universität wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit/Title of the Master's Thesis

Existence of global solutions and behaviour of energy in the Yang-Mills system coupled to a scalar field

verfasst von/ submitted by

Kevin Islami, BSc BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2020 / Vienna, 2020

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von / Supervisor:	Univ.-Prof. Adrian Constantin, PhD

Abstract

A Yang-Mills theory has the purpose of extending the abelian $U(1)$ gauge principle of Quantum Electrodynamics to the case of a non-abelian $SU(n)$ gauge principal. The Yang-Mills equations of a Yang-Mills field coupled to a scalar field arise as the equations of motion of the Lagrangian density of the corresponding field. Some global solutions of this system of equations are studied for both the presence and the absence of a scalar field ϕ in section 3. Results on the energy such as the possibility of splitting the energy in different parts and corresponding estimates in the system together with conserved quantities are presented in section 4 and 5. Results on the asymptotic behaviour of the system for the possibility of the scalar field having mass zero are presented in section 6. Section 7 collects estimates and their proofs in the case the scalar field has positive mass $m > 0$.

Zusammenfassung

Eine Yang-Mills Theorie hat den Zweck, das abelsche $U(1)$ Eichprinzip der Quantenelektrodynamik für den Fall eines nicht-abelschen $SU(n)$ Eichprinzips zu erweitern. Die Yang-Mills Gleichungen eines Yang-Mills Feldes, welches an ein skalares Feld gekoppelt, ist erhält man als Bewegungsgleichungen der Lagrangedichte des dazugehörigen Feldes. Einige globale Lösungen dieses Systems von Gleichungen werden für beide Fälle, nämlich die Anwesenheit und Abwesenheit eines skalaren Feldes, in Kapitel 3 studiert. Resultate bezüglich der Energie, wie beispielsweise die Möglichkeit die Energie in verschiedene Bestandteile aufzuteilen, und dazugehörige Abschätzungen in dem System werden zusammen mit Erhaltungsgrößen in den Kapiteln 4 und 5 behandelt. Das asymptotische Verhalten des Systems für die Möglichkeit eines skalaren Feldes mit Masse null wird in Kapitel 6 analysiert. In Kapitel 7 werden Abschätzungen für den Fall, dass das skalare Feld eine positive Masse $m > 0$ hat, behandelt.

Contents

1 Introduction to Yang-Mills theory

First, we start with a brief introduction to Yang-Mills theory itself to gain a first impression of why one would like to delve deeper into this area of physics and investigate the mathematics behind it.

We begin with the definition of the action and why we need the Euler-Lagrange equations.[6]

Definition. (Lagrange function and the action)
Let $L = L(x_1(t), x_2(t), \cdots, x_n(t), \dot{x}_1(t), \dot{x}_2(t), \cdots, \dot{x}_n(t))$ be a first order function (Lagrange function). The action S is defined by

$$S = \int_{t_0}^{t_1} L(x_1(t), x_2(t), \cdots, x_n(t), \dot{x}_1(t), \dot{x}_2(t), \cdots, \dot{x}_n(t))dt \quad t_0, t_1 \in \mathbb{R}.$$

Now, we want to obtain stationary points of the action S under compactly supported variations. The following Theorem is the first step obtaining these points.

Theorem 1.1. *A field is a stationary point of the action if and only if it satisfies the Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_n} - \frac{\partial L}{\partial x_n} = 0, \;\; n = 1, 2, ...$$

Next, we need the definition of a Lie group.

Definition. (Lie group)
A Lie group is a group G, which is simultaneously a differentiable manifold and such that the group multiplication

$$\mu : G \times G \mapsto G,$$
$$\mu(g, h) = g \cdot h$$

is smooth.

Remark: The smoothness of the inverse operation follows from the smoothness of the multiplication.

Now we can explain the principal idea of a Yang-Mills theory (YM) with this setup[7]. In physics, there are four fundamental interactions: the strong interaction, the electromagnetic interaction, the weak interaction and gravity. A theory states that at very high energies, all of these interactions were unified into fewer combinations of these four. Physicists managed to formulate a theory where the electromagnetic interaction could be combined with the weak one to obtain the electroweak interaction. Gauge groups play an essential role in all these theories because knowing the corresponding group of the interaction allows mathematics to get a deeper insight of the theory. A Yang-Mills theory is a non-abelian gauge theory, i.e., a field theory, where the Lagrangian is invariant under certain Lie groups of local transformations[1]. Its purpose is, roughly speaking, to study the unification of the electromagnetic and weak interaction and, in future, it may also allow to add the strong interaction in this picture. Whether gravity plays a role in this theory or not is currently studied. Summarizing, the aim of the Yang-Mills theory is

[1] The global gauge group of a Yang-Mills theory is in fact a connected semisimple compact Lie group whose Lie algebra consists of skew symmetric matrices. For more information on that topic, see [1].

to extend the abelian $U(1)$ gauge principle of Quantum Electrodynamics (quantum field theory of the electromagnetic interaction) to the case of a non-abelian $SU(n)$ gauge principle whose goal is the construction of a theory that is invariant under $SU(n)$ gauge transformations. With the purpose of the theory known, we can now continue with obtaining the equations of motions via the Yang-Mills Lagrangian.

2 The Yang-Mills Lagrangian and the equations of motion

First we need to acquaint ourselves with the notation that will be used throughout the next chapters which is based on [3] and [5]. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. The Lie multiplication will be denoted by $\times$ instead of the common $[\cdot,\cdot]$, i.e., for $A, B \in \mathfrak{g}$ we have $A \times B = -B \times A$, $A \times A = 0$ and the Jacobi identity is written as follows:

$$A \times (B \times C) + C \times (A \times B) + B \times (C \times A) = 0.$$

Since G was supposed to be compact, $\mathfrak{g}$ posses a natural inner product denoted by $A \cdot B$ with the properties $(A \times B) \cdot C = A \cdot (B \times C)$. Let $|A|^2 = A \cdot A$. The choice of this notation comes from the fact that if we take $SU(2)$ to be our Lie group, its Lie algebra is three-dimensional whose elements can be regarded as ordinary vectors with the ordinary cross and dot product as the operations on it. The space-variables are $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and the time variable is $t \in \mathbb{R}$. The partial derivatives are written as

$$\partial^k = \frac{\partial}{\partial x_k} \quad (k = 1, 2, 3), \;\; \partial^0 = \frac{\partial}{\partial t}.$$

The functions A^μ, $(\mu = 0, 1, 2, 3)$, which we call gauge potentials, are functions on $\mathbb{R}^4$ with values in $\mathfrak{g}$. ϕ shall be a scalar field which also is a function on space-time with values in $\mathfrak{g}$. (This function is called scalar field not because it is a function into $\mathbb{R}$, but rather due to the fact that it transforms under Lorentz-transformations the way a scalar does[7]).
The covariant derivatives are defined as follows

$$D^0 = \partial^0 - gA^0 \times \tag{2.1}$$

$$D^k = \partial^k + gA^k \times \qquad (k = 1, 2, 3), \;\; g \in \mathbb{R}. \tag{2.2}$$

The following identity will be used in computations that will follow

$$D^\mu A \cdot B + A \cdot D^\mu B = \partial^\mu (A \cdot B) \qquad (\mu = 0, 1, 2, 3) \tag{2.3}$$

for C^1 vector fields A, B.

The Yang-Mills field strengths are given by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + gA^\mu \times A^\nu, \quad (\mu, \nu = 1, 2, 3) \tag{2.4}$$

for some real constant g.
We denote

$$E^1 = F^{10}, \quad E^2 = F^{20}, \quad E^3 = F^{30} \tag{2.5}$$

$$H^1 = F^{32}, \quad H^2 = F^{13}, \quad H^3 = F^{21}. \tag{2.6}$$

The physical meaning of the quantities E^i and H^i is that they are the components of the electric and magnetic field respectively. ($F^{\mu\nu}$ is sometimes called field strength tensor)[4].
Next, we put

$$\psi^\mu = D^\mu \phi \quad (\mu = 0, 1, 2, 3). \tag{2.7}$$

Let E be the matrix with columns E^k where the $E^k \in \mathfrak{g}$ for $(k = 1, 2, 3)$. If we have a vector $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, then we write $E\alpha = \sum_k \alpha_k E^k$. By denoting $|E|^2 = \sum_k |E^k|^2$, we have the inequality $|E\alpha| \leq |E||\alpha|$. We do the same procedure with $H = (H^1, H^2, H^3)$ and $\Psi = (\psi^1, \psi^2, \psi^3)$. Let V_0 be a real function of a real variable, set $V(\phi) = V_0(|\phi|^2)$ and let $V'(\phi)$ denote $2\phi V_0'(|\phi|^2)$.

We write the Lagrangian density for a Yang-Mills field coupled to a scalar field (YMS) as

$$L = \frac{1}{2}\sum_k |E^k|^2 - \frac{1}{2}\sum_k |H^k|^2 + \frac{1}{2}|D^0\phi|^2 - \frac{1}{2}|D^k\phi|^2 - V(\phi). \tag{2.8}$$

The Euler-Lagrange equations for this Lagrangian are of the following form:

$$D^0H^1 = D^3E^2 - D^2E^3, \tag{2.9}$$
$$D^0H^2 = D^1E^3 - D^3E^1, \tag{2.10}$$
$$D^0H^3 = D^2E^1 - D^1E^2, \tag{2.11}$$
$$D^0E^1 = D^2H^3 - D^3H^2 + g\psi^1 \times \phi, \tag{2.12}$$
$$D^0E^2 = D^3H^1 - D^1H^3 + g\psi^2 \times \phi, \tag{2.13}$$
$$D^0E^3 = D^1H^2 - D^2H^1 + g\psi^3 \times \phi, \tag{2.14}$$
$$\sum_k D^kH^k = 0, \tag{2.15}$$
$$\sum_k D^kE^k = g\psi^0 \times \phi, \tag{2.16}$$
$$D^0\psi^0 - \sum_k D^k\psi^k = -V'(\phi), \tag{2.17}$$
$$D^0\psi^j - D^j\psi^0 = gE^j \times \phi, , \qquad (j = 1, 2, 3) \tag{2.18}$$
$$D^1\psi^2 - D^2\psi^1 = -gH^3 \times \phi, \tag{2.19}$$
$$D^2\psi^3 - D^3\psi^2 = -gH^1 \times \phi, \tag{2.20}$$
$$D^3\psi^1 - D^1\psi^3 = -gH^2 \times \phi, \qquad g \in \mathbb{R}. \tag{2.21}$$

The interesting thing about these equations is that only 2.12-2.14, 2.16 and 2.17 result directly from the Lagrangian L, whereas the other equations appear as constraint equations which follow from the definitions of the field strengths and the ψ^μ which we will be dealing with now.

