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abstract

This thesis is dedicated to model equations for atmospheric flows in the tropics, employing the $\beta-$plane approximation. Besides the general equations for compressible flows and those for shallow water in the equatorial region, we focus on the weak temperature gradient approximation and further discuss the Matsuno-Gill model and the equatorial long-wave equations.
We then investigate the relationship between those models with formal asymptotic expansions utilizing multiple scales, based on the seminal paper by Rupert Klein and Andrew Majda [12]. We give a detailed explanation of the method of multiple scales with a particular emphasis placed on the interplay between modelling assumptions and the choice of the formal ansatz.
The topic of rigorous analysis of multiple scales asymptotics is also included: we present a recent convergence proof from the paper [3] concerning the formal relationship between the shallow water equations and the equatorial long-wave equations.

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Introductory remarks

This thesis deals with time dependent PDEs (partial differential equations) as they appear to describe “flows” that model the (earths) atmosphere, i.e. in climate and/or weather research. The earliest such PDE models date back to Euler, then Navier and Stokes, to name just the most famous ones, modeling (in)compressible (in)viscid fluid dynamics. The analysis e.g. of existence of global (in time) solutions of such PDEs as well as their rigorous justification in model hierarchies is largely open, with contributions of many excellent mathematicians. Of course, the numerical solution of such PDEs is a considerable and very important task, too in this thesis we put emphasis on analysis and modeling, dealing with the very complex situation of realistic models for atmospheric flows near the equator.
The method of multiple scales, which belongs to the broader mathematical field of “asymptotic analysis” (“perturbation theory” in physics), has its origins in the treatment of physical problems that depend on two or more different (time) scales. These two scales are linked through a physical parameter that can be viewed as “asymptotically small” (the famous ϵ that can be sent to zero: $\epsilon \to 0$ to obtain a “reduced problem”.) We consider a multiple scales expansion of functions that are allowed to explicitly depend on two scales with respect to small parameters.
The basic idea is historically well known as “formal asymptotics”, more rigorous mathematical results start from 1960 on dealing with its application to nonlinear ordinary differential equations, specifically the nonlinear oscillator. More recent is the use of multiple scales in the rigorous analysis of partial differential equations; in particular multiple scale techniques have proven to be a powerful tool in establishing and analysing model hierarchies for geophysical flows. The reason for this is that atmospheric motions as well as oceanic currents are characterized by the simultaneous occurrence of phenomena acting on vastly different scales in time and space; in a sense, the method of multiple scales is therefore naturally adapted to the challenge of finding reduced model equations that are more accessible to rigorous mathematical and numerical analysis. This challenge is a very timely one, given that current general circulation models that we use to predict the evolution of the earth’s climate are ultimately based on the PDEs of geophysical fluid dynamics. Deriving new approximations to those equations and assessing their accuracy is therefore an important task.
In the present thesis, we review developments in this field pertaining to the equatorial region. The main focus is on understanding the method, specifically the subtleties and pitfalls involved in the choice of the correct modelling ansatz, but we also devote one chapter to a convergence theorem that illustrates how the formal derivation of a reduced model can be translated to a rigorous proof.

Chapter 1

Near-equatorial flows

We start by introducing the governing equations for atmospheric flows in the tropics. In order to provide sufficient context, we also sketch the derivation of the equatorial long-wave equations, the Matsuno-Gill model and the weak temperature gradient approximation, employing traditional scaling techniques. These approximations will later be rederived in a unified, systematic fashion from a multiscale ansatz.
The purpose of this first derivation is twofold:

1. We familiarise ourselves with the original physical motivation for the various models.

2. We observe how common it is for equivalent systems to be formulated and scaled in drastically different ways, which can make it difficult to establish connections between the various flow regimes. This highlights the main strength of the multiscale ansatz: therein, all model equations are derived from the same basic scaling and questions regarding self-consistency or interaction between different regimes can be readily explored.

1.1 Equations for compressible flow

1.1.1 The β-plane approximation

Since the earth rotates around its own axis, it is customary in meteorology to introduce a corresponding *rotating frame of reference* for the equations of motion; any given point on the earth's surface is then stationary in that frame of reference, while the equations of motion are altered: in particular, as described for example in [18], the equations for conservation of momentum have to incorporate the *Coriolis acceleration*

$$2\mathbf{\Omega} \times \mathbf{v}$$

in order to correctly describe the motion of fluids in the atmosphere. ($\mathbf{v}$ denotes the fluid velocity; $\mathbf{\Omega}$ is the earth rotation vector)
Throughout this thesis, we will follow the convention that

- x denotes *zonal variation,* that is the eastwards distance from the prime meridian,

- y denotes *meridional variation,* that is the northwards distance from the equator and
- z denotes the vertical distance from the surface of the earth.

Since the globe can be viewed as an approximate sphere, (x, y, z) constitute a spherical coordinate system; however, whenever we are only interested in a region around a given latitude, it is common to employ a so-called *tangent plane approximation:* the curvilinear coordinates x and y are then replaced by cartesian coordinates in the tangent plane at reference latitude θ_0; colloquially, one could speak of a "locally flattened" earth.
In such a tangent plane approximation, sphericity is not completely neglected, however: the Coriolis force accounts for its effects in an indirect manner, and it is therefore important to assess its strength. Generally, with $\Omega := |\mathbf{\Omega}|$, we have $\mathbf{\Omega} = \Omega\left(\begin{smallmatrix} 0 \\ \cos\theta \\ \sin\theta \end{smallmatrix}\right)$ and

$$2\mathbf{\Omega} \times \mathbf{v} = 2\Omega \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 2\Omega \begin{pmatrix} w \cdot \cos\theta - v \cdot \sin\theta \\ u \cdot \sin\theta \\ -u \cdot \cos\theta \end{pmatrix}.$$

In the equatorial region, our reference latitude is $\theta_0 = 0$. The sin-terms in the above are then small and we seek an approximation via linearization around the reference latitude; with a denoting the earth radius, we get

$$\sin\theta = \sin\Big(\frac{y}{a}\Big) = \sin(0) + \frac{y}{a}\cos(0) + O(\Big(\frac{y}{a}\Big)^2) = \frac{y}{a} + O(\Big(\frac{y}{a}\Big)^2);$$

replacing $\sin\theta$ by $\frac{y}{a}$ in the Coriolis acceleration constitutes the core of the *equatorial β-plane approximation* that we will use throughout this thesis - the *Coriolis parameter* $f = 2\Omega\sin\theta$ can then be estimated by $f \approx \beta y$, with $\beta = \frac{2\Omega}{a}$. In its classical form, it also postulates that the term $w \cdot \cos\theta$ can be dropped; we will later obtain this as a natural consequence of our scaling assumptions.

1.1.2 The governing equations

The most general mathematical setting for our subject matter is provided by the equations for three-dimensional compressible flow in the equatorial β-plane:

$$\mathbf{u}_t - \mathbf{u} \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \mathbf{u}_z + 2(\mathbf{\Omega} \times \mathbf{v})_{\parallel} + \frac{1}{\rho}\nabla p = S_{\mathbf{u}}, \tag{1.1}$$

$$w_t + \mathbf{u} \cdot \nabla w + w w_z + 2(\mathbf{\Omega} \times \mathbf{v})_{\perp} + \frac{1}{\rho} p_z = S_w - g, \tag{1.2}$$

$$p_t + \mathbf{u} \cdot \nabla p + w p_z + \gamma p(\nabla \cdot \mathbf{u} + w_z) = \rho S_p, \tag{1.3}$$

$$\theta_t + \mathbf{u} \cdot \nabla\theta + w\theta_z = S_\theta. \tag{1.4}$$

Here, $\mathbf{v} = (u, v, w)$ is the full flow velocity, where $\mathbf{u} = (u, v)$ and w denote its horizontal and vertical components, respectively. $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ denotes the horizontal gradient and the subscripts $\parallel$ and $\perp$ indicate the horizontal and vertical projections

of a vector, respectively.
$\mathbf{\Omega}$ is the earth rotation vector, p denotes the pressure and θ the potential temperature. ρ is the density of the fluid and g the acceleration of gravity. Finally, γ is the isentropic exponent and it holds the relation $\rho = A\frac{p^{\frac{1}{\gamma}}}{\theta}$, where A is a physical constant. This is the *isentropic equation.* Any term of the form S_x denotes a source term for the quantity x that accounts for the effects of turbulent transport, radiation, moisture etc.

For the purposes of scale analysis, we need to write those equations in nondimensional form: Choosing a reference length l_{ref}, a reference time t_{ref}, and repeating this process for all the other physical quantities involved (v_{ref} is given by $v_{ref} = \frac{l_{ref}}{t_{ref}}$), we obtain the system

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \mathbf{u}_z + \frac{2}{Ro_B}(\mathbf{f} \times \mathbf{v})_{\parallel} + \frac{1}{M^2}\frac{1}{\rho}\nabla p = S_{\mathbf{u}}, \tag{1.5}$$

$$w_t + \mathbf{u} \cdot \nabla w + w w_z + \frac{2}{Ro_B}(\mathbf{f} \times \mathbf{v})_{\perp} + \frac{1}{M^2}\frac{1}{\rho}p_z = S_w - \frac{1}{\bar{Fr}^2}, \tag{1.6}$$

$$p_t + \mathbf{u} \cdot \nabla p + w p_z + \gamma p(\nabla \cdot \mathbf{u} + w_z) = \rho S_p, \tag{1.7}$$

$$\theta_t + \mathbf{u} \cdot \nabla \theta + w\theta_z = S_\theta, \tag{1.8}$$

as presented in Klein and Majda's paper [12]. We keep the same notation for the nondimensional variables for simplicity. Here, $\mathbf{f}$ is an earth rotation unit vector; the isentropic relation in its nondimensional form now reads $\rho = \frac{p^{\frac{1}{\gamma}}}{\theta}$. The characteristic dimensionless numbers, defined as in [12], are:

- The *bulk microscale Rossby number* $Ro_B = \frac{v_{ref}}{\Omega l_{ref}}$.
- The *Mach number* $M = \frac{v_{ref}}{\sqrt{p_{ref}/\rho_{ref}}}$.
- The *barotropic Froude number* $\bar{Fr} = \frac{v_{ref}}{\sqrt{g \cdot l_{ref}}}$.

We note that the Rossby number depends on the chosen basic length scale; this distinction is very important, since Rossby numbers for different scales will exhibit qualitatively different behaviour when deriving reduced model equations. Equations (1.5)-(1.8) will serve as the starting point for the multiple-scale analysis in chapter 3; before that, however, let us consider a simpler set of equations that lends itself more readily to mathematical analysis:

1.2 The equatorial shallow water equations

For the study of certain large-scale atmospheric motions, the effects of density stratification can be ignored. This fundamental assumption leads to a strongly simplified mathematical model called the *shallow water equations* - [18] gives a detailed derivation from the compressible flow equations.
In the equatorial β-plane, where the Coriolis parameter f is given by $f = \beta y$, those equations have the form

$$\frac{D\mathbf{v}}{Dt} + \beta y \mathbf{v}^{\perp} + g\nabla H = S_{\mathbf{v}} \tag{1.9}$$

$$\frac{DH}{Dt} + H\mathrm{div}\mathbf{v} = S_h. \tag{1.10}$$

Here, $\mathbf{v} = (u, v)$ is the *horizontal* velocity, whereas H is the geopotential height; $\mathbf{v}^{\perp} = (-v, u)$ and the right-hand side comprises the source terms, as above. The operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ denotes the *material derivative*, which constitutes the time derivative of a moving parcel of fluid.

The derivation of those equations from the compressible flow equations is not our focus here, but it is instructive to sketch their relationship: Ignoring the source terms for now and identifying H with ρ, equation (1.10) is the shallow water variant of the continuity equation

$$\frac{D\rho}{Dt} + \rho\mathrm{div}\mathbf{v} = 0,$$

while equation (1.9) is the momentum equation corresponding to (1.1). If we now recall that for an isentropic ideal gas, the formula $p = A\rho^{\gamma}$ holds, the corresponding equations can actually be shown to be equivalent in this case for suitable values of A and γ.

Again, we want to analyse the equations in nondimensional form; Choosing a reference length, time and velocity as before and writing the geopotential height in the form $H = H_0(1 + Fh)$, where the parameter F gives the nondimensional strength of geopotential height perturbations, we obtain

$$\frac{D\mathbf{v}}{Dt} + \tilde{\beta} y \mathbf{v}^{\perp} + F(Fr)^{-2}\nabla h = S_{\mathbf{v}} \tag{1.11}$$

and

$$\frac{Dh}{Dt} + h\,\mathrm{div}\mathbf{v} + F^{-1}\mathrm{div}\mathbf{v} = S_h. \tag{1.12}$$

Here, $\tilde{\beta} = \beta l_{ref} t_{ref}$, while $Fr = \frac{v_{ref}}{\sqrt{gH_0}}$ is the *Froude number.* Note that H_0 here takes the place of l_{ref} in the previous definition; since the convention $H_0 = l_{ref}$ will be used in our analysis of compressible flow, the two definitions are equivalent.

This is the formulation of the shallow water equations we will use in chapter 2.

If we specify

- $l_{ref} = (\frac{c}{\beta})^{\frac{1}{2}}$
- $t_{ref} = (c\beta)^{-\frac{1}{2}}$
- $v_{ref} = c$
- $H_0 = \frac{c^2}{g}$,

with some reference velocity c, and choose $F = 1$ in the definition of H, the nondimensional equations read

$$\frac{D\mathbf{v}}{Dt} + y\mathbf{v}^{\perp} + \nabla h = S_{\mathbf{v}} \tag{1.13}$$

and

$$\frac{Dh}{Dt} + h\ \mathrm{div}\mathbf{v} + \mathrm{div}\mathbf{v} = S_h. \tag{1.14}$$

This is the nondimensionalisation employed in the section to follow.

Remark: Under the thermodynamic assumptions of shallow water theory, the above equations can be restated essentially equivalently in terms of $\mathbf{v}$ and one the following variables:

- The already introduced geopotential height.
- The (ordinary) height, measured from the surface of the earth to the fluid top layer.
- The *fluid depth*, given by the top-to-bottom thickness of the fluid.
- The pressure.
- The potential temperature.

1.3 The equatorial long-wave equations

Near-equatorial flows are characterized by the smallness of the Coriolis parameter, which vanishes at the equator. This leads to the emergence of equatorial trapped waves that exhibit qualitatively different behaviour in the (zonal) x-direction and the (meridional) y-direction, as described in chapter 9 of Andrew Majda's book [15]; in particular, geostrophic balance is enforced only in the meridional direction. It is therefore important to find reduced model equations that capture those phenomena, and in a manner already resembling the more advanced multiscale ansatz, we will now rescale the shallow water equations in order to arrive at a reduced set of equations as a formal limit:
Our scaling is based on the observation that for a standard value of $c = 50\,{}^{m}/_{s}$, the length scale defined above is $l_{ref} \approx 1500\,km$. However, the circumference of the earth is $\approx 40000\,km$, implying that the zonal length scale is greater than the meridional length scale by an order of magnitude. Additionally, meridional velocities are low, leading to the following rescaling:

- $x' = \epsilon x$
- $t' = \epsilon t$
- $\epsilon v' = v$,

while all remaining variables remain unchanged; ϵ is a small parameter that stands for the ratio of our two different length scales.
We introduce accordingly rescaled source terms $S'_{\mathbf{v}} = \epsilon \cdot S_{\mathbf{v}}$and $S'_h = \epsilon \cdot S_h$. - It should be noted that the validity of this scaling procedure generally depends on the physical effects subsumed in those terms; we content ourselves with the remark that it is indeed valid when only dissipation and thermal forcing are considered.
Plugging the new variables into (1.13)-(1.14) and dropping the primes fr notational simplicity results in the system of *long-wave scaled equatorial shallow water equations* from [15], chapter 9:

$$\frac{Du}{Dt} - yv + h_x = S_u \tag{1.15}$$

$$\epsilon\frac{Dv}{Dt} + yu + h_y = \epsilon S_v \tag{1.16}$$

$$\frac{Dh}{Dt} + h\ \mathrm{div}\mathbf{v} + \mathrm{div}\mathbf{v} = S_h. \tag{1.17}$$

Further ignoring terms of order ϵ and ϵ^2 in equation (1.16) - which amounts to enforcing meridional geostrophic balance -, we get the *nonlinear equatorial long-wave equations (NLELWE)*

$$\frac{Dh}{Dt} + h\ \mathrm{div}\mathbf{v} + \mathrm{div}\mathbf{v} = S_h \tag{1.18}$$

$$\frac{Du}{Dt} - yv + h_x = S_u \tag{1.19}$$

$$yu + h_y = 0. \tag{1.20}$$

Setting the source terms to zero and linearising this system around $(u, h) = (0, 0)$, which in this case simply means dropping the advective term in $\frac{D}{Dt}$, we finally obtain the *linear equatorial long-wave equations (LELWE)*

$$h_t + h\ \mathrm{div}\mathbf{v} + \mathrm{div}\mathbf{v} = 0 \tag{1.21}$$

$$u_t - yv + h_x = 0 \tag{1.22}$$

$$yu + h_y = 0. \tag{1.23}$$

A discussion of basic properties of those equations can be found in [15], chapter 9.

1.4 The Matsuno-Gill model

In 1980, A.E. Gill published an article with the aim of studying the response of the tropical atmosphere to a heating source, using the simplest mathematical model that could still be considered viable; this motivated the choice of the following variant of the *linearised* shallow water equations:

$$\mathbf{v}_t + \frac{1}{2} y \mathbf{v}^{\perp} + \nabla p = 0 \tag{1.24}$$

$$p_t + \mathrm{div}\mathbf{v} = -S. \tag{1.25}$$

The equations are taken from [6] in nondimensional form; instead of the height, the equations are given in terms of the nondimensional pressure perturbation p and S is the heating rate. The reference length $l_{ref} = (\frac{c}{2\beta})^{\frac{1}{2}}$ and time $t_{ref} = (w\beta c)^{-\frac{1}{2}}$ produce the factor $\frac{1}{2}$ in the momentum equation (1.24).
The inclusion of dissipative processes in the form of Rayleigh friction and Newtonian cooling can be expressed by simply replacing the time derivative $\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial t} + \epsilon$ with a small parameter ϵ, provided the effects of friction and cooling have the same magnitude. Further assuming a *stationary* flow, the time derivatives vanish and (1.24)-(1.25) then look like the following:

$$\epsilon u - \frac{1}{2} y v + p_x = 0 \tag{1.26}$$

$$\epsilon v + \frac{1}{2} y u + p_y = 0 \tag{1.27}$$

$$\epsilon p + \mathrm{div}\mathbf{v} = -S \tag{1.28}$$

These model equations were first investigated by Matsuno in 1966.
Gill in his 1980 article derived, for certain given S, explicit solutions for equations (26)-(28) under the additional assumption of geostrophic balance in the y-momentum equation, i.e. $\epsilon = 0$ in (27). Here, we are not concerned with the structure of those solutions that describe geophysical phenomena like the Walker circulation over the Pacific ocean, but again point to the fact that the model equations just derived will reappear, together with LELWE, as a leading order limit to the shallow water system in chapter 2.

