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Abstract

Two main constructions of local Fourier bases are presented. The first consists in constructing a simple orthonormal Wilson basis $\psi_{m,n}$ where both $\psi_{m,n}$ and its Fourier transform $\hat{\psi}_{m,n}$ have exponential decay. These are modifications of a Gabor system in a way that the redundancy of a Gabor frame is removed but the good time-frequency localization is preserved. We describe three equivalent ways to show that $\psi_{m,n}$ is an orthonormal basis, two of them using frame theory and Gabor analysis.

The second construction is based on the idea of finding smooth orthogonal projections of functions over intervals. For every interval I we can construct several orthonormal bases for $L^2(I)$ consisting of trigonometric functions multiplied by the characteristic function of I and, given a partition $\{\alpha_k\}$ of $\mathbb{R}$, these bases can be patched together to form an orthonormal basis of $L^2(\mathbb{R})$. To improve the frequency localization, we show that we can replace the characteristic function of each interval in the partition by a smooth bell function $b_k \in \mathcal{C}^N(\mathbb{R})$ with compact support for $N \in \mathbb{N} \cup \{\infty\}$. These bases are called "local Fourier bases".

In the case of a uniform partition and considering a suitable parity, Wilson bases are a particular case of local Fourier bases.

Two applications of local Fourier bases are presented. Firstly, we show that local Fourier bases are unconditional bases for the modulation spaces on $\mathbb{R}$ and, as a consequence, the function spaces defined by the approximation with respect to a local Fourier bases are the modulation spaces. The second application consists in the use of local Fourier bases for the extraction of a gravitational waves.

Zusammenfassung

Es werden zwei Hauptkonstruktionen von lokalen Fourierbasen erläutert. Die erste besteht darin, eine einfache Wilson-Orthonormalbasis $\psi_{m,n}$ zu bilden, wobei sowohl $\psi_{m,n}$ als auch die entsprechende Fouriertransformation $\hat{\psi}_{m,n}$ exponentiell abfallen. Solche Basen entstehen aus der Modifikation eines Gaborframes, sodass die Redundanz des Gaborsystems behoben wird, aber die gute Zeitlokalisierung erhalten bleibt. Wir präsentieren drei äquivalente Beweise dafür, dass $\psi_{m,n}$ eine Orthonormalbasis ist; in zwei Fällen werden Frametheorie und Gaboranalysis verwendet.

Die zweite Konstruktion beruht auf der Grundidee, glatte orthogonale Projektionen von Funktionen auf Intervallen zu finden. Für jedes Intervall I können wir mehrere Orthonormalbasen von $L^2(I)$ konstruieren, die aus trigonometrischen Funktionen multipliziert mit der charakteristischen Funktion von I bestehen. Für eine gegebene Partition $\{\alpha_k\}$ von $\mathbb{R}$ können diese Basen zu einer Orthonomalbasis von $L^2(\mathbb{R})$ „zusammengeklebt“ werden. Um die Lokalisierungsfrequenz zu verbessern, zeigen wir, dass man die charakteristische Funktion auf jedem Intervall in der Partition durch eine glatte Funktion $b_k \in \mathcal{C}^N(\mathbb{R})$ mit kompaktem Träger für $N \in \mathbb{N} \cup \{\infty\}$ ersetzen kann. Diese Basen werden „lokale Fourierbasen“ genannt.

Im Falle einer gleichmäßigen Partition mit passender Parität sind Wilsonbasen ein Spezialfall von lokalen Fourierbasen.

Schließlich werden zwei Anwendungen von lokalen Fourierbasen präsentiert. Erstens wird gezeigt, dass lokale Fourierbasen unbedingte Basen von den Modulationsräumen auf $\mathbb{R}$ darstellen; es folgt daraus, dass Funktionsräume, die durch die Approximation bezüglich einer lokalen Fourierbasis definiert sind, die Modulationsräume sind. Die zweite Anwendung besteht in der Verwendung von lokalen Fourierbasen bei der Messung von Gravitationswellen.

Contents

Introduction

One of the goals in signal processing and time-frequency analysis is to find a convenient series expansion of functions in $L^2(\mathbb{R})$. A first attempt could be the so called Gabor system $\{e^{2\pi imx}\chi_{[0,1]}(x-n)\}_{m,n\in\mathbb{Z}} = \{M_mT_n\chi_{[0,1]}(x)\}_{m,n\in\mathbb{Z}}$ which forms an orthonormal basis of $L^2(\mathbb{R})$. Note that all elements in the basis consists of translation and modulation of $\chi_{[0,1]}$. This example presents one of the limitations on the properties we expect from a Gabor basis. In fact, computing the Fourier transform of $\chi_{[0,1]}$, we see that

$$\hat{\chi}_{[0,1]}(\omega) = \int_0^1 e^{-2\pi ix\omega}dx = e^{-\pi i\omega}\frac{\sin(\pi\omega)}{\pi\omega}$$

oscillates and has slow decay. These considerations and the discontinuity of $\chi_{[0,1]}$, make the orthonormal basis $\{M_mT_n\chi_{[0,1]}(x)\}_{m,n\in\mathbb{Z}}$ unattractive for its application in time-frequency analysis.

A natural consequence is to ask if we can replace the characteristic function by a continuous function g and still obtain an orthonormal basis. For this aim, in the 1946, in his famous article “Theory of communication”, Gabor proposed to decompose every signal with respect to time-frequency shifts of the Gaussian $g(t) = 2^{\frac{1}{4}}e^{-\pi t^2}$ as

$$M_{mb}T_{na}g(x) = e^{2\pi imbx}g(x-na),\ m,n\in\mathbb{Z}. \tag{1}$$

with $a = b = 1$.

Unfortunately, in the 1980s, the Balian-Low theorem states that functions of type (1) can only be orthonormal bases if

$$\left(\int_{\mathbb{R}} x^2|g(x)|dx\right)\left(\int_{\mathbb{R}} \omega^2|\hat{g}(\omega)|d\omega\right) = \infty.$$

This means that a function g generating a Gabor orthonormal basis cannot be well localized in both time and frequency. Moreover, the proof of the Ron-Shen duality principle for Gabor frames in the mid 1990s, had some important consequence in stating the necessary condition for a Gabor system of type (1) to be a frame. In particular, $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame only if $ab \leq 1$, and, at $ab = 1$ a normalized tight Gabor frame is an orthonormal basis. Hence, if we abandon the requirement that (1) is a basis to improve the time-frequency localization, we obtain a highly redundant expansion with $ab < 1$.

Several researchers tried to overcome the negative aspect of this theorem, in particular, two different constructions coming from distinct motivations are probably the most striking and constitute the core of this thesis. The first construction came from quantum mechanics and followed the idea of K. Wilson [30]. In 1987, he proposed an idea on how to overcome the Balian-Low barrier: replacing the exponential $e^{2\pi imbx}$ with sines and cosines, allowing the localization of the functions of the basis around two frequencies with opposite sign. This construction has been simplified by Ingrid Daubechies, Stéphane Jaffard and Jean-Lin Journé in [15] to yield what is now called a Wilson basis $\{\Psi_{l,n}\}_{l\in\mathbb{N},n\in\mathbb{Z}}$ defined by

$$\begin{cases} \Psi_{0,n}(x) = \phi(x-n), \\ \Psi_{l,n}(x) = \frac{1}{\sqrt{2}} e^{\pi iln}[M_l + (-1)^{l+n} M_{-l}] T_{\frac{n}{2}} \phi(x) \\ \qquad\qquad = \sqrt{2}\phi\left(x - \frac{n}{2}\right) \begin{cases} \cos(2\pi lx), & \text{if } l+n \text{ is even,} \\ \sin(2\pi lx), & \text{if } l+n \text{ is odd.} \end{cases} \\ l \in \mathbb{N}\setminus\{0\},\ n \in \mathbb{Z} \end{cases} \tag{2}$$

where ϕ is real and even, both ϕ and its Fourier transform $\hat{\phi}$ have exponential decay and can be constructed as a rapidly converging superposition of Gaussians. Moreover, both $\Psi_{l,n}$ and $\hat{\Psi}_{l,n}$ have exponential decay. The construction produces an orthonormal basis that possesses the desired time-frequency localization while keeping much of the structure of a Gabor system. In Chapter 2 we will reformulate this approach following [15].

In particular, it is proved that for every $\phi \in L^2(\mathbb{R})$ such that $||\phi(x)||_2 = 1$, $\{M_{\frac{m}{2}} T_n \phi\}_{m,n\in\mathbb{Z}}$ being a tight frame is equivalent to $\{\Psi_{l,n}\}_{l\in\mathbb{N},n\in\mathbb{Z}}$ being an orthonormal basis. We have that the transition from a tight Gabor frame of redundancy 2 to the Wilson system (2) removes incredibly the redundancy and leads to a basis for $L^2(\mathbb{R})$. Moreover, there exists $\phi \in \mathcal{C}^N(\mathbb{R})$ with $N \in \mathbb{N}\cup\{\infty\}$ and compact support and $\phi \in \mathcal{S}(\mathbb{R})$ such that $\{\Psi_{l,n}\}_{l\in\mathbb{N},n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.

The second construction was first made by Malvar [24] and emerged in the context of subband coding theory: he wanted to eliminate the aliasing effect of subband coding when two blocks overlap in the block by block discrete cosine transform. His solution consists in a modulated lapped transform which cancels the aliasing effects and allows a perfect reconstruction. These bases were independently formulated in a generalized form by Coifman and Meyer [13]. The paper [2] by Auscher, Weiss and Wickerhauser shows that these two apparently different constructions are actually particular cases of a family called "local Fourier bases". They proved that for a given partition $\{\alpha_j\}_{j\in\mathbb{Z}}$ with interval length $l_j = \alpha_{j+1} - \alpha_j$ and a sequence $\varepsilon_j > 0$ such that $\alpha_j + \varepsilon_j \leq \alpha_{j+1} - \varepsilon_{j+1}$, then for any smoothness $N \in \mathbb{N}\cup\{\infty\}$, bell functions $b_{[\alpha_j,\alpha_{j+1}]} \in \mathcal{C}^N(\mathbb{R})$ with $\operatorname{supp} b_{[\alpha_j,\alpha_{j+1}]} \subseteq [\alpha_j - \varepsilon_j, \alpha_{j+1} + \varepsilon_{j+1}]$ can be constructed such that with a suitable choice of "parity" each of the systems

(i) $\left\{\sqrt{\frac{2}{l_j}} b_{[\alpha_j,\alpha_{j+1}]}(x) \sin\left(\frac{2k+1}{2}\frac{\pi}{l_j}(x-\alpha_j)\right),\ k \in \mathbb{N}\cup\{0\},\ j \in \mathbb{Z}\right\}$;

(ii) $\left\{\sqrt{\frac{2}{l_j}} b_{[\alpha_j,\alpha_{j+1}]}(x) \sin\left(k\frac{\pi}{l_j}(x-\alpha_j)\right),\ k \in \mathbb{N},\ j \in \mathbb{Z}\right\}$;

(iii) $\left\{\sqrt{\frac{2}{l_j}}b_{[\alpha_j,\alpha_{j+1}]}(x)\cos\left(\frac{2k+1}{2}\frac{\pi}{l_j}(x-\alpha_j)\right),\ k\in\mathbb{N}\cup\{0\},\ j\in\mathbb{Z}\right\}$;

(iv) $\left\{\sqrt{\frac{1}{l_j}}b_{[\alpha_j,\alpha_{j+1}]}(x), \sqrt{\frac{2}{l_j}}b_{[\alpha_j,\alpha_{j+1}]}(x)\cos\left(k\frac{\pi}{l_j}(x-\alpha_j)\right),\ k\in\mathbb{N},\ j\in\mathbb{Z}\right\}$

forms an orthonormal basis for $L^2(\mathbb{R})$. Each of (i)-(iv) is called local Fourier basis. Thanks to the support condition, these functions are well localized on $[\alpha_j, \alpha_{j+1}]$ and by the smoothness of $b_{[\alpha_j,\alpha_{j+1}]}$, they are well localized in the frequency-plane. This construction will be discussed in Chapter 3.

This thesis is organized as follows: Chapter 1 presents a collection of preliminaries. We establish the main definitions and results about the Fourier transform and the basics of frame theory in Hilbert spaces. In addition to some characterization of Riesz bases and Riesz sequences, a short introduction to Gabor analysis and the duality principle is presented. Finally, the Zak transform and the modulation spaces are introduced. Chapter 2 consists in the construction of Wilson bases by Daubechies, Jaffard and Journé. Chapter 3 includes the construction of local Fourier bases by Auscher, Weiss and Wickerhauser and, in addition, we will show that the Wilson bases constructed by Daubechies, Jaffard and Journé are actually a special case of those constructed by Auscher, Weiss and Wickerhauser following [1].

Chapter 4 and Chapter 5 describe two interesting applications of the local Fourier bases. In Chapter 4, we will show that Wilson orthonomal bases as constructed by Daubechies, Jaffard and Journé are unconditional bases for the modulation spaces on $\mathbb{R}$ and, as a consequence, the abstract function spaces defined by the approximation properties with respect to a local Fourier basis are the modulation spaces. In Chapter 5, following the paper of Chassande-Mottin, Jaffard and Meyer [11], we explain how local Fourier bases played an important role in the algorithm for the extraction of the first detected gravitational wave in September 2015.

Chapter 1

Prerequisites

In this chapter we present a collection of results which will be useful to understand the most important topic of this thesis. In the first section we review the main points about the Fourier transform, in the second we recall the basics of frame theory in Hilbert spaces and in the third section we consider some important results in Gabor analysis such as the duality principle. In the fourth section we present the Zak transform and its properties and in the last section we recall the basics of modulation spaces.

1.1 Fourier Transform

In this first section we refer to Chapter 1 of the book by Gröchenig "Foundations of Time-Frequency Analysis" [19] and we present the fundamentals of Fourier analysis needed to understand the main results of this thesis. Most of the theorems are stated without proof.

Definition 1.1. Let $f \in L^1(\mathbb{R}^d)$. For $\omega \in \mathbb{R}^d$, we define the *Fourier transform of f at ω* by

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \omega} dx.$$

Note that, since $|\hat{f}(\omega)| \leq \int_{\mathbb{R}^d} |f(x)| = ||f||_1$, then $\hat{f}$ is well-defined for any ω. We can now present some basic properties of the Fourier transform.

Lemma 1.1 (Properties of the Fourier Transform). *Let $f \in L^1(\mathbb{R}^d)$, $\mu, \eta \in \mathbb{R}^d$, $s \in (0, \infty)$, $A \in GL(d, \mathbb{R})$. The following holds:*

(i) Let $T_\mu f(x) = f(x - \mu)$ be the translation operator. Then $\widehat{T_\mu f}(\omega) = e^{-2\pi i \mu \cdot \omega} \hat{f}(\omega)$.

(ii) Let $M_\eta f(x) = e^{2\pi i \eta \cdot x} f(x)$ be the modulation operator. Then $\widehat{M_\eta f}(\omega) = T_\eta \hat{f}(\omega)$.

(iii) Let $D_s f(x) = s^{-\frac{d}{2}} f(\frac{x}{s})$ be the dilation operator. Then $\widehat{D_s f}(\omega) = D_{\frac{1}{s}} \hat{f}(\omega)$.

(iv) Let $f^*(x) = \overline{f(-x)}$ be the involution. Then $\widehat{f^*}(\omega) = \overline{\hat{f}(\omega)}$.

(v) Let $U_A f(x) = |\det(A)|^{-1} f(A^{-1}x)$. *Then* $\widehat{U_A f}(\omega) = \hat{f}(A^T\omega)$.

From the definition it is easy to see that $||\hat{f}||_\infty \leq ||f||_1$ and the following results hold:

Lemma 1.2 (Riemann-Lebesgue). *If* $f \in L^1(\mathbb{R}^d)$*, then* $\hat{f}$ *is uniformly continuous and* $\lim_{|\omega|\to\infty} |\hat{f}(\omega)| = 0$.

Lemma 1.3. *Let* $f, g \in L^1(\mathbb{R}^d)$. *Then* $\int_{\mathbb{R}^d} \hat{f}(\omega)g(\omega)d\omega = \int_{\mathbb{R}^d} f(y)\hat{g}(y)dy$.

We can state an inversion formula for the Fourier transform.

Theorem 1.4 (Inversion Formula). *Assume* $f \in L^1(\mathbb{R}^d)$ *and* $\hat{f} \in L^1(\mathbb{R}^d)$. *Then*

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\omega) e^{2\pi i x\cdot\omega} d\omega \quad \text{for a.e. } x \in \mathbb{R}^d.$$

Proof (Sketch). Choose $g(\omega) = e^{-\delta\pi|\omega|^2} e^{2\pi i x\cdot\omega} = M_x e^{-\delta\pi|\omega|^2}$. Then, $\hat{g}(y) = \delta^{-\frac{d}{2}} e^{-\frac{\pi|y-x|^2}{\delta}}$. We apply the previous lemma to a given f to obtain

$$\underbrace{\int_{\mathbb{R}^d} \hat{f}(\omega) e^{-\delta\pi|\omega|^2} e^{2\pi i x\cdot\omega} d\omega}_{LHS} = \int_{\mathbb{R}^d} \hat{f}(\omega) g(\omega) d\omega = \int_{\mathbb{R}^d} f(y)\hat{g}(y)dy = \underbrace{\delta^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y) e^{-\frac{\pi|y-x|^2}{\delta}} dy}_{RHS}.$$

Choose $\delta_n > 0$, $\delta_n \longrightarrow 0$ and consider the LHS: since the integrand is bounded by $|\hat{f}(\omega)| \in L^1(\mathbb{R}^d)$ and converges pointwise to $\hat{f}(\omega)e^{2\pi i x\cdot\omega}$, then by the dominated convergence the LHS tends to $\int_{\mathbb{R}^d} \hat{f}(\omega)e^{2\pi i x\cdot\omega}d\omega$. Since the RHS is a convolution between f and an approximate identity, it tends to $f(x)$ almost everywhere and $f(x) = \int_{\mathbb{R}^d} \hat{f}(\omega)e^{2\pi i x\cdot\omega}d\omega$ almost everywhere. □

If we abandon the requirement that the Fourier transform is defined pointwise by Definition 1.1, we can extend it to other spaces.

Theorem 1.5 (Plancherel). *If* $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ *then*

$$||f||_2 = ||\hat{f}||_2.$$

Moreover, the Fourier transform extends to a unitary operator on $L^2(\mathbb{R}^d)$ *and satisfies Parseval's formula*

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle \quad \text{for all } f, g \in L^2(\mathbb{R}^d). \tag{1.1}$$

Finally we recall the Poisson summation formula which relates the Fourier series with the Fourier transform on $\mathbb{R}^d$.

Theorem 1.6. *Assume that for* $C, \varepsilon > 0$, *we have* $|f(x)| \leq C(1+|x|)^{-d-\varepsilon}$ *and* $|\hat{f}(\omega)| \leq C(1+|\omega|)^{-d-\varepsilon}$ *for all* $x, \omega \in \mathbb{R}^d$. *Then,* f *and* $\hat{f}$ *are continuous and*

$$\sum_{k\in\mathbb{Z}^d} f(x+k) = \sum_{k\in\mathbb{Z}^d} \hat{f}(k) e^{2\pi i k\cdot x}. \tag{1.2}$$

The identity holds pointwise for all $x \in \mathbb{R}^d$*, and both sums converge absolutely for all* $x \in \mathbb{R}^d$.

The conditions on the dacay of f and $\hat{f}$ are needed for the absolute convergence of the sums and the pointwise validity of (1.2). A weaker version of Poisson summation formula is obtained by replacing the absolute convergence of the first sum by convergence in $L^2(\mathbb{R}^d)$ and pointwise equality by equality almost everywhere. We have that: if $\sum_{k\in\mathbb{Z}^d} f(x+k) \in L^2(\mathbb{T}^d)$ and $\sum_{k\in\mathbb{Z}^d} |\hat{f}(k)|^2 < \infty$, then (1.2) holds almost everywhere.

1.1.1 Periodic Analytic Functions

It will be useful to recall the following theorem from Chapter 3 of the textbook by Simon [27] which is a consequence of the Cauchy integral formula related to Fourier series. In particular we will use Theorem 1.7 in Chapter 2.

Theorem 1.7. *Let $a, b > 0$ and $\Omega_{a,b} = \{z \in \mathbb{C} : -a < Im(z) < b\}$. Then any analytic function f on $\Omega_{a,b}$ satisfying $f(z+1) = f(z)$ has an expansion*

$$f(z) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ikz} \tag{1.3}$$

converging uniformly on compact subsets of $\Omega_{a,b}$ and such that for any $y \in (-a, b)$

$$c_k = \int_0^1 f(x+iy)e^{-2\pi ik(x+iy)}dx.$$

Moreover, for any $\varepsilon > 0$, there exists a C_ε such that

$$|c_k| \le C_\varepsilon \min\{e^{-2\pi(a-\varepsilon)k}, e^{2\pi(b-\varepsilon)k}\} \text{ for all } k. \tag{1.4}$$

Conversely, if $\{c_k\}_{k\in\mathbb{Z}}$ is a sequence obeying (1.4) for all k and ε, then the series (1.3) converges on compact subsets of $\Omega_{a,b}$ and defines an analytic function f obeying the periodicity property $f(z+1) = f(z)$.

Remark 1.1. For periodic functions, analyticity conditions are equivalent to exponential decay hypotheses on its Fourier series coefficients. If one drops the periodicity requirement and replaces Fourier series by Fourier transform, there are analogous theorems associated with the work of Paley and Wiener.

1.2 Frame Theory

Bases are very important when studying Banach spaces and Hilbert spaces. The main feature of a basis $\{e_k\}_{k\in\mathbb{Z}}$ of elements in a Banach space B is that every element f of the space has a unique expansion in terms of the elements of the basis.

Definition 1.2. Let B be a Banach space. A sequence $\{e_k\}_{k\in\mathbb{Z}}$ in B is a *(Schauder) basis* for B if, for every $f \in B$, there exist unique coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ such that

$$f = \sum_{k\in\mathbb{Z}} c_k(f)e_k. \tag{1.5}$$

In particular, (1.5) means that the series $f = \sum_{k\in\mathbb{Z}} c_k(f)e_k$ converges with respect to the chosen order of the elements.

There is another important type of convergence for which rearrangements of the elements or interchanging a summation with the action of a linear operator are permitted.

Definition 1.3. Let $\{f_k\}_{k\in\mathbb{Z}}$ be a countable set in a Banach space B. The series $\sum_{k\in\mathbb{Z}} f_k$ is said to *converge unconditionally* to $f \in B$ if for every $\varepsilon > 0$ there exists a finite set $F_0 \subseteq \mathbb{Z}$ such that

$$\Big|\Big|f - \sum_{k\in F} f_k\Big|\Big|_B < \varepsilon \quad \text{for all finite sets } F \supseteq F_0.$$

This means that the net of partial sums $s_F = \sum_{k\in F} f_k$ converges to f.

If (1.5) converges unconditionally for each $f \in B$, we call $\{e_k\}_{k\in\mathbb{Z}}$ an *unconditional basis.*

Definition 1.4. A countable set $\{e_k\}_{k\in\mathbb{Z}}$ of vectors in a Banach space B is called an *unconditional basis* for B if

(i) the finite linear combinations of e_k's span a dense subspace of B, and

(ii) there exists $C > 0$, such that for all $\mu = \{\mu_k\}_{k\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ and all finite sequences $\{c_k\}_{k\in\mathbb{Z}}$,

$$\left|\left|\sum_{k\in\mathbb{Z}} c_k\mu_k e_k\right|\right| \leq C||\mu||_\infty \left|\left|\sum_{k\in\mathbb{Z}} c_k e_k\right|\right|.$$

In other words, if $\{e_k\}_{k\in\mathbb{Z}}$ is a basis which is not unconditional, there exists a permutation σ for which $\{e_{\sigma(k)}\}_{k\in\mathbb{Z}}$ is not a basis.

Unfortunately, the conditions to a basis can be very difficult to satisfy and for this reason it can be convenient to use frames.
A frame is a sequence of elements $\{f_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$ which satisfies (1.5), but the coefficients need not be unique. Frames are widely use in signal analysis to reconstruct the information from a signal. Here we will give a brief review on the main concepts about frames and Gabor frames, stated without proof. For more details, see the book of Ole Christensen [12].

1.2.1 Bases and Bessel sequences

Lemma 1.8. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a sequence in $\mathcal{H}$ and suppose that $\sum_{k\in\mathbb{Z}} c_k f_k$ is convergent for all $\{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Then*

$$T : \ell^2(\mathbb{Z}) \to \mathcal{H}, \quad T\{c_k\}_{k\in\mathbb{Z}} = \sum_{k\in\mathbb{Z}} c_k f_k \tag{1.6}$$

is a well-defined bounded operator. The adjoint operator is given by

$$T^* : \mathcal{H} \to \ell^2(\mathbb{Z}), \quad T^* f = \{\langle f, f_k \rangle\}_{k\in\mathbb{Z}}. \tag{1.7}$$

Furthermore,

$$\sum_{k\in\mathbb{Z}} |\langle f, f_k \rangle|^2 \leq ||T||^2 \; ||f||^2, \; \forall f \in \mathcal{H}.$$

Definition 1.5. A sequence $\{f_k\}_{k\in\mathbb{Z}}$ of elements in a Hilbert space $\mathcal{H}$ is called a *Bessel sequence* if there exists a constant $B > 0$ such that

$$\sum_{k\in\mathbb{Z}} |\langle f, f_k \rangle|^2 \leq B||f||^2, \; \forall f \in \mathcal{H},$$

The next theorem shows that the Bessel condition can be expressed in terms of the operator T in (1.6).

Theorem 1.9. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a sequence in $\mathcal{H}$ and $B > 0$ be given. Then $\{f_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence with bound B if and only if*

$$T : \{c_k\}_{k\in\mathbb{Z}} \to \sum_{k\in\mathbb{Z}} c_k f_k$$

is a well-defined bounded operator from $\ell^2(\mathbb{Z})$ into $\mathcal{H}$ and $||T|| \leq \sqrt{B}$.

Definition 1.6. A sequence $\{e_k\}_{k\in\mathbb{Z}}$ of elements in a Hilbert space $\mathcal{H}$ is an *orthonormal system* if

$$\langle e_k, e_j \rangle = \delta_{k,j}.$$

An *orthonormal basis* is an orthonormal system $\{e_k\}_{k\in\mathbb{Z}}$ which is a basis for $\mathcal{H}$.

Remark 1.2. Note that an orthonormal system $\{e_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence.

