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Combinatorics and Definability on the Real Line and the Higher Continuum

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Abstract

The central topic of this thesis concerns the definability of various types of combinatorial families of reals. Among these families, we study in detail the definability of towers and of ultrafilters at the low projective levels. We provide positive definability results in the constructible universe L and show how they fail in other models such as forcing extensions of L or Solovay's model, in which every set of reals is Lebesgue measurable. Among other things, we show that, although coanalytic bases for P- and Q-points exist in L, a base for a Ramsey ultrafilter can never be coanalytic. In another chapter, we prove that after forcing over L with countable support iterations of a large class of posets, including e.g Sacks forcing, most types of "maximal" families of reals have $\mathbf{\Delta}^1_2$ witnesses. This can be used to solve an open problem of Brendle, Fischer and Khomskii.

In a second part, we study the generalized pseudointersection and tower numbers $\mathfrak{p}(\kappa)$ and $\mathfrak{t}(\kappa)$ at uncountable regular cardinals κ and provide results towards a possible generalization of Malliaris' and Shelah's proof that $\mathfrak{p} = \mathfrak{t}$. We also give a natural way to force $\mathfrak{p}(\kappa) < \mathfrak{b}(\kappa)$.

Zusammenfassung

Das zentrale Thema dieser Arbeit betrifft die Definierbarkeit verschiedener Typen kombinatorischer Familien reeller Zahlen. Unter diesen Familien untersuchen wir im Detail die Definierbarkeit von Türmen und Ultrafiltern bezüglich niedrig projektiver Komplexität. Wir liefern positive Definierbarkeitsergebnisse im konstruierbaren Universum L und zeigen, wie sie in anderen Modellen versagen, z.B. in Forcingerweiterungen von L oder im Solovay-Modell, in dem jede Menge reeller Zahlen Lebesgue-messbar ist. Unter anderem zeigen wir, dass, obwohl koanalytische Basen für P- und Q-Punkte in L existieren, eine Basis für einen Ramsey-Ultrafilter niemals koanalytisch sein kann. In einem anderen Kapitel beweisen wir, dass nach dem Forcen über L mit einer abzählbar gestützten Iteration von partiellen Ordnungen einer großen Klasse, einschließlich z.B. dem Sackforcing, die meisten Typen von "maximalen" Familien $\mathbf{\Delta}^1_2$-Definitionen haben. Dies kann zur Lösung eines offenen Problems von Brendle, Fischer und Khomskii verwendet werden.

In einem zweiten Teil untersuchen wir die verallgemeinerten Pseudodurchschnitts- und Turmzahlen $\mathfrak{p}(\kappa)$ und $\mathfrak{t}(\kappa)$ auf überabzählbaren regulären Kardinalzahlen κ und liefern Ergebnisse im Hinblick auf eine mögliche Verallgemeinerung von Malliaris' und Shelahs Beweis, dass $\mathfrak{p} = \mathfrak{t}$. Wir geben außerdem eine natürliche Weise an, $\mathfrak{p}(\kappa) < \mathfrak{b}(\kappa)$ zu forcen.

Contents

CHAPTER 1

Introduction

This thesis can be divided into two thematic parts. The first part and main body of this thesis consists of Chapters 2-4 and deals with the definability of special families of reals, prominent in topology, algebra, combinatorics or measure theory, that can typically only be obtained by use of the Axiom of Choice. The second part, Chapter 5, studies the pseudointersection and tower numbers in the higher Baire space κ^κ for κ a regular uncountable cardinal. The research leading to this thesis resulted in four articles each corresponding to a section. Two of them, [16], joint with V. Fischer, and [15], joint with V. Fischer, D.C. Montoya and D.T. Soukup, are accepted and the other two, [47] and [49], are currently under review.

We will provide first a historical introduction to the subject up until the current state of the art. For readers that are not familiar with the required prerequisites, we give a survey over some of the main notions in descriptive set theory and forcing necessary to make sense of the results.

1.1 Historical overview

The reals are among the most fundamental and most important objects in mathematics. Coming after the integers and the rationals, they are the first objects of an infinite nature. The importance of their existence for proving strong theorems, even about just the finite realm, is indisputable. Since its beginnings, the reals were set theory's main object of study. One of the earliest results is *Cantor's Theorem* (see [10]) saying that the set $\mathbb{R}$ of all real numbers cannot be put in a one to one correspondence with the naturals. This did not only have a tremendous amount of implications, but it also lead to the revolutionary concept of infinite cardinality. Cantor was effectively showing that only a few very rudimentary facts about sets inevitably lead to different sizes of infinity. It was the first time in history that the concept of actual infinity, which surely has been subject to human thought for millennia, could be dealt with in a completely formal mathematical setting. The cardinality[1] of a set A, which in the infinite case is an abstraction of the concept of “the number of elements", is usually denoted by $|A|$. Thus Cantor was showing that $|\mathbb{N}| < |\mathbb{R}|$. An immediate question that can be asked following this observation is whether, there could be A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$. In fact, this very question became the first on Hilbert’s prominent list of open problem’s [27] from 1900. The assertion that no such A exists was named the *Continuum Hypothesis*, abbreviated as *CH*. The general tendency, at least that of Cantor and of Hilbert[2], was to conjecture that CH is true.

In quest of finding a counterexample to CH or showing that there is none, mathematicians started to look at many natural types of subsets of $\mathbb{R}$. Obviously open sets cannot satisfy the inequality above but the argument for closed sets is already less trivial. The *Cantor-Bendixson Theorem* [33, Theorem 6.4] devises a fine analysis on closed sets which can be used to settle the question and show that closed sets cannot be used as such counterexamples. A larger class of sets encompassing that of closed ones is formed by the *Borel sets*. These are sets that can be formed successively, starting from open sets and taking countable unions, intersections and complements. What about those sets? What about continuous images of Borel sets? These are called *analytic sets*. Or their complements? These are the *co-analytic sets*. Answering these questions required completely new tools and new insights about the deeper topological and combinatorial structure of the reals. This layed the foundation of a field called *descriptive set theory*, which studies these kinds of sets.

Although descriptive set theory seems, from our description, mainly topological in nature, it has a strong connection to logic, especially to the notion of *definability*. In most generality, a set A is definable from parameters $a_0, \dots, a_{n-1}$ if there is a formula $\varphi(x, a_0, \dots, a_{n-1})$ in the language of mathematics, that holds true exactly of those x which are members of A. It turns out that the Borel, analytic and co-analytic sets, which were defined in a purely topological manner, each correspond precisely to a class of logical definability in second-order arithmetic. In this way, they can be viewed as complexity classes for subsets of reals, similar to the complexity of subsets of the naturals according to computability theory. For the reader which is unfamiliar with this,

[1] Formally defined as the smallest ordinal that can be put in bijection to A, at least in contexts where the Axiom of Choice is assumed.

[2] cf. “*Die Untersuchungen von Cantor [...] machen einen Satz sehr wahrscheinlich, dessen Beweis jedoch trotz eifrigster Bemühungen bisher noch niemandem gelungen ist*” in [27].

we provide a preliminaries section below in which the main notions of descriptive set theory are explained.

Another major development in the early 20th century was Zermelo's formulation [71] of seven independent "principles" through which set theory, and in fact, mathematics as a whole, could be treated axiomatically. Other than basing the foundational work done by Cantor, and e.g. Russell (see [45]), on a few accessible axioms, it made it possible for Zermelo to formalize his much contested proof of the *Well-Ordering Theorem* [70], making it indisputable on the grounds of his assumptions. For this, he formulated the *Axiom of Choice*[3] which, by itself, corresponded to mathematical practice, making it a natural assumption[4]. Later, in [19], Fraenkel proposed the *Axiom of Replacement* as an addition to Zermelo's list, in order to include other mathematical constructions, most notably those involving *transfinite recursion*[5]. Zermelo's axioms together with Fraenkel's Axiom of Replacement are known as *Zermelo-Fraenkel-Choice* (short *ZFC*).

The Axiom of Choice is much known for the controversy it created. A common criticism is that it allows to construct objects without explicitly defining them. Even more, it may produce counter-intuitive theorems such as the *Banach-Tarski Paradox*. For this reason, it is often dealt with particular care and mentioned in every instance it is used. Ironically though, it is a common experience to see non-logicians mentioning it at times where it in fact can be avoided and in contrast not being aware of when it is used in more subtle ways[6]. Nevertheless, throughout mathematics, the existence of various kinds of maximal sets can typically only be obtained by an appeal to the Axiom of Choice or one of its popular forms, such as *Zorn's Lemma*. Under certain circumstances, it is possible though, to explicitly define such objects.

In 1938, Gödel made the first step to a solution of CH by showing that the negation of CH cannot be proven on basis of ZFC alone (see [22], [23]). For this he defined what is known as the constructible universe L. It turned out that L is not just useful for questions surrounding CH but also for many others, especially ones related to definability. The earliest result in this direction is probably due to Gödel who noted in [22, p. 67] that in the constructible universe L, there is a Δ^1_2-definable well-order of the reals (see [31, 25] for a modern treatment). Loosely speaking, we can exhibit a concrete well-order of the continuum when the structure of the reals is not too rich.

[3]"*Axiom der Auswahl*". It is worthwhile to note that the standard modern proof of the Well-Ordering Theorem in addition uses the *Axiom of Replacement*, which was not included in Zermelo's original list.

[4]cf. Bona's ever repeated joke "*The Axiom of Choice is obviously true, the Well-Ordering Principle obviously false, and who can tell about Zorn's Lemma?*".

[5]As an example, Fraenkel mentions that the existence of the set $\{\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathcal{P}(\mathcal{P}(\mathbb{N})), \ldots\}$ cannot be be argued based on Zermelo's axioms alone.

[6]For instance, when we use that the countable union of countable sets is countable.

Using similar ideas, many other special sets of reals, such as *Vitali sets*, *Hamel bases* or *mad families*, just to name a few, can be constructed in L in a Δ^1_2 fashion. In particular, such sets can be continuous images of coanalytic sets. This has become by now a standard set theoretic technique that is so general that details can be usually omitted. In many cases, these results also give an optimal bound for the complexity of such a set. For example, a Vitali set cannot be Lebesgue measurable and in particular cannot have a $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$ definition. In other cases, one can get stronger results by constructing Π^1_1 witnesses. This is typically done using a coding technique, originally developped by Erdős, Kunen and Mauldin in [13], later streamlined by Miller (see [38]) and further generalized by Vidnyánszky (see [69]). For example, Miller showed that there are Π^1_1 Hamel bases and mad families in L.

Recently, another phenomenon has been discovered that leads a path to Π^1_1 witnesses. In [65], Törnquist showed in ZFC that the existence of $\Sigma^1_2(r)$ mad families already implies that of $\Pi^1_1(r)$ ones. This turned out to be a general tendency. For instance, Brendle, Fischer and Khomskii showed that the same holds true for maximal independent families (see [8]). We are going to provide similar results.

On the other hand, families of the kind we mentioned usually do not admit analytic witnesses. Mathias showed in [37], that there are no analytic mad families. Miller proved the same for maximal independent families and Hamel bases (see [38]). In contrast, it was shown recently by Horowitz and Shelah ([28], [29]), very surpsisingly, that Borel maximal eventually different families and maximal cofinitary groups do exist.

In 1963, Cohen gave the second part of the solution to CH by showing that CH has no proof using the axioms of ZFC alone. For this he devised a revolutionary new method that would become a major tool in set theory. His method is called forcing and it shows how to extend models of ZFC by adding new sets (e.g. new reals) to it and preserve ZFC. This is similar to when we form a field extension by adding a new element to it and then adding “anything that there shoud be“. In particular, forcing can be used to create models which are different and much richer than L. Since the assumption that $V = L$ is quite restrictive, it is interesting to know in what forcing extensions of L, definable witnesses for the above mentioned kinds of sets still exist. By now, various such results exist in the literature, e.g. in [9], [14], [18], [51] or [17]. Typically, witnesses in L are preserved directly in a forcing extension and Shoenfield absoluteness ensures that they keep the same definitions. Only a few exceptions to this exist so far, most notably [9], where the authors preserve the definition a mad family, while its version in L is destroyed. We will provide a similar result in Chapter 4.

1.2 Preliminaries

1.2.1 Descriptive set theory

Descriptive set theory studies definable subsets of the reals, or more generally, of Polish spaces. It is usually much more convenient to work with spaces such as *Baire space* ω^ω or *Cantor space* 2^ω instead of $\mathbb{R}$, or any other Polish space, directly. ω^ω is the set of functions from naturals to naturals and 2^ω is the set of functions from naturals to $\{0, 1\}$. For concrete Polish spaces X, there are usually many effective ways to associate members of 2^ω or ω^ω with elements of X. In full generality, we have that:

Fact ([33, Theorem 7.9])**.** *Let X be a Polish space. Then there is a continuous surjection $\phi\colon \omega^\omega \to X$. Moreover, there is a closed set $C \subseteq \omega^\omega$ and a continuous bijection $\psi\colon C \to X$.*

In the context of $X = \mathbb{R}$ for instance, ϕ may be a computable function. More precisely, there is an algorithm which, given an arbitrary long finite initial segment of $x \in \omega^\omega$ and $n \in \omega$, computes a rational approximation up to the n'th decimal place of $\phi(x)$. Closed subsets of ω^ω (or of 2^ω) are paticularly nice to work with since they have the following representation theorem:

Fact ([33, Proposition 2.4])**.** *Let $T \subseteq \omega^{<\omega}$ be a tree, then the set of branches through T, $[T] = \{x \in \omega^\omega : \forall n \in \omega(x \restriction n \in T)\}$, is a closed set. For any closed set $C \subseteq \omega^\omega$, there is a tree $T \subseteq \omega^{<\omega}$ so that $C = [T]$.*

Definition 1.2.1. Consider the language of second-order arithmetic. It has two sorts of variables, those for reals, usually using the letters u, v, w, x, y, z, and those for natural numbers, usually i, j, k, l, m, n. It has the common constant, function and relation symbols $0, 1, +, \cdot, <$ for naturals and an additional evaluation predicate $x(n) = m$. The semantics of this logic should be clear. A formula φ is this language is called

- arithmetic, if quantifiers are only bound to natural number variables,
- Σ^1_1, if it is of the form $\exists x_0, \dots, x_n \psi(x_0, \dots, x_n)$, where ψ is arithmetic without parameters,
- Π^1_1, if it is of the form $\forall x_0, \dots, x_n \psi(x_0, \dots, x_n)$, where ψ is arithmetic without parameters,
- Σ^1_{n+1}, if it is of the form $\exists x_0, \dots, x_n \psi(x_0, \dots, x_n)$, where ψ is Π^1_n,

– Π^1_{n+1}, if it is of the form $\forall x_0, \dots, x_n \psi(x_0, \dots, x_n)$, where ψ is Σ^1_n.

Moreover, a formula is $\Sigma^1_n(r)$, resp. $\Pi^1_n(r)$ for $r \in \omega^\omega$, if it is of the form $\varphi(r)$ for a Σ^1_n, resp. Π^1_n formula φ. And it is $\mathbf{\Sigma}^1_n$, resp. $\mathbf{\Pi}^1_n$ (boldface) if it is $\Sigma^1_n(r)$, resp. $\Pi^1_n(r)$ for some $r \in \omega^\omega$.

For any of the classes Γ of formulas defined above, we say that a set $A \subseteq \omega^\omega$ is Γ, or sometimes, Γ-definable, if there is a formula $\varphi(x) \in \Gamma$ in one free variable x so that $A = \{x \in \omega^\omega : \varphi(x)\}$. We say that A is Δ^1_n, $\Delta^1_n(r)$ or $\mathbf{\Delta}^1_n$ if A is both Σ^1_n, resp. $\Sigma^1_n(r)$, resp. $\mathbf{\Sigma}^1_n$ and Π^1_n, resp. $\Pi^1_n(r)$, resp. $\mathbf{\Pi}^1_n$.

This description can be easily adapted to concrete Polish spaces, other than ω^ω, such as $\mathbb{R}$, changing the semantics. In this case for instance, we may stipulate that $x(0) = m$ says that $\lfloor x \rfloor = m$ and for $n > 0$, $x(n) = m$ says that the n'th digit in the (or rather a canonically chosen) decimal expansion of x is m. More generally, if $\{O_n : n \in \omega\}$ is a basis for X, $x(n) = 1$ could be understood as saysing $x \in O_n$ and $x(n) \neq 1$ as $x \notin O_n$. In the statements below, the specific semantics (i.e. the specific choice of a basis) for arbitrary Polish spaces is irrelevant.

Fact. *Let X be a Polish space and $A \subseteq X$. Then TFAE:*

1. *A is analytic, i.e. $A = f''B$ for some $f \colon Y \to X$ continuous, Y a Polish space and $B \subseteq Y$ Borel.*
2. *$A = f''C$ for some $f \colon \omega^\omega \to X$ continuous, $C \subseteq \omega^\omega$ closed.*
3. *$A = f''\omega^\omega$ for some $f \colon \omega^\omega \to X$ continuous.*
4. *A is $\mathbf{\Sigma}^1_1$.*
5. *A is the projection of a closed set $C \subseteq X \times \omega^\omega$.*

And in case $X = \omega^\omega$,

6. *$A = p[T]$, for T a tree on $\omega \times \omega$, where $p[T]$ is the projection of $[T]$ to the first coordinate.*

Fact. *Let X be a Polish space and $A \subseteq X$. Then TFAE:*

1. *A is Borel.*
2. *A is analytic and coanalytic, i.e. the complement of an analytic set.*

3. *A is $\mathbf{\Delta}^1_1$.*

We fix from now on a relatively small, but still strong enough finite fragment of ZFC which we call ZFC*. We do not specify it further but let us say that it does not include the *Powerset Axiom* so that models of the form $H(\theta)$ satisfy it.

Fact (Shoenfield Absoluteness)**.** *Let $M \subseteq N$ be transitive models (possibly proper classes) of ZFC* and $r \in M$. Then, if φ is $\Sigma^1_1(r)$ or $\Pi^1_1(r)$,*

$$M \models \varphi \leftrightarrow N \models \varphi.$$

Moreover, if $(\omega_1)^N \in M$ and φ is $\Sigma^1_2(r)$ or $\Pi^1_2(r)$, then

$$M \models \varphi \leftrightarrow N \models \varphi.$$

1.2.2 Forcing

Forcing is a technique that shows how to extend given models of set theory, by adding specially chosen objects to it. It can be used to show that certain mathematical questions cannot be settled on the basis of ZFC alone. Simply put and skipping the logical details, this is done by providing a model of ZFC in which the question has a positive answer and another one in which it is negative. But let us say that this is a rather superficial description and does not always correspond to how set theorists think about these results in practice. Forcing constructions usually reveal something much deeper than merely saying that something does not have a proof. Forcing can also often be used to prove results in ZFC, e.g. via Shoenfield absoluteness.

Definition 1.2.2. Let $(\mathbb{P}, \leq)$ be a partial order with a greatest element $\mathbb{1}$. Then we call $\mathbb{P}$ a *forcing poset* or *forcing notion*.

- The elements of $\mathbb{P}$ are often called *conditions* and denoted with letters p, q, r. $\mathbb{1}$ is called the *trivial condition*.
- When $p \leq q$, we say that p *extends* q or sometimes that p *is stronger than* q.
- A set $D \subseteq \mathbb{P}$ is called *dense* if for every $p \in \mathbb{P}$ there is $q \in D$ such that $q \leq p$.
- A set $G \subseteq \mathbb{P}$ is called a *filter* if for any p, q, if $p \in G$ and $p \leq q$, then $q \in G$, and if $p, q \in G$, then there is $r \in G$ so that $r \leq p, q$.

Let $(M, \in) \models$ ZFC*, $\mathbb{P} \in M$ and G a filter on $\mathbb{P}$.

– We say that G is $\mathbb{P}$-*generic over* M if for every dense subset $D \in M$ of $\mathbb{P}$, $M \cap G \cap D \neq \emptyset$.

Whenever G is $\mathbb{P}$-generic over M, we can form a model $M[G]$ which corresponds to the smallest model extending M and containing G. The way $M[G]$ is defined is via the notion of $\mathbb{P}$-names.

Definition 1.2.3. The class $V^{\mathbb{P}}$ of $\mathbb{P}$-names is defined recursively on ranks, stipulating that every $\tau \in V^{\mathbb{P}}$ consists of elements of the form (p, σ) for $p \in \mathbb{P}$ and $\sigma \in V^{\mathbb{P}}$. For example, $\emptyset$ is the $\mathbb{P}$-name of lowest rank. Whenever $G \subseteq \mathbb{P}$ and τ is a $\mathbb{P}$-name, we define recursively the evaluation of τ by G as $\tau[G] := \{\sigma[G] : \exists p \in G((p, \sigma) \in \tau)\}$. For a class M, we define $M[G] := \{\tau[G] : \tau \in V^{\mathbb{P}} \cap M\}$.

Fact. *Let $(M, \in) \models$ ZFC* be transitive, $\mathbb{P} \in M$ and let G be $\mathbb{P}$-generic over M. Let Λ be an arbitrary finite fragment of ZFC. Then there is a finite fragment Σ of ZFC, so that*

$$(M, \in) \models \Sigma \to (M[G], \in) \models \Lambda.$$

Moreover, $M[G]$ is the smallest transitive model satisfying ZFC with $G \in M[G]$ and $M \subseteq M[G]$. Also, M and $M[G]$ have the same ordinals.*

Fact. *Let $(M, \in) \models$ ZFC* be transitive. Whenever $\varphi(x_0, \dots, x_n)$ is a formula in the language of set theory, there is a formula $\psi(y, z, x_0, \dots, x_n)$ so that for any forcing poset $\mathbb{P} \in M$, any $\mathbb{P}$-names $\tau_0, \dots, \tau_n \in M$ and $p \in \mathbb{P}$, $(M, \in) \models \psi(\mathbb{P}, p, \tau_0, \dots, \tau_n)$ iff for every G, $\mathbb{P}$-generic over M with $p \in G$,*

$$(M[G], \in) \models \varphi(\tau_0[G], \dots, \tau_n[G]).$$

Moreover, for any $\mathbb{P}$-generic filter G over M, $(M[G], \in) \models \varphi(\tau_0[G], \dots, \tau_n[G]))$ iff there is $p \in G$ such that $(M, \in) \models \psi(\mathbb{P}, p, \tau_0, \dots, \tau_n)$.

Finally, generic filters exist when M is countable.

Fact. *Let $M, \mathbb{P}$ be as before, $p \in \mathbb{P}$ and let M be countable. Then there is a $\mathbb{P}$-generic filter G over M with $p \in G$.*

1.3 Structure of the thesis

In Chapter 2, we study the definability of maximal towers and of inextendible linearly ordered towers (ilt's), a notion that is more general than that of a maximal tower. We show that there is, in the constructible universe, a Π^1_1 definable maximal tower that is indestructible by any proper Suslin poset. Resembling earlier results in the literature, we prove that the existence of a Σ^1_2 ilt implies that the universe is close to L in the sense that $\omega_1^L = \omega_1$. Moreover, we show that analogous results hold for other combinatorial families of reals. We prove that there is no ilt in Solovay's famous model, in which every set of reals is Lebesgue measurable. And finally we show that the existence of a Σ^1_2 ilt is equivalent to that of a Π^1_1 ilt.

The next chapter deals with the definability of ultrafilters and ultrafilter bases on the naturals. As a main result we show that there is no coanalytic base for a Ramsey ultrafilter, while in contrast we can construct Π^1_1 P-point and Q-point bases in L. This is interesting since a Ramsey ultrafilter is exactly an ultrafilter that is a P- and Q-point at the same time. We also show that the existence of a Δ^1_{n+1} ultrafilter is equivalent to that of a Π^1_n base, for $n \in \omega$. Moreover we introduce a Borel version $\mathfrak{u}_B$ of the classical ultrafilter number $\mathfrak{u}$ and make some observations.

In Chapter 4, we prove a fairly general result that applies to a large number of examples of special families of reals. We show that after forcing with a countable support iteration or a finite product of Sacks or splitting forcing over L, every analytic hypergraph on a Polish space admits a $\mathbf{\Delta}^1_2$ maximal independent set. This extends an earlier result by Schrittesser. As a main application we get the consistency of $\mathfrak{r} = \mathfrak{u} = \mathfrak{i} = \omega_2$ together with the existence of a Δ^1_2 ultrafilter, a Π^1_1 maximal independent family and a Δ^1_2 Hamel basis. This solves open problems of Brendle, Fischer and Khomskii. We also show in ZFC that $\mathfrak{d} \leq \mathfrak{i}_{cl}$.

In the last chapter, which corresponds to the second part of the thesis, our goal is to study the pseudointersection and tower numbers on uncountable regular cardinals. First, we prove that either $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa)$ or there is a $(\mathfrak{p}(\kappa), \lambda)$-gap of club-supported slaloms for some $\lambda < \mathfrak{p}(\kappa)$. While the existence of such gaps is unclear, this is a promising step to lift Malliaris and Shelah's proof of $\mathfrak{p} = \mathfrak{t}$ to uncountable cardinals. We analyze gaps of slaloms and in particular, show that $\mathfrak{p}(\kappa)$ is always regular. This extends results of Garti. Finally, we present a new model for the inequality $\mathfrak{p}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa) = 2^\kappa$. In contrast to earlier arguments by Shelah and Spasojevic, we achieve this by adding κ-Cohen reals and then successively diagonalising the club filter which is shown to preserve a Cohen witness to $\mathfrak{p}(\kappa) = \kappa^+$.

CHAPTER 2

Inextendible linearly ordered towers

2.1 Introduction

A tower will be, as usual, a set $X \subseteq [\omega]^\omega$ which is well ordered with respect to reverse almost inclusion, i.e. the relation $x \leq y$ given by $\exists n \in \omega(y \setminus n \subseteq x)$. A tower is maximal if it has no pseudointersection. In the definition of a linearly ordered tower we drop the requirement that the order is well-founded. An inextendible linearly ordered tower is one that has no top-extension, i.e. has no pseudointersection.

The questions that we will ask and answer for towers are inspired to a great extent by those that appeared in relation to mad families. Recall that two sets $x, y \in [\omega]^\omega$ are called almost disjoint whenever $x \cap y$ is finite. An almost disjoint family is a subset of $[\omega]^\omega$ all of whose elements are pairwise almost disjoint. A maximal almost disjoint family (mad family) is an infinite almost disjoint family that cannot be properly extended to a larger one. For mad families, the story begins with Mathias' influential work [37] in which he showed that mad families cannot be analytic.

In Section 2.2 we will show that neither maximal towers nor ilt's can be analytic (Theorem 2.2.2 and Theorem 2.2.5). On the other hand we prove in Section 2.3, as a main result, that Π^1_1 maximal towers do exist in L (Theorem 2.3.2), using the technique developped by Miller in [38].

Another topic that has been studied extensively for mad families is the existence of Π^1_1 examples in various forcing extensions. For instance it has been shown in [9] that there is a Π^1_1 mad family in a model obtained by adding Hechler reals. In Section 2.4 we will outshadow all these questions for towers by showing that in L there is a Π^1_1

maximal tower that is indestructible by any proper Suslin partial order (Theorem 2.4.3).

Section 2.5 deals with the value of ω_1 in models where ilt's can have simple definitions. As a main result we show that the existence of a $\Sigma^1_2(x)$ ilt implies that $\omega_1 = \omega_1^{L[x]}$ (Theorem 2.5.3). The same has been shown for mad families in [66]. Using similar ideas we show that this holds analogously for maximal independent families, Hamel bases and ultrafilters (Theorem 2.5.7, 2.5.9 and 2.5.11). In [9] Brendle and Khomskii ask whether there is some notion of transcendence over L that is equivalent to the non-existence of a Π^1_1 mad family. The same question can be asked for other families and our observations contribute to this question by giving a sufficient condition of this kind.

In Section 2.6 we show that there is no ilt in Solovay's model (Theorem 2.6.1). For mad families this was a long standing open question first asked by Mathias in [37] and solved by Törnquist in [66].

In Section 2.7 we show that the existence of a Σ^1_2 ilt is equivalent to that of a Π^1_1 ilt (Theorem 2.7.1). This theorem fits into a series of results stating that we can canonicaly construct Π^1_1 objects from given Σ^1_2 ones. For mad families this was shown in [65]. For maximal independent families see [8] and for maximal eventually different families see [17].

We will always stress the difference between lightface $(\Pi^1_1, \Sigma^1_1, \Sigma^1_2)$ and boldface $(\boldsymbol{\Pi}^1_1, \boldsymbol{\Sigma}^1_1, \boldsymbol{\Pi}^1_2)$ definitions as well as definitions relative to a fixed real parameter $(\Pi^1_1(x), \Sigma^1_1(x), \Sigma^1_2(x))$ to stay as general as possible.

2.2 Towers and Definability

Definition 2.2.1. A tower is a set $X \subseteq [\omega]^\omega$ which is well ordered with respect to the relation defined by $x \leq y$ iff $y \subseteq^* x$. It is called maximal if it cannot be further extended, i.e. it has no pseudointersection.

Theorem 2.2.2. *A tower contains no (uncountable) perfect set, i.e. is thin. In particular there is no $\boldsymbol{\Sigma}^1_1$ maximal tower.*

Proof. Assume $X \subseteq [\omega]^\omega$ is a tower and $P \subseteq X$ is a perfect set. The set $R = \{(x, y) : x, y \in P \wedge y \subseteq^* x\}$ is Borel. P is an uncountable Polish space and R is Borel as a subset of $P \times P$. But R is a well order of P, which contradicts R having the Baire property by [33, Theorem 8.48]. A maximal tower must be uncountable and an uncountable analytic set has a perfect subset by the Perfect Set Theorem. Thus there is no analytic maximal tower. $\square$

Theorem 2.2.3. *Every $\Sigma^1_2(x)$ tower is a subset of $L[x]$ and thus of size at most $\omega_1^{L[x]}$.*

Proof. If X is a $\Sigma^1_2(x)$ tower then it contains no perfect set and is thus a subset of $L[x]$ by the Mansfield-Solovay Theorem [39, Theorem 21.1]. $\square$

Corollary 2.2.4. The existence of a $\Sigma^1_2(x)$ maximal tower implies that $\omega_1 = \omega_1^{L[x]}$.

All of the proofs above rely mostly on the fact that towers exhibit a well ordered structure and the maximality is inessential. Thus it is natural to ask for a more general version of a tower which is not trivially ruled out by an analytic definition. We call a set $X \subseteq [\omega]^\omega$ an inextendible linearly ordered tower (abbreviated as ilt) if it is linearly ordered with respect to $\subseteq^*$ and has no pseudointersection. We call $Y \subseteq X$ cofinal in X if $\forall x \in X \exists y \in Y (y \subseteq^* x)$.

Theorem 2.2.5. *There is no $\mathbf{\Sigma}^1_1$ definable inextendible linearly ordered tower.*

Proof. Assume $X = p[T]$ is an ilt where T is a tree on $2 \times \omega$.

Claim 2.2.6. *There is $T' \subseteq T$ so that for every $(s,t) \in T'$, $p[T'_{(s,t)}]$ is cofinal in X.*

Proof. Let $T' = \{(s,t) : p[T_{(s,t)}]$ is cofinal in $X\}$. For every $(u,v) \in T \setminus T'$, we let $x_{u,v} \in X$ be such that $\forall y \in p[T_{(u,v)}](x_{u,v} \subseteq^* y)$. The collection $\{x_{u,v} : (u,v) \in T \setminus T'\}$ is countable and therefore there is $x \in X$ so that $x \subsetneq^* x_{u,v}$ for every $(u,v) \in T \setminus T'$. Now let $(s,t) \in T'$ be arbitrary and $x' \in X$ such that $x' \subseteq^* x$. As $p[T_{(s,t)}]$ is cofinal in X, there is $y \in p[T_{(s,t)}]$ so that $y \subseteq^* x'$. Say $(y,z) \in [T_{(s,t)}]$. For every $n \in \omega$, $(y \restriction n, z \restriction n) \in T'$ because else we get a contradiction to $y \subseteq^* x$. Thus $y \in p[T'_{(s,t)}]$. $\square$

By the claim we can wlog assume that for every $(s,t) \in T$, $p[T_{(s,t)}]$ is cofinal in X. Now consider T as a forcing notion (which is equivalent to Cohen forcing). The generic real will be a new element of $p[T]$ together with a witness. Let $\dot{c}$ be a name for the generic real. Notice that the statement that $p[T]$ is linearly ordered by $\subseteq^*$ is absolute. Thus for every $y \in X$ there is a condition $(s,t) \in T$ and $n \in \omega$ so that either

$$(s,t) \Vdash \dot{c} \subseteq y \setminus n$$

or

$$(s,t) \Vdash y \subseteq \dot{c} \setminus n.$$

The second option is impossible, because $p[T_{(s,t)}]$ is cofinal in X. We can thus find (s,t), $n \in \omega$ and $Y \subseteq X$ cofinal in X, so that for every $y \in Y$, $(s,t) \Vdash \dot{c} \subseteq y \setminus n$. Let $(x,z) \in [T_{(s,t)}]$ be arbitrary. As Y is cofinal in X, there is $y \in Y$ so that $y \subsetneq^* x$. But this clearly contradicts $(s,t) \Vdash \dot{c} \subseteq y \setminus n$. $\square$

Corollary 2.2.7. Every $\mathbf{\Sigma}^1_2$ inextendible linearly ordered tower has a cofinal subset of size ω_1.

Proof. Assume X is $\mathbf{\Sigma}^1_2$. Then it is the union of ω_1 many Borel sets (see e.g. [40]). By Theorem 2.2.5 each of these Borel sets has a lower bound in X. □

Note that the above results can be applied similarly to inextendible linearly ordered subsets of $(\omega^\omega, \leq^*)$.

2.3 A Π^1_1 definable maximal tower in L

In this section we will show how to construct in L a maximal tower with a Π^1_1 definition. For this we apply the coding technique that has been developed by A. Miller in [38] in order to show the existence of various nicely definable combinatorial objects in L.

Let O be the set of odd and E the set of even natural numbers.

Lemma 2.3.1. *Suppose $z \in 2^\omega$, $y \in [\omega]^\omega$ and $\langle x_\alpha : \alpha < \gamma \rangle$ is a tower, where $\gamma < \omega_1$, so that $\forall \alpha < \gamma(|x_\alpha \cap O| = \omega \wedge |x_\alpha \cap E| = \omega)$. Then there is $x \in [\omega]^\omega$ so that $\forall \alpha < \gamma(x \subseteq^* x_\alpha)$, $|x \cap O| = \omega$, $|x \cap E| = \omega$, $z \leq_T x$ and $|y \cap \omega \setminus x| = \omega$.*

Proof. It is a standard diagonalization to find x so that $\forall \alpha < \gamma(x \subseteq^* x_\alpha)$, $|x \cap O| = \omega$, $|x \cap E| = \omega$ and $|y \cap \omega \setminus x| = \omega$. We assume that z is not eventually constant, else there is nothing to do. Now given x find $\langle n_i \rangle_{i \in \omega}$ increasing in x so that $n_i \in O$ iff $z(i) = 0$. Let $x' = \{n_i : i < \omega\}$. Then x' works. □

Theorem 2.3.2. *Assume $V = L$. Then there is a Π^1_1 definable maximal tower.*

In the rest of the paper, $<_L$ will always stand for the canonical global L well-order. Whenever $r \in 2^\omega$, we write $E_r \subseteq \omega^2$ for the relation defined by

$$mE_rn \text{ iff } r(2^m 3^n) = 0.$$

If E_r is a well-founded and extensional relation then we denote with M_r the unique transitive $\in$-model isomorphic to (ω, E_r). Notice that $\{r \in 2^\omega : E_r$ is well-founded and extensional$\}$ is Π^1_1.

If E_r is a well-order on ω then $\|r\|$ denotes the unique countable ordinal α so that (ω, E_r) is isomorphic to $(\alpha, \in)$. We also say that r codes α. The set of r so that E_r is a well-order is called WO. WO is obviously Π^1_1.

For any real $x \in 2^\omega$ we define ω_1^x to be the least countable ordinal which has no recursive code in x, i.e. the least ordinal α so that for any recursive function $r\colon 2^\omega \to 2^\omega$, $r(x)$ does not code α.

Proof of Theorem 2.3.2. Let $\langle y_\xi : \xi < \omega_1 \rangle$ enumerate $[\omega]^\omega$ via the canonical well order of L. We will construct a sequence $\langle \delta(\xi), z_\xi, x_\xi : \xi < \omega_1 \rangle$, where for every $\xi < \omega_1$:

- $\delta(\xi)$ is a countable ordinal
- $z_\xi \in 2^\omega \cap L_{\delta(\xi)+\omega}$
- $x_\xi \in [\omega]^\omega \cap L_{\delta(\xi)+\omega}$

The sequence is defined by the following requirements for each $\xi < \omega_1$:

1. $\delta(\xi)$ is the least ordinal δ greater than $\sup_{\nu<\xi} \delta(\nu)$ so that $y_\xi, \langle \delta(\nu), z_\nu, x_\nu : \nu < \xi \rangle \in L_\delta$ and L_δ projects to ω[1].
2. z_ξ is the $<_L$ least code for the ordinal $\delta(\xi)$.
3. $\langle x_\nu : \nu < \xi \rangle$ is a tower and $\forall \nu < \xi (|x_\nu \cap O| = \omega \wedge |x_\nu \cap E| = \omega)$.
4. x_ξ is $<_L$ least so that $\forall \nu < \xi (x_\xi \subseteq^* x_\nu)$, $|x_\xi \cap O| = \omega$, $|x_\xi \cap E| = \omega$, $z_\xi \leq_T x$ and $|y_\xi \cap \omega \setminus x| = \omega$.

Notice that z_ξ and x_ξ indeed can be found in $L_{\delta(\xi)+\omega}$ given that $y_\xi, \langle x_\nu : \nu < \xi \rangle \in L_{\delta(\xi)}$, and that $L_{\delta(\xi)}$ projects to ω. It is then straightforward to check that (1)-(4) uniquely determine a sequence $\langle \delta(\xi), z_\xi, x_\xi : \xi < \omega_1 \rangle$ for which $\langle x_\xi : \xi < \omega_1 \rangle$ is a maximal tower.

Claim 2.3.3. $\{x_\xi : \xi < \omega_1\}$ *is a* Π^1_1 *subset of* 2^ω.

Proof. Let $\Psi(v)$ be the formula expressing that for some $\xi < \omega_1$, $v = \langle \delta(\nu), z_\nu, x_\nu : \nu \leq \xi \rangle$. More precisely, $\Psi(v)$ says that v is a sequence $\langle \rho_\nu, \zeta_\nu, \tau_\nu : \nu \leq \xi \rangle$ of some length $\xi + 1$, that satisfies the clauses (1)-(4) for every $\nu \leq \xi$.

The formula $\Psi(v)$ is absolute for transitive models of some finite fragment Th of ZFC which holds at limit stages of the L hierarchy. Namely we need absoluteness of the formula $\varphi_1(\xi, y)$ expressing that $y = y_\xi$, $\varphi_2(\delta, M)$ expressing that $M = L_\delta$ projects to ω and $\varphi_3(z, \delta)$ expressing that z is the $<_L$ least code for δ.

Moreover we have that $\langle \delta(\nu), z_\nu, x_\nu : \nu \leq \xi \rangle \in L_{\delta(\xi)+\omega}$ and that

$$L_{\delta(\xi)+\omega} \models \Psi(\langle \delta(\nu), z_\nu, x_\nu : \nu \leq \xi \rangle)$$

for every $\xi < \omega_1$.

[1]This means that over L_δ there is a definable surjection to ω. The set of such δ is unbounded in ω_1.

Now let $\Phi(r,x)$ be a formula expressing that E_r is a well founded and extensional relation, $M_r \models$ Th and for some $v \in M_r$,

$$M_r \models v \text{ is a sequence } \langle \rho_\nu, \zeta_\nu, \tau_\nu : \nu \leq \xi \rangle \wedge \Psi(v) \wedge \tau_\xi = x.$$

We thus have that $x = x_\xi$ for some $\xi < \omega_1$ iff $\exists r \in 2^\omega \Phi(r,x)$. $\Phi(r,x)$ can clearly be taken as a Π^1_1 formula.

For any $\xi < \omega_1$, the well order $\delta(\xi)$ is coded by z_ξ and $z_\xi \leq_T x_\xi$. Thus $\delta(\xi) + \omega < \omega_1^{x_\xi}$ and there is $r \in L_{\omega_1^{x_\xi}}$ so that $M_r = L_{\delta(\xi)+\omega}$. In particular

$$\exists r \in L_{\omega_1^{x_\xi}} \cap 2^\omega(\Phi(r, x_\xi)).$$

We get that

$$\exists \xi < \omega_1(x = x_\xi) \leftrightarrow \exists r \in L_{\omega_1^x} \cap 2^\omega(\Phi(r,x)).$$

The right hand side can be expressed by a Π^1_1 formula. □

□

Remark 2.3.4. By Theorem 2.2.3 the Π^1_1 tower constructed above is a subset of L. This implies that its definition will evaluate to the same set in any extension of L. As an immediate corollary, we obtain that the existence of a Π^1_1 definable tower is consistent with $\mathfrak{c} > \aleph_1$ (here $\mathfrak{c}$ denotes the continuum), a question which has been of interest for many combinatorial objects of the real line. For some more recent results in this direction regarding maximal independent families and maximal eventually different families of functions, see [8] and [17] respectively.

Corollary 2.3.5. The existence of a coanalytic tower is consistent with the bounding number $\mathfrak{b}$ being arbitrarily large.

Recall that the bounding number is defined as the least size of an unbounded family in $(\omega^\omega, <^*)$. It is a natural lower bound for many other classical cardinal characteristics.

Proof. It is well known that finite support iterations of Hechler forcing for adding a dominating real preserve all ground model maximal towers to be maximal (see [2] for more details). □

2.4 Indestructible Towers

Recall that the pseudointersection number $\mathfrak{p}$ is the least cardinal κ so that any set $\mathcal{F} \subseteq [\omega]^\omega$ with the finite intersection property and $|\mathcal{F}| < \kappa$ has a pseudointersection. $\mathcal{F}$ has the finite intersection property if for any $\mathcal{F}' \in [\mathcal{F}]^{<\omega}$, $\bigcap \mathcal{F}'$ is infinite. We obtain the following result.

Theorem 2.4.1. *Assume* $\mathfrak{p} = \mathfrak{c}$. *Let* $\mathcal{P}$ *be a collection of at most* $\mathfrak{c}$ *many proper posets of size* $\mathfrak{c}$. *Then there is a maximal tower indestructible by any* $\mathbb{P} \in \mathcal{P}$.

Proof. Let us call a $\mathbb{P}$ name $\dot{x}$ for a real a nice name whenever it has the form $\bigcup_{n\in\omega}\{(p, \check{n}) : p \in A_n\}$ where the A_n's are countable antichains in $\mathbb{P}$. Remember that when $\mathbb{P}$ is proper, then for any $\mathbb{P}$ name $\dot{x}$ for a real and any $p \in \mathbb{P}$, there is a nice name $\dot{y}$ and $q \leq p$ such that $q \Vdash \dot{y} = \dot{x}$. The number of nice $\mathbb{P}$ names is $|\mathbb{P}|^{\aleph_0}$.

Let us enumerate all pairs $\langle(\mathbb{P}_\alpha, p_\alpha, \dot{y}_\alpha) : \alpha < \mathfrak{c}\rangle$ where $p_\alpha \in \mathbb{P}_\alpha$, $\mathbb{P}_\alpha \in \mathcal{P}$ and $\dot{y}_\alpha$ is a nice $\mathbb{P}_\alpha$ name such that $p_\alpha \Vdash \dot{y}_\alpha \in [\omega]^\omega$.

We construct a tower $\langle x_\alpha : \alpha < \mathfrak{c}\rangle$ recursively. At step α we first choose a pseudointersection x of $\langle x_\xi : \xi < \alpha\rangle$ (here we use $\alpha < \mathfrak{p}$). Next we partition x into two disjoint infinite subsets x^0, x^1. Now note that $p_\alpha \Vdash_{\mathbb{P}_\alpha} (\dot{y}_\alpha \subseteq^* x^0 \wedge \dot{y}_\alpha \subseteq^* x^1)$ is impossible. Thus we find $i \in 2$ and $q_\alpha \leq p_\alpha$ such that $q_\alpha \Vdash_{\mathbb{P}_\alpha} \dot{y}_\alpha \not\subseteq^* x^i$. Let $x_\alpha = x^i$.

Now let $\dot{x}$ be an arbitrary $\mathbb{P}$ name for a real for some $\mathbb{P} \in \mathcal{P}$. We see easily that the set $D = \{q \in \mathbb{P} : \exists\alpha < \mathfrak{c}(q \Vdash \dot{x} \not\subseteq^* x_\alpha)\}$ is dense. Namely for any p we find $(\mathbb{P}_\alpha, p_\alpha, \dot{y}_\alpha)$ where $p_\alpha \leq p$ and $p_\alpha \Vdash \dot{x} = \dot{y}_\alpha$. Then we have $q_\alpha \leq p$ with $q_\alpha \in D$. $\square$

Definition 2.4.2. A forcing notion $(\mathbb{P}, \leq)$ is Suslin if

1. $\mathbb{P} \subseteq 2^\omega$ is analytic,
2. $\leq \subseteq 2^\omega \times 2^\omega$ is analytic,
3. the incompatibility relation $\perp \subseteq 2^\omega \times 2^\omega$ is analytic (and in particular Borel).

The next thing we want to show is that (in L) for $\mathcal{P}$ the collection of all proper Suslin posets, we can get an indestructible maximal tower which is coanalytic.

Theorem 2.4.3. *(V=L) There is a* Π^1_1 *maximal tower indestructible by any proper Suslin poset.*

Proof. First let us note that there is a recursive map $f\colon \mathrm{Tree} \times [\omega]^\omega \to 2^\omega$, where Tree is the set of trees on $\omega \times \omega$, such that $f(T, y) \in \mathrm{WO}$ iff $\forall x \in p[T](|x \cap (\omega \setminus y)| = \omega)$ (see [40, Theorem 4A.3]). Fix this map f.

For the construction of our tower we now enumerate via the canonical well order of L all trees $\langle T_\alpha : \alpha < \omega_1\rangle$ on $\omega \times \omega$. Now as in the proof of Theorem 2.3.2 we define a sequence $\langle \delta(\xi), z_\xi, x_\xi : \xi < \omega_1\rangle$ with

- $\delta(\xi)$ is a countable ordinal
- $z_\xi \in 2^\omega \cap L_{\delta(\xi)+\omega}$
- $x_\xi \in [\omega]^\omega \cap L_{\delta(\xi)+\omega}$

and the following properties:

1. $\langle x_\nu : \nu < \xi\rangle$ is a tower and $\forall \nu < \xi(|x_\nu \cap O| = \omega \wedge |x_\nu \cap E| = \omega)$.
2. $\delta(\xi)$ is the least ordinal δ greater than $\sup_{\nu<\xi} \delta(\nu)$ so that
   - $\langle \delta(\nu), z_\nu, x_\nu : \nu < \xi\rangle, T_\xi \in L_\delta$,
   - there are disjoint pseudointersections $x^0, x^1 \in L_\delta$ of $\langle x_\nu : \nu < \xi\rangle$ both hitting O and E infinitely,
   - either (a) there is $(x, w) \in [T_\xi] \cap L_\delta$ such that $x \subseteq^* x^0$ or (b) $f(T_\xi, x^0) \in$ WO, $\|f(T_\xi, x^0)\| < \delta$ and there is in L_δ an order preserving map $(\omega, E_{f(T_\xi,x^0)}) \to \|f(T_\xi, x^0)\|$,
   - and L_δ projects to ω.
3. z_ξ is the $<_L$ least code for the ordinal $\delta(\xi)$.
4. x_ξ is $<_L$ least so that $x_\xi \subseteq^* x^1$ or $x_\xi \subseteq^* x^0$ depending on whether (a) or (b) holds true, $|x_\xi \cap O| = \omega$, $|x_\xi \cap E| = \omega$ and $z_\xi \leq_T x_\xi$.

As in the proof of Theorem 2.3.2 we see that this definition determines a tower $\langle x_\xi : \xi < \omega_1\rangle$ which is Π^1_1.

Now let us note the following for a proper Suslin poset $\mathbb{P}$. Whenever $\dot{x}$ is a nice $\mathbb{P}$ name for a real and $p \in \mathbb{P}$, then the set

$$\{z \in [\omega]^\omega : \exists q \leq p(n \in z \leftrightarrow q \nVdash n \in \omega \setminus \dot{x})\}$$

is analytic ($q \nVdash n \in \omega \setminus \dot{x}$ iff $\exists r \in \operatorname{dom} \dot{x}[(r, n) \in \dot{x} \wedge r \not\perp q]$).

Thus for any $\mathbb{P}, p \in \mathbb{P}$ and $\dot{x}$ a nice name there is $\alpha < \omega_1$ so that

$$p[T_\alpha] = \{z \in [\omega]^\omega : \exists q \leq p(n \in z \leftrightarrow q \nVdash n \in \omega \setminus \dot{x})\}.$$

Consider x_α and the respective disjoint sets x^0 and x^1 at stage α of the construction. There are two options:

(a) There is $(x, w) \in [T_\alpha]$ such that $x \subseteq^* x^0$. In this case we have chosen $x_\alpha \subseteq^* x^1$ and there is $q \leq p$ so that $|\{n \in \omega : q \not\Vdash n \notin \dot{x}\} \cap x^1| < \omega$. In particular $p \not\Vdash \dot{x} \subseteq^* x_\alpha$.

(b) Or $L_{\delta(\alpha)} \models$ "$(\omega, E_{f(T_\xi, x^0)})$ is isomorphic to an ordinal". This means that $L \models$ "$(\omega, E_{f(T_\xi, x^0)})$ is isomorphic to an ordinal" and this means that for any $x \in p[T_\alpha]$, x has infinite intersection with $\omega \setminus x^0$. In this case we chose $x_\alpha \subseteq^* x^0$. Now if $q \leq p$ and $n \in \omega$ are arbitrary we can find $r \leq q$ and $m \geq n$ such that $r \Vdash m \in \dot{x} \setminus x_\alpha$. This means again that $p \not\Vdash \dot{x} \subseteq^* x_\alpha$.

Thus we have shown that for any proper Suslin poset $\mathbb{P}$, $\dot{x}$ an arbitrary $\mathbb{P}$ name for a real and $p \in \mathbb{P}$, $p \not\Vdash \dot{x}$ is a pseudointersection of $\langle x_\xi : \xi < \omega_1 \rangle$. $\square$

2.5 ω_1 and $\mathbf{\Sigma}^1_2$ definitions

Definition 2.5.1. Let $\mathcal{F}$ be a filter on ω containing all cofinite sets. Then Mathias forcing relative to $\mathcal{F}$ is the poset $\mathbb{M}(\mathcal{F})$ consisting of pairs $(s, F) \in [\omega]^{<\omega} \times \mathcal{F}$ such that $\max s < \min F$. The extension relation is defined by $(s, F) \leq (t, E)$ iff $t \subseteq s$, $F \subseteq E$ and $t \setminus s \subseteq E$.

Lemma 2.5.2. *Assume that X is a $\mathbf{\Sigma}^1_2$ definable subset of $[\omega]^\omega$, linearly ordered with respect to $\subseteq^*$. Then there is a ccc forcing notion $\mathbb{Q}$ consisting of reals so that for any transitive model $V' \supseteq V^{\mathbb{Q}}$ (with the same ordinals), the reinterpretation of X in V' is not an ilt in V'.*

Proof. As X is $\mathbf{\Sigma}^1_2$, X can be written as a union $\bigcup_{\xi<\omega_1} X_\xi$ of analytic sets. Namely whenever $X = p[Y]$ where $Y \subseteq [\omega]^\omega \times 2^\omega$ is coanalytic then Y can be written as $\{(x, w) : f(x, w) \in \mathrm{WO}\}$ for some fixed continuous function f related to the definition of Y (see [40] for more details). Then X_ξ is defined as $\{x \in [\omega]^\omega : \exists w \in 2^\omega(\|f(x, w)\| = \xi)\}$.

Moreover we see that in any model $W \supseteq V$ where $\omega_1^W = \omega_1^V$, the reinterpretation of X is the union of the reinterpretations of the X_ξ.

If X has a pseudointersection x in V, then x will stay a pseudointersection of (the reinterpretation of) X in any extension by absoluteness. The statement $\forall y(y \notin X \vee x \subseteq^* y)$ is $\mathbf{\Pi}^1_2$. In this case let $\mathbb{Q}$ be the trivial poset.

If X is inextendible in V, then for any $\xi < \omega_1$ there is $x_\xi \in X$ so that $\forall y \in X_\xi(x_\xi \subseteq^* y)$. As X is linearly ordered with respect to $\subseteq^*$, $\{x_\alpha : \alpha < \omega_1\}$ generates a non-principal filter $\mathcal{F}$. Let $\mathbb{Q} = \mathbb{M}(\mathcal{F})$. Then in $V^{\mathbb{Q}}$ there is a real x so that $x \subseteq^* x_\alpha$

for every $\alpha < \omega_1$. By absoluteness $\forall y \in X_\xi(x_\xi \subseteq^* y)$ will still hold true in $V^{\mathbb{Q}}$. In particular $\forall y \in X_\xi(x \subseteq^* y)$ will hold true for any $\xi < \omega_1^V$.