1.5 The weak temperature gradient approximation

It is another characteristic feature of the tropical atmosphere that horizontal temperature gradients are small. As in the case of the long-wave equations, this observation

translates to a scaling ansatz in the model equations that involves a small dimensionless parameter; the derivation below follows the presentation in [21] by Adam Sobel et al.
Sobel and his co-authors take the shallow water equations on an f-plane, where the Coriolis parameter is taken to be constant, as their starting point; source terms incorporate (Rayleigh) friction and mass:

$$\frac{DH}{Dt} + H\ \mathrm{div}\mathbf{v} = S \tag{1.29}$$

$$\frac{D\zeta}{Dt} + \zeta\ \mathrm{div}\mathbf{v} + f\ \mathrm{div}\mathbf{v} = -\alpha\zeta \tag{1.30}$$

$$\delta_t + \frac{1}{2}\Delta|\mathbf{v}|^2 + g\Delta H - \mathbf{k}\cdot\nabla\times(\mathbf{v}(\zeta+f)) = -\alpha\delta. \tag{1.31}$$

The drastically different appearance of the latter two equations stems from a reformulation of the momentum equations in terms of the *vertical vorticity* $\zeta = -u_y + v_x$ and the divergence $\delta = \mathrm{div}\mathbf{v}$ (it follows from Helmholtz's theorem that the two formulations are actually equivalent). H is the fluid depth, $\mathbf{k}$ is a vertical unit vector, S a mass source (corresponding to heating) and α the dissipation rate.
The nondimensionalization employed by Sobel et al. is the following: Given reference values l_{ref}, t_{ref}, and S_0 for length, time and the mass source, respectively, and separating the depth $H = H_0 + h(\mathbf{x}, t)$ into a mean and a perturbation around it, we choose

- $v_{ref} = \frac{S_0 l_{ref}}{H_0}$ and
- $\zeta_{ref} = \delta_{ref} = \frac{S_0}{H_0}$,

i.e. vorticity and divergence are supposed to be of the same order of magnitude; for the height perturbation, we choose

$$h_{ref} = \frac{S_0 f l_{ref}^2}{g H_0}.$$

The nondimensional form of (1.29)-(1.31) is then given by

$$Bu\left(\frac{1}{f t_{ref}} h_t + Ro\ \mathrm{div}(\mathbf{v}h)\right) + \delta = S, \tag{1.32}$$

$$\frac{1}{f t_{ref}}\zeta_t + \mathrm{div}(\mathbf{v}(Ro\zeta + 1)) = -\frac{\alpha}{f}\zeta, \tag{1.33}$$

$$\frac{1}{f t_{ref}}\delta_t + Ro\frac{1}{2}\Delta(|\mathbf{v}|^2) + \Delta h - \mathbf{k}\cdot\nabla\times(\mathbf{v}(Ro\zeta+1)) = -\frac{\alpha}{f}\delta, \tag{1.34}$$

where $Bu = (\frac{l_{ref}}{l_R})^2$ is the *Burger number*, with the *Rossby radius* $l_R = \frac{\sqrt{gH}}{f}$; the Rossby number is given by $Ro = \frac{v_{ref}}{f l_{ref}}$ (recall that the Coriolis parameter here is taken to be constant!)

Remark: In the standard references, the Burger number is defined as the inverse ratio $\left(\frac{l_R}{l_{ref}}\right)^2$.
We now want to motivate the characteristic form of the *weak temperature gradient (WTG) approximation* in the shallow water context: recalling that H can be seen as a stand-in for the potential temperature, it is natural to assume small height perturbations h, corresponding to weak temperature gradients. The leading-order approximation to (1.29) is then given by $H_0\delta \approx S$; in the nondimensional equation (1.32), this translates to $\delta \approx S$.
This approximation now arises if we let $Bu \to 0$ formally, under the assumption that $\frac{1}{ft_{ref}}$ and Ro are both at most $O(1)$. In order to ensure balance in the vorticity and divergence equations, we further rescale $\zeta'(\mathbf{x}, ft_{ref}t) = \zeta(\mathbf{x}, t)$ and $h'(\mathbf{x}, ft_{ref}t) = h(\mathbf{x}, t)$. Changing variables, dropping the primes and taking the formal limit yields equations (10)-(12) in [21]:

$$\delta = S \tag{1.35}$$

$$\zeta_t + S(Ro\zeta + 1) + \mathbf{v} \cdot \nabla(Ro\zeta + 1) = -\frac{\alpha}{f}\zeta \tag{1.36}$$

$$\Delta h = -S_t - Ro\frac{1}{2}\Delta(|\mathbf{v}|^2) + \mathbf{k} \cdot \nabla \times (\mathbf{v}(Ro\zeta + 1)) - \frac{\alpha}{f}S. \tag{1.37}$$

it should be pointed out that, in the present regime, the first two equations already fully determine the flow and (1.37) is only given for completeness; (1.35)-(1.36) are what we will call the WTG approximation from here on out.

Chapter 2

Modeling with multiple scales

2.1 Outline of the general approach

Motions in the earth's atmosphere - and the ocean - occur on a large variety of different length - and timescales, from small-scale phenomena such as dust devils and tornadoes to the flows that make up the global atmospheric circulation, which include the westerlies in the midlatitudes, the easterlies in the tropical region and many others.
For scales that exceed a few kilometers or minutes, the fundamental assumption that permits the application of the method of multiple scales is *scale separation* induced by the thermal stratification of the atmosphere - as described in [11]; in mathematical terms, the ratio between two scales can then be expressed in terms of a nondimensional parameter ϵ, which motivates a *formal asymptotic expansion* ansatz for the dependent variables in the governing equations:

$$\mathbf{U} = \sum_i \epsilon^i \mathbf{U}^{(i)}.$$

If multiple time and/or length scales are involved, the functions $U^{(i)}$ generally depend on all of them; taking two length scales $\mathbf{X}_1, \mathbf{X}_2$ and two time scales T_1, T_2 each as an example, we would have $U^{(i)} = U^{(i)}(\mathbf{X}_1, \mathbf{X}_2, T_1, T_2)$, where $\mathbf{X}_j = \epsilon^{\alpha_j}\mathbf{x}$ and $T_j = \epsilon^{\beta_j}t$ for some - not necessarily integer-valued - α_j, β_j, respectively. We should also remark that the asymptotic expansion itself is not always restricted to integer powers of ϵ, as we will see later on in this chapter.
The physical assumption of scale separation can now be translated to the mathematical assumption that *the expansion just described is unique;* in the case of multiple scales, this has farther-reaching consequences that will be discussed in the next section.
The application of the method of multiple scales to a physical problem can now be sketched as follows:

1. Having chosen a system of equations, such as the compressible flow equations or the shallow water equations described in the introduction, the equations are nondimensionalized by an appropriate choice of scales for time, length and the involved dependent variables such as flow velocity, pressure or potential temperature.

2. In most cases, several nondimensional numbers will occur in the resulting system; those are then expressed as powers of *one single* small parameter ϵ. The limit as $\epsilon \to 0$ formally is then called a *distinguished limit.* The "correct" choice of a distinguished limit generally is nontrivial and requires a solid empirical understanding of the problem at hand; in the realm of meteorology, Rupert Klein in [9] and [10] proposed a universally applicable distinguished limit that we will employ in the derivations to come.

3. Additional length and time scales are chosen and an expansion ansatz as described above is inserted into the equations that now only depend on one parameter.

4. The leading-order and higer-order equations, as needed, are derived in the usual fashion of collecting terms of equal order in ϵ. Further reduced dynamics are obtained by considering solutions that depend on only one set of variables in space and time.

The remainder of this chapter will be devoted to the models obtained in the already mentioned seminal paper [12] by Rupert Klein and Andrew Majda by the method of multiple scales; its first showcase will be the systematic re-derivation of the classical approximations to the shallow water equations discussed in chapter 1. The scope of this thesis does not permit going beyond Klein and Majda's results, but we will expound the mathematical aspects of their work in more detail than the original article provides.

2.2 The shallow water case

Remark: Since our derivations in this chapter are formal, no discussion of appropriate function spaces, convergence, smoothness or interchange of limits will take place; a rigorous treatment of those derivations is generally possible, but very demanding, as demonstrated to some degree in chapter 3.

Let us recall the equatorial shallow water equations in nondimensional form, as given in [12]:

$$\frac{D\mathbf{v}}{Dt} + \tilde{\beta} y \mathbf{v}^{\perp} + F(Fr)^{-2}\nabla h = S_{\mathbf{v}}$$

and

$$\frac{Dh}{Dt} + h \ \mathrm{div}\mathbf{v} + F^{-1} \ \mathrm{div}\mathbf{v} = S_h,$$

where F denotes the nondimensional strength of geopotential height perturbations and $\tilde{\beta} = \beta l_{ref} t_{ref}$, $Fr = \frac{v_{ref}}{\sqrt{gH_0}}$.
The standard choice for the reference velocities here is $v_{ref} = 5\, m\, s^{-1}$ and $C_{ref} := \sqrt{gH_0} = 50\, m\, s^{-1}$, which represents the wave speed of the first baroclinic mode for dry gravity waves; with an estimate of perturbations of geopotential height being in the range $\pm 10\%$ and the assumption that v_{ref} is small compared to C_{ref}, we can set $\epsilon := Fr = F = 0.1$ to investigate in the following the reduced dynamics that result from the formal limit $\epsilon \to 0$:

2.2.1 The IPESD models

The standard length and time units in the equatorial region on the synoptic scale are l_s, the *synoptic length scale:*

$$l_s = (\frac{C_{ref}}{\beta})^{\frac{1}{2}} = 1500\,km$$

and T_{ES}, the *equatorial synoptic timescale:*

$$T_{ES} = (C_{ref}\beta)^{-\frac{1}{2}} = 8\,h.$$

Our first regime of interest, the *intraseasonal planetary equatorial synoptic-scale dynamics (IPESD),* occur on the equatorial synoptic length scale, but not on the standard timescale; instead, we consider the corresponding *advective timescale* T_I :

$$T_I = \frac{l_s}{v_{ref}} = \epsilon^{-1} T_{ES} = 80\,h,$$

10 units of which make up more than a month; for that reason, T_I is called an *intraseasonal timescale.*

Owing to the shape of the equatorial region, it is natural to expect zonal variations on a larger scale; therefore, we introduce $l_p = \epsilon^{-1} l_s = 15000\,km$ as a *planetary length scale* and the corresponding variable $X_p = \epsilon x$; with those considerations and the above definition of ϵ, we are therefore looking for solutions to the system

$$\frac{D\mathbf{v}}{Dt} + \epsilon^{-1}(y\mathbf{v}^{\perp} + \nabla h) = \epsilon^{-1}\hat{S}_{\mathbf{v}}, \tag{2.1}$$

$$\frac{Dh}{Dt} + h \text{ div}\mathbf{v} + \epsilon^{-1} \text{ div}\mathbf{v} = \epsilon^{-1}\hat{S}_h, \tag{2.2}$$

where $\epsilon^{-1}\hat{S}_{\mathbf{v}} = S_{\mathbf{v}}$, $\epsilon^{-1}\hat{S}_h = S_h$, i.e. we assume source terms of order up to $O(\epsilon^{-1})$. The nondimensional velocity and height perturbations are $\mathbf{v} = \mathbf{v}(X_p, x, y, t)$ and $h = h(X_p, x, y, t)$, with zonal variations on both the synoptic and the planetary scale.

In order to isolate the dynamics on the planetary scale, we will make use of the *zonal average* $\overline{f}$ of any function $f = f(X_p, x, y, t)$, which is defined as follows:

$$\overline{f}(X_p, y, t) = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} f(X_p, x, y, t) dx.$$

Evidently, we can decompose any such f in the manner $f = \overline{f} + f'$, where $\overline{f'} = 0$. With this definition, we are now ready to state the expansion ansatz for the IPESD models:

$$\mathbf{v} = \mathbf{V}^{(0)}(X_p, y, t) + \mathbf{v}^{(0)}(X_p, x, y, t) + \epsilon\mathbf{v}^{(1)}(X_p, x, y, t) + O(\epsilon^2), \tag{2.3}$$

$$h = H^{(0)}(X_p, y, t) + h^{(0)}(X_p, x, y, t) + \epsilon h^{(1)}(X_p, x, y, t) + O(\epsilon^2), \tag{2.4}$$

where the leading-order terms are split into the zonal averages $\mathbf{V}^{(0)}, H^{(0)}$ and the remainders $\mathbf{v}^{(0)}, h^{(0)}$, which satisfy $\overline{\mathbf{v}^{(0)}} = 0, \overline{h^{(0)}} = 0$. This a-priori decomposition is done purely for the purpose of more compact notation.
Before we start with the derivation of our model equations, we need to point to a crucial requirement for the formal validity of our multiscale ansatz: in order to guarantee that the terms of leading order in (2.3)-(2.4) actually describe the leading-order behaviour in our formal setting, $\mathbf{v}^{(1)}, h^{(1)}$ need to fulfil the *sublinear growth conditions*

$$\lim_{x\to\infty}\left(\frac{\mathbf{v}^{(1)}(X_p,x,y,t)}{|x|+1}\right) = \lim_{x\to\infty}\left(\frac{h^{(1)}(X_p,x,y,t)}{|x|+1}\right) = 0. \tag{2.5}$$

See [7], chapter 4, for a thorough discussion of the mathematical implications of this condition.
If those conditions were not fulfilled, $\epsilon\mathbf{v}^{(1)}, \epsilon h^{(1)}$ would attain the same magnitude as the leading-order terms for $|x| = O(\epsilon^{-1})$.
Further, we clarify that X_p at every step of the procedure is regarded as an independent variable initially, which is *then* restricted to $X_p = \epsilon x$; the chain rule then yields for any $f(X_p, x, y, t)$ that $\frac{df}{dx} = \epsilon\frac{\partial f}{\partial X_p} + \frac{\partial f}{\partial x}$. Finally, we assume an expansion for the source terms:

$$\hat{S}_{\mathbf{v}} = \hat{S}_{\mathbf{v}}^{(0)} + \epsilon\hat{S}_{\mathbf{v}}^{(1)} + O(\epsilon^2),$$
$$\hat{S}_h = \hat{S}_h^{(0)} + \epsilon\hat{S}_h^{(1)} + O(\epsilon^2).$$

We are now ready to begin:
Inserting (2.3)-(2.4) into (2.1), we collect terms of equal order in ϵ as usual to identify the resulting regimes. In this, we need to treat the zonal and meridional momentum equations separately, due to the split terms in our expansion ansatz. At leading order $O(\epsilon^{-1})$, we obtain

$$-yV^{(0)} - yv^{(0)} + \frac{\partial h^{(0)}}{\partial x} = \hat{S}_u^{(0)}, \tag{2.6}$$

$$yU^{(0)} + yu^{(0)} + \frac{\partial H^{(0)}}{\partial y} + \frac{\partial h^{(0)}}{\partial y} = \hat{S}_v^{(0)}. \tag{2.7}$$

The same ansatz in (2.2) yields

$$\frac{\partial V^{(0)}}{\partial y} + \mathrm{div}v^{(0)} = \hat{S}_h^{(0)}. \tag{2.8}$$

Next, we apply the zonal averaging operator $\overline{(\cdot)}$: By the fundamental theorem of analysis, $\overline{\left(\frac{\partial f}{\partial x}\right)} = 0$ for any bounded f; since the leading-order terms have to be bounded for obvious physical reasons, we can make use of this relation in equation (2.5). Keeping in mind that $V^{(0)}$ is independent of x, $\overline{v^{(0)}} = 0$ and partial derivatives other than ∂_x commute with $\overline{(\cdot)}$, the zonally averaged versions of (2.6) and (2.8) then are:

$$-yV^{(0)} = \overline{\hat{S}_u^{(0)}}, \tag{2.9}$$

$$\frac{\partial V^{(0)}}{\partial y} = \overline{\hat{S}_h^{(0)}}. \tag{2.10}$$

We note that these equations impose a constraint on the source terms: $\overline{\hat{S}_u^{(0)}} = -y \int \overline{\hat{S}_h^{(0)}} - Cy$ for some constant C.
Equation (2.7) becomes

$$yU^{(0)} + \frac{\partial H^{(0)}}{\partial y} = \overline{\hat{S}_v^{(0)}}. \tag{2.11}$$

Substracting (2.9)-(2.11) from the respective original equations gives

$$-yv^{(0)} + \frac{\partial h^{(0)}}{\partial x} = \hat{S}_u^{(0)} - \overline{\hat{S}_u^{(0)}}, \tag{2.12}$$

$$yu^{(0)} + \frac{\partial h^{(0)}}{\partial y} = \hat{S}_v^{(0)} - \overline{\hat{S}_v^{(0)}}, \tag{2.13}$$

$$\mathrm{div}\mathbf{v}^{(0)} = \hat{S}_h^{(0)} - \overline{\hat{S}_h^{(0)}}. \tag{2.14}$$

In this system, we recognize the structure of the *Matsuno-Gill model* (1.26)-(1.28) from chapter 1, where the $\epsilon-$terms on the left-hand side can be equated with the source terms in the present regime.

