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Numerical algorithms for structured nonsmooth and nonconvex optimization problems

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Abstract

A large number of applications in real-world can be formulated and designed as optimization problems. These models are usually large-scale, complexly structured, and exhibit features like nonsmoothness and nonconvexity, which require specific solution methods when addressing them. Such numerical algorithms are preferable first-order methods, due to their simplicity and low iteration and memory storage costs, but also to be formulated in a full splitting spirit, meaning that every element involved in the formulation of the underlying optimization problem is evaluated separately and in an efficient way.

The main purpose of this thesis is to formulate and investigate the convergence properties of full splitting algorithms for different nonsmooth optimization problems, ranging from bilevel convex to structured nonconvex. We focus in particular on the study of the convergence behavior of the developed algorithms and, in some situations, on their rate of convergence.

In the preliminaries, we introduce basic notions and results of convex analysis, maximal monotone operators, variational and nonsmooth analysis, which are of relevance for the thesis. Further, we propose a forward-backward splitting algorithm of penalty type with inertial effects for a complexly structured monotone inclusion problem, which provides a general setting for solving convex bilevel minimization problems. The last three chapters of the thesis are dedicated to the design and analysis of algorithms for nonsmooth and nonconvex optimization problems. They share the feature that, along with the subsequence convergence analysis, the global convergence and converge rates are discussed in the setting of the Kurdyka-Łojasiewicz property. In this context, we first propose a projected gradient algorithm for the factorization of a completely positive matrix with parameters that take into account the effects of relaxation and inertia. Then we consider the proximal and the proximal linearized alternating direction method of multipliers for a nonsmooth and nonconvex optimization problem involving compositions with linear operators. Finally, we develop a proximal approach for nonsmooth problems with block structure coupled by a smooth function.

Zusammenfassung

Viele Anwendungen können als Optimierungsprobleme formuliert werden. Diese Modelle sind in der Regel hochdimensional, komplex strukturiert und weisen Merkmale wie Nichtglattheit und Nichtkonvexität auf, für deren Behandlung spezielle Lösungsmethoden erforderlich sind. Solche numerische Algorithmen sind, aufgrund ihrer Einfachheit und geringen Iterations- und Speicherkosten, vorzugsweise Verfahren erster Ordnung. Des weitern liegt unser Hauptaugenmerk auf sogenannten *full splitting* Verfahren, was bedeutet, dass jedes Element, das an der Formulierung des zugrunde liegenden Optimierungsproblems beteiligt ist, separat und auf effiziente Weise ausgewertet wird.

Der Hauptzweck dieser Arbeit ist die Formulierung und Untersuchung der Konvergenzeigenschaften solcher Algorithmen für verschiedene nicht glatte Optimierungsprobleme, die von konvexen bilevel bis hin zu strukturierten nicht-konvexen Problemen reichen. Wir konzentrieren uns insbesondere auf die Untersuchung des Konvergenzverhaltens der entwickelten Algorithmen und in einigen Situationen auf ihre Konvergenzrate.

Nach einer Einleitung stellen wir Grundbegriffe und Ergebnisse der konvexen Analysis, der maximalmonotonen Operatoren, der Variations- und der nicht-glatten Analysis vor, die für die Arbeit relevant sind. Ferner schlagen wir ein *Forward-Backward-Splitting* Verfahren der *penalty* Art mit Inertialeffekten für ein komplex strukturiertes monotones Inklusionsproblem vor. Dies bietet einen allgemeine Rahmen zur Lösung konvexer bilevel Minimierungsprobleme. Die letzten drei Kapitel der Arbeit befassen sich mit dem Entwurf und der Analyse von Algorithmen für nicht-glatte und nicht-konvexe Optimierungsprobleme. Sie teilen das Merkmal, dass zusammen mit der Konvergenzanalyse der Teilfolgen die globalen Konvergenz- und Konvergenzraten unter der Kurdyka-Łojasiewicz-Eigenschaft diskutiert werden. In diesem Zusammenhang schlagen wir zunächst einen projizierten Gradientenalgorithmus zur Faktorisierung einer vollständig positiven Matrix mit Parametern vor, die die Auswirkungen von Relaxation und Inertia berücksichtigen. Dann betrachten wir die proximale und die proximale linearisierte Version der *alternating direction method of multipliers* für ein nicht-glattes und nicht-konvexes Optimierungsproblem, das Hintereinanderausführungen mit linearen Operatoren beinhaltet. Schließlich entwickeln wir einen proximalen Ansatz für nicht glatte Probleme mit Blockstruktur, die durch eine glatte Funktion gekoppelt sind.

Contents

Chapter 1

Introduction

A large number of real-world applications, from engineering, economics to image and signal processing and machine learning, can be formulated and designed as optimization problems. In order to capture the desired phenomena, these models are usually large-scaled and complexly structured and share features like nonsmoothness and nonconvexity. As a result, the obtained optimization problems are challenging, and specific solution methods are required when addressing them. Such numerical algorithms are preferable first-order methods and should be formulated in a fully splitting spirit. First-order methods exploit only the information provided by function values and gradients/subgradients but not second-order information like the Hessians. They are attractive in modern optimization due to their simplicity and low iteration and memory storage costs. A fully splitting scheme means every element involved in the formulation of the underlying optimization problem is evaluated separately and efficiently. In addition, there is no expensive performance regarding the operator's inversion, and evaluating the sum or composition of the operators/functions is not needed.

The notion of the proximal operator of a convex function, introduced about half a century ago by Moreau [108], is a vital object for full splitting schemes. This fundamental regularization process gave rise to the so-called proximal minimization algorithm by Martinet [105], followed by its extension in Rockafellar [118] for solving monotone inclusions. The proximal operator of a convex function is also the resolvent of the subdifferential associated with the convex function, which is a maximally monotone operator. This is the most direct connection between monotone operator theory and convex optimization. The operator splitting methods were motivated by applications in mechanics and partial differential equations. In 1956, Douglas and Rachford proposed a numerical method to study heat conduction problems [76]. Later on, when considering the monotone inclusions consisting sum of two maximally monotone operators in [100], Lions and Mercier extended this method and proved weak convergence of the algorithm to a solution. For a recent extension of this result, see [122]. In case one of the two maximally monotone operators in the inclusion is single-valued and cocoercive, the forward-backward algorithm [61, 86] can be applied. The principle of this algorithm is to use at every iteration a forward (explicit) step on the single-valued mapping, followed by a backward (implicit) step on the other. For the optimization context, this algorithm is also known as the proximal-gradient algorithm, and the convergence rate for functional value can be derived. If the cocoercivity of the single-valued operator is further relaxed to monotone and Lipschitz continuous, we can use Tseng's forward-backward-forward algorithm [123]. A class of complex optimization problems in which the functions being composed with a bounded linear operator is a good example for the benefit of splitting scheme. They have been successfully used to reduce complex problems into a series of simpler subproblems. In this context, we mention the Proximal Alternating Direction Method of Multipliers, or Proximal ADMM, see [23, 120]. The classical Alternating Direction Method of Multipliers [84, 85] or the primal-dual splitting algorithms [49, 65, 70, 124] are the particular instances of this iterative scheme. Besides the weak convergence of the iterates, one can also obtain the rate for primal-dual gaps in the ergodic sense. In the seminal paper [110], Nesterov proposed an accelerated gradient method. Later on, it has been further extended to the composite minimization problem by Beck and Teboulle in [28], known as FISTA. Since the introduction of Nesterov's scheme, the first-order accelerating methods have become a subject of active research. Accelerated primal-dual schemes can also be obtained, provided some additional conditions on the function are fulfilled, see for example [44, 65].

In the absence of convexity, one of the first papers to study the global convergence of the iterates of the proximal point algorithm was [5] by Attouch and Bolte. This work is a starting point for many papers that study the convergence of various algorithms in the nonconvex setting such as the proximal-gradient, and the Gauss-Seidel method [7, 8] as well as some inertial variants [43, 51, 111]. All the above work rely on the Kurdyka-Łojasiewicz property. The origins of this notion go back to the pioneering work of Kurdyka, who introduced in [93] a general form of the Łojasiewicz inequality [103]. Further extensions to the nonsmooth setting can be found in the works of Attouch, Bolte, and their co-authors [7, 33, 34, 35]. Li and Pong studied some calculus rules in [98]. One of the remarkable properties of the Kurdyka-Łojasiewicz functions is their ubiquity in applications, including semi-algebraic, real sub-analytic, uniformly convex and convex functions satisfying a growth condition. For nonconvex block-structured optimization problem, we mention the Proximal Alternating Linearized Minimization (PALM) of Bolte, Sabach and Teboulle [36]. Li and Pong study in [96] the ADMM for minimizing the sum of a smooth function with a bounded Hessian and a nonsmooth one, the latter being the composition of a proper lower semicontinuous function and a linear operator. The Douglas-Rachford algorithm in the nonconvex setting was also obtained by the same authors [97].

The main purpose of this thesis is to formulate and investigate the convergence properties of full splitting algorithms for different nonsmooth optimization problems, ranging from bilevel convex to structured nonconvex. We focus in particular on the study of the convergence behavior of the sequences of iterates and function values generated by the developed algorithms and, in some situations, on their rate of convergence.

The organization of this thesis is as follows.

We first introduce in the preliminaries basic notions and results of convex analysis, monotone operators theory, variational and nonsmooth analysis, which are of relevance for the thesis. We then present the definition of the Kurdyka-Łojasiewicz property and finally some results regarding the convergence of real sequences.

In Chapter 3, we focus on a complexly structured monotone inclusion problem, consisting of the sum of a maximally monotone operator and a cocoercive one and the convex normal cone to the set of zeroes of another cocoercive operator. This problem also provides a general setting for solving convex bilevel minimization problems containing smooth function in the lower level. To solve this problem, we propose an algorithm that combines the forward-backward splitting with a penalization technique; inertial effects are also considered. We show weak ergodic convergence of the generated sequence of iterates to a solution of the monotone inclusion problem. In the context of bilevel optimization, weak nonergodic and strong convergence can be achieved under further assumptions for the involved functions.

The last three chapters of the thesis are dedicated to the design and analysis of algorithms for nonsmooth and nonconvex optimization problems. Asides from the subsequence convergence, which is the best one can expect in a general nonconvex setting, we can prove global convergence and derive convergence rates by using the Kurdyka-Łojasiewicz property. We also provide sufficient conditions for the boundedness of the generated sequence. In the nonconvex setting, the boundedness of the sequence of generated iterates plays a central role in the convergence analysis, as it would guarantee the existence of cluster points. Cluster points are usually expected to be critical points of the underlying problem.

In Chapter 4, we aim to factorize a completely positive matrix by using an optimization approach. Our model leads to a projected gradient type algorithm with parameters that take into account the effects of relaxation and inertia. Both projection and gradient steps are simple because they have explicit formulas and do not require inner loops. Related approaches in the literature are the ones proposed by Groetzner and Dür [87] or by Chen, Pong, Tan and Zeng [66]. These schemes require in each iteration the performance of a singular value decomposition in the calculation of the projection, which is expensive when the dimension of the matrix to decompose increase. Furthermore, a straightforward step can be performed to find an appropriate starting point for our algorithm, which is another advantage over the methods mentioned above.

Chapter 5 is devoted to the minimization of the sum of a smooth function and the composition of a nonsmooth function with a linear operator in the fully nonconvex setting, similar to the setting in [96]. We propose two numerical algorithms and carry out a parallel convergence analysis for both algorithms. By appropriate choices of the matrix sequences, these two schemes can be formulated in the spirit of the proximal and, respectively, proximal linearized alternating direction method of multipliers.

In the final chapter, we develop a proximal type algorithm for minimizing objective functions consisting of three summands: the composition of a nonsmooth function with a linear operator, another nonsmooth function, each of the nonsmooth summands depending on an independent block variable, and a smooth function which couples the two block variables. We carry out for this scheme a convergence analysis. If the linear operator is merely the identity, our problem becomes the model in [36].

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Chapter 2

Preliminaries

2.1 Basic notions of monotone operators and of convex analysis

Let $\mathcal{H}$ be a real Hilbert space with *inner product* $\langle \cdot, \cdot \rangle$ and associated *norm* $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$.

For an arbitrary set-value operator $A \colon \mathcal{H} \rightrightarrows \mathcal{H}$ we denote by

$$\begin{aligned}
\operatorname{gph} A &:= \left\{ (x, v) \in \mathcal{H} \times \mathcal{H} \colon v \in Ax \right\}, \\
\operatorname{dom} A &:= \left\{ x \in \mathcal{H} \colon Ax \neq \varnothing \right\}, \\
\operatorname{ran} A &:= \left\{ v \in \mathcal{H} \colon \exists x \in \mathcal{H} \text{ with } v \in Ax \right\}, \\
\operatorname{zer} A &:= \left\{ x \in \mathcal{H} \colon 0 \in Ax \right\},
\end{aligned}$$

its *graph*, *domain*, *range* and *set of zeros*, respectively. The *inverse operator* of A is denoted by $A^{-1} \colon \mathcal{H} \rightrightarrows \mathcal{H}$ and defined by $(v, x) \in \operatorname{gph} A^{-1}$ if and only if $(x, v) \in \operatorname{gph} A$. Obviously, $\operatorname{zer} A = A^{-1}(0)$.

Definition 2.1.1. Let $A \colon \mathcal{H} \rightrightarrows \mathcal{H}$ be a set-valued operator.

(i) The operator A is said to be *monotone*, if

$$\langle x - y, v - w \rangle \geqslant 0 \text{ for every } (x, v), (y, w) \in \operatorname{gph} A.$$

(ii) The monotone operator A is said to be *maximally monotone*, if there exists no other monotone operator $A' \colon \mathcal{H} \rightrightarrows \mathcal{H}$ such that $\operatorname{gph} A' \supsetneq \operatorname{gph} A$.

(iii) The operator A is said to be γ*−strongly monotone* for $\gamma > 0$, if

$$\langle x - y, v - w \rangle \geqslant \|x - y\|^2 \text{ for every } (x, v), (y, w) \in \operatorname{gph} A.$$

Let us mention that if A is maximally monotone, then $\operatorname{zer} A$ is a convex and closed set, [24, Proposition 23.39]. We refer to [24, Section 23.4] for conditions ensuring that $\operatorname{zer} A$ is nonempty. If A is maximally monotone, then one has the following characterization for the set of its zeros

$$z \in \operatorname{zer} A \text{ if and only if } \langle u - z, y \rangle \geqslant 0 \text{ for every } (u, y) \in \operatorname{gph} A. \tag{2.1.1}$$

If A is maximally monotone and strongly monotone, then $\operatorname{zer} A$ is a singleton, thus nonempty, [24, Corollary 23.37].

Definition 2.1.2. Let $A \colon \mathcal{H} \to \mathcal{H}$ be a single-valued operator. The operator A is said to be *cocoercive* with constant $\mu > 0$ if its inverse is μ-strongly monotone, that is,

$$\langle x - y, Bx - By \rangle \geqslant \mu \|Bx - By\|^2 \text{ for every } x, y \in \mathcal{H}.$$

A typical example of a cocoercive operator is the gradient of a Fréchet differentiable convex function such that its gradient is Lipschitz continuous. In particular, according to the Baillon-Haddad theorem (see e.g. [24, Corollary 18.17]), if $\Psi\colon \mathcal{H} \to \mathbb{R}$ is a Fréchet differentiable convex function, then $\nabla\Psi$ is Lipschitz continuous with modulus $L > 0$ if and only if it is L^{-1}-cocoercive.

Another beneficial single-valued Lipschitz continuous operator is the resolvent associated with a maximally monotone operator.

Definition 2.1.3. Let $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ be a set-valued operator. The *resolvent* of A, $J_A\colon \mathcal{H} \rightrightarrows \mathcal{H}$, is defined by

$$J_A := (\mathrm{Id} + A)^{-1},$$

where $\mathrm{Id}\colon \mathcal{H} \to \mathcal{H}$ denotes the *identity operator* on $\mathcal{H}$.

This operator enjoys many important properties that make it a central tool in monotone operator theory and its applications. The Theorem of Minty states that it is defined everywhere in $\mathcal{H}$, i.e. $\mathrm{ran}\,(\mathrm{Id} + A) = \mathcal{H}$, if and only if A is maximally monotone ([24, Corollary 23.10]). In particular, it is 1-cocoercive, therefore 1-Lipschitz continuous, and single-valued.

For an arbitrary $\gamma > 0$, we have the following identity ([24, Proposition 23.18])

$$J_{\gamma A} + \gamma J_{\gamma^{-1}A^{-1}} \circ \gamma^{-1}\mathrm{Id} = \mathrm{Id}.$$

Now we consider functions with values in the extended real line $\mathbb{R} \cup \{\pm\infty\}$.

Definition 2.1.4. Let $\Psi\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be an extended-real valued function.

(i) The *effective domain* of Ψ is defined as

$$\mathrm{dom}\Psi := \{x \in \mathcal{H}\colon \Psi(x) < +\infty\}.$$

(ii) The function Ψ is called *proper*, if $\mathrm{dom}\Psi \neq \varnothing$ for all $x \in \mathcal{H}$.

(iii) The function Ψ is called *convex*, if for every $x, y \in \mathcal{H}$ and $0 \leqslant \theta \leqslant 1$

$$\Psi((1-\theta)x + \theta y) \leqslant (1-\theta)\Psi(x) + \theta\Psi(y).$$

(iv) The function Ψ is called *lower semi-continuous* at $x \in \mathcal{H}$ if

$$\liminf_{y \to x} \Psi(y) \geqslant \Psi(x).$$

The function Ψ is called *lower semi-continuous* if it is lower semi-continuous at every $x \in \mathcal{H}$.

Definition 2.1.5. Let $\Psi\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a given function. The *convex subdifferential* of Ψ at the point $x \in \mathcal{H}$ is the set

$$\partial\Psi(x) := \{v \in \mathcal{H}\colon \Psi(y) \geqslant \Psi(x) + \langle v, y - x\rangle\, \forall y \in \mathcal{H}\},$$

whenever $\Psi(x) \in \mathbb{R}$. We take by convention $\partial\Psi(x) = \varnothing$, if $\Psi(x) = +\infty$.

The proximal operator of a proper, convex and lower semicontinuous function is the most direct connection between monotone operator theory and convex optimization. Let $\Psi\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and convex function, $\mathrm{prox}_{\gamma\Psi}\colon \mathcal{H} \to \mathcal{H}$ is a single-valued operator defined as

$$\mathrm{prox}_{\gamma\Psi} = J_{\gamma\partial\Psi} = (\mathrm{Id} + \gamma\partial\Psi)^{-1}.$$

Definition 2.1.6. Let $\Psi\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a function. The *conjugate function* of Ψ is $\Psi^*\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\Psi^*(u) = \sup_{x\in\mathcal{H}} \{\langle x, u\rangle - \Psi(x)\}$$

and it is a proper, convex and lower semicontinuous.

Notice that if Ψ is proper, convex and lower semicontinuous, then $\partial\Psi$ is a maximally monotone operator and it holds $(\partial\Psi)^{-1} = \partial\Psi^*$. We have the so-called *Moreau's decomposition formula*:

$$\mathrm{prox}_{\gamma\Psi} + \gamma\mathrm{prox}_{\gamma^{-1}\Psi^*} \circ \gamma^{-1}\mathrm{Id} = \mathrm{Id}.$$

The function $\Psi\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is said to be $\gamma-$*strongly convex* with $\gamma > 0$, if $\Psi - \frac{\gamma}{2}\|\cdot\|^2$ is a convex function. This property implies that $\partial\Psi$ is a $\gamma-$strongly monotone operator.

Definition 2.1.7. Let M be a nonempty subset of $\mathcal{H}$.

(i) The *indicator function* of the set M is defined by

$$\delta_M(x) := \begin{cases} 0, x \in M \\ +\infty, x \notin M \end{cases}.$$

(ii) The *normal cone* of M is the convex subdifferential of its indicator function. In particular

$$\mathcal{N}_M(x) := \begin{cases} \{v \in \mathcal{H}\colon \langle y - x, v\rangle \leqslant 0 \ \forall y \in \mathcal{H}\}, x \in M \\ \varnothing, x \notin M \end{cases}.$$

Notice that for $x \in M$ we have

$$v \in \mathcal{N}_M(x) \Leftrightarrow \sigma_M(x) = \langle x, v\rangle,$$

where $\sigma_M = \delta_M^*$ is the *support function* of M.

Definition 2.1.8. Let M be a nonempty closed subset of $\mathcal{H}$. We say that an element $z \in M$ is a *projection* of an element x onto a nonempty closed subset M of $\mathcal{H}$, if

$$\|x - z\| = \inf_{y\in M} \|x - y\|.$$

If the set M is also convex, then the projection of an element x onto M is uniquely defined and we will denote it by $\mathbf{Pr}_M(x)$. The projection is also characterized by

$$\mathbf{Pr}_M(x) \in M \qquad \text{and} \qquad \langle x - \mathbf{Pr}_M(x), y - \mathbf{Pr}_M(x)\rangle \leqslant 0 \qquad \forall y \in M.$$

If $M \subseteq \mathcal{H}$ is a nonempty convex closed set and $x \in \mathcal{H}$, then

$$z = \mathbf{Pr}_M(x) \Leftrightarrow x - z \in \mathcal{N}_M(z). \tag{2.1.2}$$

Moreover, notice that for every $x \in \mathcal{H}$ it holds $\mathrm{Pr}_M(x) = \mathrm{prox}_{\delta_M}(x)$.

Introduced by Fitzpatrick in [80], the notion below opened the gate towards the employment of convex analysis specific tools when investigating the maximality of monotone operators (see [24, 41] and the references therein).

Definition 2.1.9. The *Fitzpatrick function* associated to a monotone operator A is defined as

$$\varphi_A\colon \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}, \quad \varphi_A(x, u) := \sup_{(y,v)\in\mathrm{gph}A} \{\langle x, v\rangle + \langle y, u\rangle - \langle y, v\rangle\}$$

and it is a convex and lower semicontinuous function.

For insights in the outstanding role played by the Fitzpatrick function in the convex analysis with the theory of monotone operators we refer to [24, 26, 41, 59, 62] and the references therein. If A is maximally monotone, then φ_A is proper and it fulfills

$$\varphi_A(x, u) \geqslant \langle x, u \rangle \text{ for every } (x, u) \in \mathcal{H} \times \mathcal{H},$$

with equality if and only if $(x, u) \in \text{gph}A$. The following inequality is true when $A := \partial\Psi$ (see [26]):

$$\varphi_{\partial\Psi}(x, u) \leqslant \Psi(x) + \Psi^*(u) \text{ for every } (x, v) \in \mathcal{H} \times \mathcal{H}. \tag{2.1.3}$$

2.2 Variational analysis tools

In the following we will introduce some tools from variational analysis which will play an important role in this thesis.

Definition 2.2.1. Let $\Psi\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semicontinuous function and $x \in \text{dom}\Psi := \{y \in \mathcal{H}\colon \Psi(y) < +\infty\}$. The *Fréchet (viscosity) subdifferential* of Ψ at x is

$$\widehat{\partial}\Psi(x) := \left\{ v \in \mathcal{H}\colon \liminf_{y \to x} \frac{\Psi(y) - \Psi(x) - \langle v, y - x \rangle}{\|y - x\|} \geqslant 0 \right\}$$

and the *limiting (Mordukhovich) subdifferential* of Ψ at x is

$$\partial\Psi(x) := \{v \in \mathcal{H}\colon \text{exist sequences } x_k \to x \text{ and } v_k \to d \text{ as } k \to +\infty \text{ such that } \Psi(x_k) \to \Psi(x) \text{ as } k \to +\infty \text{ and } v_k \in \widehat{\partial}\Psi(x_k) \text{ for any } k \geqslant 0\}.$$

For $x \notin \text{dom}\Psi$, we set $\widehat{\partial}\Psi(x) = \partial\Psi(x) := \varnothing$.

The inclusion $\widehat{\partial}\Psi(x) \subseteq \Psi(x)$ holds for each $x \in \mathcal{H}$. If Ψ is convex, then the two subdifferentials coincide with the convex subdifferential of Ψ. If $x \in \mathcal{H}$ is a local minimum of Ψ, then $0 \in \partial\Psi(x)$. We denote by

$$\text{crit}(\Psi) := \{x \in \mathcal{H} : 0 \in \partial\Psi(x)\}$$

the set of *critical points* of Ψ.

The limiting subdifferential fulfils the following *closedness criterion*: if $\{x_k\}_{k\geqslant 0}$ and $\{v_k\}_{k\geqslant 0}$ are sequence in $\mathcal{H}$ such that

$$v_k \in \partial\Psi(x_k) \text{ for any } k \geqslant 0, (x_k, v_k) \to (x, v) \text{ and } \Psi(x_k) \to \Psi(x) \text{ as } k \to +\infty,$$

then $v \in \partial\Psi(x)$.

We also have the following subdifferential sum formula (see [107, Proposition 1.107], [119, Exercise 8.8]): if $\Phi\colon \mathcal{H} \to \mathbb{R}$ is a continuously differentiable function, then $\partial(\Psi + \Phi)(x) = \partial\Psi(x) + \nabla\Phi(x)$ for any $x \in \mathcal{H}$; and also a formula for the subdifferential of the composition of Ψ with a linear operator $A\colon \mathcal{G} \to \mathcal{H}$ (see [107, Proposition 1.112], [119, Exercise 10.7]): if A is injective, then $\partial(\Psi \circ A)(x) = A^*\partial\Psi(Ax)$ for any $x \in \mathcal{G}$.

Definition 2.2.2. The *proximal point operator with parameter* $\gamma > 0$ of a proper and lower semicontinuous function $\Psi\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is the set-valued operator defined as ([108])

$$\text{prox}_{\gamma\Psi} : \mathcal{H} \rightrightarrows \mathcal{H}, \quad \text{prox}_{\gamma\Psi}(x) = \arg\min_{y \in \mathcal{H}} \left\{ \Psi(y) + \frac{1}{2\gamma}\|x - y\|^2 \right\}.$$

If Ψ is bounded from below, then the prox operator is nonempty for every $x \in \mathcal{H}$. Exact formulas for the proximal operator are available not only for large classes of convex functions ([24, 27, 69]), but also for various nonconvex functions ([7, 89, 95]).

The following proposition collects some important properties of a (not necessarily convex) Fréchet differentiable function with Lipschitz continuous gradient.

Proposition 2.2.1. *Let $\Psi\colon \mathcal{H} \to \mathbb{R}$ be Fréchet differentiable such that its gradient is Lipschitz continuous with constant $L > 0$. Then the following statements are true:*

(i) For every $x, y \in \mathcal{H}$ and every $z \in [x, y] = \{(1-t)x + ty : t \in [0,1]\}$ it holds

$$\Psi(y) \leqslant \Psi(x) + \langle \nabla\Psi(z), y - x\rangle + \frac{L}{2}\|y - x\|^2; \tag{2.2.1}$$

(ii) If Ψ is bounded from below, then for every $\gamma > 0$ it holds

$$\inf_{x\in\mathcal{H}} \left\{ \Psi(x) - \left(\frac{1}{\gamma} - \frac{L}{2\gamma^2}\right)\|\nabla\Psi(x)\|^2 \right\} > -\infty.$$

Proof. (i) Let be $x, y \in \mathcal{H}$ and $z := (1-t)x + ty$ for $t \in [0,1]$. By the fundamental theorem of differentiation and integration we have

$$\begin{aligned}\Psi(y) - \Psi(x) &= \int_0^1 \langle \nabla\Psi((1-s)x + sy), y - x\rangle ds \\ &= \int_0^1 \langle \nabla\Psi((1-s)x + sy) - \nabla\Psi(z), y - x\rangle ds + \langle \nabla\Psi(z), y - x\rangle. \end{aligned} \tag{2.2.2}$$

Since

$$\begin{aligned} &\left| \int_0^1 \langle \nabla\Psi((1-s)x + sy) - \nabla\Psi(z), y - x\rangle ds \right| \\ \leqslant &\int_0^1 \|\nabla\Psi((1-s)x + sy) - \nabla\Psi(z)\| \cdot \|y - x\| \, ds \leqslant L\|x - y\|^2 \int_0^1 |s - t| \, ds \\ = &\, L\|x-y\|^2 \left(\int_0^t (-s + t)\, ds + \int_t^1 (s - t)\, ds \right) = L\left(\frac{1}{2} - t(1-t)\right)\|x - y\|^2, \end{aligned} \tag{2.2.3}$$

the inequality in (2.2.1) follows by combining (2.2.2) and (2.2.3) and by using that $0 \leqslant t \leqslant 1$.

(ii) The inequality in (2.2.1) gives for every $x \in \mathcal{H}$

$$\begin{aligned} -\infty < \inf_{y\in\mathcal{H}} \Psi(y) &\leqslant \Psi\left(x - \frac{1}{\gamma}\nabla\Psi(x)\right) \\ &\leqslant \Psi(x) + \left\langle \left(x - \frac{1}{\gamma}\nabla\Psi(x)\right) - x, \nabla\Psi(x) \right\rangle + \frac{L}{2}\left\| \left(x - \frac{1}{\gamma}\nabla\Psi(x)\right) - x \right\|^2 \\ &= \Psi(x) - \left(\frac{1}{\gamma} - \frac{L}{2\gamma^2}\right)\|\nabla\Psi(x)\|^2, \end{aligned}$$

which leads to the desired conclusion. □

Remark 2.2.1. (i) The Descent Lemma, which says that for a Fréchet differentiable function $\Psi\colon \mathcal{H} \to \mathbb{R}$ having a Lipschitz continuous gradient with constant $L > 0$ it holds

$$\Psi(y) \leqslant \Psi(x) + \langle \nabla\Psi(x), y - x\rangle + \frac{L}{2}\|y - x\|^2 \quad \forall x, y \in \mathcal{H}, \tag{2.2.4}$$

follows from (2.2.1) for $z := x$.

(ii) In addition, by taking in (2.2.1) $z := y$ we obtain

$$\Psi(x) \geqslant \Psi(y) + \langle \nabla\Psi(y), x - y\rangle - \frac{L}{2}\|x - y\|^2 \quad \forall x, y \in \mathcal{H}.$$

This is equivalent to the fact that $\Psi + \frac{L}{2}\|\cdot\|^2$ is a convex function. Such a function is called L-weakly convex. In other words, a consequence of Proposition 2.2.1 is, that a Fréchet differentiable function with L-Lipschitz continuous gradient is L-weakly convex.

2.3 Kurdyka-Łojasiewicz property

In this section let $\mathcal{H}$ be a *finite-dimentional* real Hilbert space.

The origins of this notion go back to the pioneering work of Kurdyka who introduced in [93] a general form of the Łojasiewicz inequality [103]. An extension to the nonsmooth setting has been proposed and studied in the works of Attouch, Bolte, and their co-authors [7, 33, 34, 35].

Definition 2.3.1. Let $\eta \in (0, +\infty]$. We denote by Φ_η the set of all concave and continuous functions $\varphi\colon [0, \eta) \to [0, +\infty)$ which satisfy the following conditions:

(i) $\varphi(0) = 0$;

(ii) φ is $\mathcal{C}^1$ on $(0, \eta)$ and continuous at 0;

(iii) for any $s \in (0, \eta) : \varphi'(s) > 0$.

Definition 2.3.2. Let $\Psi\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be proper and lower semicontinuous.

(i) The function Ψ is said to have the Kurdyka-Łojasiewicz (KŁ) property at a point $\widehat{v} \in \mathrm{dom}\partial\Psi := \{v \in \mathcal{H}\colon \partial\Psi(v) \neq \varnothing\}$, if there exists $\eta \in (0, +\infty]$, a neighborhood V of $\widehat{v}$ and a function $\varphi \in \Phi_\eta$ such that for any

$$v \in V \cap [\Psi(\widehat{v}) < \Psi(v) < \Psi(\widehat{v}) + \eta]$$

the following inequality holds

$$\varphi'(\Psi(v) - \Psi(\widehat{v})) \cdot \mathrm{dist}(0, \partial\Psi(v)) \geqslant 1.$$

(ii) If Ψ satisfies the KŁ property at each point of $\mathrm{dom}\partial\Psi$, then Ψ is called KŁ function.

The functions φ belonging to the set Φ_η for $\eta \in (0, +\infty]$ are called desingularization functions. The KŁ property reveals the possibility to reparametrize the values of Ψ in order to avoid flatness around the critical points. To the class of KŁ functions belong semialgebraic, real subanalytic, uniformly convex functions and convex functions satisfying a growth condition. Recall that a function is called semialgebraic if its graph can be expressed as a semialgebraic set

$$\bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in \mathcal{H}\colon P_{i,j} = 0, Q_{i,j} < 0\},$$

where $P_{i,j}, Q_{i,j}\colon \mathcal{H} \to \mathbb{R}$ are polynomials for all $1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q$. The real polynomial functions, indicator functions of semi-algebraic sets; finite sum and product/composition of semi-algebraic sets are all semialgebraic functions. It worth to also mention the counting norm:

$$\|x\|_0 = \text{number of nonzero coordinates of } x.$$

and ℓ_p norm for rational p.

We recall the following definition of *Łojasiewicz property* from [5] (see, also, [103]).

Definition 2.3.3. Let $\Psi\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be proper and lower semicontinuous. Then Ψ satisfies the *Łojasiewicz property* if for any critical point $\widehat{u}$ of Ψ, there exists $C_L > 0$, $\theta \in [0, 1)$ and $\varepsilon > 0$ such that

$$|\Psi(u) - \Psi(\widehat{u})|^\theta \leqslant C_L \cdot \mathrm{dist}(0, \partial\Psi(u)) \ \forall u \in \mathbb{B}(\widehat{u}, \varepsilon),$$

where $\mathbb{B}(\widehat{u}, \varepsilon)$ denotes the open ball with centre $\widehat{u}$ and radius ε.

Obviously, Ψ is a KŁ function with desingularization function

$$\varphi : [0, +\infty) \to [0, +\infty),\ \varphi(s) := \frac{1}{1-\theta} C_L s^{1-\theta}.$$

We refer to the works of Attouch, Bolte, and their co-authors [5, 7, 8, 33, 34, 35, 36] for more properties of KŁ functions and illustrating examples.

Bolte, Sabach and Teboulle proved the following result in [36, Lemma 6]. We will use this result in the convergence analysis for many algorithms in this thesis.

Lemma 2.3.1. ***(Uniformized KŁ property)*** *Let Ω be a compact set and $\Psi \colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semicontinuous function. Assume that Ψ is constant on Ω and satisfies the KŁ property at each point of Ω. Then there exist $\varepsilon > 0, \eta > 0$ and $\varphi \in \Phi_\eta$ such that for every $\widehat{u} \in \Omega$ and every element u in the intersection*

$$\{u \in \mathcal{H} \colon \operatorname{dist}(u, \Omega) < \varepsilon\} \cap [\Psi(\widehat{u}) < \Psi(u) < \Psi(\widehat{u}) + \eta]$$

it holds

$$\varphi'(\Psi(u) - \Psi(\widehat{u})) \cdot \operatorname{dist}(0, \partial\Psi(u)) \geqslant 1.$$

2.4 Convergence results for real sequences

We close this chapter by presenting some convergence results for real sequences that will be used in what follows in the convergence analysis.

The following result can be found in the paper of Alvarez and Attouch [3], see also [46].

Lemma 2.4.1. *Let $\{\theta_k\}_{k\geqslant 0}, \{\xi_k\}_{k\geqslant 1}$ and $\{d_k\}_{k\geqslant 1}$ be nonnegative real sequences with $\sum_{k\geqslant 1} d_k < +\infty$. If there exists $k_0 \geqslant 1$ such that*

$$\theta_{k+1} - \theta_k \leqslant \alpha_k(\theta_k - \theta_{k-1}) - \xi_k + d_k \quad \forall k \geqslant k_0$$

and α such that

$$0 \leqslant \alpha_k \leqslant \alpha_+ < 1 \quad \forall k \geqslant 1,$$

then the following statements are true:

(i) $\sum_{k\geqslant 1} [\theta_k - \theta_{k-1}]_+ < +\infty$, *where* $[s]_+ := \max\{s, 0\}$*;*

(ii) the limit $\lim_{k\to\infty} \theta_k$ *exists.*

(iii) it holds $\sum_{k\geqslant 1} \xi_k < +\infty$.

As a consequence, we get the following statement, which follows from Lemma 2.4.1, applied in case $\alpha_k := 0$ and $\theta_k := \rho_k - \rho$ for all $k \geqslant 1$, where ρ is a lower bound of a sequence $\{\rho_k\}_{k\geqslant 1}$.

Lemma 2.4.2. *Let $\{\rho_k\}_{k\geqslant 1}$ be a real sequence, which is bounded from below, and $\{\xi_k\}_{k\geqslant 1}$, $\{d_k\}_{k\geqslant 1}$ be nonnegative sequences with $\sum_{k\geqslant 1} d_k < +\infty$. If there exists $k_0 \geqslant 1$ such that*

$$\rho_{k+1} \leqslant \rho_k - \xi_k + d_k \quad \forall k \geqslant k_0,$$

then the following statements are true:

(i) the sequence $\{\rho_k\}_{k\geqslant 1}$ is convergent.

(ii) it holds $\sum_{k \geqslant 1} \xi_k < +\infty$.

The following result, which will be useful in this work, shows that statement (ii) in Lemma 2.4.2 can be obtained also when $\{\rho_k\}_{k \geqslant 1}$ is not bounded by below, but has a particular form (see also [57, Lemma 1.4]).

Lemma 2.4.3. *Let* $\{\rho_k\}_{k \geqslant 1}$ *be a real sequence and* $\{\xi_k\}_{k \geqslant 1}$, $\{d_k\}_{k \geqslant 1}$ *be nonnegative real sequences with* $\sum_{k \geqslant 1} d_k < +\infty$ *and*

$$\rho_k := \theta_k - \alpha_k \theta_{k-1} + \delta_k \quad \forall k \geqslant 1,$$

where $\{\theta_k\}_{k \geqslant 0}, \{\delta_k\}_{k \geqslant 1}$ *are nonnegative sequences and there exists* α *such that*

$$0 \leqslant \alpha_k \leqslant \alpha_+ < 1 \quad \forall k \geqslant 1.$$

If there exists $k_0 \geqslant 1$ *such that*

$$\rho_{k+1} \leqslant \rho_k - \xi_k + d_k \quad \forall k \geqslant k_0, \tag{2.4.1}$$

then it holds $\sum_{k \geqslant 1} \xi_k < +\infty$.

Proof. We fix an integer $\bar{K} \geqslant k_0$, sum up the inequalities in (2.4.1) for $k = k_0, k_0 + 1, \cdots, \bar{K}$ and obtain

$$\rho_{\bar{K}+1} - \rho_{k_0} \leqslant - \sum_{k=k_0}^{\bar{K}} \xi_k + \sum_{k=k_0}^{\bar{K}} d_k \leqslant \sum_{k \geqslant 1} d_k < +\infty. \tag{2.4.2}$$

Hence the sequence $\{\rho_k\}_{k \geqslant 1}$ is bounded from above. Let $\bar{\rho} > 0$ be an upper bound of this sequence. For all $k \geqslant 1$ it holds

$$\theta_k - \alpha_+ \theta_{k-1} \leqslant \theta_k - \alpha_k \theta_{k-1} + \delta_k = \rho_k \leqslant \bar{\rho},$$

from which we deduce that

$$- \rho_k \leqslant -\theta_k + \alpha_+ \theta_{k-1} \leqslant \alpha_+ \theta_{k-1}. \tag{2.4.3}$$

By induction we obtain for all $k \geqslant k_0 + 1$

$$\theta_k \leqslant \alpha_+ \theta_{k-1} + \bar{\rho} \leqslant \cdots \leqslant \alpha_+^{k-k_0} \theta_{k_0} + \bar{\rho} \sum_{k=1}^{k-k_0} \alpha^{k-1} \leqslant \alpha_+^{k-k_0} \theta_{k_0} + \frac{\bar{\rho}}{1-\alpha}. \tag{2.4.4}$$

Then inequality (2.4.2) combined with (2.4.3) and (2.4.4) leads to

$$\begin{aligned} \sum_{k=k_0}^{\bar{K}} \xi_k \leqslant \rho_{k_0} - \rho_{\bar{K}+1} + \sum_{k=k_0}^{\bar{K}} d_k &\leqslant \rho_{k_0} + \alpha_+ \theta_{\bar{K}} + \sum_{k \geqslant 1} d_k \\ &\leqslant \rho_{k_0} + \alpha_+^{\bar{K}-k_0+1} \theta_{k_0} + \frac{\alpha_+ \bar{\rho}}{1-\alpha} + \sum_{k \geqslant 1} d_k < +\infty. \end{aligned} \tag{2.4.5}$$

We let $\bar{K}$ converge to $+\infty$ and obtain that $\sum_{k \geqslant 1} \xi_k < +\infty$. □

The following lemma is a simplified version of [56, Lemma 3].

Lemma 2.4.4. *Let $\{a_k\}_{k\geqslant 0}$ be a nonnegative sequence and $\{d_k\}_{k\geqslant 0}$ a real sequence such that*

$$a_{k+1} \leqslant \chi_0 \cdot a_k + \chi_1 \cdot a_{k-1} + \chi_2 \cdot a_{k-2} + d_k \ \forall k \geqslant 2, \tag{2.4.6}$$

where $\chi_0 \in \mathbb{R}$, $\chi_1, \chi_2 \in \mathbb{R}_+$ fulfill $\chi_0 + \chi_1 + \chi_2 < 1$. Assume further that there exists $\bar{d} \geqslant 0$ such that for every $\overline{K} \geqslant \underline{K} \geqslant 2$

$$\sum_{k=\underline{K}}^{\overline{K}} d_k \leqslant \bar{d}.$$

Then, it holds

$$\sum_{k\geqslant 0} a_k < +\infty.$$

In particular, for every $i = 1, \dots, N$ and every $\overline{K} \geqslant \underline{K} \geqslant 2$, it holds

$$\sum_{k=\underline{K}}^{\overline{K}} a_k \leqslant \frac{(1-\chi_0-\chi_1)\, a_{\underline{K}} + (1-\chi_0)\, a_{\underline{K}+1} + a_{\underline{K}+2} + \bar{d}}{1-\chi_0-\chi_1-\chi_2}. \tag{2.4.7}$$

Proof. Fix $\overline{K} \geqslant \underline{K} \geqslant 2$. If $\overline{K} = \underline{K}$ or $\overline{K} = \underline{K}+1$, then (2.4.7) holds automatically. Assume now that $\overline{K} \geqslant \underline{K}+2$. Summing up the inequality in (2.4.6) for $k = \underline{K}+2, \cdots, \overline{K}$, we obtain

$$\sum_{k=\underline{K}+2}^{\overline{K}} a_{k+1} \leqslant \chi_0 \sum_{k=\underline{K}+2}^{\overline{K}} a_k + \chi_1 \cdot \sum_{k=\underline{K}+2}^{\overline{K}} a_{k-1} + \chi_2 \cdot \sum_{k=\underline{K}+2}^{\overline{K}} a_{k-2} + \sum_{k=\underline{K}+2}^{\overline{K}} d_k. \tag{2.4.8}$$

Since

$$\begin{aligned}
\sum_{k=\underline{K}+2}^{\overline{K}} a_{k+1} &= \sum_{k=\underline{K}+3}^{\overline{K}+1} a_k = \sum_{k=\underline{K}}^{\overline{K}} a_k + a_{\overline{K}+1} - a_{\underline{K}} - a_{\underline{K}+1} - a_{\underline{K}+2} \\
\sum_{k=\underline{K}+2}^{\overline{K}} a_k &= \sum_{k=\underline{K}}^{\overline{K}} a_k - \left(a_{\underline{K}} + a_{\underline{K}+1}\right) \\
\sum_{k=\underline{K}+2}^{\overline{K}} a_{k-1} &= \sum_{k=\underline{K}+1}^{\overline{K}-1} a_k = \sum_{k=\underline{K}}^{\overline{K}} a_k - \left(a_{\underline{K}} + a_{\overline{K}}\right) \\
\sum_{k=\underline{K}+2}^{\overline{K}} a_{k-2} &= \sum_{k=\underline{K}}^{\overline{K}-2} a_k = \sum_{k=\underline{K}}^{\overline{K}} a_k - \left(a_{\overline{K}-1} + a_{\overline{K}}\right),
\end{aligned}$$

the inequality in (2.4.8) can be rewritten as

$$\begin{aligned}
\sum_{k=\underline{K}}^{\overline{K}} a_k + a_{\overline{K}+1} - a_{\underline{K}} - a_{\underline{K}+1} - a_{\underline{K}+2} &\leqslant \chi_0 \sum_{k=\underline{K}}^{\overline{K}} a_k - \chi_0 \left(a_{\underline{K}} + a_{\underline{K}+1}\right) \\
&+ \chi_1 \sum_{k=\underline{K}}^{\overline{K}} a_k - \chi_1 \left(a_{\underline{K}} + a_{\overline{K}}\right) + \chi_2 \cdot \sum_{k=\underline{K}}^{\overline{K}} a_k - \chi_2 \left(a_{\overline{K}-1} + a_{\overline{K}}\right) + \sum_{k=\underline{K}+2}^{\overline{K}} d_k,
\end{aligned}$$

which further implies

$$
\begin{aligned}
(1-\chi_0-\chi_1-\chi_2)\sum_{k=\underline{K}}^{\overline{K}} a_k &= (1-\chi_0-\chi_1-\chi_2)\sum_{k=\underline{K}}^{\overline{K}} a_k \\
&\leqslant (1-\chi_0-\chi_1)\, a_{\underline{K}} + (1-\chi_0)\, a_{\underline{K}+1} + a_{\underline{K}+2} + \sum_{k=\underline{K}+2}^{\overline{K}} d_k \\
&= (1-\chi_0-\chi_1)\, a_{\underline{K}} + (1-\chi_0)\, a_{\underline{K}+1} + a_{\underline{K}+2} + \sum_{k=\underline{K}+2}^{\overline{K}} d_k .
\end{aligned}
$$

Hence, it holds

$$
(1-\chi_0-\chi_1-\chi_2)\sum_{k=\underline{K}}^{\overline{K}} a_k \leqslant (1-\chi_0-\chi_1)\, a_{\underline{K}} + (1-\chi_0)\, a_{\underline{K}+1} + a_{\underline{K}+2} + \bar{d}
$$

and the conclusion follows by taking into consideration that $\chi_0 + \chi_1 + \chi_2 < 1$. □

The following lemma will provide convergence rates for a particular class of monotonically decreasing sequences converging to 0 (see also [56, Lemma 15]).

Lemma 2.4.5. *Let $\{\varepsilon_k\}_{k\geqslant 0}$ be a monotonically decreasing sequence in $\mathbb{R}_+$ converging to 0. Assume further that there exists natural numbers $k_0 \geqslant l_0 \geqslant 1$ such that for every $k \geqslant k_0$*

$$
\varepsilon_{k-l_0} - \varepsilon_k \geqslant C_\varepsilon \varepsilon_k^{2\theta}, \tag{2.4.9}
$$

where $C_\varepsilon > 0$ is some constant and $\theta \in [0,1)$. Then following statements are true:

(i) if $\theta = 0$, then $\{\varepsilon_k\}_{k\geqslant 0}$ converges in finite time;

(ii) if $\theta \in (0, 1/2]$, then there exists $C_{\varepsilon,0} > 0$ and $Q \in [0,1)$ such that for every $k \geqslant k_0$

$$
0 \leqslant \varepsilon_k \leqslant C_{\varepsilon,0} Q^k;
$$

(iii) if $\theta \in (1/2, 1)$, then there exists $C_{\varepsilon,1} > 0$ such that for every $k \geqslant k_0 + l_0$

$$
0 \leqslant \varepsilon_k \leqslant C_{\varepsilon,1} (k - l_0 + 1)^{-\frac{1}{2\theta-1}} .
$$

Proof. Fix an integer $k \geqslant k_0$. Since $k_0 \geqslant l_0 \geqslant 0$, the recurrence inequality (2.4.9) is well defined for every $k \geqslant k_0$.

(i) The case when $\theta = 0$. We assume that $\varepsilon_k > 0$ for every $k \geqslant 0$. From (2.4.9) we get

$$
\varepsilon_{k-l_0} - \varepsilon_k \geqslant C_\varepsilon > 0
$$

for every $k \geqslant k_0$, which actually contradicts the fact that $\{\varepsilon_k\}_{k\geqslant 0}$ converges to 0 as $k \to +\infty$. Consequently, there exists $k' \geqslant 0$ such that $\varepsilon_{k'} = 0$ for every $k \geqslant k'$ and thus the conclusion follows.

For the proof of (ii) and (iii) we can assume that $\varepsilon_k > 0$ for every $k \geqslant 0$. Otherwise, as $\{\varepsilon_k\}_{k\geqslant 0}$ is monotonically decreasing and convergent to 0, the sequence is constant beginning with a given index, which means that both statements are true.

(ii) The case when $\theta \in (0, 1/2]$. We have $\varepsilon_k \leqslant \varepsilon_0$, which leads to

$$\varepsilon_{k-l_0} - \varepsilon_k \geqslant C_\varepsilon \varepsilon_k^{2\theta} \geqslant C_\varepsilon \varepsilon_0^{2\theta-1} \varepsilon_k$$

for every $k \geqslant k_0$. Therefore,

$$\varepsilon_k \leqslant \left(\frac{1}{C_\varepsilon \varepsilon_0^{2\theta-1} + 1} \right)^{\frac{k}{l_0} - \frac{k_0}{l_0} - 1} \varepsilon_0 = \varepsilon_0 \left(C_\varepsilon \varepsilon_0^{2\theta-1} + 1 \right)^{\frac{k_0}{l_0}+1} \left(\frac{1}{\sqrt[l_0]{C_\varepsilon \varepsilon_0^{2\theta-1} + 1}} \right)^k.$$

(iii) The case when $\theta \in (1/2, 1)$. From (2.4.9) we get

$$C_\varepsilon \leqslant (\varepsilon_{k-l_0} - \varepsilon_k) \, \varepsilon_k^{-2\theta}. \tag{2.4.10}$$

Define $\zeta \colon (0, +\infty) \to \mathbb{R}, \zeta(s) = s^{-2\theta}$. We have that

$$\frac{d}{ds} \left(\frac{1}{1-2\theta} s^{1-2\theta} \right) = s^{-2\theta} = \zeta(s) \text{ and } \zeta'(s) = -2\theta s^{-2\theta-1} < 0 \; \forall s \in (0, +\infty).$$

Consequently, $\zeta(\varepsilon_{k-l_0}) \leqslant \zeta(s)$ for all $s \in [\varepsilon_k, \varepsilon_{k-l_0}]$.

- *Assume that* $\zeta(\varepsilon_k) \leqslant 2\zeta(\varepsilon_{k-l_0})$. Then (2.4.10) gives

  $$C_\varepsilon \leqslant 2\zeta(\varepsilon_{k-l_0}) \int_{\varepsilon_k}^{\varepsilon_{k-l_0}} 1 ds \leqslant 2 \int_{\varepsilon_k}^{\varepsilon_{k-l_0}} \zeta(s) \, ds = \frac{2}{2\theta-1} \left(\varepsilon_k^{1-2\theta} - \varepsilon_{k-l_0}^{1-2\theta} \right)$$

  or, equivalently,

  $$\varepsilon_k^{1-2\theta} - \varepsilon_{k-l_0}^{1-2\theta} \geqslant C_1', \quad \text{where } C_1' := \frac{(2\theta-1)\, C_\varepsilon}{2} > 0. \tag{2.4.11}$$

- *Assume that* $\zeta(\varepsilon_k) > 2\zeta(\varepsilon_{k-l_0})$. For $\nu := 2^{-\frac{1}{2\theta}} \in (0,1)$ this is equivalent to

  $$\left(\nu^{1-2\theta} - 1 \right) \varepsilon_{k-l_0}^{1-2\theta} \leqslant \varepsilon_k^{1-2\theta} - \varepsilon_{k-l_0}^{1-2\theta},$$

  thus,

  $$\varepsilon_k^{1-2\theta} - \varepsilon_{k-l_0}^{1-2\theta} \geqslant \left(\nu^{1-2\theta} - 1 \right) \varepsilon_{k-l_0}^{1-2\theta} \geqslant C_2', \quad \text{where } C_2' := \left(\nu^{1-2\theta} - 1 \right) \varepsilon_0^{2\theta-1} > 0. \tag{2.4.12}$$

In both situations we get for every $i \geqslant k_0$

$$\varepsilon_i^{1-2\theta} - \varepsilon_{i-l_0}^{1-2\theta} \geqslant C' := \min \{C_1', C_2'\} > 0, \tag{2.4.13}$$

where C_1' and C_2' are defined as in (2.4.11) and (2.4.12), respectively. For every $k \geqslant k_0 + 2l_0$, by summing up the inequalities (2.4.13) for $i = k_0 + l_0, \cdots, k$, we get

$$\sum_{j=0}^{l_0-1} \left(\varepsilon_{k-j}^{1-2\theta} - \varepsilon_{k_0+j}^{1-2\theta} \right) \geqslant (k - k_0 - l_0 + 1)\, C' > 0.$$

Since

$$l_0 \left(\varepsilon_k^{1-2\theta} - \varepsilon_{k_0}^{1-2\theta} \right) \geqslant \sum_{j=0}^{l_0-1} \left(\varepsilon_{k-j}^{1-2\theta} - \varepsilon_{k_0+j}^{1-2\theta} \right) \geqslant C' (k - k_0 - l_0 + 1),$$

we have

$$\varepsilon_k^{1-2\theta} \geqslant \varepsilon_{k_0}^{1-2\theta} + \frac{k - k_0 - l_0 + 1}{l_0} C'. \tag{2.4.14}$$

We obtain from (2.4.13) that

$$\varepsilon_{k_0}^{1-2\theta} \geqslant \left\lfloor \frac{k_0 + l_0}{l_0} \right\rfloor C' \geqslant \left(\frac{k_0 + l_0}{l_0} - 1 \right) C' = \frac{k_0}{l_0} C', \tag{2.4.15}$$

where $\lfloor p \rfloor$ denotes the greatest integer that is less than or equal to the real number p. By plugging (2.4.15) into (2.4.14) we obtain

$$\varepsilon_k^{1-2\theta} \geqslant \frac{k - l_0 + 1}{l_0} C',$$

which implies

$$\varepsilon_k \leqslant \left(\frac{C'}{l_0} \right)^{-\frac{1}{2\theta - 1}} (k - l_0 + 1)^{-\frac{1}{2\theta-1}} . \tag{2.4.16}$$

This concludes the proof. □

Remark 2.4.1. The inequality in Lemma 2.4.5 (iii) can be writen for k large enough in terms of k instead of $k - l_0 + 1$. If, for instance, $k \geqslant 2\,(l_0 + 1)$, then $k - l_0 + 1 \geqslant \frac{1}{2} k$ and thus from (2.4.16) we get

$$\varepsilon_k \leqslant \left(\frac{C'}{l_0} \right)^{-\frac{1}{2\theta - 1}} (k - l_0 + 1)^{-\frac{1}{2\theta-1}} \leqslant \left(\frac{C'}{2l_0} \right)^{-\frac{1}{2\theta - 1}} k^{-\frac{1}{2\theta-1}} .$$

Chapter 3

A forward-backward penalty scheme with inertial effects for montone inclusions

This chapter follows our work [57].

We investigate forward-backward splitting algorithm of penalty type with inertial effects for finding a zero of the sum of a maximally monotone operator, a cocoercive operator and the convex normal cone to the set of zeroes of an another cocoercive operator. Weak ergodic convergence is obtained for the generated iterates, provided that a condition express via the Fitzpatrick function of the operator describing the underlying set of the normal cone is verified. Under strong monotonicity assumptions, strong convergence for the sequence of generated iterates is proved. As a particular instance we consider a convex bilevel minimization problem including the sum of a nonsmooth and a smooth function in the upper level and another smooth function in the lower level. We show that in this context weak nonergodic and strong convergence of the iterates can be also achieved under inf-compactness assumptions for the involved functions.

3.1 Problem formulation and motivation

In the last years one could observe an increasing interest in numerical schemes for solving variational inequalities expressed as monotone inclusion problems of the form

$$0 \in Ax + \mathcal{N}_M(x), \tag{3.1.1}$$

where $\mathcal{H}$ is a real Hilbert space, $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone operator, $M := \arg\min h$ is the set of global minima of a proper, convex and lower semicontinuous function $h\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ and $\mathcal{N}_M\colon \mathcal{H} \rightrightarrows \mathcal{H}$ is the normal cone of the set M. The article [14] of Attouch and Czarnecki was the starting point for a series of papers [13, 16, 17, 21, 45, 46, 82, 109, 114] addressing this topic or related ones. All these papers share the common feature that the proposed iterative schemes use penalization strategies, namely, the evaluate a penalization of h by its gradient, in case the function is smooth (see, for instance, [16]), and by its proximal operator, in case it is nonsmooth (see, for instance,[17]).

