value theorem and the Cauchy Schwarz inequality, we obtain:

$$\log Z(J(t)) \leq \sqrt{\frac{3}{2}} \|J(t)\|$$

and by Young's inequality ($ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy$):

$$\frac{1}{4} \|J(t)\|^2 \leq V(J_0) + 3\rho^2.$$

Since the potential V is an analytic function, a result in [20] implies that the solution to the ODE (12) converges to an equilibrium.

3.4 Rescaling of the equation

In this section we want to find out about the large scale behavior of the system. To achieve this, we perform a hyperbolic rescaling of the equation. For this, we now use the spatially inhomogeneous model (see (2)). The next step is, to rescale the kinetic equation in the Vicsek-BGK Model to obtain the macroscopic dynamics and observe phase transitions,

$$t' = \varepsilon t,\ x' = \varepsilon x.$$

(t', x') are the temporal and spatial macroscopic variables and $0 < \varepsilon \ll 1$ is a scale parameter. This rescaling corresponds to speeding up time ($t = \frac{t'}{\varepsilon}$) and zooming out in space ($x = \frac{x'}{\varepsilon}$), meaning the distance between two collisions becomes very small. The function f^ε of the macroscopic variables is defined as:

$$f^\varepsilon(t', x', \omega) = f(t, x, \omega).$$

Now we can rewrite the non-normalized Vicsek-BGK equation (3) with the rescaled parameters:

$$\begin{aligned} \partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon &= \frac{1}{\varepsilon}\left(\left(\int_{\mathbb{S}} f^\varepsilon d\omega\right) M_{J_f^\varepsilon}(\omega) - f^\varepsilon\right), \\ J_f^\varepsilon &= \int_{\mathbb{S}} \omega f(t, x, \omega)\ d\omega. \end{aligned} \tag{19}$$

We rename the right hand side of the equation:

$$\Gamma(f^\varepsilon) := \left(\int_{\mathbb{S}} f^\varepsilon d\omega\right) M_{J_f^\varepsilon}(\omega) - f^\varepsilon.$$

The goal is, to analyze the limit of f^ε as ε approaches 0 under the assumption that f^ε converges to a stable equilibrium, and to determine the hydrodynamic equations for the density $\rho(t, x)$ and the orientation $\Omega(t, x)$.
Because of the results in the previous section, we can conclude that formally, as ε converges to 0 , f^ε approaches $f^0 = \rho(t, x) M_{J_0}$ with $J_0 = \kappa(\rho)\Omega(t, x)$.
Since we assume the limit to be a stable equilibrium, there are two options, the first one being $\kappa = 0$, that means $\rho < n$ and f^ε converges to the uniform distribution in terms of the orientation and the limit is $\rho(t, x)$. The second stable equilibrium is $\kappa = \kappa(\rho) > 0$ with $\rho > n$. In that case the limit is $\rho M_{\kappa(\rho)\Omega}$.

3.4.1 Equation for the density ρ

To obtain the equation of the density ρ, we integrate the rescaled equation (19) with respect to the orientation ω. Most of the calculations in this section follow along the lines of [11].

$$\begin{aligned} \partial_t \int_{\mathbb{S}} f^\varepsilon d\omega + \nabla_x \cdot \int_{\mathbb{S}} \omega f^\varepsilon d\omega = 0, \\ \partial_t \rho^\varepsilon + \nabla_x \cdot \int_{\mathbb{S}} \omega f^\varepsilon d\omega = 0. \end{aligned} \tag{20}$$

From the first to the second line we used the fact that $\rho^\varepsilon = \int f^\varepsilon d\omega$. The right hand side of the equation is zero when integrating, since the total mass is conserved and the number of particles stays the same throughout the interactions:

$$\int_{\mathbb{S}} \left[\left(\int_{\mathbb{S}} f^\varepsilon d\omega \right) M_{J_f^\varepsilon}(\omega) - f^\varepsilon \right] d\omega = \rho^\varepsilon \int_{\mathbb{S}} M_{J_f^\varepsilon}(\omega) d\omega - \int_{\mathbb{S}} f^\varepsilon d\omega = 0.$$

The integral over $M_{J_f^\varepsilon}(\omega) = 1$ and since $\rho^\varepsilon = \int f^\varepsilon d\omega$ mass is conserved and the whole integral is equal to 0.
Looking at $\Gamma(f^\varepsilon)$ we see that since f^ε converges to a stable equilibrium of the BGK equation, as $\varepsilon \to 0$ the limit of f^ε is

$$f^0 = \rho M_{J_0}(\omega).$$

So formally taking the limit $\varepsilon \to 0$ in (20) we get:

$$\partial_t \left(\int_{\mathbb{S}} \rho M_{J_0}(\omega) d\omega \right) + \nabla_x \cdot \left(\int_{\mathbb{S}} \omega \rho M_{J_0}(\omega) d\omega \right) = 0.$$

Since ρ does not depend on ω, $\int_{\mathbb{S}} M_{J_0} = 1$ and because of the compatibility condition (5) and (7) we can rewrite this as:

$$0 = \partial_t \rho \left(\int_{\mathbb{S}} M_{J_0}(\omega) d\omega \right) + \nabla_x \cdot \left(\rho \int_{\mathbb{S}} \omega M_{J_0}(\omega) d\omega \right) = \partial_t \rho + \nabla_x \left(\rho c(\kappa) \Omega(t, x) \right),$$

resulting in the rescaled equation for the density ρ :

$$\partial_t \rho + \nabla_x \left(c(\kappa) \rho \Omega \right) = 0. \tag{21}$$

Next, we compute the evolution for $\Omega(t, x)$

3.4.2 Equation for the orientation Ω

To obtain the equation for the orientation Ω, we first multiply the rescaled equation for f (19) with ω, the we integrate with respect to ω. This way, we obtain:

$$\partial_t \int_{\mathbb{S}} \omega f^\varepsilon d\omega + \int_{\mathbb{S}} \left[\omega \otimes \omega \cdot \nabla_x f^\varepsilon\right] d\omega = \frac{1}{\varepsilon} \int_{\mathbb{S}} \omega \Gamma(f^\varepsilon) d\omega + \mathcal{O}(\varepsilon).$$

Taking the limit in the first term on the left hand side, we get:

$$\partial_t \int_{\mathbb{S}} \omega \rho M_{J_0}(\omega) d\omega = c(\kappa) \partial_t \left(\rho \Omega\right).$$

However, we are confronted with the problem that for the right hand side, in general

$$\int_{\mathbb{S}} \omega \Gamma(f^\varepsilon) d\omega = \int_{\mathbb{S}} \omega \left[\left(\int_{\mathbb{S}} f^\varepsilon d\omega \right) M_{J_f^\varepsilon}(\omega) - f^\varepsilon \right] d\omega \neq 0,$$

since the orientation is not a conserved quantity and momentum is not conserved in the context of self-propelled particles.
To tackle this problem, we don't want $\int_{\mathbb{S}} \omega \Gamma(f^\varepsilon) d\omega = 0$ for all functions f but we restrict ourselves to a particular set of functions. This leads to the next section about the Generalized Collision Invariant, which has been introduced to establish a link between kinetic and macroscopic equations when dealing with self-propelled particles.

3.4.3 The generalized collision invariant ψ

As a reminder to the reader, we first define a classical collision invariant, see [5]:

Definition 2. *Collision Invariant*

A collision invariant is any function $\psi = \psi(\omega)$ such that:

$$\int Q(f) \psi(\omega) d\omega = 0, \ \forall \ f.$$

Definition 3. *The Operator $\overline{\Gamma}$.*

Let $\overline{\Gamma} \in \mathbb{S} \times L^2(\mathbb{S})$. $\overline{\Gamma}(\Omega_0, f) = M_{\kappa,\Omega_0}(\omega) \int_{\mathbb{S}} f d\omega - f$, where $\Omega_0 \in \mathbb{S}$ and $f \in L^2(\mathbb{S})$.
See [5]

Definition 4. *The Generalized Collision Invariant.*

A function ψ_{Ω_0} is a Generalized Collision Invariant (GCI) associated with $\Omega_0 \in \mathbb{S}$ of the operator Γ if it holds that:

$$\int_{\mathbb{S}} \overline{\Gamma}(\Omega_0, f)\psi_{\Omega_0} d\omega = 0 \ \forall \ f : P_{\Omega_0^\perp}\left(\int_{\mathbb{S}} \omega f d\omega\right) = 0.$$

See [5]

This GCI is very useful to help us obtain the desired equations for the orientation. Therefore we start by multiplying our initial rescaled equation (19) with the GCI associated with $\Omega_{f^\varepsilon}, \psi_{\Omega_{f^\varepsilon}}$. Integrating with respect to ω gives:

$$\varepsilon \int \left(\partial_t f^\varepsilon + (\omega \cdot \nabla_x)(f^\varepsilon)\right) \psi_{\Omega_{f^\varepsilon}} d\omega = 0.$$

Assuming, f^ε satisfies the condition

$$P_{\Omega_{f^\varepsilon}^\perp}\left(\int_{\mathbb{S}} \omega f^\varepsilon d\omega\right) = P_{\Omega_{f^\varepsilon}^\perp} \lambda_{f^\varepsilon},$$

the right hand side of equation (19) vanishes, since in that case:

$$\int_{\mathbb{S}} \Gamma(f^\varepsilon)\psi_{\Omega_{f^\varepsilon}} d\omega = 0.$$

Now we can divide by ε and take the limit $\varepsilon \to 0$ in (19). Remembering that $f^\varepsilon \to \rho M_{J_0}$, as $\varepsilon \to 0$ we get:

$$\int_{\mathbb{S}} \left(\partial_t(\rho \tilde{M}_{J_0}) + \omega \cdot \nabla_x(\rho \tilde{M}_{J_0})\right) \psi_{\Omega_{f^\varepsilon}} d\omega = 0. \tag{22}$$

We now see that if we can compute the Generalized Collision Invariant explicitly, similarly as in [5], we obtain the explicit limit in the previous equation, which is what we want.

Lemma 5. *Characterization of the GCI.*
A function $\psi_{\Omega_0} = \psi_{\Omega_0}(\omega)$ is a Generalized Collision Invariant (GCI) associated with $\Omega_0 \in \mathbb{S} \iff \exists \beta \in \Omega_0^\perp : \psi_{\Omega_0}$ is solution of:

$$\int_{\mathbb{S}} \psi_{\Omega_0} M_{\kappa,\Omega_0}(\omega) d\omega - \psi_{\Omega_0}(\omega) = \beta \cdot \omega. \tag{23}$$

Beweis. See [5] p.102 and following. □

Proposition 2. *Description of the GCI [5].*
Let $\psi_{\Omega_0} \in \mathbb{S}$. Then the following holds:

$$GCI(\Omega_0) = span\left\{1, \cup_{\beta\in\Omega_0^\perp}\psi_{\Omega_0}^{\beta}\right\},$$

where

$$\psi_{\Omega_0}^{\beta} = \beta\cdot\omega$$

are unique solutions of (23) in the space

$$\left\{\psi \ s.t. \int_{\mathbb{S}}\psi(\omega) = 0\right\}.$$

Beweis. Assume $\psi_{\Omega_0} = -\beta\cdot\omega$ and substitute in (23):

$$\int_{\mathbb{S}} -\beta\cdot\omega M_{\kappa,\Omega_0}(\omega)d\omega + \beta\cdot\omega = \beta\cdot\omega.$$

It follows that:

$$-\beta\cdot\int_{\mathbb{S}}\omega M_{\kappa,\Omega_0}(\omega)d\omega = 0.$$

And since $\int_{\mathbb{S}}\omega M_{\kappa,\Omega_0}(\omega)d\omega = c(\kappa)\Omega_0$ we get:

$$-\left(\beta\cdot\Omega_0\right)c(\kappa) = 0.$$

Which is true, since $\beta\in\Omega_0^\perp$ and therefore $(\beta\cdot\Omega_0) = 0$.
The general solutions are of the form $\beta\cdot\omega + c,\ \beta\in\Omega_0^\perp,\ c = const.$ since $\beta\in\Omega_0^\perp$ and therefore also $-\beta\in\Omega_0^\perp$. For this reason we can neglect the sign.

$$\int_{\mathbb{S}}\left(-\beta\cdot\omega + c\right)M_{\kappa,\Omega_0}(\omega)d\omega + \beta\cdot\omega - c = \beta\cdot\omega.$$

It follows that

$$-\beta\cdot\int_{\mathbb{S}}\omega M_{\kappa,\Omega_0}(\omega)d\omega + c\int_{\mathbb{S}}M_{\kappa,\Omega_0}(\omega)d\omega - c = 0,$$

and since $\int_{\mathbb{S}}M_{\kappa,\Omega_0}(\omega)d\omega = 1$, we get:

$$-\left(\beta\cdot\Omega_0\right)c(\kappa) + c - c = 0.$$

Which is true, since $(\beta\cdot\Omega_0) = 0$. □

Now that we have found the explicit form of the GCI we can go back and plug $\psi_{\Omega_0} = \beta \cdot \omega$ into equation (22) we obtained by multiplying the rescaled equation with the GCI. This way we obtain a macroscopic equation:

$$\int_{\mathbb{S}} \left(\partial_t(\rho \tilde{M}_{J_0}) + \omega \cdot \nabla_x(\rho \tilde{M}_{J_0}) \right) \beta \cdot \omega \; d\omega = 0, \; \beta \in \Omega_0^{\perp}.$$

Since $\beta \in \Omega_0^{\perp}$ we can generalize and use the projection on $\Omega_0^{\perp}$ instead of the dot product with β. So the equation:

$$\beta \cdot \left[\int_{\mathbb{S}} \partial_t(\rho \tilde{M}_{J_0})\omega + \omega \cdot \nabla_x(\rho \tilde{M}_{J_0})\omega \; d\omega \right] = 0,$$

becomes:

$$P_{\Omega_0^{\perp}} \left[\int_{\mathbb{S}} \left[\partial_t(\rho \tilde{M}_{J_0}) \right] \omega \; d\omega + \int_{\mathbb{S}} \left[\omega \cdot \nabla_x(\rho \tilde{M}_{J_0}) \right] \omega \; d\omega \right] = 0.$$

We will split this equation into two parts, to calculate the integrals separately. So for clarity we shall call

$$\begin{aligned} \mathrm{I} &= \int_{\mathbb{S}} \left[\partial_t(\rho \tilde{M}_{J_0}) \right] \omega \; d\omega, \\ \mathrm{II} &= \int_{\mathbb{S}} \left[\omega \cdot \nabla_x(\rho \tilde{M}_{J_0}) \right] \omega \; d\omega. \end{aligned}$$

The two integrals can be solved a little further, yielding four integrals.

$$\begin{aligned} \mathrm{I}_1 &= \int_{\mathbb{S}} (\partial_t \rho) \tilde{M}_{J_0}, \\ \mathrm{I}_2 &= \int_{\mathbb{S}} \rho (\partial_t \tilde{M}_{J_0}) \omega \; d\omega, \\ \mathrm{II}_1 &= \int_{\mathbb{S}} (\omega \cdot \nabla_x \rho) \tilde{M}_{J_0}, \\ \mathrm{II}_2 &= \int_{\mathbb{S}} (\omega \cdot \nabla_x \tilde{M}_{J_0}) \rho \; d\omega. \end{aligned}$$

The first one can be solved quickly by making use of (9) and the orthogonality.

$$\begin{aligned}
\mathrm{I}_1 = \; & P_{\Omega_0^\perp} \int_{\mathbb{S}} (\partial_t \rho) \tilde{M}_{J_0} \omega \; d\omega \\
= \; & \partial_t \rho \; P_{\Omega_0^\perp} \int_{\mathbb{S}} \omega \tilde{M}_{J_0} \; d\omega \\
= \; & \partial_t \rho \; P_{\Omega_0^\perp} \; c(\kappa) \Omega_0 = 0,
\end{aligned}$$

since we project on $\Omega_0^\perp$. It follows that $\mathrm{I}_1 = 0$.

The second integral needs a little more transformations to solve:

$$\begin{aligned}
\mathrm{I}_2 = \; & P_{\Omega_0^\perp} \int_{\mathbb{S}} \rho (\partial_t \tilde{M}_{J_0}) \omega \; d\omega \\
= \; & \rho \; P_{\Omega_0^\perp} \int_{\mathbb{S}} (\partial_t \tilde{M}_{J_0}) \omega \; d\omega \\
= \; & \rho \; P_{\Omega_0^\perp} \int_{\mathbb{S}} \left[\partial_t \left(\kappa(\rho) \Omega_0 \right) \cdot \omega \right] \omega \tilde{M}_{J_0} \; d\omega \\
= \; & \rho \; P_{\Omega_0^\perp} \left[\int_{\mathbb{S}} (\omega \otimes \omega) \tilde{M}_{J_0} \; d\omega \right] \partial_t \left(\kappa(\rho) \Omega_0 \right).
\end{aligned}$$

The third equality stems from the fact that $\partial_t \tilde{M}_{J_0} = \partial_t \left(\frac{e^{\kappa(\rho)\Omega_0 \cdot \omega}}{z} \right) = \partial_t \left(\kappa(\rho) \Omega_0 \right) \cdot \omega \tilde{M}_{J_0}$.
Bear in mind that Ω depends on t, since $\Omega = \frac{J_f}{|J_f|}$ and J_f depends on t.
The last line is derived since $\kappa(\rho)$ and Ω_0 don't depend on ω, so they can be taken out of the integral in the last equation and since ω and the von Mises distribution do not depend on time, they need not be included in the partial derivative with respect to time.

We now substitute ω with $\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp$. For $\omega \otimes \omega$ this gives:

$$\begin{aligned}
\omega \otimes \omega = \; & \cos^2(\theta) \Omega_0 \otimes \Omega_0 \\
& + \cos(\theta)\sin(\theta) \Omega_0 \otimes \Omega_0^\perp \\
& + \cos(\theta)\sin(\theta) \Omega_0^\perp \otimes \Omega_0 \\
& + \sin^2(\theta) \Omega_0^\perp \otimes \Omega_0^\perp.
\end{aligned} \tag{24}$$

The first two terms of (24) will give 0 under the projection onto $\Omega_0^\perp$ whereas the last two terms will remain unchanged under this projection. $\tilde{M}_{J_0}$ also changes with this substitution

$$\tilde{M}_{J_0} = \frac{e^{\kappa(\rho)\Omega_0 \cdot \omega}}{z}.$$

Substituting according to (24) this gives:

$$\frac{e^{\kappa(\rho)\Omega_0\cdot(\cos\theta\Omega_0+\sin\theta\Omega_0^\perp)}}{z}=\frac{e^{\kappa(\rho)\cdot\cos\theta}}{z}=M_\theta.$$

since $\Omega_0\cdot\Omega_0=1$ and $\Omega_0\cdot\Omega_0^\perp=0$.
We continue with the calculation.

$$\begin{aligned}
\mathrm{I}_2 &= \rho\int_{\mathbb{S}}\left[\partial_t\left(\kappa(\rho)\Omega_0\right)\cdot\left(\cos(\theta)\Omega_0+\sin(\theta)\Omega_0^\perp\right)\right]\sin(\theta)\Omega_0^\perp M_\theta\ d\theta,\\
&= \rho\left(\int_{-\pi}^{\pi}\left[\partial_t\kappa(\rho)\right]\Omega_0\cdot\cos(\theta)\Omega_0\sin(\theta)\Omega_0^\perp M_\theta\ d\theta\right)\\
&+ \rho\left(\int_{-\pi}^{\pi}\left[\partial_t\kappa(\rho)\right]\Omega_0\cdot\sin(\theta)\Omega_0^\perp\sin(\theta)\Omega_0^\perp M_\theta\ d\theta\right)\\
&+ \rho\left(\int_{-\pi}^{\pi}\kappa(\rho)\left(\partial_t\Omega_0\right)\cdot\cos(\theta)\Omega_0\sin(\theta)\Omega_0^\perp M_\theta\ d\theta\right)\\
&+ \rho\left(\int_{-\pi}^{\pi}\kappa(\rho)\left(\partial_t\Omega_0\right)\cdot\sin(\theta)\Omega_0^\perp\sin(\theta)\Omega_0^\perp M_\theta\ d\theta\right).
\end{aligned}$$

By applying the product rule to $\partial_t\left(\kappa(\rho)\Omega_0\right)$ and solving the parentheses we get these four integral terms. The first Integral is zero, since $\Omega_0\cdot\Omega_0=1$ and since both $\cos(\theta)$ and M_θ are even functions and $\sin(\theta)$ is an odd function. The second integral is zero since $\Omega_0\cdot\Omega_0^\perp=0$. The third integral is zero again since we integrate over an odd product of functions. So all that remains is the fourth integral:

$$\begin{aligned}
\mathrm{I}_2 &= \rho\left(\int_{-\pi}^{\pi}\kappa(\rho)\left(\partial_t\Omega_0\right)\cdot\Omega_0^\perp\sin^2(\theta)\Omega_0^\perp M_\theta\ d\theta\right)\\
&= \rho\kappa(\rho)\left(\partial_t\Omega_0\cdot\Omega_0^\perp\right)\Omega_0^\perp\left(\int_{-\pi}^{\pi}\sin^2(\theta)M_\theta\ d\theta\right).
\end{aligned}$$

To further calculate the result, we look at $\left(\partial_t\Omega_0\right)\cdot\Omega_0^\perp$:

$$0=\partial_t 1=\partial_t\Omega_0\cdot\Omega_0=\partial_t|\Omega_0|^2=2\Omega_0\cdot\partial_t\Omega_0.$$

It follows that $\Omega_0\cdot\partial_t\Omega_0=0$ and therefore $\partial_t\Omega_0\in\ \Omega_0^\perp$.