So far, we have derived equations that determine $V^{(0)}$ as well as $u^{(0)}$, $v^{(0)}$ and $h^{(0)}$. To obtain a closed system for $U^{(0)}$ and $H^{(0)}$, we will need to consider the first-order perturbations $O(\epsilon^0)$ of (2.1) and (2.2):
The u-component of (1) at $O(\epsilon^0)$ reads

$$\begin{aligned} U_t^{(0)} + u_t^{(0)} + (U^{(0)} + u^{(0)}) \cdot u_x^{(0)} + (V^{(0)} + v^{(0)}) \cdot (U_y^{(0)} + u_y^{(0)}) \\ -yv^{(1)} + H_{X_p}^{(0)} + h_{X_p}^{(0)} + h_x^{(1)} = \hat{S}_u^{(1)}, \end{aligned} \tag{2.15}$$

where we have abbreviated partial derivatives by a subscript for brevity; we shall keep this notation as long as it does not create room for confusion.
Once more, we are interested in the zonal average, and it is here that the aforementioned sublinear growth condition comes into play: another straightforward application of the fundamental theorem of analysis yields that $\overline{u_x^{(1)}} = \overline{v_x^{(1)}} = \overline{h_x^{(1)}} = 0$, provided that the sublinear growth condition is fulfilled! With this remark and the observation that $u^{(0)}u_x^{(0)} = [\frac{u^{(0)2}}{2}]_x$, meaning that the zonal average of this term vanishes, the result is quickly seen to be

$$U_t^{(0)} + V^{(0)}U_y^{(0)} + \overline{v^{(0)}u_y^{(0)}} - yV^{(1)} + H_{X_p}^{(0)} = \overline{\hat{S}_u^{(1)}}, \tag{2.16}$$

where $V^{(1)} := \overline{v^{(1)}}$.
Collecting all $O(\epsilon^0)-$terms in (2.3) yields

$$
\begin{aligned}
H_t^{(0)} + h_t^{(0)} + (U^{(0)} + u^{(0)})h_x^{(0)} + (V^{(0)} + v^{(0)})(H_y^{(0)} + h_y^{(0)}) \\
+(H^{(0)} + h^{(0)}) \cdot (u_x^{(0)} + V_y^{(0)} + v_y^{(0)}) + U_{X_p}^{(0)} + u_{X_p}^{(0)} + u_x^{(1)} + v_y^{(1)} = \hat{S}_h^{(1)}. \qquad (2.17)
\end{aligned}
$$

Here, we remark that $u^{(0)}h_x^{(0)} + h^{(0)}u_x^{(0)} = [u^{(0)}h^{(0)}]_x$; together with all previous indications, the zonal average of (2.18) then is

$$
H_t^{(0)} + V^{(0)}H_y^{(0)} + \overline{v^{(0)}h_y^{(0)}} + H^{(0)}V_y^{(0)} + \overline{h^{(0)}v_y^{(0)}} + U_{X_p}^{(0)} + V_y^{(1)} = \overline{\hat{S}_h^{(1)}}. \qquad (2.18)
$$

Combined with (2.11), we get the system

$$
\begin{aligned}
U_t^{(0)} + V^{(0)}U_y^{(0)} + \overline{v^{(0)}u_y^{(0)}} - yV^{(1)} + H_{X_p}^{(0)} &= \overline{\hat{S}_u^{(1)}}, \qquad (2.19)\\
H_t^{(0)} + V^{(0)}H_y^{(0)} + \overline{v^{(0)}h_y^{(0)}} + H^{(0)}V_y^{(0)} + \overline{h^{(0)}v_y^{(0)}} + U_{X_p}^{(0)} + V_y^{(1)} &= \overline{\hat{S}_h^{(1)}}, \qquad (2.20)\\
yU^{(0)} + \frac{\partial H^{(0)}}{\partial y} &= \overline{\hat{S}_v^{(0)}}. \qquad (2.21)
\end{aligned}
$$

These equations are called the *quasi-linear equatorial long-wave equations (QLELWE)*, according to [12]. A side-by-side comparison with equations (1.18)-(1.23) from before reveals that they are indeed reduced versions of the nonlinear variant, but considerably more complex than LELWE.

Remark: The term $H^{(0)}V_y^{(0)}$ in QLELWE is omitted in the original article; without specific assumptions on the source terms, it needs to be included.

(2.9)-(2.10), the Matsuno-Gill-type equations (2.12)-(2.14) and QLELWE all combined constitute the IPESD regime as defined by Klein and Majda. When we ignore variation on the planetary scale, that is, we look for solutions independent of X_p, the resulting equations are called the *synoptic-scale equatorial weak temperature gradient (SEWTG)* equations. Our derivation is now complete.

Remark: We can work with the exact same setup, but consider two timescales instead of two length scales: looking for solutions that vary only on the synoptic scale l_s spatially, but on the synoptic timescale $T_{ES} = \epsilon T_I$ as well as the intraseasonal scale T_I, the corresponding new time *variable* is $T_{ES} = t/\epsilon$ and we have $\mathbf{v} = \mathbf{v}(x, y, t, T_{ES})$ and $h = h(x, y, t, T_{ES})$. Assuming a regular expansion of the form

$$
\begin{aligned}
\mathbf{v} &= \mathbf{v}^{(0)}(x, y, t, T_{ES}) + \epsilon \mathbf{v}^{(1)}(x, y, t, T_{ES}) + O(\epsilon^2),\\
h &= h^{(0)}(x, y, t, T_{ES}) + \epsilon h^{(1)}(x, y, t, T_{ES}) + O(\epsilon^2),
\end{aligned}
$$

the resulting equations at $O(\epsilon^{-1})$ then are

$$\mathbf{v}^{(0)}_{T_{ES}} + y\mathbf{v}^{(0)\perp} + \nabla h^{(0)} = \hat{S}^{(0)}_{\mathbf{v}}, \tag{2.22}$$

$$h^{(0)}_{T_{ES}} + \text{div}\mathbf{v}^{(0)} = \hat{S}^{(0)}_{h}, \tag{2.23}$$

which make up the *standard* linear equatorial wave equations (LEWE). The derivation itself is straighforward enough to be skipped in this instance; we only mention the result for comparison purposes in the next section.

2.2.2 The MEWTG models

We now derive the classical *mesoscale equatorial weak temperature gradient (MEWTG)* dynamics. In order to obtain meaningful results, we first need to go back to the drawing board and determine new scales for our physical quantities:
With v_{ref} and C_{ref} as well as ϵ defined as before, we choose as a standard mesoscale length $l_m = \left(\frac{v_{ref}}{\beta}\right)^{\frac{1}{2}}$, the so-called *Charney inertial scale.* The corresponding advective timescale T_m is defined by $T_m = \frac{l_m}{v_{ref}}$. It is instructive to observe that, in the present context, $l_m = (\text{Fr})^{\frac{1}{2}} l_s = \epsilon^{\frac{1}{2}} l_s$, i.e. the ratio between the previously used length scale and this one is a fractional power of ϵ. The concrete values for our units are $l_m = 500$ km, whereas T_m is a little more than a day; these are mesoscales in the context of geophysical fluid dynamics.
Inserting $l_m = l_{ref}$ and $T_m = t_{ref}$ in the nondimensional shallow water equations yields

$$\frac{D\mathbf{v}}{Dt} + y\mathbf{v}^{\perp} + \epsilon^{-1}\nabla h = \epsilon^{-1}\hat{S}_{\mathbf{v}}, \tag{2.24}$$

$$\frac{Dh}{Dt} + h\ \text{div}\mathbf{v} + \epsilon^{-1}\ \text{div}\mathbf{v} = \epsilon^{-1}\hat{S}_h \tag{2.25}$$

as our scaled system; we note that the Coriolis term $y\mathbf{v}^{\perp}$ now is $O(\epsilon^0)$ as the crucial deviation compared to (2.1)-(2.2).
The expansion ansatz itself here is considerably simpler; no multiple scales are involved and we only consider the approximation at leading order, so we can write

$$\mathbf{v} = \mathbf{v}^{(0)}(x, y, t) + O(\epsilon), \tag{2.26}$$

$$h = h^{(0)}(x, y, t) + O(\epsilon). \tag{2.27}$$

Here, we start with the $O(\epsilon^{-1})-$equation for h:

$$\text{div}\mathbf{v}^{(0)} = \hat{S}^{(0)}_h. \tag{2.28}$$

This is the characteristic weak temperature gradient approximation, as discussed in chapter 1.
The treatment of the momentum equation here is somewhat different - we start by writing down the equations for $u^{(0)}$, $v^{(0)}$ individually:

$$u_t^{(0)} + u^{(0)}u_x^{(0)} + v^{(0)}u_y^{(0)} - yv^{(0)} + \epsilon^{-1}h_x^{(0)} = \epsilon^{-1}\hat{S}_u^{(0)},$$
$$v_t^{(0)} + u^{(0)}v_x^{(0)} + v^{(0)}v_y^{(0)} + yu^{(0)} + \epsilon^{-1}h_y^{(0)} = \epsilon^{-1}\hat{S}_v^{(0)}.$$

We can eliminate the h-terms from those equations by applying the (vertical) curl: $\mathrm{curl}(a,b) := -\partial_y a + \partial_x b$; after some algebraic manipulation, the resulting equation can be written as

$$[-u_y^{(0)} + v_x^{(0)}]_t + \partial_x[u^{(0)}(-u_y^{(0)} + v_x^{(0)})] + \partial_y[v^{(0)}(-u_y^{(0)} + v_x^{(0)})]$$
$$+\partial_x[y \cdot u^{(0)}] + \partial_y[yv^{(0)}] = \epsilon^{-1}\mathrm{curl}\hat{S}_{\mathbf{v}}^{(0)}. \qquad (2.29)$$

Setting $\omega := -u_y^{(0)} + v_x^{(0)}$, we can introduce the (scaled) *potential vorticity*

$$Q^{(0)} := \omega + y. \qquad (2.30)$$

The potential vorticity is a very important quantity in theoretical meteorology; for a comprehensive discussion, we refer to chapter II of [18]. Equation (2.29) now reads

$$Q_t^{(0)} + \mathrm{div}(\mathbf{v}^{(0)}Q^{(0)}) = \epsilon^{-1}\mathrm{curl}\hat{S}_{\mathbf{v}}^{(0)} \qquad (2.31)$$

(2.28) and (2.30)-(2.31) can now through a side-by-side comparison be seen to be the classical WTG equations from chapter 1, as introduced by [21].

2.2.3 The SPEWTG models

The *seasonal subplanetary equatorial weak temperature gradient (SPEWTG)* regime takes the MEWTG scaling as a starting point, but assumes zonal fluctuations on the larger subplanetary scale: we choose $l_{sp} = \epsilon^{-1}l_m = 5000$ km as a unit length and the corresponding zonal advection timescale, defined by $T_S = \frac{v_{ref}}{l_{sp}} \approx 11$ days. This is a seasonal timescale, since 10 units span (more than) a season. Furthermore, we assume stronger variation in the zonal direction than in meridional flow or height; this motivates the expansion ansatz

$$\mathbf{v} = \begin{pmatrix} u^{(0)}(X_{sp}, y, T_S) \\ 0 \end{pmatrix} + \begin{pmatrix} \epsilon u^{(1)}(X_{sp}, y, T_S) \\ \epsilon v^{(1)}(X_{sp}, y, T_S) \end{pmatrix} + O(\epsilon^2), \qquad (2.32)$$
$$h = \epsilon h^{(1)}(X_{sp}, y, T_S) + O(\epsilon^2), \qquad (2.33)$$

where $X_{sp} = \epsilon x$ and $T_S = \epsilon t$. Since the variables x, y, t are scaled as in MEWTG, equations (2.24)-(2.25) remain unchanged.
Plugging (2.32)-(2.33) into (2.24)-(2.25) now yields only one nontrivial equation at order $O(\epsilon^1)$:

$$\frac{\tilde{D}u^{(0)}}{DT_S} - yv^{(1)} + h_{X_{sp}}^{(1)} = \hat{S}_u^{(2)}, \qquad (2.34)$$

where $\frac{\tilde{D}}{DT_S} := \partial_{T_S} + u^{(0)}\partial_{X_{sp}} + v^{(1)}\partial_y$ could be dubbed the "seasonal-planetary advective derivative".
Collecting the $O(\epsilon^0)-$terms in the remaining equations gives us

$$yu^{(0)} + h_y^{(1)} = \hat{S}_v^{(1)} \tag{2.35}$$

and

$$u_{X_{sp}}^{(0)} + v_y^{(1)} = \hat{S}_h^{(1)}. \tag{2.36}$$

The system (2.34)-(2.36) comprises the SPEWTG equations - which were actually introduced in our reference paper [12] for the very first time. (2.35) can be interpreted as geostrophic balance in the meridional direction, while (2.36) is the by now familiar WTG approximation. This completes our discussion of WTG regimes derived from the equatorial shallow water equations.

2.3 3D compressible flows

We now move on to the treatment of the full compressible flow equations. Our main goal in this section is to show that there are apparent 3D-analogues to the regimes derived in the shallow water context; furthermore, those regimes all arise from one and the same distinguished limit and the same basic scaling, only differentiating by the respective length and timescales used in the expansion ansatz. As pointed out in [12], this crucially underlines the mutual compatibility of those diverse dynamics and also shows how they interact with one another.
We recall the nondimensionalized equations for compressible flow in the near-equatorial region:

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\mathbf{u}_z + \frac{2}{Ro_B}(\mathbf{f}\times\mathbf{v})_\parallel + \frac{1}{M^2}\frac{1}{\rho}\nabla p = S_{\mathbf{u}}, \tag{2.37}$$

$$w_t + \mathbf{u}\cdot\nabla w + ww_z + \frac{2}{Ro_B}(\mathbf{f}\times\mathbf{v})_\perp + \frac{1}{M^2}\frac{1}{\rho}p_z = S_w - \frac{1}{\bar{Fr}^2}, \tag{2.38}$$

$$p_t + \mathbf{u}\cdot\nabla p + wp_z + \gamma p(\nabla\cdot\mathbf{u} + w_z) = \rho S_p, \tag{2.39}$$

$$\theta_t + \mathbf{u}\cdot\nabla\theta + w\theta_z = S_\theta, \tag{2.40}$$

with the isentropic relation

$$\rho = \frac{p^{\frac{1}{\gamma}}}{\theta}. \tag{2.41}$$

We now specify the reference quantities for those equations; they are the fundamental units for the so-called *bulk microscale*:

- $h_{scale} \sim l_{ref} = 10$ km
- $v_{ref} = 5$ m s^{-1}

- $t_{ref} = l_{ref}/v_{ref} = 30$ min
- $\rho_{ref} = 1$ kg m^{-1}
- $p_{ref} = 10^5$ kg m^{-1} s^{-2}

In order to conduct an asymptotic analysis, we need to fix the aforementioned distinguished limit, expressing the Mach number, Froude number and (bulk microscale) Rossby number all in terms of one small parameter ϵ. One way of motivating our choice is provided by the evaluation of those numbers in terms of the above: plugging in the concrete values for l_{ref}, v_{ref} and so forth, we get $\mathrm{Fr} = \mathrm{M} \sim \frac{1}{64}$ and $\mathrm{Ro}_B \sim 5$. With a representative physical value $\sim \frac{1}{8}$ for ϵ, the following ansatz is plausible:

$$\epsilon = K_1\sqrt{\bar{\mathrm{Fr}}} = K_2\sqrt{\mathrm{M}} = K_3\frac{1}{\mathrm{Ro}_B}, \tag{2.42}$$

where $K_i = O(1)$ as $\epsilon \to 0$ for all i.
Remark: We would be remiss not to concede that other choices are not excluded by the strength of this argument; in fact, slight variations on this particular approach have also been used with success. The chief reason to use (2.42) is its universal applicability: it has been used to derive the quasi-geostrophic approximation and WTG regimes in the midlatitudes, for the present WTG approximations in the near-equatorial region and many more; Klein's review article [11] provides an overview.
For the sake of notational simplicity, we set $K_i = 1$ for all i - which is of course *not* an accurate assumption for any and all real-life geophysical flows! In a more narrowly defined setting, one would have to incorporate suitable values for those parameters, since they impact the actual solutions to our limit equations.
Before we get started, we need to determine the correct scaling for the potential temperature; in order to do that, consider the *Brunt-Väisälä frequency* N, which provides a measure of an atmosphere's stability and is defined by

$$N^2 = \frac{g}{\theta}\frac{\partial\theta}{\partial z}.$$

A typical reference value for N is $N = 2 \cdot 10^{-2}$ s^{-2}, which yields

$$\left(\frac{l_{ref}N}{v_{ref}}\right)^2 = 100 \sim \epsilon^{-2}$$

for the nondimensionalized frequency; the definition of N then yields

$$\begin{aligned}\left(\frac{l_{ref}N}{v_{ref}}\right)^2 = \frac{l_{ref}}{v_{ref}}\frac{g}{\theta}\frac{\partial\theta}{\partial z} &\sim \frac{l_{ref}^2}{v_{ref}^2}\frac{g}{\theta_{ref}}\frac{\delta\theta}{h_{scale}} \\ &= \frac{l_{ref}g}{v_{ref}^2}\frac{\delta\theta}{\theta_{ref}} \sim \epsilon^{-2}.\end{aligned}$$

Recalling $\frac{l_{ref}g}{v_{ref}^2} = \bar{\mathrm{Fr}}^{-2} \sim \epsilon^{-4}$, this implies

$$\frac{\delta\theta}{\theta_{ref}} \sim \epsilon^2.$$

Therefore the nondimensional potential temperature satisfies $\theta = 1 + O(\theta^2)$. This explains the particular ansatz for the thermodynamic variables in the following calculations. - First, however, let us restate (2.37)-(2.41) in terms of ϵ as defined by (2.42):

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\mathbf{u}_z + (\epsilon^4\beta y\mathbf{k}\times\mathbf{u} - \epsilon k\mathbf{k}\times\mathbf{f})_{\|} + \epsilon^{-4}\frac{1}{\rho}\nabla p = S_{\mathbf{u}}, \tag{2.43}$$

$$w_t + \mathbf{u}\cdot\nabla w + ww_z + \epsilon(\mathbf{f}\times\mathbf{u})_{\perp} + \epsilon^{-4}\frac{1}{\rho}p_z = S_w - \epsilon^{-4}, \tag{2.44}$$

$$p_t + \mathbf{u}\cdot\nabla p + wp_z + \gamma p(\nabla\cdot\mathbf{u} + w_z) = \rho S_p, \tag{2.45}$$

$$\theta_t + \mathbf{u}\cdot\nabla\theta + w\theta_z = S_\theta, \tag{2.46}$$

with ρ given by (41). In this, the structure of the Coriolis term in (2.43) may not be obvious: in the β-plane approximation for $\mathbf{f}$, we split $2(\mathbf{f}\times\mathbf{v})_{\|} = \beta' y\mathbf{k}\times\mathbf{u} - 2w\mathbf{k}\times\mathbf{f}$ with $\beta' := 2\frac{h_{scale}}{a}$ and set $\beta' = \epsilon^3\beta$. The latter replacement is due to the estimate $\frac{h_{scale}}{a} = \frac{1}{600} \sim \epsilon^3$. In the term $\mathbf{k}\times\mathbf{u}$, $\mathbf{u}$ is identified with the vector $(u, v, 0)^T$.