Weak ergodic convergence has been obtained in [16, 17] under the hypothesis:

$$\text{For all } p \in \operatorname{ran}\mathcal{N}_M, \sum_{k\geqslant 1} \lambda_k\beta_k \left[h^*\left(\frac{p}{\beta_k}\right) - \sigma_M\left(\frac{p}{\beta_k}\right)\right] < +\infty, \tag{3.1.2}$$

with $\{\lambda_k\}_{k\geqslant 1}$, the sequence of step sizes, $\{\beta_k\}_{k\geqslant 1}$, the sequence of penalty parameters, $h^*\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, the Fenchel conjugate function of h, and $\operatorname{ran}\mathcal{N}_M$ the range of the normal cone operator $\mathcal{N}_M \colon \mathcal{H} \rightrightarrows \mathcal{H}$. Let us mention that (3.1.2) is the discretized counterpart of a condition introduced in [14] for continuous-time nonautonomous differential inclusions.

One motivation for studying numerical algorithms for monotone inclusions of type (3.1.1) comes from the fact that, when $A \equiv \partial f$ is the convex subdifferential of a proper, convex and lower semicontinuous function $f \colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, they furnish iterative methods for solving bilevel optimization problems of the form

$$\min_{x \in \mathcal{H}} \{f(x) : x \in \arg\min h\}. \tag{3.1.3}$$

Among the applications where bilevel programming problems play an important role we mention the modelling of Stackelberg games, the determination of Wardrop equilibria for network flows, convex feasibility problems [9], domain decomposition methods for PDEs [6], image processing problems [45], and optimal control problems [17].

Later on, in [46], the following monotone inclusion problem, which turned out to be more suitable for applications, has been addressed in the same spirit of penalty algorithms

$$0 \in Ax + Dx + \mathcal{N}_M(x), \tag{3.1.4}$$

where $A \colon \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone operator, $D \colon \mathcal{H} \to \mathcal{H}$ is cocoercive operator and the constraint set M is the set of zeros of another cocoercive operator $B \colon \mathcal{H} \to \mathcal{H}$. The provided algorithm of forward-backward type evaluates the operator A by a backward step and the two single-valued operators by forward steps. For the convergence analysis, (3.1.2) has been replaced by a condition formulated in terms of the Fitzpatrick function associated with the operator B, which we will also use in this chapter. In [21], several particular situations for which this condition is fulfilled have been provided.

In this chapter, we will endow the forward-backward penalty scheme for solving (3.1.4) from [46] with inertial effects, which means that every new iterate will be defined in terms of the previous two iterates. Inertial algorithms have their roots in the time discretization of second order differential systems [3]. They can accelerate the convergence of iterates when minimizing a differentiable function [116] and the convergence of the objective function values when minimizing the sum of a convex nonsmooth and a convex smooth function [28, 64]. Moreover, as emphasized in [29], see also [51], algorithms with inertial effects may detect optimal solutions of minimization problems which cannot be found by their noninertial variants. In the last years, a huge interest in inertial algorithms can be noticed (see, for instance, [1, 2, 3, 15, 20, 47, 48, 50, 53, 54]).

In particular, we will prove weak ergodic convergence of the sequence generated by the inertial forward-backward penalty algorithm to a solution of the monotone inclusion problem (3.1.4), under reasonable assumptions for the sequences of step sizes, penalty and inertial parameters. When the operator A is assumed to be strongly monotone, we will also prove strong convergence of the generated iterates to the unique solution of (3.1.4).

In Section 3.3, we will address the minimization of the sum of a convex nonsmooth and a convex smooth function with respect to the set of minimizes of another convex and smooth function. Besides the convergence results obtained from the general case, we achieve weak nonergodic and strong convergence statements under inf-compactness assumptions for the involved functions. The weak nonergodic theorem is an useful alternative to the one in [54], where a similar statement has been obtained for the inertial forward-backward penalty algorithm with constant inertial parameters under assumptions which are quite complicated and hard to verify (see also [109, 114]).

3.2 The general monotone inclusion problem

The monotone inclusion problem we will consider in this chapter is the following.

Let $\mathcal{H}$ be a real Hilbert space, $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ a maximally monotone operator, $D\colon \mathcal{H} \to \mathcal{H}$ an $\eta-$cocoercive with $\eta > 0$, $B\colon \mathcal{H} \to \mathcal{H}$ a $\mu-$cocoercive with $\mu > 0$ and assume that $M := \operatorname{zer} B \neq \varnothing$. The monotone inclusion problem to solve reads

$$0 \in Ax + Dx + \mathcal{N}_M(x). \tag{3.2.1}$$

The following forward-backward penalty algorithm with inertial effects for solving (3.2.1) will be in the focus of our investigations in this chapter.

Algorithm 3.2.1. *Let $\{\alpha_k\}_{k\geqslant 1}, \{\lambda_k\}_{k\geqslant 1}$ and $\{\beta_k\}_{k\geqslant 1}$ be sequences of positive real numbers such that*

(C_1) *$\{\lambda_k\}_{k\geqslant 1} \in \ell^2 \setminus \ell^1$, that is $\sum_{k\geqslant 1} \lambda_k^2 < +\infty$ and $\sum_{k\geqslant 1} \lambda_k = +\infty$;*

(C_2) *$\{\alpha_k\}_{k\geqslant 1}$ is nondecreasing;*

(C_3) *there exists α with $0 \leqslant \alpha_k \leqslant \alpha_+ < 1/3$ for all $k \geqslant 1$.*

Let $x_0, x_1 \in \mathcal{H}$. For all $k \geqslant 1$ we set

$$x_{k+1} := J_{\lambda_k A}\left(x_k - \lambda_k D x_k - \lambda_k \beta_k B x_k + \alpha_k (x_k - x_{k-1})\right).$$

When $D = 0$ and $B = \nabla h$, where $h : \mathcal{H} \to \mathbb{R}$ is a convex and differentiable function with $\mu^{-1}-$Lipschitz continuous gradient with $\mu > 0$ fulfilling $\min h = 0$, then (3.2.1) recovers the monotone inclusion problem addressed in [16, Section 3] and Algorithm 3.2.1 can be seen as an inertial version of the iterative scheme considered. When $B = 0$, we have that $\mathcal{N}_M = \{0\}$ and Algorithm 3.2.1 is nothing else than the inertial version of the classical forward-backward algorithm (see for instance [24, 67]).

Hypothesis 3.2.1. *The convergence analysis will be carried out in the following hypotheses (see also [46]):*

$(\mathrm{H}_1^{\mathrm{fitz}})$ *$A + \mathcal{N}_M$ is maximally monotone and* $\operatorname{zer}(A + D + \mathcal{N}_M) \neq \varnothing$;

$(\mathrm{H}_2^{\mathrm{fitz}})$ *for every $p \in \operatorname{ran}\mathcal{N}_M$, $\sum_{k\geqslant 1} \lambda_k \beta_k \left[\sup_{u\in M} \varphi_B\left(u, \frac{p}{\beta_k}\right) - \sigma_M\left(\frac{p}{\beta_k}\right)\right] < +\infty$, where φ_B denotes the Fitzpatrick function of B.*

Since A and $\mathcal{N}_M$ are maximally monotone operators, the sum $A + \mathcal{N}_M$ is maximally monotone, provided some conditions are fulfilled (see [24, 41, 59, 130]). Furthermore, since D is also maximally monotone and $\operatorname{dom} D \equiv \mathcal{H}$, if $A + \mathcal{N}_M$ is maximally monotone, then $A + D + \mathcal{N}_M$ is also maximally monotone.

Let us also notice that for $p \in \operatorname{ran}\mathcal{N}_M$ there exists $\widehat{u} \in M$ such that $p \in \mathcal{N}_M(\widehat{u})$, hence, for every $\beta > 0$ it holds

$$\sup_{u\in M} \varphi_B\left(u, \frac{p}{\beta}\right) - \sigma_M\left(\frac{p}{\beta}\right) \geqslant \left\langle \widehat{u}, \frac{p}{\beta}\right\rangle - \sigma_M\left(\frac{p}{\beta}\right) = 0.$$

Example 3.2.1. Here we discuss a particular instance for which $(\mathrm{H}_2^{\mathrm{fitz}})$ is verified. Given a convex and closed set $\varnothing \neq M \subseteq \mathcal{H}$, consider

$$h(x) := \frac{1}{2} \inf_{y\in M} \|x - y\|^2 = \frac{1}{2}\|x - \mathbf{Pr}_M x\|^2 \qquad \forall x \in \mathcal{H}.$$

Then h is differentiable, $\nabla h(x) = x - \mathbf{Pr}_M x$ for all $x \in \mathcal{H}$ and $B := \nabla h$ is Lipschitz continuous, thus cocoercive. In addition, the definition of h^* and σ_M yields $h^* = \sigma_M + \frac{1}{2}\|\cdot\|^2$. Since $h(x) = 0$ for every $x \in M$, we get from (2.1.3)

$$\sum_{k\geqslant 1} \lambda_k \beta_k \left[\sup_{u\in M} \varphi_{\nabla h}\left(u, \frac{p}{\beta_k}\right) - \sigma_M\left(\frac{p}{\beta_k}\right)\right] \leqslant \sum_{k\geqslant 1} \lambda_k \beta_k \left[h^*\left(\frac{p}{\beta_k}\right) - \sigma_M\left(\frac{p}{\beta_k}\right)\right]$$
$$= \sum_{k\geqslant 1} \lambda_k \beta_k \left\|\frac{p}{\beta_k}\right\|^2 = \|p\|^2 \sum_{k\geqslant 1} \frac{\lambda_k}{\beta_k}.$$

For every positive sequence $\{\lambda_k\}_{k\geqslant 1} \in \ell^2 \setminus \ell^1$, if we take

$$\beta_k := \frac{1}{\lambda_k},$$

then $\sum_{k\geqslant 1} \frac{\lambda_k}{\beta_k} = \sum_{k\geqslant 1} \lambda_k^2 < +\infty$

For further particular situations where $(\mathrm{H}_2^{\mathrm{fitz}})$ is satisfied we refer the reader [21, 53, 54, 109].

Before formulating the main theorem of this section, we will prove some useful technical results.

Lemma 3.2.2. *Let $\{x_k\}_{k\geqslant 0}$ be the sequence generated by* Algorithm 3.2.1 *and (u, y) be an element in* $\operatorname{gph}(A + D + \mathcal{N}_M)$ *such that*

$$y = v + Du + p \text{ with } v \in Au \text{ and } p \in \mathcal{N}_M(u).$$

Furthermore, let $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ be such that $1 - \varepsilon_3 > 0$. Then the following inequality holds for every $k \geqslant 1$

$$\begin{aligned}
&\|x_{k+1} - u\|^2 - \|x_k - u\|^2 \\
\leqslant\ & \alpha_k \|x_k - u\|^2 - \alpha_k \|x_{k-1} - u\|^2 - (1 - 4\varepsilon_1 - \varepsilon_2)\|x_{k+1} - x_k\|^2 \\
&+ \left(\alpha_k + \frac{\alpha_k^2}{4\varepsilon_1}\right)\|x_k - x_{k-1}\|^2 + \left(\frac{2}{\varepsilon_2}\lambda_k^2\beta_k^2 - 2\mu(1 - \varepsilon_3)\lambda_k\beta_k\right)\|Bx_k\|^2 \\
&+ \left(\frac{4}{\varepsilon_2}\lambda_k^2 - 2\eta\lambda_k\right)\|Dx_k - Du\|^2 + \frac{4}{\varepsilon_2}\lambda_k^2\|Du + v\|^2 \\
&+ 2\varepsilon_3\lambda_k\beta_k\left[\sup_{u\in M}\varphi_B\left(u, \frac{p}{\varepsilon_3\beta_k}\right) - \sigma_M\left(\frac{p}{\varepsilon_3\beta_k}\right)\right] + 2\lambda_k\langle u - x_k, y\rangle.
\end{aligned} \tag{3.2.2}$$

Proof. Let $k \geqslant 1$ be fixed. According to definition of the resolvent of the operator A we have

$$x_k - x_{k+1} - \lambda_k(Dx_k + \beta_k Bx_k) + \alpha_k(x_k - x_{k-1}) \in \lambda_k A x_{k+1} \tag{3.2.3}$$

and, since $\lambda_k v \in \lambda_k Au$, the monotonicity of A guarantees

$$\langle x_{k+1} - u, x_k - x_{k+1} - \lambda_k(Dx_k + \beta_k Bx_k + v) + \alpha_k(x_k - x_{k-1})\rangle \geqslant 0 \tag{3.2.4}$$

or, equivalently,

$$2\langle u - x_{k+1}, x_k - x_{k+1}\rangle \leqslant 2\lambda_k\langle u - x_{k+1}, \beta_k Bx_k + Dx_k + v\rangle - 2\alpha_k\langle u - x_{k+1}, x_k - x_{k-1}\rangle. \tag{3.2.5}$$

For the term in the left-hand side of (3.2.5) we have

$$2\langle u - x_{k+1}, x_k - x_{k+1}\rangle = \|x_{k+1} - u\|^2 + \|x_{k+1} - x_k\|^2 - \|x_k - u\|^2. \tag{3.2.6}$$

Since

$$-2\alpha_k \langle u - x_k, x_k - x_{k-1}\rangle = -\alpha_k \|u - x_{k-1}\|^2 + \alpha_k \|u - x_k\|^2 + \alpha_k \|x_k - x_{k-1}\|^2$$

and

$$2\langle x_{k+1} - x_k, \alpha_k (x_k - x_{k-1})\rangle \leqslant 4\varepsilon_1 \|x_{k+1} - x_k\|^2 + \frac{\alpha_k^2}{4\varepsilon_1} \|x_k - x_{k-1}\|^2,$$

by adding the two inequalities, we obtain the following estimation for the second term in the right-hand side of (3.2.5)

$$\begin{aligned} & -2\alpha_k \langle u - x_{k+1}, x_k - x_{k-1}\rangle \\ & \leqslant \alpha_k \|x_k - u\|^2 - \alpha_k \|x_{k-1} - u\|^2 + 4\varepsilon_1 \|x_{k+1} - x_k\|^2 + \left(\alpha_k + \frac{\alpha_k^2}{4\varepsilon_1}\right) \|x_k - x_{k-1}\|^2. \end{aligned} \tag{3.2.7}$$

We turn now our attention to the first term in the right-hand side of (3.2.5), which can be written as

$$\begin{aligned} & 2\lambda_k \langle u - x_{k+1}, \beta_k Bx_k + Dx_k + v\rangle \\ & = 2\lambda_k \langle u - x_k, \beta_k Bx_k + Dx_k + v\rangle + 2\lambda_k\beta_k \langle x_k - x_{k+1}, Bx_k\rangle + 2\lambda_k \langle x_k - x_{k+1}, Dx_k + v\rangle. \end{aligned} \tag{3.2.8}$$

We have

$$2\lambda_k\beta_k \langle x_k - x_{k+1}, Bx_k\rangle \leqslant \frac{\varepsilon_2}{2} \|x_{k+1} - x_k\|^2 + \frac{2}{\varepsilon_2}\lambda_k^2\beta_k^2 \|Bx_k\|^2 \tag{3.2.9}$$

and

$$\begin{aligned} 2\lambda_k \langle x_k - x_{k+1}, Dx_k + v\rangle & \leqslant \frac{\varepsilon_2}{2} \|x_{k+1} - x_k\|^2 + \frac{2}{\varepsilon_2}\lambda_k^2 \|Dx_k + v\|^2 \\ & \leqslant \frac{\varepsilon_2}{2} \|x_{k+1} - x_k\|^2 + \frac{4}{\varepsilon_2}\lambda_k^2 \|Dx_k - Du\|^2 + \frac{4}{\varepsilon_2}\lambda_k^2 \|Du + v\|^2. \end{aligned} \tag{3.2.10}$$

On the other hand, we have

$$\begin{aligned} & 2\lambda_k \langle u - x_k, \beta_k Bx_k + Dx_k + v\rangle \\ & = 2\lambda_k\beta_k \langle u - x_k, Bx_k\rangle + 2\lambda_k \langle u - x_k, Dx_k - Du\rangle + 2\lambda_k \langle u - x_k, Du + v\rangle. \end{aligned} \tag{3.2.11}$$

Since $0 < \varepsilon_3 < 1$ and $Bu = 0$, the cocoercivity of B gives us

$$2\lambda_k\beta_k \langle u - x_k, Bx_k\rangle \leqslant -2\mu(1 - \varepsilon_3)\lambda_k\beta_k \|Bx_k\|^2 + 2\varepsilon_3\lambda_k\beta_k \langle u - x_k, Bx_k\rangle. \tag{3.2.12}$$

Similarly, the cocoercivity of D gives us

$$2\lambda_k \langle u - x_k, Dx_k - Du\rangle \leqslant -2\eta\lambda_k \|Dx_k - Du\|^2. \tag{3.2.13}$$

Combining (3.2.12) - (3.2.13) with (3.2.11) and by using the definition Fitzpatrick function and the fact that $\sigma_M\left(\frac{p}{\varepsilon_3\beta_k}\right) = \left\langle u, \frac{p}{\varepsilon_3\beta_k}\right\rangle$, we obtain

$$\begin{aligned}
&2\lambda_k \langle u - x_k, \beta_k B x_k + D x_k + v\rangle \\
\leqslant& -2\mu(1-\varepsilon_3)\lambda_k\beta_k \|Bx_k\|^2 + 2\varepsilon_3\lambda_k\beta_k\langle u - x_k, Bx_k\rangle - 2\eta\lambda_k \|Dx_k - Du\|^2 \\
&+ 2\lambda_k\langle u - x_k, Du + v\rangle \\
=& -2\mu(1-\varepsilon_3)\lambda_k\beta_k \|Bx_k\|^2 + 2\varepsilon_3\lambda_k\beta_k\langle u - x_k, Bx_k\rangle - 2\eta\lambda_k \|Dx_k - Du\|^2 \\
&+ 2\lambda_k\langle u - x_k, y - p\rangle \\
=& -2\mu(1-\varepsilon_3)\lambda_k\beta_k \|Bx_k\|^2 - 2\eta\lambda_k\|Dx_k - Du\|^2 + 2\lambda_k\langle u - x_k, y\rangle \\
&+ 2\varepsilon_3\lambda_k\beta_k\left(\langle u, Bx_k\rangle + \left\langle x_k, \frac{p}{\varepsilon_3\beta_k}\right\rangle - \langle x_k, Bx_k\rangle - \left\langle u, \frac{p}{\varepsilon_3\beta_k}\right\rangle\right) \\
\leqslant& -2\mu(1-\varepsilon_3)\lambda_k\beta_k \|Bx_k\|^2 - 2\eta\lambda_k\|Dx_k - Du\|^2 + 2\lambda_k\langle u - x_k, y\rangle \\
&+ 2\varepsilon_3\lambda_k\beta_k\left[\sup_{u\in M}\varphi_B\left(u, \frac{p}{\varepsilon_3\beta_k}\right) - \sigma_M\left(\frac{p}{\varepsilon_3\beta_k}\right)\right].
\end{aligned} \tag{3.2.14}$$

The inequalities (3.2.9), (3.2.10) and (3.2.14) lead to

$$\begin{aligned}
&2\lambda_k\langle u - x_{k+1}, \beta_k Bx_k + Dx_k + v\rangle \\
\leqslant& \left(\frac{2}{\varepsilon_2}\lambda_k^2\beta_k^2 - 2\mu(1-\varepsilon_3)\lambda_k\beta_k\right)\|Bx_k\|^2 + \left(\frac{4}{\varepsilon_2}\lambda_k^2 - 2\eta\lambda_k\right)\|Dx_k - Du\|^2 + \varepsilon_2\|x_{k+1} - x_k\|^2 \\
&+ \frac{4}{\varepsilon_2}\lambda_k^2\|Du + v\|^2 + 2\varepsilon_3\lambda_k\beta_k\left[\sup_{u\in M}\varphi_B\left(u, \frac{p}{\varepsilon_3\beta_k}\right) - \sigma_M\left(\frac{p}{\varepsilon_3\beta_k}\right)\right] + 2\lambda_k\langle u - x_k, y\rangle.
\end{aligned} \tag{3.2.15}$$

Finally, by combining (3.2.6), (3.2.7) and (3.2.15), we obtain (3.2.2). □

From now on we will assume that for $0 < \alpha_+ < \frac{1}{3}$ the constants $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and the sequences $\{\lambda_k\}_{k\geqslant 1}$ and $\{\beta_k\}_{k\geqslant 1}$ are chosen such that

$$(\mathrm{C}_4) \quad 1 - \varepsilon_3 > 0, \quad \varepsilon_2 < 1 - 4\varepsilon_1 - \alpha_+ - \frac{\alpha^2}{4\varepsilon_1} \quad \text{and} \quad \sup_{k\geqslant 1}\lambda_k\beta_k < \mu\varepsilon_2(1-\varepsilon_3).$$

As a consequence, there exists

$$0 < s \leqslant 1 - \frac{\varepsilon_1}{1 - 3\varepsilon_1 - \varepsilon_2}\left(1 + \frac{\alpha}{2\varepsilon_1}\right)^2,$$

which means that for all $k \geqslant 1$ it holds

$$\alpha_{k+1} + \frac{\alpha_{k+1}^2}{4\varepsilon_1} - (1 - 4\varepsilon_1 - \varepsilon_3) \leqslant \alpha_+ + \frac{\alpha^2}{4\varepsilon_1} - (1 - 4\varepsilon_1 - \varepsilon_3) < -s. \tag{3.2.16}$$

On the other hand, there exists

$$0 < t \leqslant \mu(1 - \varepsilon_2) - \frac{1}{\varepsilon_3}\sup_{k\geqslant 0}\lambda_k\beta_k,$$

which means that for all $k \geqslant 1$ it holds

$$\frac{1}{\varepsilon_3}\lambda_k\beta_k - \mu(1-\varepsilon_2) \leqslant -t. \tag{3.2.17}$$

Remark 3.2.1. (i) Since $0 < \alpha_+ < \dfrac{1}{3}$, one can always find $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\varepsilon_2 < 1 - 4\varepsilon_1 - \alpha_+ - \frac{\alpha^2}{4\varepsilon_1}.$$

One possible choice is

$$\varepsilon_1 = \frac{\alpha}{4} \text{ and } 0 < \varepsilon_2 < 1 - 3\alpha.$$

From the second inequality in (C_4) it follows that

$$1 - 3\varepsilon_1 - \varepsilon_2 > \varepsilon_1 + \alpha_+ + \frac{\alpha^2}{4\varepsilon_1} > 0.$$

(ii) As

$$1 - \frac{\varepsilon_1}{1 - 3\varepsilon_1 - \varepsilon_2}\left(1 + \frac{\alpha}{2\varepsilon_1}\right)^2 = \frac{1}{1 - 3\varepsilon_1 - \varepsilon_2}\left(1 - 4\varepsilon_1 - \varepsilon_2 - \alpha_+ - \frac{\alpha^2}{4\varepsilon_1}\right) > 0,$$

it is always possible to choose s such that

$$0 < s \leqslant 1 - \frac{\varepsilon_1}{1 - 3\varepsilon_1 - \varepsilon}\left(1 + \frac{\alpha}{2\varepsilon_1}\right)^2.$$

Since in this case

$$s < 1 - 4\varepsilon_1 - \varepsilon_2 - \alpha_+ - \frac{\alpha^2}{4\varepsilon_1},$$

one has (3.2.16).

The following proposition brings us closer to the convergence result.

Proposition 3.2.3. *Let $0 < \alpha_+ < \frac{1}{3}$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and the sequences $\{\lambda_k\}_{k\geqslant 1}$ and $\{\beta_k\}_{k\geqslant 1}$ satisfy condition (C_4). Let $\{x_k\}_{k\geqslant 0}$ be the sequence generated by Algorithm 3.2.1 and assume that the Hypotheses 3.2.1 are verified. Then the following statements are true:*

(i) the sequence $\{\|x_{k+1} - x_k\|\}_{k\geqslant 0}$ belongs to ℓ^2 and the sequence $\left\{\lambda_k\beta_k\|Bx_k\|^2\right\}_{k\geqslant 1}$ belongs to ℓ^1;

(ii) if, moreover, $\liminf_{k\to+\infty} \lambda_k\beta_k > 0$, then $\lim_{k\to+\infty}\|Bx_k\| = 0$ and thus every cluster point of the sequence $\{x_k\}_{k\geqslant 0}$ lies in M.

(iii) for every $u \in \operatorname{zer}(A + D + N_M)$, the limit $\lim_{k\to+\infty}\|x_k - u\|$ exists.

Proof. Since $\lim_{k\to+\infty}\lambda_k = 0$, there exists a integer $k_1 \geqslant 1$ such that $\lambda_k \leqslant \dfrac{2}{\varepsilon_2}\eta$ for any integer $k \geqslant k_0$. According to Lemma 3.2.2, for every $(u, y) \in \operatorname{gph}(A + D + N_M)$ such that $y = v + Du + p$, with $v \in Au$ and $p \in N_M(u)$, and all $k \geqslant k_0$ the following inequality holds

$$\begin{aligned}
&\|x_{k+1} - u\|^2 - \|x_k - u\|^2 \\
&\leqslant \alpha_k\|x_k - u\|^2 - \alpha_k\|x_{k-1} - u\|^2 - (1 - 4\varepsilon_1 - \varepsilon_2)\|x_{k+1} - x_k\|^2 \\
&\quad + \left(\alpha_k + \frac{\alpha_k^2}{4\varepsilon_1}\right)\|x_k - x_{k-1}\|^2 + \left(\frac{2}{\varepsilon_2}\lambda_k\beta_k - 2\mu(1 - \varepsilon_3)\right)\lambda_k\beta_k\|Bx_k\|^2 \\
&\quad + \frac{4}{\varepsilon_2}\lambda_k^2\|Du + v\|^2 + 2\varepsilon_3\lambda_k\beta_k\left[\sup_{u\in M}\varphi_B\left(u, \frac{p}{\varepsilon_3\beta_k}\right) - \sigma_M\left(\frac{p}{\varepsilon_3\beta_k}\right)\right] + 2\lambda_k\langle u - x_k, y\rangle .
\end{aligned} \tag{3.2.18}$$

We consider $u \in \operatorname{zer}(A + D + \mathcal{N}_M)$, which means that we can take $y = 0$ in (3.2.18). For all $k \geqslant 1$ we denote

$$\theta_k := \|x_k - u\|^2, \quad \rho_k := \theta_k - \alpha_k \theta_{k-1} + \left(\alpha_k + \frac{\alpha_k^2}{4\varepsilon_1}\right) \|x_k - x_{k-1}\|^2 \tag{3.2.19}$$

and

$$\delta_k := \frac{4}{\varepsilon_2} \lambda_k^2 \|Du + v\|^2 + 2\varepsilon_3 \lambda_k \beta_k \left[\sup_{u \in M} \varphi_B \left(u, \frac{p}{\varepsilon_3 \beta_k}\right) - \sigma_M \left(\frac{p}{\varepsilon_3 \beta_k}\right)\right]. \tag{3.2.20}$$

Using that $(\alpha_k)_{k \geqslant 1}$ is nondecreasing, for all $k \geqslant k_0$ it yields

$$\begin{aligned} \rho_{k+1} - \rho_k &\leqslant \left(\alpha_{k+1} + \frac{\alpha_{k+1}^2}{4\varepsilon_1} - (1 - 4\varepsilon_1 - \varepsilon_2)\right) \|x_{k+1} - x_k\|^2 \\ &\quad + \left(\frac{2}{\varepsilon_3} \lambda_k \beta_k - 2\mu (1 - \varepsilon_2)\right) \lambda_k \beta_k \|Bx_k\|^2 + \delta_k \\ &\leqslant -s \|x_{k+1} - x_k\|^2 - 2t \lambda_k \beta_k \|Bx_k\|^2 + \delta_k, \end{aligned} \tag{3.2.21}$$

where $s, t > 0$ are chosen according to (3.2.16) and (3.2.17), respectively.

Thanks to $(\mathrm{H}_2^{\text{fitz}})$ and (C_1) it holds

$$\sum_{k \geqslant 1} \delta_k = \frac{4}{\varepsilon_2} \|Du + v\|^2 \sum_{k \geqslant 1} \lambda_k^2 + 2 \sum_{k \geqslant 1} \varepsilon_3 \lambda_k \beta_k \left[\sup_{u \in M} \varphi_B \left(u, \frac{p}{\varepsilon_3 \beta_k}\right) - \sigma_M \left(\frac{p}{\varepsilon_3 \beta_k}\right)\right] < +\infty. \tag{3.2.22}$$

Hence, according to Lemma 2.4.3, we obtain

$$\sum_{k \geqslant 0} \|x_{k+1} - x_k\|^2 < +\infty \text{ and } \sum_{k \geqslant 1} \lambda_k \beta_k \|Bx_k\|^2 < +\infty, \tag{3.2.23}$$

which proves (i). If, in addtion $\liminf_{k \to \infty} \lambda_k \beta_k > 0$, then $\lim_{k \to +\infty} \|Bx_k\| = 0$, which means every cluster point of the sequence $\{x_k\}_{k \geqslant 0}$ lies in $\operatorname{zer} B = M$.

In order to prove (iii), we consider again the inequality (3.2.18) for an arbitrary element $u \in \operatorname{zer}(A + D + \mathcal{N}_M)$ and $y = 0$. With the notations in (3.2.19) and (3.2.20), we get for all $k \geqslant k_0$

$$\theta_{k+1} - \theta_k \leqslant \alpha_k (\theta_k - \theta_{k-1}) + \left(\alpha_k + \frac{\alpha_k^2}{4\varepsilon_1}\right) \|x_k - x_{k-1}\|^2 + \delta_k. \tag{3.2.24}$$

According to (3.2.22) and (3.2.23) we have

$$\sum_{k \geqslant 1} \left(\alpha_k + \frac{\alpha_k^2}{4\varepsilon_1}\right) \|x_k - x_{k-1}\|^2 + \sum_{k \geqslant 1} \delta_k \leqslant \left(\alpha_+ + \frac{\alpha^2}{4\varepsilon_1}\right) \sum_{k \geqslant 1} \|x_k - x_{k-1}\|^2 + \sum_{k \geqslant 1} \delta_k < +\infty, \tag{3.2.25}$$

therefore, by Lemma 2.4.1, the limit $\lim_{k \to +\infty} \theta_k = \lim_{k \to +\infty} \|x_k - u\|^2$ exists, which means that the limit $\lim_{k \to +\infty} \|x_k - u\|$ exists, too. $\square$

Remark 3.2.2. The condition (C_3) that we imposed on the sequence of inertial parameters $\{\alpha_k\}_{k \geqslant 1}$ is similar with the one proposed in [3, Proposition 2.1] when addressing the convergence of the inertial proximal point algorithm. However, the statements in the proposition above and in the following convergence theorem remain valid if one alternatively assumes that there exists α'_+ such that $0 \leqslant \alpha_k \leqslant \alpha'_+ < 1$ for all $k \geqslant 1$ and

$$\sum_{k \geqslant 1} \left(\alpha_k + \frac{\alpha_k^2}{4\varepsilon_1}\right) \|x_k - x_{k-1}\|^2 < +\infty.$$

This can be realized if one chooses for a fixed $q > 1$

$$\alpha_k \leqslant \min \left\{ \alpha'_+, 2\varepsilon_1 \left(-1 + \sqrt{1 + k^{-q} \left\| x_k - x_{k-1} \right\|^{-2}} \right) \right\} \ \forall k \geqslant 1.$$

Indeed, in this situation we have that $\dfrac{\alpha_k^2}{4\varepsilon_1} + \alpha_k - \dfrac{1}{k^q \left\| x_k - x_{k-1} \right\|^2} \leqslant 0$ for all $k \geqslant 1$, which gives

$$\sum_{k \geqslant 1} \left(\alpha_k + \frac{\alpha_k^2}{4\varepsilon_1} \right) \left\| x_k - x_{k-1} \right\|^2 \leqslant \sum_{k \geqslant 1} \frac{1}{k^q} < +\infty.$$

The *sequence of weighted averages* $\{z_k\}_{k \geqslant 1}$ is defined for every $k \geqslant 1$ as

$$z_k := \frac{1}{\tau_k} \sum_{n=1}^{k} \lambda_n x_n, \ \text{where } \tau_k := \sum_{n=1}^{k} \lambda_n. \tag{3.2.26}$$

Lemma 3.2.4 (**Opial-Passty**). *Let Z be a nonempty subset of $\mathcal{H}$ and assume that the limit $\lim_{k \to +\infty} \|x_k - u\|$ exists for every element $u \in Z$. If every sequential weak cluster point of $\{x_k\}_{k \geqslant 0}$, respectively $\{z_k\}_{k \geqslant 1}$, lies in Z, then the sequence $\{x_k\}_{k \geqslant 0}$, respectively $\{z_k\}_{k \geqslant 1}$, converges weakly to an element in Z as $k \to +\infty$.*

Now we are ready to prove the main theorem of this section, which addresses the convergence of the sequence generated by Algorithm 3.2.1.

Theorem 3.2.5. *Let $0 < \alpha_+ < \frac{1}{3}$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and the sequences $\{\lambda_k\}_{k \geqslant 1}$ and $\{\beta_k\}_{k \geqslant 1}$ satisfy condition* (C_4). *Let $\{x_k\}_{k \geqslant 0}$ be the sequence generated by Algorithm 3.2.1, $\{z_k\}_{k \geqslant 1}$ be the sequence defined in* (3.2.26) *and assume that the Hypotheses 3.2.1 are verified. Then the following statements are true:*

(i) the sequence $\{z_k\}_{k \geqslant 1}$ converges weakly to an element in $\operatorname{zer}(A + D + \mathcal{N}_M)$ *as $k \to +\infty$.*

(ii) if A is $\gamma-$strongly monotone with $\gamma > 0$, then $\{x_k\}_{k \geqslant 0}$ converges strongly to the unique element in $\operatorname{zer}(A + D + \mathcal{N}_M)$ *as $k \to +\infty$.*

Proof. (i) According to Proposition 3.2.3 (iii), the limit $\lim_{k \to +\infty} \|x_k - u\|$ exisits for every $u \in \operatorname{zer}(A + D + \mathcal{N}_M)$. Let z be a sequential weak cluster point of $(z_k)_{k \geqslant 1}$. We will show that $z \in \operatorname{zer}(A + D + \mathcal{N}_M)$, by using the characterization (2.1.1) of the maximal monotonicity, and the conclusion will follow by Lemma 3.2.4.

To this end we consider an arbitrary $(u, y) \in \operatorname{gph}(A + D + \mathcal{N}_M)$ such that $y = v + Du + p$, where $v \in Au$ and $p \in \mathcal{N}_M(u)$. From (3.2.18), with the notations (3.2.19) and (3.2.20), we have for all $k \geqslant k_0$

$$\begin{aligned} & \rho_{k+1} - \rho_k \\ & \leqslant - s \left\| x_{k+1} - x_k \right\|^2 - 2t\lambda_k \beta_k \left\| B x_k \right\|^2 + \delta_k + 2\lambda_k \langle u - x_k, y \rangle \leqslant \delta_k + 2\lambda_k \langle u - x_k, y \rangle. \end{aligned} \tag{3.2.27}$$

Recall that from (3.2.22) that $\sum_{k \geqslant 1} \delta_k < +\infty$. Since $(x_k)_{k \geqslant 0}$ is bounded, the sequence $(\rho_k)_{k \geqslant 1}$ is also bounded.

We fix an arbitrary integer $\bar{K} \geqslant k_0$ and sum up the inequalities in (3.2.27) for $n = k_0 + 1, k_0 + 2, \cdots, \bar{K}$. This yields

$$\rho_{\bar{K}+1} - \rho_{k_0+1} \leqslant \sum_{k \geqslant 1} \delta_k + 2 \left\langle - \sum_{k=1}^{k_0} \lambda_k u + \sum_{k=1}^{k_0} \lambda_k x_k, y \right\rangle + 2 \left\langle \tau_{\bar{K}} u - \sum_{k=1}^{\bar{K}} \lambda_k x_k, y \right\rangle.$$

After dividing this last inequality by $2\tau_{\bar{K}} = 2\sum_{k=1}^{\bar{K}} \lambda_k$, we obtain

$$\frac{1}{2\tau_{\bar{K}}}\left(\rho_{\bar{K}+1} - \rho_{k_0+1}\right) \leqslant \frac{1}{2\tau_{\bar{K}}} T + 2\left\langle u - z_{\bar{K}}, y\right\rangle, \tag{3.2.28}$$

where $T := \sum_{k\geqslant 1} \delta_k + 2\left\langle -\sum_{k=1}^{k_0} \lambda_k u + \sum_{k=1}^{k_0} \lambda_k x_k, y\right\rangle \in \mathbb{R}$. By passing in (3.2.28) to the limit and by using that $\lim_{k\to\infty} \tau_{\bar{K}} = \lim_{\bar{K}\to\infty} \sum_{k=1}^{\bar{K}} \lambda_k = +\infty$, we get

$$\liminf_{\bar{K}\to\infty} \left\langle u - z_{\bar{K}}, y\right\rangle \geqslant 0.$$

As z is a sequential weak cluster point of $(z_k)_{k\geqslant 1}$, the above inequality gives us $\langle u - z, y\rangle \geqslant 0$, which finally means that $z \in \operatorname{zer}(A + D + N_M)$.

(ii) Let $u \in \mathcal{H}$ be the unique element in $\operatorname{zer}(A + D + N_M)$. Since A is $\gamma-$strongly monotone with $\gamma > 0$, the formula in (3.2.4) reads for all $k \geqslant 1$

$$\left\langle x_{k+1} - u, x_k - x_{k+1} - \lambda_k\left(Dx_k + \beta_k Bx_k + v\right) + \alpha_k\left(x_k - x_{k-1}\right)\right\rangle \geqslant \gamma\lambda_k \left\|x_{k+1} - u\right\|^2$$

or, equivalently,

$$\begin{aligned} &2\gamma\lambda_k \left\|x_{k+1} - u\right\|^2 + 2\left\langle u - x_{k+1}, x_k - x_{k+1}\right\rangle \\ &\leqslant 2\lambda_k \left\langle u - x_{k+1}, \beta_k Bx_k + Dx_k + v\right\rangle - 2\alpha_k\left\langle u - x_{k+1}, x_k - x_{k-1}\right\rangle. \end{aligned}$$

By using again (3.2.6), (3.2.7) and (3.2.15) we obtain for all $k \geqslant 1$

$$\begin{aligned} &2\gamma\lambda_k \left\|x_{k+1} - u\right\|^2 + \left\|x_{k+1} - u\right\|^2 - \left\|x_k - u\right\|^2 \\ &\leqslant \alpha_k \left\|x_k - u\right\|^2 - \alpha_k \left\|x_{k-1} - u\right\|^2 - \left(1 - 4\varepsilon_1 - \varepsilon_2\right)\left\|x_{k+1} - x_k\right\|^2 \\ &\quad + \left(\alpha_k + \frac{\alpha_k^2}{4\varepsilon_1}\right)\left\|x_k - x_{k-1}\right\|^2 + \left(\frac{2}{\varepsilon_2}\lambda_k^2\beta_k^2 - 2\mu\left(1 - \varepsilon_3\right)\lambda_k\beta_k\right)\left\|Bx_k\right\|^2 \\ &\quad + \left(\frac{4}{\varepsilon_2}\lambda_k^2 - 2\eta\lambda_k\right)\left\|Dx_k - Du\right\|^2 + \frac{4}{\varepsilon_2}\lambda_k^2\left\|Du + v\right\|^2 \\ &\quad + 2\varepsilon_3\lambda_k\beta_k\left[\sup_{u\in M} \varphi_B\left(u, \frac{p}{\varepsilon_3\beta_k}\right) - \sigma_M\left(\frac{p}{\varepsilon_3\beta_k}\right)\right] + 2\lambda_k\left\langle u - x_k, y\right\rangle. \end{aligned}$$

By using the notations in (3.2.19) and (3.2.20), this yields for all $k \geqslant 1$

$$2\gamma\lambda_k \left\|x_{k+1} - u\right\|^2 + \theta_{k+1} - \theta_k \leqslant \alpha_k\left(\theta_k - \theta_{k-1}\right) + \left(\alpha_k + \frac{\alpha_k^2}{4\varepsilon_1}\right)\left\|x_k - x_{k-1}\right\|^2 + \delta_k$$

By taking into account (3.2.25), from Lemma 2.4.1 we get

$$2\gamma\sum_{k\geqslant 1} \lambda_k \left\|x_k - u\right\|^2 < +\infty.$$

According to (C_1) we have $\sum_{k\geqslant 1} \lambda_k = +\infty$, which implies that the limit $\lim_{k\to\infty} \left\|x_k - u\right\|$ must be equal to zero. This provides the desired conclusion. □

3.3 Applications to convex bilevel programming

We will employ the results obtained in the previous section in the context of monotone inclusions to the solving of convex bilevel programming problems.

Let $\mathcal{H}$ be a real Hilbert space, $f\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ a proper, convex and lower semicontinuous function and $g, h\colon \mathcal{H} \to \mathbb{R}$ differentiable functions with L_g-Lipschitz continuous and, respectively, L_h-Lipschitz continuous gradients. Suppose that $\arg\min h \neq \varnothing$ and $\min h = 0$. The bilevel programming problem to solve reads

$$\min_{x \in \arg\min h} f(x) + g(x). \tag{3.3.1}$$

The assumption $\min h = 0$ is not restrictive as, otherwise, one can replace h with $h - \min h$.

Hypothesis 3.3.1. *The convergence analysis will be carry out in the following hypotheses:*

($\mathrm{H}_1^{\mathrm{prog}}$) $\partial f + \mathcal{N}_{\arg\min h}$ *is maximally monotone and* $\mathcal{S} := \arg \min_{x \in \arg\min h} \{f(x) + g(x)\} \neq \varnothing$*;*

($\mathrm{H}_2^{\mathrm{prog}}$) *for every* $p \in \operatorname{ran}\mathcal{N}_{\arg\min h}$, $\sum_{k \geqslant 1} \lambda_k \beta_k \left[h^*\left(\frac{p}{\beta_k}\right) - \sigma_{\arg\min h}\left(\frac{p}{\beta_k}\right)\right] < +\infty.$

In the above hypotheses, we have that $\partial f + \nabla g + \mathcal{N}_{\arg\min h} = \partial\left(f + g + \delta_{\arg\min h}\right)$ and hence $\mathcal{S} = \operatorname{zer}\left(\partial f + \nabla g + \mathcal{N}_{\arg\min h}\right) \neq \varnothing$. Since ∇g and ∇h are L_g^{-1}-cocoercive and, respectively, L_h^{-1}- cocoercive, and $\arg\min h = \operatorname{zer}\nabla h$ solving the bilevel programming problem in (3.3.1) reduces to solving the monotone inclusion

$$0 \in \partial f(x) + \nabla g(x) + \mathcal{N}_{\arg\min h}(x).$$

By using to this end Algorithm 3.2.1, we recieve the following iterative scheme.

Algorithm 3.3.1. *Let* $\{\alpha_k\}_{k \geqslant 1}, \{\lambda_k\}_{k \geqslant 1}$ *and* $\{\beta_k\}_{k \geqslant 1}$ *be sequences of positive real numbers such that*

(C_1) $\{\lambda_k\}_{k \geqslant 1} \in \ell^2 \setminus \ell^1$;

(C_2) $\{\alpha_k\}_{k \geqslant 1}$ *is nondecreasing;*

(C_3) *there exists* α *with* $0 \leqslant \alpha_k \leqslant \alpha_+ < 1/3$ *for all* $k \geqslant 1$.

Let $x_0, x_1 \in \mathcal{H}$. *For all* $k \geqslant 1$ *we set*

$$x_{k+1} := \operatorname{prox}_{\lambda_k f}\left(x_k - \lambda_k \nabla g(x_k) - \lambda_k \beta_k \nabla h(x_k) + \alpha_k (x_k - x_{k-1})\right).$$

By using the inequality (2.1.3), one can easily notice, that ($\mathrm{H}_2^{\mathrm{prog}}$) implies ($\mathrm{H}_2^{\mathrm{fitz}}$), which means that the convergence statements for Algorithm 3.3.1 can be derived as particular instances of the ones derived in the previous section.

Alternatively, one can use to this end the following lemma and employ the same ideas and techniques as in Section 3.2. Lemma 3.3.1 is similar to Lemma 3.2.2, however, it will allow us to provide convergence statements also for the sequence of function values $(h(x_k))_{k \geqslant 0}$.

Lemma 3.3.1. *Let* $\{x_k\}_{k \geqslant 0}$ *be the sequence generated by* Algorithm 3.3.1 *and* (u, y) *be an element in* $\operatorname{gph}\left(\partial f + \nabla g + \mathcal{N}_{\arg\min h}\right)$ *such that*

$$y = v + \nabla g(u) + p \text{ with } v \in \partial f(u) \text{ and } p \in N_{\arg\min h}(u).$$

Further, let $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ be such that $1 - \varepsilon_3 > 0$. Then the following inequality holds for all $k \geqslant 1$

$$
\begin{aligned}
& \|x_{k+1} - u\|^2 - \|x_k - u\|^2 \\
\leqslant\ & \alpha_k \|x_k - u\|^2 - \alpha_k \|x_{k-1} - u\|^2 - (1 - 4\varepsilon_1 - \varepsilon_2) \|x_{k+1} - x_k\|^2 + \left(\alpha_k + \frac{\alpha_k^2}{4\varepsilon_1}\right) \|x_k - x_{k-1}\|^2 \\
& \left(\frac{2}{\varepsilon_2}\lambda_k^2\beta_k^2 - 2\mu(1-\varepsilon_3)\lambda_k\beta_k\right) \|\nabla h(x_k)\|^2 + \left(\frac{4}{\varepsilon_2}\lambda_k^2 - 2\eta\lambda_k\right) \|\nabla g(x_k) - \nabla g(u)\|^2 \\
& + \lambda_k\beta_k [h(u) - h(x_k)] + \frac{4}{\varepsilon_2}\lambda_k^2 \|v + \nabla g(u)\|^2 \\
& + \varepsilon_3\lambda_k\beta_k \left[h^*\left(\frac{2p}{\varepsilon_3\beta_k}\right) - \sigma_{\arg\min h}\left(\frac{2p}{\varepsilon_3\beta_k}\right)\right] + 2\lambda_k \langle u - x_k, y\rangle .
\end{aligned}
$$

Proof. Let be $k \geqslant 1$ fixed. The proof follows by combining the estimates used in the proof of Lemma 3.2.2 with some inequalities which better exploits the convexity of h. From (3.2.12) we have

$$
2\lambda_k\beta_k \langle u - x_k, \nabla h(x_k)\rangle \leqslant -2\mu(1-\varepsilon_3)\lambda_k\beta_k \|\nabla h(x_k)\|^2 + 2\varepsilon_3\lambda_k\beta_k \langle u - x_k, \nabla h(x_k)\rangle .
$$

Since h is convex, the following relation also hold

$$
2\lambda_k\beta_k \langle u - x_k, \nabla h(x_k)\rangle \leqslant 2\lambda_k\beta_k [h(u) - h(x_k)] .
$$

Summing up the two inequalities above give us

$$
\begin{aligned}
2\lambda_k\beta_k \langle u - x_k, \nabla h(x_k)\rangle \leqslant & -\mu(1-\varepsilon_3)\lambda_k\beta_k \|\nabla h(x_k)\|^2 + \varepsilon_3\lambda_k\beta_k \langle u - x_k, \nabla h(x_k)\rangle \\
& + \lambda_k\beta_k [h(u) - h(x_k)] .
\end{aligned}
$$

Using the same techniques as in the derivation of (3.2.14), we get

$$
\begin{aligned}
& 2\lambda_k \langle u - x_k, v + \nabla g(x_k) + \beta_k \nabla h(x_k)\rangle \\
\leqslant & -\mu(1-\varepsilon_3)\lambda_k\beta_k \|\nabla h(x_k)\|^2 - 2\eta\lambda_k \|\nabla g(x_k) - \nabla g(u)\|^2 + \lambda_k\beta_k [h(u) - h(x_k)] \\
& + 2\lambda_k \langle u - x_k, y\rangle + \varepsilon_3\lambda_k\beta_k \left[h^*\left(u, \frac{2p}{\varepsilon_3\beta_k}\right) - \sigma_{\arg\min h}\left(\frac{2p}{\varepsilon_3\beta_k}\right)\right] .
\end{aligned}
$$

With this improved estimates, the conclusion follows as in the proof of Lemma 3.2.2. □

By using now Lemma 3.3.1, one obains, after slightly adapting the proof of Proposition 3.2.3, the following result.

Proposition 3.3.2. *Let $0 < \alpha_+ < \frac{1}{3}$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and the sequences $\{\lambda_k\}_{k\geqslant 1}$ and $\{\beta_k\}_{k\geqslant 1}$ satisfy condition* (C_4). *Let $\{x_k\}_{k\geqslant 0}$ be the sequence generated by Algorithm 3.3.1 and assume that the Hypotheses 3.3.1 are verified. Then the following statements are true:*

(i) the sequence $\{\|x_{k+1} - x_k\|\}_{k\geqslant 0}$ belongs to ℓ^2 and the sequences $\left\{\lambda_k\beta_k \|\nabla h(x_k)\|^2\right\}_{k\geqslant 1}$ and $\{\lambda_k\beta_k h(x_k)\}_{k\geqslant 1}$ belong to ℓ^1;

(ii) if, moreover, $\liminf_{k\to+\infty} \lambda_k\beta_k > 0$, then $\lim_{k\to+\infty} \|\nabla h(x_k)\| = \lim_{k\to+\infty} h(x_k) = 0$ and thus every cluster point of the sequence $\{x_k\}_{k\geqslant 0}$ lies in $\arg\min h$.

(iii) for every $u \in \mathcal{S}$, the limit $\lim_{k\to+\infty} \|x_k - u\|$ exists.

Finally, the above proposition leads to the following convergence result.

Theorem 3.3.3. *Let $0 < \alpha_+ < \frac{1}{3}$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and the sequences $\{\lambda_k\}_{k\geqslant 1}$ and $\{\beta_k\}_{k\geqslant 1}$ satisfy condition* (C$_4$). *Let $\{x_k\}_{k\geqslant 0}$ be the sequence generated by Algorithm 3.3.1, $\{z_k\}_{k\geqslant 1}$ be the sequence defined in* (3.2.26) *and assume that the Hypotheses 3.3.1 are verified. Then the following statements are true:*

(i) the sequence $\{z_k\}_{k\geqslant 1}$ converges weakly to an element in $\mathcal{S}$ as $k \to +\infty$.

(ii) if f is $\gamma-$strongly convex with $\gamma > 0$, then $\{x_k\}_{k\geqslant 0}$ converges strongly to the unique element in $\mathcal{S}$ as $k \to +\infty$.

As follows we will show that under inf-compactness assumptions one can achieve weak nonergodic convergence for the sequence $\{x_k\}_{k\geqslant 0}$. Weak nonergodic convergence has been obtained for Algorithm 3.3.1 in [54] when $\alpha_k = \alpha$ for all $k \geqslant 1$ and for restrictive choices for both the sequence of step sizes and penalty parameters.

We denote by $(f+g)_* = \min_{x\in\arg\min h}(f(x)+g(x))$. For every element x in $\mathcal{H}$, we denote by $\operatorname{dist}(x,\mathcal{S}) = \inf_{u\in\mathcal{S}} \|x-u\|$ the distance from x to $\mathcal{S}$. In particular, $\operatorname{dist}(x,\mathcal{S}) = \|x - \mathbf{Pr}_{\mathcal{S}}x\|$, where $\mathbf{Pr}_{\mathcal{S}}x$ denotes the projection of x onto $\mathcal{S}$. The projection operator $\mathbf{Pr}_{\mathcal{S}}$ is *firmly nonexpansive* ([24, Proposition 4.8]), this means

$$\|\mathbf{Pr}_{\mathcal{S}}(x) - \mathbf{Pr}_{\mathcal{S}}(y)\|^2 + \|[\mathrm{Id} - \mathbf{Pr}_{\mathcal{S}}](x) - [\mathrm{Id} - \mathbf{Pr}_{\mathcal{S}}](y)\|^2 \leqslant \|x-y\|^2 \quad \forall x,y \in \mathcal{H}. \tag{3.3.2}$$

Denoting $d(x) = \frac{1}{2}\operatorname{dist}(x,\mathcal{S})^2 = \frac{1}{2}\|x - \mathbf{Pr}_{\mathcal{S}}x\|^2$ for all $x \in \mathcal{H}$, one has that $x \mapsto d(x)$ is differentiable and it holds $\nabla d(x) = x - \mathbf{Pr}_{\mathcal{S}}x$ for all $x \in \mathcal{H}$.

Lemma 3.3.4. *Let $\{x_k\}_{k\geqslant 0}$ be the sequence generated by Algorithm 3.3.1 and assume that the Hypotheses 3.3.1 are verified. Then the following inequality holds for all $k \geqslant 1$*

$$\begin{aligned} & d(x_{k+1}) - d(x_k) - \alpha_k (d(x_k) - d(x_{k-1})) + \lambda_k [(f+g)(x_{k+1}) - (f+g)_*] \\ \leqslant & \left(\frac{L_g}{2}\lambda_k + \frac{L_h}{4}\lambda_k\beta_k + \frac{\alpha_k}{2}\right)\|x_{k+1}-x_k\|^2 + \alpha_k\|x_k - x_{k-1}\|^2. \end{aligned} \tag{3.3.3}$$

Proof. Let $k \geqslant 1$ be fixed. Since d is convex, we have

$$d(x_{k+1}) - d(x_k) \leqslant \langle x_{k+1} - \mathbf{Pr}_{\mathcal{S}}(x_{k+1}), x_{k+1} - x_k\rangle. \tag{3.3.4}$$

Then there exists $v_{k+1} \in \partial f(x_{k+1})$ such that (see (3.2.3))

$$x_k - x_{k+1} - \lambda_k(\nabla g(x_k) + \beta_k \nabla h(x_k)) + \alpha_k(x_k - x_{k-1}) = \lambda_k v_{k+1}$$

and, so,

$$\begin{aligned} & \langle x_{k+1} - \mathbf{Pr}_{\mathcal{S}}(x_{k+1}), x_{k+1} - x_k\rangle \\ = & \langle x_{k+1} - \mathbf{Pr}_{\mathcal{S}}(x_{k+1}), -\lambda_k v_{k+1} - \lambda_k \nabla g(x_k) - \lambda_k\beta_k\nabla h(x_k) + \alpha_k(x_k - x_{k-1})\rangle \\ & - \lambda_k\beta_k\langle x_{k+1} - \mathbf{Pr}_{\mathcal{S}}(x_{k+1}), \nabla h(x_k)\rangle + \alpha_k\langle x_{k+1} - \mathbf{Pr}_{\mathcal{S}}(x_{k+1}), x_k - x_{k-1}\rangle. \end{aligned} \tag{3.3.5}$$

Since $v_{k+1} \in \partial f(x_{k+1})$, we get

$$-\lambda_k\langle x_{k+1} - \mathbf{Pr}_{\mathcal{S}}(x_{k+1}), v_{k+1}\rangle \leqslant \lambda_k[f(\mathbf{Pr}_{\mathcal{S}}(x_{k+1})) - f(x_{k+1})]. \tag{3.3.6}$$

Using the convexity of g it follows

$$g(x_k) - g(\mathbf{Pr}_{\mathcal{S}}(x_{k+1})) \leqslant \langle \nabla g(x_k), x_k - \mathbf{Pr}_{\mathcal{S}}(x_{k+1})\rangle. \tag{3.3.7}$$

On the other hand, the Descent Lemma (2.2.4) gives

$$g(x_{k+1}) \leqslant g(x_k) + \langle \nabla g(x_k), x_{k+1} - x_k\rangle + \frac{L_g}{2}\|x_{k+1} - x_k\|^2. \tag{3.3.8}$$

By adding (3.3.7) and (3.3.8), it yields

$$-\lambda_k \langle x_{k+1} - \mathbf{Pr}_{\mathcal{S}}(x_{k+1}), \nabla g(x_k)\rangle \leqslant \lambda_k [g(\mathbf{Pr}_{\mathcal{S}}(x_{k+1})) - g(x_{k+1})] + \frac{L_g \lambda_k}{2} \|x_{k+1} - x_k\|^2 . \tag{3.3.9}$$

Using the $\frac{1}{L_h}-$cocoercivity of ∇h combined with the fact that $\nabla h(\mathbf{Pr}_{\mathcal{S}}(x_{k+1})) = 0$ (as $\mathbf{Pr}_{\mathcal{S}}(x_{k+1})$ belongs to $\mathcal{S}$), it yields

$$-\langle x_k - \mathbf{Pr}_{\mathcal{S}}(x_{k+1}), \nabla h(x_k)\rangle \leqslant -\frac{1}{L_h} \|\nabla h(x_k)\|^2 .$$

Therefore

$$\begin{aligned} -\lambda_k \beta_k \langle x_{k+1} - \mathbf{Pr}_{\mathcal{S}}(x_{k+1}), \nabla h(x_k)\rangle &\leqslant \lambda_k \beta_k \left(\langle x_k - x_{k+1}, \nabla h(x_k)\rangle - \frac{1}{L_h} \|\nabla h(x_k)\|^2 \right) \\ &\leqslant \lambda_k \beta_k \frac{L_h}{4} \|x_{k+1} - x_k\|^2 . \end{aligned} \tag{3.3.10}$$

Further, we have

$$\begin{aligned} &\alpha_k \langle x_{k+1} - \mathbf{Pr}_{\mathcal{S}}(x_{k+1}) - (x_k - \mathbf{Pr}_{\mathcal{S}}(x_k)), x_k - x_{k-1}\rangle \\ \leqslant\ & \frac{\alpha_k}{2} \|[\mathrm{Id} - \mathbf{Pr}_{\mathcal{S}}](x_{k+1}) - [\mathrm{Id} - \mathbf{Pr}_{\mathcal{S}}](x_k)\|^2 + \frac{\alpha_k}{2} \|x_k - x_{k-1}\|^2 \\ \leqslant\ & \frac{\alpha_k}{2} \|x_{k+1} - x_k\|^2 + \frac{\alpha_k}{2} \|x_k - x_{k-1}\|^2 , \end{aligned}$$

and

$$\begin{aligned} &\alpha_k \langle x_k - \mathbf{Pr}_{\mathcal{S}}(x_k), x_k - x_{k-1}\rangle \\ =\ & \alpha_k d(x_k) + \frac{\alpha_k}{2} \|x_k - x_{k-1}\|^2 - \frac{\alpha_k}{2} \|x_{k-1} - \mathbf{Pr}_{\mathcal{S}}(x_k)\|^2 \\ \leqslant\ & \alpha_k d(x_k) + \frac{\alpha_k}{2} \|x_k - x_{k-1}\|^2 - \alpha_k d(x_{k-1}) . \end{aligned}$$

By adding two relations above, we obtain

$$\begin{aligned} &\alpha_k \langle x_{k+1} - \mathbf{Pr}_{\mathcal{S}}(x_{k+1}), x_k - x_{k-1}\rangle \\ =\ & \alpha_k \langle x_{k+1} - \mathbf{Pr}_{\mathcal{S}}(x_{k+1}) - (x_k - \mathbf{Pr}_{\mathcal{S}}(x_k)), x_k - x_{k-1}\rangle + \alpha_k \langle x_k - \mathbf{Pr}_{\mathcal{S}}(x_k), x_k - x_{k-1}\rangle \\ \leqslant\ & \frac{\alpha_k}{2} \|x_{k+1} - x_k\|^2 + \alpha_k \|x_k - x_{k-1}\|^2 + \alpha_k (d(x_k) - d(x_{k-1})) . \end{aligned} \tag{3.3.11}$$

By combining (3.3.6) , (3.3.9) , (3.3.10) and (3.3.11) with (3.3.5) we obtain the desired conclusion. □

Definition 3.3.1. A function $\Psi \colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is said to be inf-compact if for every $r > 0$ and every $\kappa \in \mathbb{R}$ the set

$$\mathrm{lev}_{\kappa}^{r}(\Psi) := \{x \in \mathcal{H} \colon \|x\| \leqslant r, \Psi(x) \leqslant \kappa\}$$

is relatively compact in $\mathcal{H}$.