Now we can use the fact that $\partial_t\Omega_0\in\Omega_0^\perp$ for our calculation of I_2:

$$\mathrm{I}_2=\rho\kappa(\rho)\left(\partial_t\Omega_0\cdot\Omega_0^\perp\right)\Omega_0^\perp\left(\int_{-\pi}^{\pi}\sin^2(\theta)M_\theta\ d\theta\right)$$

$$= \rho\kappa(\rho)\partial_t\Omega_0 \left(\int_{-\pi}^{\pi} \sin^2(\theta) M_\theta \, d\theta \right).$$

Here we used that $\left(\partial_t\Omega_0 \cdot \Omega_0^\perp\right)\Omega_0^\perp = P_{\Omega_0^\perp}(\partial_t\Omega_0) = \partial_t\Omega_0$.

By setting $\sin^2(\theta) = c_0 \neq 0$, we get an expression for I_2 :

$$\mathrm{I}_2 = \rho\kappa(\rho)\partial_t\Omega_0 c_0.$$

We have now computed the first two terms, I_1 and I_2, of the rescaled equation (22) with the GCI plugged in. The next steps are to compute II_1 and II_2 .

$$\mathrm{II}_1 = \; P_{\Omega_0^\perp} \int_{\mathbb{S}} \omega \left[(\omega \cdot \nabla_x \rho) \tilde{M}_{J_0} \right] \, d\omega = P_{\Omega_0^\perp} \left[\int_{\mathbb{S}} \omega \otimes \omega \tilde{M}_{J_0} \, d\omega \right] \nabla_x \rho,$$

using the substitution described in (24), this gives:

$$\begin{aligned} &\int_{\mathbb{S}} \left[\nabla_x \rho \cdot (\cos(\theta)\Omega_0 + \sin(\theta)\Omega_0^\perp) \right] \sin(\theta)\Omega_0^\perp \tilde{M}_\theta \, d\theta \\ &= \left[(\nabla_x \rho \cdot \Omega_0)\Omega_0^\perp \right] \int_{\mathbb{S}} \cos(\theta)\sin(\theta) \; \tilde{M}_\theta \, d\theta + \left[(\nabla_x \rho \cdot \Omega_0^\perp)\Omega_0^\perp \right] \int_{\mathbb{S}} \sin^2(\theta) \; \tilde{M}_\theta \, d\theta. \end{aligned}$$

Lastly, we use that $(\nabla_x \rho \cdot \Omega_0^\perp)\Omega_0^\perp = P_{\Omega_0^\perp} \nabla_x \rho$ and the fact that $\int_{\mathbb{S}} \cos(\theta)\sin(\theta) \; \tilde{M}_\theta \, d\theta$ is an integral over an odd function to obtain the equation for II_1 :

$$\mathrm{II}_1 = c_0 \; P_{\Omega_0^\perp} \nabla_x \rho. \tag{25}$$

What is left is the last integral, where we set $\int_{\mathbb{S}} \sin^2(\theta) \; \tilde{M}_\theta \, d\theta = c_0$ just like in I_2.

Finally we can solve the last of the four integrals:

$$\begin{aligned} \mathrm{II}_2 = \; & P_{\Omega_0^\perp} \int_{\mathbb{S}} \omega \left[(\omega \cdot \nabla_x \tilde{M}_{J_0}) \rho \right] \, d\omega \\ &= \rho P_{\Omega_0^\perp} \int_{\mathbb{S}} \omega \otimes \omega \nabla_x \tilde{M}_{J_0}. \end{aligned}$$

For further calculation, we need to perform a few side-computations:

$$\tilde{M}_{J_0} = \frac{e^{\kappa(\rho)\Omega\cdot\omega}}{z},$$

$$\partial_{x_i}\tilde{M}_{J_0} = \left[\partial_{x_i}(\kappa(\rho)\Omega)\cdot\omega\right]\tilde{M}_{J_0},$$

$$\begin{aligned}\omega\cdot\nabla_x\tilde{M}_{J_0} &= \sum_{i=1}^{N}\omega_i\partial_{x_i}\tilde{M}_{J_0} = \sum_{i=1}^{N}\omega_i\left[\partial_{x_i}(\kappa(\rho)\Omega)\cdot\omega\right]\tilde{M}_{J_0}\\ &= \omega\cdot\left[\sum_{i=1}^{N}\left(\omega_i\partial_{x_i}\right)\left(\kappa(\rho)\Omega\right)\right]\tilde{M}_{J_0}\\ &= \omega\cdot\left[\left(\omega\cdot\nabla_x\right)\left(\kappa(\rho)\Omega\right)\right]\tilde{M}_{J_0},\end{aligned}$$

$$\begin{aligned}\omega\left[\left(\omega\cdot\nabla_x\tilde{M}_{J_0}\right)\right] &= \omega\big[\omega\cdot\left[\left(\omega\cdot\nabla_x\right)\left(\kappa(\rho)\Omega\right)\right]\big]\tilde{M}_{J_0}\\ &= \omega\left[\omega\cdot\left[\left(\sum_{i=1}^{N}\omega_i\partial_{x_i}\right)\left(\kappa(\rho)\Omega\right)\right]\right]\tilde{M}_{J_0}\\ &= \omega\left[\sum_{j=1}^{N}\omega_j\left(\sum_{i=1}^{N}\omega_i\partial_{x_i}\right)\left(\kappa(\rho)\Omega_j\right)\right]\tilde{M}_{J_0}\\ &= \omega\left[\sum_{j=1}^{N}\sum_{i=1}^{N}\omega_j\omega_i\partial_{x_i}(\kappa(\rho)\Omega_j)\right]\tilde{M}_{J_0}\\ &= \omega\left[\sum_{j=1}^{N}\sum_{i=1}^{N}(\omega\otimes\omega)_{ji}\nabla_x(\kappa(\rho)\Omega)_{ji}\right]\tilde{M}_{J_0}\\ &= \omega\left[\omega\otimes\omega:\nabla_x(\kappa(\rho)\Omega)\right]\ \tilde{M}_{J_0}\\ &= \left[\omega\otimes\omega\otimes\omega\ \tilde{M}_{J_0}:\nabla_x(\kappa(\rho)\Omega)\right]_{2:3,1:2}.\end{aligned} \tag{26}$$

This means:

$$\mathrm{II}_2 = P_{\Omega_0^\perp}\int_{\mathbb{S}}\left[\omega\otimes\omega\otimes\omega\ \tilde{M}_{J_0}:\nabla_x(\kappa(\rho)\Omega)\right]d\omega.$$

The notation $M:Q$ for matrices M, Q means a tensor contraction:

$$M:Q = \sum_{i,j=1}^{N} m_{ij}q_{ij},$$

which, for matrices, gives a scalar value as a result. In the last line of (26) the contraction reduces a $2:3$ tensor to a vector.

Remembering the substitution described in (24): $\omega = \cos\theta\Omega_0 + \sin\theta\Omega_0^\perp$ and the facts that $\cos\theta\Omega_0$ projected on $\Omega_0^\perp$ is zero, as well as that cosine is an even function, sine is odd and $M_\theta = \frac{e^{c(\rho)\cos(\theta)}}{z}$ is even as well, we get:

$$\mathrm{II}_2 = \rho P_{\Omega_0^\perp}\left[\int_{-\pi}^{\pi}\left[(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp)\otimes(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp)\otimes(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp)\right] M_\theta \; d\theta : \nabla_x\left(\kappa(\rho)\Omega\right)\right].$$

We call the whole integral on the left-hand-side of the contraction A and compute the integral:

$$\begin{aligned} A = &\left[\int_{-\pi}^{\pi}\cos^3\theta M_\theta \; d\theta\right](\Omega_0\otimes\Omega_0\otimes\Omega_0) \\ &+\left[\int_{-\pi}^{\pi}\cos\theta\sin^2\theta M_\theta d\theta\right](\Omega_0\otimes\Omega_0^\perp\otimes\Omega_0^\perp + \Omega_0^\perp\otimes\Omega_0\otimes\Omega_0^\perp + \Omega_0^\perp\otimes\Omega_0^\perp\otimes\Omega_0). \end{aligned}$$

Only these two terms remain, since all other terms are products of even and odd functions that result in an odd function, that is an odd power of $\sin$.
Because the projection of Ω_0 onto $\Omega_0^\perp$ is zero as well, II_2 is reduced to:

$$\mathrm{II}_2 = \rho P_{\Omega_0^\perp}\left[\left(\int_{-\pi}^{\pi}\sin^2(\theta)\cos(\theta)M_\theta d\theta\right)(\Omega_0^\perp\otimes\Omega_0\otimes\Omega_0^\perp + \Omega_0^\perp\otimes\Omega_0^\perp\otimes\Omega_0) : \nabla_x\left(\kappa(\rho)\Omega\right)\right].$$

We call the integral $\int_{-\pi}^{\pi}\sin^2(\theta)\cos(\theta)M_\theta d\theta = c_1$.

We now look at the contraction part of II_2 i.e.:

$$X = c_1\ (\Omega_0^\perp\otimes\Omega_0\otimes\Omega_0^\perp + \Omega_0^\perp\otimes\Omega_0^\perp\otimes\Omega_0) : \nabla_x\left(\kappa(\rho)\Omega\right),$$

since the components of the gradient ∇_x are:

$$\partial_{x_i}(\kappa(\rho)\Omega_j) = [\partial_{x_i}\kappa(\rho)]\Omega_j + \kappa(\rho)[\partial_{x_i}\Omega_j],$$

we get:

$$\nabla_x\left(\kappa(\rho)\Omega\right) = \nabla_x\kappa(\rho)\otimes\Omega + \kappa(\rho)\nabla_x\Omega. \tag{27}$$

Both of these terms correspond to matrices.

Therefore, we see the dimensions in order to do the contraction fit together, and

$$X = c_1(\Omega_0^\perp \otimes \Omega_0 \otimes \Omega_0^\perp + \Omega_0^\perp \otimes \Omega_0^\perp \otimes \Omega_0) : (\nabla_x \kappa(\rho) \otimes \Omega + \kappa(\rho)\nabla_x \Omega).$$

This tensor contraction can now be split into four parts, eventually leading to the final integral for II_2. We leave out $\kappa(\rho)$ when performing the contraction, since it is a scalar and can simply be added again after the computations are completed, we also use the fact that

$$\left[\left(\Omega_0^\perp \otimes \Omega_0 \otimes \Omega_0^\perp\right) : A\right] = \Omega_0^\perp \left[\left(\Omega_0 \otimes \Omega_0^\perp\right) : A\right],$$

and

$$\left[\left(\Omega_0^\perp \otimes \Omega_0^\perp \otimes \Omega_0\right) : A\right] = \Omega_0^\perp \left[\left(\Omega_0^\perp \otimes \Omega_0\right) : A\right].$$

$$(i) \quad \left(\Omega_0 \otimes \Omega_0^\perp\right) : \nabla_x \kappa(\rho) \otimes \Omega = \sum_{i,j=1}^{N} \Omega_i \Omega_j^\perp \partial_{x_i} \kappa(\rho) \Omega_j = 0,$$

$$(ii) \quad \left(\Omega_0 \otimes \Omega_0^\perp\right) : \nabla_x \Omega = \sum_{i,j=1}^{N} \Omega_i \Omega_j^\perp (\partial_{x_i} \Omega_j) = \Omega^\perp \cdot [(\Omega \cdot \nabla_x)\Omega],$$

$$(iii) \quad \left(\Omega_0^\perp \otimes \Omega_0\right) : \nabla_x \kappa(\rho) \otimes \Omega = \sum_{i,j=1}^{N} \Omega_i^\perp \Omega_j \partial_{x_i} \kappa(\rho) \Omega_j = \Omega^\perp \cdot \nabla_x \kappa(\rho),$$

$$(iv) \quad \left(\Omega_0^\perp \otimes \Omega_0\right) : \nabla_x \Omega = \sum_{i,j=1}^{N} \Omega_i^\perp \Omega_j (\partial_{x_i} \Omega_j) = 0.$$

The first contraction, $(i) = 0$ since $\Omega^\perp \cdot \Omega = 0$, for (ii) we simply applied the dot product, for (iii) we used the fact that $\Omega \cdot \Omega = 1$ and in (iv) we get the result since $\partial_x \Omega \in \Omega^\perp$ and therefore we have $\Omega^\perp \cdot \Omega = 0$.

Now we get for X:

$$X = c_1 \left[\Omega^\perp (\Omega^\perp \cdot \nabla_x \kappa(\rho)) + \Omega^\perp \left(\kappa(\rho)\Omega^\perp \cdot [(\Omega \cdot \nabla_x)\Omega]\right)\right].$$

And since $\mathrm{II}_2 = \rho P_{\Omega_0^\perp}(X)$, this means:

$$\begin{aligned} \mathrm{II}_2 &= \rho c_1 P_{\Omega^\perp} \left[\Omega^\perp (\Omega^\perp \cdot \nabla_x \kappa(\rho)) + \kappa(\rho)\Omega^\perp \left(\Omega^\perp \cdot [(\Omega \cdot \nabla_x)\Omega]\right)\right] \\ &= \rho c_1 P_{\Omega^\perp} \left[\Omega^\perp (\Omega^\perp \cdot \nabla_x \kappa(\rho))\right] + \rho c_1 \kappa(\rho) P_{\Omega^\perp} \left[\Omega^\perp \left(\Omega^\perp \cdot [(\Omega \cdot \nabla_x)\Omega]\right)\right], \end{aligned}$$

where $\Omega^\perp (\Omega^\perp \cdot \nabla_x \kappa(\rho)) = P_{\Omega^\perp}(\nabla_x \kappa(\rho))$ and $\Omega^\perp \left(\Omega^\perp \cdot [(\Omega \cdot \nabla_x)\Omega]\right) = P_{\Omega^\perp}\left((\Omega \cdot \nabla_x)\Omega\right)$.

Since projections are idempotent, i.e. $P^2 = P$ and $P_{\Omega^\perp}\left[(\Omega \cdot \nabla_x)\Omega\right] = (\Omega \cdot \nabla_x)\Omega$, because $\partial_x \Omega \in \Omega^\perp$ we get the following integral:

$$\mathrm{II}_2 = \rho c_1 P_{\Omega^\perp}\big[\nabla_x \kappa(\rho)\big] + \rho c_1 \kappa(\rho)\big[(\Omega\cdot\nabla_x)\Omega\big]. \tag{28}$$

Summarizing all this we get:

$$\begin{aligned}
\mathrm{I}_1 &= 0\\
\mathrm{I}_2 &= c_0\ \rho\kappa(\rho)\partial_t\Omega_0, \quad \text{with } c_0 = \int_{-\pi}^{\pi} \sin^2(\theta) M_\theta\ d\theta\\
\mathrm{II}_1 &= c_0\ P_{\Omega_0^\perp}\nabla_x\rho, \quad \text{with } c_0 = \int_{-\pi}^{\pi} \sin^2(\theta) M_\theta\ d\theta\\
\mathrm{II}_2 &= \rho c_1 P_{\Omega^\perp}(\nabla_x\kappa(\rho)) + \rho c_1\kappa(\rho)(\Omega\cdot\nabla_x)\Omega, \quad \text{with } c_1 = \int_{-\pi}^{\pi} \sin^2(\theta)\cos(\theta) M_\theta d\theta
\end{aligned}$$

Putting all this together, we can finally formulate the rescaled equation for Ω.

$$\begin{aligned}
&\int_{\mathbb{S}} \Big(\partial_t(\rho\tilde{M}_{J_0}) + \omega\cdot\nabla_x(\rho\tilde{M}_{J_0})\Big)\,\beta\cdot\omega\ d\omega = 0\\
&\int_{\mathbb{S}} \Big(\partial_t(\rho\tilde{M}_{J_0}) + \omega\cdot\nabla_x(\rho\tilde{M}_{J_0})\Big)\,\beta\cdot\omega\ d\omega = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{II}_1 + \mathrm{II}_2 = 0.
\end{aligned}$$

$$\begin{aligned}
0 &= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{II}_1 + \mathrm{II}_2\\
&= \rho\kappa(\rho)\partial_t\Omega_0 c_0 + P_{\Omega_0^\perp}\nabla_x\rho c_0 + \rho c_1 P_{\Omega^\perp}(\nabla_x\kappa(\rho)) + \rho c_1\kappa(\rho)(\Omega\cdot\nabla_x)\Omega,
\end{aligned}$$

dividing this equation by $\kappa(\rho)$ and by c_0, we obtain the equation for the orientation Ω:

$$\rho\partial_t\Omega_0 + \frac{1}{\kappa(\rho)} P_{\Omega_0^\perp}\nabla_x\rho + \frac{c_1}{c_0}\rho(\Omega\cdot\nabla_x)\Omega = 0.$$

This gives rise to the following theorem, following along the lines of Theorem 12.1.8 in [5]:

Theorem 7. *Self-Organized Hydrodynamic equations, macroscopic equations.*
We state the theorem in dimension 2. Let f_ε be a solution of the rescaled Vicsek-BGK equation (19). Supposing f_ε converges strong enough as $\varepsilon \to 0$. Then $f_\varepsilon \to \rho M_{\kappa,\Omega}$ where $\rho = \rho(t,x)$ describes the mass of the particles and $\Omega = \Omega(t,x) \in \mathbb{S}$ gives the mean direction of the particles. These two quantities satisfy the following system of equations:

$$\begin{cases}
\partial_t\rho + \nabla_x\left(c(\kappa)\rho\Omega\right) = 0,\\
\rho\partial_t\Omega_0 + \frac{1}{\kappa(\rho)} P_{\Omega_0^\perp}\nabla_x\rho + \frac{c_1}{c_0}\rho(\Omega\cdot\nabla_x)\Omega = 0,
\end{cases}$$

with

$$
\begin{aligned}
c_0 &= \int_{-\pi}^{\pi} \sin^2(\theta) M_\theta \; d\theta, \\
c_1 &= \int_{-\pi}^{\pi} \sin^2(\theta)\cos(\theta) M_\theta d\theta, \\
c(\kappa) &= \int_{-\pi}^{\pi} \cos(\theta) M_\theta d\theta, \\
M_\theta &= \frac{e^{\kappa(\rho)\cos(\theta)}}{\int_{-\pi}^{\pi} e^{\kappa(\rho)\cos(\theta)} d\theta}.
\end{aligned}
$$

3.4.4 Diffusion model in a disordered region [1]

We consider a region where f^ε converges to a stable, uniform equilibrium $\rho(t,x)$ as $\varepsilon \to 0$. We remember that, depending on the density, there exist two stable equilibria. One being the uniform distribution with $f = \rho$ for $0 < \rho \leq n$ and the other being a von-Mises distribution $f = \rho M_{\kappa(\rho)\Omega}$. The hydrodynamic equation satisfied by the density ρ (21) gives the following conservation law (conservation of mass, see (20)):

$$\partial_t \rho^\varepsilon + \nabla_x \cdot \int_{\mathbb{S}} \omega f^\varepsilon d\omega = 0.$$

Keeping in mind that $\int_{\mathbb{S}} \omega f^\varepsilon d\omega = J^\varepsilon$ and considering the Chapman–Enskog expansion, that is:

$$f^\varepsilon = f_0 + R^\varepsilon, \; R^\varepsilon = \varepsilon h^\varepsilon,$$

where R^ε is the remainder, it follows:

$$f^\varepsilon = f_0 + \varepsilon f_1^\varepsilon + \varepsilon^2 f_2^\varepsilon + \dots.$$

We now look at the next higher order in ε of this expansion, to obtain more precise information. Remembering that $f^\varepsilon \to f_0 = \rho(t,x) M_{J_0}$ and that in a disordered region, where particles are uniformly distributed, $J = 0$, we write: $f^\varepsilon = \rho^\varepsilon + \varepsilon f_1^\varepsilon$, since $f_0 = \rho^\varepsilon$ in that case. This gives:

$$J_{f^\varepsilon} = \varepsilon J_{f_1^\varepsilon},$$

since

$$
\begin{aligned}
J_{f^\varepsilon} &= \int_{\mathbb{S}} \omega f^\varepsilon d\omega = \int_{\mathbb{S}} \omega \left(\rho^\varepsilon + \varepsilon f_1^\varepsilon\right) d\omega \\
&= \rho^\varepsilon \int_{\mathbb{S}} \omega d\omega + \varepsilon \int_{\mathbb{S}} \omega f_1^\varepsilon d\omega = \varepsilon J_{f_1^\varepsilon}.
\end{aligned}
$$

Here we can take ρ^ε and ε out of the integrals, since they both do not depend on ω. Also, the integral over ω is zero, since ω is an odd function.