2.3.1 MEWTG and SPEWTG in 3D

In this subsection, we investigate regimes on the meso - and seasonal/subplanetary scales, respectively, as well as their interaction. Thereby, we will obtain analogous results to subsections 2.2.2 and 2.2.3 - this time, however, with an expansion utilizing multiple scales:
Bearing in mind that $\epsilon \sim \frac{1}{8}$,

$$\mathbf{X}_M = \epsilon^2\mathbf{x}$$

and

$$T_M = \epsilon^2 t$$

define appropriate mesoscale variables for horizontal length and time, while

$$\mathbf{X}_{sp} = \epsilon^3\mathbf{x}$$

and

$$T_{sea} = \epsilon^3 t$$

determine the corresponding subplanetary length and seasonal timescales, which are "longer" by one order of ϵ.
The multiscale ansatz therefore reads

$$(\mathbf{u}, w, p, \theta) =: \mathbf{U} = \mathbf{U}(\mathbf{x}, z, t; \epsilon) = \sum \epsilon^i \mathbf{U}^{(i)}(X_M, X_{sp}, z, T_M, T_{sea}). \tag{2.47}$$

More specifically, the horizontal and vertical velocities are expanded as

$$\mathbf{u} = \mathbf{u}^{(0)} + \epsilon \mathbf{u}^{(1)} + \epsilon^2 \mathbf{u}^{(2)} + \epsilon^3 \mathbf{u}^{(3)} + O(\epsilon^4) \tag{2.48}$$

and

$$w = w^{(0)} + \epsilon w^{(1)} + \epsilon^2 w^{(2)} + \epsilon^3 w^{(3)} + O(\epsilon^4), \tag{2.49}$$

while the pressure reads

$$p = P_0(z) + \rho_0(z)[\epsilon P_1(z) + \epsilon^2 P_2(z) + \epsilon^3 \pi^{(3)} + \dots]; \tag{2.50}$$

the potential temperature has the expansion

$$\theta = 1 + \epsilon^2 \Theta_2(z) + \epsilon^3 \theta^{(3)} + \dots \tag{2.51}$$

All expansion terms with a purely vertical profile are indicated by the use of subscripts instead of superscripts.
This ansatz is consistent with our introductory remarks; additionally, it implies time-independent potential temperature distribution that varies only vertically up to $O(\epsilon^2)$. In (2.50), the choice $p^{(i)} = \rho^{(i)}\pi^{(i)}$ for $i \geq 1$ has purely technical reasons: it helps with some of the thornier details in the calculations that come next.
The source terms have regular expansions with varying strength; specifically, we assume

$$S_{\mathbf{u}} = O(\epsilon^2), \tag{2.52}$$
$$S_w = O(\epsilon), \tag{2.53}$$
$$S_p = O(\epsilon^4), \tag{2.54}$$
$$S_\theta = O(\epsilon^4). \tag{2.55}$$

Remark: Generally speaking, such assumptions are due to the cancellation of balancing terms in the various equations; to give an example, it would not make much sense for a pressure term to solely constitute a momentum source. In other instances, the purely mathematical structure already enforces vanishing source terms at leading order.

Finally, the partial derivatives in the new variables, transformed by the chain rule, read:

$$\partial_t = \epsilon^2 \partial_{T_M} + \epsilon^3 \partial_{T_{sea}}$$

and

$$\nabla = \epsilon^2 \nabla_M + \epsilon^3 \nabla_{sp}.$$

Vertical momentum and expansion of the isentropic relation At various points, we shall require higher-order terms in the expansion of $\rho = \sum \epsilon^i \rho^{(i)}$; these are determined by the expansions of p and θ, respectively: inserting those in (2.42), utilizing Taylor expansions for $p^{\frac{1}{\gamma}}$ and $\frac{1}{\theta}$ around the respective leading-order terms P_0 and 1 and then collecting terms of equal order yields $\rho^{(i)}$ as a function of $p^{(i)}$ and $\theta^{(i)}$. To leading order, we immediately obtain for $\rho^{(0)} =: \rho_0$

$$\rho_0 = P_0^{\frac{1}{\gamma}}; \tag{2.56}$$

in particular, $\rho_0 = \rho_0(z)$, as already stated in the expansion ansatz (2.50). We now turn to the leading-order equation derived from the vertical momentum balance (2.44): At order $O(\epsilon^{-4})$, it reads

$$\frac{1}{\rho_0} P_0'(z) = -1. \tag{2.57}$$

Inserting (2.56) yields an ordinary differential equation that can be solved by separation of variables. With $P_0(0) = 1$, the solution is

$$P_0(z) = (1 - \frac{\gamma}{\gamma - 1} z)^{\frac{\gamma}{\gamma-1}} \tag{2.58}$$

and accordingly,

$$\rho_0(z) = (1 - \frac{\gamma}{\gamma - 1} z)^{\frac{1}{\gamma-1}}. \tag{2.59}$$

Remark: Purely mathematically, those solutions only make sense for $z \leq \frac{\gamma}{\gamma-1}$; with a realistic value of $\gamma \approx 1.4$, this means that our derivations are valid up to a height of about 35 km, which accounts for most of the troposphere.

Going back to the expansion of ρ, the first-order perturbation $\rho^{(1)} =: \rho_1$ is given by the corresponding Taylor expansion of $p^{\frac{1}{\gamma}}$ to order $O(\epsilon)$:

$$p^{\frac{1}{\gamma}} = P_0^{\frac{1}{\gamma}} + \frac{1}{\gamma} P_0^{\frac{1-\gamma}{\gamma}} (p - P_0) + \cdots = P_0^{\frac{1}{\gamma}} + \frac{1}{\gamma} P_0^{\frac{1-\gamma}{\gamma}} \epsilon \rho_0 P_1 + O(\epsilon^2). \tag{2.60}$$

This implies

$$\rho_1 = \frac{1}{\gamma} P_0^{\frac{1-\gamma}{\gamma}} \rho_0 P_1. \tag{2.61}$$

For the $O(\epsilon^{-3})-$equation for vertical momentum, we expand $\frac{1}{\rho} = \frac{1}{\rho_0} - \frac{1}{\rho_0^2}(\rho - \rho_0) + \cdots = \frac{1}{\rho_0} - \epsilon \frac{\rho_1}{\rho_0^2} + O(\epsilon^2)$ to obtain

$$\frac{1}{\rho_0} [\rho_0 \cdot P_1]_z - \frac{\rho_1}{\rho_0^2} P_0' = 0. \tag{2.62}$$

This simplifies to

$$\rho_0' P_1 + \rho_0 P_1' + \rho_1 = 0 \tag{2.63}$$

and, using $\rho_0' = -\frac{1}{\gamma} P_0^{\frac{1-\gamma}{\gamma}} \rho_0$, we see that $\rho_0' P_1 = -\rho_1$. (63) now reduces to

$$\rho_0 P_1' = 0, \tag{2.64}$$

implying

$$P_1(z) \equiv 0, \tag{2.65}$$

which in turn means

$$\rho_1 \equiv 0. \tag{2.66}$$

(2.66) immediately simplifies the calculation of higher-order terms in the pressure expansion: for $\rho^{(2)} =: \rho_2$, we now get

$$\rho_2 = -P_0^{\frac{1}{\gamma}} \cdot \Theta_2 + \frac{1}{\gamma} P_0^{\frac{1-\gamma}{\gamma}} \rho_0 P_2. \tag{2.67}$$

The $O(\epsilon^{-2})-$equation is derived in the same manner as the preceding one:

$$\frac{1}{\rho_0} [\rho_0 \cdot P_2]_z - \frac{\rho_2}{\rho_0^2} P_0' = 0. \tag{2.68}$$

Inserting (2.67), a tedious but straighforward calculation yields

$$P_2'(z) = \Theta_2(z); \tag{2.69}$$

in addition, we note that $\rho_2 = \rho_2(z)$.
For the next term in the expansion - the last one required for our purposes - we point to the fact that $P_1 \equiv 0$ and $\rho_1 \equiv 0$ enforce the same structure as in (2.67):

$$\rho^{(3)} = -P_0^{\frac{1}{\gamma}} \theta^{(3)} + \frac{1}{\gamma} P_0^{\frac{1-\gamma}{\gamma}} \rho_0 \pi^{(3)}. \tag{2.70}$$

By the same token, vertical momentum at order $O(\epsilon^{-1})$ has the exact same structure as (2.68); we therefore allow ourselves to skip to the final result:

$$\pi_z^{(3)} = \theta^{(3)}. \tag{2.71}$$

Summarily, we can state: to the orders considered, the pressure is in hydrostatic balance.

Pressure equation Plugging the leading-order terms in (2.49) and (2.50) into the pressure equation (2.45), we arrive at the relation

$$w^{(0)}[P_0]_z + \gamma P_0 w_z^{(0)} = 0. \tag{2.72}$$

With the known expressions (2.58) and (2.59) plus some algebraic manipulation, this can be shown to be equivalent to

$$\begin{aligned} w^{(0)}[\rho_0]_z + \rho_0 w_z^{(0)} &= 0 \\ \iff [\rho_0 w^{(0)}]_z &= 0. \end{aligned} \tag{2.73}$$

Taking into account that $w^{(0)}$ must be bounded and $\rho_0 \to 0$ as $z \to \frac{\gamma}{\gamma-1}$, the only solution is

$$w^{(0)} \equiv 0. \tag{2.74}$$

By the same token, keeping in mind that $P_1 \equiv 0$, (2.45) at order $O(\epsilon^1)$ reads

$$w^{(1)}[P_0]_z + \gamma P_0 w_z^{(1)} = 0, \tag{2.75}$$

which leads to the same conclusion for $w^{(1)}$:

$$w^{(1)} \equiv 0. \tag{2.76}$$

At the next order $O(\epsilon^2)$, horizontal divergence starts to have an effect and we obtain the equation

$$w^{(2)}[P_0]_z + \gamma P_0(\nabla_m \cdot \mathbf{u}^{(0)} + w_z^{(2)}) = 0. \tag{2.77}$$

Writing this in terms of ρ_0, we get

$$\begin{aligned} w^{(2)}(-\rho_0) + \gamma\rho_0^\gamma(\nabla_M \cdot \mathbf{u}^{(0)} + w_z^{(2)}) &= 0 \\ \implies \nabla_M \cdot \mathbf{u}^{(0)} + w^{(2)}\frac{1}{\gamma}(-\rho_0^{1-\gamma}) + w_z^{(2)} &= 0 \\ \implies \nabla_M \cdot \mathbf{u}^{(0)} + \frac{1}{\rho_0}[w^{(2)}\frac{1}{\gamma}(-\rho_0^{2-\gamma}) + w_z^{(2)}\rho_0] &= 0 \\ \implies \nabla_M \cdot \mathbf{u}^{(0)} + \frac{1}{\rho_0}[\rho_0 w^{(2)}]_z &= 0. \end{aligned} \tag{2.78}$$

Due to $P_2 = P_2(z)$, the $O(\epsilon^3)-$equation still possesses the same structure: an analogous derivation yields

$$\nabla_{sp} \cdot \mathbf{u}^{(0)} + \nabla_M \cdot \mathbf{u}^{(1)} + \frac{1}{\rho_0}[\rho_0 w^{(3)}]_z = 0. \tag{2.79}$$

This completes our discussion of the pressure equation.

Horizontal momentum balance *Introductory remark:* Up to and including $O(\epsilon^0)$, (2.43) is trivial; however, it is useful to know that the independence of $\pi^{(i)}$ from the horizontal variables for $i \leq 2$ could also be obtained *without any prior assumptions* by applying sublinear growth conditions - in the manner explained in section 2.1 - to the equations at order $O(\epsilon^{-2})$, $O(\epsilon^{-1})$ and so on: to give an illustrative example, we immediately obtain at $O(\epsilon^{-2})$:

$$\nabla_M P_0 = 0; \tag{2.80}$$

the next equation is

$$\frac{1}{\rho_0}[\nabla_{sp} P_0 + \nabla_M(\rho_0 P_1)] = 0. \tag{2.81}$$

Application of an averaging operator for the mesoscale variables combined with (2.80) and sublinear growth then yields

$$\nabla_{sp} P_0 = 0, \tag{2.82}$$

and this procedure can be iterated to give the same result for P_1, P_2.

Now, let us turn to the derivation proper: with (2.52), the $O(\epsilon^1)-$equation for $\mathbf{u}$ is

$$\frac{1}{\rho_0}\nabla_M[\rho_0 \pi^{(3)}] = \nabla_M \pi^{(3)} = 0, \tag{2.83}$$

so $\pi^{(3)}$ does not depend on the mesoscale variables either.
The next order then yields (remember that $Y_M = \epsilon^2 y$!)

$$\begin{aligned} \mathbf{u}^{(0)}_{T_M} + \mathbf{u}^{(0)} \cdot \nabla_M \mathbf{u}^{(0)} + w^{(2)} \mathbf{u}^{(0)}_z + \beta Y_M \mathbf{k} \times \mathbf{u}^{(0)} \\ + \nabla_M \pi^{(4)} + \nabla_{sp} \pi^{(3)} = S^{(2)}_{\mathbf{u}}. \end{aligned} \tag{2.84}$$

At third order, the horizontal momentum equation is

$$\begin{aligned} \mathbf{u}^{(1)}_{T_M} + \mathbf{u}^{(0)}_{T_{sea}} + u^{(1)} \cdot \nabla_M \mathbf{u}^{(0)} \\ + \mathbf{u}^{(0)} \cdot \nabla_{sp} \mathbf{u}^{(0)} + \mathbf{u}^{(0)} \cdot \nabla_M \mathbf{u}^{(1)} + w^{(2)} \mathbf{u}^{(1)}_z + w^{(3)} \mathbf{u}^{(0)}_z \\ + \beta Y_M \mathbf{k} \times \mathbf{u}^{(1)} + 2 w^{(2)} \mathbf{k} \times \mathbf{f} + \nabla_M \pi^{(5)} + \nabla_{sp} \pi^{(4)} = S^{(3)}_{\mathbf{u}}. \end{aligned} \tag{2.85}$$

Potential temperature transport Up to and including $O(\epsilon^3)$, the potential temperature transport equation is trivial. The first nontrivial relation at next order is

$$w^{(2)}[\Theta_2]_z = S^{(4)}_\theta; \tag{2.86}$$

this expresses the fundamental WTG balance. To quote directly from [12]:

"It states that mass elements move quasiinstantaneously toward their new vertical level of neutral buoyancy under local heat addition."

For $\theta^{(3)}$, we have at order $O(\epsilon^5)$:

$$\theta^{(3)}_{T_M} + u^{(0)} \cdot \nabla_M \theta^{(3)} + w^{(2)} \theta^{(3)}_z + w^{(3)} [\Theta_2]_z = S^{(5)}_\theta. \tag{2.87}$$

(2.86) and (2.87) are the final equations that we will need in order to formulate the MEWTG and SPEWTG regimes in three dimensions.

The 3D-MEWTG regime The three-dimensional MEWTG regime arises from the leading-order equations just derived, specializing to solutions that only act on the mesoscales; with the replacements

$$\begin{aligned}
[\mathbf{u}^{(0)}, w^{(2)}, \pi^{(4)}] &\to [\mathbf{u}, w, \pi], \\
[S^{(2)}_{\mathbf{u}}, S^{(4)}_\theta] &\to [S_{\mathbf{u}}, S_\theta], \\
[T_M, \mathbf{X}_M] &\to [t, \mathbf{x}], \\
\nabla_M &\to \nabla,
\end{aligned}$$

chosen as in [12], we readily obtain from (2.84), (2.78) and (2.86), respectively:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + w \mathbf{u}_z + \beta y \mathbf{k} \times \mathbf{u} + \nabla \pi = S_{\mathbf{u}}, \tag{2.88}$$

$$\nabla \cdot (\rho_0 \mathbf{u}) + [\rho_0 w]_z = 0, \tag{2.89}$$

$$w \Theta_2'(z) = S_\theta. \tag{2.90}$$

These equations form a three-dimensional analogue to the "classical" WTG equations described in chapter 1 and section 2.2 before.

The 3D-SPEWTG regime Here, the same reasoning as in section 1.3 applies: to avoid the inclusion of midlatitude effects, we choose a smaller length scale for the meridional direction. Furthermore, the meridional and vertical velocities are assumed to be one order of magnitude weaker than the zonal one. Hence, we require $v^{(0)} = w^{(2)} = 0$, and consider solutions that depend only on X_{sp}, Y_M, T_{sea} and z. With the replacements

$$\begin{aligned}
[u^{(0)}, v^{(1)}, w^{(3)}, \pi^{(4)}] &\to [u, v, w, \pi], \\
[S^{(3)}_u, S^{(2)}_v, S^{(5)}_\theta] &\to [S_u, S_v, S_\theta] \text{ and} \\
[T_{sea}, X_{sp}, Y_M] &\to [t, x, y],
\end{aligned}$$

again chosen as in [12], the zonal component of equation (2.85), the meridional component of (2.84), (2.79) and (2.87) translate to

$$u_t + uu_x + vu_y + wu_z - \beta yv + \pi_x = S_u, \tag{2.91}$$
$$\beta yu + \pi_y = S_v, \tag{2.92}$$
$$u_x + v_y + \frac{1}{\rho_0}[\rho_0 w]_z = 0, \tag{2.93}$$
$$w\Theta_2'(z) = S_\theta, \tag{2.94}$$

where the heat source here is weaker, i.e. higher-order in ϵ, than in (2.90).
This is the 3D-version of the SPEWTG regime from section 2.3.

