universität wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

"A Local Markov's Theorem"

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2020 / Vienna, 2020

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von / Supervisor:	Ass.-Prof. Dr. Anton Mellit, Privatdoz.

Abstract

We explain how the One-Move Markov Theorem from [LR97] simplifies braid equivalence from the two classical moves to one move called the L-move. Using this simplification, we apply the L-move to tangles obtaining braided tangles and show their significance by outlining the relationship between oriented tangles and R-matrices as seen in [Tur90]. The outline provides the incentive to look at the relationship between braided tangles and R-matrices.

Zusammenfassung

Wir erklären, wie das Ein-Operation Markov Satz von [LR97] die Zopf-Äquivalenz von den zwei klassischen Operationen zu einer Operation, die L-Operation genannt wird, vereinfacht. Unter Verwendung dieser Vereinfachung setzen wir die L-Operation für Tangles ein, wodurch wir verflochtene Tangles erhalten. Ihre Signifikanz wird durch die Kurzdarstellung der Beziehung zwischen ausgerichteten Tangles und R-Matrize wie in [Tur90] zu sehen ist, dargelegt. Die Kurzdarstellung bietet den Anreiz, die Beziehung zwischen verflochtenen Tangles und R-Matrize zu untersuchen.

Acknowledgments

I have to start by thanking my supervisor Ass.-Prof Dr. Anton Mellit, Privatdoz. who introduced me to this topic and has been patient with my progress and questions throughout the way. I cannot understate how important his feedback was towards helping me think more specifically as opposed to my instinctive comfort zone of thinking conceptually.

Then I would like to thank the program *Illustrating Mathematics* at ICERM (Institute for Computational and Experimental Mathematics) for providing a graduate student lecture series during which I could work on explaining “braided tangles.” Conversations with Prof. J. Scott Carter who gave me conceptual overviews of diagrammatic algebra was a nice addition.

Of course, I have to thank my family and friends for being my community of boundless support in whichever endeavor.

Contents

Introduction

Chapter 1 involves proving Alexander's Theorem, a result constructing a braid given any link diagram. In Section 1.1 and Section 1.2 we give some brief definitions and results that are fundamental to knot theory and braid theory. Then we show how the L-move introduced in Section 1.4, the main operation we study, stems from the sawtooth construction in Section 1.3. Both operations are involved in two versions of Alexander's Theorem which we prove in Theorem 1.25 and Theorem 1.39. For people familiar with knot theory and braid theory, one can skip straight to Section 1.4 to learn about the L-move and the criteria needed for a link diagram ensuring that L-moves do not affect the braid isotopy class with respect to the link diagram equivalence class.

Chapter 2's main result is proving the One-Move Markov Theorem (Theorem 2.19). It is another proof of Markov's Theorem (Theorem 2.9), except it replaces braid equivalence with L-equivalence. The general outline of the proof for Markov's Theorem using the classical Markov equivalence is shown in Section 2.1. Markov equivalence consists of stabilization and conjugation moves as shown by Weinberg (for Markov's original proof, see [Mar36] for the three-move Markov theorem). Using Theorem 2.9, we deduce that L-equivalence and Markov equivalence of braids yield the same equivalence classes. We only use the results of Section 2.2 for Chapter 3 where we deal with tangles instead of links.

Chapter 3 introduces braided tangles in Defintion 3.15. Given a tangle T, they are obtained by applying L-moves to a smaller tangle, T', within T. Then we define the modified tangle diagram in Definition 3.14 for the same reason we defined the modified link diagram in Definition 1.35, since the latter was essential to define before braiding the link diagram. The fundamentals of tangles that we use are stated in Section 3.1, then braided tangles and their operations are defined in Section 3.2, and finally we prove results concerning braided tangles in Section 3.3. The main result of the section is Theorem 3.21 but the remaining results of Section 3.3 are important for Chapter 4.

Chapter 4 provides an outline from Turaev in [Tur90] regarding why we want to define braided tangles, as a strict monoidal category denoted by BTa. Basic category theory and some abstract algebra is assumed. From algebra we use properties associated to rings, modules and tensor product.

1 Alexander's Theorem

The first two subsections are fundamental objects and results in knot theory and braid theory needed to construct a sawtooth in Section 1.3 and a special case of it denoted by the L-move in Section 1.4. We state Reidemeister's Theorem (Theorem 1.12) which provides criteria we need to check for links when proving Markov's Theorem (Theorem 2.9) and the One-Move Markov's Theorem (Theorem 2.19). Lastly we show respective results of Alexander's Theorem using sawteeth (Theorem 1.25 and L-moves (Theorem 1.39). Both theorems explicitly show how to obtain a braid given any link diagram. We use these very L-moves to obtain a braided tangle diagram of Definition 3.15 given any oriented tangle diagram.

1.1 Basic Knot Theory

Definition 1.1. A **link** L is the embedding of $n \in \mathbb{N}$ mutually disjoint simple closed polygonal curves into $\mathbb{R}^3$. We call n the **components** of a link L and we call a link a **knot** if $n = 1$.

Remark 1.2. The standard definition of a link uses "smooth" instead of "polygonal" curves. A classical result of topology is that the combinatorial isotopy of polygonal links is equivalent to the smooth isotopy of smooth links. Assume we only work in the combinatorial category i.e. the piecewise linear category. However in sections involving L-moves, we alternate between smooth and combinatorial settings, sometimes even combining them for the sake of illustration.

Suppose $a_1, \ldots, a_n$ are a finite number of points in $\mathbb{R}^3$. Then $[a_1, \ldots, a_n]$ denotes the convex hull of those points.

Definition 1.3. Let $[a, b]$ be an arc of link L and let c be a vertex that does not lie on L. Suppose $[a, b, c] \cap L = [a, b]$. Then we define a **$\Delta$-move** on L as $\mathcal{E}^c_{a,b} L = L - [a, b] + [a, c] + [b, c]$.

Remark 1.4. The purpose of using explicit notation, $\mathcal{E}^c_{a,b} L$, denoting a Δ-move is only a handy aid when we describe constructing a **sawtooth** in Definition 1.21.

Definition 1.5. Two links denoted by L and L' are **combinatorially isotopic** if there is a chain of links $L := L_0, L_1, \ldots, L_n =: L'$ such that every pair L_i and L_{i+1} is related by a Δ-move for $i = 0, \ldots, n - 1$. This defines equivalence classes of links.

Remark 1.6. We can also give an orientation to each component of a link, it imposes an additional requirement that a link isotopy is orientation-preserving. Since we are working with braids, assume all link components have an orientation.

Definition 1.7. A **link diagram**, D , is the orthogonal projection of a link, L, onto $\mathbb{R}^2$. We say D is in **general position** if:

1. the projection direction is not parallel to any arc of L,
2. every vertex in D lifts up to at most two vertices of L,
3. every vertex of D lifting up to exactly two vertices in L belong to two edges of D that intersect transversely at this vertex,
4. for every transverse intersection in D, we respect the differing heights of the corresponding edges of L by denoting over-arcs and under-arcs in D.

Vertices of D lifting up to two vertices of L are called **double points**.

Remark 1.8. Assume that every link diagram is in general position by a classical general positioning argument seen in [Bir75].

Definition 1.9. Let the new vertices of a Δ-move of L be projected onto D in general position and label the projected vertices of the move a', b', c'. In the special case when $[a', c'] \cap b' \neq \varnothing$ and $[b', c'] \cap a' \neq \varnothing$, then our Δ-move introduces or deletes the vertex c' on the arc $[a', b']$. This case is called **subdivision** in D

Definition 1.10. For link diagrams, we can perform **Reidemeister moves** denoted Ω_1, Ω_2 and Ω_3 seen in Figure 1.

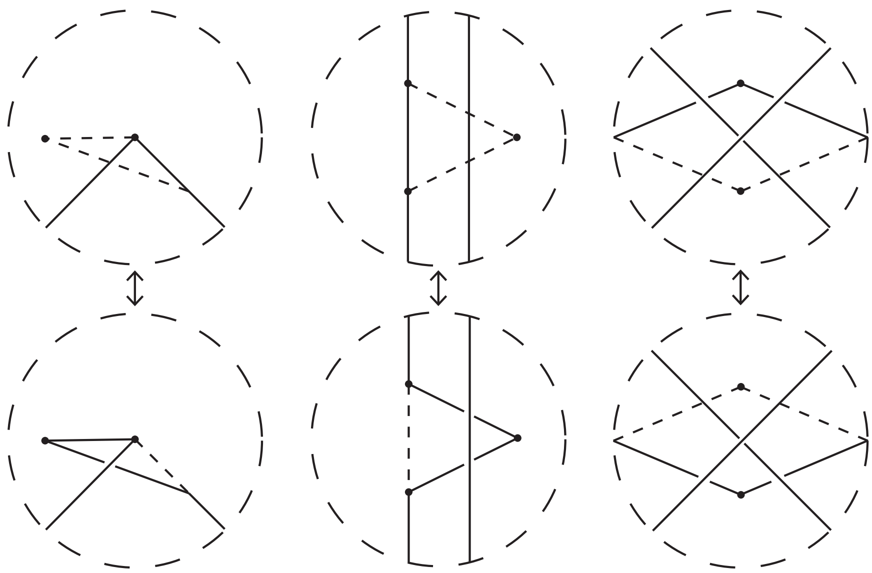


Figure 1: Ω_1, Ω_2 and Ω_3 respectively shown by the diagrams starting from the left.

Definition 1.11. Two link diagrams D and D' are **equivalent** if there is a chain of diagrams $D := D_0, D_1, \ldots, D_n =: D'$ such that every pair D_i and D_{i+1} is related by a planar Δ-move or a Reidemeister move.

Theorem 1.12. *(Reidemeister's Theorem) Two links L and L' are isotopic if and only if their corresponding link diagrams D and D' are equivalent.*

Proof. See [Rei48] for the classical proof. □

1.2 Braid Theory

Definition 1.13. Suppose we have $2n$ points denoted $A_i = (i, 0, 0)$ and $B_i = (i, 0, 1)$ in $\mathbb{R}^3$ for $i = 1, \ldots, n$. Then a polygonal line joining one of the A_i and B_j will be called a **down arc** if it intersects with every plane $(x, 0, z)$ for $x \in \mathbb{R}$ and $z \in [0, 1] \subset \mathbb{R}$ at exactly one point. A **(geometric) braid** on n arcs is a set of n mutually disjoint down arcs joining n distinct points of A_i to n distinct points of B_i.

Remark 1.14. Braids can also be defined smoothly, see Remark 1.2. One can naturally deduce the braid diagram in general position.

Definition 1.15. Braid isotopy is analogous to that of link isotopy as in Definition 1.5 with the exception that our braids are oriented and all relevant arcs involved in a Δ-move are down arcs. This equivalence relation is a definition for the equivalence class of a braid.

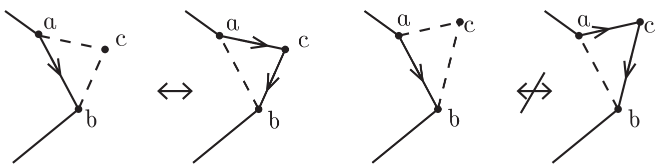


Figure 2: On the left is an admissible Δ-move but $[a, c]$ is an up arc on the right.

Definition 1.16. The equivalence class of braids on n arcs is a group with the presentation

$$B_n := \langle b_1, \ldots, b_{n-1} | b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \text{ for } i = 1 \ldots, n-2 \text{ and } b_i b_j = b_j b_i \text{ for } |i-j| \geq 2 \rangle$$

The generators $b_i^{\pm}$ is an element that has a positive (or negative) crossing as seen in Figure 3. The operation in this group is called **concatentation** where the product

$b_i b_j$ places the element b_i vertically above b_j. The first relation in the group presentation is called **conjugation** and the second relation is called **far commutativity**.

Remark 1.17. Concatenation in the braid group is an associative operation.

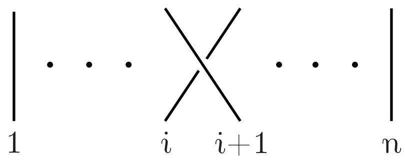


Figure 3: An elementary braid b_i with a positive crossing.

Theorem 1.18. *(Artin's Theorem) The braid isotopy class on n arcs is isomorphic to the braid group B_n.*

Proof. See [Bir75] for a full proof. □

Definition 1.19. A **closed braid** of a braid B on n arcs is given by connecting the n endpoints on the top to the n endpoints on bottom of the braid directly by n arcs. The closure of a braid B is a link denoted $\mathcal{C}(B)$.

Remark 1.20. We are usually working with braid diagram as opposed to geometric braids. So when we form the closure of a braid diagram, the arcs connecting the respective $2n$ endpoints lie "outside" of the braid diagram area.

1.3 Sawtooth Construction

(based on [Bir75])

Represent the orthogonal projection of a link, L, onto $\mathbb{R}^2$ from Definition 1.7 by a line $l \in \mathbb{R}^3$ such that $L \cap l = \varnothing$ and the plane of projection intersects l at only one point denoted by $\hat{l}$. The line l is oriented counterclockwise (positively) and is called the **axis** of L. Let D refer to the corresponding link diagram.

Assume that L is oriented, we want to show that D represents a closed braid by making sure all the arcs of D are down arcs with respect to the axis l.