As $\mathbb{Q}$ is ccc we have that $\omega_1^{V^{\mathbb{Q}}} = \omega_1^V$. This implies that x is actually a pseudointersection of X in $V^{\mathbb{Q}}$. Again, this will hold true in any extension. □

Theorem 2.5.3. *If there is a Σ^1_2 ilt, then $\omega_1 = \omega_1^L$. More generally, the existence of a $\Sigma^1_2(x)$ ilt implies $\omega_1 = \omega_1^{L[x]}$.*

Proof. We only prove the first part as the rest follows similarly.

Suppose that X is a Σ^1_2 ilt and $\omega_1^L < \omega_1$. Apply Lemma 2.5.2 to (the definition of) X in L to get the respective poset $\mathbb{Q}$ in L. As $\omega_1^L < \omega_1$, $V \models |\mathcal{P}(\omega) \cap L| = \omega$. But this means there is a $\mathbb{Q}$ generic $x \in V$ over L. $L[x] \subseteq V$, thus by Lemma 2.5.2 X has a pseudointersection in V, contradicting our assumption. □

Remark 2.5.4. We think that the proofs of Lemma 2.5.2 and Theorem 2.5.3 showcase something interesting about Schoenfield absoluteness. Recall that Schoenfield's absoluteness theorem says that Σ^1_2 formulas are absolute between any inner models $W \subseteq W'$, but it does not say anything about the relationship between ω_1^W and $\omega_1^{W'}$. In fact in many applications of Σ^1_2 absoluteness W and W' have the same ω_1 (e.g. when W' is a ccc or proper forcing extension of W). But in this case it can be deduced directly from analytic absoluteness and the representation of Σ^1_2 sets as the same ω_1 union of analytic set in any extension with the same ω_1. The reason is that the existential quantifier $\exists \alpha < \omega_1$ stays the same. So the full strength of Schoenfield absoluteness is only needed in the case where ω_1^W is countable in W' and this is the case that we crucially used in the proof of Theorem 2.5.3.

We also want to remark that the proofs of Lemma 2.5.2 and Theorem 2.5.3 are very general and can be applied to many other maximal combinatorial families. For example A. Törnquist has shown the following theorem in [66], using a similar argument.

Theorem 2.5.5. *If there is a Σ^1_2 mad family, then $\omega_1 = \omega_1^L$. More generally, the existence of a $\Sigma^1_2(x)$ mad family implies $\omega_1 = \omega_1^{L[x]}$.*

The argument for maximal independent families is a bit different. Let us recall the definition of a maximal independent family.

Definition 2.5.6. A set $X \subseteq [\omega]^\omega$ is called independent if for any $F \in [X]^{<\omega}$ and $G \in [X]^{<\omega}$ where $F \cap G = \emptyset$, $\bigcap_{x \in F} x \cap \bigcap_{y \in G}(\omega \setminus y)$ is infinite. An independent family is called maximal if it is maximal under inclusion.

The set $\bigcap_{x\in F} x \cap \bigcap_{y\in G}(\omega \setminus y)$ is often denoted $\sigma(F,G)$. We will also use this notation below. Note that an independent family X is not maximal iff there is a real x so that $x \cap \sigma(F,G)$ and $(\omega \setminus x) \cap \sigma(F,G)$ are infinite for all $F, G \in [X]^{<\omega}$ where $F \cap G = \emptyset$. Such a real will be called independent over X.

We obtain the following result.

Theorem 2.5.7. *If there is a Σ^1_2 maximal independent family, then $\omega_1 = \omega_1^L$. More generally, the existence of a $\Sigma^1_2(x)$ maximal independent family implies $\omega_1 = \omega_1^{L[x]}$.*

In [38] Miller basically proved that a Cohen real is independent over any ground model coded analytic independent family. He did not put his theorem in these words, so before we go on let us repeat his argument.

Lemma 2.5.8 ([38, Proof of Theorem 10.28])**.** *Let $\varphi(x)$ be a $\mathbf{\Sigma}^1_1$ formula defining an independent family and let c be a Cohen real over V. Then in $V[c]$, c is independent over the family defined by $\varphi(x)$.*

Proof. Let X denote the set $\{x \in [\omega]^\omega : \varphi(x)\}$ in any model extending V. Note that in any model X is an independent family by Schoenfield absolutness. Let

$$K = \{x \in [\omega]^\omega : \exists F \in [X]^{<\omega} \exists G \in [X]^{<\omega}(F \cap G = \emptyset \wedge |\sigma(F,G) \cap x| < \omega)\}$$

and

$$H = \{x \in [\omega]^\omega : \exists F \in [X]^{<\omega} \exists G \in [X]^{<\omega}(F \cap G = \emptyset \wedge |\sigma(F,G) \cap (\omega \setminus x)| < \omega)\}.$$

These sets are both analytic. Note that x is independent over X iff $x \notin H \cup K$. To show that any Cohen real c is independent over X, i.e. $c \notin H \cup K$ it suffices to prove that H and K are meager. Why? When $H \cup K$ is meager then there is a meager F_σ set C so that $H \cup K \subseteq C$ and this statement is absolute ($\forall x(x \in H \cup K \to x \in C)$). As c is Cohen, $V[c] \models c \notin C$ and thus $V[c] \models c \notin H \cup K$ which implies that in $V[c]$, c is independent over X.

So let us prove:

Claim. *K and H are meager.*

Proof. Suppose e.g. that H is nonmeager. The argument for K will follow similarly. Because H is analytic it has the Baire property and is thus comeager somewhere. It is well known and easy to see that any comeager set contains a perfect set of almost disjoint reals. So let $P \subseteq H$ be a perfect almost disjoint family. For each $x \in P$ we have F_x and G_x so that $\sigma(F_x, G_x) \subseteq^* x$. By the Delta system lemma, there is a set $S \in [P]^{\omega_1}$ and $R \in [P]^{<\omega}$ so that

$$\forall x \neq y \in S((F_x \cup G_x) \cap (F_y \cup G_y) = R).$$

For any $x \in S$ we define $R^0_x = R \cap F_x$ and $R^1_x = R \cap G_x$. As S is uncountable there is an uncountable $S' \subseteq S$ so that

$$\forall x, y \in S'(R^0_x = R^0_y \wedge R^1_x = R^1_y).$$

But now note that for any $x \neq y \in S'$, $F_x \cap G_y = (R \cap F_x) \cap (R \cap G_y) = R^0_x \cap R^1_y = R^0_x \cap R^1_x = \emptyset$. By symmetry we also have that $F_y \cap G_x = \emptyset$ and this implies that

$$(F_x \cup F_y) \cap (G_x \cup G_y) = \emptyset.$$

In particular we can form $\sigma(F_x \cup F_y, G_x \cup G_y)$. By choice of F_x, G_x, F_y, G_y we have that

$$\sigma(F_x \cup F_y, G_x \cup G_y) \subseteq^* x \cap y =^* \emptyset$$

as P was an almost disjoint family. But this contradicts the independence of X. $\square$

$\square$

Proof of Theorem 2.5.7. Assume X is a Σ^1_2 maximal independent family. Then in L, X is also independent and it can be written as a union $\bigcup_{\xi < \omega_1^L} X_\xi$ of analytic sets X_ξ. Assuming for a contradiction $\omega_1^L < \omega_1$, there is a Cohen real c over L. We have that $\omega_1^{L[c]} = \omega_1^L$ and in $L[c]$, X still corresponds to the union $\bigcup_{\xi < \omega_1^L} X_\xi$. By the above lemma c is independent over all the X_ξ so in particular c is independent over X. This statement is Π^1_2 and thus absolute between any inner models containing c. In particular in V, X is not maximal. $\square$

Theorem 2.5.9. *If there is a Σ^1_2 Hamel basis of $\mathbb{R}$, then $\omega_1 = \omega_1^L$. More generally, the existence of a $\Sigma^1_2(x)$ Hamel basis of $\mathbb{R}$ implies $\omega_1 = \omega_1^{L[x]}$.*

A Hamel basis of $\mathbb{R}$ is a maximal set of linearly independent reals over the rationals $\mathbb{Q}$. Again it was Miller who first showed that a Cohen real in $\mathbb{R}$ is independent over any ground model coded analytic linearly independent family.

Lemma 2.5.10 ([38, Proof of Theorem 9.25])**.** *Assume $A \subseteq \mathbb{R}$ is an analytic set of reals that are linearly independent over the field of rationals. Assume $c \in \mathbb{R}$ is a Cohen real over V. Then in $V[c]$, c is linearly independent over (the reinterpretation of) A.*

Proof. We assume that $A \neq \emptyset$, else the argument is trivial. Let $x \in A \cap V$ be arbitrary. Suppose that $U \Vdash$ "$\dot{c}$ is not independent over A" where $U \subseteq \mathbb{R}$ is some basic open set. Say $x_0, \dots, x_k \in A \cap V$ and $q_0, \dots, q_k \in \mathbb{Q}$ are such that

$$U \Vdash \exists x_{k+1}, \dots, x_n \in A \setminus V \exists q_{k+1}, \dots, q_n \in \mathbb{Q}(\dot{c} = q_0 x_0 + \dots + q_n x_n)$$

for some $n \in \omega$. Now let $c \in U$ be Cohen over V and $x_{k+1}, \dots, x_n \in A \setminus V, q_{k+1}, \dots q_n \in \mathbb{Q}$ so that

$$c = q_0 x_0 + \dots + q_n x_n.$$

Let $s \neq 0$ be a small enough rational number so that $c + sx \in U$. Recall that, as $x \in V$, $c + sx$ is also a Cohen real over V. Thus let $y_{k+1}, \dots, y_n \in A \setminus V, r_{k+1}, \dots, r_n \in \mathbb{Q}$ so that

$$c + sx = q_0 x_0 + \dots + q_k x_k + r_{k+1} y_{k+1} + \dots + r_n y_n.$$

But now we have that

$$r_{k+1} y_{k+1} + \dots + r_n y_n - (q_{k+1} x_{k+1} + \dots + q_n x_n) = sx$$

and so A is not linearly independent in $V[c]$. But this is impossible by absoluteness. □

Proof of Theorem 2.5.9. Same as the proof of Theorem 2.5.7. □

For ultrafilters the proof is not much different. We give a proof in Chapter 3.

Theorem 2.5.11. *If there is a Σ^1_2 ultrafilter, then $\omega_1 = \omega_1^L$. More generally, the existence of a $\Sigma^1_2(x)$ ultrafilter implies $\omega_1 = \omega_1^{L[x]}$.*

We want to remark the ideas above can also be used to show that under Martin's Axiom none of the families above have $\mathbf{\Sigma}^1_2$ witnesses.

Theorem 2.5.12. *MA(ω_1) implies that there is no $\mathbf{\Sigma}^1_2$ ilt, mad family, maximal independent family, Hamel basis or ultrafilter.*

Proof. For mad families this was proven in [66]. For ilt's Theorem 2.2.3 is enough. For ultrafilters it suffices to note that under MA(ω_1) every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable (see [32]) and an ultrafilter cannot be Lebesgue measurable. The argument for independent families and Hamel bases is the same. Write $X = \bigcup_{\xi < \omega_1} B_\xi$ where the B_ξ's are analytic. Let M be an elementary submodel of size ω_1 containing all the parameters defining the B_ξ's. Then let $c \in V$ be Cohen over M and use Lemma 2.5.8 or Lemma 2.5.10 to conclude that c is independent or linearly independent over X. □

2.6 Solovay's model

In this section we prove the following result.

Theorem 2.6.1. *There is no ilt in Solovay's model.*

Let us review some basics about Solovay's model. A good presentation of Solovay's model can be found in [31, Chapter 26]. Assuming κ is an inaccessible cardinal in the constructible universe L we first form an extension V of L in which $\omega_1 = \kappa$ using the Lévy collapse (see again [31, Chapter 26]). Then we let $W \subseteq V$ consist of all sets which are hereditarily definable from ordinals and reals as the only parameters. W is then called Solovay's model. The only facts that we use about W are listed below and are well-known.

Suppose $a \in 2^\omega \cap W$ is arbitrary, then

1. for every poset $\mathbb{P} \in H(\kappa)^{L[a]}$, there is a $\mathbb{P}$ generic filter over $L[a]$ in W,
2. whenever $x \in 2^\omega \cap W$, there is a poset $\mathbb{P} \in H(\kappa)^{L[a]}$, $\sigma \in H(\kappa)^{L[a]}$ a $\mathbb{P}$ name and $G \in W$ a $\mathbb{P}$ generic over $L[a]$ so that $x = \sigma[G]$.

Suppose $X \in \mathcal{P}(2^\omega) \cap W$. Then there is $a \in 2^\omega \cap W$ and a formula $\varphi(x)$ in the language of set theory using only a and ordinals as parameters so that

3. for any poset $\mathbb{P} \in H(\kappa)^{L[a]}$, $\sigma \in H(\kappa)^{L[a]}$ a $\mathbb{P}$ name and $G \in W$, $\mathbb{P}$ generic over $L[a]$,
$$\sigma[G] \in X \leftrightarrow \exists p \in G(p \Vdash \varphi(\sigma)).$$

Until the end of the section we are occupied with proving Theorem 2.6.1. To prove Theorem 2.6.1, assume that $X \in \mathcal{P}(2^\omega) \cap W$ is linearly ordered with respect to $\subseteq^*$. We will show that X cannot be an ilt. Let $a \in 2^\omega \cap W$ and $\varphi(x)$ be as in (3). To simplify notation we will assume that $a \in L$ and thus $L[a] = L$. From now on let us work in L.

Lemma 2.6.2. *Let $\mathbb{P} \in H(\kappa)$, $p \in \mathbb{P}$ and σ a $\mathbb{P}$ name so that $p \Vdash \varphi(\sigma)$. Then there is $p_0, p_1 \leq p$ and $n \in \omega$ so that for any $m \geq n$,*

$$\exists r \leq p_0(r \Vdash m \in \sigma) \rightarrow p_1 \Vdash m \in \sigma.$$

Proof. Consider $\mathbb{P} \times \mathbb{P} \in H(\kappa)$ and σ_0 and σ_1 the $\mathbb{P} \times \mathbb{P}$ names so that whenever $G_0 \times G_1$ is $\mathbb{P} \times \mathbb{P}$ generic over V then $\sigma_0[G_0 \times G_1] = \sigma[G_0]$, $\sigma_1[G_0 \times G_1] = \sigma[G_1]$.

Note that $(p, p) \Vdash \varphi(\sigma_0) \wedge \varphi(\sigma_1)$, because whenever $G_0 \times G_1$ is $\mathbb{P} \times \mathbb{P}$ generic over V with $(p, p) \in G_0 \times G_1$ then G_0 and G_1 are $\mathbb{P}$ generic over V with $p \in G_0, G_1$. But then there must be $(p_0, p_1) \leq (p, p)$ and $n \in \omega$ so that either,

$$(p_0, p_1) \Vdash \sigma_0 \setminus n \subseteq \sigma_1$$

or

$$(p_0, p_1) \Vdash \sigma_1 \setminus n \subseteq \sigma_0.$$

Say wlog that $(p_0, p_1) \Vdash \sigma_0 \setminus n \subseteq \sigma_1$. Note that whenever $\exists r_0 \leq p_0(r_0 \Vdash m \in \sigma)$ for some $m \geq n$ then $p_1 \Vdash m \in \sigma$. Suppose this was not the case. Then there is $r_1 \leq p_1$ so that $r_1 \Vdash m \notin \sigma$. But then $(r_0, r_1) \Vdash \exists m \geq n(m \in \sigma_0 \wedge m \notin \sigma_1)$ which is a contradiction to $(r_0, r_1) \leq (p_0, p_1)$. $\square$

Still in L, let $\langle \mathbb{P}_\xi, p_\xi, \sigma_\xi : \xi < \kappa \rangle$ enumerate all triples $\langle \mathbb{P}, p, \sigma \rangle$, where $\mathbb{P} \in H(\kappa)$, $p \in \mathbb{P}$ and $\sigma \in H(\kappa)$ is a $\mathbb{P}$ name so that $p \Vdash \varphi(\sigma)$. This is possible as $|H(\kappa)| = \kappa$.

For every $\xi < \kappa$ we find $p^0_\xi, p^1_\xi \leq p_\xi$ in $\mathbb{P}_\xi$ and $n \in \omega$ so that for every $m \geq n$

$$\exists r \leq p^0_\xi (r \Vdash m \in \sigma_\xi) \to p^1_\xi \Vdash m \in \sigma_\xi.$$

Let $x_\xi = \{m \in \omega : p^1_\xi \Vdash m \in \sigma_\xi\}$ for every $\xi < \kappa$.

Claim. *$\{x_\xi : \xi < \kappa\}$ has the finite intersection property.*

Proof of Claim. Suppose $x_{\xi_0}, \dots x_{\xi_{k-1}}$ are such that $\bigcap_{i<k} x_{\xi_i}$ is finite, say $\bigcap_{i<k} x_{\xi_i} \subseteq n$. Consider the poset $\mathbb{Q} = \prod_{i<k} \mathbb{P}_{\xi_i} \in H(\kappa)$, $(p^0_{\xi_0}, \dots, p^0_{\xi_{k-1}}) \in \mathbb{Q}$ and for every $i < k$, σ_i the $\mathbb{Q}$ name so that whenever $(G_0, \dots, G_{k-1})$ is $\mathbb{Q}$ generic then $\sigma_i[G_0 \times \dots \times G_{k-1}] = \sigma_{\xi_i}[G_i]$.

We have that $(p^0_{\xi_0}, \dots, p^0_{\xi_{k-1}}) \Vdash \varphi(\sigma_0) \wedge \dots \wedge \varphi(\sigma_{k-1})$ and thus, as X has the finite intersection property, there is $m \geq n$ and $(r_0, \dots, r_{k-1}) \leq (p^0_{\xi_0}, \dots, p^0_{\xi_{k-1}})$ so that

$$(r_0, \dots, r_{k-1}) \Vdash m \in \bigcap_{i<k} \sigma_i.$$

But this means that $r_i \Vdash m \in \sigma_i$ and thus $m \in x_{\xi_i}$ for each individual i. This contradicts $\bigcap_{i<k} x_{\xi_i} \subseteq n$ as $m \geq n$. $\square$

Let $\mathcal{F}$ be the filter generated by $\{x_\xi : \xi < \kappa\}$. We have that $\mathcal{F} \in \mathcal{P}([\omega]^\omega)$ and thus $\mathcal{F} \in H(\kappa)$. Moreover we have that $\mathbb{M}(\mathcal{F}) \in H(\kappa)$. Thus in W there is $y \in [\omega]^\omega$ a $\mathbb{M}(\mathcal{F})$ generic real over L.

Claim. *For every $x \in X$, $y \subseteq^* x$. In particular X is not an ilt.*

Proof of Claim. Let $x \in X$ be arbitrary. Then we have in L a poset $\mathbb{P} \in H(\kappa)$ and a $\mathbb{P}$ name σ so that there is in W a $\mathbb{P}$ generic G over V so that $x = \sigma[G]$. Moreover there is $p \in G$ so that $p \Vdash \varphi(\sigma)$.

It suffices to show that there is some $\xi < \kappa$ and $q \in G$ so that $q \Vdash x_\xi \subseteq^* \sigma$. To see this we simply show that the set of conditions $q \in \mathbb{P}$ so that $\exists \xi < \kappa (q \Vdash x_\xi \subseteq^* \sigma)$ is dense below p. To show this fix $p' \leq p$ arbitrary. Let ξ be such that $\langle \mathbb{P}, p', \sigma \rangle = \langle \mathbb{P}_\xi, p_\xi, \sigma_\xi \rangle$. But then $p^1_\xi \leq p_\xi$ and $p^1_\xi \Vdash x_\xi \subseteq^* \sigma_\xi$. $\square$

This finishes the proof of Theorem 2.6.1.

2.7 Σ^1_2 versus Π^1_1

Theorem 2.7.1. *The existence of a $\Sigma^1_2(x)$ ilt implies the existence of a $\Pi^1_1(x)$ ilt.*

Proof. We are going to prove the statement only for lightface Σ^1_2 as everything will relativize. So let X be a Σ^1_2 ilt.

Claim. *$X \cap L$ is cofinal in X (where L is the constructible universe).*

Proof. By Theorem 2.5.3 we have that $\omega_1 = \omega_1^L$ must be the case. Thus X can be written as a union $\bigcup_{\xi<\omega_1} X_\xi$ of analytic sets X_ξ which are coded in L (see the proof of Lemma 2.5.2). Note that $X \cap L$ is an ilt in L by a downwards absoluteness argument. This implies that for every $\xi < \omega_1$ there is $x \in L \cap X$ which is a pseudointersection of X_ξ. The statement "x is a pseudointersection of X_ξ" is absolute. Thus $X \cap L$ is indeed cofinal in X. $\square$

As $[\omega]^\omega \cap L$ is Σ^1_2 we may just assume that $X \subseteq L$. Let $\varphi(x, w)$ be Π^1_1 such that $x \in X$ iff $\exists w \varphi(x, w)$. Using Π^1_1 uniformization we can further assume that $x \in X$ iff $\exists! w \varphi(x, w)$.

The idea will now be to get a linearly ordered tower that basically consists of $x \in X$ together with their unique witness w. To do this we have to introduce some notation.

- For $y \subseteq [\omega \times \omega]^\omega$, we write y_n for y's n'th vertical section.
- For $x \in [\omega]^\omega$, we write $x(n)$ for the n'th element of x.

We now define the new ilt Y which lives on $\omega \times \omega$. A set $y \in [\omega \times \omega]^\omega$ is in Y iff the following are satisfied:

1. For every $n \geq 1$, $y_n = y_0 \setminus y_0(n)$ or $y_n = y_0 \setminus y_0(n + 1)$.

2. If $w \in 2^\omega$ is such that $w(n) = \begin{cases} 0 \text{ if } y_{n+1} = y_0 \setminus y_0(n+1) \\ 1 \text{ if } y_{n+1} = y_0 \setminus y_0(n+2) \end{cases}$ then $\varphi(y_0, w)$ and in particular $y_0 \in X$.

Claim. *Y is Π^1_1 ilt.*

Proof. (i) Checking whether $y \in [\omega \times \omega]^\omega$ is as described in (1) is Δ^1_1. Checking whether for the function $w \in 2^\omega$ as in (2), $\varphi(y_0, w)$ holds is Π^1_1.

(ii) Y is linearly ordered by $\subseteq^*$: Let us note first that whenever $x \subsetneq^* y$ then eventually $x(n) > y(n)$. Why is this the case? As $x \subsetneq^* y$ (so $x \neq^* y$), there is a big enough $n \in \omega$ so that $\forall m \geq n(|y \cap x(m)| > m)$. But this means that $x(m) > y(m)$ for all $m \geq n$.

Now let's assume that $x \neq y \in Y$ and without loss of generality that $x_0 \subsetneq^* y_0$. By the observation above there is an n so that for every $m \geq n$, $x_0(m) > y_0(m)$ and $x_0(m) \in y_0$. But this also means that $\forall m \geq n$,

$$x_m \subseteq x_0 \setminus x_0(m) \subseteq y_0 \setminus y_0(m+1) \subseteq y_m.$$

In particular $x_m \subseteq y_m$ for $m \geq n$. For $k < n$ we have that $x_k \subsetneq^* y_k$. Thus all together we have that $x \subsetneq^* y$.

(iii) Y has no pseudointersection: Suppose z is a pseudointersection of Y. If there is $n \in \omega$ so that $|z_n| = \omega$, then z_n is a pseudointersection of X. Else let $x = \{\min z_n : n \in \omega \wedge z_n \neq \emptyset\}$. It is easy to see that x must be infinite (else z would not be $\subseteq^*$ below any member of Y). We claim that x is a pseudointersection of X. Namely let $y_0 \in X$ be arbitrary where $y \in Y$. As $z \subseteq^* y$, there is an n so that $\forall m \geq n(z_m \neq \emptyset \rightarrow (m, \min z_m) \in y)$. This means in particular that $\forall m \geq n(z_m \neq \emptyset \rightarrow \min z_m \in y_0)$. □

□

CHAPTER **3**

The definability of ultrafilters

3.1 Introduction

In this chapter we will study the definability of ultrafilters and more specifically ultrafilter bases. Filters will always live on ω and contain all cofinite sets. Thus a filter is a subset of $\mathcal{P}(\omega)$ and we can study its definability. It is well known that an ultrafilter can neither have the Baire property nor be Lebesgue measurable. This already rules out the existence of analytic ultrafilter generating sets as the generated filter will also be analytic and thus have the Baire property. But this still leaves open the possibility of a coanalytic ultrafilter base since a priori the generated set will only be $\mathbf{\Delta}^1_2$. Recall that for $x, y \in [\omega]^\omega$ we write $x \subseteq^* y$ whenever $x \setminus y$ is finite. An ultrafilter $\mathcal{U}$ is called a *P-point* if for any countable $\mathcal{F} \subseteq \mathcal{U}$, there is $x \in \mathcal{U}$ so that $\forall y \in \mathcal{F}(x \subseteq^* y)$. $\mathcal{U}$ is a *Q-point* if for any partition $\langle a_n : n \in \omega\rangle$ of ω into finite sets a_n, there is $x \in \mathcal{U}$ so that $\forall n \in \omega(|x \cap a_n| \leq 1)$. A *Ramsey* ultrafilter is an ultrafilter that is both a P- and a Q-point. A more commonly known and equivalent definition for Ramsey ultrafilters $\mathcal{U}$ is that for any coloring $c\colon [\omega]^2 \to 2$, there is $x \in \mathcal{U}$ so that c is homogeneous on x, i.e. $c \restriction [x]^2$ is constant. In fact we will show in Section 3.3 that:

Theorem 3.1.1. *There is a Π^1_1 base for a P-point in the constructible universe L.*

Theorem 3.1.2. *There is a Π^1_1 base for a Q-point in the constructible universe L.*

In Section 3.2 we will take another look at Miller's coding technique which is used for the above results. In strong contrast we will show in Section 3.4 that:

Theorem 3.1.3. *There is no $\mathbf{\Pi}^1_1$ base for a Ramsey ultrafilter.*

Notice that any ultrafilter that is $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ is already $\mathbf{\Delta}^1_n$. Namely suppose that φ defines an ultrafilter, then we have that $\varphi(x) \leftrightarrow \neg\varphi(\omega \setminus x)$. Moreover any base for an ultrafilter that is $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ generates a $\mathbf{\Delta}^1_n$ or respectively a $\mathbf{\Delta}^1_{n+1}$ ultrafilter.

In Section 3.5 we will compare $\mathbf{\Delta}^1_2$ ultrafilters to $\mathbf{\Pi}^1_1$ bases. As a main result we find that:

Theorem 3.1.4. *The following are equivalent for any $r \in 2^\omega$, $n \in \omega$.*

1. *There is a $\Delta^1_{n+1}(r)$ ultrafilter.*
2. *There is a $\Pi^1_n(r)$ ultrafilter base.*

In Section 3.6 we study the effects of adding reals to the definability of utrafilters. In Section 3.7 we introduce a new cardinal invariant that is a Borel version of the classical ultrafilter number $\mathfrak{u}$ and make some observations.

3.2 Miller's coding technique revisited

When we say that z codes the ordinal α, we mean the following. To any real $z \in 2^\omega$ we associate a relation E_z on ω defined by

$$E_z(n, m) \leftrightarrow z(2^n 3^m) = 1.$$

This relation may be a linear order and if it is a well-order and isomorphic to α we say that it codes α. Such α is unique and we define $\|z\| := \alpha$. More generally we say that z codes M if (ω, E_z) is isomorphic to $(M, \in)$. The set of $z \in 2^\omega$ coding an ordinal is denoted WO. The set WO is tightly connected to coanalytic sets. On one hand side, WO is itself Π^1_1 and on the other, for any $\mathbf{\Pi}^1_1$ set $X \subseteq 2^\omega$, there is a continuous function $f\colon 2^\omega \to 2^\omega$ so that $X = f^{-1}(\mathrm{WO})$.

There is a very canonical way of defining in L various combinatorial subsets X of reals in a $\mathbf{\Delta}^1_2$ fashion. Typically the elements are found recursively by making adequate choices which are absolute between models of the form L_α (e.g. taking the $<_L$ least candidate which has some simple property holding with respect to the previously chosen reals).

Then $x \in X$ can be written as

$$\underbrace{\exists M}_{\exists}\,[\,\underbrace{M \text{ is well-founded}}_{\forall},\, \underbrace{x \in M}_{\Delta^1_1} \text{ and } \underbrace{M \models V = L \wedge \varphi(x)}_{\Delta^1_1}\,] \tag{3.1}$$

or as

$$\underbrace{\forall M \models V = L, x \in M}_{\forall + \Delta^1_1}[\underbrace{M \text{ is not well-founded}}_{\exists} \text{ or } \underbrace{M \models \varphi(x)}_{\Delta^1_1}]. \tag{3.2}$$

Quantifying over models is shorthand for quantifying over codes in 2^ω of countable models satisfying some basic set theoretic axioms. Thus e.g. (3.1) can be recast as "$\exists z \in 2^\omega((\omega, E_z)$ is well-founded, $x \in (\omega, E_z)$ and $(\omega, E_z) \models V = L \wedge \varphi(x))$", where $x \in (\omega, E_z)$ means that $x \in M$ for M the Mostowski collapse of (ω, E_z). It is not difficult to see that this can be expressed in a Δ^1_1 way.

As such, finding a Δ^1_2 ultrafilter base in L is very simple. The major improvement in Miller's technique is to get rid of the first existential quantifier in (3.1). This is done by letting x already encode a relevant well-founded model M in a Borel or even in a recursive way. Then if C is the Borel coding relation used, the definition usually looks as follows:

$$\underbrace{x \in Y}_{\forall} \text{ and } \underbrace{\forall z \in 2^\omega}_{\forall}[\underbrace{\neg C(x, z)}_{\Delta^1_1} \text{ or } \underbrace{(\omega, E_z) \models V = L \wedge \varphi(x)}_{\Delta^1_1}], \tag{3.3}$$

for some known coanalytic Y.

Lemma 3.2.1. *There is a lightface Borel set $C \subseteq (2^\omega)^3$ so that whenever z codes $\alpha < \omega_1$ and $r, y \in 2^\omega$ then $(z, r, y) \in C$ iff y codes $L_\alpha[r]$.*

Proof. The claim is easy to verify by noting that an adequate E_y can be constructed by recursion on α. Thus $(z, r, y) \in C$ can be defined by formulas of the form "$\exists/\forall \langle E_k : k \in \omega\rangle$ a sequence indexed via the order coded by z satisfying certain recursive assumptions, E_y is the union of all E_k". This definition is uniform on z and r. □

Lemma 3.2.2. *There is a recursive function $(\cdot)^{+\omega} \colon 2^\omega \to 2^\omega$ so that whenever z codes α, then $(z)^{+\omega}$ codes $\alpha + \omega$.*

Proof. Let $(z)^{+\omega} = y$ such that $y(2^n 3^m) = 1$ iff $\begin{cases} n \text{ even } \wedge m \text{ even } \wedge z(2^{\frac{n}{2}} 3^{\frac{m}{2}}) = 1 \\ n \text{ even } \wedge m \text{ odd} \\ n \text{ odd } \wedge m \text{ odd } \wedge n < m. \end{cases}$

□

3.3 Π^1_1 bases for P- and Q-points

In Chapter 2 we constructed, using Miller's technique, a coanalytic tower (i.e. a set $X \subseteq [\omega]^\omega$ well-ordered wrt $^* \supseteq$ and with no pseudointersection). A crucial property of the tower was that all its elements were split by the set of even natural numbers. In particular this meant that the tower could not generate an ultrafilter. We will construct in L a tower generating an ultrafilter and thus generating a P-point.

Before we start to construct the Π^1_1 P-point base, we need some ingredients.

Definition 3.3.1. We call $\mathcal{W}^+$ the set of $x \in [\omega]^\omega$ containing arbitrary long arithmetic progressions, i.e. $\forall k \in \omega \exists a, b \in \omega(\{a \cdot l + b : l < k\} \subseteq x)$.

The following fact follows from Van der Waerden's Theorem which is well known.

Fact. *The set $\mathcal{W} = \mathcal{P}(\omega) \setminus \mathcal{W}^+$ is a proper ideal on ω. It is called the Van der Waerden ideal.*

Proof of Theorem 3.1.1. Let $(y_\alpha)_{\alpha<\omega_1}$ enumerate $[\omega]^\omega$ via the global L well-order $<_L$. The statement "y is the α'th element according to $<_L$" is absolute between L_β's with $y \in L_\beta$ and $\alpha \in L_\beta$. Let $O\colon 2^\omega \to 2^\omega$ be the following lightface Borel function: If $x \subseteq \omega$ we want to define a unique sequence $(i_n)_{n\in\omega}$ of subsets of ω so that $\max i_n < \min i_{n+1}$ and i_{n+1} is the next maximal arithmetic progression in x of length ≥ 3 above $\max i_n$ (note that any pair of natural numbers forms an arithmetic progression). Now if this sequence can be defined up to ω (in particular every i_n is finite), then we define $O(x)(n) = 1$ iff i_n has even length. Else we let $O(x)(n) = 0$.

We construct a sequence $(x_\xi, \delta_\xi)_{\xi<\omega_1}$ where $x_\xi \in [\omega]^\omega$, $\delta_\xi < \omega_1$ as follows.

Given $(x_\xi, \delta_\xi)_{\xi<\alpha}$ we let δ_α be the least limit ordinal such that $\sup_{\xi<\alpha} \delta_\xi < \delta_\alpha$, $y_\alpha \in L_{\delta_\alpha}$ and δ_α projects to ω, i.e. $L_{\delta_\alpha+\omega} \models \delta_\alpha$ is countable. It is not difficult to see that the set of ordinals projecting to ω is unbounded in ω_1. $x_\alpha = x$ is chosen least in the $<_L$ well-order so that

(a) $x \subseteq^* x_\xi$ for every $\xi < \alpha$,

(b) $x \in \mathcal{W}^+$

(c) $x \subseteq y_\alpha$ or $x \subseteq \omega \setminus y_\alpha$.

(d) $O(x)$ codes δ_α.

Note that any sequence $(x_\xi)_{\xi<\omega_1}$ defined as above is a tower generating an ultrafilter.

Claim 3.3.2. *x_α can be found in $L_{\delta_\alpha+\omega}$.*

Proof. Note that the definition of $(x_\xi)_{\xi<\alpha}$ is absolute between L_β's. In particular $(x_\xi)_{\xi<\alpha}$ can be defined over L_{δ_α}. As δ_α projects to ω, there is an enumeration $(x^n)_{n\in\omega}$ of $\{x_\xi : \xi < \alpha\}$ in $L_{\delta_\alpha+\omega}$. Given y_α we have that, as $\mathcal{W}$ is an ideal, that for every $\xi < \alpha$, $y_\alpha \cap x_\xi \in \mathcal{W}^+$ or $\omega \setminus y_\alpha \cap x_\xi \in \mathcal{W}^+$. Assume wlog that for cofinally many x_ξ, $y_\alpha \cap x_\xi \in \mathcal{W}^+$ is the case. This implies that for all x_ξ this is the case as $(x_\xi)_{\xi<\alpha}$ forms a tower. Again as δ_α projects to ω, there is a real $z \in L_{\delta_\alpha+\omega} \cap 2^\omega$ coding δ_α. Now we define a sequence $(i_n)_{n\in\omega}$ of finite subsets of ω so that $\max i_n < \min i_{n+1}$, $i_n \subseteq y_\alpha \cap \bigcap_{k\leq n} x^k$, i_n consists of an arithmetic progression so that its length is $\geq n$ and it is even iff $z(n) = 1$. Moreover $\min i_n$ is chosen large enough so that $i_{n-1} \cup i_n$ cannot form an arithmetic progression. $x := \bigcup_{n\in\omega} i_n$ can be defined in $L_{\delta_\alpha+\omega}$ and satisfies (a)-(d). Thus in particular the $<_L$-least such x exists in $L_{\delta_\alpha+\omega}$. □

Remark 3.3.3. There is a formula $\varphi(x)$ in the language of set theory so that $\varphi(x)$ iff $\exists\xi(x = x_\xi)$ and $L_\beta \models \varphi(x)$ for some β implies that $\varphi(x)$ is true. Moreover $L_{\delta_\xi+\omega} \models \varphi(x_\xi)$ for every ξ.

Proof. $\varphi(x)$ expresses that there is an ordinal α and a sequence $(x_\xi, \delta_\xi)_{\xi\leq\alpha}$ according to the recursive definitions given above so that $x = x_\alpha$. □

Now we can check that the set $X = \{x_\xi : \xi \in \omega_1\}$ is Π^1_1. Let C and $(\cdot)^{+\omega}$ be as in Lemma 3.2.1 and Lemma 3.2.2. Then $x \in X$ iff

$$O(x) \in \mathrm{WO} \text{ and } \forall z[\neg C(O(x)^{+\omega}, 0, z) \text{ or } (\omega, E_z) \models \varphi(x)].$$

□

Definition 3.3.4. The ideal Fin^2 on $\omega \times \omega$ consists of $x \in \mathcal{P}(\omega \times \omega)$ so that $\forall^\infty n \in \omega \forall^\infty m \in \omega(\langle n, m\rangle \notin x)$

Proof of Theorem 3.1.2. The ultrafilter that we construct will live on $\omega \times \omega$. Let $O\colon (\mathrm{Fin}^2)^+ \to 2^\omega$ be the following Borel function. Given $x \in (\mathrm{Fin}^2)^+$ let x_0, x_1 be the first two infinite vertical sections of x. We denote with $x_0(n)$ or $x_1(n)$ the n'th element of x_0 or x_1. Then

$$O(x)(n) = \begin{cases} 0 \text{ if } x_0(n) \geq x_1(n) \\ 1 \text{ if } x_1(n) > x_0(n). \end{cases}$$

As in the proof of Theorem 3.1.1 we let $(y_\alpha)_{\alpha<\omega_1}$ enumerate $[\omega \times \omega]^\omega$ and $(P_\alpha)_{\alpha<\omega_1}$ enumerate all partitions of $\omega \times \omega$ into finite sets via the well-ordering $<_L$.

Similarly to the proof of Theorem 3.1.1 we construct a sequence $(x_\xi, \delta_\xi)_{\xi<\omega_1}$ where $x_\xi \in (\mathrm{Fin}^2)^+$, intersections of finitely many elements in $\{x_\xi : \xi < \omega_1\}$ are in $(\mathrm{Fin}^2)^+$ and $\delta_\xi < \omega_1$ as follows.

Given $(x_\xi, \delta_\xi)_{\xi<\alpha}$ we let δ_α be the least limit ordinal such that $\sup_{\xi<\alpha} \delta_\xi < \delta_\alpha$, $y_\alpha, P_\alpha \in L_{\delta_\alpha}$ and δ_α projects to ω, i.e. $L_{\delta_\alpha+\omega} \models \delta_\alpha$ is countable. $x_\alpha = x$ is then chosen least in the $<_L$ well-order so that

(a) $\{x\} \cup \{x_\xi : \xi < \alpha\}$ has all finite intersections in $(\mathrm{Fin}^2)^+$,

(b) $x \in (\mathrm{Fin}^2)^+$,

(c) $x \subseteq y_\alpha$ or $x \subseteq \omega \setminus y_\alpha$,

(d) for every $a \in P_\alpha$, $|a \cap x| \leq 1$,

(e) $O(x)$ codes δ_α.

Again we show that such an x_α exists and can be found in $L_{\delta_\alpha+\omega}$.

Claim 3.3.5. *x_α can be found in $L_{\delta_\alpha+\omega}$.*

Proof. We have that if $(x_\xi)_{\xi<\alpha}$ exists then it must be definable over L_{δ_α}. As δ_α projects to ω there is in $L_{\delta_\alpha+\omega}$ an enumeration $(x^n)_{n\in\omega}$ of all finite intersections of elements in $\{x_\xi : \xi < \alpha\}$. We are given $y_\alpha \in L_{\delta_\alpha}$. It is not hard to see that either y_α or $(\omega\times\omega) \setminus y_\alpha$ is in $(\mathrm{Fin}^2)^+$ and has $(\mathrm{Fin}^2)^+$ intersection with all x^n. Without loss of generality we assume y_α has this property. Let $P_\alpha = \{a_i : i \in \omega\}$ and $z \in 2^\omega \cap L_{\delta_\alpha+\omega}$ code δ_α. Further let $k_0 < k_1$ be first so that the k_0'th and k_1'th vertical section of y_α is infinite. Let $(p_j)_{j\in\omega}$ enumerate $\omega \times \omega$ in a way that every pair (n, m) appears infinitely often. Given $(p_j)_{j\in\omega}$ we define recursively a sequence $\langle m_i^0, m_i^1\rangle_{i\in\omega}$ and auxiliarily $(n_i)_{i\in\omega}$ as follows:

– for every i, $\langle m_i^0, m_i^1\rangle \in y_\alpha$, $\langle m_i^0, m_i^1\rangle \notin \bigcup_{j<i} a_{n_j}$ and $\langle m_i^0, m_i^1\rangle \in a_{n_i}$,

– if $i = 3j$ for $j \in \omega$, then $\langle m_i^0, m_i^1\rangle$ is in the $p_j(0)$'th infinite vertical section of $y_\alpha \cap x^{p_j(1)}$ greater than k_1,

– if $i = 1 \mod 3$ then $m_i^0 = k_0$ and $m_{i+1}^0 = k_1$ and $m_i^1 \geq m_{i+1}^1$ or $m_{i+1}^1 > m_i^1$ depending on whether $z(i) = 0$ or $z(i) = 1$.

Now the set $\{\langle m_i^0, m_i^1\rangle : i \in \omega\} \in L_{\delta_\alpha+\omega}$ satisfies (a)-(e) as can be seen from the construction. In particular $L_{\delta_\alpha+\omega}$ contains the $<_L$-least such set. $\square$

The set $\{x_\xi : \xi < \omega_1\}$ is now a base for a Q-Point and as in the proof of Theorem 3.1.1 it is Π^1_1. $\square$

3.4 There are no $\mathbf{\Pi}^1_1$ Ramsey ultrafilter bases

Definition 3.4.1. Let $\mathcal{F}$ be a filter. Then the forcing $\mathbb{M}(\mathcal{F})$ consists of pairs $(a, F) \in [\omega]^{<\omega} \times \mathcal{F}$ such that $\max a < \min F$. A condition (b, E) extends (a, F) if b is an end-extension of a, $E \subseteq F$ and $b \setminus a \subseteq F$.

$\mathbb{M}(\mathcal{F})$ is the natural forcing to add a pseudointersection of $\mathcal{F}$.

Definition 3.4.2. Let $\mathcal{F}$ be a filter. Then we define the game $G(\mathcal{F})$ as follows:

Player I	$F_0 \in \mathcal{F}$		$F_1 \in \mathcal{F}$		$\dots$
Player II		$a_0 \in [F_0]^{<\omega} \setminus \{\emptyset\}$		$a_1 \in [F_1]^{<\omega} \setminus \{\emptyset\}$	$\dots$

Player II wins iff $\bigcup_{n\in\omega} a_n \in \mathcal{F}$.

Lemma 3.4.3. *Let $\mathcal{F}$ be a filter on ω. Then TFAE:*

(i) For any countable model M, $\mathcal{F} \in M$, of enough set theory, there is $x \in \mathcal{F}$, $\mathbb{M}(\mathcal{F})$ generic over M.

(ii) I has no winning strategy in $G(\mathcal{F})$.

Proof. (i) implies (ii): Suppose σ is a winning strategy for I in $G(\mathcal{F})$ and let $\sigma, \mathcal{F} \in M$. Wlog we assume that $\sigma(\langle\rangle) = \omega$. Thus Player II is allowed to play any a_0 as his first move and then σ carries on as if a_0 had not been played. In particular this means that any initial play a_0 of II is a legal move, i.e. $\langle a_0\rangle \in \mathrm{dom}(\sigma)$. Consider the dense sets $D_n := \{(s, F) : F \subseteq \bigcap\{\sigma(\langle s_0, \dots, s_{n-1}\rangle) : \langle s_0, \dots, s_{n-1}\rangle \in \mathrm{dom}(\sigma), \bigcup_{i<k} s_i = s\}\}$ for $n \in \omega$. $D_n \in M$ for every $n \in \omega$. By (i) there is $x \in \mathcal{F}$, $\mathbb{M}(\mathcal{F})$ generic over M. This means that for every $n \in \omega$ there is s an initial segement of x and $F \in \mathcal{F}$ so that $(s, F) \in D_n$ and $x \setminus s \subseteq F$. Now using this construct a sequence $\langle s_i\rangle_{i\in\omega}$ and $\langle F_i\rangle_{i\in\omega}$ recursively so that:

1. $\bigcup_{i<n} s_i$ is an initial segment of x for every $n \in \omega$,

2. $\max s_i < \min s_{i+1}$ for every $i \in \omega$,

3. $x \setminus \bigcup_{i<n} s_i \subseteq F_n$ for every $n \in \omega$,

4. $(\bigcup_{i<n} s_i, F_n) \in D_n$.

We find recursively that $\langle s_i \rangle_{i<n} \in \mathrm{dom}(\sigma)$, i.e. $\langle s_i \rangle_{i<n}$ is a legal move. But $\bigcup_{i\in\omega} s_i = x \in \mathcal{F}$ contradicting σ being a winning strategy for I.

(ii) implies (i): Let $M \ni \mathcal{F}$ be countable and $\langle D_n \rangle_{n\in\omega}$ enumerate all dense subsets of $\mathbb{M}(\mathcal{F})$ in M. We describe a strategy for Player I: I starts by playing some F_0 so that there is $(t_0, F_0) \in D_0$. Then Player II will play $a_0 \subseteq F_0$, i.e. $(t_0 \cup a_0, F_0) \leq (t_0, F_0)$. Now I plays F_1 so that there is $(t_0 \cup a_0 \cup t_1, F_1) \in D_1$, $(t_0 \cup a_0 \cup t_1, F_1) \leq (t_0 \cup a_0, F_0)$...

By assumption there is a winning run $\langle a_i \rangle_{i\in\omega}$ for II according to this strategy. This means that $\bigcup a_i \in \mathcal{F}$ and moreover $x = \bigcup a_i \cup \bigcup t_i \in \mathcal{F}$ where t_i are as described. But x is now $\mathbb{M}(\mathcal{F})$ generic over M. $\square$

It is a well known theorem that for ultrafilters $\mathcal{U}$, I not having a winning strategy in $G(\mathcal{U})$ is equivalent to $\mathcal{U}$ being a P-point. For sake of completeness we prove a more general (in light of Lemma 3.4.3) version of this below. Recall that $\mathfrak{p}$ is the pseudointersection number, i.e. the least size of a set $\mathcal{B} \subseteq [\omega]^\omega$ with the finite intersection property and no pseudointersection, a set $x \in [\omega]^\omega$ such that $x \subseteq^* y$ for all $y \in \mathcal{B}$. The bounding number $\mathfrak{b}$ is the least size of a family $\mathcal{B} \subseteq \omega^\omega$ such that there is no $f \in \omega^\omega$ eventually dominating every member of $\mathcal{B}$. It is well known that $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{b}$. An ultrafilter $\mathcal{U}$ is called a P_κ point if for any $\mathcal{B} \in [\mathcal{U}]^{<\kappa}$ there is a pseudointersection $x \in \mathcal{U}$ of $\mathcal{B}$. In particular a P-point is the same as a $P_{\aleph_1}$-point.

Lemma 3.4.4. *Assume $\kappa \leq \mathfrak{p}$ and $\mathcal{U}$ is an ultrafilter. Then TFAE:*

(i) $\mathcal{U}$ is a P_κ-point.

(ii) For every M a model of enough set theory with $|M| < \kappa$ and $\mathcal{U} \in M$, there is $x \in \mathcal{U}$ which is $\mathbb{M}(\mathcal{U})$ generic over M.

Proof. (ii) implies (i) is trivial.

(i) implies (ii): Let $|M| < \kappa \leq \mathfrak{p}$. Then as $\mathcal{U}$ is a P_κ-point, there is $U \in \mathcal{U}$ so that $U \subseteq^* V$ for every $V \in M \cap \mathcal{U}$. Define for every $D \in M$, which is a dense open subset of $\mathbb{M}(\mathcal{U})$ and every $V \in M \cap \mathcal{U}$ a function $f_{D,V} : \omega \to \omega$ so that for $n \in \omega$:

$$\forall a \subseteq n \exists b \subseteq [n, f_{D,V}(n)) \exists V' \in M \cap \mathcal{U}((a \cup b, V') \leq (a, V) \wedge (a \cup b, V') \in D \wedge U \setminus f_{D,V}(n) \subseteq V').$$

The set of functions $f_{D,V}$ is smaller than $\kappa \leq \mathfrak{p} \leq \mathfrak{b}$. Thus there is one $f \in \omega^\omega$ dominating all of them. Let $i_0 = 0$, $i_{n+1} = f(i_n)$. We write $I_n = [i_n, i_{n+1})$. As $\mathcal{U}$ is an ultrafilter, either $U_0 = \bigcup_{n\in\omega} I_{2n} \cap U$ or $U_1 = \bigcup_{n\in\omega} I_{2n+1} \cap U$ is in $\mathcal{U}$. Assume wlog that $U_0 \in \mathcal{U}$.

We define a σ-centered partial order $\mathbb{P}$ as follows. $\mathbb{P}$ consists of pairs (s, F) where

1. $s\colon n \to [\omega]^{<\omega}$ for some $n \in \omega$,
2. $s(i) \subseteq I_i$ for every $i < n$,
3. $s(i) = U \cap I_i$ when i is even,
4. $F \in \mathcal{U} \cap M$.

A condition (t, F) extends (s, E) iff $t \supseteq s$, $F \subseteq E$ and $(t(i) \subseteq E)$ whenever $i \in \operatorname{dom} t \setminus \operatorname{dom} s$ is odd. For any $D \in M$ which is dense in $\mathbb{M}(\mathcal{U})$ we define a subset of $\mathbb{P}$, $\tilde{D}$ as follows:

$$\tilde{D} = \{(t, F) : (\bigcup_{i \in \operatorname{dom} t} t(i), F) \in D\}.$$

We claim that $\tilde{D}$ is dense in $\mathbb{P}$. Let $(s, E) \in \mathbb{P}$ be arbitrary. Then as $f_{D,E} <^* f$ there is $n \in \omega$ so that $[i_{2n+1}, f_{D,E}(i_{2n+1})) \subseteq [i_{2n+1}, i_{2n+2})$ and $2n + 1 \geq \operatorname{dom} s$. Now extend s to s_0 so that $\operatorname{dom} s_0 = 2n + 1$ and $s_0(i) = \emptyset$ for $i \in 2n + 1 \setminus \operatorname{dom} s$ odd and $s_0(i) = U \cap I_i$ for i even. By definition of $f_{D,E}$ there is $b \subseteq I_{2n+1}$ so that $\exists F \subseteq E$ with $(a \cup b, F) \in D$ where $a = \bigcup_{i<2n+1} s_0(i)$, $(a \cup b, F) \leq (a, E)$ and $U \setminus i_{2n+2} \subseteq F$. Let $t = s_0 \cup \{(2n + 1, b)\}$. Then $(t, F) \leq (s, E)$ in $\mathbb{P}$ and $(t, F) \in \tilde{D}$.

Now as $\kappa \leq \mathfrak{p}$ and by Bell's theorem (see [6]) there is a $\mathbb{P}$ generic real $g\colon \omega \to [\omega]^{<\omega}$ over M. But then $x := \bigcup_{i\in\omega} g(i) \in \mathcal{U}$ as $U_0 \subseteq x$ and x is $\mathbb{M}(\mathcal{U})$ generic over M. $\square$

Corollary 3.4.5. Suppose $\mathcal{U}$ is a P-point, M countable and $\mathcal{U} \in M$. Then there is $x \in \mathcal{U}$, $\mathbb{M}(\mathcal{U})$ generic over M.

Lemma 3.4.6 (see [24, Chapter 24])**.** *Assume $\mathcal{U} \in M$ is a Ramsey ultrafilter and x is $\mathbb{M}(\mathcal{U})$ generic over M. Then every $y \subseteq^* x$ is $\mathbb{M}(\mathcal{U})$ generic over M.*

Proof of Theorem 3.1.3. Suppose $\mathcal{U}$ is a Ramsey ultrafilter with a coanalytic base $X \subseteq [\omega]^\omega$. As X is coanalytic, there is a continuous function $f : 2^\omega \to 2^\omega$ so that

$$x \in X \leftrightarrow f(x) \in \mathrm{WO}\,.$$

Let M be a countable model elementary in some $H(\theta)$ where θ is large enough and $\mathcal{U}, f \in M$. As $\mathcal{U}$ is a P-point and by Corollary 3.4.5, there is $x \in \mathcal{U}$ that is $\mathbb{M}(\mathcal{U})$ generic over M. Moreover as $\mathcal{U}$ is Ramsey and by Lemma 3.4.6, any $y \subseteq^* x$ is also generic over M. Let $\alpha = M \cap \omega_1$ and let $y \in X$ be arbitrary such that $y \subseteq^* x$. Let $\beta = \|f(y)\|$, then $\beta \in M[y]$. Thus $\beta < \alpha = M[y] \cap \omega_1$. As y was arbitrary, we have shown that the set $X' = \{y : f(y) \in WO \wedge \|f(y)\| \leq \alpha\} \subseteq X$ contains

$\{y \subseteq^* x : y \in X\}$. This means that X' also generates $\mathcal{U}$. But X' is Borel and cannot generate an ultrafilter. □

3.5 $\mathbf{\Delta}^1_2$ versus $\mathbf{\Pi}^1_1$

Using a result of Shelah we can show the following.