The next theorem gives equivalent conditions for an orthonormal system $\{e_k\}_{k\in\mathbb{Z}}$ to be an orthonormal basis.

Theorem 1.10. *Let $\{e_k\}_{k\in\mathbb{Z}}$ be an orthonormal system in a Hilbert space $\mathcal{H}$, then the following are equivalent:*

(i) $\{e_k\}_{k\in\mathbb{Z}}$ *is an orthonormal basis.*

(ii) $f = \sum_{k\in\mathbb{Z}} \langle f, e_k \rangle e_k, \; \forall f \in \mathcal{H}.$

(iii) $\langle f, g \rangle = \sum_{k\in\mathbb{Z}} \langle f, e_k \rangle \langle e_k, g \rangle, \; \forall f, g \in \mathcal{H}.$

(iv) $\sum_{k\in\mathbb{Z}} |\langle f, e_k \rangle|^2 = ||f||^2, \; \forall f \in \mathcal{H}.$

(v) $\overline{span}\{e_k\}_{k\in\mathbb{Z}} = \mathcal{H}.$

(vi) If $\langle f, e_k \rangle = 0, \; \forall k \in \mathbb{N},$ *then* $f = 0.$

Lemma 1.11. *Let* $\{e_k\}_{k\in\mathbb{Z}}$ *be a sequence of elements in a Hilbert space* $\mathcal{H}$*. Suppose the following properties hold:*

$$||e_k|| = 1, \ k \in \mathbb{Z} \quad \textit{and} \tag{1.8}$$

$$\sum_{k\in\mathbb{Z}} \langle g, e_k\rangle\langle e_k, h\rangle = \langle g, h\rangle, \ \forall g, h \in L^2(\mathbb{R}). \tag{1.9}$$

Then $\{e_k\}_{k\in\mathbb{Z}}$ *is an orthonormal basis.*

Proof. We first show that $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal system. Consider (1.8) and (1.9) with $g = h = e_j$, we have:

$$1 \overset{(1.8)}{=} ||e_j||^2 \overset{(1.9)}{=} \sum_{k\in\mathbb{Z}} |\langle e_j, e_k\rangle|^2 = 1 + \sum_{k\neq j} |\langle e_j, e_k\rangle|^2.$$

We have $\sum_{k\neq j} |\langle e_j, e_k\rangle|^2 = 0$ which implies $\langle e_k, e_j\rangle = \delta_{k,j}$. By Definition 1.6, $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal system. By Theorem 1.10, condition (1.9) is equivalent to $\{e_k\}_{k\in\mathbb{Z}}$ being an orthonormal basis if $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal system. □

Definition 1.7. Two sequences $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$ are called *biorthogonal* if

$$\langle g_k, f_j\rangle = \delta_{k,j}, \ \forall j, k \in \mathbb{Z}.$$

In particular, if a biorthogonal sequence for $\{f_k\}_{k\in\mathbb{Z}}$ exists, it is uniquely determined if and only if $\{f_k\}_{k\in\mathbb{Z}}$ is complete in $\mathcal{H}$.

Theorem 1.12. *Assume that* $\{e_k\}_{k\in\mathbb{Z}}$ *is a (Schauder) basis for the Hilbert space* $\mathcal{H}$*. Then there exists a unique family* $\{g_k\}_{k\in\mathbb{Z}}$ *in* $\mathcal{H}$ *such that*

$$f = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle e_k, \ \forall f \in \mathcal{H}.$$

Moreover, $\{g_k\}_{k\in\mathbb{Z}}$ *is a basis for* $\mathcal{H}$*, and* $\{e_k\}_{k\in\mathbb{Z}}$ *and* $\{g_k\}_{k\in\mathbb{Z}}$ *are biorthogonal.*

1.2.2 Riesz basis

Definition 1.8. A sequence $\{f_k\}_{k\in\mathbb{Z}}$ of a Hilbert space $\mathcal{H}$ is a *Riesz sequence* if there exist bounds $A, B > 0$ such that for all finite sequences $c \in \ell^2(\mathbb{Z})$,

$$A||c||^2 \leq \left|\left|\sum_{k\in\mathbb{Z}} c_k f_k\right|\right|^2 \leq B||c||^2.$$

A Riesz sequence which generates the whole space $\mathcal{H}$ is called a *Riesz basis* for $\mathcal{H}$.

The following lemma gives equivalent conditions for a sequence to be a Riesz basis.

Lemma 1.13. *For a sequence $\{f_k\}_{k\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$, the following conditions are equivalent:*

(i) $\{f_k\}_{k\in\mathbb{Z}}$ *is a Riesz basis for $\mathcal{H}$.*

(ii) There is an equivalent inner product on $\mathcal{H}$ for which $\{f_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{H}$.

(iii) $\{f_k\}_{k\in\mathbb{Z}}$ *is a complete Bessel sequence, and it has a complete biorthogonal sequence $\{g_k\}_{k\in\mathbb{Z}}$ which is also a Bessel sequence.*

An interesting result about Riesz sequences that will come in handy later is the following:

Proposition 1.14. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a Bessel sequence in $\mathcal{H}$. Then the following are equivalent:*

(i) $\{f_k\}_{k\in\mathbb{Z}}$ *is a Riesz sequence with lower bound A;*

(ii) $\{f_k\}_{k\in\mathbb{Z}}$ *has a biorthogonal system $\{g_k\}_{k\in\mathbb{Z}}$ which is a Bessel sequence with bound A^{-1}.*

1.2.3 Frames

Definition 1.9. A sequence $\{f_k\}_{k\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A||f||^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leq B||f||^2, \ \forall f \in \mathcal{H}, \tag{1.10}$$

where A, B are called *frame bounds.*

If $A = B$, then $\{f_k\}_{k\in\mathbb{Z}}$ is called a *tight frame.*
Since a frame $\{f_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence, the operator T defined by (1.6) is well-defined and T is called the *synthesis operator* or the *pre-frame operator.* The adjoint T^* defined by (1.7) is called the *analysis operator.*
By composing T and T^*, we obtain the *frame operator*

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = TT^*f = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle f_k \tag{1.11}$$

which is positive, bounded, invertible and, by (1.10), satisfies

$$A\langle f, f\rangle \leq \langle Sf, f\rangle \leq B\langle f, f\rangle, \quad \forall f \in \mathcal{H}. \tag{1.12}$$

Moreover, its inverse S^{-1} is positive and has a self-adjoint square root $\{S^{-\frac{1}{2}}\}$ such that $\{S^{-\frac{1}{2}} f_k\}_{k\in\mathbb{Z}}$ is a tight frame. The sequence $\{S^{-1}f_k\}_{k\in\mathbb{Z}}$ is again a frame and is called a *canonical dual frame* for $\{f_k\}_{k\in\mathbb{Z}}$ in the sense that

$$f = \sum_{k\in\mathbb{Z}} \langle f, S^{-1}f_k\rangle f_k = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle S^{-1} f_k, \quad \forall f \in \mathcal{H}, \tag{1.13}$$

and both series converge unconditionally for all $f \in \mathcal{H}$.

1.3 Gabor Analysis in $L^2(\mathbb{R})$

The purpose of Gabor analysis is to represent functions $f \in L^2(\mathbb{R})$ as a superposition of translated and modulated versions of a fixed function $g \in L^2(\mathbb{R})$, where the translation and modulation operators are those described in Lemma 1.1.

Definition 1.10. Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ we call *Gabor system* a collection of the form $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, more explicitly

$$M_{mb}T_{na}g(x) = e^{2\pi imbx}g(x-na),\ x \in \mathbb{R}.$$

The Gabor system $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ only involves translates with parameters na, $n \in \mathbb{Z}$ and modulations with parameters mb, $m \in \mathbb{Z}$. The points $\{(na, mb)\}_{m,n\in\mathbb{Z}}$ form a *lattice* in $\mathbb{R}^2$.

Definition 1.11. A *Gabor frame* is a frame for $L^2(\mathbb{R})$ of the form $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with $a, b > 0$ and a fixed function $g \in L^2(\mathbb{R})$.

We state now a necessary condition for a Gabor system to be a frame.

Theorem 1.15. *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the following hold:*

(i) Assume $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then $ab \leq 1$.

(ii) Assume that $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Then $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Riesz basis if and only if $ab = 1$.

Note that the assumption $ab \leq 1$ is not enough for $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame and, in particular, the Theorem shows that it is only possible for $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame if $ab \leq 1$; and, assuming that $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame, it is overcomplete if and only if $ab < 1$. As a quantitative measure of the overcompleteness we use the redundancy:

Definition 1.12. Given a Gabor frame $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, the number $(ab)^{-1}$ is called the *redundancy.*

A result that relates the parameters a and b with the frame bounds is the following, proved by Daubechies in [14]:

Proposition 1.16. *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. If $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame with bounds A, B, then*

$$A \leq \frac{||g||^2}{ab} \leq B.$$

If $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a tight Gabor frame, then $A = \frac{||g||^2}{ab}$.

For a Gabor frame $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with associated frame operator S, the frame decomposition in (1.13) shows that, for every $f \in L^2(\mathbb{R})$,

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, S^{-1}M_{mb}T_{na}g\rangle M_{mb}T_{na}g. \tag{1.14}$$

We know that frames are particularly useful when the frame decomposition takes a simple form, which is the case if either the frame is tight or we have access to a convenient dual frame.

Theorem 1.17. *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given, and assume $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame with frame operator S. Then the following hold:*

(i) The canonical dual frame also has the structure of Gabor frame and is given by $\{M_{mb}T_{na}S^{-1}g\}_{m,n\in\mathbb{Z}}$.

(ii) The canonical tight frame associated with $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is given by $\{M_{mb}T_{na}S^{-\frac{1}{2}}g\}_{m,n\in\mathbb{Z}}$.

The function $S^{-1}g$ is called the *canonical dual window* or the *canonical dual generator*. By the previous theorem, the frame decomposition (1.14) associated with the Gabor frame $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ takes the form

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, M_{mb}T_{na}S^{-1}g\rangle M_{mb}T_{na}g \quad \forall f \in L^2(\mathbb{R}).$$

We consider now one of the most important results in Gabor analysis, known as the *duality principle*. The duality principle concerns the relationship between frame properties for a function g with respect to the lattice $\{(na, mb)\}_{m,n\in\mathbb{Z}}$ and with respect to the adjoint lattice $\{(\frac{n}{b}, \frac{m}{a})\}_{m,n\in\mathbb{Z}}$. It was discovered independently between the 1995 and the 1997 by three groups of researchers: Janssen [22], Daubechies, Landau, and Landau [16], and Ron and Shen [26].

Theorem 1.18 (Duality Principle). *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the following are equivalent:*

(i) $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B;

(ii) $\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}$ is a Riesz sequence with bounds abA, abB.

The importance of Theorem 1.18 lies in the fact that it often is easier to prove that $\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}$ is a Riesz sequence than to prove directly that $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame.

We present the following corollary of the duality principle which will be useful in Chapter 2 to prove Proposition 2.5.

Corollary 1.19. *Let $g \in L^2(\mathbb{R})$ satisfying $\int |g(x)|^2 dx = 1$. Then the following are equivalent:*

(i) The Gabor system $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *constitutes a tight frame.*

(ii) The Gabor system $\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}$ *is an orthonormal system.*

Proof.
(ii) $\Longrightarrow$ (i) Suppose $\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}$ is an orthonormal system. By Remark 1.2, any orthonormal system is a Bessel sequence with bound $B = 1$ and is biorthogonal to itself. By Proposition 1.14 $\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}$ is a Riesz sequence with bounds $A = B = 1$. By the duality principle for Gabor frame, expressed by Theorem 1.18, we have that $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a tight frame with constant $\tilde{A} = \frac{A}{ab}$.
(i) $\Longrightarrow$ (ii) Suppose $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a tight frame. By Proposition 1.16 the constant $A = \frac{||g||^2}{ab} = \frac{1}{ab}$. By the duality principle, $\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}$ is a Riesz sequence with constant $abA = ||g||^2 = 1$. Recall that a Riesz sequence is a Riesz basis for its closed span, hence

$$\sum_{m,n} |\langle f, M_{\frac{m}{a}}T_{\frac{n}{b}}g\rangle|^2 = ||f||^2 \text{ for all } f \in \overline{\text{span}}\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}.$$

By Theorem 1.13, there exists a unique sequence $\{\tilde{g}_{m,n}\}_{m,n\in\mathbb{Z}}$ in $\overline{\text{span}}\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}$ such that $\{\tilde{g}_{m,n}\}_{m,n\in\mathbb{Z}}$ is a Riesz basis for $\overline{\text{span}}\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}$, $\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}$ and $\{\tilde{g}_{m,n}\}_{m,n\in\mathbb{Z}}$ are biorthogonal and $f = \sum_{m,n}\langle f, \tilde{g}_{m,n}\rangle M_{\frac{m}{a}}T_{\frac{n}{b}}g$ for all $f \in \overline{\text{span}}\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}$. Thus, we have that for every $f \in \overline{\text{span}}\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}$

$$||f||^2 = \langle f, f\rangle = \langle \sum_{m,n}\langle f, \tilde{g}_{m,n}\rangle M_{\frac{m}{a}}T_{\frac{n}{b}}g, f\rangle = \sum_{m,n}\langle f, \tilde{g}_{m,n}\rangle\langle M_{\frac{m}{a}}T_{\frac{n}{b}}g, f\rangle$$
$$||f||^2 = \sum_{m,n}|\langle f, M_{\frac{m}{a}}T_{\frac{n}{b}}g\rangle|^2 = \sum_{m,n}\langle f, M_{\frac{m}{a}}T_{\frac{n}{b}}g\rangle\langle M_{\frac{m}{a}}T_{\frac{n}{b}}g, f\rangle$$

and by uniqueness of the biorthogonal basis in $\overline{\text{span}}\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}$ it must be $\tilde{g}_{m,n} = M_{\frac{m}{a}}T_{\frac{n}{b}}g$. Then $\{M_{\frac{m}{a}}T_{\frac{n}{b}}g\}_{m,n\in\mathbb{Z}}$ is biorthogonal to itself and hence it is a orthonormal system. □

1.4 The Zak Transform

The Zak transform is one of the most used tools for the analysis of Gabor systems $\{M_{mb}T_{na}\}_{m,n\in\mathbb{Z}}$ for the case $ab \in \mathbb{Q}$. Applications of the Zak transform to Gabor analysis can be found in the book by Gröchenig "Foundations of Time-Frequency Analysis" [19]. In this section we will refer to the book by Christensen [12] and the article of Janssen [21].

Definition 1.13. Let $\alpha > 0$ be a given parameter, the *Zak transform* $Z_\alpha f$ of $f \in L^2(\mathbb{R})$ is defined by

$$(Z_\alpha f)(s,t) = \sqrt{\alpha}\sum_{k\in\mathbb{Z}} f(\alpha(s-k))e^{2\pi ikt}, \text{ for } s,t \in \mathbb{R}. \tag{1.15}$$

The next lemma summarizes two fundamental properties of the Zak transform.

Lemma 1.20. *Consider the Zak transform Z_α, $\alpha > 0$, and $f \in L^2(\mathbb{R})$. Then the following two properties hold almost everywhere:*

(i) quasiperiodicity:

$$Z_\alpha f(s+1,t) = e^{2\pi it} Z_\alpha f(s,t), \tag{1.16}$$

(ii) periodicity:

$$Z_\alpha f(s,t+1) = Z_\alpha f(s,t). \tag{1.17}$$

Proof.
(i) *Quasiperiodicity.* Consider the change of variable $\tilde{k} = k-1$ then

$$\begin{aligned} Z_\alpha f(s+1,t) &= \sqrt{\alpha} \sum_{k\in\mathbb{Z}} f(\alpha(s+1-k)) e^{2\pi ikt} \\ &= \sqrt{\alpha} \sum_{\tilde{k}\in\mathbb{Z}} f(\alpha(s-\tilde{k})) e^{2\pi i(\tilde{k}+1)t} \\ &= e^{2\pi it} Z_\alpha f(s,t). \end{aligned}$$

(ii) *Periodicity.*

$$Z_\alpha f(s,t+1) = \sqrt{\alpha} \sum_{k\in\mathbb{Z}} f(\alpha(s-k)) e^{2\pi ik(t+1)} = Z_\alpha f(s,t).$$

□

In Chapter 2 we will use the following property.

Proposition 1.21. *Let the Zak transform Z_2 with $\alpha = 2$ and consider the Gabor system $\{M_{\frac{m}{2}} T_n f\}_{m,n\in\mathbb{Z}}$. Then:*

(i) $Z_2 M_{\frac{m}{2}} T_{2n} f(s,t) = e^{2\pi ims} e^{-2\pi int} Z_2 f(s,t)$,

(ii) $Z_2 M_{\frac{m}{2}} T_{2n-1} f(s,t) = e^{2\pi ims} e^{-2\pi int} Z_2 f(s+\frac{1}{2},t)$.

Proof.

$$\begin{aligned} \text{(i)} \quad Z_2 M_{\frac{m}{2}} T_{2n} f(s,t) &= \sqrt{2} \sum_{k\in\mathbb{Z}} e^{2\pi itk} e^{2\pi im(s-k)} f(2(s-k)-2n) \\ &= e^{2\pi ims} \sqrt{2} \sum_{k\in\mathbb{Z}} e^{2\pi itk} f(2(s-n-k)) \\ &= e^{2\pi ims} Z_2 f(s-n,t) \\ &= e^{2\pi ims} e^{-2\pi itn} Z_2 f(s,t); \end{aligned}$$

$$
\begin{aligned}
\text{(ii)} \quad Z_2 M_{\frac{m}{2}} T_{2n-1} f(s,t) &= \sqrt{2} \sum_{k\in\mathbb{Z}} e^{2\pi itk} e^{2\pi im(s-k)} f(2(s-k)-(2n-1)) \\
&= e^{2\pi ims} \sqrt{2} \sum_{k\in\mathbb{Z}} e^{2\pi itk} f(2(s-n+\frac{1}{2}-k)) \\
&= e^{2\pi ims} Z_2 f(s-n+\frac{1}{2},t) \\
&= e^{2\pi ims} e^{-2\pi itn} Z_2 f(s+\frac{1}{2},t).
\end{aligned}
$$

□

For functions in $L^2(\mathbb{R})$ the Zak transform is defined almost everywhere, in fact $\{f(\alpha(s-k))\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ for almost all s. We consider the following interpretation of the Zak transform.

Lemma 1.22. *Let $\alpha > 0$ and $Q = [0,1) \times [0,1)$, the Zak transform Z_α is a unitary map of $L^2(\mathbb{R})$ onto $L^2(Q)$, i.e., $Z_\alpha f$ converges almost everywhere on Q.*

Proof. Consider first the case $\alpha = 1$ and let $f \in L^2(\mathbb{R})$. To show that Z_1 is well-defined as a function in $L^2(Q)$, we define

$$F_k(s,t) = f(s-k)e^{2\pi ikt}, \ k \in \mathbb{Z},$$

then $F_k \in L^2(Q)$. We observe that

$$
\begin{aligned}
\left|\left|\sum_{k\in\mathbb{Z}} F_k\right|\right|^2_{L^2(Q)} &= \left\langle \sum_{k\in\mathbb{Z}} F_k, \sum_{j\in\mathbb{Z}} F_j \right\rangle_{L^2(Q)} = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \langle F_k, F_j\rangle_{L^2(Q)} \\
&= \sum_{k\in\mathbb{Z}} \langle F_k, F_k\rangle_{L^2(Q)} + \sum_{k\in\mathbb{Z}} \sum_{\substack{j\in\mathbb{Z}\\ j\neq k}} \langle F_k, F_j\rangle_{L^2(Q)} \\
&= \sum_{k\in\mathbb{Z}} ||F_k||^2_{L^2(Q)} + \underbrace{\sum_{k\in\mathbb{Z}} \sum_{\substack{j\in\mathbb{Z}\\ j\neq k}} \int_0^1 f(s-k)\overline{f(s-j)} \left(\int_0^1 e^{2\pi i(k-j)t} dt\right) ds}_{=0} \\
&= \sum_{k\in\mathbb{Z}} \int_0^1 \int_0^1 |F_k(s,t)|^2 dt ds = \sum_{k\in\mathbb{Z}} \int_0^1 |f(s-k)|^2 ds = ||f||_2^2.
\end{aligned}
$$

Hence, $\sum_{k\in\mathbb{Z}} F_k$ converges in $L^2(Q)$ and Z_1 is an isometry from $L^2(\mathbb{R})$ into $L^2(Q)$. To prove that Z_1 is unitary, we show that Z_1 maps the orthonormal basis $\{M_m T_n \chi_{[0,1]}\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ onto the orthonormal basis $\{e^{2\pi ims}e^{-2\pi int}\}_{m,n\in\mathbb{Z}}$ for $L^2(Q)$. We consider the Gabor basis $\{M_m T_n \chi_{[0,1]}\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ and we apply the Zak transform for $(s,t) \in Q$:

$$
\begin{aligned}
(Z_1 M_m T_n \chi_{[0,1]})(s,t) &= \sum_{k\in\mathbb{Z}} e^{2\pi im(s-k)} \chi_{[0,1]}(s-n-k) e^{2\pi ikt} \\
&= e^{2\pi ims} e^{-2\pi int} \sum_{k\in\mathbb{Z}} \chi_{[0,1]}(s-k) e^{2\pi ikt} \\
&= e^{2\pi ims} e^{-2\pi int}.
\end{aligned}
$$

For the general case $\alpha > 0$, we use the dilation $Z_\alpha f = Z_1(D_{\alpha^{-1}} f)$. In particular, since the dilation is a unitary operator and we have proved that Z_1 is unitary, then Z_α is itself unitary. $\square$

Remark 1.3. Since the Zak transform is unitary, then $Z_\alpha^{-1} = Z_\alpha^*$ and for every $f, g \in L^2(\mathbb{R})$ we have

$$\langle f, g\rangle = \langle Z_\alpha^{-1} Z_\alpha f, g\rangle = \langle Z_\alpha^* Z_\alpha f, g\rangle = \langle Z_\alpha f, Z_\alpha g\rangle.$$

We formulate an inverse of the Zak transform for $Z_\alpha f \in L^2(Q)$.

Proposition 1.23. *Let $Z_\alpha f \in L^2(Q)$ be the Zak transform of f. Then:*

$$f(x) = Z_\alpha^{-1}(Z_\alpha f)(x) = \frac{1}{\sqrt{\alpha}} \int_0^1 Z_\alpha f\left(\frac{x}{\alpha}, t\right) dt. \tag{1.18}$$

Another useful property is the relation between the Zak transform and the Schwartz space.

Theorem 1.24. *If $f \in \mathcal{S}(\mathbb{R})$, then $Z_\alpha f \in \mathcal{C}^\infty(\mathbb{R}^2)$. Conversely, if $F \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that the quasiperiodicity* (1.16) *and periodicity* (1.17) *conditions are satisfied, then $F = Z_\alpha f$ for a (unique) $f \in \mathcal{S}(\mathbb{R})$.*

1.4.1 Fourier Transform and Zak Transform

It is sometimes useful to define the Zak transform $Z_\alpha f$ by using the Fourier transform of f instead of f itself. In fact, using Poisson summation formula and Properties 1.1 we have that, for $f \in W(\mathbb{R}^d)$ and $\hat{f} \in W(\mathbb{R}^d)$ the following equality holds for all s, t.

$$\begin{aligned}
Z_\alpha f(s,t) &= \sqrt{\alpha} \sum_{k\in\mathbb{Z}} f(\alpha s - \alpha k) e^{2\pi i k t} \\
&= \sqrt{\alpha} \sum_{k\in\mathbb{Z}} f(\alpha s + \alpha k) e^{2\pi i k(-t)} \\
&= \sum_{k\in\mathbb{Z}} (M_{-t} T_{-s} D_{\frac{1}{\alpha}} f)(k) \\
&= \sum_{k\in\mathbb{Z}} e^{2\pi i t s} M_s T_{-t} D_\alpha \hat{f}(k) \\
&= \sqrt{\alpha}^{-1} \sum_{k\in\mathbb{Z}} e^{2\pi i t s} e^{2\pi i s k} \hat{f}\left(\frac{1}{\alpha}(k+t)\right) \\
&= e^{2\pi i t s} Z_{\frac{1}{\alpha}} \hat{f}(t, -s).
\end{aligned} \tag{1.19}$$

With slight modifications we can show that the formula holds for almost all s, t for $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$.

1.5 Modulation Spaces

In this section we recall some basic facts about the short-time Fourier transform from Chapter 3 of [19] and the main properties of the modulation spaces from Chapter 11 of [19] and the article [20].

1.5.1 Short-time Fourier Transform

For a signal $f(x)$, the variable x usually represents the time, while its Fourier transform $\hat{f}$ evaluated at a point ω provides information about the content of oscillations with frequency ω. The time information is lost when applying the Fourier transform, and it is not possible to know which frequencies appear at which time. A way to address this problem is to multiply the signal f by a window function g such that g is constant on a small interval I and decays fast and smooth to zero outside I, and then apply the Fourier transform. We obtain information about the frequency content on I. We define in this way the *short-time Fourier transform.*

Definition 1.14. Let $g \in L^2(\mathbb{R}^d) \setminus \{0\}$ be a window function and $f \in L^2(\mathbb{R}^d)$. Then the *short-time Fourier transform* (STFT) of f with respect to g is defined as

$$V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}dt, \quad \text{for } x, \omega \in \mathbb{R}^d.$$

Note that, applying Parseval's identity (1.1), we can rewrite the STFT as

$$V_g f(x, \omega) = (f \cdot T_x \overline{g})\widehat{\ }(\omega) = \langle f, M_\omega T_x g \rangle = \langle \hat{f}, T_\omega M_{-x} \hat{g} \rangle = e^{-2\pi i x\cdot\omega} V_{\hat{g}} \hat{f}(\omega, -x).$$

Evidently, if $g, f \in L^2(\mathbb{R}^d)$ then $V_g f(x, \omega) = (f \cdot T_x \overline{g})\widehat{\ }(\omega)$ is defined pointwise. The representation as an inner product makes it possible to generalize the STFT beyond $L^2(\mathbb{R}^d)$ via duality.

Definition 1.15. The *Schwartz class* of functions $\mathcal{S}(\mathbb{R}^d)$ consists of all $\mathcal{C}^\infty$-functions f on $\mathbb{R}^d$ such that

$$||f||_{\alpha,\beta} = \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta f(x)| \leq \infty \text{ for all multi-indices } \alpha, \beta \in \mathbb{N}^d.$$

Let $\mathcal{S}'(\mathbb{R}^d)$ be its topological dual. In particular, $V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle$ is well-defined for all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ if $g \in \mathcal{S}(\mathbb{R}^d)$. Practically, a good choice for g can be any non-zero Schwartz function, such as the Gaussian $g(x) = e^{-\pi x^2}$. In fact, the Gaussian is a rapidly decreasing and Fourier invariant function.