The Euler-Lagrange equations of 2.8 are of the following form

$$\frac{\partial L}{\partial \phi} = \sum_\mu \partial^\mu \frac{\partial L}{\partial(\partial^\mu \phi)} \tag{2.22}$$

and

$$\frac{\partial L}{\partial A^\mu} = \sum_\nu \partial^\nu \frac{\partial L}{\partial(\partial^\nu A^\mu)} \qquad (\mu = 0, 1, 2, 3). \tag{2.23}$$

The values of the derivatives of L lie in $\mathfrak{g}$, hence, we can write

$$\frac{\partial L}{\partial \phi} \cdot \tilde{\phi} = \frac{d}{d\epsilon} L(\phi + \epsilon\tilde{\phi}|_{\epsilon=0}).$$

For evaluating equations 2.22 and 2.23, we will need the following two identities:

$$(i) \quad \frac{\partial}{\partial A}\frac{1}{2}|A + (B \times C)|^2 = A + (B \times C)$$

and

$$(ii) \quad \frac{\partial}{\partial B}\frac{1}{2}|A + (B \times C)|^2 = C \times (A + (B \times C)).$$

For (ii), observe that

$$\frac{d}{d\epsilon}\frac{1}{2}|A+((B+\epsilon\tilde{B})\times C)|^2 = \tilde{B}\times C\cdot(A+(B\times C))$$
$$= \tilde{B}\cdot C\times(A+(B\times C)) \;\; at \;\; \epsilon=0, \text{ for } C^1 \text{ vector fields } A, B, \tilde{B}, C$$

while (i) is clear.
Now, we evaluate equation 2.22.

$$\frac{\partial L}{\partial\phi} = \frac{1}{2}\frac{\partial}{\partial\phi}(|\partial^0\phi - gA^0\times\phi|^2 - \sum_k|\partial^k\phi + gA^k\times\phi|^2 - 2V(\phi))$$
$$= gA^0\times\psi^0 + g\sum_k A^k\times\psi^k - V'(\phi).$$

Note that

$$\frac{\partial L}{\partial(\partial^0\phi)} = \psi^0, \text{ and } \frac{\partial L}{\partial(\partial^j\phi)} = -\psi^j \qquad (j=1,2,3) \text{ by (i)}.$$

Therefore, 2.22 implies that

$$\partial^0\psi^0 - \sum_k\partial^k\psi^k = gA^0\times\psi^0 + g\sum_k A^k\times\psi^k - V'(\phi)$$

which is equation 2.17.
Now we look at equation 2.23 for $\mu=0$. Using identity (ii) we find

$$\frac{\partial L}{\partial A^0} = \frac{1}{2}\frac{\partial}{\partial A^0}(\sum_k|\partial^k A^0 + \partial^0 A^k - gA^0\times A^k|^2 + |\partial^0\phi - gA^0\times\phi|^2)$$
$$= -g\sum_k A^k\times E^k - g\phi\times\psi^0.$$

Since L does not depend on $\partial^0 A^0$ and

$$\frac{\partial L}{\partial(\partial^j A^0)} = E^j \text{ by } (i)$$

we get that 2.23 implies

$$\sum_k\partial^k E^k = -g\sum_k A^k\times E^k - g\phi\times\psi^0$$

which is nothing but equation 2.16.
For equations 2.12-2.14 we consider 2.23 for the case $\mu=1$. We can write

$$\frac{\partial L}{\partial A^1} = \frac{1}{2}\frac{\partial}{\partial A^1}(|\partial^1 A^0 + \partial^0 A^1 + gA^1\times A^0|^2 - |\partial^1 A^3 - \partial^3 A^1 + gA^1\times A^3|^2$$
$$- |\partial^2 A^1 - \partial^1 A^2 - gA^1\times A^2|^2 - |\partial^1\phi + gA^1\times\phi|^2).$$

Using (ii), we obtain

$$\frac{\partial L}{\partial A^1} = gA^0\times E^1 - gA^3\times F^{13} + gA^2\times F^{21} - g\phi\times\psi^1$$
$$= gA^0\times E^1 - gA^3\times H^2 + gA^2\times H^3 - g\phi\times\psi^1.$$

Moreover, by using identity (i), we obtain the following results

$$\frac{\partial L}{\partial(\partial^0 A^1)} = E^1, \ \frac{\partial L}{\partial(\partial^2 A^1)} = -F^{21} = -H^3, \frac{\partial L}{\partial(\partial^3 A^1)} = F^{13} = H^2.$$

Since the Lagrangian is independent of $\partial^1 A^1$ equation 2.23 yields for $\mu = 1$

$$\partial^0 E^1 - \partial^2 H^3 + \partial^3 H^2 = gA^0 \times E^1 - gA^3 \times H^2 + gA^1 \times H^3 - g\phi \times \psi^1,$$

which is exactly 2.12. The other two equations of this form are obtained in a similar way.
Now, since we have computed the part of the Yang-Mills equations that can be derived directly from the Lagrangian, we will take a closer look at the constraint equations and how they arise where each in case we will make use of the Jacobi identity.
Consider first equation 2.9. By inserting the definition of H^1 we get

$$\begin{aligned} D^0 H^1 = D^0 F^{32} =& (\partial^0 - gA^0 \times)(\partial^3 A^2 - \partial^2 A^3 + gA^3 \times A^2) \\ =& \partial^3 \partial^0 A^2 - \partial^2 \partial^0 A^3 + g\partial^0 (A^3 \times A^2) - gA^0 \times \partial^3 A^2 \\ &+ gA^0 \times \partial^2 A^3 - g^2 A^0 \times (A^3 \times A^2). \end{aligned}$$

Now we substitute in the second line

$$\partial^0 A^2 \text{ by } E^2 - \partial^2 A^0 - gA^2 \times A^0$$

and

$$\partial^0 A^3 \text{ by } E^3 - \partial^3 A^0 - gA^3 \times A^0.$$

Consequently, we obtain

$$\begin{aligned} D^0 H^1 =& \partial^3 E^2 - \partial^2 E^3 - g\partial^3 (A^2 \times A^0) + g\partial^2 (A^3 \times A^0) \\ &+ g\partial^0 (A^3 \times A^2) - gA^0 \times \partial^3 A^2 + gA^0 \times \partial^2 A^3 - g^2 A^0 \times (A^3 \times A^2). \end{aligned}$$

The next step is writing the first two terms as

$$\partial^3 E^2 - \partial^2 E^3 = D^3 E^2 - gA^3 \times E^2 - D^2 E^3 + gA^2 \times E^3,$$

after which we then evaluate the appearing crossproducts using 2.4 for $\mu = k, \ \nu = 0$ and 2.5. In doing so, we see that in the expression for $D^0 H^1$ the quadratic terms cancel pairwise and the three cubic terms sum to zero by the Jacobi identity. Therefore this establishes equation 2.9 and, hence, by cyclic permutation of the indices, 2.10 and 2.11 as well.
Now, we will have a closer look at equation 2.15. We find

$$\begin{aligned} D^1 H^1 =& \partial^1 H^1 + gA^1 \times H^1 = \partial^1 F^{32} + gA^1 \times F^{32} \\ =& \partial^1 (\partial^3 A^2 - \partial^2 A^3 + gA^3 \times A^2) \\ &+ gA^1 \times \partial^3 A^2 - gA^1 \times \partial^2 A^3 + g^2 A^1 \times (A^3 \times A^2) \end{aligned}$$

$D^2 H^2$ and $D^3 H^3$ can be then determined by cyclic permutation of the indices of the above equation. Taking the sum $\sum_k D^k H^k$, the second-derivative terms cancel pairwise just as the quadratic ones do. Again, the Jacobi identity shows that the cubic terms vanish upon summation.

Therefore, $\sum_k D^k H^k = 0$ which is the equation we wanted to obtain.
For equation 2.18 with $(j = 1, 2, 3)$ we have

$$\begin{aligned}
D^0\psi^j - D^j\psi^0 &= D^0\psi^j - \partial^j\psi^0 - gA^0 \times \psi^j - gA^j \times \psi^0 \\
&= \partial^0(\partial^j\phi + gA^j \times \phi) - \partial^j(\partial^0\phi - gA^0 \times \phi) - gA^0 \times \psi^j - gA^j \times \psi^0 \\
&= g(\partial^0 A^j + \partial^j A^0) \times \phi + gA^j \times (\partial^0\phi - \psi^0) + gA^0 \times (\partial^j\phi - \psi^j).
\end{aligned}$$

This equation reduces with the identities 2.5 and 2.7 to

$$g(E^j - gA^j \times A^0) \times \phi + g^2 A^j \times (A^0 \times \phi) - g^2 A^0 \times (A^j \times \phi) = gE^j \times \phi.$$

For the final equations, consider 2.19. We get

$$\begin{aligned}
D^1\psi^2 - D^2\psi^1 =&\partial^1\psi^2 + gA^1 \times \psi^2 - \partial^2\psi^1 - gA^2 \times \psi^1 \\
=&\partial^1(\partial^2\phi + gA^2 \times \phi) - \partial^2(\partial^1\phi + gA^1 \times \phi) \\
&+ gA^1 \times (\partial^2\phi + gA^2 \times \phi) - gA^2 \times (\partial^1\phi + gA^1 \times \phi).
\end{aligned}$$

This simplifies to

$$g(\partial^1 A^2 - \partial^2 A^1) \times \phi + g^2 A^1 \times (A^2 \times \phi) - g^2 A^2 \times (A^1 \times \phi).$$

Using the Jacobi identity together with the substitution

$$\partial^1 A^2 - \partial^2 A^1 = -H^3 - gA^1 \times A^2$$

we obtain equation 2.19 and, by permutation, the remaining equations as well to finish the derivation of the Yang-Mills equations. Having derived the equations, the next step is proving some global existence Theorems.

3 Existence of global solutions

First, we need to add some new definitions in addition to those of chapter 2 before we can state and proof two existence Theorems that are covered in [5]. For the beginning, we will only consider a pure Yang-Mills field, i.e. the scalar field $\phi = 0$.
We define the energy $\mathcal{E}$ as

$$\mathcal{E} = \frac{1}{2} \int (|E|^2 + |H|^2) dx. \tag{3.1}$$

Furthermore, put

$$\omega = x/r, \ (r = |x|, x \in \mathbb{R}^3)$$
$$e^1 = (1,0,0), e^2 = (0,1,0), e^3 = (0,0,1) \text{ the standard basis and}$$
$$v^k = e^k \times \omega \quad (k = 1,2,3). \tag{3.2}$$

Next, we need the concept of radial functions.

Definition. A function defined on $\mathbb{R}^3$ is called radial if it depends only on the radius r. $\tilde{H}^1_r$ is defined to be the completion of C_c^∞-functions which are radial, under the Dirichlet norm

$$||\nabla\phi||_2 = (\int_{\mathbb{R}^3} |\nabla\phi|^2 dx)^{1/2}.$$

Furthermore, we define L^2_r to be the space of radial L^2 functions.

The following Theorem is the first of our two existence Theorems of global solutions for the Yang-Mills equations with the absence of a scalar field ϕ.

Theorem 3.1. *Let $\alpha_0 \in \tilde{H}^1_r$ and $\alpha_1 \in L^2_r$. Let $g \in \mathbb{R}$. Then there exists a unique solutions of the Yang-Mills equations in all space-time with the following properties:*

$$\begin{aligned} &A^0(x,t) = 0, \\ &A^k(x,0) = \alpha_0(r) v^k, \quad (k = 1,2,3) \\ &\partial_t A^k(x,0) = \alpha_1(r) v^k, \\ &A^k \in C(\mathbb{R}; \tilde{H}^1_r), \\ &\partial_t A^k \in C(\mathbb{R}; L^2_r). \end{aligned} \tag{3.3}$$

The energy is a constant,

$$\begin{aligned} &A^k(x,t) = \alpha(r,t) v^k \quad (k = 1,2,3) \\ &\alpha(r,t) \text{ is a real scalar function and} \\ &\frac{\alpha^2}{r}(g\alpha - \frac{2}{r})^2 \text{ is integrable over all spacetime.} \end{aligned} \tag{3.4}$$

From now on, we will let the coupling constant $g = 1$ without loss of generality.
To prove this Theorem, we will claim four Lemmata which together prove Theorem 3.1.

Lemma 3.2. *For every $\phi \in C_c^\infty(\mathbb{R}^3)$ therefore for all $\phi \in \tilde{H}_r^1$ we have the following inequalities:*

$$\int \phi^6 dx \leq c(\int |\nabla\phi|^2 dx)^3, \textit{ and} \tag{3.5}$$

$$\int \phi^2 r^{-2} dx \leq 4 \int |\nabla\phi|^2 dx, \tag{3.6}$$

where c is a constant. The first of these inequalities is the Sobolev inequality and the second one is Hardy's inequality. Furthermore, every $\phi \in \tilde{H}_r^1$ is almost everywhere equal to a function which is continuous for $x \neq 0$,

$$4\pi r\phi^2(r) \leq \int |\nabla\phi|^2 dx \tag{3.7}$$

and

$$r\phi^2 \to 0 \textit{ as } r \to 0 \textit{ and } r \to \infty. \tag{3.8}$$

For a proof of these standard PDE inequalities see e.g. [5] or [2].

The next Lemma allows a simplification of the YM system into a single scalar equation.