Note that this condition is automatically fulfilled in the finite-dimensional Hilbert space.

An useful property of inf-compact functions follow.

Lemma 3.3.5. *Let $\Psi \colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be inf-compact and $\{x_k\}_{k \geqslant 0}$ be a bounded sequence in $\mathcal{H}$ such that $\{\Psi(x_k)\}_{k \geqslant 0}$ is bounded as well. If the sequence $\{x_k\}_{k \geqslant 0}$ converges weakly to an element in $\widehat{x}$ as $k \to +\infty$, then it converges strongly to this element.*

Proof. Let $\bar{r} > 0$ and $\bar{\kappa} \in \mathbb{R}$ be such that for all $k \geqslant 1$

$$\|x_k\| \leqslant \bar{r} \quad \text{and} \quad \Psi(x_k) \leqslant \bar{\kappa}.$$

Hence, $\{x_k\}_{k\geqslant 0}$ belongs to the set $\operatorname{lev}_{\bar{\kappa}}^{\bar{r}}(\Psi)$, which is relatively compact. Then $\{x_k\}_{k\geqslant 0}$ has at least one strongly convergente subsequence. Since every strongly convergent subsequence $\{x_{k_l}\}_{l\geqslant 0}$ of $\{x_k\}_{k\geqslant 0}$ has as limit $\widehat{x}$, the conclusion follows. □

Now we can formulate the weak nonergodic convergence result.

Theorem 3.3.6. *Let the sequences $\{\lambda_k\}_{k\geqslant 1}$ and $\{\beta_k\}_{k\geqslant 1}$ satisfy the condition $0 < \liminf_{k\to\infty} \lambda_k\beta_k \leqslant \sup_{k\geqslant 0} \lambda_k\beta_k \leqslant \mu$, $\{x_k\}_{k\geqslant 0}$ be the sequence generated by Algorithm 3.3.1, assume that the Hypotheses 3.3.1 are verified and that either $f+g$ or h is inf-compact. Then the following statements are true:*

(i) $\lim_{k\to+\infty} d(x_k) = 0$;

(ii) the sequence $\{x_k\}_{k\geqslant 0}$ converges weakly to an element in $\mathcal{S}$ as $k \to +\infty$;

(iii) if h is inf-compact, then the sequence $\{x_k\}_{k\geqslant 0}$ converges strongly to an element in $\mathcal{S}$ as $k \to +\infty$.

Proof. (i) Thanks to Lemma 3.3.4 , for all $k \geqslant 1$ we have

$$d(x_{k+1}) - d(x_k) + \lambda_k\left[(f+g)(x_{k+1}) - (f+g)_*\right] \leqslant \alpha_k\left(d(x_k) - d(x_{k-1})\right) + \zeta_k, \quad (3.3.12)$$

where

$$\zeta_k := \left(\frac{L_g}{2}\lambda_k + \frac{L_h}{4}\lambda_k\beta_k + \frac{\alpha_k}{2}\right)\|x_{k+1} - x_k\|^2 + \alpha_k\|x_k - x_{k-1}\|^2.$$

From Proposition 3.3.2 (i), combined with the fact that both sequences $\{\lambda_k\}_{k\geqslant 1}$ and $\{\beta_k\}_{k\geqslant 1}$ are bounded, it follows that $\sum_{k\geqslant 1} \zeta_k < +\infty$.

In general, since $\{x_k\}_{k\geqslant 0}$ is not necessarily included in $\arg\min h$, we have to treat two different cases.

Case 1: There exists an integer $k_1 \geqslant 1$ such that $(f+g)(x_k) \geqslant (f+g)_*$ for all $k \geqslant k_1$. In this case, we obtain from Lemma 2.4.1 that:

- the limit $\lim_{k\to+\infty} d(x_k)$ exists.
- $\sum_{k\geqslant k_1} \lambda_k\left[(f+g)(x_{k+1}) - (f+g)_*\right] < +\infty$. Moreover, since $\{\lambda_k\}_{k\geqslant 1} \notin \ell^1$, we must have

$$\liminf_{k\to+\infty}(f+g)(x_k) \leqslant (f+g)_*. \quad (3.3.13)$$

Consider a subsequence $\{x_{k_n}\}_{n\geqslant 1}$ of $\{x_k\}_{k\geqslant 0}$ such that

$$\lim_{n\to+\infty}(f+g)(x_{k_n}) = \liminf_{k\to+\infty}(f+g)(x_k)$$

and note that, thanks to (3.3.13), the sequence $\{(f+g)(x_{k_n})\}_{n\geqslant 1}$ is bounded. From Proposition 3.3.2 (ii)-(iii) we get that also $\{x_{k_n}\}_{n\geqslant 1}$ and $\{h(x_{k_n})\}_{n\geqslant 0}$ are bounded. Thus, since either $f+g$ or h is inf-compact, there exists a subsequence $\{x_{k_l}\}_{l\geqslant 0}$ of $\{x_{k_n}\}_{n\geqslant 1}$, which converges strongly to an element $\widehat{x}$ as $l \to +\infty$. According to Proposition 3.3.2 (ii)-(iii), $\widehat{x}$ belongs to $\arg\min h$. On the other hand,

$$\lim_{l\to+\infty}(f+g)(x_{k_l}) = \liminf_{k\to+\infty}(f+g)(x_k) \geqslant (f+g)(\widehat{x}) \geqslant (f+g)_*. \quad (3.3.14)$$

We deduce from (3.3.13) - (3.3.14) that $(f+g)(\widehat{x}) = (f+g)_*$, or in other words, that $\widehat{x} \in \mathcal{S}$. In conclusion, thanks to the continuity of d,

$$\lim_{k\to+\infty} d(x_k) = \lim_{l\to\infty} d(x_{k_l}) = d(\widehat{x}) = 0.$$

Case 2: for all $k \geqslant 1$ there exists some $k' > k$ such that $(f+g)(x_{k'}) < (f+g)_*$. We define the set

$$V = \left\{k' \geqslant 1 \colon (f+g)(x_{k'}) < (f+g)_*\right\}.$$

There exist an integer $k_2 \geqslant 2$ such that for all $k \geqslant k_2$ the set $\{n \leqslant k \colon n \in V\}$ is nonempty. Hence, for all $k \geqslant k_2$ the number

$$t_k := \max\{n \leqslant k \colon n \in V\}$$

is well-defined. By definition $t_k \leqslant k$ for all $k \geqslant k_3$ and moreover the sequence $\{t_k\}_{k\geqslant k_2}$ is nondecreasing and $\lim_{k\to+\infty} t_k = +\infty$. Indeed, if $\lim_{k\to\infty} t_k = t \in \mathbb{R}$, then for all $k' > t$ it holds $(f+g)(x_{k'}) \geqslant (f+g)_*$, contradiction. Choose an integer $N \geqslant k_2$.

- If $t_k < N$, then, for all $k = t_k, \cdots, N-1$, since $(f+g)(x_k) \geqslant (f+g)_*$, the inequality (3.3.12) gives

$$\begin{aligned} d(x_{k+1}) - d(x_k) &\leqslant d(x_{k+1}) - d(x_k) + \lambda_k [F(x_{k+1}) - F_*] \\ &\leqslant \alpha_k (d(x_k) - d(x_{k-1})) + \zeta_k. \end{aligned} \tag{3.3.15}$$

  Summing (3.3.15) for $k = t_k, \cdots, N-1$ and using that $\{\alpha_k\}_{k\geqslant 1}$ is nondecreasing, it yields

$$\begin{aligned} d(x_N) - d(x_{t_N}) &\leqslant \sum_{k=t_N}^{N-1} (\alpha_k d(x_k) - \alpha_{k-1} d(x_{k-1})) + \sum_{k=t_N}^{N-1} \zeta_k \\ &\leqslant \alpha_+ d(x_{k-1}) + \sum_{k\geqslant t_N} \zeta_k. \end{aligned} \tag{3.3.16}$$

- If $t_k = N$, then $d(x_N) = d(x_{t_N})$ and we have

$$d(x_N) - \alpha_+ d(x_{N-1}) \leqslant d(x_{t_N}) + \sum_{k\geqslant t_N} \zeta_k. \tag{3.3.17}$$

for all $k \geqslant 1$ we define $a_k := d(x_k) - \alpha_+ d(x_{k-1})$. In both cases it yields

$$a_N \leqslant d(x_{t_N}) + \sum_{k=t_N}^{N} \zeta_k \leqslant d(x_{t_N}) + \sum_{k\geqslant t_N} \zeta_k. \tag{3.3.18}$$

Passing in (3.3.18) to limit as $N \to +\infty$ we obtain that

$$\limsup_{k\to+\infty} a_k \leqslant \limsup_{k\to+\infty} d(x_{t_k}). \tag{3.3.19}$$

Let be $u \in \mathcal{S}$. for all $k \geqslant 1$ we have

$$d(x_k) = \frac{1}{2}\operatorname{dist}(x_k, \mathcal{S})^2 \leqslant \frac{1}{2}\|x_k - u\|^2,$$

which shows that $\{d(x_k)\}_{k\geqslant 0}$ is bounded, as $\lim_{k\to+\infty} \|x_k - u\|$ exists. We obtain

$$\limsup_{k\to\infty} a_k = \limsup_{k\to\infty} [d(x_k) - \alpha_+ d(x_{k-1})] \geqslant (1-\alpha_+) \limsup_{k\to\infty} d(x_k) \geqslant 0. \tag{3.3.20}$$

Further, for all $k \geqslant 1$ we have $(f+g)(x_{t_k}) < (f+g)_*$, which gives

$$\limsup_{k \to +\infty} (f+g)(x_{t_k}) \leqslant (f+g)_* . \tag{3.3.21}$$

This means that the sequence $\{(f+g)(x_{t_k})\}_{k \geqslant 0}$ is bounded from above. Consider a subsequence $\{x_{t_q}\}_{q \geqslant 0}$ of $\{x_{t_k}\}_{k \geqslant 0}$ such that

$$\lim_{q \to +\infty} d(x_{t_q}) = \limsup_{k \to +\infty} d(x_{t_k}) .$$

From Proposition 3.3.2 (ii)-(iii) we get that also $\{x_{t_q}\}_{q \geqslant 0}$ and $\left(h(x_{t_q})\right)_{q \geqslant 0}$ are bounded. Thus, since either $f+g$ or h is inf-compact, there exists a subsequence $(x_{t_l})_{l \geqslant 0}$ of $\{x_{t_q}\}_{q \geqslant 0}$, which converges strongly to an element $\widehat{x}$ as $l \to +\infty$. According to Proposition 3.3.2 (ii)-(iii), $\widehat{x}$ belongs to $\arg\min h$. Furthermore, it holds

$$\liminf_{l \to +\infty} (f+g)(x_{t_l}) \geqslant (f+g)(\widehat{x}) \geqslant (f+g)_* . \tag{3.3.22}$$

We deduce from (3.3.21) and (3.3.22) that

$$(f+g)_* \leqslant (f+g)(\widehat{x}) \leqslant \limsup_{l \to +\infty} (f+g)(x_{t_l}) \leqslant \limsup_{k \to +\infty} (f+g)(x_{t_k}) \leqslant (f+g)_* ,$$

which gives $\widehat{x} \in \mathcal{S}$. Thanks to the continuity of d we get

$$\limsup_{k \to +\infty} d(x_{t_k}) = \lim_{l \to +\infty} d(x_{t_l}) = d(\widehat{x}) = 0. \tag{3.3.23}$$

By combining (3.3.19), (3.3.20) and (3.3.23), it yields

$$0 \leqslant (1-\alpha_+) \limsup_{k \to +\infty} d(x_k) \leqslant \limsup_{k \to +\infty} a_k \leqslant \limsup_{k \to +\infty} d(x_{t_k}) = 0,$$

which implies $\limsup_{k \to +\infty} d(x_k) = 0$ and thus

$$\lim_{k \to +\infty} d(x_k) = \liminf_{k \to +\infty} d(x_k) = \limsup_{k \to +\infty} d(x_k) = 0.$$

(ii) According to (i) we have $\lim_{k \to \infty} d(x_k) = 0$, thus every weak cluster point of the sequence $\{x_k\}_{k \geqslant 0}$ belongs to $\mathcal{S}$. From Lemma 3.2.4 it follows that $\{x_k\}_{k \geqslant 0}$ converges weakly to a point in $\mathcal{S}$ as $k \to +\infty$.

(iii) Since $\liminf_{k \to \infty} \lambda_k \beta_k > 0$, from Proposition 3.3.2(ii) we have that

$$\lim_{k \to +\infty} \|\nabla h(x_k)\| = \lim_{k \to +\infty} h(x_k) = 0.$$

Since $\{x_k\}_{k \geqslant 0}$ is bounded, there exist $\bar{r} > 0$ and $\bar{\kappa} \in \mathbb{R}$ such that for all $k \geqslant 1$

$$\|x_k\| \leqslant \bar{r} \quad \text{ and } \quad h(x_k) \leqslant \bar{\kappa}.$$

Thanks to (ii) the sequence $\{x_k\}_{k \geqslant 0}$ converges weakly to an element in $\mathcal{S}$. Therefore, according to Lemma 3.3.5, it converges strongly to this element in $\mathcal{S}$. □

3.4 Further perspectives

It would be interesting to extend the interval value of $\{\alpha_k\}_{k\geqslant 1}$ from $[0, 1/3)$ to $[0, 1]$. One possible strategy is to insert a relaxation factor into the scheme, similar to the paper [10] of Attouch and Cabot, inspired by a technique recently introduced by Attouch and Peypouquet in [19] and to study the interplay of the relaxation and inertial parameters. The continuous counterpart of the presented algorithm expressed as a second-order dynamical system would also be interesting to consider.

For unconstrained optimization problems, which correspond to the situation when $h = 0$ in (3.3.1), one can obtain convergence rates of $o\left(1/k^2\right)$ for the sequence of function values, see for instance [12, 28, 110]. This is a setting which is not covered by our analysis, however, it is a topic which might be of interest.

Another interesting direction for the bilevel optimization problem is to study the convergence behavior of the generated sequence in the absence of convexity. Using the Kurdyka-Łojasiewicz property, several results for unconstrained nonconvex optimization have been obtained, while the constrained setting has not been so much considered.

Chapter 4

Factorization of completely positive matrices using iterative projected gradient steps

This chapter follows our work [58].

We aim to factorize a completely positive matrix by using an optimization approach which consists of the minimization of a nonconvex smooth function over a convex and compact set. To solve this problem we propose a projected gradient algorithm with parameters that take into account the effects of relaxation and inertia. Both projection and gradient steps are simple in the sense that they have explicit formulas and do not require inner loops. Furthermore, no expensive procedure to find an appropriate starting point is needed. The convergence analysis shows that the whole sequence of generated iterates converges to a critical point of the objective function, and it makes use of the Łojasiewicz inequality. Its rate of convergence expressed in terms of the Łojasiewicz exponent of a regularization of the objective function is also provided. Numerical experiments demonstrate the efficiency of the proposed method, in particular in comparison to other factorization algorithms, and emphasize the role of the relaxation and inertial parameters.

4.1 Problem formulation and motivation

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called *completely positive* if there exists an entrywise nonnegative matrix $X \in \mathbb{R}_+^{n\times r}$ such that

$$A = XX^T.$$

Let

$$\mathcal{CP}_n := \left\{A \in \mathbb{R}^{n\times n} \colon A = XX^T \text{ with } X \in \mathbb{R}_+^{n\times r}, r \geqslant 1\right\}$$

denote the set of $n \times n$ completely positive matrices. This set is a *proper cone* whose extreme rays are the rank-one matrices xx^T with $x \in \mathbb{R}_+^n$ (see [31]), thus

$$\mathcal{CP}_n = \operatorname{conv}\left\{xx^T \colon x \in \mathbb{R}_+^n\right\},$$

where conv stands for the convex hull operator.

Closely related to the completely positive matrices is the class of *copositive matrices*

$$\mathcal{COP}_n := \left\{A \in \mathbb{S}^{n\times n} \colon x^T A x \geqslant 0 \quad \forall x \in \mathbb{R}_+^n\right\},$$

where $\mathbb{S}^{n\times n}$ denotes the set of $n\times n$ symmetric matrices. In fact, $\mathcal{CP}_n$ is the *dual cone* of $\mathcal{COP}_n$ (see, for instance, [31]), namely,

$$\mathcal{CP}_n = (\mathcal{COP}_n)^* := \left\{A \in \mathbb{S}^{n\times n} \colon \langle A, B\rangle \geqslant 0 \quad \forall B \in \mathcal{COP}_n\right\}.$$

Here, $\langle \cdot, \cdot \rangle$ denotes the *Frobenius inner product* (see Section 4.2 for the precise definition).

Many relaxations of combinatorial optimization problems and of nonconvex quadratic optimization problems can be formulated as linear problems over $\mathcal{CP}_n$ or $\mathcal{COP}_n$. Since the objective function and the constraint functions are linear, the challenge when addressing these is entirely transferred in the proper handling of the cone constraints. Consequently, copositive and completely positive matrices have received considerable attention in recent years (see, for instance, [38, 63, 78]). The application fields, where copositive and completely positive matrices appear, include block design, complementarity problems, projections in energy demand, the Markovian modelling of DNA evolutions, and maximin efficiency robust tests, see [31] and the references therein.

We illustrate this approach for a nonconvex quadratic programming problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & x^T M x \\ \text{s.t. } & \mathbf{j}_n^T x = 1 \\ & x \in \mathbb{R}^n_+ \end{aligned} \tag{4.1.1}$$

where $M \in \mathbb{S}^{n \times n}$ and $\mathbf{j}_n$ denotes the all-ones vector in $\mathbb{R}^n$. If M is not a positive semidefinite matrix, then (4.1.1) is a nonconvex optimization problem which is usually NP-hard and exhibits numerous local minima. Observe that the objective function of (4.1.1) can be rewritten in terms of the Frobenius inner product as $x^T M x = \langle M, xx^T \rangle$. In the same fashion, the constraint $\mathbf{j}_n^T x = 1$ implies $\langle \mathbf{j}_n \mathbf{j}_n^T, X \rangle = 1$, for $X = xx^T$. Therefore, the optimization problem

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times n}} \ & \langle M, X \rangle \\ \text{s.t. } & \langle \mathbf{j}_n \mathbf{j}_n^T, X \rangle = 1 \\ & X \in \mathcal{CP}_n \end{aligned} \tag{4.1.2}$$

is a convex relaxation of the nonconvex quadratic problem (4.1.1). In [40] it has been shown how optimal solutions of (4.1.2) can be related to optimal solutions of (4.1.1). Let X_* be an optimal solution of (4.1.2). If X_* is of rank one, then it can be expressed as $X_* = x_* x_*^T$ and therefore x_* is an optimal solution of (4.1.1). If $\operatorname{rank}(X_*) > 1$, then X_* can be factorized as $X_* = \sum_{i=1}^r x_i x_i^T$ and it can be shown that an appropriately scaled version of each x_i is an optimal solution of (4.1.1).

One of the main challenge when dealing with completely positive matrices is their efficient factorization ([31, 74, 87]). This is a question of high relevance in many applications, as, for example, in the statistics of multivariate extremes. Cooley and Thibaud have shown in [71] that the tail dependence of a multi-variate regularly-varying random vector can be summarized in a so-called tail pairwise dependence matrix Σ of pairwise dependence metrics. This matrix Σ can be shown to be completely positive, and a nonnegative factorization of it can be used to estimate probabilities of extreme events or to simulate realizations with pairwise dependence summarized by Σ. This approach has been used in [71] to study data describing daily precipitation measurements. Further applications of the nonnegative factorization of completely positive matrices can be found in data mining and clustering ([75]), and in automatic control ([32, 106]).

Recently, Groetzner and Dür proposed in [87] a novel approach to the nonnegative factorization problem which consists of formulating it as a nonconvex split feasibility problem and, consequently, of solving it via the *method of alternating projections*. It is known that when the initial point is sufficiently close to the feasible set, then the sequence generated by the nonconvex method of alternating projections convergences to an feasible element. The drawback of this algorithm is that it requires in every iteration two projections, which both have in general to be approximately calculated via inner loops, since they amount to solve a second order cone problem (SOCP) and to find a singular value decomposition of a matrix, respectively. In the same article, a modification of this method has been suggested, which replaces the solving of the SOCP by a simple projection on the nonnegative orthant, but keeps the singular value decomposition, however, without a theoretical evidence of its convergence. Also very recently, Chen, Pong, Tan and Zeng proposed in [66] another approach which consists of reformulating the split feasibility problem as a *difference-of-convex optimization problem* and, consequently, in solving it via a specific algorithm, which also requires the singular valued decomposition of a matrix in every iteration. We will present these approaches in more detail later.

In this chapter we develop a different approach for the nonnegative factorization of a completely positive matrix, which amounts to the minimization of a nonconvex smooth function over a convex and compact set. To solve this problem we propose a *projected gradient algorithm* with parameters that take into account the effects of *relaxation* and *inertia*. The gradient and the projection steps are expressed by simple explicit formulas and thus do not require any inner loops. We prove the global convergence of the generated sequence for any starting point, which is another advantage over the methods discussed above, that make use of expensive computing procedures to find the points where the algorithms start. We provide rates of convergence for both the sequences of objective function values and of iterates in terms of the Łojasiewicz exponent of a regularization of the objective function. Numerical experiments show that our algorithm outperforms the other iterative factorization methods and emphasizes the influence of the relaxation and inertial parameters on its performances.

Relaxation techniques have been introduced to provide more flexibility to iterative schemes ([24]), while inertial effects in order to accelerate the convergence of numerical methods ([110, 28, 18]) and to allow the detection of various critical points ([116]). Inertial proximal gradient algorithms for nonconvex optimization problems have been proposed and studied in [43, 51, 111, 115]; their global convergence has been shown in the framework of the Kurdyka-Łojasiewicz property ([5, 8, 33, 36, 93, 103]). For convex optimization problems, *relaxed inertial algorithms* have been proved to combine the advantages of both relaxation techniques and inertial effects (see [10, 11, 92]). One of the aims of this chapter is to investigate, also in the nonconvex setting, to which extent the interplay between relaxation and inertial parameters influence the numerical performances of projected/proximal gradient algorithms.

4.2 Preliminaries

4.2.1 Notations

We will write for a $n \times r$ matrix $X := (x_{i,j})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant r}$ if we want to specify its elements, and neglect the subscripts if there is no risk of confusion. The *Frobenius inner product* of $X, Y \in \mathbb{R}^{n \times r}$ is defined by $\langle X, Y \rangle := \operatorname{trace}\left(X^T Y\right) = \sum_{i=1}^{n} \sum_{j=1}^{r} x_{i,j} y_{i,j}$. Due to the definition of trace operator it holds

$$\operatorname{trace}\left(X^T Y\right) = \operatorname{trace}\left(X Y^T\right) = \operatorname{trace}\left(Y^T X\right) = \operatorname{trace}\left(Y X^T\right). \tag{4.2.1}$$

For $X \in \mathbb{R}^{n \times r}$ we will denote its *Frobenius norm* by

$$\|X\|_{\mathcal{F}} := \sqrt{\langle X, X \rangle} = \sqrt{\operatorname{trace}\left(X^T X\right)} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{r} |x_{i,j}|^2}, \tag{4.2.2}$$

and its *2-norm* by

$$\|X\|_2 := \sup_{\|\xi\| \neq 0} \frac{\|X\xi\|}{\|\xi\|},$$

where $\|\cdot\|$ denotes the usual *Euclidean norm* of a vector. If $X := \left[X_1\middle|\cdots\middle|X_r\right]$ is the column representation of the matrix X, then we have

$$\|X\|_{\mathcal{F}} = \sqrt{\sum_{j=1}^{r} \|X_j\|^2}.$$

For every $X, Y \in \mathbb{R}^{n \times r}$ we have

$$\|X+Y\|_{\mathcal{F}}^2 = \|X\|_{\mathcal{F}}^2 + \|Y\|_{\mathcal{F}}^2 + 2\langle X, Y\rangle, \tag{4.2.3a}$$

$$\|X\|_2 \leqslant \|X\|_{\mathcal{F}}, \tag{4.2.3b}$$

$$\left\|X^T Y\right\|_2 \leqslant \|X\|_2 \cdot \|Y\|_2, \tag{4.2.3c}$$

$$\left\|X^T Y\right\|_{\mathcal{F}} \leqslant \|X\|_{\mathcal{F}} \cdot \|Y\|_{\mathcal{F}}. \tag{4.2.3d}$$

In addition, for every $\eta \in \mathbb{R}$, it holds

$$\|\eta X + (1-\eta) Y\|_{\mathcal{F}}^2 = \eta \|X\|_{\mathcal{F}}^2 + (1-\eta)\|Y\|_{\mathcal{F}}^2 - \eta(1-\eta)\|X-Y\|_{\mathcal{F}}^2. \tag{4.2.4}$$

For a symmetric positive semidefinite matrix $A \in \mathbb{R}^{n \times n}$ we denote by

$$\lambda_{\max}(A) := \lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A) := \lambda_{\min}(A) \geqslant 0$$

its *eigenvalues*. Therefore,

$$\operatorname{trace}(A) = \sum_{i=1}^{n} \lambda_i(A) \geqslant \lambda_{\max}(A) = \|A\|_2 \geqslant \lambda_{\min}(A). \tag{4.2.5}$$

The following two estimates, which we also prove for the sake of completeness, will be useful later on.

Lemma 4.2.1. *Let $X, Y \in \mathbb{R}^{n \times r}$.*

(i) It holds

$$\left\|X^T Y\right\|_{\mathcal{F}} \leqslant \|X\|_2 \cdot \|Y\|_{\mathcal{F}}. \tag{4.2.6}$$

(ii) If $A \in \mathbb{R}^{n \times n}$ is a symmetric positive semidefinite matrix, then

$$\lambda_{\min}(A)\|X\|_{\mathcal{F}}^2 \leqslant \left\langle A, XX^T\right\rangle \leqslant \|A\|_2 \cdot \|X\|_{\mathcal{F}}^2. \tag{4.2.7}$$

Proof. (i) Using the column representation of $Y := \left[Y_1\middle|\cdots\middle|Y_r\right]$, we have

$$X^T Y = \left[X^T Y_1\middle|\cdots\middle|X^T Y_r\right].$$

Thus

$$\left\|X^T Y\right\|_{\mathcal{F}}^2 = \sum_{j=1}^{r} \left\|X^T Y_j\right\|^2 \leqslant \|X\|_2^2 \sum_{j=1}^{r} \|Y_j\|^2 = \|X\|_2^2 \|Y\|_{\mathcal{F}}^2.$$

Notice that, in view of (4.2.3b), inequality (4.2.6) is sharper than (4.2.3d).

(ii) For two positive semidefinite matrices $A, B \in \mathbb{R}^{n \times n}$ we have the following consequence of the Von Neumann's trace inequality (see [104, pp. 340–341])

$$\sum_{i=1}^{n} \lambda_i(A)\lambda_{n+1-i}(B) \leqslant \operatorname{trace}(AB) \leqslant \sum_{i=1}^{n} \lambda_i(A)\lambda_i(B). \tag{4.2.8}$$

The inequality (4.2.7) follows by applying (4.2.8) for the positive semidefinite matrices A and XX^T, and by noticing further that $\sum_{i=1}^{n} \lambda_i\left(XX^T\right) = \operatorname{trace}\left(XX^T\right) = \|X\|_{\mathcal{F}}^2$. □

We denote by

$$\mathring{\mathbb{B}}_{\mathcal{F}}(X;\varepsilon):=\left\{Y\in\mathbb{R}^{n\times r}\colon \|X-Y\|_{\mathcal{F}}<\varepsilon\right\}$$

the *open ball* around $X\in\mathbb{R}^{n\times r}$ with radius $\varepsilon>0$ is and the *closed ball* by $\mathbb{B}_{\mathcal{F}}(X;\varepsilon):=\operatorname{cl}\left(\mathring{\mathbb{B}}_{\mathcal{F}}(X;\varepsilon)\right)$, where the closure is taken with respect to the topology induced by the Frobenius norm. In this chapter, $\mathbf{Pr}_{\mathcal{D}}(X)$ is the projection of an element X onto a nonempty closed convex subset $\mathcal{D}$ with respect to the Frobenius norm. Recall that it is characterized by

$$\mathbf{Pr}_{\mathcal{D}}(X)\in\mathcal{D} \qquad \text{and} \qquad \langle X-\mathbf{Pr}_{\mathcal{D}}(X), Y-\mathbf{Pr}_{\mathcal{D}}(X)\rangle\leqslant 0 \qquad \forall Y\in\mathcal{D}. \tag{4.2.9}$$

Example 4.2.2. For every $X\in\mathbb{R}^{n\times r}$,

(i) if $\mathcal{D}:=\mathbb{R}^{n\times r}_{+}$, then it holds

$$\mathbf{Pr}_{\mathcal{D}}(X)=[X]_{+}:=\max\{X,0\},$$

where the max operator is understood entrywise;

(ii) if $\mathcal{D}:=\mathbb{B}_{\mathcal{F}}(0;\varepsilon)$ for $\varepsilon>0$, we have

$$\mathbf{Pr}_{\mathcal{D}}(X)=\frac{\varepsilon}{\max\{\|X\|_{\mathcal{F}},\varepsilon\}}X.$$

In general, it is challenging to compute the projection onto the intersection of two sets, even if these are both convex and explicit forms for the projections onto each of the sets are available. In the following example we provide one particular pair of two convex sets for which the projection onto their intersection can expressed by a closed formula.

Example 4.2.3. Let $\varepsilon>0$ and K be a nonempty closed convex cone in $\mathbb{R}^{n\times r}$. Then the projection onto the intersection $K\cap\mathbb{B}_{\mathcal{F}}(0,\varepsilon)$ is given by (see [25, Theorem 7.1])

$$\mathbf{Pr}_{K\cap\mathbb{B}_{\mathcal{F}}(0,\varepsilon)}(X)=\mathbf{Pr}_{\mathbb{B}_{\mathcal{F}}(0,\varepsilon)}\circ\mathbf{Pr}_{K}(X)=\frac{\varepsilon}{\max\{\|\mathbf{Pr}_{K}(X)\|_{\mathcal{F}},\varepsilon\}}\mathbf{Pr}_{K}(X) \quad \forall X\in\mathbb{R}^{n\times r}. \tag{4.2.10}$$

Notice that in general $\mathbf{Pr}_{\mathbb{B}_{\mathcal{F}}(0,\varepsilon)}\circ\mathbf{Pr}_{K}(X)\neq\mathbf{Pr}_{K}(X)\circ\mathbf{Pr}_{\mathbb{B}_{\mathcal{F}}(0,\varepsilon)}$ (see [25, Example 7.5]).

For later comparison we discuss two more examples of projections on some particular sets which were used in the nonnegative factorization of completely positive matrices.

Example 4.2.4. Let $B\in\mathbb{R}^{n\times r}$ and consider the following set associated to B

$$\mathcal{P}(B):=\left\{X\in\mathbb{R}^{r\times r}\colon BX\in\mathbb{R}^{n\times r}_{+}\right\}. \tag{4.2.11}$$

The set $\mathcal{P}(B)$ is a polyhedral cone and thus a closed convex subset of $\mathbb{R}^{r\times r}$. The projection of $X\in\mathbb{R}^{r\times r}$ onto the set $\mathcal{P}(B)$ is the unique solution of the optimization problem

$$\begin{aligned} \min_{Y\in\mathbb{R}^{r\times r}} \ & \|Y-X\|_{\mathcal{F}}. \\ \text{s.t.}\ & BY\in\mathbb{R}^{n\times r}_{+}. \end{aligned} \tag{4.2.12}$$

It was shown in [87] that (4.2.12) is equivalent to the second order cone problem (SOCP)

$$\begin{aligned} \min_{t\in\mathbb{R},Z\in\mathbb{R}^{r\times r}} \ & t. \\ \text{s.t.}\ & B(X+Z)\in\mathbb{R}^{n\times r}_{+}, \\ & \|Z\|_{\mathcal{F}}\leqslant t. \end{aligned} \tag{SOCP}$$

Second order cone problems have been intensively studied in the literature from both theoretical and numerical perspectives.

Example 4.2.5. Let $\mathcal{O}_r$ be the set of orthogonal matrices in $\mathbb{R}^{r\times r}$

$$\mathcal{O}_r := \left\{X \in \mathbb{R}^{r\times r} \colon XX^T = X^T X = \mathrm{Id}_r\right\}, \tag{4.2.13}$$

where Id_r denotes $r \times r$ *identity matrix.* The set $\mathcal{O}_r$ is compact but nonconvex, so projections on this set always exist, but may not be unique. A projection of an element $X \in \mathbb{R}^{r\times r}$ on $\mathcal{O}_r$ can be found by polar decomposition of X (see, for instance, [87, Lemma 4.1]). In particular, for every $X \in \mathbb{R}^{r\times r}$, there exist a positive semidefinite matrix $T \in \mathbb{R}^{r\times r}$ and an orthogonal matrix $Y \in \mathbb{R}^{r\times r}$ such that

$$X = TY \qquad \text{and} \qquad \|X - Y\|_{\mathcal{F}} \leqslant \|X - Z\|_{\mathcal{F}} \qquad \forall Z \in \mathcal{O}_r.$$

Therefore, the matrix Y is a projection of X onto $\mathcal{O}_r$ and it can be computed by means of the singular value decomposition of $X = U\Sigma V^T$. Indeed, for $T := U\Sigma U^T$ and $Y := UV^T$ it holds $X = U\Sigma V^T = U\Sigma U^T UV^T = TY$.

4.2.2 Properties of factorizations

We first recall some fundamental properties of the factorizations. The factorization of a completely positive matrix $A \neq 0$ is never unique. We illustrate this with an example by Dickinson [73].

Example 4.2.6. Consider the matrix

$$A := \begin{pmatrix} 18 & 9 & 9 \\ 9 & 18 & 9 \\ 9 & 9 & 18 \end{pmatrix}.$$

Then $A = B_i B_i^T$ for each of the following matrices:

$$B_1 := \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}, \qquad B_2 := \begin{pmatrix} 3 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 3 & 0 & 0 & 3 \end{pmatrix},$$

$$B_3 := \begin{pmatrix} 3 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 3 \end{pmatrix}, \qquad B_4 := \begin{pmatrix} -1.2030 & 2.1337 & 3.4641 \\ 2.4494 & 0.0250 & 3.4641 \\ -1.2463 & -2.1087 & 3.4641 \end{pmatrix}.$$

The number of columns of the factors B_i varies, which gives rise to the following definitions.

Definition 4.2.1. Let $A \in \mathbb{R}^{n\times n}$. The *cp-rank* of A is defined as

$$\mathrm{cpr}\,(A) := \inf\left\{r \geqslant 0 \colon \exists X \in \mathbb{R}_+^{n\times r}, A = XX^T\right\}.$$

The *cp*$^+$*-rank* of A is defined as

$$\mathrm{cpr}^+\,(A) := \inf\left\{r \geqslant 0 \colon \exists X \in \mathbb{R}_{++}^{n\times r}, A = XX^T\right\},$$

here $\mathbb{R}_{++}^{n\times r}$ denoting the set of matrices in $\mathbb{R}_+^{n\times r}$ which have at least one column with positive entries.

The notion of *cp*$^+$-rank is useful for the matrix belongs to the interior of $\mathcal{CP}_n$. Recall that, Dickinson showed in [73, Theorem 3.8] that the interior of $\mathcal{CP}_n$ can be characterized as follows

$$\mathrm{int}\,(\mathcal{CP}_n) = \left\{A \in \mathbb{R}^{n\times n} \colon \mathrm{rank}\,(A) = n, A = XX^T, X \in \mathbb{R}_{++}^{n\times r}\right\}.$$

Until now, we can only derive an upper bound for this value rank, which we will recall in the following lemma. The problem of computing the cp-rank of a matrix in general remains open (see [30]).

Lemma 4.2.7. *[39, Theorem 4.1] For all* $A \in \mathcal{CP}_n$ *we have*

$$\operatorname{cpr}(A) \leqslant \operatorname{cp}_n := \begin{cases} n & \text{for } n \in \{2,3,4\}, \\ \frac{1}{2}n(n+1) - 4 & \text{for } n \geqslant 5. \end{cases}$$

If $A \in \operatorname{int}(\mathcal{CP}_n)$*, then*

$$\operatorname{cpr}^+(A) \leqslant \operatorname{cp}_n^+ := \begin{cases} n+1 & \text{for } n \in \{2,3,4\}, \\ \frac{1}{2}n(n+1) - 3 & \text{for } n \geqslant 5. \end{cases}$$

Notice that there exists matrices $A \in \operatorname{int}(\mathcal{CP}_n)$ *such that* $\operatorname{cpr}(A) \neq \operatorname{cpr}^+(A)$.

In the numerical experiments, we will often choose r as n up to a multiplicative constant, which is smaller than $\operatorname{cpr}(A)$ and $\operatorname{cpr}^+(A)$ when n is large and still obtains reasonable results.

4.2.3 Nonnegative factorization of completely positive matrices via projection onto the orthogonal set $\mathcal{O}_r$

In the following we will revisit some recent iterative approaches from the literature for finding a nonnegative factorization of completely positive matrices.

In [87] this problem was reformulated as a feasibility problem. For a given matrix $A \in \mathbb{R}^{n\times n}$, in a first step, a not necessarily entrywise nonnegative matrix $B \in \mathbb{R}^{n\times r}$ such that $A = BB^T$ was considered. The aim was

$$\text{to find a } r \times r \text{ square matrix } Q \text{ such that } Q \in \mathcal{P}(B) \cap \mathcal{O}_r, \tag{4.2.14}$$

where $\mathcal{P}(B)$ and $\mathcal{O}_r$ are the polyhedral cone associated to B and the set of $r \times r$ orthogonal matrices given in (4.2.11) and in (4.2.13), respectively. This approach was motivated by the observation that, for every $B_1, B_2 \in \mathbb{R}^{n\times r}$, it holds $B_1 B_1^T = B_2 B_2^T$ if and only if there exists $Q \in \mathcal{O}_r$ such that $B_1 Q = B_2$ (see [87, Lemma 2.6]).

To solve (4.2.14), naturally, the method of alternating projections was used, which, given $B \in \mathbb{R}^{n\times r}$ such that $A = BB^T$ and an initial point $Q_0 \in \mathcal{O}_r$, generates a sequence $\{Q_k\}_{k\geqslant 0}$ as follows:

$$(\forall k \geqslant 0) \qquad \begin{cases} P_k & := \mathbf{Pr}_{\mathcal{P}(B)}(Q_k), \\ Q_{k+1} & \in \mathbf{Pr}_{\mathcal{O}_r}(P_k). \end{cases} \tag{4.2.15}$$

The nonconvex method of alternating projections is known to converge *locally*, which means that convergence can be guaranteed if the initial point is sufficiently close to $\mathcal{P}(B) \cap \mathcal{O}_r$.

As noticed in Example 4.2.4, the first step in (4.2.15) amounts to solve a second-order cone problem, which usually can be done only in an approximate way and requires an inner loop. To avoid this drawback, another algorithm was proposed in [87], which, in every iteration, calculates an approximation of $\mathbf{Pr}_{\mathcal{P}(B)}(Q_k)$. This is done by using the projection on $\mathbb{R}_+^{n\times r}$, for which an exact formula exists, and an update step which uses the *Moore-Penrose-Inverse* of B, that is $B^+ := B^T \left(BB^T\right)^{-1}$. Given $B \in \mathbb{R}^{n\times r}$ such that $A = BB^T$ and an initial point $Q_0 \in \mathcal{O}_r$, this second algorithm generates a sequence $\{Q_k\}_{k\geqslant 0}$ as follows:

$$(\forall k \geqslant 0) \qquad \begin{cases} R_k & := \mathbf{Pr}_{\mathbb{R}_+^{n\times r}}(BQ_k), \\ \widehat{P}_k & := B^+ R_k + (\mathrm{Id}_r - B^+ B) Q_k, \\ Q_{k+1} & \in \mathbf{Pr}_{\mathcal{O}_r}\left(\widehat{P}_k\right). \end{cases} \tag{4.2.16}$$

In [66], an alternative approach to (4.2.14) was considered, by reformulating the nonnegative factorization problem as a difference-of-convex optimization problem and by solving the latter via a nonmonotone linesearch algorithm. This can be found in [66, Section 6.1], here we present for easy reference the iterative scheme with a fixed stepsize. Let $B \in \mathbb{R}^{n\times r}$ such that $A = BB^T$, $L_B > \lambda_{\max}\left(B^T B\right)$, and an initial point $Q_0 \in \mathcal{O}_r$. The algorithm generates the sequence $\{Q_k\}_{k\geqslant 0}$ as follows

$$(\forall k \geqslant 0) \qquad \begin{cases} W_k & := \mathbf{Pr}_{\mathbb{R}^{n\times r}_+}\left(BQ_k\right), \\ Q_{k+1} & \in \mathbf{Pr}_{\mathcal{O}_r}\left(Q_k - \dfrac{1}{L_B} B^T\left(BQ_k - W_k\right)\right). \end{cases} \tag{4.2.17}$$

One can notice that all three iterative schemes require in every iteration the calculation of a projection onto the orthogonal set $\mathcal{O}_r$. To do this one basically needs to carry out a singular value decomposition of a matrix, as discussed in Example 4.2.5, which can be done in a subroutine that needs $\mathcal{O}\left(r^3\right)$ steps. Furthermore, all three algorithms ask for finding a matrix $B \in \mathbb{R}^{n\times r}$ such that $A = BB^T$. This can be done, for instance, by the Cholesky decomposition of A, in which case B is a lower triangular matrix, or by the spectral decomposition $A = V\Sigma V^T$ and then by setting $B := V\Sigma^{\frac{1}{2}}$. In either case, one needs an additional procedure to find an appropriate initial matrix B.

4.3 An optimization model with convergence guarantees

In this section we will propose a new approach for the nonnegative factorization of completely positive matrices, which consists of solving a nonconvex optimization problem by means of a projected gradient algorithm. We will also carry out for the iterative method a comprehensive convergence analysis, and even derive convergence rates.

4.3.1 The optimization model

For a given nonzero completely positive matrix $A \in \mathbb{R}^{n\times n}$, finding a factorization $A = XX^T$, where $X \in \mathbb{R}^{n\times r}_+$, can be cast as an optimization problem

$$\begin{aligned} \min_{X\in\mathbb{R}^{n\times r}} \ & \mathcal{E}\left(X\right) := \frac{1}{2}\left\|A - XX^T\right\|^2_{\mathcal{F}}. \\ \text{s.t. } & X \in \mathcal{D} := \mathbb{R}^{n\times r}_+ \cap \mathbb{B}_{\mathcal{F}}\left(0, \sqrt{\operatorname{trace}\left(A\right)}\right) \end{aligned} \tag{4.3.1}$$

Denoting by $\mathcal{E}_* := \inf_{X\in\mathcal{D}} \mathcal{E}\left(X\right)$ the optimal objective value of (4.3.1), it holds

$$A = X_* X_*^T \quad \text{with} \quad X_* \in \mathbb{R}^{n\times r}_+ \Leftrightarrow \left[X_* \text{ solves (4.3.1)} \quad \text{and} \quad \mathcal{E}_* = 0\right].$$

Notice that $\mathcal{E}$ is a nonconvex objective function with continuous gradient

$$\nabla\mathcal{E}\left(X\right) = -2\left(A - XX^T\right)X,$$

which is however not Lipschitz continuous, but *locally* Lipschitz continuous. In order to be able to handle this situation in a proper way in the convergence analysis, we minimize the objective function $\mathcal{E}\left(X\right)$ over a meaningfully chosen bounded set, which, however, does not pose any restriction on the model. Indeed, if X satisfies $A = XX^T$, then

$$\left\|X\right\|_{\mathcal{F}} \leqslant \sqrt{\operatorname{trace}\left(A\right)}.$$

By the definition of the Frobenius norm and (4.2.1) - (4.2.2), we have

$$\left\|X\right\|_{\mathcal{F}} = \sqrt{\operatorname{trace}\left(X^T X\right)} = \sqrt{\operatorname{trace}\left(XX^T\right)} = \sqrt{\operatorname{trace}\left(A\right)}.$$

This explains the choice of $\mathcal{D}$ as the intersection of $\mathbb{R}^{n\times r}_+$ and $\mathbb{B}_{\mathcal{F}}\left(0, \sqrt{\operatorname{trace}\left(A\right)}\right)$. Furthermore, thanks to its specific structure, we have an exact formula for the projection on $\mathcal{D}$.

Proposition 4.3.1. *Let* $A \in \mathcal{CP}_n$.

(i) The set $\mathcal{D}$ *is nonempty convex and closed, and for any* $X \in \mathbb{R}^{n\times r}$ *it holds*

$$\mathbf{Pr}_{\mathcal{D}}(X) := \frac{\sqrt{\operatorname{trace}(A)}}{\max\left\{\left\|[X]_+\right\|_{\mathcal{F}}, \sqrt{\operatorname{trace}(A)}\right\}}[X]_+, \tag{4.3.2}$$

where $[X]_+ := \max\{X,0\}$ *and the* max *operator is understood entrywise.*

(ii) For $X, Y \in \mathbb{R}^{n\times r}$, *the following inequalities are true*

$$-\|A\|_2 \cdot \|X-Y\|_{\mathcal{F}}^2 \leqslant \mathcal{E}(X) - \mathcal{E}(Y) - \langle \nabla\mathcal{E}(Y), X-Y\rangle \leqslant \frac{L(X,Y)}{2}\|X-Y\|_{\mathcal{F}}^2, \tag{4.3.3}$$

where

$$L(X,Y) := 2\left(\|Y\|_2^2 - \lambda_{\min}(A)\right) + \left(\|X\|_2 + \|Y\|_2\right)^2. \tag{4.3.4}$$

Proof. (i) Since $\mathcal{D}$ is the intersection of the cone $K := \mathbb{R}_+^{n\times r}$ with the ball $\mathbb{B}_{\mathcal{F}}\left(0, \sqrt{\operatorname{trace}(A)}\right)$, it follows from (4.2.10) that

$$\mathbf{Pr}_{\mathcal{D}}(X) = \frac{\sqrt{\operatorname{trace}(A)}}{\max\left\{\|\mathbf{Pr}_K(X)\|_{\mathcal{F}}, \sqrt{\operatorname{trace}(A)}\right\}}\mathbf{Pr}_K(X).$$

For $K = \mathbb{R}_+^{n\times r}$ it holds $\mathbf{Pr}_K(X) = \mathbf{Pr}_{\mathbb{R}_+^{n\times r}}(X) = [X]_+ = \max\{X, 0\}$.

(ii) We introduce the auxiliary function $\mathcal{Q}\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ defined as

$$\mathcal{Q}(Z) := \frac{1}{2}\|A-Z\|_{\mathcal{F}}^2 \qquad \forall Z \in \mathbb{R}^{n\times n}.$$

By the definition, $\mathcal{E}(X) = \mathcal{Q}\left(XX^T\right)$ for every $X \in \mathbb{R}^{n\times n}$. Since $\nabla\mathcal{Q}(Z) = -(A-Z)$, the following relation is true for every $Z, W \in \mathbb{R}^{n\times n}$

$$\mathcal{Q}(W) = \mathcal{Q}(Z) + \langle \nabla\mathcal{Q}(Z), W-Z\rangle + \frac{1}{2}\|W-Z\|_{\mathcal{F}}^2. \tag{4.3.5}$$

Moreover, if Z is symmetric, then so is $\nabla\mathcal{Q}(Z)$.

Let $X, Y \in \mathbb{R}^{n\times r}$ be fixed. One can easily verify that

$$XX^T - YY^T = (X-Y)Y^T + Y(X-Y)^T + (X-Y)(X-Y)^T. \tag{4.3.6}$$

Applying (4.3.5) with $W := XX^T$ and $Z := YY^T$ and by taking into consideration (4.3.6), we get

$$\begin{aligned}
\mathcal{Q}\left(XX^T\right) - \mathcal{Q}\left(YY^T\right) &= \left\langle \nabla\mathcal{Q}\left(YY^T\right), XX^T - YY^T\right\rangle + \frac{1}{2}\left\|XX^T - YY^T\right\|_{\mathcal{F}}^2 \\
&= \left\langle \nabla\mathcal{Q}\left(YY^T\right), (X-Y)Y^T\right\rangle + \left\langle \nabla\mathcal{Q}\left(YY^T\right), Y(X-Y)^T\right\rangle \\
&\quad + \left\langle \nabla\mathcal{Q}\left(YY^T\right), (X-Y)(X-Y)^T\right\rangle + \frac{1}{2}\left\|XX^T - YY^T\right\|_{\mathcal{F}}^2 \\
&= 2\left\langle \nabla\mathcal{Q}\left(YY^T\right)Y, (X-Y)\right\rangle + \left\langle \nabla\mathcal{Q}\left(YY^T\right), (X-Y)(X-Y)^T\right\rangle \\
&\quad + \frac{1}{2}\left\|XX^T - YY^T\right\|_{\mathcal{F}}^2.
\end{aligned} \tag{4.3.7}$$

Since $2\nabla\mathcal{Q}\left(YY^T\right)Y = -2\left(A - YY^T\right)Y = \nabla\mathcal{E}\left(Y\right)$, it remains to estimate the two last terms in (4.3.7). Observe that

$$\begin{aligned}
&\left\langle \nabla\mathcal{Q}\left(YY^T\right), \left(X-Y\right)\left(X-Y\right)^T\right\rangle + \frac{1}{2}\left\|XX^T - YY^T\right\|_{\mathcal{F}}^2 \\
=\ & -\left\langle A - YY^T, \left(X-Y\right)\left(X-Y\right)^T\right\rangle + \frac{1}{2}\left\|XX^T - YY^T\right\|_{\mathcal{F}}^2 \\
=\ & -\left\langle A, \left(X-Y\right)\left(X-Y\right)^T\right\rangle + \left\|Y^T\left(X-Y\right)\right\|_{\mathcal{F}}^2 + \frac{1}{2}\left\|XX^T - YY^T\right\|_{\mathcal{F}}^2,
\end{aligned} \tag{4.3.8}$$

where the last equation comes from the fact that trace operator is invariant under cyclic permutations, as we see below

$$\begin{aligned}
\left\langle YY^T, \left(X-Y\right)\left(X-Y\right)^T\right\rangle &= \operatorname{trace}\left[\left(YY^T\right)^T\left(X-Y\right)\left(X-Y\right)^T\right] \\
&= \operatorname{trace}\left[YY^T\left(X-Y\right)\left(X-Y\right)^T\right] \\
&= \operatorname{trace}\left[\left(X-Y\right)^T YY^T\left(X-Y\right)\right] \\
&= \operatorname{trace}\left[\left(Y^T\left(X-Y\right)\right)^T Y^T\left(X-Y\right)\right] \\
&= \left\|Y^T\left(X-Y\right)\right\|_{\mathcal{F}}^2.
\end{aligned}$$

Notice that, thanks to (4.2.7), $\left\langle A, \left(X-Y\right)\left(X-Y\right)^T\right\rangle \leqslant \left\|A\right\|_2\left\|X-Y\right\|_{\mathcal{F}}^2$. Plugging this estimate into (4.3.8), also neglecting the last two nonnegative terms, we obtain the left-hand side inequality in (4.3.3).

By applying (4.2.6) we can derive an upper bound for the last term in (4.3.8)

$$\begin{aligned}
\left\|XX^T - YY^T\right\|_{\mathcal{F}} &\leqslant \left\|\left(X-Y\right)X^T\right\|_{\mathcal{F}} + \left\|Y\left(X-Y\right)^T\right\|_{\mathcal{F}} \\
&\leqslant \left\|X\right\|_2\left\|X-Y\right\|_{\mathcal{F}} + \left\|Y\right\|_2\left\|X-Y\right\|_{\mathcal{F}} = \left(\left\|X\right\|_2 + \left\|Y\right\|_2\right)\left\|X-Y\right\|_{\mathcal{F}}.
\end{aligned} \tag{4.3.9}$$

By plugging (4.3.9) into (4.3.8) and recalling the inequalities (4.2.7) and (4.2.6), we get the right-hand side inequality in (4.3.3) with $L\left(X,Y\right)$ defined as in (4.3.4). □

4.3.2 A projected gradient algorithm with relaxation and inertial parameters

We are now in the position to formulate the projected gradient algorithm we propose in this chapter to solve (4.3.1).

Algorithm 4.3.1. *Let* $\{\alpha_k\}_{k\geqslant 1} \subseteq [0,1]$ *and, for* $\alpha_+ := \sup_{k\geqslant 0}\alpha_k$*, set*

$$L_{\mathcal{F}}(\alpha_+) := 2\left[\left(3 + 8\alpha_+ + 6\alpha_+^2\right)\operatorname{trace}\left(A\right) - \lambda_{\min}\left(A\right)\right] > 0.$$

Choose $\rho \in (0,1]$ *such that*

$$0 < \frac{\sqrt{L_{\mathcal{F}}(\alpha_+) + 2\left\|A\right\|_2}}{\sqrt{L_{\mathcal{F}}(\alpha_+) + 2\left\|A\right\|_2} + \sqrt{L_{\mathcal{F}}(\alpha_+)}} < \rho < \frac{\sqrt{L_{\mathcal{F}}(\alpha_+) + 2\left\|A\right\|_2}}{\left(1+\alpha_+\right)\sqrt{L_{\mathcal{F}}(\alpha_+) + 2\left\|A\right\|_2} - \sqrt{L_{\mathcal{F}}(\alpha_+)}}. \tag{4.3.10}$$

For a given starting point $X_1 := X_0 \in \mathcal{D}$ *generate the sequence* $\{X_k\}_{k\geqslant 0}$ *as follows*

$$Y_k := X_k + \alpha_k\left(X_k - X_{k-1}\right), \tag{4.3.11a}$$

$$Z_{k+1} := \mathbf{Pr}_{\mathcal{D}}\left(Y_k - \frac{1}{L_{\mathcal{F}}}\nabla\mathcal{E}\left(Y_k\right)\right), \tag{4.3.11b}$$

$$X_{k+1} := \left(1-\rho\right)X_k + \rho Z_{k+1}. \tag{4.3.11c}$$

Recall that the formula of $\mathbf{Pr}_{\mathcal{D}}$ is given in (4.3.2) explicitly. For any $k \geqslant 1$, the following equivalent formulation of (4.3.11c) will be useful in the analysis

$$X_{k+1} = (1-\rho) X_k + \rho Z_{k+1} \Leftrightarrow Z_{k+1} - X_k = \frac{1}{\rho}(X_{k+1} - X_k) \tag{4.3.12a}$$

$$\Leftrightarrow Z_{k+1} - X_{k+1} = \left(\frac{1}{\rho} - 1\right)(X_{k+1} - X_k). \tag{4.3.12b}$$

To help the readers to understand the choice of the parameters, we give the following results first and postpone the discussion on the feasibility of ρ in (4.3.10) to Remark 4.3.2. In the following we will use, to ease the reading, $L_{\mathcal{F}}$ instead of $L_{\mathcal{F}}(\alpha_+)$, however, we will return to this notation in the last section, where we will consider some particular choices of the sequence of inertial parameter.