2.3.2 IPESD in 3D

For our final asymptotic regime, we once again consider motions on the synoptic and planetary scales; contrasting with section 2.1, we differentiate between two scales for time as well as length; the length scales correspond to those previously introduced. With $\epsilon \sim \frac{1}{8}$, we achieve the correct scaling by fractional powers of ϵ :

$$\mathbf{X}_S = \epsilon^{5/2} x,$$
$$\mathbf{X}_P = \epsilon^{7/2} x$$

are the spatial variables that resolve processes on the synoptic scale (up to 5000 km) and the planetary scale (up to 40000 km), respectively.
For time, one natural choice is the synoptic advection timescale, with the time variable

$$T_S = \epsilon^{5/2} t,$$

corresponding to the intraseasonal timescale in section 2.1; additionally, we introduce the *synoptic gravity wave timescale* with time variable

$$T_{S,g} = \epsilon^{3/2} t;$$

it is worth mentioning that this is *not* the advection timescale corresponding to $\mathbf{X}_P$, which might appear to be the "logical" choice; some familiarity with equatorial waves is needed to come up with the physically correct multiscale ansatz.
As far as the multiple scales expansion is concerned, most of our discussion in subsection 2.3.1 carries over: all functions - unless stated otherwise - depend on $(\mathbf{X}_S, X_P, z, T_{S,g}, T_S)$, where only the zonal component of $\mathbf{X}_P$ is included in order to exclude midlatitude phenomena; the individual expansions are given by

$$\mathbf{u} = \mathbf{u}^{(0)} + \epsilon \mathbf{u}^{(1)} + O(\epsilon^2), \tag{2.95}$$
$$w = \epsilon^{3/2} w^{(3/2)} + \epsilon^{5/2} w^{(5/2)} + \epsilon^{7/2} w^{(7/2)} + O(\epsilon^{9/2}), \tag{2.96}$$
$$p = P_0(z) + \rho_0(z)[\epsilon P_1(z) + \epsilon^2 P_2(z) + \epsilon^3 \pi^{(3)} + \dots], \tag{2.97}$$
$$\theta = 1 + \epsilon^2 \Theta_2(z) + \epsilon^3 \theta^{(3)} + \dots. \tag{2.98}$$

Here, the vertical velocity w has to be expanded in fractional powers of ϵ because the terms in the asymptotic expansion of the momentum equations otherwise would not match. Finally, the source terms have the following strengths:

$$S_{\mathbf{u}} = O(\epsilon^{3/2}), \tag{2.99}$$
$$S_w = O(\epsilon^{3/2}), \tag{2.100}$$
$$S_p = O(\epsilon^{9/2}), \tag{2.101}$$
$$S_\theta = O(\epsilon^{9/2}); \tag{2.102}$$

partial derivatives are given by

$$\nabla = \epsilon^{5/2}\nabla_S + \epsilon^{7/2}\nabla_P \text{ and}$$
$$\partial_t = \epsilon^{3/2}\partial_{T_{S,g}} + \epsilon^{5/2}\partial_{T_S}.$$

We are now ready to proceed with the derivations:

Vertical momentum balance Since no derivatives with respect to rescaled variables are involved, the results for vertical momentum in section 3.1 carry over term-by-term: in particular, $P_1 = \rho_1 \equiv 0$.

Pressure equation The leading-order equation for $w^{(3/2)}$ leads to the exact same result as (2.72)-(2.73); therefore, $w^{(3/2)} \equiv 0$.
At next order $O(\epsilon^{5/2})$, the equation reads

$$w^{(5/2)}P_0'(z) + \gamma P_0(\nabla_S \cdot \mathbf{u}^{(0)} + w_z^{(5/2)}) = 0; \tag{2.103}$$

expressing P_0 in terms of ρ_0 again yields

$$\nabla_S \cdot \mathbf{u}^{(0)} + \frac{1}{\rho_0}[\rho_0 w^{(5/2)}]_z = 0. \tag{2.104}$$

The equation at next order is obtained in the same manner - we only need to remember that there is no meridional variation on the planetary scale, hence the presence of ∂_{X_P} only:

$$u_{X_P}^{(0)} + \nabla_S \cdot \mathbf{u}^{(1)} + \frac{1}{\rho_0}[\rho_0 w^{(7/2)}]_z = 0. \tag{2.105}$$

Horizontal momentum balance The first nontrivial equation here is of order $O(\epsilon^{3/2})$: a straightforward evaluation of the individual terms at leading order yields

$$\mathbf{u}_{T_{S,g}}^{(0)} + \beta Y_S \mathbf{k} \times \mathbf{u}^{(0)} + \nabla_S \pi^{(3)} = S_{\mathbf{u}}^{(3/2)}. \tag{2.106}$$

At next order, we obtain

$$\mathbf{u}_{T_S}^{(0)} + \mathbf{u}_{T_{S,g}}^{(1)} + \mathbf{u}^{(0)} \cdot \nabla_S \mathbf{u}^{(0)} + w^{(5/2)} \mathbf{u}_z^{(0)} \\ + \beta Y_S \mathbf{k} \times \mathbf{u}^{(1)} + \nabla_S \pi^{(4)} + \mathbf{i} \pi_{X_P}^{(3)} = S_{\mathbf{u}}^{(5/2)}, \qquad (2.107)$$

with $\mathbf{i} = (1, 0, 0)^T$ being the zonal unit vector.

Potential temperature transport The first nontrivial equation occurs at order $O(\epsilon^{9/2})$:

$$\theta_{T_{S,g}}^{(3)} + w^{(5/2)} \Theta_2'(z) = S_\theta^{(9/2)}. \qquad (2.108)$$

In contrast to the earlier derivation for MEWTG and SPEWTG, the presence of the fast gravity wave timescale $T_{S,g}$ here produces an equation for the evolution of potential temperature in time. At next order, we get

$$\theta_{T_{S,g}}^{(4)} + \theta_{T_S}^{(3)} + \mathbf{u}^{(0)} \cdot \nabla_S \theta^{(3)} + w^{(5/2)} \theta_z^{(3)} + w^{(7/2)} \Theta_2'(z) = S_\theta^{(11/2)}. \qquad (2.109)$$

The set of equations required for our asymptotic regimes is now complete.

IPESD model equations in 3D We first introduce the three-dimensional version of LEWE from (2.22)-(2.23); for this purpose, we will neglect dependencies on the planetary scale. We then present the generalization of the principal IPESD equations derived in section 2.1 to the 3D case. The replacements for brevity, adapted from [12], are

$$\begin{aligned} [\mathbf{u}^{(0)}, \mathbf{u}^{(1)}, w^{(5/2)}, w^{(7/2)}, \pi^{(3)}, \pi^{(4)}, \theta^{(3)}] &\to [\mathbf{u}, \mathbf{u}', w, w', \pi, \pi', \theta], \\ [S_{\mathbf{u}}^{(3/2)}, S_{\mathbf{u}}^{(5/2)}, S_\theta^{(9/2)}, S_\theta^{(11/2)}] &\to [S_{\mathbf{u}}, S_{\mathbf{u}}', S_\theta, S_\theta'], \\ [T_{S,g}, T_S] &\to [\tau, t], \\ [\mathbf{X}_S, X_P] &\to [x, y, X] \text{ and} \\ \nabla_S &\to \nabla. \end{aligned}$$

The leading-order equations (2.106), (2.71), (2.103) and (2.108) now yield

$$u_\tau - \beta y v + \pi_x = S_u, \qquad (2.110)$$

$$v_\tau + \beta y u + \pi_y = S_v, \qquad (2.111)$$

$$\pi_z = \theta, \qquad (2.112)$$

$$\nabla \cdot \mathbf{u} + \frac{1}{\rho_0} (\rho_0 w)_z = 0, \qquad (2.113)$$

$$\theta_\tau + w \Theta_2'(z) = S_\theta, \qquad (2.114)$$

the *3D linear equatorial wave equations.*

In order to formulate IPESD in 3D, we first consider (2.110)-(2.114) without variation on the fast scale τ :

$$-\beta y v + \pi_x = S_u, \tag{2.115}$$
$$\beta y u + \pi_y = S_v, \tag{2.116}$$
$$\pi_z = \theta, \tag{2.117}$$
$$\nabla \cdot \mathbf{u} + \frac{1}{\rho_0}(\rho_0 w)_z = 0, \tag{2.118}$$
$$w\Theta_2'(z) = S_\theta. \tag{2.119}$$

(2.115)and (2.116) now express horizontal geostrophic balance, while (2.117) tells us that the pressure is in hydrostatic balance; (2.119) is a WTG equation.
Next, we want to describe a particular solution $(u^p, v^p, w^p, \pi^p, \theta^p)$ of (2.115)-(2.119); for simplicity, we assume that the source terms are given externally, meaning that they do not depend on u, v and so forth. The equations are then linear.
(2.119) immediately yields an explicit formula for vertical velocity:

$$w^p = \frac{S_\theta}{\Theta_2'(z)}. \tag{2.120}$$

The vertical curl of our geostrophic balance equations, i.e. ∂_x of (2.116) minus ∂_y of (2.115) gives us the same for v^p :

$$\begin{aligned}
\beta y u_x^p + \beta v^p + \beta y v_y^p &= [S_v]_x - [S_u]_y \\
\implies v^p &= \frac{1}{\beta}([S_v]_x - [S_u]_y) - y(u_x^p + v_y^p) \\
\implies v^p &= \frac{1}{\beta}([S_v]_x - [S_u]_y) + y\frac{1}{\rho_0}[\rho_0 w^p]_z \\
\implies v^p &= \frac{1}{\beta}([S_v]_x - [S_u]_y) + y\frac{1}{\rho_0}[\frac{\rho_0 S_\theta}{\Theta_2'(z)}]_z,
\end{aligned} \tag{2.121}$$

by (2.118) and (2.120).
For the zonal velocity, we first recall the zonal averaging operator $\overline{(.)}$, which here denotes the average with respect to x. u^p is then selected by requiring

$$\beta y \overline{u^p} = \overline{S_v}; \tag{2.122}$$

together with (2.118) restated as

$$u_x^p = -v_y^p - \frac{1}{\rho_0}[\rho_0 w^p]_z, \tag{2.123}$$

this specifies a *unique* solution - contrasting with w^p and v^p, we need the additional assumption (2.122)!

(2.115)-(2.116) now yield explicit formulas or π^p_x and π^p_y; (2.122) further implies $[\overline{\pi^p}]_y = 0$. Therefore, the additional constraint

$$\overline{\pi^p}(t, 0, z) = 0 \tag{2.124}$$

is sufficient to uniquely determine π^p.
Lastly, the potential temperature is given by (2.117):

$$\theta^p = [\pi^p]_z. \tag{2.125}$$

We note that the zonal average of (2.115) combined with (2.121) imposes a constraint on S_u :

$$\overline{S_u} + y(-[\overline{S_u}]_y + \frac{\beta y}{\rho_0}[\frac{\rho_0 \overline{S_\theta}}{\Theta_2'(z)}]_z) = 0. \tag{2.126}$$

This can be viewed as analogous to the constraint on the source terms for (2.9)-(2.10) in the shallow water case, and the system (2.115)-(2.119) in fact corresponds to the latter, in spite of its vastly greater complexity.
It is straightforward to see that *homogeneous* solutions to (2.115)-(2.119) are given by

$$v = w = 0, \tag{2.127}$$

while $(u, \pi, \theta) = (U, P, \Theta)(t, y, z)$ do not depend on x but are otherwise arbitrary except for hydrostatic balance

$$P_z = \Theta \tag{2.128}$$

and meridional geostrophic balance

$$P_y = -\beta y U. \tag{2.129}$$

Thanks to our assumptions on the source terms, we can now write any solution to (2.115)-(2.119) in the form $u = U + u^p$, $v = v^p$ and so forth.

The IPESD model equations are now obtained by zonally averaging the higher-order equations that we derived earlier in this section; we give the details in only one instance:
horizontal momentum (2.107) becomes

$$\begin{aligned} &\overline{u_t} + \overline{uu_x} + \overline{vu_y} + \overline{wu_z} - \beta y \overline{v'} + \overline{\pi'_x} + \overline{\pi_X} = \overline{S'_u} \\ \implies &\overline{u_t} + \overline{vu_y} + \overline{wu_z} - \beta y \overline{v'} + \overline{\pi_X} = \overline{S'_u}, \end{aligned} \tag{2.130}$$

due to boundedness and sublinear growth conditions. With $u = U + u^p$, this further yields

$$
\begin{aligned}
&U_t + \overline{u^p}_t + \overline{v^p}U_y + \overline{v^p u^p_y} \\
&+\overline{w^p}U_z + \overline{w^p u^p_z} - \beta y\overline{v'} + P_X = \overline{S'_u} \\
&\Longrightarrow\ U_t + \overline{u^p}_t + \overline{v^p}U_y + \overline{v^p u^p_y} \\
&+\overline{w^p}U_z + \overline{w^p u^p_z} - \beta y\overline{v'} + P_X = \overline{S'_u}.
\end{aligned}
\tag{2.131}
$$

Introducing (averaged) material derivatives with respect to our particular solutions given by

$$
\begin{aligned}
D^p_t &= \partial_t + u^p\partial_x + v^p\partial_y + w^p\partial_z \text{ and} \\
\overline{D^p_t} &= \partial_t + \overline{v^p}\partial_y + \overline{w^p}\partial_z,
\end{aligned}
$$

(2.131) can be rewritten in a more compact form:

$$
\overline{D^p_t}U + P_X - \beta y\overline{v'} = \overline{S'_u} - \overline{D^p_t u^p}. \tag{2.132}
$$

A similar derivation from (2.109) yields

$$
\overline{D^p_t}\Theta + \overline{w'}\Theta'_2(z) = \overline{S'_\theta} - \overline{D^p_t\theta^p}, \tag{2.133}
$$

and the zonal average of (2.105) becomes

$$
U_X + [\overline{v'}]_y + \frac{1}{\rho_0}[\rho_0 w']_z = 0. \tag{2.134}
$$

(2.132)-(2.134) plus the constraints (2.128) and (2.129) constitute the three-dimensional IPESD model equations.
To round out our discussion, we mention that ignoring variation on the planetary scale X yields the equivalent to the SEWTG equations from section 2.1; if, on the other hand, we instead prescribe weak source terms $S_u = S_v = S_\theta \equiv 0$, the particular solutions also vanish; the material derivatives then reduce to simple time derivatives and the resulting equations are the 3D-generalization of QLELWE.

Chapter 3

Rigorous Justification: The Linear Equatorial Long-Wave Equations as a Singular Limit

As we laid out in the preceding chapter, the validity of formal limits can be justified in a variety of ways; the conscientious mathematician, however, will still try to prove beyond any doubt what intuition and formal reasoning make plausible - namely that the governing equations actually *converge* to their respective leading-order approximations. Generally, this is a very difficult task, especially in the scenario where multiple scales are involved; The books [14] and [15] by Andrew Majda provide some classical results.
As far as our derivations in chapter 2 are concerned, there is an elegant proof for the convergence of the equatorial shallow water equations to LELWE. More specifically, we prove that (2.1)-(2.2) converges to (2.19)-(2.21) when the source terms vanish (which implies that the equations become linear, since $V^{(0)} = v^{(0)} = 0$).
In the first section of this chapter, we review a few properties of modified Sobolev spaces and symmetric hyperbolic systems of partial differential equations. Then, we present a proof for the aforementioned theorem that was originally published in [3]. Basic facts about Sobolev spaces are summarized in Appendix A.

3.1 Preparatory Results

Looking back to the setup for subsection 2.2.1, we have equations that depend on two scales zonally and a single meridional scale; since the equatorial region covers the whole circumference of the earth in the zonal direction, with limited meridional variation, we take the zonal variables X and x to be periodic and view them as elements of $\mathbb{T} = \{z \in \mathbb{C} | |z| = 1\}$. The meridional variable y varies in $\mathbb{R}$.
We therefore consider solutions on the set $\Omega := \mathbb{T} \times \mathbb{T} \times \mathbb{R}$. We can define Sobolev spaces on Ω in the usual manner and prove the standard embedding theorems for them, as laid out in the appendix; however, the special mathematical structure of the shallow water equations in long-wave scaling requires that we use spaces with more restrictive regularity assumptions, the *y-weighted Sobolev spaces* $\tilde{H}^{2n}$:

Definition. $\tilde{H}^{2n}(\Omega)$ *is the space of all functions* $w : \Omega \to \mathbb{R}$ *in* $H^{2n}(\Omega)$ *such that the norm defined by*

$$\|w\|^2_{\tilde{H}^{2n}} = \sum_{j+k+l+m\leq 2n} \left\| y^j \partial_X^k \partial_x^l \partial_y^m w \right\|^2_{L^2}$$

is finite.

Equipped with this norm, $\tilde{H}^{2n}$ is a Banach space, and replacing the norms on the right-hand side with L^2-inner products yields the corresponding inner product that makes $\tilde{H}^{2n}$ a Hilbert space. Since we defined those spaces for the very narrow purpose of proving a theorem that only makes use of spaces of even order, we included only even exponents in the definition.
We shall need the following technical lemma:

Lemma 1. *Bounded sets of $\tilde{H}^{2n}$ are precompact in $\tilde{H}^{2(n-1)}$. In other words: the embedding of $\tilde{H}^{2n}$ in $\tilde{H}^{2(n-1)}$ is compact.*

Proof. It is sufficient to prove that every bounded sequence in $\tilde{H}^{2n}$ has a convergent subsequence in $\tilde{H}^{2(n-1)}$. In order to do that, we consider so-called *cutoff functions:* By convolution of a suitable characteristic function with a mollifier (see Appendix A), we can postulate the existence of a smooth function $\phi_1(y)$ with the following properties:

- $\phi_1 \geq 0$.
- $\phi_1 = 1$ on the interval [-1,1].
- $\phi_1 = 0$ outside of (-2,2).