Furthermore, by doing the Taylor expansion of $M_{\varepsilon J_1^\varepsilon}(\omega)$, this gives:

$$M_{J_f^\varepsilon}(\omega) = M_{\varepsilon J_1^\varepsilon}(\omega) = \frac{e^{\varepsilon\omega\cdot J_{f_1^\varepsilon}}}{z} = \frac{1}{z}\left(1 + \varepsilon\omega\cdot J_{f_1^\varepsilon} + \mathcal{O}(\varepsilon^2)\right).$$

And then also expanding the denominator:

$$\begin{aligned} z &= \int e^{\varepsilon\omega\cdot J_{f_1^\varepsilon}} d\omega = \int \left(1 + \varepsilon\omega\cdot J_{f_1^\varepsilon} + \mathcal{O}(\varepsilon^2)\right) d\omega \\ &= \int 1 d\omega + \int \varepsilon\omega\cdot J_{f_1^\varepsilon}\ d\omega + \mathcal{O}(\varepsilon^2) = 1 + \varepsilon J_{f_1^\varepsilon}\cdot\int \omega\ d\omega + \mathcal{O}(\varepsilon^2) = 1 + 0 + \mathcal{O}(\varepsilon^2), \end{aligned}$$

we get: $M_{J_f^\varepsilon}(\omega) \approx 1 + \varepsilon\omega\cdot J_{f_1^\varepsilon} + \mathcal{O}(\varepsilon^2)$

We can now plug $f^\varepsilon = \rho^\varepsilon + \varepsilon f_1^\varepsilon$ and $M_{J_f^\varepsilon}(\omega) = 1 + \varepsilon\omega\cdot J_{f_1^\varepsilon} + \mathcal{O}(\varepsilon^2)$ into the rescaled equation (19). This gives:

$$\begin{aligned} &\partial_t\rho^\varepsilon + \omega\cdot\nabla_x\rho^\varepsilon + \varepsilon(\partial_t + \omega\cdot\nabla_x)f_1^\varepsilon = \frac{1}{\varepsilon}\left(\rho^\varepsilon\left(1 + \varepsilon\omega\cdot J_{f_1^\varepsilon} + \mathcal{O}(\varepsilon^2)\right) - \rho^\varepsilon - \varepsilon f_1^\varepsilon\right) \\ =&\frac{1}{\varepsilon}\left(\rho^\varepsilon + \varepsilon\rho^\varepsilon\omega\cdot J_{f_1^\varepsilon} - \rho^\varepsilon - \varepsilon f_1^\varepsilon + \mathcal{O}(\varepsilon^2)\right) = \rho^\varepsilon\omega\cdot J_{f_1^\varepsilon} - f_1^\varepsilon + \mathcal{O}(\varepsilon) \end{aligned}$$

Now we can multiply both the left hand and the right hand side by ω and integrate as we did to obtain the equation for the density (21) in section 3.4.1:

$$\begin{aligned} &\partial_t\rho^\varepsilon\int \omega\ d\omega + \left(\int \omega\otimes\omega\ d\omega\right)\nabla_x\rho^\varepsilon + \varepsilon\left(\partial_t\int \omega f_1^\varepsilon\ d\omega + \left(\int \omega\otimes\omega\ d\omega\right)\nabla_x f_1^\varepsilon\right) \\ &= \rho^\varepsilon\int \omega\left(\omega\cdot J_{f_1^\varepsilon}\right)\ d\omega - \int \omega f_1^\varepsilon\ d\omega + \mathcal{O}(\varepsilon). \end{aligned}$$

Since $\int \omega\ d\omega = 0$ and $\int \omega f_1^\varepsilon\ d\omega = J_{f_1^\varepsilon}$ this gives:

$$\begin{aligned} &\left(\int \omega\otimes\omega\ d\omega\right)\nabla_x\rho^\varepsilon + \varepsilon\left(\partial_t\int \omega f_1^\varepsilon\ d\omega + \left(\int \omega\otimes\omega\ d\omega\right)\nabla_x f_1^\varepsilon\right) \\ &= \rho^\varepsilon\int \omega\left(\omega\cdot J_{f_1^\varepsilon}\right)\ d\omega - J_{f_1^\varepsilon}\ d\omega + \mathcal{O}(\varepsilon). \end{aligned}$$

So we get

$$J_{f_1^\varepsilon} = \rho^\varepsilon \int \omega \left(\omega \cdot J_{f_1^\varepsilon}\right)\ d\omega - \left(\int \omega \otimes \omega\ d\omega\right) \nabla_x \rho^\varepsilon + \mathcal{O}(\varepsilon), \tag{29}$$

because

$$\varepsilon \left(\partial_t \int \omega f_1^\varepsilon\ d\omega + \left(\int \omega \otimes \omega\ d\omega\right) \nabla_x f_1^\varepsilon\right) \in \mathcal{O}(\varepsilon).$$

The first integral on the right hand side is:

$$\int \omega \left(\omega \cdot J_{f_1^\varepsilon}\right)\ d\omega = \left[\int \omega \otimes \omega\ d\omega\right] J_{f_1^\varepsilon} = \int \begin{bmatrix} \omega_1^2 & \omega_1\omega_2 \\ \omega_2\omega_1 & \omega_2^2 \end{bmatrix}\ d\omega\ J_{f_1^\varepsilon}.$$

Since $\omega_1\omega_2$ is odd and $\int \omega_1^2\ d\omega = \int \omega_2^2\ d\omega = \int \frac{\omega_1^2+\omega_2^2}{2}\ d\omega = \frac{1}{2}\int d\omega = \frac{1}{2}$, this gives:

$$\int \omega \left(\omega \cdot J_{f_1^\varepsilon}\right)\ d\omega = \int \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\ d\omega\ J_{f_1^\varepsilon} = \frac{1}{2} J_{f_1^\varepsilon}.$$

Applying this in the second integral of (29) as well, we get:

$$J_{f_1^\varepsilon} = \frac{\rho^\varepsilon}{2} J_{f_1^\varepsilon} - \frac{1}{2} \nabla_x \rho^\varepsilon + \mathcal{O}(\varepsilon),$$

and eventually:

$$J_{f_1^\varepsilon} = -\frac{1}{2} \left(\frac{\nabla_x \rho^\varepsilon}{1 - \frac{\rho^\varepsilon}{2}}\right) + \mathcal{O}(\varepsilon). \tag{30}$$

Recalling the rescaled equation for the density ρ (which we got by integrating (19) w.r.t. ω), we can now plug in (30) in lieu of $\int \omega f^\varepsilon\ d\omega$. This leads to:

$$\partial_t \rho^\varepsilon - \frac{\varepsilon}{2} \nabla_x \cdot \left(\frac{\nabla_x \rho^\varepsilon}{(1 - \frac{\rho^\varepsilon}{2})}\right) = \mathcal{O}(\varepsilon^2). \tag{31}$$

We can now formulate these results in a Theorem following closely the steps in [3]:

Theorem 8. *Diffusion Equation.*
Considering the rescaled Vicsek BGK equation (19), we assume that in a disordered region, f^ε converges towards the density ρ^ε as ε approaches 0. This density $\rho^\varepsilon \equiv \rho_{f_\varepsilon}$ with $f^\varepsilon = \rho^\varepsilon + \varepsilon f_1^\varepsilon$ satisfies formally at first order the following diffusion equation:

$$\partial_t \rho^\varepsilon = \frac{\varepsilon}{2} \nabla_x \cdot \left(\frac{\nabla_x \rho^\varepsilon}{(1 - \frac{\rho^\varepsilon}{2})} \right). \tag{32}$$

4 Coupling the Vicsek-BGK with the Stokes equation

In this section the aim is to investigate self-organized motion of self-propelled particles in a viscous fluid. In this case, we refer to particles as 'swimmers'. When investigating such swimmers in a fluid, we no longer only have interaction between the single particles, but we also have interplay between the swimmers and the fluid. We have to keep in mind that swimmers perturb the surrounding fluid when moving, which has an effect on neighboring swimmers. These swimmer-fluid-swimmer interactions are highly non-linear, which adds to the complexity of the model [2].
It is still assumed that the swimmers align their direction of motion. Therefore the Vicsek model for self-propelled particles undergoing local alignment can be used to model the interactions between the swimmers in a phenomenological way. In order to account for the interactions between the agents and the fluid, we couple this model with the Stokes equation for the surrounding viscous fluid. It is assumed that the fluid-density remains constant throughout time [2].

The dynamics of the agents are given by the evolution of their position and orientation over time that is, the distribution of swimmers $f = f(t, x, \omega)$ with $\omega_i \in \mathbb{R}^2$ and $x_i \in \mathbb{S}$ as before (see (2)). The velocity of the fluid at position x and time t is denoted by $v = v(x, t) \in \mathbb{R}^2$. By $p = p(x, t) \in \mathbb{R}$ the fluid's pressure is denoted. It is assumed that the fluid-density remains constant throughout time [2].

$$\begin{cases} \partial_t f + \nabla_x \cdot \left(u_{(f,v)} f\right) + \nabla_\omega \cdot \left[P_{\omega^\perp} \left[(\lambda S(v) + A(v))\,\omega\right] f\right] = \rho_f M_{J_f} - f \\ u_{(f,v)} = v + \omega \\ -\Delta_x v + \nabla_x p = -b \nabla_x \cdot Q_f \\ \nabla_x \cdot v = 0 \text{ (incompressibility)} \end{cases} \tag{33}$$

where $\left[P_{\omega^\perp} \left[(\lambda S(v) + A(v))\,\omega\right] f\right]$ is Jeffery's Equation, $Q_f = \int \left(\omega \otimes \omega - \frac{1}{2} Id\right) f d\omega$ with b, λ *const.*, an $\omega \otimes \omega - \frac{1}{2} Id$ is a tracefree matrix (i.e. the trace is zero).

The first equation in (33) without the terms involving v is the Vicsek equation, since $u_{(f,v)}$ simply is ω when neglecting the velocity of the fluid. The last two equations describe the Stokes equation for the velocity of the fluid, with the incompressibility condition stating that we consider a fluid which does not change density when the pressure changes.

1. Effect of the fluid on the particles.

   - Velocity:
     The term $u_{(f,v)} = v + \omega$ describes the total velocity of the individual particles by summarizing the velocity of the fluid and the orientation vector of the particle. So this $u_{(f,v)}$ term gives the effect the fluid has on the particle velocity. This roots in the observation that if we consider zero inertia particles (i.e. no resistance to velocity change) the particle velocity without self propulsion relaxes toward the fluid velocity.
   - Orientation:
     The effect the fluid has on the orientation of the particles is expressed by Jeffery's equation which describes the motion of passive ellipsoidal particles in a viscous fluid, i.e. how the fluid moves the particle (without self propulsion). More precisely the term $(\lambda S(v) + A(v))\,\omega$ is responsible for the effect the fluid has on the orientation of the particles.
     $A(v)$ and $S(v)$ are matrices. A is the antisymmetric part of the linear flow $\nabla_x v$ and S is the symmetric part. They are defined as:

     $$A(v) = \frac{1}{2}\left(\nabla_x v - (\nabla_x v)^T\right),$$

     $$S(v) = \frac{1}{2}\left(\nabla_x v + (\nabla_x v)^T\right),$$

     $A(v)$ describes the rotational effect of the fluid on the swimmers and $S(v)$ describes the forces in the fluid that make a passive particle align with a certain preferred direction.

2. Effect of the particles on the fluid.

   - Drag:
     Generally the motion of particles, or swimmers, will produce some drag on the fluid. Since we consider zero inertia particles we can neglect this force.
   - Self-propulsion:
     The right hand side term of the Stokes equation for v describes how the self propulsion of the swimmers affects the fluid. If we assume our swimmers to be so called pushersï.e. they swim by pushing with their tail, they push with a force $\vec{F}$ in the opposite direction of motion. This means, the tail exerts a force $\vec{F}$ on the fluid in the direction of motion, whereas the head exerts the exact opposite force $-\vec{F}$. However these two forces do not cancel, since they are not applied at the same position.

3. Other effects.
   Since in the present work only a very simplified model is considered, it should be mentioned that generally various other effects, such as noise (alignment ërrors") or effects on the fluid that appear when considering particles that are not infinitesimally small and where inertia (both particle and fluid) is not zero, cannot be neglected [2].

4.1 Rescaling the coupled dynamics

As in the previous section, we will try to rescale the equation to obtain macroscopic coupled dynamics. Again we rescale the distribution of the swimmers $f(t,x,\omega)$ by rescaling time and space, however since we consider the coupled dynamics, we also have to rescale the velocity and the pressure of the fluid, $v(t,x)$ and $p(t,x)$ respectively.
So as before, we have:

$$t' = \varepsilon t,\ x' = \varepsilon x$$

However, in the coupled model this does not only impact $f(t,x,\omega)$ but $v(t,x)$ and $p(t,x)$ as well. The functions f^ε, v^ε and p^ε of the macroscopic variables is defined as:

$$\begin{aligned} f^\varepsilon(t',x',\omega) &= f(t,x,\omega) \\ v^\varepsilon(t',x') &= v(t,x) \\ p^\varepsilon(t',x') &= p(t,x) \end{aligned}$$

After rescaling the coupled model in this way, we obtain the following system:

$$\partial_t f^\varepsilon + \nabla_x \cdot \left(u_{(f^\varepsilon,v^\varepsilon)} f^\varepsilon\right) + \nabla_\omega \cdot \left[P_{\omega^\perp}(B\omega) f^\varepsilon\right] = \frac{1}{\varepsilon}\left(\rho_f^\varepsilon M_{J_{f^\varepsilon}} - f^\varepsilon\right), \tag{34a}$$

$$u_{(f^\varepsilon,v^\varepsilon)} = v^\varepsilon + \omega, \tag{34b}$$

$$-\Delta_x v^\varepsilon + \nabla_x p^\varepsilon = -b\nabla_x \cdot Q_{f^\varepsilon}, \tag{34c}$$

$$\nabla_x \cdot v^\varepsilon = 0. \tag{34d}$$

We call $\left(\rho_f^\varepsilon M_{J_{f^\varepsilon}} - f^\varepsilon\right) = Q(f^\varepsilon)$, but bear in mind:

$$Q(f^\varepsilon) \neq Q_{f^\varepsilon}\Big(= \int_{\mathbb{S}^2}\left(\omega\otimes\omega - \frac{1}{3}Id\right) f^\varepsilon d\omega\Big),$$

and

$$B\omega = \left[(\lambda S(v) + A(v))\,\omega\right].$$

The factor $\frac{1}{\varepsilon}$ on the right hand side stems from the fact that in all three terms on the left hand side a factor ε comes from the inner derivative with respect to x, t or v. In the third term the derivation w.r.t. v is within the definition of the matrices $A(v)$ and $S(v)$.

4.1.1 Equation for the density ρ

We will follow the same steps as in section 3.4.1. So first, we integrate equation (34a) with respect to the self propelled part of the movement (the orientation) ω.

$$\partial_t \int_{\mathbb{S}} f^\varepsilon d\omega + \nabla_x \cdot \int_{\mathbb{S}} \left(u_{(f^\varepsilon, v^\varepsilon)} f^\varepsilon\right) d\omega + \int_{\mathbb{S}} \nabla_\omega \cdot \left[P_{\omega^\perp} \left[\left(\lambda S(v^\varepsilon) + A(v^\varepsilon)\right)\omega\right] f^\varepsilon\right] d\omega = 0,$$

With $\rho^\varepsilon = \int f^\varepsilon d\omega$ it follows::

$$\partial_t \rho^\varepsilon + \nabla_x \cdot \int_{\mathbb{S}} \left(u_{(f^\varepsilon, v^\varepsilon)} f^\varepsilon\right) d\omega + \int_{\mathbb{S}} \nabla_\omega \cdot \left[P_{\omega^\perp} \left[\left(\lambda S(v^\varepsilon) + A(v^\varepsilon)\right)\omega\right] f^\varepsilon\right] d\omega = 0. \tag{35}$$

The right hand side is still zero as in section 3.4.1, since it has not changed. Therefore, when we take the limit of $f^\varepsilon,\ v^\varepsilon,\ p^\varepsilon$ with $\varepsilon \to 0$, we obtain the same limit for f^ε, i.e.:

$$f^0 = \rho M_{J_0}(\omega).$$

Note that v^ε and p^ε converge to v and p respectively as $\varepsilon \to 0$ [2].
Now we can formally take the limit in (35). This way we get:

$$\begin{aligned}\partial_t \int_{\mathbb{S}} \rho M_{J_0}(\omega) d\omega + \nabla_x \cdot \int_{\mathbb{S}} \left(u_{(\rho M_{J_0}(\omega), v)} \rho M_{J_0}(\omega)\right) d\omega \\ + \int_{\mathbb{S}} \nabla_\omega \cdot \left[P_{\omega^\perp} \left[\left(\lambda S(v) + A(v)\right)\omega\right] \rho M_{J_0}(\omega)\right] d\omega = 0.\end{aligned}$$

As in section 3.4.1, ρ does not depend on ω and $\int_{\mathbb{S}} M_{J_0} = 1$, since M_J is a probability density function. The integral $\int_{\mathbb{S}} \omega M_{J_0}(\omega) d\omega = \Omega_0 c(\kappa)$ (see (7)) since the compatibility condition (5) has to be fulfilled. The right hand side of the equation is zero, since 1 is a collision invariant for $Q(f^\varepsilon)$.

$$\begin{aligned}0 &= \partial_t \rho + \nabla_x \cdot \left(\rho \int_{\mathbb{S}} (v + \omega) M_{J_0}(\omega)\right) d\omega + \rho \int_{\mathbb{S}} \nabla_\omega \cdot \left[P_{\omega^\perp} \left[\left(\lambda S(v) + A(v)\right)\omega\right] M_{J_0}(\omega)\right] d\omega \\ &= \partial_t \rho + \nabla_x \cdot \left(\rho v + \rho \int_{\mathbb{S}} \omega M_{J_0}(\omega) d\omega\right) = \partial_t \rho + \nabla_x \cdot \left(\rho v + \rho \Omega_0 c(\kappa)\right).\end{aligned}$$

This gives the equation describing the time-evolution of the spatial density of the swimmers:

$$\partial_t \rho + \nabla_x \cdot \rho \left(v + \Omega_0 c(\kappa) \right) = 0. \tag{36}$$

4.1.2 Equation for the orientation Ω

To obtain the equation for the mean direction of the swimmers, we also proceed as we did in the BGK-Vicsek model, see section 3.4.2. We multiply the equation (34a) with ω and integrate with respect to ω as well.