Let $[a, b]$ be the edge of D and assume $[a, b]$ is oriented from a to b. Then $[a, b]$ is a down arc if the vector connecting a to $\hat{l}$ moving to the vector connecting b and $\hat{l}$ moves positively about l (counterclockwise). For notation, we will then say $[a, b] > 0$ if it is a down arc, otherwise $[a, b] < 0$.

Definition 1.21. Let $[a_0, a_m] < 0$ be an edge of a link L. Then suppose $a_0, \dots, a_m$ and $b_1, \dots, b_m$ are points satisfying $[a_i, a_{i+1}] < 0$, $[a_i, b_{i+1}] > 0$, $[a_{i+1}, b_{i+1}] > 0$ and $a_i \in [a_0, a_m]$ for $i = 0, \dots, m-1$. If, in addition

$$\sum_{i=0}^{m-1} [a_i, b_{i+1}, a_{i+1}] \cap L = [a_0, a_m]$$

then we construct a **sawtooth** consisting of $[a_i, b_{i+1}]$ and $[b_{i+1}, a_{i+1}]$ on $[a_0, a_m]$ given by

$$\begin{aligned}\mathcal{S}^{b_1,\dots,b_m}_{a_0,\dots,a_m} L &= (\prod_{i=0}^{m-1} \mathcal{E}^{b_{i+1}}_{a_i,a_{i=1}})(\prod_{i=0}^{m-1} \mathcal{E}^{a_{i+1}}_{a_i,a_m} L) \\ &= L - [a_0, a_m] + \sum_{i=0}^{m-1} ([a_i, b_{i+1}] + [b_{i+1}, a_{i+1}])\end{aligned}$$

Remark 1.22. The sawtooth constructed uses more than one tooth if there is a double point on $[a_0, a_m]$ in D as seen in Lemma 1.23. Figure 4 illustrates how sawteeth appear in the link diagram.

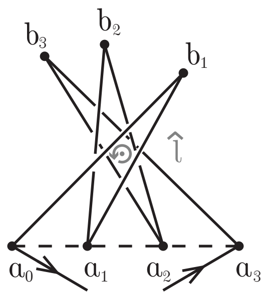


Figure 4: A sawtooth constructed on $[a_0, a_3]$. The two arcs of the diagram connected to $[a_0, a_3]$ are down arcs while $[a_0, a_3]$ was an up arc with respect to $\hat{l}$.

Lemma 1.23. *Let L be an oriented link with axis l and $[a_0, a_m] < 0$ be an edge of L. Then we can construct a sawtooth on $[a_0, a_m]$.*

Proof. For any pairs of vertices $a_i, a_{i+1} \in [a_0, a_m]$ such that $[a_i, a_{i+1}] < 0$, we can partition $\mathbb{R}^3$ into four regions bounded by two planes. One of the two planes is determined by the vertex a_i and line l while the other is determined the vertex a_{i+1} and line l so that both planes intersect only on l. The four regions are labelled $I(a_i, a_{i+1})$, $II(a_i, a_{i+1})$, $III(a_i, a_{i+1})$ and $IV(a_i, a_{i+1})$. Region $III(a_i, a_{i+1})$ contains $[a_i, a_{i+1}] \setminus \{a_i, a_{i+1}\}$, $IV(a_i, a_{i+1})$ is to the right of it and enumerate the other two accordingly. If b_{i+1} is a point in the region $I(a_i, a_{i+1})$ then a_i, b_{i+1}, a_{i+1} satisfies the initial conditions of constructing a sawtooth on $[a_i, a_{i+1}]$ of Definition 1.21.

Case 1: Let the projection of $[a_0, a_m]$ onto $\mathbb{R}^2$ contain no double points. Then we can choose $b \in I(a_0, a_m)$ such that $[a_0, b, a_m] \cap L = [a_0, a_m]$ and construct $\mathcal{S}^{b}_{a_0,a_m}L$. Let B be the line through b which is perpendicular to the projecting plane. Replacing b by any other b' on B does not affect whether or not $\mathcal{S}^{b'}_{a_0,a_m}L$ can still be constructed.

Case 2: Let the projection of $[a_0, a_m]$ onto $\mathbb{R}^2$ contain one double point. Without loss of generality, let $[a_0, a_m]$ be an under-arc with respect to the double point. Then we can still choose $b \in I(a_0, a_m)$ but we move b downward on the line B (defined in Case 1) sufficiently enough so that $[a_0, b]$ and $[b, a_m]$ project to under-arcs onto the projection plane. When $[a_0, a_m]$ is an over-arc, we naturally move b upward along B so that $[a_0, b]$ and $[b, a_m]$ project to over-arcs. The choice of b must also satisfy that $[a_0, b, a_m] \cap L = [a_0, a_m]$.

Case 3: Let the projection of $[a_0, a_m]$ onto $\mathbb{R}^2$ contain the double points $p_1, \ldots, p_k$ for $k > 1$. Then we can subdivide $[a_0, a_m]$ to include points $a_1, \ldots, a_{k-1}$ such that $[a_0, a_m] = \coprod_{i=0}^{m-1}[a_i, a_{i+1}]$ with $m := k+1$ and $p_i \in [a_{i-1}, a_i]$ for $i = 1, \ldots, k$. This ensures that each new edge contains only one double point. Then each new edge is either an under-arc or over-arc, the setting of Case 2.

At the end, we have constructed a sawtooth $\mathcal{S}^{b_1 \ldots b_k}_{a_0,\ldots,a_m}L$. ☐

Remark 1.24. Regarding the technical aspect of how we choose our subdividing edges of $[a_0, a_m]$ and the selection of the tips of the sawtooth, we can introduce the following: For any 3-simplex S, the sawtooth **avoids** S if $\bigcup_{i=0}^{m-1}[a_i, b_{i+1}, a_{i+1}] \cap S = \varnothing$ or a_0 or a_m. This condition is used in the technical proof seen in [Bir75].

Theorem 1.25. *(Alexander's Theorem, I) Let $\mathcal{C}$ be the map sending any representative of the braid equivalence class to a link using the closure operation. Then $\mathcal{C}$ is a surjective map which means any link is the closure of some braid.*

Proof. Given a link L oriented with respect to its axis l, let k denote the number of up arcs of L. If $k = 0$ then L is the closure of some braid which can be read by cutting the diagram of L by a vector with its initial point at $\hat{l}$ as seen in Figure 5.

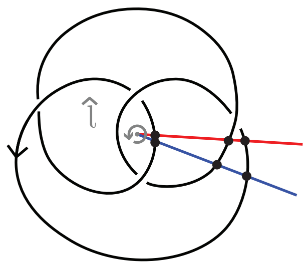


Figure 5: We get a braid on three strands, the red line is the top boundary of the braid and the blue line is the bottom boundary of the braid. This example is taken from [PS97].

If $k = 1$ then we can construct a sawtooth on the up arc $[a_0, a_m]$ of L according to Lemma 1.23 such that the sawtooth respects the under-arc and over-arc projections of our link diagram. Induct on k. $\square$

1.4 L-moves

(based on [LR97])

Once again, similar to dealing with constructing a sawtooth, assume that L is oriented, but this time the down arcs we want to obtain as in Definition 1.13 are arcs that are oriented downwards on the plane with respect to the x and y axes.

Definition 1.26. Let D be an oriented link diagram of a link L and p a point in D that may or may not be a vertex of D. Make sure that p is not vertically aligned with any double points and with any other vertices of D. Then we perform a **L-move** at p as seen in Figure 6. First we cut the arc at p and bend the resulting smaller arcs vertically so that the vertical parts of each of these arcs is oriented downwards. Extend these vertical arcs either over or under all other arcs of D - this depends on if our original arc was an under arc or an over arc. These new vertical arcs are connected and thus identified to each other in the same way of braid closure as in Definition 1.19.

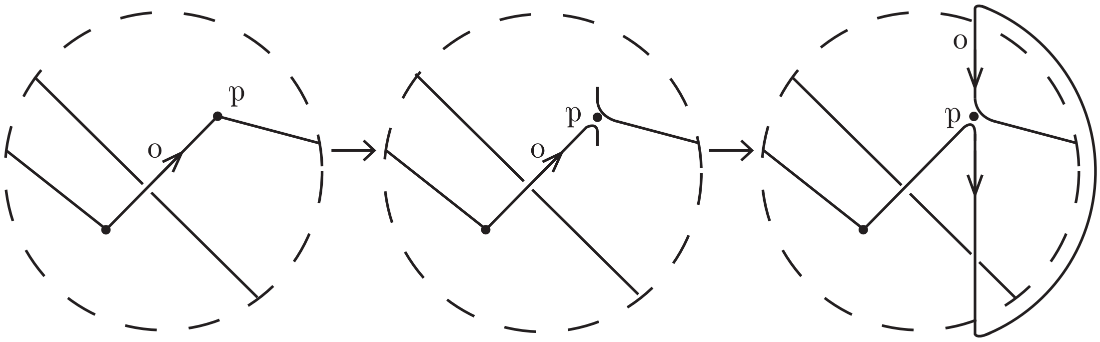


Figure 6: An "over" L-move requires that we label our up arc 'o' in order to indicate whether the new vertical arcs go over or under all other arcs of D. For short, we call it a L_o-move. We show how vertical arcs connect on the far right diagram, but in the future, this connection is assumed and not depicted.

Remark 1.27. Unlike Definition 1.21, the L-move operation takes place in the diagram of the link. A L-move can be thought of as a sawtooth constructed on an edge with the axis at infinity. Then the sawtooth is seen by the vertical lines after the L-move without the need for selecting vertices $b_1, \ldots b_m$.

In Definition 1.26 we assumed that the up arc contains one or no double points and did not interfere with any other L-move of the diagram. But we will show how to modify our link diagram of Definition 1.35 which ensures that all L-moves for up arcs can be performed simultaneously without altering our braid isotopy class. This results in giving us a closed braid and proving Alexander's Theorem (Theorem 1.39).

We want to label each up arc "over" or "under" if it is respectively an over or under arc. If the up arc contains no double point, then it is a **free arc** and can be labelled "over" or "under." Suppose that an up arc has more than one double point that has the up arc alternate between an under and over arc. Then we subdivide the up arc so that each up arc can only be labelled "over" or "under."

Definition 1.28. Let $[a, b]$ be an up arc of a link diagram D with a the initial vertex and b being the terminal vertex of the arc. Let $T(b)$ be a right triangle with hypotenuse $[a, b]$ and the right angle below the hypotenuse, moreover let a' be the third vertex of $T(b)$. Define a'' so that it is vertically lower than a' and $[a, a'']$ does not intersect any other arc of D. The choice of a'' is to avoid forming any horizontal arcs in D. Perform a **sliding triangle move** as seen in Figure 7. First, perform a L-move at b and then a Δ-move replacing $[a, b] \bigcup [b, a'']$ by $[a, a'']$. We call $T(b)$ the **sliding triangle**. Label a sliding triangle "over" or "under" depending on if the up arc in question is an over or under arc.

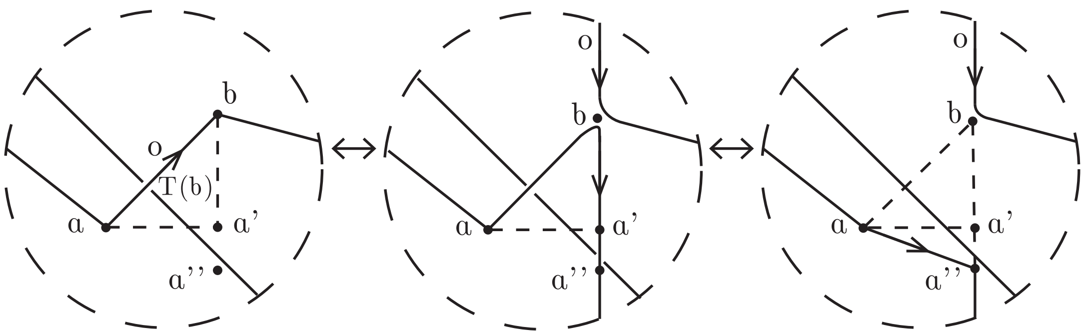


Figure 7: A sliding triangle move at $T(b)$: a L_o-move at b and then a Δ-move.

Remark 1.29. Often we depict the piece of an arc after applying a sliding triangle move that is only slightly horizontal for efficiency of illustration.

Before stating the definition of a modified link diagram in Definition 1.35, we need to prove Lemma 1.30 and Lemma 1.33 adjusting the relationships between our sliding triangles.

Lemma 1.30. *Let D be a link diagram with no horizontal arcs, then we can subdivide D to get D' such that for every two overlapping sliding triangles which do not share a common vertex, those sliding triangles have opposite labels.*

Proof. Let $d_1 := \min\{d(x,y)|x$ and y are double points of $D\}$ and let $\{B^{p_i}_{d_1/2}\}^n_{i=1}$ be n balls of radius $d_1/2$ centered at n double points of D. Then define

$$D' := D \setminus B^{p_1}_{d_1/2} \bigcup \cdots \bigcup B^{p_n}_{d_1/2}$$

Now let $d_2 := \min d(x,y)|x$ and y are vertices of D' and set $d_3 := \frac{1}{2}\min d_1, d_2$. We can subdivide D to obtain D'' such that every up arc is of length less than d_3 and ensure that no two vertices vertically align. Then D'' satisfies the triangle condition. □

Remark 1.31. The point of taking distances between vertices and double points is to find the minimum length between the vertices of sliding triangles of the same type that overlap, then the resulting new vertices of the up arcs are within this minimum distance determining the size of their sliding triangles.