Lemma 3.3. *The formulas 3.2, 3.3 and 3.4 together reduce the YM system to the scalar equation*

$$\partial_t^2\alpha - \Delta\alpha + F(r,\alpha) = 0, \tag{3.9}$$

with

$$F(r,\alpha) = 2r^{-2}\alpha - 3r^{-1}\alpha^2 + \alpha^3 \tag{3.10}$$

Proof. First, we will make use of the following properties of the vectors v^k:

$$\begin{aligned}
&\sum_k \partial^k v^k = \sum_k \omega_k v^k = 0, \quad (\omega_k = x_k/r)\\
&\sum_k |v^k|^2 = 2 = \sum_{j,k} |v^j \times v^k|^2,\\
&v^k \cdot v^l = \delta_{kl} - \omega_k\omega_l,\\
&\sum_{j,k} |\partial^j v^k|^2 = r^{-2},\\
&\sum_{j,k} \partial^j v^k \cdot v^j \times v^k = -2r^{-1}.
\end{aligned}$$

Since $A^0 = 0$, we have that

$$\begin{aligned}
E^k = \partial^0 A^k = (\partial^0\alpha)v^k, &\text{ therefore we find}\\
\sum_k D^k E^k = \sum_k ((\partial_r\partial^0\alpha)\omega_k v^k &+ (\partial^0\alpha)\partial^k v^k + (\partial^0\alpha)\alpha v^k \times v^k) = 0,
\end{aligned}$$

which is precisely 2.16 with $\phi = 0$.
Now, using the definition of H^k, we have

$$\begin{aligned}
H^2 &= \frac{2\alpha}{r}e^2 + (\alpha_r - \frac{\alpha}{r})(\omega_1 v^3 - \omega_3 v^1) - \omega_2\alpha^2\omega,\\
H^3 &= \frac{2\alpha}{r}e^3 + (\alpha_r - \frac{\alpha}{r})(\omega_2 v^1 - \omega_1 v^2) - \omega_3\alpha^2\omega.
\end{aligned}$$

Taking now the vector product of H^2, H^3 with A^2, A^3 from the respective side we find

$$A^2 \times H^3 = \frac{2\alpha^2}{r} v^2 \times e^3 + \alpha(\alpha_r - \frac{\alpha}{r})\omega_2 v^2 \times v^1 - \omega_3 \alpha^3 v^2 \times \omega,$$
$$H^2 \times A^3 = \frac{-2}{r} \alpha^2 v^3 \times e^2 + \alpha(\alpha_r - \frac{\alpha}{r})\omega_3 v^3 \times v^1 + \omega_2 \alpha^3 v^3 \times \omega.$$

Taking the sum of the above expressions, we get

$$A^2 \times H^3 + H^2 \times A^3 = (\frac{2}{r}\alpha^2 - \alpha^3)v^1.$$

These terms are exactly the nonlinear ones in 2.12 again with $\phi = 0$. The linear terms are $\partial^2 H^3 - \partial^3 H^2$ which contribute after a tedious computation the following terms

$$(\alpha_{rr} + 2r^{-1}\alpha_r - 2r^{-2}\alpha + r^{-1}\alpha^2)v^1.$$

By cyclic permutation of the indices we then obtain equations 2.13 and 2.14 in the other directions v^2 and v^3 respectively. This shows that for α satisfying equation 3.9 the equations 2.12-2.14 are valid. By definition of the H^k all the other equations follow by lengthy computations.
We find with the help of the expressions of H^1, H^2, H^3 the following

$$r^2|H|^2 = 2r^2\alpha_r^2 + (2\alpha - r\alpha^2)^2 + (2r\alpha^2)_r.$$

Therefore we can rewrite the energy in terms of α as follows

$$\mathcal{E} = \int (\alpha_t^2 + \alpha_r^2 + \frac{1}{2}\alpha^2(\frac{2}{r} - \alpha)^2)dx \tag{3.11}$$

provided $r\alpha^2(r) \to 0$ as $r \to 0$ and $r \to \infty$. □

Now, we will discuss the existence of solutions of equation 3.9 which shows that there exists a solution of the Yang-Mills equations (in $\mathbb{R}^4$).

Lemma 3.4. *The scalar equation 3.9 posses a global solution with given Cauchy data α_0, α_1.*

Proof. Since we have a singularity in this equation at $r = 0$, we consider an approximate equation for $\epsilon > 0$

$$\partial_{tt}\alpha_\epsilon - \Delta\alpha_\epsilon + \frac{r^2}{r^2 + \epsilon}F(r, \alpha_\epsilon) = 0. \tag{3.12}$$

The nonlinear term in this equation is locally Lipschitz from H^1 into L^2 due to Sobolev's and Hardy's inequalities. Next, we choose smooth functions with compact support $\alpha_{0\epsilon}$ and $\alpha_{1\epsilon}$ which converge to α_0 and α_1 in $\tilde{H}_r^1$ and L_r^2 respectively. We know that if we fix $\epsilon > 0$, then equation 3.12 has a unique C_c^∞ solution $\alpha_\epsilon(x, t)$ at fixed time with given Cauchy data $\alpha_{0\epsilon}$ and $\alpha_{1\epsilon}$. It can be shown that this solution exists in a time interval $|t| \leq T$. For this, see [10]
Let now $G(r, \alpha)$ be the primitive of $F(r, \alpha)$ with $G(r, 0) = 0$. Now, we multiply the approximate equation by the first time derivative of α_ϵ to obtain the following equation

$$\partial_t(\frac{1}{2}(\partial_t\alpha_\epsilon)^2 + \frac{1}{2}|\nabla\alpha_\epsilon|^2 + \frac{r^2}{r^2 + \epsilon}G(r, \alpha_\epsilon)) = \nabla \cdot (\partial_t\alpha_\epsilon \nabla\alpha_\epsilon)). \tag{3.13}$$

Now, if we integrate over all space, we get an approximate energy identity which is time independent and given by

$$\mathcal{E}_\epsilon = \int (\frac{1}{2}(\partial_t\alpha_\epsilon)^2 + \frac{1}{2}|\nabla\alpha_\epsilon|^2 + \frac{r^2}{r^2 + \epsilon}G(r, \alpha_\epsilon))dx. \tag{3.14}$$

By letting $t = 0$, we see that this expression is determined by the initial data and therefore independently of ϵ bounded.
Now, we see that G can be written as a perfect square in the following form:

$$G(r, \alpha_\epsilon) = \frac{1}{4}\alpha_\epsilon^2(\alpha_\epsilon - \frac{2}{r})^2 \geq 0.$$

This then leads to the conclusion that the approximate solution α_ϵ exists for all time. For a proof of this conclusion, see [10].
Next, we will use a so called Morawetz' radial estimate[2]. Let $u = r\alpha_\epsilon$ such that

$$\partial_t^2 u - \partial_r^2 u + (r^2 + \epsilon)^{-1}(2u - 3u^2 + u^3) = 0.$$

If we now multiply this equation by $\partial_r u$ and integrate over the radius, we get

$$\frac{d}{dt}\int_0^\infty \partial_t u \cdot \partial_r u dr + \frac{1}{2}(\partial_r u(0,t))^2 + \int_0^\infty \frac{1}{2} r(r^2+\epsilon)^{-2}u^2(u-2)^2 dr = 0$$

because $u(0, t) = 0$. Now, if we integrate over time and substitute u by $r\alpha$, the last term yields the bound

$$\int_0^\infty \int \frac{r^4}{(r^2+\epsilon)^2}\alpha_\epsilon^2(\alpha_\epsilon - \frac{2}{r})^2 \frac{dx}{r} dt \leq 2\mathcal{E}_\epsilon. \tag{3.15}$$

With the setup we have, we can now pass to the limit. There exists a sequence of ϵ's converging to zero such that we have

$$\alpha_\epsilon \to \alpha \text{ weakly* in } L^\infty(\mathbb{R}, \tilde{H}_r^1) \text{ and}$$
$$\partial_t \alpha_\epsilon \to \partial_t \alpha \text{ in } L^\infty(\mathbb{R}, L_r^2),$$

but this means precisely that the derivative terms in equation 3.12 converge in the sense of distributions. The sequence now may be chosen due to reasons of compactness and diagonalization, such that $\alpha_\epsilon \to \alpha$ almost everywhere in space-time. Therefore we have $\alpha_\epsilon^3 \to \alpha^3$ a.e. and α_ϵ^3 is bounded in $L^2_{loc}(\mathbb{R}^4)$, but this means that $r^2(r^2+\epsilon)^{-1}\alpha_\epsilon^3 \to \alpha^3$ weakly in $L^2_{loc}(\mathbb{R}^4)$.We find in a completely similar fashion that $r^2(r^2+\epsilon)^{-1}\alpha_\epsilon^2 \to \alpha^2$ weakly in $L^3_{loc}(\mathbb{R})$, while $r^{-1} \in L^{3/2}_{loc}$. Finally, $r^2(r^2+\epsilon)^{-1}\alpha_\epsilon \to \alpha$ weakly in $L^6_{loc}(\mathbb{R}^4)$ while $r^{-2} \in L^{6/5}_{loc}$. What we have shown is that every term in equation 3.12 converges to their proper limits in the sense of distributions in space-time. This means that equation 3.9 is valid for α in all of space-time. It now follows that α and α_t are continuous functions of time with values in $\mathcal{D}'(\mathbb{R}^3)$ and that $\alpha(x, 0) = \alpha_0(r)$ and $\alpha_t(x, 0) = \alpha_1(r)$ which completes the proof. □

The last Lemma we need for the proof of Theorem 3.1 is the following.

Lemma 3.5. *Let $f \in L^1_{loc}(\mathbb{R}, L_r^2), \beta_0 \in \tilde{H}_r^1$ and $\beta_1 \in L_r^2$. Then there is a unique solution of the linear problem*

$$\partial_t^2 \beta - \Delta\beta + 2r^{-2}\beta = f, \tag{3.16}$$
$$\beta = \beta_0 \textit{ and } \partial_t\beta = \beta_1 \textit{ when } t = 0,$$

such that $\beta \in C(\mathbb{R}, \tilde{H}_r^1)$ and $\partial_t\beta \in C(\mathbb{R}, L_r^2)$ and

$$\int (\frac{1}{2}(\partial_t\beta)^2 + \frac{1}{2}|\nabla\beta|^2 + r^{-2}\beta^2)dx|_0^T = \int_0^T \int f \cdot \partial_t\beta dx dt. \tag{3.17}$$

[2]These type of estimates were used by Cathleen S. Morawetz in [8].

The Proof is based on the theory of linear contraction semigroups and the fact that the operator $-\Delta + r^{-2}$ restricted to $C_c^\infty(\mathbb{R}^3\{0\})$, hence,it will not be proven here. More information regarding this proof, however, can be found in [9].
The last thing to show is that the solution we found in Lemma 3.4 satisfies the required properties of Theorem 3.1.
First of all, we get the following inequality with aid of Fatou's Lemma and 3.15

$$\int_0^\infty \int \alpha^2(\alpha - \frac{2}{r}^2 \frac{dx}{r} dz \leq 2\mathcal{E}(0)). \tag{3.18}$$

The next step is using Lemma 3.5 with $\beta = \alpha$ and the function f being

$$f = \frac{3}{r}\alpha^2 - \alpha^3 = \frac{3}{2}\alpha(\frac{2}{r}\alpha - \alpha^2) + \frac{1}{2}\alpha^3.$$

Then, we obtain the bound

$$\int f^2 dx \leq 3 \sup_r (r\alpha^2) \cdot \int \alpha^2(\alpha - \frac{2}{r})^2 \frac{dx}{r} + \int \alpha^6 dx.$$

By 3.7 and 3.18 the first term on the right hand side is integrable over time. By Sobolev's inequality and 3.11 the last term is bounded over time as well. This means that $f \in L^1_{loc}(\mathbb{R}, L^2_r)$ and we have the requirements to apply Lemma 3.5. Now, we only have to prove uniqueness. Let α and α_* be two solutions with the required properties of Theorem 3.1 and let $\beta = \alpha - \alpha_*$. Then β satisfies 3.16 with

$$f = \frac{3}{r}(\alpha^2 - \alpha_*^2) - (\alpha^3 - \alpha_*^3)$$

with Cauchy data that vanishes. We find

$$f = \frac{3}{2}\beta(\frac{2}{r}\alpha - \alpha^2) + \frac{3}{2}\beta(\frac{2}{r}\alpha_* - \alpha_*^2) + \frac{1}{2}\beta^3,$$

such that

$$\int f^2 dx \leq 3 \sup_r (r\beta^2) \int ((\alpha^2 - \frac{2}{r}\alpha)^2 + (\alpha_*^2 - \frac{2}{r}\alpha_*)^2)\frac{dx}{r} + \int \beta^6 dx \leq (l(t) + c) \int |\nabla\beta|^2 dx$$

using 3.5, 3.7 and 3.18, where $l(t)$ is integrable over time and c a numerical constant.
If we denote the left hand side of 3.17 by $E(T)$, we find

$$E(T) \leq \int_0^T (l(t) + c)^{1/2} E(t) dt$$

which means that $E(T) = 0$, hence $\beta = 0$ and $\alpha = \alpha_*$. Now, the scalar equation in Lemma 3.3 is invariant under time reversal and since the solution of it is unique, we conclude that the consveration of Energy must be valid and, hence, the proof is complete.