We then define for arbitrary $N \in \mathbb{N}$:

$$\phi_N(y) = \begin{cases} \phi_1(y-N+1),\ y \in (N, N+1) \\ \phi_1(y+N-1),\ y \in (-N-1, -N) \\ 1,\ y \in [-N, N] \\ 0 \text{ otherwise.} \end{cases} \tag{3.1}$$

By construction, ϕ_N is a smooth function with compact support for all N, and since all ϕ_N are just shifted versions of ϕ_1, their derivatives up to any given order are uniformly bounded. We further define

$$\psi_N(y) := 1 - \phi_N(y.)$$

We obviously have $\psi_N = 0$ on [-N,N].
Let now (w_i) be a bounded sequence in $\tilde{H}^{2n}$; for every N, $(\phi_N w_i)$ is then a bounded sequence in $\tilde{H}^{2n}_0(\tilde{\Omega})$ for some bounded $\tilde{\Omega} \subset \Omega$ and the variant of Rellich's embedding theorem discussed in the appendix shows that $(\phi_N w_i)$ has a $\tilde{H}^{2(n-1)}$-convergent subsequence. Furthermore, we can estimate any term of the form $\left\| y^j \partial_X^k \partial_x^l \partial_y^m (\psi_N w) \right\|_{L^2}$ with $j + k + l + m \leq 2(n-1)$ as follows:

$$\begin{aligned}
\left\|y^j\partial_X^k\partial_x^l\partial_y^m(\psi_N w)\right\|_{L^2}^2 &= \int \left|y^j\partial_X^k\partial_x^l\partial_y^m(\psi_N w)\right|^2 \\
= N^{-2}\int \left|Ny^j\partial_X^k\partial_x^l\partial_y^m(\psi_N w)\right|^2 &\le N^{-2}\int \left|y^{j+1}\partial_X^k\partial_x^l\partial_y^m(\psi_N w)\right|^2 \\
\le mN^{-2}\left[\int \left|y^{j+1}\partial_X^k\partial_x^l\partial_y^m(w)\psi_N\right|^2 + \int \left|y^{j+1}\partial_X^k\partial_x^l\partial_y^{m-1}(w)\partial_y(\psi_N)\right|^2 + \dots\right], \quad (3.2)
\end{aligned}$$

where we applied the product rule and the algebraic estimate $\left|\sum_{i=1}^m x_i\right|^2 \le m\sum_{i=1}^m |x_i|^2$. Every term on the right-hand side is of the form

$$\int \left|y^{j+1}\partial_X^k\partial_x^l\partial_y^{m-m'}(w)\partial_y^{m'}(\psi_N)\right|^2.$$

Since derivatives of ψ_N are uniformly bounded up to order $2n$, say with constant C, and the remaining terms make up a part of the $\tilde{H}^{2n}-$norm of w_i, we can further estimate (3.2) by

$$\frac{C}{N^2}\|w_i\|_{\tilde{H}^{2n}}^2; \tag{3.3}$$

summing up, this implies

$$\|\psi_N w_i\|_{\tilde{H}^{2(n-1)}}^2 \le \frac{\tilde{C}}{N^2}\|w_i\|_{\tilde{H}^{2n}}^2 \le \frac{\tilde{\tilde{C}}}{N^2} \tag{3.4}$$

for some constant $\tilde{\tilde{C}}$ that depends only on n (remember that (w_i) is bounded in $\tilde{H}^{2n}$!) With that in mind, we can proceed:
We first choose a subsequence (w_{i_1}) such that $(\phi_1 w_{i_1})$ has a $\tilde{H}^{2(n-1)}$-convergent subsequence. Then

$$\begin{aligned}
\|w_{i_1} - w_{j_1}\|_{\tilde{H}^{2(n-1)}}^2 &= \|\phi_1(w_{i_1} - w_{j_1}) + \psi_1(w_{i_1} - w_{j_1})\|_{\tilde{H}^{2(n-1)}}^2 \\
&\le 2\|\phi_1 w_{i_1} - \phi_1 w_{j_1}\|_{\tilde{H}^{2(n-1)}}^2 + 2\|\psi_1(w_{i_1} - w_{j_1})\|_{\tilde{H}^{2(n-1)}}^2 \\
&\le 2\|\phi_1 w_{i_1} - \phi_1 w_{j_1}\|_{\tilde{H}^{2(n-1)}}^2 + 2\tilde{\tilde{C}}. \qquad (3.5)
\end{aligned}$$

The sequence $(\phi_1 w_{i_1})$ is Cauchy in $\tilde{H}^{2(n-1)}$, therefore there is an index I_1 such that

$$\|\phi_1 w_{i_1} - \phi_1 w_{j_1}\|_{\tilde{H}^{2(n-1)}}^2 \le \tilde{\tilde{C}}\,\forall\, i_1, j_1 \ge I_1. \tag{3.6}$$

Inductively, we now refine our subsequence for $N \ge 1$ as follows:

- $(w_{i_{N+1}}) \subset (w_{i_N})$.
- $(w_{i_{N+1}})$ is such that $(\phi_{N+1} w_{i_{N+1}})$ converges in $\tilde{H}^{2(n-1)}$.

For every such sequence, we can choose an index I_{N+1} such that $I_{N+1} > I_N$ and

$$\left\|\phi_{N+1}w_{i_{N+1}} - \phi_{N+1}w_{j_{N+1}}\right\|^2_{\tilde{H}^{2(n-1)}} \leq \frac{\tilde{\tilde{C}}}{N^2}\,\forall\, i_{N+1}, j_{N+1} \geq I_{N+1}. \tag{3.7}$$

Then we can derive as in (3.5):

$$\left\|w_{i_{N+1}} - w_{j_{N+1}}\right\|^2_{\tilde{H}^{2(n-1)}} \leq \left\|\phi_{N+1}w_{i_{N+1}} - \phi_{N+1}w_{j_{N+1}}\right\|^2_{\tilde{H}^{2(n-1)}} + \frac{\tilde{\tilde{C}}}{N^2} \leq \frac{2\tilde{\tilde{C}}}{N^2} \tag{3.8}$$

for all i_{N+1}, j_{N+1} as specified above.
In particular:

$$\left\|w_{I_N} - w_{I_{N+k}}\right\|^2_{\tilde{H}^{2(n-1)}} \leq \frac{2\tilde{\tilde{C}}}{N^2}\,\forall\, k \geq 0 \tag{3.9}$$

due to our construction; this means that the subsequence (w_{I_N}) is Cauchy and therefore converges in $\tilde{H}^{2(n-1)}$. □

So far, we have only treated the spatial dependence of our solutions; since smoothness requirements with respect to time are typically different in the study of partial differential equations, we introduce the following notation:

Definition. *$C^k([0,T],B)$ denotes the space of functions $w : [0,T] \to B$ with values in the Banach space B that are k times continuously differentiable with respect to the norm of B.*

The compact embedding shown in Lemma 1 now permits us to state the following version of the *Aubin-Lions compactness lemma* analogous to [15], p. 72:

Lemma 2. *Let (w_i) be a sequence of functions in $C([0,T],\tilde{H}^{2n}(\Omega)) \cap C^1([0,T],\tilde{H}^{2(n-1)}(\Omega))$ that satisfies the following:*

1. *(w_i) is uniformly bounded in $C([0,T],\tilde{H}^{2n}(\Omega))$: there is a constant C_1 such that $\max_{0\leq t\leq T} \|w_i\|_{\tilde{H}^{2n}} \leq C_1$.*

2. *The sequence of time derivatives $(\partial_t w_i)$ is uniformly bounded in $C([0,T],\tilde{H}^{2(n-1)}(\Omega))$: there is a C_2 such that $\max_{0\leq t\leq T} \|\partial_t w_i\|_{\tilde{H}^{2(n-1)}} \leq C_2$.*

Then (w_i) has a subsequence that converges in $C([0,T],\tilde{H}^{2(n-1)}(\Omega))$.

This is one of the main technical tools that we will need for our convergence theorem. Finally, we rely on the following general existence and uniqueness theorem for symmetric hyperbolic systems of conservation laws, proved in chapter 2 of [14]:

Theorem 1. *Consider the system*

$$\mathbf{v}_t + \sum_{i=1}^{N} \partial_{x_i} F_i(\mathbf{v}) = S(\mathbf{v}, \mathbf{x}, t) \tag{3.10}$$

in N space dimensions, with smooth functions F_i and S and initial state $\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$. Assume that the linearized, source-free system

$$\mathbf{v}_t + \sum_{i=1}^{N} A_i(v_0) \cdot \mathbf{v}_{x_i} = 0,$$
$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}),$$

where $A_i(\mathbf{v}) = \partial_u F_i$, can be symmetrized in the following way:
There is a symmetric, positive definite matrix $A_0(\mathbf{v})$, smooth as a function of $\mathbf{v}$, with the properties

- *$cI \leq A_0(\mathbf{v}) \leq c^{-1}I$ (elementwise) for some constant c uniformly for all $\mathbf{v}$ in the state space.*
- *$A_0(\mathbf{v})A_i(\mathbf{v})$ is symmetric for all i.*

Assume further $\mathbf{v}_0 \in H^k(\mathbb{R}^N)$, with $k > \frac{N}{2} + 1$. Then there is a time $T > 0$ such that (3.10) has a unique solution

$$\mathbf{v} \in C([0, T], H^k(\mathbb{R}^N)) \cap C^1([0, T], H^{k-1}(\mathbb{R}^N)).$$

We will shortly see that the shallow water equations in long-wave scaling satisfy those conditions.

Remark: Theorem 1 applies to functions on $\mathbb{R}^N$. Its proof, however, carries over to our space Ω without any difficulty.

3.2 The convergence theorem

Recall the shallow water equations scaled for the derivation of the IPESD models (with vanishing source terms):

$$\frac{D\mathbf{v}}{Dt} + \epsilon^{-1}(y\mathbf{v}^{\perp} + \nabla h) = 0,$$
$$\frac{Dh}{Dt} + h \ \mathrm{div}\mathbf{v} + \epsilon^{-1} \ \mathrm{div}\mathbf{v} = 0.$$

We consider solutions that depend on two zonal length scales x and $X = \epsilon x$:

$$\begin{pmatrix} u \\ v \\ h \end{pmatrix} = \begin{pmatrix} u \\ v \\ h \end{pmatrix} (x, X, y, t).$$

Written term by term, our system reads

$$u_t + uu_x + \epsilon uu_X + vu_y + h_x + \epsilon^{-1}(-yv + h_x) = 0, \tag{3.11}$$
$$v_t + uv_x + \epsilon uv_X + vv_y + h_y + \epsilon^{-1}(yu + h_y) = 0, \tag{3.12}$$
$$h_t + uh_x + \epsilon uh_X + vh_y + (1+\epsilon h)u_X + h(u_x + v_y) + \epsilon^{-1}(u_x + v_y) = 0, \tag{3.13}$$

and we supplement it with initial conditions

$$\begin{aligned} u_{|t=0} &= u_{0,\epsilon}, \\ v_{|t=0} &= v_{0,\epsilon}, \\ h_{|t=0} &= h_{0,\epsilon}. \end{aligned} \tag{3.14}$$

Our aim is to prove that the solutions to (3.11)-(3.14) exist and converge as $\epsilon \to 0$ to the solutions of

$$u_t + h_X - yV = 0, \tag{3.15}$$
$$h_t + u_X + V_y = 0, \tag{3.16}$$
$$yu + h_y = 0, \tag{3.17}$$

with initial conditions

$$\begin{aligned} u_{|t=0} &= u_0, \\ h_{|t=0} &= h_0, \end{aligned} \tag{3.18}$$

which constitute LELWE; the relationship of V and v will be specified later.
With the results from section 3.1, we are now ready to state and prove the main theorem of [3],

Theorem 2. *Assume that for some $n \geq 3$, $\epsilon_0 > 0$ and all $\epsilon \leq \epsilon_0$, $(u_{0,\epsilon}, v_{0,\epsilon}, h_{0,\epsilon}) \in \tilde{H}^{2n}$ and*

$$(-yv_{0,\epsilon} + \partial_x h_{0,\epsilon}, yu_{0,\epsilon} + \partial_y h_{0,\epsilon}, \partial_x u_{0,\epsilon} + \partial_y v_{0,\epsilon}) = O(\epsilon) \tag{3.19}$$

as $\epsilon \to 0$ in $\tilde{H}^{2(n-1)}$. Then there is a time $T > 0$ independent of ϵ such that there are solutions

$$(u^\epsilon, v^\epsilon, h^\epsilon) \in C([0,T], \tilde{H}^{2n}) \cap C^1([0,T], \tilde{H}^{2(n-1)})$$

of (3.11)-(3.14). Those solutions converge in $C([0,T], \tilde{H}^{2(n-1)})$ to $(u^0, 0, h^0)$, where u^0, h^0 do not depend on x and together with some function V, given in terms of u^0 and h^0, solve LELWE.

The proof of Theorem 2, adapted from [3], will be structured as follows:

1. We symmetrize the equations.

2. We prove that two differential operators occurring in the symmetrized system commute.
3. We find an equivalent norm for $\tilde{H}^{2n}$.
4. We prove existence and uniqueness of solutions for all $\epsilon > 0$, with a uniform time of existence $T > 0$.
5. We prove that those solutions converge to solutions of LELWE as $\epsilon \to 0$.

3.2.1 Symmetrization

We begin with a substitution for the height perturbation h :
Let $\tilde{h}$ be defined by

$$h = \frac{(1 + \frac{\epsilon\tilde{h}}{2})^2 - 1}{\epsilon}. \tag{3.20}$$

This transforms equations (3.11)-(3.13) into

$$u_t + uu_x + \epsilon uu_X + vu_y + \left(1 + \frac{\epsilon\tilde{h}}{2}\right)\tilde{h}_X + \epsilon^{-1}\left(-yv + \left(1 + \frac{\epsilon\tilde{h}}{2}\right)\tilde{h}_X\right) = 0, \tag{3.21}$$

$$v_t + uv_x + \epsilon uv_X + vv_y + \epsilon^{-1}\left(yu + \left(1 + \frac{\epsilon\tilde{h}}{2}\right)\tilde{h}_y\right) = 0, \tag{3.22}$$

$$\begin{aligned}\left(1 + \frac{\epsilon\tilde{h}}{2}\right)\tilde{h}_t + u\left(1 + \frac{\epsilon\tilde{h}}{2}\right)\tilde{h}_x + \epsilon u\left(1 + \frac{\epsilon\tilde{h}}{2}\right)\tilde{h}_X + v\left(1 + \frac{\epsilon\tilde{h}}{2}\right)\tilde{h}_y \\ + \left(1 + \frac{\epsilon\tilde{h}}{2}\right)^2 u_X + \left[\frac{\left(1 + \frac{\epsilon\tilde{h}}{2}\right)^2 - 1}{\epsilon}\right](u_x + v_y) + \epsilon^{-1}(u_x + v_y) = 0.\end{aligned} \tag{3.23}$$

Expanding and simplifying then yields

$$\begin{aligned}u_t + uu_x + \epsilon uu_X + vu_y + \tilde{h}_X + \epsilon\frac{1}{2}\tilde{h}\tilde{h}_X + \frac{1}{2}\tilde{h}\tilde{h}_x \\ + \epsilon^{-1}(-yv + \tilde{h}_x) = 0,\end{aligned} \tag{3.24}$$

$$v_t + uv_x + \epsilon uv_X + vv_y + \frac{1}{2}\tilde{h}\tilde{h}_y + \epsilon^{-1}(yu + \tilde{h}_y) = 0, \tag{3.25}$$

$$\begin{aligned}\tilde{h}_t + u\tilde{h}_x + \epsilon u\tilde{h}_X + v\tilde{h}_y + \left(1 + \frac{\epsilon\tilde{h}}{2}\right)u_X \\ + \frac{1}{2}\tilde{h}(u_x + v_y) + \epsilon^{-1}(u_x + v_y) = 0.\end{aligned} \tag{3.26}$$

We further define r and l by

$$u = \frac{r-l}{\sqrt{2}}, \tag{3.27}$$

$$\tilde{h} = \frac{r+l}{\sqrt{2}}, \tag{3.28}$$

which lets us rewrite (3.24) as

$$\begin{aligned} \frac{r_t - l_t}{\sqrt{2}} + \frac{r-l}{\sqrt{2}} \cdot \frac{r_x - l_x}{\sqrt{2}} + \epsilon \frac{r-l}{\sqrt{2}} \cdot \frac{r_X - l_X}{\sqrt{2}} + v \frac{r_y - l_y}{\sqrt{2}} \\ + \frac{1}{2} \frac{r+l}{\sqrt{2}} \cdot \frac{r_x + l_x}{\sqrt{2}} + \left(1 + \frac{1}{2} \epsilon \frac{r+l}{\sqrt{2}}\right) \frac{r_X - l_X}{\sqrt{2}} + \epsilon^{-1}(-yv + \frac{r_x + l_x}{\sqrt{2}}) = 0. \end{aligned} \tag{3.29}$$

Equation (3.25) becomes

$$\begin{aligned} v_t + \frac{r-l}{\sqrt{2}} v_x + \epsilon \frac{r-l}{\sqrt{2}} v_X + v v_y + \frac{1}{2} \frac{r+l}{\sqrt{2}} \cdot \frac{r_y + l_y}{\sqrt{2}} \\ + \epsilon^{-1} \left(y \frac{r-l}{\sqrt{2}} + \frac{r_y + l_y}{\sqrt{2}} \right) = 0 \end{aligned} \tag{3.30}$$

and (3.26) now reads

$$\begin{aligned} \frac{r_t + l_t}{\sqrt{2}} + \frac{r-l}{\sqrt{2}} \cdot \frac{r_x + l_x}{\sqrt{2}} + \epsilon \frac{r-l}{\sqrt{2}} \cdot \frac{r_X + l_X}{\sqrt{2}} + v \frac{r_y + l_y}{\sqrt{2}} \\ + \frac{1}{2} \frac{r+l}{\sqrt{2}} \left(\frac{r_x - l_x}{\sqrt{2}} + v_y \right) + \left(1 + \frac{1}{2} \epsilon \frac{r+l}{\sqrt{2}}\right) \frac{r_X - l_X}{\sqrt{2}} + \epsilon^{-1} \left(\frac{r_x - l_x}{\sqrt{2}} + v_y \right) = 0. \end{aligned} \tag{3.31}$$

From the sum of (3.29) and (3.31) we obtain

$$\begin{aligned} r_t + \frac{r-l}{\sqrt{2}}(r_x + \epsilon r_X) + v r_y + \frac{r+l}{2\sqrt{2}}(r_x + \epsilon r_X) + \frac{r+l}{4} v_y + r_X \\ + \epsilon^{-1} \left(r_x + \frac{v_y - yv}{\sqrt{2}} \right) = 0, \end{aligned} \tag{3.32}$$

whereas (3.29) subtracted from (3.31) yields

$$\begin{aligned} l_t + \frac{r-l}{\sqrt{2}}(l_x + \epsilon l_X) + v l_y + \frac{r+l}{2\sqrt{2}}(-l_x - \epsilon l_X) + \frac{r+l}{4} v_y - l_X \\ + \epsilon^{-1} \left(-l_x + \frac{v_y + yv}{\sqrt{2}} \right) = 0. \end{aligned} \tag{3.33}$$

Simplifying one more time finally gives us

$$r_t + \frac{3r-l}{2\sqrt{2}}(r_x + \epsilon r_X) + r_X + v r_y + \frac{r+l}{4} v_y + \epsilon^{-1}\left(r_x + \frac{v_y - yv}{2\sqrt{2}}\right) = 0, \tag{3.34}$$

$$l_t + \frac{r-3l}{2\sqrt{2}}(l_x + \epsilon l_X) - l_X + v l_y + \frac{r+l}{4} v_y + \epsilon^{-1}\left(-l_x + \frac{v_y + yv}{2\sqrt{2}}\right) = 0, \tag{3.35}$$

$$v_t + \frac{r-l}{\sqrt{2}}(v_x + \epsilon v_X) + v v_y + \frac{r+l}{4}(r_y + l_y) + \epsilon^{-1}\left(\frac{r_y + yr}{\sqrt{2}} + \frac{l_y - yl}{\sqrt{2}}\right) = 0. \tag{3.36}$$

in our analysis, we will from now on use this transformed system as our starting point.