To summarize, if g is centered at 0 and has most of its content on a small interval, then $V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle$ measures the magnitude of f near x. At the same time, $V_g f(x, \omega) = \langle \hat{f}, T_\omega M_{-x} \hat{g} \rangle$ and $V_g f$ measures the magnitude of $\hat{f}$ near ω.

The next property is called the *covariance property* of the STFT.

Lemma 1.25. *Whenever $V_g f$ is well define, we have*

$$V_g(T_\mu M_\eta f)(x,\omega) = e^{-2\pi i\mu\cdot\omega} V_g f(x-\mu, \omega-\eta) \text{ for } x, \mu, \omega, \eta \in \mathbb{R}^d.$$

Moreover,

$$|V_g(T_\mu M_\eta f)(x,\omega)| = |V_g f(x-\mu, \omega-\eta)|. \tag{1.20}$$

1.5.2 Modulation spaces

The short-time Fourier transform describes the global time-frequency distribution of a signal but, even assuming $V_g f \in L^2(\mathbb{R}^d)$, we are not able to estimate precisely the time-frequency localization of the function f. Considering "weights" in the time-frequency plane makes the measurement of the decay of the short-time Fourier transform more precise.

Definition 1.16. Let w be a nonnegative continuous function on $\mathbb{R}^{2d}$. We call w a *weight* if, for some constants $C > 0$ and $s \geq 0$, it holds

$$w(x_1+\omega_1, x_2+\omega_2) \leq C(1+|x|)^s w(\omega_1, \omega_2)$$

for all $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix},$ and $x_1, x_2, \omega_1, \omega_2 \in \mathbb{R}^d$.

Modulation spaces are mathematical tool to measure the joint time-frequency distribution of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$.

Definition 1.17. Let $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, let w be a weight on $\mathbb{R}^{2d}$ and $1 \leq p, q \leq \infty$. Then the *modulation space* $M_w^{p,q}(\mathbb{R}^d)$ is the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the norm

$$||f||_{M_w^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p w(x,\omega)^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}} < \infty$$

is finite.

Note that the norm on $M_w^{p,q}$ is $||f||_{M_w^{p,q}} = ||V_g f||_{L_w^{p,q}}$. It can be shown that the definition is independent of the window function; in fact, different non-zero windows $g \in \mathcal{S}(\mathbb{R}^d)$ yield equivalent norms.

The duality between the spaces $L_w^{p,q}$ and $L_{\frac{1}{w}}^{p',q'}$ suggests a similar statement for modulation spaces.

Theorem 1.26. *Let $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. If $1 \leq p, q < \infty$, then $(M_w^{p,q})^* = M_{\frac{1}{w}}^{p',q'}$ under the duality*

$$\langle f, h \rangle = \int \int_{\mathbb{R}^d} V_g f(x,\omega) \overline{V_g h(x,\omega)} dx d\omega$$

for $f \in M_w^{p,q}$ and $h \in M_{\frac{1}{w}}^{p',q'}$.

In this thesis we will consider the case $d = 1$ and $p = q$ and we will write M^p_w instead of $M^{p,p}_w$.

The next result shows that we characterize the modulation spaces by means of Gabor frames and hence the modulation spaces have a nice Gabor expansion.

Theorem 1.27. *Given $g \in \mathcal{S}(\mathbb{R})$ and $a, b > 0$ small enough, then there exists a dual window $h \in \mathcal{S}(\mathbb{R})$, such that every $f \in \mathcal{S}'(\mathbb{R})$ can be written as*

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, M_{mb}T_{na}h\rangle M_{mb}T_{na}g. \tag{1.21}$$

Moreover,

$$f \in M^p_w \iff \left(\sum_{m,n\in\mathbb{Z}} |\langle f, M_{mb}T_{na}h\rangle|^p w(na, mb)^p\right)^{\frac{1}{p}} < \infty. \tag{1.22}$$

and the sequence space of norm in (1.22) is equivalent to the norm in M^p_w. Furthermore, if $1 \le p, q < \infty$, the Gabor expansion (1.21) converges unconditionally in $M^{p,q}_w$.

Chapter 2

Wilson Bases

In 1991, Ingrid Daubechies, Stéphane Jaffard and Jean-Lin Journé in [15], following the basic idea of Wilson orthonormal bases for $L^2(\mathbb{R})$, constructed what now are known as Wilson bases. They were able to overcome the barrier presented by the Balian-Low theorem which states that it is not possible to have good time-frequency localization of Gabor frames at the critical density. In particular, their construction gives a modification of Gabor systems in a way that the redundancy of a Gabor frame is deleted and the time-frequency localization is preserved. Basically, they constructed a real function ϕ with $\phi_{m,n}(x) = M_{\frac{m}{2}} T_n \phi(x)$ such that with the definitions

$$\begin{cases} \hat{\psi}_{1,n}(\omega) = M_{-n}\phi(\omega) = \phi_{-2n,0}(\omega), \\ \hat{\psi}_{2l+\kappa,n}(\omega) = \frac{1}{\sqrt{2}} e^{-2\pi i n\omega} e^{i\pi\kappa\omega}[\phi(\omega - l) + (-1)^{l+\kappa}\phi(\omega + l)] \\ \qquad\qquad = \frac{1}{\sqrt{2}} e^{i\pi\kappa\omega} M_{-n}[T_l + (-1)^{l+\kappa} T_{-l}]\phi(\omega) \\ \qquad\qquad = \frac{1}{\sqrt{2}}[\phi_{-(2n-\kappa),l} + (-1)^{l+\kappa}\phi_{-(2n-\kappa),-l}](\omega) \\ l \in \mathbb{N} \setminus \{0\}, \ \kappa = 0 \text{ or } 1. \end{cases} \tag{2.1}$$

the family

$$\psi_{m,n} \text{ with } m \in \mathbb{N} \setminus \{0\} \text{ and } n \in \mathbb{Z} \tag{2.2}$$

forms an orthonormal basis.

We can relabel the $\psi_{m,n}$ to make the notation simpler:

$$\begin{aligned} &\Psi_{0,n} = \psi_{1,n} \\ &\Psi_{l,2n-\kappa} = \psi_{2l+\kappa,n}, \ l \neq 0, \ \kappa = 0 \text{ or } 1 \end{aligned}$$

to obtain

$$\begin{cases} \hat{\Psi}_{0,n}(\omega) = \phi_{-2n,0}(\omega), \\ \hat{\Psi}_{l,n}(\omega) = \hat{\psi}_{2l+\kappa,\frac{n+\kappa}{2}}(\omega) = \frac{1}{\sqrt{2}}[\phi_{-n,l} + (-1)^{l+n}\phi_{-n,-l}](\omega), \ l \neq 0. \end{cases}$$

Note that the functions $\psi_{m,n}$ in (2.1) can be obtained by the inverse Fourier transform of ϕ. If ϕ is real and even, then the inverse Fourier transform of ϕ and $\hat{\phi}$

coincide and we can rewrite

$$\begin{cases} \psi_{1,n}(x) = \hat{\phi}(x-n) = T_n\hat{\phi}(x), \\ \psi_{2l+\kappa,n}(x) = \frac{1}{\sqrt{2}}\hat{\phi}(x+\frac{\kappa}{2}-n)e^{\pi i l\kappa}[e^{2\pi i l x} + (-1)^{l+\kappa}e^{-2\pi i l x}] \\ \qquad\qquad\qquad = \frac{1}{\sqrt{2}}e^{\pi i l\kappa}[M_l + (-1)^{l+\kappa}M_{-l}]T_{n-\frac{\kappa}{2}}\hat{\phi}(x). \end{cases}$$

Relabelling as before, one gets

$$\begin{cases} \Psi_{0,n}(x) = T_n\hat{\phi}(x), \\ \Psi_{l,n}(x) = \frac{1}{\sqrt{2}}e^{\pi i l n}[M_l + (-1)^{l+n}M_{-l}]T_{\frac{n}{2}}\hat{\phi}(x) \\ \qquad\qquad = \sqrt{2}\hat{\phi}(x-\frac{n}{2})\begin{cases} \cos(2\pi l x), & \text{if } l+n \text{ is even,} \\ \sin(2\pi l x), & \text{if } l+n \text{ is odd.} \end{cases} \\ l \in \mathbb{N}\setminus\{0\},\ n\in\mathbb{Z} \end{cases} \tag{2.3}$$

It is surprising that both $\Psi_{l,n}$ and $\hat{\Psi}_{l,n}$ constructed in this way have the same structure.

It is still mysterious how they came up with this formula, and the paper they wrote is entirely dedicated to prove that formulation (2.1) has all the properties we are looking for. In addition, they presented a way to construct ϕ where both ϕ and its Fourier transform $\hat{\phi}$ have exponential decay and can be constructed as a rapidly converging superposition of Gaussians.

We follow the presentation of [15]. The first result is the following.

Proposition 2.1. *Let $\phi \in L^2(\mathbb{R})$ be a real-valued function. Then the functions $\psi_{m,n}$ defined by* (2.1) *and* (2.2) *form an orthonormal basis for $L^2(\mathbb{R})$ if and only if*

$$\sum_{l\in\mathbb{Z}} T_{-l}\phi(\omega)T_{-l}\phi(\omega+2j) = \delta_{j0} \ \text{a.e.}, \tag{2.4}$$

where δ_{j0} is the Kronecker delta.

Proof. By Lemma 1.11, $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ is an orthonormal basis, if properties (1.8) and (1.9) hold.

1. *Claim: $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ satisfies* (1.9) *if and only if $\sum_{l=-\infty}^{\infty} T_{-l}\phi(\omega)T_{-l}\phi(\omega+2j) = \delta_{j0}$ a.e.*

Suppose (1.9) holds true, then apply Parseval's identity to the first inner product and Poisson summation formula (1.2) to $M_{-\omega}T_{-x}\psi_{m,0}$ in the eighth equality. We obtain that for every $g, h \in L^2(\mathbb{R})$:

$$\begin{aligned} \langle g,h\rangle = & \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\langle g,\psi_{m,n}\rangle\langle\psi_{m,n},h\rangle \\ \overset{(1.1)}{=} & \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\langle \hat{g},\hat{\psi}_{m,n}\rangle\langle\psi_{m,n},h\rangle \\ = & \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\int \hat{g}(\omega)\overline{\hat{\psi}_{m,n}(\omega)}\left(\int \psi_{m,n}(x)\overline{h(x)}dx\right)d\omega \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\int \hat{g}(\omega)\overline{M_{-n}M_n\hat{\psi}_{m,n}(\omega)}\left(\int \psi_{m,n}(x)\overline{h(x)}dx\right)d\omega \\
&= \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\int \hat{g}(\omega)\overline{M_n\hat{\psi}_{m,n}(\omega)}\left(\int e^{2\pi i n\omega}\psi_{m,n}(x)\overline{h(x)}dx\right)d\omega \\
&= \sum_{m=1}^{\infty}\int \hat{g}(\omega)\overline{\hat{\psi}_{m,0}(\omega)}\left(\int \sum_{n=-\infty}^{\infty}\psi_{m,0}(x+n)e^{-2\pi i n\omega}\overline{h(x)}dx\right)d\omega \\
&= \sum_{m=1}^{\infty}\int \hat{g}(\omega)\overline{\hat{\psi}_{m,0}(\omega)}\left(\int \sum_{n=-\infty}^{\infty}M_{-\omega}T_{-x}\psi_{m,0}(n)\overline{h(x)}dx\right)d\omega \\
&= \sum_{m=1}^{\infty}\int \hat{g}(\omega)\overline{\hat{\psi}_{m,0}(\omega)}\left(\int \sum_{k=-\infty}^{\infty}e^{2\pi i\omega x}M_xT_{-\omega}\hat{\psi}_{m,0}(k)\overline{h(x)}dx\right)d\omega \\
&= \sum_{m=1}^{\infty}\int \hat{g}(\omega)\overline{\hat{\psi}_{m,0}(\omega)}\left(\int \sum_{k=-\infty}^{\infty}\hat{\psi}_{m,0}(\omega+k)e^{2\pi i x(\omega+k)}\overline{h(x)}dx\right)d\omega \\
&= \sum_{m=1}^{\infty}\sum_{k=-\infty}^{\infty}\int \hat{g}(\omega)\overline{\hat{\psi}_{m,0}(\omega)}\hat{\psi}_{m,0}(\omega+k)\underbrace{\left(\int e^{2\pi i x(\omega+k)}\overline{h(x)}dx\right)}_{=\overline{\hat{h}(\omega+k)}}d\omega \\
&= \sum_{m=1}^{\infty}\sum_{k=-\infty}^{\infty}\int \hat{g}(\omega)\overline{\hat{\psi}_{m,0}(\omega)}\hat{\psi}_{m,0}(\omega+k)\overline{\hat{h}(\omega+k)}d\omega \\
&= \sum_{k=-\infty}^{\infty}\int \hat{g}(\omega)\overline{\hat{h}(\omega+k)}\sum_{m=1}^{\infty}\overline{\hat{\psi}_{m,0}(\omega)}\hat{\psi}_{m,0}(\omega+k)d\omega.
\end{aligned}
$$

Since we assumed $\phi \in L^2(\mathbb{R})$ then $\hat{\psi}_{m,n} \in L^2(\mathbb{R})$. Using the periodization trick we have that $\sum ||\hat{\psi}_{m,n}||^2 \in L^1(\mathbb{T})$ and then, applying Fubini, summation and integration commute. Moreover, since $\phi \in L^2(\mathbb{R})$, the Poisson summation formula is defined almost everywhere and the series converges uniformly almost everywhere. From the previous computations we have that

$$\int \hat{g}(\omega)\overline{\hat{h}(\omega)}d\omega = \sum_{k=-\infty}^{\infty}\int \hat{g}(\omega)\overline{\hat{h}(\omega+k)}\sum_{m=1}^{\infty}\overline{\hat{\psi}_{m,0}(\omega)}\hat{\psi}_{m,0}(\omega+k)d\omega \text{ a.e.}$$

Hence, condition (1.9) holds true if and only if

$$\sum_{m=1}^{\infty}\overline{\hat{\psi}_{m,0}(\omega)}\hat{\psi}_{m,0}(\omega+k) = \delta_{k0} \text{ a.e.} \tag{2.5}$$

Since ϕ is real, $\phi(\omega) = \overline{\phi(\omega)}$ and, using definition in (2.1), replace $\hat{\psi}_{m,n}$ in (2.5). We have

$$\begin{aligned}
&\sum_{m=1}^{\infty} \overline{\hat{\psi}_{m,0}(\omega)}\hat{\psi}_{m,0}(\omega+k) \\
&\quad = \overline{\phi(\omega)}\phi(\omega+k) + \frac{1}{2}\sum_{l=1}^{\infty}\sum_{\kappa=0}^{1} \overline{M_{\frac{\kappa}{2}}[T_l + (-1)^{l+\kappa}T_{-l}]\phi(\omega)} \cdot M_{\frac{\kappa}{2}}[T_l + (-1)^{l+\kappa}T_{-l}]\phi(\omega+k) \\
&\quad = \phi(\omega)\phi(\omega+k) + \frac{1}{2}\sum_{l=1}^{\infty}\{[T_l + (-1)^{l}T_{-l}]\phi(\omega)\cdot[T_l + (-1)^{l}T_{-l}]\phi(\omega+k) \\
&\qquad\quad + (-1)^k[T_l + (-1)^{l+1}T_{-l}]\phi(\omega)\cdot[T_l + (-1)^{l+1}T_{-l}]\phi(\omega+k)\} \\
&\quad = \phi(\omega)\phi(\omega+k) + \frac{1}{2}\underbrace{\sum_{l=1}^{\infty}[T_l\phi(\omega)T_l\phi(\omega+k) + T_{-l}\phi(\omega)T_{-l}\phi(\omega+k)]}_{=\sum_{l\in\mathbb{Z},l\neq 0} T_{-l}\phi(\omega)T_{-l}\phi(\omega+k)}[1+(-1)^k] \\
&\qquad\quad + \frac{1}{2}\underbrace{\sum_{l=1}^{\infty}[T_l\phi(\omega)T_{-l}\phi(\omega+k) + T_{-l}\phi(\omega)T_l\phi(\omega+k)]}_{=\sum_{l\in\mathbb{Z},l\neq 0} T_{-l}\phi(\omega)T_l\phi(\omega+k)}(-1)^l[1-(-1)^k] \\
&\quad = \phi(\omega)\phi(\omega+k) + \frac{1}{2}\sum_{\substack{l\in\mathbb{Z},\\ l\neq 0}}[T_{-l}\phi(\omega)T_{-l}\phi(\omega+k)][1+(-1)^k] \\
&\qquad\quad + \frac{1}{2}\sum_{\substack{l\in\mathbb{Z},\\ l\neq 0}}(-1)^l[T_{-l}\phi(\omega)T_l\phi(\omega+k)][1-(-1)^k].
\end{aligned}$$

We notice that if k is even, then $k = 2j$ and

$$\sum_{m=1}^{\infty} \overline{\hat{\psi}_{m,n}(\omega)}\hat{\psi}_{m,n}(\omega+2j) = \sum_{l=-\infty}^{\infty} T_{-l}\phi(\omega)T_{-l}\phi(\omega+2j).$$

If k is odd, then $k = 2j+1$ and

$$\sum_{m=1}^{\infty} \overline{\hat{\psi}_{m,n}(\omega)}\hat{\psi}_{m,n}(\omega+2j+1) = \sum_{l=-\infty}^{\infty} (-1)^l T_{-l}\phi(\omega)T_l\phi(\omega+2j+1) = 0$$

since taking $l' = -l+2j+1$ and substituting we have

$$\sum_{l=-\infty}^{\infty} (-1)^l T_{-l}\phi(\omega)T_l\phi(\omega+2j+1) = \sum_{l'=-\infty}^{\infty} (-1)^{-l'+2j+1} T_{l'}\phi(\omega+2j+1)T_{-l'}\phi(\omega) = 0.$$

We obtain that (2.5) holds if and only if

$$\sum_{l=-\infty}^{\infty} T_{-l}\phi(\omega)T_{-l}\phi(\omega+2j) = \delta_{j0} \text{ a.e.}$$

2. *Claim:* $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ *satisfies* (1.8) *if and only if* $\int_{-\infty}^{+\infty}\phi(\omega)\phi(\omega+2l)d\omega = \delta_{l0}$.
Consider condition (1.8): we have to show $||\psi_{m,n}|| = 1$ for $m \in \mathbb{N}\setminus\{0\}$ and $n \in \mathbb{Z}$.
For $m = 1$, then $||\psi_{1,n}||^2 = ||\hat{\psi}_{1,n}||^2 = \int_{-\infty}^{+\infty}|\phi(\omega)|^2 d\omega$;
for $m = 2l+\kappa$, $l \geq 1$ and $\kappa = 0$ or 1, then

$$||\hat{\psi}_{m,n}||^2 = \frac{1}{2}\int_{-\infty}^{+\infty}|T_l\phi(\omega)+(-1)^{l+\kappa}T_{-l}\phi(\omega)|^2 d\omega = 1+(-1)^{l+\kappa}\int_{-\infty}^{+\infty}\phi(\omega)\phi(\omega+2l)d\omega.$$

From the previous consideration we have that $||\psi_{m,n}||^2 = 1$ if and only if

$$\int_{-\infty}^{+\infty}\phi(\omega)\phi(\omega+2l)d\omega = \delta_{l0}. \tag{2.6}$$

3. *Claim:* $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ *forms an orthonormal basis if and only if* $\sum_{l\in\mathbb{Z}}\phi(\omega+l)\phi(\omega+l+2j) = \delta_{j0}$ *a.e.*
Note that, if condition (2.4) holds true, then (2.6) is satisfied, in fact:

$$\int_{-\infty}^{+\infty}\phi(\omega)\phi(\omega+2l)d\omega = \sum_{k\in\mathbb{Z}}\int_0^1 T_{-k}\phi(\omega)T_{-k}\phi(\omega+2l)d\omega = \int_0^1 \delta_{l0}d\omega = \delta_{l0}.$$

To summarize, if condition (2.4) is satisfied then, respectively by claims 1 and 2, condition (1.9) and $||\psi_{m,n}|| = 1$ hold true. Hence, by Lemma 1.11, $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ forms an orthonormal basis. Conversely, if $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ forms an orthonormal basis, i.e. conditions (1.8) and (1.9) hold, then, by item 1, (2.4) holds. Thus, we have proved that $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ forms an orthonormal basis if and only if $\sum_{l\in\mathbb{Z}}\phi(\omega+l)\phi(\omega+l+2j) = \delta_{j0}$ a.e. □

2.1 Equivalent formulation using Zak Transform

We can simplify the condition of Proposition 2.1 by using the Zak transform, which has been introduced in Section 1.4 to rewrite our problem into a different form.
For our aim we choose $\alpha = 2$ in the definition of the Zak transform (1.15):

$$Z_2\phi(s,t) = \sqrt{2}\sum_{k\in\mathbb{Z}}e^{2\pi itk}\phi(2(s-k)). \tag{2.7}$$

We recall that the function $Z_2 g$ is periodic in the second and quasiperiodic in the first variable from Proposition 1.20:

$$Z_2\phi(s,t+1) = Z_2\phi(s,t),$$
$$Z_2\phi(s+1,t) = Z_2\phi(s,t)e^{2\pi it}.$$

Moreover, the inverse transform of (2.7) is given by

$$\phi(x) = \frac{1}{\sqrt{2}}\int_0^1 (Z_2\phi)\left(\frac{x}{2},t\right)dt. \tag{2.8}$$

We can rewrite the previous proposition in the following way.

Proposition 2.2. *Let $\phi \in L^2(\mathbb{R})$ be a real-valued function. Then $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ defined by* (2.1) *and* (2.2) *forms an orthonormal basis for $L^2(\mathbb{R})$ if and only if the Zak transform $Z_2\phi$ of ϕ, as defined by* (2.7) *satisfies*

$$|Z_2\phi(s,t)|^2 + |Z_2\phi(s+\frac{1}{2},t)|^2 = 2 \tag{2.9}$$

for almost all $s, t \in [0,1]^2$.

Proof. We consider the inverse Zak transform and we rewrite equation (2.4) in terms of $Z_2\phi$. We divide the sum in odd and even part, apply the quasiperiodicity property of the Zak transform and use the delta point measure. Recall that since ϕ is real, then $\overline{Z_2\phi(s,t)} = Z_2\phi(s,-t)$.

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}} T_{-l}\phi(\omega)T_{-l}\phi(\omega+2j)\\
&= \frac{1}{2}\sum_{l\in\mathbb{Z}}\int_0^1\left[\int_0^1 Z_2\phi\left(\frac{\omega+l}{2},t\right)Z_2\phi\left(\frac{\omega+l}{2}+j,t'\right)dt'\right]dt\\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^1\left[\int_0^1 Z_2\phi\Big(\frac{\omega}{2}+k,t\Big)Z_2\phi\Big(\frac{\omega}{2}+k+j,t'\Big)\right.\\
&\qquad \left.+ Z_2\phi\Big(\frac{\omega+1}{2}+k,t\Big)Z_2\phi\Big(\frac{\omega+1}{2}+k+j,t'\Big)dt'\right]dt\\
&= \frac{1}{2}\int_0^1\left[\int_0^1 \underbrace{\sum_{k\in\mathbb{Z}} e^{2\pi ik(t+t')}}_{\sum_{n\in\mathbb{Z}}\delta(t'-(n-t))} e^{2\pi ijt'}\left\{Z_2\phi\Big(\frac{\omega}{2},t\Big)Z_2\phi\Big(\frac{\omega}{2},t'\Big)\right.\right.\\
&\qquad \left.\left.+ Z_2\phi\Big(\frac{\omega+1}{2},t\Big)Z_2\phi\Big(\frac{\omega+1}{2},t'\Big)\right\}dt'\right]dt\\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^1 e^{2\pi ij(n-t)}\left[Z_2\phi\Big(\frac{\omega}{2},t\Big)Z_2\phi\Big(\frac{\omega}{2},n-t\Big)\right.\\
&\qquad \left.+ Z_2\phi\Big(\frac{\omega+1}{2},t\Big)Z_2\phi\Big(\frac{\omega+1}{2},n-t\Big)\right]dt\\
&= \frac{1}{2}\int_0^1 e^{-2\pi ijt}\left[\left|Z_2\phi\Big(\frac{\omega}{2},t\Big)\right|^2 + \left|Z_2\phi\Big(\frac{\omega+1}{2},t\Big)\right|^2\right]dt
\end{aligned}$$

Note that $Z_2\phi(s,\cdot)$ is square integrable for almost all s. In fact, by the definition of Zak transform this is equivalent to $\sum_k |\phi(2s-2k)|^2$ being summable for almost all s and this is satisfied for $\phi \in L^2(\mathbb{R})$. Impose that

$$\delta_{j0} = \frac{1}{2}\int_0^1 e^{-2\pi ijt}\left[\left|Z_2\phi\Big(\frac{\omega}{2},t\Big)\right|^2 + \left|Z_2\phi\Big(\frac{\omega+1}{2},t\Big)\right|^2\right]dt$$

Hence, the equality is satisfied if and only if $|Z_2\phi(s,t)|^2 + |Z_2\phi(s+\frac{1}{2},t)|^2 = 2$. Thus, we have proved that $\sum_{l=-\infty}^{\infty} T_{-l}\phi(\omega)T_{-l}\phi(\omega+2j) = \delta_{j0}$ if and only if $|Z_2\phi(s,t)|^2 + |Z_2\phi(s+\frac{1}{2},t)|^2 = 2$. ∎

2.2 Wilson basis with good localization

The next step is to construct ϕ satisfying (2.4) where both ϕ and $\hat{\phi}$ have exponential decay.
We prove the following theorem which gives us a recipe to construct the desired function ϕ.

Theorem 2.3. *Consider a real-valued function g such that both g and its Fourier transform $\hat{g}$ have exponential decay, i.e.,*

$$|g(x)| \leq Ce^{-\zeta|x|}, \tag{2.10}$$

$$|\hat{g}(\omega)| \leq Ce^{-\gamma|\omega|}. \tag{2.11}$$

Consider Z_2g and assume that

$$\inf_{t,s\in[0,1]}[|(Z_2g)(s,t)|^2 + |(Z_2g)(s+\frac{1}{2},t)|^2] > 0. \tag{2.12}$$

Define

$$\Phi(s,t) = \sqrt{2}\frac{(Z_2g)(s,t)}{[|(Z_2g)(s,t)|^2 + |(Z_2g)(s+\frac{1}{2},t)|^2]^{\frac{1}{2}}} \tag{2.13}$$

and, using (2.8), define the function ϕ by $\phi = Z_2^{-1}\Phi$. Then ϕ is real, both ϕ and its Fourier transform $\hat{\phi}$ have exponential decay, and $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ defined by (2.1) and (2.2) forms an orthonormal basis for $L^2(\mathbb{R})$.