Having discussed the existence and uniqueness of solutions of the Yang-Mills equations, we will now state a similar Theorem considering the YM equations coupled to a scalar field.
The energy in this case is given by

$$\mathcal{E} = \frac{1}{2} \int (|E|^2 + |H|^2 + |\psi^0|^2 + \sum_k |\psi^k|^2 + 2V(\phi)) dx.$$

Theorem 3.6. *Let $\alpha_0, \beta_0 \in \tilde{H}^1_r$ and $\alpha_1, \beta_1 \in L^2_r$. Let $g \in \mathbb{R}$ and let the Potential V be $V(\phi) = c_2|\phi|^2 + c_4|\phi|^4$ with $c_2 \geq 0, c_4 \geq 0$. Then there exists a unique solution of YMS with the following properties:*

$$A^0 = (x,t) = 0, A^k(x,t) = \alpha(r,t)v^k(k = 1,2,3) \tag{3.19}$$
$$\phi(x,t) = \beta(r,t)\omega \quad (\omega = x_k/r) \tag{3.20}$$
$$\alpha(x,0) = \alpha_0(r), \partial_t\alpha(x,0) = \alpha_1(r), \beta(x,0) = \beta_0(r), \partial_t\beta(x,0) = \beta_1(r)$$
$$\phi, A^k \in C(\mathbb{R}, \tilde{H}^1_r), \partial_t\phi, \partial_t A^k \in C(\mathbb{R}, L^2_r.,)$$

The energy $\mathcal{E}$ is a constant and, furthermore, $c_4\beta^{4/r}$ and the following expression are integrable over all space time :

$$\frac{\alpha^2}{r}(g\alpha - \frac{2}{r}) \text{ and } \frac{\beta^2}{r}(g\alpha - \frac{1}{r})^2. \tag{3.21}$$

Proof. In contrast to the existence proof with the absence of a scalar field, the substitution results in two scalar equations of the form

$$\alpha_{tt} - \Delta\alpha + \frac{2}{r^2}\alpha - \frac{3}{r}\alpha^2 + \alpha^3 + \beta^2(\alpha - \frac{1}{r}) = 0 \tag{3.22}$$
$$\beta_{tt} - \Delta\beta + \frac{2}{r^2}\beta + 2\alpha\beta(\alpha - \frac{2}{r}) + 2\beta V_0'(\beta^2) = 0 \tag{3.23}$$

For the energy we have

$$\mathcal{E} = \int(\alpha_t^2 + \alpha_r^2 + \frac{1}{2}\alpha^2(\alpha - \frac{2}{r})^2 + \frac{1}{2}\beta_t^2 + \frac{1}{2}\beta_r^2 + \beta^2(\alpha - \frac{1}{r})^2 + V_0(\beta^2))dx.$$

By the assumptions on V_0, each term in the energy expression is bounded. We introduce in a similar way to Lemma 3.4 the factor $r^2(r^2+\epsilon)^{-1}$. Passing to the limit is possible, since the degree of the potential V_0 is 4. 3.21 is a conclusion of Theorem 7.2 that will be proven in the last section. The estimates and the uniqueness are proven by applying the same procedure as in proving Theorem 3.1. □

Remark. Theorem 3.6 is also true for $c_2 < 0$ with the difference that 3.21 is only valid for finite time integrals and the assumption that $\alpha_0, \beta_0 \in L^2$.
The following Theorem is a more general existence Theorem but also one where uniqueness is not given.

Theorem 3.7. *Assume that V_0 is C^1 and such that*

$$V_0(s) \geq 0 \text{ and } V_0'(s) \geq 0 \text{ for } s \geq s_0.$$

Let $\alpha_0, \beta_0, \alpha_1, \beta_1$ be as in Theorem 3.6. Then there exists a solution of the Yang-Mills equation coupled to a scalar field that satisfies the requirements of Theorem 3.6 with the exception that ϕ, A^k are weakly continuous with values in $\tilde{H}^1_r$,$\partial_t\phi, \partial_t A^k$ ware weakly continuous with values in L^2_r, the energy satisfies $\mathcal{E}(t) \leq \mathcal{E}(0)$ and 3.21 are integrable over all space with the restriction of taking only finite time intervals.

For a proof of this Theorem, see [5] and [12].

4 The role of energy in the Yang-Mills system

In this section, we will talk about the propagation of the energy and underline the importance of analysing the energy on its own, since it contains valuable information in studying the Yang-Mills system[3].

Theorem 4.1. *The Yang-Mills system is causal. The Yang-Mills system coupled to a scalar field is causal if the potential V is non-negative.*

Let us take any C^∞ solution of the Yang-Mills system with a scalar field. The law of conservation of energy reads

$$\partial^0 e = \sum \partial^k p^k,$$

where e denotes the energy density and p^k the momentum densities (more detail on these expressions will follow in section 5). Integrating this over a piece of the solid light cone with base B, top T and side K and applying the divergence Theorem, we obtain the following expression

$$\int_T e - \int_B e + \frac{1}{\sqrt{2}} \int_K (\sum_k \omega_k p^k + e) = 0.$$

It can be shown (e.g.[4]) that

$$|\sum_k \omega_k p^k| \leq e$$

which means the integral over K is non-negative, thus, yielding the following inequality

$$\int_T e dx \leq \int_B e dx. \tag{4.1}$$

So if $e = 0$ on B, the Cauchy-data vanishes in the solid cone depending on B, which expresses the causality. More precisely, the solutions constructed in the previous section satisfy 4.1. This inequality is valid for the approximate solutions for each ϵ that is smooth, see [10]. The validity for the exact solution is implied by the passage to the weak limit in the previous section if $B \subseteq \{t = 0\}$, i.e. the time the Cauchy data are prescribed.Therefore, 4.1 is valid for any of the solutions described by Theorems 3.1,3.6,3.7.

Now, we will state and prove a Theorem that lets us split the energy into three different parts.

Theorem 4.2. *(Energy splitting)*
Consider a solution satisfying any of the conditions (I)-(III) below. Then, there exists a decomposition of the energy density e into non-negative parts

$$e = e_{for} + e_{back} + e_{ang}$$

such that

$$\int_{\mathbb{R}^3} (e_{back} + e_{ang}) dx \to 0, \ \ as\ t \to \infty$$
$$\int_{\mathbb{R}^3} (e_{for} + e_{ang}) dx \to 0, \ \ as\ t \to -\infty.$$

The expressions in the Theorem will be defined in its proof. e_{for} carries all the energy forward in time and e_{back} carries all the energy backwards in time, asymptotically. Now, assume that the initial data satisfy $\int (r^2+1)edx < \infty =: (*)$ and one of the following:
(I): The solution is of class C^2 of the YMS where $(*)$ holds at all times and where G is a compact Lie group and, furthermore, $0 \leq 4V(\phi) \leq \phi \cdot V'(\phi)$.
(II)It is a solution of YM given in Theorem 3.1.
(III)It is a solution of YMS given in Theorem 3.6 where $V(\phi) = C_4|\phi|^4, C_4 \geq 0$.

Proof. This result is a consequence of the invariance of the equations under the conformal group[3], specifically the first inversional identity, which reads

$$\int ((t^2+r^2)e + 2Tr(\sum_k \omega_k p^k) + 2t\phi \cdot \psi^0 - \phi \cdot \phi)dx \leq \text{const.} \tag{4.2}$$

under assumption (I) with

$$p^k = p^k_{YM} + \psi^0 \cdot \psi^k \text{ , (k=1,2,3) and}$$
$$p^1_{YM} = H^2 \cdot E^3 - H^3 \cdot E^2 \text{ etc.}$$

Now we split the integrand in 4.2 into a Yang-Mills part I_{YM} and a scalar part I_s, i.e. $I = I_{YM} + I_s$, where

$$I_{YM} = \frac{1}{2}(t^2+r^2)(|E|^2+|H|^2) + 2Tr(\sum_k \omega_k p^k_{YM})$$

and

$$I_s = \frac{t^2+r^2}{2}(|\psi^0|^2 + \sum_1^3)|\psi^k|^2 + 2V(\phi)) + 2t\psi^0 \cdot (\Psi x + \phi) - |\phi|^2.$$

Define $X^k = \psi^k + x_k r^{-2}\phi$ and let Ξ be the matrix with columns X^1, X^2, X^3. We will use the following identity, which will be proven in section 6.

$$|\Xi|^2 = |\Psi|^2 + r^{-2}\partial_r(r|\phi|^2).$$

This allows us to write I_s as follows

$$I_s = I_s^* - \frac{1}{2r^2}\partial_r((t^2+r^2)r|\phi|^2),$$

where

$$I_s^* = \frac{t^2+r^2}{2}(|\psi^0|^2 + |\Xi|^2 + 2V(\phi)) + 2t\psi^0 \cdot \Xi x.$$

Now, take a look at I_{YM}. First, define $(E,H) = Tr(E^T H)$so, that $|E|^2 = (E,E)$. Then, define $\omega \times E$ as the matrix with columns $\omega_2 E^3 - \omega_3 E^2, \omega_3 E^1 - \omega_1 E^3, \omega_1 E^2 - \omega_2 E^1$ and note the identity

$$|E|^2 = |E\omega|^2 + |\omega \times E|^2.$$

Using this, we can write I_{YM} as follows

$$\begin{aligned} I_{YM} &= \frac{r^2+t^2}{2}(|E|^2+|H|^2) + 2Tr(\sum_1^3 \omega_k p^k_{YM}) \\ &= \frac{1}{2}(r^2+t^2)(|E|^2+|H|^2) + rt(E,\omega\times H) - rt(H,\omega\times E) \\ &= \frac{1}{4}(r^2+t^2)(|E\omega|^2+|H\omega|^2) + \frac{1}{8}(r+t)^2(|E+\omega\times H|^2 \\ &+ |H-\omega\times E|^2) + \frac{1}{8}(r-t)^2(|E-\omega\times H|^2+|H+\omega\times E|^2). \end{aligned} \tag{4.3}$$

For I_s^*, we recall the following identities

$$\frac{1}{4}(r\pm t)^2|\psi^0 \pm \Xi\omega|^2 = \frac{1}{4}(r^2 \pm 2rt + t^2)(|\psi^0|^2 + |\Xi\omega|^2 \pm 2\psi^0\cdot\Xi\omega)$$

and obtain

$$\begin{aligned} &\frac{1}{4}(r+t)^2|\psi^0+\Xi\omega|^2 + \frac{1}{4}(r-t)^2|\psi^0-\Xi\omega|^2 \\ &= \frac{1}{2}(r^2+t^2)(|\psi^0|^2+|\Xi\omega|^2) + 2rt\psi^0\cdot\Xi\omega \\ &= I_s^* - \frac{(r^2+t^2)}{2}(|\Xi|^2 - |\Xi\omega|^2 + 2V(\phi)). \end{aligned} \tag{4.4}$$