$$\int_{\mathbb{S}} \partial_t f^\varepsilon \omega \; d\omega + \nabla_x \cdot \int_{\mathbb{S}} \left(u_{(f^\varepsilon, v^\varepsilon)} f^\varepsilon\right) \omega \; d\omega + \int_{\mathbb{S}} \nabla_\omega \cdot \left[P_{\omega^\perp} \left[\left(\lambda S(v^\varepsilon) + A(v^\varepsilon)\right) \omega\right] f^\varepsilon\right] \omega \; d\omega$$
$$= \frac{1}{\varepsilon} \int_{\mathbb{S}} \left(\rho_f^\varepsilon M_{J_{f^\varepsilon}} - f^\varepsilon\right) \omega \; d\omega.$$

Again we are confronted with the problem that the right hand side, which has not changed in the coupled model, in this case is not necessarily zero. So, again we need to multiply with the GCI, which is the same as in section 3.4.3, since the right hand side of the equation has not changed; see (23).
The GCI is: $\psi_{\Omega_0} = \beta \cdot \omega$ and taking the limit $(f^\varepsilon, v^\varepsilon, p^\varepsilon) \xrightarrow{\varepsilon \to 0} (\rho M_{J_0}, v, p)$, we get:

$$\int_{\mathbb{S}} \left(\partial_t(\rho M_{J_0}) + \nabla_x \cdot u_{(\rho M_{J_0}, v)} \rho M_{J_0} + \nabla_\omega \cdot \left[P_{\omega^\perp} \left[\left(\lambda S(v) + A(v)\right) \omega\right] \rho M_{J_0}\right]\right) \beta \cdot \omega d\omega = 0.$$

Since $\beta \in \Omega_0^\perp$ (see section 3.4.2), we can rewrite this equation as:

$$P_{\Omega_0^\perp} \left[\int_{\mathbb{S}} \left(\partial_t(\rho M_{J_0}) + \nabla_x \cdot u_{(\rho M_{J_0}, v)} (\rho M_{J_0}) + \nabla_\omega \cdot \left[P_{\omega^\perp} \left[\left(\lambda S(v) + A(v)\right) \omega\right] \rho M_{J_0}\right]\right) \omega d\omega\right] = 0.$$

To make things a little clearer, we split the integral into 3 terms:

$$\mathrm{I} = \int_{\mathbb{S}} \left[\partial_t(\rho M_{J_0}) \cdot \omega + \omega \cdot \nabla_x (\rho M_{J_0}) \omega\right] \; d\omega, \tag{37a}$$
$$\mathrm{II} = \int_{\mathbb{S}} v \cdot \nabla_x (\rho M_{J_0}) \omega \; d\omega, \tag{37b}$$
$$\mathrm{III} = \int_{\mathbb{S}} \nabla_\omega \cdot \left[P_{\omega^\perp} \left[\left(\lambda S(v) + A(v)\right) \omega\right] \rho M_{J_0}\right] \omega \; d\omega. \tag{37c}$$

We notice that equation (37a) is exactly the equation for the orientation we used to calculate the evolution of the orientation in section 3.4.2. So this equation is already taken care of.
That means:

$$\boxed{P_{\Omega_0^\perp} \mathrm{I} = \rho \partial_t \Omega_0 + \frac{1}{\kappa(\rho)} P_{\Omega_0^\perp} \nabla_x \rho + \frac{c_1}{c_0} \rho (\Omega_0 \cdot \nabla_x) \Omega_0}$$

Equations (37b) and (37c) correspond to the coupling terms and have to be treated separately. For (37b) we have to bear in mind that $v \cdot \nabla_x(\rho M_{J_0}) = (v \cdot \nabla_x \rho) M_{J_0} + (v \cdot \nabla_x M_{J_0})\rho$:

$$\int_{\mathbb{S}} v \cdot \nabla_x(\rho M_{J_0})\omega \; d\omega = \int_{\mathbb{S}} \left[(v \cdot \nabla_x \rho) M_{J_0} + (v \cdot \nabla_x M_{J_0})\rho\right]\omega \; d\omega.$$

We can split II into two integrals:

$$\begin{aligned} \mathrm{II}_1 &= \int_{\mathbb{S}} (v \cdot \nabla_x \rho) M_{J_0} \omega \; d\omega, \\ \mathrm{II}_2 &= \int_{\mathbb{S}} (v \cdot \nabla_x M_{J_0}) \rho\omega \; d\omega. \end{aligned}$$

$$\mathrm{II}_1 = \int_{\mathbb{S}} (v \cdot \nabla_x \rho) M_{J_0} \omega \; d\omega = \left(\int_{\mathbb{S}} M_{J_0} \omega \; d\omega \right) (v \cdot \nabla_x \rho) = c(\kappa)\Omega_0 (v \cdot \nabla_x \rho),$$

since $\int_{\mathbb{S}} M_{J_0} \omega \; d\omega = c(\kappa)\Omega_0$.

$$P_{\Omega_0^\perp} \mathrm{II}_1 = P_{\Omega_0^\perp} \left(c(\kappa)\Omega_0 (v \cdot \nabla_x \rho) \right) = (v \cdot \nabla_x \rho) P_{\Omega_0^\perp} \left(c(\kappa)\Omega_0 \right) = 0,$$

since Ω_0 is projected onto $\Omega_0^\perp$.

$$\mathrm{II}_2 = \int_{\mathbb{S}} (v \cdot \nabla_x M_{J_0}) \rho\omega \; d\omega = \rho \int_{\mathbb{S}} (v \cdot \nabla_x M_{J_0}) \omega \; d\omega = \rho \int_{\mathbb{S}} \left[\omega \cdot (v \cdot \nabla_x) \left(\kappa(\rho)\Omega_0\right)\right] M_{J_0} \omega \; d\omega,$$

since $v \cdot \nabla_x M_{J_0} = \left[\omega \cdot (v \cdot \nabla_x) \left(\kappa(\rho)\Omega_0\right)\right] M_{J_0}$ (see (26)).

Calculating the term within the integral gives:

$$\begin{aligned} &\omega \left[\omega \cdot (v \cdot \nabla_x) \left(\kappa(\rho)\Omega_0\right)\right] M_{J_0} = \omega \left[\omega \cdot \left[\left(\sum_{i=1}^{N} v_i \partial_{x_i} \right) \left(\kappa(\rho)\Omega_0\right) \right] \right] M_{J_0} \\ =&\omega \left[\sum_{j=1}^{N} \omega_j \left(\sum_{i=1}^{N} v_i \partial_{x_i} \right) \left(\kappa(\rho)\Omega_{0j}\right) \right] M_{J_0} = \omega \left[\sum_{j=1}^{N} \sum_{i=1}^{N} \omega_j v_i \partial_{x_i} (\kappa(\rho)\Omega_{0j}) \right] M_{J_0} \\ =&\omega \left[\sum_{j=1}^{N} \sum_{i=1}^{N} (\omega \otimes v)_{ji} \nabla_x (\kappa(\rho)\Omega_0)_{ji} \right] M_{J_0} = \left[\omega \otimes \omega \otimes v \; M_{J_0} : \nabla_x (\kappa(\rho)\Omega_0)\right]. \end{aligned}$$

So it follows:

$$\mathrm{II}_2 = \rho \int_{\mathbb{S}} [\omega \otimes \omega \otimes v \; M_{J_0} : \nabla_x(\kappa(\rho)\Omega_0)] \; d\omega.$$

Since $\int(\omega \otimes \omega \otimes v)_{ijk} M_{J_0} \; d\omega = \int \omega_i\omega_j v_k M_{J_0} \; d\omega = \int \omega_i\omega_j M_{J_0} \; d\omega v_k = \int(\omega \otimes \omega M_{J_0} \; d\omega \otimes v)_{ijk}$ we can take the v out and obtain

$$\mathrm{II}_2 = \left(\rho \int_{\mathbb{S}} [\omega \otimes \omega \; M_{J_0} : \nabla_x(\kappa(\rho)\Omega_0)] \; d\omega\right) v$$

When doing the projection of II_2 we follow the same procedure as in (25) and we also use the same substitution as before (See (24)):

$$\begin{aligned} P_{\Omega_0^\perp}\mathrm{II}_2 &= \rho P_{\Omega_0^\perp}\left(\int_{\mathbb{S}} [\omega \otimes \omega] \; M_{J_0} \; d\omega : \nabla_x(\kappa(\rho)\Omega)\right) v \\ &= \rho P_{\Omega_0^\perp}\left(\int_{-\pi}^{\pi} \left[(\cos(\theta)\Omega_0 + \sin(\theta)\Omega_0^\perp) \otimes (\cos(\theta)\Omega_0 + \sin(\theta)\Omega_0^\perp)\right] \; M_\theta \; d\theta : \nabla_x(\kappa(\rho)\Omega_0)\right) v. \end{aligned}$$

We call the right hand side of the contraction A and proceed to compute that:

$$\begin{aligned} A = &\int_{-\pi}^{\pi} \left[\cos^2(\theta) M_\theta \; d\theta\right] (\Omega_0 \otimes \Omega_0) v \\ &+ \int_{-\pi}^{\pi} \left[\sin^2(\theta) M_\theta \; d\theta\right] (\Omega_0^\perp \otimes \Omega_0^\perp) v \\ &+ \int_{-\pi}^{\pi} [\cos(\theta)\sin(\theta) M_\theta \; d\theta] \, (\Omega_0 \otimes \Omega_0^\perp \otimes v + \Omega_0^\perp \otimes \Omega_0) v. \end{aligned}$$

The last integral is zero, since the product $\cos(\theta)\sin(\theta)M_\theta$ is odd. Keeping in mind that the projection of Ω_0 onto $\Omega_0^\perp$ is zero, we get:

$$P_{\Omega_0^\perp}\mathrm{II}_2 = P_{\Omega_0^\perp}\mathrm{II} = \rho P_{\Omega_0^\perp}\left(\int_{-\pi}^{\pi} \left[\sin^2(\theta) M_\theta \; d\theta\right] (\Omega_0^\perp \otimes \Omega_0^\perp) : \nabla_x(\kappa(\rho)\Omega_0)\right) v.$$

Still following the steps from section 3.4.2 we now look at the gradient part of the contraction: $\nabla_x(\kappa(\rho)\Omega_0)$.
As in (27) this gives:

$$\nabla_x \left(\kappa(\rho)\Omega_0\right) = \nabla_x\kappa(\rho) \otimes \Omega_0 + \kappa(\rho)\nabla_x\Omega_0.$$

We now split the tensor contraction $(\Omega_0^\perp \otimes \Omega_0^\perp) : (\nabla_x \kappa(\rho) \otimes \Omega_0 + \kappa(\rho)\nabla_x \Omega_0)$ into its parts. We can leave out $\kappa(\rho)$ again since it is a scalar.

$$(i) \quad \left(\Omega_0^\perp \otimes \Omega_0^\perp\right) : \nabla_x \kappa(\rho) \otimes \Omega_0 = \sum_{i,j=1}^{N} \Omega_i^\perp \Omega_j^\perp \partial_{x_i} \kappa(\rho) \Omega_j = 0,$$

$$(ii) \quad \left(\Omega_0^\perp \otimes \Omega_0^\perp\right) : \nabla_x \Omega_0 = \sum_{i,j=1}^{N} \Omega_i^\perp \Omega_j^\perp (\partial_{x_i} \Omega_j) = \Omega_0^\perp \cdot \left[(\Omega_0^\perp \cdot \nabla_x)\Omega_0\right] = P_{\Omega_0^\perp}(\nabla_x \Omega_0) = \nabla_x \Omega_0.$$

The first contraction is zero, since $\Omega_0^\perp \cdot \Omega_0 = 0$ and in the second one we use that $\partial_x \Omega_0 \in \Omega_0^\perp$ for the last equality.

Putting all the pieces together, we get:

$$P_{\Omega_0^\perp} \mathrm{II}_2 = \rho P_{\Omega_0^\perp} \left(\int_{-\pi}^{\pi} \left[\sin^2(\theta) M_\theta \; d\theta\right] \nabla_x \Omega_0 \right) v.$$

Since $\int_{-\pi}^{\pi} \left[\sin^2(\theta) M_\theta \; d\theta\right] := c_0$ this means:

$$\boxed{P_{\Omega_0^\perp} \mathrm{II}_2 = \rho c_0 \kappa(\rho) \nabla_x \Omega_0 \cdot v}$$

Now we look at the last part of the equation for the orientation Ω_0,

$$\mathrm{III} = \int_{\mathbb{S}} \nabla_\omega \cdot \left[P_{\omega^\perp} \left[(\lambda S(v) + A(v))\,\omega\right] \rho M_{J_0}\right] \omega \; d\omega.$$

First, we rename $(\lambda S(v) + A(v)) = B$ so we obtain:

$$\begin{aligned} \nabla_\omega \cdot (P_{\omega^\perp}(B\omega)\rho M_{J_0}) &= \nabla_\omega \cdot (P_{\omega^\perp} B\omega)\rho M_{J_0} + (P_{\omega^\perp} B\omega) \cdot \nabla_\omega(\rho M_{J_0}) \\ &= B : (Id - 3\omega \otimes \omega)\rho M_{J_0} + \kappa\rho M_{J_0} \left[(\omega \cdot B^T \Omega_0) - (\omega \cdot B\omega)(\omega \cdot \Omega_0)\right], \end{aligned}$$

by applying the product rule and since $\nabla M_{J_0} = \kappa P_{\omega^\perp} \Omega_0 M_{J_0}$ and $\nabla_\omega \cdot (P_{\omega\perp} B\omega) = B : (Id - 3\omega \otimes \omega)$ for any matrix independent of ω. See [2], [21] Ap.A.2.
Now we decompose III yet another time and obtain:

$$\mathrm{III}_1 = \rho \int_{\mathbb{S}} B : (Id - 3\omega \otimes \omega) M_{J_0} \omega \; d\omega,$$

$$\mathrm{III}_2 = \kappa\rho \int_{\mathbb{S}} (\omega \cdot B^T \Omega_0) M_{J_0} \omega \; d\omega,$$
$$\mathrm{III}_3 = -\kappa\rho \int_{\mathbb{S}} (\omega \cdot B\omega)(\omega \cdot \Omega_0) M_{J_0} \omega \; d\omega.$$

We notice that in III_1 $(Id - 3\omega \otimes \omega)$ is a symmetric matrix. We also recall that $B = (\lambda S(v) + A(v))$ and that $A(v)$ is an antisymmetric matrix. The contraction between a symmetric and an antisymmetric matrix is zero, so we are left with:

$$B : (Id - 3\omega \otimes \omega) = \lambda S(v) : (Id - 3\omega \otimes \omega).$$

First, we notice that everything is scalar, except ω, so the projection is only performed on ω. Then we also perform the change of variables $\omega = \cos\theta\Omega_0 + \sin\theta\Omega_0^\perp$ as before, keeping in mind that the projection of Ω_0 onto $\Omega_0^\perp$ is zero, integration over an odd function gives zero as well, and using the fact that $S(v) : (\Omega_0^\perp \otimes \Omega_0 + \Omega_0 \otimes \Omega_0^\perp) = 2\Omega_0^\perp \cdot S(v)\Omega_0$ since both matrices of the contraction are symmetric.

$$\begin{aligned}
P_{\Omega_0^\perp}\mathrm{III}_1 &= \rho \int_{\mathbb{S}} \left(\lambda S(v) : (Id - 3\omega \otimes \omega)\right) M_{J_0} P_{\Omega_0^\perp}(\omega) \; d\omega \\
&= \rho\lambda \int_{-\pi}^{\pi} \Big(S(v) : (Id - 3(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp) \otimes (\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp)) \Big) \\
&\quad M_\theta P_{\Omega_0^\perp}(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp) \; d\theta \\
&= \rho\lambda \int_{-\pi}^{\pi} \Big(S(v) : (Id - 3(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp) \otimes (\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp)) \Big) \\
&\quad M_\theta \sin\theta\Omega_0^\perp \; d\theta \\
&= 3\rho\lambda \int_{-\pi}^{\pi} \cos\theta \sin^2\theta M_\theta \; d\theta \left[S(v) : (Id - (\Omega_0 \otimes \Omega_0^\perp + \Omega_0^\perp \otimes \Omega_0)) \right] \Omega_0^\perp \\
&= -3\rho\lambda \int_{-\pi}^{\pi} \cos\theta \sin^2\theta M_\theta \; d\theta \left[S(v) : (\Omega_0^\perp \otimes \Omega_0 + \Omega_0 \otimes \Omega_0^\perp) \right] \Omega_0^\perp \\
&= -6\rho\lambda c_1 \; (\Omega_0^\perp \cdot S(v)\Omega_0)\Omega_0^\perp = -6\rho\lambda c_1 P_{\Omega_0^\perp}(S(v)\Omega_0).
\end{aligned}$$

Since we call $\int_{-\pi}^{\pi} \cos\theta \sin^2\theta M_\theta \; d\theta := c_1$.

Again keeping in mind that integration over an odd function gives zero, we obtain the projection of III_2 :

$$P_{\Omega_0^\perp}\mathrm{III}_2 = \kappa\rho P_{\Omega_0^\perp} \int_{\mathbb{S}} (\omega \cdot B^T \Omega_0) M_{J_0} \omega \; d\omega$$

$$= \kappa\rho P_{\Omega_0^\perp} \int_{-\pi}^{\pi} ((\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp)\cdot(\lambda S(v) - A(v))\Omega_0) M_\theta (\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp)\ d\theta$$
$$= \kappa\rho \int_{-\pi}^{\pi} \sin^2\theta M_\theta d\theta\ \Omega_0^\perp \otimes \Omega_0^\perp (\lambda S(v) - A(v))\Omega_0 = \kappa\rho c_0 P_{\Omega_0^\perp}(\lambda S(v) - A(v))\Omega_0,$$

Since $\Omega_0^\perp \otimes \Omega_0^\perp = P_{\Omega_0^\perp}$ and we call $\int_{-\pi}^{\pi} \sin^2\theta M_\theta\ d\theta := c_0$.

We proceed similarly as for III_1 to obtain the projection of III_3 :

$$\begin{aligned}
P_{\Omega_0^\perp}\mathrm{III}_3 &= -\kappa\rho P_{\Omega_0^\perp} \int_{\mathbb{S}} (\omega\cdot B\omega)(\omega\cdot\Omega_0) M_{J_0}\omega\ d\omega \\
&= -\kappa\rho \int_{\mathbb{S}} (\omega\cdot B\omega)(\omega\cdot\Omega_0) M_{J_0} P_{\Omega_0^\perp}(\omega)\ d\omega \\
&= -\kappa\rho \int_{-\pi}^{\pi} \left[(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp)\cdot B(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp)\right] \\
&\quad \left[(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp)\cdot\Omega_0\right] M_\theta P_{\Omega_0^\perp}(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp)\ d\theta \\
&= -\kappa\rho \int_{-\pi}^{\pi} \left[(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp)\cdot B(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp)\right] \cos\theta M_\theta \sin\theta\Omega_0^\perp\ d\theta \\
&= -\kappa\rho \int_{-\pi}^{\pi} \cos^2\theta\sin^2\theta M_\theta d\theta\ \left[(\Omega_0^\perp\cdot B\Omega_0)\Omega_0^\perp + (\Omega_0\cdot B\Omega_0^\perp)\Omega_0^\perp\right] \\
&= -\kappa\rho \int_{-\pi}^{\pi} \cos^2\theta\sin^2\theta M_\theta d\theta\ (\Omega_0^\perp\cdot(B+B^T)\Omega_0)\Omega_0^\perp \\
&= -\kappa\rho\lambda \int_{-\pi}^{\pi} \cos^2\theta\sin^2\theta M_\theta d\theta\ (\Omega_0^\perp\otimes\Omega_0^\perp) 2S(v)\Omega_0 \\
&= -\kappa\rho\lambda c_2\ P_{\Omega_0^\perp} S(v)\Omega_0,
\end{aligned}$$

with $c_2 := 2\int_{-\pi}^{\pi} \cos^2\theta\sin^2\theta M_\theta d\theta$.