Theorem 3.5.1. *It is consistent that every P-point is $\mathbf{\Delta}^1_2$ and has no $\mathbf{\Pi}^1_1$ base.*

Proof. This follows immediately by [56, Theorem XVIII.4.1] and the subsequent remark, which states that starting from L we can choose any Ramsey ultrafilter $\mathcal{U}$ and pass to an extension in which $\mathcal{U}$ generates the unique P-point up to permutation of ω. Moreover this ultrafilter will stay Ramsey.

Thus let $\mathcal{U}$ be any (definition of a) Δ^1_2 Ramsey ultrafilter in L. Now apply Shelah's theorem to this ultrafilter and pass to an extension V of L in which $\mathcal{U}^L$ generates the unique P-point and is Ramsey. In $V, \mathcal{U}^V$ will still have the finite intersection property and $\mathcal{U}^L \subseteq \mathcal{U}^V$ by Shoenfield-absolutness. Thus in $V, \mathcal{U}^V$ generates the same ultrafilter as $\mathcal{U}^L$. As $\mathcal{U}^V$ is Δ^1_2 the ultrafilter it generates will be Δ^1_2 as well. We know that in V there is for every P-point $\mathcal{V}$ a permutation f of ω so that $V \in \mathcal{V} \leftrightarrow f(V) \in \mathcal{U}$. In particular $\mathcal{V}$ has a $\Delta^1_2(f)$ definition. On the other hand, every P-point is a Ramsey ultrafilter so none of them can have a $\mathbf{\Pi}^1_1$ base by Theorem 3.1.3. □

Proof of Theorem 3.1.4. To simplify notation we assume that $r = 0$. Let $\mathcal{U}$ be a Δ^1_{n+1} ultrafilter. Let us introduce the following notation. For $y \in [\omega \times \omega]^\omega$, we let y_n be y's n'th vertical section. We let $z(y) = \{n \in \omega : y_n \neq \emptyset\}$. When $z(y)$ is infinite then we denote with y^n, the n'th nonempty vertical section of y.

The Fubini product of $\mathcal{U}, \mathcal{U} \otimes \mathcal{U}$, consists of all $y \in [\omega \times \omega]^\omega$ so that

$$\{n \in \omega : y_n \in \mathcal{U}\} \in \mathcal{U}.$$

$\mathcal{U} \otimes \mathcal{U}$ is again an ultrafilter. We will show that it has a Π^1_n base. Let $\varphi(x, w)$ be Π^1_1 so that

$$x \in \mathcal{U} \leftrightarrow \exists w \in 2^\omega(\varphi(x, w)).$$

Let $r : \omega \times 2^\omega \to 2^\omega$ be a recursive function such that for any sequence $\langle w_n \rangle_{n \in \omega}$ there is $w \in 2^\omega$, which is not eventually constant, so that $r(n, w) = w_n$ for every $n \in \omega$.

Let $O\colon [\omega \times \omega]^\omega \to 2^\omega$ be the function defined by

$$O(y)(n) = \begin{cases} 0 \text{ if } |z(y)| < \omega \\ 0 \text{ if } \min y^n \geq \min y^{n+1} \\ 1 \text{ if } \min y^n < \min y^{n+1}. \end{cases}$$

O is obviously lightface Borel. Let us define $X \subseteq [\omega \times \omega]^\omega$ as follows:

$$y \in X \leftrightarrow |z(y)| = \omega \wedge \varphi(z(y), r(0, O(y))) \wedge \forall n \in \omega \exists s \in [\omega]^{<\omega}[\varphi(s \cup y^n, r(n+1, O(y)))].$$

X is obviously Π^1_n. Moreover $X \subseteq \mathcal{U} \otimes \mathcal{U}$. To see this let us decode what $y \in X$ means. The first clause in the definition of X says that y has infinitely many nonempty vertical sections. The next clause ensures that $z(y) \in \mathcal{U}$ as witnessed by $r(0, O(y))$, the 0'th real coded by $O(y)$. The last clause ensures that for every nonempty vertical section y^n of y, $s \cup y^n$ is in $\mathcal{U}$ for some finite s as witnessed by $r(n+1, O(y))$, the $n+1$'th real coded by $O(y)$. In particular $y^n \in \mathcal{U}$. Thus we indeed have that $y \in X \to y \in \mathcal{U} \otimes \mathcal{U}$.

Moreover we have that X is a base for $\mathcal{U} \otimes \mathcal{U}$. To see this fix $u \in \mathcal{U} \otimes \mathcal{U}$ and we show that there is $y \in X$ so that $y \subseteq u$. First let $y_0 = \bigcup\{\{n\} \times u_n : n \in \omega, u_n \in \mathcal{U}\}$, i.e. we remove from u the vertical sections that are not in $\mathcal{U}$. Then we let w_0 be such that $\varphi(z(y_0), w_0)$ holds true. Further we let w_{n+1} be such that $\varphi(y_0^n, w_{n+1})$ holds true. Let $w \in 2^\omega$ be a single real coding the sequence $\langle w_n \rangle_{n\in\omega}$ via r, i.e. $r(n, w) = w_n$ for every $n \in \omega$. Find a sequence $\langle m_n \rangle_{n\in\omega}$ so that $m_n \in y_0^n$ for every n and $w(n) = 1$ iff $m_{n+1} > m_n$. Such a sequence can be constructed recursively. Whenever $w(n) = 1$ we can simply find $m_{n+1} \in y_0^{n+1}$ large enough such that $m_{n+1} > m_n$ and if additionally $w(n+1), \dots, w(n+k)$ is a maximal block of 0s in w then we let $m_{n+1} = \dots = m_{n+k+1} \in y^{n+1} \cap \dots \cap y^{n+k+1}$. Finally given the sequence $\langle m_n \rangle_{n\in\omega}$ let $y = \bigcup\{\{z(y_0)(n)\} \times (y_0^n \setminus m_n) : n \in \omega\}$, where $z(y_0)(n)$ is the n'th element of $z(y_0)$. We see that $y \subseteq y_0 \subseteq u$, that $z(y) = z(y_0)$, that $y^n =^* y_0^n$ for every n and that $O(y) = w$. In particular $y \in X$ by definition of X.

□

3.6 Adding reals

Let $A \subseteq V$. A set $X \in V$ is called $\mathrm{OD}(A)$ if it is definable over V from ordinals and elements of A as parameters. Recall that a poset $\mathbb{P}$ is weakly homogeneous if for any $p, q \in \mathbb{P}$, there is an automorphism $\pi\colon \mathbb{P} \to \mathbb{P}$ so that $\pi(p)$ is compatible to q. In this section we will denote with $\mathcal{P}_A$ the collection of weakly homogeneous $\mathrm{OD}(A)$ posets.

Theorem 3.6.1. *Let c be a Cohen real over V, $\mathbb{P} \in (\mathcal{P}_V)^{V[c]}$ and G a $\mathbb{P}$-generic filter over $V[c]$. Then in $V[c][G]$, c is splitting over any set of reals with the finite intersection property that is* $\mathrm{OD}(V)$.

Proof. Let $X \in V[c][G]$ be an $\mathrm{OD}(V)$ set of reals with the finite intersection property, say $V[c][G] \models$ "$\dot{X} = \{x \in [\omega]^\omega : \varphi(x, a, \bar{\alpha})\}$" where $a \in V$ and $\bar{\alpha}$ is a finite sequence of ordinals. Wlog we may assume that X is a filter, since the filter generated by X is also $\mathrm{OD}(V)$. Suppose c does not split X. This means exactly that $c \in X$ or $\omega \setminus c \in X$. Thus there is $s \subseteq c$, deciding the formula and parameters defining $\mathbb{P}$, and $\dot{p}$ with $\dot{p}[c] \in G$, $(s, \dot{p}) \Vdash$ "φ defines a filter" so that either

$$(s, \dot{p}) \Vdash \dot{c} \in \dot{X}$$

or

$$(s, \dot{p}) \Vdash \omega \setminus \dot{c} \in \dot{X}.$$

But now notice that $c' = s \cup \{(n, 1-m) : (n, m) \in c, n \geq |s|\}$ is also Cohen over V with $s \subseteq c'$ (we identify c as a subset of ω with its characteristic function). Moreover $V[c] = V[c']$ and thus $\dot{\mathbb{P}}[c] = \dot{\mathbb{P}}[c']$. Let $p_0 := \dot{p}[c]$ and $p_1 := \dot{p}[c']$. Working in $V[c]$ we find that $p_0, p_1 \in \mathbb{P}$, so there is an automorphism π of $\mathbb{P}$ so that $\pi(p_1)$ is compatible to p_0. Let H be $\mathbb{P}$-generic over $V[c]$ containing p_0 and $\pi(p_1)$. In either of the above cases, $V[c][H] \models \varphi(c, a, \bar{\alpha}) \wedge \varphi(c', a, \bar{\alpha})$. This is a contradiction to $(s, \dot{p}) \Vdash$ "φ defines a filter". □

Theorem 3.6.2. *Let r be a random real over V, $\mathbb{P} \in (\mathcal{P}_V)^{V[r]}$ and G a $\mathbb{P}$-generic filter over $V[r]$. Then in $V[r][G]$, r is splitting over any set of reals with the finite intersection property that is* $\mathrm{OD}(V)$.

Proof. Let us assume that $\mathbb{P}$ is simply the trivial forcing, since this part of the argument is essentially the same as in the last proof. As before we fix $X \in V[r]$ an $\mathrm{OD}(V)$ set with the finite intersection property and we assume that it is already a filter.

First note that any finite modification of r is still a random real. Moreover, as complementation is a measure preserving homeomorphism of 2^ω, the complement of a random real is still random. Thus any $r' =^* \omega \setminus r$ is still random.

Now similarly as in the proof for Cohen forcing we find that there is Borel set B of positive measure coded in V so that $r \in B$ and

$$B \Vdash \dot{r} \in X$$

or

$$B \Vdash \omega \setminus \dot{r} \in X.$$

Recall that for any Borel set A of positive measure, its E_0 closure $\tilde{A} = \{x \in 2^\omega : \exists y \in A(x =^* y)\}$ has full measure. To see this Let $\varepsilon > 0$ be arbitrarily small. Apply Lebesgue's density theorem to find a basic open set $[s] \subseteq 2^\omega$ so that $\frac{\mu(A \cap [s])}{\mu([s])} > 1 - \varepsilon$. Follow from this that $\mu(\tilde{A}) > 1 - \varepsilon$.

Now let $C := \{\omega \setminus x : x \in \tilde{B}\}$. C is coded in V and has full measure. Thus we have that $r \in B \cap C$. By definition of C, there is $r' \in B$ so that $r' =^* \omega \setminus r$. Moreover r' is also a random real over V by our first remark. $r, r' \in X$ and $\omega \setminus r, \omega \setminus r' \in X$ are both contradictions to X having the finite intersection property. $\square$

Recall that Silver forcing consists of partial functions $p \colon \omega \to 2$ so that $\omega \setminus \operatorname{dom}(p)$ is infinite.

Theorem 3.6.3. *Let s be a Silver real over V, $\mathbb{P} \in (\mathcal{P}_V)^{V[s]}$ and G a $\mathbb{P}$-generic filter over $V[s]$. Then, in $V[s]$, there is a real splitting over any set of reals that is* $\mathrm{OD}(V)$ *in* $V[s][G]$.

Proof. Again we only consider the case when $\mathbb{P}$ is trivial. Let $X \in V[s]$ be an $\mathrm{OD}(V)$ filter. Let $S_s = \{n \in \omega : |\{m < n : s(m) = 1\}| \text{ is even}\}$. As before assume $p \subseteq s$ is such that either

$$p \Vdash S_{\dot{s}} \in X$$

or

$$p \Vdash \omega \setminus S_{\dot{s}} \in X.$$

Let $n = \min(\omega \setminus \operatorname{dom}(p))$ and note that s' defined by $s'(i) = s(i)$ for all $i \neq n$ and $s'(n) = 1 - s(n)$ is also Silver and $p \subseteq s'$. But $S_{s'} =^* \omega \setminus S_s$. We get the same contradiction as in the last two proofs. $\square$

Corollary 3.6.4. Let $r \in 2^\omega$ and assume that there is a Cohen, a random or a Silver real over $L[r]$. Then there is no $\Delta^1_2(r)$ ultrafilter.

In particular, the existence of a $\Delta^1_2(r)$ ultrafilter implies that $\omega_1 = \omega_1^{L[r]}$.

Proof. Suppose that φ is a $\Sigma^1_2(r)$ definition for an ultrafilter and that c is a Cohen, random or Silver real over $L[r]$. In $L[r][c]$, the set defined by φ will have the finite intersection property by downwards absoluteness. Thus by Theorem 3.6.1, 3.6.2 or 3.6.3 respectively, $L[r][c] \models \exists x \in [\omega]^\omega \forall y \in [\omega]^\omega (\neg\varphi(y) \vee (|x \cap y| = \omega \wedge |x \cap \omega \setminus y| = \omega))$. This is a $\Sigma^1_3(x, c)$ statement, so by upwards Shoenfield absoluteness it holds true in $V \supseteq L[x][c]$. Thus φ cannot define an ultrafilter in V.

The second part follows, since whenever $\omega_1^{L[r]} < \omega_1$, there is a Cohen real in V over $L[r]$. □

Another way of seeing the above for Cohen or random forcing is to use the classical result of Judah and Shelah (see [30]), saying that the existence of a Cohen or random real over $L[r]$ is equivalent to every $\Delta^1_2(r)$ set having the Baire property or being Lebesgue measurable respectively.

Corollary 3.6.5. There is no $OD(\mathbb{R})$ ultrafilter, in particular no projective one, after adding ω_1 many Cohen reals in a finite support iteration, random reals using a product of Lebesgue measure or Silver reals in a countable support iteration.

Proof. Let $\langle c_\alpha : \alpha < \omega_1 \rangle$ be Cohen reals added via a finite support iteration over a ground model V and suppose that in $V[\langle c_\alpha : \alpha < \omega_1 \rangle]$ there is an ultrafilter $\mathcal{U}$ definable from a real a and ordinals. It is well known that there is $\xi < \omega_1$ so that $a \in V[\langle c_\alpha : \alpha \in \omega_1 \setminus \{\xi\}\rangle]$. But then, by Theorem 3.6.1, c_ξ is splitting over $\mathcal{U}$, since $V[\langle c_\alpha : \alpha < \omega_1 \rangle] = V[\langle c_\alpha : \alpha \in \omega_1 \setminus \{\xi\}\rangle][c_\xi]$.

The argument for random reals is essentially the same.

Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \omega_1 \rangle$ be the ω_1-length countable support iteration of Silver forcing. Any real a appears in $V^{\mathbb{P}_\xi}$ for some $\xi < \omega_1$. But now note that $\mathbb{P}_{\omega_1}$ is $\mathrm{OD}(V)$ and weakly homogeneous. Moreover, $\mathbb{P}_{\omega_1} \cong \mathbb{P}_\xi * \dot{\mathbb{P}}_{\omega_1}$. Thus applying Theorem 3.6.3, we find that there is no ultrafilter definable from parameters in $V^{\mathbb{P}_\xi}$ over $V^{\mathbb{P}_{\omega_1}}$. In particular there is no $\mathrm{OD}(\{a\})$ ultrafilter in $V^{\mathbb{P}_{\omega_1}}$. □

3.7 The Borel ultrafilter number

The ultrafilter number $\mathfrak{u}$ is the least size of a base for an ultrafilter. As with mad families (see [41]) and maximal independent families (see [8]) it makes sense to introduce a Borel version of the ultrafilter number that is closely related to the definability of ultrafilters.

Definition 3.7.1. The Borel ultrafilter number is defined as

$$\mathfrak{u}_B := \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathbf{\Delta}^1_1, \bigcup \mathcal{B} \text{ is an ultrafilter}\}.$$

Note that $\aleph_1 \leq \mathfrak{u}_B$, as a countable union of Borel sets is Borel.

Remark 3.7.2. Let $\mathfrak{u}'_B = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathbf{\Delta}^1_1, \bigcup \mathcal{B} \text{ is an ultrafilter base}\}$ and $\mathfrak{u}''_B = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathbf{\Delta}^1_1, \bigcup \mathcal{B} \text{ generates an ultrafilter}\}$. Then $\mathfrak{u}''_B = \mathfrak{u}'_B = \mathfrak{u}_B$.

Proof. Obviously, $\mathfrak{u}''_B \leq \mathfrak{u}'_B \leq \mathfrak{u}_B$. Remember that whenever B is Borel, then the filter F_B that it generates is analytic. Thus $\mathfrak{u}''_B$ is uncountable as well. Now let $\mathcal{B}$ be a collection of Borel sets, whose union generates an ultrafilter. We may assume that $\mathcal{B}$ is closed under finite unions. For every $B \in \mathcal{B}$, let F_B be the filter generated by B. Since F_B is analytic, we can write it as an ω_1-union $F_B = \bigcup_{\alpha<\omega_1} F^\alpha_B$ of Borel sets. Now consider $\{F^\alpha_B : B \in \mathcal{B}, \alpha < \omega_1\}$. It has the same size as $\mathcal{B}$ and is a witness for $\mathfrak{u}_B$. $\square$

Any coanalytic set is an ω_1-union of Borel sets. Thus the existence of a coanalytic ultrafilter base implies that $\mathfrak{u}_B = \aleph_1$.

Theorem 3.7.3. $\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N}), \mathfrak{b} \leq \mathfrak{u}_B \leq \mathfrak{u}$.

Proof. Let $\mathcal{B}$ be a collection of $< \mathrm{cov}(\mathcal{M})$ many Borel sets and assume that $\bigcup \mathcal{B}$ has the finite intersection property. Let $M \preccurlyeq H(\theta)$ for some large θ, so that $|M| < \mathrm{cov}(\mathcal{M})$ and $\mathcal{B} \subseteq M$. Then there is a Cohen real c over M. But then in $M[c]$, c is splitting over every $B \in \mathcal{B}$. Moreover in V it is true that c is splitting over B, by Σ_1-upwards-absoluteness. Thus c is splitting over $\bigcup \mathcal{B}$ which cannot be an ultrafilter. The argument for random forcing is exactly the same.

For $\mathfrak{b} \leq \mathfrak{u}_B$, note that any Borel filter is meager. By a classical result of Talagrand (see [64]), meager filters $\mathcal{F}$ are exactly those for which there is $f \in \omega^\omega$ so that $\forall x \in \mathcal{F} \forall^\infty n \in \omega (x \cap [n, f(n)) \neq \emptyset)$. For $\mathcal{B}$ a collection of Borel filters, we let f_B be such a function for every $B \in \mathcal{B}$. If $\mathcal{B}$ has size smaller than $\mathfrak{b}$, then there is a single function $f \in \omega^\omega$ so that $f_B <^* f$ for each $B \in \mathcal{B}$. Now note that $x_0 \cup x_1 = \omega$, where $x_0 := \bigcup_{n\in\omega} [f^{2n}(0), f^{2n+1}(0))$ and $x_1 := \bigcup_{n\in\omega} [f^{2n+1}(0), f^{2n+2}(0))$. But neither x_0 nor x_1 can be in $\bigcup \mathcal{B}$. $\square$

Question 3.7.1. Is it consistent that $\mathfrak{u}_B < \mathfrak{u}$? Is it consistent that there is a Π^1_1 ultrafilter base while $\aleph_1 < \mathfrak{u}$?

We will give a positive answer in the next chapter.

CHAPTER **4**

Hypergraphs and definability in tree forcing extensions

4.1 Introduction

The starting observation for this chapter is that almost all examples of maximal families that we considered in the introduction can be treated in the same framework, as *maximal independent sets in hypergraphs*.

Definition 4.1.1. A *hypergraph* E on a set X is a collection of finite non-empty subsets of X, i.e. $E \subseteq [X]^{<\omega} \setminus \{\emptyset\}$. Whenever $Y \subseteq X$, we say that Y is *E-independent* if $[Y]^{<\omega} \cap E = \emptyset$. Moreover, we say that Y is *maximal E-independent* if Y is maximal under inclusion as an E-independent subset of X.

Whenever X is a topological space, $[X]^{<\omega}$ is the disjoint sum of the spaces $[X]^n$ for $n \in \omega$. Here, as usual, $[X]^n$ is endowed with the natural quotient topology induced by the equivalence relation $(x_0, \ldots, x_{n-1}) \sim (y_0, \ldots, y_{n-1})$ iff $\{x_0, \ldots, x_{n-1}\} = \{y_0, \ldots, y_{n-1}\}$ on the space of injective n-tuples on X. Whenever X is Polish, $[X]^{<\omega}$ is Polish as well and we can study its definable subsets. In particular, we can study definable hypergraphs on Polish spaces.

The main result of this paper is the following theorem.

Theorem 4.1.2. *After forcing with the ω_2-length csi of Sacks or splitting forcing over L, every analytic hypergraph on a Polish space has a $\mathbf{\Delta}^1_2$ maximal independent set.*

This extends a result by Schrittesser [50], who proved the above for Sacks forcing, which we denote by $\mathbb{S}$, and ordinary 2-dimensional graphs (see also [51]). We will also prove the case of finite products but our main focus will be on the countable support iteration. *Splitting forcing* $\mathbb{SP}$ (Definition 4.4.1) is a less-known forcing notion that was originally introduced by Shelah in [55] and has been studied in more detail recently ([61], [62], [25] and [35]). Although it is very natural and gives a minimal way to add a *splitting real* (see more below), it has not been exploited a lot and to our knowledge, there is no major set theoretic text treating it in more detail.

Our three guiding examples for Theorem 4.1.2 will be *ultrafilters*, *maximal independent families* and *Hamel bases*.

Recall that an *ultrafilter* on ω is a maximal subset $\mathcal{U}$ of $\mathcal{P}(\omega)$ with the *strong finite intersection property*, i.e. the property that for any $\mathcal{A} \in [\mathcal{U}]^{<\omega}$, $|\bigcap \mathcal{A}| = \omega$. Thus, letting $E_u := \{\mathcal{A} \in [\mathcal{P}(\omega)]^{<\omega} : |\bigcap \mathcal{A}| < \omega\}$, an ultrafilter is a maximal E_u-independent set. In the last chapter, we studied the projective definability of ultrafilters and introduced the cardinal invariant $\mathfrak{u}_B$, which is the smallest size of a collection of Borel subsets of $\mathcal{P}(\omega)$ whose union is an ultrafilter. If there is a $\mathbf{\Sigma}^1_2$ ultrafilter, then $\mathfrak{u}_B = \omega_1$, since every $\mathbf{\Sigma}^1_2$ set is the union of ω_1 many Borel sets. Recall that the classical ultrafilter number $\mathfrak{u}$ is the smallest size of an ultrafilter base. We showed in the last chapter, that $\mathfrak{u}_B \leq \mathfrak{u}$ and asked whether it is consistent that $\mathfrak{u}_B < \mathfrak{u}$ or even whether a Δ^1_2 ultrafilter can exist while $\omega_1 < \mathfrak{u}$. The difficulty is that we have to preserve a definition for an ultrafilter, while its interpretation in L must be destroyed. This has been achieved before for mad families (see [9]).

An *independent family* is a subset $\mathcal{I}$ of $\mathcal{P}(\omega)$ so that for any disjoint $\mathcal{A}_0, \mathcal{A}_1 \in [\mathcal{I}]^{<\omega}$, $|\bigcap_{x\in\mathcal{A}_0} x \cap \bigcap_{x\in\mathcal{A}_1} \omega \setminus x| = \omega$. It is called *maximal independent family* if it is additionally maximal under inclusion. Thus, letting $E_i = \{\mathcal{A}_0 \dot\cup \mathcal{A}_1 \in [\mathcal{P}(\omega)]^{<\omega} : |\bigcap_{x\in\mathcal{A}_0} x \cap \bigcap_{x\in\mathcal{A}_1} \omega \setminus x| < \omega\}$, a maximal independent family is a maximal E_i-independent set. The definability of maximal independent families was studied by Miller in [38], who showed that they cannot be analytic, and recently by Brendle, Fischer and Khomskii in [8], where they introduced the invariant $\mathfrak{i}_B$, the least size of a collection of Borel sets whose union is a maximal independent family. The classical independence number $\mathfrak{i}$ is simply the smallest size of a maximal independent family. In [8], it was asked whether $\mathfrak{i}_B < \mathfrak{i}$ is consistent and whether there can be a Π^1_1 maximal independent family while $\omega_1 < \mathfrak{i}$. Here, Π^1_1 can be changed to Δ^1_2, as shown in [8]. The difficulty in the problem is similar to that before.

A *Hamel basis* is a vector-space basis of $\mathbb{R}$ over the field of rationals $\mathbb{Q}$. Thus, letting $E_h := \{\mathcal{A} \in [\mathbb{R}]^{<\omega} : \mathcal{A} \text{ is linearly dependent over } \mathbb{Q}\}$, a Hamel basis is a maximal E_h-independent set. A Hamel basis must be as large as the continuum itself. This is reflected in the fact that, when adding a real, every ground-model Hamel basis is destroyed. But still it makes sense to ask how many Borel sets are needed to get one. Miller, also in [38], showed that a Hamel basis can never be analytic. As before, we may ask whether there can be a Δ^1_2 Hamel basis while CH fails. Again, destroying ground-model Hamel bases, seems to pose a major obstruction.

The most natural way to increase $\mathfrak{u}$ and $\mathfrak{i}$ is by iteratively adding *splitting reals*. Recall that for $x, y \in \mathcal{P}(\omega)$, we say that x *splits* y iff $|x \cap y| = \omega$ and $|y \setminus x| = \omega$. A real x is called *splitting over* V iff for every $y \in \mathcal{P}(\omega) \cap V$, x splits y. The classical forcing notions adding splitting reals are *Cohen*, *Random* and *Silver forcing* and forcings that add so called *dominating reals*. It was showed in Chapter 3, that all of these forcing notions fail in preserving definitions for ultrafilters and the same argument can be applied to independent families. For this reason, we are going to use the forcing notion $\mathbb{SP}$ that we mentioned above. As an immediate corollary of Theorem 4.1.2, we get the following.

Theorem 4.1.3. *It is consistent that* $\mathfrak{r} = \mathfrak{u} = \mathfrak{i} = \omega_2$ *while there is a* Δ^1_2 *ultrafilter, a* Π^1_1 *maximal independent family and a* Δ^1_2 *Hamel basis. In particular, we get the consistency of* $\mathfrak{i}_B, \mathfrak{u}_B < \mathfrak{r}, \mathfrak{i}, \mathfrak{u}$.

Here, $\mathfrak{r}$ is the reaping number, the least size of a set $\mathcal{S} \subseteq \mathcal{P}(\omega)$ so that there is no splitting real over $\mathcal{S}$. This solves Question 3.7.1 and the above mentioned question from [8]. Moreover, Theorem 4.1.2 gives a “black-box" way to get many results, saying that certain definable families exists in the Sacks model.

In [8], another cardinal invariant $\mathfrak{i}_{cl}$ is introduced, which is the smallest size of a collection of closed sets, whose union is a maximal independent family. Here, it is irrelevant whether we consider them as closed subsets of $[\omega]^\omega$ or $\mathcal{P}(\omega)$, since every closed subset of $[\omega]^\omega$ with the strong finite intersection property is σ-compact (see Lemma 4.5.15). In the model of Theorem 4.1.3, we have that $\mathfrak{i}_{cl} = \mathfrak{i}_B$, further answering the questions of Brendle, Fischer and Khomskii. On the other hand we show that $\mathfrak{d} \leq \mathfrak{i}_{cl}$, mirroring Shelah's result that $\mathfrak{d} \leq \mathfrak{i}$ (see [68]). Here, $\mathfrak{d}$ is the dominating number, the least size of a dominating family in $(\omega^\omega, <^*)$.

Theorem 4.1.4. *(ZFC)* $\mathfrak{d} \leq \mathfrak{i}_{cl}$.

The paper is organized as follows. In Section 4.2, we will consider basic results concerning iterations of tree forcings. This section is interesting in its own right and can be read independently from the rest. More specifically, we prove a version of continuous reading of names for countable support iterations that is widely applicable (Lemma 4.2.2). In Section 4.3, we prove our main combinatorial lemma (Main

Lemma 4.3.4 and 4.3.14) which is at the heart of Theorem 4.1.2. As for Section 4.2, Section 4.3 can be read independently of the rest, since our result is purely descriptive set theoretical. In Section 4.4, we introduce splitting and Sacks forcing and place it in bigger class of forcings to which we can apply the main lemma. This combines the results from Section 4.2 and 4.3. In Section 4.5, we bring everything together and prove Theorem 4.1.2, 4.1.3 and 4.1.4. We end with concluding remarks concerning the further outlook of our technique and pose some questions.

4.2 Tree forcing

Let A be a fixed countable set, usually ω or 2.

(a) A *tree* T on A is a subset of $A^{<\omega}$ so that for every $t \in T$ and $n < |t|$, $t \restriction n \in T$.

(b) T is *perfect* if for every $t \in T$ there are $s_0, s_1 \in T$ so that $s_0, s_1 \supseteq t$ and $s_0 \perp s_1$.

(c) A node $t \in T$ is called a *splitting node*, if there are $i \neq j \in A$ so that $t^\frown i, t^\frown j \in T$. The set of splitting nodes in T is denoted $\mathrm{split}(T)$.

(d) For any $t \in T$ we define the restriction of T to t as $T_t = \{s \in T : s \not\perp t\}$.

(e) The set of branches through T is denoted by $[T] = \{x \in A^\omega : \forall n \in \omega(x \restriction n \in T)\}$.

(f) A^ω carries a natural Polish topology generated by the clopen sets $[t] = \{x \in A^\omega : t \subseteq x\}$ for $t \in A^{<\omega}$. Then $[T]$ is closed in A^ω.

(g) Whenever $X \subseteq A^\omega$ is closed, there is a continuous *retract* $\varphi\colon A^\omega \to X$, i.e. $\varphi'' A^\omega = X$ and $\varphi \restriction X$ is the identity.

(h) A *tree forcing* is a collection $\mathbb{P}$ of perfect trees ordered by inclusion.

(i) By convention, all tree forcings are closed under restrictions, i.e. if $T \in \mathbb{P}$ and $t \in T$, then $T_t \in \mathbb{P}$, and the trivial condition is $A^{<\omega}$.

(j) The set $\mathcal{T}$ of perfect subtrees of $A^{<\omega}$ is a G_δ subset of $\mathcal{P}(A^{<\omega})$ and thus carries a natural Polish topology. It is not hard to see that it is homeomorphic to ω^ω, when $|A| \geq 2$.

(k) Let $\langle T_i : i < \alpha \rangle$ be a sequence of trees where α is an arbitrary ordinal. Then we write $\bigotimes_{i<\alpha} T_i$ for the set of finite partial sequences $\bar{s}$ where $\mathrm{dom}\,\bar{s} \in [\alpha]^{<\omega}$ and for every $i \in \mathrm{dom}\,\bar{s}$, $s(i) \in T_i$.

(l) $(A^\omega)^\alpha$ carries a topology generated by the sets $[\bar{s}] = \{\bar{x} \in (A^\omega)^\alpha : \forall i \in \operatorname{dom} \bar{s}(x(i) \in [s(i)])\}$ for $\bar{s} \in \bigotimes_{i<\alpha} A^{<\omega}$.

(m) Whenever $X \subseteq (A^\omega)^\alpha$ and $C \subseteq \alpha$, we define the *projection of X to C* as $X \restriction C = \{\bar{x} \restriction C : \bar{x} \in X\}$.

Fact. *Let $\mathbb{P}$ be a tree forcing and G a $\mathbb{P}$-generic filter over V. Then $\mathbb{P}$ adds a real $x_G := \bigcup\{s \in A^{<\omega} : \forall T \in G(s \in T)\} \in A^\omega$.*

Definition 4.2.1. We say that $(\mathbb{P}, \leq)$ is *Axiom A* if there is a decreasing sequence of partial orders $\langle \leq_n : n \in \omega\rangle$ refining $\leq$ on $\mathbb{P}$ so that

1. for any $n \in \omega$ and $T, S \in \mathbb{P}$, if $S \leq_n T$, then $S \cap A^{<n} = T \cap A^{<n}$,
2. for any fusion sequence, i.e. a sequence $\langle p_n : n \in \omega\rangle$ where $p_{n+1} \leq_n p_n$ for every n, $p = \bigcap_{n\in\omega} p_n \in \mathbb{P}$ and $p \leq_n p_n$ for every n,
3. and for any maximal antichain $D \subseteq \mathbb{P}$, $p \in \mathbb{P}$, $n \in \omega$, there is $q \leq_n p$ so that $\{r \in D : r \not\perp q\}$ is countable.

Moreover we say that $(\mathbb{P}, \leq)$ is *Axiom A with continuous reading of names* (*crn*) if there is such a sequence of partial orders so that additionally,

4. for every $p \in \mathbb{P}$, $n \in \omega$ and $\dot{y}$ a $\mathbb{P}$-name for an element of a Polish space[1] X, there is $q \leq_n p$ and a continuous function $f \colon [q] \to X$ so that

$$q \Vdash \dot{y}[G] = f(x_G).$$

Although (1) is typically not part of the definition of Axiom A, we include it for technical reasons. The only classical example that we are aware of, in which it is not clear whether (1) can be realized, is Mathias forcing.

Let $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \alpha\rangle$ be a countable support iteration of tree forcings that are Axiom A with crn, where for each $\beta < \alpha$,

$$\Vdash_{\mathbb{P}_\beta} \text{“}\langle \dot{\leq}_{\beta,n} : n \in \omega\rangle \text{ witnesses that } \dot{\mathbb{Q}}_\beta \text{ is Axiom A with crn”}.$$

(n) For each $n \in \omega, a \subseteq \alpha$, we define $\leq_{n,a}$ on $\mathbb{P}_\alpha$, where

$$\bar{q} \leq_{n,a} \bar{p} \leftrightarrow \left(\bar{q} \leq \bar{p} \wedge \forall \beta \in a(\bar{q} \restriction \beta \Vdash_{\mathbb{P}_\beta} q(\beta) \dot{\leq}_{\beta,n} p(\beta))\right).$$

[1] In the generic extension $V[G]$ we reinterpret X as the completion of $(X)^V$. Similarly, we reinterpret spaces $(A^\omega)^\alpha$, continuous functions, open and closed sets on these spaces. This should be standard.

(o) The support of $\bar{p} \in \mathbb{P}_\alpha$ is the set $\operatorname{supp}(\bar{p}) = \{\beta < \alpha : \bar{p} \Vdash \dot{p}(\beta) \neq \mathbb{1}\}$.

Recall that a condition q is called a master condition over a model M if for any maximal antichain $D \in M$, $\{p \in D : q \not\perp p\} \subseteq M$. Equivalently, it means that for every generic filter G over V containing q, G is generic over M as well. Throughout this paper, when we say that M *is elementary*, we mean that it is elementary in a large enough model of the form $H(\theta)$. Sometimes, we will say that M *is a model of set theory* or just that M *is a model*. In most generality, this just mean that $(M, \in)$ satisfies a strong enough fragment of ZFC. But this is a way to general notion for our purposes. For instance, such M may not even be correct about what ω is. Thus, let us clarify that in all our instances this will mean, that M is either elementary or a ccc forcing extensions of an elementary model. In particular, some basic absoluteness (e.g. for Σ^1_1 or Π^1_1 formulas) holds true between M and V, M is transitive below ω_1 and ω_1 is computed correctly.

Fact (Fusion Lemma, see e.g. [3, Lemma 1.2, 2.3])**.** *If $\langle a_n : n \in \omega \rangle$ is $\subseteq$-increasing, $\langle \bar{p}_n : n \in \omega \rangle$ is such that $\forall n \in \omega(\bar{p}_{n+1} \leq_{n,a_n} \bar{p}_n)$ and $\bigcup_{n\in\omega} \operatorname{supp}(\bar{p}_n) \subseteq \bigcup_{n\in\omega} a_n \subseteq \alpha$, then there is a condition $\bar{p} \in \mathbb{P}_\alpha$ so that for every $n \in \omega$, $\bar{p} \leq_{n,a_n} \bar{p}_n$; in fact, for every $\beta < \alpha$, $\bar{p} \restriction \beta \Vdash \dot{p}(\beta) = \bigcap_{n\in\omega} \dot{p}_n(\beta)$.*

Moreover, let M be a countable elementary model, $\bar{p} \in M \cap \mathbb{P}_\alpha$, $n \in \omega$, $a \subseteq M \cap \alpha$ finite and $\langle \alpha_i : i \in \omega \rangle$ a cofinal increasing sequence in $M \cap \alpha$. Then there is $\bar{q} \leq_{n,a} \bar{p}$ a master condition over M so that for every name $\dot{y} \in M$ for an element of ω^ω and $j \in \omega$, there is $i \in \omega$ so that below $\bar{q}$, the value of $\dot{y} \restriction j$ only depends on the $\mathbb{P}_{\alpha_i}$-generic.

(p) For G a $\mathbb{P}_\alpha$-generic, we write $\bar{x}_G$ for the generic element of $\prod_{\beta<\alpha} A^\omega$ added by $\mathbb{P}_\alpha$.

Let us from now on assume that for each $\beta < \alpha$ and $n \in \omega$, $\mathbb{Q}_\beta$ and $\leq_{\beta,n}$ are fixed analytic subsets subsets of $\mathcal{T}$ and $\mathcal{T}^2$ respectively, coded in V. Although the theory that we develop below can be extended to a large extend to non-definable iterands, we will only focus on this case, since we need stronger results later on.

Lemma 4.2.2. *For any $\bar{p} \in \mathbb{P}_\alpha$, M a countable elementary model so that $\mathbb{P}_\alpha, \bar{p} \in M$ and $n \in \omega, a \subseteq M \cap \alpha$ finite, there is $\bar{q} \leq_{n,a} \bar{p}$ a master condition over M and a closed set $[\bar{q}] \subseteq (A^\omega)^\alpha$ so that*

1. $\bar{q} \Vdash \bar{x}_G \in [\bar{q}]$,

for every $\beta < \alpha$,

2. *$\bar{q} \Vdash \dot{q}(\beta) = \{s \in A^{<\omega} : \exists \bar{z} \in [\bar{q}](\bar{z} \restriction \beta = \bar{x}_G \restriction \beta \wedge s \subseteq z(\beta))\}$,*

3. *the map sending $\bar{x} \in [\bar{q}] \restriction \beta$ to $\{s \in A^{<\omega} : \exists \bar{z} \in [\bar{q}](\bar{z} \restriction \beta = \bar{x} \wedge s \subseteq z(\beta))\}$ is continuous and maps to $\mathbb{Q}_\beta$,*

and for every name $\dot{y} \in M$ for an element of a Polish space X,

4. *there is a continuous function $f\colon [\bar{q}] \to X$ so that $\bar{q} \Vdash \dot{y} = f(\bar{x}_G)$.*

(q) We call such $\bar{q}$ as in Lemma 4.2.2 a *good master condition over M.*

Before we prove Lemma 4.2.2, let us draw some consequences from the definition of a good master condition.

Lemma 4.2.3. *Let $\bar{q} \in \mathbb{P}_\alpha$ be a good master condition over a model M and $\dot{y} \in M$ a name for an element of a Polish space X.*

(i) Then $[\bar{q}]$ is unique, in fact it is the closure of $\{\bar{x}_G : G \ni \bar{q}$ is generic over $V\}$.

(ii) The continuous map $f\colon [\bar{q}] \to X$ given by (4) is unique and

(iii) whenever $Y \in M$ is an analytic subset of X and $\bar{q} \Vdash \dot{y} \in Y$, then $f''[\bar{q}] \subseteq Y$.

Moreover, there is a countable set $C \subseteq \alpha$, not depending on $\dot{y}$, so that

(iv) $[\bar{q}] \restriction C$ is a closed subset of the Polish space $(A^\omega)^C$ and $[\bar{q}] = ([\bar{q}] \restriction C) \times (A^\omega)^{\alpha \setminus C}$,

(v) for every $\beta \in C$, there is a continuous function $g\colon [\bar{q}] \restriction (C \cap \beta) \to \mathbb{Q}_\beta$, so that for every $\bar{x} \in [\bar{q}]$,

$$g(\bar{x} \restriction (C \cap \beta)) = \{s \in A^{<\omega} : \exists \bar{z} \in [\bar{q}](\bar{z} \restriction \beta = \bar{x} \restriction \beta \wedge s \subseteq z(\beta))\},$$

(vi) there is a continuous function $f\colon [\bar{q}] \restriction C \to X$, so that

$$\bar{q} \Vdash \dot{y} = f(\bar{x}_G \restriction C).$$

Proof. Let us write, for every $\beta < \alpha$ and $\bar{x} \in [\bar{q}] \restriction \beta$,

$$T_{\bar{x}} := \{s \in A^{<\omega} : \exists \bar{z} \in [\bar{q}](\bar{z} \restriction \beta = \bar{x} \wedge s \subseteq z(\beta))\}.$$

For (i), let $\bar{s} \in \bigotimes_{i<\alpha} A^{<\omega}$ be arbitrary so that $[\bar{s}] \cap [\bar{q}]$ is non-empty. We claim that there is a generic G over V containing $\bar{q}$ so that $\bar{x}_G \in [\bar{s}]$. This is shown by induction on $\max(\mathrm{dom}(\bar{s}))$. For $\bar{s} = \emptyset$ the claim is obvious. Now assume $\max(\mathrm{dom}(\bar{s})) = \beta$, for $\beta < \alpha$. Then, by (3), $O := \{\bar{x} \in [\bar{q}] : s(\beta) \in T_{\bar{x} \restriction \beta}\}$ is open and it is non-empty since $[\bar{s}] \cap [\bar{q}] \neq \emptyset$. Applying the inductive hypothesis, there is a generic $G \ni \bar{q}$ so that $\bar{x}_G \in O$. In $V[G \restriction \beta]$ we have, by (2), that $T_{\bar{x}_G \restriction \beta} = \dot{q}(\beta)[G]$. Moreover, since $\bar{x}_G \in O$, we have that $s(\beta) \in \dot{q}(\beta)[G]$. Then it is easy to force over $V[G \restriction \beta]$, to get a full $\mathbb{P}_\alpha$ generic $H \supseteq G \restriction \beta$ containing $\bar{q}$ so that $\bar{x}_H \restriction \beta = \bar{x}_G \restriction \beta$ and $s(\beta) \subseteq \bar{x}_H(\beta)$. By (1), for every generic G over V containing $\bar{q}$, $\bar{x}_G \in [\bar{q}]$. Thus we have shown that the set of such $\bar{x}_G$ is dense in $[\bar{q}]$. Uniqueness follows from $[\bar{q}]$ being closed.

Now (ii) follows easily since any two continuous functions given by (4) have to agree on a dense set.

For (iii), let us consider the analytic space $Z = \{0\} \times X \cup \{1\} \times Y$, which is the disjoint union of the spaces X and Y. Then there is a continuous surjection $F \colon \omega^\omega \to Z$ and by elementarity we can assume it is in M. Let us find in M a name $\dot{z}$ for an element of ω^ω so that in $V[G]$, if $\dot{y}[G] \in Y$, then $F(\dot{z}[G]) = (1, \dot{y}[G])$, and if $\dot{y}[G] \notin Y$, then $F(\dot{z}[G]) = (0, \dot{y}[G])$. By (4), there is a continuous function $g \colon [\bar{q}] \to \omega^\omega$ so that $\bar{q} \Vdash \dot{z} = g(\bar{x}_G)$. Since $\bar{q} \Vdash \dot{y} \in Y$, we have that for any generic G containing $\bar{q}$, $F(g(\bar{x}_G)) = (1, f(\bar{x}_G))$. By density, for every $\bar{x} \in [\bar{q}]$, $F(g(\bar{x})) = (1, f(\bar{x}))$ and in particular $f(\bar{x}) \in Y$.

Now let us say that the support of a function $g \colon [\bar{q}] \to X$ is the smallest set $C_g \subseteq \alpha$ so that the value of $g(\bar{x})$ only depends on $\bar{x} \restriction C_g$. The results of [7] imply that if g is continuous, then g has countable support. Note that for all $\beta \notin \mathrm{supp}(\bar{q})$, the map in (3) is constant on the set of generics and by continuity it is constant everywhere. Thus it has empty support. Let C be the union of $\mathrm{supp}(\bar{q})$ with all the countable supports given by instances of (3) and (4). Then C is a countable set. For (iv), (v) and (vi), note that $[\bar{q}] \restriction C = \{\bar{y} \in (A^\omega)^C : \bar{y}^\frown(\bar{x} \restriction \alpha \setminus C) \in [\bar{q}]\}$ for $\bar{x} \in [\bar{q}]$ arbitrary, and recall that in a product, sections of closed sets are closed and continuous functions are coordinate-wise continuous. $\square$

Proof of Lemma 4.2.2. Let us fix for each $\beta < \alpha$ a continuous surjection $F_\beta \colon \omega^\omega \to \mathbb{Q}_\beta$. The proof is by induction on α. If $\alpha = \beta + 1$, then $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$. Let $\bar{q}_0 \leq_{n,a} \bar{p} \restriction \beta$ be a master condition over M and $H \ni \bar{q}_0$ a $\mathbb{P}_\beta$ generic over V. Then, applying a standard fusion argument using Axiom A with continuous reading of names in $V[H]$ to $\mathbb{Q}_\beta$, we find $q(\beta) \leq_{\beta,n} p(\beta)$ a master condition over $M[H]$ (note that H is also M generic since $\bar{q}_0$ is a master condition over M) so that for each name $\dot{y} \in M[H]$ for an element of a Polish space X there is a continuous function $f \colon [q(\beta)] \to X$ so that $q(\beta) \Vdash \dot{y} = f(\dot{x}_G)$. Thus we find in V, a $\mathbb{P}_\beta$-name $\dot{q}(\beta)$ so that $\bar{q}_0$ forces that it is such a condition. Let $M^+ \ni M$ be a countable elementary model containing $\dot{q}(\beta)$ and $\bar{q}_0$, and let $\bar{q}_{1/2} \leq_{n,a} \bar{q}_0$ be a master condition over M^+. Again let $M^{++} \ni M^+$ be a countable elementary model containing $\bar{q}_{1/2}$. By the induction hypothesis we find $\bar{q}_1 \leq_{n,a} \bar{q}_{1/2}$ a good master condition over M^{++}. Finally, let $\bar{q} = \bar{q}_1^\frown \dot{q}(\beta)$. Then $\bar{q} \leq_{n,a} \bar{p}$ and $\bar{q}$ is a master condition over M. Since $\dot{q}(\beta) \in M^+ \subseteq M^{++}$, there is a continuous function $f\colon [\bar{q}_1] \to \omega^\omega$, so that $\bar{q}_1 \Vdash_\beta F_\beta(f(\bar{x}_H)) = \dot{q}(\beta)$. Here note that F_β is in M by elementarity and we indeed find a name $\dot{z}$ in M^+ so that $\bar{q}_0 \Vdash F_\beta(\dot{z}) = \dot{q}(\beta)$. Let $[\bar{q}] = \{\bar{x} \in (A^\omega)^\alpha : \bar{x} \restriction \beta \in [\bar{q}_1] \wedge x(\beta) \in [F_\beta(f(\bar{x} \restriction \beta))]\}$. Then $[\bar{q}]$ is closed and (1), (2), (3) hold true. To see that $[\bar{q}]$ is closed, note that the graph of a continuous function is always closed, when the codomain is a Hausdorff space. For (4), let $\dot{y} \in M$ be a $\mathbb{P}_\alpha$-name for an element of a Polish space X. If $H \ni \bar{q}_1$ is V-generic, then there is a continuous function $g\colon [q(\beta)] \to X$ in $V[H]$ so that $V[H] \models q(\beta) \Vdash g(\dot{x}_G) = \dot{y}$, where we view $\dot{y}$ as a $\mathbb{Q}_\beta$-name in $M[H]$. Moreover there is a continuous retract $\varphi\colon A^\omega \to [q(\beta)]$ in $V[H]$. Since M^+ was chosen elementary enough, we find names $\dot{g}$ and $\dot{\varphi}$ for g and φ in M^+. The function $g \circ \varphi$ is an element of the space[2] $C(A^\omega, X)$, but this is not a Polish space when A is infinite, i.e. when A^ω is not compact. It is though, always a coanalytic space (consult e.g. [33, 12, 2.6] to see how $C(A^\omega, X)$ is a coanalytic subspace of a suitable Polish space). Thus there is an increasing sequence $\langle Y_\xi : \xi < \omega_1\rangle$ of analytic subspaces such that $\bigcup_{\xi<\omega_1} Y_\xi = C(A^\omega, X)$ and the same equality holds in any ω_1-preserving extension. Since $\bar{q}_{1/2}$ is a master condition over M^+, we have that $\bar{q}_{1/2} \Vdash \dot{g} \circ \dot{\varphi} \in Y_\xi$, where $\xi = M^+ \cap \omega_1$. Since $\bar{q}_1$ is a good master condition over M^{++} and $Y_\xi \in M^{++}$, by Lemma 4.2.3, there is a continuous function $g' \in V$, $g'\colon [\bar{q}_1] \to Y_\xi$, so that $\bar{q}_1 \Vdash g'(\bar{x}_H) = \dot{g} \circ \dot{\varphi}$. Altogether we have that $\bar{q} \Vdash \dot{y} = g'(\bar{x}_G \restriction \beta)(x_G(\beta))$.

For α limit, let $\langle \alpha_i : i \in \omega\rangle$ be a strictly increasing sequence cofinal in $M \cap \alpha$ and let $\bar{q}_0 \leq_{n,a} \bar{p}$ be a master condition over M so that for every name $\dot{y} \in M$ for an element of ω^ω, $j \in \omega$, the value of $\dot{y} \restriction j$ only depends on the generic restricted to $\mathbb{P}_{\alpha_i}$ for some $i \in \omega$. Let us fix a "big" countable elementary model N, with $\bar{q}_0, M \in N$. Let $\langle a_i : i \in \omega\rangle$ be an increasing sequence of finite subsets of $N \cap \alpha$ so that $a_0 = a$ and $\bigcup_{i\in\omega} a_i = N \cap \alpha$. Now inductively define sequences $\langle M_i : i \in \omega\rangle$, $\langle \bar{r}_i : i \in \omega\rangle$, initial segments lying in N, so that for every $i \in \omega$,

- $M_0 = M$, $\bar{r}_0 = \bar{q}_0 \restriction \alpha_0$,
- $M_{i+1} \ni \bar{q}_0$ is a countable model,

[2]The topology is such that for any continuous h mapping to $C(A^\omega, X)$, $(x, y) \mapsto h(x)(y)$ is continuous.

- $M_i, \bar{r}_i, a_i \in M_{i+1}$

- $\bar{r}_i$ is a good $\mathbb{P}_{\alpha_i}$ master condition over M_i,

- $r_{i+1} \leq_{n+i, a_i \cap \alpha_i} r_i^\frown \bar{q}_0 \restriction [\alpha_i, \alpha_{i+1})$.

Define for each $i \in \omega$, $\bar{q}_i = \bar{r}_i^\frown \bar{q}_0 \restriction [\alpha_i, \alpha)$. Then $\langle \bar{q}_i : i \in \omega \rangle$ is a fusion sequence in $\mathbb{P}_\alpha$ and we can find a condition $\bar{q} \leq_{n,a} \bar{q}_0 \leq_{n,a} \bar{p}$, where for each $\beta < \alpha$, $\bar{q} \restriction \beta \Vdash \dot{q}(\beta) = \bigcap_{i \in \omega} \dot{q}_i(\beta)$. Finally let $[\bar{q}] := \bigcap_{i \in \omega}([\bar{r}_i] \times (A^\omega)^{[\alpha_i, \alpha)})$. Then (1) is easy to check. For (4), we can assume without loss of generality that $\dot{y}$ is a name for an element of ω^ω since for any Polish space X, there is a continuous surjection from ω^ω to X. Now let $(i_j)_{j \in \omega}$ be increasing so that $\dot{y} \restriction j$ is determined on $\mathbb{P}_{\alpha_{i_j}}$ for every $j \in \omega$. Since $\bar{r}_{i_j}$ is a good master condition over M, there is a continuous function $f_j \colon [\bar{r}_{i_j}] \to \omega^j$ so that $\bar{r}_{i_j} \Vdash \dot{y} \restriction j = f_j(\bar{x}_{G_{\alpha_{i_j}}})$ for every $j \in \omega$. It is easy to put these functions together to a continuous function $f \colon [\bar{q}] \to 2^\omega$, so that $f(\bar{x}) \restriction j = f_j(\bar{x} \restriction \alpha_{i_j})$. Then we obviously have that $\bar{q} \Vdash \dot{y} = f(\bar{x}_G)$.