Lemma 4.3.2. *Let $\{X_k\}_{k\geqslant 0}$ be the sequence generated by Algorithm 4.3.1. The following statements are true for any $k \geqslant 1$*

(i) $X_{k+1} \in \mathcal{D}$ *and* $\|Y_k\|_{\mathcal{F}} \leqslant (1+2\alpha_+)\sqrt{\operatorname{trace}(A)}$;

(ii)

$$L(Z_{k+1}, Y_k) \leqslant L_{\mathcal{F}} = 2\left[\left(3 + 8\alpha_+ + 6\alpha_+^2\right)\operatorname{trace}(A) - \lambda_{\min}(A)\right], \tag{4.3.13}$$

where $(X,Y) \mapsto L(X,Y)$ *is defined in* (4.3.4).

Proof. (i) Notice that $\{Z_k\}_{k\geqslant 2} \subseteq \mathcal{D}$ due to (4.3.11b). If we assume that $X_1 \in \mathcal{D}$, then, by induction arguments, $X_{k+1} \in \mathcal{D}$, since it is a convex combination of X_k and Z_{k+1}. Consequently, $\|X_k\|_{\mathcal{F}} \leqslant \sqrt{\operatorname{trace}(A)}$ for any $k \geqslant 0$. By the definition of Y_k in (4.3.11a), we have

$$\|Y_k\|_{\mathcal{F}} \leqslant (1+\alpha_k)\|X_k\|_{\mathcal{F}} + \alpha_k \|X_{k-1}\|_{\mathcal{F}} \leqslant (1+2\alpha_+)\sqrt{\operatorname{trace}(A)} \qquad \forall k \geqslant 1.$$

(ii) Since $\{Z_k\}_{k\geqslant 2} \subseteq \mathcal{D} \subseteq \mathbb{B}_{\mathcal{F}}\left(0; \sqrt{\operatorname{trace}(A)}\right)$ and $\{Y_k\}_{k\geqslant 1} \subseteq \mathbb{B}_{\mathcal{F}}\left(0; (1+2\alpha_+)\sqrt{\operatorname{trace}(A)}\right)$ it follows from the definition of $(X,Y) \mapsto L(X,Y)$ in (4.3.4) that

$$\begin{aligned} L(Z_{k+1}, Y_k) &= 2\left(\|Y_k\|_2^2 - \lambda_{\min}(A)\right) + \left(\|Z_{k+1}\|_2 + \|Y_k\|_2\right)^2 \\ &= 3\|Y_k\|_2^2 + \|Z_{k+1}\|_2^2 + 2\|Z_{k+1}\|_2 \cdot \|Y_k\|_2 - 2\lambda_{\min}(A) \\ &\leqslant \left[3(1+2\alpha_+)^2 + 1 + 2(1+2\alpha_+)\right]\operatorname{trace}(A) - 2\lambda_{\min}(A). \end{aligned}$$

□

Remark 4.3.1. In the nonconvex setting, the boundedness of the sequence of iterates plays an important role in the convergence analysis. As seen in Lemma 4.3.2 (i), the nature of Algorithm 4.3.1 ensures that $X_k \in \mathcal{D}$ for every $k \geqslant 0$, and thus the sequence $\{X_k\}_{k\geqslant 0}$ is bounded.

For readers' convenience we denote the objective function of (4.3.1) by $\Psi := \mathcal{E} + \delta_{\mathcal{D}}$.

Lemma 4.3.3. *Let $\{X_k\}_{k\geqslant 0}$ be the sequence generated by Algorithm 4.3.1. For every $k \geqslant 2$ it holds*

$$\Psi(Z_{k+1}) + \left(\frac{L_{\mathcal{F}} - (L_{\mathcal{F}} + 2\|A\|_2)\gamma}{2} + \frac{\tau}{2}\right)\|X_{k+1} - X_k\|^2 \leqslant \Psi(Z_k) + \frac{\tau}{2}\|X_k - X_{k-1}\|^2, \tag{4.3.14}$$

where

$$\gamma := \max\left\{\left(\frac{1}{\rho} - 1\right)^2, \left(1 + \alpha_+ - \frac{1}{\rho}\right)^2\right\}, \tag{4.3.15a}$$

$$\tau := \frac{L_{\mathcal{F}}(1-\rho)}{\rho} + (L_{\mathcal{F}} + 2\|A\|_2)\gamma. \tag{4.3.15b}$$

Proof. Let $k \geqslant 2$ be fixed. We first show that

$$\Psi\left(Z_{k+1}\right)+\frac{L_{\mathcal{F}}}{2}\left\|Z_{k+1}-Z_k\right\|_{\mathcal{F}}^2 \leqslant \Psi\left(Z_k\right)+\frac{L_{\mathcal{F}}+2\|A\|_2}{2}\left\|Z_k-Y_k\right\|_{\mathcal{F}}^2 . \tag{4.3.16}$$

The characterization of the projection (4.2.9) ensures that

$$\left\langle Y_k-\frac{1}{L_{\mathcal{F}}} \nabla \mathcal{E}\left(Y_k\right)-Z_{k+1}, X-Z_{k+1}\right\rangle \leqslant 0 \qquad \forall X \in \mathcal{D}. \tag{4.3.17}$$

In view of (4.3.11b), it is clear that $Z_k \in \mathcal{D}$, thus, setting $X := Z_k$ in (4.3.17) yields

$$\begin{aligned} 0 &\leqslant \left\langle\nabla \mathcal{E}\left(Y_k\right), Z_k-Z_{k+1}\right\rangle+L_{\mathcal{F}}\left\langle Z_{k+1}-Y_k, Z_k-Z_{k+1}\right\rangle \\ &=\left\langle\nabla \mathcal{E}\left(Y_k\right), Z_k-Z_{k+1}\right\rangle-\frac{L_{\mathcal{F}}}{2}\left\|Z_{k+1}-Y_k\right\|_{\mathcal{F}}^2-\frac{L_{\mathcal{F}}}{2}\left\|Z_{k+1}-Z_k\right\|_{\mathcal{F}}^2+\frac{L_{\mathcal{F}}}{2}\left\|Z_k-Y_k\right\|_{\mathcal{F}}^2 . \end{aligned} \tag{4.3.18}$$

The left-hand side inequality in (4.3.3) implies that

$$\mathcal{E}\left(Z_k\right) \geqslant \mathcal{E}\left(Y_k\right)+\left\langle\nabla \mathcal{E}\left(Y_k\right), Z_k-Y_k\right\rangle-\|A\|_2 \cdot\left\|Y_k-Z_k\right\|_{\mathcal{F}}^2, \tag{4.3.19}$$

while the right-hand side inequality in (4.3.3) and (4.3.13) imply

$$\mathcal{E}\left(Z_{k+1}\right) \leqslant \mathcal{E}\left(Y_k\right)+\left\langle\nabla \mathcal{E}\left(Y_k\right), Z_{k+1}-Y_k\right\rangle+\frac{L_{\mathcal{F}}}{2}\left\|Z_{k+1}-Y_k\right\|_{\mathcal{F}}^2 . \tag{4.3.20}$$

Summing up (4.3.18), (4.3.20) and (4.3.19), and noticing that $\delta_{\mathcal{D}}\left(Z_{k+1}\right)=\delta_{\mathcal{D}}\left(Z_k\right)=0$, yield (4.3.16).

Next we will study the term $\left\|Z_{k+1}-Z_k\right\|_{\mathcal{F}}^2$ in detail. From (4.3.12a) we have that

$$Z_{k+1}=\frac{1}{\rho}\left(X_{k+1}-X_k\right)+X_k,$$

and

$$Z_k=\frac{1}{\rho}\left(X_k-X_{k-1}\right)+X_{k-1},$$

thus

$$Z_{k+1}-Z_k=\frac{1}{\rho}\left(X_{k+1}-X_k\right)+\left(1-\frac{1}{\rho}\right)\left(X_k-X_{k-1}\right) . \tag{4.3.21}$$

Then, by using identity (4.2.4), it holds

$$\begin{aligned} \left\|Z_{k+1}-Z_k\right\|_{\mathcal{F}}^2 &=\left\|\frac{1}{\rho}\left(X_{k+1}-X_k\right)+\left(1-\frac{1}{\rho}\right)\left(X_k-X_{k-1}\right)\right\|_{\mathcal{F}}^2 \\ &=\frac{1}{\rho}\left\|X_{k+1}-X_k\right\|_{\mathcal{F}}^2+\left(1-\frac{1}{\rho}\right)\left\|X_k-X_{k-1}\right\|_{\mathcal{F}}^2 \\ &\quad -\frac{1}{\rho}\left(1-\frac{1}{\rho}\right)\left\|\left(X_{k+1}-X_k\right)-\left(X_k-X_{k-1}\right)\right\|_{\mathcal{F}}^2 \\ &\geqslant \frac{1}{\rho}\left\|X_{k+1}-X_k\right\|_{\mathcal{F}}^2-\left(\frac{1}{\rho}-1\right)\left\|X_k-X_{k-1}\right\|_{\mathcal{F}}^2 . \end{aligned} \tag{4.3.22}$$

Combining (4.3.11a) and (4.3.12b) gives us further

$$Z_k-Y_k=Z_k-X_k-\alpha_k\left(X_k-X_{k-1}\right)=\left(\frac{1}{\rho}-1-\alpha_k\right)\left(X_k-X_{k-1}\right) . \tag{4.3.23}$$

By plugging (4.3.22) and (4.3.23) into (4.3.16), we get

$$\begin{aligned}
&\Psi\left(Z_{k+1}\right) + \frac{L_{\mathcal{F}}}{2\rho}\left\|X_{k+1} - X_k\right\|_{\mathcal{F}}^2 \\
&= \Psi\left(Z_{k+1}\right) + \left(\frac{L_{\mathcal{F}}\left(1-\rho\right)}{2\rho} + \frac{L_{\mathcal{F}}}{2}\right)\left\|X_{k+1} - X_k\right\|_{\mathcal{F}}^2 \\
&\leqslant \Psi\left(Z_k\right) + \left(\frac{L_{\mathcal{F}}\left(1-\rho\right)}{2\rho} + \frac{L_{\mathcal{F}} + 2\left\|A\right\|_2}{2}\left(\frac{1}{\rho} - 1 - \alpha_k\right)^2\right)\left\|X_k - X_{k-1}\right\|_{\mathcal{F}}^2 \\
&\leqslant \Psi\left(Z_k\right) + \left(\frac{L_{\mathcal{F}}\left(1-\rho\right)}{2\rho} + \frac{\left(L_{\mathcal{F}} + 2\left\|A\right\|_2\right)\gamma}{2}\right)\left\|X_k - X_{k-1}\right\|_{\mathcal{F}}^2, \qquad (4.3.24)
\end{aligned}$$

which is nothing else than (4.3.14) with the constants τ and γ as defined in (4.3.15). Notice that (4.3.24) is true since γ is an upper bound for $\left(\frac{1}{\rho} - 1 - \alpha_k\right)^2$. Indeed, if $\frac{1}{\rho} - 1 \geqslant \alpha_k$, then

$$0 \leqslant \frac{1}{\rho} - 1 - \alpha_k \leqslant \frac{1}{\rho} - 1 \Rightarrow \left(\frac{1}{\rho} - 1 - \alpha_k\right)^2 \leqslant \left(\frac{1}{\rho} - 1\right)^2 \leqslant \gamma.$$

Otherwise, we have

$$0 < 1 + \alpha_k - \frac{1}{\rho} \leqslant 1 + \alpha_+ - \frac{1}{\rho} \Rightarrow \left(\frac{1}{\rho} - 1 - \alpha_k\right)^2 \leqslant \left(1 + \alpha_+ - \frac{1}{\rho}\right)^2 \leqslant \gamma,$$

which leads to the desired statement. □

The estimate above remains true if we replace Ψ by $\mathcal{E}$. In fact, the indicator function was artificially inserted in the decreasing property (4.3.14), as it will help us to prove the convergence of the iterates later on. Now, with $\tau \geqslant 0$ introduced in (4.3.15b), we define the following function

$$\Psi_\tau \colon \mathbb{R}^{n\times r} \times \mathbb{R}^{n\times r} \to \mathbb{R} \cup \{+\infty\}, \quad \Psi_\tau\left(Z, X\right) := \Psi\left(Z\right) + \frac{\rho^2\tau}{2}\left\|Z - X\right\|_{\mathcal{F}}^2. \qquad (4.3.25)$$

The objective function Ψ of (4.3.1) is closely related to Ψ_τ in terms of their critical point. Indeed, if $\tau = 0$, which is the case when $\rho = 1$ and $\alpha_+ = 0$, then $\Psi_\tau(Z,X) = \Psi(Z)$ for any $(Z,X) \in \mathbb{R}^{n\times r} \times \mathbb{R}^{n\times r}$, thus $X_* \in \operatorname{crit}\Psi$ if and only if $(Z_*, X_*) \in \operatorname{crit}\Psi_\tau$ for $Z_* \in \mathbb{R}^{n\times r}$. On the other hand, one can easily verify that for every $\tau > 0$ we have

$$X_* \in \operatorname{crit}\Psi \Leftrightarrow (X_*, X_*) \in \operatorname{crit}\Psi_\tau. \qquad (4.3.26)$$

Remark 4.3.2. In the view of (4.3.11c), it holds $X_{k+1} - X_k = \rho\left(Z_{k+1} - X_k\right)$ for every $k \geqslant 1$. Therefore, using the definition (4.3.25), the inequality (4.3.14) can be rewritten for any $k \geqslant 2$ as

$$\Psi_\tau\left(Z_{k+1}, X_k\right) + C_0\left\|X_{k+1} - X_k\right\|_{\mathcal{F}}^2 \leqslant \Psi_\tau\left(Z_k, X_{k-1}\right), \quad \text{where } C_0 := \frac{L_{\mathcal{F}} - \left(L_{\mathcal{F}} + 2\left\|A\right\|_2\right)\gamma}{2}. \qquad (4.3.27)$$

We will show that $C_0 > 0$. It holds

$$L_{\mathcal{F}} - \left(L_{\mathcal{F}} + 2\left\|A\right\|_2\right)\gamma > 0 \Leftrightarrow \begin{cases} \left(\frac{1}{\rho} - 1\right)^2 < \frac{L_{\mathcal{F}}}{L_{\mathcal{F}} + 2\left\|A\right\|_2}, \\ \left(1 + \alpha_+ - \frac{1}{\rho}\right)^2 < \frac{L_{\mathcal{F}}}{L_{\mathcal{F}} + 2\left\|A\right\|_2}. \end{cases} \qquad (4.3.28)$$

On the one hand, since $0 < \rho \leqslant 1$, we have

$$0 \leqslant \frac{1}{\rho} - 1 < \sqrt{\frac{L_{\mathcal{F}}}{L_{\mathcal{F}} + 2\left\|A\right\|_2}} \Leftrightarrow 1 \leqslant \frac{1}{\rho} < \frac{\sqrt{L_{\mathcal{F}} + 2\left\|A\right\|_2} + \sqrt{L_{\mathcal{F}}}}{\sqrt{L_{\mathcal{F}} + 2\left\|A\right\|_2}}.$$

This is further equivalent to

$$\frac{\sqrt{L_{\mathcal{F}}+2\|A\|_2}}{\sqrt{L_{\mathcal{F}}+2\|A\|_2}+\sqrt{L_{\mathcal{F}}}}<\rho\leqslant 1. \tag{4.3.29}$$

On the other hand, by setting $\xi := \frac{1}{\rho} > 0$, the second inequality in (4.3.28) can be equivalently expressed as

$$\xi^2-2(1+\alpha_+)\xi+(1+\alpha_+)^2-\frac{L_{\mathcal{F}}}{L_{\mathcal{F}}+2\|A\|_2}<0. \tag{4.3.30}$$

Its reduced discriminant reads

$$\Delta' := (1+\alpha_+)^2-\left((1+\alpha_+)^2-\frac{L_{\mathcal{F}}}{L_{\mathcal{F}}+2\|A\|_2}\right)=\frac{L_{\mathcal{F}}}{L_{\mathcal{F}}+2\|A\|_2}>0.$$

Thus, the inequality (4.3.30) is equivalent to

$$\begin{aligned}1+\alpha_+-\sqrt{\frac{L_{\mathcal{F}}}{L_{\mathcal{F}}+2\|A\|_2}}&=\frac{(1+\alpha_+)\sqrt{L_{\mathcal{F}}+2\|A\|_2}-\sqrt{L_{\mathcal{F}}}}{\sqrt{L_{\mathcal{F}}+2\|A\|_2}}\\&<\xi=\frac{1}{\rho}<1+\alpha_++\sqrt{\frac{L_{\mathcal{F}}}{L_{\mathcal{F}}+2\|A\|_2}}=\frac{(1+\alpha_+)\sqrt{L_{\mathcal{F}}+2\|A\|_2}+\sqrt{L_{\mathcal{F}}}}{\sqrt{L_{\mathcal{F}}+2\|A\|_2}},\end{aligned}$$

which means

$$\frac{\sqrt{L_{\mathcal{F}}+2\|A\|_2}}{(1+\alpha_+)\sqrt{L_{\mathcal{F}}+2\|A\|_2}+\sqrt{L_{\mathcal{F}}}}<\rho<\frac{\sqrt{L_{\mathcal{F}}+2\|A\|_2}}{(1+\alpha_+)\sqrt{L_{\mathcal{F}}+2\|A\|_2}-\sqrt{L_{\mathcal{F}}}}. \tag{4.3.31}$$

Combining (4.3.29) and (4.3.31), we observe further that

$$\frac{\sqrt{L_{\mathcal{F}}+2\|A\|_2}}{(1+\alpha_+)\sqrt{L_{\mathcal{F}}+2\|A\|_2}+\sqrt{L_{\mathcal{F}}}}\leqslant\frac{\sqrt{L_{\mathcal{F}}+2\|A\|_2}}{\sqrt{L_{\mathcal{F}}+2\|A\|_2}+\sqrt{L_{\mathcal{F}}}}.$$

Thus, in view of (4.3.10), $C_0 > 0$.

A direct consequence of Lemma 4.3.3 follows.

Proposition 4.3.4. *Let $\{X_k\}_{k\geqslant 0}$ be the sequence generated by Algorithm 4.3.1. The following statements are true:*

(i) the sequence $\{\Psi_\tau(Z_k, X_{k-1})\}_{k\geqslant 2}$ is monotonically decreasing and convergent;

(ii) $X_{k+1}-X_k \to 0$ as $k\to+\infty$, and so $X_{k+1}-Y_k\to 0$ and $Z_{k+1}-Y_k\to 0$ as $k\to+\infty$.

Proof. Let $k\geqslant 2$ be fixed. In view of (4.3.27) we have

$$\Psi_\tau(Z_{k+1}, X_k)+C_0\|X_{k+1}-X_k\|^2\leqslant\Psi_\tau(Z_k, X_{k-1}).$$

It is clear that the sequence $\{\Psi(Z_k, X_{k-1})\}_{k\geqslant 2}$ is monotonically decreasing and, since it is nonnegative, is convergent. The fact that $C_0 > 0$ and telescoping arguments (see, for instance, [24, Lemma 5.31]) give $\sum_{k\geqslant 1}\|X_{k+1}-X_k\|^2<+\infty$, thus $X_{k+1}-X_k\to 0$ as $k\to+\infty$. By taking into consideration (4.3.21), we deduce that $Z_{k+1}-Z_k\to 0$ as $k\to+\infty$. Using further (4.3.12a) and (4.3.11a), we have $Z_{k+1}-Y_k\to 0$ as $k\to+\infty$, and the proof is completed. □

Now we show that every cluster point of $\{X_k\}_{k\geqslant 0}$ is a critical point of Ψ.

Theorem 4.3.5. *Let $\{X_k\}_{k\geqslant 0}$ be the sequence generated by Algorithm 4.3.1. Then every cluster point of $\{X_k\}_{k\geqslant 0}$ is a critical point of Ψ.*

Proof. Let $\bar{X}$ be a cluster point of $\{X_k\}_{k\geqslant 0}$, which means that there exists a subsequence $\{X_{k_i}\}_{i\geqslant 1}$ such that $X_{k_i} \to \bar{X}$ as $i \to +\infty$. We deduce further that $Z_{k_i} \to \bar{X}$ as $i \to +\infty$, due to (4.3.12b). By the characterization of the projection (2.1.2) and (4.3.11b), we get that for every $i \geqslant 1$

$$W_{k_i} := Y_{k_i-1} - Z_{k_i} - \frac{1}{L_{\mathcal{F}}}\nabla\mathcal{E}\left(Y_{k_i-1}\right) \in \mathcal{N}_{\mathcal{D}}\left(Z_{k_i}\right).$$

From here,

$$L_{\mathcal{F}}W_{k_i} = L_{\mathcal{F}}\left(Y_{k_i-1} - Z_{k_i}\right) + \nabla\mathcal{E}\left(Z_{k_i}\right) - \nabla\mathcal{E}\left(Y_{k_i-1}\right) - \nabla\mathcal{E}\left(Z_{k_i}\right) \in \mathcal{N}_{\mathcal{D}}\left(Z_{k_i}\right) \quad \forall i \geqslant 1.$$

By passing to limit as $i \to +\infty$, and by taking into consideration the continuity of $\nabla\mathcal{E}$ and the fact that $Z_{k+1} - Y_k \to 0$ as $k \to +\infty$ (see Proposition 4.3.4 (ii)), we get

$$L_{\mathcal{F}}W_{k_i} \to -\nabla\mathcal{E}\left(\bar{X}\right).$$

The closedness of the graph of the normal cone gives $-\nabla\mathcal{E}\left(\bar{X}\right) \in \mathcal{N}_{\mathcal{D}}\left(\bar{X}\right)$. In other words, $\bar{X} \in \operatorname{crit}\Psi$. □

4.3.3 Global convergence thanks to the Łojasiewicz property

In this subsection we will prove that actually the whole sequence of iterates $\{X_k\}_{k\geqslant 0}$ generated by Algorithm 4.3.1 converges to a critical point of the objective function Ψ and even establish its rate of convergence. To this end we will use that the regularized objective function Ψ_τ fulfills the *Łojasiewicz property* (see [103]), since it is a *semialgebraic function* (see [5, Example 1], [33]).

If Ω is a connected and compact subset of $\operatorname{crit}\Psi_\tau$, then, according to Lemma 2.3.1, Ψ_τ fulfills the *uniform Łojasiewicz property*, which means that there exist (global constants) $C, \varepsilon > 0$ and $\theta \in [0, 1)$ such that for all $(\bar{Z}, \bar{X}) \in \Omega$

$$\left|\Psi_\tau(Z, X) - \Psi_\tau\left(\bar{Z}, \bar{X}\right)\right|^\theta \leqslant C \cdot \operatorname{dist}\left(0, \partial\Psi_\tau(Z, X)\right) \qquad \forall (Z, X) \in \mathbb{R}^{n\times r} \times \mathbb{R}^{n\times r} \text{ with } \operatorname{dist}\left((Z, X), \Omega\right) < \varepsilon.$$

Next we will see that, for $\Omega := \Omega\left(\{(Z_k, X_{k-1})\}_{k\geqslant 2}\right)$ the set of cluster points of the sequence $\{(Z_k, X_{k-1})\}_{k\geqslant 2}$, we actually are in the setting of the uniform Łojasiewicz property. Notice that $\Omega \neq \varnothing$ thanks to the boundedness of the sequences $\{X_k\}_{k\geqslant 0}$ and $\{Z_k\}_{k\geqslant 2}$.

Lemma 4.3.6. *Let $\{X_k\}_{k\geqslant 0}$ be the sequence generated by Algorithm 4.3.1. The following statements are true:*

(i) $\Omega \subseteq \operatorname{crit}\Psi_\tau = \{(X_*, X_*) \in \mathbb{R}^{n\times r} \times \mathbb{R}^{n\times r} : X_* \in \operatorname{crit}\Psi\}$;

(ii) it holds $\lim\limits_{k\to+\infty} \operatorname{dist}\left[\left(Z_k, X_{k-1}\right), \Omega\right] = 0$;

(iii) the set Ω is nonempty, connected and compact;

(iv) the function Ψ_τ takes on Ω the value $\Psi_* := \lim\limits_{k\to+\infty} \Psi_\tau\left(Z_k, X_{k-1}\right)$.

Proof. The item (i) follows from Theorem 4.3.5 and (4.3.26). The proof of (ii) - (iii) follows in the lines of [36, Theorem 5 (ii)-(iii)], by taking into consideration [36, Remark 5], according to which the properties in (ii) - (iii) are generic for sequences satisfying $Z_k - Z_{k-1} \to 0$ and $X_k - X_{k-1} \to 0$ as $k \to +\infty$, which is indeed our case due to Proposition 4.3.4 (ii).

Finally, to prove (iv), we consider an arbitrary element $\left(\bar{X}, \bar{X}\right)$ in Ω, that is, there exists a subsequence $(Z_{k_i}, X_{k_i-1}) \to \left(\bar{X}, \bar{X}\right)$ as $i \to +\infty$. It holds $\bar{X} \in \mathcal{D}$ and

$$\lim_{i\to+\infty} \Psi_\tau\left(Z_{k_i}, X_{k_i-1}\right) = \Psi_\tau\left(\bar{X}, \bar{X}\right).$$

As a consequence, since $\{\Psi\left(Z_k, X_{k-1}\right)\}_{k\geqslant 2}$ converges due to Proposition 4.3.4 (i), it follows that Ψ_τ is a constant on Ω, namely, $\Psi_\tau\left(\bar{X}, \bar{X}\right) = \Psi_* = \lim\limits_{k\to+\infty} \Psi_\tau\left(Z_k, X_{k-1}\right)$ for every $\left(\bar{X}, \bar{X}\right) \in \Omega$. □

As a last preparatory step we derive an upper bound for a subgradient of Ψ_τ.

Lemma 4.3.7. *Let $\{X_k\}_{k\geqslant 0}$ be a sequence generated by Algorithm 4.3.1. For any $k \geqslant 2$ we have*

$$V_k := \left(V_k', V_k''\right) \in \partial\Psi_\tau\left(Z_k, X_{k-1}\right), \tag{4.3.32}$$

where

$$\begin{aligned} V_k' &:= L_{\mathcal{F}}\left(Y_{k-1} - Z_k\right) + \nabla\mathcal{E}\left(Z_k\right) - \nabla\mathcal{E}\left(Y_{k-1}\right) + \rho^2\tau\left(Z_k - X_{k-1}\right) \\ V_k'' &:= -\rho^2\tau\left(Z_k - X_{k-1}\right). \end{aligned}$$

In addition,

$$\|V_k\|_{\mathcal{F}} \leqslant C_1 \|X_k - X_{k-1}\|_{\mathcal{F}} + C_2 \|X_{k-1} - X_{k-2}\|_{\mathcal{F}} \qquad \forall k \geqslant 2, \tag{4.3.33}$$

where

$$\begin{aligned} L_{\mathcal{E}} &:= 2\left(\|A\|_2 + \left(3 + 6\alpha_+ + 4\alpha_+^2\right)\operatorname{trace}(A)\right), \\ C_1 &:= \frac{L_{\mathcal{F}} + L_{\mathcal{E}} + 2\rho^2\tau}{\rho} > 0, \\ C_2 &:= \left(L_{\mathcal{F}} + L_{\mathcal{E}}\right)\alpha_+ \geqslant 0. \end{aligned}$$

Proof. Let $k \geqslant 2$ be fixed. The calculus rules of the limiting subdifferential give for every $(Z, X) \in \mathbb{R}^{n\times r} \times \mathbb{R}^{n\times r}$

$$\begin{aligned} \partial_Z\Psi_\tau(Z, X) &= \partial\Psi(Z) + \rho^2\tau(Z - X) = \nabla\mathcal{E}(Z) + N_{\mathcal{D}}(Z) + \rho^2\tau(Z - X) \\ \nabla_X\Psi_\tau(Z, X) &= -\rho^2\tau(Z - X). \end{aligned}$$

By the characterization of the projection (2.1.2) and (4.3.11b), we have

$$W_k := Y_{k-1} - Z_k - \frac{1}{L_{\mathcal{F}}}\nabla\mathcal{E}\left(Y_{k-1}\right) \in \mathcal{N}_{\mathcal{D}}\left(Z_k\right).$$

From this we deduce

$$L_{\mathcal{F}}W_k = L_{\mathcal{F}}\left(Y_{k-1} - Z_k\right) + \nabla\mathcal{E}\left(Z_k\right) - \nabla\mathcal{E}\left(Y_{k-1}\right) \in \nabla\mathcal{E}\left(Z_k\right) + \mathcal{N}_{\mathcal{D}}\left(Z_k\right),$$

which proves (4.3.32).

Further, we observe that

$$\begin{aligned} \|\nabla\mathcal{E}(Z_k) - \nabla\mathcal{E}(Y_{k-1})\|_{\mathcal{F}} &= 2\left\|\left(A - Z_kZ_k^T\right)Z_k - \left(A - Y_{k-1}Y_{k-1}^T\right)Y_{k-1}\right\|_{\mathcal{F}} \\ &\leqslant 2\|A\|_2\|Z_k - Y_{k-1}\|_{\mathcal{F}} + 2\left\|Z_kZ_k^TZ_k - Y_{k-1}Y_{k-1}^TY_{k-1}\right\|_{\mathcal{F}} \\ &\leqslant 2\|A\|_2\|Z_k - Y_{k-1}\|_{\mathcal{F}} + 2\left\|Z_kZ_k^T\right\|_2\|Z_k - Y_{k-1}\|_{\mathcal{F}} \\ &\qquad + 2\|Z_k\|_2\|Y_{k-1}\|_2\|Z_k - Y_{k-1}\|_{\mathcal{F}} + 2\left\|Y_{k-1}Y_{k-1}^T\right\|_2\|Z_k - Y_{k-1}\|_{\mathcal{F}} \\ &\leqslant 2\left(\|A\|_2 + \left(3 + 6\alpha_+ + 4\alpha_+^2\right)\operatorname{trace}(A)\right)\|Z_k - Y_{k-1}\|_{\mathcal{F}} \\ &= L_{\mathcal{E}}\|Z_k - Y_{k-1}\|_{\mathcal{F}}, \end{aligned}$$

where the last inequality follows from (4.2.3b) - (4.2.3c) and the fact that $\{Z_k\}_{k\geqslant 1} \subseteq \mathcal{D} \subseteq \mathbb{B}\left(0; \sqrt{\text{trace}\,(A)}\right)$ and $\{Y_k\}_{k\geqslant 0} \subseteq \mathbb{B}\left(0; (1+\alpha_+)\sqrt{\text{trace}\,(A)}\right)$ (see Lemma 4.3.2). From here we derive the following estimate which holds for all $k \geqslant 2$

$$\begin{aligned}
\|V_k\|_{\mathcal{F}} &= \sqrt{\|V_k'\|_{\mathcal{F}}^2 + \|V_k''\|_{\mathcal{F}}^2} \leqslant \|V_k'\|_{\mathcal{F}} + \|V_k''\|_{\mathcal{F}} \\
&= \left\|L_{\mathcal{F}}(Y_{k-1} - Z_k) + \nabla\mathcal{E}(Z_k) - \nabla\mathcal{E}(Y_{k-1}) + \rho^2\tau(Z_k - X_{k-1})\right\|_{\mathcal{F}} + \rho^2\tau\|Z_k - X_{k-1}\|_{\mathcal{F}} \\
&\leqslant L_{\mathcal{F}}\|Z_k - Y_{k-1}\|_{\mathcal{F}} + \|\nabla\mathcal{E}(Z_k) - \nabla\mathcal{E}(Y_{k-1})\|_{\mathcal{F}} + 2\rho^2\tau\|Z_k - X_{k-1}\|_{\mathcal{F}} \\
&= (L_{\mathcal{F}} + L_{\mathcal{E}})\|Z_k - Y_{k-1}\|_{\mathcal{F}} + 2\rho^2\tau\|Z_k - X_{k-1}\|_{\mathcal{F}} \\
&\leqslant \left(L_{\mathcal{F}} + L_{\mathcal{E}} + 2\rho^2\tau\right)\|Z_k - X_{k-1}\|_{\mathcal{F}} + (L_{\mathcal{F}} + L_{\mathcal{E}})\,\alpha_k\|X_{k-1} - X_{k-2}\|_{\mathcal{F}} \\
&\leqslant \frac{L_{\mathcal{F}} + L_{\mathcal{E}} + 2\rho^2\tau}{\rho}\|X_k - X_{k-1}\|_{\mathcal{F}} + (L_{\mathcal{F}} + L_{\mathcal{E}})\,\alpha_+\|X_{k-1} - X_{k-2}\|_{\mathcal{F}},
\end{aligned}$$

which yields the inequality (4.3.33). □

We are now in the position to prove the convergence of the whole sequence generated by Algorithm 4.3.1. To simplify the notation, let us define for every $k \geqslant 2$

$$\zeta_k := \Psi_\tau(Z_k, X_{k-1}) - \Psi_*, \tag{4.3.34}$$

where $\Psi_* = \lim_{k\to+\infty} \Psi_\tau(Z_k, X_{k-1})$. According to Proposition 4.3.4 (i), the sequence $\{\zeta_k\}_{k\geqslant 0}$ converges monotonically decreasing to 0.

Theorem 4.3.8. *Let $\{X_k\}_{k\geqslant 0}$ be the sequence generated by Algorithm 4.3.1. The sequence $\{X_k\}_{k\geqslant 0}$ converges to a critical point of Ψ.*

Proof. Let $\left(\bar{X}, \bar{X}\right) \in \Omega$. Then, according to Lemma 4.3.6 (iv), $\Psi_\tau\left(\bar{X}, \bar{X}\right) = \Psi_*$ and, for every $k \geqslant 2$, we have $\Psi_\tau(Z_k, X_{k-1}) - \Psi_\tau\left(\bar{X}, \bar{X}\right) = \zeta_k$. We will show that $\{X_k\}_{k\geqslant 0}$ has finite length, namely,

$$\sum_{k\geqslant 0}\|X_{k+1} - X_k\|_{\mathcal{F}} < +\infty. \tag{4.3.35}$$

Form here it will follow that $\{X_k\}_{k\geqslant 0}$ is a Cauchy sequence, thus it converges to some X_*, which, according to Theorem 4.3.5, will be a critical point of (4.3.1).

In order to prove (4.3.35) we will consider two cases:

Case 1. There exists an integer $k_1 \geqslant 2$ such that $\zeta_k = 0 \Leftrightarrow \Psi_\tau(Z_{k_1}, X_{k_1-1}) = \Psi_*$. The monotonicity of $\{\zeta_k\}_{k\geqslant 0}$ implies that $\zeta_k = 0$ for all $k \geqslant k_1$ and, further, in view of (4.3.27) and (4.3.10), that $X_{k+1} - X_k = 0$ for all $k \geqslant k_1$. Hence

$$\sum_{k\geqslant 0}\|X_{k+1} - X_k\|_{\mathcal{F}} = \sum_{k=0}^{k_1-1}\|X_{k+1} - X_k\|_{\mathcal{F}} < +\infty.$$

Case 2. It holds $\zeta_k > 0$ for every $k \geqslant 2$. As Ψ_τ fulfills the uniform Łojasiewicz property, there exist $C, \varepsilon > 0$ and $\theta \in [0,1)$ such that

$$\left|\Psi_\tau(Z, X) - \Psi_\tau\left(\bar{X}, \bar{X}\right)\right|^\theta \leqslant C \cdot \text{dist}\,(0, \partial\Psi_\tau(Z, X)) \tag{4.3.36}$$

for all $(Z, X) \in \mathbb{R}^{n\times r} \times \mathbb{R}^{n\times r}$ with $\text{dist}\,[(Z, X), \Omega] < \varepsilon$. Since $\lim_{k\to+\infty} \text{dist}\,[(Z_k, X_{k-1}), \Omega] = 0$ (see Lemma 4.3.6 (ii)), there exists an interger $k_2 \geqslant 2$ such that

$$\text{dist}\,[(Z_k, X_{k-1}), \Omega] < \varepsilon \qquad \forall k \geqslant k_2. \tag{4.3.37}$$

Combining (4.3.36) and (4.3.37), we deduce that for every $k \geqslant k_2$ it holds

$$\begin{aligned}\left|\Psi_\tau\left(Z_k, X_{k-1}\right)-\Psi_\tau\left(\bar{X}, \bar{X}\right)\right|^\theta=\left|\zeta_k\right|^\theta & \leqslant C \cdot \operatorname{dist}\left(0, \partial \Psi_\tau\left(Z_k, X_{k-1}\right)\right) \\ & \leqslant C\left\|V_k\right\|_{\mathcal{F}} \\ & \leqslant C \cdot C_1\left\|X_k-X_{k-1}\right\|_{\mathcal{F}}+C \cdot C_2\left\|X_{k-1}-X_{k-2}\right\|_{\mathcal{F}},\end{aligned} \tag{4.3.38}$$

where the last two inequalities follow from Lemma 4.3.7. For the given exponent $\theta \in [0,1)$, we define

$$\varphi\colon \mathbb{R}_+ \to \mathbb{R}, \qquad s \mapsto s^{1-\theta}, \tag{4.3.39}$$

which is a nondecreasing function as $\varphi'(s) = \dfrac{s^{-\theta}}{1-\theta} > 0$. The concavity of φ gives, by taking into consideration (4.3.27), for all $k \geqslant 2$

$$\begin{aligned}\varphi\left(\zeta_k\right)-\varphi\left(\zeta_{k+1}\right) & \geqslant \varphi'\left(\zeta_k\right) \cdot\left(\zeta_k-\zeta_{k+1}\right) \\ & =\frac{\left(\zeta_k\right)^{-\theta}}{1-\theta}\left(\Psi_\tau\left(Z_k, X_{k-1}\right)-\Psi_\tau\left(Z_{k+1}, X_k\right)\right) \\ & \geqslant \frac{\left(\zeta_k\right)^{-\theta}}{1-\theta} \cdot C_0\left\|X_{k+1}-X_k\right\|_{\mathcal{F}}^2 .\end{aligned}$$

From here we get that for every $k \geqslant k_2$

$$\begin{aligned}\left\|X_{k+1}-X_k\right\|_{\mathcal{F}} & \leqslant \sqrt{\frac{1-\theta}{C_0}\left(\zeta_k\right)^\theta\left(\varphi\left(\zeta_k\right)-\varphi\left(\zeta_{k+1}\right)\right)} \\ & \leqslant \frac{1}{2 C \cdot\left(C_1+C_2\right)}\left(\zeta_k\right)^\theta+\frac{(1-\theta) C \cdot\left(C_1+C_2\right)}{2 C_0}\left(\varphi\left(\zeta_k\right)-\varphi\left(\zeta_{k+1}\right)\right) \\ & \leqslant \frac{C_1}{2\left(C_1+C_2\right)}\left\|X_k-X_{k-1}\right\|_{\mathcal{F}}+\frac{C_2}{2\left(C_1+C_2\right)}\left\|X_{k-1}-X_{k-2}\right\|_{\mathcal{F}} \\ & \quad+\frac{(1-\theta) C \cdot\left(C_1+C_2\right)}{2 C_0}\left(\varphi\left(\zeta_k\right)-\varphi\left(\zeta_{k+1}\right)\right) .\end{aligned} \tag{4.3.40}$$

By setting for every $k \geqslant k_2$

$$\begin{aligned} a_k &:= \left\|X_k - X_{k-1}\right\|_{\mathcal{F}}, \\ d_k &:= C_3\left(\varphi\left(\zeta_k\right)-\varphi\left(\zeta_{k+1}\right)\right), \\ C_3 &:= \frac{(1-\theta) C \cdot\left(C_1+C_2\right)}{2 C_0},\end{aligned}$$

the inequality (4.3.40) becomes

$$a_{k+1} \leqslant \chi_0 a_k + \chi_1 a_{k-1} + d_k,$$

with

$$\chi_0 := \frac{C_1}{2\left(C_1+C_2\right)} \in (0,1) \qquad \text{and} \qquad \chi_1 := \frac{C_2}{2\left(C_1+C_2\right)} \in [0,1).$$

Since $\chi_0 + \chi_1 = \dfrac{1}{2} < 1$, by Lemma 2.4.4 we obtain that $\sum_{k \geqslant k_2} \left\|X_k - X_{k-1}\right\|_{\mathcal{F}} < +\infty$. This leads to (4.3.35) and the proof is completed. □

We will close this section by discussing the rates of convergence of the projected gradient algorithm with relaxation and inertial parameters. The nature of the rates is determined by the *Łojasiewicz exponent* θ, which we cannot calculate exactly. This is why we will cover in our statements all possible situations. Some discussions about the values the Łojasiewicz exponent take will be made in the last section of the chapter in the context of some numerical experiments.

We will show that the sequence $\{\zeta_k\}_{k \geqslant 0}$ defined in (4.3.34) satisfies the recursion inequality (2.4.9) in Lemma 2.4.5.

Lemma 4.3.9. *Let $\{X_k\}_{k\geqslant 0}$ be the sequence generated by Algorithm 4.3.1 and $\{\zeta_k\}_{k\geqslant 2}$ the sequence defined in* (4.3.34). *Then there exists $k_3 \geqslant 2$ such that for any $k \geqslant k_3$*

$$\zeta_{k-2} - \zeta_k \geqslant C_4 \cdot \zeta_k^{2\theta}, \qquad \textit{where } C_4 := \frac{C_0}{2\left(C \cdot C_1\right)^2} > 0.$$

Proof. From (4.3.27) we get for any $k \geqslant 4$

$$\begin{aligned}
\zeta_{k-2} - \zeta_k &= \Psi_\tau\left(Z_{k-2}, X_{k-3}\right) - \Psi_\tau\left(Z_{k-1}, X_{k-2}\right) + \Psi_\tau\left(Z_{k-1}, X_{k-2}\right) - \Psi_\tau\left(Z_k, X_{k-1}\right) \\
&\geqslant C_0 \left\|X_{k-1} - X_{k-2}\right\|_{\mathcal{F}}^2 + C_0 \left\|X_k - X_{k-1}\right\|_{\mathcal{F}}^2 \\
&\geqslant \frac{C_0}{2}\left(\left\|X_k - X_{k-1}\right\|_{\mathcal{F}} + \left\|X_{k-1} - X_{k-2}\right\|_{\mathcal{F}}\right)^2 \\
&\geqslant \frac{C_0}{2C_1^2}\left(C_1 \left\|X_k - X_{k-1}\right\|_{\mathcal{F}} + C_2 \left\|X_{k-1} - X_{k-2}\right\|_{\mathcal{F}}\right)^2 \qquad (4.3.41) \\
&\geqslant \frac{C_0}{2C_1^2} \left\|V_k\right\|_{\mathcal{F}}^2, \qquad (4.3.42)
\end{aligned}$$

where $V_k \in \partial\Psi_\tau\left(Z_k, X_{k-1}\right)$ is the element defined in Lemma 4.3.7 and(4.3.41) holds true by taking into account further that $0 \leqslant \rho\alpha_+ \leqslant 1$, hence

$$C_1 = \frac{L_{\mathcal{F}} + L_{\mathcal{E}} + 2\rho^2\tau}{\rho} \geqslant \frac{L_{\mathcal{F}} + L_{\mathcal{E}}}{\rho} \geqslant (L_{\mathcal{F}} + L_{\mathcal{E}})\alpha_+ = C_2.$$

By the same argument as in the proof of Theorem 4.3.8, if we take $k_3 := k_2 \geqslant 2$ for which (4.3.37) holds, then according to (4.3.36) the following inequality holds for every $k \geqslant k_3$

$$\left|\Psi_\tau\left(Z_k, X_{k-1}\right) - \Psi_*\right|^\theta = \zeta_k^\theta \leqslant C \cdot \operatorname{dist}\left(0, \partial\Psi_\tau\left(Z_k, X_{k-1}\right)\right) \leqslant C \left\|V_k\right\|_{\mathcal{F}}.$$

The desired statement is a combination of this estimate and (4.3.42). □

In order to transfer the convergence rates from $\{\zeta_k\}_{k\geqslant 0}$ to the sequence $\{X_k\}_{k\geqslant 0}$, we will need the following lemma.

Lemma 4.3.10. *Let $\{X_k\}_{k\geqslant 0}$ be the sequence generated by Algorithm 4.3.1 and $\{\zeta_k\}_{k\geqslant 2}$ the sequence defined in* (4.3.34). *Let X_* be the critical point of* (4.3.1) *to which the sequence $\{X_k\}_{k\geqslant 0}$ converges as $k \to +\infty$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}, \varphi(s) = s^{1-\theta}$. Then there exists $k_3 \geqslant 2$ such that for any $k \geqslant k_3$*

$$\left\|X_k - X_*\right\|_{\mathcal{F}} \leqslant C_5 \max\left\{\sqrt{\zeta_k}, \varphi\left(\zeta_k\right)\right\}, \qquad \textit{where } C_5 := \frac{4}{\sqrt{C_0}} + 2C_3 > 0. \qquad (4.3.43)$$

Proof. By using the same arguments as in the proof of Theorem 4.3.8, there exists $k_3 \geqslant 2$ such that for any $k \geqslant k_3$ the following inequality is true

$$\begin{aligned}
\left\|X_{k+1} - X_k\right\|_{\mathcal{F}} \leqslant{}& \frac{C_1}{2\left(C_1 + C_2\right)} \left\|X_k - X_{k-1}\right\|_{\mathcal{F}} + \frac{C_2}{2\left(C_1 + C_2\right)} \left\|X_{k-1} - X_{k-2}\right\|_{\mathcal{F}} \\
&+ C_3\left(\varphi\left(\zeta_k\right) - \varphi\left(\zeta_{k+1}\right)\right). \qquad (4.3.44)
\end{aligned}$$

Let $k \geqslant k_3$ be fixed. By an induction argument one can prove that

$$\left\|X_k - X_*\right\|_{\mathcal{F}} \leqslant \left\|X_{k+1} - X_*\right\|_{\mathcal{F}} + \left\|X_{k+1} - X_k\right\|_{\mathcal{F}} \leqslant \cdots \leqslant \sum_{i\geqslant k} \left\|X_{i+1} - X_i\right\|_{\mathcal{F}}. \qquad (4.3.45)$$

For any $K \geqslant k+2 \geqslant k_3$, by summing up (4.3.44) for $i = k+2, \cdots, K$, we get

$$\begin{aligned}
\sum_{i=k+2}^{K} \left\|X_{i+1} - X_i\right\|_{\mathcal{F}} \leqslant{}& \frac{C_1}{2\left(C_1 + C_2\right)} \sum_{i=k+2}^{K} \left\|X_i - X_{i-1}\right\|_{\mathcal{F}} + \frac{C_2}{2\left(C_1 + C_2\right)} \sum_{i=k+2}^{K} \left\|X_{i-1} - X_{i-2}\right\|_{\mathcal{F}} \\
&+ C_3 \sum_{i=k+2}^{K} \left(\varphi\left(\zeta_i\right) - \varphi\left(\zeta_{i+1}\right)\right). \qquad (4.3.46)
\end{aligned}$$

Notice that

$$\sum_{i=k+2}^{K} \|X_{i+1} - X_i\|_{\mathcal{F}} = \sum_{i=k}^{K} \|X_{i+1} - X_i\|_{\mathcal{F}} - \|X_{k+2} - X_{k+1}\|_{\mathcal{F}} - \|X_{k+1} - X_k\|_{\mathcal{F}}, \quad (4.3.47a)$$

$$\sum_{i=k+2}^{K} \|X_i - X_{i-1}\|_{\mathcal{F}} = \sum_{i=k+1}^{K-1} \|X_{i+1} - X_i\|_{\mathcal{F}}$$

$$= \sum_{i=k}^{K} \|X_{i+1} - X_i\|_{\mathcal{F}} - \|X_{k+1} - X_k\|_{\mathcal{F}} - \|X_{K+1} - X_K\|_{\mathcal{F}}, \quad (4.3.47b)$$

$$\sum_{i=k+2}^{K} \|X_{i-1} - X_{i-2}\|_{\mathcal{F}} = \sum_{i=k}^{K-2} \|X_{i+1} - X_i\|_{\mathcal{F}},$$

$$= \sum_{i=k}^{K} \|X_{i+1} - X_i\|_{\mathcal{F}} - \|X_K - X_{K-1}\|_{\mathcal{F}} - \|X_{K+1} - X_K\|_{\mathcal{F}}. \quad (4.3.47c)$$

Plugging these relations into (4.3.46), neglecting the last two negative terms in (4.3.47b) and (4.3.47c), we get

$$\begin{aligned}
\sum_{i=k}^{K} \|X_{i+1} - X_i\|_{\mathcal{F}} \leqslant{}& \frac{C_1}{2(C_1 + C_2)} \sum_{i=k+1}^{K} \|X_i - X_{i-1}\|_{\mathcal{F}} + \frac{C_2}{2(C_1 + C_2)} \sum_{i=k+1}^{K} \|X_{i-1} - X_{i-2}\|_{\mathcal{F}} \\
&+ \|X_{k+2} - X_{k+1}\|_{\mathcal{F}} + \|X_{k+1} - X_k\|_{\mathcal{F}} + C_3 \sum_{i=k+1}^{K} (\varphi(\zeta_i) - \varphi(\zeta_{i+1})) \\
\leqslant{}& \frac{1}{2} \sum_{i=k}^{K} \|X_{i+1} - X_i\|_{\mathcal{F}} + \|X_{k+2} - X_{k+1}\|_{\mathcal{F}} + \|X_{k+1} - X_k\|_{\mathcal{F}} \\
&+ C_3 (\varphi(\zeta_{k+1}) - \varphi(\zeta_{K+1})).
\end{aligned}$$

Thanks to (4.3.27) we can deduce that

$$\begin{aligned}
\sum_{i=k}^{K} \|X_{i+1} - X_i\|_{\mathcal{F}} &\leqslant 2\|X_{k+2} - X_{k+1}\|_{\mathcal{F}} + 2\|X_{k+1} - X_k\|_{\mathcal{F}} + 2C_3 (\varphi(\zeta_{k+1}) - \varphi(\zeta_{K+1})) \\
&\leqslant \frac{2}{\sqrt{C_0}} \sqrt{\zeta_{k+1} - \zeta_{k+2}} + \frac{2}{\sqrt{C_0}} \sqrt{\zeta_k - \zeta_{k+1}} + 2C_3 (\varphi(\zeta_{k+1}) - \varphi(\zeta_{K+1})) \\
&\leqslant \frac{2}{\sqrt{C_0}} \sqrt{\zeta_{k+1}} + \frac{2}{\sqrt{C_0}} \sqrt{\zeta_k} + 2C_3 \varphi(\zeta_{k+1}). \quad (4.3.48)
\end{aligned}$$

The fact that $\{\zeta_k\}_{k \geqslant 0}$ is monotonically decreasing implies $\sqrt{\zeta_{k+1}} \leqslant \sqrt{\zeta_k}$ and $\varphi(\zeta_{k+1}) \leqslant \varphi(\zeta_k)$. By passing $K \to +\infty$ in (4.3.48) and by using (4.3.45), we get the desired statement. □

We can now formulate the rates of convergence for the sequences of *objective function values* and *iterates*.

Theorem 4.3.11. *Let $\{X_k\}_{k \geqslant 0}$ be the sequence generated by Algorithm 4.3.1 and $\{\zeta_k\}_{k \geqslant 2}$ the sequence defined in (4.3.34). Let X_* be the critical point of (4.3.1) to which the sequence $\{X_k\}_{k \geqslant 0}$ converges as $k \to +\infty$. Then there exists $k_4 \geqslant 2$ such that the following statements are true:*

(i) if $\theta = 0$, then $\{\zeta_k\}_{k \geqslant 2}$ and $\{X_k\}_{k \geqslant 0}$ converge in finite time;

(ii) if $\theta \in (0, 1/2]$, then there exist $C_1', C_2' > 0$ and $Q_1, Q_2 \in [0, 1)$ such that for any $k \geqslant k_4$

$$0 \leqslant \mathcal{E}(Z_k) - \Psi_* \leqslant C_1' Q_1^k \qquad \text{and} \qquad \|X_k - X_*\|_{\mathcal{F}} \leqslant C_2' Q_2^k;$$

(iii) if $\theta \in (1/2, 1)$, then there exist $C_3', C_4' > 0$ such that for any $k \geqslant k_4 + 2$

$$0 \leqslant \mathcal{E}(Z_k) - \Psi_* \leqslant C_3' (k-1)^{-\frac{1}{2\theta-1}} \qquad \textit{and} \qquad \|X_k - X_*\|_{\mathcal{F}} \leqslant C_4' (k-1)^{-\frac{1-\theta}{2\theta-1}}.$$

Proof. Let $k_3 \geqslant 2$ be the index provided by previous lemma with the property that (4.3.43) holds for any $k \geqslant k_3$. Since $\{\zeta_k\}_{k \geqslant 0}$ converges to 0, there exists $k_4 \geqslant k_3$ such that for any $k \geqslant k_4$

$$\|X_k - X_*\|_{\mathcal{F}} \leqslant C_5 \max\left\{\sqrt{\zeta_k}, \varphi(\zeta_k)\right\}, \tag{4.3.49}$$

$$\zeta_k \leqslant 1. \tag{4.3.50}$$

(i) If $\theta = 0$, then $\{\zeta_k\}_{k \geqslant 1}$ converges in finite time. By similar arguments as in the proof of Theorem 4.3.8, we get that the sequence $\{X_k\}_{k \geqslant 0}$ becomes identical to X_* starting from a given index. In other words, the sequence $\{X_k\}_{k \geqslant 0}$ converges in finite time, too.

(ii) If $\theta \in (0, 1/2]$, then, according to Lemma 2.4.5 (ii), there exist $C_1' > 0$ and $Q_1 \in [0, 1)$ such that for any $k \geqslant k_4$

$$0 \leqslant \mathcal{E}(Z_k) - \Psi_* \leqslant \zeta_k \leqslant C_1' Q_1^k.$$

Moreover, as $1 - 2\theta \geqslant 0$, due to (4.3.50) it holds

$$\zeta_k^{\frac{1-2\theta}{2}} = \zeta_k^{\frac{1}{2}-\theta} \leqslant 1 \Leftrightarrow \zeta_k^{1-\theta} \leqslant \sqrt{\zeta_k}.$$

Consequently, Lemma 4.3.10 implies that

$$\|X_k - X_*\|_{\mathcal{F}} \leqslant C_5 \sqrt{\zeta_k} \leqslant C_5 \sqrt{C_1'} \left(\sqrt{Q_1}\right)^k \quad \forall k \geqslant k_4,$$

which is nothing else than the second inequality of (ii) with $C_2' := C_5 \sqrt{C_1'} > 0$ and $Q_2 := \sqrt{Q_1} \in (0, 1)$.

(iii) If $\theta \in (1/2, 1)$, then we can use Lemma 2.4.5 (iii) to ensure that there exist $C_3' > 0$ such that for any $k \geqslant k_4$

$$0 \leqslant \mathcal{E}(Z_k) - \Psi_* \leqslant \zeta_k \leqslant C_3' (k-1)^{-\frac{1}{2\theta-1}}.$$

Since $2\theta - 1 > 0$ and $\zeta_k \leqslant 1$ due to (4.3.50), we have

$$\zeta_k^{\frac{2\theta-1}{2}} = \zeta_k^{\theta-\frac{1}{2}} \leqslant 1 \Leftrightarrow \sqrt{\zeta_k} \leqslant \zeta_k^{1-\theta}.$$

Then the second statement follows from (4.3.49) with $C_4' := C_5 C_3'^{1-\theta} > 0$. ☐

4.4 Particular instances and numerical experiments

4.4.1 Some particular instances of Algorithm 4.3.1

In the following we will discuss some particular instances of Algorithm 4.3.1. To this aim we will use again the notation $L_{\mathcal{F}}(\alpha_+)$, which will allow us to better underline the dependence of the step size from the inertial parameters.