The first equality here comes from the fact that only ω is non-scalar and therefore can be projected. In the second equality the substitution was performed. The third equality is due to the projection of $\cos\theta\Omega_0$ onto $\Omega_0^\perp$ being zero and to the fact that $\Omega_0\cdot\Omega_0 = 1$ and $\Omega_0\cdot\Omega_0^\perp = 0$. In the next line we consider even functions only. We also use the fact that $(B + B^T) = S(v)$ since only the symmetric part survives.
Putting the $\mathrm{III}_1 - \mathrm{III}_3$ together, the projection of III is obtained. Then putting $\mathrm{I} - \mathrm{III}$ together, the equation for the orientation is obtained:

$$\begin{aligned}
P_{\Omega_0^\perp}\mathrm{III} &= -6\rho\lambda c_1 P_{\Omega_0^\perp}(S(v)\Omega_0) + \kappa\rho c_0 P_{\Omega_0^\perp}(\lambda S(v) - A(v))\Omega_0 - \kappa\rho\lambda c_2\ P_{\Omega_0^\perp} S(v)\Omega_0 \\
&= \rho\left[P_{\Omega_0^\perp} S(v)\Omega_0\,(-6\lambda c_1 + \kappa c_0\lambda - \kappa\lambda c_2) + \kappa c_0 P_{\Omega_0^\perp}\,(-A(v))\,\Omega_0\right]
\end{aligned}$$

$$= \rho \left[\lambda P_{\Omega_0^\perp} S(v)\Omega_0 \left(\kappa c_0 - 6c_1 - \kappa c_2\right) - \kappa c_0 P_{\Omega_0^\perp} A(v)\Omega_0 \right].$$

Now for equations I – III:

$$\begin{aligned} P_{\Omega_0^\perp} \mathrm{I} &= \rho c_0 \kappa(\rho)\partial_t \Omega_0 + c_0 P_{\Omega_0^\perp} \nabla_x \rho + \kappa(\rho) c_1 \rho (\Omega_0 \cdot \nabla_x)\Omega_0, \\ P_{\Omega_0^\perp} \mathrm{II} &= \rho c_0 \nabla_x \Omega_0 \cdot v, \\ P_{\Omega_0^\perp} \mathrm{III} &= \rho \left[\lambda P_{\Omega_0^\perp} S(v)\Omega_0 \left(\kappa c_0 - 6c_1 - \kappa c_2\right) - \kappa c_0 P_{\Omega_0^\perp} A(v)\Omega_0 \right]. \end{aligned}$$

So it follows for the equation for Ω:

$$\begin{aligned} \rho(c_0\kappa(\rho)\partial_t\Omega_0 + \kappa(\rho)c_1(\Omega_0\cdot\nabla_x)\Omega_0) + c_0 P_{\Omega_0^\perp}\nabla_x\rho + \rho c_0 \nabla_x \Omega_0 \cdot v \\ + \rho \left[\lambda P_{\Omega_0^\perp} S(v)\Omega_0 \left(\kappa c_0 - 6c_1 - \kappa c_2\right) - \kappa c_0 P_{\Omega_0^\perp} A(v)\Omega_0 \right]. \end{aligned} \tag{38}$$

We are done with the rescaling of the parameters of the Vicsek Model when combined with the Stokes equation, summarized below in Lemma 6, following along the lines of [5], [2].

Lemma 6. *Macroscopic equations for the coupled BGK equation.*
We state the theorem in dimension 2. Let f_ε be a solution of the rescaled coupled BGK equation (34a)-(34b). Supposing f_ε converges strong enough as $\varepsilon \to 0$. Then $f_\varepsilon \to \rho M_{\kappa,\Omega}$ where $\rho = \rho(t,x)$ describes the mass of the particles and $\Omega = \Omega(t,x) \in \mathbb{S}$ gives the mean direction of the particles. These two quantities satisfy the following system of equations:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot \rho(v + \Omega_0 c(\kappa)) = 0, \\ \rho(c_0\kappa(\rho)\partial_t\Omega_0) + \kappa(\rho)c_1(\Omega_0\cdot\nabla_x)\Omega_0) + c_0 P_{\Omega_0^\perp}\nabla_x\rho + \rho c_0 \nabla_x \Omega_0 \cdot v \\ +\rho \left[\lambda P_{\Omega_0^\perp} S(v)\Omega_0 \left(\kappa c_0 - 6c_1 - \kappa c_2\right) - \kappa c_0 P_{\Omega_0^\perp} A(v)\Omega_0 \right], \end{cases}$$

with

$$\begin{aligned} c_0 &= \int_{-\pi}^{\pi} \sin^2(\theta) M_\theta \; d\theta, \\ c_1 &= \int_{-\pi}^{\pi} \sin^2(\theta)\cos(\theta) M_\theta d\theta, \\ c_2 &= 2\int_{-\pi}^{\pi} \cos^2(\theta)\sin^2(\theta) M_\theta d\theta \\ M_\theta &= \frac{e^{\kappa(\rho)\cos(\theta)}}{\int_{-\pi}^{\pi} e^{\kappa(\rho)\cos(\theta)} d\theta}. \end{aligned}$$

Now we still have to take care of the newparameters that appear in the Stokes equation.

4.1.3 Limit for the Stokes equation

Let's recall the rescaled Stokes Equation (34c):

$$-\Delta_x v^\varepsilon + \nabla_x p^\varepsilon = -b\nabla_x \cdot \int_{\mathbb{S}^2} \left(\omega \otimes \omega - \frac{1}{2} Id\right) f^\varepsilon d\omega$$

Here we only have to compute the right hand side of the equation, since $(v^\varepsilon, p^\varepsilon) \longrightarrow (v, p)$ as εapproaches 0.
Remember that $f^\varepsilon \longrightarrow \rho M_{\Omega_0}(\omega)$ as $\varepsilon \to 0$.

$$-b\nabla_x \cdot \int_{\mathbb{S}^2} \left(\omega \otimes \omega - \frac{1}{2} Id\right) f^\varepsilon d\omega = -b\nabla_x \cdot \int_{\mathbb{S}^2} \left(\omega \otimes \omega - \frac{1}{2} Id\right) \rho M_{\Omega_0}(\omega) d\omega$$

First we will tackle the integral, making use of $\Omega^\perp \otimes \Omega^\perp = (Id - \Omega \otimes \Omega)$:

$$\begin{aligned}
&\int_{\mathbb{S}^2} \left(\omega \otimes \omega - \frac{1}{2} Id\right) \rho M_{\Omega_0}(\omega) d\omega \\
&\quad = \int_{-\pi}^{\pi} \left[\left(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp\right) \otimes \left(\cos\theta\Omega_0 + \sin\theta\Omega_0^\perp\right) - \frac{1}{2} Id\right] \rho M_\theta d\theta \\
&\quad = \rho \left[\left(\int_{-\pi}^{\pi} \cos^2\theta M_\theta d\theta\right) \Omega_0 \otimes \Omega_0 + \left(\int_{-\pi}^{\pi} \sin^2\theta M_\theta d\theta\right) \Omega_0^\perp \otimes \Omega_0^\perp - \int_{-\pi}^{\pi} \frac{1}{2} Id M_\theta d\theta\right] \\
&\quad = \rho \left(\int_{-\pi}^{\pi} \cos^2\theta M_\theta d\theta\right) \Omega_0 \otimes \Omega_0 + \rho \left(\int_{-\pi}^{\pi} \sin^2\theta M_\theta d\theta\right) (Id - \Omega_0 \otimes \Omega_0) - \rho \frac{1}{2} Id \int_{-\pi}^{\pi} M_\theta d\theta \\
&\quad = \rho \left(\int_{-\pi}^{\pi} \cos^2\theta M_\theta d\theta - \int_{-\pi}^{\pi} \sin^2\theta M_\theta d\theta\right) \Omega_0 \otimes \Omega_0 + \rho \left(\int_{-\pi}^{\pi} \sin^2\theta M_\theta d\theta - \frac{1}{2}\right) Id \\
&\quad = \rho \left[c_4 \left(\Omega_0 \otimes \Omega_0 - \frac{1}{2} Id\right) + c_4 \frac{1}{2} Id + \left(c_0 - \frac{1}{2}\right) Id\right] \\
&\quad = \rho \left[c_4 \left(\Omega_0 \otimes \Omega_0 - \frac{1}{2} Id\right) + \left[c_4 \frac{1}{2} + \left(c_0 - \frac{1}{2}\right)\right] Id\right] \\
&\quad = \rho \left[c_4 \left(\Omega_0 \otimes \Omega_0 - \frac{1}{2} Id\right) + c_5 Id\right].
\end{aligned}$$

We followed the usual steps to obtain the results above. First we do the substitution according to (24), then, we only consider even functions. Next we use the previously mentioned fact that $\Omega^\perp \otimes \Omega^\perp = (Id - \Omega \otimes \Omega)$ and eventually, we simply factorize the results in a convenient way, with

$$
\begin{aligned}
c_0 &= \int_{-\pi}^{\pi} \sin^2\theta M_\theta d\theta, \\
c_4 &= \left(\int_{-\pi}^{\pi} \cos^2\theta M_\theta d\theta - \int_{-\pi}^{\pi} \sin^2\theta M_\theta d\theta \right) = -c_0 + \int_{-\pi}^{\pi} \cos^2\theta M_\theta d\theta, \\
c_5 &= c_4 \frac{1}{2} + \left(c_0 - \frac{1}{2} \right).
\end{aligned}
$$

A short calculation will show, $c_5 = 0$:

$$
\begin{aligned}
c_5 &= c_4 \frac{1}{2} + \left(c_0 - \frac{1}{2} \right) \\
&= \frac{1}{2} \left(\int_{-\pi}^{\pi} \cos^2\theta M_\theta d\theta - \int_{-\pi}^{\pi} \sin^2\theta M_\theta d\theta \right) + \left(\int_{-\pi}^{\pi} \sin^2\theta M_\theta d\theta - \frac{1}{2} \right) \\
&= \int_{-\pi}^{\pi} M_\theta \left(\frac{1}{2}\cos^2\theta - \frac{1}{2}\sin^2\theta + \sin^2\theta \right) - \frac{1}{2} \\
&= \frac{1}{2} \int_{-\pi}^{\pi} M_\theta \left(\cos^2\theta + \sin^2\theta \right) - \frac{1}{2} \\
&= 0.
\end{aligned}
$$

We now obtain the fully rescaled Stokes Equation:

$$
\Delta_x v + \nabla_x p = -b\ c_4 \nabla_x \cdot \left[\rho \left(\Omega_0 \otimes \Omega_0 - \frac{1}{2} Id \right) \right]. \tag{39}
$$

See [2].

To conclude this section we shall summarize our finding in a theorem, following closely Theorem 4.1 in [2] and giving the equations for the time-evolution of the spatial density of swimmers $\rho = \rho(t,x)$, their orientation $\Omega = \Omega(t,x)$, the velocity of the fluid $v = v(t,x)$ and the fluid pressure $p = p(t,x)$:

Theorem 9. *Self-Organized Hydrodynamic- Stokes equations, fully macroscopic system. We state the theorem in dimension 2. Let f_ε be a solution of the rescaled coupled BGK equation (34a)-(34d). Supposing f_ε converges strong enough as $\varepsilon \to 0$. Then $f_\varepsilon \to \rho M_{\kappa,\Omega}$ where $\rho = \rho(t,x)$ describes the density of the particles, and is large enough ($\rho > 2$), since we are in the ordered regime. $\Omega = \Omega(t,x) \in \mathbb{S}$ gives the mean direction of the particles, $v = v(t,x)$ describes the velocity of the fluid and $p = p(t,x)$ is the fluid pressure. These four quantities satisfy the following system of equations:*

$$\begin{cases} \partial_t \rho + \nabla_x \cdot \rho \left(v + \Omega_0 c(\kappa) \right) = 0, \\ \rho(c_0 \kappa(\rho) \partial_t \Omega_0 + \kappa(\rho) c_1 (\Omega_0 \cdot \nabla_x) \Omega_0) + c_0 P_{\Omega_0^\perp} \nabla_x \rho + \rho c_0 \nabla_x \Omega_0 \cdot v, \\ \qquad + \rho \left[\lambda P_{\Omega_0^\perp} S(v) \Omega_0 \left(\kappa c_0 - 6 c_1 - \kappa c_2 \right) - \kappa c_0 P_{\Omega_0^\perp} A(v) \Omega_0 \right], \\ \Delta_x v + \nabla_x p = -b \ c_4 \nabla_x \cdot \left[\rho \left(\Omega_0 \otimes \Omega_0 - \frac{1}{2} Id \right) \right], \\ \nabla_x \cdot v = 0. \end{cases}$$

with

$$\begin{aligned} c_0 &= \int_{-\pi}^{\pi} \sin^2(\theta) M_\theta \ d\theta, \\ c_1 &= \int_{-\pi}^{\pi} \sin^2(\theta) \cos(\theta) M_\theta d\theta, \\ c_2 &= 2 \int_{-\pi}^{\pi} \cos^2 \theta \sin^2 \theta M_\theta d\theta, \\ c_4 &= -c_0 + \int_{-\pi}^{\pi} \cos^2 \theta M_\theta d\theta, \\ M_\theta &= \frac{e^{\kappa(\rho) \cos(\theta)}}{\int_{-\pi}^{\pi} e^{\kappa(\rho) \cos(\theta)} d\theta}. \end{aligned}$$

4.1.4 Diffusion equation for the density in the disordered region [1] & [2]

In this section we attempt to find a diffusion equation for the density ρ of the coupled model. Again, as in section 3.4.4, we consider a region where f^ε converges to a stable, uniform equilibrium, when $\varepsilon \to 0$. The equilibria in the coupled model have stayed the same, because the compatibility condition (5) has not changed, since the right hand side of the kinetic equation (33) is still $\int f M_{J_f} - f$.
We are trying to establish a diffusion model in a disordered region, that means, when we do not have alignment. Therefore we shall consider the uniform distribution - equilibrium, where $f = \rho$ and $J = 0$. Again, there is conservation of mass in the hydrodynamic equation (35):

$$\partial_t \rho^\varepsilon + \nabla_x \cdot \int_{\mathbb{S}} \left(u_{(f^\varepsilon, v^\varepsilon)} f^\varepsilon \right) d\omega + \int_{\mathbb{S}} \nabla_\omega \cdot \left[P_{\omega^\perp} \left[(\lambda S(v^\varepsilon) + A(v^\varepsilon)) \, \omega \right] f^\varepsilon \right] d\omega = 0.$$

For the sake of notational-compactness, we shall write $B = (\lambda S(v^\varepsilon) + A(v^\varepsilon))$ from now on. Doing some further computations and keeping in mind that $J = 0$ in a disordered region, for the last equality, we get:

$$
\begin{aligned}
0 &= \partial_t \rho^\varepsilon + \nabla_x \cdot \int_{\mathbb{S}} \left(u_{(f^\varepsilon, v^\varepsilon)} f^\varepsilon\right) d\omega + \int_{\mathbb{S}} \nabla_\omega \cdot \left[P_{\omega^\perp}\left(B\omega\right) f^\varepsilon\right] d\omega \\
&= \partial_t \rho^\varepsilon + \nabla_x \cdot \int_{\mathbb{S}} v^\varepsilon f^\varepsilon + \omega f^\varepsilon d\omega + \int_{\mathbb{S}} \nabla_\omega \cdot \left[P_{\omega^\perp}\left(B\omega\right) f^\varepsilon\right] d\omega \\
&= \partial_t \rho^\varepsilon + \nabla_x \cdot \left(\int_{\mathbb{S}} v^\varepsilon f^\varepsilon d\omega + \int_{\mathbb{S}} \omega f^\varepsilon d\omega\right) \\
&= \partial_t \rho^\varepsilon + \nabla_x \cdot \left(v^\varepsilon \rho + J^\varepsilon\right) \\
&= \partial_t \rho^\varepsilon + \nabla_x \cdot \left(v^\varepsilon \rho\right).
\end{aligned}
$$

The second integral, $\int_{\mathbb{S}} \nabla_\omega \cdot \left[P_{\omega^\perp}\left[B\omega\right] f^\varepsilon\right] d\omega = 0$, because of the Divergence Theorem:

$$
\begin{aligned}
\partial V &= S, \\
\int_V \nabla \cdot \vec{\boldsymbol{F}}\, dV &= \int_{\mathcal{S}} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\, dS,
\end{aligned}
$$

and because of the fact that the sphere has no boundary. Therefore the integral is zero. So in a disordered region, when $J = 0$, we get:

$$
\partial_t \rho^\varepsilon + \nabla_x \cdot \left(v^\varepsilon \rho\right) = 0.
$$

This result makes sense, since the velocity of the fluid is relevant in diffusing the particles, when the self propelled motion leads to uniform distribution.
As before in section 3.4.4 we expand f^ε in order to obtain $f^\varepsilon = \rho^\varepsilon + \varepsilon f_1^\varepsilon$. This again gives $J_{f^\varepsilon} = \varepsilon J_{f_1^\varepsilon}$ and also $M_{J_f^\varepsilon}(\omega) \approx 1 + \varepsilon\omega \cdot J_{f_1^\varepsilon} + \mathcal{O}(\varepsilon^2)$.
The next step is to substitute $f^\varepsilon = \rho^\varepsilon + \varepsilon f_1^\varepsilon$ and $M_{J_f^\varepsilon}(\omega) = 1 + \varepsilon\omega \cdot J_{f_1^\varepsilon} + \mathcal{O}(\varepsilon^2)$ on the left-hand side of the rescaled equation (34a):

$$
\begin{aligned}
&\partial_t(\rho^\varepsilon + \varepsilon f_1^\varepsilon) + \nabla_x \cdot ((v+\omega)(\rho^\varepsilon + \varepsilon f_1^\varepsilon)) + \nabla_\omega \cdot \left[P_{\omega^\perp}\left(B\omega\right)\left(\rho^\varepsilon + \varepsilon f_1^\varepsilon\right)\right] \\
&= \partial_t \rho^\varepsilon + \omega \cdot \nabla_x \rho^\varepsilon + v \cdot \nabla_x \rho^\varepsilon + P_{\omega^\perp}\left(B\omega\right) \cdot \nabla_\omega \rho^\varepsilon \\
&+ \varepsilon\Big(\partial_t + \omega \cdot \nabla_x + v \cdot \nabla_x + P_{\omega^\perp}\left(B\omega\right) \cdot \nabla_\omega\Big) f_1^\varepsilon,
\end{aligned}
$$

as well as on the right-hand-side:

$$
\frac{1}{\varepsilon}\left(\rho^\varepsilon(1 + \varepsilon\omega \cdot J_{f_1^\varepsilon} + \mathcal{O}(\varepsilon^2)) - (\rho^\varepsilon + \varepsilon f_1^\varepsilon)\right) = \rho^\varepsilon \omega \cdot J_{f_1^\varepsilon} - f_1^\varepsilon + \mathcal{O}(\varepsilon).
$$

it follows:

$$\begin{aligned}&\partial_t\rho^\varepsilon + \omega\cdot\nabla_x\rho^\varepsilon + v\cdot\nabla_x\rho^\varepsilon + P_{\omega^\perp}(B\omega)\cdot\nabla_\omega\rho^\varepsilon\\&+\varepsilon\Big(\partial_t + \omega\cdot\nabla_x + v\cdot\nabla_x + P_{\omega^\perp}(B\omega)\cdot\nabla_\omega\Big)f_1^\varepsilon\\&=\rho^\varepsilon\omega\cdot J_{f_1^\varepsilon} - f_1^\varepsilon + \mathcal{O}(\varepsilon).\end{aligned}$$

Where $P_{\omega^\perp}(B\omega)\cdot\nabla_\omega\rho^\varepsilon = 0$, since ρ^ε does not depend on ω. Now we proceed by multiplying by ω and integrate with respect to ω. We start with the left-hand-side of the equation, the right-hand-side did not change through coupling, so the result after integration will be the same as in section 3.4.4. We keep in mind that ω is an odd function, so the integral $\int_{\mathbb{S}}\omega\ d\omega = 0$, as well as the facts that ρ^ε does *not* depend on ω, whereas f_1^ε *does* depend on it.

$$\begin{aligned}&\partial_t\rho^\varepsilon\int_{\mathbb{S}}\omega\ d\omega + \left(\int_{\mathbb{S}}\omega\otimes\omega\ d\omega\right)\nabla_x\rho^\varepsilon + v\cdot\nabla_x\rho^\varepsilon\ \int_{\mathbb{S}}\omega\ d\omega\\&+\varepsilon\Big(\partial_t\int_{\mathbb{S}}\omega f_1^\varepsilon\ d\omega + \int_{\mathbb{S}}(\omega\otimes\omega)\nabla_x f_1^\varepsilon\ d\omega + v\int_{\mathbb{S}}\nabla_x f_1^\varepsilon\cdot\omega\ d\omega\Big)\\&+\varepsilon\int_{\mathbb{S}}\Big(P_{\omega^\perp}(B\omega)\cdot\nabla_\omega f_1^\varepsilon\Big)\omega\ d\omega\\&=\left(\int_{\mathbb{S}}\omega\otimes\omega\ d\omega\right)\nabla_x\rho^\varepsilon +\\&\varepsilon\Big(\partial_t\int_{\mathbb{S}}\omega f_1^\varepsilon\ d\omega + \int_{\mathbb{S}}(\omega\otimes\omega)\nabla_x f_1^\varepsilon\ d\omega + v\int_{\mathbb{S}}\nabla_x f_1^\varepsilon\cdot\omega\ d\omega + \int_{\mathbb{S}}\Big(P_{\omega^\perp}(B\omega)\cdot\nabla_\omega f_1^\varepsilon\Big)\omega\ d\omega\Big).\end{aligned}$$

As stated before, the right hand side of the equation has not changed so it is the same as in section 3.4.4:

$$\rho^\varepsilon\int_{\mathbb{S}}\omega\left(\omega\cdot J_{f_1^\varepsilon}\right)\ d\omega - \int_{\mathbb{S}}\omega f_1^\varepsilon\ d\omega + \mathcal{O}(\varepsilon).$$

This gives:

$$\begin{aligned}&\left(\int_{\mathbb{S}}\omega\otimes\omega\ d\omega\right)\nabla_x\rho^\varepsilon\\&+\varepsilon\Big(\partial_t\int_{\mathbb{S}}\omega f_1^\varepsilon\ d\omega + \int_{\mathbb{S}}(\omega\otimes\omega)\nabla_x f_1^\varepsilon\ d\omega + v\int_{\mathbb{S}}\nabla_x f_1^\varepsilon\cdot\omega\ d\omega + \int_{\mathbb{S}}\Big(P_{\omega^\perp}(B\omega)\cdot\nabla_\omega f_1^\varepsilon\Big)\omega\ d\omega\Big)\\&=\rho^\varepsilon\int_{\mathbb{S}}\omega\left(\omega\cdot J_{f_1^\varepsilon}\right)\ d\omega - \int_{\mathbb{S}}\omega f_1^\varepsilon\ d\omega\ + \mathcal{O}(\varepsilon),\end{aligned}$$

where the whole expression $\varepsilon\left(\partial_t \int_{\mathbb{S}} \omega f_1^\varepsilon \, d\omega + \cdots\right) \in \mathcal{O}(\varepsilon)$ and $\int_{\mathbb{S}} \omega f_1^\varepsilon \, d\omega = J_{f_1^\varepsilon}$.