Now let us fix for each $i \in \omega$, $C_i \subseteq \alpha_i$ a countable set as given by Lemma 4.2.3 applied to $\bar{r}_i$, M_i, which by elementarity exists in N. Let $C = \bigcup_{i \in \omega} C_i$. Then $[\bar{q}] = [\bar{q}] \restriction C \times (A^\omega)^{\alpha \setminus C}$ and $[\bar{q}] \restriction C$ is closed. For every $\beta \in \alpha \setminus C$, the map given in (3) is constant and maps to $\mathbb{Q}_\beta$, as $A^{<\omega}$ is the trivial condition. Thus we may restrict our attention to $\beta \in C$. Let us write $X_i = ([\bar{r}_i] \times (A^\omega)^{[\alpha_i, \alpha)}) \restriction C$ for every $i \in \omega$ and note that $\bigcap_{i \in \omega} X_i = [\bar{q}] \restriction C$. For every $\beta \in C$, $\bar{x} \in [\bar{q}] \restriction (C \cap \beta)$ and $i \in \omega$, we write

$$T_{\bar{x}} := \{s \in A^{<\omega} : \exists \bar{z} \in [\bar{q}] \restriction C(\bar{z} \restriction \beta = \bar{x} \wedge s \subseteq z(\beta))\}$$

and

$$T^i_{\bar{x}} = \{s \in A^{<\omega} : \exists \bar{z} \in X_i(\bar{z} \restriction \beta = \bar{x} \wedge s \subseteq z(\beta))\}.$$

Claim 4.2.4. *For every $i \in \omega$, where $\beta \in a_i$, $T^{i+1}_{\bar{x}} \leq_{\beta,i} T^i_{\bar{x}}$. In particular, $\bigcap_{i \in \omega} T^i_{\bar{x}} \in \mathbb{Q}_\beta$.*

Proof. If $\alpha_{i+1} \leq \beta$, then $T^{i+1}_{\bar{x}} = T^i_{\bar{x}} = A^{<\omega}$. Else consider a $\mathbb{P}_{\alpha_{i+2}}$-name for $(T^{i+1}_{\bar{y}}, T^i_{\bar{y}}) \in \mathcal{T}^2$, where $\bar{y} = \bar{x}_G \restriction (C \cap \beta)$. Such a name exists in M_{i+2} and $\beta \in a_i \subseteq M_{i+2}$. Thus $\leq_{\beta,i} \in M_{i+2}$ and by Lemma 4.2.3, we have that for every $\bar{y} \in [\bar{r}_{i+2}] \restriction (C \cap \beta)$, $(T^{i+1}_{\bar{y}}, T^i_{\bar{y}}) \in \leq_{\beta,i}$, thus also for $\bar{y} = \bar{x}$. The rest follows from the fact that the statement, that for any fusion sequence in $\mathbb{Q}_\beta$, its intersection is in $\mathbb{Q}_\beta$, is $\mathbf{\Pi}^1_2$ and thus absolute. □

Claim 4.2.5. $T_{\bar{x}} = \bigcap_{i \in \omega} T^i_{\bar{x}}$.

Proof. Let $\bar{s} \in \bigotimes_{i \in C} A^{<\omega}$ and $j \in \omega$ be so that $\mathrm{dom}(\bar{s}) \subseteq a_j$, $\max_{i \in \mathrm{dom}(\bar{s})} |s_i| \leq j$ and $[\bar{s}] \cap X_j \neq \emptyset$. Then we have that for every $i \in \omega$, $[\bar{s}] \cap X_i \neq \emptyset$. This is shown by induction on $\max(\mathrm{dom}(\bar{s}))$. If $\max(\mathrm{dom}(\bar{s})) = \min C \setminus \xi$, then the set $O = \{\bar{y} \in X_j \restriction \xi : \bar{y} \in [\bar{s} \restriction \xi], s(\xi) \in T^j_{\bar{y}}\}$ is open non-empty by continuity of the map in (3) for $\bar{r}_j$. Applying the inductive hypothesis to O, we get for every $i \geq j$, some $\bar{z}_i \in O \cap (X_i \restriction \xi)$. Since $T^i_{\bar{z}_i} \leq_{\xi,j} T^j_{\bar{z}_i}$ and $|s(\xi)| \leq j$, we have that $s(\xi) \in T^i_{\bar{z}_i}$ and we can extend $\bar{z}_i$ to $\bar{z} \in X_i \cap [\bar{s}]$. For $i \leq j$, there is nothing to show since then $X_j \subseteq X_i$.

That $T_{\bar{x}} \subseteq \bigcap_{i \in \omega} T^i_{\bar{x}}$ is clear. Thus let $s \in \bigcap_{i \in \omega} T^i_{\bar{x}}$, say $|s| = j$. The claim is proven by constructing recursively a sequence $\langle \bar{s}_i : i \geq j \rangle$ so that for every $i \in \omega$, $\mathrm{dom}(\bar{s}_i) = a_i \cap C$, $\forall \xi \in a_i \cap C(|s_i(\xi)| = i)$, $s_i(\beta) \supseteq s$, $\bar{x} \in [\bar{s} \restriction \beta]$ and $[\bar{s}_i] \cap X_i \neq \emptyset$. Starting with $\bar{s}_0 = \{(\beta, s)\}$, this sequence is easy to construct via the statement that we just proved. Then $\bigcap_{i \geq j} [\bar{s}_i]$ is a singleton $\{\bar{z}\}$ so that $\bar{z} \restriction \beta = \bar{x}$, $z(\beta) \supseteq s$ and $\bar{z} \in [\bar{q}] \restriction C$. □

Now (2) follows easily. For the continuity of $\bar{x} \mapsto T_{\bar{x}}$, let $t \in A^{<\omega}$ be arbitrary and j large enough so that $|t| \leq j$ and $\beta \in a_j$. Then $\{\bar{x} \in [\bar{q}] \restriction \beta : t \notin T_{\bar{x}}\} = \{\bar{x} \in [\bar{q}] \restriction \beta : t \notin T^j_{\bar{x}}\}$ and $\{\bar{x} \in [\bar{q}] \restriction \beta : t \in T_{\bar{x}}\} = \{\bar{x} \in [\bar{q}] \restriction \beta : t \in T^j_{\bar{x}}\}$ which are both open. Thus we have shown (3). □

Lemma 4.2.6. *Let $C \subseteq \alpha$ be countable and $X \subseteq (A^\omega)^C$ be a closed set so that for every $\beta \in C$ and $\bar{x} \in X \restriction \beta$,*

$$\{s \in A^{<\omega} : \exists \bar{z} \in X(\bar{z} \restriction \beta = \bar{x} \wedge s \subseteq z(\beta))\} \in \mathbb{Q}_\beta.$$

Let $M \ni X$ be countable elementary. Then there is a good master condition $\bar{r}$ over M so that $[\bar{r}] \restriction C \subseteq X$.

Proof. It is easy to construct $\bar{q} \in M$ recursively so that $\bar{q} \Vdash \bar{x}_G \restriction C \in X$. By Lemma 4.2.2, we can extend $\bar{q}$ to a good master condition $\bar{r}$ over M. The unique continuous function $f : [\bar{r}] \to (A^\omega)^C$ so that for generic G, $f(\bar{x}_G) = \bar{x}_G \restriction C$, is so that $f(\bar{x}) = \bar{x} \restriction C$ for every $\bar{x} \in [\bar{r}]$. Since f maps to X, $[\bar{r}] \restriction C \subseteq X$. □

4.3 The Main Lemma

4.3.1 Mutual Cohen Genericity

Let X be a Polish space and M a model of set theory with $X \in M$. Recall that $x \in X$ is *Cohen generic in* X over M if for any open dense $O \subseteq X$, such that $O \in M$, $x \in O$.

Let $x_0, \ldots, x_{n-1} \in X$. Then we say that $x_0, \ldots x_{n-1}$ are *mutually Cohen generic (mCg) in X over M* if $(y_0, \ldots, y_{K-1})$ is Cohen generic in X^K over M, where $\langle y_i : i < K\rangle$ is some, equivalently any, enumeration of $\{x_0, \ldots, x_{n-1}\}$. In particular, we allow for repetition in the definition of mutual genericity.

Definition 4.3.1. Let $\langle X_l : l < k\rangle \in M$ be Polish spaces. Then we say that $\bar{x}_0, \ldots, \bar{x}_{n-1} \in \prod_{l<k} X_l$ are *mutually Cohen generic (mCg) with respect to the product $\prod_{l<k} X_l$ over M*, if

$$(y_0^0, \ldots, y_0^{K_0}, \ldots, y_{k-1}^0, \ldots, y_{k-1}^{K_{k-1}}) \in \prod_{l<k} X_l^{K_l} \text{ is Cohen generic in } \prod_{l<k} X_l^{K_l} \text{ over } M,$$

where $\langle y_l^i : i < K_l\rangle$ is some, equivalently any, enumeration of $\{x_i(l) : i < n\}$ for each $l < k$.

Definition 4.3.2. Let X be a Polish space with a fixed countable basis $\mathcal{B}$. Then we define the forcing poset $\mathbb{C}(2^\omega, X)$ consisting of functions $h\colon 2^{\leq n} \to \mathcal{B} \setminus \{\emptyset\}$ for some $n \in \omega$ such that $\forall \sigma \subseteq \tau \in 2^{\leq n}(h(\sigma) \supseteq h(\tau))$. The poset is ordered by function extension.

The poset $\mathbb{C}(2^\omega, X)$ adds generically a continuous function $\chi\colon 2^\omega \to X$, given by $\chi(x) = y$ where $\bigcap_{n\in\omega} h(x \restriction n) = \{y\}$ and $h = \bigcup G$ for G the generic filter. This forcing will be useful in this section several times. Note for instance that if G is generic over M, then for any $x \in 2^\omega$, $\chi(x)$ is Cohen generic in X over M, and moreover, for any $x_0, \ldots, x_{n-1} \in 2^\omega$, $\chi(x_0), \ldots, \chi(x_{n-1})$ are mutually Cohen generic in X over M. Sometimes we will use $\mathbb{C}(2^\omega, X)$ to force a continuous function from a space homeomorphic to 2^ω, such as $(2^\omega)^\alpha$ for $\alpha < \omega_1$.

Lemma 4.3.3. *Let M be a model of set theory, $K, n \in \omega$, $X_j \in M$ a Polish space for every $j < n$ and G a $\prod_{j<n} \mathbb{C}(2^\omega, X_j)$-generic over M yielding $\chi_j\colon 2^\omega \to X_j$ for every $j < n$. Then, whenever $\bar{x}$ is Cohen generic in $(2^\omega)^K$ over $M[G]$ and $u_0, \ldots, u_{n-1} \in 2^\omega \cap M[\bar{x}]$ are pairwise distinct,*

$$\bar{x}^\frown\langle \chi_j(u_i) : i < n, j < n\rangle$$

is Cohen generic in

$$(2^\omega)^K \times \prod_{i<n} X_i$$

over M.

Proof. Since $\bar{x}$ is generic over M it suffices to show that $\langle \chi_j(u_i) : i < n, j \rangle$ is generic over $M[\bar{x}]$. Let $\dot{O} \in M$ be a $(2^{<\omega})^K$-name for a dense open subset of $\prod_{j<n}(X_j)^n$ and $\dot{u}_i$ a $(2^{<\omega})^K$-name for u_i, $i < n$, pairwise distinct. Then consider the set

$$D := \{(\bar{h}, \bar{s}) \in \prod_{i<n} \mathbb{C}(2^\omega, X_i) \times (2^{<\omega})^K : \exists t_0, \ldots, t_{n-1} \in 2^{<\omega}$$

$$(\forall i < n(\bar{s} \Vdash t_i \subseteq \dot{u}_i) \wedge \bar{s} \Vdash \prod_{i,j<n} h_j(t_i) \subseteq \dot{O})\}.$$

We claim that this set is dense in $\prod_{i<n} \mathbb{C}(2^\omega, X_i) \times (2^{<\omega})^K$ which finishes the proof. Namely let $(\bar{h}, \bar{s})$ be arbitrary, wlog $\operatorname{dom} h_j = 2^{\leq n_0}$ for every $j < n$. Then we can extend $\bar{s}$ to $\bar{s}'$ so that there are incompatible t_i, with $|t_i| \geq n_0$, so that $\bar{s}' \Vdash t_i \subseteq \dot{u}_i$ and there are $U_{i,j} \subseteq h_j(t_i \restriction n_0)$ basic open subsets of X_j in M for every $i < n$ and $j < n$, so that $\bar{s}' \Vdash \prod_{i,j<n} U_{i,j} \subseteq \dot{O}$. Then we can extend $\bar{h}$ to $\bar{h}'$ so that $h'_j(t_i) = U_{i,j}$ for every $i, j < n$. We see that $(\bar{h}', \bar{s}') \in D$. $\square$

4.3.2 Finite products

This subsection can be skipped entirely if one is only interested in the results for the countable support iteration. Nevertheless, the following lemma is interesting in its own right and can be seen as a preparation for Main Lemma 4.3.14.

Main Lemma 4.3.4. Let $k \in \omega$ and $E \subseteq [(2^\omega)^k]^{<\omega} \setminus \{\emptyset\}$ an analytic hypergraph on $(2^\omega)^k$. Then there is a countable model M so that either

1. for any $\bar{x}_0, \ldots, \bar{x}_{n-1} \in (2^\omega)^k$ that are mCg wrt $\prod_{l<k} 2^\omega$ over M,

$$\{\bar{x}_0, \ldots, \bar{x}_{n-1}\} \text{ is } E\text{-independent}$$

or for some $N \in \omega$,

2. there are $\phi_0, \ldots, \phi_{N-1} \colon (2^\omega)^k \to (2^\omega)^k$ continuous, $\bar{s} \in \bigotimes_{l<k} 2^{<\omega}$ so that for any $\bar{x}_0, \ldots, \bar{x}_{n-1} \in (2^\omega)^k \cap [\bar{s}]$, that are mCg wrt $\prod_{l<k} 2^\omega$ over M,

$$\{\phi_j(\bar{x}_i) : j < N, i < n\} \text{ is } E\text{-independent but } \{\bar{x}_0\} \cup \{\phi_j(\bar{x}_0) : j < N\} \in E.$$

Remark 4.3.5. Note that $N = 0$ is possible in the second option. For example whenever $[(2^\omega)^k]^1 \subseteq E$, then $\emptyset$ is the only E-independent set. In this case the last line simplifies to "$\{\bar{x}_0\} \in E$".

Proof. Let $\kappa = \beth_{2k-1}(\aleph_0)^+$. Recall that by Erdős-Rado (see [31, Thm 9.6]), for any $c\colon [\kappa]^{2k} \to H(\omega)$, there is $B \in [\kappa]^{\aleph_1}$ which is monochromatic for c, i.e. $c \restriction [B]^{2k}$ is constant. Let $\mathbb{Q}$ be the forcing adding κ many Cohen reals

$$\langle z_{(l,\alpha)} : \alpha < \kappa \rangle \text{ in } 2^\omega \text{ for each } l < k$$

with finite conditions, i.e. $\mathbb{Q} = \prod_\kappa^{<\omega} (2^{<\omega})^k$. We will use the notational convention that elements of $[\kappa]^d$, for $d \in \omega$, are sequences $\bar{\alpha} = (\alpha_0, \ldots, \alpha_{d-1})$ ordered increasingly. For any $\bar{\alpha} \in [\kappa]^k$ we define $\bar{z}_{\bar{\alpha}} := (z_{(0,\alpha_0)}, \ldots, z_{(k-1,\alpha_{k-1})}) \in (2^\omega)^k$.

Let $\dot{\mathcal{A}}$ be a $\mathbb{Q}$-name for a maximal E-independent subset of $\{\bar{z}_{\bar{\alpha}} : \bar{\alpha} \in [\kappa]^k\}$, reinterpreting E in the extension by $\mathbb{Q}$. For any $\bar{\alpha} \in [\kappa]^k$, we fix $p_{\bar{\alpha}} \in \mathbb{Q}$ so that either

$$p_{\bar{\alpha}} = \mathbb{1} \wedge p_{\bar{\alpha}} \Vdash \bar{z}_{\bar{\alpha}} \in \dot{\mathcal{A}} \tag{1}$$

or

$$p_{\bar{\alpha}} \Vdash \bar{z}_{\bar{\alpha}} \notin \dot{\mathcal{A}}. \tag{2}$$

In case (2) we additionally fix $N_{\bar{\alpha}} < \omega$ and $(\bar{\beta}^i)_{i<N_{\bar{\alpha}}} = (\bar{\beta}^i(\bar{\alpha}))_{i<N_{\bar{\alpha}}}$, and we assume that

$$p_{\bar{\alpha}} \Vdash \{\bar{z}_{\bar{\beta}^i} : i < N_{\bar{\alpha}}\} \subseteq \dot{\mathcal{A}} \wedge \{\bar{z}_{\bar{\alpha}}\} \cup \{\bar{z}_{\bar{\beta}^i} : i < N_{\bar{\alpha}}\} \in E.$$

We also define $H_l(\bar{\alpha}) = \{\beta^i_l : i < N_{\bar{\alpha}}\} \cup \{\alpha_l\} \in [\kappa]^{<\omega}$ for each $l < k$.

Now for $\bar{\alpha} \in [\kappa]^{2k}$ we collect the following information:

(i) whether $p_{\bar{\alpha}\restriction k} = p_{\alpha_0,\ldots,\alpha_{k-1}} \Vdash \bar{z}_{\bar{\alpha}\restriction k} \in \dot{\mathcal{A}}$ or not,

(ii) $\bar{s} = (p_{\bar{\alpha}\restriction k}(0,\alpha_0), \ldots, p_{\bar{\alpha}\restriction k}(k-1,\alpha_{k-1})) \in (2^{<\omega})^k$,

(iii) the relative position of the $p_{\bar{\gamma}}$ for $\bar{\gamma} \in \Gamma := \prod_{l<k}\{\alpha_{2l}, \alpha_{2l+1}\}$ to each other. More precisely consider $\bigcup_{\bar{\gamma}\in\Gamma} \operatorname{dom} p_{\bar{\gamma}} = \{0\} \times d_0 \cup \cdots \cup \{k-1\} \times d_{k-1}$ where $d_0, \ldots, d_{k-1} \subseteq \kappa$. Let $M_l = |d_l|$ for $l < k$ and for each $\bar{\gamma} = (\alpha_{j_0}, \ldots, \alpha_{j_{k-1}})$ collect $r_{\bar{\jmath}}$ with $\operatorname{dom} r_{\bar{\jmath}} \subseteq \{0\} \times M_0 \cup \cdots \cup \{k-1\} \times M_{k-1}$ and $r_{\bar{\jmath}}(l,m) = p_{\bar{\gamma}}(l,\beta_m)$ whenever β_m is the m'th element of d_l.

In case $p_{\bar{\alpha}\restriction k} \Vdash \bar{z}_{\bar{\alpha}\restriction k} \notin \dot{\mathcal{A}}$ we additionally remember

(iv) $N = N_{\bar{\alpha}\restriction k}$,

(v) $N_l = |H_l(\bar{\alpha} \restriction k)|$, for each $l < k$,

(vi) $\bar{b}^i \in \prod_{l<k} N_l$ so that β^i_l is the b^i_l'th element of $H_l(\bar{\alpha} \restriction k)$, for each $i < N$,

(vii) $\bar{a} \in \prod_{l<k} N_l$ so that α_l is the a_l'th member of $H_l(\bar{\alpha} \restriction k)$,

(viii) the partial function r with domain a subset of $\bigcup_{l<k}\{l\} \times N_l$, so that $r(l,m) = t \in 2^{<\omega}$ iff $p_{\bar{\alpha}\restriction k}(l,\beta) = t$ where β is the m'th element of $H_l(\bar{\alpha} \restriction k)$.

And finally we also remember

(ix) for each pair $\bar{\gamma}, \bar{\delta} \in \prod_{l<k}\{\alpha_{2l}, \alpha_{2l+1}\}$, where $\bar{\gamma} = (\alpha_{j_l})_{l<k}$ and $\bar{\delta} = (\alpha_{j'_l})_{l<k}$, finite partial injections $e_{l,\bar{j},\bar{j}'} \colon N_l \to N_l$ so that $e_{l,\bar{j},\bar{j}'}(m) = m'$ iff the m'th element of $H_l(\bar{\gamma})$ equals the m''th element of $H_l(\bar{\delta})$.

This information is finite and defines a coloring $c\colon [\kappa]^{2k} \to H(\omega)$. Let $B \in [\kappa]^{\omega_1}$ be monochromatic for c. Let $M \preccurlyeq H(\theta)$ be countable for θ large enough so that $\kappa, c, B, \langle p_{\bar{\alpha}} : \bar{\alpha} \in [\kappa]^k\rangle, E, \dot{\mathcal{A}} \in M$.

Claim 4.3.6. *If for every $\bar{\alpha} \in [B]^k$, $p_{\bar{\alpha}} \Vdash \bar{z}_{\bar{\alpha}} \in \dot{\mathcal{A}}$, then (1) of the main lemma holds true.*

Proof. Let $\bar{x}_0, \dots, \bar{x}_{n-1}$ be arbitrary mCg over M. Say $\{x_i(l) : i < n\}$ is enumerated by $\langle y^i_l : i < K_l\rangle$ for every $l < k$. Now find

$$\alpha^0_0 < \dots < \alpha^{K_0-1}_0 < \dots < \alpha^0_{k-1} < \dots < \alpha^{K_{k-1}-1}_{k-1}$$

in $M \cap B$. Then there is a $\mathbb{Q}$-generic G over M so that for any $\bar{j} \in \prod_{l<k} K_l$,

$$\bar{z}_{\bar{\beta}}[G] = (y^{j_0}_0, \dots, y^{j_{k-1}}_{k-1}),$$

where $\bar{\beta} = (\alpha^{j_0}_0, \dots, \alpha^{j_{k-1}}_{k-1})$. In particular, for each $i < n$, there is $\bar{\beta}_i \in [B \cap M]^k$ so that $\bar{z}_{\bar{\beta}_i}[G] = \bar{x}_i$. Since $p_{\bar{\beta}_i} = \mathbb{1} \in G$ for every $\bar{\beta}_i$ we have that

$$M[G] \models \bar{x}_i \in \dot{\mathcal{A}}[G]$$

for every $i < n$ and in particular

$$M[G] \models \{\bar{x}_i : i < n\} \text{ is } E\text{-independent.}$$

By absoluteness $\{\bar{x}_i : i < n\}$ is indeed E-independent. □

Assume from now on that $p_{\bar{\alpha}} \Vdash \bar{z}_{\bar{\alpha}} \notin \dot{\mathcal{A}}$ for every $\bar{\alpha} \in [B]^k$. Then we may fix $\bar{s}$, N, $(N_l)_{l<k}$, $\bar{b}^i$ for $i < N$, $\bar{a}$, r and $e_{l,\bar{j},\bar{j}'}$ for all $l < k$ and $\bar{j}, \bar{j}' \in \prod_{l'<k}\{2l', 2l'+1\}$ corresponding to the coloring on $[B]^{2k}$.

Claim 4.3.7. *For any $\bar{\alpha} \in [B]^{2k}$ and $\bar{\gamma}, \bar{\delta} \in \prod_{l<k}\{\alpha_{2l}, \alpha_{2l+1}\}$,*

$$p_{\bar{\gamma}} \restriction (\operatorname{dom} p_{\bar{\gamma}} \cap \operatorname{dom} p_{\bar{\delta}}) = p_{\bar{\delta}} \restriction (\operatorname{dom} p_{\bar{\gamma}} \cap \operatorname{dom} p_{\bar{\delta}}).$$

Proof. Suppose not. By homogeneity we find a counterexample $\bar{\alpha}$, $\bar{\gamma}$, $\bar{\delta}$ where $B \cap (\alpha_{2l'}, \alpha_{2l'+1})$ is non-empty for every $l' < k$. So let $(l, \beta) \in \operatorname{dom} p_{\bar{\gamma}} \cap \operatorname{dom} p_{\bar{\delta}}$ such that $p_{\bar{\gamma}}(l, \beta) = u \neq v = p_{\bar{\delta}}(l, \beta)$. Let $\bar{\rho} \in [B]^k$ be such that for every $l' < k$,

$$\begin{cases} \rho_{l'} \in (\gamma_{l'}, \delta_{l'}) & \text{if } \gamma_{l'} < \delta_{l'} \\ \rho_{l'} \in (\delta_{l'}, \gamma_{l'}) & \text{if } \delta_{l'} < \gamma_{l'} \\ \rho_{l'} = \gamma_{l'} & \text{if } \gamma_{l'} = \delta_{l'}. \end{cases}$$

Now note that $\bar{\rho}$'s relative position to $\bar{\gamma}$ is the same as that of $\bar{\delta}$ to $\bar{\gamma}$. More precisely, let $\bar{j}, \bar{j}' \in \prod_{l'<k}\{2l', 2l'+1\}$ so that $\bar{\gamma} = (\alpha_{j_0}, \ldots, \alpha_{j_{k-1}})$, $\bar{\delta} = (\alpha_{j'_0}, \ldots, \alpha_{j'_{k-1}})$. Then there is $\bar{\beta} \in [B]^{2k}$ so that $\bar{\gamma} = (\beta_{j_0}, \ldots, \beta_{j_{k-1}})$ and $\bar{\rho} = (\beta_{j'_0}, \ldots, \beta_{j'_{k-1}})$. Thus by homogeneity of $[B]^{2k}$ via c, $p_{\bar{\rho}}(l, \beta) = v$. Similarly $\bar{\delta}$ is in the same position relative to $\bar{\rho}$ as to $\bar{\gamma}$. Thus also $p_{\bar{\rho}}(l, \beta) = u$ and we find that $v = u$ – we get a contradiction. □

Claim 4.3.8. *For any $l < k$ and $\bar{j}, \bar{j}' \in \prod_{l'<k}\{2l', 2l'+1\}$, $e_{l,\bar{j},\bar{j}'}(m) = m$ for every $m \in \operatorname{dom} e_{l,\bar{j},\bar{j}'}$.*

Proof. Let $\alpha_0 < \cdots < \alpha_{2k} \in B$ so that $(\alpha_{2l'}, \alpha_{2l'+1}) \cap B \neq \emptyset$ for every $l' < k$. Consider $\bar{\gamma} = (\alpha_{j_{l'}})_{l'<k}$, $\bar{\delta} = (\alpha_{j'_{l'}})_{l'<k}$ and again we find $\bar{\rho} \in [B]^k$ so that $\rho_{l'}$ is between (possibly equal to) $\alpha_{j_{l'}}$ and $\alpha_{j'_{l'}}$. If $e_{l,\bar{j},\bar{j}'}(m) = m'$, then if β is the m'th element of $H_l(\bar{\gamma})$, then β is m''th element of $H_l(\bar{\delta})$ aswell as of $H_l(\bar{\rho})$. But also β is the m'th element of $H_l(\bar{\rho})$, thus $m = m'$. □

Note that by the above claim $e_{l,\bar{j},\bar{j}'} = (e_{l,\bar{j}',\bar{j}})^{-1} = e_{l,\bar{j}',\bar{j}}$ and the essential information given by $e_{l,\bar{j},\bar{j}'}$ is it's domain.

Next let us introduce some notation. Let L be an arbitrary linear order. For any $g \in \{-1, 0, 1\}^k$ we naturally define a relation $\tilde{R}_g$ on L^k as follows:

$$\bar{\nu} \tilde{R}_g \bar{\mu} \leftrightarrow \forall l < k \begin{cases} \nu_l < \mu_l & \text{if } g(l) = -1 \\ \nu_l = \mu_l & \text{if } g(l) = 0 \\ \nu_l > \mu_l & \text{if } g(l) = 1. \end{cases}$$

Further we write $\bar{\nu} R_g \bar{\mu}$ iff $\bar{\nu} \tilde{R}_g \bar{\mu}$ or $\bar{\mu} \tilde{R}_g \bar{\nu}$. Enumerate $\{R_g : g \in \{-1, 0, 1\}^k\}$ without repetition as $\langle R_i : i < K \rangle$ (it is easy to see that $K = \frac{3^k+1}{2}$). Note that for any $\bar{\nu}, \bar{\mu}$ there is a unique $i < K$ so that $\bar{\nu} R_i \bar{\mu}$. Now for each $l < k$ and $i < K$, we let

$$I_{l,i} := \operatorname{dom} e_{l,\bar{j},\bar{j}'} \subseteq N_l,$$

where $\bar{j}R_i\bar{j}'$. By homogeneity of $[B]^{2k}$ and the observation that $e_{l,\bar{j},\bar{j}'} = e_{l,\bar{j}',\bar{j}}$, we see that $I_{l,i}$ does not depend on the particular choice of $\bar{j}, \bar{j}'$, such that $\bar{j}R_i\bar{j}'$.

We consider the $<_{\text{lex}}$ order on 2^ω. For each $l < k$ and $m < N_l$, we define a relation $E_{l,m}$ on $(2^\omega)^k$ as follows:

$$\bar{x}E_{l,m}\bar{y} \leftrightarrow m \in I_{l,i} \text{ where } i \text{ is such that } \bar{x}R_i\bar{y}.$$

Claim 4.3.9. *$E_{l,m}$ is an equivalence relation.*

Proof. The reflexivity and symmetry of $E_{l,m}$ is obvious. Assume that $\bar{x}_0E_{l,m}\bar{x}_1$ and $\bar{x}_1E_{l,m}\bar{x}_2$, and say $\bar{x}_0R_{i_0}\bar{x}_1$, $\bar{x}_1R_{i_1}\bar{x}_2$ and $\bar{x}_0R_{i_2}\bar{x}_2$. Find $\bar{\gamma}^0, \bar{\gamma}^1, \bar{\gamma}^2 \in [B]^k$ so that

$$\{\gamma_0^i : i < 3\} < \cdots < \{\gamma_{k-1}^i : i < 3\}$$

and

$$\bar{\gamma}^0R_{i_0}\bar{\gamma}^1, \bar{\gamma}^1R_{i_1}\bar{\gamma}^2, \bar{\gamma}^0R_{i_2}\bar{\gamma}^2.$$

If β is the m'th element of $H_l(\bar{\gamma}^0)$, then β is also the m'th element of $H_l(\bar{\gamma}^1)$, since we can find an appropriate $\bar{\alpha} \in [B]^{2k}$ and $\bar{j}, \bar{j}'$ so that $\bar{\gamma}^0 = (\alpha_{j_l})_{l<k}$ and $\bar{\gamma}^1 = (\alpha_{j'_l})_{l<k}$, $\bar{j}R_{i_0}\bar{j}'$ and we have that $m \in I_{l,i_0}$. Similarly β is the m'th element of $H_l(\bar{\gamma}^2)$.

But now we find again $\bar{\alpha} \in [B]^{2k}$ and $\bar{j}, \bar{j}'$ so that $\bar{\gamma}^0 = (\alpha_{j_l})_{l<k}$ and $\bar{\gamma}^2 = (\alpha_{j'_l})_{l<k}$. Thus $m \in I_{l,i_2}$, as $e_{l,\bar{j},\bar{j}'}(m) = m$ and $\bar{x}_0E_{l,m}\bar{x}_2$. □

Claim 4.3.10. *$E_{l,m}$ is smooth as witnessed by a continuous function, i.e. there is a continuous map $\varphi_{l,m} \colon (2^\omega)^k \to 2^\omega$ so that $\bar{x}E_{l,m}\bar{y}$ iff $\varphi_{l,m}(\bar{x}) = \varphi_{l,m}(\bar{y})$.*

Proof. We will check the following:

(a) For every open $O \subseteq (2^\omega)^k$, the $E_{l,m}$ saturation of O is Borel,

(b) every $E_{l,m}$ equivalence class is G_δ.

By a theorem of Srivastava ([63, Thm 4.1.]), (a) and (b) imply that $E_{l,m}$ is smooth, i.e. we can find $\varphi_{l,m}$ Borel.

(a) The $E_{l,m}$ saturation of O is the set $\{\bar{x} : \exists \bar{y} \in O(\bar{x}E_{l,m}\bar{y})\}$. It suffices to check for each $g \in \{-1,0,1\}^k$ that the set $X = \{\bar{x} : \exists \bar{y} \in O(\bar{x}\tilde{R}_g\bar{y})\}$ is Borel. Let $\mathcal{S} = \{\bar{\sigma} \in (2^{<\omega})^k : [\sigma_0] \times \cdots \times [\sigma_{k-1}] \subseteq O\}$. Consider

$$\varphi(\bar{x}) :\leftrightarrow \exists \bar{\sigma} \in \mathcal{S} \forall l' < k \begin{cases} x_{l'} <_{\text{lex}} \sigma_{l'}{}^\frown 0^\omega & \text{if } g(l') = -1 \\ x_{l'} \in [\sigma_{l'}] & \text{if } g(l') = 0 \\ \sigma_{l'}{}^\frown 1^\omega <_{\text{lex}} x_{l'} & \text{if } g(l') = 1 \end{cases}.$$

If $\varphi(\bar{x})$ holds true then let $\bar{\sigma}$ witness this. We then see that there is $\bar{y} \in [\sigma_0] \times \cdots \times [\sigma_{k-1}]$ with $\bar{x}\tilde{R}_g\bar{y}$. On the other hand, if $\bar{y} \in O$ is such that $\bar{x}R_g\bar{y}$, then we find $\bar{\sigma} \in \mathcal{S}$ defining a neighborhood of $\bar{y}$ witnessing $\varphi(\bar{x})$. Thus X is defined by φ and is thus Borel.

(b) Since finite unions of G_δ's are G_δ it suffices to check that $\{\bar{x} : \bar{x}\tilde{R}_g\bar{y}\}$ is G_δ for every $\bar{y}$ and $g \in \{-1, 0, 1\}^k$. But this is obvious from the definition.

Now note that given $\varphi_{l,m}$ Borel, we can find perfect $X_0, \ldots X_{k-1} \subseteq 2^\omega$ so that $\varphi_{l,m}$ is continuous on $X_0 \times \cdots \times X_{k-1}$ ($\varphi_{l,m}$ is continuous on a dense G_δ). But there is a $<_{\text{lex}}$ preserving homeomorphism from X_l to 2^ω for each $l < k$ so we may simply assume $X_l = 2^\omega$. □

Fix such $\varphi_{l,m}$ for every $l < k$, $m < N_l$, so that $\varphi_{l,a_l}(\bar{x}) = x_l$ (note that $\bar{x}E_{l,a_l}\bar{y}$ iff $x_l = y_l$). Now let $M_0 \preccurlyeq H(\theta)$ countable for θ large, containing all relevant information and $\varphi_{l,m} \in M_0$ for every $l < k$, $m < N_l$. Let $\chi_{l,m} \colon 2^\omega \to [r(l,m)]$ for $l < k$ and $m \neq a_l$ be generic continuous functions over M_0, i.e. the sequence $(\chi_{l,m})_{l<k, m \in N_l \setminus \{a_l\}}$ is $\prod_{l<k, m\in N_l\setminus\{a_l\}} \mathbb{C}(2^\omega, [r(l,m)])$ generic over M_0. Let us denote with M the generic extension of M_0. Also let $\chi_{l,m}$ for $m = a_l$ be the identity and $\psi_{l,m} = \chi_{l,m} \circ \varphi_{l,m}$ for all l, m. Finally we set

$$\phi_i(\bar{x}) = (\psi_{l,b^i_l}(\bar{x}))_{l<k}$$

for each $i < N$.

Claim 4.3.11. *(2) of the main lemma holds true with M, $\bar{s}$ and ϕ_i, $i < N$, that we just defined.*

Proof. Let $\bar{x}_0 \ldots, \bar{x}_{n-1} \in [\bar{s}]$ be mCg wrt $\prod_{l<k} 2^\omega$ over M. Let us write $\{\bar{x}_i(l) : i < n\} = \{y^i_l : i < K_l\}$ for every $l < k$, where $y^0_l <_{\text{lex}} \cdots <_{\text{lex}} y^{K_l-1}_l$. Now find

$$\alpha^0_0 < \cdots < \alpha^{K_0-1}_0 < \cdots < \alpha^0_{k-1} < \cdots < \alpha^{K_{k-1}-1}_{k-1}$$

in $B \cap M$. For every $\bar{j} \in \prod_{l<k} K_l$, define $\bar{y}_{\bar{j}} := (y^{j(0)}_0, \ldots, y^{j(k-1)}_{k-1})$ and $\bar{\alpha}_{\bar{j}} := (\alpha^{j(0)}_0, \ldots, \alpha^{j(k-1)}_{k-1})$. Then, for each $i < n$, we have $\bar{j}_i \in \prod_{l<k} K_l$ so that $\bar{x}_i = \bar{y}_{\bar{j}_i}$. For each $i < n$ define the function $g_i \colon \bigcup_{l<k}\{l\} \times H_l(\bar{\alpha}_{\bar{j}_i}) \to 2^\omega$, setting

$$g_i(l, \beta) = \psi_{l,m}(\bar{x}_i),$$

whenever β is the m'th element of $H_l(\bar{\alpha}_{\bar{j}_i})$.

Now we have that the g_i agree on their common domain. Namely let $i_0, i_1 < n$ and $(l, \beta) \in \operatorname{dom} g_{i_0} \cap \operatorname{dom} g_{i_1}$. Then if we set i to be so that $\bar{x}_{i_0}R_i\bar{x}_{i_1}$, we have that $m \in I_{l,i}$, where β is the m'th element of $H_l(\bar{\alpha}_{\bar{j}_{i_0}})$ and of $H_l(\bar{\alpha}_{\bar{j}_{i_1}})$. In particular $\bar{x}_{i_0} E_{l,m} \bar{x}_{i_1}$ and $\varphi_{l,m}(\bar{x}_{i_0}) = \varphi_{l,m}(\bar{x}_{i_1})$ and thus

$$g_{i_0}(l,\beta) = \psi_{l,m}(\bar{x}_{i_0}) = \chi_{l,m}(\varphi_{l,m}(\bar{x}_{i_0})) = \chi_{l,m}(\varphi_{l,m}(\bar{x}_{i_1})) = \psi_{l,m}(\bar{x}_{i_0}) = g_{i_1}(l,\beta).$$

Let $g := \bigcup_{i<n} g_i$. Then we see by Lemma 4.3.3, that g is Cohen generic in $\prod_{(l,\beta)\in \operatorname{dom} g} 2^\omega$ over M. Namely consider $K = \sum_{l<k} K_l$ and $(y_0^0, \dots, y_{k-1}^{K_{k-1}-1})$ as a $(2^{<\omega})^K$-generic over M. Then, if $\langle u_i : i < n' \rangle$ enumerates $\{\varphi_{l,m}(\bar{x}_i) : i < n, l < k, m < N_l\}$, we have that every value of g is contained in $\{\chi_{l,m}(u_i) : i < n', l < k, m < N_l\}$. Also note that by construction for every $i < n$, $p_{\bar{\alpha}_{\bar{j}_i}} \restriction \operatorname{dom} g$ is in the generic filter defined by g. Since $\{p_{\bar{\alpha}_{\bar{j}_i}} : i < n\}$ is centered we can extend the generic filter of g to a $\mathbb{Q}$-generic G over M so that $p_{\bar{\alpha}_{j_i}} \in G$ for every $i < n$.

Now we have that

$$\bar{z}_{\bar{\alpha}_{\bar{j}_i}}[G] = \bar{x}_i \text{ and } \bar{z}_{\bar{\beta}^j(\bar{\alpha}_{\bar{j}_i})}[G] = \phi_j(\bar{x}_i)$$

for every $i < n$ and $j < N$. Thus we get that

$$M[G] \models \bigcup_{i<n} \{\phi_j(\bar{x}_i) : j < N\} \subseteq \dot{A}[G] \wedge \{\bar{x}_0\} \cup \{\phi_j(\bar{x}_0) : j < N\} \in E.$$

Again, by absoluteness, we get the required result. $\square$

$\square$

4.3.3 Infinite products

Definition 4.3.12. Let $\langle X_i : i < \alpha \rangle \in M$ be Polish spaces indexed by a countable ordinal α. Then we say that $\bar{x}_0, \dots, \bar{x}_{n-1} \in \prod_{i<\alpha} X_i$ are *mutually Cohen generic (mCg) with respect to the product* $\prod_{i<\alpha} X_i$ *over* M if there are $\xi_0 = 0 < \dots < \xi_k = \alpha$ for some $k \in \omega$ so that

$$\bar{x}_0, \dots, \bar{x}_{n-1} \text{ are mutually Cohen generic with respect to } \prod_{l<k} Y_l \text{ over } M,$$

where $Y_l = \prod_{i\in[\xi_l,\xi_{l+1})} X_i$ for every $l < k$.

Note that whenever $\bar{x}_0, \dots, \bar{x}_{n-1}$ are mCg over M with respect to $\prod_{i<\alpha} X_i$ and $\beta \leq \alpha$, then $\bar{x}_0 \restriction \beta, \dots, \bar{x}_{n-1} \restriction \beta$ are mCg over M with respect to $\prod_{i<\beta} X_i$.

Definition 4.3.13. We say that $\bar{x}_0, \dots, \bar{x}_{n-1} \in \prod_{i<\alpha} X_i$ are *strongly mCg* over M with respect to $\prod_{i<\alpha} X_i$ if they are mCg over M with respect to $\prod_{i<\alpha} X_i$ and for any $i, j < n$, if $\xi = \min\{\beta < \alpha : x_i(\beta) \neq x_j(\beta)\}$, then $x_i(\beta) \neq x_j(\beta)$ for all $\beta \geq \xi$.

Main Lemma 4.3.14. Let $\alpha < \omega_1$ and $E \subseteq [(2^\omega)^\alpha]^{<\omega} \setminus \{\emptyset\}$ be an analytic hypergraph. Then there is a countable model M, $\alpha + 1 \subseteq M$, so that either

1. for any $\bar{x}_0, \dots, \bar{x}_{n-1} \in (2^\omega)^\alpha$ that are strongly mCg with respect to $\prod_{i<\alpha} 2^\omega$ over M,
$$\{\bar{x}_0, \dots, \bar{x}_{n-1}\} \text{ is } E\text{-independent}$$

or for some $N \in \omega$,

2. there are $\phi_0, \dots, \phi_{N-1} \colon (2^\omega)^\alpha \to (2^\omega)^\alpha$ continuous, $\bar{s} \in \bigotimes_{i<\alpha} 2^{<\omega}$ so that for any $\bar{x}_0, \dots, \bar{x}_{n-1} \in (2^\omega)^\alpha \cap [\bar{s}]$ that are strongly mCg over M,
$$\{\phi_j(\bar{x}_i) : j < N, i < n\} \text{ is } E\text{-independent but } \{\bar{x}_0\} \cup \{\phi_j(\bar{x}_0) : j < N\} \in E.$$

Proof. We are going to show something slightly stronger. Let R be an analytic hypergraph on $(2^\omega)^\alpha \times \omega$, M a countable model with $R \in M, \alpha + 1 \subseteq M$ and $k \in \omega$. Then consider the following two statements.

$(1)_{R,M,k}$: For any pairwise distinct $\bar{x}_0, \dots, \bar{x}_{n-1}$ that are strongly mCg with respect to $\prod_{i<\alpha} 2^\omega$ over M, and any $k_0, \dots, k_{n-1} < k$,
$$\{\bar{x}_0{}^\frown k_0, \dots, \bar{x}_{n-1}{}^\frown k_{n-1}\} \text{ is } R\text{-independent.}$$

$(2)_{R,M,k}$: There is $N \in \omega$, there are $\phi_0, \dots, \phi_{N-1} \colon (2^\omega)^\alpha \to (2^\omega)^\alpha$ continuous, such that for every $\bar{x} \in (2^\omega)^\alpha$ and $j_0 < j_1 < N$, $\phi_{j_0}(\bar{x}) \neq \phi_{j_1}(\bar{x})$ and $\phi_{j_0}(\bar{x}) \neq \bar{x}$, there are $k_0, \dots, k_{N-1} \leq k$ and $\bar{s} \in \bigotimes_{i<\alpha} 2^{<\omega}$, so that for any pairwise distinct $\bar{x}_0, \dots, \bar{x}_{n-1} \in (2^\omega)^\alpha \cap [\bar{s}]$ that are strongly mCg with respect to $\prod_{i<\alpha} 2^\omega$ over M,
$$\{\phi_j(\bar{x}_i){}^\frown k_j : j < N, i < n\} \text{ is } R\text{-independent, but}$$
$$\{\bar{x}_0{}^\frown k\} \cup \{\phi_j(\bar{x}_0){}^\frown k_j : j < N\} \in R.$$

In fact, if $k > 0$,
$$\{\bar{x}_i{}^\frown (k-1) : i < n\} \cup \{\phi_j(\bar{x}_i){}^\frown k_j : j < N, i < n\} \text{ is } R\text{-independent.}$$

We are going to show that whenever $(1)_{R,M,k}$ is satisfied, then either $(1)_{R,M,k+1}$ or there is a countable model $M^+ \supseteq M$ so that $(2)_{R,M^+,k}$. From this we easily follow the statement of the main lemma. Namely, whenever E is a hypergraph on $(2^\omega)^\alpha$, consider the hypergraph R on $(2^\omega)^\alpha \times \omega$ where $\{\bar{x}_0{}^\frown k_0, \dots, \bar{x}_{n-1}{}^\frown k_{n-1}\} \in R$ iff

$\{\bar{x}_0, \ldots, \bar{x}_{n-1}\} \in R$. Then, if M is an arbitrary countable elementary model with $R, \alpha \in M$ and if $k = 0$, $(1)_{R,M,k}$ holds vacuously true. Applying the claim we find M^+ so that either $(1)_{R,M,1}$ or $(2)_{R,M^+,0}$. The two options easily translate to the conclusion of the main lemma.

Let us first consider the successor step. Assume that $\alpha = \beta + 1$, R is an analytic hypergraph on $(2^\omega)^\alpha \times \omega$ and M a countable model with $R \in M, \alpha + 1 \subseteq M$ so that $(1)_{R,M,k}$ holds true for some given $k \in \omega$. Let $\mathbb{Q}$ be the forcing adding mutual Cohen reals $\langle z_{0,i,j}, z_{1,i,j} : i, j \in \omega \rangle$ in 2^ω. Then we define the hypergraph $\tilde{R}$ on $(2^\omega)^\beta \times \omega$ where $\{\bar{y}_0{}^\frown m_0, \ldots, \bar{y}_{n-1}{}^\frown m_{n-1}\} \in \tilde{R} \cap [(2^\omega)^\beta \times \omega]^n$ iff there is $p \in \mathbb{Q}$ and there are $K_i \in \omega$, $k_{i,0}, \ldots, k_{i,K_i-1} < k$ for every $i < n$, so that

$$p \Vdash_{\mathbb{Q}} \bigcup_{i<n} \{\bar{y}_i{}^\frown \dot{z}_{0,i,j}{}^\frown k_{i,j} : j < K_i\} \cup \{\bar{y}_i{}^\frown \dot{z}_{1,i,j}{}^\frown k : j < m_i\} \in R.$$

Then $\tilde{R}$ is analytic (see e.g. [33, 29.22]).

Claim 4.3.15. $(1)_{\tilde{R},M,1}$ *is satisfied.*

Proof. Suppose $\bar{y}_0, \ldots, \bar{y}_{n-1}$ are pairwise distinct and strongly mCg over M, but $\{\bar{y}_0{}^\frown 0, \ldots, \bar{y}_{n-1}{}^\frown 0\} \in \tilde{R}$ as witnessed by $p \in \mathbb{Q}$, $\langle K_i : i < n\rangle$ and $\langle k_{i,j} : i < n, j < K_i\rangle$, each $k_{i,j} < k$. More precisely,

$$p \Vdash_{\mathbb{Q}} \bigcup_{i<n} \{\bar{y}_i{}^\frown \dot{z}_{0,i,j}{}^\frown k_{i,j} : j < K_i\} \in R. \tag{$*_0$}$$

By absoluteness, $(*_0)$ is satisfied in $M[\bar{y}_0, \ldots, \bar{y}_{n-1}]$. Thus, let $\langle z_{0,i,j}, z_{1,i,j} : i, j \in \omega\rangle$ be generic over $M[\bar{y}_0, \ldots, \bar{y}_{n-1}]$ with p in the associated generic filter. Then $\langle \bar{y}_i{}^\frown z_{(0,i,j)} : i < n, j < K_i\rangle$ are pairwise distinct and strongly mCg over M, but

$$\bigcup_{i<n} \{\bar{y}_i{}^\frown z_{0,i,j}{}^\frown k_{i,j} : j < K_i\} \in R.$$

This poses a contradiction to $(1)_{R,M,k}$. □

Claim 4.3.16. *If* $(1)_{\tilde{R},M,m}$ *is satisfied for every* $m \in \omega$*, then also* $(1)_{R,M,k+1}$.

Proof. Let $\bar{x}_0, \ldots, \bar{x}_{n-1} \in (2^\omega)^\alpha$ be pairwise distinct, strongly mCg over M and let $k_0, \ldots, k_{n-1} \leq k$. Then we may write $\{\bar{x}_0{}^\frown k_0, \ldots, \bar{x}_{n-1}{}^\frown k_{n-1}\}$ as

$$\bigcup_{i<n'} \{\bar{y}_i{}^\frown z_{0,i,j}{}^\frown k_{i,j} : j < K_i\} \cup \{\bar{y}_i{}^\frown z_{1,i,j}{}^\frown k : j < m_i\}, \tag{$*_1$}$$

for some pairwise distinct $\bar{y}_0, \ldots, \bar{y}_{n'-1}$, $\langle K_i : i < n'\rangle$, $\langle k_{i,j} : i < n', j < K_i\rangle$, $\langle m_i : i < n'\rangle$ and $\langle z_{0,i,j} : i, j \in \omega\rangle$, $\langle z_{1,i,j} : i, j \in \omega\rangle$ mutually Cohen generic in 2^ω

over $M[\bar y_0, \dots, \bar y_{n'-1}]$. Letting $m = \max_{i<n'} m_i + 1$, we follow the R-independence of the set in $(*_1)$ from $(1)_{\tilde R, M, m}$. $\square$

Claim 4.3.17. *If there is $m \in \omega$ so that $(1)_{\tilde R,M,m}$ fails, then there is a countable model $M^+ \supseteq M$ so that $(2)_{R,M^+,k}$.*

Proof. Let $m \geq 1$ be least so that $(2)_{\tilde R, M_0, m}$ for some countable model $M_0 \supseteq M$. We know that such m exists, since from $(1)_{\tilde R,M,1}$ we follow that either $(1)_{\tilde R,M,2}$ or $(2)_{\tilde R,M_0,1}$ for some M_0, then, if $(1)_{\tilde R,M,2}$, either $(1)_{\tilde R,M,3}$ or $(2)_{\tilde R,M_0,2}$ for some M_0, and so on. Let $\phi_0, \dots, \phi_{N-1}, m_0, \dots, m_{N-1} \leq m$ and $\bar s \in \bigotimes_{i<\beta}(2^{<\omega})$ witness $(2)_{\tilde R, M_0, m}$. Let M_1 be a countable elementary model such that $\phi_0, \dots, \phi_{N-1}, M_0 \in M_1$. Then we have that for any $\bar y$ that is Cohen generic in $(2^\omega)^\beta \cap [\bar s]$ over M_1, in particular over M_0, that

$$\{\bar y^\frown m\} \cup \{\phi_j(\bar y)^\frown m_j : j < N\} \in \tilde R,$$

i.e. there is $p \in \mathbb{Q}$, there are $K_i \in \omega$, $k_{i,0}, \dots, k_{i,K_i-1} < k$ for every $i \leq N$, so that

$$p \Vdash_{\mathbb{Q}} \bigcup_{i<N} \{\phi_i(\bar y)^\frown \dot z_{0,i,j}{}^\frown k_{i,j} : j < K_i\} \cup \{\phi_i(\bar y)^\frown \dot z_{1,i,j}{}^\frown k : j < m_i\}$$
$$\cup \{\bar y^\frown \dot z_{0,N,j}{}^\frown k_{N,j} : j < K_N\} \cup \{\bar y^\frown \dot z_{1,N,j}{}^\frown k : j < m\} \in R. \quad (*_2)$$

By extending $\bar s$, we can assume wlog that p, $\langle K_i : i \leq N\rangle$, $\langle k_{i,j} : i \leq N, j < K_i\rangle$ are the same for each $\bar y \in [\bar s]$ generic over M_1, since $(*_2)$ can be forced over M_1. Also, from the fact that ϕ_j is continuous for every $j < N$, that $\phi_j(\bar y) \neq \bar y$ for every $j < N$, and that $\phi_{j_0}(\bar y) \neq \phi_{j_1}(\bar y)$ for every $j_0 < j_1 < N$, we can assume wlog that for any $\bar y_0, \bar y_1 \in [\bar s]$ and $j_0 < j_1 < N$,

$$\phi_{j_0}(\bar y_0) \neq \bar y_1 \text{ and } \phi_{j_0}(\bar y_0) \neq \phi_{j_1}(\bar y_1). \quad (*_3)$$

Let us force in a finite support product over M_1 continuous functions $\chi_{0,i,j} \colon (2^\omega)^\beta \to [p(0,i,j)]$ and $\chi_{1,i,j} \colon (2^\omega)^\beta \to [p(1,i,j)]$ for $i,j \in \omega$ and write $M^+ = M_1[\langle \chi_{0,i,j}, \chi_{1,i,j} : i,j \in \omega\rangle]$. For every $i < N$ and $j < K_i$ and $\bar x \in (2^\omega)^\alpha$, define

$$\phi_{0,i,j}(\bar x) := \phi_i(\bar x \restriction \beta)^\frown \chi_{0,i,j}(\phi_i(\bar x \restriction \beta)) \text{ and } k_{0,i,j} = k_{i,j}.$$

For every $i < N$ and $j < m_i$ and $\bar x \in (2^\omega)^\alpha$, define

$$\phi_{1,i,j}(\bar x) := \phi_i(\bar x \restriction \beta)^\frown \chi_{1,i,j}(\phi_i(\bar x \restriction \beta)) \text{ and } k_{1,i,j} = k.$$

For every $j < K_N$ and $\bar x \in (2^\omega)^\alpha$, define

$$\phi_{0,N,j}(\bar x) := \bar x \restriction \beta^\frown \chi_{0,N,j}(\bar x \restriction \beta) \text{ and } k_{0,N,j} = k_{N,j}.$$

At last, define for every $j < m - 1$ and $\bar{x} \in (2^\omega)^\alpha$,

$$\phi_{1,N,j}(\bar{x}) := \bar{x} \restriction \beta^\frown \chi_{1,N,j}(\bar{x} \restriction \beta) \text{ and } k_{1,N,j} = k.$$

Let $\bar{t} \in \bigotimes_{i<\alpha} 2^{<\alpha}$ be $\bar{s}$ with $p(1, N, m-1)$ added in coordinate β. Now we have that for any $\bar{x} \in [\bar{t}]$ that is Cohen generic in $(2^\omega)^\alpha$ over M^+,

$$\{\bar{x}^\frown k\} \cup \{\phi_{0,i,j}(\bar{x})^\frown k_{0,i,j} : i \leq N, j < K_N\} \cup \{\phi_{1,i,j}(\bar{x})^\frown k_{1,i,j} : i < N, j < m_i\} \\ \cup \{\phi_{1,N,j}(\bar{x})^\frown k_{1,N,j} : j < m - 1\} \in R.$$

This follows from $(*)_2$ and applying Lemma 4.3.3 to see that the $\chi_{0,i,j}(\phi_i(\bar{x} \restriction \beta))$, $\chi_{1,i,j}(\phi_i(\bar{x} \restriction \beta))$, $\chi_{0,N,j}(\bar{x} \restriction \beta)$, $\chi_{1,N,j}(\bar{x} \restriction \beta)$ and $x(\beta)$ are mutually Cohen generic over $M_1[\bar{x} \restriction \beta]$. Moreover they correspond to the reals $z_{0,i,j}$, $z_{1,i,j}$ added by a $\mathbb{Q}$-generic over $M_1[\bar{x} \restriction \beta]$, containing p in its generic filter. Also, remember that $(*)_2$ is absolute between models containing the relevant parameters, which $M_1[\bar{y}]$ is, with $\bar{y} = \bar{x} \restriction \beta$.