Using the last two equations 3.10 and 3.11, we find

$$I = I_{YM} + I_s = I_{YM} + I_s^* - \frac{1}{2r^2}\partial_r((r^2+t^2)r|\phi|^2)$$

which we can further write as follows

$$\begin{aligned} I &= \frac{r^2+t^2}{4}(|E\omega|^2+|H\omega|^2+2(|\Xi|^2-|\Xi\omega|^2+2V(\phi))) \\ &+ \frac{(r+t)^2}{8}(|E+\omega\times H|^2+|H-\omega\times E|^2+2|\psi^0+\Xi\omega|^2) \\ &+ \frac{(r-t)^2}{8}(|E-\omega\times H|^2+|H+\omega\times E|^2+2|\psi^0-\Xi\omega|^2) \\ &- \frac{1}{2r^2}\partial_r((r^2+t^2)r|\phi|^2). \end{aligned} \tag{4.5}$$

Now, if we define the energy parts as follows

$$\begin{aligned} 4e_{ang} &= |E\omega|^2+|H\omega|^2+2(|\Xi|^2-|\Xi\omega|^2)+4V(\phi), \\ 8e_{back} &= |E+\omega\times H|^2+|H-\omega\times E|^2+2|\psi^0+\Xi\omega|^2, \\ 8e_{for} &= |E-\omega\times H|^2+|H+\omega\times E|^2+2|\psi^0-\Xi\omega|^2, \end{aligned}$$

then equation 4.5 implies that these three expressions do indeed have the claimed properties and thus proves Theorem 4.2 under assumption (I).
Next we will show the proof of Theorem 4.2 under the assumption of (III), since (II) is only a special case of (III) for $\phi = 0$. Our task here is to derive an analogous form of 4.2 for the

approximate equation and the prove will be them completed by passing to the limit.
We recall from the previous section the equations for α, β

$$\alpha_{tt} - \Delta\alpha + F_1(r, \alpha, \beta) = 0, \tag{4.6}$$
$$\beta_{tt} - \Delta\beta + F_2(r, \alpha, \beta) = 0, \tag{4.7}$$

with

$$F_1(r, \alpha, \beta) = \frac{2}{r^2}\alpha - \frac{3}{r}\alpha^2 + \alpha^3 + \beta^2(\alpha - \frac{1}{r}) \tag{4.8}$$
$$F_2(r, \alpha, \beta) = \frac{2}{r^2}\beta + 2\alpha\beta(\alpha - \frac{2}{r}) + 2\beta V_0'(\beta^2). \tag{4.9}$$

Furthermore we recall the approximate equations

$$\alpha_{tt} - \Delta\alpha - \frac{r^2}{r^2 + \epsilon}F_1(r, \alpha, \beta) = 0 \tag{4.10}$$
$$\beta_{tt} - \Delta\beta + \frac{r^2}{r^2 + \epsilon}F_2(r, \alpha, \beta) = 0 \tag{4.11}$$

with solutions $\alpha = \alpha_\epsilon, \beta = \beta_\epsilon$ that also have smooth Cauchy data with compact support.
The energy density of this approximation is given by

$$\begin{aligned} e_\epsilon = &\frac{1}{2}(\alpha_t^2 + \alpha_r^2) + \frac{1}{4}(\beta_t^2 + \beta_r^2) \\ &+ \frac{r^2}{r^2 + \epsilon}(\frac{1}{4}\alpha^2(\alpha - \frac{2}{r})^2 + \frac{1}{2}\beta^2(\alpha - \frac{1}{r})^2 + \frac{1}{2}V_0(\beta^2)). \end{aligned} \tag{4.12}$$

We only sketch the derivation of the inversional identity for the approximate equations since it is a special case of 4.2 [3].
First, we multiply 4.10 by

$$(r^2 + t^2)\alpha_t + 2Tr(\alpha_r) + 2t\alpha$$

and use the multiplier

$$\frac{1}{2}(r^2t^2)\beta_t + tr\beta_r + t\beta$$

on equation 4.11. These two expressions are then summed and integrated over $\mathbb{R}^3$ with the Lebesgue measure $dx = 4\pi r^2 dr$ to obtain the following identity

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^3}&((r^2 + t^2)(\frac{1}{2}\alpha_t^2 + \frac{1}{2}\alpha_r^2 + \frac{1}{4}\beta_t^2 + \frac{1}{4}\beta_r^2 + 2rt\alpha_t\alpha_r + rt\beta_t\beta_r \\ &+ 2t\alpha\alpha_t + t\beta\beta_t - \alpha^2 - \frac{1}{2}\beta^2)dx \\ &+ \int_{\mathbb{R}^3}\frac{r^2(r^2 + t^2)}{r^2 + \epsilon}(\alpha_t F_1 + \frac{1}{2}\beta_t F_2)dx \\ &+ \int_{\mathbb{R}^3}\frac{r^3 t}{r^2 + \epsilon}(2\alpha_r F_1 + \beta_r F_2)dx \\ &+ \int_{\mathbb{R}^3}\frac{r^2 t}{r^2 + \epsilon}(2\alpha F_1 + \beta F_2)dx = 0. \end{aligned} \tag{4.13}$$

[3]This calculation is tedious but it can be found in more detail in [11].

Using the expressions for F_1 and F_2, we get

$$\alpha_t F_1 + \frac{1}{2}\beta_t F_2 = \frac{\partial}{\partial_t}(\frac{\alpha^2}{r^2} - \frac{\alpha^3}{r} + \frac{\alpha^4}{4} + \frac{\alpha^2\beta^2}{2} - \frac{\alpha\beta^2}{r} + \frac{\beta^2}{2r^2} + \frac{1}{2}V_0(\beta^2))$$

and

$$2\alpha F_1 + \beta F_2 = \frac{4\alpha^2}{r^2} + \frac{2\beta^2}{r^2} - \frac{6\alpha^3}{r} + 2\alpha^4 + 4\alpha^2\beta^2 - \frac{6\alpha\beta^2}{r} + 2\beta^2 V_0'(\beta^2)$$

and similarly

$$\begin{aligned}2\alpha_r F_1 + \beta_r F_2 = &\frac{\partial}{\partial_r}(\frac{2\alpha^2}{r^2} - \frac{2\alpha^3}{r} + \frac{\alpha^4}{2} - \frac{2}{r}\alpha\beta^2 + \frac{\beta^2}{r^2} + \alpha^2\beta^2 + V_0(\beta^2)) \\ &+ \frac{4}{r^3}\alpha^2 - \frac{2}{r^2}\alpha^3 - \frac{2}{r^2}\alpha\beta^2 + \frac{2}{r^3}\beta^2.\end{aligned}$$

Now we can compute the third term of 4.13 using integration by parts to find

$$\begin{aligned}&\int_{\mathbb{R}^3} \frac{r^3 t}{r^3+\epsilon}(2\alpha_r F_1 + \beta_r F_2)dx \\ &= \int_{\mathbb{R}^3} \frac{r^2 t}{r^2+\epsilon}(-\frac{2\alpha^2}{r^2} + \frac{4\alpha^3}{r} - \frac{\beta^2}{r^2} + \frac{4\alpha\beta^2}{r} - \frac{3\alpha^4}{2} - 3\alpha^2\beta^2 - 3V_0(\beta^2))dx \\ &- 2\epsilon \int_{\mathbb{R}^3} \frac{r^2 t}{r^2+\epsilon}(\frac{2\alpha^2}{r^2} - \frac{2\alpha^3}{r} + \frac{\alpha^4}{2} - \frac{2}{r}\alpha\beta^2 + \frac{\beta^2}{r^2} + \alpha^2\beta^2 + V_0(\beta^2))dx\end{aligned}$$

Using the above computations, we now can obtain the first inversional identity

$$\frac{d}{dt}\int I dx = J_\epsilon = 2\epsilon t \int \frac{r^2}{(r^2+\epsilon)^2} Q dx, \tag{4.14}$$

with

$$Q = \frac{1}{2}\alpha^2(\alpha - \frac{2}{r})^2 + \beta^2(\alpha - \frac{1}{r})^2 + V_0(\beta^2),$$

and

$$\begin{aligned}I = &\frac{1}{2}(r^2+t^2)(\alpha_t^2 + \alpha_r^2 + \frac{1}{2}(\beta_t^2 + \beta_r^2)) \\ &+ rt(2\alpha_t\alpha_r + \beta_t\beta_r) + t(2\alpha\alpha_t + \beta\beta_t) - \alpha^2 - \frac{1}{2}\beta^2 \\ &+ \frac{r^2(r^2+t^2)}{2(r^2+\epsilon)} Q.\end{aligned} \tag{4.15}$$

Considering the case of YMS, we have an analogous estimate to 3.15 which is

$$\int_0^\infty \int \frac{r^3}{(r^2+\epsilon)^2} Q(\alpha_\epsilon, \beta_\epsilon) dx dt \le const.$$

The next step is estimating J_ϵ with the above inequality. For this, let $\delta = \epsilon^{3/4}$. Then

$$\begin{aligned}A := \int_0^T 2\epsilon t \int_{r>\delta} \frac{r^2}{(r^2+\epsilon)^2} Q dx dt &\le 2\epsilon^{1/4} T \int_0^T \int \frac{r^3}{(r^2+\epsilon)^2} Q dx dt \\ &\le 2\epsilon^{1/4} T c.\end{aligned}$$

We also have

$$B := \int_0^T 2\epsilon t \int_{r>\delta} \frac{r^2}{(r^2+\epsilon)^2} Q dx dt \leq \frac{1}{\epsilon} T^2 \sup_t \int_{r<\delta} r^2 Q dx$$

Furthermore, we have $Q \leq cr^{-3}$ for $r < 1$ by the definition of Q and using equation 3.7, therefore

$$B \leq c\epsilon^{-1} T^2 \int_0^\delta r^2 r^{-3} r^2 \leq c\epsilon^{-1} T^2 \delta^2 = cT^2 \sqrt{\epsilon}.$$

So, in conclusion, for any T we have

$$\int_0^T J_\epsilon(t) dt \to 0 \text{ as } \epsilon \to 0.$$

From 4.14, it now follows that

$$\lim_{\epsilon\to 0} \int I(\alpha_\epsilon, \beta_\epsilon)|_{t=T} dx = \lim_{\epsilon \to 0} \int I(\alpha_\epsilon, \beta_\epsilon)|_{t=0} dx. \tag{4.16}$$

As before, we express I as a sum of squares as follows

$$\begin{aligned} I(\alpha_\epsilon, \beta_\epsilon) = & \frac{1}{4}(r+t)^2(\partial_t \alpha_\epsilon + \frac{1}{r}\partial_r(r\alpha_\epsilon))^2 + \frac{1}{4}(r-t)^2(\partial_t\alpha_\epsilon - \frac{1}{r}\partial_r(r\alpha_\epsilon))^2 \\ & + \frac{1}{8}(r+t)^2(\partial_t\beta_\epsilon + \frac{1}{r}\partial_r(r\beta_\epsilon))^2 + \frac{1}{8}(r-t)^2(\partial_t\beta_\epsilon - \frac{1}{r}\partial_r(r\beta_\epsilon))^2 \\ & + \frac{r^2(r^2+t^2)}{2(r^2+\epsilon)}(\frac{1}{2}\alpha_\epsilon^2(\alpha_\epsilon - \frac{2}{r})^2 + \beta_\epsilon^2(\alpha_\epsilon - \frac{1}{r})^2 + V_0(\beta_\epsilon^2)) \\ & - \frac{1}{2r^2}\partial_r((t^2+r^2)(r\alpha_\epsilon^2 + \frac{1}{2}r\beta_\epsilon^2)) \end{aligned} \tag{4.17}$$