Example 1.32. Let D satisfy the criteria of Lemma 1.30. Suppose we have two up arcs such that they are non-adjacent and have non-overlapping sliding triangles with the same label. Without loss of generality, suppose they are both forced to be labelled "over" and suppose we are in one of the configurations (apart from (e)) of Figure 8. If we apply the sliding triangle moves to the up arcs, we have overlapping vertical strands that are labelled "over" and thus we may still get different braids depending on the order of braiding the given up arcs.

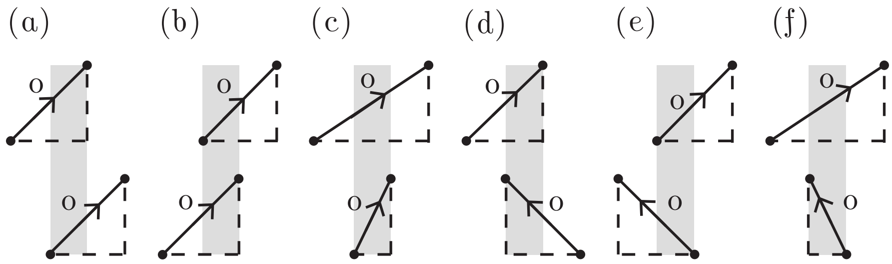


Figure 8: Apart from (e), the configuration of the sliding triangles create vertical strands of the same type after sliding triangle moves.

Lemma 1.33. *Let D be a link diagram with no horizontal arcs, then we can subdivide D to get D' such that there is no pair of non-overlapping sliding triangles that are in one of the configurations of Example 1.32.*

Proof. Using Lemma 1.30 we subdivide D to get D' such that the triangle condition is satisfied. For every up arc, label its initial vertex q_i and its terminal vertex p_i, so that we get a set of vertices $p_1, \dots, p_n$ and $q_1, \dots, q_n$. Let $O := \{[q_i, p_i] : [q_i, p_i] \text{ is labelled "over"}\}$ and $U := \{[q_i, p_i] : [q_i, p_i] \text{ is labelled "under"}\}$ and relabel indices so that the first $k \in \mathbb{N}$ pairs of indices are in O and then $k+1, \dots, n$ in U.

Project $[q_i, p_i]$ from O to $\mathbb{R} \times \{0\}$ and call the projected edge $[q_i', p_i']$. Fix some $i \in 1, \dots, k$, $[q_i', p_i'] \cap [q_j', q_j'] \neq$ for some $j \in (\{1, \dots, k\} \smallsetminus \{i\})$. Given every such pair i, j that exists, let $[q_i, p_i], [q_j, p_j] \in O'$ for a new set $O' \subset U$ i.e. they overlap like one of the configurations in Figure 8 apart from condition (e). For every pair of up arc vertices $i \neq j$ in O', let

$$d_{i,j} := \min\{\text{distance between two inner vertices of the four projected vertices}\}$$

and let $m := \min\{\{d_{i,j}\}_{i \neq j \in \{1, \dots, k\}}^{k}\}$. Let $\epsilon > 0$ be fixed and sufficiently small enough so that $m - \epsilon > 0$. Then returning to each pair of up arcs that intersect at their projection to the x-axis, we subdivide one of the up arcs by a distance $m - \epsilon$ to the left of their overlapping intersection as seen in Figure 9 and then relabel in order to avoid overlapping vertical arcs of the same type.

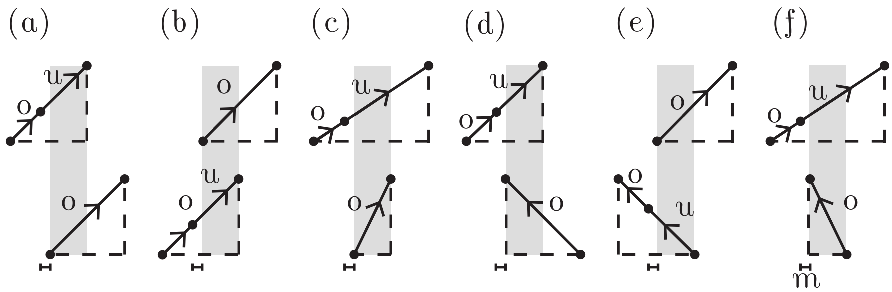


Figure 9: Subdividing and relabelling up arcs by $m - \epsilon$. For configuration (e), this is overkill in terms of subdivision.

Repeat subdivision and relabelling in the same manner for the set U and take care for both O and U to not let new vertices vertically align with other vertices or double points. $\square$

Remark 1.34. In the case that one of the up arcs was a free up arc and if changing its label does not violate the condition of Lemma 1.30, then we can simply relabel the arc.

Definition 1.35. A **modified link diagram** D is a generic link diagram satisfying the additional conditions:

1. there are no horizontal arcs in D,
2. there are no vertically aligned vertices in D and no vertex of D is vertically aligned with a double point,
3. D satisfies the condition of D' in Lemma 1.30,
4. D satisfies the condition of D' in Lemma 1.33 and
5. all up arcs are only labelled “free,” “over” or “under.

Remark 1.36. A modified link diagram allows for us to apply sliding triangle moves in any order to all up arcs since any order of elimination yields the same braid isotopy class.

Lemma 1.37. *Two modified link diagrams D and D' are isotopic if and only if there is a chain $D := D_0, \ldots, D_n =: D'$ such that each D_i is a modified link diagram. This means that D_i and D_{i+1} are related by a Δ-move in which D_i and D_{i+1} satisfy Definition 1.35 for $i = 0, \ldots, n-1$.*

Proof. If we have such a chain, then it is clear that D and D' are isotopic. To prove the other direction, suppose D and D' are isotopic. If at some diagram D_i of the chain violates condition 1 of Definition 1.35, then we can replace one of the vertices of the horizontal edge by a nearby point arbitrarily close to our original point. We use this same general positioning argument if some D_i violates condition 2 while maintaining that the new vertex does not violate any of the other conditions of Definition 1.35. For condition 5, we simply subdivide the diagram so as not violate condition 2 and any further subdivision of the diagram still satisfies condition 5. New vertices in all of those cases are carried throughout the diagram chain.

If at some point condition 3 or 4 is violated, then we will describe how to subdivide participating up arcs. First, suppose condition 3 is violated, this means that after a Δ-move, we have sliding triangles of the same type overlapping.

If the sliding triangles intersect on a point, then by definition of an isotopy, an up arc's vertex will not meet any interior part of another up arc which leaves only three admissible cases of sliding triangles of the same type intersecting on a point as seen in (d), (e) and (f) of Figure 10. However, some of these cases violate condition 2 and so we use the same general positioning argument and if vertices are not vertically aligned, we still use a general positioning argument replacing the point of intersection by another vertex arbitrarily close.

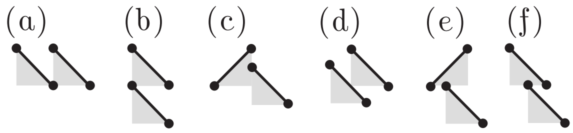


Figure 10: Non admissible cases of intersection on a point are (a) - (c) and the admissible ones are (d) - (f).

Now, if sliding triangles of the same type intersect - but not on a point, then these intersections are nonessential or essential. A nonessential intersection is when the hypotenuses of the sliding triangles do not intersect, here we use subdivision as seen in Figure 11.

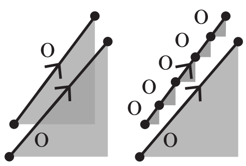


Figure 11: For nonessential intersections between two sliding triangles of the same label, subdivide and relabel by carrying over the label of the previous bigger up arc.

An essential intersection is when the hypotenuse of the sliding triangle forms a double point after some Δ-move, then we subdivide the up arc to obtain a free up arc as in Figure 12. Relabel the free up arc to maintain condition 3. Such subdivision on essential and nonessential intersections is possible by Lemma 1.30. Once again, we can ensure the subdividing points do not violate condition 2 and we carry out these subdivisions throughout the diagram chain.

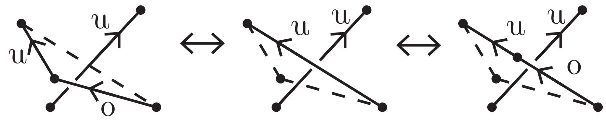


Figure 12: A Δ-move on the left side causes an essential intersection of two sliding triangles of the same type. The right side shows a subdivision and relabelling to fix the essential intersection.

Lastly, suppose condition 4 is violated at some point, then we can always choose a subdivision by Lemma 1.33 taking care that condition 2 is also not violated. □

Remark 1.38.

1. If we have a diagram satisfying condition 3, then any further subdivision also satisfies condition 3.
2. The purpose of this result is that we can assume all link diagrams are modified since the set of all modified link diagrams are dense in the set of all diagrams in general position. Such assumption is needed for Theorem 1.39.
3. We can define a **modified Δ-move** to be a Δ-move between modified link diagrams. The result states that each diagram in our chain is related by a modified Δ-move.

Theorem 1.39. *(Alexander's Theorem, II) Let D be an oriented modified link diagram. Then D is isotopic to $\mathcal{C}(B)$ for some braid B.*

Proof. Given D, let k denote the number of up arc of L. Once again we induct on k like in the proof of Theorem 1.25. Note that eliminating an up arc requires a sliding triangle move which consists of a L-move and isotopy. Moreover, a L-move on a closed braid diagram i.e. a link diagram is an isotopy proving our statement. □

2 Markov's Theorem

We replace "braid equivalence" with "Markov equivalence" and "L-equivalence" for braid isotopy classes and their bijection to their corresponding link isotopy classes in Theorem 2.9 and Theorem 2.19. This naturally concludes that there is a bijection between braid isotopy classes with respect to Markov equivalence and braid isotopy classes with respect to L-equivalence.

2.1 Markov equivalence

We want to replace Definition 1.15 that uses combinatorial equivalence to define braid equivalence by **Markov equivalence** as in Definition 2.1. Combinatorial equivalence is defined in the geometric braid but Markov equivalence is defined in the braid group.

Definition 2.1. Two braids B and B' are **Markov equivalent** if there is a chain of braids $B := B_0, \dots, B_n =: B'$ such that every pair B_i and B_{i+1} is related by a **conjugation** move in the braid group given by $aba \leftrightarrow a$ for $a, b \in B_n$ or a **stabilizing** move in the braid group given by $a \leftrightarrow ab$ for $b \in B_{n+1}$. Note that in the latter move, $ab \in B_n$ if we choose $b^{-1} \in B_n$. Both moves are more generally called **Markov moves**.

Remark 2.2. While these two moves in Markov equivalence are seen in the braid group, this definition is obtained as a corollary of proving Markov's Theorem using so-called **R-moves** and **W-moves** that are applied to geometric braids as seen in [Bir75]. Moves in the braid group are derived from moves on the geometric braid.

Definition 2.3. Let $[a, b]$ be an arc of a link L and let c be a vertex that does not lie on L. Suppose $[a, b, c] \cap L = [a, b]$ and $[a, b]$, $[a, c]$ and $[c, b] > 0$ with respect to some axis l. Then we define a **$\mathcal{R}$-move** as $\mathcal{R}^c_{a,b}L := L - [a, b] + [a, c] + [b, c]$. This is just the restricted Δ-move defined for braids in Definition 1.15.

Definition 2.4. Let $[a, b]$ be an arc of a link L with c and d vertices such that $[a, b, c, d] \cap V = [a, b]$. Suppose $[a, c]$, $[c, d]$, $[d, b]$, $[d, a]$, $[b, c]$, $[a, b] > 0$ with respect to some axis l where $\mathcal{E}^c_{[a,b]}L$ and $\mathcal{E}^d_{b,c}(\mathcal{E}^c_{[a,b]}L)$ are admissible. Then we define a **$\mathcal{W}$-move** as $\mathcal{W}^{cd}_{ab}L := L - [a, b] + [a, c] + [c, d] + [d, b]$.

Remark 2.5. Assume the result that a sawtooth constructed on the arc of a link can be realized as a combination of $\mathcal{R}$-moves and $\mathcal{W}$-moves. For the proof, see [Bir75].

The main purpose of Markov equivalence and eventually **L-equivalence** (see Defintion 2.11) is that suppose we have two Markov equivalent braids B and B' and $\mathcal{C}(B)$ and $\mathcal{C}(B')$ are their corresponding closed braids i.e. link diagrams. Then there is a deformation chain of equivalent link diagrams $\mathcal{C}(B) \coloneqq D_0, \dots, D_n \eqqcolon \mathcal{C}(B')$ such that D_i and D_{i+1} are related by a Markov move for $i = 0, \dots, n-1$. This means that for every pair D_i and D_{i+1}, we have link diagrams $D_i = \mathcal{C}(B'')$ and $D_{i+1} = \mathcal{C}(B''')$ for some braids B'' and B'''.

Lemma 2.7 and 2.8 are the main results in proving Theorem 2.9 (based on [Bir75]). Their proofs are laborious and involve eliminating and selecting moves from $\mathcal{E}$, $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{W}$ on a case by case basis. We will not give their proofs.