On the other hand, whenever $\bar{x}_0, \ldots, \bar{x}_{n-1} \in (2^\omega)^\alpha \cap [\bar{t}]$ are pairwise distinct and strongly mCg over M^+, letting $\bar{y}_0, \ldots, \bar{y}_{n'-1}$ enumerate $\{\bar{x}_i \restriction \beta : i < n\}$, we have that

$$\{\bar{y}_i{}^\frown(m-1) : i < n'\} \cup \{\phi_j(\bar{y}_i)^\frown m_j : i < n', j < N\} \text{ is } \tilde{R}\text{-independent.} \qquad (*_4)$$

According to the definition of $\tilde{R}$, $(*_4)$ is saying e.g. that whenever $A \cup B \subseteq (2^\omega)^\alpha$ is an arbitrary set of strongly mCg reals over M_1, where $A \restriction \beta, B \restriction \beta \subseteq \{\bar{y}_i, \phi_j(\bar{y}_i) : i < n', j < N\}$ and in B, $\bar{y}_i$ is extended at most $m-1$ many times and $\phi_j(\bar{y}_i)$ at most m_j many times for every $i < n'$, $j < N$, and, assuming for now that $k > 0$, if $f\colon A \to k$, then

$$\{\bar{x}^\frown f(\bar{x}) : \bar{x} \in A\} \cup (B \times \{k\}) \text{ is } R\text{-independent.}$$

As an example for such sets A and B we have,

$$A = \{\phi_{0,i,j}(\bar{x}_l) : l < n, i \leq N, j < K_i\} \cup \{\bar{x}_l : l < n'\}, \text{ and}$$

$$B = \{\phi_{1,i,j}(\bar{x}_l) : l < n, i < N, j < m_i\} \cup \{\phi_{1,N,j}(\bar{x}_l) : l < n', j < m - 1\}.$$

Again, to see this we apply Lemma 4.3.3 to show that the relevant reals are mutually generic over the model $M_1[\bar{y}_0, \ldots, \bar{y}_{n'-1}]$. Also, remember from the definition of $\phi_{1,i,j}$ for $i < N$ and $j < m_i$ that, if $\phi_i(\bar{x}_{l_0} \restriction \beta) = \phi_i(\bar{x}_{l_1} \restriction \beta)$, then also $\phi_{1,i,j}(\bar{x}_{l_0}) = \phi_{1,i,j}(\bar{x}_{l_1})$, for all $l_0, l_1 < n$. Equally, if $\bar{x}_{l_0} \restriction \beta = \bar{x}_{l_1} \restriction \beta$, then $\phi_{1,N,j}(\bar{x}_{l_0}) = \phi_{1,N,j}(\bar{x}_{l_1})$ for every $j < m - 1$. Use $(*_3)$ to note that $\{\bar{y}_i : i < n'\}$, $\{\phi_0(\bar{y}_i) : i < n'\}, \ldots,$ $\{\phi_{N-1}(\bar{y}_i) : i < n'\}$ are pairwise disjoint. From this we can follow that indeed, each $\bar{y}_i$

is extended at most $m-1$ many times in B and $\phi_j(\bar{y}_i)$ at most m_i many times. In total, we get that

$$\{\phi_{0,i,j}(\bar{x}_l)^\frown k_{0,i,j} : l < n, i \leq N, j < K_i\} \cup \{\bar{x}_l{}^\frown(k-1) : l < n'\} \cup$$
$$\{\phi_{1,i,j}(\bar{x}_l)^\frown k : l < n, i < N, j < m_i\} \cup \{\phi_{1,N,j}(\bar{x}_l)^\frown k : l < n', j < m-1\}$$
$$\text{is } R\text{-independent.}$$

It is now easy to check that we have the witnesses required in the statement of $(2)_{R,M^+,k}$. For example, $\phi_{0,i,j}(\bar{x}) \neq \bar{x}$ when $i < N$, follows from $\phi_i(\bar{x}) \neq \bar{x}$. For the values $\phi_{0,N,j}(\bar{x})$ we simply have that $\chi_{0,N,j}(\bar{x} \restriction \beta) \neq x(\beta)$, as the two values are mutually generic. Everything else is similar and consists only of a few case distinctions. Also, the continuity of the functions is clear.

If $k = 0$, then we can simply forget the set A above, since K_i must be 0 for every $i \leq N$. In this case we just get that

$$\{\phi_{1,i,j}(\bar{x}_l)^\frown k : l < n, i < N, j < m_i\} \cup \{\phi_{1,N,j}(\bar{x}_l)^\frown k : l < n', j < m-1\}$$
$$\text{is } R\text{-independent,}$$

which then yields $(2)_{R,M^+,k}$. $\square$

This finishes the successor step. Now assume that α is a limit ordinal. We fix some arbitrary tree $T \subseteq \omega^{<\omega}$ such that for every $t \in T$, $|\{n \in \omega : t^\frown n \in T\}| = \omega$ and for any branches $x \neq y \in [T]$, if $d = \min\{i \in \omega : x(i) \neq y(i)\}$ then $x(j) \neq x(j)$ for every $j \geq d$. We will use T only for national purposes. For every sequence $\xi_0 < \cdots < \xi_{k'} = \alpha$, we let $\mathbb{Q}_{\xi_0,\ldots,\xi_{k'}} = \left(\prod_{l<k'}(\bigotimes_{i\in[\xi_l,\xi_{l+1})} 2^{<\omega})^{<\omega}\right) \times (\bigotimes_{i\in[\xi_0,\alpha)} 2^{<\omega})^{<\omega}$. $\mathbb{Q}_{\xi_0,\ldots,\xi_{k'}}$ adds, in the natural way, reals $\langle \bar{z}^0_{l,i} : l < k', i \in \omega\rangle$ and $\langle \bar{z}^0_i : i \in \omega\rangle$, where $\bar{z}^0_{l,i} \in (2^\omega)^{[\xi_l,\xi_{l+1})}$ and $\bar{z}^1_i \in (2^\omega)^{[\xi_0,\alpha)}$ for every $l < k'$, $i \in \omega$. Whenever $t \in T \cap \omega^{k'}$, we write $\bar{z}^0_t = \bar{z}^0_{0,t(0)}{}^\frown \cdots {}^\frown \bar{z}^0_{k'-1,t(k'-1)}$.Note that for generic $\langle \bar{z}^0_{l,i} : i \in \omega, l < k'\rangle$, the reals $\langle \bar{z}^0_t : t \in T \cap \omega^{k'}\rangle$ are strongly mCg with respect to $\prod_{i\in[\xi_0,\alpha)} 2^\omega$.

Now, let us define for each $\xi < \alpha$ an analytic hypergraph R_ξ on $(2^\omega)^\xi \times 2$ so that $\{\bar{y}^0_i{}^\frown 0 : i < n_0\} \cup \{\bar{y}^1_i{}^\frown 1 : i < n_1\} \in R_\xi \cap [(2^\omega)^\xi \times 2]^{n_0+n_1}$, where $|\{\bar{y}^0_i{}^\frown 0 : i < n_0\}| = n_0$ and $|\{\bar{y}^1_i{}^\frown 1 : i < n_1\}| = n_1$, iff there are $\xi_0 = \xi < \cdots < \xi_{k'} = \alpha$, $(p,q) \in \mathbb{Q}_{\xi_0,\ldots,\xi_{k'}}$, $K_i \in \omega$, $k_{i,0},\ldots,k_{i,K_i-1} < k$ and distinct $t_{i,0},\ldots,t_{i,K_i-1} \in T \cap \omega^{k'}$ for every $i < n_0$, so that $t_{i_0,j_0}(0) \neq t_{i_1,j_1}(0)$ for all $i_0 < i_1 < n_0$ and $j_0 < K_{i_0}, j_1 < K_{i_1}$, and

$$(p,q) \Vdash_{\mathbb{Q}_{\bar{\xi}}} \bigcup_{i<n_0} \{\bar{y}^0_i{}^\frown \bar{z}^0_{t_{i,j}}{}^\frown k_{i,j} : j < K_i\} \cup \{\bar{y}^1_i{}^\frown \bar{z}^1_i{}^\frown k : i < n_1\} \in R.$$

Note that each R_ξ can be defined within M. It should be clear, similar to the proof of Claim 4.3.15, that from $(1)_{R,M,k}$, we can show the following.

Claim 4.3.18. *For every $\xi < \alpha$, $(1)_{R_\xi,M,1}$.*

Claim 4.3.19. *Assume that for every $\xi < \alpha$, $(1)_{R_\xi,M,2}$. Then also $(1)_{R,M,k+1}$.*

Proof. Let $\bar{x}^0_0, \dots, \bar{x}^0_{n_0-1}, \bar{x}^1_0, \dots, \bar{x}^1_{n_1-1}$ be pairwise distinct and strongly mCg over M and $k_0, \dots, k_{n_0-1} < k$. Then there is $\xi < \alpha$ large enough so that $\bar{x}^0_0 \restriction \xi, \dots, \bar{x}^0_{n_0-1} \restriction \xi, \bar{x}^1_0 \restriction \xi, \dots, \bar{x}^1_{n_1-1} \restriction \xi$ are pairwise distinct and in particular, $\bar{x}^0_0 \restriction [\xi,\alpha), \dots, \bar{x}^0_{n_0-1} \restriction [\xi,\alpha), \bar{x}^1_0 \restriction [\xi,\alpha), \dots, \bar{x}^1_{n_1-1} \restriction [\xi,\alpha)$ are pairwise different in every coordinate. Let $\xi_0 = \xi$, $\xi_1 = \alpha$, $K_i = 1$ for every $i < n_0$ and $t_{0,0}, \dots, t_{n_0-1,0} \in T \cap \omega^1$ pairwise distinct. Also, write $k_{0,0} = k_0$, ..., $k_{n_0-1,0} = k_{n_0-1}$. Then, from $(1)_{R_\xi,M,2}$, we have that

$$\mathbb{1} \Vdash_{\xi_0,\xi_1} \{(\bar{x}^0_i \restriction \xi)^\frown \bar{z}^0_{t_{i,0}}{}^\frown k_{i,0} : i < n_0\} \cup \{(\bar{x}^1_i \restriction \xi)^\frown \bar{z}^1_i{}^\frown k : i < n_1\} \text{ is } R\text{-independent.}$$

By absoluteness, this holds true in $M[\langle \bar{x}^0_i \restriction \xi, \bar{x}^1_j \restriction \xi : i < n_0, j < n_1\rangle]$ and we find that

$$\{\bar{x}^0_i{}^\frown k_i : i < n_0\} \cup \{\bar{x}^1_i{}^\frown k\} \text{ is } R\text{-independent,}$$

as required. □

Claim 4.3.20. *If there is $\xi < \alpha$ so that $(1)_{R_\xi,M,2}$ fails, then there is a countable model $M^+ \supseteq M$ so that $(2)_{R,M^+,k}$.*

Proof. If $(1)_{R_\xi,M,2}$ fails, then there is a countable model $M_0 \supseteq M$ so that $(2)_{R_\xi,M,1}$ holds true as witnessed by $\bar{s} \in \bigotimes_{i<\xi} 2^{<\omega}$, $\phi^0_0, \dots, \phi^0_{N_0-1}, \phi^1_0, \dots, \phi^1_{N_1-1} \colon (2^\omega)^\xi \to (2^\omega)^\xi$ such that for any pairwise distinct $\bar{y}_0, \dots, \bar{y}_{n-1} \in (2^\omega)^\xi \cap [\bar{s}]$ that are strongly mCg over M_0,

$$\{\bar{y}_i{}^\frown 0 : i < n\} \cup \{\phi^0_j(\bar{y}_i)^\frown 0 : i < n, j < N_0\} \cup \{\phi^1_j(\bar{y}_i)^\frown 1 : i < n, j < N_1\} \quad (*_5)$$

is R_ξ-independent, but

$$\{\bar{y}_0{}^\frown 1\} \cup \{\phi^0_j(\bar{y}_0)^\frown 0 : j < N_0\} \cup \{\phi^1_j(\bar{y}_0)^\frown 1 : j < N_1\} \in R_\xi. \quad (*_6)$$

As before, we may pick $M_1 \ni M_0$ elementary containing all relevant information, assume that $(*_6)$ is witnessed by fixed $\xi_0 = \xi < \dots < \xi_{k'} = \alpha$, $(p,q) \in \mathbb{Q}_{\xi_0,\dots,\xi_{k'}}$, $K_0, \dots, K_{N_0-1}$, $k_{i,0}, \dots, k_{i,K_i-1}$ and $t_{i,0}, \dots, t_{i,K_i-1} \in T \cap \omega^{k'}$ for every $i < N_0$, so that for every generic $\bar{y}_0 \in (2^\omega)^\xi \cap [\bar{s}]$ over M_1,

$$(p,q) \Vdash_{\mathbb{Q}_{\bar{\xi}}} \{\bar{y}_0{}^\frown \bar{z}^1_{N_1}{}^\frown k\} \cup \bigcup_{i<N_0} \{\phi^0_i(\bar{y}_0)^\frown \bar{z}^0_{t_{i,j}}{}^\frown k_{i,j} : j < K_i\} \cup$$
$$\{\phi^1_j(\bar{y}_0)^\frown \bar{z}^1_j{}^\frown k : j < N_1\} \in R. \quad (*_7)$$

As before, we may also assume that $\bar{y}_0 \neq \phi_{i_0}^{j_0}(\bar{y}_0) \neq \phi_{i_1}^{j_1}(\bar{y}_1)$ for every $\bar{y}_0, \bar{y}_1 \in [\bar{s}]$ and $(j_0, i_1) \neq (j_1, i_1)$. We let $\bar{s}' = \bar{s}^\frown q(N_1)$. Now we force continuous functions $\chi_{l,i}^0 \colon (2^\omega)^\xi \to (2^\omega)^{[\xi_l, \xi_{l+1})} \cap [p(l,i)]$ and $\chi_i^1 \colon (2^\omega)^\xi \to (2^\omega)^{[\xi,\alpha)} \cap [q(i)]$ over M_1 for every $i \in \omega$, $l < k'$ and we let $M^+ = M_1[\langle \chi_{l,i}^0, \chi_i^1 : i \in \omega, l < k' \rangle]$. Finally we let

$$\phi_{0,i,j}(\bar{x}) := \phi_i^0(\bar{x} \restriction \xi)^\frown \chi_{0,t_{i,j}(0)}(\phi_i^0(\bar{x} \restriction \xi))^\frown \dots {}^\frown \chi_{k'-1,t_{i,j}(k'-1)}(\phi_i^0(\bar{x} \restriction \xi))$$

for every $i < N_0$ and $j < K_i$, $\bar{x} \in (2^\omega)^\alpha$, and

$$\phi_{1,i}(\bar{x}) := \phi_i^1(\bar{x} \restriction \xi)^\frown \chi_{1,i}(\phi_i^1(\bar{x} \restriction \xi))$$

for every $i < N_1$, $\bar{x} \in (2^\omega)^\alpha$.

We get from $(*_7)$, and, as usual, applying Lemma 4.3.3, that for any $\bar{x} \in (2^\omega)^\alpha \cap [\bar{s}']$ which is generic over M^+,

$$\{\bar{x}^\frown k\} \cup \bigcup_{i<N_0} \{\phi_{0,i,j}(\bar{x})^\frown k_{i,j} : j < K_i\} \cup \{\phi_{1,i}(\bar{x})^\frown k : i < N_1\} \in R.$$

On the other hand, whenever $\bar{x}_0, \dots, \bar{x}_{n'-1} \in (2^\omega)^\alpha \cap [\bar{s}']$ are strongly mCg over M^+, and letting $\bar{y}_0, \dots, \bar{y}_{n-1}$ enumerate $\{\bar{x}_i \restriction \xi : i < n'\}$, knowing that the set in $(*_5)$ is R_ξ-independent, we get that

$$\{\bar{x}_l^\frown (k-1) : l < n'\} \cup \bigcup_{i<N_0} \{\phi_{0,i,j}(\bar{x}_l)^\frown k_{i,j} : j < K_i, l < n'\} \cup$$
$$\{\phi_{1,i}(\bar{x}_l)^\frown k : i < N_1, l < n'\} \text{ is } R\text{-independent},$$

in case $k > 0$. To see this, we let $\eta_0 < \dots < \eta_{k''}$ be a partition refining $\xi_0 < \dots, \xi_{k'}$ witnessing the mCg of $\bar{x}_0 \restriction [\xi, \alpha), \dots, \bar{x}_{n'-1} \restriction [\xi, \alpha)$ and we find appropriate $u_{0,0}, \dots, u_{0,L_0-1}, \dots, u_{n-1,0}, \dots, u_{n-1,L_{n-1}-1} \in T \cap \omega^{k''}$ and $v_{i,j} \in T \cap \omega^{k''}$ for $i < N_0, j < K_i$ to interpret the above set in the form

$$\{\bar{y}_l^\frown \bar{z}_{u_{l,i}}^0{}^\frown (k-1) : l < n, i < L_i\} \cup \bigcup_{i<N_0} \{\phi_i^0(\bar{y}_l)^\frown \bar{z}_{v_{i,j}}^0{}^\frown k_{i,j} : i < N_0, j < K_i, l < n\}$$
$$\cup \{\phi_i^1(\bar{y}_l)^\frown \bar{z}_i^1{}^\frown k : i < N_1, l < n\},$$

for $\mathbb{Q}_{\eta_0,\dots,\eta_{k''-1}}$-generic $\langle \bar{z}_{l,i}^0, \bar{z}_i^1 : l < k'', i \in \omega \rangle$ over $M_1[\bar{y}_0, \dots, \bar{y}_{n-1}]$. We leave the details to the reader. In case $k = 0$, all K_i are 0 and we get that

$$\{\phi_{1,i}(\bar{x}_l)^\frown k : i < N_1, l < n'\} \text{ is } R\text{-independent}.$$

Everything that remains, namely showing e.g. that $\bar{x} \neq \phi_{1,i}(\bar{x})$ is clear. $\square$

As a final note, let us observe that the case $\alpha = 0$ is trivial, since $(2^\omega)^\alpha$ has only one element. $\square$

Remark 4.3.21. If we replace “strong mCg" with “‘mCg" in the above Lemma, then it already becomes false for $\alpha = \omega$. Namely consider the equivalence relation E on $(2^\omega)^\omega$, where $\bar{x} E \bar{y}$ if they eventually agree, i.e. if $\exists n \in \omega \forall m \geq n(x(n) = y(n))$. Then we can never be in case (1) since we can always find two distinct $\bar{x}$ and $\bar{y}$ that are mCg and $\bar{x} E \bar{y}$. On the other hand, in case (2) we get a continuous selector ϕ_0 for E (note that $N = 0$ is not possible). More precisely we have that for any $\bar{x}$, $\bar{y}$ that are mCg, $\bar{x} E \phi_0(\bar{x})$ and $\phi_0(\bar{x}) = \phi_0(\bar{y})$ iff $\bar{x} E \bar{y}$. But for arbitrary mCg $\bar{x}$ and $\bar{y}$ so that $\bar{x} \neg E \bar{y}$, we easily find a sequence $\langle \bar{x}_n : n \in \omega \rangle$ so that $\bar{x}$ and $\bar{x}_n$ are mCg and $\bar{x} E \bar{x}_n$, but $\bar{x}_n \restriction n = \bar{y} \restriction n$ for all n. In particular $\lim_{n \in \omega} \bar{x}_n = \bar{y}$. Then $\phi_0(\bar{y}) = \lim_{n \in \omega} \phi_0(\bar{x}_n) = \lim_{n \in \omega} \phi_0(\bar{x}) = \phi_0(\bar{x})$.

Remark 4.3.22. The proofs of Main Lemma 4.3.4 and 4.3.14 can be generalized to E that is ω-universally Baire. For this, additional details are required, for example related to the complexity of the forcing relation in Cohen forcing.

Definition 4.3.23. For $\bar{x}_0, \dots, \bar{x}_{n-1} \in \prod_{i<\alpha} X_i$, we define

$$\Delta(\bar{x}_0, \dots, \bar{y}_{n-1}) := \{\Delta_{\bar{x}_i, \bar{x}_j} : i \neq j < n\} \cup \{0, \alpha\},$$

where $\Delta_{\bar{x}_i, \bar{x}_j} := \min\{\xi < \alpha : x_i(\xi) \neq x_j(\xi)\}$ if this exists and $\Delta_{\bar{x}_i, \bar{x}_j} = \alpha$ if $\bar{x}_i = \bar{x}_j$.

Remark 4.3.24. Whenever $\bar{x}_0, \dots, \bar{x}_{n-1}$ are strongly mCg, then they are mCg as witnessed by the partition $\xi_0 < \cdots < \xi_k$, where $\{\xi_0, \dots, \xi_k\} = \Delta(\bar{x}_0, \dots, \bar{x}_{n-1})$.

4.4 Sacks and splitting forcing

4.4.1 Splitting Forcing

Definition 4.4.1. We say that $S \subseteq 2^{<\omega}$ is *fat* if there is $m \in \omega$ so that for all $n \geq m$, there are $s, t \in S$ so that $s(n) = 0$ and $t(n) = 1$. A tree T on 2 is called *splitting tree* if for every $s \in T$, T_s is fat. We call *splitting forcing* the tree forcing $\mathbb{SP}$ consisting of splitting trees.

Note that for $T \in \mathbb{SP}$ and $s \in T$, T_s is again a splitting tree. Recall that $x \in 2^\omega$ is called splitting over V, if for every $y \in 2^\omega \cap V$, $\{n \in \omega : y(n) = x(n) = 1\}$ and $\{n \in \omega : x(n) = 1 \wedge y(n) = 0\}$ are infinite. The following is easy to see.

Fact. *Let G be $\mathbb{SP}$-generic over V. Then x_G, the generic real added by $\mathbb{SP}$, is splitting over V.*

Whenever S is fat let us write $m(S)$ for the minimal $m \in \omega$ witnessing this.

Definition 4.4.2. Let S, T be splitting trees and $n \in \omega$. Then we write $S \leq_n T$ iff $S \leq T$, $\text{split}_{\leq n}(S) = \text{split}_{\leq n}(T)$ and $\forall s \in \text{split}_{\leq n}(S)(m(S_s) = m(T_s))$.

Proposition 4.4.3. *The sequence $\langle \leq_n : n \in \omega \rangle$ witnesses that $\mathbb{SP}$ has Axiom A with continuous reading of names.*

Proof. It is clear that $\leq_n$ is a partial order refining $\leq$ and that $\leq_{n+1} \subseteq \leq_n$ for every $n \in \omega$. Let $\langle T_n : n \in \omega \rangle$ be a fusion sequence in $\mathbb{SP}$, i.e. for every n, $T_{n+1} \leq_n T_n$. Then we claim that $T := \bigcap_{n \in \omega} T_n$ is a splitting tree. More precisely, for $s \in T$, we claim that $m := m((T_{|s|})_s)$ witnesses that T_s is fat. To see this, let $n \geq m$ be arbitrary and note that $n \geq m \geq |s|$ must be the case. Then, since $\text{split}_{\leq n+1}(T_{n+1}) \subseteq T$ we have that $s \in \text{split}_{\leq n+1}(T_{n+1})$ and $m((T_{n+1})_s) = m$. So find $t_0, t_1 \in T_{n+1}$ so that $t_0(n) = 0$, $t_1(n) = 1$ and $|t_0| = |t_1| = n + 1$. But then $t_0, t_1 \in T$, because $t_0, t_1 \in \text{split}_{\leq n+1}(T_{n+1}) \subseteq T$.

Now let $D \subseteq \mathbb{SP}$ be open dense, $T \in \mathbb{SP}$ and $n \in \omega$. We will show that there is $S \leq_n T$ so that for every $x \in [S]$, there is $t \subseteq x$, with $S_t \in D$. This implies condition (3) in Definition 4.2.1.

Claim 4.4.4. *Let S be a splitting tree. Then there is $A \subseteq S$ an antichain (seen as a subset of $2^{<\omega}$) so that for every $k \in \omega, j \in 2$, if $\exists s \in S(s(k) = j)$, then $\exists t \in A(t(k) = j)$.*

Proof. Start with $\{s_i : i \in \omega\} \subseteq S$ an arbitrary infinite antichain and let $m_i := m(S_{s_i})$ for every $i \in \omega$. Then find for each $i \in \omega$, a finite set $H_i \subseteq S_{s_i}$ so that for all $k \in [m_i, m_{i+1})$, there are $t_0, t_1 \in H_i$, so that $t_0(k) = 0$ and $t_1(k) = 1$. Moreover let $H \subseteq S$ be finite so that for all $k \in [0, m_0)$ and $j \in 2$, if $\exists s \in S(s(k) = j)$, then $\exists t \in H(t(k) = j)$. Then define $F_i = H_i \cup (H \cap S_{s_i})$ for each $i \in \omega$ and let $F_{-1} := H \setminus \bigcup_{i \in \omega} F_i$. Since F_i is finite for every $i \in \omega$, it is easy to extend each of its elements to get a set F'_i that is an additionally an antichain in S_{s_i}. Also extend the elements of F_{-1} to get an antichain F'_{-1} in S. It is easy to see that $A := \bigcup_{i \in [-1, \omega)} F'_i$ works. $\square$

Now enumerate $\text{split}_n(T)$ as $\langle \sigma_i : i < N \rangle$, $N := 2^n$. For each $i < N$, let $A_i \subseteq T_{\sigma_i}$ be an antichain as in the claim applied to $S = T_{\sigma_i}$. For every $i < N$ and

$t \in A_i$, let $S^t \in D$ be so that $S^t \leq T_t$. For every $i < N$ pick $t_i \in A_i$ arbitrarily and $F_i \subseteq A_i$ a finite set so that for every $k \in [0, m(S^{t_i}))$ and $j \in 2$, if $\exists s \in A_i(s(k) = j)$, then $\exists t \in F_i(t(k) = j)$. Then we see that $S := \bigcup_{i<N}(\bigcup_{t \in F_i} S^t \cup S^{t_i})$ works. We constructed S so that $S \leq_n T$. Moreover, whenever $x \in [S]$, then there is $i < N$ be so that $\sigma_i \subseteq x$. Then $x \in [\bigcup_{t \in F_i} S_t \cup S_{t_i}]$ and since F_i is finite, there is $t \in F_i \cup \{t_i\}$ so that $t \subseteq x$. But then $S_t \leq S^t \in D$.

Finally, in order to show the continuous reading of names, let $\dot{y}$ be a name for an element of ω^ω, $n \in \omega$ and $T \in \mathbb{SP}$. It suffices to consider such names, since for every Polish space X, there is a continuous surjection $F\colon \omega^\omega \to X$. Then we have that for each $i \in \omega$, $D_i := \{S \in \mathbb{SP} : \exists s \in \omega^i(S \Vdash \dot{y} \restriction i = s)\}$ is dense open. Let $\langle T_i : i \in \omega\rangle$ be so that $T_0 \leq_n T$, $T_{i+1} \leq_{n+i} T_i$ and for every $x \in [T_i]$, there is $t \subseteq x$ so that $(T_i)_t \in D_i$. Then $S = \bigcap_{i \in \omega} T_i \leq_n T$. For every $x \in [S]$, define $f(x) = \bigcup\{s \in \omega^{<\omega} : \exists t \subseteq x(S_t \Vdash s \subseteq \dot{y})\}$. Then $f\colon [S] \to \omega^\omega$ is continuous and $S \Vdash \dot{y} = f(x_G)$. $\square$

Corollary 4.4.5. $\mathbb{SP}$ is proper and ω^ω-bounding.

4.4.2 Weighted tree forcing

Definition 4.4.6. Let T be a perfect tree. A *weight* on T is a map $\rho\colon T \times T \to [T]^{<\omega}$ so that $\rho(s,t) \subseteq T_s \setminus T_t$ for all $s, t \in T$. Whenever ρ_0, ρ_1 are weights on T we write $\rho_0 \subseteq \rho_1$ to say that for all $s, t \in T$, $\rho_0(s,t) \subseteq \rho_1(s,t)$.

Note that if $t \subseteq s$ then $\rho(s,t) = \emptyset$ must be the case.

Definition 4.4.7. Let T be a perfect tree, ρ a weight on T and S a tree. Then we write $S \leq_\rho T$ if $S \subseteq T$ and there is a dense set of $s_0 \in S$ with an injective sequence $(s_n)_{n \in \omega}$ in S_{s_0} such that $\forall n \in \omega(\rho(s_n, s_{n+1}) \subseteq S)$.

Remark 4.4.8. Whenever $\rho_0 \subseteq \rho_1$, we have that $S \leq_{\rho_1} T$ implies $S \leq_{\rho_0} T$.

Definition 4.4.9. Let $\mathbb{P}$ be a tree forcing. Then we say that $\mathbb{P}$ is *weighted* if for any $T \in \mathbb{P}$ there is a weight ρ on T so that for any tree S, if $S \leq_\rho T$ then $S \in \mathbb{P}$.

Lemma 4.4.10. $\mathbb{SP}$ *is weighted.*

Proof. Let $T \in \mathbb{SP}$. For any $s, t \in T$ let $\rho(s,t) \subseteq T_s \setminus T_t$ be finite so that for any $k \in \omega$ and $i \in 2$, if there is $r \in T_s$ so that $r(k) = i$ and there is no such $r \in T_t$, then there is such r in $\rho(s,t)$. This is possible since T_t is fat. Let us show that ρ works. Assume that $S \leq_\rho T$ and let $s \in S$ be arbitrary. Then there is $s_0 \supseteq s$ in S with a sequence

$(s_n)_{n\in\omega}$ as in the definition of $\leq_\rho$. Let $k \geq m(T_{s_0})$ and $i \in 2$ and suppose there is no $r \in S_{s_0}$ with $r(k) = i$. In particular this means that no such r is in $\rho(s_n, s_{n+1})$ for any $n \in \omega$, since $\rho(s_n, s_{n+1}) \subseteq S_{s_0}$. But then, using the definition of ρ and $m(T_{s_0})$, we see inductively that for each $n \in \omega$ such r must be found in T_{s_n}. Letting n large enough so that $k < |s_n|$, $s_n(k) = i$ must be the case. But $s_n \in S_{s_0}$, which is a contradiction. $\square$

Definition 4.4.11. *Sacks forcing* is the tree forcing $\mathbb{S}$ consisting of all perfect subtrees of $2^{<\omega}$. It is well-known that it is Axiom A with continuous reading of names.

Lemma 4.4.12. $\mathbb{S}$ *is weighted.*

Proof. Let $T \in \mathbb{S}$. For $s, t \in T$, we let $\rho(s,t)$ contain all $r^\frown i \in T_s \setminus T_t$ such that $r^\frown(1-i) \in T$ and where $|r|$ is minimal with this property. $\square$

Recall that for finite trees T_0, T_1 we say that T_1 is an end-extension of T_0, written as $T_0 \sqsubset T_1$, if $T_0 \subsetneq T_1$ and for every $t \in T_1 \setminus T_0$ there is a terminal node $\sigma \in \mathrm{term}(T_0)$ so that $\sigma \subseteq t$. A node $\sigma \in T_0$ is called terminal if it has no proper extension in T_0.

Definition 4.4.13. Let T be a perfect tree, ρ a weight on T and T_0, T_1 finite subtrees of T. Then we write $T_0 \lhd_\rho T_1$ iff $T_0 \sqsubset T_1$ and

$$\forall \sigma \in \mathrm{term}(T_0) \exists N \geq 2 \exists \langle s_i \rangle_{i<N} \in ((T_1)_\sigma)^N \text{ injective}$$
$$\big(s_0 = \sigma \wedge s_{N-1} \in \mathrm{term}(T_1) \wedge \forall i < N(\rho(s_i, s_{i+1}) \subseteq T_1)\big). \quad (*_0)$$

Lemma 4.4.14. *Let T be a perfect tree, ρ a weight on T and $\langle T_n : n \in \omega\rangle$ be a sequence of finite subtrees of T so that $T_n \lhd_\rho T_{n+1}$ for every $n \in \omega$. Then $\bigcup_{n\in\omega} T_n \leq_\rho T$.*

Proof. Let $S := \bigcup_{n\in\omega} T_n$. To see that $S \leq_\rho T$ note that $\bigcup_{n\in\omega} \mathrm{term}(T_n)$ is dense in S, in a very strong sense. Let $\sigma \in \mathrm{term}(T_n)$ for some $n \in \omega$, then let $s_0, \dots, s_{N_0-1}$ be as in $(*_0)$ for T_n, T_{n+1}. Since $s_{N_0-1} \in \mathrm{term}(T_{n+1})$ we again find $s_{N_0-1}, \dots, s_{N_1-1}$ as in $(*_0)$ for T_{n+1}, T_{n+2}. Continuing like this, we find a sequence $\langle s_i : i \in \omega\rangle$ in S starting with $s_0 = \sigma$ so that $\rho(s_i, s_{i+1}) \subseteq S$ for all $i \in \omega$, as required. $\square$

Lemma 4.4.15. *Let T be a perfect tree, ρ a weight on T and T_0 a finite subtree of T. Moreover, let $k \in \omega$ and $D \subseteq (T)^k$ be dense open. Then there is $T_1 \rhd_\rho T_0$ so that*

$$\forall \{\sigma_0, \dots, \sigma_{k-1}\} \in [\mathrm{term}(T_0)]^k \forall \sigma'_0, \dots, \sigma'_{k-1} \in \mathrm{term}(T_1)$$
$$\big(\forall l < k(\sigma_l \subseteq \sigma'_l) \to (\sigma'_0, \dots, \sigma'_{k-1}) \in D\big). \quad (*_1)$$

Proof. First let us enumerate $\mathrm{term}(T_0)$ by $\sigma_0, \ldots, \sigma_{K-1}$. We put $s_0^l = \sigma_l$ for each $l < K$. Next find for each $l < K$, $s_1^l \in T, s_0^l \subsetneq s_1^l$ above a splitting node in $T_{s_0^l}$. Moreover we find $s_2^l \in T_{s_0^l}$ so that $s_2^l \perp s_1^l$ and s_2^l is longer than any node appearing in $\rho(s_0^l, s_1^l)$. This is possible since we chose s_1^l to be above a splitting node in $T_{s_0^l}$. For each $l < K$ we let $\tilde{T}_2^l$ be the tree generated by (i.e. the downwards closure of) $\{s_1^l, s_2^l\} \cup \rho(s_0^l, s_1^l) \cup \rho(s_1^l, s_2^l)$. Note that $s_2^l \in \mathrm{term}(\tilde{T}_2^l)$ as $\rho(s_1^l, s_2^l) \perp s_2^l$.

Let us enumerate by $(f_j)_{2 \leq j < N}$ all functions $f \colon K \to \{1, 2\}$ starting with f_2 the constant function mapping to 1. We are going to construct recursively a sequence $\langle \tilde{T}_j^l : 2 \leq j \leq N \rangle$ where $\tilde{T}_j^l \sqsubseteq \tilde{T}_{j+1}^l$, and $\langle s_j^l : 2 \leq j \leq N \rangle$ without repetitions, for each $l < K$ such that at any step $j < N$:

1. for every $l < K$, $s_j^l \in \mathrm{term}(\tilde{T}_j^l)$ and $\begin{cases} s_2^l \subseteq s_j^l & \text{if } f_j(l) = 1 \\ s_1^l \subseteq s_j^l & \text{if } f_j(l) = 2. \end{cases}$

2. for any $\{l_i : i < k\} \in [K]^k$ and $(t_i)_{i<k}$ where $t_i \in \mathrm{term}(\tilde{T}_{j+1}^{l_i})$ and $\begin{cases} s_1^{l_i} \subseteq t_i & \text{if } f_j(l_i) = 1 \\ s_1^{l_i} \perp t_i & \text{if } f_j(l_i) = 2 \end{cases}$ for every $i < k$, $(t_0, \ldots, t_{k-1}) \in D$

3. for every $l < K$, $\rho(s_j^l, s_{j+1}^l) \subseteq \tilde{T}_{j+1}^l$.

Note that (1) holds true at the initial step $j = 2$ since $f_2(l) = 1$, $s_2^l \subseteq s_2^l$ and $s_2^l \in \mathrm{term}(\tilde{T}_2^l)$ for each $l < K$. Given $\tilde{T}_j^l$ and s_j^l for each l with (1) holding true we proceed as follows. Let $\{t_i^l : i < N_l\}$ enumerate $\{t : t \in \mathrm{term}(\tilde{T}_j^l) \wedge s_1^l \subseteq t \text{ if } f_j(l) = 1 \wedge s_1^l \perp t \text{ if } f_j(l) = 2\}$ for each $l < K$. Now it is simple to find $r_i^l \in T$, $t_i^l \subseteq r_i^l$ for each $i < N_l, l < K$ so that $[\{r_i^l : i < N_l, l < K\}]^k \subseteq D$.

Let R_l be the tree generated by $\tilde{T}_j^l$ and $\{r_i^l : i < N_l\}$ for each $l < K$. It is easy to see that $\tilde{T}_j^l \sqsubseteq R_l$ since we only extended elements from $\mathrm{term}(\tilde{T}_j^l)$ (namely the t_i^l's). Note that it is still the case that $s_j^l \in \mathrm{term}(R_l)$ since $s_j^l \perp t_i^l$ for all $i < N_l$. Next we choose s_{j+1}^l extending an element of $\mathrm{term}(R_l)$, distinct from all previous choices and so that $s_2^l \subseteq s_j^l$ if $f_{j+1}(l) = 1$ and $s_1^l \subseteq s_j^l$ if $f_{j+1}(l) = 2$.

Taking $\tilde{T}_{j+1}^l$ to be the tree generated by $R_l \cup \{s_{j+1}^l\} \cup \rho(s_j^l, s_{j+1}^l)$ gives the next step of the construction. Again $R_l \sqsubseteq \tilde{T}_{j+1}^l$, as we only extended terminal nodes of R_l. Then (3) obviously holds true and $s_{j+1}^l \in \mathrm{term}(\tilde{T}_{j+1}^l)$ since $\rho(s_j^l, s_{j+1}^l) \perp s_{j+1}^l$. It follows from the construction that (2) holds true for each $\tilde{T}_{j+1}^l$ replaced by R_l. Since $R_l \sqsubseteq \tilde{T}_{j+1}^l$ we easily see that (2) is satisfied.

Finally we put $T_1 = \bigcup_{l<K} \tilde{T}_N^l$. It is clear that $(*_0)$ is true, in particular that $T_0 \lhd_\rho T_1$. For $(*_1)$ let $\{l_i : i < k\} \in [K]^k$ be arbitrary and assume that $t_i \in \mathrm{term}(\tilde{T}_N^{l_i})$ for each

$i < k$. Let $f \colon K \to \{1, 2\}$ be so that for each $i < k$ if $s_1^{l_i} \subseteq t_i$ then $f(l_i) = 1$, and if $s_1^{l_i} \perp t_i$ then $f(l_i) = 2$. Then there is $j \in [2, N)$ so that $f_j = f$. Clause (2) ensured that for initial segments $t'_i \subseteq t_i$ where $t'_i \in \operatorname{term}(\tilde{T}^{l_i}_{j+1})$, $(t'_0, \dots, t'_{k-1}) \in D$. In particular $(t_0, \dots, t_{k-1}) \in D$ which proves $(*_1)$. $\square$

Proposition 4.4.16. *Let M be a countable model of set theory, $R_l \in M$ a perfect tree and ρ_l a weight on R_l for every $l < k \in \omega$. Then there is $S_l \leq_{\rho_l} R_l$ for every $l < k$ so that any $\bar{x}_0, \dots, \bar{x}_{n-1} \in \prod_{l<k}[S_l]$ are mutually Cohen generic with respect to $\prod_{l<k}[R_l]$ over M.*

Proof. Let $T := \{\emptyset\} \cup \{\langle l \rangle^\frown s : s \in R_l, l < k\}$ be the disjoint sum of the trees R_l for $l < k$. Also let ρ be a weight on T extending arbitrarily the weights ρ_l defined on the copy of R_l in T. As M is countable, let $(D_n, k_n)_{n \in \omega}$ enumerate all pairs $(D, m) \in M$, such that D is a dense open subset of T^m and $m \in \omega \setminus \{0\}$, infinitely often. Let us find a sequence $(T_n)_{n \in \omega}$ of finite subtrees of T, such that for each $n \in \omega$, $T_n \lhd_\rho T_{n+1}$ and

$$\forall \{\sigma_0, \dots, \sigma_{k_n - 1}\} \in [\operatorname{term}(T_n)]^{k_n} \forall \sigma'_0, \dots, \sigma'_{k_n-1} \in \operatorname{term}(T_{n+1}) \\ [\forall l < k(\sigma_l \subseteq \sigma'_l) \to (\sigma'_0, \dots, \sigma'_{k_n-1}) \in D_n]. \quad (*_1)$$

We start with $T_0 = k^{<2} = \{\emptyset\} \cup \{\langle l \rangle : l < k\}$ and then apply Lemma 4.4.15 recursively. Let $S := \bigcup_{n \in \omega} T_n$. Then we have that $S \leq_\rho T$.

Claim 4.4.17. *For any $m \in \omega$ and distinct $x_0, \dots, x_{m-1} \in [S]$, $(x_0, \dots, x_{m-1})$ is T^m-generic over M.*

Proof. Let $D \subseteq T^m$ be open dense with $D \in M$. Then there is a large enough $n \in \omega$ with $(D_n, k_n) = (D, m)$ and $\sigma_0, \dots, \sigma_{m-1} \in \operatorname{term}(T_n)$ distinct such that $\sigma_0 \subseteq x_0, \dots, \sigma_{m-1} \subseteq x_{m-1}$. Then there are unique $\sigma'_0, \dots, \sigma'_{m-1} \in \operatorname{term}(T_{n+1})$ such that $\sigma'_0 \subseteq x_0, \dots, \sigma'_{m-1} \subseteq x_{m-1}$. By $(*_1)$, $(\sigma'_0, \dots, \sigma'_{m-1}) \in D$. $\square$

Finally let $S_l = \{s : \langle l \rangle^\frown s \in S\}$ and note that $S_l \leq_{\rho_l} R_l$ for every $l < k$. The above claim clearly implies the statement of the proposition. $\square$

Remark 4.4.18. Proposition 4.4.16 implies directly the main result of [62]. A modification of the above construction for splitting forcing can be used to show that for $T \in M$, we can in fact find a master condition $S \leq T$ so that for any distinct $x_0, \dots, x_{n-1} \in [S]$, $(x_0, \dots, x_{n-1})$ is $\mathbb{SP}^n$-generic over M. In that case $(S, \dots, S) \in \mathbb{SP}^n$ is a $\mathbb{SP}^n$-master condition over M. We won't provide a proof of this since our only application is

Corollary 4.4.21 below, which seems to be implicit in [62]. The analogous statement for Sacks forcing is a standard fusion argument.

Corollary 4.4.19. Let $\mathbb{P}$ be a weighted tree forcing and let G be $\mathbb{P}$-generic over V. Then $G = \{S \in \mathbb{P} \cap V : x_G \in [S]\}$. Thus we may write $V[x_G]$ instead of $V[G]$.

Proof. Obviously $G \subseteq H := \{S \in \mathbb{P} \cap V : x_G \in [S]\}$. Suppose that $S \in H \setminus G$ and $T \in G$ is such that $T \Vdash S \in \dot{H} \setminus \dot{G}$. Let $M \preccurlyeq H(\theta)$ be so that $T, S, \mathbb{P} \in M$, for θ large enough. By Proposition 4.4.16, there is $T' \leq T$ so that any $x \in [T']$ is Cohen generic in $[T]$ over M. If there is some $x \in [T'] \cap [S]$, then there is $t \subseteq x$ so that $M \models t \Vdash_T \dot{c} \in [S]$, where $\dot{c}$ is a name for the generic branch added by T. But then $(T')_t \subseteq S$ contradicting that $(T')_t \Vdash S \notin G$. Thus $[T'] \cap [S] = \emptyset$, implying that $T' \Vdash S \notin H$. Again this is a contradiction. $\square$

Corollary 4.4.20. Let $\mathbb{P}$ be a weighted tree forcing with continuous reading of names. Then $\mathbb{P}$ adds a minimal real in the sense that, for any $\mathbb{P}$-generic G, if $y \in 2^\omega \cap V[G] \setminus V$, then there is a Borel map $f \colon 2^\omega \to A^\omega$ in V so that $x_G = f(y)$.

Proof. Using the continuous reading of names let $T \in G$ be so that there is a continuous map $g \colon [T] \to 2^\omega$ with $T \Vdash \dot{y} = g(x_G)$. Moreover let $M \preccurlyeq H(\theta)$ be countable for large enough θ with $g, T \in M$. Now let $S \leq T$ be so that any $x_0, x_1 \in [S]$ are mCg in $[T]$ over M.

Suppose that there are $x_0 \neq x_1 \in [S]$, with $g(x_0) = g(x_1)$. Then there must be $s \subseteq x_0$ and $t \subseteq x_1$, so that $M \models (s,t) \Vdash_{T^2} g(\dot{c}_0) = g(\dot{c}_1)$, where $\dot{c}_0$, $\dot{c}_1$ are names for the generic branches added by T^2. But then note that for any $x \in S_t$, since x and x_0 are mCg and $s \subseteq x_0$, $t \subseteq x$, we have that $g(x) = g(x_0)$. In particular g is constant on S_t and $S_t \Vdash g(x_G) = g(\check{x}_0) \in V$.

On the other hand, if g is injective on $[S]$, it is easy to extend g^{-1} to a Borel function $f \colon A^\omega \to 2^\omega$. $\square$

Corollary 4.4.21. $V^{\mathbb{SP}}$ is a minimal extension of V, i.e. whenever W is a model of ZFC so that $V \subseteq W \subseteq V^{\mathbb{SP}}$, then $W = V$ or $W = V^{\mathbb{SP}}$.

Proof. Let G be an $\mathbb{SP}$-generic filter over V. By Corollary 4.4.19, it suffices to show that if $\langle \alpha_\xi : \xi < \delta \rangle \in W \setminus V$ is an increasing sequence of ordinals, then $x_G \in W$ (see also [31, Theorem 13.28]). So let $\langle \dot{\alpha}_\xi : \xi < \delta \rangle$ be a name for such a sequence of ordinals and $T \in \mathbb{SP}$ be such that $T \Vdash \langle \dot{\alpha}_\xi : \xi < \delta \rangle \notin V$. Note that this is in fact equivalent to saying that $(T,T) \Vdash_{\mathbb{SP}^2} \langle \dot{\alpha}_\xi[\dot{x}_0] : \xi < \delta \rangle \neq \langle \dot{\alpha}_\xi[\dot{x}_1] : \xi < \delta \rangle$, where $\dot{x}_0, \dot{x}_1$ are names for the generic reals added by $\mathbb{SP}^2$. Let M be a countable elementary model so that $T, \langle \dot\alpha_\xi : \xi < \delta \rangle \in M$ and let $T' \leq T$ be a master condition over M as in Remark 4.4.18. Then also $T' \Vdash \langle \dot\alpha_\xi : \xi \in \delta \cap M \rangle \notin V$. Namely, suppose towards a contradiction that there are $x_0, x_1 \in [T']$ generic over V so that $\langle \dot\alpha_\xi[x_0] : \xi \in \delta \cap M \rangle = \langle \dot\alpha_\xi[x_1] : \xi \in \delta \cap M \rangle$, then (x_0, x_1) is $\mathbb{SP}^2$-generic over M and $M[x_0][x_1] \models \langle \dot\alpha_\xi[x_0] : \xi < \delta \rangle = \langle \dot\alpha_\xi[x_1] : \xi < \delta \rangle$ which yields a contradiction to the sufficient elementarity of M. Since $T' \Vdash \langle \dot\alpha_\xi : \xi \in \delta \cap M \rangle \subseteq M$ we can view $\langle \dot\alpha_\xi : \xi \in \delta \cap M \rangle$ as a name for a real, for M is countable. Back in W, we can define $\langle \alpha_\xi : \xi \in \delta \cap M \rangle$ since $M \in V \subseteq W$. But then, applying Corollary 4.4.20, we find that $x_G \in W$. $\square$

4.4.3 The countable support iteration

Recall that for any perfect subtree T of $2^{<\omega}$, $\operatorname{split}(T)$ is order-isomorphic to $2^{<\omega}$ in a canonical way, via a map $\eta_T \colon \operatorname{split}(T) \to 2^{<\omega}$. This map induces a homeomorphism $\tilde\eta_T \colon [T] \to 2^\omega$ and note that the value of $\tilde\eta_T(x)$ depends continuously on T and x. Whenever ρ is a weight on T, η_T also induces a weight $\tilde\rho$ on $2^{<\omega}$, so that whenever $S \leq_{\tilde\rho} 2^{<\omega}$, then $\eta_T^{-1}(S)$ generates a tree S' with $S' \leq_\rho T$.

Let $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \lambda \rangle$ be a countable support iteration where for each $\beta < \lambda$, $\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{Q}}_\beta = \mathbb{P}$, for some $\mathbb{P} \in \{\mathbb{SP}, \mathbb{S}\}$. We fix in this section a $\mathbb{P}_\lambda$ name $\dot y$ for an element of a Polish space X, $\bar p \in \mathbb{P}_\lambda$ a good master condition over a countable model M_0, where $\dot y, X \in M_0$, and let $C \subseteq \lambda$ be a countable set as in Lemma 4.2.3. For every $\beta \in C$ and $\bar y \in [\bar p] \restriction (C \cap \beta)$, let us write

$$T_{\bar y} = \{ s \in 2^{<\omega} : \exists \bar x \in [\bar p]\, [\bar x \restriction (C \cap \beta) = \bar y \wedge s \subseteq x(\beta)]\}.$$

According to Lemma 4.2.3, the map $\bar y \mapsto T_{\bar y}$ is a continuous function from $[\bar p] \restriction (C \cap \beta)$ to $\mathcal{T}$. Let $\alpha := \operatorname{otp}(C) < \omega_1$ as witnessed by an order-isomorphism $\iota \colon \alpha \to C$. Then we define the homeomorphism $\Phi \colon [\bar p] \restriction C \to (2^\omega)^\alpha$ so that for every $\bar y \in [\bar p] \restriction C$ and every $\delta < \alpha$,

$$\Phi(\bar y) \restriction (\delta + 1) = \Phi(\bar y) \restriction \delta^\frown \tilde\eta_{T_{\bar y \restriction \iota(\delta)}}(y(\iota(\delta))).$$

Note that for $\mathbb{P} \in \{\mathbb{SP}, \mathbb{S}\}$, the map sending $T \in \mathbb{P}$ to the weight ρ_T defined in Lemma 4.4.10 or Lemma 4.4.12 is a Borel function from $\mathbb{P}$ to the Polish space of partial functions from $(2^{<\omega})^2$ to $[2^{<\omega}]^{<\omega}$. Thus for $\beta \in C$ and $\bar x \in [\bar p] \restriction (C \cap \beta)$, letting $\rho_{\bar x} := \rho_{T_{\bar x}}$, we get that $\bar x \mapsto \rho_{\bar x}$ is a Borel function on $[\bar p] \restriction (C \cap \beta)$. For each $\delta < \alpha$ and $\bar y \in (2^\omega)^\delta$, we may then define $\tilde\rho_{\bar y}$ a weight on $2^{<\omega}$, induced by $\rho_{\bar x}$ and $\eta_{T_{\bar x}}$, where $\bar x = \Phi^{-1}(\bar y^\frown \bar z) \restriction \beta$ for arbitrary, equivalently for every, $\bar z \in (2^\omega)^{\alpha \setminus \delta}$. The map sending $\bar y \in (2^\omega)^\delta$ to $\tilde\rho_{\bar y}$ is then Borel as well.