Proof.
1. *Claim: ϕ is real.*
First of all, by decreasing properties of g, Z_2g is continuous and well defined. Since g is real, $Z_2g(s,-t) = \overline{Z_2g(s,t)}$. Using (2.13), we notice that $\Phi(s,-t) = \overline{\Phi(s,t)}$. Then, by equation (2.8)

$$\begin{aligned}\overline{\phi(x)} &= \frac{1}{\sqrt{2}}\int_0^1 \overline{\Phi\left(\frac{x}{2},t\right)}dt = \frac{1}{\sqrt{2}}\int_0^1 \Phi\left(\frac{x}{2},-t\right)dt = \frac{1}{\sqrt{2}}\int_{-1}^0 \Phi\left(\frac{x}{2},t\right)dt \\ &= \frac{1}{\sqrt{2}}\int_0^1 \Phi\left(\frac{x}{2},t\right)dt = \phi(x),\end{aligned}$$

where we have used a change of variable and the periodicity property of the Zak transform.
2. *Claim: ϕ has exponential decay.*
We want to find an extension of Φ to $\mathbb{R}\times(\mathbb{R}+i(-\tilde{\zeta},\tilde{\zeta}))$ which is analytic for every fixed s so that we can apply Theorem 1.7 and show that the Fourier coefficients $\phi(l)$ of the series defined by Φ have exponential decay.
Firstly, we extend the domain of Z_2g from $\mathbb{R}^2$ to $\mathbb{R}\times(\mathbb{R}+i(-\frac{\zeta}{\pi},\frac{\zeta}{\pi}))$.
Consider the series $Z_2g(s,t+i\tau) = \sqrt{2}\sum_{l\in\mathbb{Z}} e^{2\pi i(t+i\tau)l}g(2(s-l))$. By the decay

property (2.10) of g, this expression converges absolutely for $|\tau| < \frac{\zeta}{\pi}$. The function $Z_2g(s,z)$ is continuous on $\mathbb{R} \times (\mathbb{R} + i(-\frac{\zeta}{\pi}, \frac{\zeta}{\pi}))$, $Z_2g(s,\cdot)$ is analytic on $(\mathbb{R} + i(-\frac{\zeta}{\pi}, \frac{\zeta}{\pi}))$ for every $s \in \mathbb{R}$ and, the properties of periodicity and quasiperiodicity hold. Define

$$G(s,z) = Z_2g(s,z)Z_2g(s,-z) + Z_2g(s+\frac{1}{2},z)Z_2g(s+\frac{1}{2},-z).$$

$G(s,\cdot)$ is analytic for every $s \in \mathbb{R}$ on $\mathbb{R} + i(-\frac{\zeta}{\pi}, \frac{\zeta}{\pi})$, and, for all $z \in \mathbb{R} + i(-\frac{\zeta}{\pi}, \frac{\zeta}{\pi})$, and $s \in \mathbb{R}$, G satisfies the following periodic properties:

$$G(s,z+1) = G(s,z) = G(s+\frac{1}{2},z). \tag{2.14}$$

Thanks to (2.11), $\hat{g} \in L^1(\mathbb{R})$ and using the inverse Fourier transform we can see that g is uniformly continuous on $\mathbb{R}$. Moreover, it can also be shown that Z_2g is uniformly continuous on $[0,1] \times (\mathbb{R} + i(-\frac{\zeta}{\pi}, \frac{\zeta}{\pi}))$, and, by (2.14), G is uniformly continuous on $\mathbb{R} \times (\mathbb{R} + i(-\frac{\zeta}{\pi}, \frac{\zeta}{\pi}))$. Finally, we note that $G\big|_{\mathbb{R}\times\mathbb{R}}$ is real and, by (2.12), it is bounded away from 0. Hence, there exists $\tilde{\zeta} > 0$ such that $|G|$ is bounded from below away from 0 on $\mathbb{R} \times (\mathbb{R} + i(-\tilde{\zeta}, \tilde{\zeta}))$. We can define the uniformly continuous function $G^{-\frac{1}{2}}$ on $\mathbb{R} \times (\mathbb{R} + i(-\tilde{\zeta}, \tilde{\zeta}))$, which is analytic in $z \in \mathbb{R} + i(-\tilde{\zeta}, \tilde{\zeta})$, for every $s \in \mathbb{R}$. We can rewrite the definition (2.13) as

$$\Phi(s,z) = \sqrt{2}G(s,z)^{-\frac{1}{2}}Z_2g(s,z).$$

This is an analytic in $\mathbb{R} \times (\mathbb{R} + i(-\tilde{\zeta}, \tilde{\zeta}))$ and satisfies the periodicity and quasiperiodicity conditions. Recall that, by definition of Zak transform

$$\Phi(s,z) = \sqrt{2}\sum_{l\in\mathbb{Z}} e^{2\pi izl}\phi(2(s-l))$$

and recalling Theorem 1.7 with $a, b = \tilde{\zeta}$, $f = \Phi$ and $c_l = \phi(2(s-l))$ we then have that for all $\tilde{\zeta} > 0$ such that

$$\tilde{\zeta} < \min\left(\frac{\zeta}{\pi}, \inf\{|\tau| : G(s,t+i\tau) = 0 \text{ for some } s,t \in [0,1]\}\right),$$

there exists a constant $C_{\tilde{\zeta}}$ such that

$$|\phi(x)| \leq C_{\tilde{\zeta}}e^{-\pi\tilde{\zeta}|x|}.$$

3. *Claim: $\hat{\phi}$ has exponential decay.*
We apply the same argument as in claim 2 to find an extension of Φ to $(\mathbb{R}+i(-\tilde{\gamma},\tilde{\gamma}))\times\mathbb{R}$ which is analytic for every fixed t so that we can apply Theorem 1.7. Using the exponential decay of $\hat{g}$ expressed by (2.11) and recalling from the Prerequisites in 1.4.1 the relation between the Zak transform of a function and the Zak transform of its Fourier transform we have that the relation $Z_2g(s,t) = e^{2\pi ist}(Z_{\frac{1}{2}}\hat{g})(t,-s)$

extends analytically in s to $Z_2 g(w,t) = e^{2\pi iwt}(Z_{\frac{1}{2}}\hat{g})(t,-w)$ for every $t \in \mathbb{R}$ and $w = s + i\sigma \in \mathbb{R} + i(-\frac{\gamma}{4\pi}, \frac{\gamma}{4\pi})$. We can define

$$\Gamma(w,t) = Z_2 g(w,t) Z_2 g(w,-t) + Z_2 g(w+\frac{1}{2},t) Z_2 g(w+\frac{1}{2},-t),$$

for $w = s + i\sigma \in \mathbb{R} + i(-\frac{\gamma}{4\pi}, \frac{\gamma}{4\pi})$ and for $t \in \mathbb{R}$. Following the same approach as before, $\Gamma(w,t)$ is analytic and there exists $\tilde{\gamma} > 0$ such that $|\Gamma|$ is bounded below away from 0 on $(\mathbb{R} + i(-\tilde{\gamma}, \tilde{\gamma})) \times \mathbb{R}$. Hence, Φ can be extended to an analytic function on $(\mathbb{R} + i(-\tilde{\gamma}, \tilde{\gamma})) \times \mathbb{R}$ by

$$\Phi(w,t) = \sqrt{2} Z_2 g(w,t) \Gamma(w,t)^{-\frac{1}{2}},$$

which satisfies the periodicity and quasiperiodicity conditions. We use again 1.4.1 and we have

$$\Phi(w,t) = e^{2\pi iwt}(Z_{\frac{1}{2}}\hat{\phi})(t,-w).$$

Applying Theorem 1.7 we obtain that for all $\tilde{\gamma}$ such that

$$\tilde{\gamma} < \min\left(\frac{\gamma}{\pi}, 4\inf\{|\sigma| : \Gamma(s+i\sigma,t) = 0 \text{ for some } s,t \in [0,1]\}\right),$$

there exists a constant $C_{\tilde{\gamma}}$ such that

$$|\hat{\phi}(\omega)| \le C_{\tilde{\gamma}} e^{-\pi\tilde{\gamma}|\omega|}.$$

4. *Claim:* $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ *forms an orthonormal basis for* $L^2(\mathbb{R})$.
By Proposition 2.2, $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$ if and only if $|\Phi(s,t)|^2 + |\Phi(s+\frac{1}{2},t)|^2 = 2$ for almost all $s,t \in [0,1]^2$.

$$\begin{aligned}
&|\Phi(s,t)|^2 + |\Phi(s+\frac{1}{2},t)|^2 \\
&= 2\left[\frac{|Z_2 g(s,t)|^2}{|Z_2 g(s,t)|^2 + |Z_2 g(s+\frac{1}{2},t)|^2} + \frac{|Z_2 g(s+\frac{1}{2},t)|^2}{|Z_2 g(s+\frac{1}{2},t)|^2 + \underbrace{|Z_2 g(s+1,t)|^2}_{=|Z_2 g(s,t)|^2}}\right] = 2.
\end{aligned}$$

Hence, we have construct a function ϕ with exponentially decreasing ϕ and $\hat{\phi}$ such that $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$. □

Remark 2.1. It must be underlined that the definition of $Z_2\phi$ in (2.13) allowed the exponential decay of g and $\hat{g}$ to be preserved in ϕ and $\hat{\phi}$.

With the ingredients of Theorem 2.3 we can then construct an orthonormal Wilson basis of the type described by (2.1) and (2.2). We end this section giving an explicit example.

Example 2.1. Consider the Gaussian $g(x) = (2\nu)^{\frac{1}{4}} e^{-\nu\pi x^2}$ and its Fourier transform $\hat{g}(y) = (\frac{2}{\nu})^{\frac{1}{4}} e^{-\frac{\pi}{\nu}y^2}$, they both are exponentially decreasing. Compute the Zak transform of g

$$Z_2 g(s,t) = \sqrt{2}(2\nu)^{\frac{1}{4}} e^{-4\nu\pi s^2} \sum_{l\in\mathbb{Z}} e^{-4\nu\pi l^2} e^{2\pi l(4\nu s + it)}. \tag{2.15}$$

To construct an orthonormal Wilson basis, we need to show that condition (2.12) is satisfied: it can be done by proving that $Z_2g(s,t) = 0$ only once in $[0,1]^2$. Recall from [29] that the third Jacobi theta function is defined by

$$\theta_3(z|\tau) = 1 + 2\sum_{l=1}^{\infty} \cos(2\pi l z) e^{i\pi\tau l^2} = \sum_{l\in\mathbb{Z}} e^{i\pi\tau l^2} e^{2\pi i l z}. \tag{2.16}$$

Comparing (2.15) and (2.16) we see that

$$Z_2g(s,t) = \sqrt{2}(2\nu)^{\frac{1}{4}} e^{-4\nu\pi s^2} \theta_3(t - 4i\nu s|4i\nu).$$

With this definition, Z_2g has only one zero in $[0,1]^2$ at $s = \frac{1}{2}$ and $t = \frac{1}{2}$ by [29]. Hence, condition (2.12) is satisfied. The construction in Theorem 2.3 leads to a Wilson basis with exponential phase space localization. In particular, the decay rates of ϕ and $\hat{\phi}$ can be adjusted by taking different ν in the Gaussian. In fact, taking $\nu = \frac{1}{2}$ we obtain that, $\forall\varepsilon > 0$, there exists C_ε so that $|\phi(x)| \leq C_\varepsilon e^{-\frac{(\pi-\varepsilon)|x|}{2}}$ and $|\hat{\phi}(y)| \leq C_\varepsilon e^{-(\pi-\varepsilon)|x|}$. While, taking $\nu = \frac{1}{\sqrt{2}}$, we can bound ϕ and $\hat{\phi}$ by $|\phi(x)| \leq C_\varepsilon e^{-\frac{(\pi-\varepsilon)|x|}{\sqrt{2}}}$ and $|\hat{\phi}(y)| \leq C_\varepsilon e^{-\frac{(\pi-\varepsilon)|y|}{\sqrt{2}}}$.

2.3 Equivalent formulation using tight frames

Another important result is the interpretation of Proposition 2.1 in terms of frames. In [15], it is shown that $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$ if and only if $\{M_{\frac{m}{2}}T_n\phi\}_{m,n\in\mathbb{Z}}$ is a tight frame with redundancy 2. Geometrically, this means that a tight frame contains twice as many vectors as an orthonormal basis and the formula (2.1) provides a way to eliminate the redundancy and to form an orthonormal basis. Hence, starting from a tight Gabor frame of redundancy 2, it is possible to construct an orthonormal basis for $L^2(\mathbb{R})$ whose generator is well localized in time and frequency, for example the window function can be chosen to be a Schwartz function or a $\mathcal{C}^\infty$-function with compact support.

Before showing the main result of this section, we will prove the following proposition that provides a starting point in the construction of ϕ as in Theorem 2.3 considering any well-localized g so that $\{g_{\frac{m}{2},n}\}_{m,n\in\mathbb{Z}}$ is a frame.

Proposition 2.4. *Consider the Gabor system with $a = 1$ and $b = \frac{1}{2}$ defined by $\{g_{\frac{m}{2},n}(x) = M_{\frac{m}{2}}T_n g(x), m,n\in\mathbb{Z}\}$ with $g \in L^2(\mathbb{R})$ and real-valued. Then $\{g_{\frac{m}{2},n}\}_{m,n\in\mathbb{Z}}$ is a frame with lower bound A and upper bound B if and only if the Zak transform Z_2g of g, as defined by (2.7), satisfies*

$$A = \inf_{t,s\in[0,1]} [|Z_2g(s,t)|^2 + |Z_2g(s+\frac{1}{2},t)|^2] > 0 \quad \textit{and}$$

$$B = \sup_{t,s\in[0,1]} [|Z_2g(s,t)|^2 + |Z_2g(s+\frac{1}{2},t)|^2] < \infty$$

for almost all $s,t \in [0,1]^2$.

Proof. We want to find which conditions on g make $\{M_{\frac{m}{2}}T_n g\}_{m,n\in\mathbb{Z}}$ a frame. Recall the division in n even and odd of the Zak transform in Proposition 1.21 and consider Remark 1.3. First study the case $n = 2n'$. For $h \in L^2(\mathbb{R})$

$$\begin{aligned}\langle h, g_{\frac{m}{2},2n'}\rangle &= \langle Z_2 h, Z_2 g_{\frac{m}{2},2n'}\rangle \\ &= \int_0^1 \left(\int_0^1 Z_2 h(s,t)\overline{Z_2 g(s,t)} e^{2\pi i t n'} e^{-2\pi i m s} ds\right) dt \\ &= (Z_2 h \cdot \overline{Z_2 g})\hat{}\,(m, -n')\end{aligned}$$

We have that $(Z_2 h \cdot \overline{Z_2 g})\hat{}\,(m,-n')$ are the Fourier coefficients of the periodic function $Z_2 h \cdot \overline{Z_2 g}$. Recall the frame operator in (1.11). Define $S(g) = \sum_{m,n} S_{\frac{m}{2},n}$ and $S_{\frac{m}{2},n} f = \langle g_{\frac{m}{2},n}, f\rangle g_{\frac{m}{2},n}$. Using Parseval's theorem, for $h_1, h_2 \in L^2(\mathbb{R})$ we have

$$\begin{aligned}\sum_{m,n'\in\mathbb{Z}} \langle h_1, S_{\frac{m}{2},2n'} h_2\rangle &= \sum_{m,n'\in\mathbb{Z}} \langle h_1, g_{\frac{m}{2},2n'}\rangle \langle g_{\frac{m}{2},2n'}, h_2\rangle \\ &= \sum_{m,n'\in\mathbb{Z}} \langle Z_2 h_1, Z_2 g_{\frac{m}{2},2n'}\rangle \langle Z_2 g_{\frac{m}{2},2n'}, Z_2 h_2\rangle \\ &= \sum_{m,n'\in\mathbb{Z}} (Z_2 h_1 \cdot \overline{Z_2 g})\hat{}\,(m,-n') \cdot \overline{(Z_2 h_2 \cdot \overline{Z_2 g})\hat{}}\,(m,-n') \\ &= \int_0^1 \left(\int_0^1 Z_2 h_1(s,t)\overline{Z_2 h_2(s,t)} |Z_2 g(s,t)|^2 ds\right) dt.\end{aligned}$$

We obtain that $Z_2[\sum_{m,n\in\mathbb{Z}} S_{\frac{m}{2},2n'}]Z_2^{-1}$ is multiplication by $|Z_2 g(s,t)|^2$ in $L^2(Q)$ and, analogously, $Z_2[\sum_{m,n\in\mathbb{Z}} S_{\frac{m}{2},2n'-1}]Z_2^{-1}$ corresponds to multiplication by $|Z_2 g(s+\frac{1}{2},t)|^2$. Hence, $S(g) = \sum_{m,n} S_{\frac{m}{2},n}$ is unitarily equivalent to multiplication by $|Z_2 g(s,t)|^2 + |Z_2 g(s+\frac{1}{2},t)|^2$ on $L^2(Q)$. We have that $S(g)$ satisfies (1.12), i.e. $\{g_{\frac{m}{2},n}\}_{m,n\in\mathbb{Z}}$ is a frame, if and only if

$$0 < A \leq |Z_2 g(s,t)|^2 + |Z_2 g(s+\frac{1}{2},t)|^2 \leq B < \infty$$

for almost all $s, t \in [0,1]$. □

Note that if we choose g in Proposition 2.4 such that both g and $\hat{g}$ have exponential decay and such that $\{M_{\frac{m}{2}}T_n g\}_{m,n\in\mathbb{Z}}$ is a frame, then condition (2.12) of Theorem 2.3 is satisfied. Hence with the definition (2.13) for $Z_2\phi$, we meet the request of Theorem 2.3 and we can construct an exponentially decreasing function ϕ to form the orthonormal basis $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ defined by (2.1) and (2.2).

We summarize our findings so far.

Proposition 2.5. *Let $\phi \in L^2(\mathbb{R})$ be real-valued such that $\int |\phi(x)|^2 dx = 1$. Then the following are equivalent:*

(i) $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$*, as defined by* (2.1) *and* (2.2)*, constitutes an orthonormal basis.*

(ii) The Zak transform $Z_2\phi$ of ϕ satisfies $|Z_2\phi(s,t)|^2 + |Z_2\phi(s+\frac{1}{2},t)|^2 = 2$ a.e.

(iii) The Gabor system $\{\phi_{\frac{m}{2},n}(x) = M_{\frac{m}{2}}T_n\phi(x),\ m,n \in \mathbb{Z}\}$ constitutes a tight frame.

(iv) The Gabor system $\{\phi_{m,2n}(x) = M_mT_{2n}\phi(x),\ m,n \in \mathbb{Z}\}$ is an orthonormal system.

Proof.
(i) $\iff$ (ii) Follows directly from Propositions 2.1 and 2.2.
(ii) $\Longrightarrow$ (iii) By Proposition 2.4 with $g = \phi$ we have

$$S(\phi) = Z_2^{-1}\{\text{multiplication by } [|Z_2\phi(s,t)|^2 + |Z_2\phi(s+\frac{1}{2},t)|^2]\}Z_2. \tag{2.17}$$

If (ii) holds, then $|Z_2\phi(s,t)|^2 + |Z_2\phi(s+\frac{1}{2},t)|^2 = 2$ and $S(\phi) = 2\mathrm{Id}$. Hence,

$$\sum_{m,n} |\langle \phi_{\frac{m}{2},n}, f\rangle|^2 = 2||f||^2$$

and, by definition, $\{\phi_{\frac{m}{2},n}\}_{m,n\in\mathbb{Z}}$ is a tight frame.
(iii) $\Longrightarrow$ (ii) If $\{\phi_{\frac{m}{2},n}\}_{m,n\in\mathbb{Z}}$ is a tight frame, then there exists a constant $A > 0$ such that $\sum_{m,n} |\langle \phi_{\frac{m}{2},n}, f\rangle|^2 = A||f||^2$. By Proposition 1.16, we have that $A = (ab)^{-1}||\phi||^2$ and, since $a = 1$, $b = \frac{1}{2}$ and $||\phi||^2 = 1$, then $A = 2$. The frame operator in (1.11) is $S(\phi) = 2\mathrm{Id}$ and using (2.17) we have $|Z_2\phi(s,t)|^2 + |Z_2\phi(s+\frac{1}{2},t)|^2 = 2$.
(iii) $\iff$ (iv) Follows directly from Corollary 1.19 with $a = 1$ and $b = \frac{1}{2}$. □

Since one can prove the existence of oversampled Gabor frames with good time-frequency localization, we can impose smoothness and decay conditions on the window. In this way $\psi_{m,n}$, defined by (2.1) and (2.2), form an orthonormal basis with the required time-frequency localization while preserving much of the structure of a Gabor system. Hence, for Wilson bases there is compatibility between good time-frequency localization and non-redundancy.

Remark 2.2. It is important to notice that we can use Proposition 2.5 to construct tight frames with exponential localization in both time and frequency.

Remark 2.3. It must be underlined that the equivalence (iv) of Proposition 2.5 was not proved by Daubechies, Jaffard, and Journé in [15] since the main key to prove this equivalence is the duality principle for Gabor frames. In fact, the duality principle was discovered only betweent the 1995 and the 1997 by three groups of researchers: Janssen [22], Daubechies, Landau, and Landau [16], and Ron and Shen [26] and hence this equivalence was not available in the 1991 when [15] was published.

The following result shows the existence of other useful window functions ϕ different to the one described in Theorem 2.3.

Corollary 2.6. *(i) There exists a window function $\phi \in \mathcal{S}(\mathbb{R})$ such that $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$ as defined by (2.1) and (2.2), forms an orthonormal basis for $L^2(\mathbb{R})$.*

(ii) There exists a window function $\phi \in \mathcal{C}^\infty(\mathbb{R})$ with compact support such that $\{\psi_{m,n}\}_{m\in\mathbb{N}\setminus\{0\},n\in\mathbb{Z}}$, as defined by (2.1) *and* (2.2)*, forms an orthonormal basis for $L^2(\mathbb{R})$.*

Proof. The idea is to construct ϕ in $\mathcal{S}(\mathbb{R})$ or in $\mathcal{C}^\infty(\mathbb{R})$ with compact support such that one of the conditions in Proposition 2.5 is satisfied.
(i) Consider g as in Theorem 2.3, then $g \in \mathcal{S}(\mathbb{R})$. By Theorem 2.3, ϕ and $\hat{\phi}$ are of exponential decay and hence $\phi \in \mathcal{S}(\mathbb{R})$.
(ii) Recall Proposition 2.1, we want to construct $\phi \in \mathcal{C}^\infty(\mathbb{R})$ with compact support such that condition (2.4) is satisfied. Let $\operatorname{supp}(\phi) \subseteq [-1,1]$, if we consider condition (2.6) in the proof of Proposition 2.1, we have that if $l = 0$ then $\phi(\omega)\phi(\omega+2l) = 0$ for every $\omega \in \mathbb{R}$. By computations in claim 3 of Proposition 2.1, condition (2.4) is satisfied if $\sum_{k\in\mathbb{Z}} \phi(\omega+k)^2 = 1$. The sum is periodic in ω with period 1, hence we only need to check the condition for $\omega \in [0,1]$. Moreover, ϕ has support in $[-1,1]$, then it suffices to find ϕ such that $\phi(\omega)^2 + \phi(\omega-1)^2 = 1$ for $\omega \in [0,1]$. Take $g \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$g(x) = \begin{cases} 0, & x \le 0, \\ 1, & x \ge 1, \end{cases}$$
$$0 \le g(x) \le 1 \text{ for all } x.$$

Define ϕ as

$$\phi(\omega) = \begin{cases} \sin\left[\frac{\pi}{2} g(\omega+1)\right], & \omega \le 0, \\ \cos\left[\frac{\pi}{2} g(\omega)\right], & \omega \ge 0. \end{cases}$$

Since $g \in \mathcal{C}^\infty(\mathbb{R})$, then $\phi \in \mathcal{C}^\infty(\mathbb{R})$ with support in $[-1,1]$ and satisfies condition (2.4) of Proposition 2.1 which is equivalent to (ii) of Proposition (2.5). Thus, $\psi_{m,n}$ form an orthonormal basis for $L^2(\mathbb{R})$. □

Remark 2.4. It is important to notice that in point (ii) of the previous proof, the regularity of ϕ strongly depends on the regularity of the function g. In fact, for $g \in \mathcal{C}^k(\mathbb{R})$, then $\phi \in \mathcal{C}^k(\mathbb{R})$.

Remark 2.5. It might be interesting to ask whether it is possible to construct the analogue of a Wilson system for a tight Gabor frame with redundancy different from 2. In 1997, in [9] and [8], Bölcskei, Gröchenig, Hlawatsch and Feichtinger constructed the analogue of a Wilson system for a tight Gabor frame with even redundancy $2N$ for $N \in \mathbb{Z}$ being under certain condition a tight frame with the frame bound reduced by factor 2. Later Gröchenig posed a challenging question, whether there exists a Wilson basis for the case of redundancy 3. A positive answer to this question has been not given yet but the results of Wojdiłło in [31] seem promising: in fact, he constructed a system whose elements are the combinations of the time-frequency shifts with redundancy 3.

Remark 2.6. It easily follow that a tensor product of orthonormal Wilson basis forms an orthonormal basis for $L^2(\mathbb{R}^d)$. Unfortunately, tensoring Wilson bases has some undesirable side effects, among others, they are associated with highly redundant Gabor frames of redundancy 2^d. In [6] Bownik, Jakobsen, Lemvig and Okoudjou presented a construction which improved the tensoring method by showing that we can even construct multi-dimensional orthonormal Wilson bases starting from tight Gabor frames of redundancy 2^k with $k = 1, 2, ..., d$ where $k = d$ corresponds to the tensoring method.