Example 4.4.1. Choosing $\alpha_k = 0$ for all $k \geqslant 1$, Algorithm 4.3.1 reduces to the *relaxed projected gradient algorithm*

$$Z_{k+1} := \mathbf{Pr}_{\mathcal{D}}\left(X_k - \frac{1}{L_{\mathcal{F}}(0)} \nabla \mathcal{E}(X_k)\right),$$
$$X_{k+1} := (1-\rho) X_k + \rho Z_{k+1}.$$

In this case, $\alpha_+ = 0$ and condition (4.3.10) becomes

$$\frac{\sqrt{L_{\mathcal{F}}(0) + 2\|A\|_2}}{\sqrt{L_{\mathcal{F}}(0) + 2\|A\|_2} + \sqrt{L_{\mathcal{F}}(0)}} = \frac{\sqrt{3\mathrm{trace}(A) + \|A\|_2 - \lambda_{\min}(A)}}{\sqrt{3\mathrm{trace}(A) + \|A\|_2 - \lambda_{\min}(A)} + \sqrt{3\mathrm{trace}(A) - \lambda_{\min}(A)}}$$
$$< \rho \leqslant 1 < \frac{\sqrt{L_{\mathcal{F}}(0) + 2\|A\|_2}}{\sqrt{L_{\mathcal{F}}(0) + 2\|A\|_2} - \sqrt{L_{\mathcal{F}}(0)}}. \tag{4.4.1}$$

Notice that, according to (4.4.1), the the choice $\rho = 1$ is allowed, which leads to the classical *projected gradient algorithm*.

Example 4.4.2. For $\rho = 1$, Algorithm 4.3.1 reduces to the *inertial projected gradient algorithm*

$$Y_k := X_k + \alpha_k (X_k - X_{k-1}),$$
$$X_{k+1} := \mathbf{Pr}_{\mathcal{D}}\left(Y_k - \frac{1}{L_{\mathcal{F}}(\alpha_+)}\nabla\mathcal{E}(Y_k)\right).$$

In the nonconvex setting, algorithms with inertial effects proved to be helpful to detect critical points of minimization problems which cannot be found by their non-inertial variants (see, for instance, [51, 94]). For constant inertial parameters $\alpha_k = \alpha_+ \in [0,1]$ for any $k \geqslant 1$, condition (4.3.10) is equivalent to

$$1 < \frac{\sqrt{L_{\mathcal{F}}(\alpha_+) + 2\|A\|_2}}{(1+\alpha_+)\sqrt{L_{\mathcal{F}}(\alpha_+) + 2\|A\|_2} - \sqrt{L_{\mathcal{F}}(\alpha_+)}}$$

and further to

$$0 \leqslant \alpha_+ < \sqrt{\frac{L_{\mathcal{F}}(\alpha_+)}{L_{\mathcal{F}}(\alpha_+) + 2\|A\|_2}}. \tag{4.4.2}$$

Condition (4.4.2) is in implicit form, however, one can show that it is satisfied for every $0 < \alpha_+ \leqslant 0.967$. In order to find a larger α_+, which fulfills (4.4.2), one can use a bisection routine starting from 0.967, as we did in our numerical experiments and will explain in the next subsection.

In order to see that for every $0 < \alpha_+ \leqslant 0.967$ the inequality (4.4.2) always holds true, one can rewrite (4.4.2) equivalently as

$$\alpha_+^2\left(\|A\|_2 + \left(3 + 8\alpha_+ + 6\alpha_+^2\right)\mathrm{trace}(A) - \lambda_{\min}(A)\right) \leqslant \left(3 + 8\alpha_+ + 6\alpha_+^2\right)\mathrm{trace}(A) - \lambda_{\min}(A). \tag{4.4.3}$$

Relation (4.4.3) is definitively fulfilled if

$$w(\alpha_+) \leqslant 0,$$

where

$$w(\xi) := 6\mathrm{trace}(A)\,\xi^4 + 8\mathrm{trace}(A)\,\xi^3 - (\lambda_{\min}(A) + 2\mathrm{trace}(A))\,\xi^2$$
$$- 8\xi\mathrm{trace}(A) - 3\mathrm{trace}(A) - \lambda_{\min}(A).$$

We have

$$w(\alpha_+) \leqslant \mathrm{trace}(A)\,\phi(\alpha_+) - \lambda_{\min}(A)\,\alpha_+^2 - \lambda_{\min}(A) \leqslant \mathrm{trace}(A)\,\phi(\alpha_+),$$

where

$$\phi(\xi) := 6\xi^4 + 8\xi^3 - 2\xi^2 - 8\xi - 3,$$

and this is why we will solve a more restricted yet easier inequality $\phi(\xi) \leqslant 0$ instead of (4.4.3). The derivative of ϕ reads

$$\phi'(\xi) = 24\xi^3 + 24\xi^2 - 4\xi - 8$$

and has exactly one root

$$\nu = \frac{1}{18}\sqrt[3]{594 - 54\sqrt{67}} + \frac{1}{6}\sqrt[3]{2\left(11 + \sqrt{67}\right)} - \frac{1}{3} \approx 0.5253.$$

Since $\phi'(0) = -8 < 0$ and $\phi'(1) = 36 > 0$, we have that ϕ is decreasing on $(0, \nu)$ and increasing on $(\nu, 1)$. Moreover, as $\phi(0.967) = -0.00458574 < 0$, $\phi(0) = -3 < 0$ and $\phi(1) = 1 > 0$, we can conclude that $\phi(\xi) < 0$ for every $\xi \in [0, 0.967]$, which implies that (4.4.3) is fulfilled as a strict inequality for every $\alpha_+ \in [0, 0.967]$ as well. Since in the above approach we weakened (4.4.3) in order to simplify the computations, one cannot expect 0.967 to be the largest value for which this inequality is fulfilled. However, we will use in our numerical experiments 0.0967 as the starting point for a bisection procedure aimed to find larger values of α_+ which fulfill (4.4.3).

Example 4.4.3. An interesting choice of the variable inertial parameters $\{\alpha_k\}_{k\geqslant 1}$ in the context of the *inertial projected gradient algorithm* discussed in Example 4.4.2 is

$$\alpha_k := \kappa \cdot \frac{t_k - 1}{t_{k+1}}, \qquad \text{where } \begin{cases} t_1 & := 1 \\ t_{k+1} & := \dfrac{1 + \sqrt{1 + 4t_k^2}}{2} \end{cases} \quad \forall k \geqslant 1. \tag{4.4.4}$$

Notice that, for $\kappa := 1$, this is exactly the update rule of the celebrated Nesterov/FISTA algorithm [110, 28]. This iterative scheme have attracted the interest of the optimization community and of many practitioners due to the fact that, in the convex setting, it improves for the sequence of objective function values the convergence rate over the one of the standard non-inertial method. In the nonconvex setting, no theoretical results, which emphasize an improvement in the convergence behaviour through this update rule, have been obtained so far, however, some empirical studies suggest that this might be the case (see, for instance, [115]).

Since $\alpha_+ = \sup_{k\geqslant 0} \alpha_k = \kappa$, one can find κ such that (4.3.10) holds by solving (see (4.4.2))

$$0 \leqslant \kappa < \sqrt{\frac{L_{\mathcal{F}}(\kappa)}{L_{\mathcal{F}}(\kappa) + 2\|A\|_2}}. \tag{4.4.5}$$

If one wants to choose larger values for κ, for instance to take κ close to 1, a restart mechanism can be adapted into the scheme (4.4.4), like, for example, in [112].

Example 4.4.4. If we set, again in the context of the *inertial projected gradient algorithm*,

$$\alpha_k := \frac{\kappa k}{k+3} \ \forall k \geqslant 1, \qquad \text{where } \kappa \in (0, 1),$$

then it holds $\alpha_+ = \kappa$. This is a setting considered by László in [94] for the inertial gradient algorithm, which is the scheme in Example 4.4.2 without the projection step. Our algorithm can be considered as an extension of the one in [94]. To guarantee convergence, in [94] is required that the step size fulfills

$$0 < \mu < \frac{2(1-\kappa)}{L_{\mathcal{F}}},$$

where $L_{\mathcal{F}}$ denotes the Lipschitz constant of the gradient of the objective function. This condition excludes the case $\kappa = 1$ and allows $\mu = 1/L_{\mathcal{F}}$ as stepsize when $\kappa = 1/2$. In our setting, we can have larger values of κ in combination with the stepsize $1/L_{\mathcal{F}}$, namely, those for which (4.4.5) is fulfilled (see also the discussion at the end of Example 4.4.2).

Example 4.4.5. Other than for the classical inertial algorithms for convex optimization problems and monotone inclusions, for which the inertial parameters were not allowed to take values greater than 1/3, the interplay between relaxation and inertia gives us much more freedom when it comes to the choice of the latter. We have seen that as far as α_+ satisfies (4.4.2) we can choose $\rho = 1$. For α_+ close to 1 such that (4.4.2) is not satisfied, in other words

$$\sqrt{\frac{L_{\mathcal{F}}(\alpha_+)}{L_{\mathcal{F}}(\alpha_+) + 2\|A\|_2}} \leqslant \alpha_+,$$

we can take

$$0 < \frac{\sqrt{L_{\mathcal{F}}(\alpha_+) + 2\|A\|_2}}{\sqrt{L_{\mathcal{F}}(\alpha_+) + 2\|A\|_2} + \sqrt{L_{\mathcal{F}}(\alpha_+)}} < \rho < \frac{\sqrt{L_{\mathcal{F}}(\alpha_+) + 2\|A\|_2}}{(1+\alpha_+)\sqrt{L_{\mathcal{F}}(\alpha_+) + 2\|A\|_2} - \sqrt{L_{\mathcal{F}}(\alpha_+)}} < 1. \tag{4.4.6}$$

This applies also for the case when $\alpha_k = 1$ for any $k \geqslant 1$, and thus $\alpha_+ = 1$, for which Algorithm 4.3.1 becomes

$$\begin{aligned} Z_{k+1} &:= \mathbf{Pr}_{\mathcal{D}}\left(2X_k - X_{k-1} - \frac{1}{L_{\mathcal{F}}(1)}\nabla\mathcal{E}\left(2X_k - X_{k-1}\right)\right), \\ X_{k+1} &:= (1-\rho)X_k + \rho Z_{k+1}. \end{aligned}$$

As we will see in the numerical results, the strategy of choosing α_+ close to 1 and ρ according to (4.4.6) yields to the best performances of the algorithm.

4.4.2 Numerical experiments

The aim of the numerical experiments we will present in this subsection is twofold: to compare the performances of our algorithm with those of other numerical methods for the nonnegative factorization of completely positive matrices, as are (4.2.16) and (4.2.17) from [87] and [66], respectively, and to show in which way and to which extent the algorithm parameters influence these performances.

A particular attention will be paid to the nonnegative factorization of matrices not belonging to the interior of $\mathcal{CP}_n$, for which the algorithms in [87, 66] perform rather poor.

Number of runs and starting points. In every numerical experiment, for $A \in \mathbb{R}^{n\times n}$ with $n < 100$, we run Algorithm 4.3.1 100 times for 100 randomly chosen initial matrices in $\mathcal{D}$ (for instance, by chosing a random matrix in $\mathbb{R}^{n\times r}$ and then by using the projection formula (4.3.2)), and run the algorithms (4.2.16) and (4.2.17) also 100 times for 100 randomly chosen initial matrices in $\mathcal{O}_r$ (for instance, by chosing a random matrix in $\mathbb{R}^{r\times r}$ and by computing a SVD decomposition); if $n \geqslant 100$, then we do this for each of the algorithms 10 times.

As noticed in Section 4.2.3, the algorithm (4.2.16) and (4.2.17) require, in addition, a matrix B, which we compute by the Cholesky decomposition. If the Cholesky decomposition fails, then we use the eigenvalue decomposition. Here we follow the approach described in [87, Section 3], see also [66, Section 6].

Parameter choice. We will choose the constant α_+, which will then determine the sequence of inertial parameters $\{\alpha_k\}_{k\geqslant 1}$, with two different aims:

- by running a simple bisection routine which starts at 0.967 in order to find greater values for α_+ that satisfy (4.4.2), namely,

  $$0 \leqslant \alpha_+ < \sqrt{\frac{L_{\mathcal{F}}(\alpha_+)}{L_{\mathcal{F}}(\alpha_+) + 2\|A\|_2}}.$$

  Instead of using the midpoint rule, we will use as update rule for the bisection routine $\alpha_+ := (3\alpha_+ + 1)/4$, which seemingly gives better results. We will then choose $\alpha_+ := \widehat{\alpha}_+$, which is the last value at which (4.4.2) holds. As seen in the previous subsection, as long as (4.4.2) is fulfilled, we can and do choose $\rho = 1$.

- by taking $\widehat{\alpha}_1 := (3\widehat{\alpha}_+ + 1)/4$, $\widehat{\alpha}_2 := (\widehat{\alpha}_+ + 1)/2$, and $\widehat{\alpha}_3 := (\widehat{\alpha}_+ + 3)/4$, which, when $\widehat{\alpha}_+$ is obtained as above, all violate (4.4.2). The corresponding relaxation parameters will be denoted by $\rho(\widehat{\alpha}_1)$, $\rho(\widehat{\alpha}_2)$ and $\rho(\widehat{\alpha}_3)$, respectively, and chosen to satisfy (4.4.6). Another value of α_+ which violates (4.4.2) is 1, which we will also use in the experiments in combination with a relaxation parameter $\rho(1)$ fulfilling (4.4.6) as well.

Stopping criteria. For $A \in \mathbb{R}^{n\times n}$, we will run each of the algorithms at most 10000 iterations if $n < 100$ and 50000 otherwise. We count the algorithms (4.2.16) and (4.2.17) as "success" if the stopping criterion

$$\min\{(BQ_k)_{i,j}\} \geqslant -\mathtt{Tol}_{\mathtt{fea}}$$

is reached before the maximal number of iterations is attained. This is nothing else than the stopping criterion used in [87, 66]. For (4.2.17), we will set $\mathtt{Tol}_{\mathtt{val}} := 10^{-16}$ if the matrix A belongs to $\operatorname{int}(\mathcal{CP}_n)$, and $\mathtt{Tol}_{\mathtt{val}} := 10^{-7}$ otherwise. For (4.2.16) we will take as threshold $10 \times \mathtt{Tol}_{\mathtt{fea}}$. On the other hand, for all instances of Algorithm 4.3.1 we will use as stopping criterion the relative error condition

$$\frac{\left\|A - X_k X_k^T\right\|_{\mathcal{F}}^2}{\|A\|_{\mathcal{F}}^2} < \mathtt{Tol}_{\mathtt{val}}.$$

Also here, we will set $\mathtt{Tol}_{\mathtt{val}} := 10^{-16}$ if A belongs to $\operatorname{int}(\mathcal{CP}_n)$, and $\mathtt{Tol}_{\mathtt{val}} := 10^{-7}$ otherwise.

Tables. In the tables with numerical results, we report the (rounded) successful rate over the total number of trials (**Rate**), the average CPU time in seconds for both successful (**Time (s)**) and failed (**Time (f)**) trials, and the average number of iterations (**Iter.**) needed to reach the stopping criteria for the successful trials. We also use boldfaces to highlight the best results among all methods that have successful rate 1.

Plots. We plot for some particular instances the sequences of function values $\{\mathcal{E}(Z_k) - \mathcal{E}_{\min}\}_{k\geqslant 2}$ and of distances $\{\frac{1}{2}\|X_k - X_{\text{sol}}\|_{\mathcal{F}}^2\}_{k\geqslant 0}$ in logarithmic scale, where $\mathcal{E}_{\min}$ denotes the smallest objective function value over all methods and X_{sol} is the last iterate X_k for each method. With the plots we want to emphasize that the sequences of both function values and iterates have linear rates of convergence.

Algorithms. We summarize here the different variants of Algorithm 4.3.1 with corresponding parameter choices we will use in the numerical experiments:

(i) `PG`: the classical projected gradient algorithm formulated in Example 4.4.1 in case $\rho = 1$;

(ii) `FISTA`: the FISTA/Nesterov algorithm from [110, 28];

(iii) `IPG-const`: the inertial projected gradient algorithm formulated in Example 4.4.2 (for $\rho = 1$) with constant inertial parameters $\alpha_k = \alpha_+$ for any $k \geqslant 1$ and $\widehat{\alpha}_+$ chosen to satisfy (4.4.2);

(iv) `IPG-sFISTA`: the inertial projected gradient algorithm formulated in Example 4.4.3 (for $\rho = 1$) with inertial parameters fulfilling (4.4.4) for $\kappa := \widehat{\alpha}_+$;

(v) `IPG-mod`: the modification of Nesterov's scheme from [94] discussed in Example 4.4.4 with $\kappa := \widehat{\alpha}_+$ and step size $\mu := 1/L_{\mathcal{F}}$. The setting goes beyond the one in which convergence was proved in [94], but it is within the one for which our convergence result holds.

(vi) `RIPG-const`, `RIPG-sFISTA` and `RIPG-mod`: the relaxed versions of `IPG-const`, `IPG-sFISTA` and `IPG-mod`, respectively, for different values of α_+ that violate (4.4.2), as in Example 4.4.5, and with corresponding relaxation parameters ρ satisfying (4.4.6).

Numerical experiment 4.4.1. In our first experiment, we use randomly generated completely positive matrices as in [87, Section 7.8]. Precisely, in each test we generate a random $n \times 2n$ matrix B_0 and then we set $A := |B_0| |B_0|^T$; here the absolute value operator $|\cdot|$ is understood entrywise. We test the algorithms on 50 randomly generated 40×40 matrices, 10 randomly generated 100×100 matrices, and 10 randomly generated 500×500 matrices, all via the approach described above. For the nonnegative factorization we use in each case $r := 1.5n + 1$ and $r := 3n + 1$. The performances of the different numerical methods on the for the different instances are captured in the Tables 4.4.1 - 4.4.6.

One can notice that (4.2.17) outperforms the other methods with respect to the number of iterations, which is due the fact that (4.2.17) uses a linesearch routine to improve the step size, while the other methods have quite conservative step size rules. However, some of the instances of Algorithm 4.3.1 can compete with (4.2.17) in terms of computational time. This is due to the fact that the latter runs in every iteration a SVD routine, which is much more time expensive than the simple projection step made by Algorithm 4.3.1. In particular with growing dimensions our algorithm becomes faster than (4.2.17).

Method	**Rate**	**Time (s)**	**Time (f)**	**Iter.**
Algorithm (4.2.16)	0.80	2.5137×10^0	7.0416×10^0	3467.08
Algorithm (4.2.17)	1.00	$\mathbf{4.1259 \times 10^{-2}}$	−//−	**38.51**
`PG`	0.00	−//−	4.5239×10^{-1}	−//−
`IPG-const`: $\alpha = \widehat{\alpha}_+$	1.00	1.3017×10^{-1}	−//−	2554.45
`IPG-sFISTA`: $\alpha = \widehat{\alpha}_+$	1.00	1.2994×10^{-1}	−//−	2561.51
`IPG-mod`: $\alpha = \widehat{\alpha}_+$	1.00	1.3122×10^{-1}	−//−	2562.88
`RIPG-const`: $(\alpha, \rho) = (\widehat{\alpha}_2, \rho(\widehat{\alpha}_2))$	1.00	2.8331×10^{-1}	−//−	5490.14
`RIPG-const`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	2.8589×10^{-1}	−//−	5532.32
`RIPG-sFISTA`: $(\alpha, \rho) = (\widehat{\alpha}_2, \rho(\widehat{\alpha}_2))$	1.00	$\mathbf{8.8411 \times 10^{-2}}$	−//−	**1752.14**
`RIPG-sFISTA`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	1.4610×10^{-1}	−//−	2906.58
`RIPG-mod`: $(\alpha, \rho) = (\widehat{\alpha}_2, \rho(\widehat{\alpha}_2))$	1.00	$\mathbf{8.9617 \times 10^{-2}}$	−//−	**1751.66**
`RIPG-mod`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	1.4798×10^{-1}	−//−	2904.48

Table 4.4.1: The nonnegative factorization of random completely positive matrices for $n = 40$ and $r = 61$.

Method	**Rate**	**Time (s)**	**Time (f)**	**Iter.**
Algorithm (4.2.16)	0.90	8.3492×10^0	2.1794×10^1	3883.03
Algorithm (4.2.17)	1.00	$\mathbf{6.3118 \times 10^{-2}}$	−//−	**19.22**
`PG`	0.00	−//−	8.4875×10^{-1}	−//−
`IPG-const`: $\alpha = \widehat{\alpha}_+$	1.00	1.9973×10^{-1}	−//−	2020.26
`IPG-sFISTA`: $\alpha = \widehat{\alpha}_+$	1.00	2.5665×10^{-1}	−//−	2589.74
`IPG-mod`: $\alpha = \widehat{\alpha}_+$	1.00	2.6477×10^{-1}	−//−	2591.06
`RIPG-const`: $(\alpha, \rho) = (\widehat{\alpha}_2, \rho(\widehat{\alpha}_2))$	1.00	5.0055×10^{-1}	−//−	4964.26
`RIPG-const`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	5.0620×10^{-1}	−//−	5014.23
`RIPG-sFISTA`: $(\alpha, \rho) = (\widehat{\alpha}_2, \rho(\widehat{\alpha}_2))$	1.00	$\mathbf{1.6188 \times 10^{-1}}$	−//−	**1634.78**
`RIPG-sFISTA`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	2.7420×10^{-1}	−//−	2760.50
`RIPG-mod`: $(\alpha, \rho) = (\widehat{\alpha}_2, \rho(\widehat{\alpha}_2))$	1.00	$\mathbf{1.6681 \times 10^{-1}}$	−//−	**1633.88**
`RIPG-mod`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	2.8115×10^{-1}	−//−	2756.80

Table 4.4.2: The nonnegative factorization of random completely positive matrices for $n = 40$ and $r = 121$.

Method	**Rate**	**Time (s)**	**Time (f)**	**Iter.**
Algorithm (4.2.16)	0.62	6.4857×10^1	1.3183×10^2	24245.13
Algorithm (4.2.17)	1.00	$\mathbf{5.3558 \times 10^{-1}}$	−//−	**109.72**
`PG`	0.68	1.0220×10^1	1.0925×10^1	47216.68
`IPG-const`: $\alpha = \widehat{\alpha}_+$	1.00	1.9569×10^0	−//−	7948.22
`IPG-sFISTA`: $\alpha = \widehat{\alpha}_+$	1.00	1.6213×10^0	−//−	6606.02
`IPG-mod`: $\alpha = \widehat{\alpha}_+$	1.00	1.6379×10^0	−//−	6607.08
`RIPG-const`: $(\alpha, \rho) = (\widehat{\alpha}_2, \rho(\widehat{\alpha}_2))$	1.00	3.4802×10^0	−//−	14271.40
`RIPG-const`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	3.5571×10^0	−//−	14465.50
`RIPG-sFISTA`: $(\alpha, \rho) = (\widehat{\alpha}_2, \rho(\widehat{\alpha}_2))$	1.00	8.3203×10^{-1}	−//−	**3160.96**
`RIPG-sFISTA`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	$\mathbf{8.1442 \times 10^{-1}}$	−//−	3216.90
`RIPG-mod`: $(\alpha, \rho) = (\widehat{\alpha}_2, \rho(\widehat{\alpha}_2))$	1.00	8.2046×10^{-1}	−//−	**3163.08**
`RIPG-mod`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	$\mathbf{7.9077 \times 10^{-1}}$	−//−	3215.90

Table 4.4.3: The nonnegative factorization of random completely positive matrices for $n = 100$ and $r = 151$.

Method	**Rate**	**Time (s)**	**Time (f)**	**Iter.**
Algorithm (4.2.16)	0.16	6.1287×10^2	9.1004×10^2	34943.88
Algorithm (4.2.17)	1.00	2.1906×10^0	−//−	**96.08**
`PG`	0.80	2.4696×10^1	2.3458×10^1	47725.30
`IPG-const`: $\alpha = \widehat{\alpha}_+$	1.00	1.9569×10^0	−//−	7948.22
`IPG-sFISTA`: $\alpha = \widehat{\alpha}_+$	1.00	1.6213×10^0	−//−	6606.02
`IPG-mod`: $\alpha = \widehat{\alpha}_+$	1.00	1.6379×10^0	−//−	6607.08
`RIPG-const`: $(\alpha, \rho) = (\widehat{\alpha}_2, \rho(\widehat{\alpha}_2))$	1.00	3.8786×10^0	−//−	13377.24
`RIPG-const`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	3.7777×10^0	−//−	13551.98
`RIPG-sFISTA`: $(\alpha, \rho) = (\widehat{\alpha}_2, \rho(\widehat{\alpha}_2))$	1.00	2.0073×10^0	−//−	3232.04
`RIPG-sFISTA`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	$\mathbf{1.7938 \times 10^0}$	−//−	**3021.04**
`RIPG-mod`: $(\alpha, \rho) = (\widehat{\alpha}_2, \rho(\widehat{\alpha}_2))$	1.00	$\mathbf{1.9433 \times 10^0}$	−//−	3234.30
`RIPG-mod`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	$\mathbf{1.7880 \times 10^0}$	−//−	**3018.80**

Table 4.4.4: The nonnegative factorization of random completely positive matrices for $n = 100$ and $r = 301$.

Method	**Rate**	**Time (s)**	**Time (f)**	**Iter.**
Algorithm (4.2.17)	1.00	$1.6557e \times 10^2$	−//−	**929.38**
`RIPG-sFISTA`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	$\mathbf{1.4526 \times 10^2}$	−//−	7919.40
`RIPG-mod`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	1.4861×10^2	−//−	7921.64

Table 4.4.5: The nonnegative factorization of random completely positive matrices for $n = 500$ and $r = 751$.

Method	**Rate**	**Time (s)**	**Time (f)**	**Iter.**
Algorithm (4.2.17)	1.00	1.3813×10^3	−//−	**914.15**
`RIPG-sFISTA`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	$\mathbf{2.2975 \times 10^2}$	−//−	7776.30
`RIPG-mod`: $(\alpha, \rho) = (\widehat{\alpha}_3, \rho(\widehat{\alpha}_3))$	1.00	2.3037×10^2	−//−	7779.60

Table 4.4.6: The nonnegative factorization of random completely positive matrices for $n = 500$ and $r = 1501$.

Method	Rate	Time (s)	Time (f)	Iter.
Algorithm (4.2.16)	0.00	−//−	4.7649×10^{-1}	−//−
Algorithm (4.2.17)	0.02	7.0223×10^{-1}	7.5259×10^{-1}	9220.50
`PG`	0.27	1.8571×10^{-2}	2.7675×10^{-2}	7069.00
`FISTA`	1.00	$\mathbf{2.1624 \times 10^{-3}}$	−//−	**728.32**
`IPG-const`: $\alpha_+ = 0.9814$	1.00	7.2203×10^{-3}	−//−	2385.20
`IPG-sFISTA`: $\alpha_+ = 0.9814$	1.00	7.9190×10^{-3}	−//−	2474.65
`IPG-mod`: $\alpha_+ = 0.9814$	1.00	7.7214×10^{-3}	−//−	2473.84
`RIPG-const`: $(\alpha, \rho) = (0.9954, 0.9705)$	0.93	1.3141×10^{-2}	3.1291×10^{-2}	4383.86
`RIPG-const`: $(\alpha, \rho) = (1.0000, 0.9661)$	0.94	1.3217×10^{-2}	3.2318×10^{-2}	4446.59
`RIPG-sFISTA`: $(\alpha, \rho) = (0.9954, 0.9705)$	1.00	3.5561×10^{-3}	−//−	1050.93
`RIPG-sFISTA`: $(\alpha, \rho) = (1.0000, 0.9661)$	1.00	$\mathbf{2.5225 \times 10^{-3}}$	−//−	**742.12**
`RIPG-mod`: $(\alpha, \rho) = (0.9954, 0.9705)$	1.00	3.5350×10^{-3}	−//−	1056.10
`RIPG-mod`: $(\alpha, \rho) = (1.0000, 0.9661)$	1.00	$\mathbf{2.4953 \times 10^{-3}}$	−//−	**744.37**

Table 4.4.7: The nonnegative factortization of $A_{0.99}$ given by (4.4.7) - (4.4.8) for $r = 12$.

Numerical experiment 4.4.2. In the second numerical experiment, we consider the perturbed matrix A_ω defined by

$$A_\omega := \omega A + (1 - \omega) P, \quad \text{for } \omega \in [0, 1], \tag{4.4.7}$$

where

$$A := \begin{pmatrix} 8 & 5 & 1 & 1 & 5 \\ 5 & 8 & 5 & 1 & 1 \\ 1 & 5 & 8 & 5 & 1 \\ 1 & 1 & 5 & 8 & 5 \\ 5 & 1 & 1 & 5 & 8 \end{pmatrix} \quad \text{and} \quad P := \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}. \tag{4.4.8}$$

Both A and A_ω belong to $\mathcal{CP}_5$ for all $\omega \in [0, 1]$. Furthermore, $A_\omega \in \text{int}(\mathcal{CP}_5)$ whenever $0 \leqslant \omega < 1$, since $P = \left[\mathbf{j}_5 | \text{Id}_5\right]\left[\mathbf{j}_5 | \text{Id}_5\right]^T \in \text{int}(\mathcal{CP}_5)$, while $A \in \mathcal{CP}_5 \backslash \text{int}(\mathcal{CP}_5)$. It has been observed in [87, 66] that it is much more difficult to perform a nonnegative factorization of A than of A_ω when $\omega < 1$. In particular, the rate of success for (4.2.16) and (4.2.17) decreases to zero when ω to 1, that is, when A_ω becomes nearly identical to A. For this experiment, we set, as suggested in [39, Theorem 4.1], $r := 11$ for $\omega := 1$ and $r := 12$ otherwise. We present in Table 4.4.7 and in Table 4.4.8 the numerical performances of the algorithms applied to the nonnegative factorization of the matrices $A_{0.99}$ and $A_{1.00} = A$, respectively. One can see that both (4.2.16) and (4.2.17) practically fail to factorize the two matrices, a fact which was noticed in [87, 66]. In what concerns the inertial methods `IPG-const`, `IPG-sFISTA` and `IPG-mod`, they also seem to face some difficulties in solving these matrices, as the rate of success is not for every initial matrix equal to 1. On the other hand, the methods `RIPG-sFISTA` and `RIPG-mod` combining inertial and relaxation parameters always return nonnegative factorizations for α_+ taken equal to $\widehat{\alpha}_3$ and equal to 1. This emphasizes the importance of the interplay between the inertial and relaxation parameters, as mentioned in Example 4.4.5, and provides a strong motivation for the approach proposed in this chapter.

Method	Rate	Time (s)	Time (f)	Iter.
Algorithm (4.2.16)	0.00	−//−	5.0659×10^{-1}	−//−
Algorithm (4.2.17)	0.00	−//−	9.1030×10^{-1}	−//−
PG	0.01	1.7454×10^{-2}	2.7524×10^{-2}	7531.00
FISTA	1.00	$\mathbf{3.1237 \times 10^{-3}}$	−//−	**1067.09**
IPG-const: $\alpha_+ = 0.9814$	0.99	1.1232×10^{-2}	2.9201×10^{-2}	3785.31
IPG-sFISTA: $\alpha_+ = 0.9814$	0.95	1.2694×10^{-2}	3.3234×10^{-2}	4052.98
IPG-mod: $\alpha_+ = 0.9814$	0.95	1.2337×10^{-2}	3.0064×10^{-2}	4041.04
RIPG-const: $(\alpha, \rho) = (0.9954, 0.9705)$	0.76	1.7583×10^{-2}	2.9249×10^{-2}	5882.72
RIPG-const: $(\alpha, \rho) = (1.0000, 0.9661)$	0.76	1.7549×10^{-2}	2.9381×10^{-2}	5908.16
RIPG-sFISTA: $(\alpha, \rho) = (0.9954, 0.9705)$	1.00	6.0671×10^{-3}	−//−	1835.64
RIPG-sFISTA: $(\alpha, \rho) = (1.0000, 0.9661)$	1.00	$\mathbf{3.6109 \times 10^{-3}}$	−//−	**1083.75**
RIPG-mod: $(\alpha, \rho) = (0.9954, 0.9705)$	1.00	6.0041×10^{-3}	−//−	1850.06
RIPG-mod: $(\alpha, \rho) = (1.0000, 0.9661)$	1.00	$\mathbf{3.6073 \times 10^{-3}}$	−//−	**1084.20**

Table 4.4.8: The nonnegative factortization of $A_1 = A$ given by (4.4.7) - (4.4.8) for $r = 11$.

Numerical experiment 4.4.3. Let Id_n and $\mathbf{J}_n$ denote the identity matrix and the all-ones-matrix in $\mathbb{R}^{n \times n}$, respectively, and define

$$A_{2n} := \begin{pmatrix} n\mathrm{Id}_n & \mathbf{J}_n \\ \mathbf{J}_n & n\mathrm{Id}_n \end{pmatrix}. \tag{4.4.9}$$

This family of matrices, that lie on the boundary of $\mathcal{CP}_{2n}$, has been also considered in [87]. The authors report that the algorithms they propose fail to factorize matrices in this family, which is also the case with (4.2.17), as we have seen in our experiments. We exemplify this in Table 4.4.9 for $n = 15$ and $r = 30$. On the other hand, the methods RIPG-sFISTA and RIPG-mod combining inertial and relaxation parameters provide a factorization in reasonable time, as it is also the case for $n = 50$ and $r = 100$ on which we report in Table 4.4.10. It is also interesting to notice that, for this family of matrices, FISTA outperforms all the other methods, despite of the fact that the parameter choice for this method does not fail into the setting for which convergence was proved.

Method	Rate	Time (s)	Time (f)	Iter.
Algorithm (4.2.16)	0.00	−//−	3.4746×10^{2}	−//−
Algorithm (4.2.17)	0.00	−//−	5.8390×10^{2}	−//−
PG	0.00	−//−	1.3049×10^{0}	−//−
FISTA	1.00	$\mathbf{9.9557 \times 10^{-1}}$	−//−	**6959.95**
IPG-const: $\alpha_+ = 0.9861$	0.00	−//−	1.5734×10^{0}	−//−
IPG-sFISTA: $\alpha_+ = 0.9861$	0.00	−//−	1.5584×10^{0}	−//−
IPG-mod: $\alpha_+ = 0.9861$	0.00	−//−	1.5747×10^{0}	−//−
RIPG-const: $(\alpha, \rho) = (0.9965, 0.9730)$	0.00	−//−	1.6052×10^{0}	−//−
RIPG-const: $(\alpha, \rho) = (1.0000, 0.9697)$	0.00	−//−	1.6032×10^{0}	−//−
RIPG-sFISTA: $(\alpha, \rho) = (0.9965, 0.9730)$	1.00	1.4735×10^{0}	−//−	7719.29
RIPG-sFISTA: $(\alpha, \rho) = (1.0000, 0.9697)$	1.00	1.4564×10^{0}	−//−	7037.52
RIPG-mod: $(\alpha, \rho) = (0.9965, 0.9730)$	1.00	1.4998×10^{0}	−//−	7728.84
RIPG-mod: $(\alpha, \rho) = (1.0000, 0.9697)$	1.00	1.4641×10^{0}	−//−	7036.06

Table 4.4.9: The nonnegative factorization of A_{30} given by (4.4.9) for $r = 30$.

Method	Rate	Time (s)	Time (f)	Iter.
FISTA	1.00	$\mathbf{1.9818 \times 10^2}$	$-//-$	**22246.50**
RIPG-sFISTA: $(\alpha, \rho) = (0.9998, 0.9796)$	1.00	2.3743×10^2	$-//-$	23125.20
RIPG-sFISTA: $(\alpha, \rho) = (1.0000, 0.9794)$	1.00	2.3330×10^2	$-//-$	22467.40
RIPG-mod: $(\alpha, \rho) = (0.9998, 0.9794)$	1.00	2.3752×10^2	$-//-$	23130.90
RIPG-mod: $(\alpha, \rho) = (1.0000, 0.9794)$	1.00	2.3290×10^2	$-//-$	22463.90

Table 4.4.10: The nonnegative factorization of A_{100} given by (4.4.9) for $r = 100$.

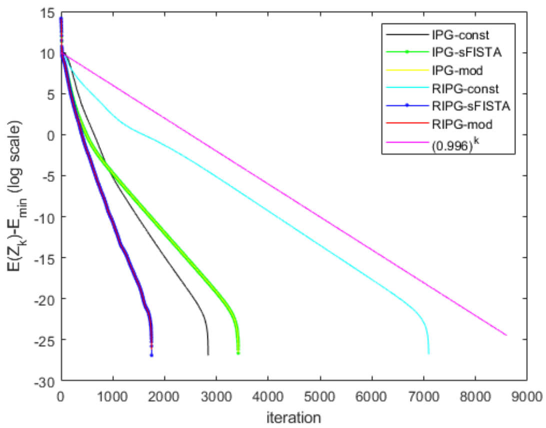

Figure 4.4.1: The sequence $\mathcal{E}(Z_k) - \mathcal{E}_{\min}$ for a particular instance of A in case $n = 40$ and $r = 61$.

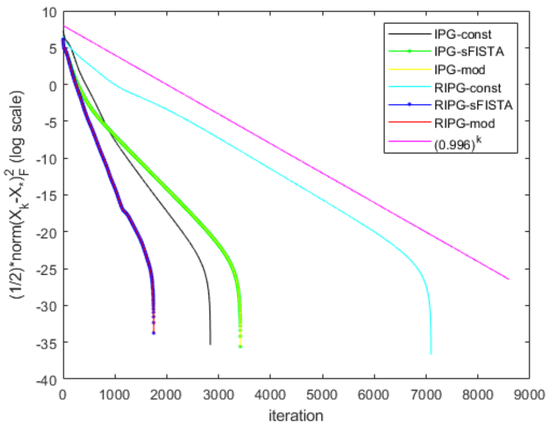

Figure 4.4.2: The sequence $\frac{1}{2} \|X_k - X_{\mathrm{sol}}\|_{\mathcal{F}}^2$ for a particular instance of A in case $n = 40$ and $r = 61$.

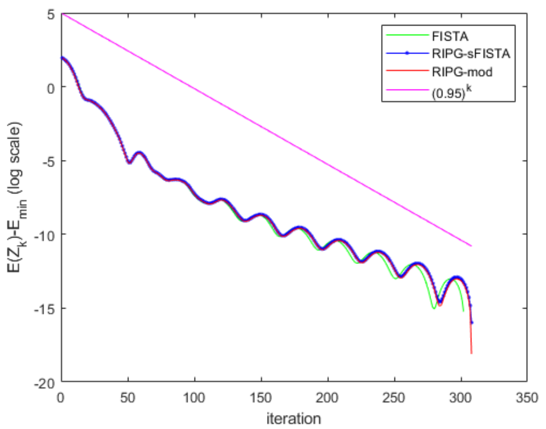

Figure 4.4.3: The sequence $\mathcal{E}(Z_k) - \mathcal{E}_{\min}$ for the factorization of $A_{0.99}$ given by (4.4.7) - (4.4.8).

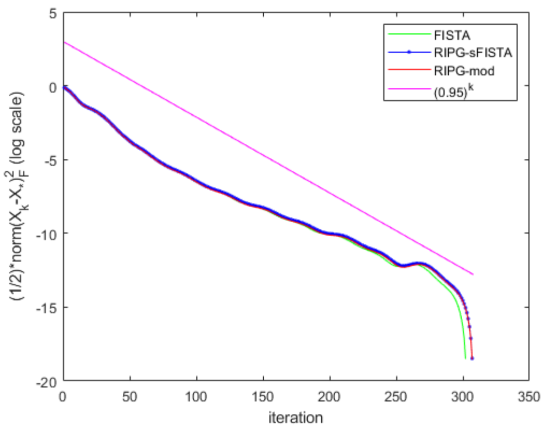

Figure 4.4.4: The sequence $\frac{1}{2} \|X_k - X_{\mathrm{sol}}\|_{\mathcal{F}}^2$ for the factorization of $A_{0.99}$ given by (4.4.7) - (4.4.8).

4.5 Further perspectives

Numerical evidence in all three experiments (see Figures 4.4.1 - 4.4.6) suggests that the convergence rates of our model are linear. This suggests that the Łojasiewicz exponent of the function Ψ_τ is at most 1/2. Even though the Łojasiewicz exponent has played an important role in the derivation of many convergence rates results, too little is known about the calculation of its exact values for functions with complex structure. Some calculus rules for the Łojasiewicz exponent have been provided in [96] and in [102] for some simple models, however, it is not yet clear how to calculate it for Ψ_τ. This is an interesting topic of future research.

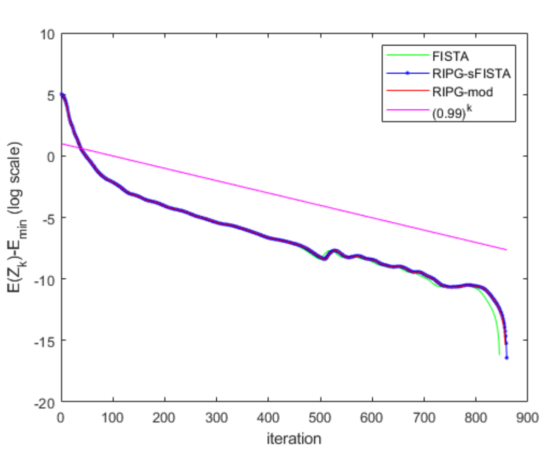

Figure 4.4.5: The sequence $\mathcal{E}(Z_k) - \mathcal{E}_{\min}$ for the factorization of $A_1 = A$ given by (4.4.7) - (4.4.8).

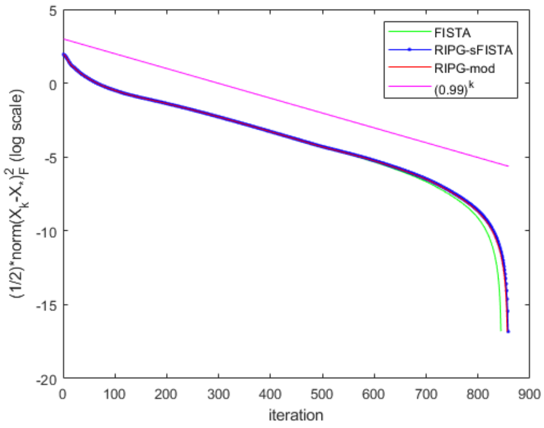

Figure 4.4.6: The sequence $\frac{1}{2}\|X_k - X_{\mathrm{sol}}\|_{\mathcal{F}}^2$ for the factorization of $A_1 = A$ given by (4.4.7) - (4.4.8).

The empirical evidence on the benefit of using linesearch techniques gives rise to the interesting question of studying the theoretical convergence guarantees of the iterates generated by Algorithm 4.3.1 enhanced with such a procedure. Another topic of further research is related to the extension of the convergence analysis beyond the current setting, in order to cover the parameter choice of the `FISTA` method, which, for the numerical experiments 2 and 3, proves to have excellent numerical performances.

Last but not least, one can replace in the formulation of the optimization problem (4.3.1) the closed ball with radius $\sqrt{\mathrm{trace}\,(A)}$ by the sphere of the same radius, formulate a projected gradient algorithm with relaxation and inertial parameters (by using the formula of the projection on the intersection of a cone and a sphere from [25]), determine a parameter setting for which convergence can be guaranteed and convergence rates can be derived (in the spirit of the analysis for inertial proximal gradient algorithms in the fully nonconvex setting from [51]), and, of course, investigate its numerical performances.

Chapter 5

The proximal alternating direction method of multipliers in the nonconvex setting

This chapter follows our work [56].

We propose two numerical algorithms for minimizing the sum of a smooth function and the composition of a nonsmooth function with a linear operator in the fully nonconvex setting. The iterative schemes are formulated in the spirit of the proximal and, respectively, proximal linearized alternating direction method of multipliers. The proximal terms are introduced through variable metrics, which facilitates the derivation of proximal splitting algorithms for nonconvex complexly structured optimization problems as particular instances of the general schemes. Convergence of the iterates to a KKT point of the objective function is proved under mild conditions on the sequence of variable metrics and by assuming that a regularization of the associated augmented Lagrangian has the Kurdyka-Łojasiewicz property. If the augmented Lagrangian has the Łojasiewicz property, then convergence rates of both augmented Lagrangian and iterates are derived.

5.1 Introduction

5.1.1 Problem formulation and motivation

Let $\mathcal{H}$ and $\mathcal{G}$ be real finite-dimensional Hilbert spaces. In this chapter we deal with the solving of optimization problems of the form

$$\min_{x \in \mathcal{H}} \left\{ g\left(Ax\right) + h\left(x\right) \right\}, \tag{5.1.1}$$

where $g \colon \mathcal{G} \to \mathbb{R} \cup \{+\infty\}$ is a proper and lower semicontinuous function, $h \colon \mathcal{H} \to \mathbb{R}$ is a Fréchet differentiable function with L-Lipschitz continuous gradient and $A \colon \mathcal{H} \to \mathcal{G}$ is a linear operator. The spaces $\mathcal{H}$ and $\mathcal{G}$ are equipped with Euclidean inner products $\langle \cdot, \cdot \rangle$ and associated norms $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$, which are both denoted in the same way, as there is no risk of confusion.

We propose a proximal ADMM (P-ADMM) algorithm and a proximal linearized ADMM (PL-ADMM) algorithm for solving the optimization problem (5.1.1) and carry out a parallel convergence analysis for both algorithms. We first prove, under not very restrictive assumptions on the problem data, that the sequence of generated iterates is bounded. Given these premises we show that the cluster points of the sequence are *KKT points* of the problem (5.1.1). Provided that a regularization of the augmented Lagrangian satisfies the Kurdyka-Łojasiewicz property, we show global convergence of the generated sequence of iterates. When this regularization of the augmented Lagrangian has the Łojasiewicz property, we derive rates of convergence for the sequence of iterates. To the best of our knowledge, these are the first results in the literature that deal with convergence rates for the nonconvex ADMM.

We prove under quite general assumptions that the sequence $\{(x_k, z_k, y_k)\}_{k \geqslant 0}$ is bounded. In the nonconvex setting, the boundedness of the sequence of generated iterates plays a central role in the convergence analysis. In fact, the reason, why we assume that the function g is smooth, is exclusively given by the fact that only in this setting we can prove boundedness for this sequence under general assumptions.

We also prove convergence for relaxed variants of the nonconvex ADMM algorithms, which allow to chose in the update of the dual sequence $\sigma \in (0, 2)$. We notice that $\sigma = 1$ is the standard choice in the literature ([4, 23, 44, 96, 120, 126]). Gabay and Mercier proved in [85] in the convex setting that σ may be chosen in $(0, 2)$, however, the majority of the extensions of the convex relaxed ADMM algorithm assume that $\sigma \in \left(0, \dfrac{1+\sqrt{5}}{2}\right)$ (see [72, 79, 84, 121, 127, 128]) or ask for a particular choice of σ, which is interpreted as a step size (see [90]). In [128], an alternating minimization algorithm for the minimization of the sum of a simple nonsmooth function and a smooth function in the nonconvex setting, which allows for a parameter σ different from 1, has been proposed.

By appropriate choices of the matrix sequences, we derive from the proposed iterative schemes full splitting algorithms for solving the nonconvex complexly structured optimization problem (5.1.1). More precisely, (P-ADMM) gives rise to an iterative scheme formulated only in terms of proximal steps for the functions g and h and of forward evaluations of the matrix A, while (PL-ADMM) gives rise to an iterative scheme in which the function h is performed via a gradient step. The fruitful idea to linearize the step involving the smooth term has been used in the past in the context of ADMM algorithms mostly in the convex setting (see [99, 113, 117, 127, 129]), but also in the nonconvex setting (see [101]).

5.1.2 Notations

Let p be a positive integer. Every space $\mathcal{H}_i$ for $i = 1, \cdots, p$ is assumed to be equipped with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. The Cartesian product $\mathcal{H}_1 \times \mathcal{H}_2 \times \ldots \times \mathcal{H}_p$ of the Euclidean spaces $\mathcal{H}_i, i = 1, \cdots, p$, will be endowed with inner product and associated norm defined for $x := (x_1, \ldots, x_p), y := (y_1, \ldots, y_p) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \ldots \times \mathcal{H}_p$ by

$$\langle\langle x, y \rangle\rangle = \sum_{i=1}^{p} \langle x_i, y_i \rangle \quad \text{and} \quad |||x||| = \sqrt{\sum_{i=1}^{p} \|x_i\|^2},$$

respectively. For every $x := (x_1, \ldots, x_p) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \ldots \times \mathcal{H}_p$ we have

$$\frac{1}{\sqrt{p}} \sum_{i=1}^{p} \|x_i\| \leqslant |||x||| = \sqrt{\sum_{i=1}^{p} \|x_i\|^2} \leqslant \sum_{i=1}^{p} \|x_i\|. \tag{5.1.2}$$

We denote by $\mathbb{S}_+(\mathcal{H})$ the family of symmetric and positive semidefinite matrices $\mathcal{M} \in \mathcal{H}$. Every $\mathcal{M} \in \mathbb{S}_+(\mathcal{H})$ induces a *semi-norm* defined by

$$\|x\|_{\mathcal{M}}^2 := \langle \mathcal{M}x, x \rangle \ \forall x \in \mathcal{H}.$$

The *Loewner partial ordering* on $\mathbb{S}_+(\mathcal{H})$ is defined for $\mathcal{M}, \mathcal{M}' \in \mathbb{S}_+(\mathcal{H})$ as

$$\mathcal{M} \geqslant \mathcal{M}' \Leftrightarrow \|x\|_M^2 \geqslant \|x\|_{\mathcal{M}'}^2 \ \forall x \in \mathcal{H}.$$

Thus $\mathcal{M} \in \mathbb{S}_+(\mathcal{H})$ is nothing else than $\mathcal{M} \geqslant 0$. For $\rho > 0$ we set

$$\mathcal{P}_\rho(\mathcal{H}) := \{\mathcal{M} \in \mathbb{S}_+(\mathcal{H}) : \mathcal{M} \geqslant \rho \mathrm{Id}\},$$

where Id denotes as usual the identity operator in $\mathcal{H}$. If $\mathcal{M} \in \mathcal{P}_\rho(\mathcal{H})$, then the semi-norm $\|\cdot\|_{\mathcal{M}}$ becomes a norm.

The linear operator A is *surjective* if and only if its associated matrix has full row rank. This assumption is further equivalent to the fact that the matrix associated to AA^*, where A^* denotes the *adjoint operator* of A, is positively definite. Since

$$\lambda_{\min}(AA^*)\|y\|^2 \leqslant \|y\|_{AA^*}^2 = \langle AA^*y, y\rangle = \|A^*y\|^2 \ \ \forall y \in \mathcal{G},$$

this is further equivalent to $\lambda_{\min}(AA^*) > 0$ (and $AA^* \in \mathcal{P}_{\lambda_{\min}(AA^*)}(\mathcal{H})$), where $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of a matrix. Similarly, A is injective if and only if $\lambda_{\min}(A^*A) > 0$ (and $A^*A \in \mathcal{P}_{\lambda_{\min}(A^*A)}(\mathcal{G})$).

5.2 Related works

We start by briefly describing the Alternating Direction Method of Multipliers (ADMM) designed to solve optimization problems of the form

$$\min_{x\in\mathcal{H}} \{f(x) + g(Ax) + h(x)\}, \tag{5.2.1}$$

where g and h are assumed to be also *convex* and $f\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is another proper, convex and lower semicontinuous function. By introducing an auxiliary variable, one can rewrite problem (5.2.1) as

$$\min_{\substack{(x,z)\in\mathcal{H}\times\mathcal{G}\\ Ax-z=0}} \{f(x) + g(z) + h(x)\}. \tag{5.2.2}$$

The *Lagrangian* associated with problem (5.2.2) is

$$\mathcal{L}\colon \mathcal{H}\times\mathcal{G}\times\mathcal{G} \to \mathbb{R}\cup\{+\infty\}, \ \mathcal{L}(x,z,y) = f(x) + g(z) + h(x) + \langle y, Ax - z\rangle,$$

and we say that $(\tilde{x}, \tilde{z}, \tilde{y})$ is a *saddle point* of $\mathcal{L}$ if

$$\mathcal{L}(\tilde{x}, \tilde{z}, y) \leqslant \mathcal{L}(\tilde{x}, \tilde{z}, \tilde{y}) \leqslant \mathcal{L}(x, z, \tilde{y}) \qquad \forall (x,z,y) \in \mathcal{H}\times\mathcal{G}\times\mathcal{G}.$$

It is known that $(\tilde{x}, \tilde{z}, \tilde{y})$ is a saddle point of $\mathcal{L}$ if and only if $\tilde{z} = A\tilde{x}$ and $(\tilde{x}, \tilde{z})$ is an optimal solution of (5.2.2), $\tilde{y}$ is an optimal solution of the Fenchel-Rockafellar dual problem (see [24, 41, 130]) to (5.1.1), namely

$$\max_{y\in\mathcal{G}} \{-(f+h)^*(-A^*y) - g(y)\}. \tag{5.2.3}$$

and the optimal objective values of (5.1.1) and (5.2.3) coincide.

For a fixed real number $\beta > 0$, the *augmented Lagrangian* associated with problem (5.2.2) reads

$$\mathcal{L}_\beta\colon \mathcal{H}\times\mathcal{G}\times\mathcal{G} \to \mathbb{R}\cup\{+\infty\}, \ \mathcal{L}_\beta(x,z,y) = f(x) + g(z) + h(x) + \langle y, Ax - z\rangle + \frac{\beta}{2}\|Ax - z\|^2.$$

Given a starting vector $(x_0, z_0, y_0) \in \mathcal{H}\times\mathcal{G}\times\mathcal{G}$ and $\{\mathcal{M}_1^k\}_{k\geqslant 0} \subseteq \mathbb{S}_+(\mathcal{H}), \{\mathcal{M}_2^k\}_{k\geqslant 0} \subseteq \mathbb{S}_+(\mathcal{G})$, two sequences of symmetric and positive semidefinite matrices, the following proximal ADMM algorithm formulated in the presence of a smooth function and involving variable metrics has been proposed and investigated in [23]: generate the sequence $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ for every $k \geqslant 0$ as

$$x_{k+1} \in \arg\min_{x\in\mathcal{H}} \left\{ f(x) + \langle x - x_k, \nabla h(x_k)\rangle + \frac{\beta}{2}\left\|Ax - z_k + \frac{1}{\beta}y_k\right\|^2 + \frac{1}{2}\|x - x_k\|_{\mathcal{M}_1^k}^2 \right\}, \tag{5.2.4a}$$

$$z_{k+1} := \arg\min_{z\in\mathcal{G}} \left\{ g(z) + \frac{\beta}{2}\left\|Ax_{k+1} - z + \frac{1}{\beta}y_k\right\|^2 + \frac{1}{2}\|z - z_k\|_{\mathcal{M}_2^k}^2 \right\}, \tag{5.2.4b}$$

$$y_{k+1} := y_k + \sigma\beta(Ax_{k+1} - z_{k+1}). \tag{5.2.4c}$$

It has been proved in [23] that, if $\sigma = 1$ and the set of the saddle points of the Lagrangian associated with (5.2.2) (which is nothing else than $\mathcal{L}_\beta$ when $\beta = 0$) is nonempty, and the two matrix sequences and the operator A fulfill mild additional assumptions, then the sequence $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ converges to a saddle point of the Lagrangian associated with problem (5.2.2) and provides in this way both an optimal solution of (5.1.1) and an optimal solution of its Fenchel dual problem. Furthermore, an ergodic primal-dual gap convergence rate result has been proved.

In case $h = 0$, the above iterative scheme encompasses as special cases different numerical algorithms considered in the literature. If $\mathcal{M}_1^k = \mathcal{M}_2^k = 0$ for all $k \geqslant 0$, then (5.2.4a)-(5.2.4c) becomes the *classical ADMM algorithm* ([60, 81, 84, 85]), which lately gained a huge popularity in the optimization community, despite its poor implementation properties caused by the fact that, in general, the calculation of the sequence of primal variables $\{x_k\}_{k\geqslant 0}$ does not correspond to a proximal step. For an *inertial version* of the classical ADMM algorithm we refer the reader to [42]. On the other hand, if $\mathcal{M}_1^k = \mathcal{M}_1$ and $\mathcal{M}_2^k = \mathcal{M}_2$ for all $k \geqslant 0$, then (5.2.4a)-(5.2.4c) recovers the *proximal ADMM algorithm* investigated by Shefi and Teboulle in [120] (see also [72, 79]). It has been pointed out in [120] that, for suitable choices of the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$, the proximal ADMM algorithm becomes a primal-dual splitting algorithm in the sense of those considered in [49, 65, 70, 124], and which, due to its full splitting character, overcomes the drawbacks of the classical ADMM algorithm. Recently, in [44] it has been shown that, if f is strongly convex, then suitable choices of the non-constant sequences $\left\{\mathcal{M}_1^k\right\}_{k\geqslant 0}$ and $\left\{\mathcal{M}_2^k\right\}_{k\geqslant 0}$ lead to a rate of convergence of $\mathcal{O}\left(1/k\right)$ for the sequence of primal iterates.