Lemma 4.4.22. *Let M_1 be a countable elementary model with $M_0, \bar{p}, \mathbb{P}_\lambda \in M_1$ and let $\bar{s} \in \bigotimes_{i<\alpha} 2^{<\omega}$. Then there is $\bar{q} \leq \bar{p}$, a good master condition over M_0, so that*

$$\forall \bar{x}_0, \dots, \bar{x}_{n-1} \in [\bar{q}]\big(\Phi(\bar{x}_0 \restriction C), \dots, \Phi(\bar{x}_{n-1} \restriction C) \in (2^\omega)^\alpha \cap [\bar{s}] \text{ are strongly mCg wrt } \prod_{i<\alpha} 2^\omega \text{ over } M_1\big).$$

Moreover $[\bar{q}] \restriction C$ is a closed subset of $[\bar{p}] \restriction C$ and $[\bar{q}] = ([\bar{q}] \restriction C) \times (2^\omega)^{\lambda \setminus C}$ (cf. Lemma 4.2.3).

Proof. We can assume without loss of generality that $\bar{s} = \emptyset$, i.e. $[\bar{s}] = (2^\omega)^\alpha$. It will be obvious that this assumption is inessential. Next, let us introduce some notation. For any $\delta \leq \alpha$ and $\bar{y}_0, \dots, \bar{y}_{n-1} \in (2^\omega)^\delta$, recall that we defined

$$\Delta(\bar{y}_0, \dots, \bar{y}_{n-1}) := \{\Delta_{\bar{y}_i, \bar{y}_j} : i \neq j < n\} \cup \{0, \delta\}.$$

Let us write

$$\operatorname{tp}(\bar{y}_0, \dots, \bar{y}_{n-1}) := (\langle \xi_l : l \leq k\rangle, \langle K_l : l < k\rangle, \langle U_i : i < n\rangle),$$

where $\{\xi_0 < \dots < \xi_k\} = \Delta(\bar{y}_0, \dots, \bar{y}_{n-1})$, $K_l = |\{\bar{y}_i \restriction [\xi_l, \xi_{l+1}) : i < n\}|$ for every $l < k$ and $\langle U_i : i < n\rangle$ are the clopen subsets of $(2^\omega)^\delta$ of the form $U_i = [\bar{s}_i]$ for $\bar{s}_i \in \bigotimes_{\xi<\delta} 2^{<\omega}$ with $\bar{s}_i$ minimal in the order of $\bigotimes_{\xi<\delta} 2^{<\omega}$ so that

$$\bar{y}_i \in [\bar{s}_i] \text{ and } \forall j < n(\bar{y}_j \neq \bar{y}_i \to \bar{y}_j \notin [\bar{s}_i]),$$

for every $i < n$.

Note that for any $\delta_0 \leq \delta$, if

$$\operatorname{tp}(\bar{y}_0 \restriction \delta_0, \dots, \bar{y}_{n-1} \restriction \delta_0) := (\langle \eta_l : l \leq k'\rangle, \langle M_l : l < k'\rangle, \langle V_i : i < n\rangle),$$

then $V_i = U_i \restriction \delta_0$ for every $i < n$. Moreover, for any $\bar{y}'_0, \dots, \bar{y}'_{n-1} \in (2^\omega)^\delta$ with

$$\operatorname{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \xi_l : l \leq k\rangle, \langle K_l : l < k\rangle, \langle U_i : i < n\rangle),$$

we have that

$$\operatorname{tp}(\bar{y}'_0 \restriction \delta_0, \dots, \bar{y}'_{n-1} \restriction \delta_0) = (\langle \eta_l : l \leq k'\rangle, \langle M_l : l < k'\rangle, \langle V_i : i < n\rangle).$$

Any $\bar{y}_0, \dots, \bar{y}_{n-1}$, with $\operatorname{tp}(\bar{y}_0, \dots, \bar{y}_{n-1}) := (\langle \xi_l : l \leq k\rangle, \langle K_l : l < k\rangle, \langle U_i : i < n\rangle)$, that are mutually Cohen generic with respect to $\prod_{i<\delta} 2^\omega$ over M_1 as witnessed by $\xi_0 < \dots < \xi_k$, induce a $\prod_{l<k}(\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l}$-generic and vice-versa. Thus whenever τ is a $\prod_{l<k}(\bigotimes_{\xi\in[\xi_l,\xi_{l+1})} 2^{<\omega})^{K_l}$-name, we may write $\tau[\bar{y}_0,\dots,\bar{y}_{n-1}]$ for the evaluation of τ via the induced generic. It will not matter in what particular way we define the $\prod_{l<k}(\bigotimes_{\xi\in[\xi_l,\xi_{l+1})} 2^{<\omega})^{K_l}$-generic from given $\bar{y}_0,\dots,\bar{y}_{n-1}$. We may stipulate for instance, that the generic induced by $\bar{y}_0,\dots,\bar{y}_{n-1}$ is $\langle \bar{z}_{l,j} : l<k, j<K_l\rangle$, where for each fixed $l<k$, $\langle \bar{z}_{l,j} : l<k, j<K_l\rangle$ enumerates $\{\bar{y}_i \restriction [\xi_l,\xi_{l+1}) : i<n\}$ in lexicographic order.

Let us get to the bulk of the proof. We will define a finite support iteration $\langle \mathbb{R}_\delta, \dot{\mathbb{S}}_\delta : \delta\leq\alpha\rangle$ in M_1, together with, for each $\delta\leq\alpha$, an $\mathbb{R}_\delta$-name $\dot{X}_\delta$ for a closed subspace of $(2^\omega)^\delta$, where $\Vdash_{\mathbb{R}_{\delta_1}} \dot{X}_{\delta_0} = \dot{X}_{\delta_1} \restriction \delta_0$ for every $\delta_0<\delta_1\leq\alpha$. This uniquely determines the limit steps of the construction. Additionally we will make the following inductive assumptions $(1)_\delta$ and $(2)_\delta$ for all $\delta\leq\alpha$ and any $\mathbb{R}_\delta$-generic G. Let $\bar{y}_0,\dots,\bar{y}_{n-1}\in\dot{X}_\delta[G]$ be arbitrary and $\operatorname{tp}(\bar{y}_0,\dots,\bar{y}_{n-1}) = (\langle\xi_l : l\leq k\rangle, \langle K_l : l<k\rangle, \langle U_i : i<n\rangle)$. Then

$(1)_\delta$ $\bar{y}_0,\dots,\bar{y}_{n-1}$ are strongly mCg over M_1 with respect to $\prod_{i<\delta} 2^\omega$,

$(2)_\delta$ and for any $\prod_{l<k}(\bigotimes_{\xi\in[\xi_l,\xi_{l+1})} 2^{<\omega})^{K_l}$-name $\dot{D}\in M_1$ for an open dense subset of a countable poset $\mathbb{Q}\in M_1$,

$$\bigcap\Big\{\dot{D}[\bar{y}'_0,\dots,\bar{y}'_{n-1}] : \bar{y}'_0,\dots,\bar{y}'_{n-1}\in X_\delta,$$
$$\operatorname{tp}(\bar{y}'_0,\dots,\bar{y}'_{n-1}) = (\langle\xi_l : l\leq k\rangle, \langle K_l : l<k\rangle, \langle U_i : i<n\rangle)\Big\}$$

is open dense in $\mathbb{Q}$.

Having defined $\mathbb{R}_\delta$ and $\dot{X}_\delta$, for $\delta<\alpha$, we proceed as follows. Fix for now G an $\mathbb{R}_\delta$-generic over M_1 and $X_\delta := \dot{X}_\delta[G]$. Then we define a forcing $\mathbb{S}_\delta\in M_1[G]$ which generically adds a continuous map $F\colon X_\delta\to\mathcal{T}$, so that for each $\bar{y}\in X_\delta$, $S_{\bar{y}} := F(\bar{y}) \leq_{\tilde{\rho}_{\bar{y}}} 2^{<\omega}$. In $M_1[G][F]$, we then define $X_{\delta+1}\subseteq(2^\omega)^{\delta+1}$ to be $\{\bar{y}^\frown z : \bar{y}\in X_\delta, z\in[S_{\bar{y}}]\}$. The definition of $\mathbb{S}_\delta$ is as follows.

Work in $M_1[G]$. Since the map $\bar{y}\in(2^\omega)^\delta\mapsto\tilde{\rho}_{\bar{y}}$ is Borel and an element of M_1 and by $(1)_\delta$ any $\bar{y}\in X_\delta$ is Cohen generic over M_1, it is continuous on X_δ. Since X_δ is compact we find a single weight $\tilde{\rho}$ on $2^{<\omega}$, so that $\tilde{\rho}_{\bar{y}}\subseteq\tilde{\rho}$ for every $\bar{y}\in X_\delta$. Let $\{O_s : s\in 2^{<\omega}\}$ be a basis of X_δ so that $O_s\subseteq O_t$ for $t\subseteq s$ and $O_s\cap O_t=\emptyset$ for $s\perp t$. This is possible since X_δ is homeomorphic to 2^ω. Let $\mathcal{FT}$ be the set of finite subtrees of $2^{<\omega}$. Then $\mathbb{S}_\delta$ consists of functions $h\colon 2^{\leq n}\to\mathcal{FT}$, for some $n\in\omega$, so that for every $s\subseteq t\in 2^{\leq n}$, $(h(s)\trianglelefteq_{\tilde{\rho}} h(t))$. The extension relation is defined by function extension. Note that $\mathbb{S}_\delta$ is indeed a forcing poset with trivial condition $\emptyset$.

Given H, an $\mathbb{S}_\delta$-generic over $M_1[G]$, we let $F\colon X_\delta \to \mathcal{T}$ be defined as

$$F(\bar{y}) := \bigcup_{\substack{s \in 2^{<\omega}, \bar{y} \in O_s \\ h \in H}} h(s).$$

Claim 4.4.23. *For every $\bar{y} \in X_\delta$, $F(\bar{y}) = S_{\bar{y}} \leq_{\tilde{\rho}} 2^{<\omega}$, in particular $S_{\bar{y}} \leq_{\tilde{\rho}_{\bar{y}}} 2^{<\omega}$. For any $\bar{y}_0, \bar{y}_1 \in X_\delta$, $[S_{\bar{y}_0}] \cap [S_{\bar{y}_1}] \neq \emptyset$. Any $z_0, \dots, z_{n-1} \in \bigcup_{\bar{y} \in X_\delta}[S_{\bar{y}}]$ are mutually Cohen generic in 2^ω over $M_1[G]$. And for any countable poset $\mathbb{Q} \in M_1$, any $m \in \omega$ and any dense open $E \subseteq (2^{<\omega})^n \times \mathbb{Q}$ in $M_1[G]$, there is $r \in \mathbb{Q}$ and $m_0 \geq m$ so that for any $z_0, \dots, z_{n-1} \in \bigcup_{\bar{y} \in X_\delta}[S_{\bar{y}}]$ where $z_0 \restriction m, \dots, z_{n-1} \restriction m$ are pairwise distinct, $((z_0 \restriction m_0, \dots, z_{n-1} \restriction m_0), r) \in E$.*

Proof. We will make a genericity argument over $M_1[G]$. Let $h \in \mathbb{S}_\delta$ be arbitrary. Then it is easy to find $h' \leq h$, say with $\mathrm{dom}(h') = 2^{\leq a_0}$, so that for every $s \in 2^{a_0}$ and every $t \in \mathrm{term}(h(s))$, $|t| \geq m$. For the first claim, it suffices through Lemma 4.4.14 to find $h'' \leq h'$, say with $\mathrm{dom}(h'') = 2^{\leq a_1}$, $a_0 < a_1$, so that for every $s \in 2^{a_0}$ and $t \in 2^{a_1}$, with $s \subseteq t$, $h''(s) \lhd_{\tilde{\rho}} h''(t)$. Finding h'' so that additionally $\mathrm{term}(h''(t_0)) \cap \mathrm{term}(h''(t_1)) = \emptyset$ for every $t_0 \neq t_1 \in 2^{a_1}$ proves the second claim. For the last two claims, given a fixed dense open subset $E \subseteq (2^{<\omega})^n \times \mathbb{Q}$ in $M_1[G]$, it suffices to find $r \in \mathbb{Q}$ and to ensure that for any pairwise distinct $s_0, \dots, s_{n-1} \in \bigcup_{s \in 2^{a_0}} \mathrm{term}(h''(s))$ and $t_0 \supseteq s_0, \dots, t_{n-1} \supseteq s_{n-1}$ with $t_0, \dots, t_{n-1} \in \bigcup_{t \in 2^{a_1}} \mathrm{term}(h''(t))$, $((t_0, \dots, t_{n-1}), r) \in E$. Then we may put $m_0 = \max\{|t| : t \in \bigcup_{s \in 2^{a_1}} \mathrm{term}(h''(s))\}$. We may also assume wlog that $\mathbb{Q} = 2^{<\omega}$.

To find such h'' we apply Lemma 4.4.15 as in the proof of Proposition 4.4.16. More precisely, for every $s \in 2^{a_0}$, we find $T^0_s, T^1_s \rhd_{\tilde{\rho}} h'(s)$, and we find $T \subseteq 2^{<\omega}$ finite, so that for any pairwise distinct $s_0, \dots, s_{n-1} \in \bigcup_{s \in 2^{a_0}} \mathrm{term}(h'(s))$, any $t_0 \supseteq s_0, \dots, t_{n-1} \supseteq s_{n-1}$ with $t_0, \dots, t_{n-1} \in \bigcup_{s \in 2^{a_0}, i \in 2} \mathrm{term}(T^i_s)$ and any $\sigma \in \mathrm{term}(T)$, $((t_0, \dots, t_{n-1}), \sigma) \in E$ and $\mathrm{term}(T^i_s) \cap \mathrm{term}(T^j_t) = \emptyset$ for every $i, j \in 2$, $s, t \in 2^{a_0}$. Then simply define $h'' \leq h'$ with $\mathrm{dom}(h'') = 2^{a_0+1}$, where $h''(s^\frown i) = T^i_s$ for $s \in 2^{a_0}$, $i \in 2$. $\square$

The function F is obviously continuous and $X_{\delta+1}$ is a closed subset of $(2^\omega)^{\delta+1}$, with $X_{\delta+1} \restriction \delta_0 = (X_{\delta+1} \restriction \delta) \restriction \delta_0 = X_\delta \restriction \delta_0 = X_{\delta_0}$ for every $\delta_0 < \delta + 1$.

Proof of $(1)_{\delta+1}$, $(2)_{\delta+1}$. Let G be $\mathbb{R}_{\delta+1}$ generic over M_1 and $\bar{y}_0, \dots, \bar{y}_{n-1} \in \dot{X}_{\delta+1}[G] = X_{\delta+1}$ be arbitrary. By the inductive assumption we have that $\bar{y}_0 \restriction \delta, \dots, \bar{y}_{n-1} \restriction \delta$ are strongly mCg over M_1 with respect to $\prod_{i<\delta} 2^\omega$. By the above claim, whenever $\bar{y}_i \restriction \delta \neq \bar{y}_j \restriction \delta$, then $\bar{y}_i(\delta) \neq \bar{y}_j(\delta)$. Thus, for $(1)_{\delta+1}$, we only need to show that

$\bar{y}_0, \ldots, \bar{y}_{n-1}$ are mCg. Let $\operatorname{tp}(\bar{y}_0, \ldots, \bar{y}_{n-1}) = (\langle \xi_l : l < k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle)$, $\operatorname{tp}(\bar{y}_0 \restriction \delta, \ldots, \bar{y}_{n-1} \restriction \delta) = (\langle \eta_l : l \leq k' \rangle, \langle M_l : l < k \rangle, \langle U_i \restriction \delta : i < n \rangle)$ and $n' = |\{y_i(\delta) : i < n\}| = K_{k-1}$. Then we may view a dense open subset of $\prod_{l<k}(\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l}$ as a $\prod_{l<k'}(\bigotimes_{\xi \in [\eta_l, \eta_{l+1})} 2^{<\omega})^{M_l}$-name for a dense open subset of $(2^{<\omega})^{n'}$. To this end, let $\dot{D} \in M_1$ be a $\prod_{l<k'}(\bigotimes_{\xi \in [\eta_l, \eta_{l+1})} 2^{<\omega})^{M_l}$ name for a dense open subset of $(2^{<\omega})^{n'}$. Then we have, by $(2)_\delta$, that

$$\tilde{D} = \bigcap \Big\{ \dot{D}[\bar{y}'_0, \ldots, \bar{y}'_{n-1}] : \bar{y}'_0, \ldots, \bar{y}'_{n-1} \in X_\delta, \\ \operatorname{tp}(\bar{y}'_0, \ldots, \bar{y}'_{n-1}) = (\langle \eta_l : l \leq k' \rangle, \langle M_l : l < k' \rangle, \langle U_i \restriction \delta : i < n \rangle) \Big\}$$

is a dense open subset of $(2^{<\omega})^{n'}$ and $\tilde{D} \in M_1[G \restriction \delta]$. By the above claim, $y_0(\delta), \ldots, y_{n-1}(\delta)$ are mCg over $M_1[G \restriction \delta]$ in 2^ω. Altogether, this shows that $\bar{y}_0, \ldots, \bar{y}_{n-1}$ are mCg over M_1 with respect to $\prod_{i<\delta+1} 2^\omega$.

For $(2)_{\delta+1}$, let $\dot{D} \in M_1$ now be a $\prod_{l<k}(\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l}$-name for a dense open subset of $\mathbb{Q}$. Consider a name $\dot{E}$ in M_1 for the dense open subset of $(2^{<\omega})^{n'} \times \mathbb{Q}$, where for any $\bar{y}'_0, \ldots, \bar{y}'_{n-1} \in X_\delta$, with $\operatorname{tp}(\bar{y}'_0, \ldots, \bar{y}'_{n-1}) = (\langle \eta_l : l \leq k' \rangle, \langle M_l : l < k \rangle, \langle U_i \restriction \delta : i < n \rangle)$,

$$\dot{E}[\bar{y}'_0, \ldots, \bar{y}'_{n-1}] = \{ (\bar{t}, r) : M_1[\bar{y}'_0, \ldots, \bar{y}'_{n-1}] \models \\ \bar{t} \Vdash r \in \dot{D}[\bar{y}'_0, \ldots, \bar{y}'_{n-1}][\dot{z}_0, \ldots, \dot{z}_{n'-1}] \},$$

where $(\dot{z}_0, \ldots, \dot{z}_{n'-1})$ is a name for the $(2^{<\omega})^{n'}$-generic. By $(2)_\delta$, we have that

$$\tilde{E} = \bigcap \Big\{ \dot{E}[\bar{y}'_0, \ldots, \bar{y}'_{n-1}] : \bar{y}'_0, \ldots, \bar{y}'_{n-1} \in X_\delta, \\ \operatorname{tp}(\bar{y}'_0, \ldots, \bar{y}'_{n-1}) = (\langle \eta_l : l \leq k' \rangle, \langle M_l : l < k \rangle, \langle U_i \restriction \delta : i < n \rangle) \Big\}$$

is a dense open subset of $(2^{<\omega})^{n'} \times \mathbb{Q}$ and $\tilde{E} \in M_1[G \restriction \delta]$. Let $m \in \omega$ be large enough so that for any $i, j < n$, if $U_i \neq U_j$, then $\forall \bar{y}'_i \in U_i \cap X_{\delta+1}, \bar{y}'_j \in U_j \cap X_{\delta+1} (y'_i(\delta) \restriction m \neq y'_j(\delta) \restriction m)$. To see that such m exists, note that if $U_i \neq U_j$, then $U_i \cap X_{\delta+1}$ and $U_j \cap X_{\delta+1}$ are disjoint compact subsets of $X_{\delta+1}$. By the claim, there is $r \in \mathbb{Q}$ and $m_0 \geq m$ so that for any $z_0, \ldots, z_{n'-1} \in \bigcup_{\bar{y} \in X_\delta} [S_{\bar{y}}]$, if $z_0 \restriction m, \ldots, z_{n'-1} \restriction m$ are pairwise different, then $((z_0 \restriction m_0, \ldots, z_{n'-1} \restriction m_0), r) \in \tilde{E}$. Altogether we find that

$$r \in \bigcap \Big\{ \dot{D}[\bar{y}'_0, \ldots, \bar{y}'_{n-1}] : \bar{y}'_0, \ldots, \bar{y}'_{n-1} \in X_{\delta+1}, \\ \operatorname{tp}(\bar{y}'_0, \ldots, \bar{y}'_{n-1}) = (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle) \Big\}.$$

Of course the same argument can be carried out below any condition in $\mathbb{Q}$, showing that this set is dense. That it is open is also clear since it is the intersection of open subsets of a partial order. $\square$

Now let $\delta \leq \alpha$ be a limit ordinal.

Proof of $(1)_\delta$ *and* $(2)_\delta$. Let G be $\mathbb{R}_\delta$-generic over M_1, $\bar{y}_0, \dots, \bar{y}_{n-1} \in \dot{X}_\delta[G] = X_\delta$, this time wlog pairwise distinct, and $\mathrm{tp}(\bar{y}_0, \dots, \bar{y}_{n-1}) = (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle)$. We will make a genericity argument over M_1 to show $(1)_\delta$ and $(2)_\delta$. To this end, let $D_0 \subseteq \prod_{l<k}(\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l}$ be dense open, $D_0 \in M_1$, and let $\dot{D}_1 \in M_1$ be a $\prod_{l<k}(\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l}$-name for a dense open subset of $\mathbb{Q}$. Then consider the dense open subset $D_2 \subseteq \prod_{l<k}(\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l} \times \mathbb{Q}$ in M_1, where

$$D_2 = \{(r_0, r_1) : r_0 \in D_0 \wedge r_0 \Vdash r_1 \in \dot{D}_1\}.$$

Also let $\bar{h}_0 \in G$ be an arbitrary condition so that

$$\bar{h}_0 \Vdash \forall i < n (U_i \cap \dot{X}_\delta \neq \emptyset).$$

Then there is $\delta_0 < \delta$ so that $\mathrm{supp}(\bar{h}_0), \xi_{k-1} + 1 \subseteq \delta_0$. We may equally well view D_2 as a $\prod_{l<k-1}(\bigotimes_{\xi \in [\xi_l, \xi_{l+1})} 2^{<\omega})^{K_l} \times (\bigotimes_{\xi \in [\xi_{k-1}, \delta_0)} 2^{<\omega})^{K_{k-1}}$-name $\dot{E} \in M_1$ for a dense open subset

$$E \subseteq (\bigotimes_{\xi \in [\delta_0, \xi_k)} 2^{<\omega})^{K_{k-1}} \times \mathbb{Q} = (\bigotimes_{\xi \in [\delta_0, \delta)} 2^{<\omega})^n \times \mathbb{Q}.$$

We follow again from $(2)_{\delta_0}$, that the set $\tilde{E} \in M_1[G \cap \mathbb{R}_{\delta_0}]$, where

$$\begin{aligned} \tilde{E} = \bigcap \Big\{ & \dot{E}[\bar{y}'_0, \dots, \bar{y}'_{n-1}] : \bar{y}'_0, \dots, \bar{y}'_{n-1} \in X_{\delta_0}, \\ & \mathrm{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \xi_0 < \dots < \xi_{k-1} < \delta_0 \rangle, \langle K_l : l < k \rangle, \langle U_i \restriction \delta_0 : i < n \rangle) \Big\}, \end{aligned}$$

is dense open. Let $((\bar{t}_0, \dots, \bar{t}_{n-1}), r) \in \tilde{E}$ be arbitrary and $\bar{h}_1 \in G \cap \mathbb{R}_{\delta_0}$, $\bar{h}_1 \leq \bar{h}_0$, so that $\bar{h}_1 \Vdash ((\bar{t}_0, \dots, \bar{t}_{n-1}), r) \in \tilde{E}$.

Let us show by induction on $\xi \in [\delta_0, \delta)$, $\xi > \sup\big(\bigcup_{i<n} \mathrm{dom}(\bar{t}_i)\big)$, that there is a condition $\bar{h}_2 \in \mathbb{R}_\xi$, $\bar{h}_2 \leq \bar{h}_1$, so that

$$\begin{aligned} \bar{h}_2 \Vdash \forall \bar{y}'_0, \dots, \bar{y}'_{n-1} \in \dot{X}_\delta \big(& \mathrm{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle) \\ & \rightarrow \bar{y}'_0 \in [\bar{t}_0] \wedge \dots \wedge \bar{y}'_{n-1} \in [\bar{t}_{n-1}] \big) \end{aligned}$$

and in particular, if $\bar{h}_2 \in G$, then for all $\bar{y}'_0, \dots, \bar{y}'_{n-1} \in X_\delta$ with $\mathrm{tp}(\bar{y}'_0, \dots, \bar{y}'_{n-1}) = (\langle \xi_l : l \leq k \rangle, \langle K_l : l < k \rangle, \langle U_i : i < n \rangle)$, the generic corresponding to $\bar{y}'_0, \dots, \bar{y}'_{n-1}$ hits D_0, and $r \in \dot{D}_1[\bar{y}'_0, \dots, \bar{y}'_{n-1}]$. Since $\bar{h}_0 \in G$ was arbitrary, genericity finishes the argument.

The limit step of the induction follows directly from the earlier steps since if $\mathrm{dom}(\bar{t}_i) \subseteq \xi$, with ξ limit, then there is $\eta < \xi$ so that $\mathrm{dom}(\bar{t}_i) \subseteq \eta$. So let us consider step $\xi + 1$. Then there is, by the inductive assumption, $\bar{h}'_2 \in \mathbb{R}_\xi$, $\bar{h}'_2 \leq \bar{h}_1$, so that

$$\bar{h}_2' \Vdash \forall \bar{y}_0', \dots, \bar{y}_{n-1}' \in \dot{X}_\delta\big(\operatorname{tp}(\bar{y}_0', \dots, \bar{y}_{n-1}') = (\langle \xi_l : l \leq k\rangle, \langle K_l : l < k\rangle, \langle U_i : i < n\rangle) \\ \to (\bar{y}_0' \in [\bar{t}_0 \restriction \xi] \wedge \dots \wedge \bar{y}_{n-1}' \in [\bar{t}_{n-1} \restriction \xi]\big).$$

Now extend $\bar{h}_2'$ to $\bar{h}_2''$ in $\mathbb{R}_\xi$, so that there is $m \in \omega$ such that for every $s \in 2^m$ and every $i < n$, either $\bar{h}_2'' \Vdash \dot{O}_s \subseteq U_i \restriction \xi$ or $\bar{h}_2'' \Vdash \dot{O}_s \cap (U_i \restriction \xi) = \emptyset$, where $\langle \dot{O}_s : s \in 2^{<\omega}\rangle$ is a name for the base of $\dot{X}_\xi$ used to define $\dot{\mathbb{S}}_\xi$. The reason why this is possible, is that in any extension by $\mathbb{R}_\xi$ and for every $i < n$, by compactness of $X_\xi \cap (U_i \restriction \xi)$, there is a finite set $a \subseteq 2^{<\omega}$ so that $X_\xi \cap (U_i \restriction \xi) = \bigcup_{s \in a} O_s$. Let us define $h \colon 2^{\leq m} \to \mathcal{FT}$, where

$$h(s) = \begin{cases} \emptyset & \text{if } \forall i < n(\bar{h}_2'' \Vdash \dot{O}_s \cap U_i \restriction \xi = \emptyset) \\ \{t \in 2^{<\omega} : t \subseteq t_i(\xi)\} & \text{if } \bar{h}_2'' \Vdash \dot{O}_s \subseteq U_i \restriction \xi \text{ and } i < n. \end{cases}$$

Note that h is well-defined as $(U_i \restriction \xi) \cap (U_j \restriction \xi) = \emptyset$ for every $i \neq j < n$. Since $\emptyset \trianglelefteq_\rho T$ and $T \trianglelefteq_\rho T$ for any weight ρ and any finite tree T, we have that $\bar{h}_2'' \Vdash h \in \dot{\mathbb{S}}_\xi$ and $\bar{h}_2 = \bar{h}_2''{}^\frown h \in \mathbb{R}_{\xi+1}$ is as required. □

This finishes the definition of $\mathbb{R}_\alpha$ and $\dot{X}_\alpha$. Finally let G be $\mathbb{R}_\alpha$-generic over M_1 and $X_\alpha = \dot{X}_\alpha[G]$. Now let us define $\bar{q} \leq \bar{p}$ recursively so that for every $\delta \leq \alpha$,

$$\forall \bar{x} \in [\bar{q}](\Phi(\bar{x} \restriction C) \restriction \delta \in X_\alpha \restriction \delta).$$

If $\beta \notin C$ we let $\dot{q}(\beta)$ be a name for the trivial condition $2^{<\omega}$, say e.g. $\dot{q}(\beta) = \dot{p}(\beta)$. If $\beta \in C$, say $\beta = \iota(\delta)$, we define $\dot{q}(\beta)$ to be a name for the tree generated by

$$\eta^{-1}_{T_{\bar{x}_G \restriction (C \cap \beta)}}(S_{\bar{y}}),$$

where $\bar{x}_G$ is the generic sequence added by $\mathbb{P}_\lambda$ and $\bar{y} = \Phi(\bar{x}_G \restriction C) \restriction \delta$. This ensures that $\bar{q} \restriction \beta \Vdash \dot{q}(\beta) \in \mathbb{Q}_\beta \wedge \dot{q}(\beta) \leq \dot{p}(\beta)$. Inductively we see that $\bar{q} \restriction \beta^\frown \bar{p} \restriction (\lambda \setminus \beta) \Vdash \Phi(\bar{x}_G \restriction C) \restriction \delta \in X_\alpha \restriction \delta$. Having defined $\bar{q}$, it is also easy to check that it is a good master condition over M_0, with $[\bar{q}] = \Phi^{-1}(X_\alpha) \times (2^\omega)^{\lambda \setminus C}$. Since for every $\bar{x} \in [\bar{q}]$, $\Phi(\bar{x} \restriction C) \in X_\alpha$ and by $(1)_\alpha$, $\bar{q}$ is as required. □

Proposition 4.4.24. *Let $E \subseteq [X]^{<\omega} \setminus \{\emptyset\}$ be an analytic hypergraph on X, say E is the projection of a closed set $F \subseteq [X]^{<\omega} \times \omega^\omega$, and let $f \colon [\bar{p}] \restriction C \to X$ be continuous so that $\bar{p} \Vdash \dot{y} = f(\bar{x}_G \restriction C)$ (cf. Lemma 4.2.3). Then there is a good master condition $\bar{q} \leq \bar{p}$, with $[\bar{q}] \restriction C$ a closed subset of $[\bar{p}] \restriction C$ and $[\bar{q}] = ([\bar{q}] \restriction C) \times (2^\omega)^{\lambda \setminus C}$, a compact E-independent set $Y \subseteq X$, $N \in \omega$ and continuous functions $\phi \colon [\bar{q}] \restriction C \to [Y]^{<N}$, $w \colon [\bar{q}] \restriction C \to \omega^\omega$, so that*

(i) either $f''([\bar{q}] \restriction C) \subseteq Y$, thus $\bar{q} \Vdash \dot{y} \in Y$,

(ii) or $\forall \bar{x} \in [\bar{q}] \restriction C((\phi(\bar{x}) \cup \{f(\bar{x})\}, w(\bar{x})) \in F)$, thus $\bar{q} \Vdash \{\dot{y}\} \cup Y$ is not E-independent.

Proof. On $(2^\omega)^\alpha$ let us define the analytic hypergraph $\tilde{E}$, where

$$\{\bar{y}_0, \dots, \bar{y}_{n-1}\} \in \tilde{E} \leftrightarrow \{f(\Phi^{-1}(\bar{y}_0), \dots, f(\Phi^{-1}(\bar{y}_{n-1}))\} \in E.$$

By Main Lemma 4.3.14, there is a countable model M and $\bar{s} \in \bigotimes_{i<\alpha} 2^{<\omega}$ so that either

1. for any $\bar{y}_0, \dots, \bar{y}_{n-1} \in (2^\omega)^\alpha \cap [\bar{s}]$ that are strongly mCg wrt $\prod_{i<\alpha} 2^\omega$ over M, $\{\bar{y}_0, \dots, \bar{y}_{n-1}\}$ is E-independent,

or for some $N \in \omega$,

2. there are $\phi_0, \dots, \phi_{N-1} \colon (2^\omega)^\alpha \to (2^\omega)^\alpha$ continuous so that for any $\bar{y}_0, \dots, \bar{y}_{n-1} \in (2^\omega)^\alpha \cap [\bar{s}]$ that are strongly mCg over M, $\{\phi_j(\bar{y}_i) : j < N, i < n\}$ is E-independent but $\{\bar{y}_0\} \cup \{\phi_j(\bar{y}_0) : j < N\} \in E$.

Let M_1 be a countable elementary model with $M_0, M, \bar{p}, \mathbb{P}_\lambda \in M_1$ and apply Lemma 4.4.22 to get the condition $\bar{q} \leq \bar{p}$. In case (1), let $Y := f''([\bar{q}] \restriction C)$. Then (i) is satisfied. To see that Y is E-independent let $\bar{x}_0, \dots, \bar{x}_{n-1} \in [\bar{q}]$ be arbitrary and suppose that $\{f(\bar{x}_0 \restriction C), \dots, f(\bar{x}_{n-1} \restriction C)\} \in E$. By definition of $\tilde{E}$ this implies that $\{\Phi(\bar{x}_0 \restriction C), \dots, \Phi(\bar{x}_{n-1} \restriction C)\} \in \tilde{E}$ but this is a contradiction to (1) and the conclusion of Lemma 4.4.22. In case (2), by elementarity, the ϕ_j are in M_1 and there is a continuous function $\tilde{w} \in M_1$, with domain some dense G_δ subset of $(2^\omega)^\alpha$, so that $\bar{s} \Vdash (\{f(\bar{z}), \phi_j(\bar{z}) : j < N\}, \tilde{w}(\bar{z})) \in F$, where $\bar{z}$ is a name for the Cohen generic. Let $\phi(\bar{x}) = \{f(\Phi^{-1}(\phi_j(\Phi(\bar{x})))) : j < N\}$, $w(\bar{x}) = \tilde{w}(\Phi(\bar{x}))$ for $\bar{x} \in [\bar{q}] \restriction C$ and $Y := \bigcup_{\bar{x} \in [\bar{q}] \restriction C} \phi(\bar{x})$. Since $\Phi(\bar{x})$ is generic over M_1, we indeed have that $(\phi(\bar{x}), w(\bar{x})) \in F$ for every $\bar{x} \in [\bar{q}] \restriction C$. Seeing that Y is E-independent is as before. $\square$

4.5 Main results and applications

4.5.1 Definable maximal independent sets

Theorem 4.5.1. *(V=L) Let $\mathbb{P}$ be a countable support iteration of Sacks or splitting forcing of arbitrary length. Let X be a Polish space and $E \subseteq [X]^{<\omega} \setminus \{\emptyset\}$ be an analytic hypergraph. Then there is a $\mathbf{\Delta}^1_2$ maximal E-independent set in $V^{\mathbb{P}}$. If $X = 2^\omega$, $r \in 2^\omega$ and E is $\Sigma^1_1(r)$, then we can find a $\Delta^1_2(r)$ such set.*

Proof. We will only prove the second part since the first one follows easily from the fact that there is a Borel isomorphism from 2^ω to any uncountable Polish space X. If X is countable, then the statement is trivial. Also, let us only consider splitting forcing. The proof for Sacks forcing is the same.

First let us us mention some well-known facts and introduce some notation. Recall that a set $Y \subseteq 2^\omega$ is $\Sigma^1_2(x)$-definable if and only if it is $\Sigma_1(x)$-definable over $H(\omega_1)$ (see e.g. [31, Lemma 25.25]). Also recall that there is a Σ^1_1 set $A \subseteq 2^\omega \times 2^\omega$ that is universal for analytic sets, i.e. for every analytic $B \subseteq 2^\omega$, there is some $x \in 2^\omega$ so that $B = A_x$, where $A_x = \{y \in 2^\omega : (x, y) \in A\}$. In the same way, there is a universal Π^0_1 set $F \subseteq 2^\omega \times [2^\omega]^{<\omega} \times \omega^\omega$ ([33, 22.3, 26.1]). For any $x \in 2^\omega$, let E_x be the analytic hypergraph on 2^ω consisting of $a \in [2^\omega]^{<\omega} \setminus \{\emptyset\}$ so that there is $b \in [A_x]^{<\omega}$ with $a \cup b \in E$. Then there is $y \in 2^\omega$ so that E_x is the projection of F_y. Moreover, it is standard to note, from the way A and F are defined, that for every x, $y = e(x, r)$ for some fixed recursive function e. Whenever $\alpha < \omega_1$ and $Z \subseteq (2^\omega)^\alpha$ is closed, it can be coded naturally by the set $S \subseteq \bigotimes_{i<\alpha} 2^{<\omega}$, where

$$S = \{(\bar{x} \restriction a) \restriction n : \bar{x} \in Z, a \in [\alpha]^{<\omega}, n \in \omega\}$$

and we write $Z = Z_S$. Similarly, any continuous function $f \colon Z \to \omega^\omega$ can be coded by a function $\zeta \colon S \to \omega^{<\omega}$, where

$$f(\bar{x}) = \bigcup_{\bar{s} \in S, \bar{x} \in [\bar{s}]} \zeta(\bar{s})$$

and we write $f = f_\zeta$. For any $\beta < \alpha$ and $\bar{x} \in Z \restriction \beta$, let us write $T_{\bar{x},Z} = \{s \in 2^{<\omega} : \exists \bar{z} \in Z(\bar{z} \restriction \delta = \bar{x} \wedge s \subseteq z(\delta))\}$. The set Ψ_0 of pairs (α, S), where S codes a closed set $Z \subseteq (2^\omega)^\alpha$ so that for every $\beta < \alpha$ and $\bar{x} \in Z \restriction \beta$, $T_{\bar{x},Z} \in \mathbb{SP}$ is then Δ_1 over $H(\omega_1)$. This follows since the set of such S is Π^1_1, seen as a subset of $\mathcal{P}(\bigotimes_{i<\alpha} 2^{<\omega})$, uniformly on α. Similarly, the set Ψ_1 of triples (α, S, ζ), where $(\alpha, S) \in \Psi_0$ and ζ codes a continuous function $f \colon Z_S \to \omega^\omega$, is Δ_1.

Now let $\langle \alpha_\xi, S_\xi, \zeta_\xi : \xi < \omega_1 \rangle$ be a Δ_1-definable enumeration of all triples $(\alpha, S, \zeta) \in \Psi_1$. This is possible since we assume $V = L$ (cf. [31, Theorem 25.26]). Let us recursively construct a sequence $\langle x_\xi, y_\xi, T_\xi, \bar{\eta}_\xi, \theta_\xi : \xi < \omega_1 \rangle$, where for each $\xi < \omega_1$,

1. $\bigcup_{\xi' < \xi} A_{x_{\xi'}} = A_{y_\xi}$ and $A_{y_\xi} \cup A_{x_\xi}$ is E-independent,

2. $\bar{\eta}_\xi = \langle \eta_{\xi,j} : j < N \rangle$ for some $N \in \omega$,

3. $T_\xi \subseteq S_\xi$, $(\alpha_\xi, T_\xi, \eta_{\xi,j}) \in \Psi_1$ for every $j < N$ and $(\alpha_\xi, T_\xi, \theta_\xi) \in \Psi_1$,

4. either $\forall \bar{x} \in Z_{T_\xi}(f_{\zeta_\xi}(\bar{x}) \in A_{x_\xi})$ or $\forall \bar{x} \in Z_{T_\xi}\big(\forall n < N(f_{\eta_{\xi,n}}(\bar{x}) \in A_{x_\xi}) \wedge (\{f_{\eta_{\xi,n}}(\bar{x}), f_{\zeta_\xi}(\bar{x}) : n < N\}, f_{\theta_\xi}(\bar{x})) \in F_{e(y_\xi,r)}\big)$,

and $(x_\xi, y_\xi, T_\xi, \bar{\eta}_\xi, \theta_\xi)$ is $<_L$-least such that (1)-(4), where $<_L$ is the Δ_1-good global well-order of L. That $<_L$ is Δ_1-good means that for every $z \in L$, the set $\{z' : z' <_L z\}$ is $\Delta_1(z)$ uniformly on the parameter z. In particular, quantifying over this set does not increase the complexity of a Σ_n-formula. Note that (1)-(4) are all $\Delta_1(r)$ in the given variables. E.g. the second part of (1) is uniformly $\Pi^1_1(r)$ in the variables x_ξ, y_ξ, similarly for (4).

Claim 4.5.2. *For every $\xi < \omega_1$, $(x_\xi, y_\xi, T_\xi, \bar{\eta}_\xi, \theta_\xi)$ exists.*

Proof. Assume we succeeded in constructing the sequence up to ξ. Then there is y_ξ so that $\bigcup_{\xi'<\xi} A_{x_{\xi'}} = A_{y_\xi}$. By Lemma 4.2.6, there is a good master condition $\bar{r} \in \mathbb{P}_{\alpha_\xi}$ so that $[\bar{r}] \subseteq Z_{S_\xi}$, where $\mathbb{P}_{\alpha_\xi}$ is the α_ξ-long csi of splitting forcing. Then f_{ζ_ξ} corresponds to a $\mathbb{P}_{\alpha_\xi}$-name $\dot{y}$ so that $\bar{r} \Vdash \dot{y} = f_{\zeta_\xi}(\bar{x}_G)$. Let M_0 be a countable elementary model with $\dot{y}, \mathbb{P}_{\alpha_\xi}, \bar{r} \in M_0$ and $\bar{p} \leq \bar{r}$ a good master condition over M_0. Applying Proposition 4.4.24 to E_{y_ξ}, we get $\bar{q} \leq \bar{p}$ and $T_\xi \subseteq S_\xi$ with $[\bar{q}] = Z_{T_\xi}$, $x_\xi \in 2^\omega$, $N \in \omega$ and continuous functions $f_{\eta_{\xi,j}}, f_{\theta_\xi}$, for $j < N$, as required. □

Let $Y = \bigcup_{\xi<\omega_1} A_{x_\xi}$. Then Y is $\Sigma_1(r)$-definable over $H(\omega_1)$, namely $x \in Y$ iff there is a sequence $\langle x_\xi, y_\xi, T_\xi, \bar{\eta}_\xi, \theta_\xi : \xi \leq \alpha < \omega_1\rangle$ so that for every $\xi \leq \alpha$, (1)-(4), for every $(x, y, T, \bar{\eta}, \theta) <_L (x_\xi, y_\xi, T_\xi, \bar{\eta}_\xi, \theta_\xi)$, not (1)-(4), and $x \in A_{x_\alpha}$.

Claim 4.5.3. *In $V^{\mathbb{P}}$, the reinterpretation of Y is maximal E-independent.*

Proof. Let $\bar{p} \in \mathbb{P}$ and $\dot{y} \in M_0$ be a $\mathbb{P}$-name for an element of 2^ω, $M_0 \ni \mathbb{P}, \bar{p}$ a countable elementary model. Then let $\bar{q} \leq \bar{p}$ be a good master condition over M_0 and C countable, $f\colon [\bar{q}] \restriction C \to 2^\omega$ continuous according to Lemma 4.2.3. Now $(2^\omega)^C$ is canonically homeomorphic to $(2^\omega)^\alpha$, $\alpha = \operatorname{otp}(C)$, via the map $\Phi\colon (2^\omega)^C \to (2^\omega)^\alpha$. Then we find some $\xi < \omega_1$ so that $\alpha_\xi = \alpha$, $\Phi''([\bar{q}] \restriction C) = Z_{S_\xi}$ and $f_{\zeta_\xi} \circ \Phi = f$. On the other hand, $\Phi^{-1}(Z_{T_\xi})$ is a subset of $[\bar{q}] \restriction C$ conforming to the assumptions of Lemma 4.2.6. Thus we get $\bar{r} \leq \bar{q}$ so that $[\bar{r}] \restriction C \subseteq \Phi^{-1}(Z_{T_\xi})$. According to (4), either $\bar{r} \Vdash \dot{y} \in A_{x_\xi}$ or $\bar{r} \Vdash \{\dot{y}\} \cup A_{x_\xi} \cup A_{y_\xi}$ is not E-independent. Thus we can not have that $\bar{p} \Vdash \dot{y} \notin Y \wedge \{\dot{y}\} \cup Y$ is E-independent. This finishes the proof of the claim, as $\bar{p}$ and $\dot{y}$ were arbitrary. □

To see that Y is $\Delta^1_2(r)$ in $V^{\mathbb{P}}$ it suffices to observe that any $\Sigma^1_2(r)$ set that is maximal E-independent is already $\Pi^1_2(r)$. □

A priori, Theorem 4.5.1 only works for hypergraphs that are defined in the ground model. But note that there is a universal analytic hypergraph on $2^\omega \times 2^\omega$, whereby we can follow the more general statement of Theorem 4.1.2.

Theorem 4.5.4. *After forcing with the ω_2-length countable support iteration of $\mathbb{SP}$ over L, there is a Δ^1_2 ultrafilter, a Π^1_1 maximal independent family and a Δ^1_2 Hamel basis, and in particular, $\mathfrak{i}_B = \mathfrak{i}_{cl} = \mathfrak{u}_B = \omega_1 < \mathfrak{r} = \mathfrak{i} = \mathfrak{u} = \omega_2$.*

Proof. Apply Theorem 4.5.1 to E_u, E_i and E_h from the introduction. To see that $\mathfrak{i}_{cl} = \omega_1$ note that every analytic set is the union of $\mathfrak{d}$ many compact sets and that $\mathfrak{d} = \omega_1$, since $\mathbb{SP}$ is ω^ω-bounding. □

Theorem 4.5.5. *(V=L) Let $\mathbb{P}$ be either Sacks or splitting forcing and $k \in \omega$. Let X be a Polish space and $E \subseteq [X]^{<\omega} \setminus \{\emptyset\}$ be an analytic hypergraph. Then there is a $\boldsymbol{\Delta}^1_2$ maximal E-independent set in $V^{\mathbb{P}^k}$.*

Proof. This is similar to the proof of Theorem 4.5.1, using Main Lemma 4.3.4 and Proposition 4.4.16 to get an analogue of Proposition 4.4.24. □

4.5.2 P-points

An interesting corollary of the construction in the proof of Theorem 4.5.1 is the following.

Theorem 4.5.6. *There is a Δ^1_2 P-point after forcing with the countable support iteration of $\mathbb{S}$ or $\mathbb{SP}$ over L.*

This is well-known for Sacks forcing, which preserves all ground model P-points, in the sense that they generate a P-point in the extension again. The key observation is the following.

Lemma 4.5.7. *Let $A \subseteq \mathcal{P}(\omega)$ be a σ-compact filter. Then there is a compact set $K \subseteq \mathcal{P}(\omega)$ so that $A \cup K$ generates a filter and for every $C \in [A]^\omega$, there is $x \in K$ a pseudointersection of C.*

Proof. Let us write $A = \bigcup_{n\in\omega} K_n$, each K_n compact. We claim that there is an increasing sequence $\langle i_n : n \in \omega\rangle$ so that for every $n \in \omega$ and any $x_0, \dots, x_{n^2-1} \in \bigcup_{m\leq n} K_m$, $\bigcap_{m<n^2} x_m \cap [i_n, i_{n+1}) \neq \emptyset$. To see this note that $C_n := \{\bigcap F : F \in [\bigcup_{m\leq n} K_m]^{\leq n^2}\} \subseteq [\omega]^\omega$ is compact for every $n \in \omega$. Then it follows that for each $i \in \omega$ there is i^+ so that for every $x \in C_n$, $x \cap [i, i^+) \neq \emptyset$, since $\{\{x \in \mathcal{P}(\omega) : j \in x\} : j \geq i\}$

must have a finite subcover of C_n. Now let K be the set of $x \in \mathcal{P}(\omega)$ of the form $\bigcup_{n\in\omega} \bigcap F_n \cap [i_n, i_{n+1})$, where for each n, $F_n \in [\bigcup_{m\leq n} K_m]^{\leq n}$. It is not hard to check that K is as required. $\square$

Proof of Theorem 4.5.6. Instead of constructing a sequence of analytic sets as in the proof of Theorem 4.5.1, we construct a sequence of compact sets $\langle K_{x_\xi} : \xi < \omega_1\rangle$ using a universal closed subset $K \subseteq 2^\omega \times \mathcal{P}(\omega)$. At every second step ξ we find the $<_L$-least x_ξ so that K_{x_ξ} is as in the above lemma applied to $A = \bigcup_{\xi'<\xi} K_{x_{\xi'}}$. Here note that the filter generated by a K_σ set is itself K_σ. In the other steps we proceed as usual with regards to the hypergraph E_u. According to Proposition 4.4.24, the relevant set can be found compact and the construction can continue. In the end, we have ensured that the resulting ultrafilter is a P-point and it will keep this property by an absoluteness argument. $\square$

If we drop the definability requirement in Theorem 4.5.6, a similar construction shows that there is a P-point if we force over a model CH. This is interesting, since it has been shown in [12] that there is no P-point after iterating with Silver forcing, which is another tree forcing adding splitting reals.

4.5.3 Separating families and Borel chromatic numbers

The following is another interesting application of mutual genericity.

Definition 4.5.8. Let X be any set and $\mathcal{B} \subseteq \mathcal{P}(X)$. Then we say that $\mathcal{B}$ is $(\aleph_1, 2)$-separating if for any countable $A \subseteq X$ and $x \in X \setminus A$, there is $B \in \mathcal{B}$ so that $x \in B$ and $A \cap B = \emptyset$.

$(\aleph_1, 2)$-separating families appear in [26], where it was shown e.g. that if $|X| = 2^{\aleph_0} = \aleph_2$, then there is an $(\aleph_1, 2)$-separating family $\mathcal{B} \subseteq \mathcal{P}(X)$ of size $\aleph_1$ (see also [34, 3.1]). We show that in the Sacks or splitting extensions, if X is a Polish space, $\mathcal{B}$ can consist of compact sets alone.

Theorem 4.5.9. *After forcing with the countable support iteration of Sacks or splitting forcing over a model of CH, for any Polish space X there is $\mathcal{B} \subseteq K(X)$, $|\mathcal{B}| = \aleph_1$, an $(\aleph_1, 2)$ separating family.*

Here, $K(X)$ denotes the collection of compact subsets of X.

Proof. Let $\mathbb{P}_\lambda = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \lambda\rangle$ be the λ-length countable support iteration of Sacks or splitting forcing for some λ. Let $\dot{x}$ and $\langle \dot{y}_i : i \in \omega\rangle$ be $\mathbb{P}$-names for distinct elements of X and $\bar{p}_0 \in \mathbb{P}_\lambda$. Then, according to Section 4.4.3, there is a good master condition $\bar{p} \leq \bar{p}_0$, $C \subseteq \lambda$ countable, $\Phi\colon [\bar{p}] \restriction C \to (2^\omega)^\alpha$ the canonical homeomorphism, $\iota\colon \alpha \to C$ an order isomorphism, $f, g_i\colon (2^\omega)^\alpha \to X$ continuous so that $\bar{p} \Vdash \dot{x} = f(\Phi(\bar{x}_G \restriction C)) \wedge \dot{y}_i = g_i(\Phi(\bar{x}_G \restriction C))$ for every $i \in \omega$, and a countable elementary model M_1 containing all these objects. According to Lemma 4.4.22, there is $\bar{q} \leq \bar{p}$ a good master condition so that for any $\bar{y}_0, \bar{y}_1 \in [\bar{q}] \restriction C$, $\Phi(\bar{y}_0), \Phi(\bar{y}_1)$ are strongly mCg over M_1.

Claim 4.5.10. *There is $\bar{r} \leq \bar{q}$, a good master condition so that for any $\bar{y}_0, \bar{y}_1 \in [\bar{r}] \restriction C$ and $i \in \omega$, $f(\Phi(\bar{y}_0)) \neq g_i(\Phi(\bar{y}_1))$.*

Proof. Suppose that there are $\bar{y}_0, \bar{y}_1 \in [\bar{q}] \restriction C$ so that $f(\bar{x}_0) = g_i(\bar{x}_1)$ for some $i \in \omega$, where $\bar{x}_0 = \Phi(\bar{y}_0)$ and $\bar{x}_1 = \Phi(\bar{y}_1)$. If not we are simply done. Then note that $\bar{y}_0 \neq \bar{y}_1$, by Lemma 4.2.3 (iii) and since $\bar{q} \Vdash \dot{x} \neq \dot{y}_j$. In particular, $\bar{x}_0 \neq \bar{x}_1$, so let $\xi_0 = \Delta_{\bar{x}_0, \bar{x}_1}$. Then $\bar{x}_0$ is $\bigotimes_{i<\alpha} 2^{<\omega}$-generic over M_1 and $\bar{x}_1 \restriction [\xi_0, \alpha)$ is $\bigotimes_{i \in [\xi,\alpha)} 2^{<\omega}$-generic over $M_1[\bar{x}_0]$. In particular, there is $\bar{s}_0 \in \bigotimes_{i \in [\xi,\alpha)} 2^{<\omega}$, $\bar{x}_1 \restriction [\xi_0, \alpha) \in [\bar{s}_0]$, forcing over $M_1[\bar{x}_0]$ that $f_i(\bar{x}_0) = g_j(\bar{x}_1)$. By the continuity of g_j, we find that g_j is constant on $\{\bar{x}_0 \restriction \xi^\frown \bar{z} : \bar{z} \in [\bar{s}_0] \restriction [\xi, \alpha)\}$. Again, there is $t_0 \in \bigotimes_{i<\xi_0} 2^{<\omega}$, $\bar{x}_0 \restriction \xi_0 \in [\bar{t}_0]$, forcing this over M_1. Let $O_0 \subseteq [\bar{q}] \restriction C$ be an open non-empty set so that $\Phi(\bar{y}) \in [t_0 \cup s_0]$ for every $\bar{y} \in O_0$. In particular, varying over $\bar{y} \in O_0$, $f(\Phi(\bar{y}))$ only depends on $\Phi(\bar{y}) \restriction \xi_0$.