The last term integrates to zero. The right-hand side of 4.16 can be chosen to converge to $\int I(\alpha,\beta)dx$ at $t = 0$, since it only depends on the intial data. Fix now a time T, then the following expressions

$$(r \pm T)(\partial_t \alpha_\epsilon \pm \frac{1}{r}\partial_r(r\alpha_\epsilon)),$$
$$(r \pm T)(\partial_t \beta_\epsilon \pm \frac{1}{r}\partial_r(r\beta_\epsilon))$$

at time T converge in the sense of distributions on $\mathbb{R}^3$ to the same expressions with ϵ omitted similar as in section 2. Therefore 4.16 and 4.17 combined imply that each of them does indeed converge weekly in $L^2(\mathbb{R}^3)$ to the same limit. Furthermore, the last term in 4.17 converges a.e. which implies that

$$\int I(\alpha,\beta)|_{t=T} dx \leq \liminf_{\epsilon\to 0} \int I(\alpha_\epsilon, \beta_\epsilon)|_{t=T} dx = \int I(\alpha,\beta)|_{t=0} dx$$

which is exactly the integrated term of 4.5 and completes the proof. □

Corollary 4.3. *Under the assumptions of Theorem 4.2,* $\int(|E\omega|^2 + |H\omega|^2)dx = \mathcal{O}(t^{-2})$*.If* $A^0 = 0$*, then* $\int |A\omega|^2 dx = \mathcal{O}(log|t|^2)$ *as* $t \to \pm\infty$*.*

Proof. The first statements follows immediately, since $|E\omega|^2 + |H\omega|^2 \leq 4e_{ang}$. Now, we do a gauge transformation where $A^0 = 0$[4]. Then $E^k = \partial^0 A^k$, so that $E\omega = \partial^0 A\omega$ and

$$\frac{d}{dt}\int |A\omega|^2 dx = \int A\omega \cdot E\omega dx \leq c((l+t^2)^{-1} \int |A\omega|^2 dx)^{1/2}$$

with l being integrable over time. This proves the corollary. □

[4] The temporal gauge condition $A^0 = 0$ can always be assumed. See e.g [7].

5 Conservation laws

This section is dedicated to the discussion of conservation laws and the invariance of YMS-system under the conformal group[3].

We recall that the energy is obtained by multiplying 2.9 with H^1, 2.10 with H^2 and 2.11 with H^3. Next we do the same thing with 2.12-2.14 with E^j, $j = 1, 2, 3$ and sum the resulting six equations. Together with 2.3, we obtain the following identity

$$\partial^0 e_{YM} = \sum_k (\partial^k p^k_{YM} + g\psi^k \times \phi \cdot E^k), \tag{5.1}$$

where $e_{YM} = \frac{1}{2}(|E|^2 + |H|^2)$ and

$$\begin{aligned} p^1_{YM} &= H^2 \cdot E^3 - E^2 \cdot H^3, \\ P^2_{YM} &= H^3 \cdot E^1 - E^3 \cdot H^1, \\ p^3_{YM} &= H^1 \cdot E^2 - E^1 \cdot H^2. \end{aligned}$$

Now, we multiply 2.17 by ψ^0 and 2.18 by ψ^j, $(j = 0, 1, 2, 3)$ and sum the resulting four equations to get

$$\begin{aligned} \frac{1}{2}\partial^0 \sum_k |\psi^\mu|^2 &= -V'(\phi) \cdot \psi^0 + \sum_k \partial^k(\psi^0 \cdot \psi^k) + g\sum_k E^k \times \phi \cdot \psi^k \\ &= -\partial^0 V(\phi) + \sum_k (\partial^k(\psi^0 \cdot \psi^k) + gE^k \times \phi \cdot \psi^k) \end{aligned} \tag{5.2}$$

Then, we obtain the energy conservation law

$$\partial^0 e = \sum_k \partial^k p^k \tag{5.3}$$

by adding equations 5.1 and 5.2, where

$$e = \frac{1}{2}(|E|^2 + |H|^2 + \sum_\mu |\psi^\mu|^2) + V(\phi) \tag{5.4}$$

and

$$p^k = p^k_{YM} + \psi^0 \cdot \psi^k, \quad (k = 1, 2, 3). \tag{5.5}$$

Our next step is the computation of the momenta. Consider

$$p^1 = H^2 \cdot E^3 - H^3 \cdot E^2 + \psi^0 \cdot \psi^1.$$

Using 2.3 we find

$$\partial^0 p^1 = I + II,$$

with

$$\begin{aligned} I &= D^0 H^2 \cdot E^3 + H^2 \cdot D^0 E^3 - D^0 H^3 \cdot E^2 - H^3 \cdot D^0 E^2, \\ II &= D^0 \psi^0 \cdot \psi^1 + \psi^0 \cdot D^0 \psi^1. \end{aligned}$$

Together with equations 2.9-2.14 and 2.17-2.18, we obtain

$$\begin{aligned}
I &= (D^1E^3 - D^3E^1)\cdot E^3 + H^2\cdot(D^1H^2 - D^2H^1 + g\psi^3\times\phi) \\
&\quad - (D^2E^1 - D^1E^2)\cdot E^2 - H^3\cdot(D^3H^1 - D^1H^3 + g\psi^2\times\phi), \\
II &= (\sum_k D^k\psi^k - V'(\phi))\cdot\psi^1 + \psi^0\cdot(D^1\psi^0 + gE^1\times\phi).
\end{aligned}$$

Next, we use 2.3 to rewrite the previous equations as

$$\begin{aligned}
I &= \frac{1}{2}\partial^1(|E^2|^2 + |E^3|^2 + |H^2|^2 + |H^3|^2) \\
&\quad - \partial^2(E^1\cdot E^2 + H^1\cdot H^2) - \partial^3(E^1\cdot E^3 + H^1\cdot H^3) \\
&\quad + E^1\cdot(D^2E^2 + D^3E^3) + (D^2H^2 + D^3H^3)\cdot H^1 \\
&\quad + gH^2\cdot\psi^3\times\phi - gH^3\cdot\psi^2\times\phi; \\
II &= \frac{1}{2}\partial^1(|\psi^0|^2 + |\psi^1|^2 - 2V(\phi)) + \partial^2(\psi^1\cdot\psi^2) + \partial^3(\psi^1\cdot\psi^3) \\
&\quad - \psi^2\cdot D^2\psi^1 - \psi^3\cdot D^3\psi^1 + g\psi^0\cdot E^1\times\phi.
\end{aligned}$$

Using 2.16, the third line of I becomes

$$-\frac{1}{2}\partial^1(|E^1|^2 + |H|^2) + gE^1\cdot\psi^0\times\phi.$$

Using 2.19-2.21, we rewrite the second line of II as

$$-\frac{1}{2}\partial^1(|\psi^2|^2 + |\psi^3|^2) - g\psi^2\cdot H^3\times\phi + g\psi^3\cdot H^2\times\phi + g\psi^0\cdot E^1\times\phi.$$

The cubic terms vanish and for $j = 1$ we obtain the Momentum Conservation Law

$$\partial^0 p^j = \partial^j f + \sum_k \partial^k q^{jk}, \tag{5.6}$$

with

$$f = e - |\Psi|^2 - 2V(\phi),$$

and

$$q^{jk} = -E^j\cdot E^k - H^j\cdot H^k + \psi^j\cdot\psi^k.$$

The equations for $j = 2, 3$ are obtained in a similar way.
The other eleven identities are derived from 5.3 and 5.6 and are as follows:

$$\partial^0(x_j e + tp^j) = \partial^j(tf) + \sum_k \partial^k(x_j p^k + tq^{jk}), (j = 1, 2, 3), \tag{5.7}$$

$$\partial^0(x_2p^1 - x_1p^2) = \partial^1(x_2 f) - \partial^2(x_1 f) + \sum_k \partial^k(x_2 q^{1k} - x_1 q^{2k}). \tag{5.8}$$

By cyclic permutation, two similar laws are obtained.

$$\partial^0(te + \sum_k x_k p^k + \psi^0 \cdot \phi) + V'(\phi) \cdot \phi - 4V(\phi)$$
$$= \sum_k \partial^k (tp^k + x_k f + \sum_j x_j q^{jk} + \psi^k \cdot \phi), \tag{5.9}$$
$$\partial^0((t^2 + r^2)e + 2t \sum_k x_k p^k + 2t\psi^0 \cdot \phi - |\phi|^2) + 2t(V'(\phi) \cdot \phi - 4V(\phi))$$
$$= \sum_k \partial^k ((t^2 + r^2)p^k + 2t(x_k f + \sum_j x_j q^{jk} + \psi^k \cdot \phi)), \tag{5.10}$$
$$\partial^0(tx_j e + \frac{1}{2}(t^2 - r^2)p^j + x_j \sum_k x_k p^k + x_j \psi^0 \cdot \phi) + x_j(\phi \cdot V'(\phi) - 4V(\phi))$$
$$= \partial^j(\frac{t^2 - r^2}{2} f) + \sum_k \partial^k (tx_j e - x_j \phi \cdot \psi^k + \frac{t^2 - r^2}{2} q^{jk} + x_j x_k f + x_j \sum_m x_m q^{mk}) \tag{5.11}$$

$(j = 1, 2, 3)$.

Now, we have the tools to work with the main Theorem of this chapter.

Theorem 5.1. *5.1 If $V(\phi) = c|\phi|^4$, the system is invariant under the conformal group.*

Note, that $4V(\phi) = \phi \cdot V'(\phi)$, so all 15 identities are conservation laws. In fact, they are exactly those conservation laws that follow from the invariance via Noether's Theorem[6]. We will not prove the invariance directly as it is a very tedious computation.
We present a detailed derivation of the first inversional law 5.10 as an example now.
First, multiply 5.3 by $r^2 + t^2$ and 5.6 by $2tx_j$. This gives the sum of four equations which can be written as follows

$$\partial^0((r^2 + t^2)e + 2t \sum_j x_j p^j) =$$
$$= \sum_k \partial^k ((r^2 + t^2)p^k + 2tx_k f + 2t \sum_j x_j q^{jk}) + 2t(e - 3f - \sum_k q^{kk}). \tag{5.12}$$

The last expression reads

$$e - 3f - \sum_k q^{kk} = -2e + |E|^2 + |H|^2 + 2|\Psi|^2 + 6V(\phi) = -|\psi^0|^2 + |\Psi|^2 + 4V(\phi).$$

Now, we multiply 2.17 by ϕ and use 2.3 to get

$$\partial^0(\psi^0 \cdot \phi) - \psi^0 \cdot D^0\phi - \sum_k (\partial^k(\psi^k \cdot \phi) - \psi^k \cdot D^k\phi) = -\phi \cdot V'(\phi).$$

This can be written as follows by using 2.7

$$\partial^0(\psi^0 \cdot \phi) + V'(\phi) \cdot \phi = |\psi^0|^2 - |\Psi|^2 + \sum_k \partial^k(\psi^k \cdot \phi). \tag{5.13}$$

Multiplying this by $2t$, we get

$$\partial^0((2t\psi^0 - \phi) \cdot \phi) + 2t\phi \cdot V'(\phi) = 2t(|\psi^0|^2 - |\Psi|^2) + \sum_k \partial^k(2t\psi^k \cdot \phi).$$

5.10 is then obtained by adding this result to 5.12.

6 Asymptotics for the zero mass case

The following sections will be dealing with the asymptotic behaviour of the YMS-system, i.e., the case where the scalar field has zero mass and the other one being the possibility of the scalar field having positive mass[3].
First, we start with a Theorem which gives us information about the behaviour of the energy density. After proving it, we will state two corollaries which give extra information about the growth and the non-existence of finite energy solutions of a specific form.

Theorem 6.1. *Assume that V satisfies the inequalities*

$$0 \leq 4V(s) \leq s \cdot V'(s)$$

Let $R > 0$ and $0 < \epsilon \leq 1$. Then, as $t \to \infty$

$$\int_{|x|<R+(1-\epsilon)t} edx = \mathcal{O}(t^{-2}),$$

where e is the energy density 5.4, provided that $\int r^2 edx < \infty$ at all times.