In the following we will comment on previous works addressing the ADMM algorithm in the nonconvex setting. None of the papers which have addressed nonconvex optimization problems involving compositions with linear operators propose and investigate iterative schemes designed in the spirit of full splitting algorithms. In [96], the convergence of the ADMM algorithm for solving the problem (5.1.1) is studied under the assumption that h is twice continuously differentiable with bounded Hessian. In [91], the ADMM algorithm is used to minimize the sum of finitely many smooth nonconvex functions and a nonsmooth convex function, by rewriting it as an general consensus problem. No linear operator occurs in the formulation of the optimization problem under investigation. In [4], the ADMM algorithm is used to solve a DC optimization problem over the unit ball which occurs in the penalized zero-variance linear discriminant analysis. In [125], a nonconvex ADMM algorithm involving proximal terms induced via Bregman distances is introduced and investigated, however, without addressing the question of the boundedness of the generated iterates. On the other hand, in [88], in order to guarantee boundedness of the iterates a strong assumption on g is made, which is proved to hold for the normed-squared function. In [126], a lot of efforts are made to guarantee boundedness for the generated iterates of the nonconvex ADMM algorithm, which is an essential component of the convergence analysis, however, this is done by assuming that the objective function is continuous and *coercive over the feasible set*, while its nonsmooth part is either *restricted prox-regular* or *piecewise linear*. Similar ingredients are used in [101] in the convergence analysis of a nonconvex linearized ADMM algorithm.

Recently, Bolte, Sabach and Teboulle have proposed in [37] a generic iterative scheme for solving a general optimization problem of the form (5.1.1), but by replacing the linear operator A with a general continuously differentiable operator. A global convergence analysis relying on the use of the Kurdyka-Łojasiewicz property is carried out. It is also shown that the generic iterative scheme encompasses several Lagrangian based algorithms, including the proximal alternating direction method of multipliers and the proximal alternating linearized minimization method. The latter is analysed into detail in the particular case when g is composed with a linear operator, which coincides with the one in this chapter. The two algorithms we propose are formulated in the same spirit, however, they lead for some particular choices of the variable metrics to full splitting algorithms. In addition, we carefully address the issue of the boundedness of the sequence of generated iterates and complement the convergence analysis with the derivation of convergence rates.

5.3 A proximal ADMM algorithm and a proximal linearized ADMM algorithm in the nonconvex setting

In this section we propose two proximal ADMM algorithms for solving the optimization problem (5.1.1) and study their convergence behaviour. A central role will be played by the augmented Lagrangian associated with problem (5.1.1), which is defined for every $\beta > 0$ as

$$\mathcal{L}_\beta \colon \mathcal{H} \times \mathcal{G} \times \mathcal{G} \to \mathbb{R} \cup \{+\infty\}, \ \mathcal{L}_\beta (x, z, y) = g(z) + h(x) + \langle y, Ax - z \rangle + \frac{\beta}{2} \| Ax - z \|^2 .$$

5.3.1 General formulations and full proximal splitting algorithms as particular instances

Algorithm 5.3.1. *Let be the matrix sequences* $\{\mathcal{M}_1^k\}_{k \geqslant 0} \in \mathbb{S}_+(\mathcal{H})$, $\{\mathcal{M}_2^k\}_{k \geqslant 0} \in \mathbb{S}_+(\mathcal{G})$, $\beta > 0$ *and* $0 < \sigma < 2$. *For a given starting vector* $(x_0, z_0, y_0) \in \mathcal{H} \times \mathcal{G} \times \mathcal{G}$, *generate the sequence* $\{(x_k, z_k, y_k)\}_{k \geqslant 0}$ *for every* $k \geqslant 0$ *as:*

$$\begin{aligned} z_{k+1} &\in \arg\min_{z \in \mathcal{G}} \left\{ \mathcal{L}_\beta (x_k, z, y_k) + \frac{1}{2} \| z - z_k \|^2_{\mathcal{M}_2^k} \right\} \\ &= \arg\min_{z \in \mathcal{G}} \left\{ g(z) + \langle y_k, Ax_k - z \rangle + \frac{\beta}{2} \| Ax_k - z \|^2 + \frac{1}{2} \| z - z_k \|^2_{\mathcal{M}_2^k} \right\}, \end{aligned} \tag{5.3.1a}$$

$$\begin{aligned} x_{k+1} &\in \arg\min_{x \in \mathcal{H}} \left\{ \mathcal{L}_\beta (x, z_{k+1}, y_k) + \frac{1}{2} \| x - x_k \|^2_{\mathcal{M}_1^k} \right\} \\ &= \arg\min_{x \in \mathcal{H}} \left\{ h(x) + \langle y_k, Ax - z_{k+1} \rangle + \frac{\beta}{2} \| Ax - z_{k+1} \|^2 + \frac{1}{2} \| x - x_k \|^2_{\mathcal{M}_1^k} \right\}, \end{aligned} \tag{5.3.1b}$$

$$y_{k+1} := y_k + \sigma\beta (Ax_{k+1} - z_{k+1}) . \tag{5.3.1c}$$

The above particular instance of Algorithm 5.3.1 is an iterative scheme formulated in the spirit of full splitting numerical methods; in other words, the functions g and h are evaluated by their proximal operators, while the linear operator A and its adjoint operator are evaluated by simple forward steps. Exact formulas for the proximal operator are available not only for large classes of convex functions ([27, 69]), but also for many nonconvex functions occurring in applications ([7, 89, 95]).

Let $\{t_k\}_{k \geqslant 0}$ be a sequence of positive real numbers such that $t_k \geqslant \beta \|A\|^2$, and $\mathcal{M}_1^k := t_k \mathrm{Id} - \beta A^* A$ and $\mathcal{M}_2^k := 0$ for every $k \geqslant 0$. In this particular case Algorithm 5.3.1 becomes an iterative scheme which generates a sequence $\{(x_k, z_k, y_k)\}_{k \geqslant 0}$ for every $k \geqslant 0$ as:

$$\begin{aligned} z_{k+1} &\in \arg\min_{z \in \mathcal{G}} \left\{ g(z) + \frac{\beta}{2} \left\| z - Ax_k - \frac{1}{\beta} y_k \right\|^2 \right\}, \\ x_{k+1} &\in \arg\min_{x \in \mathcal{H}} \left\{ h(x) + \frac{t_k}{2} \left\| x - x_k + \frac{1}{t_k} A^* \left[y_k + r (Ax_k - z_{k+1}) \right] \right\|^2 \right\}, \\ y_{k+1} &:= y_k + \sigma\beta (Ax_{k+1} - z_{k+1}) . \end{aligned}$$

The second algorithm that we propose replaces for every $k \geqslant 0$ the function h in the definition of x_{k+1} by its linearization at x_k.

Algorithm 5.3.2. *Let be the matrix sequences* $\{\mathcal{M}_1^k\}_{k\geqslant 0} \in \mathbb{S}_+(\mathcal{H})$, $\{\mathcal{M}_2^k\}_{k\geqslant 0} \in \mathbb{S}_+(\mathcal{G})$, $\beta > 0$ *and* $0 < \sigma < 2$. *For a given starting vector* $(x_0, z_0, y_0) \in \mathcal{H} \times \mathcal{G} \times \mathcal{G}$, *generate the sequence* $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ *for every* $k \geqslant 0$ *as:*

$$z_{k+1} \in \arg\min_{z\in\mathcal{G}} \left\{ g(z) + \langle y_k, Ax_k - z\rangle + \frac{\beta}{2}\|Ax_k - z\|^2 + \frac{1}{2}\|z - z_k\|^2_{\mathcal{M}_2^k} \right\}, \tag{5.3.2a}$$

$$x_{k+1} \in \arg\min_{x\in\mathcal{H}} \left\{ \langle x - x_k, \nabla h(x_k)\rangle + \langle y_k, Ax - z_{k+1}\rangle + \frac{\beta}{2}\|Ax - z_{k+1}\|^2 + \frac{1}{2}\|x - x_k\|^2_{\mathcal{M}_1^k} \right\}, \tag{5.3.2b}$$

$$y_{k+1} := y_k + \sigma\beta(Ax_{k+1} - z_{k+1}). \tag{5.3.2c}$$

Due to the presence of the variable metric inducing matrix sequences, Algorithm 5.3.2 represents a unifying scheme for several linearized ADMM algorithms from the literature (see [99, 101, 113, 117, 127, 129]). By choosing as above $\mathcal{M}_1^k := t_k\mathrm{Id} - \beta A^*A$, where t_k is positive such that $t_k \geqslant \beta\|A\|^2$, and $\mathcal{M}_2^k := 0$, for every $k \geqslant 0$, Algorithm 5.3.2 translates for every $k \geqslant 0$ into:

$$\begin{aligned}
z_{k+1} &\in \arg\min_{z\in\mathcal{G}} \left\{ g(z) + \frac{\beta}{2}\left\| z - Ax_k - \frac{1}{\beta}y_k \right\|^2 \right\},\\
x_{k+1} &:= x_k - \frac{1}{t_k}\left(\nabla h(x_k) + A^*\left[y_k + r(Ax_k - z_{k+1})\right]\right),\\
y_{k+1} &:= y_k + \sigma\beta(Ax_{k+1} - z_{k+1}).
\end{aligned}$$

In this iterative scheme the smooth term is evaluated via a gradient step, which is an improvement with respect to other nonconvex ADMM algorithms, such as [126, 128], where the smooth function is involved in a subproblem, which may be difficult to solve, unless it can be reformulated as a proximal step (see [96]).

We will carry out a parallel convergence analysis for Algorithm 5.3.1 and Algorithm 5.3.2 in the following setting.

Assumption 5.3.1. *We assume that*

(i) g *and* h *are bounded from below;*

(ii) A *is surjective and thus the constant*

$$T_0 := \begin{cases} \dfrac{1}{\sigma\lambda_{\min}(AA^*)}, & \text{if } 0 < \sigma \leqslant 1,\\[2ex] \dfrac{\sigma}{(2-\sigma)^2\lambda_{\min}(AA^*)}, & \text{if } 1 < \sigma < 2, \end{cases}$$

is well-defined;

(iii) $\mu_1 := \sup_{k\geqslant 0}\|\mathcal{M}_1^k\| < +\infty$ *and* $\mu_2 := \sup_{k\geqslant 0}\|\mathcal{M}_2^k\| < +\infty$*;*

(iv) $\beta > 0, \sigma \in (0,2)$ *and* $\mu_1 \geqslant 0$ *are such that*

$$\beta \geqslant 4T_0L > 0 \tag{5.3.3}$$

and

$$2\mathcal{M}_1^k + \beta A^*A \succcurlyeq \left(L + \frac{C_{\mathcal{M}}}{\beta}\right)\mathrm{Id} \quad \forall k \geqslant 0, \tag{5.3.4}$$

where

$$C_{\mathcal{M}} := \begin{cases} \left(6\mu_1^2 + 4(L+\mu_1)^2\right)T_0, & \text{for Algorithm 5.3.1},\\ \left(4\mu_1^2 + 6(L+\mu_1)^2\right)T_0, & \text{for Algorithm 5.3.2}. \end{cases}$$

Example 5.3.1. In the following we discuss possible choices of the matrix sequence $\left\{\mathcal{M}_1^k\right\}_{k\geqslant 0}$ which fulfil Assumption 5.3.1:

(i) If $\sup_{k\geqslant 0}\left\|\mathcal{M}_1^k\right\| = \mu_1 > \frac{L}{2}$, then, for every

$$\beta \geqslant \max\left\{4T_0L, \frac{C_{\mathcal{M}}}{2\mu_1 - L}\right\} > 0,$$

there exists $\rho_1 > 0$ such that

$$\mu_1 \geqslant \rho_1 \geqslant \frac{1}{2}\left(L + \frac{C_{\mathcal{M}}}{\beta}\right) > 0.$$

The inequality in (5.3.4) is ensured for $\mathcal{M}_1^k$ chosen such that

$$\mu_1 \mathrm{Id} \succcurlyeq \mathcal{M}_1^k \succcurlyeq \rho_1 \mathrm{Id} \quad \forall k \geqslant 0.$$

(ii) If A is assumed to be also injective, then $\lambda_{\min}(A^*A) > 0$. By choosing

$$\beta \geqslant \max\left\{4T_0L, \frac{L + \sqrt{L^2 + 4\lambda_{\min}(A^*A)\, C_{\mathcal{M}}}}{2\lambda_{\min}(A^*A)}\right\} > 0,$$

it follows that $\beta^2\lambda_{\min}(A^*A) - L\beta - C_{\mathcal{M}} \geqslant 0$. Thus,

$$\beta A^*A - \left(L + \beta^{-1}C_{\mathcal{M}}\right)\mathrm{Id} \succcurlyeq 0,$$

and (5.3.4) holds for an arbitrary sequence of symmetric and positive semidefinite matrices $\left\{\mathcal{M}_1^k\right\}_{k\geqslant 0}$. A possible choice is $\mathcal{M}_1^k = 0$, which, together $\mathcal{M}_2^k = 0$, for every $k \geqslant 0$, allows us to recover the classical ADMM algorithm and the linearized ADMM algorithm as particular instances of our iterative schemes.

(iii) For $t > 0$, we take $\mathcal{M}_1^k := t\mathrm{Id} - \beta A^*A$ for every $k \geqslant 0$. Then

$$\mu_1 = \|t\mathrm{Id} - \beta A^*A\| = \lambda_{\max}(t\mathrm{Id} - \beta A^*A) = t - \beta\lambda_{\min}(A^*A).$$

Condition (5.3.4) is equivalent to

$$2t - \beta\|A\|^2 - \left(L + \frac{C_{\mathcal{M}}}{\beta}\right) \geqslant 0$$

and is guaranteed for both algorithms when

$$2t - \beta\|A\|^2 - \left(L + \frac{\left(4\mu_1^2 + 6(L+\mu_1)^2\right)T_0}{\beta}\right) \geqslant 0$$

or, equivalently,

$$10T_0\mu_1^2 - 2(\beta - 6T_0L)\mu_1 + 6T_0L^2 + \beta^2\left(\|A\|^2 - 2\lambda_{\min}(A^*A)\right) - L\beta \leqslant 0.$$

This quadratic inequality in $\mu_1 \geqslant 0$ has nonnegative solutions if, for instance, $\beta \geqslant 6T_0L$ (thus (5.3.3) holds) and the reduced discriminant

$$\begin{aligned}\Delta &:= (\beta - 6T_0L)^2 - 60T_0^2L^2 - 10T_0\beta^2\left(\|A\|^2 - 2\lambda_{\min}(A^*A)\right) + 10T_0L\beta \\ &= \left[1 + 10T_0\left(2\lambda_{\min}(A^*A) - \|A\|^2\right)\right]\beta^2 - 2T_0L\beta - 24T_0^2L^2\end{aligned}$$

is nonnegative. This holds true if the condition number of the matrix A^*A fulfils

$$\kappa\left(A^*A\right) := \frac{\lambda_{\max}\left(A^*A\right)}{\lambda_{\min}\left(A^*A\right)} = \frac{\|A\|^2}{\lambda_{\min}\left(A^*A\right)} \leqslant 2.$$

In conclusions, if the latter is given, then we can chose an arbitrary

$$\beta \geqslant 6T_0L$$

and t such that

$$\beta\lambda_{\min}\left(A^*A\right) \leqslant t \leqslant \beta\lambda_{\min}\left(A^*A\right) + \frac{1}{10T_0}\left(\beta - 6T_0L + \sqrt{\Delta}\right).$$

Remark 5.3.1. (i) It has been noticed also by other authors (see, for instance, [37, 96]) that the surjectivity of the linear operator is an assumption which at this moment cannot be omitted when aiming to prove convergence for nonconvex Lagrangian based algorithms.

(ii) When proving convergence and deriving convergence rates for variable metric algorithms designed for convex optimization problems one usually assumes monotonicity for the matrix sequences inducing the variable metrics (see, for instance, [68, 23, 44]). It is worth to mention that the convergence analysis for both Algorithm 5.3.1 and Algorithm 5.3.2 does not require monotonicity assumptions on $\left\{\mathcal{M}_1^k\right\}_{k\geqslant 0}$ or $\left\{\mathcal{M}_2^k\right\}_{k\geqslant 0}$.

5.3.2 Preliminaries of the convergence analysis

Within the setting of Assumption 5.3.1 we will make use of the following constants:

$$C_0 := \begin{cases} L + \dfrac{4T_0\left(L+\mu_1\right)^2}{\beta}, & \text{for Algorithm 5.3.1,} \\ L + \dfrac{4T_0\mu_1^2}{\beta}, & \text{for Algorithm 5.3.2,} \end{cases}$$

$$C_1 := \begin{cases} \dfrac{4T_0\mu_1^2}{\beta}, & \text{for Algorithm 5.3.1,} \\ \dfrac{4T_0\left(L+\mu_1\right)^2}{\beta}, & \text{for Algorithm 5.3.2,} \end{cases}$$

$$T_1 := \begin{cases} \dfrac{1-\sigma}{\lambda_{\min}\left(AA^*\right)\sigma^2\beta}, & \text{if } 0 < \sigma \leqslant 1, \\ \dfrac{\sigma-1}{\lambda_{\min}\left(AA^*\right)\left(2-\sigma\right)\sigma\beta}, & \text{if } 1 < \sigma < 2, \end{cases}$$

and we will denote for every $k \geqslant 0$

$$\mathcal{M}_3^k := 2\mathcal{M}_1^k + \beta A^*A - C_0\text{Id}.$$

The following result of Fejér monotonicity type will play a fundamental role in our convergence analysis.

Lemma 5.3.2. *Let Assumption 5.3.1 be satisfied and $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2. Then for every $k \geqslant 1$ it holds:*

$$\begin{aligned} &\mathcal{L}_\beta\left(x_{k+1}, z_{k+1}, y_{k+1}\right) + T_1\left\|A^*\left(y_{k+1}-y_k\right)\right\|^2 + \frac{1}{2}\left\|x_{k+1}-x_k\right\|^2_{\mathcal{M}_3^k} + \frac{1}{2}\left\|z_{k+1}-z_k\right\|^2_{\mathcal{M}_2^k} \\ &\leqslant \mathcal{L}_\beta\left(x_k, z_k, y_k\right) + T_1\left\|A^*\left(y_k-y_{k-1}\right)\right\|^2 + \frac{C_1}{2}\left\|x_k-x_{k-1}\right\|^2. \end{aligned} \quad (5.3.5)$$

Proof. Let $k \geqslant 1$ be fixed. In both cases the proof builds on showing that the following inequality

$$\mathcal{L}_\beta(x_{k+1}, z_{k+1}, y_{k+1}) + \frac{1}{2}\|x_{k+1} - x_k\|^2_{2\mathcal{M}_1^k + \beta A^*A} - \frac{L}{2}\|x_{k+1} - x_k\|^2 + \frac{1}{2}\|z_{k+1} - z_k\|^2_{\mathcal{M}_2^k}$$
$$\leqslant \mathcal{L}_\beta(x_k, z_k, y_k) + \frac{1}{\sigma\beta}\|y_{k+1} - y_k\|^2 \tag{5.3.6}$$

is true and on providing afterwards an upper bound for $\frac{1}{\sigma\beta}\|y_{k+1} - y_k\|^2$.

(i) For *Algorithm 5.3.1*: From (5.3.1a) we have

$$g(z_{k+1}) + \langle y_k, Ax_k - z_{k+1}\rangle + \frac{\beta}{2}\|Ax_k - z_{k+1}\|^2 + \frac{1}{2}\|z_{k+1} - z_k\|^2_{\mathcal{M}_2^k}$$
$$\leqslant g(z_k) + \langle y_k, Ax_k - z_k\rangle + \frac{\beta}{2}\|Ax_k - z_k\|^2. \tag{5.3.7}$$

The optimality criterion of (5.3.1b) is

$$\nabla h(x_{k+1}) = -A^* y_k - rA^*(Ax_{k+1} - z_{k+1}) + \mathcal{M}_1^k(x_k - x_{k+1}). \tag{5.3.8}$$

From (2.2.1) (applied for $z := x_{k+1}$) we get

$$h(x_{k+1}) \leqslant h(x_k) + \langle y_k, Ax_k - Ax_{k+1}\rangle + r\langle Ax_{k+1} - z_{k+1}, Ax_k - Ax_{k+1}\rangle$$
$$- \|x_{k+1} - x_k\|^2_{\mathcal{M}_1^k} + \frac{L}{2}\|x_{k+1} - x_k\|^2. \tag{5.3.9}$$

By combining (5.3.7), (5.3.9) and (5.3.1c), after some rearrangements, we obtain (5.3.6). By using the notation

$$u_1^k := -\nabla h(x_k) + \mathcal{M}_1^{k-1}(x_{k-1} - x_k) \ \ \forall k \geqslant 1 \tag{5.3.10}$$

and by taking into consideration (5.3.1c), we can rewrite (5.3.8) as

$$A^* y_{k+1} = \sigma u_1^{k+1} + (1-\sigma) A^* y_k \ \ \forall k \geqslant 0. \tag{5.3.11}$$

- *The case* $0 < \sigma \leqslant 1$. We have

$$A^*(y_{k+1} - y_k) = \sigma\left(u_1^{k+1} - u_1^k\right) + (1-\sigma)A^*(y_k - y_{k-1}).$$

Since $0 < \sigma \leqslant 1$, the convexity of $\|\cdot\|^2$ gives

$$\|A^*(y_{k+1} - y_k)\|^2 \leqslant \sigma\left\|u_1^{k+1} - u_1^k\right\|^2 + (1-\sigma)\|A^*(y_k - y_{k-1})\|^2$$

and from here we get

$$\lambda_{\min}(AA^*)\sigma\|y_{k+1} - y_k\|^2 \leqslant \sigma\|A^*(y_{k+1} - y_k)\|^2$$
$$\leqslant \sigma\left\|u_1^{k+1} - u_1^k\right\|^2 + (1-\sigma)\|A^*(y_k - y_{k-1})\|^2 - (1-\sigma)\|A^*(y_{k+1} - y_k)\|^2. \tag{5.3.12}$$

By using the Lipschitz continuity of ∇h we have

$$\left\|u_1^{k+1} - u_1^k\right\| \leqslant (L + \mu_1)\|x_{k+1} - x_k\| + \mu_1\|x_k - x_{k-1}\|, \tag{5.3.13}$$

thus

$$\left\|u_1^{k+1} - u_1^k\right\|^2 \leqslant 2(L + \mu_1)^2\|x_{k+1} - x_k\|^2 + 2\mu_1^2\|x_k - x_{k-1}\|^2. \tag{5.3.14}$$

After plugging (5.3.14) into (5.3.12) we get

$$\frac{1}{\sigma\beta}\left\|y_{k+1}-y_k\right\|^2 \leqslant \frac{2\left(L+\mu_1\right)^2}{\lambda_{\min}\left(AA^*\right)\sigma\beta}\left\|x_{k+1}-x_k\right\|^2+\frac{2\mu_1^2}{\lambda_{\min}\left(AA^*\right)\sigma\beta}\left\|x_k-x_{k-1}\right\|^2 \\ +\frac{(1-\sigma)}{\lambda_{\min}\left(AA^*\right)\sigma^2\beta}\left(\left\|A^*\left(y_k-y_{k-1}\right)\right\|^2-\left\|A^*\left(y_{k+1}-y_k\right)\right\|^2\right), \tag{5.3.15}$$

which, combined with (5.3.6), provides (5.3.5).

- *The case* $1<\sigma<2$. This time we have from (5.3.11) that

$$A^*\left(y_{k+1}-y_k\right)=(2-\sigma)\frac{\sigma}{2-\sigma}\left(u_1^{k+1}-u_1^k\right)+(\sigma-1)A^*\left(y_{k-1}-y_k\right).$$

As $1<\sigma<2$, the convexity of $\|\cdot\|^2$ gives

$$\left\|A^*\left(y_{k+1}-y_k\right)\right\|^2 \leqslant \frac{\sigma^2}{2-\sigma}\left\|u_1^{k+1}-u_1^k\right\|^2+(\sigma-1)\left\|A^*\left(y_k-y_{k-1}\right)\right\|^2$$

and from here it follows

$$\lambda_{\min}\left(AA^*\right)(2-\sigma)\left\|y_{k+1}-y_k\right\|^2 \leqslant(2-\sigma)\left\|A^*\left(y_{k+1}-y_k\right)\right\|^2 \\ \leqslant \frac{\sigma^2}{2-\sigma}\left\|u_1^{k+1}-u_1^k\right\|^2+(\sigma-1)\left\|A^*\left(y_k-y_{k-1}\right)\right\|^2-(\sigma-1)\left\|A^*\left(y_{k+1}-y_k\right)\right\|^2. \tag{5.3.16}$$

After plugging (5.3.14) into (5.3.16) we get

$$\frac{1}{\sigma\beta}\left\|y_{k+1}-y_k\right\|^2 \\ \leqslant \frac{2\sigma\left(L+\mu_1\right)^2}{\lambda_{\min}\left(AA^*\right)(2-\sigma)^2 r}\left\|x_{k+1}-x_k\right\|^2+\frac{2\sigma\mu_1^2}{\lambda_{\min}\left(AA^*\right)(2-\sigma)^2 r}\left\|x_k-x_{k-1}\right\|^2 \\ +\frac{(\sigma-1)}{\lambda_{\min}\left(AA^*\right)(2-\sigma)\sigma\beta}\left(\left\|A^*\left(y_k-y_{k-1}\right)\right\|^2-\left\|A^*\left(y_{k+1}-y_k\right)\right\|^2\right), \tag{5.3.17}$$

which, combined with (5.3.6), provides (5.3.5).

(ii) For *Algorithm 5.3.2*: The optimality criterion of (5.3.2b) is

$$\nabla h\left(x_k\right)=-A^*y_k-rA^*\left(Ax_{k+1}-z_{k+1}\right)+\mathcal{M}_1^k\left(x_k-x_{k+1}\right). \tag{5.3.18}$$

From (2.2.1) (applied for $z:=x_k$) we get

$$h\left(x_{k+1}\right) \leqslant h\left(x_k\right)+\left\langle y_k, Ax_k-Ax_{k+1}\right\rangle+r\left\langle Ax_{k+1}-z_{k+1}, Ax_k-Ax_{k+1}\right\rangle \\ -\left\|x_{k+1}-x_k\right\|_{\mathcal{M}_1^k}^2+\frac{L}{2}\left\|x_{k+1}-x_k\right\|^2. \tag{5.3.19}$$

Since the definition of z_{k+1} in (5.3.2a) leads also to (5.3.7), by combining this inequality with (5.3.19) and (5.3.2c), after some rearrangments, (5.3.6) follows. By using this time the notation

$$u_2^k:=-\nabla h\left(x_{k-1}\right)+\mathcal{M}_1^{k-1}\left(x_{k-1}-x_k\right)\ \forall k \geqslant 1 \tag{5.3.20}$$

and by taking into consideration (5.3.2c), we can rewrite (5.3.18) as

$$A^*y_{k+1}=\sigma u_2^{k+1}+(1-\sigma)A^*y_k\ \forall k \geqslant 0. \tag{5.3.21}$$

- *The case* $0 < \sigma \leqslant 1$. As in (5.3.12) we obtain

$$\begin{aligned} &\lambda_{\min}\left(AA^*\right)\sigma\left\|y_{k+1}-y_k\right\|^2 \leqslant \sigma\left\|A^*\left(y_{k+1}-y_k\right)\right\|^2 \\ &\leqslant \sigma\left\|u_2^{k+1}-u_2^k\right\|^2 + (1-\sigma)\left\|A^*\left(y_k-y_{k-1}\right)\right\|^2 - (1-\sigma)\left\|A^*\left(y_{k+1}-y_k\right)\right\|^2. \end{aligned} \tag{5.3.22}$$

  By using the Lipschitz continuity of ∇h we have

$$\left\|u_2^{k+1}-u_2^k\right\| \leqslant \mu_1\left\|x_{k+1}-x_k\right\| + (L+\mu_1)\left\|x_k-x_{k-1}\right\|, \tag{5.3.23}$$

  thus

$$\left\|u_2^{k+1}-u_2^k\right\|^2 \leqslant 2\mu_1^2\left\|x_{k+1}-x_k\right\|^2 + 2\left(L+\mu_1\right)^2\left\|x_k-x_{k-1}\right\|^2. \tag{5.3.24}$$

  After plugging (5.3.24) into (5.3.22) it follows

$$\begin{aligned} \frac{1}{\sigma\beta}\left\|y_{k+1}-y_k\right\|^2 \leqslant{}& \frac{2\mu_1^2}{\lambda_{\min}\left(AA^*\right)\sigma\beta}\left\|x_{k+1}-x_k\right\|^2 + \frac{2\left(L+\mu_1\right)^2}{\lambda_{\min}\left(AA^*\right)\sigma\beta}\left\|x_k-x_{k-1}\right\|^2 \\ &+ \frac{(1-\sigma)}{\lambda_{\min}\left(AA^*\right)\sigma^2\beta}\left(\left\|A^*\left(y_k-y_{k-1}\right)\right\|^2 - \left\|A^*\left(y_{k+1}-y_k\right)\right\|^2\right), \end{aligned} \tag{5.3.25}$$

  which, combined with (5.3.6), provides (5.3.5).

- *The case* $1 < \sigma < 2$. As in (5.3.16) we obtain

$$\begin{aligned} &\lambda_{\min}\left(AA^*\right)(2-\sigma)\left\|y_{k+1}-y_k\right\|^2 \leqslant (2-\sigma)\left\|A^*\left(y_{k+1}-y_k\right)\right\|^2 \\ &\leqslant \frac{\sigma^2}{2-\sigma}\left\|u_2^{k+1}-u_2^k\right\|^2 + (\sigma-1)\left\|A^*\left(y_k-y_{k-1}\right)\right\|^2 - (\sigma-1)\left\|A^*\left(y_{k+1}-y_k\right)\right\|^2. \end{aligned} \tag{5.3.26}$$

  After plugging (5.3.24) into (5.3.26) it follows

$$\begin{aligned} &\frac{1}{\sigma\beta}\left\|y_{k+1}-y_k\right\|^2 \\ &\leqslant \frac{2\sigma\mu_1^2}{\lambda_{\min}\left(AA^*\right)(2-\sigma)^2 r}\left\|x_{k+1}-x_k\right\|^2 + \frac{2\sigma\left(L+\mu_1\right)^2}{\lambda_{\min}\left(AA^*\right)(2-\sigma)^2 r}\left\|x_k-x_{k-1}\right\|^2 \\ &\quad + \frac{(\sigma-1)}{\lambda_{\min}\left(AA^*\right)(2-\sigma)\sigma\beta}\left(\left\|A^*\left(y_k-y_{k-1}\right)\right\|^2 - \left\|A^*\left(y_{k+1}-y_k\right)\right\|^2\right). \end{aligned} \tag{5.3.27}$$

  which, combined with (5.3.6), provides (5.3.5).

This completes the proof. □

The following three estimates will be useful in the sequel.

Lemma 5.3.3. *Let Assumption 5.3.1 be satisfied and* $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ *be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2. Then the following statements are true:*

(i) for every $k \geqslant 1$ *it holds*

$$\begin{aligned} \left\|z_{k+1}-z_k\right\| &\leqslant \|A\|\cdot\left\|x_{k+1}-x_k\right\| + \left\|Ax_{k+1}-z_{k+1}\right\| + \left\|Ax_k-z_k\right\| \\ &= \|A\|\cdot\left\|x_{k+1}-x_k\right\| + \frac{1}{\sigma\beta}\left\|y_{k+1}-y_k\right\| + \frac{1}{\sigma\beta}\left\|y_k-y_{k-1}\right\|; \end{aligned} \tag{5.3.28}$$

(ii) for every $k \geqslant 0$ *it holds*

$$\frac{1}{2\beta}\left\|y_{k+1}\right\|^2 \leqslant \frac{T_1}{2}\left\|A^*\left(y_{k+1}-y_k\right)\right\|^2 + \frac{T_0}{\beta}\left\|\nabla h\left(x_{k+1}\right)\right\|^2 + \frac{C_1}{4}\left\|x_{k+1}-x_k\right\|^2; \tag{5.3.29}$$

(iii) for every $k \geqslant 1$ it holds

$$\begin{aligned}\|y_{k+1} - y_k\| \leqslant C_3 \|x_{k+1} - x_k\| + C_4 \|x_k - x_{k-1}\| \\ + T_2 \left(\|A^* (y_k - y_{k-1})\| - \|A^* (y_{k+1} - y_k)\|\right),\end{aligned} \tag{5.3.30}$$

where

$$C_3 := \begin{cases} \dfrac{\sigma (L + \mu_1)}{\sqrt{\lambda_{\min} (AA^*)} (1 - |1 - \sigma|)}, & \textit{for Algorithm } 5.3.1, \\ \\ \dfrac{\sigma \mu_1}{\sqrt{\lambda_{\min} (AA^*)} (1 - |1 - \sigma|)}, & \textit{for Algorithm } 5.3.2, \end{cases}$$

$$C_4 := \begin{cases} \dfrac{\sigma \mu_1}{\sqrt{\lambda_{\min} (AA^*)} (1 - |1 - \sigma|)}, & \textit{for Algorithm } 5.3.1, \\ \\ \dfrac{\sigma (L + \mu_1)}{\sqrt{\lambda_{\min} (AA^*)} (1 - |1 - \sigma|)}, & \textit{for Algorithm } 5.3.2, \end{cases}$$

$$T_2 := \frac{|1 - \sigma|}{\sqrt{\lambda_{\min} (AA^*)} (1 - |1 - \sigma|)}.$$

Proof. The statement in (5.3.28) is straightforward.

From (5.3.11) and (5.3.21) we have for every $k \geqslant 0$

$$A^* y_{k+1} = \sigma u^{k+1} + (1 - \sigma) A^* y_k$$

or, equivalently,

$$\sigma A^* y_{k+1} = \sigma u^{k+1} + (1 - \sigma) A^* (y_k - y_{k+1}),$$

where u^{k+1} is defined as being equal to u_1^{k+1} in (5.3.10), for Algorithm 5.3.1, and, respectively, to u_2^{k+1} in (5.3.20), for Algorithm 5.3.2.

For $0 < \sigma \leqslant 1$ we have

$$\lambda_{\min} (AA^*) \sigma^2 \|y_{k+1}\|^2 \leqslant \sigma^2 \|A^* y_{k+1}\|^2 \leqslant \sigma \left\|u^{k+1}\right\|^2 + (1 - \sigma) \|A^* (y_{k+1} - y_k)\|^2, \tag{5.3.31}$$

while, for $1 < \sigma < 2$, we have

$$\lambda_{\min} (AA^*) \sigma^2 \|y_{k+1}\|^2 \leqslant \sigma^2 \|A^* y_{k+1}\|^2 \leqslant \frac{\sigma^2}{2 - \sigma} \left\|u^{k+1}\right\|^2 + (\sigma - 1) \|A^* (y_{k+1} - y_k)\|^2. \tag{5.3.32}$$

Notice further that for $1 < \sigma < 2$ we have $\dfrac{1}{\sigma} < 1$ and $1 < \dfrac{\sigma}{2 - \sigma}$.

In case u^{k+1} is defined as in (5.3.10) it holds

$$\left\|u^{k+1}\right\|^2 = \left\|u_1^{k+1}\right\|^2 \leqslant 2 \|\nabla h (x_{k+1})\|^2 + 2\mu_1^2 \|x_{k+1} - x_k\|^2 \ \ \forall k \geqslant 0, \tag{5.3.33}$$

while, in case u_2^{k+1} is defined as in (5.3.20), it holds

$$\left\|u^{k+1}\right\|^2 = \left\|u_2^{k+1}\right\|^2 \leqslant 2 \|\nabla h (x_{k+1})\|^2 + 2 (L + \mu_1)^2 \|x_{k+1} - x_k\|^2 \ \ \forall k \geqslant 0. \tag{5.3.34}$$

We divide (5.3.31) and (5.3.32) by $2\lambda_{\min} (AA^*) \sigma^2 \beta > 0$ and plug (5.3.33) and, respectively, (5.3.34) into the resulting inequalities. This gives us (5.3.29).

Finally, in order to prove (5.3.30), we notice that for every $k \geqslant 1$ it holds

$$\|A^* (y_{k+1} - y_k)\| \leqslant \sigma \left\|u^{k+1} - u^k\right\| + |1 - \sigma| \|A^* (y_k - y_{k-1})\|,$$

so,

$$\begin{aligned}\sqrt{\lambda_{\min}(AA^*)}\,(1-|1-\sigma|)\,\|y_{k+1}-y_k\| &\leqslant (1-|1-\sigma|)\,\|A^*(y_{k+1}-y_k)\| \\ &\leqslant \sigma\left\|u^{k+1}-u^k\right\| + |1-\sigma|\,\|A^*(y_k-y_{k-1})\| - |1-\sigma|\,\|A^*(y_{k+1}-y_k)\|. \end{aligned} \tag{5.3.35}$$

We plug into (5.3.35) the estimates for $\left\|u^{k+1}-u^k\right\|$ derived in (5.3.13) and, respectively, (5.3.23) and divide the resulting inequality by $\sqrt{\lambda_{\min}(AA^*)}\,(1-|1-\sigma|)>0$. This furnishes the desired statement. □

The following regularization of the augmented Lagrangian will play an important role in the convergence analysis of the nonconvex proximal ADMM algorithms

$$\Psi_\beta\colon \mathcal{H}\times\mathcal{G}\times\mathcal{G}\times\mathcal{H}\times\mathcal{G}\to\mathbb{R}\cup\{+\infty\},$$
$$\Psi_\beta(x,z,y,x',y') = \mathcal{L}_\beta(x,z,y) + T_1\left\|A^*\left(y-y'\right)\right\|^2 + \frac{C_1}{2}\left\|x-x'\right\|^2,$$

where T_1 and C_1 are defined in Assumption 5.3.1. For every $k\geqslant 1$ we denote

$$\Psi_k := \Psi_\beta(x_k,z_k,y_k,x_{k-1},y_{k-1}) = \mathcal{L}_\beta(x_k,z_k,y_k) + T_1\|A^*(y_k-y_{k-1})\|^2 + \frac{C_1}{2}\|x_k-x_{k-1}\|^2. \tag{5.3.36}$$

Since the convergence analysis will rely on the fact that the set of cluster points of the sequence $\{(x_k,z_k,y_k)\}_{k\geqslant 0}$ is nonempty, we will present first two situations which guarantee that this sequence is bounded. They make use of standard coercivity assumptions for the functions g and h, respectively. Recall that a function $\Psi:\mathcal{H}\to\mathbb{R}\cup\{+\infty\}$ is called *coercive*, if $\lim\limits_{\|x\|\to+\infty}\Psi(x)=+\infty$.

Theorem 5.3.4. *Let Assumption 5.3.1 be satisfied and $\{(x_k,z_k,y_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2. Suppose that one of the following conditions holds:*

(B-I) A is invertible and g is coercive;

(B-II) h is coercive.

Then the sequence $\{(x_k,z_k,y_k)\}_{k\geqslant 0}$ is bounded.

Proof. From Lemma 5.3.2 we have that for every $k\geqslant 1$

$$\Psi_{k+1} + \frac{1}{2}\|x_{k+1}-x_k\|^2_{\mathcal{M}_3^k-C_1\mathrm{Id}} + \frac{1}{2}\|z_{k+1}-z_k\|^2_{\mathcal{M}_2^k} \leqslant \Psi_k \tag{5.3.37}$$

which shows, according to (5.3.4), that $\{\Psi_k\}_{k\geqslant 1}$ is monotonically decreasing. Consequently, for every $k\geqslant 1$ we have

$$\begin{aligned}\Psi_1 &\geqslant \Psi_{k+1} + \frac{1}{2}\|x_{k+1}-x_k\|^2_{\mathcal{M}_3^k-C_1\mathrm{Id}} + \frac{1}{2}\|z_{k+1}-z_k\|^2_{\mathcal{M}_2^k} \\ &= h(x_{k+1}) + g(z_{k+1}) - \frac{1}{2\beta}\|y_{k+1}\|^2 + \frac{\beta}{2}\left\|Ax_{k+1}-z_{k+1}+\frac{1}{\beta}y_{k+1}\right\|^2 \\ &\quad + T_1\|A^*(y_{k+1}-y_k)\|^2 + \frac{1}{2}\|x_{k+1}-x_k\|^2_{\mathcal{M}_3^k-C_0\mathrm{Id}} + \frac{1}{2}\|z_{k+1}-z_k\|^2_{\mathcal{M}_2^k} + \frac{C_1}{2}\|x_{k+1}-x_k\|,\end{aligned}$$

which, thanks to (5.3.29), leads to

$$\begin{aligned}\Psi_1 &\geqslant h(x_{k+1}) + g(z_{k+1}) - \frac{T_0}{\beta}\|\nabla h(x_{k+1})\|^2 + \frac{\beta}{2}\left\|Ax_{k+1}-z_{k+1}+\frac{1}{\beta}y_{k+1}\right\|^2 \\ &\quad + \frac{T_1}{2}\|A^*(y_{k+1}-y_k)\|^2 + \frac{1}{2}\|x_{k+1}-x_k\|^2_{\mathcal{M}_3^k-C_1\mathrm{Id}} + \frac{1}{2}\|z_{k+1}-z_k\|^2_{\mathcal{M}_2^k} + \frac{C_1}{4}\|x_{k+1}-x_k\|^2.\end{aligned} \tag{5.3.38}$$

Next we will prove the boundedness of $\{(x_k,z_k,y_k)\}_{k\geqslant 0}$ under each of the two scenarios.

(B-I) Since $\beta \geqslant 4T_0L$, there exists $\gamma > 0$ such that

$$\frac{1}{\gamma} - \frac{L}{2\gamma^2} = \frac{T_0}{\beta}.$$

From Proposition 2.2.1 and the relation (5.3.38) we see that for every $k \geqslant 1$

$$\begin{aligned} & g\left(z_{k+1}\right) + \frac{\beta}{2}\left\|Ax_{k+1} - z_{k+1} + \frac{1}{\beta}y_{k+1}\right\|^2 + \frac{C_1}{4}\|x_{k+1} - x_k\|^2 \\ \leqslant\ & \Psi_1 - \inf_{x\in\mathcal{H}}\left\{h\left(x\right) - \frac{T_0}{\beta}\left\|\nabla h\left(x\right)\right\|^2\right\} < +\infty. \end{aligned}$$

Since g is coercive, it follows that the sequence $\{z_k\}_{k\geqslant 0}$ is bounded. On the other hand, since g is bounded from below, it follows that the sequences $\left\{Ax_k - z_k + \beta^{-1}y_k\right\}_{k\geqslant 0}$ and $\{x_{k+1} - x_k\}_{k\geqslant 0}$ are bounded as well. In addition, since for every $k \geqslant 0$ it holds

$$\left\|A\left(x_{k+1} - x_k\right) - \left(z_{k+1} - z_k\right)\right\| \leqslant \|A\| \cdot \|x_{k+1} - x_k\| + \|z_{k+1}\| + \|z_k\|$$

it follows that $\left\{A\left(x_{k+1} - x_k\right) - \left(z_{k+1} - z_k\right)\right\}_{k\geqslant 0}$ is bounded, thus so is $\left\{\beta^{-1}\left(y_{k+1} - y_k\right)\right\}_{k\geqslant 0}$. According to the third update in the iterative scheme we obtain that $\{Ax_k - z_k\}_{k\geqslant 0}$ is bounded and from here that $\{y_k\}_{k\geqslant 0}$ is also bounded. This implies the boundedness of $\{Ax_k\}_{k\geqslant 0}$ and, finally, since A is invertible, the boundedness of $\{x_k\}_{k\geqslant 0}$.

(B-II) Again thanks to (5.3.3) there exists $\gamma > 0$ such that

$$\frac{1}{\gamma} - \frac{L}{2\gamma^2} = \frac{3T_0}{2\beta}.$$

We assume first that $\sigma \neq 1$ or, equivalently, $T_1 \neq 0$. From Proposition 2.2.1 and (5.3.38) we see that for every $k \geqslant 1$

$$\begin{aligned} & \frac{1}{2}h\left(x_{k+1}\right) + \frac{T_0}{4\beta}\left\|\nabla h\left(x_{k+1}\right)\right\|^2 + \frac{\beta}{2}\left\|Ax_{k+1} - z_{k+1} + \frac{1}{\beta}y_{k+1}\right\|^2 + \frac{T_1}{2}\left\|A^*\left(y_{k+1} - y_k\right)\right\| \\ \leqslant\ & \Psi_1 - \inf_{z\in\mathcal{G}} g\left(z\right) - \frac{1}{2}\inf_{x\in\mathcal{H}}\left\{h\left(x\right) - \frac{3T_0}{2\beta}\left\|\nabla h\left(x\right)\right\|^2\right\} < +\infty. \end{aligned}$$

Since h is coercive and bounded from below, we obtain that $\{x_k\}_{k\geqslant 0}$, $\left\{Ax_k - z_k + \beta^{-1}y_k\right\}_{k\geqslant 0}$ and $\left\{A^*\left(y_{k+1} - y_k\right)\right\}_{k\geqslant 0}$ are bounded. For every $k \geqslant 0$ we have that

$$\lambda_{\min}\left(A^*A\right)\sigma^2\beta^2\left\|Ax_{k+1} - z_{k+1}\right\|^2 = \lambda_{\min}\left(A^*A\right)\left\|y_{k+1} - y_k\right\|^2 \leqslant \left\|A^*\left(y_{k+1} - y_k\right)\right\|^2,$$

thus $\{Ax_k - z_k\}_{k\geqslant 0}$ is bounded. Consequently, $\{y_k\}_{k\geqslant 0}$ and $\{z_k\}_{k\geqslant 0}$ are bounded.

In case $\sigma = 1$ or, equivalently, $T_1 = 0$, we have that for every $k \geqslant 1$

$$\begin{aligned} & \frac{1}{2}h\left(x_{k+1}\right) + \frac{T_0}{4\beta}\left\|\nabla h\left(x_{k+1}\right)\right\|^2 + \frac{\beta}{2}\left\|Ax_{k+1} - z_{k+1} + \frac{1}{\beta}y_{k+1}\right\|^2 \\ \leqslant\ & \Psi_1 - \inf_{z\in\mathcal{G}} g\left(z\right) - \frac{1}{2}\inf_{x\in\mathcal{H}}\left\{h\left(x\right) - \frac{3T_0}{2\beta}\left\|\nabla h\left(x\right)\right\|^2\right\} < +\infty \end{aligned}$$

from which we deduce that $\{x_k\}_{k\geqslant 0}$ and $\left\{Ax_k - z_k + \beta^{-1}y_k\right\}_{k\geqslant 0}$ are bounded. From Lemma 5.3.3 (iii), which now reads

$$\|y_{k+1} - y_k\| \leqslant C_3\|x_{k+1} - x_k\| + C_4\|x_k - x_{k-1}\| \ \ \forall k \geqslant 1,$$

it yields that $\{y_{k+1} - y_k\}_{k\geqslant 0}$ is bounded, thus, $\{Ax_k - z_k\}_{k\geqslant 0}$ is bounded. Consequently, $\{y_k\}_{k\geqslant 0}$ and $\{z_k\}_{k\geqslant 0}$ are bounded.

Both considered scenarios lead to the conclusion that the sequence $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ is bounded.
□

Remark 5.3.2. Guarantee the boundedness of $\{(x_k, z_k, y_k)\}_{k \geqslant 0}$ is an essential issue in the convergence analysis. In contrast to what we usually have in the convex setting (see e.g. [23, 44]), it is not clear whether the sequence of multiplier $\{y_k\}_{k \geqslant 0}$ is bounded in general.

Theorem 5.3.5. *Let Assumption 5.3.1 be satisfied and $\{(x_k, z_k, y_k)\}_{k \geqslant 0}$ be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2, which is assumed to be bounded. The following statements are true:*

(i) for every $k \geqslant 1$ it holds

$$\Psi_{k+1} + \frac{C_1}{4} \|x_{k+1} - x_k\|^2 + \frac{1}{2} \|z_{k+1} - z_k\|^2_{\mathcal{M}_2^k} \leqslant \Psi_k; \tag{5.3.39}$$

(ii) the sequence $\{\Psi_k\}_{k \geqslant 0}$ is bounded from below and convergent. In addition,

$$x_{k+1} - x_k \to 0, \ z_{k+1} - z_k \to 0 \text{ and } y_{k+1} - y_k \to 0 \text{ as } k \to +\infty; \tag{5.3.40}$$

(iii) the sequences $\{\Psi_k\}_{k \geqslant 0}$, $\{\mathcal{L}_\beta (x_k, z_k, y_k)\}_{k \geqslant 0}$ and $\{h (x_k) + g (z_k)\}_{k \geqslant 0}$ have the same limit, which we denote by $\Psi_* \in \mathbb{R}$.

Proof. (i) According to (5.3.4) we have that $\mathcal{M}_3^k - C_1 \mathrm{Id} \in \mathcal{P}^n_{\frac{C_1}{2}}$ and thus (5.3.37) implies (5.3.39).

(ii) We will show that $\{\mathcal{L}_\beta (x_k, z_k, y_k)\}_{k \geqslant 0}$ is bounded from below, which will imply that $\{\Psi_k\}_{k \geqslant 0}$ is bounded from below as well. Assuming the contrary, as $\{(x_k, z_k, y_k)\}_{k \geqslant 0}$ is bounded, there exists a subsequence $\left\{\left(x_{k_q}, z_{k_q}, y_{k_q}\right)\right\}_{q \geqslant 0}$ converging to an element $(\widehat{x}, \widehat{z}, \widehat{y}) \in \mathcal{H} \times \mathcal{G} \times \mathcal{G}$ such that $\left\{\mathcal{L}_\beta \left(x_{k_q}, z_{k_q}, y_{k_q}\right)\right\}_{q \geqslant 0}$ converges to $-\infty$ as $q \to +\infty$. However, using the lower semicontinuity of g and the continuity of h, we obtain

$$\liminf_{q \to +\infty} \mathcal{L}_\beta \left(x_{k_q}, z_{k_q}, y_{k_q}\right) \geqslant h(\widehat{x}) + g(\widehat{z}) + \langle \widehat{y}, A\widehat{x} - \widehat{z} \rangle + \frac{\beta}{2} \|A\widehat{x} - \widehat{z}\|^2,$$

which leads to a contradiction. From Lemma 2.4.2 we conclude that $\{\Psi_k\}_{k \geqslant 1}$ is convergent and

$$\sum_{k \geqslant 0} \|x_{k+1} - x_k\|^2 < +\infty,$$

thus $x_{k+1} - x_k \to 0$ as $k \to +\infty$.

We proved in (5.3.15), (5.3.17), (5.3.25) and (5.3.27) that for every $k \geqslant 1$

$$\begin{aligned} \frac{1}{\sigma\beta} \|y_{k+1} - y_k\|^2 \leqslant {} & \frac{C_0 - L}{2} \|x_{k+1} - x_k\|^2 + \frac{C_1}{2} \|x_k - x_{k-1}\|^2 \\ & + T_1 \|A^* (y_k - y_{k-1})\|^2 - T_1 \|A^* (y_{k+1} - y_k)\|^2 . \end{aligned}$$

Summing up the above inequality for $k = 1, \ldots, K$, for $K > 1$, we get

$$\begin{aligned} \frac{1}{\sigma\beta} \sum_{k=1}^{K} \|y_{k+1} - y_k\|^2 \leqslant {} & \frac{C_0 - L}{2} \sum_{k=1}^{K} \|x_{k+1} - x_k\|^2 + \frac{C_1}{2} \sum_{k=1}^{K} \|x_k - x_{k-1}\|^2 \\ & + T_1 \|A^* (y_1 - y_0)\|^2 - T_1 \|A^* (y_{K+1} - y_K)\|^2 \\ \leqslant {} & \frac{C_0 - L}{2} \sum_{k=1}^{K} \|x_{k+1} - x_k\|^2 + \frac{C_1}{2} \sum_{k=1}^{K} \|x_k - x_{k-1}\|^2 \\ & + T_1 \|A^* (y_1 - y_0)\|^2 . \end{aligned}$$

We let K converge to $+\infty$ and conclude

$$\sigma\beta \sum_{k\geqslant 0} \|Ax_{k+1} - z_{k+1}\|^2 = \frac{1}{\sigma\beta} \sum_{k\geqslant 0} \|y_{k+1} - y_k\|^2 < +\infty,$$

thus $Ax_{k+1} - z_{k+1} \to 0$ and $y_{k+1} - y_k \to 0$ as $k \to +\infty$. Since $x_{k+1} - x_k \to 0$ as $k \to +\infty$, it follows that $z_{k+1} - z_k \to 0$ as $k \to +\infty$.

(iii) By using (5.3.40) and the fact that $\{y_k\}_{k\geqslant 0}$ is bounded, it follows

$$\Psi_* = \lim_{k\to+\infty} \Psi_k = \lim_{k\to+\infty} \mathcal{L}_\beta (x_k, z_k, y_k) = \lim_{k\to+\infty} \{h(x_k) + g(z_k)\},$$

which is the desired statement. □

The following lemmas provides upper estimates in terms of the iterates for limiting subgradients of the augmented Lagrangian and the regularized augmented Lagrangian Ψ_β, respectively.

Lemma 5.3.6. *Let Assumption 5.3.1 be satisfied and* $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ *be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2. For every* $k \geqslant 0$ *we have*

$$v^{k+1} := \left(v_x^{k+1}, v_z^{k+1}, v_y^{k+1}\right) \in \partial\mathcal{L}_\beta (x_{k+1}, z_{k+1}, y_{k+1}), \tag{5.3.41}$$

where

$$v_x^{k+1} := C_2 (\nabla h(x_{k+1}) - \nabla h(x_k)) + A^* (y_{k+1} - y_k) + \mathcal{M}_1^k (x_k - x_{k+1}), \tag{5.3.42a}$$

$$v_z^{k+1} := y_k - y_{k+1} + rA(x_k - x_{k+1}) + \mathcal{M}_2^k (z_k - z_{k+1}), \tag{5.3.42b}$$

$$v_y^{k+1} := \frac{1}{\sigma\beta} (y_{k+1} - y_k). \tag{5.3.42c}$$

and

$$C_2 := \begin{cases} 0, & \text{for Algorithm } 5.3.1, \\ 1, & \text{for Algorithm } 5.3.2. \end{cases}$$

Moreover, for every $k \geqslant 0$ *it holds*

$$|||v^{k+1}||| \leqslant C_5 \|x_{k+1} - x_k\| + C_6 \|z_{k+1} - z_k\| + C_7 \|y_{k+1} - y_k\|, \tag{5.3.43}$$

where

$$C_5 := C_2 L + \mu_1 + \beta \|A\|, \quad C_6 := \mu_2, \quad C_7 := 1 + \|A\| + \frac{1}{\sigma\beta}.$$

Proof. Let $k \geqslant 0$ be fixed. Applying the calculus rules of the limiting subdifferential, we obtain

$$\nabla_x \mathcal{L}_\beta (x_{k+1}, z_{k+1}, y_{k+1}) = \nabla h(x_{k+1}) + A^* y_{k+1} + rA^* (Ax_{k+1} - z_{k+1}), \tag{5.3.44a}$$

$$\partial_z \mathcal{L}_\beta (x_{k+1}, z_{k+1}, y_{k+1}) = \partial g(z_{k+1}) - y_{k+1} - r(Ax_{k+1} - z_{k+1}), \tag{5.3.44b}$$

$$\nabla_y \mathcal{L}_\beta (x_{k+1}, z_{k+1}, y_{k+1}) = Ax_{k+1} - z_{k+1}. \tag{5.3.44c}$$

Then (5.3.42c) follows directly from (5.3.44c) and (5.3.1c), respectively, (5.3.2c), while (5.3.42b) follows from

$$y_k + r(Ax_k - z_{k+1}) + \mathcal{M}_2^k (z_k - z_{k+1}) \in \partial g(z_{k+1}),$$

which is a consequence of the optimality criterion of (5.3.1a) and (5.3.2a), respectively. In order to derive (5.3.42a), let us notice that for Algorithm 5.3.1 we have (see (5.3.8))

$$-A^* y_k + \mathcal{M}_1^k (x_k - x_{k+1}) = \nabla h(x_{k+1}) + rA^* (Ax_{k+1} - z_{k+1}), \tag{5.3.45}$$

while for Algorithm 5.3.2 we have (see (5.3.18))

$$-\nabla h\left(x_k\right)-A^* y_k+\mathcal{M}_1^k\left(x_k-x_{k+1}\right)=r A^*\left(A x_{k+1}-z_{k+1}\right). \tag{5.3.46}$$

By using (5.3.44a) we get the desired statement.