Definition 2.6. Let $h(L)$ denote the number of up arcs for a given link L, in [Bir75], this function is called the **height** of L. Assume $h(D) = h(L)$ for the corresponding link diagram D.

Lemma 2.7. *Suppose we have link L and L' such that we have an (isotopic) deformation chain connecting the links and $h(L) = h(L') > 0$. Then we can replace our initial deformation chain by another (possibly longer) deformation chain $L \coloneqq L_0, \dots, L_n \eqqcolon L''$ for $n \geq 1$ and $h(L_i) < h(L')$ for all $1 \leq i \leq n-1$.*

Proof. See [Bir75] for a full proof. □

Lemma 2.8. *Suppose we have links L, L' and L'' such that we have an (isotopic) deformation chain connecting the links $L \coloneqq L_0, \dots, L' \coloneqq L_i, \dots, L_n \eqqcolon L''$ for $n > 1$ and some $0 < i < n$. If $h(L) < h(L')$ and $h(L'') < h(L')$, then we replace our deformation chain by another (possibly longer) deformation chain $L \coloneqq L_0, \dots, L_s \eqqcolon L''$ for $s \geq 1$ such that $h(L_j) < h(L')$ for all $1 \leq j \leq s-1$.*

Proof. See [Bir75] for a full proof. □

In the proof, geometric closed braids refer to corresponding links in $\mathbb{R}^3$ for close braids in the plane.

Theorem 2.9. *(Markov's Theorem) Two closed braids $D \coloneqq \mathcal{C}(B)$ and $D' \coloneqq \mathcal{C}(B')$ are isotopic for some braids B and B' if and only if there is a chain $B \coloneqq B_0, \dots, B_m \eqqcolon B'$ such that B_i and B_{i+1} are Markov equivalent for $i = 0, \dots, k-1$.*

Proof. The easier direction is the "only if" direction. Assume we have such a chain of braids. Then we must show how conjugation and stabilization give braid closures that are isotopic.

Now, the non-trivial direction is the "if" direction. Assuming we have isotopic closed braids means we have a deformation chain $D := D_0, \ldots, D_n =: D'$.

In the case $h(D_i) = 0$ for $1 \leq i \leq n-1$, then each diagram in the chain must be related by a Markov move. Each D_i and D_{i+1} for $i = 0, \ldots, n-1$ is a closed braid by definition. The only moves that alter the number of up arcs of geometric closed braids are $\mathcal{E} \neq \mathcal{R}$ and $\mathcal{S}$ moves, this is not possible since there are zero up arcs. Hence the closed braids must be related by $\mathcal{R}$ and $\mathcal{W}$ moves corresponding to Markov moves in the respective braid groups by Remark 2.2.

Suppose now that not all $h(D_i) = 0$ and set $H := \max h(D_i)$ among $i = 1, \ldots, m-1$. If there is some j satisfying $1 \leq j \leq m-1$ and $h(D_j) = h(D_{j+1}) = H$, then we are in the case of Lemma 2.7 to this smaller chain. As a result, we have a new deformation chain $D := D_0, \ldots, D_{m'} =: D'$ for $m' \geq 1$ and $h(D_i) < h(D')$ for $1 \leq i \leq m'-1$. While we still have $H := \max h(D_i) > 0$ for $i = 1, \ldots, m'-1$, we now no longer have the condition that for some j we have $h(D_j) = h(D_{j+1}) = H$. Continue to apply Lemma 2.7 to each such pair of diagrams so that there is only one such diagram D_l in our final chain with $h(D_l) = H$.

The diagram D_l is not D or D' by construction due to our indices, this allows us to apply Lemma 2.8 where $h(D_l) > h(D)$ and $h(D_l) > h(D')$. Our new deformation chain $D := D_0, \ldots, D_{m''} =: D''$ for $m'' \geq 1$ such that $h(D_j) < h(D')$ for all $1 \leq j \leq m''-1$. But we reach our desired result that $h(D_j) < H$ for all given j. We continue applying Lemma 2.8 until we get a chain $D := D_0, \ldots, D_{m'''} =: D'$ such that $h(D_j) = 0$ and then we are in the initial case which we have already shown has each consecutive pair in the deformation chain connected by a Markov move. □

Remark 2.10. See [Bir75] for a full complete proof of all technical lemmas involved.

2.2 L-equivalence

(based on [LR97])

We prove One Move Markov's Theorem (Theorem 2.19 very differently from Theorem 2.9. Rather than looking at the heights of diagram chains, we check that L-equivalence of braids are independent of our choice of given operations and that link diagram equivalence corresponds to L-equivalence. For link diagram equivalence, we check Δ-moves and Reidemeister moves by Theorem 1.12.

Definition 2.11. Two braids B and B' are **L-equivalent** if there is a chain of braids $B := B_0, \ldots, B_n =: B'$ such that every pair B_i and B_{i+1} is related by a L-move or a braid isotopy for $i = 0, \ldots, n-1$.

Remark 2.12. We are using this definition to replace Definition 1.15.

Definition 1.35 for link diagram D helps us define a map $\mathcal{B}$ which maps any isotopy class of a link diagram to the L-equivalent braid isotopy class, denoted $\mathcal{B}(D)$, via eliminating all up arcs by sliding triangle moves. Note that $\mathcal{B} := \mathcal{C}^{-1}$.

Lemma 2.13. *Given a link diagram with an up arc, α, let c be a subdividing point of α forming up arcs α_1 and α_2. Then the braid obtained from the diagram containing α is L-equivalent to the braid obtained from the diagram containing α_1 and α_2.*

Proof. Without loss of generality, assume α is labelled "over" and let C be the vertical line intersecting c. After braiding α, C defines the location of c', a subdividing point on the new arcs after braiding α. We take a neighborhood N' about c' and this allows us to show our desired result in series of equivalent diagrams seen in Figure 13.

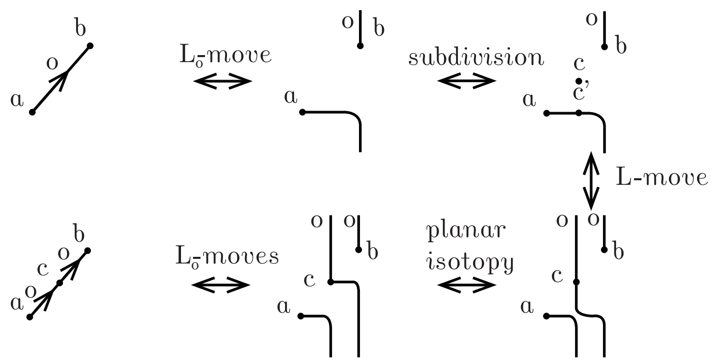


Figure 13: A chain showing how subdivision in the diagram maintains L-equivalent braids. Planar isotopy here is also braid isotopy.

$\square$

Lemma 2.14. *Given a link diagram with a free up arc, α the braid obtained by labelling α "over" is L-equivalent to the braid obtained by labelling α "under."*

Proof. Label the terminal vertex of α as b and the initial vertex as a. Without loss of generality, let the sliding triangle $T(b)$ associated to α be labelled "over."

Case 1: Assume that $T(b)$ contains no other arcs of the diagram. After braiding α, we define a new point b' near the vertical projection of b onto the new arc after braiding such that for vertical lines B and B' going through b and b', we have no other vertex of the diagram. After applying a L_u-move at b' we have two new vertices c and d. Note that ensuring there is no other vertex of the diagram between B and B' means that $[c,d]$is not forced to be labelled "under." Only vertical arcs of the diagram can be in $T(b)$. Applying a braid isotopy and the inverse of a L_o-move to obtain a down arc $[c,b]$, we get a braided local diagram if we had instead labelled $T(b)$ "under." This is is our desired result and illustrated in Figure 14.

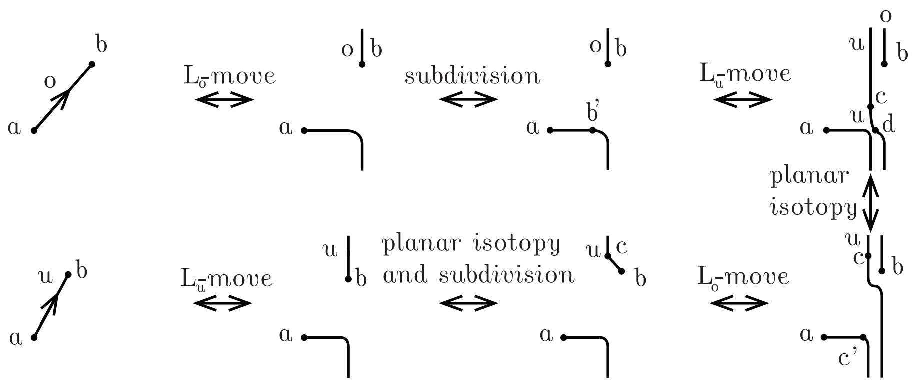


Figure 14: Case 1 showing how both types of labels yield L-equivalent braids for free up arcs.

Case 2: Assume that $T(b)$ contains over and under arcs of the diagram. Subdivide α by $n \in \mathbb{N}$ new points $a := a_1, \dots, a_n$ with $a_i \neq b$ for all i and each sliding triangle $T(a_i)$ does not contain any other arcs of the diagram and each is labelled "over" as well. This subdivision does not affect our braiding by Lemma 2.13. Then we apply case 1 to each sliding triangle changing all labels to "over." Use Lemma 2.13 to eliminate each subdividing point ("subdivision" refers to adding/deleting a vertex) to obtain our desired result. □

Corollary 2.15. *Let D be a link diagram, D_1 a link diagram obtained from subdividing D by subdivision S_1 and $D_2 \neq D_1$ a link diagram obtained from subdividing D by subdivision S_2. Then $\mathcal{B}(D_1)$ is L-equivalent to $\mathcal{B}(D_2)$.*

Proof. Subdivide D by $S_1 \cup S_2$ to obtain a link diagram D_3. We know that if D satisfies the triangle condition, then we know that any subdivision of D still satisfies the triangle condition. By Lemma 2.13 and Lemma 2.14, we get our desired result. □

Lemma 2.16. *Let D be a link diagram containing an arc $[a,b]$ and let D' be a link diagram obtained from D after applying a Δ-move to $[a,b]$. Then $\mathcal{B}(D)$ and $\mathcal{B}(D')$ are L-equivalent.*

Proof. Without loss of generality, assume $[a, b]$ is an up arc and by symmetry, we only look at Δ-moves occurring in the top right quadrant of the plane. To emphasize, assume after each operation that D is still a modified link diagram. Note that Corollary 2.15 means that if we perform a Δ-move that results in violating Lemma 1.30, we can eliminate Δ-moves so that the condition of Lemma 1.30 is not violated. This allows the resulting braid is still L-equivalent. Let A and B be vertical lines intersecting vertices a and b. Case 1 looks at Δ-moves within the smaller region bounded by A and B. Case 2 looks at Δ-moves outside the smaller region bounded by A and B, thanks to symmetry, we only look at those moves on the right side of the strip defined by A and B. Proving the result for both cases finishes the proof.

Case 1: We have three distinct possibilities for each new vertex after the Δ-move. Label them p_1, p_2 and p_3 along a vertical line (all lying above $[a, b]$ since this suffices thanks to symmetry) as seen in Figure 15.

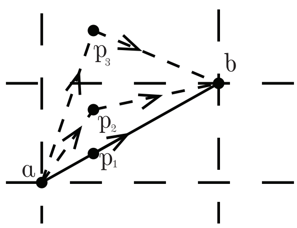


Figure 15: Three general regions for a new vertex from a Δ-move of the Case 1 type.

For p_1 lying on $[a, b]$ we just apply Lemma 2.13. For p_2 and p_3 we show L-equivalence in Figure 16.

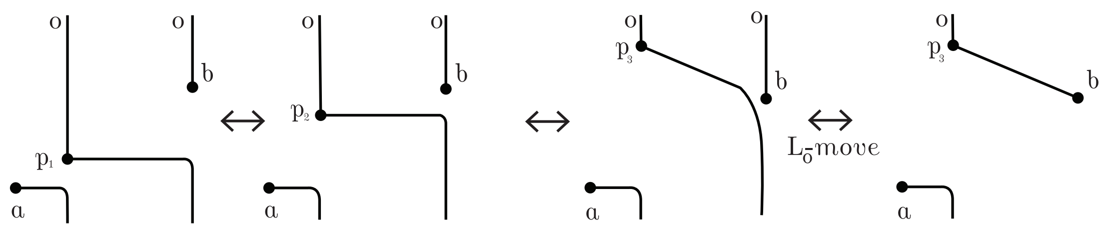


Figure 16: We show how a diagram after Δ-move at p_1, another at p_2 and a third at p_3 are all L-equivalent.

If we instead subdivide at some point $q \neq p_1$ with $q \in [a, b]$, then Lemma 2.13 and Lemma 2.14 still give us L-equivalent diagrams.

Case 2: Once again there three distinct possibilities for each new vertex after the Δ-move labelled p_1, p_2 and p_3 along a vertical line as seen in Figure 17.

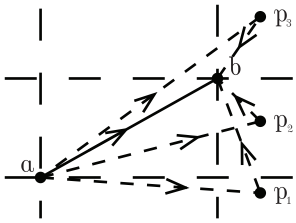


Figure 17: Three general regions for a new vertex from a Δ-move of the Case 2 type.