Proof. We integrate 5.10 first over all space. Assume that the solution is smooth and satisfies $\int r^2 edx < \infty$ at all times. This lets the right side vanish and we obtain

$$\int ((r^2 + t^2)e + 2t\sum_k x_k p^k + 2t\phi \cdot \psi^0 - \phi \cdot \phi)dx \leq const = C.$$

Next, we split this integrand into two parts, one being the the pure Yang-Mills field I_{YM} and the other one the scalar field I_s. Let $\omega = \dfrac{x}{r}$ and take unit vectors α, β such that α, β, ω form an orthonormal basis for $\mathbb{R}^3$ with $\alpha \times \beta = \omega$. For orthonormal basis vectors, we have

$$|E|^2 = |E\alpha|^2 + |E\beta|^2 + |E\omega|^2$$

and a similar one with E replaced by H. Then we have the following equation[5]

$$e_Y \pm \sum_k \omega_k p^k_{YM} = \frac{1}{2}(|E\omega|^2 + |H\omega|^2 + |E\alpha \mp H\beta|^2 + |E\beta \pm H\alpha|^2). \tag{6.1}$$

In particular, $|\sum_k \omega_k p^k_{YM}| \leq e_{YM}$, therefore

$$I_{YM} \geq (t-r)^2 e_{YM}. \tag{6.2}$$

The next step is expressing I_s as a sum of squares. Let $\chi^k = \psi^k + \dfrac{x_k}{r^2}\phi$ and define Ξ to be the matrix with columns χ^1, χ^2, χ^3. By the spatial components of 2.7, we get

$$|\Xi|^2 = |\Psi|^2 + \frac{1}{r^2}\partial^r(r|\phi|^2), \tag{6.3}$$

where $r = |x|$ and $r\partial^r = \sum_k x_k \partial^k$.

[5] A derivation can be found in [4].

Thus we find

$$
\begin{aligned}
I_s &= \frac{t^2+r^2}{2}(|\psi^0|^2+|\Psi|^2+2V(\phi))+2t\psi^0\cdot(\Psi x+\phi)-|\phi|^2 \\
&= I_s^* - \frac{1}{2r^2}\partial^r((t^2+r^2)r|\phi|^2),
\end{aligned} \tag{6.4}
$$

with

$$
\begin{aligned}
I_s^* &= \frac{t^2+r^2}{2}(|\psi^0|^2+|\Xi|^2+2V(\phi))+2t\psi^0\cdot\Xi x \\
&\geq \frac{(t-r)^2}{2}(|\psi^0|^2+|\Xi|^2+2V(\phi))\geq 0.
\end{aligned} \tag{6.5}
$$

This means that for any subset B of space we have

$$\int_B (I_{YM}+I_s^*)dx \leq \int (I_{YM}+I_s)dx \leq C.$$

By choosing $B=\{x:|x|<R+(1-\epsilon)t\}$, we obtain from 6.2 and 6.5

$$(\epsilon t-R)^2\int_B (e_{YM}+\frac{1}{2}|\psi^0|^2+\frac{1}{2}|\Xi|^2+V(\phi))dx \leq C$$

for $t>R\epsilon^{-1}$.Using 6.3 again, we find

$$\int_B |\Psi|^2 dx \leq \int_B |\Xi|^2 dx \leq 2C(\epsilon t-R)^{-2}$$

for $t>R\epsilon^{-1}$. □

The method on how to prove the following two corollaries can be found in [4].

Corollary 6.2. *Assume that any finite energy solution can be approximated by cut-off solutions in the energy norm, uniformly in time. Then, for any finite energy solution and for each $R>0$ and $0<\epsilon\leq 1$, we have*

$$\lim_{t\to\infty}\int_{|x|<R+(1-\epsilon)t} e dx = 0.$$

Corollary 6.3. *Under the same assumptions, there is no finite energy solution of the form*

$$E=E(x-ct),\quad H=H(x-bt),\quad \phi=\phi(x-at),$$

where a,b,c are constant vectors of norm less than one, except for the trivial solution

$$E=H=0,\quad \phi=const,\quad V(\phi)=0.$$

The method of how to prove these corollaries can be found in [4].
Remark: By 6.1 and 6.4, the integral over *all* space of certain components of e is $\mathcal{O}(t^{-2})$.

7 Estimates for the case the scalar field has positive mass

The following Theorem shows us that certain components of the fields are square integrable over light cones[3].

Theorem 7.1. *For any finite energy solution,*

$$\int_{|x|=t} (2V(\phi) + |\psi^0 + \Psi\omega|^2 + |\Psi\alpha|^2 + |\Psi\beta|^2 \\ + |E\omega|^2 + |H\omega|^2 + |E\alpha - H\beta|^2 + |E\beta + H\alpha|^2) dS \le const,$$

where dS is the usual surface measure on $\{|x| = t\}$. If $V = 0$, each term is positive and therefore integrable on the cone.

Proof. We integrate 5.3 over the four dimensional region $\{|x| < t < T\}$ and then let $T \to \infty$ to obtain

$$2\int_{|x|=t} (e + \sum_k \omega_k p^k) dS \le 2\sqrt{2} \int e dx = const,$$

with $\omega_k = \dfrac{x_k}{r}$. The YM terms in the integrand are written as in 6.1. The other terms are $2V(\phi)$ and

$$|\psi^0|^2 + |\Psi|^2 + 2\psi^0 \cdot \Psi\omega = |\psi^0 + \psi\omega|^2 + |\Psi\alpha|^2 + |\psi\beta|^2.$$

This finishes the proof. □

Now, with that result, we can turn our focus again to the study of the asymptotic behaviour in case the potential $V(\phi)$ includes a mass term. Typically, one works with

$$V(\phi) = m_0^2|\phi|^2 + c|\phi|^{p+1} (m > 0, c > 0, p > 1).$$

For this reason, assume $\phi \cdot V'(\phi) \ge 2V(\phi) \ge 0$. For the next steps, we will use the summation convention. Multiply first 5.6 by a function $2l_j(x)$ $(j = 1, 2, 3)$ and sum over j to obtain

$$2\partial^0(l_j p^j) + 2(\partial^j l_j)f + 2\partial^k l_j \cdot q^{jk} = 2\partial^j(l_j f) + 2\partial^k(l_j q^{jk}).$$

Let $m = \partial^j l_j$. Multiply 5.13 by m to obtain

$$\begin{aligned} &\partial^0(m\psi^0 \cdot \phi) + m(|\Psi|^2 - |\psi^0|^2 + V'(\phi) \cdot \phi) \\ &\quad = \partial^k(m\psi^k \cdot \phi) - (\partial^k m)\psi^k \cdot \phi \\ &\quad = \partial^k(m\psi^k \cdot \phi - \frac{1}{2}(\partial^k m)|\phi|^2) + \frac{1}{2}(\partial^k\partial^k m)|\phi|^2. \end{aligned}$$

By summing these two equations, we find

$$\begin{aligned} &\partial^0(2l_j p^j + m\psi^0 \cdot \phi) \\ &\quad + 2\partial^k l_j \cdot q^{jk} + m(2f + |\Psi|^2 - |\psi^0|^2 + V'(\phi) \cdot \phi) - \frac{1}{2}(\partial^k\partial^k m)|\phi|^2 \\ &\quad = \partial^k(2l_k f + 2l_j q^{jk} + m\psi^k \cdot \phi - \frac{1}{2}(\partial^k m)|\phi|^2). \end{aligned} \tag{7.1}$$

The last equation 7.1 can be written as $\partial^0 X + Z = \partial^k Y^k$. Then, we evaluate Z by substituting the expressions for f and q^{jk}:

$$Z =(m\delta_{jk} - 2\partial^k l_j)(E^j \cdot E^k + H^j \cdot H^k) + 2(\partial^k l_j)\psi^j \cdot \psi^k \\ + m(\phi \cdot V'(\phi) - 2V(\phi)) - \frac{1}{2}(\partial^k\partial^k m)|\phi|^2.$$

In 7.1 we have

$$X = 2l_j p^j + m\psi^0 \cdot \phi = 2l_j p^j_{YM} + 2\psi^0 \cdot (l_j\psi^j + \frac{m}{2}\phi).$$

Therefore,

$$|X| \leq 2|l_j p^j_{YM}| + |\psi^0|^2 + |l_j\psi^j + \frac{m}{2}\phi|^2. \tag{7.2}$$

The last term can be written as

$$|l_j\psi^j + \frac{m}{2}\phi|^2 = |l_j\psi^j|^2 + \frac{1}{2}ml_j\partial^j(|\phi|^2) + \frac{m^2}{4}|\phi|^2 \\ = |l_j\psi^j|^2 + \partial^j(\frac{1}{2}l_j m|\phi|^2) - \frac{1}{2}(l_j\partial^j m + \frac{1}{2}m^2)|\phi|^2. \tag{7.3}$$

Choose $l_j(x) = \frac{x_j}{r}$. Then

$$\partial^k l_j = \delta_{jk}/r - x_j x_k/r^3 \text{ and } m = \partial^j l_j = \frac{2}{r}.$$

Therefore, $m\delta_{jk} - 2\partial^k l_j = 2x_j x_k/r^3$,

$$l_j\partial^j m + \frac{1}{2}m^2 = 0 \text{ and } \partial^k\partial^k m = 0 \text{ for } x \neq 0.$$

From 7.2 and 7.3, it follows that

$$|X| \leq 2e_{YM} + |\psi^0|^2 + |\Psi|^2 + \partial^j(\frac{1}{2}l_j m|\phi|^2).$$

This means that $\int |X| dx$ is bounded by twice the energy. Now, we integrate 7.1 over the exterior of a small sphere $\{|x| > \epsilon\}$ and let $\epsilon \to 0$. Then, on the right side of the resulting equation, the terms in 7.1 of the form $\partial^k Y^k$ drop out except for the last term which yields

$$-\frac{1}{2}\int_{|x|>\epsilon} \partial^k((\partial^k m)|\phi|^2)dx = \int_{|x|>\epsilon} \partial^k(\frac{x_k}{r^3}|\phi|^2)dx \\ = -\frac{1}{\epsilon^2}\int_{|x|=\epsilon} |\phi(x,t)|^2 dS_x \to -4\pi|\phi(0,t)|^2 \leq 0$$

as $\epsilon \to 0$. On the left side, we have the integral of

$$Z = \frac{2}{r}(|E_r|^2 + |H_r|^2 + |\Psi|^2 - |\Psi_r|^2 + V'(\phi)\cdot\phi - 2V(\phi)),$$

with

$$E_r = \frac{1}{r}Ex, \quad H_r = \frac{1}{r}Hx, \quad \Psi_r = \frac{1}{r}\Psi x.$$

Thus, we have proven that

$$\int_{-\infty}^{\infty}\int Z dx dt + 4\pi\int_{-\infty}^{\infty} |\phi(0,t)^2 dt \leq 4\int e dx = 4e_0. \tag{7.4}$$

Theorem 7.2. *Assume that $\phi \cdot V'(\phi) \geq 2V(\phi) \geq 0$. Consider a smooth solution of finite energy. Then*

$$\int\int (|E_r|^2 + |H_r|^2)\frac{1}{r}dxdt < \infty, \tag{7.5}$$

$$\int\int (|\Psi|^2 - |\Psi_r|^2)\frac{1}{r}dxdt < \infty, \tag{7.6}$$

$$\int\int (\phi \cdot V'(\phi) - 2V(\phi))\frac{1}{r}dxdt < \infty. \tag{7.7}$$

These integrals, and the ones below, are taken over all space and time. Furthermore, if $\delta > 0$,

$$\int\int (|E|^2 + |H|^2 + |\Psi|^2)\frac{1}{(1+r)^{1+\delta}}dxdt < \infty \tag{7.8}$$

$$\int\int |\phi|^2 \frac{1}{(1+r)^{3+\delta}}dxdt < \infty. \tag{7.9}$$

Proof. The first three estimates follow immediately from the integrability of Z in 7.4. Using the second term in 7.4 and translating the origin $x = 0$ to any other point, we obtain

$$\pi \int_{-\infty}^{\infty} |\phi(x,t)|^2 dt \leq e_0 \text{ for all x }.$$

Hence,

$$\int (\int |\phi(x,t)|^2 dt)(1+r)^{-3-\delta}dx \leq \pi^{-1}e_0 \int (1+r)^{-3-\delta}dx < \infty.$$