Relation (5.3.43) follows by combining the inequalities

$$\begin{aligned}
\left\|v_x^{k+1}\right\| &\leqslant \left(C_2 L+\mu_1\right)\left\|x_{k+1}-x_k\right\|+\|A\| \cdot\left\|y_{k+1}-y_k\right\|, \\
\left\|v_z^{k+1}\right\| &\leqslant \left\|y_k-y_{k+1}\right\|+\beta\|A\| \cdot\left\|x_{k+1}-x_k\right\|+\mu_2\left\|z_{k+1}-z_k\right\|
\end{aligned}$$

and (5.1.2). □

Lemma 5.3.7. *Let Assumption 5.3.1 be satisfied and* $\left\{\left(x_k, z_k, y_k\right)\right\}_{k \geqslant 0}$ *be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2. For every* $k \geqslant 0$ *we have*

$$D^{k+1}:=\left(D_x^{k+1}, D_z^{k+1}, D_y^{k+1}, D_{x'}^{k+1}, D_{y'}^{k+1}\right) \in \partial \Psi_\beta\left(x_{k+1}, z_{k+1}, y_{k+1}, x_k, y_k\right) \tag{5.3.47}$$

where

$$\begin{aligned}
D_x^{k+1}:=v_x^{k+1}+C_1\left(x_{k+1}-x_k\right), \quad D_z^{k+1}:=v_z^{k+1}, \quad D_y^{k+1}:=v_y^{k+1}+2 T_1 A A^*\left(y_{k+1}-y_k\right), \\
D_{x'}^{k+1}:=-C_1\left(x_{k+1}-x_k\right), \quad D_{y'}^{k+1}:=-2 T_1 A A^*\left(y_{k+1}-y_k\right).
\end{aligned} \tag{5.3.48}$$

Moreover, for every $k \geqslant 0$ *it holds*

$$\left|\left|\left|D^{k+1}\right|\right|\right| \leqslant C_8\left\|x_{k+1}-x_k\right\|+C_9\left\|z_{k+1}-z_k\right\|+C_{10}\left\|y_{k+1}-y_k\right\|, \tag{5.3.49}$$

where

$$C_8:=2 C_1+C_5, \quad C_9:=C_6, \quad C_{10}:=C_7+4 T_1\|A\|^2 .$$

Proof. Let $k \geqslant 0$ be fixed. Applying the calculus rules of the limiting subdifferential it follows

$$\nabla_x \Psi_\beta\left(x_{k+1}, z_{k+1}, y_{k+1}, x_k, y_k\right):=\nabla_x \mathcal{L}_\beta\left(x_{k+1}, z_{k+1}, y_{k+1}\right)+C_1\left(x_{k+1}-x_k\right), \tag{5.3.50a}$$

$$\partial_z \Psi_\beta\left(x_{k+1}, z_{k+1}, y_{k+1}, x_k, y_k\right):=\partial_z \mathcal{L}_\beta\left(x_{k+1}, z_{k+1}, y_{k+1}\right) \tag{5.3.50b}$$

$$\nabla_y \Psi_\beta\left(x_{k+1}, z_{k+1}, y_{k+1}, x_k, y_k\right):=\nabla_y \mathcal{L}_\beta\left(x_{k+1}, z_{k+1}, y_{k+1}\right)+2 T_1 A A^*\left(y_{k+1}-y_k\right), \tag{5.3.50c}$$

$$\nabla_{x'} \Psi_\beta\left(x_{k+1}, z_{k+1}, y_{k+1}, x_k, y_k\right):=-C_1\left(x_{k+1}-x_k\right), \tag{5.3.50d}$$

$$\nabla_{y'} \Psi_\beta\left(x_{k+1}, z_{k+1}, y_{k+1}, x_k, y_k\right):=-2 T_1 A A^*\left(y_{k+1}-y_k\right). \tag{5.3.50e}$$

Then (5.3.47) follows directly from the above relations and (5.3.41). Inequality (5.3.49) follows by combining

$$\begin{aligned}
\left\|D_x^{k+1}\right\| &\leqslant \left\|v_x^{k+1}\right\|+C_1\left\|x_{k+1}-x_k\right\|, \\
\left\|D_y^{k+1}\right\| &\leqslant \left\|v_y^{k+1}\right\|+2 T_1\|A\|^2 \cdot\left\|y_{k+1}-y_k\right\| .
\end{aligned}$$

and (5.1.2). □

The following result is a straightforward consequence of Lemma 5.3.3 and Lemma 5.3.7.

Corollary 5.3.8. *Let Assumption 5.3.1 be satisfied and* $\left\{\left(x_k, z_k, y_k\right)\right\}_{k \geqslant 0}$ *be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2. Then the norm of* $D^{k+1} \in \partial \Psi_\beta\left(x_{k+1}, z_{k+1}, y_{k+1}, x_k, y_k\right)$ *defined in the previous lemma verifies for every* $k \geqslant 2$ *the following estimate*

$$\begin{aligned}
\left|\left|\left|D^{k+1}\right|\right|\right| \leqslant\; & C_{11}\left(\left\|x_{k+1}-x_k\right\|+\left\|x_k-x_{k-1}\right\|+\left\|x_{k-1}-x_{k-2}\right\|\right) \\
& +C_{12}\left(\left\|A^*\left(y_k-y_{k-1}\right)\right\|-\left\|A^*\left(y_{k+1}-y_k\right)\right\|\right) \\
& +C_{13}\left(\left\|A^*\left(y_{k-1}-y_{k-2}\right)\right\|-\left\|A^*\left(y_k-y_{k-1}\right)\right\|\right),
\end{aligned} \tag{5.3.51}$$

where

$$C_{11} := \max\left\{C_8 + C_9\|A\| + C_3C_{10} + \frac{C_3C_9}{\sigma\beta}, C_4C_{10} + \frac{C_3C_9}{\sigma\beta}, \frac{C_4C_9}{\sigma\beta}\right\},$$
$$C_{12} := \left(C_{10} + \frac{C_9}{\sigma\beta}\right)T_2, \qquad C_{13} := \frac{C_9T_2}{\sigma\beta}.$$

In the following, we denote by $\omega\left(\{u_k\}_{k\geqslant 0}\right)$ the set of *cluster points* of the sequence $\{u_k\}_{k\geqslant 0}$.

Lemma 5.3.9. *Let Assumption 5.3.1 be satisfied and $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2, which is assumed to be bounded. The following statements are true:*

(i) if $\left\{\left(x_{k_q}, z_{k_q}, y_{k_q}\right)\right\}_{q\geqslant 0}$ is a subsequence of $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ which converges to $(\widehat{x}, \widehat{z}, \widehat{y})$ as $q \to +\infty$, then
$$\lim_{q\to\infty} \mathcal{L}_\beta\left(x_{k_q}, z_{k_q}, y_{k_q}\right) = \mathcal{L}_\beta\left(\widehat{x}, \widehat{z}, \widehat{y}\right);$$

(ii) it holds
$$\begin{aligned}\omega\left(\{(x_k, z_k, y_k)\}_{k\geqslant 0}\right) &\subseteq \operatorname{crit}\left(\mathcal{L}_\beta\right)\\ &\subseteq \left\{(\widehat{x}, \widehat{z}, \widehat{y}) \in \mathcal{H}\times\mathcal{G}\times\mathcal{G}\colon -A^*\widehat{y} = \nabla h\left(\widehat{x}\right), \widehat{y} \in \partial g\left(\widehat{z}\right), \widehat{z} = A\widehat{x}\right\};\end{aligned}$$

(iii) we have $\lim_{k\to+\infty} \operatorname{dist}\left[(x_k, z_k, y_k), \omega\left(\{(x_k, z_k, y_k)\}_{k\geqslant 0}\right)\right] = 0$;

(iv) the set $\omega\left(\{(x_k, z_k, y_k)\}_{k\geqslant 0}\right)$ is nonempty, connected and compact;

(v) the function $\mathcal{L}_\beta$ takes on $\omega\left(\{(x_k, z_k, y_k)\}_{k\geqslant 0}\right)$ the value $\Psi_* = \lim_{k\to+\infty}\mathcal{L}_\beta\left(x_k, z_k, y_k\right)$, as the objective function $g\circ A + h$ does on the projection of the set $\omega\left(\{(x_k, z_k, y_k)\}_{k\geqslant 0}\right)$ onto the space $\mathcal{H}$ corresponding to the first component.

Proof. Let $(\widehat{x}, \widehat{z}, \widehat{y}) \in \omega\left(\{(x_k, z_k, y_k)\}_{k\geqslant 0}\right)$, which exists since we assumed $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ is bounded. Let $\left\{\left(x_{k_q}, z_{k_q}, y_{k_q}\right)\right\}_{q\geqslant 0}$ be a subsequence of $\{x_k, z_k, y_k\}_{k\geqslant 0}$ converging to $(\widehat{x}, \widehat{z}, \widehat{y})$ as $q \to +\infty$.

(i) From either (5.3.1a) or (5.3.2a) we obtain for every $q \geqslant 1$

$$\begin{aligned}&g\left(z_{k_q}\right) + \left\langle y_{k_q-1}, Ax_{k_q-1} - z_{k_q}\right\rangle + \frac{\beta}{2}\left\|Ax_{k_q-1} - z_{k_q}\right\|^2 + \frac{1}{2}\left\|z_{k_q} - z_{k_q-1}\right\|^2_{\mathcal{M}_2^{k_q-1}}\\ \leqslant\ &g\left(\widehat{z}\right) + \left\langle y_{k_q-1}, Ax_{k_q-1} - \widehat{z}\right\rangle + \frac{\beta}{2}\left\|Ax_{k_q-1} - \widehat{z}\right\|^2 + \frac{1}{2}\left\|\widehat{z} - z_{k_q-1}\right\|^2_{\mathcal{M}_2^{k_q-1}}.\end{aligned}$$

Taking the limit superior on both sides of the above inequalities we get

$$\limsup_{q\to+\infty} g\left(z_{k_q}\right) \leqslant g\left(\widehat{z}\right),$$

which, combined with the lower semicontinuity of g, leads to

$$\lim_{q\to+\infty} g\left(z_{k_q}\right) = g\left(\widehat{z}\right).$$

Since h is continuous, we further obtain

$$\begin{aligned}\lim_{q\to+\infty}\mathcal{L}_\beta\left(x_{k_q}, z_{k_q}, y_{k_q}\right) &= \lim_{q\to+\infty}\left[g\left(z_{k_q}\right) + h\left(x_{k_q}\right) + \left\langle y_{k_q}, Ax_{k_q} - z_{k_q}\right\rangle + \frac{\beta}{2}\left\|Ax_{k_q} - z_{k_q}\right\|^2\right]\\ &= g\left(\widehat{z}\right) + h\left(\widehat{x}\right) + \left\langle\widehat{y}, A\widehat{x} - \widehat{z}\right\rangle + \frac{\beta}{2}\left\|A\widehat{x} - \widehat{z}\right\|^2 = \mathcal{L}_\beta\left(\widehat{x}, \widehat{z}, \widehat{y}\right).\end{aligned}$$

(ii) For the sequence $\{d^k\}_{k\geqslant 0}$ defined in (5.3.42a)-(5.3.42c) we have that $d^{k_q} \in \partial \mathcal{L}_\beta(x_{k_q}, z_{k_q}, y_{k_q})$ for every $q \geqslant 1$ and $d^{k_q} \to 0$ as $q \to +\infty$, while $\left(x_{k_q}, z_{k_q}, y_{k_q}\right) \to (\widehat{x}, \widehat{z}, \widehat{y})$ and $\mathcal{L}_\beta\left(x_{k_q}, z_{k_q}, y_{k_q}\right) \to \mathcal{L}_\beta\left(\widehat{x}, \widehat{z}, \widehat{y}\right)$ as $q \to +\infty$. The closedness criterion of the limiting subdifferential guarantees that $0 \in \partial \mathcal{L}_\beta\left(\widehat{x}, \widehat{z}, \widehat{y}\right)$ or, in other words, $(\widehat{x}, \widehat{z}, \widehat{y}) \in \operatorname{crit}\left(\mathcal{L}_\beta\right)$. Choosing now an element $(\widehat{x}, \widehat{z}, \widehat{y}) \in \operatorname{crit}\left(\mathcal{L}_\beta\right)$ it holds

$$\begin{aligned} 0 &= \nabla h\left(\widehat{x}\right) + A^*\widehat{y} + rA^*\left(A\widehat{x} - \widehat{z}\right) \\ 0 &\in \partial g\left(\widehat{z}\right) - \widehat{y} - r\left(A\widehat{x} - \widehat{z}\right) \\ 0 &= A\widehat{x} - \widehat{z}, \end{aligned}$$

which is further equivalent to

$$-A^*\widehat{y} = \nabla h\left(\widehat{x}\right), \ \widehat{y} \in \partial g\left(\widehat{z}\right), \ \widehat{z} = A\widehat{x}.$$

(iii)-(iv) The proof follows in the lines of the proof of Theorem 5 (ii)-(iii) in [36], also by taking into consideration [36, Remark 5], according to which the properties in (iii) and (iv) are generic for sequences satisfying $(x_{k+1}, z_{k+1}, y_{k+1}) - (x_k, z_k, y_k) \to 0$ as $k \to +\infty$, which is indeed the case due to (5.3.40).

(v) The conclusion follows according to the first two statements of this theorem and of the third statement of Theorem 5.3.5. □

Remark 5.3.3. An element $(\widehat{x}, \widehat{z}, \widehat{y}) \in \mathcal{H} \times \mathcal{G} \times \mathcal{G}$ fulfilling

$$-A^*\widehat{y} = \nabla h\left(\widehat{x}\right), \ \widehat{y} \in \partial g\left(\widehat{z}\right), \ \widehat{z} = A\widehat{x}$$

is a so-called *KKT point* of the optimization problem (5.1.1). For such a KKT point we have

$$0 = A^*\partial g\left(A\widehat{x}\right) + \nabla h\left(\widehat{x}\right). \tag{5.3.52}$$

When A is injective this is further equivalent to

$$0 \in \partial (g \circ A)(\widehat{x}) + \nabla h\left(\widehat{x}\right) = \partial\left(g \circ A + h\right)\left(\widehat{x}\right), \tag{5.3.53}$$

in other words, $\widehat{x}$ is a *critical point* of the optimization problem (5.1.1).

If the functions g and h are convex, then (5.3.52) and (5.3.53) are equivalent, which means that $\widehat{x}$ is a *global optimal solution* of the optimization problem (5.1.1). In this case, $\widehat{y}$ is a *global optimal solution* of the Fenchel dual problem of (5.1.1).

By combining Lemma 5.3.7, Theorem 5.3.5 and Lemma 5.3.9, one obtains the following result.

Lemma 5.3.10. *Let Assumption 5.3.1 be satisfied and $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2, which is assumed to be bounded. Denote by*

$$\Omega := \omega\left(\{(x_k, z_k, y_k, x_{k-1}, y_{k-1})\}_{k\geqslant 1}\right).$$

The following statements are true:

(i) it holds

$$\Omega \subseteq \{(\widehat{x}, \widehat{z}, \widehat{y}, \widehat{x}, \widehat{y}) \in \mathcal{H} \times \mathcal{G} \times \mathcal{G} \times \mathcal{H} \times \mathcal{G} \colon \ (\widehat{x}, \widehat{z}, \widehat{y}) \in \operatorname{crit}\left(\mathcal{L}_\beta\right)\};$$

(ii) we have

$$\lim_{k\to+\infty} \operatorname{dist}\left[\left(x_k, z_k, y_k, x_{k-1}, y_{k-1}\right), \Omega\right] = 0;$$

(iii) the set Ω is nonempty, connected and compact;

(iv) the regularized augmented Lagrangian Ψ_β takes on Ω the value $\Psi_* = \lim_{k\to+\infty} \Psi_k$, as the objective function $g \circ A + h$ does on the projection of the set Ω onto the space $\mathcal{H}$ corresponding to the first component.

5.3.3 Convergence analysis under Kurdyka-Łojasiewicz assumptions

In this subsection we will prove global convergence for the sequence $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ generated by the two nonconvex proximal ADMM algorithms in the context of *KŁ property.*

Working in the hypotheses of Lemma 5.3.10 we define for every $k \geqslant 1$

$$\mathcal{E}_k := \Psi\left(x_k, z_k, y_k, x_{k-1}, y_{k-1}\right) - \Psi_* = \Psi_k - \Psi_* \geqslant 0,$$

where Ψ_* is the limit of $\{\Psi_k\}_{k\geqslant 1}$ as $k \to +\infty$. The sequence $\{\mathcal{E}_k\}_{k\geqslant 1}$ is monotonically decreasing and it converges to 0 as $k \to +\infty$.

The next result shows that, if the regularization of the augmented Lagrangian Ψ_β is a KŁ function, then the sequence $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ converges to a KKT point of the optimization problem (5.1.1).

Theorem 5.3.11. *Let Assumption 5.3.1 be satisfied and $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2, which is assumed to be bounded. If Ψ_β is a KŁ function, then the following statements are true:*

(i) the sequence $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ has finite length, namely,

$$\sum_{k\geqslant 0} \|x_{k+1} - x_k\| < +\infty, \qquad \sum_{k\geqslant 0} \|z_{k+1} - z_k\| < +\infty, \qquad \sum_{k\geqslant 0} \|y_{k+1} - y_k\| < +\infty; \tag{5.3.54}$$

(ii) the sequence $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ converges to a KKT point of the optimization problem (5.1.1)*.*

Proof. As in Lemma 5.3.10, we denote by $\Omega := \omega\left(\{(x_k, z_k, y_k, x_{k-1}, y_{k-1})\}_{k\geqslant 1}\right)$, which is a nonempty set. Let be $(\widehat{x}, \widehat{z}, \widehat{y}, \widehat{x}, \widehat{y}) \in \Omega$, thus $\Psi_\beta(\widehat{x}, \widehat{z}, \widehat{y}, \widehat{x}, \widehat{y}) = \Psi_*$. We have seen that $\{\mathcal{E}_k = \Psi_k - \mathcal{F}^*\}_{k\geqslant 1}$ converges to 0 as $k \to +\infty$ and will consider, consequently, two cases.

We assume first that there exists an integer $k' \geqslant 0$ such that $\mathcal{E}_{k'} = 0$ or, equivalently, $\Psi_{k'} = \Psi_*$. Due to the monotonicity of $\{\mathcal{E}_k\}_{k\geqslant 1}$ it follows that $\mathcal{E}_k = 0$ or, equivalently, $\Psi_k = \Psi_*$ for all $k \geqslant k'$. Combining the inequality in (5.3.39) with Lemma 5.3.3, it yields that $x_{k+1} - x_k = 0$ for all $k \geqslant k'+1$. Using Lemma 5.3.3 (iii) and telescoping sum arguments, it yields $\sum_{k\geqslant 0} \|y_{k+1} - y_k\| < +\infty$. Finally, by using Lemma 5.3.3 (i), we obtain that $\sum_{k\geqslant 0} \|z_{k+1} - z_k\| < +\infty$.

Consider now the case when $\mathcal{E}_k > 0$ or, equivalently, $\Psi_k > \Psi_*$ for every $k \geqslant 1$. According to Lemma 2.3.1, there exist $\varepsilon > 0$, $\eta > 0$ and a desingularization function φ such that for every element u in the intersection

$$\begin{aligned} &\{u \in \mathcal{H} \times \mathcal{G} \times \mathcal{G} \times \mathcal{H} \times \mathcal{G} \colon \operatorname{dist}(u, \Omega) < \varepsilon\} \cap \\ &\{u \in \mathcal{H} \times \mathcal{G} \times \mathcal{G} \times \mathcal{H} \times \mathcal{G} \colon \Psi_* < \Psi_\beta(u) < \Psi_* + \eta\} \end{aligned} \tag{5.3.55}$$

it holds

$$\varphi'\left(\Psi_\beta(u) - \Psi_*\right) \cdot \operatorname{dist}\left(0, \partial\Psi_\beta(u)\right) \geqslant 1.$$

Let be $k_1 \geqslant 1$ such that for every $k \geqslant k_1$

$$\Psi_* < \Psi_k < \Psi_* + \eta.$$

Since $\lim_{k\to+\infty} \operatorname{dist}\left[(x_k, z_k, y_k, x_{k-1}, y_{k-1}), \Omega\right] = 0$, see Lemma 5.3.10 (ii), there exists $k_2 \geqslant 1$ such that for every $k \geqslant k_2$

$$\operatorname{dist}\left[(x_k, z_k, y_k, x_{k-1}, y_{k-1}), \Omega\right] < \varepsilon.$$

The element $(x_k, z_k, y_k, x_{k-1}, y_{k-1})$ thus belongs to the intersection in (5.3.55) for every $k \geqslant k_0 := \max\{k_1, k_2, 3\}$, which further implies

$$\begin{aligned} &\varphi'\left(\Psi_k - \Psi_*\right) \cdot \operatorname{dist}\left(0, \partial\Psi_\beta(x_k, z_k, y_k, x_{k-1}, y_{k-1})\right) \\ = \;&\varphi'\left(\mathcal{E}_k\right) \cdot \operatorname{dist}\left(0, \partial\Psi_\beta(x_k, z_k, y_k, x_{k-1}, y_{k-1})\right) \geqslant 1. \end{aligned} \tag{5.3.56}$$

Define for two arbitrary nonnegative integers p and q

$$\Delta_{p,q} := \varphi\left(\Psi_p - \Psi_*\right) - \varphi\left(\Psi_q - \Psi_*\right) = \varphi\left(\mathcal{E}_p\right) - \varphi\left(\mathcal{E}_q\right).$$

For every $K \geqslant k_0 \geqslant 1$ it holds

$$\sum_{k=k_0}^{K} \Delta_{k,k+1} = \Delta_{k_0,K+1} = \varphi\left(\mathcal{E}_{k_0}\right) - \varphi\left(\mathcal{E}_{K+1}\right) \leqslant \varphi\left(\mathcal{E}_{k_0}\right),$$

from which we get $\sum_{k\geqslant 1} \Delta_{k,k+1} < +\infty$.

By combining Theorem 5.3.5 (i) with the concavity of φ we obtain for every $k \geqslant 1$

$$\begin{aligned} \Delta_{k,k+1} = \varphi\left(\mathcal{E}_k\right) - \varphi\left(\mathcal{E}_{k+1}\right) &\geqslant \varphi'\left(\mathcal{E}_k\right)\left[\mathcal{E}_k - \mathcal{E}_{k+1}\right] = \varphi'\left(\mathcal{E}_k\right)\left[\Psi_k - \Psi_{k+1}\right] \\ &\geqslant \varphi'\left(\mathcal{E}_k\right)\frac{C_1}{4}\left\|x_{k+1} - x_k\right\|^2. \end{aligned} \tag{5.3.57}$$

The last relation combined with (5.3.56) imply

$$\begin{aligned} \left\|x_{k+1} - x_k\right\|^2 &\leqslant \varphi'\left(\mathcal{E}_k\right) \cdot \operatorname{dist}\left(0, \partial\Psi_\beta\left(x_k, z_k, y_k, x_{k-1}, y_{k-1}\right)\right)\left\|x_{k+1} - x_k\right\|^2 \\ &\leqslant \frac{4}{C_1}\Delta_{k,k+1} \cdot \operatorname{dist}\left(0, \partial\Psi_\beta\left(x_k, z_k, y_k, x_{k-1}, y_{k-1}\right)\right) \ \forall k \geqslant k_0. \end{aligned}$$

By the arithmetic mean-geometric mean inequality and Corollary 5.3.8 we have that for every $k \geqslant k_0$ and every $\nu > 0$

$$\begin{aligned} \left\|x_{k+1} - x_k\right\| &\leqslant \sqrt{\frac{4}{C_1}\Delta_{k,k+1} \cdot \operatorname{dist}\left(0, \partial\Psi_\beta\left(x_k, z_k, y_k, x_{k-1}, y_{k-1}\right)\right)} \\ &\leqslant \frac{\nu}{C_1}\Delta_{k,k+1} + \frac{1}{\nu}\operatorname{dist}\left(0, \partial\Psi_\beta\left(x_k, z_k, y_k, x_{k-1}, y_{k-1}\right)\right) \\ &\leqslant \frac{\nu}{C_1}\Delta_{k,k+1} + \frac{C_{11}}{\nu}\left(\left\|x_k - x_{k-1}\right\| + \left\|x_{k-1} - x_{k-2}\right\| + \left\|x_{k-2} - x_{k-3}\right\|\right) \\ &\quad + \frac{C_{12}}{\nu}\left(\left\|A^*\left(y_{k-1} - y_{k-2}\right)\right\| - \left\|A^*\left(y_k - y_{k-1}\right)\right\|\right) \\ &\quad + \frac{C_{13}}{\nu}\left(\left\|A^*\left(y_{k-2} - y_{k-3}\right)\right\| - \left\|A^*\left(y_{k-1} - y_{k-2}\right)\right\|\right). \end{aligned} \tag{5.3.58}$$

We denote for every $k \geqslant 3$

$$\begin{aligned} a_k &:= \left\|x_k - x_{k-1}\right\| \geqslant 0, \\ d_k &:= \frac{\nu}{C_1}\Delta_{k,k+1} + \frac{C_{12}}{\nu}\left(\left\|A^*\left(y_{k-1} - y_{k-2}\right)\right\| - \left\|A^*\left(y_k - y_{k-1}\right)\right\|\right) \\ &\quad + \frac{C_{13}}{\nu}\left(\left\|A^*\left(y_{k-2} - y_{k-3}\right)\right\| - \left\|A^*\left(y_{k-1} - y_{k-2}\right)\right\|\right). \end{aligned}$$

The inequality (5.3.58) is nothing than (2.4.6) with $\chi_0 = \chi_1 = \chi_2 := \dfrac{C_{11}}{\nu}$. Observe that for every $K \geqslant k_0$ we have

$$\sum_{k=k_0}^{K} \delta_k \leqslant \frac{\nu}{C_1}\varphi\left(\mathcal{E}_{k_0}\right) + \frac{C_{12}}{\nu}\left\|A^*\left(y_{k_0-1} - y_{k_0-2}\right)\right\| + \frac{C_{13}}{\nu}\left\|A^*\left(y_{k_0-2} - y_{k_0-3}\right)\right\|$$

and thus, by choosing $\nu > 3C_{11}$, we can use Lemma 2.4.4 to conclude that

$$\sum_{k\geqslant 0} \left\|x_{k+1} - x_k\right\| < +\infty.$$

The other two statements in (5.3.54) follow from Lemma 5.3.3. This means that the sequence $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ is Cauchy, thus it converges to an element $(\widehat{x}, \widehat{z}, \widehat{y})$ which is, according to Lemmas 5.3.9, a KKT point of the optimization problem (5.1.1). □

Remark 5.3.4. The function Ψ_β is a KŁ function if, for instance, the objective function of (5.1.1) is semi-algebraic, which is the case when the functions g and h are semi-algebraic.

5.4 Convergence rates under Łojasiewicz assumptions

In this section we derive convergence rates for the sequence $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ generated by Algorithm 5.3.1 or Algorithm 5.3.2 as well as for the regularized augmented Lagrangian function Ψ_β along this sequence, provided that the latter satisfies the Łojasiewicz property.

If Assumption 5.3.1 is fulfilled and $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ is the sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2, assumed to be bounded, then, as seen in Lemma 5.3.10, the set of cluster points $\Omega = \omega\left(\{(x_k, z_k, y_k, x_{k-1}, y_{k-1})\}_{k\geqslant 0}\right)$ is nonempty, compact and connected and Ψ_β takes on Ω the value Ψ_*; in addition, for every $(\widehat{x}, \widehat{z}, \widehat{y}, \widehat{x}, \widehat{y}) \in \Omega$, $(\widehat{x}, \widehat{z}, \widehat{y})$ belongs to $\operatorname{crit}(\mathcal{L}_\beta)$. Then there exist $C_L > 0$, $\theta \in [0, 1)$ and $\varepsilon > 0$ such that

$$\left|\Psi_\beta\left(x, z, y, x', y'\right) - \Psi_*\right|^\theta \leqslant C_L \cdot \operatorname{dist}\left(0, \partial\Psi_\beta\left(x, z, y, x', y'\right)\right) \quad \forall\left(x, z, y, x', y'\right) \in \mathbb{B}\left((\widehat{x}, \widehat{z}, \widehat{y}, \widehat{x}, \widehat{y}), \varepsilon\right). \tag{5.4.1}$$

In this case, Ψ_β is said to satisfy the Łojasiewicz property with *Łojasiewicz constant* $C_L > 0$ and *Łojasiewicz exponent* $\theta \in [0, 1)$.

We will address convergence rates for Algorithm 5.3.1 and Algorithm 5.3.2 in the context of an assumption which is slightly more restricitve than Assumption 5.3.1.

Assumption 5.4.1. *We work in the hypotheses of Assumption 5.3.1 except for* (5.3.4) *which is replaced by*

$$2\mathcal{M}_1^k + \beta A^*A \succcurlyeq \left(L + \frac{C'_{\mathcal{M}}}{\beta}\right)\operatorname{Id} \quad \forall k \geqslant 0, \tag{5.4.2}$$

where

$$C'_{\mathcal{M}} := \begin{cases} \left(10\mu_1^2 + 8\left(L + \mu_1\right)^2\right)T_0, & \text{for Algorithm 5.3.1,} \\ \left(8\mu_1^2 + 10\left(L + \mu_1\right)^2\right)T_0, & \text{for Algorithm 5.3.2.} \end{cases}$$

The condition (5.4.2) is nothing else than (5.3.4) after replacing $C_{\mathcal{M}}$ by the bigger constant $C'_{\mathcal{M}}$.

The examples in Example 5.3.1 can be all adapted to the new setting and one can provide different settings which guarantee Assumption 5.4.1. The scenarios which ensure Assumption 5.4.1 evidently satisfy Assumption 5.3.1, too, therefore the results investigated in Section 5.3 remain valid in this setting. As follows we will provide improvements of the statements used in the convergence analysis which follow thanks to Assumption 5.4.1.

Lemma 5.4.1. *Let Assumption 5.3.1 be satisfied and $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2. Then for every $k \geqslant 1$ it holds*

$$\begin{aligned}
&\mathcal{L}_\beta\left(x_{k+1}, z_{k+1}, y_{k+1}\right) + 2T_1\left\|A^*\left(y_{k+1} - y_k\right)\right\|^2 + \frac{1}{2}\left\|x_{k+1} - x_k\right\|^2_{\mathcal{M}_3^k} + \frac{1}{2}\left\|z_{k+1} - z_k\right\|^2_{\mathcal{M}_2^k} \\
&+ \frac{1}{\sigma\beta}\left\|y_{k+1} - y_k\right\|^2 \\
&\leqslant \mathcal{L}_\beta\left(x_k, z_k, y_k\right) + 2T_1\left\|A^*\left(y_k - y_{k-1}\right)\right\|^2 + C_1\left\|x_k - x_{k-1}\right\|^2.
\end{aligned} \tag{5.4.3}$$

Proof. Let $k \geqslant 1$ be fixed. By the same arguments as in Lemma 5.3.2, we have that (see (5.3.6))

$$\begin{aligned}
&\mathcal{L}_\beta\left(x_{k+1}, z_{k+1}, y_{k+1}\right) + \frac{1}{2}\left\|x_{k+1} - x_k\right\|^2_{2\mathcal{M}_1^k + \beta A^*A} - \frac{L}{2}\left\|x_{k+1} - x_k\right\|^2 + \frac{1}{2}\left\|z_{k+1} - z_k\right\|^2_{\mathcal{M}_2^k} \\
&\leqslant \mathcal{L}_\beta\left(x_k, z_k, y_k\right) + \frac{1}{\sigma\beta}\left\|y_{k+1} - y_k\right\|^2.
\end{aligned} \tag{5.4.4}$$

From (5.3.15), (5.3.17), (5.3.25) and (5.3.27) it follows that

$$\frac{1}{\sigma\beta}\left\|y_{k+1}-y_k\right\|^2 \leqslant \frac{C_0-L}{2}\left\|x_{k+1}-x_k\right\|^2+\frac{C_1}{2}\left\|x_k-x_{k-1}\right\|^2+ \\ T_1\left\|A^*\left(y_k-y_{k-1}\right)\right\|^2-T_1\left\|A^*\left(y_{k+1}-y_k\right)\right\|^2. \tag{5.4.5}$$

By multiplying (5.4.5) by 2 and by adding the resulting inequality to (5.4.4) we obtain (5.4.3). □

We replace T_1 with $2T_1$ in the definition of the regularized augmented Lagrangian Ψ_β, thus, the sequence $\{\Psi_k\}_{k\geqslant 1}$ in (5.3.36) becomes

$$\Psi_k := \mathcal{L}_\beta\left(x_k, z_k, y_k\right)+2T_1\left\|A^*\left(y_k-y_{k-1}\right)\right\|^2+C_1\left\|x_k-x_{k-1}\right\|^2 \quad \forall k \geqslant 1.$$

In this new context the inequality (5.4.3) reads for every $k \geqslant 1$

$$\Psi_{k+1}+\frac{C_1}{4}\left\|x_{k+1}-x_k\right\|^2+\frac{1}{2}\left\|z_{k+1}-z_k\right\|_{\mathcal{M}_2^k}^2+\frac{1}{\sigma\beta}\left\|y_{k+1}-y_k\right\|^2 \leqslant \Psi_k \tag{5.4.6}$$

and provides an inequality which is tighter than relation (5.3.39) in Theorem 5.3.5. Furthermore, for a subgradient D^{k+1} of Ψ_β at $(x_{k+1}, z_{k+1}, y_{k+1}, x_k, z_k)$ defined as in (5.3.48) (again by replacing T_1 by $2T_1$) we obtain for every $k \geqslant 2$ the following estimate, which is simpler than (5.3.51) in Corollary 5.3.8

$$|||D^{k+1}||| \leqslant C_{14}\left\|x_{k+1}-x_k\right\|+C_{15}\left\|y_{k+1}-y_k\right\|+C_{16}\left\|y_k-y_{k-1}\right\|,$$

where

$$C_{14} := C_8+C_9\|A\|, \quad C_{15} := C_{10}+\frac{C_9}{\sigma\beta}, \quad C_{16} := \frac{C_9}{\sigma\beta}.$$

This improvement provides, instead of inequality (5.3.57) in the proof of Theorem 5.3.11, the following very useful estimate

$$\begin{aligned}\Delta_{k,k+1} &= \varphi\left(\mathcal{E}_k\right)-\varphi\left(\mathcal{E}_{k+1}\right) \geqslant \varphi'\left(\mathcal{E}_k\right)\min\left\{\frac{C_1}{4}, \frac{1}{\sigma\beta}\right\}\left(\left\|x_{k+1}-x_k\right\|^2+\left\|y_{k+1}-y_k\right\|^2\right) \\ &\geqslant C_{17}\varphi'\left(\mathcal{E}_k\right)\left(\left\|x_{k+1}-x_k\right\|+\left\|y_{k+1}-y_k\right\|\right)^2,\end{aligned}$$

where

$$C_{17} := \frac{1}{2}\min\left\{\frac{C_1}{4}, \frac{1}{\sigma\beta}\right\}.$$

The last relation together with (5.3.56) imply that for every $k \geqslant k_0$

$$\left(\left\|x_{k+1}-x_k\right\|+\left\|y_{k+1}-y_k\right\|\right)^2 \leqslant \frac{\Delta_{k,k+1}}{C_{17}} \cdot \operatorname{dist}\left(0, \partial\Psi_\beta\left(x_k, z_k, y_k, x_{k-1}, y_{k-1}\right)\right)$$

and from here, for arbitrary $\nu > 0$,

$$\begin{aligned}&\left\|x_{k+1}-x_k\right\|+\left\|y_{k+1}-y_k\right\| \\ \leqslant\ &\frac{\nu\Delta_{k,k+1}}{4C_{17}}+\frac{\max\left\{C_{14}, C_{15}\right\}}{\nu}\left(\left\|x_k-x_{k-1}\right\|+\left\|y_k-y_{k-1}\right\|+\left\|y_{k-1}-y_{k-2}\right\|\right) \\ \leqslant\ &\frac{\nu\Delta_{k,k+1}}{4C_{17}}+\frac{\max\left\{C_{14}, C_{15}\right\}}{\nu}\left(\left\|x_k-x_{k-1}\right\|+\left\|y_k-y_{k-1}\right\|+\left\|x_{k-1}-x_{k-2}\right\|+\left\|y_{k-1}-y_{k-2}\right\|\right).\end{aligned} \tag{5.4.7}$$

By denoting

$$a_k := \left(\left\|x_k-x_{k-1}\right\|+\left\|y_k-y_{k-1}\right\|\right) \geqslant 0 \quad \text{and} \quad d_k := \frac{\nu\Delta_{k,k+1}}{4C_{17}},$$

inequality (5.4.7) can be rewritten for every $k \geqslant k_0$ as

$$a_{k+1} \leqslant \chi_0 \cdot a_k + \chi_1 \cdot a_{k-1} + d_k, \tag{5.4.8}$$

where

$$\chi_0 := \frac{\max\{C_{14}, C_{15}\}}{\nu} \quad \text{and} \quad \chi_1 := \frac{\max\{C_{14}, C_{15}\}}{\nu}.$$

Choosing $\nu > 2\max\{C_{14}, C_{15}\}$, Lemma 2.4.4 and Lemma 5.3.3 imply that $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ has finite length (see (5.3.54)).

Next we prove a recurrence inequality for the sequence $\{\mathcal{E}_k\}_{k\geqslant 0}$.

Lemma 5.4.2. *Let Assumption 5.4.1 be satisfied and $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2, which is assumed to be bounded. If Ψ_β satisfies the Łojasiewicz property with Łojasiewicz constant $C_L > 0$ and Łojasiewicz exponent $\theta \in [0, 1)$, then there exists $k_0 \geqslant 1$ such that the following estimate holds for every $k \geqslant k_0$*

$$\mathcal{E}_{k-1} - \mathcal{E}_{k+1} \geqslant C_{19}\mathcal{E}_{k+1}^{2\theta}, \quad \textit{where} \quad C_{19} := \frac{\min\left\{\frac{C_1}{4}, \frac{1}{\sigma\beta}\right\}}{3C_L^2 \max\{C_{14}, C_{15}\}^2}. \tag{5.4.9}$$

Proof. For every $k \geqslant 2$ we obtain from (5.4.6)

$$\begin{aligned}
\mathcal{E}_{k-1} - \mathcal{E}_{k+1} &= \Psi_{k-1} - \Psi_k + \Psi_k - \Psi_{k+1} \\
&\geqslant \min\left\{\frac{C_1}{4}, \frac{1}{\sigma\beta}\right\}\left(\|x_{k+1} - x_k\|^2 + \|y_{k+1} - y_k\|^2 + \|y_k - y_{k-1}\|^2\right) \\
&\geqslant \frac{1}{3}\min\left\{\frac{C_1}{4}, \frac{1}{\sigma\beta}\right\}\left(\|x_{k+1} - x_k\| + \|y_{k+1} - y_k\| + \|y_k - y_{k-1}\|\right)^2 \\
&\geqslant C_{19}C_L^2 |\!|\!| D^{k+1} |\!|\!|^2.
\end{aligned}$$

Let $\varepsilon > 0$ be such that (5.4.1) is fulfilled and choose $k_0 \geqslant 1$ such that $(x_{k+1}, z_{k+1}, y_{k+1})$ belongs to $\mathbb{B}\left((\widehat{x}, \widehat{z}, \widehat{y}), \varepsilon\right)$ for every $k \geqslant k_0$. Then (5.4.1) implies (5.4.9) for every $k \geqslant k_0$. $\square$

The following convergence rates follow by combining Lemma 2.4.5 with Lemma 5.4.2.

Theorem 5.4.3. *Let Assumption 5.4.1 be satisfied and $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2, which is assumed to be bounded. If Ψ_β satisfies the Łojasiewicz property with Łojasiewicz constant $C_L > 0$ and Łojasiewicz exponent $\theta \in [0, 1)$, then the following statements are true:*

(i) if $\theta = 0$, then $\{\Psi_k\}_{k\geqslant 1}$ converges in finite time;

(ii) if $\theta \in (0, 1/2]$, then there exist $k_0 \geqslant 1$, $\widehat{C}_0 > 0$ and $Q \in [0, 1)$ such that for every $k \geqslant k_0$

$$0 \leqslant \Psi_k - \Psi_* \leqslant \widehat{C}_0 Q^k;$$

(iii) if $\theta \in (1/2, 1)$, then there exist $k_0 \geqslant 3$ and $\widehat{C}_1 > 0$ such that for every $k \geqslant k_0$

$$0 \leqslant \Psi_k - \Psi_* \leqslant \widehat{C}_1 (k-1)^{-\frac{1}{2\theta - 1}}.$$

The next lemma will play an importat role when transferring the convergence rates for $\{\Psi_k\}_{k\geqslant 0}$ to the sequence of iterates $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ (see [83] for a similar statement).

Lemma 5.4.4. *Let Assumption 5.4.1 be satisfied and* $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ *be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2, which is assumed to be bounded. Suppose further that* Ψ_β *satisfies the Łojasiewicz property with Łojasiewicz constant* $C_L > 0$*, Łojasiewicz exponent* $\theta \in [0, 1)$ *and desingularization function*

$$\varphi : [0, +\infty) \to [0, +\infty), \varphi(s) := \frac{1}{1-\theta} C_L s^{1-\theta}.$$

Let $(\widehat{x}, \widehat{z}, \widehat{y})$ *be the KKT point of the optimization problem* (5.1.1) *to which* $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ *converges as* $k \to +\infty$*. Then there exists* $k_0 \geqslant 2$ *such that the following estimates hold for every* $k \geqslant k_0$

$$\|x_k - \widehat{x}\| \leqslant C_{20} \max\left\{\sqrt{\mathcal{E}_k}, \varphi(\mathcal{E}_k)\right\}, \quad \textit{where} \quad C_{20} := \frac{7}{\sqrt{C_{17}}} + \frac{1}{C_{17}}, \tag{5.4.10a}$$

$$\|y_k - \widehat{y}\| \leqslant C_{21} \max\left\{\sqrt{\mathcal{E}_k}, \varphi(\mathcal{E}_k)\right\}, \quad \textit{where} \quad C_{21} := \frac{7}{2\sqrt{C_{17}}} + \frac{1}{2C_{17}}, \tag{5.4.10b}$$

$$\|z_k - \widehat{z}\| \leqslant C_{22} \max\left\{\sqrt{\mathcal{E}_{k-1}}, \varphi(\mathcal{E}_{k-1})\right\}, \quad \textit{where} \quad C_{22} := C_{20}\|A\| + \frac{2C_{21}}{\sigma\beta}. \tag{5.4.10c}$$

Proof. We assume that $\mathcal{E}_k > 0$ for every $k \geqslant 0$. Otherwise, beginning with a given index, the sequence $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ becomes identical to $(\widehat{x}, \widehat{z}, \widehat{y})$ and the conclusion follows as in the proof of Theorem 5.3.11. Let $\varepsilon > 0$ be such that (5.4.1) is fulfilled and $k_0 \geqslant 2$ such that $(x_{k+1}, z_{k+1}, y_{k+1})$ belongs to $\mathbb{B}((\widehat{x}, \widehat{z}, \widehat{y}), \varepsilon)$ for every $k \geqslant k_0$. We fix $k \geqslant k_0$. One can easily notice that

$$\|x_k - \widehat{x}\| \leqslant \|x_{k+1} - x_k\| + \|x_{k+1} - \widehat{x}\| \leqslant \cdots \leqslant \sum_{l\geqslant k} \|x_{l+1} - x_l\| \tag{5.4.11a}$$

and, similarly,

$$\|z_k - \widehat{z}\| \leqslant \sum_{l\geqslant k} \|z_{l+1} - z_l\| \text{ and } \|y_k - \widehat{y}\| \leqslant \sum_{l\geqslant k} \|y_{l+1} - y_l\|. \tag{5.4.11b}$$

Recall that the inequality (5.4.7) can be rewritten as (5.4.8). For $\nu := 3\max\{C_{14}, C_{15}\} > 2\max\{C_{14}, C_{15}\}$, thanks to Lemma 2.4.4 and the estimate (5.4.6), we have that

$$\begin{aligned}
&\sum_{l\geqslant k} \|x_{l+1} - x_l\| = \sum_{l\geqslant k} a_1^{l+1} = \sum_{l\geqslant k+1} a_1^l \\
&\leqslant \|x_{k+1} - x_k\| + 2\|x_{k+2} - x_{k+1}\| + 3\|x_{k+3} - x_{k+2}\| + 2\|y_{k+1} - y_k\| \\
&\quad + 2\|y_{k+2} - y_{k+1}\| + 3\|y_{k+3} - y_{k+2}\| + \frac{\varphi(\mathcal{E}_k)}{C_{17}} \\
&\leqslant \frac{2}{\sqrt{C_{17}}}\sqrt{\Psi_k - \Psi_{k+1}} + \frac{2}{\sqrt{C_{17}}}\sqrt{\Psi_{k+1} - \Psi_{k+2}} + \frac{3}{\sqrt{C_{17}}}\sqrt{\Psi_{k+2} - \Psi_{k+3}} + \frac{\varphi(\mathcal{E}_k)}{C_{17}} \\
&\leqslant \frac{2}{\sqrt{C_{17}}}\sqrt{\mathcal{E}_k} + \frac{2}{\sqrt{C_{17}}}\sqrt{\mathcal{E}_{k+1}} + \frac{3}{\sqrt{C_{17}}}\sqrt{\mathcal{E}_{k+2}} + \frac{\varphi(\mathcal{E}_k)}{C_{17}}
\end{aligned}$$

and, similarly,

$$\sum_{l\geqslant k} \|y_{l+1} - y_l\| \leqslant \frac{1}{\sqrt{C_{17}}}\sqrt{\mathcal{E}_k} + \frac{1}{\sqrt{C_{17}}}\sqrt{\mathcal{E}_{k+1}} + \frac{3}{2\sqrt{C_{17}}}\sqrt{\mathcal{E}_{k+2}} + \frac{\varphi(\mathcal{E}_k)}{2C_{17}}.$$

By taking into account the relations above, (5.4.11a)-(5.4.11b) as well as

$$\sqrt{\mathcal{E}_{k+2}} \leqslant \sqrt{\mathcal{E}_{k+1}} \leqslant \sqrt{\mathcal{E}_k} \qquad \text{and} \qquad \varphi(\mathcal{E}_{k+1}) \leqslant \varphi(\mathcal{E}_k) \ \ \forall k \geqslant 1,$$

the estimates (5.4.10a) and (5.4.10b) follow. Statement (5.4.10c) follows from Lemma 5.3.3 and by considering (5.4.11b). □

We provide now convergence rates for the sequence $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$.

Theorem 5.4.5. *Let Assumption 5.4.1 be satisfied and $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 5.3.1 or Algorithm 5.3.2, which is assumed to be bounded. Suppose further that Ψ_β satisfies the Łojasiewicz property with Łojasiewicz constant $C_L > 0$ and Łojasiewicz exponent $\theta \in [0, 1)$. Let $(\hat{x}, \hat{z}, \hat{y})$ be the KKT point of the optimization problem (5.1.1) to which $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ converges as $k \to +\infty$. Then the following statements are true:*

(i) if $\theta = 0$, then the algorithms converge in finite time;

(ii) if $\theta \in (0, 1/2]$, then there exist $k_0 \geqslant 1$, $\hat{C}_{0,1}, \hat{C}_{0,2}, \hat{C}_{0,3} > 0$ and $\hat{Q} \in [0, 1)$ such that for every $k \geqslant k_0$

$$\|x_k - \hat{x}\| \leqslant \hat{C}_{0,1}\hat{Q}^k, \quad \|y_k - \hat{y}\| \leqslant \hat{C}_{0,2}\hat{Q}^k, \quad \|z_k - \hat{z}\| \leqslant \hat{C}_{0,3}\hat{Q}^k;$$

(iii) if $\theta \in (1/2, 1)$, then there exist $k_0 \geqslant 3$ and $\hat{C}_{1,1}, \hat{C}_{1,2}, \hat{C}_{1,3} > 0$ such that for every $k \geqslant k_0$

$$\|x_k - \hat{x}\| \leqslant \hat{C}_{1,1}(k-1)^{-\frac{1-\theta}{2\theta-1}}, \qquad \|y_k - \hat{y}\| \leqslant \hat{C}_{1,2}(k-1)^{-\frac{1-\theta}{2\theta-1}},$$

$$\|z_k - \hat{z}\| \leqslant \hat{C}_{1,3}(k-2)^{-\frac{1-\theta}{2\theta-1}}.$$

Proof. By denoting $\varphi : [0, +\infty) \to [0, +\infty), \varphi(s) := \frac{1}{1-\theta}C_L s^{1-\theta}$, the desingularization function, there exist $k_0' \geqslant 2$ such that for every $k \geqslant k_0'$ the inequalities (5.4.10a)-(5.4.10c) in Lemma 5.4.4 and $\mathcal{E}_k \leqslant \left(\frac{1}{1-\theta}C_L\right)^{\frac{2}{2\theta-1}}$ hold.

(i) If $\theta = 0$, then $\{\Psi_k\}_{k\geqslant 1}$ converges in finite time. According to (5.4.6), the sequences $\{(x_k)\}_{k\geqslant 0}$ and $\{(y_k)\}_{k\geqslant 0}$ converge also in finite time. Further, by Lemma 5.3.3, it follows that $\{(z_k)\}_{k\geqslant 0}$ converges in finite time, too. In other words, starting from a given index, the sequence $\{(x_k, z_k, y_k)\}_{k\geqslant 0}$ becomes identical to $(\hat{x}, \hat{z}, \hat{y})$ and the conclusion follows.

(ii) If $\theta \in (0, 1/2]$, then $\frac{1}{1-\theta}C_L\mathcal{E}_k^{1-\theta} \leqslant \sqrt{\mathcal{E}_k}$, for every $k \geqslant k_0'$, which implies that

$$\max\left\{\sqrt{\mathcal{E}_k}, \varphi(\mathcal{E}_k)\right\} = \sqrt{\mathcal{E}_k}.$$

By Theorem 5.4.3, there exist $k_0'' \geqslant 1$, $\hat{C}_0 > 0$ and $Q \in [0, 1)$ such that for $\hat{Q} := Q^{\frac{k}{2}}$ and every $k \geqslant k_0''$ it holds

$$\sqrt{\mathcal{E}_k} \leqslant \sqrt{\hat{C}_0 Q^{\frac{k}{2}}} = \sqrt{\hat{C}_0}\hat{Q}^k.$$

The conclusion follows from Lemma 5.4.4 for $k_0 := \max\{k_0', k_0''\}$, by noticing that

$$\sqrt{\mathcal{E}_{k-1}} \leqslant \sqrt{\hat{C}_0 Q^{\frac{k-1}{2}}} = \sqrt{\frac{\hat{C}_0}{Q}}\hat{Q}^k \quad \text{and} \quad \sqrt{\mathcal{E}_{k-2}} \leqslant \sqrt{\hat{C}_0 Q^{\frac{k-2}{2}}} = \frac{\sqrt{\hat{C}_0}}{Q}\hat{Q}^k \forall k \geqslant k_0.$$

(iii) If $\theta \in (1/2, 1)$, then $\mathcal{E}_k^{\frac{1}{2}} \leqslant \frac{1}{1-\theta}C_L\mathcal{E}_k^{1-\theta}$, for every $k \geqslant k_0'$, which implies that

$$\max\left\{\sqrt{\mathcal{E}_k}, \varphi(\mathcal{E}_k)\right\} = \varphi(\mathcal{E}_k) = \frac{1}{1-\theta}C_L\mathcal{E}_k^{1-\theta}.$$

By Theorem 5.4.3, there exist $k_0'' \geqslant 3$ and $\hat{C}_1 > 0$ such that for all $k \geqslant k_0''$

$$\frac{1}{1-\theta}C_L\mathcal{E}_k^{1-\theta} \leqslant \frac{1}{1-\theta}C_L\hat{C}_1^{1-\theta}(k-2)^{-\frac{1-\theta}{2\theta-1}}.$$

The conclusion follows again for $k_0 := \max\{k_0', k_0''\}$ from Lemma 5.4.4. □

Remark 5.4.1. For $\sigma = 1$ the same convergence rates can be obtained under the original Assumption 5.3.1. Indeed, when $\sigma = 1$ we have that $T_1 = 0$ and, as a consequence, the sequence $\{\Psi_k\}_{k\geqslant 1}$ defined in (5.3.36) becomes

$$\Psi_k = \mathcal{L}_\beta (x_k, z_k, y_k) + C_1 \|x_k - x_{k-1}\|^2 \ \ \forall k \geqslant 1.$$

In addition, the inequality (5.3.30) simplifies to

$$\|y_{k+1} - y_k\| \leqslant C_3 \|x_{k+1} - x_k\| + C_4 \|x_k - x_{k-1}\| \ \ \forall k \geqslant 1,$$

as T_2 is equal to 0. Combining this inequality with (5.3.28) and, by taking into account Lemma 5.3.7, we obtain (instead of (5.3.51))

$$|||D^{k+1}||| \leqslant \ C_{11} (\|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \|x_{k-1} - x_{k-2}\|) \ \ \forall k \geqslant 2.$$

Consequently, for every $k \geqslant 3$ we have that

$$\begin{aligned}
\mathcal{E}_{k-2} - \mathcal{E}_{k+1} &= \Psi_{k-2} - \Psi_{k-1} + \Psi_{k-1} - \Psi_k + \Psi_k - \Psi_{k+1} \\
&\geqslant \frac{C_1}{4} \left(\|x_{k-1} - x_{k-2}\|^2 + \|x_k - x_{k-1}\|^2 + \|x_{k+1} - x_k\|^2 \right) \\
&\geqslant \frac{C_1}{12} (\|x_{k-1} - x_{k-2}\| + \|x_k - x_{k-1}\| + \|x_{k+1} - x_k\|)^2 \\
&\geqslant \frac{C_1}{12 C_{11}^2} |||D^{k+1}|||^2.
\end{aligned}$$

Let $\varepsilon > 0$ be such that (5.4.1) is fulfilled and $k_0 \geqslant 3$ such that $(x_{k+1}, z_{k+1}, y_{k+1})$ belongs to the open ball $\mathbb{B} ((\widehat{x}, \widehat{z}, \widehat{y}), \varepsilon)$ for every $k \geqslant k_0$. Then (5.4.1) implies that for every $k \geqslant k_0$

$$\mathcal{E}_{k-2} - \mathcal{E}_{k+1} \geqslant C_{23} \mathcal{E}_{k+1}, \qquad \text{where} \qquad C_{23} := \frac{C_1}{12 C_L^2 C_{11}^2},$$

which is the key inequality for deriving convergence rates, as we have seen above.

5.5 Further perspectives

An interesting future research direction would be to find a setting in which convergence can be provided by avoiding the surjectivity assumption on A. One can also consider an inertial variant of (5.1.1), in order to find a setting where improvements of the convergence rates can be achieved from both theoretical and numerical perspectives.

Another challenging question is to extend the approach in this chapter to problems of the form

$$\min_{x \in \mathcal{H}} \{ f(x) + g(Ax) + h(x) \},$$

where $f \colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper and lower semicontinuous function. A major challenge will be to guarantee the boundedness of the sequence of iterates in the presence of another nonsmooth summand.

Another possibility is to go beyond the setting of compositions with linear operators. Bolte, Sabach and Teboulle have proposed in [37] a generic iterative scheme for solving a general optimization problem of the form (5.1.1) but by replacing the linear operator A with a general nonlinear continuously differentiable operator. A global convergence analysis relying on the use of the Kurdyka-Łojasiewicz property is carried out under so-called uniform regularity condition imposed on the nonlinear operator. This condition reduces to surjectivity when the operator is linear. Another approach has been studied by Drusvyatskiy and Paquette in [77], but the proposed scheme is not stated in the full splitting spirit.

Chapter 6

A proximal minimization algorithm for nonconvex and nonsmooth problems with block structured coupled by a smooth function

This chapter follows our work [52].

We propose a proximal algorithm for minimizing objective functions consisting of three summands: the composition of a nonsmooth function with a linear operator, another nonsmooth function, each of the nonsmooth summands depending on an independent block variable, and a smooth function which couples the two block variables. This can be seen as an extension of the model in [36]. The algorithm is a full splitting method, which means that the nonsmooth functions are processed via their proximal operators, the smooth function via gradient steps, and the linear operator via matrix times vector multiplication. We provide sufficient conditions for the boundedness of the generated sequence and prove that any cluster point of the latter is a KKT point of the minimization problem. In the setting of the Kurdyka-Łojasiewicz property we show global convergence, and derive convergence rates for the iterates in terms of the Łojasiewicz exponent.