Chapter 3

Malvar bases

In signal processing, it can be useful sometimes to focus on local properties of a signal. Consider a function f on $\mathbb{R}$, we are interested in its properties on a finite interval I. For every interval I there are several types of orthonormal bases consisting of trigonometric functions:

(i) $\sin\left(\frac{2k+1}{2}\frac{\pi}{|I|}x\right)\chi_I(x)$, $k = 0, 1, 2, ...$;

(ii) $\sin\left(k\frac{\pi}{|I|}x\right)\chi_I(x)$, $k = 1, 2, 3, ...$;

(iii) $\cos\left(\frac{2k+1}{2}\frac{\pi}{|I|}x\right)\chi_I(x)$, $k = 0, 1, 2, ...$;

(iv) $\cos\left(k\frac{\pi}{|I|}x\right)\chi_I(x)$, $k = 0, 1, 2, ...$

From these orthonormal bases of $L^2(I)$ we can construct an orthonormal basis of $L^2(\mathbb{R})$ by considering any partition $\{\alpha_k\}_{k\in\mathbb{R}}$ of $\mathbb{R}$ such that $\alpha_k < \alpha_{k+1}$ and $\lim_{k\to\pm\infty}\alpha_k = \pm\infty$. We can patch together these bases using that $L^2(\mathbb{R}) = \oplus_{k=-\infty}^{\infty} L^2([\alpha_k, \alpha_{k+1}])$. On one hand, these bases are well localized in x but, on the other hand, the using of the characteristic function of I produces artificial discontinuities that prevent a good frequency localization.

In this chapter we will construct the so called *local Fourier bases* or *Malvar bases*. These bases are orthonormal bases for $L^2(\mathbb{R})$ which can be constructed for any partition of $\mathbb{R}$ and such that the characteristic function of I is replaced by a $\mathcal{C}^N$-function with compact support and $N \in \mathbb{N}\cup\{\infty\}$. The idea is to find a projection that have arbitrarily smooth cut off to avoid the undesirable effects produced by multiplication by the characteristic function of I. In this section we present the construction of the Malvar bases by Auscher, Weiss and Wickerhauser in [2].

3.1 Smooth projection on the $[0, \infty)$ half ray

Consider first the special case $I = [0, \infty)$ and our goal is to construct a smooth "bell" function that approximates $\chi_{[0,\infty]}$. Since any projection is idempotent, multiplication by a function gives a projection only if the function takes the values 0 or 1 almost everywhere on $\mathbb{R}$; this proves that the projection we are looking for cannot be given simply by multiplication by a smooth function. The next lemma gives us a sufficient and necessary condition for a particular operator to be an orthogonal projection.

Lemma 3.1. *Let ρ be a non-negative function with support in $[-\varepsilon, \infty)$ for $\varepsilon > 0$ and such that*

$$\begin{aligned} &\rho(x) = 1, \ \text{if } x \geq \varepsilon \text{ and} \\ &\rho(x) + \rho(-x) = 1, \forall x \in \mathbb{R}. \end{aligned} \tag{3.1}$$

Let t be a real-valued, even function and define the operator P_I as

$$(P_I f)(x) = \rho(x) f(x) + t(x) f(-x).$$

Then P_I is an orthogonal projection if and only if $t(x) = \pm\sqrt{\rho(x)\rho(-x)}$.

Proof.
1. Firstly, P_I must be self-adjoint, i.e. $\langle P_I f, g\rangle = \langle f, P_I g\rangle$ for every $f, g \in L^2(\mathbb{R})$. Since t is real-valued and even, we have $t(-x) = \overline{t(x)}$ and recalling that ρ is a real function then we obtain

$$\begin{aligned} \langle P_I f, g\rangle &= \int_{-\infty}^{+\infty} (\rho(x) f(x) + t(x) f(-x))\overline{g(x)} dx \\ &= \int_{-\infty}^{+\infty} \rho(x) f(x)\overline{g(x)} dx + \int_{-\infty}^{+\infty} t(x) f(-x)\overline{g(x)} dx \\ &= \int_{-\infty}^{+\infty} f(x)\overline{\rho(x) g(x)} dx + \int_{-\infty}^{+\infty} t(x) f(-x)\overline{g(x)} dx \\ &= \int_{-\infty}^{+\infty} f(x)\overline{\rho(x) g(x)} dx + \int_{-\infty}^{+\infty} t(-x) f(x)\overline{g(-x)} dx \\ &= \int_{-\infty}^{+\infty} f(x)\overline{\rho(x) g(x)} dx + \int_{-\infty}^{+\infty} f(x)\overline{t(x) g(-x)} dx \\ &= \langle f, P_I g\rangle. \end{aligned}$$

2. Secondly, we ask P_I to satisfy the idempotent property $P_I^2 = P_I$:

$$\begin{aligned} (P_I^2 f)(x) &= \rho(x)(P_I f)(x) + t(x)(P_I f)(-x) \\ &= \rho(x)(\rho(x) f(x) + t(x) f(-x)) + t(x)(\rho(-x) f(-x) + t(-x) f(x)) \\ &= (\rho(x)^2 + t(x)t(-x)) f(x) + t(x) f(-x)\underbrace{(\rho(x) + \rho(-x))}_{=1 \text{ by } (3.1)} \\ &= (\rho(x)^2 + t(x)t(-x)) f(x) + t(x) f(-x). \end{aligned}$$

We ask $(P_I^2 f)(x)$ to be equal to $(P_I f)(x)$:

$$\begin{aligned} (P_I^2 f)(x) = (P_I f)(x) &\iff \rho(x) = \rho(x)^2 + t(x)t(-x) \\ &\iff \rho(x)\underbrace{(1 - \rho(x))}_{=\rho(-x)} = t(x)t(-x). \end{aligned}$$

Since t is an even function, this is equivalent to $t(x) = \pm\sqrt{\rho(x)\rho(-x)}$. $\square$

Our findings show that, under the previous assumptions on ρ and t, P_I is a projection if and only if

$$(P_I f)(x) = \rho(x)f(x) \pm \sqrt{\rho(x)\rho(-x)}f(-x). \tag{3.2}$$

The next step consists in constructing an explicit smooth function ρ that satisfies (3.1).

Lemma 3.2. *Let $\psi \in \mathcal{C}^{N-1}(\mathbb{R})$ be an even non-negative function with $N \in \mathbb{N} \cup \{\infty\}$ such that* $\operatorname{supp}(\psi) \subset [-\varepsilon, \varepsilon]$ *with $\varepsilon > 0$ and $\int_{\mathbb{R}} \psi = \frac{\pi}{2}$ and define $\theta(x) = \int_{-\infty}^{x} \psi(t)dt$. Then the function $\rho(x) = \sin^2(\theta(x))$ satisfies* (3.1).

Proof. Firstly, we notice that

$$\theta(x) + \theta(-x) = \int_{-\infty}^{x} \psi(t)dt + \underbrace{\int_{-\infty}^{-x} \psi(t)dt}_{\substack{=\int_x^{+\infty} \psi(t)dt \\ \text{since } \psi \text{ is even}}} = \int_{-\infty}^{+\infty} \psi(t)dt = \frac{\pi}{2}.$$

Define now $s_\varepsilon(x) = \sin(\theta(x))$ and $c_\varepsilon(x) = \cos(\theta(x))$, we have that

$$c_\varepsilon(x) = \cos\left[\frac{\pi}{2} - \theta(-x)\right] = \sin(\theta(-x)) = s_\varepsilon(-x)$$

and

$$s_\varepsilon^2(x) + c_\varepsilon^2(x) = 1.$$

We finally define $\rho(x) = s_\varepsilon^2(x)$. Since $\rho \in \mathcal{C}^N$, then ρ is smooth and satisfies the properties (3.1), in fact:

1. $\rho(x) = \sin^2\left(\int_{-\infty}^{x} \psi(t)dt\right) = \sin^2(\frac{\pi}{2}) = 1$ if $x \geq \varepsilon$, since $\operatorname{supp}(\psi) \subset [-\varepsilon, \varepsilon]$;
2. $\rho(x) + \rho(-x) = s_\varepsilon^2(x) + s_\varepsilon^2(-x) = s_\varepsilon^2(x) + c_\varepsilon^2(x) = 1.$

$\square$

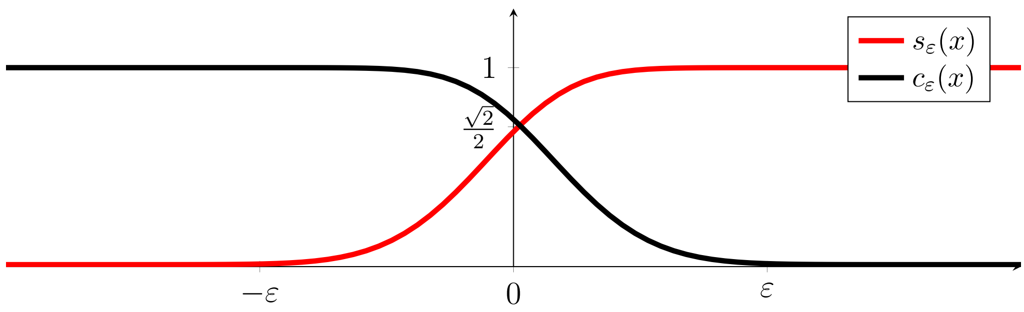


Figure (3.1). The functions s_ε and c_ε.

Thanks to Lemma 3.2 we can rewrite (3.2) as

$$(P_0 f)(x) = s_\varepsilon^2(x)f(x) \pm s_\varepsilon(x)c_\varepsilon(x)f(-x), \tag{3.3}$$

where P_0 is a smooth projection associated to $[0, \infty)$.

Remark 3.1. For the half ray $(-\infty, 0]$, a similar calculation leads to the projection

$$(P^0 f)(x) = c_{\varepsilon'}^2(x) f(x) \pm c_{\varepsilon'}(x) s_{\varepsilon'}(x) f(-x).$$

Furthermore, it is important to notice that the projections depends on the choice of the sign before the second summand and hence we have four projections

$$P_+^{0,\varepsilon'},\ P_-^{0,\varepsilon'},\ P_{0,\varepsilon}^+,\ P_{0,\varepsilon}^-. \tag{3.4}$$

3.2 Smooth projection on a bounded interval

Let $I = [\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$. First of all, we need to translate our projections P_0 and P^0 to arbitrary points α and β on $\mathbb{R}$. Define $P_\alpha = T_\alpha P_0 T_{-\alpha}$ and $P^\beta = T_\beta P^0 T_{-\beta}$ the translates of P_0 and P^0 by α and β with the translation operator $(T_\gamma f)(x) = f(x-\gamma)$ defined in Lemma 1.1. We obtain

$$\begin{aligned}
(P_\alpha f)(x) &= (T_\alpha P_0 T_{-\alpha} f)(x) = (P_0 T_{-\alpha} f)(x-\alpha) \\
&= s_\varepsilon^2(x-\alpha) T_{-\alpha} f(x-\alpha) \pm s_\varepsilon(x-\alpha) c_\varepsilon(x-\alpha) T_{-\alpha} f(-(x-\alpha)) \\
&= s_\varepsilon^2(x-\alpha) f(x) \pm s_\varepsilon(x-\alpha) c_\varepsilon(x-\alpha) f(2\alpha - x).
\end{aligned} \tag{3.5}$$

and analogously,

$$(P^\beta f)(x) = c_{\varepsilon'}^2(x-\beta) f(x) \pm s_{\varepsilon'}(x-\beta) c_{\varepsilon'}(x-\beta) f(2\beta - x). \tag{3.6}$$

Recall from functional analysis that if P is an orthogonal projections and T is a unitary operator then TPT^* is an orthogonal projection. Hence, $P_\alpha f$ and $P_\beta f$ are orthogonal projections. We observe that $2\alpha - x$ and x are symmetric with respect to α, in fact, they lie on opposite sides and are equidistant to α. This provide a motivation for the following definition:

Definition 3.1. Let f be a function on $\mathbb{R}$. f is said to be *even* with respect to α on $[\alpha - \varepsilon, \alpha + \varepsilon]$ if $f(2\alpha - x) = f(x)$ on this interval. Analogously, f is said to be *odd* with respect to α on $[\alpha - \varepsilon, \alpha + \varepsilon]$ if $f(2\alpha - x) = -f(x)$ on this interval. These definitions can be extended to all of $\mathbb{R}$ if the properties hold for all $x \in \mathbb{R}$.

Lemma 3.3. *For a general interval $I = [\alpha, \beta]$ with $-\infty < \alpha < \beta < +\infty$ such that $\alpha + \varepsilon \leq \beta - \varepsilon'$ with $\varepsilon, \varepsilon' > 0$, the operators P_α and P^β defined in* (3.5) *and* (3.6) *commute and the operator $P_I = P_{[\alpha,\beta]} = P_\alpha P^\beta = P^\beta P_\alpha$ is an orthogonal projection.*

Proof. By a general result of functional analysis: if P_α and P^β are two orthogonal projections and $[P_\alpha, P^\beta] = 0$, then $(P_\alpha P^\beta)^* = P_\alpha P^\beta$ and $(P_\alpha P^\beta)^2 = P_\alpha P^\beta$. Hence, $P_{[\alpha,\beta]} = P_\alpha P^\beta = P^\beta P_\alpha$ is an orthogonal projection. To prove the lemma it is enough to show that $P_\alpha P^\beta = P^\beta P_\alpha$. Firstly, we note that if g is an even function with respect to α, then $P_\alpha(gf) = g(P_\alpha f)$ for $g \in L^\infty(\mathbb{R})$ and $f \in L^2(\mathbb{R})$, which means that g commutes with P_α. Analogously, if g is an even function with respect to β, then $P^\beta(gf) = g(P^\beta f)$. Hence, since $\chi_{[\alpha-\varepsilon,\alpha+\varepsilon]}$ is even with respect to α and $\chi_{[\beta-\varepsilon',\beta+\varepsilon']}$

is even with respect to β, they commute with the respective projection. Moreover, by the constructions of P_I in Lemmas 3.1 and 3.2, for $x > \alpha+\varepsilon$, $P_\alpha f(x) = f(x)$ and analogously, for $x < \beta - \varepsilon'$, $P_\beta f(x) = f(x)$. We then have:

$$\begin{aligned} P_\alpha f &= P_\alpha \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]} f + P_\alpha \chi_{(\alpha+\varepsilon,\infty)} f = \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]} P_\alpha f + \chi_{(\alpha+\varepsilon,\infty)} f \\ P^\beta f &= P^\beta \chi_{(-\infty,\beta-\varepsilon')} f + P^\beta \chi_{[\beta-\varepsilon',\beta+\varepsilon']} f = \chi_{(-\infty,\beta+\varepsilon')} f + \chi_{[\beta-\varepsilon',\beta+\varepsilon']} P^\beta f. \end{aligned}$$

Since $\alpha+\varepsilon \leq \beta-\varepsilon'$, and applying P^β to the first equality and P_α to the second one, we obtain:

$$\begin{aligned} P^\beta P_\alpha f &= P^\beta\{\chi_{[\alpha-\varepsilon,\alpha+\varepsilon]} P_\alpha f + \chi_{(\alpha+\varepsilon,\infty)} f\} \\ &= P^\beta \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]} P_\alpha f + P^\beta \chi_{(\alpha+\varepsilon,\beta-\varepsilon')} f + P^\beta \chi_{[\beta-\varepsilon',\beta+\varepsilon']} f \\ &= \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]} P_\alpha f + \chi_{(\alpha+\varepsilon,\beta-\varepsilon')} f + \chi_{[\beta-\varepsilon',\beta+\varepsilon']} P^\beta f. \end{aligned}$$

Similarly,

$$\begin{aligned} P_\alpha P^\beta f &= P_\alpha\{\chi_{(-\infty,\beta+\varepsilon')} f + \chi_{[\beta-\varepsilon',\beta+\varepsilon']} P^\beta f\} \\ &= P_\alpha \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]} f + P_\alpha \chi_{(\alpha+\varepsilon,\beta-\varepsilon')} f + P_\alpha \chi_{[\beta-\varepsilon',\beta+\varepsilon']} P^\beta f \\ &= \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]} P_\alpha f + \chi_{(\alpha+\varepsilon,\beta-\varepsilon')} f + \chi_{[\beta-\varepsilon',\beta+\varepsilon']} P^\beta f \end{aligned}$$

Hence,

$$P_\alpha P^\beta f = P^\beta P_\alpha f = \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]} P_\alpha f + \chi_{(\alpha+\varepsilon,\beta-\varepsilon')} f + \chi_{[\beta-\varepsilon',\beta+\varepsilon']} P^\beta f. \tag{3.7}$$

□

We observe that $P_I = P_{[\alpha,\beta]}$ depends on $\alpha, \beta, \varepsilon, \varepsilon'$ and the sign we choose at α and β. Hence, if $\alpha, \beta, \varepsilon$ and ε' are fixed, from the choice of signs we obtain four projections. Let us now introduce a function b_I which depends on $\alpha, \beta, \varepsilon$ and ε' but not on the sign.

Definition 3.2. Let $I = [\alpha, \beta]$, with $-\infty < \alpha < \beta < +\infty$ such that $\alpha+\varepsilon \leq \beta-\varepsilon'$ with $\varepsilon, \varepsilon' > 0$ and s_ε and $c_{\varepsilon'}$ as in Lemma 3.2. We call the *bell over I* the function b_I defined by

$$b_I(x) = s_\varepsilon(x-\alpha)c_{\varepsilon'}(x-\beta), \text{ for all } x \in \mathbb{R}.$$

We have the following basic properties of b_I.

Proposition 3.4 (**Properties of** b_I)**.** *The function b_I as defined by Definition 3.2 satisfies*

(i) $\operatorname{supp}(b_I) \subseteq [\alpha-\varepsilon, \beta+\varepsilon']$.

Properties on the interval $[\alpha-\varepsilon, \alpha+\varepsilon]$*:*

(ii) $b_I(x) = s_\varepsilon(x-\alpha)$*;*

(iii) $b_I(2\alpha - x) = s_\varepsilon(\alpha - x) = c_\varepsilon(x - \alpha)$;

(iv) $b_I^2(x) + b_I^2(2\alpha - x) = s_\varepsilon^2(x - \alpha) + c_\varepsilon^2(x - \alpha) = 1$.

(v) $\operatorname{supp}(b_I(x)b_I(2\alpha - x)) \subseteq [\alpha - \varepsilon, \alpha + \varepsilon]$.

(vi) $b_I(x) = 1$ *when* $x \in [\alpha + \varepsilon, \beta - \varepsilon']$.

Properties on the interval $[\beta - \varepsilon', \beta + \varepsilon']$*:*

(vii) $b_I(x) = c_{\varepsilon'}(x - \beta)$;

(viii) $b_I(2\beta - x) = c_{\varepsilon'}(\beta - x) = s_{\varepsilon'}(x - \beta)$;

(ix) $b_I^2(x) + b_I^2(2\beta - x) = 1$.

(x) $\operatorname{supp}(b_I(x)b_I(2\beta - x)) \subseteq [\beta - \varepsilon', \beta + \varepsilon']$.

(xi) $b_I^2(x) + b_I^2(2\alpha - x) + b_I^2(2\beta - x) = 1$ *on* $\operatorname{supp}(b_I)$.

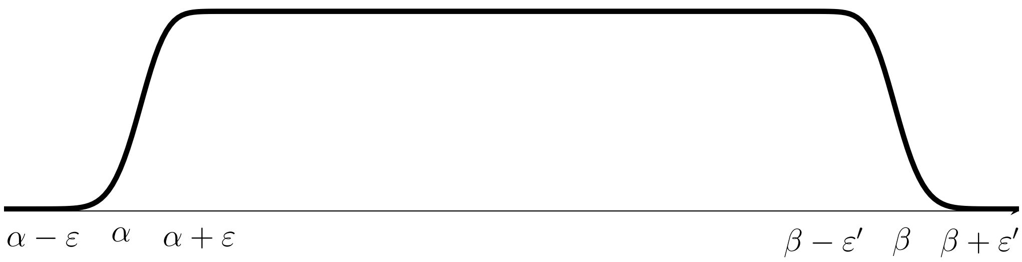


Figure (3.2). The bell b_I over $[\alpha, \beta]$.

Proof.
(i) By definition of b_I, s_ε and $c_{\varepsilon'}$ we have

$$b_I(x) = s_\varepsilon(x - \alpha)c_{\varepsilon'}(x - \beta) = \sin\left(\int_{-\infty}^{x-\alpha} \psi_\varepsilon(t)dt\right)\cos\left(\int_{-\infty}^{x-\beta} \psi_{\varepsilon'}(t)dt\right) \tag{3.8}$$

For $x \leq \alpha - \varepsilon$, the sine is 0 since $\operatorname{supp}(\psi_\varepsilon) \subset [-\varepsilon, \varepsilon]$.
For $x \geq \beta + \varepsilon'$, the cosine is 0 since the interval of integration contains all the support of $\psi_{\varepsilon'}$ and the integral is $\frac{\pi}{2}$.
(ii) By definition, $c_{\varepsilon'}(x - \beta) = \cos\left(\int_{-\infty}^{x-\beta} \psi_{\varepsilon'}(t)dt\right)$. For $x \leq \alpha + \varepsilon$, we have that $\alpha + \varepsilon - \beta < -\varepsilon'$, the integral is 0 and the cosine is 1. Thus, on the interval $[\alpha - \varepsilon, \alpha + \varepsilon]$ $b_I(x) = s_\varepsilon(x - \alpha)$.
(iii) By definition, $b_I(2\alpha - x) = s_\varepsilon(\alpha - x)c_{\varepsilon'}(2\alpha - x - \beta)$. Consider $c_{\varepsilon'}(2\alpha - x - \beta)$ and note that for $x \leq \alpha + \varepsilon$, we have $2\alpha - (\alpha + \varepsilon) - \beta < -\varepsilon'$ and hence, as for proof of (ii), $c_{\varepsilon'}(2\alpha - x - \beta) = 1$ for $x \in [\alpha - \varepsilon, \alpha + \varepsilon]$. Thus, $b_I(2\alpha - x) = s_\varepsilon(\alpha - x) = c_\varepsilon(x - \alpha)$ on the interval $[\alpha - \varepsilon, \alpha + \varepsilon]$.

(iv) Using (ii) and (iii), $b_I^2(x)+b_I^2(2\alpha-x) = s_\varepsilon^2(x-\alpha)+c_\varepsilon^2(x-\alpha) = 1$ on $[\alpha-\varepsilon, \alpha+\varepsilon]$.
(v) By definition

$$b_I(x)b_I(2\alpha-x) = s_\varepsilon(x-\alpha)c_{\varepsilon'}(x-\beta)s_\varepsilon(\alpha-x)c_{\varepsilon'}(2\alpha-x-\beta).$$

Using proof of (i), for $x \leq \alpha-\varepsilon$, $s_\varepsilon(x-\alpha) = 0$.
Consider $s_\varepsilon(\alpha-x) = c_\varepsilon(x-\alpha) = \cos\left(\int_{-\infty}^{x-\alpha}\psi_\varepsilon(t)dt\right)$. For $x \geq \alpha+\varepsilon$, the integral is $\frac{\pi}{2}$ and the cosine is 0. Hence, $\mathrm{supp}(b_I(x)b_I(2\alpha-x)) \subseteq [\alpha-\varepsilon, \alpha+\varepsilon]$.
(vi) Studying equation (3.8), we note that for $x \geq \alpha+\varepsilon$ the sine is 1, while for $x \leq \beta-\varepsilon'$ the cosine is 1. Hence, in the interval $[\alpha+\varepsilon, \beta-\varepsilon']$, $b_I(x) = 1$.
(vii), (viii), (ix), (x) can be proved similarly to (ii), (iii), (iv) and (v) respectively.
(xi) Using (iv), (vi) and (ix) we have

$$1 = \begin{cases} b_I^2(x)+b_I^2(2\alpha-x), & \text{for } x \in [\alpha-\varepsilon, \alpha+\varepsilon]; \\ b_I^2(x), & \text{for } x \in (\alpha+\varepsilon, \beta-\varepsilon'); \\ b_I^2(x)+b_I^2(2\beta-x), & \text{for } x \in [\beta-\varepsilon', \beta+\varepsilon']. \end{cases}$$

Thus, $b_I^2(x)+b_I^2(2\alpha-x)+b_I^2(2\beta-x) = 1$ on $\mathrm{supp}(b_I)$. □

The following corollary provides a new definition for P_I.

Corollary 3.5. *Let $I = [\alpha, \beta]$, with $-\infty < \alpha < \beta < +\infty$ such that $\alpha+\varepsilon \leq \beta-\varepsilon'$ with $\varepsilon, \varepsilon' > 0$ and $b_I(x) = s_\varepsilon(x-\alpha)c_{\varepsilon'}(x-\beta)$, for all $x \in \mathbb{R}$ be the bell function over I. Let P_I be defined as in Lemma 3.3, then*

$$(P_I f)(x) = b_I^2(x)f(x) \pm b_I(x)b_I(2\alpha-x)f(2\alpha-x) \pm b_I(x)b_I(2\beta-x)f(2\beta-x). \quad (3.9)$$

Proof. Using (3.5) and (3.6), we can rewrite (3.7) using Proposition 3.4 as:

$$\begin{aligned}(P_I f)(x) &= (P_\alpha P^\beta f)(x) = \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]}(P_\alpha f)(x) + \chi_{(\alpha+\varepsilon,\beta-\varepsilon')}f(x) + \chi_{[\beta-\varepsilon',\beta+\varepsilon']}(P^\beta f)(x) \\ &= \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]}(s_\varepsilon^2(x-\alpha)f(x) \pm s_\varepsilon(x-\alpha)c_\varepsilon(x-\alpha)f(2\alpha-x)) + \chi_{(\alpha+\varepsilon,\beta-\varepsilon')}f(x) \\ &\quad + \chi_{[\beta-\varepsilon',\beta+\varepsilon']}(c_{\varepsilon'}^2(x-\beta)f(x) \pm s_{\varepsilon'}(x-\beta)c_{\varepsilon'}(x-\beta)f(2\beta-x)) \\ &= \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]}(b_I^2(x)f(x) \pm b_I(x)b_I(2\alpha-x)f(2\alpha-x)) + \chi_{(\alpha+\varepsilon,\beta-\varepsilon')}b_I^2(x)f(x) \\ &\quad + \chi_{[\beta-\varepsilon',\beta+\varepsilon']}(b_I^2(x)f(x) \pm b_I(x)b_I(2\beta-x)f(2\beta-x)) \\ &= b_I^2(x)f(x) \pm b_I(x)b_I(2\alpha-x)f(2\alpha-x) \pm b_I(x)b_I(2\beta-x)f(2\beta-x).\end{aligned}$$

□

To summarize, P_I is a smooth version of the operator of pointwise multiplication by the characteristic function of I.