We assume that there exists no other braid strands between the vertical lines between B and the vertical line through each p_i. Figure 18 shows how a braid from a link diagram before and after a Δ-move at p_3 are L-equivalent to each other.

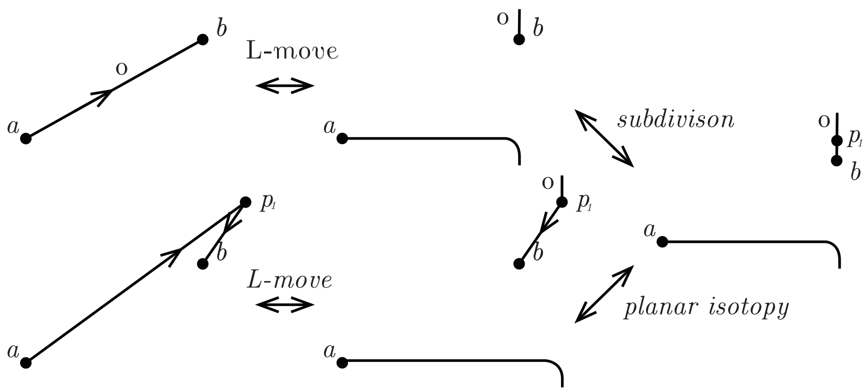


Figure 18: We show L-equivalence between both diagrams which were related by a Δ-move.

If we also perform a Δ-move at p_3 introducing a new vertex Q involving a vertical braided strand between vertical lines obtained through p_3 and q, then we can show how the diagram before the Δ-move is L-equivalent to the diagram after Δ-move. $\square$

Remark 2.17. If $[a, b]$ was instead a down arc, then we might obtain up arcs after a Δ-move which would involve us applying Lemma 2.14 and Lemma 2.13.

Lemma 2.18. *Let D and D' be link diagrams related by a Reidemeister move. Then $\mathcal{B}(D)$ and $\mathcal{B}(D')$ are L-equivalent.*

Proof. We first check the statement for a Ω_1 move in Figure 19 and without loss of generality assume our single arc is an up arc.

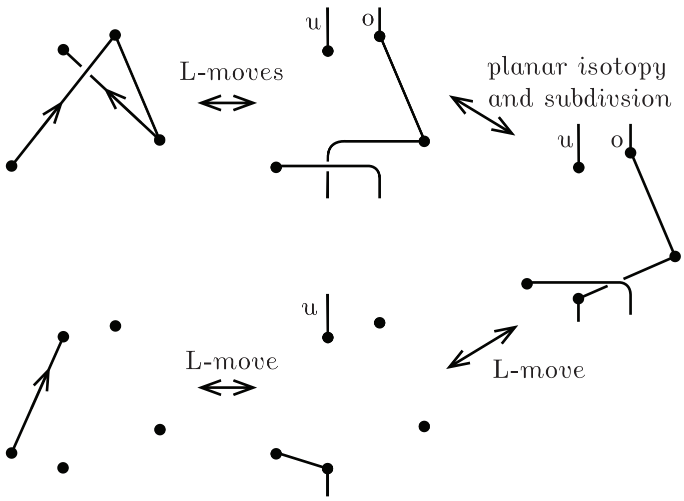


Figure 19: L-equivalence between diagrams related by Ω_1 move.

Now we check the statement for a Ω_2 move in Figure 20 and without loss of generality assume our arcs are a up arc and a down arc. The labelling of the two new up arcs after a Ω_2 are labelled "over" even thought they are under since the original up arc was labelled "over."

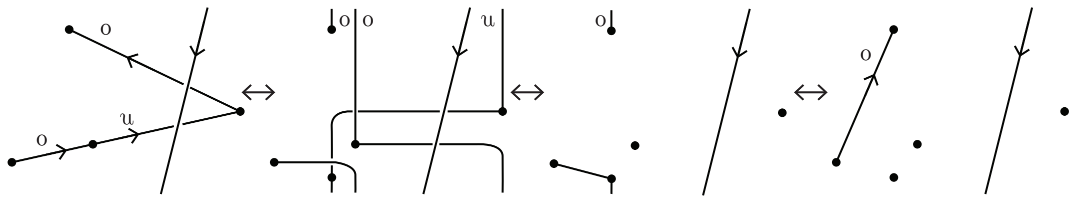


Figure 20: L-equivalence between diagrams related by Ω_2 move.

Lastly we can check the statement for a Ω_3 move but we can use a trick to minimize the work needed. We have a bottom arc, an upper arc and a middle arc involved in the move - however no matter what the top or bottom arc is, this does not matter in the braiding process. Thus, we assume the top or bottom arcs are down arcs and it remains to check when the middle arc is an up arc given by Figure 21.

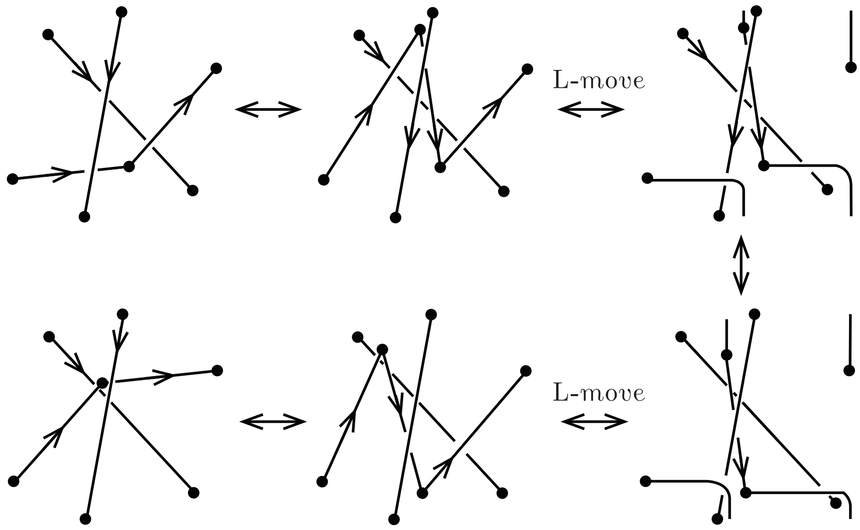


Figure 21: L-equivalence between diagrams related by Ω_2 move.

□

Theorem 2.19. *(One-move Markov's Theorem) Two closed braids* $\mathcal{C}(B)$ *and* $\mathcal{C}(B')$ *are isotopic if and only if there is a chain* $B := B_0, \ldots, B_k =: B'$ *such that each* B_i *and* B_{i+1} *are L-equivalent for* $i = 0, \ldots, k-1$.

Proof. By Lemma 2.13, Lemma 2.14, Lemma 2.16 and Lemma 2.18 we can conclude that $\mathcal{B}$ is well-defined.

Now we simply show that $\mathcal{C}$ and $\mathcal{B}$ are mutually inverse. Checking this, we get $\mathcal{C}(\mathcal{B}(D))$ is isotopic to D and $\mathcal{B}(\mathcal{C}(B)) = B$ for any braid B giving our desired results $\mathcal{B} \cdot \mathcal{D} = \mathrm{id}$ and $\mathcal{D} \cdot \mathcal{B} = \mathrm{id}$ □

3 A Local Markov's Theorem

We introduce basic definitions about tangles in Section 3.1. Then we combine oriented tangles with results about L-moves to construct braided tangles as stated in Definition 3.15 in Section 3.2. The most important results about braided tangles are in Section 3.3 which help us see braided tangles as a category in Section 4.

3.1 Oriented Tangles

We define smooth tangles, but only deal with combinatorial tangles, these two categories represent the same objects via Remark 1.2. Along with this, we define the two operations for tangles that are important for geometric and algebraic results.

Definition 3.1. A **tangle** is an embedding of n intervals and m closed curves into $\mathbb{R}^2 \times I$ such that the closed curves lie in $\mathbb{R}^2 \times [0,1]$, the endpoints of the n intervals lie mutually disjoint on the planes $\mathbb{R}^2 \times \{0,1\}$ and the endpoints are mutually disjoint.

In order to deal with tangles more specifically, we define them as **(k,l)-tangles** using the standard definition given by [Tur90].

Definition 3.2. A **(k,l)-tangle** is a smooth one dimensional compact submanifold denoted T of $\mathbb{R}^2 \times [0,1]$ such that $\partial T = T \cap (\mathbb{R}^2 \times \{0,1\})$ which is the set $\{(i,0,0) : k = 1,\dots,k\} \cup \{(j,0,1) : j = 1,\dots,l\}$ and every boundary point of T meets the upper and lower planes (respectively $\mathbb{R}^2 \times \{1,0\}$) orthogonally.

The **orientation** for a (k,l)-tangle is inherited from the orientation of T as a manifold. An orientation allows us to associate two sequences to a (k,l)-tangle. First we have $\epsilon := (\epsilon_1,\dots,\epsilon_k)$ called **source** T and we have $v := (v_1,\dots,v_l)$ called **target** T with every value in each sequence valued ± 1. We have $\epsilon_i = 1$ if the unit tangent to T at $(i,0,0) \in \partial T$ is $(0,0,1)$, it is $\epsilon_i = -1$ if the unit tangent to T at $(i,0,0) \in \partial T$ is $(0,0,-1)$. This is analgously applied to the values of v_j depending on the unit tangent to T at $(j,0,1) \in \partial T$. This is the standard definition from [Tur90].

Remark 3.3. A $(0,0)$-tangle is a link. A (n,n)-tangle is a braid when there are no disjoint polygonal curves embedded and the intervals are homeomorphically embedded into $\mathbb{R}^2 \times [0,1]$.

Definition 3.4. Two tangles T_1 and T_2 are **isotopic** in the same sense for links as in Definition 1.5 within the region $\mathbb{R}^2 \times (0,1)$ while $\mathbb{R}^2 \times \{0,1\}$ is the identity isotopy.

We can also analogously define a tangle diagram with respect to Definition 1.7. While keeping endpoints fixed, use Definition 1.11 to define equivalent tangle diagrams. With these notions defined, we can show the tangle version of Reidemeister's Theorem seen in Theorem 1.12.

Theorem 3.5. *(Reidemeister's Theorem for Tangles) Two tangles T and T' are isotopic if and only if their corresponding tangle diagrams D and D' are equivalent.*

Proof. Since we only focused on local parts of link diagrams in the proof of Theorem 1.12, then this same proof is used for the case of tangles. □

From now on, we only refer to tangle diagrams unless otherwise stated.

Definition 3.6. For two tangles T_1, a (k, l)-tangle, and T_2, a (m, n)-tangle, we can perform two operations between them. A **tensor product** is a disjoint union denoted by $T_1 \otimes T_2$ and a **product** is a composition denoted by $T_2 \circ T_2$ which is applicable only if source T_1 is the same as target T_2. Both operations are depicted in Figure 22

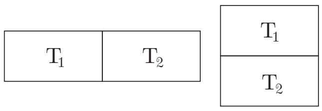
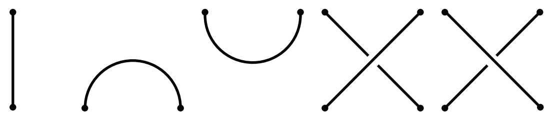


Figure 22: On the left is the tensor product and on the right is the product.

Remark 3.7. We define tensor product in the category theoretic sense in Definition 4.1.

Definition 3.8. An **elementary tangle** is an oriented tangle that is one of the given types as seen in Figure 23, they correspond to oriented written symbols: $\uparrow$, $\downarrow$, ↶, ↷, ⤺, ⤻, X_+ and X_-.



(a) (b) (c) (d) (e)

Figure 23: (a) and (b) are up and down arrows, (c) is a cap, (d) is a cup, (e) is a positive crossing and (f) is a negative crossing.

Remark 3.9. Using tangle operations, we can decompose any link or tangle into a set of elementary tangles, so it suffices to assume a tangle is an elementary tangle unless otherwise stated. For links, one way to decompose a link is given in Theorem 3.27.

3.2 Braided Tangles

Our goal is now to apply a braiding process to tangles, a process that eliminates up arcs in order to obtain only down arcs, like we did in Section 1.4. We ultimately want to obtain a diagram of a tangle that results in a braided tangle as in Definition 3.15.

Definition 3.10. Our tangle diagram T consists of edges, filled black vertices we call **original vertices**, empty black vertices we call **braided vertices** and filled red vertices we call **product vertices**.

An original vertex lies in $\mathbb{R} \times (0,1)$. A braided vertex comes in identified pairs that align vertically on $\mathbb{R} \times \{x_1, x_2\}$ for some fixed $x_1 := 0 + \epsilon_1$ and $x_2 := 1 - \epsilon_2$ for $\epsilon_1, \epsilon_2 > 0$ sufficiently small such that for each elementary tangle we obtain a smaller tangle as seen in Figure 24, one can imagine an arc with part of it lying in $T \setminus T'$ and the other part of it lying outside of T connecting each pair of braided vertices. We haven not depicted any braided tangles yet because they are the result of L-moves. A product vertex lies on $\mathbb{R} \times \{0,1\}$ and the vertex involved in the product of two braided tangles.

Remark 3.11. The sequences associated for source T and target T as in Definition 3.2 are only counted for product vertices.