Suppose there are still $\bar{y}_0, \bar{y}_1 \in O_0$ so that $f(\bar{x}_0) = g_i(\bar{x}_1)$ for some $i \in \omega$, where again $\bar{x}_0 = \Phi(\bar{y}_0)$ and $\bar{x}_1 = \Phi(\bar{y}_1)$. Then, we must have that $\bar{x}_0 \restriction \xi_0 \neq \bar{x}_1 \restriction \xi_0$. Else, by a similar argument as before, using that $f(\bar{x}_0)$ only depends on $\bar{x}_0 \restriction \xi_0 = \bar{x}_1 \restriction \xi_0$, there is $U \subseteq O_0$ so that for any $\bar{y} \in U$, $f(\Phi(\bar{y})) = g_i(\Phi(\bar{y}))$. This is impossible since we can find $\bar{r} \leq \bar{q}$ so that $\bar{r} \Vdash \bar{x}_G \restriction C \in U$, and then $\bar{r} \Vdash \dot{x} = \dot{y}_i$. Thus let $\xi_1 = \Delta_{\bar{y}_0, \bar{y}_1} < \xi_0$. As before, we find an open set $O_1 \subseteq O_0$, so that varying over $\bar{y} \in O_1$, $f(\Phi(\bar{y}))$ only depends on $\Phi(\bar{y}) \restriction \xi_1$.

Continuing in that fashion, we must be done after finitely many steps. In particular we have found a non-empty open set $O_n \subseteq [\bar{q}] \restriction C$ so that for any $i \in \omega$ and $\bar{y} \in O_n$, $f(\Phi(\bar{y})) \neq g_i(\Phi(\bar{y}))$. Now it suffices to let $\bar{r} \leq \bar{q}$ be so that $\bar{r} \Vdash \bar{x}_G \restriction C \in O_n$. □

If $\bar{r} \leq \bar{q}$ is as in the claim, then we have that $\bar{r} \Vdash \dot{x} \in B \wedge \dot{y}_i \notin B$, where $B = (f \circ \Phi)''([\bar{r}] \restriction C)$. By genericity we have shown that in $V^{\mathbb{P}}$, every countable set and a point can be separated by a ground model coded compact set, of which there are $\aleph_1$ many. □

Definition 4.5.11. Let $G \subseteq [X]^2$ be a graph on a set X. Then G is called locally countable, if the vertex degree of every $x \in X$, i.e. $|\{y \in X : \{x, y\} \in G\}|$, is at most countable.

Definition 4.5.12. Let G be a graph on a Polish space X. Then the Borel chromatic number of G, $\chi_B(G)$, is the least size of a partition of X into Borel G-independent sets.

Corollary 4.5.13. After forcing with the countable support iteration of Sacks or splitting forcing over a model of CH, $\chi_B(G) \leq \aleph_1$ for every analytic locally countable graph G on a Polish space X.

Proof. Let $\{B_\alpha : \alpha < \omega_1\}$ be the compact sets given by Theorem 4.5.9. For every $\alpha < \omega_1$, $A_\alpha = \{x \in B_\alpha : \forall y \in B_\alpha(\{x, y\} \notin G)\}$ is G-independent. Each A_α is coanalytic and thus can be written as the $\aleph_1$-union of G-independent Borel sets A^i_α, for $i < \omega_1$. Enumerate $\{A^i_\alpha : \alpha, i < \omega_1\}$ as $\langle A'_\alpha : \alpha < \omega_1\rangle$ and put $A''_\alpha = A'_\alpha \setminus \bigcup_{\xi<\alpha} A'_\xi$. Then each A''_α is Borel and G-independent and it suffices to show that $\bigcup_{\alpha<\omega_1} A_\alpha = X$. To see this, let $x \in X$ be arbitrary and $A = \{y \in X : \{x, y\} \in G\}$. Since A is countable and $x \in X \setminus A$, there is $\alpha < \omega_1$ so that $x \in B_\alpha$ and $B_\alpha \cap A = \emptyset$. But then $x \in A_\alpha$. $\square$

Using similar ideas as in the proof of Theorem 4.5.1, we easily find the following.

Corollary 4.5.14. After forcing with the countable support iteration of Sacks or splitting forcing over L, every analytic locally countable graph G on a Polish space X admits a $\mathbf{\Sigma}^1_2$-definable coloring witnessing $\chi_B(G) \leq \aleph_1$.

4.5.4 $\mathfrak{d} \leq \mathfrak{i}_{\mathrm{cl}}$

Lastly, we are going to prove Theorem 4.1.4.

Lemma 4.5.15. *Let $X \subseteq [\omega]^\omega$ be closed so that $\forall x, y \in X(|x \cap y| = \omega)$. Then X is σ-compact.*

Proof. If not, then by Hurewicz's Theorem (see [33, 7.10]), there is a superperfect tree $T \subseteq \omega^{<\omega}$ so that $[T] \subseteq X$, identifying elements of $[\omega]^\omega$ with their increasing enumeration, as usual. But then it is easy to recursively construct increasing sequences $\langle s_n : n \in \omega\rangle$, $\langle t_n : n \in \omega\rangle$ in T so that $s_0 = t_0 = \mathrm{stem}(T)$, for every $n \in \omega$, t_n and s_n are infinite-splitting nodes in T and $s_{2n+1}(|s_{2n}|) > t_{2n+1}(|t_{2n+1}| - 1)$,

$t_{2n+2}(|t_{2n}|) > s_{2n+1}(|s_{2n+1}| - 1)$. Then, letting $x = \bigcup_{n\in\omega} s_n$ and $y = \bigcup_{n\in\omega} t_n$, $x \cap y \subseteq |s_0|$, viewing x, y as elements of $[\omega]^\omega$. This contradicts that $x, y \in X$. $\square$

The proof of Theorem 4.1.4 is a modification of Shelah's proof that $\mathfrak{d} \leq \mathfrak{i}$.

Proof of Theorem 4.1.4. Let $\langle C_\alpha : \alpha < \kappa\rangle$ be compact independent families so that $\mathcal{I} = \bigcup_{\alpha<\kappa} C_\alpha$ is maximal independent and $\kappa < \mathfrak{d}$ and assume without loss of generality that $\{C_\alpha : \alpha < \kappa\}$ is closed under finite unions. Here, we will identify elements of $[\omega]^\omega$ with their characteristic function in 2^ω at several places and it should always be clear from context which representation we consider at the moment.

Claim 4.5.16. *There are $\langle x_n : n \in \omega\rangle$ pairwise distinct in $\mathcal{I}$ so that $\{x_n : n \in \omega\} \cap C_\alpha$ is finite for every $\alpha < \kappa$.*

Proof. The closure of $\mathcal{I}$ is not independent. Thus there is $x \in \bar{\mathcal{I}} \setminus \mathcal{I}$. Now we pick $\langle x_n : n \in \omega\rangle \subseteq \mathcal{I}$ converging to x. Since C_α is closed, whenever for infinitely many n, $x_n \in C_\alpha$, then also $x \in C_\alpha$ which is impossible. $\square$

Fix a sequence $\langle x_n : n \in \omega\rangle$ as above. And let $a_\alpha = \{n \in \omega : x_n \in C_\alpha\} \in [\omega]^{<\omega}$. We will say that x is a Boolean combination of a set $X \subseteq [\omega]^\omega$, if there are finite disjoint $Y, Z \subseteq X$ so that $x = (\bigcap_{y\in Y} y) \cap (\bigcap_{z\in Z} \omega \setminus z)$.

Claim 4.5.17. *For any $\alpha < \kappa$ there is $f_\alpha \colon \omega \to \omega$ so that for any $K \in [C_\alpha \setminus \{x_n : n \in a_\alpha\}]^{<\omega}$, for all but finitely many $k \in \omega$ and any Boolean combination x of $K \cup \{x_0, \dots, x_k\}$, $x \cap [k, f_\alpha(k)) \neq \emptyset$.*

Proof. We define $f_\alpha(k)$ as follows. For every $l \leq k$, we define a collection of basic open subsets of $(2^\omega)^l$, $\mathcal{O}_{0,l} := \{[\bar{s}] : \bar{s} \in (2^{<\omega})^l \wedge \forall i < l(|s_i| > k) \wedge (\exists i < l, n \in a_\alpha(s_i \subseteq x_n) \vee \exists i < j < l(s_i \not\perp s_j))\}$. Further we call any $[\bar{s}] \notin \mathcal{O}_{0,l}$ good if for any $F, G \subseteq l$ with $F \cap G = \emptyset$ and for any Boolean combination x of $\{x_0, \dots x_k\}$, there is $k' > k$ so that for every $i \in F$, $s_i(k') = 1$, for every $i \in G$, $s_i(k') = 0$ and $x(k') = 1$. Let $\mathcal{O}_{1,l}$ be the collection of all good $[\bar{s}]$. We see that $\bigcup_{l\leq k}(\mathcal{O}_{0,l} \cup \mathcal{O}_{1,l})$ is an open cover of $C_\alpha \cup (C_\alpha)^2 \cup \dots \cup (C_\alpha)^k$. Thus it has a finite subcover $\mathcal{O}'$. Now let $f_\alpha(k) := \max\{|t| : \exists [\bar{s}] \in \mathcal{O}' \exists i < k(t = s_i)\}$.

Now we want to show that f_α is as required. Let $(y_0, \dots, y_{l-1}) \in (C_\alpha \setminus \{x_n : n \in a_\alpha\})^l$ be arbitrary, $y_0, \dots, y_{l-1}$ pairwise distinct and $k \geq l$ so that $y_i \restriction k \neq x_n \restriction k$ for all $i < l, n \in a_\alpha$ and $y_i \restriction k \neq y_j \restriction k$ for all $i < j < l$. In the definition of $f_\alpha(k)$, we have the finite cover $\mathcal{O}'$ of $(C_\alpha)^l$ and thus $(y_0, \dots, y_{l-1}) \in [\bar{s}]$ for some $[\bar{s}] \in \mathcal{O}'$. We see that $[\bar{s}] \in \mathcal{O}_{0,l}$ is impossible as we chose k large enough so that for no $i < l, n \in a_\alpha$,

$s_i \subseteq x_n$ and for every $i < j < l$, $s_i \perp s_j$. Thus $[\bar{s}] \in \mathcal{O}_{1,l}$. But then, by the definition of $\mathcal{O}_{1,l}$, $f_\alpha(k)$ is as required. $\square$

As $\kappa < \mathfrak{d}$ we find $f \in \omega^\omega$ so that f is unbounded over $\{f_\alpha : \alpha < \kappa\}$. Let $x_n^0 := x_n$ and $x_n^1 := \omega \setminus x_n$ for every $n \in \omega$. For any $g \in 2^\omega$ and $n \in \omega$ we define $y_{n,g} := \bigcap_{m \leq n} x_m^{g(m)}$. Further define $y_g = \bigcup_{n\in\omega} y_{n,g} \cap f(n)$. Note $y_{n,g} \subseteq y_{m,g}$ for $m \leq n$ and that $y_g \subseteq^* y_{n,g}$ for all $n \in \omega$.

Claim 4.5.18. *For any $g \in 2^\omega$, y_g has infinite intersection with any Boolean combination of $\bigcup_{\alpha<\kappa} C_\alpha \setminus \{x_n : n \in \omega\}$.*

Proof. Let $\{y_0, \dots, y_{l-1}\} \in [C_\alpha \setminus \{x_n : n \in a_\alpha\}]^l$ for some $l \in \omega$, $\alpha < \kappa$ be arbitrary. Here, recall that $\{C_\alpha : \alpha < \kappa\}$ is closed under finite unions. We have that there is some $k_0 \in \omega$ so that for every $k \geq k_0$, any Boolean combination y of $\{y_0, \dots, y_{l-1}\}$ and x of $\{x_n : n \leq k\}$, $x \cap y \cap [k, f_\alpha(k)) \neq \emptyset$. Let y be an arbitrary Boolean combination of $\{y_0, \dots, y_{l-1}\}$ and $m \in \omega$. Then there is $k > m, k_0$ so that $f(k) > f_\alpha(k)$. But then we have that $y_{k,g}$ is a Boolean combination of $\{x_0, \dots, x_k\}$ and thus $y_{k,g} \cap y \cap [k, f(k)) \neq \emptyset$. In particular, this shows that $y \cap y_g \not\subseteq m$ and unfixing m, $|y \cap y_g| = \omega$. $\square$

Now let Q_0, Q_1 be disjoint countable dense subsets of 2^ω. We see that $|y_g \cap y_h| < \omega$ for $h \neq g \in 2^\omega$. Thus the family $\{y_g : g \in Q_0 \cup Q_1\}$ is countable almost disjoint and we can find $y'_g =^* y_g$, for every $g \in Q_0 \cup Q_1$, so that $\{y'_g : g \in Q_0 \cup Q_1\}$ is pairwise disjoint. Let $y = \bigcup_{g \in Q_0} y'_g$. We claim that any Boolean combination x of sets in $\mathcal{I}$ has infinite intersection with y and $\omega \setminus y$. To see this, assume without loss of generality that x is of the form $\tilde{x} \cap x_0^{g(0)} \cap \dots \cap x_k^{g(k)}$, where $\tilde{x}$ is a Boolean combination of sets in $\mathcal{I} \setminus \{x_n : n \in \omega\}$ and $g \in 2^\omega$. As Q_0 is dense there is some $h \in Q_0$ such that $h \restriction (k+1) = g \restriction (k+1)$. Thus we have that $y'_h \subseteq^* x_0^{g(0)} \cap \dots \cap x_k^{g(k)}$ but also $y'_h \cap \tilde{x}$ is infinite by the claim above. In particular we have that $y \cap x$ is infinite. The complement of y is handled by replacing Q_0 with Q_1. We now have a contradiction to $\mathcal{I}$ being maximal. $\square$

4.6 Concluding remarks

Our focus in this paper was on Sacks and splitting forcing but it is clear that the method presented is more general. We mostly used that our forcing has Axiom A with continuous reading of names and that it is a weighted tree forcing (Definition 4.4.9), both in a definable way. For instance, the more general versions of splitting forcing given by Shelah in [55] fall into this class. It would be interesting to know for what other tree forcings Theorem 4.5.1 holds true. In [51], the authors showed that after adding a single Miller real over L, every (2-dimensional) graph on a Polish space has a $\mathbf{\Delta}^1_2$ maximal independent set. It is very plausible that this can be extended to the countable support iteration. For instance, the following was shown by Spinas in [60].

Fact. *Let M be a countable model, then there is a superperfect tree T so that for any $x \neq y \in [T]^2$, (x, y) is $\mathbb{M}^2$ generic over M, where $\mathbb{M}$ denotes Miller forcing.*

On the other hand, $\mathbb{M}^3$ always adds a Cohen real. When trying to generalize results about Cohen genericity to Miller forcing one has to be careful though since many nice properties no longer hold true. Let us ask the following question.

Question 4.6.1. Does Theorem 4.5.1 hold true for Miller forcing?

A positive result would yield a model in which $\mathfrak{i}_B < \mathfrak{i}_{cl}$, as per $\mathfrak{d} \leq \mathfrak{i}_{cl}$. No result of this kind has been obtained so far.

Forcings adding dominating reals and preserving ω_1 destroy $\mathbf{\Delta}^1_2$ definitions for ultrafilters and the associated hypergraph E_u is F_σ. On the other hand, it was shown by Brendle and Khomskii in [9] that in the Hechler model over L (via a finite support iteration) there is a Π^1_1 mad family. Recently, Schrittesser and Törnquist showed that the same holds after adding a single Laver real (see [52]). The hypergraph associated to almost disjoint families is G_δ. Thus we may ask, very optimistically:

Question 4.6.2. Does Theorem 4.5.1 hold true for Laver and Hechler forcing and G_δ hypergraphs?

CHAPTER **5**

Towers and gaps at uncountable cardinals

5.1 Introduction

The classical pseudointersection and tower numbers ($\mathfrak{p}$ and $\mathfrak{t}$ respectively) play a significant role in the study of cardinal characteristics of the continuum and special subsets of the reals. In this chapter we take the usual convention that a tower is already implied to be maximal, i.e. not to admit a pseudointersection.

It was unknown for a long time whether these two cardinals coincide. Rothberger proved in [43] and [44] that $\mathfrak{p} \leq \mathfrak{t}$ and also that if $\mathfrak{p} = \aleph_1$ then $\mathfrak{t} = \aleph_1$ as well. The consistency of $\mathfrak{p} < \mathfrak{t}$ seemed plausible to many set theorists working in the area, hence, the groundbreaking result of Malliaris and Shelah [36] came with considerable surprise: the cardinals $\mathfrak{p}$ and $\mathfrak{t}$ are provably equal.

Meanwhile, recent years have seen an increased interest in the study of the combinatorics of the generalized Baire spaces κ^κ, when κ is an uncountable regular cardinal. This fruitful new area of research provided extensions of classical results from the $\kappa = \omega$ case often requiring the development of completely new machinery to do so. Striking new inequalities were proved as well between cardinal invariants of κ^κ which are known to fail in the classical setting. Thus a natural question becomes: Does Malliaris and Shelah's result mentioned above lift to the uncountable?

The goal of this chapter is to study the higher analogues of the tower and pseudointersection numbers. We start with some basic definitions.

Definition 5.1.1. Let κ be a regular uncountable cardinal.

1. Let $\mathcal{F}$ be a family of subsets of κ. We say that $\mathcal{F}$ has the *strong intersection property* (in short, SIP) if for any subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size $< \kappa$, the intersection $\bigcap \mathcal{F}'$ has size κ.

2. We say that $A \subseteq \kappa$ is a *pseudointersection* of $\mathcal{F}$ if $A \subseteq^* F$ for all $F \in \mathcal{F}$.[1]

3. A *tower* $\mathcal{T}$ is a $\subseteq^*$-reverse-well-ordered family of subsets of κ with the SIP that has no pseudointersection of size κ.

In the countable case, any $\subseteq^*$-well-ordered family of infinite sets has the SIP. However, for uncountable κ, the SIP requirement is necessary as there are countable $\subseteq^*$-decreasing families of subsets of κ with no pseudointersection of size κ.[2]

Definition 5.1.2 (The pseudointersection and tower number)**.**

1. The *pseudointersection number* for κ, denoted by $\mathfrak{p}(\kappa)$, is defined as the minimal size of a family $\mathcal{F} \subset [\kappa]^\kappa$ which has the SIP but no pseudointersection of size κ.

2. The *tower number* for κ, denoted by $\mathfrak{t}(\kappa)$, is defined as the minimal size of a tower $\mathcal{T} \subset [\kappa]^\kappa$ of subsets of κ.

3. $\mathfrak{p}_{\text{cl}}(\kappa)$ is the minimal size of a family $\mathcal{F}$ of *club* subsets of κ with no pseudointersection of size κ.

4. $\mathfrak{t}_{\text{cl}}(\kappa)$ the minimal size of a tower $\mathcal{T}$ of *club* subsets of κ.

Note that in the definition of $\mathfrak{p}_{\text{cl}}(\kappa)$ and $\mathfrak{t}_{\text{cl}}(\kappa)$, there is no need to assume the SIP as any family of clubs has the strong intersection property.

The study of the above cardinal invariants was initiated by Garti [21] and one of the results which motivated the work on this project is the following:

Theorem 5.1.3. *[21] Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.*

1. *If $\mathfrak{p}(\kappa) = \kappa^+$, then $\mathfrak{t}(\kappa) = \kappa^+$.*

2. *If $\mathrm{cf}(2^\kappa) \in \{\kappa^+, \kappa^{++}\}$, then $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa)$.*

3. *$\mathrm{cf}(\mathfrak{p}(\kappa)) \neq \kappa$.*

[1] As usual, $A \subseteq^* F$ means that $A \setminus F$ has size $< \kappa$.

[2] E.g., consider a partition of κ into sets $\{X_n : n < \omega\}$ and look at $\mathcal{T} = \{\bigcup_{m \geq n} X_m : n \in \omega\}$.

Related consistency results also appear in the very recent paper of Ben-Neria and Garti [5].

In Section 5.2 we introduce a natural higher analogue of the notion of a gap which gives an interesting analogue of a theorem of Malliaris-Shelah, which is central to the proof of $\mathfrak{p} = \mathfrak{t}$. More precisely, we work with club-supported gaps of slaloms[3] (see Definition 5.2.4) and prove:

Theorem 5.1.4. *Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Either $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa)$ or there is a $\lambda < \mathfrak{p}(\kappa)$ and club-supported $(\mathfrak{p}(\kappa), \lambda)$-gap of slaloms.*

In Section 5.3, we study the possible sizes of gaps of slaloms which leads in particular to the following result (see Corollary 5.3.2):

Theorem 5.1.5. *For any uncountable, regular κ, $\mathfrak{p}(\kappa)$ is regular.*

Additionally, we consider a higher analogue of Martin's Axiom (see Definition 5.3.8) and its effect on certain club-supported gaps of slaloms (see Theorem 5.3.9).

In Section 5.4, we look at the relation between $\mathfrak{p}(\kappa)$ and its restriction to the club filter, $\mathfrak{p}_{\mathrm{cl}}(\kappa)$, which has been shown to be equal to $\mathfrak{b}(\kappa)$.

Theorem 5.1.6. *(GCH) For any regular uncountable $\kappa < \lambda$, where $\kappa = \kappa^{<\kappa}$, there is a κ-closed, κ^+-cc forcing extension in which $\mathfrak{p}(\kappa) = \kappa^+ < \mathfrak{p}_{\mathrm{cl}}(\kappa) = \lambda = 2^\kappa$.*

Moreover, we extend the above result to a certain class of κ-complete filters on κ (see Theorem 5.4.8). The consistency of $\mathfrak{p}(\kappa) < \mathfrak{b}(\kappa)(= \mathfrak{p}_{\mathrm{cl}}(\kappa))$ is originally due to Shelah and Spasojević [59], however our techniques significantly differ from theirs: We add κ-Cohen reals and then successively diagonalise the club-filter while preserving a Cohen witness to $\mathfrak{p}(\kappa) = \kappa^+$.

5.1.1 Notation, terminology and preliminaries

For a function $f \in \kappa^\kappa$, we say that $C \subset \kappa$ is f-closed if for any $\xi \in C$ and $\zeta < \xi$, $f(\zeta) < \xi$. Note that for any f, there are f-closed clubs (since κ is regular). For a club $C \subseteq \kappa$, we let s_C denote the function

$$s_C(\zeta) = \min C \setminus (\zeta + 1).$$

In forcing arguments, smaller conditions are stronger.

[3]Note that there are no real gaps of function in κ^κ. Indeed, there is no infinite $<^*$-decreasing sequence of functions in κ^κ when $\kappa \geq cf(\kappa) > \omega$.

One of the main tools in the study of $\mathfrak{p}$ is Bell's theorem: for any σ-centered poset $\mathbb{P}$ and for any collection $\mathcal{D}$ of $< \mathfrak{p}$-many dense subsets of $\mathbb{P}$, there is a filter $G \subset \mathbb{P}$ that meets each element of $\mathcal{D}$. A higher analogue of Bell's theorem has been given by Schilhan in [46].

Definition 5.1.7.

– A subset $C \subseteq \mathbb{P}$ is called *κ-linked*, if given $D \in [C]^{<\kappa}$, there is a condition $q \in \mathbb{P}$ such that $q \leq p$ for every $p \in D$.

– A poset $\mathbb{P}$ is *κ-centered* if there exists a sequence $\{C_\gamma : \gamma < \kappa\}$ of κ-linked subsets of $\mathbb{P}$ so that $\mathbb{P} = \bigcup_{\gamma<\kappa} C_\gamma$.

– Assume $\mathbb{P}$ is $< \kappa$-closed and κ-centered, say $\mathbb{P} = \bigcup_{\gamma<\kappa} C_\gamma$ where all C_γ are κ-linked. Say that $\mathbb{P}$ is *κ-centered with canonical lower bounds* if there is a function $f = f^{\mathbb{P}} : \kappa^{<\kappa} \to \kappa$ such that whenever $\lambda < \kappa$ and $(p_\alpha : \alpha < \lambda)$ is a decreasing sequence with $p_\alpha \in C_{\gamma_\alpha}$, then there is $p \in C_\gamma$ with $p \leq p_\alpha$ for all $\alpha < \lambda$ and $\gamma = f(\gamma_\alpha : \alpha < \lambda)$.

For convenience of the reader, we state the higher analogue of Bells theorem mentioned above, as it appears an important tool in the analysis of $\mathfrak{p}(\kappa)$ and $\mathfrak{t}(\kappa)$. First, note that if $\mathbb{P}$ is κ-centered with lower canonical bounds, then $\mathbb{P}$ is κ-specially centered, where:

Definition 5.1.8. [48, Definition 4.2] A poset $\mathbb{P}$ is said to be κ-specially centered if $\mathbb{P} = \bigcup_{i<\kappa} C_i$ where each C_i is κ-linked and whenever $s \in {}^{<\kappa}\kappa \setminus \{\emptyset\}$, and

$$P \subseteq \mathbf{S}(s, \{C_i\}_{i\in\kappa}) = \{\langle p_\alpha : \alpha < \mathrm{lth}\, s\rangle : \forall\alpha(p_\alpha \in C_{s(\alpha)})\}$$

is of cardinality strictly smaller than κ, then there is $p \in \mathbb{P}$ which is a common lower bound of all elements of sequences in P.

Theorem 5.1.9. *[48, Theorem 4.3.3] Let $\kappa^{<\kappa} = \kappa$. Assume $\mathbb{P}$ is κ-specially centered. Then for any collection $\mathcal{D}$ of $< \mathfrak{p}(\kappa)$-many dense subsets of $\mathbb{P}$, there is a filter $G \subset \mathbb{P}$ that meets each element of $\mathcal{D}$.*

5.2 On $\mathfrak{p}(\kappa)$, $\mathfrak{t}(\kappa)$ and gaps

In their seminal work, Malliaris and Shelah [36] proved that the classical cardinal invariants $\mathfrak{p}$ and $\mathfrak{t}$ coincide, answering a longstanding open problem. By now, various interpretations of their proof surfaced (see [42, 20, 11, 46, 67]) and we shall outline an argument for $\mathfrak{p} = \mathfrak{t}$ to motivate our results presented here.

First, we need two notions of gaps. Let $\bar{y} = (y_\alpha : \alpha < \lambda), \bar{x} = (x_\beta : \beta < \kappa)$ be sequences from ω^ω. We say that $(\bar{y}, \bar{x})$ is a *pre-gap* if for every $\gamma < \alpha < \lambda$ and $\delta < \beta < \kappa$,

$$y_\gamma <^* y_\alpha <^* x_\beta <^* x_\delta.$$

Definition 5.2.1. [36, Definition 14.11] We call $(\bar{y}, \bar{x})$ a (λ, κ)-*peculiar gap* if it is a pre-gap and for any $z \in \omega^\omega$:

1. if for all $\alpha < \lambda$, $y_\alpha \leq^* z$ then there is $\beta < \kappa$ such that $x_\beta \leq^* z$,
2. if for all $\beta < \kappa$, $z \leq^* x_\beta$ then there is $\alpha < \lambda$ such that $z \leq^* y_\alpha$.

We give a short outline of the proof of $\mathfrak{p} = \mathfrak{t}$. We shall inductively aim to build a tower from a witness to $\mathfrak{p}$ using the following notion.

Definition 5.2.2. Let $\mathcal{A}$ be a family of subsets of ω with the SIP and let $\mathcal{B}$ be an $\subseteq^*$-decreasing sequence of subsets of ω, such that every element of $\mathcal{B}$ has infinite intersection with all $A \in \mathcal{A}$ (write $\mathcal{B} \parallel \mathcal{A}$). We say that $\mathcal{B}$ is a *pseudoparallel* of $\mathcal{A}$ if there is a pseudointersection of $\mathcal{B}$ that has infinite intersection with all elements of $\mathcal{A}$.

Lemma 5.2.3.

1. *(Malliaris, Shelah [36]) If $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ is not a pseudoparallel of $\mathcal{B} = \{B_\beta : \beta < \mathfrak{p}\}$ for $\kappa < \mathfrak{p}$, then there exists either a tower of length $\mathfrak{p}$ or a $(\mathfrak{p}, \kappa)$-peculiar gap.*
2. *(Shelah [57]) If there is a $(\mathfrak{p}, \kappa)$-peculiar gap, then there is a tower of length $\mathfrak{p}$.*

Let $(A_\alpha)_{\alpha<\mathfrak{p}}$ be a family of subsets of ω witnessing $\mathfrak{p}$ that is additionally closed under finite intersections. Define a sequence of sets B_α as follows. Let $B_0 = A_0$ and suppose we have constructed $\mathcal{B}_\beta = \{B_\alpha : \alpha < \beta\}$ for some $\beta < \mathfrak{p}$ such that $\mathcal{B}_\beta \parallel \mathcal{A}$. If β is a successor ordinal $\eta + 1$ put $B_\beta = B_\eta \cap A_\beta$. Then $\mathcal{B}_{\beta+1} \parallel \mathcal{A}$. If β is a limit ordinal and $\mathcal{B}_\beta$ is a pseudoparallel of $\mathcal{A}$, take B be a witness for this property and put $B_\beta = B \cap A_\beta$. Then, we have the following cases: either it is possible to carry the construction along $\mathfrak{p}$-many steps, in which case the family $\{B_\alpha : \alpha < \mathfrak{p}\}$ is a tower of length $\mathfrak{p}$; or there is some ordinal $\beta < \mathfrak{p}$ (which we can assume is regular) such that the family $\mathcal{B}_\beta = \{B_\alpha : \alpha < \beta\}$ is not a pseudoparallel of $\mathcal{A}$. Then, by Lemma 5.2.3, there is a tower of size $\mathfrak{p}$, which finishes the proof.

The following results are motivated by the question whether $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa)$ holds for an uncountable cardinal κ. Theorem 5.2.6 below is a generalized version of Lemma 5.2.3 (1) for uncountable cardinals.

Definition 5.2.4 (Slaloms)**.**

1. Suppose that $\mathcal{D} \subset [\kappa]^\kappa$ is a $< \kappa$-closed filter. A *$\mathcal{D}$-supported slalom* is a map $u : X \to [\kappa]^{<\kappa} \setminus \{\emptyset\}$ so that $X \in \mathcal{D}$. We also say that u is an X-based slalom.
2. If u is a $\mathcal{D}$-supported slalom, then let set$(u) = \bigcup_{\xi \in \mathrm{dom}(u)} u(\xi)$.
3. Whenever u, v are $\mathcal{D}$-supported slaloms and for all but $< \kappa$ many $\xi \in \mathrm{dom}\, u \cap \mathrm{dom}\, v$, $u(\xi) \subseteq v(\xi)$, we write $u \subseteq^* v$.

Definition 5.2.5. (Gaps of $\mathcal{D}$-supported slaloms) A *$\mathcal{D}$-supported (μ, λ)-gap of slaloms* is a pair of two sequences $(u_\gamma)_{\gamma<\mu}$ and $(v_\alpha)_{\alpha<\lambda}$ of $\mathcal{D}$-supported slaloms so that

1. for any $\gamma < \gamma' < \mu$ and $\alpha < \alpha' < \lambda$,
$$u_\gamma \subseteq^* u_{\gamma'} \subseteq^* v_{\alpha'} \subseteq^* v_\alpha,$$
2. there is no $\mathcal{D}$-supported slalom w so that for all $\gamma < \mu$ and $\alpha < \lambda$,
$$u_\gamma \subseteq^* w \subseteq^* v_\alpha.$$

With this, we are ready to state our main theorem.

Theorem 5.2.6. *Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Either $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa)$ or there is a $\lambda < \mathfrak{p}(\kappa)$ and club-supported $(\mathfrak{p}(\kappa), \lambda)$-gap of slaloms.*

Proof. Suppose that $(A_\alpha)_{\alpha<\mathfrak{p}(\kappa)}$ is a family with the SIP but no pseudointersection. Let E_γ denote a pseudointersection for $(A_\alpha)_{\alpha\leq\gamma}$ for $\gamma < \mathfrak{p}(\kappa)$. Further, suppose that $\mathfrak{p}(\kappa) < \mathfrak{t}(\kappa)$.

Claim. *There is a club $X \subset \kappa$ so that for all $\gamma < \mathfrak{p}(\kappa)$ and almost all $\xi \in \kappa$,*
$$E_\gamma \cap [\xi, s_X(\xi)) \neq \emptyset.$$

Proof. For each γ, let X_γ be the set of accumulation points of E_γ. Then X_γ is a club in κ and for all $\xi \in \kappa$, $E_\gamma \cap [\xi, s_{X_\gamma}(\xi)) \neq \emptyset$. Since $\mathfrak{p}(\kappa) < \mathfrak{t}(\kappa) \leq \mathfrak{t}_{cl}(\kappa) = \mathfrak{p}_{cl}(\kappa)$ (see Observation 5.4.2), we can find a single club X that is a pseudointersection of $(X_\gamma)_{\gamma<\mathfrak{p}(\kappa)}$. □

Let us try and build sequences $\{B_\alpha\}_{\alpha<\mathfrak{p}(\kappa)}$, $\{Y_\alpha\}_{\alpha<\mathfrak{p}(\kappa)}$ so that for each $\beta < \mathfrak{p}(\kappa)$,

1. Y_β is a club,
2. $B_\beta \subset^* B_\alpha$ and $Y_\beta \subset^* Y_\alpha$ for all $\alpha < \beta$,
3. $B_\beta \subset^* A_\beta$, and
4. for all $\gamma < \mathfrak{p}(\kappa)$ such that $\beta \leq \gamma$,

$$\bigcup_{\xi \in Y_\beta} E_\gamma \cap [\xi, s_X(\xi)) \subset^* B_\beta.$$

We could not succeed in constructing such a sequence of length $\mathfrak{p}(\kappa)$, as otherwise $\{B_\alpha\}_{\alpha<\mathfrak{p}(\kappa)}$ would be a tower of length $\mathfrak{p}(\kappa) < \mathfrak{t}(\kappa)$ without pseudointersection. First, note that the SIP is still preserved at any intermediate stage.

Claim. *The sequence $\{B_\alpha\}_{\alpha<\lambda}$ has the SIP.*

Proof. Suppose that $I \in [\lambda]^{<\kappa}$. Then $Y = \bigcap_{\rho\in I} Y_\rho$ is a club and for any $\gamma \in \mathfrak{p}(\kappa) \setminus \sup I$, the set $\bigcup_{\xi\in Y} E_\gamma \cap [\xi, s_X(\xi))$ has size κ and is a pseudointersection to $\{B_\rho\}_{\rho\in I}$. □

Moreover, we can only fail at some limit step $\beta < \mathfrak{p}(\kappa)$ along the construction. Indeed, if $\beta < \mathfrak{p}(\kappa)$ and both B_β and Y_β have been already constructed we can put $B_{\beta+1} = B_\beta \cap A_{\beta+1}$ and $Y_{\beta+1} = Y_\beta$.

Fix this β where the induction must fail and lets try to approximate B_β and see what goes wrong. First, take some pseudointersection club Z to the sequence $\{Y_\alpha\}_{\alpha<\beta}$.

Lemma 5.2.7. *There is a $\subseteq^*$-increasing sequence of slaloms*

$$\{u_\rho\}_{\beta\leq\rho<\mathfrak{p}(\kappa)} \subseteq \prod_{\xi\in Z} \mathcal{P}([\xi, s_X(\xi)))$$

so that $\operatorname{dom} u_\rho = Z_\rho$ *is a club such that for all ρ and all $\alpha < \beta$*

$$\bigcup_{\xi\in Z_\rho} E_\rho \cap [\xi, s_X(\xi)) \subseteq^* \textit{set}(u_\rho) \subseteq^* B_\alpha \cap A_\beta.$$

The intuition is that each slalom u_γ gives an approximation for B_β by $\mathrm{set}(u_\gamma)$ which satisfies condition (4) with this fixed γ.

Proof. The sequence is constructed inductively. Suppose we have defined $\{Z_\rho\}_{\beta\le\rho<\gamma}$ and $\{u_\rho\}_{\beta\le\rho<\gamma}$ for some $\gamma\in\mathfrak{p}(\kappa)\setminus\beta$. We will try to force to find the next slalom u_γ.

Let Z_γ^- be a club, which is a pseudointersection of $\{Z_\rho\}_{\beta\le\rho<\gamma}$ and consider the poset $\mathbb{P}_\gamma$ consisting of all triples $(\nu,\mathcal{Y},n)$ such that

1. $\mathrm{dom}(\nu)\in[Z_\gamma^-]^{<\kappa}$ is closed and $n\in\kappa$,
2. $\forall\xi\in\mathrm{dom}(\nu)$
$$E_\gamma\cap[\xi,s_X(\xi))\subseteq\nu(\xi)\subseteq A_\beta\cap[\xi,s_X(\xi)),$$
3. $\mathcal{Y}=\mathcal{Y}_0\cup\mathcal{Y}_1\in[\gamma]^{<\kappa}$ where $\mathcal{Y}_0\subseteq[\beta,\gamma)$ and $\mathcal{Y}_1\subseteq\beta$, and
4. if $\xi\in Z_\gamma^-\setminus n$ then
$$\bigcup_{\rho\in\mathcal{Y}_0}u_\rho(\xi)\subseteq\bigcap_{\rho\in\mathcal{Y}_1}B_\rho\cap A_\beta\cap[\xi,s_X(\xi)) \tag{5.1}$$
and
$$E_\gamma\cap[\xi,s_X(\xi))\subseteq\bigcap_{\rho\in\mathcal{Y}_1}B_\rho\cap A_\beta\cap[\xi,s_X(\xi)). \tag{5.2}$$

The extension relation is defined as follows: $(\mu,\mathcal{X},m)\le(\nu,\mathcal{Y},n)$ iff $\mu\supseteq\nu$, $\mathcal{X}\supseteq\mathcal{Y}$, $m\ge n$ and for all $\xi\in\mathrm{dom}(\mu)\setminus\mathrm{dom}(\nu)$:
$$\xi>n\text{ and }\bigcup_{\rho\in\mathcal{Y}_0}u_\rho(\xi)\subseteq\mu(\xi)\subseteq\bigcap_{\rho\in\mathcal{Y}_1}B_\rho.$$

Observation 5.2.8. For any pair $(\nu,\mathcal{Y})$ which satisfies condition (1)-(3) above and almost all $n\in\kappa$, $(\nu,\mathcal{Y},n)\in\mathbb{P}_\gamma$.

Proof. Using the facts that $|\mathcal{Y}_0|<\kappa$, $|\mathcal{Y}_1|<\kappa$ and $u_\rho(\xi)\subseteq[\xi,s_X(\xi))$ we can find $n(\mathcal{Y}_0,\mathcal{Y}_1)\in\kappa$ such that for each $\xi\in Z_\gamma^-\setminus n(\mathcal{Y}_0,\mathcal{Y}_1)$,
$$\bigcup_{\rho\in\mathcal{Y}_0}u_\rho(\xi)\subseteq\bigcap_{\rho\in\mathcal{Y}_1}B_\rho\cap A_\beta\cap[\xi,s_X(\xi)).$$
Moreover, by the hypothesis on $\{B_\alpha\}_{\alpha<\beta}$ for each $\rho\in\mathcal{Y}_1$, $\bigcup_{\xi\in Y_\rho}E_\gamma\cap[\xi,s_X(\xi))\subseteq^*B_\rho$. However $Z_\gamma^-\subseteq^*Y_\rho$ for each $\rho\in\mathcal{Y}_1$ and $E_\gamma\subseteq^*A_\beta$. Thus we can find $m(\mathcal{Y}_1)\in\kappa$ such that for each $\xi\in Z_\gamma^-\setminus m(\mathcal{Y}_1)$ we have
$$E_\gamma\cap[\xi,s_X(\xi))\subseteq\bigcap_{\rho\in\mathcal{Y}_1}B_\rho\cap A_\beta\cap[\xi,s_X(\xi)).$$
Now, any $n>\max\{\eta,m(\mathcal{Y}_1),n(\mathcal{Y}_0,\mathcal{Y}_1),\max\mathrm{dom}(\nu)\}$ works. $\square$

Claim. *The poset $\mathbb{P}_\gamma$ is κ-specially centered.*

Proof. Indeed, by $\kappa^{<\kappa} = \kappa$, κ-centerdness holds if $< \kappa$-many conditions with the same first coordinate are compatible. In the latter case, we can apply the above observation to see that such conditions do have common lower bounds. $\square$

Claim. *The poset $\mathbb{P}_\gamma$ is $< \kappa$-closed with canonical lower bounds.*

Proof. If $\{p_i\}_{i<j}$ is a decreasing sequence of conditions, where $j < \kappa$ and $p_i = (\nu_i, \mathcal{Y}_i, n_i)$ then let $\nu^- = \bigcup_{i<j} \nu_i$, $\mathcal{Y} = \bigcup_{i<j} \mathcal{Y}_i$ and $n = \sup_{i<j} n_i$. Now extend ν^- to ν by defining

$$\nu(\xi) = \big(E_\gamma \cap [\xi, s_X(\xi))\big) \cup \bigcup_{\rho \in \mathcal{Y}_0} u_\rho(\xi).$$

This triple $(\nu, \mathcal{Y}, n)$ is in $\mathbb{P}_\gamma$ and defines the canonical lower bound. $\square$

For each $\rho \in \gamma$ the set $D_\rho = \{(\nu, \mathcal{Y}, n) \in \mathbb{P}_\gamma : \rho \in \mathcal{Y}\}$ is dense. Indeed, given ρ and $(\nu, \mathcal{Y}, n) \in \mathbb{P}_\gamma$ we can find a large enough n^* above n so that $(\nu, \mathcal{Y} \cup \{\rho\}, n^*)$ extends $(\nu, \mathcal{Y}, n)$. Furthermore:

Claim. *For each $\eta \in \kappa$ the set $D^\eta = \{(\nu, \mathcal{Y}, n) \in \mathbb{P}_\gamma : \exists \zeta > \eta(\zeta \in \mathrm{dom}(\nu))\}$ is dense in $\mathbb{P}_\gamma$.*

Proof. For any $\zeta > \max(\eta, n)$, we can define $\mu \supset \nu$ on the set $\mathrm{dom}\, \nu \cup \{\zeta\}$ by

$$\mu(\zeta) = \big(E_\gamma \cap [\zeta, s_X(\zeta))\big) \cup \bigcup_{\rho \in \mathcal{Y}_0} u_\rho(\zeta).$$

Then $(\mu, \mathcal{Y}, n)$ belongs to D^η and extends $(\nu, \mathcal{Y}, n)$. $\square$

By the generalized Bell's theorem, there is a filter $G \subseteq \mathbb{P}_\gamma$ intersecting all the above dense sets. Thus, we can finally define

$$u_\gamma = \bigcup \{\nu : \exists \mathcal{Y} \exists n \text{ such that } (\nu, \mathcal{Y}, n) \in G\}.$$

Observe that $Z_\gamma = \mathrm{dom}\, u_\gamma$ is a club subset of Z_γ^- and hence a pseudointersection of all the other Z_β for $\beta < \gamma$.

$\square$

Note how set(u_γ) is a reasonable candidate for B_β (with Z_γ playing the role of Y_β):

Observation 5.2.9. set(u_γ) is almost contained in A_β and all B_α for $\alpha < \beta$, and also satisfies condition (4) for a particular γ.

Finally, let us take a pseudointersection club for $(Z_\gamma)_{\beta\leq\gamma<\mathfrak{p}(\kappa)}$ which we shall call Z again to ease notation. Now, we define

$$v_\alpha(\xi) = B_\alpha \cap A_\beta \cap [\xi, s_X(\xi))$$

for $\alpha < \beta$ and $\xi \in Z$. In turn, for all $\gamma < \gamma', \alpha < \alpha'$ and almost all $\xi \in Z$,

$$u_\gamma(\xi) \subset u_{\gamma'}(\xi) \subset v_{\alpha'}(\xi) \subset v_\alpha(\xi).$$

Finally, if there is a club $Y_\beta \subset Z$ and $w(\xi) \subset [\xi, s_X(\xi))$ for $\xi \in Y_\beta$ so that for all γ, α and almost all $\xi \in Y_\beta$,

$$u_\gamma(\xi) \subset w(\xi) \subset v_\alpha(\xi),$$

then $B_\beta = \text{set}(w)$ would extend $\{B_\alpha\}_{\alpha<\beta}$. Since this is impossible (the construction of the B-sets failed at step β), we must have produced a $(\mathfrak{p}(\kappa), \beta)$-gap of club-supported slaloms. □

5.3 On the sizes of gaps of slaloms

Naturally, Theorem 5.2.6 prompts us to study the existence of (λ_1, λ_2)-peculiar gaps more closely. In fact, to prove $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa)$, it would suffice to show that there are no $\mathcal{D}$-supported $(\mathfrak{p}(\kappa), \lambda)$-gaps of slaloms supported for some filter $\mathcal{D}$.

Proposition 5.3.1 together with Theorem 5.2.6 show that $\mathfrak{p}(\kappa)$ is regular. The results in this section show that in a certain sense there are no club-supported gaps of slaloms which are small on both sides. However, in Proposition 5.3.4 we show that there are short decreasing sequences of slaloms with no lower bound. Finally, in Theorem 5.3.10, we see how generalized forms of MA affect the existence of gaps.

Proposition 5.3.1. *Suppose $\kappa = \kappa^{<\kappa} \leq \lambda_1, \lambda_2$ are regular cardinals and that there is a club-supported (λ_1, λ_2)-gap of slaloms. Then $\mathfrak{p}(\kappa) \leq \max\{\lambda_1, \lambda_2\}$.*

Proof. Let $(u_\alpha : \alpha < \lambda_1)$ and $(v_\beta : \beta < \lambda_2)$ be a club-supported (λ_1, λ_2)-gap of slaloms and assume $\max\{\lambda_1, \lambda_2\} < \mathfrak{p}(\kappa)$. We can assume all the slaloms are defined on a common club C (by taking a pseudointersection for all the domains; see Observation 5.4.2). We shall find a single w that fills the gap on a club set using the generalized version of Bell's theorem (see Theorem 5.1.9).

We define a κ-specially centered poset $\mathbb{Q}$ as follows. Conditions in $\mathbb{Q}$ are triples $q = (s^q, \sigma_1^q, \sigma_2^q)$ where

1. s^q is a partial slalom defined on some closed, bounded subset of C,

2. $\sigma_i^q \in [\lambda_i]^{<\kappa}$ for $i = 1, 2$, and

3. for any $\alpha \in \sigma_1^q, \beta \in \sigma_2^q$ and $\eta > \max \operatorname{dom} s$, $u_\alpha(\eta) \subseteq v_\beta(\eta)$.

The order on $\mathbb{Q}$ is defined as follows: We say $p \leq q$ if and only if $s^p \sqsupseteq s^q$, $\sigma_i^p \supseteq \sigma_i^q$ and for all $\eta \in \operatorname{dom}(s^p) \setminus \operatorname{dom}(s^q)$,

$$\bigcup_{\alpha \in \sigma_1^q} u_\alpha(\eta) \subseteq s^p(\eta) \subseteq \bigcap_{\alpha \in \sigma_2^q} v_\beta(\eta).$$

Claim. *$\mathbb{Q}$ is a κ-closed, κ-specially centered forcing notion of size λ_2.*

Proof. For a fixed closed and bounded $s \subset C$, any subset of $\mathbb{Q}_s = \{q \in \mathbb{Q} : s^q = s\}$ has a canonical lower bound. So the partition

$$\mathbb{Q} = \bigcup \{\mathbb{Q}_s : s \in [C]^{<\kappa}, s \text{ club}\}$$

witnesses the claim. □

Claim. *For each $\eta < \kappa$, $\alpha < \lambda_1$ and $\beta < \lambda_2$ the following sets are dense in $\mathbb{Q}$:*

1. *$\mathcal{D}_\eta = \{q \in \mathbb{Q} : \eta < \max \operatorname{dom} s^q\}$, and*

2. *$\mathcal{E}_{\alpha,\beta} = \{q \in \mathbb{Q} : \alpha \in \sigma_1^q, \beta \in \sigma_2^q\}$.*

Proof. Fix $q \in \mathbb{Q}$, $\eta < \kappa$ and $\alpha < \lambda_1$, $\beta < \lambda_2$. Let $q' = (s', \sigma_1^q \cup \{\alpha\}, \sigma_2^q \cup \{\beta\})$ so that $\operatorname{dom} s' = \operatorname{dom} s \cup \{\mu\}$ and for any $\alpha' \in \sigma_1^q \cup \{\alpha\}$ and $\beta' \in \sigma_2^q \cup \{\beta\}$, if $\eta > \mu$ then $u_{\alpha'}(\eta) \subseteq v_{\beta'}(\eta)$. Moreover, pick μ to be above η and define $s'(\mu) = \bigcup_{\alpha \in \sigma_1^q} u_\alpha(\mu)$. Then q' is a condition extending q and $q' \in \mathcal{D}_\eta \cap \mathcal{E}_{\alpha,\beta}$, as desired. □

By Theorem 5.1.9, we can take a filter $G \subseteq \mathbb{Q}$ which intersects all the dense sets $\{\mathcal{D}_\eta\}_{\eta<\kappa} \cup \{\mathcal{E}_{\alpha,\beta}\}_{(\alpha,\beta)\in\lambda_1\times\lambda_2}$. Then $D = \bigcup\{\operatorname{dom} s^q : q \in G\}$ is a club and

$$w = \bigcup \{s^q : q \in G\}$$

is a slalom with domain D. Fix any $(\alpha, \beta) \in (\lambda_1, \lambda_2)$ and pick $q \in \mathcal{E}_{\alpha,\beta} \cap G$. Then for any $\eta > \max \operatorname{dom} s^q$, we have $u_\alpha(\eta) \subseteq^* w(\eta) \subseteq^* v_\beta(\eta)$ and so

$$u_\alpha \subseteq^* w \subseteq^* v_\beta,$$

which finishes the proof. □

Corollary 5.3.2. $\mathfrak{p}(\kappa)$ is regular.

Proof. This follows immediately from Theorem 5.2.6 and Proposition 5.3.1. Indeed, if $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa)$ then we are done since the latter is regular. Otherwise, there is a $(\mathfrak{p}(\kappa), \lambda_1)$-gap of slaloms with $\lambda_1 < \mathfrak{p}(\kappa)$. If $\mathfrak{p}(\kappa)$ is singular of cofinality λ_0 then we can shrink the left-hand side of the original $(\mathfrak{p}(\kappa), \lambda_1)$-gap and get a (λ_0, λ_1)-gap of slaloms. This however, contradicts Proposition 5.3.1. □

Yet another bound on the sizes of gaps is the following.

Proposition 5.3.3. *Suppose that κ is a regular, uncountable cardinal. If $\lambda < \mathfrak{b}(\kappa)$ then there is no club-supported (κ, λ)-gap of slaloms on κ.*

Proof. Let $\lambda < \mathfrak{b}(\kappa)$. Suppose that $\bar{u} = (u_\alpha : \alpha < \kappa)$ and $\bar{v} = (v_\xi : \xi < \lambda)$ are sequences of club-supported slaloms on κ, $\bar{u}$ is increasing, $\bar{v}$ is decreasing and $u_\alpha \subseteq^* v_\xi$ for all $\alpha < \kappa, \xi < \lambda$.

Let $C_\alpha = \operatorname{dom} u_\alpha$. For any club C which is a subset of the diagonal intersection $\Delta_{\alpha<\kappa} C_\alpha$, we can define a slalom w_C on C by

$$w_C(\beta) = \bigcup_{\alpha<\beta} u_\alpha(\beta).$$

It is clear that $u_\alpha \subseteq^* w_C$ for any $\alpha < \kappa$.

Given a fixed $\xi < \lambda$, there is a club D_ξ so that $\beta \in D_\xi$ and $\alpha < \beta$ implies that $u_\alpha(\beta) \subseteq v_\xi(\beta)$. The family $\{D_\xi : \xi < \lambda\}$ must have a pseudointersection D since $\lambda < \mathfrak{b}(\kappa) = \mathfrak{p}_{\mathrm{cl}}(\kappa)$.

Finally, let $w = w_C$ where $C = D \cap \Delta_{\alpha<\kappa} C_\alpha$. Now, for any $\alpha < \kappa$ and $\xi < \lambda$, $u_\alpha \subseteq^* w \subseteq^* v_\xi$ and so $\bar{u}, \bar{v}$ is not a gap. □

In particular, we proved that any κ-sequence of club-supported slaloms on κ has an upper bound. There is an interesting asymmetry here, as there are short decreasing sequences of slaloms without lower bounds.