3.2.2 A commutation relation

A glance at the terms of order $O(\epsilon^{-1})$ in (3.34)-(3.36) reveals that terms with an explicit dependence on y occur only in the form

$$\frac{1}{\sqrt{2}}(\partial_y \mp y) =: L_\pm. \tag{3.37}$$

These are known as the *raising* and *lowering* operators of the Hamiltonian of the harmonic oscillator,

$$H := L_- L_+ + L_+ L_- = \partial_y^2 - y^2. \tag{3.38}$$

Setting

$$\mathbf{U} = \begin{pmatrix} r \\ l \\ v \end{pmatrix},$$

we can write (3.34)-(3.36) in the form

$$\mathbf{U}_t + A_1(\mathbf{U})\mathbf{U}_x + A_2(\mathbf{U})\mathbf{U}_X + A_3(\mathbf{U})\mathbf{U}_y + \epsilon^{-1} M \mathbf{U} = 0, \tag{3.39}$$

with symmetric matrices A_i smoothly varying with $\mathbf{U}$ and

$$M = \begin{pmatrix} \partial_x & 0 & L_+ \\ 0 & -\partial_x & L_- \\ L_- & L_+ & 0 \end{pmatrix}. \tag{3.40}$$

Let now $[A, B] = AB - BA$ denote the commutator of the operators A and B; the following properties of $L_\pm$ can be verified by direct calculation:

- $L_\pm^\star = -L_\mp$. Therefore, M is an antisymmetric matrix operator.

- $[H, L_\pm] = \mp 2L_\pm$.

The latter identity can be written as $HL_\pm \pm 2L_\pm = L_\pm H$, or equivalently

$$(H \pm 2)L_\pm = L_\pm H. \tag{3.41}$$

If we apply, say, $(H \pm 2)$ to this relation from the left, we obtain

$$(H \pm 2)^2 L_\pm = (H \pm 2)L_\pm H = (L_\pm H)H = L_\pm H^2;$$

in the same manner, induction shows that

$$P(H \pm 2)L_\pm = L_\pm P(H) \tag{3.42}$$

for any polynomial P. Since for every such polynomial, $P(H \mp 2)$ also is a polynomial of H, it further holds

$$P(H)L_\pm = L_\pm P(H \mp 2); \tag{3.43}$$

in particular,

$$HL_\pm = L_\pm(H \mp 2). \tag{3.44}$$

Together with (3.41), this shows that the matrix operator

$$D := \begin{pmatrix} H+2 & 0 & 0 \\ 0 & H-2 & 0 \\ 0 & 0 & H \end{pmatrix} \tag{3.45}$$

commutes with M :

$$DM = MD \tag{3.46}$$

This relation will be key to obtaining a uniform estimate for our solution.

3.2.3 An alternative characterization of $\tilde{H}^{2n}$

We now want to express the $\tilde{H}^{2n}$-norm in terms of the operator H. It seems plausible that an equivalent norm would be given by

$$(\|w\|')^2_{\tilde{H}^{2n}} := \sum_{j+k+l+2p \le 2n} \left\| \partial_x^k \partial_X^l H^p w \right\|^2_{L^2}, \tag{3.47}$$

and this is indeed true:

Lemma 3. *For all $n \geq 1$, there is a constant C_n such that*

$$\|w\|^2_{\tilde{H}^{2n}} \leq C_n(\|w\|')^2_{\tilde{H}^{2n}}; \tag{3.48}$$

since $(\|w\|')_{\tilde{H}^{2n}} \leq \|w\|_{\tilde{H}^{2n}}$ by definition, this means that the two norms are equivalent.

Proof. First, observe that functions of the form $w = w_1(x, X)w_2(y)$ are dense in $\tilde{H}^{2n}$ and it therefore suffices to prove (3.48) for functions of y alone (since the L^2-norms factor in this case and the terms containing w_1 are identical on both sides). The relation that we intend to prove then reduces to

$$\sum_{j+m\leq 2n} \left\|y^j \partial_y^m w\right\|^2_{L^2} \leq C_n \sum_{p=0}^{n} \|H^p w\|^2_{L^2}. \tag{3.49}$$

We prove this for $n = 1$ first; integration by parts and the Cauchy-Schwarz inequality yield

$$\begin{aligned} \|yw\|^2_{L^2} + \|w_y\|^2_{L^2} = \|yw\|^2_{L^2} + (w_y, w_y)_{L^2} = \|yw\|^2_{L^2} - (w, w_{yy})_{L^2} = -\int_{\mathbb{R}} wHwdy \\ \leq \|w\|_{L^2}\|Hw\|_{L^2} \leq \|w\|^2_{L^2} + \|Hw\|^2_{L^2}. \end{aligned} \tag{3.50}$$

Similarly, we obtain

$$\|w_{yy}\|^2_{L^2} + 2\|yw_y\|^2_{L^2} + \left\|y^2 w\right\|^2_{L^2} = \|Hw\|^2_{L^2} + 2\|w\|^2_{L^2}. \tag{3.51}$$

Combining (3.50) and (3.51) verifies (3.49) for $n = 1$.
Applying what we just proved to higher-order terms, we can estimate for $j \geq 2$:

$$\left\|y^j \partial_y^m w\right\|^2_{L^2} \leq C_1(\left\|H(y^{j-2}\partial_y^m w)\right\|^2_{L^2} + \left\|y^{j-2}\partial_y^m w\right\|^2_{L^2}) \tag{3.52}$$

and for $m \geq 2$:

$$\left\|y^j \partial_y^m w\right\|^2_{L^2} \leq C_1(\left\|H(y^j \partial_y^{m-2} w)\right\|^2_{L^2} + \left\|y^j \partial_y^{m-2}\right\|^2_{L^2}). \tag{3.53}$$

With the additional commutation relations

$$[H, y] = 2\partial_y, \quad [H, \partial_y] = 2y \text{ and } \quad [\partial_y, y] = 1,$$

we can rearrange the H-terms on the right-hand side such that any induction step can be completed; relation (3.49) and consequently (3.48) therefore hold for all $n \geq 1$. $\square$

3.2.4 Existence and uniqueness in $\tilde{H}^{2n}$

The system (3.34)-(3.36) obviously is of the form (3.10) and its linearization yields a symmetric hyperbolic system; Theorem 1 therefore applies and it only remains to show that the solutions for $0 < \epsilon \le \epsilon_0$ are uniformly bounded in $\tilde{H}^{2n}$ for some $T > 0$. Specifically, we aim to prove

Lemma 4. *Let $n \ge 2$, $\epsilon_0 > 0$. Assume that the initial data $(r_{0,\epsilon}, l_{0,\epsilon}, v_{0,\epsilon})$ for (3.34)-(3.36) are uniformly bounded in $(\tilde{H}^{2n})^3$ for $0 \le \epsilon \le \epsilon_0$. Then there is a $T > 0$ such that (3.34)-(3.36) has a unique solution for all such ϵ on $[0, T]$, where the solution remains uniformly bounded in $(\tilde{H}^{2n})^3$.*

Proof. By Theorem 1, there is a $T > 0$ such that the solution exists and is unique in $C(, [0, T], H^{2n})$. It is therefore enough to prove a uniform estimate in the $\tilde{H}^{2n}$-norm for the solution. To do this, we first consider the general system

$$\mathbf{V}_t + A_1(\mathbf{U})\mathbf{V}_x + A_2(\mathbf{U})\mathbf{V}_X + A_3(\mathbf{U})\mathbf{V}_y + \epsilon^{-1}M\mathbf{V} = \mathbf{F}, \tag{3.54}$$

where A_i and M are defined as in (3.39)-(3.40).
Multiplying (3.54) with

$$\mathbf{V} = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix},$$

we get

$$\begin{aligned} &\sum_{i=1}^{3} v^i v^i_t + \sum_{i=1}^{3}\sum_{j=1}^{3} v^i a_1^{ij}(\mathbf{U}) v^j_x + \sum_{i=1}^{3}\sum_{j=1}^{3} v^i a_2^{ij}(\mathbf{U}) v^j_X \\ &+ \sum_{i=1}^{3}\sum_{j=1}^{3} v^i a_3^{ij}(\mathbf{U}) v^j_y + \epsilon^{-1} \sum_{i=1}^{3}\sum_{j=1}^{3} v^i m^{ij} v^j = \sum_{i=1}^{3} v^i f^i. \end{aligned} \tag{3.55}$$

We have

$$\int_\Omega \sum_i \sum_j v^i a_1^{ij}(\mathbf{U}) v^j_x = -\int_\Omega \sum_i \sum_j v^i_x a_1^{ij}(\mathbf{U}) v^j - \int_\Omega \sum_i \sum_j v^i [a_1^{ij}(\mathbf{U})]_x v^j \tag{3.56}$$

by integration over the spatial variables and integration by parts. The symmetry of A_1 then yields

$$\int_\Omega \sum_i \sum_j v^i a_1^{ij}(\mathbf{U}) v^j_x = -\frac{1}{2} \int_\Omega \sum_i \sum_j v^i [a_1^{ij}(\mathbf{U})]_x v^j. \tag{3.57}$$

Analogous identities can be derived for the terms involving A_2, A_3 and the other spatial derivatives.

In the expression $\int_\Omega \sum_i \sum_j v^i m^{ij} v^j$, all terms off the diagonal drop out due to the antisymmetry of M; the remainder $\int_\Omega v^1 v^1_x - v^2 v^2_x = 0$, by another integration by parts, meaning that the dependence on ϵ^{-1} drops out of the equation! All in all, we get:

$$\int_\Omega \sum_i [\frac{(v^i)^2}{2}]_t - \frac{1}{2}\int_\Omega \sum_i \sum_j v^i [a_1^{ij}(\mathbf{U})_x] v^j - \frac{1}{2}\int_\Omega \sum_i \sum_j v^i [a_2^{ij}(\mathbf{U})_X] v^j$$
$$-\frac{1}{2}\int_\Omega \sum_i \sum_j v^i [a_3^{ij}(\mathbf{U})_y] v^j = \int_\Omega \sum_i v^i f^i, \qquad (3.58)$$

which can be rearranged in the form

$$\int_\Omega \sum_i [(v^i)^2]_t = \int_\Omega \sum_i \sum_j v^i [a_1^{ij}(\mathbf{U})_x + a_2^{ij}(\mathbf{U})_X + a_3^{ij}(\mathbf{U})_y] v^j + 2\int_\Omega \sum_i v^i f^i$$
$$\iff \frac{d}{dt}\|\mathbf{V}\|^2_{L^2} = (\mathbf{V}, (A_1(\mathbf{U})_x + A_2(\mathbf{U})_X + A_3(\mathbf{U})_y)\mathbf{V})_{L^2} + 2(\mathbf{V}, \mathbf{F})_{L^2}. \qquad (3.59)$$

A bound for the right-hand side is obtained by Cauchy-Schwarz and pulling out the matrix norms:

$$(\mathbf{V}, (A_1(\mathbf{U})_x + A_2(\mathbf{U})_X + A_3(\mathbf{U})_y)\mathbf{V})_{L^2} + 2(\mathbf{V}, \mathbf{F})_{L^2}$$
$$\leq \sum_i \|A_i(\mathbf{U})\|_{C^1} \|\mathbf{V}\|^2_{L^2} + \|\mathbf{V}\|^2_{L^2} + \|\mathbf{F}\|^2_{L^2}$$
$$= (1 + \sum_i \|A_i(\mathbf{U})\|_{C^1}) \|\mathbf{V}\|^2_{L^2} + \|\mathbf{F}\|^2_{L^2}. \qquad (3.60)$$

Since the original system (3.39) fulfils (3.54) with $\mathbf{F} = 0$, we get as a consequence

$$\frac{d}{dt}\|\mathbf{U}\|^2_{L^2} \leq [\sum_i \|A_i(\mathbf{U})\|_{C^1}] \|\mathbf{U}\|^2_{L^2}. \qquad (3.61)$$

If we now apply $\partial_x^k \partial_X^l D^p$ to (3.39) and recall that we proved $DM = MD$, it immediately follows that $\mathbf{V} = \partial_x^k \partial_X^l D^p \mathbf{U}$ satisfies (3.54) with

$$\mathbf{F} = \mathbf{F}_{k,l,p} := [\partial_x^k \partial_X^l D^p, A_1(\mathbf{U})]\mathbf{U}_x + [\partial_x^k \partial_X^l D^p, A_2(\mathbf{U})]\mathbf{U}_X$$
$$+ [\partial_x^k \partial_X^l D^p, A_3(\mathbf{U})]\mathbf{U}_y. \qquad (3.62)$$

We now estimate $\|\mathbf{F}_{k,l,p}\|_{L^2}$:
Letting the *y-order* of any term of the form $y^j \partial_x^k \partial_X^l \partial_y^m$ be $j + m$ and the *total order* $j + k + l + m$, we observe:

The operator D can be written as

$$D = (\partial_y^2 - y^2)I + \tilde{D}, \tag{3.63}$$

where $\tilde{D}$ has lower y-order than the leading-order term, which is a *scalar* operator; accordingly,

$$D^p = (\partial_y^2 - y^2)^p I + \tilde{\tilde{D}}, \tag{3.64}$$

with $\tilde{\tilde{D}}$ having y-order $< 2n$. Recall now that the commutator of a scalar operator and a matrix operator has lower order than the sum of their individual orders, which shows that each term $[\partial_x^k \partial_X^l D^p, A_i(\mathbf{U})]$ has total order $< 2n$, and every term in $F_{k,l,p}$ consequently has maximal total order $2n$.
Out of the individual factors in each term in $F_{k,l,p}$, there can be only one with total order $\geq 2n-2$ (remember $n \geq 2$!) The Sobolev lemma for dimension three therefore guarantees that all other factors are in C^{2n-2}, which permits us to pull those factors out of the L^2-norm in the form $\|\cdot\|^2_{C^{2n-2}}$, which in turn can be estimated in terms of $\|\mathbf{U}\|^2_{\tilde{H}^{2n}}$. The remaining factor has total order $\leq 2n$ and can be estimated in terms of $\|\mathbf{U}\|^2_{\tilde{H}^{2n}}$ as well. Combining all those estimates shows that there is a smooth function G such that

$$\|F_{k,l,p}\|^2_{L^2} \leq G(\|\mathbf{U}\|^2_{\tilde{H}^{2n}}). \tag{3.65}$$

Apply now the estimate (3.59)-(3.60) to $V_{k,l,p}$ for each triple (k,l,p) with $k+l+2p \leq 2n$. Combined with the fact that $\|A_i(\mathbf{U})\|_{C^1} \leq G_1(\|\mathbf{U}\|_{H^3}) \leq G_2(\|\mathbf{U}\|_{\tilde{H}^{2n}})$ for some smooth G_1, G_2, again due to Sobolev's theorem, summing up over all terms of the form (3.65) finally yields

$$\frac{d}{dt}\|\mathbf{U}\|^2_{\tilde{H}^{2n}} \leq G_3(\|\mathbf{U}\|^2_{\tilde{H}^{2n}}) \tag{3.66}$$

for a suitable smooth function G_3. This implies that there is a $T > 0$ such that $\|\mathbf{U}\|_{\tilde{H}^{2n}}$ is uniformly bounded for $t \in [0,T]$. □

3.2.5 Convergence

It remains to show that the solutions to the transformed shallow water equations converge to a limit and that this limit satisfies LELWE. This is achieved by

Lemma 5. *Take $n \geq 3$ arbitrary and let the assumptions of Lemma 4 hold; additionally, assume*

$$\|\partial_x r_{0,\epsilon} + L_+ v_{0,\epsilon}\|_{\tilde{H}^{2(n-1)}} + \|-\partial_x l_{0,\epsilon} + L_- v_{0,\epsilon}\|_{\tilde{H}^{2(n-1)}} \\ + \|L_- r_{0,\epsilon} + L_+ l_{0,\epsilon}\|_{\tilde{H}^{2(n-1)}} \leq C\epsilon \tag{3.67}$$

*for some $C > 0$, as well as $r_{0,\epsilon} \to r_{0,0}$, $l_{0,\epsilon} \to l_{0,0}$ in $\tilde{H}^{2n}$ as $\epsilon \to 0$.
Then, as $\epsilon \to 0$, $r^\epsilon \to r^0$, $l^\epsilon \to l^0$ and $v^\epsilon \to 0$ in*

$$C([0,T],\tilde{H}^{2(n-1)}) \cap C([0,T],C^1_{loc})$$

and

$$\epsilon^{-1}v^\epsilon \rightharpoonup v^1,$$

meaning v^ϵ converges to v^1 weakly.
Furthermore, r^0, l^0 and v^1 do not depend on x and uniquely solve

$$r^0_t + r^0_X + L_+v^1 = 0, \tag{3.68}$$
$$l^0_t - l^0_X + L_-v^1 = 0, \tag{3.69}$$
$$L_-r^0 + L_+l^0 = 0 \tag{3.70}$$

with initial data $(r_{0,0}, l_{0,0})$ and the time T of uniform existence obtained in Lemma 4.