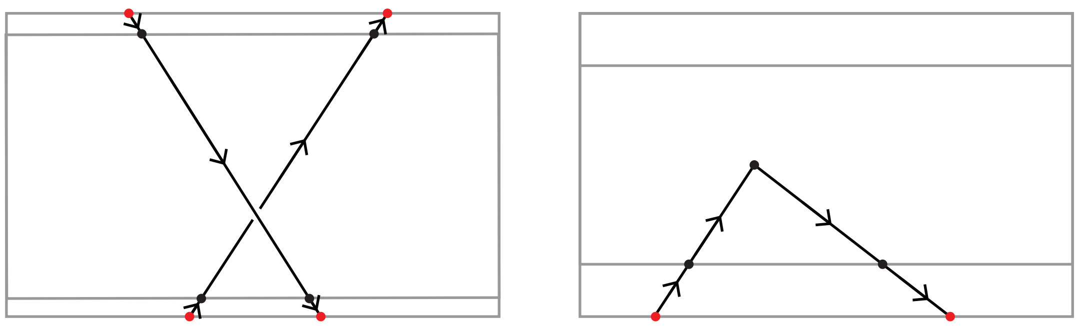


Figure 24: We have two smaller tangles T' that has boundaries given by $x_1 = x_2$ for respective values of x_1. The x_1 and x_2 in the elementary tangle on the left side is chosen so that vertices do not align vertically in T'.

Definition 3.12. For every tangle diagram T, we have an associated smaller tangle diagram T' as seen in Figure 24. There we subdivide our original tangle by the boundary lines of the smaller tangle intersecting our original tangle as seen in Figure. Label the new vertices as original vertices. Ensure that T' satisfies Definition 1.35

Definition 3.13. Given a tangle T and a smaller tangle T', we will give the following **perturbation operations** that are applied to T' so that we can eliminate up arcs simultaneously without altering our tangle isotopy type. These are given as follows:

1. (Perturbation Move 1) If there is an up arc with a vertex on the bottom boundary and it is not oriented from the bottom left to the upper right, then we perform a subdivision and a Δ-move as illustrated in Figure 25 taking care to not let vertices align.

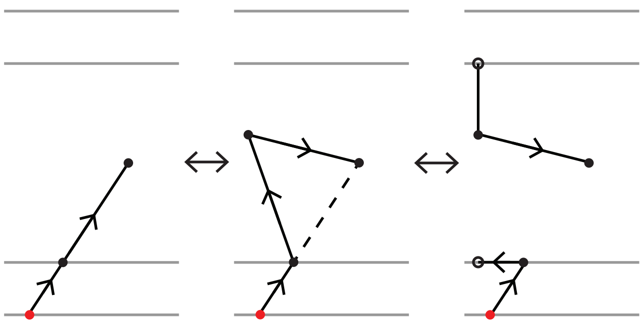


Figure 25: We apply a Δ-move so that only one of the edges of the new arcs is an up arc and then braid.

2. (Perturbation Move 2) If there is an up arc with a vertex on the upper boundary and it is not oriented from the bottom left to the upper right, then we perform a subdivision, a Δ-move, and an isotopy as illustrated in Figure 26 taking care to not let vertices align.

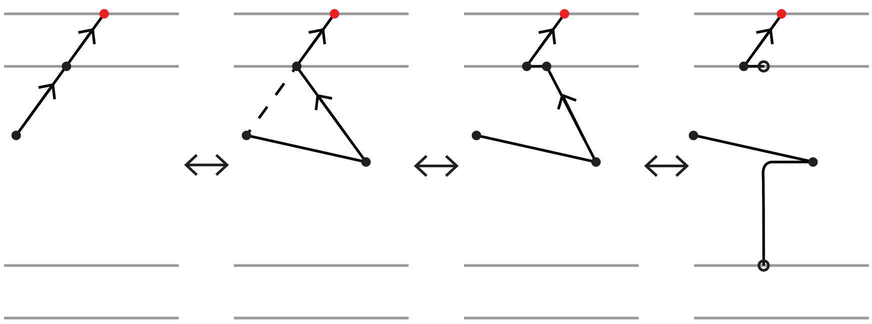


Figure 26: We apply a Δ-move so that only one of the edges of the new arcs is an up arc, shift the involved original vertex to the right on T' and connect to another original vertex at the original position, then we braid. We shifted the original vertex so that a product vertex is not connected to a braided vertex.

3. (Perturbation Move 3) Suppose we have applied Perturbation Move 1 and Perturbation Move 2 to all relevant up arcs. Now project all upper vertices of up arcs to the lower boundary of T'. In order to prevent the horizontal portion of a sliding triangle move to overlap braided and original vertices on the boundary of T', we subdivide the up arc as illustrated in Figure 27.

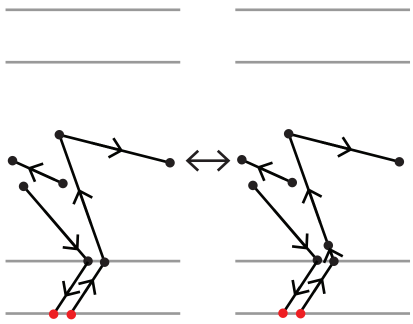


Figure 27: We subdivide so that the horizontal arc of an up arc on the boundary of T' does not intersect with any other braided vertex or original vertex.

Definition 3.14. A **modified tangle diagram** T has T' in the sense of Definition 3.12 and we have applied perturbation moves from Definition 3.13 such that they are no longer applicable.

Definition 3.15. Given a modified tangle diagram T, a **braided tangle**, $\mathcal{B}(T)$, is T with T' braided such that all up arcs are eliminated via sliding triangle moves. More specifically T' is a finite disjoint of intervals properly embedded in $\mathbb{R} \times [x_1, x_2]$ composed of the following:

1. there are braided and original vertices that are endpoints of the intervals on the lines $\mathbb{R} \times x_1$ and $\mathbb{R} \times x_2$ labelled "over" or "under,"
2. braided vertices come in pairs, a braided vertex on $\{a\} \times x_2$ for some $a \in \mathbb{R}$ is matched to a braided vertex on $\{a\} \times x_1$,
3. only original vertices are connected to product vertices that lie on $T \setminus T'$
4. a braid is formed by connecting the unpaired vertices on $\mathbb{R} \times 1$ to the unpaired vertices on $\mathbb{R} \times 0$.

Remark 3.16. While we use $\mathcal{B}(D)$ to represent a link diagram braided by sliding triangles, we also use $\mathcal{B}(T)$ to denote a braided tangle since this process also involves subdivision and L-moves and isotopy to T'. Depending on whether we are dealing with a link diagram or a tangle is how we define what it means to map it via $\mathcal{B}$.

Definition 3.17. Suppose we have two braided tangles $\mathcal{B}(T_1)$ and $\mathcal{B}(T_2)$ such that $T_1 \circ T_2$ is applicable. Then the **product** of braided tangles, $\mathcal{B}(T_1) \circ \mathcal{B}(T_2)$, occurs in the following steps:

1. Connect all product vertices by arcs that are oriented so that they cohere.
2. Suppose $\mathcal{B}(T_1)$ has a braided vertex that is not connected to a horizontal arc. Then for each pair of braided vertices, relabel the bottom braided vertex of each pair into an original vertex and connect this vertex by a vertical arc to a braided vertex at the bottom of T_2' of $\mathcal{B}(T_2)$. Eliminate the new original vertex to make sure that vertices don't vertically align.
3. Suppose $\mathcal{B}(T_2)$ has a braided vertex that is not connected to a horizontal arc. Then for each pair of braided vertices, relabel the bottom upper vertex of each pair into an original vertex and connect this vertex by a vertical arc to a braided vertex at the top of T_1' of $\mathcal{B}(T_1)$. Eliminate the new original vertex to make sure that vertices don't vertically align.
4. Suppose $\mathcal{B}(T_1)$ has a braided vertex that is connected to a horizontal arc on the upper boundary of T_1'. Then extend the bottom braided vertex of the pair as in step 2 of this definition.
5. Suppose $\mathcal{B}(T_2)$ has a braided vertex that is connected to a horizontal arc on the lower boundary of T_2'. Then extend the upper braided vertex of the pair as in step 3 of this definition.
6. We are left with the remaining braided vertices that have not been extended - these are connected to product vertices. Relabel these braided vertices as original vertices and extend them as follows: Note that this forms a chain of three consecutive arcs given by: an original vertex of T_1' of $\mathcal{B}(T_1)$, a product vertex of $\mathcal{B}(T_1)$, a product vertex of $\mathcal{B}(T_2)$ and an original vertex of T_2' of $\mathcal{B}(T_2)$. The first arc gets extended like the bottom part of a sliding triangle move and what we did in step 2. The second arc gets extended like what we did in step 3.
7. We are only left with up arcs that that can be isotoped to down arcs by a series of Δ-moves and ensure that vertices do not vertically align.

After this process $\mathcal{B}(T_1) \circ \mathcal{B}(T_2)$ is also a braided tangle.

Remark 3.18. 1. If we are performing step 2, 4 or 6 in the above process in which the "extended" vertex intersects a horizontal arc on the boundary of T_2' of $\mathcal{B}(T_2)$, then we know that horizontal arc occurred from braiding an up arc of T_2' of T_2. Apply perturbation move 3 to this up arc to make sure such intersection of a horizontal arc and a braided vertex does not occur.

2. If we are performing step 3, 5 or 6 in the above process in which the "extended" vertex intersects a horizontal arc on the boundary of T_1' of $\mathcal{B}(T_1')$, then we can shift the pair of braided vertices of question sufficiently close to the left until this intersection no longer occurs.

3. Lastly, if we are performing any steps 2 – 7 in which vertices vertically align, we can always shift pairs of vertices sufficiently close to the vertical line of intersection so that this intersection no longer occurs.

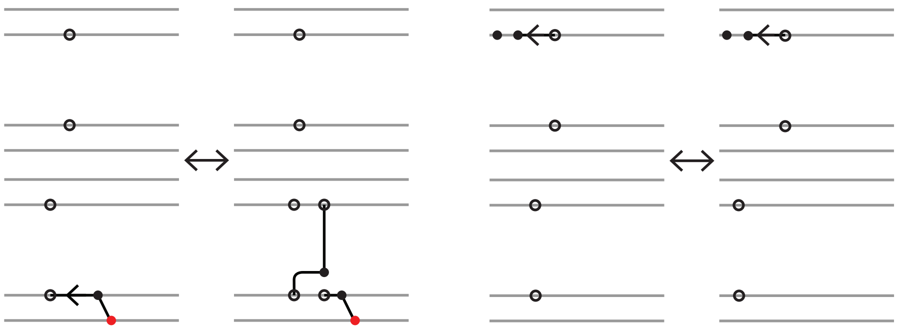


Figure 28: We only look at the involved local parts of the braided tangle diagrams. On the left is a move from (1) of Remark 3.18, on the right is a move from (2) of Remark 3.18. The latter diagram is similar to the move from (3) of Remark 3.18.

Example 3.19. Let T_1 and T_2 be the tangles on the left of Figure 29 and let $\mathcal{B}(T_1)$ and $\mathcal{B}(T_2)$ be the braided tangles on the right of Figure 29. Using Definition 3.17 detailing the steps to perform the product of two braided tangles, we want to show $\mathcal{B}(T_1) \circ \mathcal{B}(T_2)$.

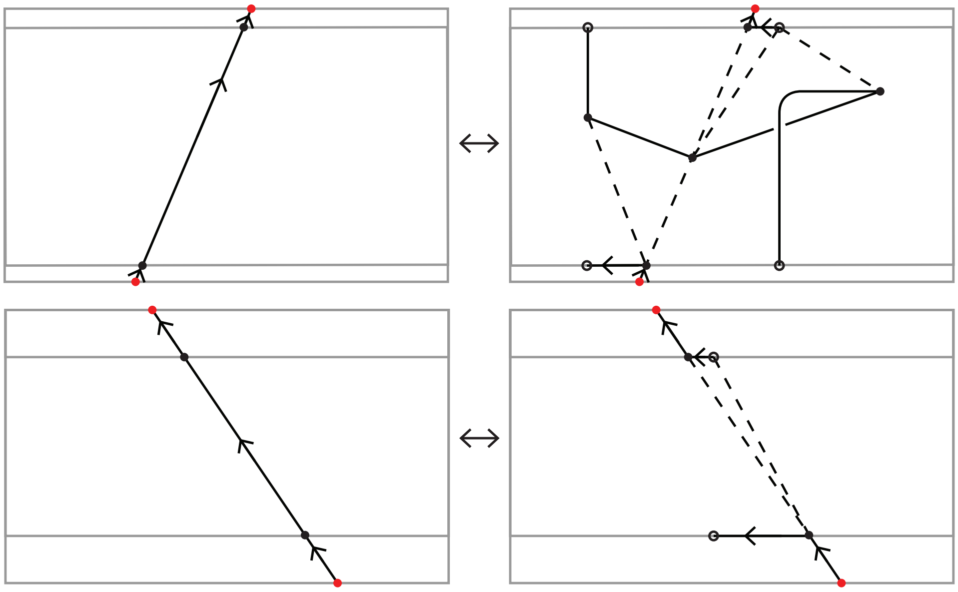


Figure 29: The above tangle on the left is T_1 and the below tangle on the left is T_2. These two elementary tangles have corresponding T_i' tangles with appropriate subdivisions for $i = 1, 2$. We braid T_i' to get the braided tangles on the right for $i = 1, 2$ according to Definition 3.14.