Proposition 5.3.4. *Suppose that $\kappa = \kappa^{<\kappa}$ is a regular, uncountable cardinal.*

1. *There is a $\subseteq^*$-decreasing, κ-sequence of club-supported slaloms on κ that has no lower bound supported on a stationary set.*

2. *Suppose λ is regular such that $\kappa \leq \lambda \leq 2^\kappa$. Then, there is a κ-specially centered poset $\mathbb{P}$ which introduces a decreasing λ-sequence of club-supported slaloms on κ with no lower bound supported on a club.*

Proof. (1) We define the decreasing sequence of slaloms $\bar{v} = (v_\beta : \beta < \kappa)$ with the following properties

1. $v_\alpha : \kappa \to [\kappa]^{<\kappa} \setminus \{\emptyset\}$,
2. for any $\alpha < \beta < \kappa$ and $\eta > \beta$, $v_\alpha(\eta) \supseteq v_\beta(\eta)$,
3. for any limit $\beta \in \kappa$, $\bigcap_{\alpha<\beta} v_\alpha(\beta) = \emptyset$.

The construction is done in κ steps: at step β, we define $v_\alpha(\beta)$ for $\alpha < \beta$ and $v_\beta \restriction \beta+1$. If β is a limit ordinal, then we make sure that the sequence of sets $\{v_\alpha(\beta) : \alpha < \beta\}$ is strictly decreasing with empty intersection. We can pick $v_\beta \restriction \beta + 1$ arbitrarily, for example, $v_\beta(\eta) = \{0\}$ for all $\eta \leq \beta$.

If $\beta = \alpha + 1$ then again we make sure that $\{v_{\alpha'}(\beta) : \alpha' \leq \alpha\}$ is strictly decreasing and we can pick $v_\beta(\eta) = \{0\}$ for all $\eta \leq \beta$.

Finally, given such a sequence $\bar{v}$, assume that $w : S \to [\kappa]^{<\kappa} \setminus \{\emptyset\}$ and $w \subseteq^* v_\alpha$ for all $\alpha < \kappa$. If S is stationary then we can find a limit $\beta \in S$ so that $\alpha < \beta$ implies that $w(\beta) \subseteq v_\alpha(\beta)$. In turn, $\bigcap_{\alpha<\beta} v_\alpha(\beta) \neq \emptyset$ and this contradiction finishes the proof.

(2) Define $\mathbb{P}$ to be the set of conditions of the form $p = (s^p_\alpha)_{\alpha\in\sigma^p}$ so that $\sigma^p \in [\lambda]^{<\kappa}$ and there is some $\mu^p < \kappa$ such that $s^p_\alpha : \mu^p \to [\kappa]^{<\kappa} \setminus \emptyset$.

Extension in $\mathbb{P}$ works as follows: $p \leq q$ if

1. $\sigma^p \supseteq \sigma^q$,
2. for any $\alpha \in \sigma^q$, $s^p_\alpha \supseteq s^q_\alpha$, and
3. for any $\alpha < \beta \in \sigma^q$ and $\eta \in \mu^p \setminus \mu^q$,

$$s^p_\beta(\eta) \subseteq s^p_\alpha(\eta).$$

First, we show that the poset $\mathbb{P}$ is κ-specially centered. Without loss of generality assume $\lambda = 2^\kappa$. Let $\mathcal{B}$ be a base for 2^κ consisting of basic open sets of the form $[t] = \{x \in 2^\kappa : t \subseteq x\}$ where $t \in {}^{<\kappa}2$. Thus $|\mathcal{B}| = \kappa$. Let $\mathcal{T}$ be the set of all non-empty subfamilies $\bar{\mathcal{U}}$ of $\mathcal{B}$ consisting of pairwise disjoint non-empty basic open sets such that $|\bar{\mathcal{U}}| < \kappa$. Thus $|\mathcal{T}| = \kappa$.

Now, for each $\bar{\mathcal{U}} = \{U_i\}_{i\in I} \in \mathcal{T}$ and each $\bar{y} = \{y_i\}_{i\in I} \in {}^I\big({}^{<\kappa}([\kappa]^{<\kappa})\big)$ let

$$C(\bar{\mathcal{U}}, \bar{y}) = \{p \in \mathbb{P} : \sigma^p \subseteq \bigcup_{i\in I} U_i, \forall i \in I(\sigma^p \cap U_i \neq \emptyset) \text{ and if } \alpha \in \sigma^p \cap U_i \text{ then } s^p_\alpha = y_i\}.$$

Then the family

$$\{C(\bar{\mathcal{U}}, \bar{y}) : \bar{\mathcal{U}} \in \mathcal{T}, \bar{y} \in {}^{|\bar{\mathcal{U}}|}\big({}^{<\kappa}([\kappa]^{<\kappa})\big)\}$$

is a partition of $\mathbb{P}$ witnessing that $\mathbb{P}$ is κ-specially centered.

The following should be straightforward to check:

(a) $\mathcal{D}_\eta = \{p \in \mathbb{P} : \eta \le \mu^p\}$ is dense in $\mathbb{P}$;

(b) $\mathcal{E}_\alpha = \{p \in \mathbb{P} : \alpha \in \sigma^p\}$ is dense in $\mathbb{P}$.

So, we can take a generic filter $G \subset \mathbb{P}$ and define

$$v_\alpha = \bigcup \{s^p_\alpha : p \in G\}.$$

Observe that if $\alpha < \beta < \lambda$ and $\alpha, \beta \in \sigma^p$ for some $p \in G$ then for any $\eta \ge \mu^p$, $v_\beta(\eta) \subseteq v_\alpha(\eta)$. Thus $(v_\alpha)_{\alpha<\lambda}$ is a decreasing sequence of slaloms.

Now, suppose $\dot{w}$ is a $\mathbb{P}$-name for a slalom defined on a club and for all $\alpha < \lambda$, $p \Vdash \dot{w} \subseteq^* v_\alpha$. Take an elementary submodel $M \prec H(\theta)$ of size $\kappa_0 < \kappa$ with all relevant parameters in M. Also, assume that $M^{<\kappa_0} \subset M$.

Construct a decreasing sequence of conditions $(p_\xi)_{\xi<\kappa_0}$ in M, so that

1. for any $\zeta < M \cap \kappa$, there is $\xi(\zeta) < \kappa_0$ and $\delta_\zeta \in M \cap \kappa \setminus \zeta$ such that $\mu^{p_{\xi(\zeta)}} \ge \zeta$ and $p_{\xi(\zeta)} \Vdash \delta_\zeta \in \operatorname{dom} \dot{w}$.

2. There is a sequence $\{\eta_n\}_{n\in\omega} \subseteq M \cap \kappa$ such that for some subsequence of indexes $\{\xi_n\}_{n\in\omega}$ we have that $p_{\xi_n} \Vdash \dot{w}(\eta) \subseteq \dot{v}_n(\eta)$ for each $\eta \ge \eta_n$.

Let $\delta = M \cap \kappa$. Thus we arranged that $\sup_{\xi<\kappa_0} \mu^{p_\xi} = \delta$ and any lower bound q for the sequence $(p_\xi)_{\xi<\kappa_0}$ will force that $\delta \in \operatorname{dom} \dot{w}$ and $\dot{w}(\delta) \subset \bigcap_{n\in\omega} v_n(\delta)$. However, we can find a lower bound q such that $q \Vdash \bigcap_{n<\omega} v_n(\delta) = \emptyset$. This contradiction finishes the proof. □

We now define another kind of gap notion for slaloms:

Definition 5.3.5. Let $(u_\alpha : \alpha < \lambda)$ and $(v_\beta : \beta < \mu)$ be two sequences of slaloms based on the same club set $C \subseteq \kappa$. We say that $\{(u_\alpha : \alpha < \lambda), (v_\beta : \beta < \mu)\}$ is a (λ, μ)-*tight gap of slaloms* if the following hold:

1. For all $\alpha < \alpha' < \lambda, \beta < \beta' < \mu$ and almost all ξ in C,

$$u_\alpha(\xi) \subset u_{\alpha'}(\xi) \subset v_{\beta'}(\xi) \subset v_\beta(\xi).$$

2. If w is a C-supported slalom such that $\forall \beta < \mu (w \subseteq^* v_\beta)$, then there is $\alpha < \lambda$ such that $w \subseteq^* u_\alpha$.

3. If w is a C-supported slalom such that $\forall\alpha < \lambda(u_\alpha \subseteq^* w)$, then there is $\beta < \mu$ such that $v_\beta \subseteq^* w$.

Question 5.3.1. Clearly, if $\{(u_\alpha : \alpha < \lambda), (v_\beta : \beta < \mu)\}$ is a (μ, λ)-tight gap of slaloms, then it is a gap. Do these notions coincide?

For the following result, we will use a higher analogue of Martin's axiom relativized to a certain class of posets. In order to do this, we will use the following definitions and results of Shelah (see Section 2.2 in [4]).

Definition 5.3.6. Let κ be an uncountable cardinal and $\mathbb{Q}$ be a forcing notion. We say that $\mathbb{Q}$ is *stationary κ^+-Knaster* if for every $\{p_i : i < \kappa^+\} \subseteq \mathbb{Q}$ there exists a club $E \subseteq \kappa^+$ and a regressive function f on $E \cap S^{\kappa^+}_\kappa$ such that for any $i, j \in E \cap S^{\kappa^+}_\kappa$, if $f(i) = f(j)$ then p_i and p_j are compatible.

Note that if a poset is stationary κ^+-Knaster then it is κ^+-cc.

Definition 5.3.7. Let κ be an uncountable cardinal. A forcing notion $\mathbb{Q}$ satisfies the $(*_\kappa)$-property, and we say it is *κ-good-Knaster*, if the following conditions hold:

1. $\mathbb{Q}$ is stationary κ^+-Knaster.
2. Any countable decreasing sequence of conditions in $\mathbb{Q}$ has a greatest lower bound.
3. Any two compatible conditions in $\mathbb{Q}$ have a greatest lower bound.
4. $\mathbb{Q}$ is $< \kappa$-closed.[4]

Finally, we can define our forcing axiom.

Definition 5.3.8. Let κ be an uncountable cardinal. We say that $\mathrm{MA}(\kappa\text{-good-Knaster})$ holds if and only if for every *κ-good-Knaster* poset $\mathbb{Q}$ and every collection $\mathcal{D}$ of dense sets of $\mathbb{Q}$ of size $< 2^\kappa$ there is a filter on $\mathbb{Q}$ intersecting all the sets in $\mathcal{D}$.

In the following, we will exploit the consistency of $\mathrm{MA}(\kappa\text{-good-Knaster})$ stated below.

Theorem 5.3.9. *Assume GCH. Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$ and $\lambda > \kappa$ such that $\lambda^{<\kappa} = \lambda$. Then, there is a cardinal preserving generic extension in which $2^\kappa = \lambda$ and $\mathrm{MA}(\kappa$-good-Knaster$)$ holds.*

[4]In the original definition of Shelah, the requirement is somewhat weaker, i.e. that $\mathbb{Q}$ is κ-strategically closed.

The proof is presented in the Appendix. We now prove that $\mathrm{MA}(\kappa$-good-Knaster$)$ implies the non-existence of certain kinds of tight gaps of slaloms.

Theorem 5.3.10. *Suppose that λ is a cardinal so that $\mathrm{cf}(\lambda) > \kappa^+$, $\lambda^{<\kappa} = \lambda$ and that $\mathrm{MA}(\kappa$-good-Knaster$)$ holds. Then there is no tight (λ, κ^+)-gap of slaloms based on a fixed club set $C \subseteq \kappa$.*

Proof. Suppose towards a contradiction that there is a (λ, κ^+)-tight gap of slaloms $\{(u_\alpha : \alpha < \lambda), (v_\beta : \beta < \kappa^+)\}$ based on (without loss of generality) κ and define the following forcing notion $\mathbb{Q}$. Conditions in $\mathbb{Q}$ are pairs $p = (\bar{s}, \sigma)$ where:

- $\sigma \subseteq \kappa^+$ and $|\sigma| < \kappa$.
- $\bar{s} = (s_i)_{i \in \sigma}$ is a sequence of partial slaloms with common domain, a fixed ordinal $\eta_p < \kappa$.
- If $i \in \sigma$, $\xi \in \eta_p$, then $s_i(\xi) \subseteq v_i(\xi)$.
- If $i^* = \sup(\sigma)$, then $i^* > |\sigma|$.

A condition $q = (\bar{t}, \tau)$ is said to extend the condition $p = (\bar{s}, \sigma)$ if:

- $\tau \supseteq \sigma$.
- For all $i \in \sigma$, $t_i \sqsupseteq s_i$.
- For all $i < i' \in \sigma$ and $\xi \in \eta_q \setminus \eta_p$, $t_i(\xi) \subset t_{i'}(\xi)$.
- For all $j \in \tau \setminus \sigma$ and $i \in \sigma$ such that $j < i$, there is $\xi \in \eta_q \setminus \eta_p$ such that $v_i(\xi) \subset t_j(\xi)$.

We want to use our assumption of $\mathrm{MA}(\kappa$-good-Knaster$)$ for this poset and some (to be defined) collection of dense sets.

Claim. *$\mathbb{Q}$ is stationary κ^+-Knaster and $< \kappa$-closed.*

Proof. Suppose $\mathcal{X} = \{p_\alpha : \alpha < \kappa^+\}$ is a sequence of conditions in $\mathbb{Q}$. We want to show that there is a club $E \subseteq \kappa^+$ and a regressive function $f : E \cap S^{\kappa^+}_\kappa \to \mathcal{X}$ such that, if $f(i) = f(j)$ then p_i and p_j are compatible.

First, we use the pigeonhole principle and the Δ-system lemma in order to assume, without loss of generality that for all $\gamma < \kappa^+$ the following hold:

- $\eta_p = \eta < \kappa$.

- $|\sigma_\gamma|= \lambda^* < \kappa$.
- $\sigma_\gamma \cap \sigma_{\gamma'} = \epsilon$.
- If $\sigma_\gamma = \{i_{\gamma,l} : l < \lambda^*\}$ (increasingly ordered), then $s^\gamma_l = s^*_\gamma$ for all $l < \lambda^*$. Here and throughout the proof s^γ_l denotes $s_{i_{\gamma,l}}$.
- The sequence $i_{\gamma,l}$ is strictly increasing in the first coordinate, for $l \notin \epsilon$.

Given $\gamma < \gamma' < \kappa^+$, we now claim that $p_\gamma = (\sigma_\gamma, \bar{s}_\gamma)$ and $p_{\gamma'} = (\sigma_{\gamma'}, \bar{s}_{\gamma'})$ are compatible. If true, we can then define $E = \kappa^+$ and $f : S^{\kappa^+}_\kappa \to \kappa^+$ to be the constant function with value 0 and we get the stationary κ^+-Knaster condition.

To prove the claim, choose an ordinal $\zeta \geq \eta$ such that, for each $\xi \geq \zeta$:

$$\{v_{i_{\rho,l}}(\xi) : \rho \in \{\gamma, \gamma'\} \wedge l < \lambda^*\}$$

is $\subset$-decreasing (this is possible because the $i_{\gamma,l}$ are increasing and the way the v's are arranged).

Moreover, we can choose ζ so that for all $\xi \geq \zeta$, $|v_{i_{\gamma',\lambda^*}}(\xi)|+\lambda^* > |v_{i_{\gamma,\lambda^*}}(\xi)|$.

Define a condition $q = (\bar{t}, \tau)$ as follows: $\tau = \sigma_\gamma \cup \sigma_{\gamma'}$ and $\bar{t} = (t_j)_{j\in\tau}$. Put $\zeta = \mathrm{dom}(t_i)$ for all i and recall the enumeration of σ_γ and $\sigma_{\gamma'}$ we have fixed above.

We consider the following cases:

- If $j \in \epsilon$, i.e $j = i_{\gamma,l}$ for $l < |\epsilon|$, then define partial slalom t_j as follows:

$$t_j(\xi) = \begin{cases} s^\gamma_j(\xi) & \text{if } \xi < \eta \\ v_j(\xi) & \text{if } \eta \leq \xi < \zeta \end{cases}$$

- If $j = i_{\gamma,l}$, for $|\epsilon|\leq l < \lambda^*$, then define partial slalom t_j as follows:

$$t_j(\xi) = \begin{cases} s^\gamma_j(\xi) & \text{if } \xi < \eta \\ v_{i_{\gamma',l'}}(\xi) & \text{if } \eta \leq \xi < \zeta \text{ and } l' < l \text{ is the supremum so that } v_{i_{\gamma',l'}} \subseteq^* v_j \end{cases}$$

- If $j = i_{\gamma',l}$, for $|\epsilon|\leq l < \lambda^*$, define analogously as in the item above, i.e.

$$t_j(\xi) = \begin{cases} s^{\gamma'}_j(\xi) & \text{if } \xi < \eta \\ v_{i_{\gamma,l'}}(\xi) & \text{if } \eta \leq \xi < \zeta \text{ and } l' < l \text{ is the supremum so that } v_{i_{\gamma,l'}} \subseteq^* v_j \\ v_j(\xi) & \text{if } \eta \leq \xi < \zeta \text{ and } \{l' < l : v_{i_{\gamma,l'}} \subseteq^* v_j\} = \emptyset \end{cases}$$

Then $q \leq p_\gamma$ and $q \leq p_{\gamma'}$. $\square$

It remains to prove that the poset $\mathbb{Q}$ has properties (2), (3) and (4) from Definition 5.3.7. Let $\{p_\alpha\}_{\alpha<\gamma}$ be a $<$-decreasing sequence of conditions in $\mathbb{Q}$, where $p_\alpha = (\bar{s}_\alpha, \sigma_\alpha)$. Then there is a canonical lower bound $p = (\bar{s}, \sigma)$ where $\sigma = \bigcup_{\alpha<\gamma} \sigma_\alpha$ (which is still a set of size $< \kappa^+$) and $\bar{s}$ is defined as follows: $\bar{s}$ is a sequence of partial slaloms $(s_i)_{i\in\sigma}$ with domain $\eta = \sup_{\alpha<\gamma} \eta_\alpha < \kappa$ such that $s_i(\xi) = \bigcup_{\alpha<\gamma} s_i^\alpha(\xi)$ when $s_i^\alpha(\xi)$ is defined (i.e. when $i \in \sigma_\alpha$). This implies that properties (2) and (4) hold. Property (3) hods, as if $p = (\bar{s}, \sigma)$ and $q = (\bar{t}, \tau)$ are compatible, then a canonical lower bound $r = (\bar{u}, \nu)$ has the form $\nu = \sigma \cup \tau$, while the third and fourth conditions in the definition of our poset determine how r must be defined.

Since by hypothesis $\mathrm{MA}(\kappa$-good-Knaster) holds, there is a generic $G \subseteq \mathbb{Q}$ intersecting the following dense sets. Let $i \in \kappa^+$ and $\eta < \kappa$.

$$\mathcal{D}_{i,\eta} = \{p \in \mathbb{Q} : \sigma_p \nsubseteq i \wedge \forall q \in \mathbb{Q}\,(q \leq p \rightarrow \sigma_q \subseteq i) \wedge \eta_p \geq \eta\}$$

The generic G adds first of all an unbounded subset of κ^+ given by $\Sigma_G = \bigcup\{\sigma_p : p \in G\}$. Also, it generically adds κ^+-many slaloms $\{w_G^i : i \in \Sigma_G\}$, where $w_G^i = \bigcup\{s_i^p : p \in G$ and $(\bar{s}_p)_i = s_i^p\}$. These slaloms satisfy that for all $i < j \in \Sigma_G$ and for almost all $\xi \in \kappa$ $w_G^i(\xi) \subset w_G^j(\xi)$.

Moreover, we have that for all $i < j \in \Sigma_G$ and for almost all $\xi \in \kappa$

$$w_G^i(\xi) \subset w_G^j(\xi) \subset v_j(\xi) \subset v_i(\xi).$$

Now, using the hypothesis that $\{(u_\alpha : \alpha < \lambda), (v_\beta : \beta < \kappa^+)\}$ is a (λ, κ^+)-tight gap of slaloms, given $i \in \Sigma_G$, we can find $\alpha(i) < \lambda$ such that, for almost all $\xi \in \kappa$ $w_G^i(\xi) \subset u_{\alpha(i)}(\xi)$.

Let $\alpha^\star = \sup\{\alpha(i) : i \in \Sigma_G\}$. Then for each $i \in \Sigma_G$ we can find $\eta_i < \kappa$ such that for all $\xi > \eta_i$:

$$w_G^i(\xi) \subset u_{\alpha^\star}(\xi) \subset v_i(\xi)$$

Again, using the pigeonhole principle, we can assume without loss of generality that $\eta_i = \eta^*$. Then we can pick a condition $p = (\sigma, \bar{s}) \in G$ so that $j \in \sigma$ where $j \in \Sigma_G$ and $|j \cap \Sigma_G| \geq \kappa$ and $\eta_p > \eta^*$.

Since $|\sigma| < \kappa$, we can choose $i \in \Sigma_G \cap (j \setminus \sigma)$ and $q = (\tau, \bar{t}) \leq p$ for which $i \in \tau$. Then, by the definition of the forcing $\mathbb{Q}$, there is $\eta_p \leq \zeta < \eta_q$ such that $v_j(\zeta) \subset t^i(\zeta) = w_G^i(\zeta)$. But then we get $v_j(\zeta) \subset w_G^i(\zeta) \subset u_{\alpha^\star}(\zeta) \subset v_j(\zeta)$ which is a contradiction. $\square$

5.4 On $\mathfrak{p}(\kappa)$ and $\mathfrak{p}_{\mathrm{cl}}(\kappa)$

The definitions of $\mathfrak{p}(\kappa)$ and $\mathfrak{t}(\kappa)$ invoke all κ-complete filters (resp. towers) on κ, without giving any additional structural information. Thus it makes sense to first consider smaller classes of filters that may be better understood. One natural way of classifying κ-complete filters is to consider larger filters in which they simultaneously embed. This leads to the following definition:

Definition 5.4.1. Let $\mathcal{F}$ be a κ-complete filter on κ. Then

$$\mathfrak{p}_{\mathcal{F}}(\kappa) := \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{F} \wedge \mathcal{B} \text{ has no pseudointersection}\}$$

and

$$\mathfrak{t}_{\mathcal{F}}(\kappa) := \min\{|\mathcal{T}| : \mathcal{T} \subseteq \mathcal{F} \wedge \mathcal{T} \text{ is a tower}\}$$

whenever these are defined.

Note that $\mathfrak{p}_{\mathcal{F}}(\kappa)$ is defined exactly when $\mathcal{F}$ has no pseudointersection. One of the most interesting examples is $\mathfrak{p}_{\mathrm{cl}}(\kappa) = \mathfrak{p}_{\mathcal{C}}(\kappa)$ where $\mathcal{C}$ is the club filter on κ. Our goal in this section is to study the relationship of $\mathfrak{p}(\kappa)$ to $\mathfrak{p}_{\mathrm{cl}}(\kappa)$. We start with some straightforward observations.

Observation 5.4.2. Let $\mathcal{F}$ be a κ-complete filter on κ such that $\mathfrak{p}_{\mathcal{F}}(\kappa)$ is defined, then

1. $\kappa^+ \leq \mathfrak{p}(\kappa) \leq \mathfrak{p}_{\mathcal{F}}(\kappa)$,
2. whenever $\mathfrak{t}_{\mathcal{F}}$ is defined, $\mathfrak{p}_{\mathcal{F}}(\kappa) \leq \mathfrak{t}_{\mathcal{F}}(\kappa) \leq \mathfrak{t}(\kappa)$,
3. $\mathfrak{p}_{\mathrm{cl}}(\kappa) = \mathfrak{t}_{\mathrm{cl}}(\kappa) = \mathfrak{b}(\kappa)$.

Proof. (1) and (2) follow immediately from the definitions. (3) has been shown in [48]. Let us recall the argument. First note that $\mathfrak{p}_{\mathrm{cl}}(\kappa)$ as well as $\mathfrak{t}_{\mathrm{cl}}(\kappa)$ are defined. To see that they are equal, let $\lambda = \mathfrak{p}_{\mathrm{cl}}(\kappa)$ and suppose that $(C_\alpha : \alpha < \lambda)$ is a family of clubs in κ with no pseudointersection of size κ. Build a sequence $(D_\alpha : \alpha < \lambda)$ of clubs so that D_β is club and a pseudointersection of $\mathcal{E}_\beta = \{D_\alpha : \alpha < \beta\} \cup \{C_\alpha : \alpha \leq \beta\}$ (note the closure of a pseudointersection is still a pseudointersection). This is possible, since $\mathcal{E}_\beta$ is a family of clubs of size $< \mathfrak{p}_{\mathrm{cl}}(\kappa)$. Now $(D_\alpha : \alpha < \lambda)$ is a witness for $\mathfrak{t}_{\mathrm{cl}}(\kappa) = \lambda$. To see that $\mathfrak{p}_{\mathrm{cl}}(\kappa) = \mathfrak{b}(\kappa)$ consider the map that sends a function $f \in \kappa^\kappa$ to $C_f = \{\alpha < \kappa : f''\alpha \subseteq \alpha\}$ and the map sending a club C to s_C. □

The consistency of $\mathfrak{p}(\kappa) < \mathfrak{b}(\kappa)$ was first shown in [59]. The argument for showing that $\mathfrak{p}(\kappa)$ stays small in the generic extension, relies on the following theorem which is the main result of the mentioned paper.

Theorem 5.4.3. *If $\kappa \leq \mu < \mathfrak{t}(\kappa)$ then $2^\mu = 2^\kappa$.*

This theorem mirrors the situation at ω. In order to keep $\mathfrak{p}(\kappa)$ smaller than μ one only needs to ensure that 2^μ will be strictly larger than 2^κ in the generic extension. Using counting of names it can be seen that this will usually not be a problem (starting with an appropriate ground model). Thus starting from GCH, having regular targets $\mu < \lambda$ for $\mathfrak{p}(\kappa)$ and $\mathfrak{b}(\kappa)$, we first use Cohen forcing to ensure that $2^\mu = \lambda^+$ and then we increase $\mathfrak{b}(\kappa)$ to λ with Hechler forcing and simultaneously diagonalize κ-complete filters of size $< \mu$. In this extension $2^\mu > 2^\kappa$ and we have ensured that $\mathfrak{p}(\kappa)$ does not blow up.

We will present a more natural approach that amounts to showing that certain witnesses for $\mathfrak{p}(\kappa)$ can be preserved while increasing $\mathfrak{b}(\kappa)$. This approach leaves more freedom for cardinal arithmetic. On the other hand, up until now, we only know how to apply it for a construction resulting in a model with $\mathfrak{p}(\kappa) = \kappa^+$.

Let us introduce the forcing used to increase $\mathfrak{b}(\kappa)$ (i.e. $\mathfrak{p}_{\mathrm{cl}}(\kappa)$) or $\mathfrak{p}_{\mathcal{F}}(\kappa)$ more generally for certain classes of $\mathcal{F}$. This poset has been used greatly in the past.

Definition 5.4.4. Let $\mathcal{F}$ be a base for a κ-complete filter on κ. The forcing $\mathbb{M}(\mathcal{F})$ consists of conditions (a, F) where $a \in [\kappa]^{<\kappa}$ and $F \in \mathcal{F}$. The order is defined by $(b, E) \leq (a, F)$ iff $E \subseteq F$ and $b \setminus a \subseteq F$.

Fact. *$\mathbb{M}(\mathcal{F})$ is κ-closed and κ^+-cc (in fact κ-centered with cannonical lower bounds).*

In what follows, $\mathcal{C}$ will always refer to the collection of clubs from a specific model, which should always be clear from context.

Our approach, that we announced earlier, will consist of showing that a $< \kappa$ support iteration of $\mathbb{M}(\mathcal{C})$ will not add a pseudointersection to a previously added collection of (more than κ many) Cohen reals. As a warm up and an introduction to the argument we will first show that this is the case when forcing with $\mathbb{M}(\mathcal{C})$ once.

Theorem 5.4.5. *Let κ be uncountable regular and $\kappa^{<\kappa} = \kappa$. Suppose $\langle y_\alpha : \alpha < \kappa^+ \rangle$ is a sequence of Cohen reals added over V and that c is a $\mathbb{M}(\mathcal{C})$ generic over $V[\bar{y}]$. Then in $V[\bar{y}][c]$, the filter generated by $\{y_\alpha : \alpha < \kappa^+\}$ has no pseudointersection.*

We write $\mathbb{C}_{\kappa^+}$ for the $< \kappa$-support product of κ^+ many copies of $2^{<\kappa}$, the forcing for adding a κ-Cohen real. Let us first check the following:

Lemma 5.4.6. *Whenever $\langle y_\alpha : \alpha < \kappa^+ \rangle$ is a $\mathbb{C}_{\kappa^+}$ generic sequence, then $\{y_\alpha : \alpha \in \kappa^+\}$ has the SIP in any further extension by κ-closed forcing.*

Proof. Let $\Gamma \in [\kappa^+]^{<\kappa}$ be in any extension of $V^{\mathbb{C}_\kappa}$ by a κ-closed forcing notion. Then $\Gamma \in V$. By genericity over V we may show that $\bigcap_{\alpha \in \Gamma} y_\alpha$ is unbounded in κ. More precisely, let $p \in \mathbb{C}_{\kappa^+}$ and $\varepsilon \in \kappa$ be arbitrary. Let $\delta > \sup_{i \in \operatorname{dom} p}(\operatorname{lth} p(i))$ and extend p to q such that $q(i)(\delta) = 1$ for every $i \in \Gamma$. □

Proof of Theorem 5.4.5. In $V[\bar{y}]$ assume $\dot{x}$ is a $\mathbb{M}(\mathcal{C})$ name for an element of $[\kappa]^\kappa$. Consider the set

$$X = \{(a, \alpha) : a \in [\kappa]^{<\kappa}, \alpha < \kappa, \exists C \in \mathcal{C}((a, C) \Vdash \alpha \in \dot{x})\}.$$

Then $X \in V[\langle y_\alpha : \alpha < \delta\rangle]$ for some $\delta < \kappa^+$. We want to show that $\dot{x}[c] \not\subseteq^* y_\delta$. First recall that y_δ is in fact Cohen over $V[\langle y_\alpha : \delta \neq \alpha < \kappa^+\rangle]$. Thus for the proof we may simply assume that $X \in V$ and show that $\dot{x}[c] \not\subseteq^* y$ where y is Cohen over V and c is $\mathbb{M}(\mathcal{C})$ generic over $V[y]$.

Suppose in $V[y]$ that (a, C) is an arbitrary condition in $\mathbb{M}(\mathcal{C})$. We have that $a \in V$ and there is some name $\dot{C} \in V$ so that $\Vdash$"$\dot{C}$ is club" in Cohen forcing and $\dot{C}[y] = C$.

Now suppose that $s \in 2^{<\kappa}$ is an arbitrary condition in Cohen forcing. Now let us define two decreasing sequences $\{p_i^0 : i < \kappa\}$ and $\{p_i^1 : i < \kappa\}$ in Cohen forcing such that the following holds:

- $p_0^0 = p_0^1 = s$,
- if $\bigcup_{i<\kappa} p_i^0 = f_0$ and $\bigcup_{i<\kappa} p_i^1 = f_1$ then $f_0^{-1}(\{1\}) \cap f_1^{-1}(\{1\}) = s^{-1}(\{1\})$,
- the sets $\tilde{C}^0 = \{\alpha : \exists i(p_i^0 \Vdash \alpha \in \dot{C})\}$ and $\tilde{C}^1 = \{\alpha : \exists i(p_i^1 \Vdash \alpha \in \dot{C})\}$ are clubs.

The sequences $\bar{p}^0$ and $\bar{p}^1$ are simply interpreting sequences for $\dot{C}$ below s. But we additionally ensure that the sets defined by $\bigcup_{i<\kappa} p_i^0$ and $\bigcup_{i<\kappa} p_i^1$ are disjoint up to their common initial part s. Call these sets $y^0 \subseteq \kappa$ and $y^1 \subseteq \kappa$

Let $\tilde{C} = \tilde{C}^0 \cap \tilde{C}^1$. Recall that $\tilde{C}$ will still be club in $V[y]$. Thus there is $b \in [\tilde{C}]^{<\kappa}$ and $\alpha > \sup\operatorname{dom}(s)$ so that $\min b > a$ and $(a \cup b, \alpha) \in X$. As $\bigcup_{i<\kappa} p_i^0$ and $\bigcup_{i<\kappa} p_i^1$ define disjoint sets there is at least one $j \in 2$ so that α is not in y^j. Say wlog $j = 0$. Now we can extend s to some $t = p_i^0$ for some i such that $p_i^0 \Vdash b \subseteq \dot{C}$, $\alpha \in \operatorname{dom}(t)$ and $t(\alpha) = 0$.

Thus by genericity we shown that back in $V[y]$ we can extend (a, C) to $(a \cup b, C')$ so that $(a \cup b, C') \Vdash \alpha \in \dot{x}$ but $\alpha \notin y$. Now by genericity of c we know that $\dot{x}[c] \not\subseteq^* y$. □

Now we are going to consider the more general case of iterating $\mathbb{M}(\mathcal{C})$ many times with $< \kappa$-support. For an ordinal i we will write $\mathbb{M}(\mathcal{C})_i$ for the i-length $< \kappa$-support iteration of $\mathbb{M}(\mathcal{C})$.

Theorem 5.4.7. *(GCH) For any regular uncountable $\kappa < \lambda$, where $\kappa = \kappa^{<\kappa}$, there is a κ-closed, κ^+-cc forcing extension in which $\mathfrak{p}(\kappa) = \kappa^+ < \mathfrak{p}_{\mathrm{cl}}(\kappa) = \lambda = 2^\kappa$.*

Proof. We are going to first add κ^+ many (κ-)Cohen reals $\langle y_\alpha : \alpha < \kappa^+\rangle$ and then iteratively diagonalize the club filter for λ many steps. Thus the poset that we are using is $\mathbb{P} = \mathbb{C}_{\kappa^+} * \dot{\mathbb{M}}(\mathcal{C})_\lambda$, where $\dot{\mathbb{M}}(\mathcal{C})_\lambda$ is a $\mathbb{C}_{\kappa^+}$ name for the $< \kappa$-support iteration of $\mathbb{M}(\mathcal{C})$ of length λ. This forcing notion is κ-closed, has the κ^+-cc and forces $2^\kappa = \lambda$ by a counting argument. Also it is clear that $V^{\mathbb{P}} \models \mathfrak{p}_{\mathrm{cl}}(\kappa) = \lambda$. Thus we are left with showing that $V^{\mathbb{P}} \models \mathfrak{p}(\kappa) = \kappa^+$.

Let us make some remarks on the notation that we will use.

– We will assume that conditions in $\mathbb{M}(\mathcal{C})_\lambda$ always have the form $(\bar{a}, q)$, where

  - $\bar{a} = \langle a_i : i \in I\rangle$, $I \in [\lambda]^{<\kappa}$, $a_i \in [\kappa]^{<\kappa}$,
  - q is a function with $\operatorname{dom} q = I$ and $q(i)$ is a $\mathbb{M}(\mathcal{C})_i$ name for a club for every $i \in I$.

  A pair $(\bar{a}, q)$ as above is naturally interpreted as the condition $\langle \check{a}_i, q(i)\rangle_{i\in I}$.

– Similarly we will assume that conditions in $\mathbb{C}_{\kappa^+} * \dot{\mathbb{M}}(\mathcal{C})_\lambda$ have the form $(p, \bar{a}, \dot{q})$, where

  - $p \in \mathbb{C}_{\kappa^+}$
  - $\bar{a} \in V$,
  - $\dot{q}$ is a $\mathbb{C}_{\kappa^+}$ name for an object as above.

  It is easy to see, using κ-closure, that conditions of this form are dense in $\mathbb{P}$.

– A nice $\mathbb{M}(\mathcal{C})_\lambda$-name $\dot{x}$ for an element of $P(\kappa)$ has the form

$$\bigcup_{\alpha<\kappa} A_\alpha \times \{\check{\alpha}\}$$

  where A_α is an antichain in $\mathbb{M}(\mathcal{C})_\lambda$ (thus has size $\leq \kappa$) and for every $(\bar{a}, q) \in A_\alpha$ and $i \in \operatorname{dom} q$, $q(i)$ is a nice $\mathbb{M}(\mathcal{C})_i$-name. Thus we define nice $\mathbb{M}(\mathcal{C})_i$-names for subsets of κ inductively on $i \in \lambda$.

– It is well known that for any $\mathbb{M}(\mathcal{C})_\lambda$-name $\dot{y}$ for a subset of κ, there is a nice name $\dot{x}$ so that $\Vdash \dot{y} = \dot{x}$.

– By induction on nice names we see that, $|\operatorname{trcl}(\dot{x})| \leq \kappa$. Namely, assume this is known for nice $\mathbb{M}(\mathcal{C})_i$-names for every $i < j$. Let $\dot{x}$ be a nice $\mathbb{M}(\mathcal{C})_j$-name. Then $\dot{x} = \bigcup_{\alpha<\kappa} A_\alpha \times \{\check{\alpha}\}$ where each A_α is a set of $\mathbb{M}(\mathcal{C})_j$ conditions of size at most κ. For each condition $(\bar{a}, p) \in A_\alpha$, $|\operatorname{dom} p| < \kappa$. For each $i \in \operatorname{dom}(p)$, $p(i)$ is a nice $\mathbb{M}(\mathcal{C})_i$-name which, by assumption, has transitive closure of size at most κ.

Claim. *$\{y_\alpha : \alpha \in \kappa^+\}$ has no pseudointersection after forcing with $\mathbb{M}(\mathcal{C})_\lambda$.*

Proof. In $V^{\mathbb{C}_{\kappa^+}} = V[\bar{y}]$, let $\dot{x}$ be a nice $\mathbb{M}(\mathcal{C})_\lambda$ name for an element of $[\kappa]^\kappa$. Then there is $\gamma < \kappa^+$, such that $\dot{x} \in V[\langle y_\alpha : \alpha \in \kappa, \alpha \neq \gamma\rangle]$. We will show that $\dot{x}$ can not be almost contained in y_γ. Without loss of generality we may assume that $\dot{x} \in V$ and that we are adding a single Cohen real $y = y_\gamma$ over V (by putting $V[\langle y_\alpha : \alpha \in \kappa, \alpha \neq \gamma\rangle]$ as the new ground model) and then we are forcing with $\mathbb{M}(\mathcal{C})_\lambda$ in $V[y]$.

Now suppose that $(p, \bar{a}, \dot{q}) \Vdash \dot{x} \setminus \varepsilon \subseteq \dot{y}$, where $(p, \bar{a}, \dot{q}) \in \mathbb{C} * \dot{\mathbb{M}}(\mathcal{C})_\lambda$ and $\varepsilon \in \kappa$. Let y be $\mathbb{C}$ generic over V with p in the generic filter. Define $y' \in 2^\kappa$ so that $y'(i) = p(i) = y(i)$ for $i \in \operatorname{dom} p$ and $y'(i) = 1 - y(i)$ for $i \in \kappa \setminus \operatorname{dom} p$. It is well known that y' is also generic over V with p in it's generic filter. Moreover $V[y] = V[y'] =: W$. But note that $q := \dot{q}[y] \neq \dot{q}[y'] =: q'$ is very much possible. Still in W, $(\bar{a}, q)$ and $(\bar{a}, q')$ are compatible. Namely we may define $r\colon \operatorname{dom} q \to W$ by putting $r(i)$ a $\mathbb{M}(\mathcal{C})_i$ name for $q(i) \cap q'(i)$. By induction we see that for any $i \in \operatorname{dom} q$,

$$(\bar{a} \restriction i, r \restriction i) \leq (\bar{a} \restriction i, q \restriction i), (\bar{a} \restriction i, q' \restriction i)$$

and that

$$r \restriction i \Vdash q(i), q'(i) \text{ are clubs.}$$

Thus indeed $r(i)$ is a $\mathbb{M}(\mathcal{C})_i$ name for a club, so $(\bar{a}, r)$ is a condition and $(\bar{a}, r) \leq (\bar{a}, q), (\bar{a}, q')$. Now let $(\bar{b}, s) \leq (\bar{a}, r)$ and $\delta \in \kappa \setminus \varepsilon$ so that

$$(\bar{b}, s) \Vdash \delta \in \dot{x}.$$

Since $y \cap y' \subseteq \varepsilon$, $\delta \notin y$ or $\delta \notin y'$. Say $\delta \notin y$. Then whenever G is $\mathbb{M}(\mathcal{C})_\lambda$ generic over W with $(\bar{b}, s) \in G$, $W[G] \models \dot{x}[G] \setminus \varepsilon \not\subseteq y$. At the same time, $(p, \bar{a}, \dot{q})$ is in the corresponding $\mathbb{C} * \mathbb{M}(\mathcal{C})_\lambda$ generic over V. This gives a contradiction. Similarly when $\delta \notin y'$. □

□

Analyzing the proof of the above result, we see that this result can be extended to a more general class of filters.

Theorem 5.4.8. *(GCH) For any regular uncountable $\kappa < \lambda$, where $\kappa = \kappa^{<\kappa}$, there is a κ-closed, κ^+-cc forcing extension in which $\mathfrak{p}(\kappa) = \kappa^+ < \mathfrak{p}_{\mathcal{F}}(\kappa) = \lambda = 2^\kappa$ for any κ-complete filter $\mathcal{F}$ on κ that is ordinal definable over $H(\kappa^+)$.*

We say that $\mathcal{F}$ is ordinal definable over $H(\kappa^+)$ if there is a formula φ in the language of set theory and finitely many ordinals $\alpha_0 < \cdots < \alpha_{n-1} < \kappa^+$ so that

$$x \in \mathcal{F} \leftrightarrow H(\kappa^+) \models \varphi(x, \bar{\alpha}).$$

For example, $\mathcal{C}$ is ordinal definable over $H(\kappa^+)$.

Proof. Let $\langle \varphi_\xi(x, \bar{\alpha}_\xi) : \xi \in \kappa^+ \rangle$ enumerate all formulas in one free variable x and parameters $\bar{\alpha} = (\alpha_0, \dots, \alpha_k) \in (\kappa^+)^{<\omega}$ in the language $\{\in\}$.

As before we first add κ^+ many Cohen reals using $\mathbb{C}_{\kappa^+}$. Then in $V^{\mathbb{C}_{\kappa^+}}$ we define an iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i : i < \lambda \rangle$ with $\mathbb{Q}_i = \prod_{\xi<\kappa^+} \mathbb{M}(\mathcal{F}_\xi)$ where

$$\mathcal{F}_\xi = \{x \in [\kappa]^\kappa : H(\kappa^+)^{V^{\mathbb{P}_i}} \models \varphi_\xi(x, \bar{\alpha}_\xi)\}$$

if this defines a κ-complete filter (in $V^{\mathbb{P}_i}$) or

$$\mathcal{F}_\xi = \{\kappa\}$$

else.

Again we consider conditions in $\mathbb{P}_\lambda$, as pairs $(\bar{a}, q)$ where $\operatorname{dom} a \in [\kappa^+ \cdot \lambda]^{<\kappa}$, $a_{\kappa^+ \cdot i + \xi} \in [\kappa]^{<\kappa}$ and q is a function with domain $\operatorname{dom} a$ so that $q(\kappa^+ \cdot i + \xi)$ is a $\mathbb{P}_i$ name for an element of $\mathcal{F}_\xi$. Similarly we define the notion of nice names.

It is crucial to note that $\mathbb{P}_\lambda$ only depends on the model $V^{\mathbb{C}_{\kappa^+}}$ and not on the particular set of generic Cohen reals. Then using the same argument as before we see that $\mathfrak{p}(\kappa) = \kappa^+$ in $V^{\mathbb{C}_{\kappa^+} * \dot{\mathbb{P}}_\lambda}$.

Now suppose $\mathcal{F}$ is ordinal definable over $H(\kappa^+)$ in $V^{\mathbb{C}_{\kappa^+} * \dot{\mathbb{P}}_\lambda}$ and $\mathfrak{p}_{\mathcal{F}}(\kappa)$ is defined. Say $\mathcal{F}$ is defined by φ_ξ. Let $\mathcal{B} \subseteq \mathcal{F}$ with $|\mathcal{B}| < \lambda$. Then there is $i < \lambda$ so that $\mathcal{B} \subseteq V^{\mathbb{C}_{\kappa^+} * \dot{\mathbb{P}}_i}$. Moreover we find $j \geq i$ so that $(H(\kappa^+)_j, \in) \preccurlyeq (H(\kappa^+)_\lambda, \in)$, where $H(\kappa^+)_j = \{x \in H(\kappa^+) : x \in V^{\mathbb{C}_{\kappa^+} * \dot{\mathbb{P}}_j}\}$. To see this just note that $|H(\kappa^+)_i| < \lambda$ for every $i < \lambda$. Thus we can find the $< \lambda$ many required Skolem-witnesses over $H(\kappa^+)_i$ in $H(\kappa^+)_{S(i)}$ for some $S(i) < \lambda$. Applying S recursively κ^+ many times, by taking suprema at limits, yields the desired situation (since no new elements of $H(\kappa^+)$ are introduced in limits of cofinality κ^+). In $V^{\mathbb{C}_{\kappa^+} * \dot{\mathbb{P}}_j}$, $\mathcal{F}_\xi$ is a κ-complete filter on κ with $\mathcal{B} \subseteq \mathcal{F}_\xi$ and $\mathbb{Q}_j$ adds a pseudointersection to $\mathcal{B}$. □

Theorem 5.4.9. *($\mathfrak{p}(\kappa) = 2^\kappa$) Let $\mathcal{P}$ be a collection of κ^+-cc forcing notions, each of size $\leq 2^\kappa$ and $|\mathcal{P}| \leq 2^\kappa$. Then there is a tower which is indestructible by any $\mathbb{P} \in \mathcal{P}$.*

Lemma 5.4.10. *Let $\mathfrak{p}(\kappa) = \lambda$. There is a map $\varphi\colon 2^{<\lambda} \to [\kappa]^\kappa$ so that for each $f \in 2^\lambda$, $\langle \varphi(f \restriction \alpha) : \alpha < \lambda \rangle$ is a tower and $\varphi(s^\frown 0) \cap \varphi(s^\frown 1) = \emptyset$ for every $s \in 2^{<\lambda}$.*

Proof. See the proof of Theorem 7 in [59]. □

Proof of Theorem 5.4.9. Let φ be as in Lemma 5.4.10 and $\lambda = 2^\kappa$. Recall that if $\mathbb{P}$ is κ^+-cc then we can assume that all $\mathbb{P}$ names for elements of $[\kappa]^\kappa$ are of size at most κ. Enumerate all triples $\langle \mathbb{P}_\alpha, p_\alpha, \dot{x}_\alpha : \alpha < \lambda \rangle$ where $\mathbb{P}_\alpha \in \mathcal{P}$, $p_\alpha \in \mathbb{P}_\alpha$ and $\dot{x}_\alpha$ is a $\mathbb{P}_\alpha$ name for an element of $[\kappa]^\kappa$. We recursively define $f \in 2^\lambda$ as follows:

Given $s_\alpha \in 2^\alpha$, let $y_0 = \varphi(s_\alpha^\frown 0)$ and $y_1 = \varphi(s_\alpha^\frown 1)$. As $y_0 \cap y_1 = \emptyset$ we have that $p_\alpha \Vdash \dot{x}_\alpha \subseteq^* y_0 \wedge \dot{x}_\alpha \subseteq^* y_1$ is impossible. Thus for some $i \in 2$ we have that there is $q_\alpha \leq p_\alpha$ so that $q_\alpha \Vdash \dot{x}_\alpha \not\subseteq^* y_i$. Let $s_{\alpha+1} = s_\alpha^\frown i$. At limits we let $s_\alpha = \bigcup_{\xi<\alpha} s_\xi$. Finally $f := \bigcup_{\alpha<\lambda} s_\alpha$.

The tower defined by f is as required. Namely given $\mathbb{P} \in \mathcal{P}$, $p \in \mathbb{P}$ and $\dot{x}$ a $\mathbb{P}$-name for an unbounded subset of κ, say $(\mathbb{P}, p, \dot{x}) = (\mathbb{P}_\alpha, p_\alpha, \dot{x}_\alpha)$, we have that $q_\alpha \leq p_\alpha$ forces that $\dot{x}$ is not almost contained in $\varphi(s_\alpha)$. □

5.5 Appendix

5.5.1 Consistency of MA(κ-good-Knaster)

Finally, we present the proof of the generalized Martin's Axiom for posets with property $(*_\kappa)$ that we applied in Section 5.3. The proof is based on the following iteration theorem but otherwise resembles the classical proof of Martin's Axiom.

Theorem 5.5.1. *(Shelah, 1976; see [58]). Let κ be an uncountable cardinal and $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ be a $< \kappa$-support iteration such that for every $\alpha < \delta$:*

$$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha \textit{ satisfies property } (*_\kappa)$$

Then $\mathbb{P}_\delta$ is stationary κ^+-Knaster.

Proof of Theorem 5.3.9. We define a $< \kappa$-support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \lambda)$ such that for all $\alpha < \lambda$:

- $\Vdash \dot{\mathbb{Q}}_\alpha$ is has the property $(*_\kappa)$.
- $\Vdash |\dot{\mathbb{Q}}_\alpha| < \lambda$.

Since by Theorem 5.5.1 the poset $\mathbb{P} = \mathbb{P}_\lambda$ is stationary κ^+-Knaster and it is $< \kappa$-closed, $\mathbb{P}$ preserves cardinals. Also, since λ is regular and $\lambda^\kappa = \lambda$, we have $|\mathbb{P}_\alpha| \leq \lambda$.

Define $\dot{\mathbb{Q}}_\alpha$ by induction on $\alpha < \lambda$ as follows. Fix a bookkeeping function $\pi : \lambda \to \lambda \times \lambda$ such that $\pi(\alpha) = (\beta, \gamma)$ implies $\beta \leq \alpha$. If we have defined $\dot{\mathbb{Q}}_\beta$ for all $\beta < \alpha$ and $\pi(\alpha) = (\beta, \gamma)$, we can look at the γ-th $\mathbb{P}_\beta$-name $\dot{\mathbb{Q}}$ in $V^{\mathbb{P}_\beta}$ for a poset of size $< \lambda$ with the property $(*_\kappa)$. Define $\mathbb{Q}_\alpha = \dot{\mathbb{Q}}$.

First, we will show that that $V^{\mathbb{P}} \models \mathrm{MA}^\kappa(\kappa\text{-good-Knaster}_{<\lambda}) \wedge 2^\kappa = \lambda$, where $\mathrm{MA}(\kappa\text{-good-Knaster}_{<\lambda})$ is the restriction of $\mathrm{MA}(\kappa\text{-good-Knaster})$ to posets of cardinality stricly smaller than λ.

Let $\dot{\mathbb{R}}$ be a $\mathbb{P}$-name for a poset with property $(*_\kappa)$ such that $\Vdash_{\mathbb{P}} |\dot{R}| < \lambda$ and let $\dot{\mathcal{D}}$ be a $\mathbb{P}$-name for family of $< \lambda$-many dense subsets of $\mathbb{R}$. Then, using the κ^+-cc, we can find $\beta < \lambda$ such that both $\dot{\mathbb{R}}$ and $\dot{\mathcal{D}}$ belong to $V^{\mathbb{P}_\beta}$. We can choose then, $\gamma < \lambda$ so that $\mathbb{R}$ is the γ-th name in $V^{\mathbb{P}_\beta}$ for a poset with property $(*_\kappa)$. Hence, in the model $V^{\mathbb{P}_{\pi(\beta,\gamma)+1}}$, the generic for $\mathbb{R}$ intersects all dense sets in $\mathcal{D}$.

The argument above is enough to obtain the full $\mathrm{MA}(\kappa\text{-good-Knaser})$ in $V^{\mathbb{P}}$:

Claim. *If* $\mathbb{R}$ *is* κ-good-Knaster *poset in* $V^{\mathbb{P}}$ *and* $\mathcal{D}$ *is a collection of* $< \lambda$*-many dense sets in* $\mathbb{R}$*, then there is* $\mathbb{R}' \subseteq \mathbb{R}$ *of cardinality* $< \lambda$ *which is also* κ-good-Knaster *such that the sets in* $\mathcal{D}$ *are dense in* $\mathbb{R}'$*.*

Proof. Given a dense set $D \in \mathcal{D}$, there exists a maximal antichain $A_D \subseteq D$ and using the stationary κ^+-Knaster condition, this antichain has size at most κ. Consider then, the poset S generated by the set of antichains $\{A_D : D \in \mathcal{D}\}$ and has size $< \lambda$ (because $\lambda^{<\kappa} = \lambda$). Now, consider the closure of S under properties (2), (3) and (4) in Definition 5.3.7 and notice that this process does not increase its size. Call the resulting poset $\mathbb{R}'$ and note that it has the desired size and it is an element of the class *κ-good-Knaster*. Finally, if $H \subseteq \mathbb{R}'$ is a generic intersecting all the dense sets in $\mathcal{D}$, we can extend it to a filter $G \supseteq H$, $G \subseteq \mathbb{R}$ meeting all sets in $\mathcal{D}$. $\square$

$\square$

There have been other attempts to get higher analogues of Martin's axiom at $\kappa = \aleph_1$. Specifically, let us mention one due to Baumgartner (see also [54, 53]):

Definition 5.5.2 (Baumgartner's axiom [1])**.** Let $\mathbb{P}$ be a partial order satisfying the following conditions:

– $\mathbb{P}$ is countably closed.

– $\mathbb{P}$ is well-met.

– $\mathbb{P}$ is $\aleph_1$-linked.

Then if $\kappa < 2^{\aleph_1}$ and $\{\mathcal{D}_\alpha : \alpha < \kappa\}$ is a collection of dense sets of $\mathbb{P}$, then there exists a generic filter $G \subseteq \mathbb{P}$ intersecting all sets $\mathcal{D}_\alpha$.

Baumgartner also proved that the former axiom is consistent with $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \kappa$, where $\kappa \geq \aleph_1$ is regular.

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