Proof. At $t=0$, applying $\|\cdot\|_{\tilde{H}^{2(n-1)}}$ to

$$\mathbf{U}_t + A_1(\mathbf{U})\mathbf{U}_x + A_2(\mathbf{U})\mathbf{U}_X + A_3(\mathbf{U})\mathbf{U}_y + \epsilon^{-1}M\mathbf{U} = 0 \tag{3.71}$$

yields

$$\|\mathbf{U}_t(0,x,X,y)\|_{\tilde{H}^{2(n-1)}} \leq G_4(\|\mathbf{U}(0,x,X,y)\|_{\tilde{H}^{2n}}) + \epsilon^{-1}\|M\mathbf{U}(0,x,X,y)\|_{\tilde{H}^{2(n-1)}}, \tag{3.72}$$

with a smooth function G_4 obtained as in the proof of Lemma 4. The bound (3.67) then implies

$$\|\mathbf{U}_t(0,x,X,y)\|_{\tilde{H}^{2(n-1)}} < \infty. \tag{3.73}$$

Application of $\partial_x^k\partial_X^l D^p\partial_t$ with $j+k+2p \leq 2(n-1)$ to (3.71) and again emulating the procedure in Lemma 4 results in the estimate

$$\frac{d}{dt}\|\mathbf{U}_t\|^2_{\tilde{H}^{2(n-1)}} \leq G_5(\|\mathbf{U}\|^2_{\tilde{H}^{2n}})\|\mathbf{U}_t\|^2_{\tilde{H}^{2(n-1)}}, \tag{3.74}$$

which tells us that $\mathbf{U}_t$ is also uniformly bounded for $t \leq T$.
It is now time to employ Lemma 2: both $\mathbf{U}$ and $\mathbf{U}_t$ fulfil the stated conditions and we can infer that $(r^\epsilon, l^\epsilon, v^\epsilon)$ converge at least along some sequence ϵ_i in $C([0,T],\tilde{H}^{2(n-1)})$ to a limit (r^0, l^0, v^0).
Multiplying (3.34)-(3.36) with ϵ and passing to the limit gives us

$$r^0_x + L_+v^0 = 0, \tag{3.75}$$
$$-l^0_x + L_-v^0 = 0 \text{ and} \tag{3.76}$$
$$L_-r^0 + L_+l^0 = 0. \tag{3.77}$$

Computing ∂_x of (3.77) plus substitution of (3.75)-(3.76) yields

$$L_- r_x^0 + L_+ l_x^0 = L_-(-L_+ v^0) + L_+(L_- v^0) = [L_+, L_-]v^0 = v^0 = 0, \tag{3.78}$$

as claimed. It follows immediately that $r_x^0 = l_x^0 = 0$, so r^0 and l^0 are independent of x.
Let now, for any function $f = f(x)$, $\overline{f} := \frac{1}{2\pi} \int\limits_{\mathbb{T}} f dx$ denote its average with respect to x. Averaging (3.34)-(3.35) then yields

$$\overline{r}_t + \overline{\frac{3r - l}{2\sqrt{2}}(r_x + \epsilon r_X)} + \overline{r}_X + \overline{v r_y} + \overline{\frac{r + l}{4} v_y} + \epsilon^{-1} L_+ \overline{v} = 0, \tag{3.79}$$

$$\overline{l}_t + \overline{\frac{r - 3l}{2\sqrt{2}}(l_x + \epsilon l_X)} - \overline{l}_X + \overline{v l_y} + \overline{\frac{r + l}{4} v_y} + \epsilon^{-1} L_- \overline{v} = 0. \tag{3.80}$$

Reminding ourselves that in the limit $\epsilon_i \to 0$, v vanishes and r and l converge to functions independent of x, we can infer from the above

$$\overline{r}_t + \overline{r}_X + \epsilon_i^{-1} L_+ \overline{v} = o(1), \tag{3.81}$$

$$\overline{l}_t - \overline{l}_X + \epsilon_i^{-1} L_- \overline{v} = o(1), \tag{3.82}$$

as $\epsilon_i \to 0$. Applying L_- and L_+ to (3.81) and (3.82), respectively, and using $[L_+, L_-] = I$, it follows

$$\partial_t (L_- \overline{r} - L_+ \overline{l}) + \partial_X (L_- \overline{r} + L_+ \overline{l}) - \epsilon_i^{-1} \overline{v} = o(1). \tag{3.83}$$

This shows that $\epsilon^{-1} v$ is bounded; remember that every bounded sequence in a Hilbert space possesses a weakly convergent subsequence, therefore there is a subsequence ϵ_{i_k} and a $v^1 \in C([0, T], \tilde{H}^{2(n-1)})$ such that

$$\epsilon_{i_k}^{-1} \rightharpoonup v^1 \tag{3.84}$$

as $\epsilon_{i_k} \to 0$; since $L_- r^0 + L_+ l^0 = 0$, we find

$$v^1 = 2 L_- r_t^0 = -2 L_+ l_t^0. \tag{3.85}$$

taking the limit $\epsilon_i \to 0$ in (3.81)-(3.82) further yields

$$r_t^0 + r_X^0 + L_+ v^1 = 0, \tag{3.86}$$

$$l_t^0 - l_X^0 + L_- v^1 = 0. \tag{3.87}$$

Together with (3.77), this shows that (r^0, l^0) indeed solve (3.68)-(3.70).
In order to finish our proof, we need to prove that the solution (r^0, l^0) is unique; due to linearity, it is sufficient to prove that (3.86)-(3.87) with vanishing initial conditions

has only the trivial solution $(0,0)$. Noting that the L^2-adjoints of L_pm are given by $L_\pm^\star = -L_\mp$, respectively, we multiply (3.86) by r^0 and integrate to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|r^0\right|^2 + \int_\Omega r_X^0 r^0 + 2\int_\Omega L_+L_-r_t^0 r^0 = 0, \tag{3.88}$$

which simplifies to

$$\frac{1}{2}\frac{d}{dt}\left\|r^0\right\|_{L^2}^2 = -2(L_+L_-r_t^0, r^0)_{L^2}$$
$$= 2(L_-r_t^0, L_-r^0)_{L^2} = \frac{d}{dt}\left\|L_-r^0\right\|_{L^2}^2. \tag{3.89}$$

By the same token, we deduce from (3.87)

$$\frac{1}{2}\frac{d}{dt}\left\|l^0\right\|_{L^2}^2 = -\frac{d}{dt}\left\|L_+l^0\right\|_{L^2}^2. \tag{3.90}$$

With $L_-r^0 = -L_+l^0$, this implies

$$\frac{d}{dt}\left\|r^0\right\|_{L^2}^2 = -\frac{d}{dt}\left\|l^0\right\|_{L^2}^2, \tag{3.91}$$

but since we assumed initial data $r_{0,0} = l_{0,0} = 0$, this necessitates $r^0 = l^0 \equiv 0$, which proves the assertion. Uniqueness further implies that the solutions converge as $\epsilon \to 0$ without restriction to a sequence. □

The proof of our theorem is now almost complete:
Reverting the substitutions by

$$r^0 = \frac{h^0 + u^0}{\sqrt{2}}, \tag{3.92}$$
$$l^0 = \frac{h^0 - u^0}{\sqrt{2}}, \tag{3.93}$$

(3.68)-(3.70) become

$$\frac{h_t^0 + u_t^0}{\sqrt{2}} + \frac{h_X^0 + u_X^0}{\sqrt{2}} + L_+v^1 = 0, \tag{3.94}$$
$$\frac{h_t^0 - u_t^0}{\sqrt{2}} - \frac{h_X^0 - u_X^0}{\sqrt{2}} + L_-v^1 = 0, \tag{3.95}$$
$$\frac{L_-h^0 + L_+u^0}{\sqrt{2}} + \frac{L_+h^0 - L_+u^0}{\sqrt{2}} = 0. \tag{3.96}$$

Simplifying (3.94) and adding and subtracting (3.92) and (3.93), respectively, yields

$$u_t^0 + h_X^0 - yv^1 = 0, \tag{3.97}$$
$$h_t^0 + u_X^0 + v_y^1 = 0 \text{ and} \tag{3.98}$$
$$h_y^0 + yu^0 = 0: \tag{3.99}$$

These are the equations for LELWE, as claimed in Theorem 2. □

Appendix A

Sobolev spaces and mollifiers

Basic definitions In chapter 3, we deal with L^2-Sobolev spaces on $\Omega = \mathbb{T}\times\mathbb{T}\times\mathbb{R}$. Their definition is the same as in $\mathbb{R}^N$: for any function $w \in L^2(\Omega)$ and multi-index α, we call $v \in L^2(\Omega)$ the *weak $\alpha-$derivative* of w if

$$\int_\Omega D^\alpha f w = (-1)^{|\alpha|} \int_\Omega f v \tag{A.1}$$

for all $f \in C_c^\infty(\Omega)$, where $C_c^\infty(\Omega)$ denotes the space of all smooth, real-valued functions that are compactly supported in Ω. The *Sobolev space* $H^n(\Omega)$ is now defined as the space of all $w \in L^2(\Omega)$ that possess weak L^2-derivatives for $|\alpha| \leq n$.
For every $n \in \mathbb{N}$, H^n is a Banach space: it is complete with respect to the norm

$$\|w\|_{H^n}^2 := \sum_{j+k+l\leq n} \left\|\partial_X^j \partial_x^k \partial_y^l w\right\|_{L^2}^2. \tag{A.2}$$

Sobolev spaces can also be defined for functions in L^p for arbitrary $p \geq 1$; the particular usefulness of the spaces H^n lies in the fact that they are - just as L^2 itself - Hilbert spaces: one only needs to replace the norms in (A.2) by L^2-inner products in order to obtain the inner product for H^n.
In $H^n(\Omega)$, $C_c^\infty-$functions are dense; this implies in particular a straightforward formula for integration by parts:

$$\int_\Omega D^\alpha w v = (-1)^{|\alpha|} \int_\Omega w D^\alpha v \tag{A.3}$$

for $w, v \in H^n(\Omega)$.

The Sobolev lemma Sobolev spaces provide a very powerful environment for the analysis of partial differential equations; when working with the immediate definition, however, there is the drawback that functions in H^n are not even defined pointwise, let alone smooth in the classical sense; the *Sobolev lemma* remedies that problem:

Lemma 6. *If $n > m + \frac{3}{2}$, then $H^n(\Omega) \subset C^m(\Omega)$ in the sense of a continuous embedding, and there is a constant C depending only on n and m such that*

$$\sup_{|\alpha|\leq m} \|\partial^\alpha w\|_\infty \leq C\|w\|_{H^n} \tag{A.4}$$

for all $w \in H^n$.

A quick proof via the L^2-Fourier transform can be found in [5], p. 194.

The Rellich-Kondrachev theorem We now consider functions in H^n that are supported in an open set $\tilde{\Omega} \subset \Omega$ in the following sense:
$H_0^n(\tilde{\Omega})$ is defined as the closure of $C_c^\infty(\tilde{\Omega})$ in $H^n(\Omega)$.
For the spaces H_0^n, the following classical result is known as the *Rellich-Kondrachev theorem:*

Theorem 3. *If $\tilde{\Omega} \subset \Omega$ is open and bounded, and $n' > n$, the embedding $H_0^{n'}(\tilde{\Omega}) \to H_0^n(\tilde{\Omega})$ is compact: we can extract from every bounded sequence in $H_0^{n'}(\tilde{\Omega})$ a subsequence that converges in $H_0^n(\tilde{\Omega})$.*

Again, we refer to [5], pp. 200-201 for a proof. In order to extend this result to the modified spaces $\tilde{H}^{2n}$ defined in section 3.1, we only need to observe the following:

- Since $\tilde{H}^{2n}(\tilde{\Omega}) \subset H^{2n}(\tilde{\Omega})$, the embedding $\tilde{H}^{2n}(\tilde{\Omega}) \to H^{2(n-1)}(\tilde{\Omega})$ is automatically compact.
- We can then use the boundedness of $\tilde{\Omega}$ to obtain convergence in $\tilde{H}^{2(n-1)}(\tilde{\Omega})$.

Mollifiers As laid out in [4], appendix C.4, for every $N \in \mathbb{N}$ there are $C^\infty(\mathbb{R}^N)-$functions $\eta_\epsilon(\mathbf{x})$ with the following properties:

- $\int_{\mathbb{R}^N} \eta_\epsilon = 1$.
- supp $\eta_\epsilon \subset B(0, \epsilon)$.

These functions are called the *standard mollifiers.* For open $V \subset \mathbb{R}^N$ and $f : V \to \mathbb{R}$ locally L^1, we can then define the *mollification*

$$f^\epsilon := f \star \eta_\epsilon, \tag{A.5}$$

with $\star$ denoting convolution of two functions. Letting $V_\epsilon := \{\mathbf{x} \in V : \text{dist}(\mathbf{x}, \partial V) > \epsilon\}$, the following then hold:

- $f^\epsilon \in C^\infty(V_\epsilon)$.
- $f^\epsilon \to f$ almost everywhere as $\epsilon \to 0$.
- For continuous f, convergence is uniform on compact subsets of V.
- For $f \in L^p_{loc}(V)$ $(1 \leq p \leq \infty)$, convergence also occurs in $L^p_{loc}(V)$.

Mollifications are important workhorses in the analysis of partial differential equations - here, we only require the very basic consequence that the mollification of an indicator function provides the existence of cutoff functions with the properties that we claimed in Lemma 2: for example,

$$\mathbb{1}_{B(0,1)} \star \eta_{\frac{1}{2}}$$

is a smooth function equal to 1 on $B(0, \frac{1}{2})$, with support contained in $B(0, \frac{3}{2})$.

Anhang B

Deutsche Zusammenfassung (german abstract)

Diese Arbeit ist Modellgleichungen für Strömungen in der tropischen Atmosphäre gewidmet, wobei wir die $\beta-$Ebenen-Näherung verwenden. Neben den allgemeinen Gleichungen für kompressible Fluide und den Flachwassergleichungen liegt der Fokus auf der sogenannten "weak temperature gradient approximation"; weiters besprechen wir das Matsuno-Gill-Modell und die äquatorialen Langwellengleichungen.
Im Folgenden untersuchen wir Beziehungen zwischen diesen Modellgleichungen mittels formaler asymptotischer Mehrskalenentwicklungen, aufbauend auf dem wegweisenden Artikel [12] von Rupert Klein und Andrew Majda. Wir erläutern die Mehrskalenmethode ausführlich, wobei besonderer Wert auf das Zusammenspiel unserer Modellannahmen mit der Wahl des formalen Ansatzes gelegt wird.
Das Themenfeld der Exaktifizierung der Mehrskalenasymptotik wird ebenfalls behandelt: Wir stellen einen aktuellen Konvergenzbeweis für die formale Beziehung zwischen den Flachwassergleichungen und den äquatorialen Langwellengleichungen vor, dem Artikel [3] entnommen.

Glossary

Froude number The (barotropic) Froude number $\mathrm{Fr} = \frac{v_{ref}}{\sqrt{g l_{ref}}}$ expresses the ratio of the (reference) fluid velocity to the barotropic gravity wave speed. 5, 6, 20

geostrophic balance A geophysical flow is in geostrophic balance when the horizontal pressure gradient is balanced by the (horizontal) Coriolis acceleration. In the equatorial $\beta-$plane, the corresponding system of equations reads $\nabla p = -\beta y \mathbf{v}^{\perp}$. 7, 8, 32, 33

hydrostatic balance Hydrostatic balance is achieved when the vertical pressure gradient is in balance with the buoyancy forces. The exact mathematical expression depends on the context: $p_z = -g \cdot \rho$ and $p_z = -\theta$ are typical. 25, 32, 33

Mach number The Mach number $\mathrm{M} = \frac{v_{ref}}{\sqrt{p_{ref}/\rho_{ref}}}$ expresses the ratio of the (reference) fluid velocity to the speed of sound waves. 5, 20

material derivative The material derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla$, also known as advective or substantial derivative, is the time derivative along the path of individual fluid elements. 6

potential temperature The temperature that would be attained by a parcel of fluid if it was adiabatically compressed to the sea level pressure. 5

Rossby number The Rossby number $\mathrm{Ro} = \frac{v_{ref}}{\Omega l_{ref}}$ expresses the ratio of inertial forces to Coriolis forces with respect to the chosen length and velocity scales. 5, 20

Acronyms

IPESD intraseasonal planetary equatorial synoptic-scale dynamics. 14, 17, 31, 33, 34

LELWE linear equatorial long-wave equations. 8, 9, 17

LEWE linear equatorial wave equations. 17, 31

MEWTG mesoscale equatorial weak temperature gradient. 17, 19, 28

NLELWE nonlinear equatorial long-wave equations. 8

QLELWE quasi-linear equatorial long-wave equations. 17, 34

SEWTG synoptic-scale equatorial weak temperature gradient. 17, 34

SPEWTG seasonal subplanetary equatorial weak temperature gradient. 19, 20, 28, 29

WTG weak temperature gradient. 11, 19, 20, 27, 28, 32

Bibliography

[1] Joseph A. Biello and Andrew J. Majda: *A New Multiscale Model for the Madden-Julian Oscillation*, Journal of the Atmospheric Sciences, Vol. 62 (2005), pp. 1694-1721.

[2] Paul Dellar: *Variations on a beta-plane: derivation of non-traditional beta-plane equations from Hamiltons principle on a sphere*, Journal of Fluid Mechanics, Vol. 674 (2011), pp. 174-195.

[3] Alexandre Dutrifoy, Andrew J. Majda and Steven Schochet: *A Simple Justification of the Singular Limit for Equatorial Shallow-Water Dynamics*, Communications on Pure and Applied Mathematics, Vol. LXII (2009), pp. 322-333.

[4] Lawrence C. Evans, *Partial Differential Equations*, American Mathematical Society, 1998.

[5] Gerald B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press (2nd ed.), 1995.

[6] A. E. Gill, *Some simple solutions for heat-induced tropical circulation*, Journal of the Royal Meteorological Society, Vol. 106 (1980), pp. 447-462.

[7] J. Kevorkian and J. D. Cole, *Multiple Scale and Singular Perturbation Methods*, Springer-Verlag, 1996.

[8] Boualem Khouider, Andrew J. Majda and Samuel N. Stechmann, *Climate science in the tropics: waves, vortices and PDEs*, Nonlinearity, Vol. 26 (2013), pp. R1-R68.

[9] Rupert Klein, *Asymptotic Analyses for Atmospheric Flows and the Construction of Asymptotically Adaptive Numerical Methods*, Zeitschrift fr Angewandte Mathematik und Mechanik, Vol. 80 (2000), pp. 765-777.

[10] Rupert Klein, *An unified approach to meteorological modelling based on multiple-scales asymptotics*, Advances in Geosciences, Vol. 15 (2008), pp. 23-33.

[11] Rupert Klein, *Scale-Dependent Models for Atmospheric Flows*, Annual Review of Fluid Mechanics, Vol. 42 (2010), pp. 249-274.

[12] Rupert Klein and Andrew J. Majda, *Systematic Multiscale Models for the Tropics*, Journal of the Atmospheric Sciences, Vol. 60 (2003), pp. 393-408.

[13] Rupert Klein and Stefan Vater, *Mathematische Modellierung in der Klimaforschung*, unpublished, 2003.

[14] Andrew J. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag, 1984.

[15] Andrew J. Majda, *Introduction to PDEs and Waves for the Atmosphere and Ocean*, American Mathematical Society, 2003.

[16] Andrew J. Majda, *New Multiscale Models and Self-Similarity in Tropical Convection*, Journal of the Atmospheric Sciences, Vol. 64 (2007), pp. 1393-1404.

[17] Herbert Oertel (edited by), *Prandtl-Führer durch die Strömungslehre*, Vieweg (10th ed.), 2001.

[18] Joseph Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag (2nd ed.), 1987.

[19] Steven Schochet, *Fast Singular Limits of Hyperbolic PDEs*, Journal of Differential Equations, Vol. 114 (1994), pp. 476-512.

[20] Steven Schochet, *The Mathematical Theory of Low Mach Number Flows*, ESAIM: Mathematical Modelling and Numerical Analysis, Vol. 39 (2005), pp. 441-458.

[21] Adam Sobel, Johan Nilsson and Lorenzo M. Polvani, *The Weak Temperature Gradient Approximation and Balanced Tropical Moisture Waves*, Journal of the Atmospheric Sciences, Vol. 58 (2001), pp. 3650-3665.

[22] Dirk Werner, *Funktionalanalysis*, Springer-Verlag (7th ed.), 2011.