Altogether this leads to following equation for $J_{f_1^\varepsilon}$ (as before):

$$J_{f_1^\varepsilon} = \rho^\varepsilon \int_{\mathbb{S}} \omega \left(\omega \cdot J_{f_1^\varepsilon}\right) \, d\omega - \left(\int_{\mathbb{S}} \omega \otimes \omega \, d\omega\right) \nabla_x \rho^\varepsilon + \mathcal{O}(\varepsilon). \tag{40}$$

We can now follow the exact same steps as in section 3.4.4 right after equation (29) to solve the integrals and obtain a more compact equation for $J_{f_1^\varepsilon}$, just like equation (30):

$$J_{f_1^\varepsilon} = -\frac{1}{2}\left(\frac{\nabla_x \rho^\varepsilon}{1 - \frac{\rho}{2}}\right) + \mathcal{O}(\varepsilon). \tag{41}$$

Recalling the rescaled equation for the density ρ (see (35)), we can now plug in $\int \omega f^\varepsilon \, d\omega = J_{f^\varepsilon} = \varepsilon J_{f_1^\varepsilon}$ (see (41)). This leads to:

$$\partial_t \rho^\varepsilon - \frac{\varepsilon}{2} \nabla_x \cdot \left(\frac{\nabla_x \rho^\varepsilon}{1 - \frac{\rho}{2}}\right) + \nabla_x \cdot v\rho + \mathcal{O}(\varepsilon^2). \tag{42}$$

Now the following diffusion equation describes the the behavior of the collective motion of the particles for a below-threshold density:

$$\partial_t \rho^\varepsilon = \nabla_x \cdot \left(\frac{\varepsilon \nabla_x \rho^\varepsilon}{2 - \rho^\varepsilon} - v\rho\right). \tag{43}$$

We shall summarize the findings of this section in one last theorem, following the steps in [11] and [3]:

Theorem 10. *Diffusion Equation.*
In a disordered region, the density ρ^ε satisfies formally at first order the following diffusion equation:

$$\partial_t \rho^\varepsilon = \nabla_x \cdot \left(\frac{\varepsilon \nabla_x \rho^\varepsilon}{2 - \rho^\varepsilon} - v\rho\right). \tag{44}$$

5 Numerical Simulation

The numerical simulations of the BGK-Vicsek model have been done using the Julia programming language. The objective of the simulation was, to notice phase transitions and observe clusters being formed. Deep numerical exploration such as parameter exploration is beyond the scope of this thesis and has therefore not been done. The idea is, to simulate N particles moving at a constant speed v_0 with a certain orientation θ. Initially, all particles are uniformly distributed on the domain, which is chosen to be $5x1$, and the orientations are uniformly distributed as well. The simulation is run for a chosen number of time-steps, in this case 1000 time-steps. At each of these times, the orientation changes and the particles choose a new orientation according to the Vicsek model. More precisely, the rule for changing orientation follows a von Mises distribution:

A particle changes its orientation at each time-step according to a von Mises distribution with parameters μ and κ, where μ is the mean and a measure of location and κ is a measure of concentration. This means, when $\kappa = 0$ the distribution is uniform, and for large κ the distribution becomes very concentrated about the mean angle μ.
The two parameters are determined as follows:
First of all, only particles within a certain chosen interaction radius R_0 of a fixed particle affect that particle's orientation. Therefore the x value of the flux J_i within that radius is the sum over $\cos(\theta)$ of all orientations θ of particles within R_0. The y value of J_i is the sum over all $\sin(\theta)$ of orientations θ within R_0. Wen already know that κ is the norm of the flux J_i therefore the concentration parameter $\kappa = \sqrt{x_{J_i}^2 + y_{J_i}^2}$. The mean angle μ is the angle corresponding to the point $(\frac{x_{J_i}}{\kappa}, \frac{y_{J_i}}{\kappa})$. This is implemented in the code by using the *atan* function, which returns a value in $(-\pi, \pi]$.

So in conclusion, at each time-step a particle picks a new orientation according to von Mises(μ, κ) which is only influenced by particles close enough to it.

5.1 Results of the Simulation

Results of the simulation with a domain of $5 \cdot 1$ over 1000 time steps with different chosen parameters, as indicated below. The images created by the simulation show the particles as red dots and their respective orientation as a blue triangle.

- Simulation 1:
  Number of particles: $N = 1000$
  Interaction radius: $R_0 = 0.05$
  Velocity: $v_0 = 0.04$
  Overall density: $\frac{1000}{5} = 200$
  Average number of particles within the interaction neighborhood: $\frac{1000 \cdot R_0^2 \pi}{5} = 200 \cdot 0.0025\pi \approx 1.5708$.

Abbildung 4: Simulation 1. Time-Step: < 50

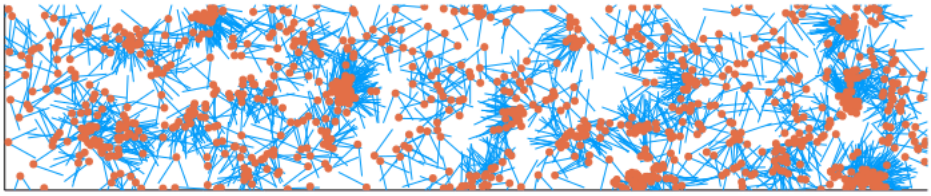


Abbildung 5: Simulation 1. Time-Step: ~200

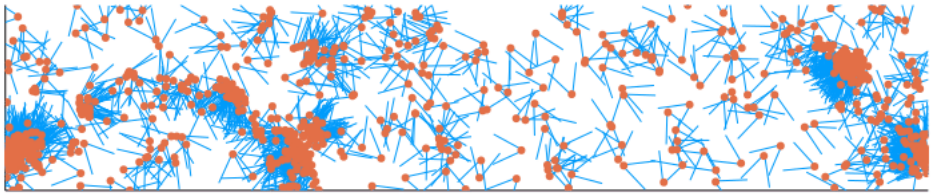


Abbildung 6: Simulation 1. Time-Step: ~400

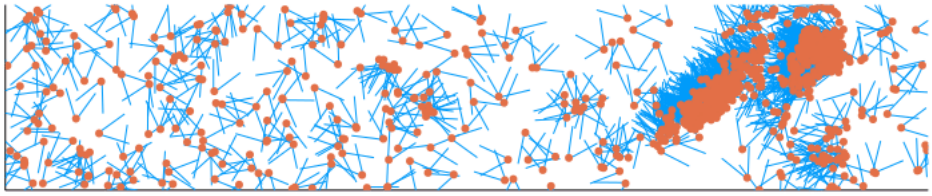


Abbildung 7: Simulation 1. Time-Step: ~800

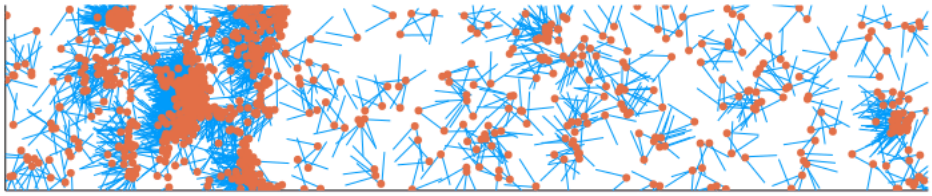


Abbildung 8: Simulation 2. Time-Step: ~1000

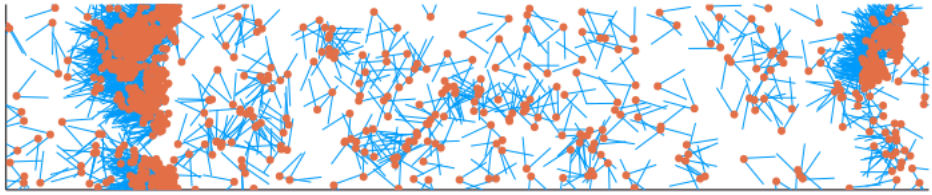


Abbildung 9: Simulation 1. Collective motion according to the Vicsek model with chosen parameters, some bigger clusters form, but many particles still move randomly.

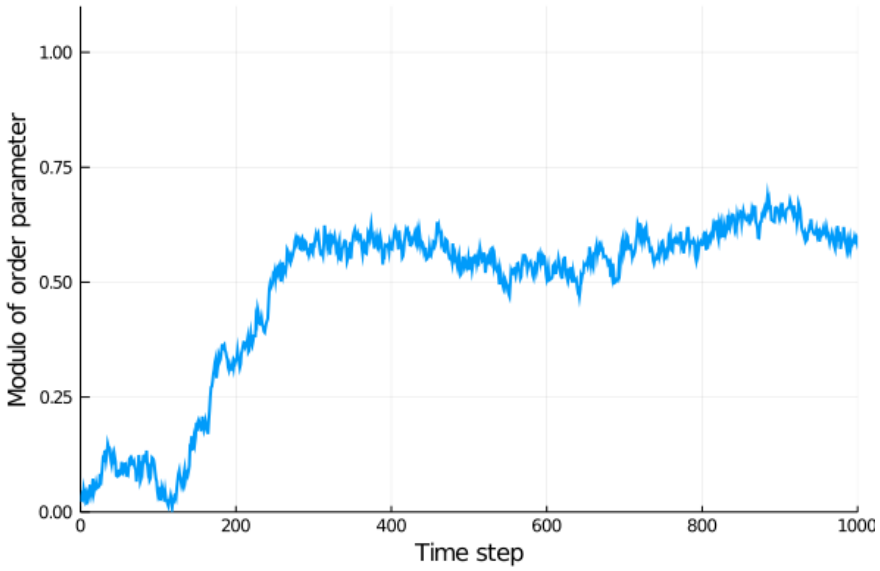


Abbildung 10: Simulation 1. Evolution of alignment of the particles

- Simulation 2:
  Number of particles: $N = 500$
  Interaction radius: $R_0 = 0.1$
  Velocity: $v_0 = 0.05$
  Overall density: $\frac{500}{5} = 100$
  Average number of particles within the interaction neighborhood: $\frac{500 \cdot R_0^2 \pi}{5} = 100 \cdot 0.01\pi = \pi \approx 3.14159$.

Abbildung 11: Simulation 2. Time-Step: << 50

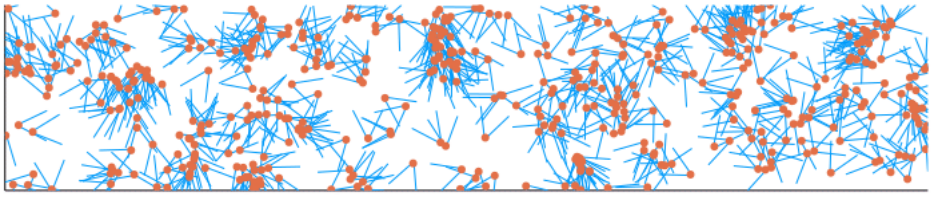


Abbildung 12: Simulation 2. Time-Step: < 50

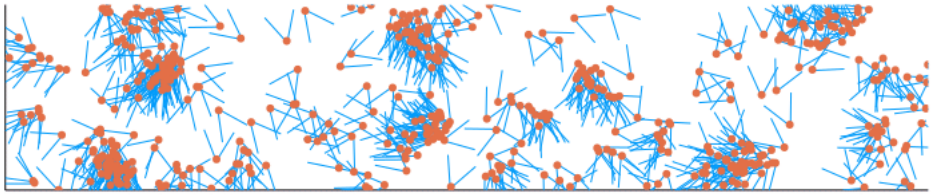


Abbildung 13: Simulation 2. Time-Step: ~300

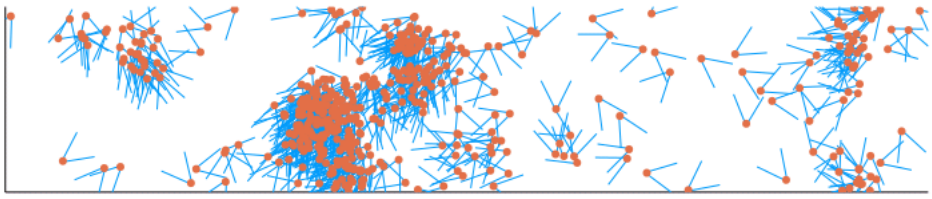


Abbildung 14: Simulation 2. Time-Step: ~500

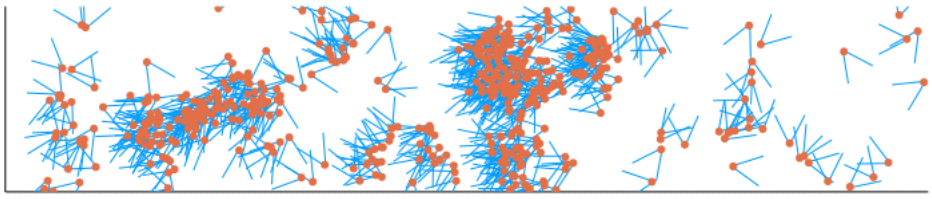


Abbildung 15: Simulation 2. Time-Step: ~700

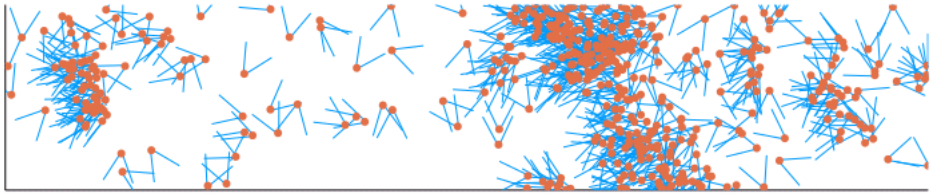


Abbildung 16: Simulation 2. Time-Step: ~1000.

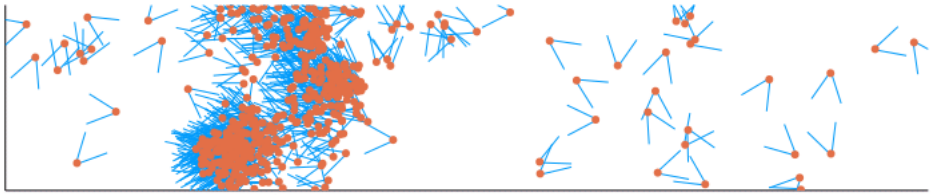


Abbildung 17: Collective motion according to the Vicsek model with chosen parameters, almost all particles flock together in big clusters.

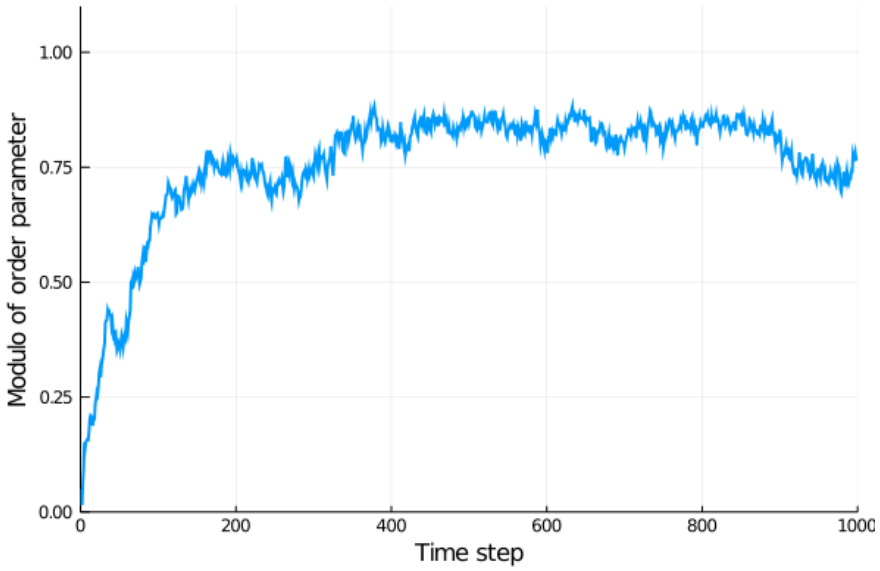


Abbildung 18: Simulation 2. Evolution of alignment of the particles

- Simulation 3:
  Number of particles: $N = 300$
  Interaction radius: $R_0 = 0.1$
  Velocity: $v_0 = 0.05$
  Overall density: $\frac{300}{5} = 60$
  Average number of particles within the interaction neighborhood: $\frac{300 \cdot R_0^2 \pi}{5} = 60 \cdot 0.001\pi \approx 1.88496$.