This proves 7.9. To prove 7.8, we need to choose this more general form for our multiplier

$$l_j(x) = \frac{x_j}{r}\zeta(r).$$

Then

$$\partial^k l_j = \frac{\zeta}{r}\delta_{jk} - (\frac{\zeta}{r} - \zeta')x_j x_k/r^2,$$
$$m = \partial^j l_j = \frac{2}{r}\zeta + \zeta',$$
$$m\delta_{jk} - 2\partial^k l_j = \zeta'\delta_{jk} + 2(\frac{\zeta}{r} - \zeta')x_j x_k/r^2.$$

The weight function ζ is chosen to satisfy the following constraints (*):

$$(i)\ \ \zeta \text{ bounded, } \frac{\zeta}{r} \geq \zeta' \geq 0$$
$$(ii)\ \ \partial^k\partial^k m = \frac{4}{r}\zeta'' + \zeta'' \leq 0$$
$$(iii)\ \ l_j\partial^j m + \frac{1}{2}m^2 = \frac{4}{r}\zeta\zeta' + \frac{1}{2}(\zeta')^2 + \zeta\zeta'' \geq 0.$$

Therefore, by 7.2 and 7.3 it follows that $\int |X|dx$ is bounded by a constant multiple of e_0. The general expression for Z leads to the following inequality

$$Z \geq \zeta'|E|^2 + 2(\frac{\zeta}{r} - \zeta')|E_r|^2 + \zeta'|H|^2 + 2(\frac{\zeta}{r} - \zeta')|H_r|^2 + 2\frac{\zeta}{r}(|\Psi|^2 - |\Psi_r|^2) + 2\zeta'|\Psi_r|^2. \tag{7.10}$$

All of these six terms are non-negative and, therefore, integrable over space-time.
Next, choose

$$\zeta(r) = 2 - (r+1)^{-\delta},$$

with $\delta > 0$. We check the constraints (*). Through explicit calculation, we find $\zeta' = \delta(r+1)^{-1-\delta}$ and

$$\begin{aligned}\zeta/r - \zeta' =& r^{-1}(r+1)^{-\delta}(2(r+1)^{\delta} - 1 + \delta r(r+1)^{-1}) \\ &> r^{-1}(r+1)^{-\delta}(1-\delta) > 0.\end{aligned}$$

This means that 7.8 follows from 7.10 which means the only thing left to do is the verification of (*). Indeed,

$$4\zeta''/r + \zeta''' = -\delta(\delta+1)r^{-1}(r+1)^{-3-\delta}((2-\delta)r+4) < 0$$

and

$$\begin{aligned}4r^{-1}\zeta\zeta' + \frac{1}{2}(\zeta')^2 + \zeta\zeta'' &\geq (4r^{-1}\zeta' + \zeta'')\zeta \\ &= \zeta\delta r^{-1}(r+1)^{-2-\delta}(4 + (3-\delta)r) \geq 0.\end{aligned}$$

□

Corollary 7.3. *If $\phi \cdot V'(\phi) \geq 2V(\phi) \geq 0$, there are no "classical lumps" of finite energy. That is, if $E(x), H(x), \phi(x)$ is a solution which is independent of time and has finite energy, then $E = H = \phi = 0$.*

Going back to 7.8, we are missing an estimate on $|\psi^0|^2$. To obtain one, we need a slightly stronger assumption on V.

Theorem 7.4. *Assume that V is of the form*

$$V(\phi) = \frac{1}{2}m_0^2|\phi|^2 + W(\phi)$$

where

$$0 \leq \alpha W(\phi) \leq \phi \cdot W'(\phi).$$

We assume m_0, δ, and R are positive constants and $\alpha > 2$. Then

$$\int\int W(\phi)\frac{1}{r}dxdt < \infty, \tag{7.11}$$

$$\int\int |\psi^0|^2 \frac{1}{(r+1)^{3+\delta}}dxdt < \infty, \tag{7.12}$$

$$\int_{-\infty}^{\infty}\int_{|x|<R} e\ dxdt < \infty, \tag{7.13}$$

$$\int_{|x|<R} e\ dx \to 0 \text{ as } |t| \to \infty. \tag{7.14}$$

Proof. We have

$$\phi \cdot V'(\phi) - 2V(\phi) = \phi \cdot W'(\phi) - 2W(\phi) \geq (\alpha - 2)W(\phi).$$

Therefore 7.7 implies 7.11. Now, we multiply 5.13 by $(r+1)^{-3-\delta} = \xi(r)$. Hence,

$$\begin{aligned}&\partial^0(\xi\psi^0\cdot\phi) + \xi(m_0^2|\phi|^2 + \phi\cdot W'(\phi))\\ &\quad = \xi(|\psi^0|^2 - |\Psi|^2) + \partial^k(\xi\psi^k\cdot\phi) + (3+\delta)(r+1)^{-4-\delta}r^{-1}x_k\psi^k\cdot\phi. \qquad (7.15)\end{aligned}$$

The last term is bounded by a constant times

$$(r+1)^{-1-\delta}|\Psi|^2 + (r+1)^{-3-\delta}|\phi|^2,$$

which is integrable over space-time. As for the first term in 7.15,

$$\int \xi\psi^0\cdot\phi dx \le \frac{1}{2}\int(|\psi^0|^2 + |\phi|^2)dx \le (1+m_0^{-2})e_0$$

due to the mass term in the energy. The terms $\xi|\Psi|^2, \xi|\phi|^2$ and $\xi\phi\cdot W'(\phi)$ are also integrable over space time since

$$\phi\cdot W'(\phi) = (\phi\cdot V'(\phi) - 2V(\phi)) + 2W(\phi).$$

This means that 7.15 implies that $\xi|\psi^0|^2$ is also integrable. This is 7.12. Since

$$2e = |E|^2 + |H|^2 + |\Psi|^2 + |\psi^0|^2 + m_0^2|\phi|^2 + 2W(\phi),$$

7.13 follows from 7.8,7.9,7.11 and 7.12 as soon as we replace the factor $(r+1)$ in the denominators by the constant $R+1$. Now we derive 7.14 from 7.13 by using a Morawetz method[8]. Let

$$f(t) = \int_R^{R+1}\int_{|x|<\rho} e\ dxd\rho \ge \int_{|x|<R} e\ dx.$$

We will show that the derivative $f'(t)$ is bounded ($t\in\mathbb{R}$). Since $f(t)$ is integrable by 7.13, we only need to show that its derivative is bounded. Now

$$\begin{aligned}f'(t) &= \int_R^{R+1}\int_{|x|<\rho}\partial^0 e\ dxd\rho\\ &= \int_R^{R+1}\int_{|x|=\rho}(\sum_k \frac{x_k}{r}p^k)dS_x d\rho \quad by(4.3)\\ &= \int_{R<|x|<R+1}(\sum_k \frac{x_k}{r}p^k)dx.\end{aligned}$$

Thus, $|f'(t)| \le \int_{R<|x|<R+1} e\ dx \le \int edx = e_0$ and the Theorem is proven. □

Finally, we establish the square integrability of the potentials A^μ.
First assume only that $V(\phi)\ge 0$. It follows, as in [4], that

$$\begin{aligned}(\sum_\mu\int|A^\mu(x,t)|^2dx)^{1/2} &\le \sum_\mu\int(|A^\mu(x,0)|^2dx)^{1/2}\\ &\quad + \int_0^t(\int|E(x,s)|^2dx)^{1/2}ds\end{aligned}$$

in, say, the Lorentz gauge. This comes from multiplying 2.4 with $\nu=0$ and $\mu=k$ by A^k, summing over $k=1,2,3$, and integrating. In particular,

$$\int\sum_\mu|A^\mu(x,t)|^2dx = \mathcal{O}(1+t^2)$$

for all t. We can also estimate $\int |\phi|^2 dx$, even though $m = 0$. Next, we integrate

$$\phi \cdot \psi^0 = \phi \cdot D^0 \phi = \frac{1}{2} \partial^0 (|\phi|^2) \tag{7.16}$$

to obtain

$$\partial^0 \int |\phi|^2 dx \leq 2 (\int |\phi|^2 dx)^{1/2} (\int |\psi^0|^2 dx)^{1/2}.$$

Therefore

$$\begin{aligned} (\int |\phi(x,t)|^2 dx)^{1/2} \leq & (\int |\phi(x,0)|^2 dx)^{1/2} \\ & + \int_0^t (\int |\psi^0(x,s)|^2 dx)^{1/2} ds. \end{aligned}$$

Since $V(\phi) \geq 0$,

$$\int |\phi(x,t)|^2 dx = \mathcal{O}(1 + t^2) \text{ for all } t.$$

We can find stronger bounds on $\int |\phi|^2 dx$ if we assume that $\varphi \cdot V'(\phi) \geq 4V(\phi) \geq 0$ and that $\int r^2 e dx < \infty$. Supposing that, we rewrite 5.10 in the following form

$$\begin{aligned} \frac{1}{2} \int (r^2 + t^2)(|E|^2 + |H|^2) dx + 2t \int r \sum_k \omega_k p^k_{YM} dx + \int (t^2 + r^2) V(\phi) dx \\ + \frac{1}{2} r^2 (|\Psi|^2 - |\Psi_r|^2) dx + t \int \varphi \cdot \psi^0 dx \\ + \frac{1}{2} \sum_k \int |t\psi^k + x_k \psi^0|^2 dx + \frac{1}{2} \int |\Psi x + t\psi^0 + \phi|^2 dx \leq C. \end{aligned}$$

It follows that

$$t \int \phi \cdot \psi^0 dx \leq C$$

so that 7.16 yields

$$\frac{1}{2} \int \partial^0 (|\phi|^2) dx \leq \frac{C}{t}.$$

for $t \geq 1$, say.
Hence, $\int |\phi(x,t)|^2 dx = \mathcal{O}(\log(t))$ as $t \to \infty$.

References

[1] Alexander Dynin. "Mathematical quantum Yang–Mills theory revisited". In: *Russian Journal of Mathematical Physics* (2017).

[2] Lawrence C. Evans. *Partial Differential Equations: Second Edition,* American Mathematical Society, 2010.

[3] Robert T. Glassey and Walter A Strauss. "Decay of a Yang-Mills field coupled to a scalar field". In: *Comm. Math. Phys. 67, no. 1, 51-67* (1979).

[4] Robert T. Glassey and Walter A Strauss. "Decay of classical Yang-Mills fields". In: *Comm. Math. Phys. 65, no. 1, 1-13* (1979).

[5] Robert T. Glassey and Walter A Strauss. "Some global solutions of the Yang-Mills equations in Minkowski space". In: *Comm. Math. Phys. 81 , no. 2, 171-187* (1981).

[6] Herbert Goldstein, C. P. Poole, and J. L. Safko. *Classical Mechanics (3rd ed.),* Pearson, 2001.

[7] David Griffiths. *Introduction to Elementary Particles (2nd ed.),* Wiley-VCH, 2008.

[8] Cathleen S. Morawetz. "Time decay for the nonlinear Klein-Gordon equation". In: *Proc. R. Soc. A306, 291-296* (1968).

[9] M. Reed and B. Simon. *Methods of modern mathematical physics (Vol. II),* New York: Academic Press, 1975.

[10] I. E. Segal. "Nonlinear semigroups". In: *Ann. of Math. 78* (1963).

[11] W. Strauss. "Nonlinear invariant wave equations". In: *Lecture notes in Physics, Vol.73, pp. 197-249 (Erice, 1977). Berlin, Heidelberg, New York: Springer* (1978).

[12] W. Strauss. "On weak solutions of semi-linear hyperbolic equtions". In: *Anais. Acad. Bras. Cienc. 42, 645-651* (1970).