Note that we used perturbation moves from Definition 3.13 before braiding the tangles. Next, follow step 1 and extend braided vertices as in steps 2 – 5 all stated in Definition 3.17 depicted in Figure 30.

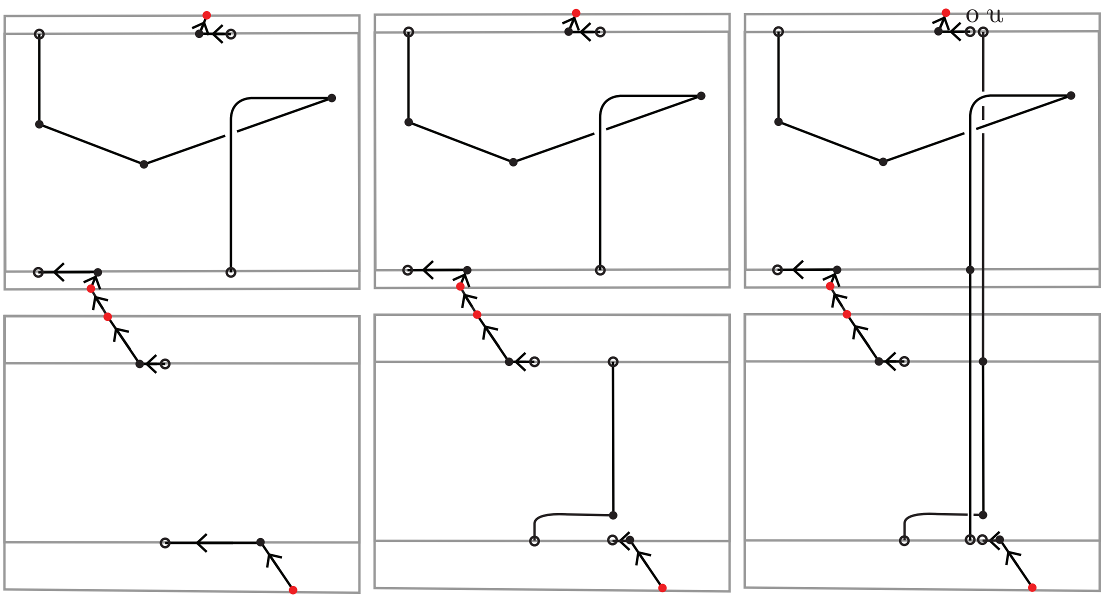


Figure 30: The left diagram is step 1. The middle diagram is applying (1) from Remark 3.18. The right diagram are steps 2 – 5 from Definition 3.17.

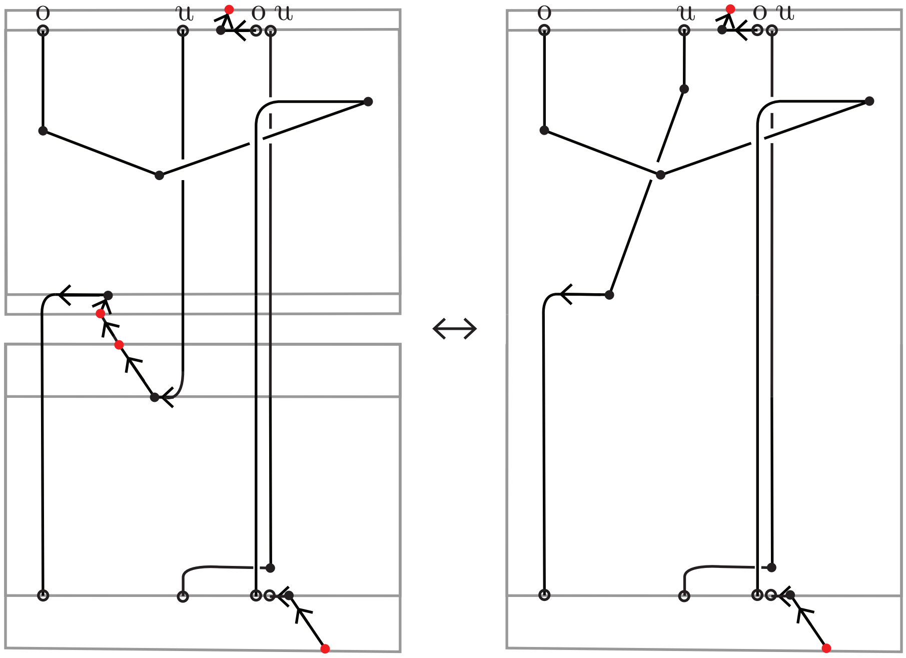


Figure 31: Lastly we perform steps 6 and 7 to eliminate remaining up arcs.

Finally we perform step 6 and then many Δ-move isotopies of step 7 on the arcs involving the remaining braided vertices that have not been "extended" obtaining the diagram on the right illustrated in Figure 31. In addition, this diagram is actually $\mathcal{B}(T_1) \circ \mathcal{B}(T_2)$.

One can begin to see how $\mathcal{B}(T_1) \circ \mathcal{B}(T_2)$ is L-equivalent to $\mathcal{B}(T_1 \circ T_2)$.

Definition 3.20. For two braided tangles $\mathcal{B}(T_1)$ and $\mathcal{B}(T_2)$, the **tensor product** and product still holds from Definition 3.6. In addition, if $\mathcal{B}(T_1) \circ \mathcal{B}(T_2)$ is still applicable, then we follow the method in Definition 3.17.

3.3 A Local Markov's Theorem

When we say that braided tangles $\mathcal{B}(T_1)$ and $\mathcal{B}(T_2)$ are L-equivalent, note that we may still have up arcs in $T_i \setminus T_i'$ for $i = 1, 2$. In the L-equivalence, if needed, we can always eliminate the up arcs in $T_i \setminus T_i'$ by a L-move if needed. The up arcs in $T_i \setminus T_i'$ are always free up arcs.

Theorem 3.21. *(A Local Markov's Theorem) Let T and T' be two isotopic tangles are isotopic if and only if there is a chain of braided tangles $\mathcal{B}(T) := B, \ldots, B_k =: \mathcal{B}(T')$ such that each B_i and B_{i+1} are L-equivalent for $i = 0, \ldots, k-1$.*

Proof. Since we only looked at local parts of a link diagram in the proof of Theorem 2.19, then the same proof is used here. The distinctions made is that we do not need a closure operation since a braided tangle is still a tangle and our L-equivalence requires that product vertices on the boundary remain fixed. □

Lemma 3.22. *Let T_1 and T_2 be two tangles that are compatible with respect to the composition operation, $T_1 \circ T_2$. Then $\mathcal{B}(T_1 \circ T_2)$ is L-equivalent to $\mathcal{B}(T_1) \circ \mathcal{B}(T_2)$.*

Proof. We know that every time we apply $\mathcal{B}$ to a tangle, we may create more braided arcs given by pairs of braided vertices, so the fewer we apply $\mathcal{B}$, then we can try to control the amount of formed braided arcs.

Then $\mathcal{B}(T_1 \circ T_2)$ has less braided arcs than $\mathcal{B}(T_1) \circ \mathcal{B}(T_2)$. Eliminate the braided arcs of $\mathcal{B}(T_1) \circ \mathcal{B}(T_2)$ that contain the product vertices of T_1 and T_2 that lie in the interior of this braded tangle. This elimination is done by inverse L-moves. Given our construction of a braided tangle, we know that $\mathcal{B}(T_1 \circ T_2)$ differs from $\mathcal{B}(T_1) \circ \mathcal{B}(T_2)$ via subdivision and diagram isotopies. Thus our desired braided tangles are L-equivalent. □

Lemma 3.23. *For every link diagram L we can decompose L into a elementary tangles that are connected via tensor products and product.*

Proof. Let the **height** of L such a height of y means we are at the line $\mathbb{R} \times \{y\}$ of $\mathbb{R}^2$ such that $(\mathbb{R} \times \{y\}) \cap L \neq$ for some $y \in \mathbb{R}$.

Let $y_1, \dots, y_n$ be the various heights of L such that $(\mathbb{R} \times \{y\}) \cap L$ is a double point of L or if it coincides with a point of L having the tangent vector 0. Modify L by an arbitrarily small isotopy to ensure that double points and points where the tangent vector are 0 lie at different heights.

Set $h_1 := \min\{y_1, \dots, y_n\}$ and then define ϵ to be the sufficient value needed so that $h_1 + \epsilon$ has not surpassed another y_i. This allows us to set a new set of heights $y'_1, \dots, y'_n$ where $y'_i := y_i + \epsilon$ along with another height $y'_{n+1} := h_1 - \epsilon$. Within each horizontal partition of the link, we use vertical lines to break up the horizontal partition into a series of elementary tangles.

The vertical lines are tensor products and horizontal partitions are how we take a product of two tangles (each composed of elementary tangles that are connected by tensor product). This provides one construction to our result. □

Remark 3.24. In the case of a combinatorial link diagram, we choose $y_1, \dots, y_n$ at double points and undefined tangent vectors.

Example 3.25. We will follow Lemma 3.23 to decompose a given link.

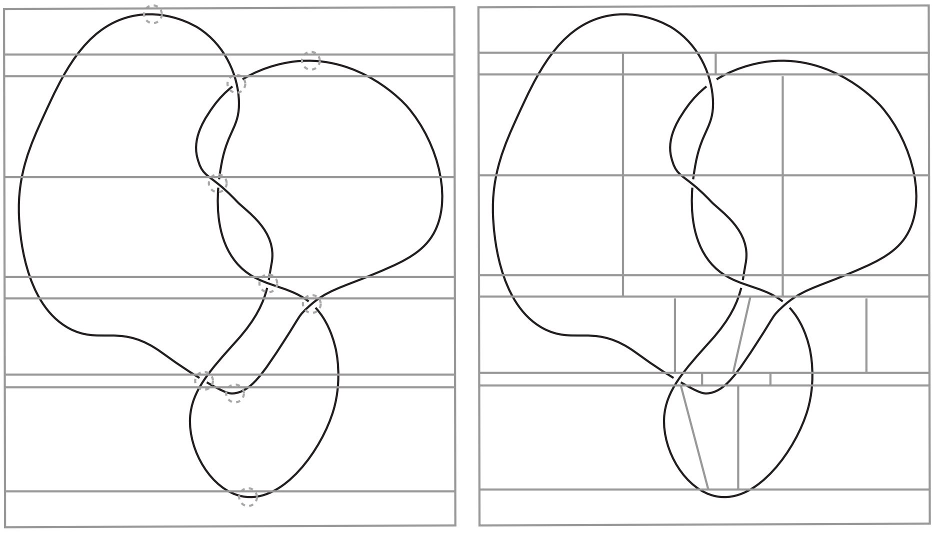


Figure 32: On the left side, we partition the plane of a knot diagram and on the right side, we separate each plane by lines so that we end up having only elementary tangles in each box.

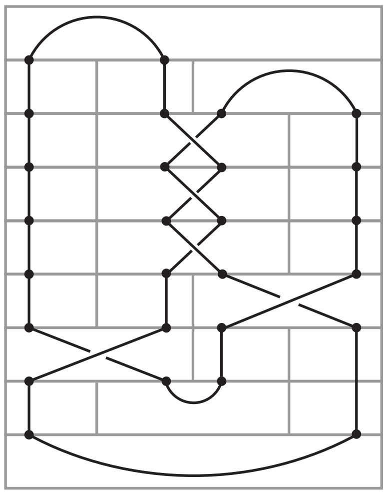


Figure 33: This is the knot decomposition into elementary tangles as seen in Figure 32 but more clearly shown.

Lemma 3.26. *Let L be a link diagram and suppose it has a decomposition into elementary tangles $T_1, \dots, T_n$. Then $\mathcal{B}(L)$ is L-equivalent to $\mathcal{B}(T_1) \circ \cdots \circ \mathcal{B}(T_n)$.*

Proof. Given Lemma 3.22, we induct on n. Since more tangles do not change that the diagrams differ by subdivision, diagram isotopies and L-moves, we still have L-equivalence. $\square$

Theorem 3.27. *Let L be a link diagram. Then $\mathcal{B}(L)$ is L-equivalent to the the corresponding operations of tensor product and product applied to braided tangles $\mathcal{B}(T_1), \ldots, \mathcal{B}(T_n)$ resulting from elementary tangles.*

Proof. Lemma 3.23 constructs how we can decompose a link diagram into a series of elementary tangles such that they reconstruct the link through tensor product and product operations. This construction has horizontal partitions of the plane where tangles in each horizontal partition are connected by tensor product operations. Grouping all respective elementary tangles by tensor product operations, we then apply product operations to each of these groups. This holds by Lemma 3.26 concluding our proof. □

Lemma 3.28. *Let $\mathcal{B}(T_1)$, $\mathcal{B}(T_2)$ and $\mathcal{B}(T_3)$ be three braided tangles that are compatible with respect to the composition operation $T_1 \circ T_2 \circ T_3$. Then $\mathcal{B}(T_1 \circ T_2) \circ \mathcal{B}(T_3)$ is L-equivalent to $\mathcal{B}(T_1) \circ \mathcal{B}(T_2 \circ T_3)$.*

Proof. As in the first section of the proof of Lemma 3.22, we know a braided tangle $\mathcal{B}(T_1 \circ T_2 \circ T_3)$, we know it has fewer braided arcs than $\mathcal{B}(T_1) \circ \mathcal{B}(T_2) \circ \mathcal{B}(T_3)$. Moreover, it may have less braided arcs than $\mathcal{B}(T_1 \circ T_2) \circ \mathcal{B}(T_3)$ and even $\mathcal{B}(T_1) \circ \mathcal{B}(T_2 \circ T_3)$.