Abbildung 19: Simulation 3. Time-Step: < 50

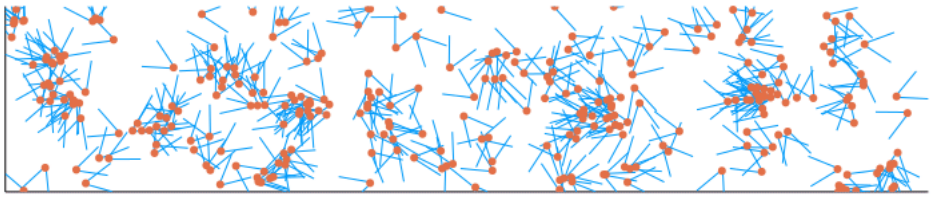


Abbildung 20: Simulation 3. Time-Step: ~300

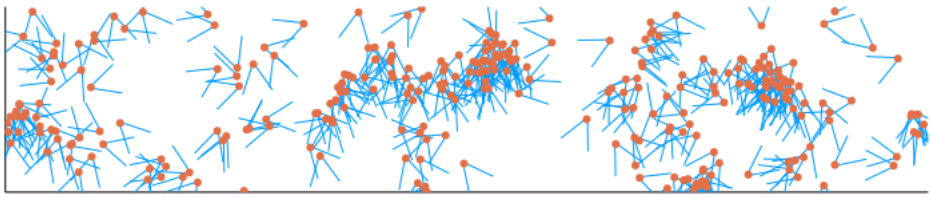


Abbildung 21: Simulation 3. Time-Step: ~600

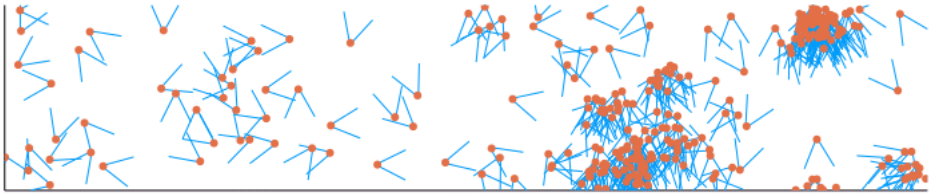


Abbildung 22: Simulation 3. Time-Step: ~700

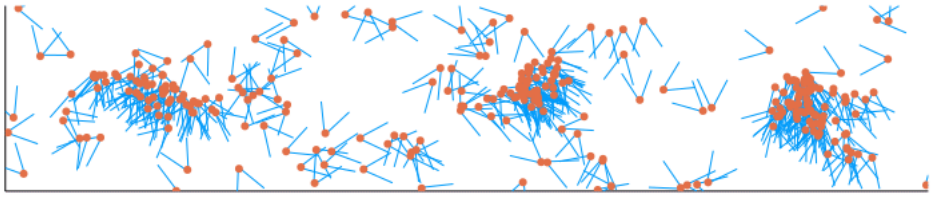


Abbildung 23: Simulation 3. Time-Step: ~1000

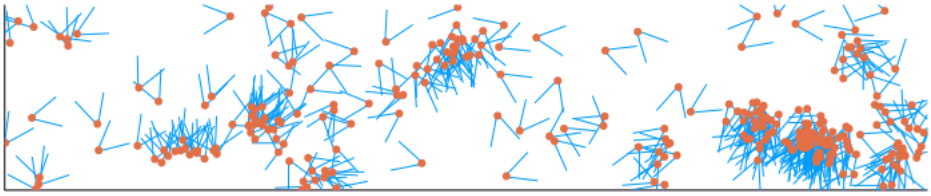


Abbildung 24: Simulation 3. Collective motion according to the Vicsek model with chosen parameters, some small clusters form over time.

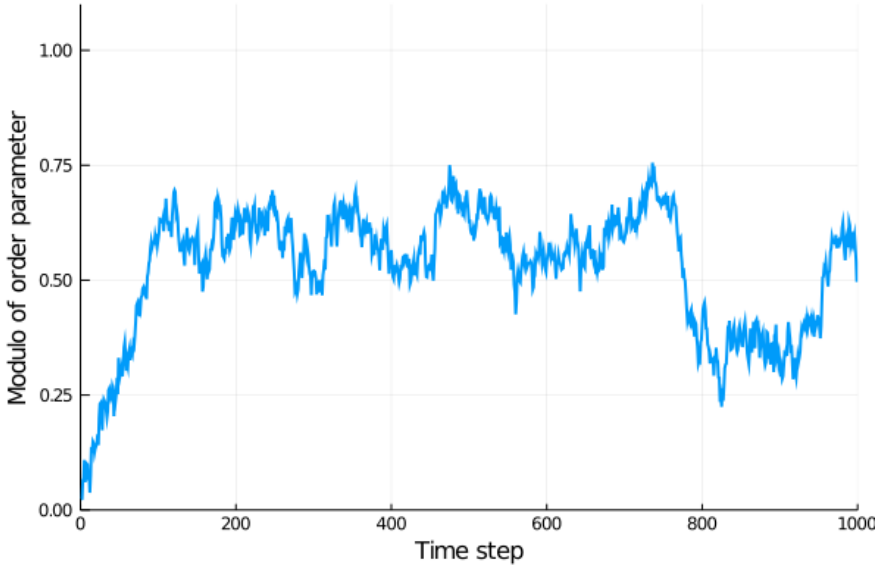


Abbildung 25: Simulation 3. Evolution of alignment of the particles

- Simulation 4:
  Number of particles: $N = 150$
  Interaction radius: $R_0 = 0.1$
  Velocity: $v_0 = 0.05$
  Overall density: $\frac{150}{5} = 30$
  Average number of particles within the interaction neighborhood: $\frac{150 \cdot R_0^2 \pi}{5} = 30 \cdot 0.001\pi \approx 0.94248$.

Abbildung 26: Simulation 4. Time-Step: < 50

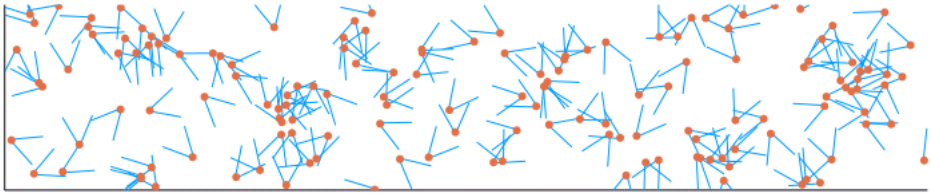


Abbildung 27: Simulation 4. Time-Step: ~200

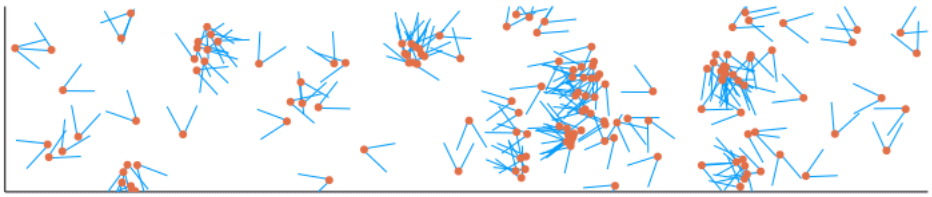


Abbildung 28: Simulation 4. Time-Step: ~500

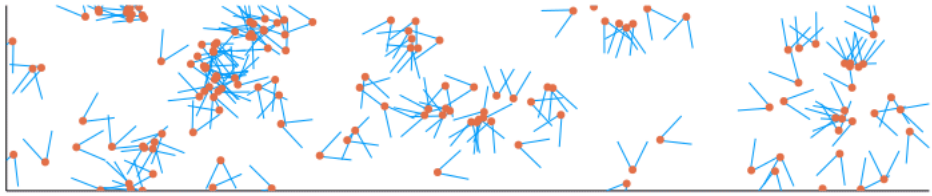


Abbildung 29: Simulation 4. Time-Step: ~800

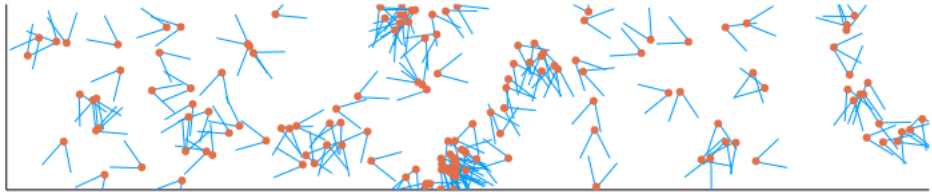


Abbildung 30: Simulation 4. Time-Step: ~1000

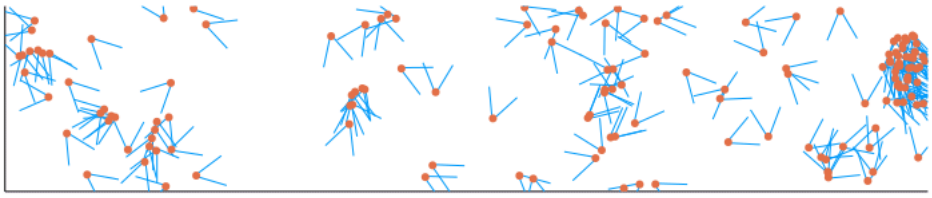


Abbildung 31: Simulation 4. Collective motion according to the Vicsek model with chosen parameters, no real clusters are visible.

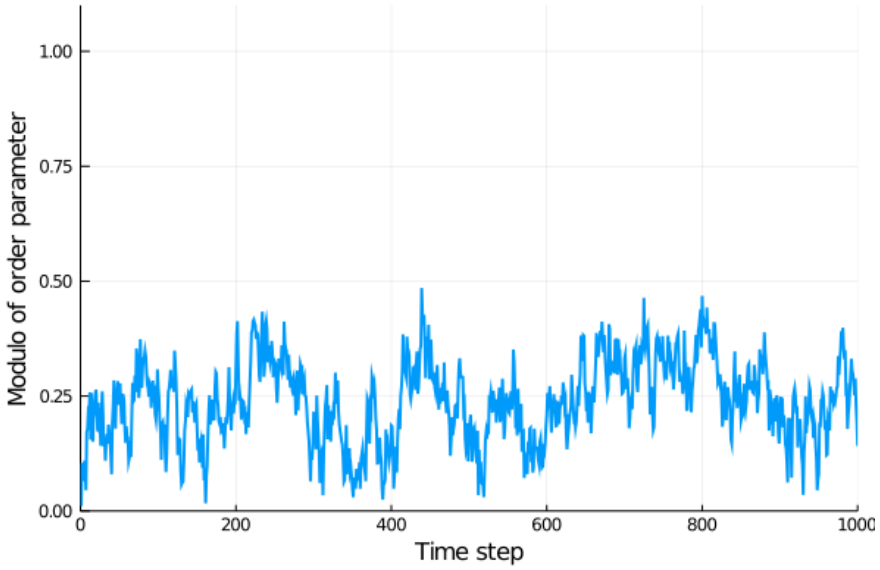


Abbildung 32: Simulation 4. Evolution of alignment of the particles

These simulations show different cases. In some, the density is just below the critical density threshold of $\rho = 2$. In simulation 1, 3 and 4 the density per interaction neighborhood is below the critical value of 2. It is immediately visible that depending on how much below the critical value the density is, the less alignment occurs. In figure 10 we have an average number of 1.57 particles per neighborhood, therefore the alignment parameter stays well below 0.75. In figure 25 the density is bigger, but still below 2 with an average of 1.88 particles per region. In that case alignment also stays below 0.75. Figure 32 shows almost no alignment with the alignment parameter below 0.5, since the density is well below 2 with only an average of 0.94 particles per neighborhood.
Only in simulation 2 the density is well above the critical value of $\rho = 2$, with an average of π particles within each interaction region. We can see in figures 11, 12, 13, 14, 15 and 16 that in this scenario alignment occurs almost immediately, with first clusters appearing even below 50 time-steps. Figure 18 shows, that the order parameter jumps to a value above 0.75 fairly quickly, after about 200 time-steps, and also stays in that area for the rest of time, which indicates strong alignment.

The figures showing the modulo of the order parameter over time, figures 10, 18, 25 and 32, display how much the orientations of individual particles align. The order parameter, φ, is defined as the sum of all $\cos(\theta)$ in the x component and the sum of all $\sin(\theta)$ in the y component. Then this vector is divided by N and eventually the norm is taken: $\varphi = \sqrt{(x^2 + y^2)}$, with $x = \frac{1}{N}\sum_{i=1}^{N}\cos(\theta_i)$ and $y = \frac{1}{N}\sum_{i=1}^{N}\sin(\theta_i)$.
It is clear that when all particles have the same orientation angle, the sum of their respective sin and $\cos(\theta)$ values is simply $N \cdot \sin(\theta)$ and $N \cdot \cos$ and therefore after dividing by N and taking the norm $\varphi = 1$. So therefore this order parameter gets closer to one, as the particles become more aligned and stays well below one, if there is no significant alignment.

6 Conclusions

In this thesis the non-normalized Vicsek-BGK Model was explored, which has not previously been done, as can be seen in Figure 1 in the introduction as well as in Figure 1 in [3].
The starting point was at the mesoscopic or kinetic level with the non-normalized, spatially homogeneous Vicsek-BGK equation. Then the equilibria of the equation were deduced and their respective stability was analyzed. This led to the emergence of phase transitions in collective behavior, triggered when a certain density threshold is crossed. The type of phase transitions occurring in this model are called ßpontaneous symmetry-breaking phase transitions". This class of phase transitions also includes many appearing in physical systems, for example ferromagnetism, liquid crystals or polymers. The gradient-flow structure of the BGK equation combined with results from other papers gives insights into the asymptotic behavior of the model
Then the macroscopic model was derived through rescaling and two regimes were identified. In the ordered region, the SOH (Self-Organized-Hydrodynamics) model for self alignment was derived. In the disordered region, the behavior of the particles is described by a diffusion equation. Next, the BGK equation was coupled with the Stokes equation for fluids to investigate the behavior of self propelled particles in a suspension. This coupling takes into account the interrelated effects between the agents and the fluid. The SOH model was derived for the coupled system as well and it follows along the lines of the SOH model derived in the first part of this thesis. However the velocity of the fluid has an influence on both the model for the ordered region as well as on the diffusion equation describing the behavior in the disordered regime.
The Vicsek-BGK model studied in this thesis is a step to complete the tree describing the models of collective behaviour, see figure 1. But it also adds to the type of collective behavior models that have been coupled with the Stokes equation. Previously only the Vicsek Fokker-Planck model has been coupled [2]. Hopefully, the tools presented in this thesis can be useful for the analysis of other collective motion models coupled with the Stokes equation.

A Appendix A

A.1 Proof that $h(\kappa)$ is strictly decreasing

Beweis. $h(\kappa) = \frac{c(\kappa)}{\kappa}$ is a strictly decreasing function.

The idea of the proof is to show that any critical point of $h(\kappa)$ has to be a maximum and that this maximum is attained at zero. Therefore for values $\kappa > 0$ the function must be at lower values.
First, we compute the first and second derivative of $h(\kappa)$:

$$
\begin{aligned}
h'(\kappa) &= \frac{c'(\kappa)\kappa - c(\kappa)}{\kappa^2} = \frac{1}{\kappa}\left(c'(\kappa) - h(\kappa)\right), \\
c(\kappa) &= \frac{\int_{-\pi}^{\pi} \cos\theta e^{\kappa\cos\theta} d\theta}{\int_{-\pi}^{\pi} e^{\kappa\cos\theta} d\theta}, \\
c'(\kappa) &= \frac{\int_{-\pi}^{\pi} \cos^2\theta e^{\kappa\cos\theta} d\theta \ \int_{-\pi}^{\pi} e^{\kappa\cos\theta} d\theta - \int_{-\pi}^{\pi} \cos\theta e^{\kappa\cos\theta} d\theta \ \int_{-\pi}^{\pi} \cos\theta e^{\kappa\cos\theta} d\theta}{\left(\int_{-\pi}^{\pi} e^{\kappa\cos\theta} d\theta\right)^2} \\
&= \frac{\int_{-\pi}^{\pi} \cos^2\theta e^{\kappa\cos\theta} d\theta \ \int_{-\pi}^{\pi} e^{\kappa\cos\theta} d\theta - \left(\int_{-\pi}^{\pi} \cos\theta e^{\kappa\cos\theta} d\theta\right)^2}{\left(\int_{-\pi}^{\pi} e^{\kappa\cos\theta} d\theta\right)^2} \\
&= \frac{\int_{-\pi}^{\pi} \cos^2\theta e^{\kappa\cos\theta} d\theta}{\int_{-\pi}^{\pi} e^{\kappa\cos\theta} d\theta} - c^2(\kappa).
\end{aligned}
$$

Now we express $\int_{-\pi}^{\pi} \cos^2\theta e^{\kappa\cos\theta} d\theta$ as $\int_{-\pi}^{\pi} (1-\sin^2\theta) e^{\kappa\cos\theta} d\theta = \int_{-\pi}^{\pi} e^{\kappa\cos\theta} d\theta - \int_{-\pi}^{\pi} \sin^2\theta e^{\kappa\cos\theta} d\theta$ and we call $\int_{-\pi}^{\pi} \sin^2\theta e^{\kappa\cos\theta} d\theta$ I and integrate by parts with $\sin\theta = u$ and $\left(\sin\theta e^{\kappa\cos\theta}\right) = v'$:

$$
\begin{aligned}
I &= \int_{-\pi}^{\pi} \sin\theta \left(\sin\theta e^{\kappa\cos\theta}\right) d\theta = \sin\theta \left(-\frac{e^{\kappa\cos\theta}}{\kappa}\right)\Bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (\cos\theta) \left(-\frac{e^{\kappa\cos\theta}}{\kappa}\right) d\theta \\
&= \int_{-\pi}^{\pi} \cos\theta \frac{e^{\kappa\cos\theta}}{\kappa} d\theta.
\end{aligned}
$$

Now our new expression for $c'(\kappa)$ is:

$$
c'(\kappa) = \frac{\int_{-\pi}^{\pi} \cos^2\theta e^{\kappa\cos\theta} d\theta}{\int_{-\pi}^{\pi} e^{\kappa\cos\theta} d\theta} - c^2(\kappa)
$$

$$
\begin{aligned}
&= \frac{\int_{-\pi}^{\pi} e^{\kappa\cos\theta}d\theta - \int_{-\pi}^{\pi}\sin^2\theta e^{\kappa\cos\theta}d\theta}{\int_{-\pi}^{\pi} e^{\kappa\cos\theta}d\theta} - c^2(\kappa) \\
&= 1 - \frac{\int_{-\pi}^{\pi}\sin^2\theta e^{\kappa\cos\theta}d\theta}{\int_{-\pi}^{\pi} e^{\kappa\cos\theta}d\theta} - c^2(\kappa) \\
&= 1 - \frac{1}{\kappa}\frac{\int_{-\pi}^{\pi}\cos\theta e^{\kappa\cos\theta}d\theta}{\int_{-\pi}^{\pi} e^{\kappa\cos\theta}d\theta} - c^2(\kappa) \\
&= 1 - \frac{c(\kappa)}{\kappa} - c^2(\kappa) = 1 - h(\kappa) - c^2(\kappa).
\end{aligned}
$$

Putting all of this together, we finally get the first derivative of $h(\kappa)$:

$$
\begin{aligned}
h'(\kappa) &= \frac{1}{\kappa}\left(c'(\kappa) - h(\kappa)\right) = \frac{1}{\kappa}\left(1 - h(\kappa) - c^2(\kappa) - h(\kappa)\right) \\
&= \frac{1}{\kappa}\left(1 - 2h(\kappa) - c^2(\kappa)\right) = -\frac{\beta}{\kappa},
\end{aligned}
$$

with $\beta = 2h(\kappa) + c^2(\kappa) - 1$.

It follows:

$$
h'(\kappa) = \frac{1}{\kappa}\left(c'(\kappa) - h(\kappa)\right) = -\frac{\beta}{\kappa},
$$

and therefore

$$
c'(\kappa) = h(\kappa) - \beta.
$$

Now we have to compute the second derivative of $h(\kappa)$, $h''(\kappa)$:

$$
\begin{aligned}
h''(\kappa) &= \frac{1}{\kappa^2}\Big(\left[-2h'(\kappa) - 2c(\kappa)c'(\kappa)\right]\kappa - \left[1 - 2h(\kappa) - c^2(\kappa)\right]\Big) \\
&= \frac{1}{\kappa}\left[-2h'(\kappa) - 2c(\kappa)c'(\kappa)\right] - \frac{1}{\kappa^2}\left[1 - 2h(\kappa) - c^2(\kappa)\right] \\
&= 2\frac{\beta}{\kappa^2} - 2\frac{c(\kappa)}{\kappa}c'(\kappa) + \frac{\beta}{\kappa^2} \\
&= 3\frac{\beta}{\kappa^2} - 2h(\kappa)c'(\kappa) \\
&= 3\frac{\beta}{\kappa^2} - 2h(\kappa)\left(h(\kappa) - \beta\right).
\end{aligned}
$$

Now we can make our argument. From the first derivative, $h'(\kappa) = -\frac{\beta}{\kappa}$ it is clear that at any critical point, $\beta = 0$. This in turn means that the second derivative $h''(\kappa) =$

$3\frac{\beta}{\kappa^2} - 2h(\kappa)\,(h(\kappa) - \beta)$ at any critical point is negative: $h''(\kappa)|_{\beta=0} = -h'(\kappa)^2 < 0$
Putting together everything we know about $h(\kappa)$ we see that the function has a maximum at $\kappa = 0$ and negative curvature which lets us conclude that $h(\kappa)$ is strictly decreasing for $\kappa > 0$.