We note that we have four choices for the sign associated with endpoints α and β of I. The choice of $\pm$ associated with α is referred to as the *polarity* of $P_{[\alpha,\beta]}$ at α, and the choice of $\pm$ associated with β is referred to as the *polarity* of $P_{[\alpha,\beta]}$ at β. In particular if we choose "+" before the second summand in (3.9), we say that the projection has *positive polarity at α*. Polarities are very important when we want to study the properties of P_I and P_J when I and J are two adjacent intervals.

Definition 3.3. Let $I = [\alpha, \beta]$ and $J = [\beta, \gamma]$ be two adjacent intervals, $\varepsilon, \varepsilon', \varepsilon'' > 0$. We say that they have *compatible bell functions* b_I and b_J if

(i) $\alpha + \varepsilon \leq \beta - \varepsilon' < \beta + \varepsilon' \leq \gamma - \varepsilon''$, and,

(ii) $b_I(x) = s_\varepsilon(x-\alpha)c_{\varepsilon'}(x-\beta), \quad b_J(x) = s_{\varepsilon'}(x-\beta)c_{\varepsilon''}(x-\gamma)$.

Remark 3.2. Considering the compatibility of the bell functions b_I and b_J we note that applying Proposition 3.4 to b_J (with β, J and ε' replacing α, I and ε) we obtain:

$$\begin{aligned}
&b_I(x) = b_J(2\beta - x), && \text{for } x \in [\beta-\varepsilon', \beta+\varepsilon'];\\
&b_I^2(x) + b_J^2(x) = 1, && \text{for } x \in [\beta-\varepsilon', \beta+\varepsilon'];\\
&b_I^2(x) + b_J^2(x) = b_{I\cup J}^2(x), && \text{for all } x \in \mathbb{R}.
\end{aligned}$$

- In the first equality we used Proposition 3.4 (iii) to obtain $b_J(2\beta - x) = c_{\varepsilon'}(x-\beta)$ for $x \in [\beta-\varepsilon', \beta+\varepsilon']$. Applying (vii) we have $b_I(x) = c_{\varepsilon'}(x-\beta)$ for $x \in [\beta-\varepsilon', \beta+\varepsilon']$.

- Using (ii) and (vii) in a similar way as before, provides us the second equality.

- The third equality follows from the second one.
  In fact, $\sqrt{b_I^2(x) + b_J^2(x)} = s_\varepsilon(x-\alpha)c_{\varepsilon''}(x-\gamma)$ for all x and this is equivalent to $b_I^2 + b_J^2 = b_{I\cup J}^2$.

The next theorem provides us the main property of these projections and allows us to decompose $L^2(\mathbb{R})$ as a direct sum of orthogonal subspaces.

Theorem 3.6. *Let $I = [\alpha, \beta]$ and $J = [\beta, \gamma]$ be two adjacent intervals with compatible bell functions and suppose P_I and P_J have opposite polarities at β. Then:*

(i) $P_I + P_J = P_{I\cup J}$,

(ii) $P_I P_J = P_J P_I = 0$, i.e. P_I and P_J are orthogonal to each other.

Proof.
(i) Let $P_I = P_\alpha P^\beta$ and let Id be the identity operator, then by (3.7) we have

$$\begin{aligned}
P_I + P_J = {}& \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]}P_\alpha + \chi_{(\alpha+\varepsilon,\beta-\varepsilon')}Id + \chi_{[\beta-\varepsilon',\beta+\varepsilon']}P^\beta\\
&+ \chi_{[\beta-\varepsilon',\beta+\varepsilon']}P_\beta + \chi_{(\beta+\varepsilon',\gamma-\varepsilon'')}Id + \chi_{[\gamma-\varepsilon'',\gamma+\varepsilon'']}P^\gamma.
\end{aligned}$$

In particular note that, since P_I and P_J have opposite polarities at β and applying (3.5) with $\alpha = \beta$ and (3.6) then:

$$\begin{aligned}
&\chi_{[\beta-\varepsilon',\beta+\varepsilon']}P^\beta f(x) + \chi_{[\beta-\varepsilon',\beta+\varepsilon']}P_\beta f(x)\\
&\quad = \chi_{[\beta-\varepsilon',\beta+\varepsilon']}(P^\beta f(x) + P_\beta f(x))\\
&\quad = \chi_{[\beta-\varepsilon',\beta+\varepsilon']}(c_{\varepsilon'}^2(x-\beta)f(x) \pm s_{\varepsilon'}(x-\beta)c_{\varepsilon'}(x-\beta)f(2\beta-x)\\
&\qquad\quad + s_{\varepsilon'}^2(x-\beta)f(x) \mp s_{\varepsilon'}(x-\beta)c_{\varepsilon'}(x-\beta)f(2\beta-x))\\
&\quad = \chi_{[\beta-\varepsilon',\beta+\varepsilon']}f(x).
\end{aligned}$$

Hence, $\chi_{[\beta-\varepsilon',\beta+\varepsilon']}P^{\beta} + \chi_{[\beta-\varepsilon',\beta+\varepsilon']}P_{\beta} = \chi_{[\beta-\varepsilon',\beta+\varepsilon']}Id$ and

$$P_I + P_J = \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]}P_{\alpha} + \chi_{(\alpha+\varepsilon,\gamma-\varepsilon'')}Id + \chi_{[\gamma-\varepsilon'',\gamma+\varepsilon'']}P^{\gamma} = P_{I\cup J}.$$

By Lemma 3.3, $P_{I\cup J}$ is an orthogonal projection.
(ii) To show that P_I and P_J are orthogonal to each other we use a general result about projections on a Hilbert space: if P and Q are orthogonal projections on a Hilbert space such that $P+Q$ is an orthogonal projection, then $PQ = QP = 0$.
Note that, since $P+Q$ is an orthogonal projection, then $(P+Q)^2 = P^2+Q^2+PQ+QP = P+Q+PQ+QP = P+Q$ and $PQ = -QP$. Applying the idempotent property of the projections we obtain: $PQ = P^2Q = -PQP = QP^2 = QP$. Thus, $PQ = -QP = QP = 0$. $\square$

Recall the following result from functional analysis.

Remark 3.3. Let $\mathcal{H}$ be an Hilbert space and $\{\mathcal{H}_k\}_{k\in\mathbb{Z}}$ be a sequence of mutually orthogonal closed subspaces. We call the *orthogonal direct sum of the spaces $\mathcal{H}_k$* the space

$$V = \bigoplus_{k=-\infty}^{\infty} \mathcal{H}_k$$

which denotes the closed subspace consisting of all $f = \sum_{k\in\mathbb{Z}} f_k$ with $f_k \in \mathcal{H}_k$ and $\sum_{k\in\mathbb{Z}} ||f_k||^2 < \infty$.

A natural consequence of Theorem 3.6 is the following theorem.

Theorem 3.7. *Let $\{\alpha_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers and $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ a sequence of positive numbers such that $\alpha_k + \varepsilon_k < \alpha_{k+1} - \varepsilon_{k+1}$. For every $k \in \mathbb{Z}$, let $I_{k-1} = [\alpha_{k-1} - \varepsilon_{k-1}, \alpha_k + \varepsilon_k]$ and $I_k = [\alpha_k - \varepsilon_k, \alpha_{k+1} + \varepsilon_{k+1}]$ be two adjacent intervals with compatible bell functions and $P_{k-1} = P_{[\alpha_{k-1},\alpha_k]}$ and $P_k = P_{[\alpha_k,\alpha_{k+1}]}$ have opposite polarity at α_k. Then $L^2(\mathbb{R})$ possesses the decomposition as orthogonal direct sum*

$$L^2(\mathbb{R}) = \bigoplus_{k=-\infty}^{\infty} P_k(L^2(\mathbb{R})).$$

Proof. Note that if $\lim_{k\to\pm\infty} \alpha_k = \pm\infty$ then $\mathbb{R} = \bigcup_{k=-\infty}^{\infty} I_k$. Since $\bigcup_{k=-N}^{N} I_k = [\alpha_{-N}, \alpha_{N+1}]$ then, by (i) in Theorem 3.6, $\sum_{k=-N}^{N} P_k = P_{[\alpha_{-N},\alpha_{N+1}]}$ and by (ii) we have that $P_k(L^2(\mathbb{R})) \perp P_j(L^2(\mathbb{R}))$ for $k \neq j$.
The idea is to apply the Lebesgue's dominated convergence theorem to show that for every $f \in L^2(\mathbb{R})$, we have $\lim_{N\to\infty} ||f - \sum_{k=-N}^{N} f_k|| = 0$ where $f_k = P_k f \in P_k(L^2(\mathbb{R}))$.
Recalling the definition of P_I for a general interval I, (3.7) we have that:

$$\begin{aligned} P_{[\alpha_{-N},\alpha_{N+1}]}f = \chi_{[\alpha_{-N}-\varepsilon_{-N},\alpha_{-N}+\varepsilon_{-N}]}P_{\alpha_{-N}}f + \chi_{(\alpha_{-N}+\varepsilon_{-N},\alpha_{N+1}-\varepsilon_{N+1})}f \\ + \chi_{[\alpha_{N+1}-\varepsilon_{N+1},\alpha_{N+1}+\varepsilon_{N+1}]}P^{\alpha_{N+1}}f. \end{aligned}$$

By definition of $P_{\alpha_{-N}}$, $|P_{\alpha_{-N}}f(x)|$ is dominated by $|f(x)| + |f(2\alpha_{-N} - x)|$ on $\chi_{[\alpha_{-N}-\varepsilon_{-N},\alpha_{-N}+\varepsilon_{-N}]}$ and $P_{\alpha_{-N}}f$ converges to 0 pointwise.

Similarly, by definition of $P^{\alpha_{N+1}}$, $|P^{\alpha_{N+1}}f(x)|$ is dominated by $|f(x)|+|f(2\alpha_{N+1}-x)|$ on $\chi_{[\alpha_{N+1}-\varepsilon_{N+1},\alpha_{N+1}+\varepsilon_{N+1}]}$ and $P_{\alpha_{N+1}}f$ converges to 0 pointwise.
Hence, by dominated convergence, both $||\chi_{[\alpha_{-N}-\varepsilon_{-N},\alpha_{-N}+\varepsilon_{-N}]}P_{\alpha_{-N}}f|| \to 0$ and $||\chi_{[\alpha_{N+1}-\varepsilon_{N+1},\alpha_{N+1}+\varepsilon_{N+1}]}P^{\alpha_{N+1}}f|| \to 0$ as $N \to \infty$. Thus, we have that

$$\begin{aligned}\lim_{N\to\infty}\left|\left|f-\sum_{k=-N}^{N}P_kf\right|\right| &= \lim_{N\to\infty}||f-P_{[\alpha_{-N},\alpha_{N+1}]}f|| \\ &= \lim_{N\to\infty}||f-\chi_{(\alpha_{-N}+\varepsilon_{-N},\alpha_{N+1}-\varepsilon_{N+1})}f|| = 0.\end{aligned}$$

We have shown that $L^2(\mathbb{R})$ is an orthogonal direct sum of the projections P_k. $\square$

Finally, we want to characterize the subspace $P_I(L^2(\mathbb{R}))$.
Let us first give another definition:
We define

$$Sf(x) = b_I(x)f(x) \pm b_I(2\alpha-x)f(2\alpha-x) \pm b_I(2\beta-x)f(2\beta-x) \tag{3.10}$$

and we rewrite equation (3.9) as

$$(P_If)(x) = b_I(x)Sf(x). \tag{3.11}$$

We note that there are four choices for $Sf(x)$ depending on the sign considered:

- $S_+^+f(x) = b_I(x)f(x) + b_I(2\alpha-x)f(2\alpha-x) + b_I(2\beta-x)f(2\beta-x)$ is even with respect to α on $[\alpha-\varepsilon,\alpha+\varepsilon]$ and even with respect to β on $[\beta-\varepsilon',\beta+\varepsilon']$.
In fact, $b_I(2\beta-x) = s_\varepsilon(2\beta-x-\alpha)c_{\varepsilon'}(\beta-x) = \sin\left(\int_{-\infty}^{2\beta-x-\alpha}\psi_\varepsilon(t)dt\right)\cos\left(\int_{-\infty}^{\beta-x}\psi_{\varepsilon'}(t)dt\right)$ and for $x \in [\alpha-\varepsilon,\alpha+\varepsilon]$, the interval of integration contains the all support of $\psi_{\varepsilon'}$ and hence the cosine is 0. Thus, $b_I(2\beta-x) = 0$ for $x \in [\alpha-\varepsilon,\alpha+\varepsilon]$.
Then, on $[\alpha-\varepsilon,\alpha+\varepsilon]$

$$\begin{aligned}S_+^+f(2\alpha-x) &= b_I(2\alpha-x)f(2\alpha-x) + b_I(2\alpha-(2\alpha-x))f(2\alpha-(2\alpha-x)) \\ &= b_I(2\alpha-x)f(2\alpha-x) + b_I(x)f(x) \\ &= S_+^+f(x).\end{aligned}$$

Hence, S_+^+f is even with respect to α on $[\alpha-\varepsilon,\alpha+\varepsilon]$. Similarly, we can prove that S_+^+f is even with respect to β on $[\beta-\varepsilon',\beta+\varepsilon']$.

- $S_-^+f(x) = b_I(x)f(x) - b_I(2\alpha-x)f(2\alpha-x) + b_I(2\beta-x)f(2\beta-x)$ is odd with respect to α on $[\alpha-\varepsilon,\alpha+\varepsilon]$ and even with respect to β on $[\beta-\varepsilon',\beta+\varepsilon']$.
To show that S_-^+f is odd with respect to α on $[\alpha-\varepsilon,\alpha+\varepsilon]$ we have from the previous point that $b_I(2\beta-x) = 0$ for $x \in [\alpha-\varepsilon,\alpha+\varepsilon]$ and in this case

$$\begin{aligned}S_-^+f(2\alpha-x) &= b_I(2\alpha-x)f(2\alpha-x) - b_I(2\alpha-(2\alpha-x))f(2\alpha-(2\alpha-x)) \\ &= b_I(2\alpha-x)f(2\alpha-x) - b_I(x)f(x) \\ &= -S_-^+f(x).\end{aligned}$$

- $S_+^- f(x) = b_I(x)f(x) + b_I(2\alpha - x)f(2\alpha - x) - b_I(2\beta - x)f(2\beta - x)$ is even with respect to α on $[\alpha - \varepsilon, \alpha + \varepsilon]$ and odd with respect to β on $[\beta - \varepsilon', \beta + \varepsilon']$.
- $S_-^- f(x) = b_I(x)f(x) - b_I(2\alpha - x)f(2\alpha - x) - b_I(2\beta - x)f(2\beta - x)$ is odd with respect to α on $[\alpha - \varepsilon, \alpha + \varepsilon]$ and odd with respect to β on $[\beta - \varepsilon', \beta + \varepsilon']$.

Theorem 3.8. *Let $I = [\alpha, \beta]$ be an interval, b_I the bell function associated with I and $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the operator defined by* (3.10) *such that it has the same polarity as I at α and β. Then $f \in P_I(L^2(\mathbb{R}))$ if and only if $f = b_I Sg$ for some $g \in L^2(\mathbb{R})$.*

Proof.
($\Longrightarrow$) From (3.11) and (3.9) it is easy to see that every element of $P_I(L^2(\mathbb{R}))$ is in the form $f = b_I Sg$.
($\Longleftarrow$) Suppose $f = b_I Sg$ with $g \in L^2(\mathbb{R})$ and recall that Sg is even or odd on $[\alpha-\varepsilon, \alpha+\varepsilon]$ according to the choice of polarity at α, and even or odd on $[\beta-\varepsilon', \beta+\varepsilon']$ according to the choice of polarity at β. Applying (3.9) to f and using (i), (iv), (vi) and (ix) of Proposition 3.4, we have:

$$
\begin{aligned}
(P_I b_I Sg)(x) &= b_I^2(x)b_I(x)Sg(x) \pm b_I(x)b_I^2(2\alpha - x)Sg(2\alpha - x) \pm b_I(x)b_I^2(2\beta - x)Sg(2\beta - x) \\
&= \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]}(x)b_I(x)[b_I^2(x)Sg(x) + b_I^2(2\alpha - x)\underbrace{Sg(2\alpha - x)}_{=Sg(x)}] \\
&\qquad + \chi_{(\alpha+\varepsilon,\beta-\varepsilon')}(x)b_I(x)Sg(x) \\
&\qquad + \chi_{[\beta-\varepsilon',\beta+\varepsilon']}(x)b_I(x)[b_I^2(x)Sg(x) + b_I^2(2\beta - x)\underbrace{Sg(2\beta - x)}_{=Sg(x)}] \\
&= b_I(x)Sg(x).
\end{aligned}
$$

$\square$

3.3 Local sine and cosine bases

In this part we present orthonormal bases for the subspace $P_I(L^2(\mathbb{R}))$ introduced in the previous section. We will see that these bases are related to certain trigonometric systems and consistent with the polarity of P_I. In fact, if P_I is chosen with alternating polarity at the endpoints of I, the elements of the basis will be locally even at the left endpoint and locally odd at the right endpoint or locally odd at the left endpoint and locally even at the right endpoint. In addition, the bases for these subspaces will be expressed in terms of trigonometric functions and the associated bell function. We start by considering $I = [0, 1]$ and we have the following result:

Proposition 3.9. *Each one of the following systems forms an orthonormal basis for $L^2([0, 1])$:*

(i) $\{\sqrt{2}\sin(\frac{2k+1}{2}\pi x),\ k \in \mathbb{N} \cup \{0\}\}$;

(ii) $\{\sqrt{2}\sin(k\pi x),\ k \in \mathbb{N}\}$;

(iii) $\{\sqrt{2}\cos(\frac{2k+1}{2}\pi x),\ k \in \mathbb{N} \cup \{0\}\}$;

(iv) $\{1, \sqrt{2}\cos(k\pi x),\ k \in \mathbb{N}\}$.

The polarities of each of the functions in the first basis are $(-,+)$, in the second basis they are $(-,-)$, in the third they are $(+,-)$ and in the fourth they are $(+,+)$.

Proof. We will prove the result for the case (i).
Consider $I = [0,1]$ and suppose P_I has polarities $-$ and $+$ at 0 and 1, respectively. Let $f \in L^2([0,1])$ and extend f to the interval $[0,2]$ such that it is even with respect to 1. Then we extend it to a new function F on $[-2,2]$ such that F is odd with respect to 0, consistently with the choice of the polarities at P_I. The function F can be developed into a Fourier series on $[-2,2]$ by means of the orthonormal basis $\left\{\frac{1}{2}, \frac{1}{\sqrt{2}}\sin\left(\frac{k\pi x}{2}\right), \frac{1}{\sqrt{2}}\cos\left(\frac{k\pi x}{2}\right)\right\}$ for $k = 1, 2, \ldots$. Since F is odd on $[-2,2]$, the cosines play no role in the Fourier expansion of F. Moreover, the functions $\sin\left(\frac{2k+1}{2}\pi x\right)$, $k = 0, 1, 2, \ldots$ are even with respect to 1. Hence, we can represent F as

$$F(x) = \sum_{k=0}^{\infty} c_k \sin\left(\frac{2k+1}{2}\pi x\right),$$

where

$$c_k = 2\int_0^1 F(x)\sin\left(\frac{2k+1}{2}\pi x\right)dx,$$

and the series converges in the norm of $L^2([-2,2])$. Note that by a deep theorem of Carleson in [10], the pointwise almost everywhere convergence of the Fourier series holds true. If we restrict to $[0,1]$ and we normalize, we find that $\left\{\sqrt{2}\sin\left(\frac{2k+1}{2}\pi x\right),\ k \in \mathbb{N}\cup\{0\}\right\}$ is an orthonormal basis for $L^2([0,1])$ with polarities of its elements at 0 and 1 that match the polarities of P_I.
The other statements can be obtained in a similar way. □

We use this result to obtain an orthonormal basis for $P_I(L^2(\mathbb{R}))$ for $I = [\alpha, \beta]$.

Theorem 3.10. *Let $I = [\alpha, \beta]$ and consider the associated bell function $b_I(x) = s_\varepsilon(x-\alpha)c_{\varepsilon'}(x-\beta)$. Suppose P_I is the smooth projection associated with negative polarity at α and positive polarity at β, then*

(i) $\left\{\sqrt{\frac{2}{|I|}}b_I(x)\sin\left(\frac{2k+1}{2}\frac{\pi}{|I|}(x-\alpha)\right),\ k \in \mathbb{N}\cup\{0\}\right\}$ *is an orthonormal basis for $P_I(L^2(\mathbb{R}))$. Moreover, if $f \in P_I(L^2(\mathbb{R}))$ then the series*

$$\sqrt{\frac{2}{|I|}}\sum_{k=0}^{\infty} c_k b_I(x)\sin\left(\frac{2k+1}{2}\frac{\pi}{|I|}(x-\alpha)\right) \tag{3.12}$$

converges to $f(x)$ in $L^2(I)$ and almost everywhere in $[\alpha-\varepsilon, \beta+\varepsilon']$.
If we choose the polarities $(-,-)$, $(+,-)$ and $(+,+)$ at (α,β), the same result is true if we use, respectively, the systems

(ii) $\left\{\sqrt{\frac{2}{|I|}}b_I(x)\sin\left(k\frac{\pi}{|I|}(x-\alpha)\right),\ k\in\mathbb{N}\right\}$;

(iii) $\left\{\sqrt{\frac{2}{|I|}}b_I(x)\cos\left(\frac{2k+1}{2}\frac{\pi}{|I|}(x-\alpha)\right),\ k\in\mathbb{N}\cup\{0\}\right\}$;

(iv) $\left\{\sqrt{\frac{1}{|I|}}b_I(x), \sqrt{\frac{2}{|I|}}b_I(x)\cos\left(k\frac{\pi}{|I|}(x-\alpha)\right),\ k\in\mathbb{N}\right\}$.

Proof. We will prove the theorem for the case (i) and in a similar way we can prove the others. Consider first for simplicity $I=[0,1]$, let $\varepsilon,\varepsilon'>0$ with $\varepsilon+\varepsilon'\leq 1$ and consider the associated bell function $b_I(x)=s_\varepsilon(x)c_{\varepsilon'}(x-1)$. Suppose that the polarities of P_I are $-$ at 0 and $+$ at 1 and in this case we can write (3.11) as

$$P_If(x)=b_I(x)\{b_I(x)f(x)-b_I(-x)f(-x)+b_I(2-x)f(2-x)\}=b_I(x)S_-^+f(x).$$

We will first show the completeness of the system.
1. *Claim: the system* $\{\sqrt{2}b_I(x)\sin(\frac{2k+1}{2}\pi x),\ k\in\mathbb{N}\cup\{0\}\}$ *is complete.*
Since $S_-^+f(x)$ is odd with respect to 0 and even with respect to 1 because of the properties of b_I, S_-^+f has the right polarity to be represent by the orthonormal basis (i) in Proposition 3.9. Hence, we can write

$$S_-^+f(x)=\sqrt{2}\sum_{k=0}^{\infty}c_k\sin\left(\frac{2k+1}{2}\pi x\right), \tag{3.13}$$

where

$$c_k=\sqrt{2}\int_0^1 S_-^+f(x)\sin\left(\frac{2k+1}{2}\pi x\right)dx,$$

with convergence in $L^2([0,1])$. Equality (3.13) is valid on $[-\varepsilon,1+\varepsilon']$ in the L^2-sense. If we multiply (3.13) on both sides by $b_I(x)$ we have that any $f\in P_I(L^2(\mathbb{R}))$ satisfies

$$f(x)=\sqrt{2}\sum_{k=0}^{\infty}c_kb_I(x)\sin\left(\frac{2k+1}{2}\pi x\right)$$

and the convergence holds in $L^2([-\varepsilon,1+\varepsilon'])$. Hence, the system $\{\sqrt{2}b_I(x)\sin\left(\frac{2k+1}{2}\pi x\right),\ k\in\mathbb{N}\cup\{0\}\}$ is complete in $P_I(L^2(\mathbb{R}))$ when P_I has polarities $(-,+)$.
2. *Claim: the system* $\{\sqrt{2}b_I(x)\sin\left(\frac{2k+1}{2}\pi x\right),\ k\in\mathbb{N}\cup\{0\}\}$ *is orthonormal.*
Let $e_k=\sin(\frac{2k+1}{2}\pi x)$, $k=0,1,2,...$, and $g(x)=\sqrt{2}b_I(x)e_l(x)$ we will show that

$$2\int_{-\varepsilon}^{1+\varepsilon'}b_I^2(x)e_k(x)e_l(x)dx=\delta_{kl},\quad k,l=0,1,2,....$$

Firstly, we study the integral in $[-\varepsilon,\varepsilon]$. We recall that on $[-\varepsilon,\varepsilon]$, the expansion of the function g is even and gb_I is odd. Using (iv) of Proposition 3.4 with $\alpha=0$, we have

$$
\begin{aligned}
&\int_{-\varepsilon}^{\varepsilon} \sqrt{2} b_I(x) e_k(x) g(x) dx \\
&= \sqrt{2} \int_0^{\varepsilon} \{g(x) b_I(x) - g(-x) b_I(-x)\} e_k(x) dx \\
&= 2 \int_0^{\varepsilon} \left\{ b_I^2(x) \sin\left(\frac{2l+1}{2}\pi x\right) + b_I^2(-x) \sin\left(\frac{2l+1}{2}\pi x\right)\right\} e_k(x) dx \\
&= 2 \int_0^{\varepsilon} \{b_I(x)^2 + b_I(-x)^2\} e_l(x) e_k(x) dx \\
&= 2 \int_0^{\varepsilon} \{s_\varepsilon^2(x) + c_\varepsilon^2(x)\} e_l(x) e_k(x) dx \\
&= 2 \int_0^{\varepsilon} e_l(x) e_k(x) dx.
\end{aligned}
$$

Using (vii) and (viii) of Proposition 3.4 and, since on $[1-\varepsilon', 1+\varepsilon']$ the expansion of the function g is even, we obtain

$$
\begin{aligned}
&\int_{1-\varepsilon'}^{1+\varepsilon'} \sqrt{2} b_I(x) e_k(x) g(x) dx \\
&= \sqrt{2} \int_{1-\varepsilon'}^{1} \{g(2-x) b_I(2-x) + g(x) b_I(x)\} e_k(x) dx \\
&= 2 \int_{1-\varepsilon'}^{1} \left\{ b_I^2(2-x) \sin\left(\frac{2l+1}{2}\pi(2-x)\right) + b_I^2(x) \sin\left(\frac{2l+1}{2}\pi x\right)\right\} e_k(x) dx \\
&= 2 \int_{1-\varepsilon'}^{1} \{b_I^2(2-x) + b_I^2(x)\} e_l(x) e_k(x) dx \\
&= 2 \int_{1-\varepsilon'}^{1} \{s_{\varepsilon'}^2(x-1) + c_{\varepsilon'}^2(x-1)\} e_l(x) e_k(x) dx \\
&= 2 \int_{1-\varepsilon'}^{1} e_l(x) e_k(x) dx.
\end{aligned}
$$

Moreover, $b_I(x) = 1$ on $[\varepsilon, 1-\varepsilon']$ and by Proposition 3.9, $\{\sqrt{2}\sin\left(\frac{2k+1}{2}\pi x\right),\ k \in \mathbb{N} \cup \{0\}\}$ is an orthonormal basis in $L^2([0,1])$. Hence, we obtain that

$$\sqrt{2} \int_{-\varepsilon}^{1+\varepsilon'} b_I(x) e_k(x) g(x) dx = 2 \int_0^1 e_l(x) e_k(x) dx = \delta_{kl}.$$

The case for the general interval $I = [\alpha, \beta]$ follows from these results by translation and dilation.
3. *Claim: the series* (3.12) *converges almost everywhere to* $f(x)$.
If $f \in P_I(L^2(\mathbb{R}))$, we have seen in claim 1 that

$$f(x) = \sqrt{\frac{2}{|I|}} \sum_{k=0}^{\infty} c_k b_I(x) \sin\left(\frac{2k+1}{2}\frac{\pi}{|I|}(x-\alpha)\right)$$

with convergence in $L^2(I)$-norm and coefficients

$$c_k = \sqrt{\frac{2}{|I|}} \int_{\alpha-\varepsilon}^{\beta+\varepsilon'} f(x) b_I(x) \sin\left(\frac{2k+1}{2}\frac{\pi}{|I|}(x-\alpha)\right) dx, \quad k = 0, 1, 2, ...$$

From computations in claim 1, the coefficients can also be calculated as

$$c_k = \sqrt{\frac{2}{|I|}} \int_{\alpha}^{\beta} S_-^+ f(x) \sin\left(\frac{2k+1}{2}\frac{\pi}{|I|}(x-\alpha)\right) dx.$$

Since $f \in L^2(\mathbb{R})$ and b_I is bounded, we can rewrite $S_-^+ f = b_I^{-1} f$ and it is square integrable over $[\alpha, \beta]$. From Carleson's theorem we have that

$$\sqrt{\frac{2}{|I|}} \sum_{k=0}^{\infty} c_k \sin\left(\frac{2k+1}{2}\frac{\pi}{|I|}(x-\alpha)\right) \longrightarrow S_-^+ f(x), \text{ almost everywhere in } [\alpha, \beta].$$

Multiplying both sides by $b_I(x)$ we have that (3.12) converges for almost every $x \in [\alpha, \beta]$ and since both sides are odd with respect to α on $[\alpha - \varepsilon, \alpha + \varepsilon]$ and even with respect to β on $[\beta - \varepsilon', \beta + \varepsilon']$ we can extend the almost everywhere convergence to $[\alpha - \varepsilon, \beta + \varepsilon']$.