We can eliminate the extra braided arcs of in a similar manner to the proof of Lemma 3.22. $\mathcal{B}(T_1 \circ T_2) \circ \mathcal{B}(T_3)$ and $\mathcal{B}(T_1) \circ \mathcal{B}(T_2 \circ T_3)$ that contain product of vertices of T_1 and T_2 along with the product vertices of T_2 and T_3 respectively which lie in the interiors of both braided tangles. Up to subdivision and tangle isotopy, both braided tangles are L-equivalent to $\mathcal{B}(T_1 \circ T_2 \circ T_3)$. Thus both desired braided tangles are L-equivalent to each other. □

Remark 3.29. Note that the results of Lemma 3.22, Lemma 3.26 and Lemma 3.28 still hold if we replace "product" by "tensor product." The proofs become trivial.

4 The Category of Braided Tangles

We outline the method of getting polynomial isotopy invariants for tangles taken from [Tur90] that is significant for how we can start to think about getting polynomial isotopy invariants for braided tangles.

4.1 The Category of Oriented Tangles

(based on [Tur90])

We define the category of oriented tangles denoted by OTa. These tangles are determines by the sequences of ϵ and v as seen in Definition 3.2.

Definition 4.1. A **tensor product** $\otimes$ for a category $\mathcal{A}$ is a covariant functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ mapping $(A, B) \to A \otimes B$ for $A, B \in \text{Ob}\ \mathcal{A}$. It also associates $(f, g) \mapsto f \otimes g$ for compatible morphisms $f, g \in \text{Mor}\ \mathcal{A}$. Moreover, this functor is universal which means let $g : (A, B) \to A \otimes B$ and so for any morphism in the category $f : (A, B) \to C$ there is a unique map $h : A \otimes B \to C$ such that $h \circ g = f$.

Definition 4.2. A **strict monoidal category** $\mathcal{A}$ is a category $\mathcal{A}$ equipped with the functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and an identity object $I \in \text{Ob}\ \mathcal{A}$. The functor is actually the tensor product defined in Definition 4.1. This tensor product must satisfy the associative property of objects and (compatible) morphisms. Lastly, the identity object must satisfy, for all $A \in \text{Ob}\ \mathcal{A}$, $A \otimes I = I \otimes A = A$ and for all morphisms $f \in \text{Mor}\ \mathcal{A}$, $f \otimes \text{Id}_I = \text{Id}_I \otimes f = f$. We usually denote a strict monoidal category by the triple $(\mathcal{A}, \otimes, I)$.

Example 4.3. The category of oriented tangles, OTa, is a strict monoidal category given by the triple $(\text{OTa}, \otimes, \varnothing)$. The objects of OTa are sequences of 1 and -1 such as the ones of ϵ and v as in Definition 3.1 representing the source and target. A morphism in OTa is a map $f : \epsilon \to v$, which is actually the isotopy equivalence class of the oriented (k, l)-tangle T where source $T = \epsilon$ and target $T = v$.

The tensor product $\otimes$ for oriented tangles is defined as:

$$\begin{aligned}
\epsilon \otimes v &= (\epsilon_1, \ldots, \epsilon_k) \otimes (v_1, \ldots, v_l) \\
&= (\epsilon_1, \ldots, \epsilon_k, v_1, \ldots, v_l)
\end{aligned}$$

The empty tangle $\varnothing \in \text{ob OTa}$ with source T = target $T = \varnothing$.

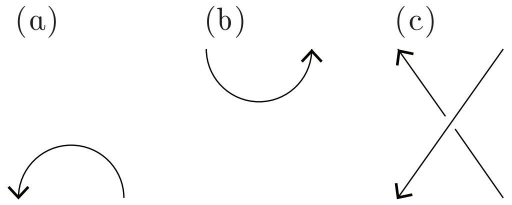


Figure 34: (a) represents morphism $(-1,1) \to \varnothing$, (b) represents morphism $\varnothing \to (-1,1)$, (c) represents morphism $(-1,1) \to (1,-1)$ and (d) represents morphism $1 \to 1$.

4.2 R-matrices

(based on [Tur90])

This section provides a brief introduction to R-matrices and their significance but do not explain how to construct one.

Definition 4.4. Given a commutative ring K, let V be a K-module, let $R : V \otimes V \to V \otimes V$ be an automorphism of the K-module $V \otimes V$ and $I : V \to V$ the identity automorphism. Then the automorphism $(R \otimes I) \circ (I \otimes R) \circ (R \otimes I) = (I \otimes R) \circ (R \otimes I) \circ (I \otimes R) : V \otimes V \otimes V \to V \otimes V \otimes V$ is called the **quantum Yang-Baxter equation** with spectral parameter zero.

Remark 4.5. The quantum Yang-Baxter equation with spectral parameter zero is one of the braid relations in the braid group presentation of Definition 1.16.

Definition 4.6. Every linear operator R satisfying the quantum Yang-Baxter equation is called a **quantum R-matrix**. This operator determines

$$R_i = I \otimes \cdots \otimes I \otimes R \otimes I \otimes \cdots \otimes I : V \otimes \cdots V \to V \otimes \cdots \otimes V$$

such that we take the first groups of tensor products of I to the $(i-1)$th tensor power, the second group of tensor products of I to the $(n-i-1)$th tensor power and so we have a map from the nth tensor power of V to the nth tensor power of V for $i = 1, \ldots, n-1$.

Remark 4.7. These R_i satisfy the same relations as the b_i for the braid group of Definition 1.16. This yields the existence of a unique homomorphism $B_n \to \mathrm{Aut}(V^{\otimes n})$ mapping $b_i \mapsto R_i$ for $i = 1, \ldots, n-1$. Here we associate a linear operator R to a braid, Turaev generalizes to associating a linear operator R to an oriented tangle. The linear operator R for tangles needs to satisfy additional properties.

4.3 Jones-Conway Polynomial

(based on [Tur90])

We outline how the Jones-Conway Polynomial for links can be obtained from the R-matrix for oriented tangles.

We set K to be the ring of Laurent polynomials $\mathbb{Z}[q^{\pm}]$ and let V be a free K-module with finite rank $m \geq 1$ and a basis $v_1, \ldots, v_m$.

Remark 4.8. As usual, let $V^* = \mathrm{Hom}_k(V, K)$ be the dual K-module with a dual basis $v_1^*, \ldots, v_m^*$.

Definition 4.9. Define the K-linear homomorphism $R : V \otimes_K V \to V \otimes_K V$ mapping

$$v_i \otimes v_j \mapsto \begin{cases} v_j \otimes v_i & i > j \\ -qv_i \otimes v_j & i = j \\ v_j \otimes v_i + (q^{-1} - q)v_i \otimes v_j & i < j \end{cases}$$

Remark 4.10.

1. R satisfies the Yang-Baxter equation.
2. $R^2 = 1 - (q - q^{-1})R$ implies that it is an isomorphism and $R^{-1} = R + (q - q^{-1}) \cdot Id_{V \otimes V}$.

Notation 4.11. The following notations are useful in Theorem 4.12 which is defines a operator that lets us create polynomial isotopy invariants for oriented tangles:

1. $\overrightarrow{b}$ denotes the K-homomorphism $K \to V^* \otimes_K V$ mapping $1 \mapsto \sum_{i=1}^m v_i^* \otimes v_i$
2. $\overleftarrow{b}$ denotes the K-homomorphism $K \to V \otimes_K V^*$ mapping $1 \mapsto \sum_{i=1}^m q^{2i-m-1} v_i \otimes v_i^*$
3. $\overrightarrow{d}$ denotes the K-homomorphism $V \otimes_K V^* \to K$ mapping $v_i \otimes v_j^* \mapsto \delta_{i,j}$
4. $\overleftarrow{d}$ denotes the K-homomorphism $V^* \otimes_K V \to K$ mapping $v_i^* \otimes v_j \mapsto q^{m+1-2i}\delta_{i,j}$

Now for every sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$ as defined for tangles in Definition 3.1, let $F(\epsilon) = (\cdots(F(\epsilon_1) \otimes_K F(\epsilon_2)) \otimes \cdots) \otimes_K F(\epsilon_k)$ with $F(1) = V$ and $F(-1) = V^*$.

Lastly we take the tensor product of the basis and dual basis in V and V^*, to form the basis of the K-module $F(\epsilon)$.

Theorem 4.12. *For $m \geq 2$, there is a unique map F_m mapping an orientable tangle T to a K-linear homomorphism $F(T) : F($ source $T) \to F($ target $T)$ satsifying:*

1. *$F(T)$ is invariant under tangle isotopy.*

2. *$F(T_1 \circ T_2) = F(T_1) \circ F(T_2)$ for all tangles T_1 and T_2 such that source T_1 = target T_2.*

3. *$F(T_1 \otimes T_2) = F(T_1) \otimes F(T_2)$ for all tangles T_1 and T_2.*

4. *$F(\uparrow) = Id_V$, $F(\downarrow) = Id_{V^*}$, $F(\curvearrowleft) = \overleftarrow{d}$, $F(\curvearrowright) = \overrightarrow{d}$, $F(\overset{\leftarrow}{\smile}) = \overleftarrow{b}$, $F(\overset{\rightarrow}{\smile}) = \overrightarrow{b}$, $F(X_+) = -q^{-m}R$ and $F(X_-) = -q^m R^{-1}$.*

Proof. The full proof is in [Tur90] using the presentation of generators and relations of OTa. □

Remark 4.13. Our map F is a linear operator which is the R-matrix from Definition 4.6. Note that F is an isotopy invariant of the tangle and that the operations of the tangle, tensor product and product, correspond to tensor product and product of the linear operator F.

Definition 4.14. The **Conway triple** as seen in Figure 35 for a link diagram L gives three links L_+, L_- and L_0 which is identical to L except for one crossing, and they differ from each other at that crossing as illustrated.

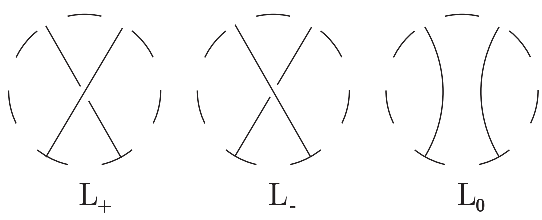


Figure 35: The Conway Triple is used for polynomial invariants of links.

Replacing oriented tangles with links, then our R-matrix operator is actually the Jones-Conway Polynomial as defined in Theorem 4.15. In short, for an orientable link L, the K-linear homomorphism $F_m(L) : K \to K$ acts by multiplying an element of the ring K and denote this new element by $F_m(L)$. We can calculate this for $m \geq 1$ via the Jones-Conway polynomial.

Theorem 4.15. *The **Jones-Conway polynomial** is the resulting polynomial of the unique map V from the set of orientable links in $\mathbb{R}^3$ to the Laurent ring in two variables $\mathbb{Z}[x^{\pm}, y^{\pm}]$. This is calculated by:*

1. *$V(L)$, often denoted by V_L is invariant under link isotopy.*

2. V maps the trivial knot to 1.

3. We use the skein formula using the Conway triple, $xV_{L_+} - x^{-1}V_{L_-} = yV_{L_0}$.

Proof. The proof using Theorem 4.12 is seen in full in [Tur90]. □

Remark 4.16. 1. The traditional Jones polynomial is replaces the third property above by $\frac{1}{t}V_{L_+} - tV_{L_-} = (\sqrt{t} + \frac{1}{\sqrt{t}})V_{L_0}$ for the Laurent ring in one variable $\mathbb{Z}[\sqrt{t}^{\pm}]$. See [Jon19] and [Jon85] regarding how the Jones polynomial derived from the trace defined on Temperley-Lieb algebras and Markov's Theorem 2.9.

2. The Jones-Conway polynomial is also called the **HOMFLY** or **HOMFLY-PT polynomial**.

The Jones-Conway polynomial can also be used to determine the K-linear homomoprhism $F_m(L) : K \to K$ as shown in [Tur90].

4.4 The Category of Braided Tangles

Using results from Section 3.3, we show how braided tangles forms a strict monoidal category similar to oriented tangles.

For tangle T, the smaller tangle T' has a distinguished sequence for the source and target from T. However, every T' has +1 for its sequence since we can always isotopy the horizontal arcs upward so that the original vertices on the boundary of T' are tilted positively. In the combinatorial setting, the vector at a point is denoted +1 if the angle of the vector lies in $(0, \pi]$, and it is −1 for $(\pi, 2\pi]$.

Thus for a tangle T, the braided tangle at its larger boundary consisting of product vertices, shares the same source and target sequences as T and we apply the product and tensor product operation similarly. This easily deduces that $(BTa, \otimes, \varnothing)$ is a strict monoidal category.

Further research involves determining a R-matrix for BTa as we did for OTa.

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