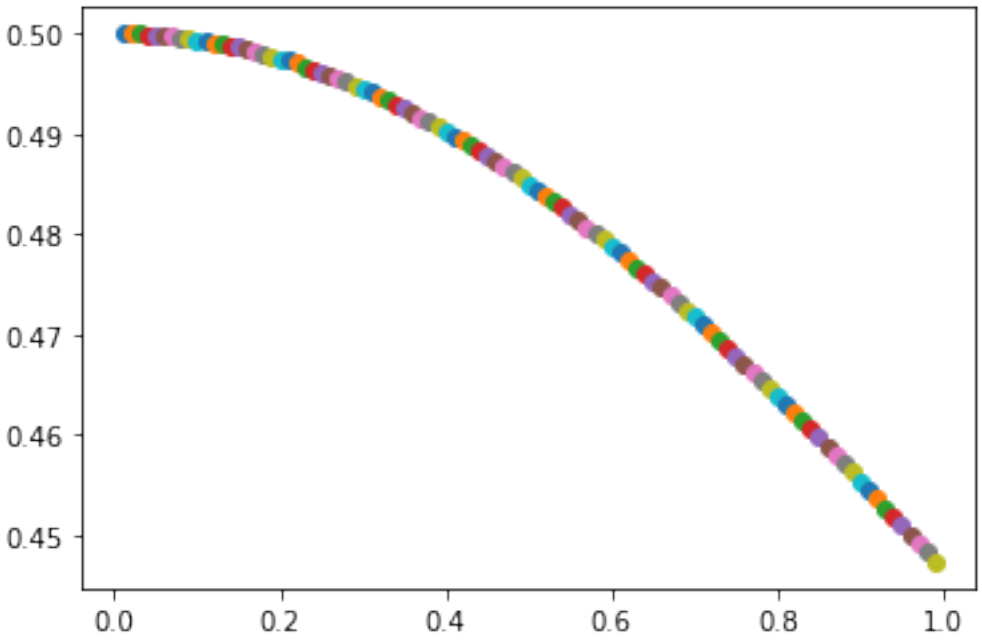
□

Alternatively there is a proof that for $\kappa > 0$ also $\beta > 0$ in [16] on page 17.

A.2 Numerical proof that $h(\kappa)$ is strictly decreasing

A quick Python plot shows that $h(\kappa)$ is decreasing between 0 and 1.

```
for j in range(1,len(kappa)):
    k= j/100
    #print(j,k)
    plt.plot(k,(kappa[j]/k), 'o')
```



B Appendix B

B.1 Numerical simulation

The julia-simulation shows the Vicsek-BGK model in two dimensions. Initially, all the particles are uniformly distributed within the domain and with randomly and uniformly

assigned orientations. The boundary conditions are periodic, meaning, the particles cannot leave the domain. The parameters such as number of particles, number of time steps, velocity, the weight of the noise and the size of the interaction radius can be chosen for each iteration of the code. Whenever particle i is within particle j's interaction radius, particle j adopts its orientation up to some noise. The noise is determined by a von Mises distribution with parameter κ, the absolute value of the flux J at the given position.

B.1.1 Code

See [22]. Changes made to the original code mostly apply to the function defining the noise, where I used a von Mises sample instead of white noise, as well as to the function defining how the new orientation is defined.

```
using Pkg
using Random, Distributions
 Random.seed!(123) # Setting the seed

 #=
Vicsek model in 2-D periodic region
=#


"""
Module for parameters and variables
"""
module  mod_param_var
    struct Parameters
        N::Int64  # Number of particles
        R_0::Float64  # Neighbour region threshold
        ::Float64  # Coefficient of white noise
        t_step::Int64 # Total iteration steps
        v0::Float64  # Velocity of particle (same for all particles)
    end

    mutable struct Variables
        itr::Int64    # Number of iteration
        r::Array{Float64, 2}    # Position of particles
        ::Array{Float64, 1}    # Angle of particles
        n_label::Array{Bool, 2} # Neighbourhood label
        ::Array{Float64, 1}    # Neighbour alignment
        ::Array{Float64, 1}    # White noise for perturbation
        #::Float64     # von Mises parameter
        r_new::Array{Float64, 2}    # New position of particles
        _new::Array{Float64, 1}    # New angle of particles
        n_sum::Array{Int64, 1}      # Number of nearby particles
    end
```

```
34
35      mutable struct StatisticalValues
36          ::Array{Float64, 2}  # Polar order parameter
37          _::Array{Float64, 1}  # Polar order parameter
38      end
39  end  # module mod_param_var
40
41
42  """
43  Module for Vicsek model time integration
44  """
45  module mod_vicsek_model
46  using Distributions
47      """
48      Initialise position of particles with [0,100][0,100] random number.
49      """
50      function set_initial_condition(param,var)
51          var.r = rand(Float64, param.N, 2) #!!! #generates N * 2 random numbers
52  ###### Totally uniform in space
53          var.r = var.r[:,:].*5
54
55  ###### Possible set of initial conditions
56  #         for i=1:param.N
57  #             var.r[i,1]=var.r[i,1]*1.5
58  #         end
59  #         for i=1:100
60  #             var.r[i,1]=var.r[i,1]+2
61  #         end
62          _ = rand(Float64, param.N)#zeros(param.N)  generates N random numbers
            ↪  <1
63          var. = 2*_ .-   # [0,1] -> [-,], maps _ to random angles in [-,], .-
            ↪  is a "vectorial" minus, orientation of the ind.particles
64      end
65
66      """
67      Calculate distance between two particles i and j,
68      if distance is below threshold R_0, n_label (neighbour_label) is true.
69      n_label of myself(tr(n_label)) is 1
70      Also, calculate number of neighbour particles and store in var.n_sum
71      """
72      function set_neighbour_list(param,var)
73
74          for i=1:param.N # iterate through particles
75              n_col = 0  # Number of colums of n_label[i,:]
76              for j=1:param.N # iterate through particles
77                  # Calculate distance between two particles
78                  dx = abs(var.r[i,1] - var.r[j,1])
79                  dx = min(dx, 5 - dx) #!!!!
80                  dy = abs(var.r[i,2] - var.r[j,2])
```

```
81                  dy = min(dy, 1 - dy) #!!!!
82                  dist = hypot(dx, dy)
83                  if dist <= param.R_0
84                      var.n_label[i,j] = true
85                      n_col += 1
86                  else
87                      var.n_label[i,j] = false
88                  end
89              end
90              var.n_sum[i] = n_col #counting "nearby neighbors"
91          end
92          # println("num. of neighbour=", n_sum.-1)  # exept myself
93      end
94
95      """
96      Calculate one particle's neighbour orientation
97       will have [-,] value defined as
98           = Arg[_j n_ij _j].
99      """
100     function set_neighbour_orientation(param,var)
101         # var. = zeros(param.N)
102         #=
103         this process uses N ^ 2
104         split into R_0 x R_0 squares, only look in neighboring
    ↪ squares
105         MPI
106         =#
107         for i=1:param.N # iterate through particles
108             tmpx = tmpy = 0.0
109             for j=1:param.N # iterate through particles  #!!!!! It should sum
                ↪ over neighbours only
110                 if var.n_label[i,j] == true  #!!!!! It should sum over
                    ↪ neighbours only --- line added
111                     tmpx += var.n_label[i,j] * cos(var.[j]) #if particle
                        ↪ nearby: x-orientation changes according to cos(),else 0
112                     tmpy += var.n_label[i,j] * sin(var.[j]) #if particle
                        ↪ nearby: y-orientation changes according to sin(),else 0
113                     end
114             end
115             avg_tmpx = tmpx #/param.N
116             avg_tmpy = tmpy  #/param.N #(define the flux)
117             J_i =(avg_tmpx, avg_tmpy)
118              = 0.0
119             square = avg_tmpx * avg_tmpx + avg_tmpy * avg_tmpy
120              = sqrt(square)
121             if iszero() == false
122                  = atan(avg_tmpy/,avg_tmpx/)
123             end
124             #println("modulus |J| = ", )
```

```
125             #println("angle = ", var.[i])
126             #println("mean angle = ", )
127             vm = VonMises(,)
128             var.[i] = rand(vm)
129         end
130     end
131
132     """
133     Calculate white noise array .
134     """
135     function set_white_noise(param,var)
136         _ = rand(Float64, param.N) # array with N random numbers <1
137         var. = 2 .* _  .-  # [0,1] -> [-,]
138     end
139
140     """
141     Calculate  at time t+t: _new
142         _new =  + .
143         _new is half old orientation, half orientation of neighbor. can be
    ↪   changed arbitrarily:
144         _new = x*+ (1-x)* + , x  [0,1]
145         can also be only new orientation
146         van mises sample instead of noise
147     """
148     function set_new_(param,var)
149         for i=1:param.N
150             var._new[i] = var.[i] # ensure _new  [-,],  new orientation
151         end
152     end
153
154     """
155     Calculate r at time t+t: r_new
156         r_new = (x_new,y_new), r = (x,y)
157         x_new = x + t*v_0*cos(_new)
158         y_new = y + t*v_0*sin(_new)
159         t = v_0 = 1
160     """
161     function set_new_r(param,var) #r=position of particles
162         for i=1:param.N
163             var.r_new[i,1] = var.r[i,1] + param.v0*cos(var._new[i]) #
                ↪  v0=velocity of particle
164             var.r_new[i,2] = var.r[i,2] + param.v0*sin(var._new[i]) # old
                ↪  position+velocity*orintation
165         #!!!!!!! here dt =1
166         end
167     end
168
169     """
170     Ensure periodic boundary condition
```

```
171     """
172     function set_periodic_bc(param,var)
173         for i=1:param.N
174                 if var.r_new[i,1] > 5 #!!!!
175                     var.r_new[i,1] -= 5 #!!!!
176                 elseif var.r_new[i,1] < 0
177                     var.r_new[i,1] += 5 #!!!!
178                 end
179                 if var.r_new[i,2] > 1 #!!!!
180                     var.r_new[i,2] -= 1 #!!!!
181                 elseif var.r_new[i,2] < 0
182                     var.r_new[i,2] += 1 #!!!!
183                 end
184         end
185     end
186 
187     """
188     Update r &
189     """
190     function set_new_r(param,var)
191         var.r = var.r_new
192         var. = var._new
193     end
194 end  # module mod_vicsek_model
195 
196 
197 """
198 Module for analysing Vicsek model
199 """
200 module mod_analysis
201     """
202     Calculate direction parameter
203          = 1/N _{i=1}^N s_i^t
204         s_i^t is a unit direction vector, implemented as (cos,sin)
205     """
206     function calc_(param,)
207         tmp_c = 0.0
208         tmp_s = 0.0
209         for i=1:param.N
210             tmp_c += cos([i]) # sums cos vals of all particles
211             tmp_s += sin([i]) # sums sin vals of all particles
212         end
213         tmp_c = tmp_c/param.N #normalize
214         tmp_s = tmp_s/param.N
215         tmp = sqrt(tmp_c^2 + tmp_s^2) #when all particles same orientation tmp=
            ↪  sqrt(cos^2+sin^2)=1
216         return [tmp_c, tmp_s], tmp
217     end
218 end  # module mod_analysis
```

```
219
220
221 """
222 Module for dat, image and movie generation
223 """
224 module mod_output
225     using Plots
226     """
227     Output snapshot image of particle distribution and direction
228     """
229     function plot_scatter_(param,var,stat,flag_out)
230         u = Array{Float64}(undef, param.N) #uninitialized array, no set values
231         v = Array{Float64}(undef, param.N)
232         for i=1:param.N
233             u[i] = 0.01 * cos(var.[i]) #initializing?
234             v[i] = 0.01 * sin(var.[i])
235         end
236         p1 = quiver(  # Vector field
237            var.r[:,1], var.r[:,2],
238            quiver=(u[:], v[:]),
239             aspect_ratio = 1,
240             xlims = (0.0, 5), #!!!!
241             ylims = (0.0, 1), #!!!!
242             xaxis=nothing,
243             yaxis=nothing,
244             color=1
245         )
246         p1 = scatter!(  # Position of particles
247             var.r[:,1],var.r[:,2],
248             markerstrokewidth = 0,
249             color=2  # Same color as quiver
250         )
251         #=
252         quiver
253
    ↪  https://discourse.julialang.org/t/plots-jl-arrows-style-in-quiver/1365
254         =#
255         p2 = plot(
256             stat._[1:var.itr],
257             xlims = (0, param.t_step),
258             ylims = (0, 1.1),
259             xaxis = ("Time step"),
260             yaxis = ("Orientation parameter"),
261             linewidth = 2)
262         """
263         empirical = ecdf(var.[:])
264         xrange = range(0,stop=1,length=200)
265         p3 = plot(xrange,empirical(xrange))
266         """
```

```
267
268         plot(p1,p2,size=(1500,480))
269         if flag_out == true
270             str_t = lpad(string(var.itr), 5, "0")  # iteration number in 5
                ↪  digit, left-padded string
271             str_N = lpad(string(param.N), 3, "0")
272             str_R = lpad(string(param.R_0), 3, "0")
273             str_ = lpad(string(param.), 3, "0")
274             str_v0 = lpad(string(param.v0), 3, "0")
275             png("img/wave_N$(str_N)_R$(str_R)_V0$(str_0)_$(str_t).png")
276         end
277     end
278
279     """
280     """
281     function make_gif(param,anim)
282         str_N = lpad(string(param.N), 3, "0")
283         str_R = lpad(string(param.R_0), 3, "0")
284         str_ = lpad(string(param.), 3, "0")
285         str_v0 = lpad(string(param.v0), 3, "0")
286         gif(anim,
287             "img/wave_N=$(str_N)_R=$(str_R)_V0=$(str_v0).gif",
288             fps=10)
289     end
290
291     function plot_(param,var,stat)
292         plot(
293             stat._[1:var.itr],
294             xlims = (0, param.t_step),
295             ylims = (0, 1.1),
296             xaxis = ("Time step"),
297             yaxis = ("Orientation parameter"),
298             linewidth = 2)
299         xaxis!("Time step")
300         yaxis!("Modulo of order parameter")
301         str_N = lpad(string(param.N), 3, "0")
302         str_R = lpad(string(param.R_0), 3, "0")
303         #str_ = lpad(string(param.), 3, "0")
304         str_v0 = lpad(string(param.v0), 3, "0")
305         str_t = lpad(string(param.t_step), 4, "0")
306         png("img/phi_N=$(str_N)_R=$(str_R)_V0=$(str_v0)_$(str_t)step.png")
307     end
308
309     """
310     Plot time-averaged _ versus noise amplitude
311     """
312     function plot__()
313         plot(
314             [1:15],
```

```
            #v[1:15],
            [1:15],
            marker= (
                :circle,  # shape of marker
                8,  # size of marker
                0  # transparency of marker
                # stroke(0,:white)  # stroke of marker
                ),
            linestyle = :solid,
            xaxis = ("Noise amplitude eta"),
            yaxis = ("Time-averaged orientation parameter varphi"),
            linewidth = 2)
        png("img/eta_phi.png")
    end
end  # module mod_output
```

```
using Random, Distributions
 Random.seed!(123) # Setting the seed

## Declare modules
using ProgressMeter
using Plots
using StatsBase
gr(
    legend = false  # Default setting for all figures
)
using .mod_param_var  # Define parameters and variables

import .mod_vicsek_model:  # Define time-integration of vicsek model
set_initial_condition,
set_neighbour_list,
set_neighbour_orientation,
set_white_noise, #!!!!! line not needed
set_new_,
set_new_r,
set_periodic_bc,
set_new_r
import .mod_analysis:  # Define functions for analysis
calc_
import .mod_output:  # Define functions for output data
plot_scatter_,
make_gif,
plot_
```

```
## Set parameter
N = 1000     #particles
R_0 = 0.05    #radius
 = 0         #noise
t_step = 1000 #timesteps
v0 = 0.04    #velocity
# the domain is 5x1
#density: 1000/5=200


#Von Mises sampling
#vm=VonMises()
#v= rand(vm)
#println(v)


param_ = mod_param_var.Parameters(N,R_0,,t_step,v0)

## Set variables
itr = 1 #number of iteration
r = Array{Float64}(undef, param_.N, 2)
 = Array{Float64}(undef, param_.N)
n_label = BitArray(undef, param_.N, param_.N)
 = Array{Float64}(undef, param_.N)
 = Array{Float64}(undef, param_.N)
# =  Array{Float64}(undef, param_.N)
r_new = Array{Float64}(undef, param_.N, 2)
_new = Array{Float64}(undef, param_.N)
n_sum = Array{Int64}(undef, param_.N)      #UNINITIALIZED ARRAYS
var_ = mod_param_var.Variables(itr,r,,n_label,,,r_new,_new,n_sum)

## Set statistical values
 = Array{Float64}(undef, param_.t_step, 2)
_ = Array{Float64}(undef, param_.t_step)    #UNINITIALIZED ARRAYS
sta_ = mod_param_var.StatisticalValues(,_)



## Main
set_initial_condition(param_,var_)
#set_neighbour_orientation(param_,var_)

progress = Progress(param_.t_step)
# for var_.itr=1:param_.t_step
anim = @animate for var_.itr=1:param_.t_step
    set_neighbour_list(param_,var_)
```

```
49      set_neighbour_orientation(param_,var_)
50      set_white_noise(param_,var_)
51      set_new_(param_,var_)
52      set_new_r(param_,var_)
53      set_periodic_bc(param_,var_)
54      set_new_r(param_,var_)
55      sta_.[var_.itr,:], sta_._[var_.itr] = calc_(param_,var_.)
56      #println("itr=",var_.itr, " [1]=", sta_.[var_.itr,1], "
        ↪  [2]=",sta_.[var_.itr,2], " _=",sta_._[var_.itr])
57      plot_scatter_(param_,var_,sta_,false)
58      next!(progress)
59  end
60
61  make_gif(param_,anim)
62  plot_(param_,var_,sta_)
63  println("")
64
65  println("average number of particles in a neighbourhood = ", N*R_0^2*pi/5)
66  println("average density = ", N/5)
67  println("time averaged _=",sum(sta_._/param_.t_step))
```

C Appendix C

C.1 Abstract in German

Diese Arbeit befasst sich mit kollektiver Bewegung von Partikeln. Zunächst im allgemeinen Fall, in dem nur die Bewegung der Partikel berücksichtigt wird. Im anschließenden zweiten Teil wird kollektive Bewegung in einer Flüssigkeit betrachtet.

Zuerst wird der Phasenübergang im nicht genormten Vicsek-BGK Modell für selbstangetriebene Partikel analysiert. Das Vicsek-BGK Modell beschreibt die zeitliche Entwicklung der Verteilung von Teilchen in kollektiver Bewegung anhand von Differentialgleichungen. In der Arbeit werden die Equilibria der Gleichungen, ebenso wie deren Stabilität in verschiedenen Regionen, untersucht. Abhängig von der Dichte in den jeweiligen Regionen sind diese Equilibria stabil oder instabil.
Des Weiteren werden die hydrodynamischen Gleichungen für die Zonen mit hoher Dichte abgeleitet, ebenso wie die Diffusionsgleichung für die Dichte in den Zonen mit niedriger Dichte.

Im zweiten Teil wird das Vicsek Modell mit den Stokes Gleichungen für Flüssigkeiten gekoppelt, um das Wechselspiel zwischen den Partikeln und der Flüssigkeit zu beschreiben. Das gekoppelte System [2], wird betrachtet und es werden erneut hydrodynamische Gleichungen in den Regionen mit hoher Dichte und die Diffusionsgleichung in den Bereichen mit niedriger Teilchen-Dichte abgeleitet. Die allgemeinen Ergebnisse verändern sich im gelkoppelten Modell nicht, allerdings werden im Vicsek-Stokes Modell neben der mittleren Orientierung und Geschwindigkeit der Teilchen auch die Geschwindigkeit und der Druck der Flüssigkeit miteinbezogen.

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