Similarly, the choice of polarities $(-,-)$, $(+,-)$ and $(+,+)$ at (α, β) yields the orthonormal basis (ii), (iii) and (iv) respectively.

□

The orthonormal bases presented in the previous theorem are also known as *local Fourier bases for the interval I*. This result, together with Theorem 3.7, can be used to obtain bases for $L^2(\mathbb{R})$.

Theorem 3.11. *Let $\{\alpha_j\}_{j\in\mathbb{Z}}$ a strictly increasing sequence of real numbers such that $\lim_{j\to\pm\infty} \alpha_j = \pm\infty$. Let $\alpha_j + \varepsilon_j \le \alpha_{j+1} - \varepsilon_{j+1}$ and $l_j = \alpha_{j+1} - \alpha_j$ for all $j \in \mathbb{Z}$. If we choose the polarities $(-,+)$ for each smooth projection $P_j = P_{[\alpha_j,\alpha_{j+1}]}$ we obtain that the system*

(i) $\left\{\sqrt{\frac{2}{l_j}} b_{[\alpha_j,\alpha_{j+1}]}(x) \sin\left(\frac{2k+1}{2}\frac{\pi}{l_j}(x-\alpha_j)\right),\ k \in \mathbb{N} \cup \{0\},\ j \in \mathbb{Z}\right\}$ *is an orthonormal basis for $L^2(\mathbb{R})$.*

Moreover, if $f \in L^2(\mathbb{R})$, then the series

$$\sum_{j\in\mathbb{Z}} \sum_{k=0}^{\infty} \sqrt{\frac{2}{l_j}} c_k b_{[\alpha_j,\alpha_{j+1}]}(x) \sin\left(\frac{2k+1}{2}\frac{\pi}{l_j}(x-\alpha_j)\right) \tag{3.14}$$

converges to $f(x)$ almost everywhere.

If we choose the polarities $(-,-)$, $(+,-)$ and $(+,+)$ at (α_j, α_{j+1}), the following systems are orthonormal bases for $L^2(\mathbb{R})$:

(ii) $\left\{\sqrt{\frac{2}{l_j}} b_{[\alpha_j,\alpha_{j+1}]}(x) \sin\left(k\frac{\pi}{l_j}(x-\alpha_j)\right),\ k \in \mathbb{N},\ j \in \mathbb{Z}\right\}$;

(iii) $\left\{\sqrt{\frac{2}{l_j}}b_{[\alpha_j,\alpha_{j+1}]}(x)\cos\left(\frac{2k+1}{2}\frac{\pi}{l_j}(x-\alpha_j)\right),\ k\in\mathbb{N}\cup\{0\},\ j\in\mathbb{Z}\right\}$;

(iv) $\left\{\sqrt{\frac{1}{l_j}}b_{[\alpha_j,\alpha_{j+1}]}(x), \sqrt{\frac{2}{l_j}}b_{[\alpha_j,\alpha_{j+1}]}(x)\cos\left(k\frac{\pi}{l_j}(x-\alpha_j)\right),\ k\in\mathbb{N},\ j\in\mathbb{Z}\right\}$.

Proof. Consider the case (i), analogously we can show the other cases.
Thanks to Theorem 3.10, we have that $\left\{\sqrt{\frac{2}{l_j}}b_{[\alpha_j,\alpha_{j+1}]}(x)\sin\left(\frac{2k+1}{2}\frac{\pi}{l_j}(x-\alpha_j)\right),\ k\in\mathbb{N}\cup\{0\}\right\}$ is an orthonormal basis for $P_{[\alpha_j,\alpha_{j+1}]}(L^2(\mathbb{R}))$. If we consider the decomposition of $L^2(\mathbb{R})$ in Theorem 3.7 we obtain that

$$\left\{\sqrt{\frac{2}{l_j}}b_{[\alpha_j,\alpha_{j+1}]}(x)\sin\left(\frac{2k+1}{2}\frac{\pi}{l_j}(x-\alpha_j)\right)\right\}_{j\in\mathbb{Z},\ k\in\mathbb{N}}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

To show the almost everywhere convergence to a function $f(x)$ of the series (3.14), we define

$$\theta_{k,j}=\sqrt{\frac{2}{l_j}}b_{[\alpha_j,\alpha_{j+1}]}(x)\sin\left(\frac{2k+1}{2}\frac{\pi}{l_j}(x-\alpha_j)\right),\quad k=0,1,2,...,\quad j\in\mathbb{Z}.\tag{3.15}$$

We have the $L^2(\mathbb{R})$-convergence of (3.14) to a function $f\in L^2(\mathbb{R})$ with respect to the basis (3.15). Applying Carleson's theorem we obtain almost everywhere convergence of (3.14) to a function $f\in L^2(\mathbb{R})$. In fact:

$$\lim_{N\to\infty}\sum_{|j|\leq N}\sum_{k=0}^{\infty}\langle f,\theta_{k,j}\rangle\theta_{k,j}(x)=f(x)\text{ for almost every }x\in\mathbb{R}.$$

□

The orthonormal bases presented in this theorem are called *local Fourier bases for* $L^2(\mathbb{R})$. In the next section we present a result by Auscher [1] which relates Malvar bases and Wilson bases.

3.4 Malvar bases and Wilson bases

In the previous section we presented what are known as *local Fourier bases for* $L^2(\mathbb{R})$. We can distinguish two types of bases: the first type is when all intervals have different polarity at their endpoints, i.e. $(-,+)$ for all or $(+,-)$ for all. These bases were formulated by Coifman and Meyer in [13]. The necessity of flexible partitions in signal segmentation for audio processing motivated Malvar to construct a discrete analogous where the basis functions are sampled at appropriate rates [24].
The second type are those bases that have the same polarity at the endpoints. Thus, the sequence of pairs of polarity must alternate $(+,+)$ and $(-,-)$. We show that Wilson bases presented in Chapter 2 are a particular case of local Fourier bases of this type.

Let $I_n = [-\frac{1}{4} + \frac{n}{2}, \frac{1}{4} + \frac{n}{2}]$ for $n \in \mathbb{Z}$. If n is even we set I_n to have polarity $(+,+)$ and if n is odd we set I_n to have polarity $(-,-)$. We obtain in this way a uniform covering of $\mathbb{R}$ with compatible adjacent intervals. Finally, let b_n be the bells over I_n and define $b_n(x) = b_0\left(x - \frac{n}{2}\right)$. Let $[u]$ be the integer part of u. We study the cases n even and n odd.

- For $n = 2n'$, we take the basis of the form (iv) in Theorem 3.10. We obtain:

$$\begin{aligned} &2b_0(x-n')\cos\left(2k\pi\left(x-\left(-\frac{1}{4}+n'\right)\right)\right) \\ &= 2b_0(x-n')\cos\left(2k\pi\left(x+\frac{1}{4}\right)\right) \\ &= 2b_0(x-n')\begin{cases}(-1)^{[k/2]}\cos(2\pi kx), & \text{if } k \text{ is even,}\\ (-1)^{[k/2]+1}\sin(2\pi kx), & \text{if } k \text{ is odd.}\end{cases} \end{aligned} \tag{3.16}$$

- For $n = 2n'+1$, we take the basis (ii) in Theorem 3.10 and in a similar way as before it holds:

$$\begin{aligned} &2b_0\left(x-\frac{2n'+1}{2}\right)\sin\left(2k\pi\left(x-\left(-\frac{1}{4}+\frac{2n'+1}{2}\right)\right)\right) \\ &= 2b_0\left(x-\frac{2n'+1}{2}\right)\sin\left(2k\pi\left(x-\frac{1}{4}\right)\right) \\ &= 2b_0\left(x-\frac{2n'+1}{2}\right)\begin{cases}(-1)^{[k/2]}\sin(2\pi kx), & \text{if } k \text{ is even,}\\ (-1)^{[k/2]+1}\cos(2\pi kx), & \text{if } k \text{ is odd.}\end{cases} \end{aligned} \tag{3.17}$$

Note that in equations (3.16) and (3.17) the first terms are elements of the Fourier local basis associated with the respective interval I_n.
Set now $\hat{\phi}(x) = \sqrt{2}b_0(x)$, we recognize on the right hand side, up to powers of (-1), elements of the Wilson basis of type (2.3), in fact if we change the index k in l we obtain

- For $n = 2n'$:

$$\tilde{\Psi}_{l,2n'}(x) = \sqrt{2}\hat{\phi}(x-n')\begin{cases}(-1)^{[l/2]}\cos(2\pi lx), & \text{if } l \text{ is even,}\\ (-1)^{[l/2]+1}\sin(2\pi lx), & \text{if } l \text{ is odd.}\end{cases} \tag{3.18}$$

- For $n = 2n'+1$:

$$\tilde{\Psi}_{l,2n'+1}(x) = \sqrt{2}\hat{\phi}\left(x-\frac{2n'+1}{2}\right)\begin{cases}(-1)^{[l/2]}\sin(2\pi lx), & \text{if } l \text{ is even,}\\ (-1)^{[l/2]+1}\cos(2\pi lx), & \text{if } l \text{ is odd.}\end{cases} \tag{3.19}$$

Thus, we can see that up to unimodular multiplicative constants, the basis $\tilde{\Psi}_{l,n}$ formed by (3.18) and (3.19) are the same as $\Psi_{l,n}$ in (2.3). Moreover, it must be underlined that this construction yields basis functions $\hat{\phi}(x) = \sqrt{2}b_0(x)$ with compact support and prescribed smoothness.

Remark 3.4. It is important to point out that E. Laeng in his article [23] constructed a family of basis of Wilson type. He considered the frequency domain covered by a family of symmetric intervals, possibly of different sizes. His construction can be shown to be a local Fourier basis of the second type we described above. Moreover, he showed that his construction can be extended for $\mathbb{R}^d$, $d > 1$.

Remark 3.5. It must be underlined that local Fourier basis of the second type can also be seen as a refinement of the first type. In fact, we can split a generic interval I following polarity $(+,-)$ at the extremal points into two parts, we obtain polarity $(+,+)$ for the left part and polarity $(-,-)$ for the right part and then we can use Theorem 3.6 to obtain the claim. In a similar way we can obtain the claim if we start with an interval with polarity $(-,+)$. This means that Fourier basis of the first type are somehow generic for all this collection of bases.

Chapter 4

Non-Linear Approximation Spaces

In Theorem 3.11 of Chapter 3 we have shown that, given a partition $\{\alpha_j\}_{j\in\mathbb{Z}}$ with interval length $l_j = \alpha_{j+1} - \alpha_j$, and a sequence $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ such that $\alpha_j + \varepsilon_j \leq \alpha_{j+1} - \varepsilon_{j+1}$, then, for any $N \in \mathbb{N} \cup \{\infty\}$, there exists a bell function $b_j = b_{[\alpha_j,\alpha_{j+1}]} \in \mathcal{C}^N(\mathbb{R})$ with $\operatorname{supp}(b_j) \subseteq [\alpha_j - \varepsilon_j, \alpha_{j+1} + \varepsilon_{j+1}]$ such that

$$\left\{ \Psi_{j,k}(x) = \sqrt{\frac{2}{l_j}} b_j(x) \sin\left(k\frac{\pi}{l_j}(x-\alpha_j)\right),\ k \in \mathbb{N},\ j \in \mathbb{Z} \right\} \tag{4.1}$$

is an orthonormal basis for $L^2(\mathbb{R})$. Recall that for every $j \in \mathbb{Z}$, the bell function $b_j(x) = s_{\varepsilon_j}(x-\alpha_j)c_{\varepsilon_{j+1}}(x-\alpha_{j+1})$ is defined following Definition 3.2. With respect to the chosen polarity three other bases with similar structure can be constructed. In this chapter we will work with (4.1) and the other cases follow considering suitable typographical modifications. Thanks to the results in Section 3.4 and considering that $\phi_j = T_{-\alpha_j} b_j$, for all $j \in \mathbb{Z}$, we can write the functions (4.1) as

$$\Psi_{j,k} = \frac{1}{2}\sqrt{\frac{2}{l_j}} T_{\alpha_j} \left[M_{\frac{k}{2l_j}} \pm M_{-\frac{k}{2l_j}} \right] \phi_j \tag{4.2}$$

Our goal in this chapter is to identify the function spaces that occur when a local Fourier basis is used while approximating. We cannot expect that the approximation spaces resulting from the use of a particular local Fourier basis are all independent of the chosen partition. We will see that the restriction

$$0 < \frac{1}{A} \leq l_j \leq A < \infty \text{ for all } j \in \mathbb{Z} \tag{4.3}$$

for $A > 1$ guarantees that the approximation properties are independent of the precise details of the basis. Moreover, we will prove that the corresponding approximation spaces are the modulation spaces.

Following the presentation in the article by Gröchenig and Samarah [20], we will show in the first section that local Fourier bases are unconditional bases for modulation spaces. In the second section, we will present the approximation properties following Section 12.4 of [19]. Most of the proofs are given as a sketch and we put more emphasis on the use of local Fourier bases.

4.1 Unconditional bases for modulation spaces

Recall from the prerequisites the Definition 1.17 of modulation space. For the purpose of nonlinear approximation, we are interested only in the case $p = q$ and we will write $M_w^{p,p} = M_w^p$. We recall that the dual space of M_w^p is $M_{\frac{1}{w}}^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

In Section 3.4 we have shown that orthonormal Wilson bases are a particular case of local Fourier bases with a uniform partition $\alpha_j = \frac{j}{2} + \frac{1}{4}$, a particular choice of polarity and where the bell function is of exponential decay. The following result about Wilson basis was proved by Feichtinger, Gröchenig and Walnut in [18].

Theorem 4.1. *The Wilson basis* (2.3) *constructed using ϕ and $\hat{\phi}$ of exponential decay is an unconditional basis for M_w^p with $1 \le p < \infty$.*

Following the paper [20], we want to show an analogous result considering local Fourier bases in the class of $\mathcal{C}^N(\mathbb{R})$.

To prove our claim we will need the following results. The first lemma is a weighted version of Schur's test.

Lemma 4.2. *Let $w_1(i)$ with $i \in I$ and $w_2(j)$ with $j \in J$ be two weight functions on index sets I and J, respectively. Let $A = (a_{j,i})_{j\in J, i\in I}$ be an infinite matrix such that*

$$\sum_{i\in I} |a_{j,i}| w_1(i)^{-1} \le C_0 w_2(j)^{-1} < \infty \quad \text{for all } j \in J \quad \text{and} \tag{4.4}$$

$$\sum_{j\in J} |a_{j,i}| w_2(j) \le C_1 w_1(i) < \infty \quad \text{for all } i \in I. \tag{4.5}$$

for some constants $C_0, C_1 > 0$. Then A is bounded from $\ell^p_{w_1}(I)$ into $\ell^p_{w_2}(J)$ for $1 \le p \le \infty$.

Proof (Sketch). Assume first $1 < p < \infty$ and let $c = \{c_i\}_{i\in I} \in \ell^p_{w_1}(I)$. Using Hölder's inequality and inequalities (4.4) and (4.5) we can show that $||Ac||^p_{\ell^p_{w_2}} \le C_0^{p/p'} C_1 ||c|^p_{\ell^p_{w_1}}$. To show the cases $p = 1$ we only need (4.5) and for $p = \infty$ only (4.4). □

The next lemma provides a pointwise estimate for the short-time Fourier transform.

Lemma 4.3. *Let $g \in \mathcal{C}^\infty(\mathbb{R})$ such that $\operatorname{supp}(g) \subseteq [-L, L]$. Let*

$$\mathcal{F} = \left\{\phi \in \mathcal{C}^N(\mathbb{R}) : \operatorname{supp}(\phi) \subseteq [-K, K] \text{ and } \max_{k=0,1,\dots,N} ||\phi^{(k)}||_1 \le B\right\}.$$

Set $C = K + L$, then there exists a constant $C_0 > 0$ depending only on B, K and N such that

$$\sup_{\phi\in\mathcal{F}} |V_g\phi(x, y)| \le C_0 \frac{1}{(1 + |y|)^N} \chi_{[-C,C]}(x), \quad \text{for all } x, y \in \mathbb{R}.$$

Proof (Sketch). Note that for $|x| > C$, $|V_g\phi(x, y)| = 0$ for all $y \in \mathbb{R}$. Hence we consider only the interval $|x| \le C$. Consider now two cases:

- $|y| \leq 1$: $|V_g\phi(x,y)| = |\langle \phi, M_y T_x g\rangle| \leq ||\phi||_1 ||g||_\infty \leq B||g||_\infty$.
- $|y| \geq 1$: apply integration by parts N times to $V_g\phi(x,y) = \int_{-K}^{K} (\phi \cdot T_x \overline{g})(t) e^{-2\pi i y t} dt$ and since $\phi \in \mathcal{F}$ we have a bound for the derivative of ϕ. We obtain an estimate for $|V_g\phi(x,y)|$ which is independent of $\phi \in \mathcal{F}$ and hence the claim.

□

A consequence of Lemma 4.3 is the following.

Lemma 4.4. *Suppose condition* (4.3) *holds and* $\inf_j \varepsilon_j > 0$ *for all* $j \in \mathbb{Z}$. *Then* $\{\phi_j = T_{-\alpha_j} b_j,\ j \in \mathbb{Z}\} \subseteq \mathcal{F}$ *for some* B, K, N.

Proof. Since $\operatorname{supp}(b_j) \subseteq [\alpha_j - \varepsilon_j, \alpha_{j+1} + \varepsilon_{j+1}]$, then $\operatorname{supp}(\phi_j) \subseteq [-\varepsilon_j, \alpha_{j+1} - \alpha_j + \varepsilon_{j+1}] \subseteq [-A, 2A]$. For $K = 2A$, we have $\operatorname{supp} \phi_j \subseteq [-K, K]$. To show the bound on the derivatives we notice that by definition and normalizing, $\theta_{\varepsilon_j}(x) = \frac{1}{\varepsilon_j} \int_{-\infty}^{x} \psi\big(\frac{t}{\varepsilon_j}\big) dt$. The derivatives of b_j are sums of products containing $\theta_{\varepsilon_j}^{(k)}(x)$, $\theta_{\varepsilon_{j+1}}^{(k)}(x)$ and sines and cosines, and, since $\psi \in \mathcal{C}^{N-1}(\mathbb{R})$ and $\inf_j \varepsilon_j > 0$, then $||\theta_{\varepsilon_j}^{(k)}||_\infty = \varepsilon_j^{-k} ||\psi^{(k-1)}||_\infty$ which is bounded for $k = 1, ..., N$. Thus, $\{\phi_j = T_{-\alpha_j} b_j,\ j \in \mathbb{Z}\} \subseteq \mathcal{F}$. □

Now we can prove the following general result.

Theorem 4.5. *Suppose* $\{\Psi_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}} \subseteq \mathcal{C}^N(\mathbb{R})$ *is a local Fourier basis such the associate partition satisfies* $0 < \frac{1}{A} \leq l_j \leq A < \infty$ *for all* $j \in \mathbb{Z}$ *and* $\inf_j \varepsilon_j > 0$. *Let* w *be a weight function as in Definition 1.16 with parameters* $C = 1$ *and* $s < N - 1$, *then* $\{\Psi_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}}$ *is an unconditional basis for* M_w^p *and every distribution* $f \in M_w^p$ *has a unique expansion*

$$f = \sum_{(j,k)\in\mathbb{Z}\times\mathbb{N}} \langle f, \Psi_{j,k}\rangle \Psi_{j,k} \tag{4.6}$$

with unconditional convergence in the norm of M_w^p. *Moreover, there exists a constant* $C > 1$ *such that*

$$\frac{1}{C}||f||_{M_w^p} \leq \left(\sum_{(j,k)\in\mathbb{Z}\times\mathbb{N}} |\langle f, \Psi_{j,k}\rangle|^p w\left(\alpha_j, \frac{k}{2l_j}\right)^p \right)^{\frac{1}{p}} \leq C||f||_{M_w^p}. \tag{4.7}$$

If $p = \infty$, *then* $\{\Psi_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}}$ *is a weak basis, i.e, the expansion* (4.6) *converges only in the weak* **-topology with respect to the predual* $M^1_{\frac{1}{w}}$.

Proof. To prove the statement we need to extend the orthonormal expansion (4.6) from $L^2(\mathbb{R})$ to M_w^p. For this we will extend the analysis operator (1.7) and synthesis operator (1.6) associated to the orthonormal basis $\{\Psi_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}}$ to other function or sequence spaces.

Since $\{\Psi_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}}$ is an orthonormal basis, $T^*(f) = \{\langle f, \Psi_{j,k}\rangle\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}}$ is a well-defined operator from $L^2(\mathbb{R})$ onto $\ell^2(\mathbb{Z}\times\mathbb{N})$ and $T(\{c_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}}) = \sum_{(j,k)\in\mathbb{Z}\times\mathbb{N}} c_{j,k}\Psi_{j,k}$ is its adjoint.

Let $\eta_{j,k} = (\alpha_j, \frac{k}{2l_j})$ with $(j,k) \in \mathbb{Z} \times \mathbb{N}$ be the points in the time-frequency plane associated to $\Psi_{j,k}$ and define $w'(j,k) = w(\eta_{j,k})$. Recall that, with this definitions, the space $\ell^p_{w'}(\mathbb{Z} \times \mathbb{N})$ is defined as

$$\ell^p_{w'}(\mathbb{Z} \times \mathbb{N}) = \left\{ \{c_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}} : \Bigg(\sum_{(j,k)\in\mathbb{Z}\times\mathbb{N}} |c_{j,k}|^p w(\eta_{j,k})^p \Bigg)^{\frac{1}{p}} < \infty \right\}.$$

1. *Claim: there exists a constant $C_1 > 0$ such that*
$|V_g\Psi_{j,k}(x,y)| \le C_1(T_{\eta_{j,k}} + T_{\eta_{j,-k}})\chi_{[-C,C]}(x)(1+|y|)^{-N}$ *for all $x, y \in \mathbb{R}$.*
Consider $\Psi_{j,k}$ as defined by (4.2) and note that by (4.3), we have $\sqrt{\frac{1}{2l_j}} \le \sqrt{\frac{A}{2}}$.
Recalling the covariance property of the STFT (1.20) and applying Lemmas 4.3 and 4.4, we have the following estimate for the STFT of a local Fourier basis.
For $\eta_{j,k} \in \mathbb{R}^2$, $x, y \in \mathbb{R}$

$$\begin{aligned} |V_g\Psi_{j,k}(x,y)| &= \sqrt{\frac{1}{2l_j}} \big|V_g\big(T_{\alpha_j}\big[M_{\frac{k}{2l_j}} \pm M_{-\frac{k}{2l_j}}\big]\phi_j\big)(x,y)\big| \\ &\le \sqrt{\frac{1}{2l_j}} \big(\big|T_{\eta_{j,k}}V_g\phi_j(x,y)\big| + \big|T_{\eta_{j,-k}}V_g\phi_j(x,y)\big|\big) \\ &\le \sqrt{\frac{A}{2}} C_0 (T_{\eta_{j,k}} + T_{\eta_{j,-k}})\chi_{[-C,C]}(x)(1+|y|)^{-N}. \end{aligned}$$

2. *Claim: the operator T^*, associated to $\Psi_{j,k}$ defined in* (1.7)*, is a bounded operator from M^p_w into $\ell^p_{w'}(\mathbb{Z} \times \mathbb{N})$ for $1 \le p \le \infty$.*
Theorem 1.27 asserts that for $g \in \mathcal{C}^\infty(\mathbb{R})$ with compact support, $a, b > 0$ small enough, there exists a dual window $h \in \mathcal{S}(\mathbb{R})$, such that every $f \in \mathcal{S}'(\mathbb{R})$ can be written as

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, M_{mb}T_{na}h\rangle M_{mb}T_{na}g.$$

Moreover, for $1 \le p < \infty$ the Gabor expansion converges unconditionally in M^p_w.
We can write

$$\{T^*f\}_{j,k} = \sum_{m,n\in\mathbb{Z}} \langle f, M_{mb}T_{na}h\rangle \langle M_{mb}T_{na}g, \Psi_{j,k}\rangle.$$

To show the statement it is enough to prove that the operator $A_{(j,k),(m,n)} = \langle M_{mb}T_{na}g, \Psi_{j,k}\rangle$ maps the sequence $c_{m,n} = \langle f, M_{mb}T_{na}h\rangle \in \ell^p_{w_1}(\mathbb{Z} \times \mathbb{Z})$ with $w_1(n,m) = w(na, mb)$, into $\ell^p_{w'}(\mathbb{Z} \times \mathbb{N})$ with $w'(j,k) = w(\eta_{j,k})$. To do that we use Lemma 4.2 and show conditions (4.4) and (4.5).

- *Condition* (4.4) *is satisfied.*
  Take Definition 1.16 with $x = (\alpha_j - na, \frac{\pm k}{2l_j} - mb)$ and $\omega = (na, mb)$, then we have

$$w\left(\alpha_j, \frac{\pm k}{2l_j}\right) \le w(na, mb)\left(1 + |\alpha_j - na| + \left|\frac{\pm k}{2l_j} - mb\right|\right)^s. \tag{4.8}$$

Considering claim 1, inequality (4.8) and recalling that $|\alpha_j - na| \le C$. Then we have

$$\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} |A_{(j,k),(m,n)}| w_1(n,m)^{-1} \\
&\quad = \sum_{m,n\in\mathbb{Z}} |\langle M_{mb}T_{na}g, \Psi_{j,k}\rangle| w(na,mb)^{-1} \\
&\quad = \sum_{m,n\in\mathbb{Z}} |V_g\Psi_{j,k}(na,mb)| w(na,mb)^{-1} \\
&\quad \le C_1 \sum_{m,n\in\mathbb{Z}} w(na,mb)^{-1}(T_{\eta_{j,k}} + T_{\eta_{j,-k}})\chi_{[-C,C]}(na)(1+|mb|)^{-N} \\
&\quad \le C_1 \sum_{m,n\in\mathbb{Z}} w\left(\alpha_j, \frac{\pm k}{2l_j}\right)^{-1} \chi_{[-C,C]}(na-\alpha_j)\left(1+\left|\frac{\pm k}{2l_j} - mb\right|\right)^{-N} \\
&\qquad \times \left(1 + |\alpha_j - na| + \left|\frac{\pm k}{2l_j} - mb\right|\right)^{s} \\
&\quad \le C_1 \frac{2C}{a} w\left(\alpha_j, \frac{\pm k}{2l_j}\right)^{-1} \sum_{m\in\mathbb{Z}} \left(1+\left|\frac{\pm k}{2l_j} - mb\right|\right)^{-N}\left(1 + C + \left|\frac{\pm k}{2l_j} - mb\right|\right)^{s}.
\end{aligned}$$

Note that since $|\alpha_j - na| \le C$, then the sum over n contains at most $\frac{2C}{a}$ terms. Since $s < N-1$, the sum is finite with bound independent of j and k.

- *Condition* (4.5) *is satisfied.*
  In a similar way as before and considering that there are at most $2CA$ terms α_j in every interval of length $2C$ we obtain

$$\begin{aligned}
&\sum_{(j,k)\in\mathbb{Z}\times\mathbb{N}} |\langle M_{mb}T_{na}g, \Psi_{j,k}\rangle| w(\eta_{j,k}) \\
&\quad \le C_1 2CAw(na,mb) \sup_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}} \left(1+\left|mb - \frac{\pm k}{2l_j}\right|\right)^{-N}\left(1 + C + \left|mb - \frac{\pm k}{2l_j}\right|\right)^{s}.
\end{aligned}$$

for the same reason as before, the sum is finite with bound independent of m and n.

3. *Claim: the operator T, associated to $\Psi_{j,k}$ defined in* (1.6)*, is a bounded operator from $\ell^p_{w'}(\mathbb{Z}\times\mathbb{N})$ into M^p_w for $1 \le p \le \infty$.*
We consider first the case $p < \infty$.
Let $\{c_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}}$ be finitely supported and $\tilde{f}$ in the dual space $M^{p'}_{1/w}$ of M^p_w. By

claim 2 we have:

$$\begin{aligned}||T\{c_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}}||_{M^p_w} &= \sup_{||\tilde{f}||_{M^{p'}_{1/w}}\le 1}\left|\left\langle \sum_{(j,k)\in\mathbb{Z}\times\mathbb{N}} c_{j,k}\Psi_{j,k}, \tilde{f}\right\rangle\right| \\ &= \sup_{||\tilde{f}||_{M^{p'}_{1/w}}\le 1}\left|\sum_{(j,k)\in\mathbb{Z}\times\mathbb{N}} c_{j,k}\overline{\{T^*(\tilde{f})\}_{j,k}}\right| \\ &\le \sup_{||\tilde{f}||_{M^{p'}_{1/w}}\le 1} ||c||_{\ell^p_{w'}}\, ||T^*(\tilde{f})||_{\ell^{p'}_{1/w'}} \le ||c||_{\ell^p_{w'}}\, ||T^*||_{op}.\end{aligned}$$

Hence, T is bounded on $\ell^p_{w'}$. Moreover, $T\{c_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}}$ converges unconditionally. In fact, for any $\varepsilon > 0$, there exists a finite subset $\mathcal{I}_\varepsilon \subseteq \mathbb{Z}\times\mathbb{N}$ such that, for all finite subsets $\mathcal{I}\supseteq\mathcal{I}_\varepsilon$, it holds:

$$\left|\left|\sum_{(j,k)\notin\mathcal{I}} c_{j,k}\Psi_{j,k}\right|\right|_{M^p_w} \le ||T^*||_{op}\left(\sum_{(j,k)\notin\mathcal{I}} |c_{j,k}|^p w(\eta_{j,k})^p\right)^{\frac{1}{p}} < \varepsilon$$

The case $p=\infty$ it is shown by taking the supremum over $M^1_{1/w}$ and showing that T is bounded on M^∞_w and the sum is w^*-convergent.

4. *Claim:* $\{\Psi_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}}$ *is an unconditional basis for* M^p_w *and inequalities* (4.7) *are satisfied.*

By point 2 and 3 we have that T and T^* are bounded on $\ell^p_{w'}$ and M^p_w and the expansion (4.6) extends from $L^2(\mathbb{R})$ to M^p_w.

Consider the case $p<\infty$. Since the series converges unconditionally, then finite linear combinations are dense in M^p_w.

Moreover, by (4.6) and $f = TT^*f$ then

$$||f||_{M^p_w} \le ||T||_{op}\, ||T^*f||_{\ell^p_{w'}} \le ||T||_{op}\, ||T^*||_{op}\, ||f||_{M^p_w}.$$

Dividing by $||T||_{op}$ and noticing that $||T^*f||_{\ell^p_{w'}} = \left(\sum_{(j,k)} |\langle f,\Psi_{j,k}\rangle|^p w\left(\alpha_j, \frac{k}{2l_j}\right)^p\right)^{\frac{1}{p}}$, we have the bound in (4.7). Let $\{\mu_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}} \in \ell^\infty$. Since in a finite linear combination $f=\sum_{(j,k)} c_{j,k}\Psi_{j,k}$ the coefficients are unique and $c_{j,k} = \langle f,\Psi_{j,k}\rangle = \{T^*f\}_{j,k}$, then we have

$$\begin{aligned}\left|\left|\sum_{(j,k)}\mu_{j,k}c_{j,k}\Psi_{j,k}\right|\right|_{M^p_w} &\le ||T||_{op}||(\mu_{j,k}c_{j,k})_{(j,k)}||_{\ell^p_{w'}} \\ &\le ||T^*||_{op}\, ||\mu||_\infty\, ||c||_{\ell^p_{w'}} \\ &\le ||T^*||_{op}\, ||T||_{op}\, ||\mu||_\infty\, ||f||_{M^p_w}.\end{aligned}$$

Hence, $\{\Psi_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}}$ is an unconditional basis for M^p_w.

The case $p=\infty$ is proved in the same way considering that the expansion (4.6) holds with w^*-convergence.

□

Remark 4.1. Note that, with a bit more effort, we can prove a similar statement for $M_w^{p,q}$ for $1 \le p, q < \infty$ and $p \neq q$. Moreover, these results can be extended to higher dimensions using the tensor product [19].

4.2 Characterization of modulation spaces

Unconditional bases are crucial for data compression whose aim is to approximate a function f by a finite linear combination of type $\sum_{n \in F} c_n e_n$. The compressed version of the data is formed by the finitely many coefficients $\{c_n : n \in F\}$. Basically, smaller is the number of coefficients needed to approximate f with a certain accuracy, better the data compression works. The idea is to study the error $\sigma_N(f)$ as $N \to \infty$ of the best approximation using N coefficients and to consider its rate of convergence to 0. If the convergence is fast, then fewer coefficients are enough to approximate f up to an error $\varepsilon > 0$ than when the convergence is slow. Hence, the rate of convergence $\sigma_N(f) \to 0$ describes how well data compression works for f. In this section we investigate data compression when approximating with local Fourier bases.
Consider the set of all linear combinations consisting of at most N elements of $\Psi_{j,k}$

$$\Sigma_N = \left\{ p = \sum_{(j,k) \in F} c_{j,k} \Psi_{j,k} : F \subseteq \mathbb{Z} \times \mathbb{N}, \mathrm{card} F \le N \right\}$$

We define the N-term approximation error in $L^2(\mathbb{R})$ by

$$\sigma_N(f) = \inf_{p \in \Sigma_N} ||f - p||_2.$$

$\sigma_N(f)$ is the error we make by approximating f with a linear combination of N functions from $\Psi_{j,k}$. Moreover, since $\Sigma_N + \Sigma_N = \Sigma_{2N}$ then Σ_N is not a linear subspace of $L^2(\mathbb{R})$ and hence this type of problem is called non-linear approximation problem.

To find the optimal approximation $p \in \Sigma_N$ we let $f = \sum_{(j,k)} c_{j,k} \Psi_{j,k}$ and we choose the N terms whose coefficients have the largest modulus using a bijection $\pi : \mathbb{N} \longmapsto \mathbb{Z} \times \mathbb{N}$ which satisfies $|c_{\pi(1)}| \ge |c_{\pi(2)}| \ge \ldots$. We call $a_n = |c_{\pi(n)}|$ the non-increasing rearrangement of c. Then the best approximation of f in L^2 by Σ_N is

$$p_{\mathrm{opt}} = \sum_{n=1}^{N} c_{\pi(n)} \Psi_{\pi(n)}.$$

Since $\{\Psi_{j,k}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, then the L^2-error is

$$\sigma_N(f) = \inf_{p \in \Sigma_N} ||f - p||_2 = ||f - p_{\mathrm{opt}}||_2 = \left(\sum_{n=N+1}^{\infty} |c_{\pi(n)}|^2 \right)^{\frac{1}{2}} = \left(\sum_{n=N+1}^{\infty} a_n^2 \right)^{\frac{1}{2}}.$$

We study first the approximation problem for non-negative and non-increasing sequences $\{a_n\}_{n \in \mathbb{N}}$. We consider $\sigma_N(a) = \left(\sum_{n=N+1}^{\infty} a_n^2 \right)^{\frac{1}{2}}$.

The next lemma of Stechkin [28] relates the rate of convergence of $\sigma_N(a)$ to the ℓ^p-norm of a for $0 < p < 2$.

Lemma 4.6. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a non-negative and non-increasing sequence, $0 < p < 2$ and $\alpha = \frac{1}{p} - \frac{1}{2}$. Then there exists a constant $C > 0$ such that*

$$\frac{1}{C}||a||_p \leq \left(\sum_{N=1}^{\infty}(N^\alpha \sigma_{N-1}(a))^p \frac{1}{N}\right)^{\frac{1}{p}} \leq C||a||_p.$$

Proof. Using Cauchy-Schwarz and the fact that $\{a_n\}_{n\in\mathbb{N}}$ is non-negative and non-increasing, we have

$$a_{2m} \leq a_{2m-1} \leq \frac{1}{m}\sum_{k=m}^{2m-1} a_k \leq \sqrt{\frac{1}{m}}\left(\sum_{k=m}^{2m-1} a_k^2\right)^{\frac{1}{2}} \leq \sqrt{\frac{1}{m}}\sigma_{m-1}(a).$$

Since $-\frac{p}{2} = p\alpha - 1$, then

$$||a||_p^p = \sum_{m=1}^{\infty}(a_{2m-1}^p + a_{2m}^p) \leq 2\sum_{m=1}^{\infty}\left(\sqrt{\frac{1}{m}}\sigma_{m-1}(a)\right)^p = 2\sum_{m=1}^{\infty}(m^\alpha \sigma_{m-1}(a))^p \frac{1}{m}.$$

The other inequality requires a little more effort. Since $\{a_n\}_{n\in\mathbb{N}}$ is decreasing, we have

$$2^k a_{2^{k+1}}^p \leq \sum_{m=2^k}^{2^{k+1}-1} a_m^p \leq 2^k a_{2^k}^p$$

and also

$$\sum_{m=k}^{\infty} 2^m a_{2^{m+1}}^p \leq \sum_{m=2^k}^{\infty} a_m^p \leq \sum_{m=k}^{\infty} 2^m a_{2^m}^p.$$

Since $\frac{1}{N}(N^\alpha \sigma_{N-1}(a))^p = N^{-\frac{p}{2}}\sigma_{N-1}(a)^p$ is decreasing and $\ell^p \subseteq \ell^2$ for $p < 2$, we have that

$$\begin{aligned}
\sum_{N=1}^{\infty} N^{-\frac{p}{2}}\sigma_{N-1}(a)^p &\leq \sum_{k=0}^{\infty} 2^{k(1-\frac{p}{2})}\sigma_{2^k-1}(a)^p \\
&= \sum_{k=0}^{\infty} 2^{k(1-\frac{p}{2})}\left(\sum_{m=2^k}^{\infty} a_m^2\right)^{\frac{p}{2}} \\
&\leq \sum_{k=0}^{\infty} 2^{k(1-\frac{p}{2})}\left(\sum_{m=k}^{\infty}(2^{\frac{m}{2}} a_{2^m})^2\right)^{\frac{p}{2}} \\
&\leq \sum_{k=0}^{\infty} 2^{k(1-\frac{p}{2})}\sum_{m=k}^{\infty}(2^{\frac{m}{2}} a_{2^m})^p \\
&= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m} 2^{k(1-\frac{p}{2})}\right) 2^{\frac{pm}{2}} a_{2^m}^p \\
&\leq C\sum_{m=0}^{\infty} 2^m a_{2^m}^p \\
&\leq C'||a||_p^p.
\end{aligned}$$

□

Remark 4.2. The previous theorem can be generalize to hold for $\sigma_{N,q}(a) = \left(\sum_{n=N+1}^{\infty} a_n^q\right)^{\frac{1}{q}}$, $\alpha = \frac{1}{p} - \frac{1}{q}$ and $0 < p < q \leq \infty$

The next theorem describes for which class of functions data compression with local Fourier bases works well. We prove that a function f can be approximated well by using a local Fourier basis if and only if $f \in M_1^p$.

Theorem 4.7. *Let* $\{\alpha_j\}_{j\in\mathbb{Z}}$ *be a partition satisfying* (4.3) *and such that* $\inf_j \varepsilon_j > 0$. *Let* $\{\Psi_{j,k}\}_{(j,k)\in\mathbb{Z}\times\mathbb{N}}$ *be an associated local Fourier basis in* $\mathcal{C}^N(\mathbb{R})$, $N > 1$. *Then for* $1 \leq p < 2$ *and* $\alpha = \frac{1}{p} - \frac{1}{2}$

$$\sum_{N=1}^{\infty} \frac{1}{N}(N^\alpha \sigma_{N-1}(f))^p < \infty \iff \int_{\mathbb{R}^2} |V_g f(x,\omega)|^p dx d\omega < \infty \tag{4.9}$$

Proof. First of all, we note that

$$\int_{\mathbb{R}^2} |V_g f(x,\omega)|^p dx d\omega < \infty \iff f \in M_1^p.$$

Let $f = \sum_{(j,k)\in\mathbb{Z}\times\mathbb{N}} c_{j,k}\Psi_{j,k} \in L^2(\mathbb{R})$ and $a_n = |c_{\pi(n)}|$ be a non-increasing rearrangement of the coefficients $c_{j,k} = \langle f, \Psi_{j,k}\rangle$. By inequality (4.7) with $w = 1$ we have that

$$\frac{1}{C}||f||_{M_1^p} \leq \left(\sum_{(j,k)\in\mathbb{Z}\times\mathbb{N}} |\langle f, \Psi_{j,k}\rangle|^p\right)^{\frac{1}{p}} = ||c||_p = ||a||_p \leq C||f||_{M_1^p}.$$

By Lemma 4.6 and since $\sigma_N(a) = \sigma_N(f)$, we have equivalence (4.9). □

Remark 4.3. As for Remark 4.2, this result can be extended to measure the approximation error in the M_w^q-norm. In fact, for $1 \leq p < q < \infty$ and $\alpha = \frac{1}{p} - \frac{1}{q}$, $f \in M_w^p$ if and only if $\sum_{N=1}^{\infty} \frac{1}{N}(N^\alpha \sigma_{N-1}(f)_{M_w^q})^p < \infty$.

Chapter 5

Gravitational Waves

One very interesting and recent application of Wilson bases is to the detection of the gravitational waves. Gravitational waves are "ripples" in space-time caused by some of the most violent and energetic processes in the universe such as colliding black holes, supernovae (massive stars exploding at the end of their lifetime), and colliding neutron stars. In 1916, the year after the final formulation of the field equations of general relativity, Albert Einstein predicted the existence of gravitational waves. Einstein showed that massive accelerating objects would modify space-time in such a way that "waves" of undulating space-time would propagate in all directions away from the source. These cosmic ripples would travel at the speed of light, carrying with them information about their origins.

Experiments to detect gravitational waves began in the 1960s and by the early 2000s, a set of initial detectors was completed, including TAMA 300 in Japan, GEO 600 in Germany, the Laser Interferometer Gravitational-Wave Observatory (LIGO) in the United States, and Virgo in Italy. Combinations of these detectors made joint observations from 2002 through 2011. Only on September 14, 2015, the Advanced LIGO became the first that physically sensed the undulations in space-time caused by gravitational waves generated by a binary black hole system merging to form a single black hole 1.3 billion light-years away.

The gravitational signal resulting from the coalescence of two black holes or two neutron stars is related to the trajectory that will lead to the union of the two components. This dynamic phenomenon and, consequently, the emitted gravitational wave can be predicted thanks to Damour, Blanchet and their collaborators [5, 4, 7]. They were able to calculate the analytic form of a gravitational wave generated by coalescence of two neutron stars and they obtained that

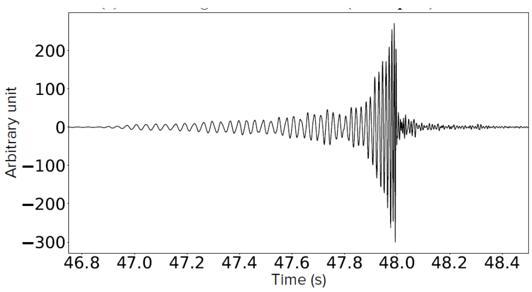
$$s(t) = c|t - t_0|^{-\frac{1}{4}} \cos(\omega|t - t_0|^{\frac{5}{8}} + \varphi)$$

where c is a constant, $\omega \gg 1$ and t_0 is the time of the coalescence of the two neutron stars. The instantaneous frequency is $\sim |t - t_0|^{-\frac{3}{8}}$. Note that the analytic form of a gravitational wave is what in signal processing is called a *chirp*, in fact, a chirp is a frequency modulated signal analytically described by

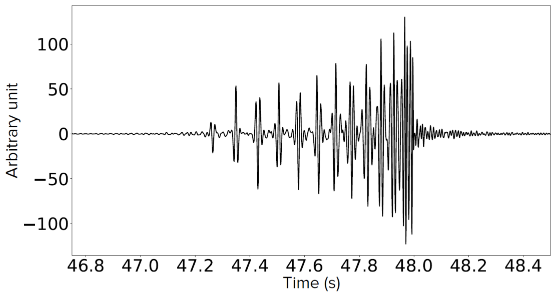
$$F(t) = A(t)e^{i\phi(t)}$$

where $A(t) = |F(t)|$. In particular chirps are defined by a strong acceleration of $\phi(t)$, i.e $|\phi(t)''| \gg 1$. This means that the problem of detection of gravitational waves become the extraction of a chirp buried inside a noisy signal.

When looking for a particular astrophysical source, the analytical form of the



(a) $20 - 20M_{\odot}$ *binary black hole waveform (spins* $s_{1z} = 0.4$ *and* $s_{2z} = 0.7$*).*



(b) $15 - 15M_{\odot}$ *binary black hole waveform (zero spins, eccentricity* $e = 0.3$*).*

Figure (5.1). Theoretical models of gravitational-wave signals emitted during the merger of two black holes. The waveform is a chirp signal with a time increasing (power-law) instantaneous frequency. Several examples are shown where the astrophysical parameters are varied, such as the component masses and spins s_{1z} and s_{2z} or the eccentricity e of the binary orbital motion. Those signals are processed through a whitening filter obtained from the detector noise power spectral distribution. This filtering discards the part of the original signal where the instrumental noise is large (low and high frequencies, below $\sim$ 30 Hz and above few kHz) and retains the frequency band where the noise is low. (Image from [3])

gravitational wave can be used to improve the search sensitivity. An approach is to search specifically for the time-frequency patterns associated with the waveform model. Such an approach is called "adaptive filtering method" and aims at selecting components that are likely to describe the gravitational wave signal and prevent the search algorithm from selecting those due to transient noise. The expected improvement is larger when the signal model can be completely characterized by a small number of time-frequency components. Many different analytic techniques have been used to detect the gravitational signal from transient sources, like binary mergers of black holes and neutron stars and, one of them was able to identify the signal in only 3 minutes. This method was based on the decomposition of the observations in the so called Wilson orthonormal bases which provided a local Fourier analysis of the signal. In the 2012, Necula, Klimenko and Mitselmakher in their article [25] proposed to use the Wilson bases for the detection of the gravitational waves. Klimenko developed a search algorithm called "Coherent WaveBurst" which has been successfully applied in this context. The idea of the algorithm consists in using a wavelet (the Meyer scaling function) such that its Fourier transform has compact support as the window and considers a collection of Wilson bases (usually seven to nine) based on different window durations and bandwidths. These bandwidths are distributed in powers of two, ranging from 1 to 64 Hz. The union of these orthonormal bases forms a redundant dictionary, which constitutes a tight frame. The main idea of the process is the following: the two LIGO detectors, one in Livingston (Louisiana) and one in Hanford (Washington), at a distance of approximately 3000 km produce two independent measurements of the gravitational waves. Coherent WaveBurst

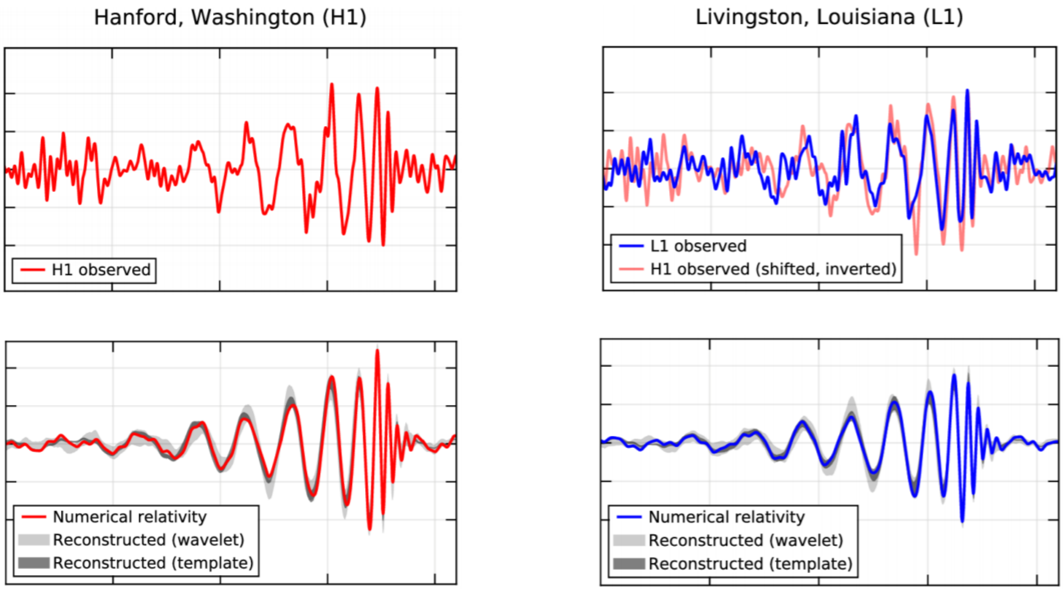


Figure (5.2). Top row, left: H1 observed. Since the gravitational wave arrived first at L1 and 6.9 (+0.5 − 0.4) ms later at H1, in the second figure L1 strain (blue) is reported and for a visual comparison the H1 data are also shown, shifted in time by this amount and inverted (to account for the detectors' relative orientations). Second row: Gravitational wave strain projected onto each detector in the 35 − 350 Hz band. Solid lines show a numerical relativity waveform for a system with parameters consistent with those recovered from the gravitational wave confirmed by an independent calculation. Shaded areas show 90% credible regions for two waveform reconstructions: one that models the signal as a set of sine-Gaussian wavelets and one that models the signal using binary black hole template waveforms. These reconstructions have a 95% overlap.
(Image from *http ://dx.doi.org/10.7935/K5MW2F23*)

maps the time series data given by each of these detectors to the time-frequency plane by projecting onto a Wilson bases: in the two decompositions the coefficients whose amplitude significantly differ from what is obtained in the presence of noise alone are identified. This allows a better separation between frequencies and the elimination of sinusoidal artifacts like, for example, the mechanical resonances in the systems for the attenuation of the seismic noise. Moreover, this representation of the data can be quickly computed and inverted by means of the Fast Fourier transform.

These two aspects lead to a great numerical efficiency of the algorithm.

This approach not only makes it possible to detect mergers of two black holes, but also of other astrophysical phenomena, including those for which we do not know the structure in advance.

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