6.1 Problem formulation and motivation

Let $\mathcal{H}, \mathcal{G}$ and $\mathcal{K}$ be real finite-dimensional Hilbert spaces. In this chapter we propose a full splitting algorithm for solving nonconvex and nonsmooth problems of the form

$$\min_{(x,y)\in\mathcal{H}\times\mathcal{K}} \left\{ f\left(Ax\right) + g\left(y\right) + h\left(x,y\right) \right\}, \tag{6.1.1}$$

where $f\colon \mathcal{G} \to \mathbb{R}\cup\{+\infty\}$ and $g\colon \mathcal{K} \to \mathbb{R}\cup\{+\infty\}$ are proper and lower semicontinuous functions, $h\colon \mathcal{H}\times\mathcal{K} \to \mathbb{R}$ is a Fréchet differentiable function with Lipschitz continuous gradient, and $A\colon \mathcal{H} \to \mathcal{G}$ is a linear operator. Neither for the nonsmooth nor for the smooth functions convexity is assumed.

In case $\mathcal{H} = \mathcal{G}$ and A is the identity operator, Bolte, Sabach and Teboulle formulated in [36], also in the nonconvex setting, a proximal alternating linearization method (PALM) for solving (6.1.1). PALM is a proximally regularized variant of the Gauss-Seidel alternating minimization scheme and basically consists of two proximal-gradient steps. It had a significant impact in the optimization community, as it can be used to solve a large variety of nonconvex and nonsmooth problems arising in applications such as: matrix factorization, image deblurring and denoising, the feasibility problem, compressed sensing, etc. An inertial version of PALM has been proposed by Pock and Sabach in [115].

A naive approach of PALM for solving (6.1.1) would require the calculation of the proximal operator of the function $f \circ A$, for which, in general, even in the convex case, a closed formula is not available. In the last decade, an impressive progress can be noticed in the field of primal-dual/proximal ADMM algorithms, designed to solve convex optimization problems involving compositions with linear operators in the spirit of the full splitting paradigm. One of the pillars of this development is the conjugate duality theory which is available for convex optimization problems.

The algorithm which we propose in this chapter for solving the nonconvex and nonsmooth problem(6.1.1) is a full splitting scheme, too; the nonsmooth functions are processed via their proximal operators, the smooth function via gradient steps, and the linear operator via matrix times vector multiplication. In case $g(y) = 0$ and $h(x, y) = h(x)$ for any $(x, y) \in \mathcal{H} \times \mathcal{K}$, where $h \colon \mathcal{H} \to \mathbb{R}$ is a Fréchet differentiable function with Lipschitz continuous gradient, it furnishes a full splitting iterative scheme for solving the nonsmooth and nonconvex optimization problem

$$\min_{x \in \mathcal{H}} \{f(Ax) + h(x)\}. \tag{6.1.2}$$

Splitting algorithms for solving problems of the form (6.1.2) have been considered in [96], under the assumption that h is twice continuously differentiable with bounded Hessian, in [128], under the assumption that one of the summands is convex and continuous on its effective domain, and in [56], as a particular case of a general nonconvex proximal ADMM algorithm. We would like to mention in this context also [37] for the case when A is nonlinear.

The convergence analysis we will carry out in this chapter relies on a descent inequality, which we prove for a regularization of the augmented Lagrangian $L_\beta : \mathcal{H} \times \mathcal{K} \times \mathcal{G} \times \mathcal{G} \to \mathbb{R} \cup \{+\infty\}$

$$L_\beta(x, y, z, u) = f(z) + g(y) + h(x, y) + \langle u, Ax - z\rangle + \frac{\beta}{2}\|Ax - z\|^2, \beta > 0,$$

associated with problem (6.1.1). This is obtained by an appropriate tuning of the parameters involved in the description of the algorithm. In addition, we provide sufficient conditions in terms of the input functions f, g and h for the boundedness of the generated sequence of iterates. We also show that any cluster point of this sequence is a KKT point of the optimization problem (6.1.1). By assuming that the above-mentioned regularization of the augmented Lagrangian satisfies the Kurdyka-Łojasiewicz property, we prove global convergence. If this function satisfies the Łojasiewicz property, then we can even derive convergence rates for the sequence of iterates formulated in terms of the Łojasiewicz exponent. For similar approaches relying on the use of the Kurdyka-Łojasiewicz property in the proof of the global convergence of nonconvex optimization algorithms we refer to the papers of Attouch and Bolte [5], Attouch, Bolte and Svaiter [8], and Bolte, Sabach and Teboulle [36].

6.2 The algorithm

The numerical algorithm we propose for solving (6.1.1) has the following formulation.

Algorithm 6.2.1. *Let $\mu, \beta, \tau > 0$ and $0 < \sigma \leqslant 1$. For a given starting point $(x_0, y_0, z_0, u_0) \in \mathcal{H} \times \mathcal{K} \times \mathcal{G} \times \mathcal{G}$ generate the sequence $\{(x_k, y_k, z_k, u_k)\}_{k \geqslant 0}$ for any $k \geqslant 0$ as follows*

$$y_{k+1} \in \arg\min_{y \in \mathcal{K}} \left\{ g(y) + \langle \nabla_y h(x_k, y_k), y\rangle + \frac{\mu}{2}\|y - y_k\|^2 \right\} \tag{6.2.1a}$$

$$z_{k+1} \in \arg\min_{z \in \mathcal{G}} \left\{ f(z) + \langle u_k, Ax_k - z\rangle + \frac{\beta}{2}\|Ax_k - z\|^2 \right\} \tag{6.2.1b}$$

$$x_{k+1} := x_k - \tau^{-1}\left(\nabla_x h(x_k, y_{k+1}) + A^* u_k + \beta A^*(Ax_k - z_{k+1})\right) \tag{6.2.1c}$$

$$u_{k+1} := u_k + \sigma\beta(Ax_{k+1} - z_{k+1}). \tag{6.2.1d}$$

In view of the proximal point, the iterative scheme (6.2.1a) - (6.2.1d) reads for every $k \geqslant 0$

$$\begin{aligned}
&y_{k+1} \in \operatorname{prox}_{\mu^{-1}g}\left(y_k - \mu^{-1}\nabla_y h\left(x_k, y_k\right)\right)\\
&z_{k+1} \in \operatorname{prox}_{\beta^{-1}f}\left(Ax_k + \beta^{-1}u_k\right)\\
&x_{k+1} := x_k - \tau^{-1}\left(\nabla_x h\left(x_k, y_{k+1}\right) + A^* u_k + \beta A^*\left(Ax_k - z_{k+1}\right)\right)\\
&u_{k+1} := u_k + \sigma\beta\left(Ax_{k+1} - z_{k+1}\right).
\end{aligned}$$

One can notice the full splitting character of Algorithm 6.2.1 and also that the first two steps can be performed in parallel.

Remark 6.2.1. (i) In case $g(y) = 0$ and $h(x,y) = h(x)$ for any $(x,y) \in \mathcal{H} \times \mathcal{K}$, where $H : \mathcal{H} \to \mathbb{R}$ is a Fréchet differentiable function with Lipschitz continuous gradient, Algorithm 6.2.1 gives rise to an iterative scheme which has been proposed in [56] for solving the optimization problem (6.1.2). This reads for any $k \geqslant 0$

$$\begin{aligned}
&z_{k+1} \in \operatorname{prox}_{\beta^{-1}f}\left(Ax_k + \beta^{-1}u_k\right)\\
&x_{k+1} := x_k - \tau^{-1}\left(\nabla_x h\left(x_k\right) + A^* u_k + \beta A^*\left(Ax_k - z_{k+1}\right)\right)\\
&u_{k+1} := u_k + \sigma\beta\left(Ax_{k+1} - z_{k+1}\right).
\end{aligned}$$

(ii) In case $\mathcal{H} = \mathcal{G}$ and $A = \text{Id}$ is the identity operator on $\mathcal{H}$, Algorithm 6.2.1 gives rise to an iterative scheme for solving

$$\min_{(x,y)\in\mathcal{H}\times\mathcal{K}} \left\{f(x) + g(y) + h(x,y)\right\}, \tag{6.2.2}$$

which reads for any $k \geqslant 0$

$$\begin{aligned}
&y_{k+1} \in \operatorname{prox}_{\mu^{-1}g}\left(y_k - \mu^{-1}\nabla_y h\left(x_k, y_k\right)\right)\\
&z_{k+1} \in \operatorname{prox}_{\beta^{-1}f}\left(x_k + \beta^{-1}u_k\right)\\
&x_{k+1} := x_k - \tau^{-1}\left(\nabla_x h\left(x_k, y_{k+1}\right) + u_k + \beta\left(x_k - z_{k+1}\right)\right)\\
&u_{k+1} := u_k + \sigma\beta\left(x_{k+1} - z_{k+1}\right).
\end{aligned}$$

This algorithm provides an alternative to PALM ([36]) for solving optimization problems of the form (6.2.2). We will give more detail in the next r emark.

(iii) In case $\mathcal{H} = \mathcal{G}$, $A = \text{Id}$, $f(x) = 0$ and $h(x,y) = h(y)$ for any $(x,y) \in \mathcal{H} \times \mathcal{K}$, where $H : \mathcal{K} \to \mathbb{R}$ is a Fréchet differentiable function with Lipschitz continuous gradient, Algorithm 6.2.1 gives rise to an iterative scheme for solving

$$\min_{y\in\mathcal{K}} \left\{g(y) + h(y)\right\}, \tag{6.2.3}$$

which reads for any $k \geqslant 0$

$$y_{k+1} \in \operatorname{prox}_{\mu^{-1}g}\left(y_k - \mu^{-1}\nabla h\left(y_k\right)\right),$$

and is nothing else than the proximal-gradient method. An inertial version of the proximal-gradient method for solving (6.2.3) in the fully nonconvex setting has been considered in [51].

Remark 6.2.2. Recall that the Proximal Alternating Linearized Minimization algorithm (or PALM) considered by Bolte, Sabach and Teboulle in [36], is designed to tackle the optimization problem (6.2.2) and it reads for every $k \geqslant 0$

$$\begin{aligned}
&y_{k+1} \in \operatorname{prox}_{\mu^{-1}g}\left(y_k - \mu^{-1}\nabla_y h\left(x_k, y_k\right)\right)\\
&z_{k+1} \in \operatorname{prox}_{\tau^{-1}f}\left(x_k + \tau^{-1}u_k\nabla_x h\left(x_k, y_{k+1}\right)\right).
\end{aligned}$$

Since the introduction of this algorithm, it hreceived a massive amount of attention due to its effectiveness and simplicity, while it still covers many fields of applications. It is, however, probably not a suitable scheme for (6.1.1) since it requires the calculation of the proximal operator of the function $f \circ A$, for which, in general, even in the convex case, a closed formula is not available.

Assumption 6.2.1. *In [36], the authors considered the convergence analysis under the following assumption:*

(i) the functions f, g and $f + g + h$ are bounded from below;

(ii) for any fixed $y \in \mathcal{K}$ there exists $L_1(y) \geqslant 0$ such that

$$\left\|\nabla_x h(x, y) - \nabla_x h(x', y)\right\| \leqslant L_1(y) \left\|x - x'\right\| \qquad \forall x, x' \in \mathcal{H}, \tag{6.2.4a}$$

and for any fixed $x \in \mathcal{H}$ there exist $L_2(x) \geqslant 0$ such that

$$\left\|\nabla_y h(x, y) - \nabla_y h(x, y')\right\| \leqslant L_2(x) \left\|y - y'\right\| \qquad \forall y, y' \in \mathcal{K}; \tag{6.2.4b}$$

(iii) there exist $L_{i,+} > 0, i = 1, 2,$ such that

$$\sup_{k \geqslant 0} L_1(y_k) \leqslant L_{1,+}, \qquad \sup_{k \geqslant 0} L_2(x_k) \leqslant L_{2,+}; \tag{6.2.5}$$

(iv) ∇H is Lipschitz continuous with constant $L > 0$ on a convex bounded subset $B_1 \times B_2 \subseteq \mathcal{H} \times \mathcal{K}$ containing $\{(x_k, y_k)\}_{k \geqslant 0}$. In other words, for any $(x, y), (x', y') \in B_1 \times B_2$ it holds

$$|||\left(\nabla_x h(x, y) - \nabla_x h(x', y'), \nabla_y h(x, y) - \nabla_y h(x', y')\right)||| \leqslant L|||(x, y) - (x', y')|||. \tag{6.2.6}$$

Together further with the KŁ property, it was shown that the sequence $\{(x_k, y_k)\}_{k \geqslant 1}$ converges to a critical point of (6.2.2). In the following, one can see that we will derive our convergence analysis under assumptionsof a similar flavors.

6.2.1 A descent inequality

We will start with the convergence analysis of Algorithm 6.2.1 by proving a descent inequality, which will play a fundamental role in our investigations. We will analyse Algorithm 6.2.1 under the following assumptions, which we will be later even weakened.

Assumption 6.2.2. *(i) the functions f, g and h are bounded from below;*

(ii) the linear operator A is surjective;

(iii) for any fixed $y \in \mathcal{K}$ there exists $L_1(y) \geqslant 0$ such that

$$\left\|\nabla_x h(x, y) - \nabla_x h(x', y)\right\| \leqslant L_1(y) \left\|x - x'\right\| \qquad \forall x, x' \in \mathcal{H}, \tag{6.2.7a}$$

and for any fixed $x \in \mathcal{H}$ there exist $L_2(x), L_3(x) \geqslant 0$ such that

$$\left\|\nabla_y h(x, y) - \nabla_y h(x, y')\right\| \leqslant L_2(x) \left\|y - y'\right\| \qquad \forall y, y' \in \mathcal{K}, \tag{6.2.7b}$$
$$\left\|\nabla_x h(x, y) - \nabla_x h(x, y')\right\| \leqslant L_3(x) \left\|y - y'\right\| \qquad \forall y, y' \in \mathcal{K}; \tag{6.2.7c}$$

(iv) there exist $L_{i,+} > 0, i = 1, 2, 3,$ such that

$$\sup_{k \geqslant 0} L_1(y_k) \leqslant L_{1,+}, \qquad \sup_{k \geqslant 0} L_2(x_k) \leqslant L_{2,+}, \qquad \sup_{k \geqslant 0} L_3(x_k) \leqslant L_{3,+}. \tag{6.2.8}$$

Remark 6.2.3. Some comments on Assumption 6.2.2 are in order.

(i) Assumption (i) ensures that the sequence generated by Algorithm 6.2.1 is well-defined. It has also as consequence that

$$\underline{\Psi} := \inf_{(x,y,z)\times\mathcal{H}\times\mathcal{K}\times\mathcal{G}} \{f(z) + g(y) + h(x,y)\} > -\infty. \tag{6.2.9}$$

(ii) Comparing the assumptions in (iii) and (iv) to the ones in [36], one can notice the presence of the additional condition (6.2.7c), which is essential in particular when proving the boundedness of the sequence of generated iterates. Notice that in iterative schemes of gradient type, proximal-gradient type or forward-backward-forward type (see [36, 43, 51]) the boundedness of the iterates follow by combining a descent inequality expressed in terms of the objective function with coercivity assumptions on the later. In our setting this undertaken is less simple, since the descent inequality which we obtain below is in terms of the augmented Lagrangian associated with problem (6.1.1).

(iii) The linear operator A is surjective if and only if its associated matrix has full row rank, which is the same with the fact that the matrix associated to AA^* is positively definite. Since

$$\lambda_{\min}(AA^*)\|z\|^2 \leqslant \langle AA^*z, z\rangle = \|A^*z\|^2 \quad \forall z \in \mathcal{G},$$

this is further equivalent to $\lambda_{\min}(AA^*) > 0$, where $\lambda_{\min}(M)$ denotes the minimal eigenvalue of a square matrix M. In addition, we denote by $\kappa(M)$ the condition number, namely the ratio between the maximal eigenvalue $\lambda_{\max}(M)$ and the minimal eigenvalue $\lambda_{\min}(M)$ of the square matrix M where the matrix norm is defined as

$$\kappa(M) := \frac{\lambda_{\max}(M)}{\lambda_{\min}(M)} = \frac{\|M\|^2}{\lambda_{\min}(M)} \geqslant 1,$$

where the matrix norm is defined as

$$\|M\| := \sup_{z\in\mathcal{G}} \frac{\|Mz\|}{\|z\|}.$$

The convergence analysis will make use of the following regularized augmented Lagrangian function

$$\Psi\colon \mathcal{H}\times\mathcal{K}\times\mathcal{G}\times\mathcal{G}\times\mathcal{H}\times\mathcal{G} \to \mathbb{R}\cup\{+\infty\},$$

defined as

$$\begin{aligned}(x,y,z,u,x',u') \mapsto\ & f(z) + g(y) + h(x,y) + \langle u, Ax - z\rangle + \frac{\beta}{2}\|Ax - z\|^2 \\ & + C_0\left\|A^*(u-u') + \sigma B(x-x')\right\|^2 + C_1\|x-x'\|^2,\end{aligned}$$

where

$$B := \tau\mathrm{Id} - \beta A^*A, \qquad C_0 := \frac{4(1-\sigma)}{\sigma^2\beta\lambda_{\min}(AA^*)} \geqslant 0 \qquad \text{and} \qquad C_1 := \frac{8(\sigma\tau + L_{1,+})^2}{\sigma\beta\lambda_{\min}(AA^*)} > 0.$$

Notice that

$$\|B\| \leqslant \tau,$$

whenever $2\tau \geqslant \beta\|A\|^2$. Indeed, this is a consequence of the relation

$$\|Bx\|^2 = \tau^2\|x\|^2 - 2\tau\beta\|Ax\|^2 + \beta^2\|A^*Ax\|^2 \leqslant \tau^2\|x\|^2 + \beta\left(\beta\|A\|^2 - 2\tau\right)\|Ax\|^2 \quad \forall x \in \mathcal{H}.$$

For simplification, we introduce the following notations

$$\begin{aligned}
\mathbf{R} &:= \mathcal{H} \times \mathcal{K} \times \mathcal{G} \times \mathcal{G} \times \mathcal{H} \times \mathcal{G} \\
\mathbf{X} &:= \left(x, y, z, u, x', u'\right) \\
\mathbf{X}_k &:= \left(x_k, y_k, z_k, u_k, x_{k-1}, u_{k-1}\right) \ \forall k \geqslant 1 \\
\Psi_k &:= \Psi\left(\mathbf{X}_k\right) \ \forall k \geqslant 1.
\end{aligned}$$

By the nature of the scheme, we can derive the following statement.

Lemma 6.2.1. *Let Assumption 6.2.2 be satisfied, $2\tau \geqslant \beta \|A\|^2$ and $\{(x_k, y_k, z_k, u_k)\}_{k \geqslant 0}$ be a sequence generated by Algorithm 6.2.1. Then for any $k \geqslant 1$ it holds*

$$\begin{aligned}
& f\left(z_{k+1}\right) + g\left(y_{k+1}\right) + h\left(x_{k+1}, y_{k+1}\right) + \left\langle u_{k+1}, Ax_{k+1} - z_{k+1}\right\rangle + \frac{\beta}{2}\left\|Ax_{k+1} - z_{k+1}\right\|^2 \\
& + \left(\tau - \frac{L_{1,+} + \beta\|A\|^2}{2}\right)\left\|x_{k+1} - x_k\right\|^2 + \frac{\mu - L_{2,+}}{2}\left\|y_{k+1} - y_k\right\|^2 + \frac{1}{\sigma\beta}\left\|u_{k+1} - u_k\right\|^2 \\
\leqslant\ & f\left(z_k\right) + g\left(y_k\right) + h\left(x_k, y_k\right) + \left\langle u_k, Ax_k - z_k\right\rangle + \frac{\beta}{2}\left\|Ax_k - z_k\right\|^2 + \frac{2}{\sigma\beta}\left\|u_{k+1} - u_k\right\|^2. \quad (6.2.10)
\end{aligned}$$

Proof. Let $k \geqslant 1$ be fixed. On the one hand, from (6.2.1a) and (6.2.1b) we obtain

$$g\left(y_{k+1}\right) + \left\langle \nabla_y h\left(x_k, y_k\right), y_{k+1} - y_k\right\rangle + \frac{\mu}{2}\left\|y_{k+1} - y_k\right\|^2 \leqslant g\left(y_k\right)$$

and

$$f\left(z_{k+1}\right) + \left\langle u_k, Ax_k - z_{k+1}\right\rangle + \frac{\beta}{2}\left\|Ax_k - z_{k+1}\right\|^2 \leqslant f\left(z_k\right) + \left\langle u_k, Ax_k - z_k\right\rangle + \frac{\beta}{2}\left\|Ax_k - z_k\right\|^2$$

respectively. Adding both sides of these relation leads to

$$\begin{aligned}
& f\left(z_{k+1}\right) + g\left(y_{k+1}\right) + \left\langle u_k, Ax_k - z_{k+1}\right\rangle + \frac{\beta}{2}\left\|Ax_k - z_{k+1}\right\|^2 + \left\langle \nabla_y h\left(x_k, y_k\right), y_{k+1} - y_k\right\rangle \\
& + \frac{\mu}{2}\left\|y_{k+1} - y_k\right\|^2 \leqslant f\left(z_k\right) + g\left(y_k\right) + \left\langle u_k, Ax_k - z_k\right\rangle + \frac{\beta}{2}\left\|Ax_k - z_k\right\|^2. \quad (6.2.11)
\end{aligned}$$

On the other hand, according to the Descent Lemma (2.2.4) we have

$$\begin{aligned}
h\left(x_k, y_{k+1}\right) &\leqslant h\left(x_k, y_k\right) + \left\langle \nabla_y h\left(x_k, y_k\right), y_{k+1} - y_k\right\rangle + \frac{L_2\left(x_k\right)}{2}\left\|y_{k+1} - y_k\right\|^2 \\
&\leqslant h\left(x_k, y_k\right) + \left\langle \nabla_y h\left(x_k, y_k\right), y_{k+1} - y_k\right\rangle + \frac{L_{2,+}}{2}\left\|y_{k+1} - y_k\right\|^2
\end{aligned}$$

and, further, by taking into consideration (6.2.1c),

$$\begin{aligned}
h\left(x_{k+1}, y_{k+1}\right) &\leqslant h\left(x_k, y_{k+1}\right) + \left\langle \nabla_x h\left(x_k, y_{k+1}\right), x_{k+1} - x_k\right\rangle + \frac{L_1\left(y_{k+1}\right)}{2}\left\|x_{k+1} - x_k\right\|^2 \\
&= h\left(x_k, y_{k+1}\right) - \left\langle u_k, Ax_{k+1} - Ax_k\right\rangle - \beta\left\langle Ax_k - z_{k+1}, Ax_{k+1} - Ax_k\right\rangle \\
&\quad - \left(\tau - \frac{L_1\left(y_{k+1}\right)}{2}\right)\left\|x_{k+1} - x_k\right\|^2 \\
&\leqslant h\left(x_k, y_{k+1}\right) - \left\langle u_k, Ax_{k+1} - Ax_k\right\rangle + \frac{\beta}{2}\left\|Ax_k - z_{k+1}\right\|^2 - \frac{\beta}{2}\left\|Ax_{k+1} - z_{k+1}\right\|^2 \\
&\quad - \left(\tau - \frac{L_{1,+} + \beta\|A\|^2}{2}\right)\left\|x_{k+1} - x_k\right\|^2.
\end{aligned}$$

Combining these above estimates we get

$$
\begin{aligned}
& h\left(x_{k+1}, y_{k+1}\right)+\left\langle u_k, A x_{k+1}-A x_k\right\rangle-\frac{\beta}{2}\left\|A x_k-z_{k+1}\right\|^2+\frac{\beta}{2}\left\|A x_{k+1}-z_{k+1}\right\|^2 \\
& \quad-\frac{L_{2,+}}{2}\left\|y_{k+1}-y_k\right\|^2\left(\tau-\frac{L_{1,+}+\beta\|A\|^2}{2}\right)\left\|x_{k+1}-x_k\right\|^2 \\
& \leqslant h\left(x_k, y_{k+1}\right)+\left\langle\nabla_y h\left(x_k, y_k\right), y_{k+1}-y_k\right\rangle . \qquad (6.2.12)
\end{aligned}
$$

Summing (6.2.11) and (6.2.12), then using the iterate (6.2.1d). After adding $\frac{2}{\sigma\beta}\left\|u_{k+1}-u_k\right\|^2$ on both side of the obtained result, we get the inequality (6.2.10). □

Next we will focus on estimating $\left\|u_{k+1}-u_k\right\|^2$.

Lemma 6.2.2. *Let Assumption 6.2.2 be satisfied,* $2\tau \geqslant \beta\|A\|^2$ *and* $\left\{\left(x_k, y_k, z_k, u_k\right)\right\}_{k \geqslant 0}$ *be a sequence generated by Algorithm 6.2.1. Then for any* $k \geqslant 1$ *it holds*

$$
\begin{aligned}
& \frac{\sigma \lambda_{\min }\left(A A^*\right)}{2}\left\|u_{k+1}-u_k\right\|^2+(1-\sigma)\left\|A^*\left(u_{k+1}-u_k\right)+\sigma B\left(x_{k+1}-x_k\right)\right\|^2 \\
& -\sigma^3 \tau^2\left\|x_{k+1}-x_k\right\|^2-2 \sigma L_{3,+}^2\left\|y_{k+1}-y_k\right\|^2 \\
& \leqslant(1-\sigma)\left\|A^*\left(u_k-u_{k-1}\right)+\sigma B\left(x_k-x_{k-1}\right)\right\|^2+2 \sigma\left(\sigma \tau+L_{1,+}\right)^2\left\|x_k-x_{k-1}\right\|^2 . \qquad (6.2.13)
\end{aligned}
$$

Proof. Let $k \geqslant 1$ be fixed. Let us now rewrite (6.2.1c)

$$
\begin{aligned}
\tau\left(x_{k+1}-x_k\right) & =\nabla_x h\left(x_k, y_{k+1}\right)+A^* u_k+\beta A^*\left(A x_{k+1}-z_{k+1}\right)+\beta A^* A\left(x_k-x_{k+1}\right) \\
& =\nabla_x h\left(x_k, y_{k+1}\right)+A^* u_k+\frac{1}{\sigma} A^*\left(u_{k+1}-u_k\right)+\beta A^* A\left(x_k-x_{k+1}\right), \qquad (6.2.14)
\end{aligned}
$$

where the last equation is due to (6.2.1d). Multiplying bothside by σ, after rearranging the terms we get

$$
A^* u_{k+1}+\sigma B\left(x_{k+1}-x_k\right)=(1-\sigma) A^* u_k-\sigma \nabla_x h\left(x_k, y_{k+1}\right)
$$

and, similarly

$$
A^* u_k+\sigma B\left(x_k-x_{k-1}\right)=(1-\sigma) A^* u_{k-1}-\sigma \nabla_x h\left(x_{k-1}, y_k\right) .
$$

Subtracting these relations and making use of the notations

$$
\begin{aligned}
w_k & :=A^*\left(u_k-u_{k-1}\right)+\sigma B\left(x_k-x_{k-1}\right) \\
v_k & :=\sigma B\left(x_k-x_{k-1}\right)+\nabla_x h\left(x_{k-1}, y_k\right)-\nabla_x h\left(x_k, y_{k+1}\right),
\end{aligned}
$$

it yields

$$
w_{k+1}=(1-\sigma) w_k+\sigma v_k .
$$

The convexity of $\|\cdot\|^2$ guarantees that (notice that $0<\sigma \leqslant 1$)

$$
\left\|w_{k+1}\right\|^2 \leqslant(1-\sigma)\left\|w_k\right\|^2+\sigma\left\|v_k\right\|^2 . \qquad (6.2.15)
$$

In addition, from the definitions of w_k and v_k, we obtain

$$
\left\|A^*\left(u_{k+1}-u_k\right)\right\| \leqslant\left\|w_{k+1}\right\|+\sigma\|B\|\left\|x_{k+1}-x_k\right\| \leqslant\left\|w_{k+1}\right\|+\sigma \tau\left\|x_{k+1}-x_k\right\| \qquad (6.2.16)
$$

and

$$
\begin{aligned}
\left\|v_k\right\| & \leqslant \sigma\|B\|\left\|x_k-x_{k-1}\right\|+\left\|\nabla_x h\left(x_{k-1}, y_k\right)-\nabla_x h\left(x_k, y_{k+1}\right)\right\| \\
& \leqslant \sigma \tau\left\|x_k-x_{k-1}\right\|+\left\|\nabla_x h\left(x_{k-1}, y_k\right)-\nabla_x h\left(x_k, y_k\right)\right\|+\left\|\nabla_x h\left(x_k, y_k\right)-\nabla_x h\left(x_k, y_{k+1}\right)\right\| \\
& \leqslant\left(\sigma \tau+L_{1,+}\right)\left\|x_k-x_{k-1}\right\|+L_{3,+}\left\|y_{k+1}-y_k\right\| \qquad (6.2.17)
\end{aligned}
$$

respectively. Using the Cauchy-Schwarz inequality, (6.2.16) yields

$$\frac{\lambda_{\min}(AA^*)}{2}\|u_{k+1}-u_k\|^2 \leqslant \frac{1}{2}\|A^*(u_{k+1}-u_k)\|^2 \leqslant \|w_{k+1}\|^2 + \sigma^2\tau^2\|x_{k+1}-x_k\|^2$$

and (6.2.17) yields

$$\|v_k\|^2 \leqslant 2(\sigma\tau + L_{1,+})^2\|x_k - x_{k-1}\|^2 + 2L_{3,+}^2\|y_{k+1}-y_k\|^2.$$

Multiplying both relations by σ. After combining the obtained results with (6.2.15), we get (6.2.13). □

The next result provides the announced descent inequality.

Lemma 6.2.3. *Let Assumption 6.2.2 be satisfied, $2\tau \geqslant \beta\|A\|^2$ and $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 6.2.1. Then for any $k \geqslant 1$ it holds*

$$\Psi_{n+1} + C_2\|x_{k+1}-x_k\|^2 + C_3\|y_{k+1}-y_k\|^2 + C_4\|u_{k+1}-u_k\|^2 \leqslant \Psi_k, \tag{6.2.18}$$

where

$$C_2 := \tau - \frac{L_{1,+} + \beta\|A\|^2}{2} - \frac{4\sigma\tau^2}{\beta\lambda_{\min}(AA^*)} - \frac{8(\sigma\tau + L_{1,+})^2}{\sigma\beta\lambda_{\min}(AA^*)}, \tag{6.2.19a}$$

$$C_3 := \frac{\mu - L_{2,+}}{2} - \frac{8L_{3,+}^2}{\sigma\beta\lambda_{\min}(AA^*)}, \tag{6.2.19b}$$

$$C_4 := \frac{1}{\sigma\beta}. \tag{6.2.19c}$$

Proof. Let $k \geqslant 1$ be fixed. We multiply the estimate (6.2.13) by $\dfrac{4}{\sigma^2\beta\lambda_{\min}(AA^*)} > 0$ to get

$$\begin{aligned}
&\frac{2}{\sigma\beta}\|u_{k+1}-u_k\|^2 + C_0\|A^*(u_{k+1}-u_k) + \sigma B(x_{k+1}-x_k)\|^2 \\
&- \frac{4\sigma\tau^2}{\beta\lambda_{\min}(AA^*)}\|x_{k+1}-x_k\|^2 - \frac{8L_{3,+}^2}{\sigma\beta\lambda_{\min}(AA^*)}\|y_{k+1}-y_k\|^2 \\
\leqslant\ & C_0\|A^*(u_k-u_{k-1}) + \sigma B(x_k - x_{k-1})\|^2 + C_1\|x_k-x_{k-1}\|^2.
\end{aligned}$$

The desired statement follows after and combine the resulting inequality with (6.2.10). □

The following result provides one possibility to choose the parameters in Algorithm 6.2.1, such that all three constants C_2, C_3 and C_4 that appear in (6.2.18) are positive.

Lemma 6.2.4. *Let*

$$0 < \sigma < \frac{1}{24\kappa(AA^*)} \tag{6.2.20a}$$

$$\beta > \frac{\nu}{1 - 24\sigma\kappa(AA^*)}\left(4 + 3\sigma + \sqrt{24 + 24\sigma + 9\sigma^2 - 192\sigma\kappa(AA^*)}\right) > 0 \tag{6.2.20b}$$

$$\max\left\{\frac{\beta\|A\|^2}{2}, \frac{\beta\lambda_{\min}(AA^*)}{24\sigma}\left(1 - \frac{4\nu}{\beta} - \sqrt{\Delta'_\tau}\right)\right\} < \tau < \frac{\beta\lambda_{\min}(AA^*)}{24\sigma}\left(1 - \frac{4\nu}{\beta} + \sqrt{\Delta'_\tau}\right) \tag{6.2.20c}$$

$$\mu > L_{2,+} + \frac{16L_{3,+}^2}{\sigma\beta\lambda_{\min}(AA^*)} > 0, \tag{6.2.20d}$$

where

$$\nu := \frac{4L_{1,+}}{\lambda_{\min}(AA^*)} > 0 \text{ and } \Delta'_\tau := 1 - \frac{8\nu}{\beta} - \frac{8\nu^2}{\beta^2} - \frac{6\nu\sigma}{\beta} - 24\sigma\kappa(AA^*) > 0. \tag{6.2.20e}$$

Then we have

$$\min\{C_2, C_3, C_4\} > 0.$$

Furthermore, there exist $\gamma_1, \gamma_2 \in \mathbb{R}\backslash\{0\}$ *such that*

$$\frac{1}{\gamma_1} - \frac{L_{1,+}}{2\gamma_1^2} = \frac{1}{\beta\lambda_{\min}(AA^*)} \quad \text{and} \quad \frac{1}{\gamma_2} - \frac{L_{1,+}}{2\gamma_2^2} = \frac{2}{\beta\lambda_{\min}(AA^*)}. \tag{6.2.21}$$

Proof. We will prove first that $C_2 > 0$, or, equivalently

$$-2C_2 = \frac{24\sigma\tau^2}{\beta\lambda_{\min}(AA^*)} - 2\left(1 - \frac{16L_{1,+}}{\beta\lambda_{\min}(AA^*)}\right)\tau + \frac{16L_{1,+}^2}{\sigma\beta\lambda_{\min}(AA^*)} + L_{1,+} + \beta\|A\|^2 < 0. \tag{6.2.22}$$

The reduced discriminant of the quadratic function in τ in the above relation reads

$$\begin{aligned}\Delta'_\tau &:= \left(1 - \frac{16L_{1,+}}{\beta\lambda_{\min}(AA^*)}\right)^2 - \frac{384L_{1,+}^2}{\beta^2\lambda_{\min}^2(AA^*)} - \frac{24L_{1,+}\sigma}{\beta\lambda_{\min}(AA^*)} - 24\sigma\kappa(AA^*)\\ &= \left(1 - \frac{4\nu}{\beta}\right)^2 - \frac{24\nu^2}{\beta^2} - \frac{6\nu\sigma}{\beta} - 24\sigma\kappa(AA^*)\\ &= 1 - \frac{8\nu}{\beta} - \frac{8\nu^2}{\beta^2} - \frac{6\nu\sigma}{\beta} - 24\sigma\kappa(AA^*) > 0,\end{aligned} \tag{6.2.23}$$

if σ and β are being chosen as in (6.2.20a) and (6.2.20b), respectively. Indeed, the inequality (6.2.23) can be rewritten as

$$(1 - 24\sigma\kappa(AA^*))\beta^2 - 2(4 + 3\sigma)\nu - 8\nu^2 > 0, \tag{6.2.24}$$

which has its discriminant reads

$$\Delta_\beta := (4 + 3\sigma)^2 + 8(1 - 24\sigma\kappa(AA^*))\nu^2 = 24 + 24\sigma + 9\sigma^2 - 192\sigma\kappa(AA^*) > 0$$

as $24 - 192\sigma\kappa(AA^*) = 16 + 8(1 - 24\sigma\kappa(AA^*)) > 0$ for every σ satisfies (6.2.20a). Hence, for every σ and β satisfy (6.2.20a) and (6.2.20b), the inequality (6.2.24) holds true and thus (6.2.23). Therefore, for

$$\frac{\beta\lambda_{\min}(AA^*)}{24\sigma}\left(1 - \frac{4\nu}{\beta} - \sqrt{\Delta'_\tau}\right) < \tau < \frac{\beta\lambda_{\min}(AA^*)}{24\sigma}\left(1 - \frac{4\nu}{\beta} + \sqrt{\Delta'_\tau}\right),$$

(6.2.22) is satisfied. It remains to verify the feasibility of τ in (6.2.20c), in other words, to prove that

$$\frac{\beta\|A\|^2}{2} < \frac{\beta\lambda_{\min}(AA^*)}{24\sigma}\left(1 - \frac{4\nu}{\beta} + \sqrt{\Delta'_\tau}\right).$$

This is easy to see, as, according to (6.2.23), we have

$$\frac{\beta\|A\|^2}{2} < \frac{\beta\lambda_{\min}(AA^*)}{24\sigma}\left(1 - \frac{4\nu}{\beta}\right) \Leftrightarrow 1 - \frac{4\nu}{\beta} - 12\sigma\kappa(AA^*) > 0.$$

The positivity of C_3 follows from the choice of μ in (6.2.20d), while, obviously, $C_4 > 0$.

Finally, two quadratic equations in (6.2.21) (in γ_1 and, respectively, γ_2) has their discriminant reads as

$$\Delta_{\gamma_1} := 1 - \frac{2L_{1,+}}{\beta\lambda_{\min}(AA^*)} = 1 - \frac{\nu}{2\beta} \quad \text{and} \quad \Delta_{\gamma_2} := 1 - \frac{L_{1,+}}{\beta\lambda_{\min}(AA^*)} = 1 - \frac{\nu}{4\beta},$$

respectively. Since

$$\beta > \frac{\nu}{1 - 24\sigma\kappa\left(AA^*\right)} > \nu > \frac{\nu}{2},$$

it follows that each of them has a nonzero real solution. □

Remark 6.2.4. Hong and Luo proved in [90] linear convergence for the iterates generated by a Lagrangian-based algorithm in the convex setting, without any strong convexity assumption. To this end a certain error bound condition must hold true and the step size of the dual update, which is also assumed to depend on the error bound constants, must be taken small. The authors also mention that this choice of the dual step size may be too conservative and cumbersome to compute unless the objective function is strongly convex. As shown in previous lemma, the step size of the dual update in our algorithm can be computed without assuming strong convexity and indeed it depends only on the linear operator A.

Theorem 6.2.5. *Let Assumption 6.2.2 be satisfied and the parameters in Algorithm 6.2.1 be such that $2\tau \geqslant \beta \left\|A\right\|^2$, and the constants defined in Lemma 6.2.3 fulfil $\min\left\{C_2, C_3, C_4\right\} > 0$. If $\left\{(x_k, y_k, z_k, u_k)\right\}_{k\geqslant 0}$ is a sequence generated by Algorithm 6.2.1, then the following statements are true:*

(i) the sequence $\left\{\Psi_k\right\}_{k\geqslant 1}$ is bounded from below and convergent;

(ii) in addition,

$$x_{k+1} - x_k \to 0,\ y_{k+1} - y_k \to 0,\ z_{k+1} - z_k \to 0 \text{ and } u_{k+1} - u_k \to 0 \text{ as } k \to +\infty. \tag{6.2.25}$$

Proof. First, we show that $\underline{\Psi}$ defined in (6.2.9) is a lower bound of $\left\{\Psi_k\right\}_{n\geqslant 2}$. Suppose the contrary, namely that there exists $k_0 \geqslant 2$ such that $\Psi_{k_0} - \underline{\Psi} < 0$. According to Lemma 6.2.3, $\left\{\Psi_k\right\}_{k\geqslant 1}$ is a nonincreasing sequence and thus for any $k \geqslant k_0$

$$\sum_{k=1}^{N}\left(\Psi_k - \underline{\Psi}\right) \leqslant \sum_{k=1}^{k_0-1}\left(\Psi_k - \underline{\Psi}\right) + \left(N - k_0 + 1\right)\left(\Psi_{k_0} - \underline{\Psi}\right),$$

which implies that

$$\lim_{N\to+\infty}\sum_{k=1}^{N}\left(\Psi_k - \underline{\Psi}\right) = -\infty.$$

On the other hand, for any $k \geqslant 1$ it holds

$$\begin{aligned}\Psi_k - \underline{\Psi} &\geqslant f\left(z_k\right) + g\left(y_k\right) + h\left(x_k, y_k\right) + \left\langle u_k, Ax_k - z_k\right\rangle - \underline{\Psi} \\ &\geqslant \left\langle u_k, Ax_k - z_k\right\rangle = \frac{1}{\sigma\beta}\left\langle u_k, u_k - u_{k-1}\right\rangle \\ &= \frac{1}{2\sigma\beta}\left\|u_k\right\|^2 + \frac{1}{2\sigma\beta}\left\|u_k - u_{k-1}\right\|^2 - \frac{1}{2\sigma\beta}\left\|u_{k-1}\right\|^2.\end{aligned}$$

Therefore, for any $k \geqslant 1$, we have

$$\sum_{k=1}^{N}\left(\Psi_k - \underline{\Psi}\right) \geqslant \frac{1}{2\sigma\beta}\sum_{k=1}^{N}\left\|u_k - u_{k-1}\right\|^2 + \frac{1}{2\sigma\beta}\left\|u_k\right\|^2 - \frac{1}{2\sigma\beta}\left\|u_0\right\|^2 \geqslant -\frac{1}{2\sigma\beta}\left\|u_0\right\|^2,$$

which leads to a contradiction. As $\left\{\Psi_k\right\}_{k\geqslant 1}$ is bounded from below, we obtain from Lemma 2.4.2 statement (i) and also that

$$x_{k+1} - x_k \to 0,\ y_{k+1} - y_k \to 0 \text{ and } u_{k+1} - u_k \to 0 \text{ as } k \to +\infty.$$

Since for any $k \geqslant 1$ it holds

$$\begin{aligned} \|z_{k+1} - z_k\| &\leqslant \|A\| \|x_{k+1} - x_k\| + \|Ax_{k+1} - z_{k+1}\| + \|Ax_k - z_k\| \\ &= \|A\| \|x_{k+1} - x_k\| + \frac{1}{\sigma\beta} \|u_{k+1} - u_k\| + \frac{1}{\sigma\beta} \|u_k - u_{k-1}\|, \end{aligned} \tag{6.2.26}$$

it follows that $z_{k+1} - z_k \to 0$ as $k \to +\infty$. □

Usually, for nonconvex algorithms, the fact that the sequences of differences of consecutive iterates converge to zero is shown by assuming that the generated sequences are bounded (see [56, 96, 128]). In our analysis the only ingredients for obtaining statement (ii) in Theorem 6.2.5 are the descent property and Lemma 2.4.2.

6.2.2 General conditions for the boundedness of $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$

In the following we will formulate general conditions in terms of the input data of the optimization problem (6.1.1) which guarantee the boundedness of the sequence $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$. Working in the setting of Theorem 6.2.5, thanks to (6.2.25), we have that the sequences $\{x_{k+1} - x_k\}_{k\geqslant 0}$, $\{y_{k+1} - y_k\}_{k\geqslant 0}$, $\{z_{k+1} - z_k\}_{k\geqslant 0}$ and $\{u_{k+1} - u_k\}_{k\geqslant 0}$ are bounded. Denote

$$s_* := \sup_{k\geqslant 0} \{\|x_{k+1} - x_k\|, \|y_{k+1} - y_k\|, \|z_{k+1} - z_k\|, \|u_{k+1} - u_k\|\} < +\infty.$$

Even though this observation does not imply immediately that $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ is bounded, this will follow under standard coercivity assumptions. Recall that a function $\psi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is called coercive, if $\lim_{\|x\|\to+\infty} \psi(x) = +\infty$.

Theorem 6.2.6. *Let Assumption 6.2.2 be satisfied and the parameters in Algorithm 6.2.1 be such that* $2\tau \geqslant \beta \|A\|^2$, *the constants defined in Lemma 6.2.3 fulfil* $\min\{C_2, C_3, C_4\} > 0$ *and there exist* $\gamma_1, \gamma_2 \in \mathbb{R}\backslash\{0\}$ *such that* (6.2.21) *holds. Suppose that one of the following conditions hold:*

(i) the function h is coercive;

(ii) the operator A is invertible, and f and g are coercive.

Then every sequence $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ *generated by Algorithm 6.2.1 is bounded.*

Proof. Let $k \geqslant 1$ be fixed. According to Lemma 6.2.3 we have that

$$\begin{aligned} \Psi_1 &\geqslant \ldots \geqslant \Psi_k \geqslant \Psi_{k+1} \\ &\geqslant f(z_{k+1}) + g(y_{k+1}) + h(x_{k+1}, y_{k+1}) - \frac{1}{2\beta} \|u_{k+1}\|^2 + \frac{\beta}{2} \left\| Ax_{k+1} - z_{k+1} + \frac{1}{\beta} u_{k+1} \right\|^2. \end{aligned} \tag{6.2.27}$$

By multiplying both sides by -1 the adding $A^* u_{k+1} + \tau(x_{k+1} - x_k)$ on both sides, we obtain

$$\begin{aligned} A^* u_{k+1} = &\left(1 - \frac{1}{\sigma}\right) A^*(u_{k+1} - u_k) + B(x_k - x_{k+1}) \\ &+ \nabla_x h(x_{k+1}, y_{k+1}) - \nabla_x h(x_k, y_{k+1}) - \nabla_x h(x_{k+1}, y_{k+1}), \end{aligned} \tag{6.2.28}$$

which implies

$$\begin{aligned} \|A^* u_{k+1}\| &\leqslant \left(\frac{1}{\sigma} - 1\right) \|A\| \|u_{k+1} - u_k\| + (\tau + L_{1,+}) \|x_{k+1} - x_k\| + \|\nabla_x h(x_{k+1}, y_{k+1})\| \\ &\leqslant \left(\left(\frac{1}{\sigma} - 1\right) \|A\| + \tau + L_{1,+}\right) s_* + \|\nabla_x h(x_{k+1}, y_{k+1})\|. \end{aligned}$$

By using the Cauchy-Schwarz inequality we further obtain

$$\begin{aligned}\lambda_{\min}(AA^*)\|u_{k+1}\|^2 &\leqslant \|A^*u_{k+1}\|^2\\ &\leqslant 2\left(\left(\frac{1}{\sigma}-1\right)\|A\|+\tau+L_{1,+}\right)^2 s_*^2+2\|\nabla_x h(x_{k+1},y_{k+1})\|^2.\end{aligned}$$

Multiplying the above relation by $\dfrac{1}{2\beta\lambda_{\min}(AA^*)}$ and combining it with (6.2.27), we get

$$\begin{aligned}\Psi_1 \geqslant\ & f(z_{k+1})+g(y_{k+1})+h(x_{k+1},y_{k+1})-\frac{1}{\beta\lambda_{\min}(AA^*)}\|\nabla_x h(x_{k+1},y_{k+1})\|^2\\ &-\frac{1}{\beta\lambda_{\min}(AA^*)}\left(\left(\frac{1}{\sigma}-1\right)\|A\|+\tau+L_{1,+}\right)^2 s_*^2+\frac{\beta}{2}\left\|Ax_{k+1}-z_{k+1}+\frac{1}{\beta}u_{k+1}\right\|^2.\end{aligned}\tag{6.2.29}$$

We will prove the boundedness of $\{(x_k,y_k,z_k,u_k)\}_{k\geqslant 0}$ in each of the two scenarios.

(i) According to (6.2.29) and Proposition 2.2.1, we have that for any $k\geqslant 1$

$$\begin{aligned}&\frac{1}{2}h(x_{k+1},y_{k+1})+\frac{\beta}{2}\left\|Ax_{k+1}-z_{k+1}+\frac{1}{\beta}u_{k+1}\right\|^2\\ &\leqslant \Psi_1+\frac{1}{\beta\lambda_{\min}(AA^*)}\left(\left(\frac{1}{\sigma}-1\right)\|A\|+\tau+L_{1,+}\right)^2 s_*^2-\inf_{z\in\mathcal{G}}f(z)-\inf_{y\in\mathcal{H}}g(y)\\ &\qquad -\frac{1}{2}\inf_{k\geqslant 1}\left\{h(x_{k+1},y_{k+1})-\left(\frac{1}{\gamma_1}-\frac{L_{1,+}}{2\gamma_1^2}\right)\|\nabla_x h(x_{k+1},y_{k+1})\|^2\right\}\\ &\leqslant \Psi_1+\frac{1}{\beta\lambda_{\min}(AA^*)}\left(\left(\frac{1}{\sigma}-1\right)\|A\|+\tau+L_{1,+}\right)^2 s_*^2\\ &\qquad -\inf_{z\in\mathcal{G}}f(z)-\inf_{y\in\mathcal{K}}g(y)-\inf_{(x,y)\in\mathcal{H}\times\mathcal{K}}h(x,y)\\ &< +\infty.\end{aligned}$$

Since h is coercive and bounded from below, it follows that $\{(x_k,y_k)\}_{k\geqslant 0}$ as well as $\left\{Ax_k-z_k+\frac{1}{\beta}u_k\right\}_{k\geqslant 0}$ are bounded. As, according to (6.2.1d), $\{Ax_k-z_k\}_{k\geqslant 0}$ is bounded, it follows that $\{u_k\}_{k\geqslant 0}$ and $\{z_k\}_{k\geqslant 0}$ are also bounded.

(ii) According to (6.2.29) and Proposition 2.2.1, we have this time that for any $k\geqslant 1$

$$\begin{aligned}&f(z_{k+1})+g(y_{k+1})+\frac{\beta}{2}\left\|Ax_{k+1}-z_{k+1}+\frac{1}{\beta}u_{k+1}\right\|^2\\ &\leqslant \Psi_1+\frac{1}{\beta\lambda_{\min}(AA^*)}\left(\left(\frac{1}{\sigma}-1\right)\|A\|+\tau+L_{1,+}\right)^2 s_*^2\\ &\qquad -\inf_{k\geqslant 1}\left\{h(x_{k+1},y_{k+1})-\left(\frac{1}{\gamma_2}-\frac{L_{1,+}}{2\gamma_2^2}\right)\|\nabla_x h(x_{k+1},y_{k+1})\|^2\right\}\\ &\leqslant \Psi_1+\frac{1}{\beta\lambda_{\min}(AA^*)}\left(\left(\frac{1}{\sigma}-1\right)\|A\|+\tau+L_{1,+}\right)^2 s_*^2-\inf_{(x,y)\in\mathcal{H}\times\mathcal{K}}h(x,y)<+\infty.\end{aligned}$$

Since f and g are coercive and bounded from below, it follows that $\{(y_k,z_k)\}_{k\geqslant 0}$ and $\left\{Ax_k-z_k+\frac{1}{\beta}u_k\right\}_{k\geqslant 0}$ are bounded sequences. As, according to (6.2.1d), the sequence $\{Ax_k-z_k\}_{k\geqslant 0}$ is bounded, it follows that $\{u_k\}_{k\geqslant 0}$ and $\{Ax_k\}_{k\geqslant 0}$ are bounded. The fact that A is invertible implies that $\{x_k\}_{k\geqslant 0}$ is bounded. □

6.2.3 The cluster points of $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ are KKT points

We will close this section dedicated to the convergence analysis of the sequence generated by Algorithm 6.2.1 in a general framework by proving that any cluster point of $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ is a KKT point of the optimization problem (6.1.1). We provided above general conditions which guarantee both the descent inequality (6.2.18), with positive constants C_2, C_3 and C_4, and the boundedness of the generated iterates. Lemma 6.2.4 and Theorem 6.2.6 provide one possible setting that ensures these two fundamental properties of the convergence analysis. We do not want to restrict ourselves to this particular setting and, therefore, we will work, from now on, under the following assumptions.

Assumption 6.2.3. *(i) the functions f, g and h are bounded from below;*

(ii) the linear operator A is surjective;

(iii) every sequence $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ generated by the Algorithm 6.2.1 is bounded:

(iv) ∇H is Lipschitz continuous with constant $L > 0$ on a convex bounded subset $B_1 \times B_2 \subseteq \mathcal{H} \times \mathcal{K}$ containing $\{(x_k, y_k)\}_{k\geqslant 0}$. In other words, for any $(x, y), (x', y') \in B_1 \times B_2$ it holds

$$|||\left(\nabla_x h(x,y) - \nabla_x h\left(x', y'\right), \nabla_y h(x,y) - \nabla_y h\left(x', y'\right)\right)||| \leqslant L|||(x,y) - \left(x', y'\right)|||; \quad (6.2.30)$$

(v) the parameters $\mu, \beta, \tau > 0$ and $0 < \sigma \leqslant 1$ are such that $2\tau \geqslant \beta \|A\|^2$ and

$$\min\{C_2, C_3, C_4\} > 0,$$

where

$$C_2 := \tau - \frac{L\sqrt{2} + \beta\|A\|^2}{2} - \frac{4\sigma\tau^2}{\beta\lambda_{\min}(AA^*)} - \frac{8\left(\sigma\tau + L\sqrt{2}\right)^2}{\sigma\beta\lambda_{\min}(AA^*)}, \quad (6.2.31a)$$

$$C_3 := \frac{\mu - L\sqrt{2}}{2} - \frac{16L^2}{\sigma\beta\lambda_{\min}(AA^*)}, \quad (6.2.31b)$$

$$C_4 := \frac{1}{\sigma\beta}. \quad (6.2.31c)$$

Remark 6.2.5. Being facilitated by the boundedness of the generated sequence, Assumption 6.2.3 (iv) not only guarantee the fulfilment of Assumption 6.2.2 (iii) and (iv) on a convex bounded set, but it also arises in a more natural way (see also [36]). Assumption 6.2.3 (iv) holds, for instance, if h is twice continuously differentiable. In addition, as (6.2.30) implies for any $(x, y), (x', y') \in B_1 \times B_2$ that

$$\left\|\nabla_x h(x,y) - \nabla_x h\left(x', y'\right)\right\| + \left\|\nabla_y h(x,y) - \nabla_y h\left(x', y'\right)\right\| \leqslant L\sqrt{2}\left(\left\|x - x'\right\| + \left\|y - y'\right\|\right),$$

we can take

$$L_{1,+} = L_{2,+} = L_{3,+} := L\sqrt{2}. \quad (6.2.32)$$

As (6.2.7a) - (6.2.7c) are valid also on a convex bounded set, the descent inequality

$$\Psi_{n+1} + C_2\|x_{k+1} - x_k\|^2 + C_3\|y_{k+1} - y_k\|^2 + C_4\|u_{k+1} - u_k\|^2 \leqslant \Psi_k \ \forall k \geqslant 1 \quad (6.2.33)$$

remains true, where the constants on the left-hand sided are given in (6.2.31) and follow from (6.2.19) under the consideration of (6.2.32). A possible choice of the parameters of the algorithm such that $\min\{C_2, C_3, C_4\} > 0$ can be obtained also from Lemma 6.2.4.

The next result provide upper estimates for the limiting subgradients of the regularized function Ψ at (x_k, y_k, z_k, u_k) for every $k \geqslant 1$.

Lemma 6.2.7. *Let Assumption 6.2.3 be satisfied and* $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ *be a sequence generated by Algorithm 6.2.1. Then for any* $k \geqslant 1$ *it holds*

$$D_k := \left(d_x^k, d_y^k, d_z^k, d_u^k, d_{x'}^k, d_{u'}^k\right) \in \partial\Psi\left(\mathbf{X}_k\right), \tag{6.2.34}$$

where

$$\begin{aligned} d_x^k := {} & \nabla_x h\left(x_k, y_k\right) + A^* u_k + \beta A^*\left(Ax_k - z_k\right) + 2C_1\left(x_k - x_{k-1}\right) \\ & + 2\sigma C_0 B^T\left(A^*\left(u_k - u_{k-1}\right) + \sigma B\left(x_k - x_{k-1}\right)\right), \end{aligned} \tag{6.2.35a}$$

$$d_y^k := \nabla_y h\left(x_k, y_k\right) - \nabla_y h\left(x_{k-1}, y_{k-1}\right) + \mu\left(y_{k-1} - y_k\right), \tag{6.2.35b}$$

$$d_z^k := u_{k-1} - u_k + \beta A\left(x_{k-1} - x_k\right), \tag{6.2.35c}$$

$$d_u^k := Ax_k - z_k + 2C_0 A\left(A^*\left(u_k - u_{k-1}\right) + \sigma B\left(x_k - x_{k-1}\right)\right), \tag{6.2.35d}$$

$$d_{x'}^k := -2\sigma C_0 B^T\left(A^*\left(u_k - u_{k-1}\right) + \sigma B\left(x_k - x_{k-1}\right)\right) - 2C_1\left(x_k - x_{k-1}\right), \tag{6.2.35e}$$

$$d_{u'}^k := -2C_0 A\left(A^*\left(u_k - u_{k-1}\right) + \sigma B\left(x_k - x_{k-1}\right)\right). \tag{6.2.35f}$$

In addition, for any $k \geqslant 1$ *it holds*

$$|||D_k||| \leqslant C_5\left\|x_k - x_{k-1}\right\| + C_6\left\|y_k - y_{k-1}\right\| + C_7\left\|u_k - u_{k-1}\right\|, \tag{6.2.36}$$

where

$$C_5 := 2\sqrt{2}\cdot L + \tau + \beta\|A\| + 4\left(\sigma\tau + \|A\|\right)\sigma\tau C_0 + 4C_1, \tag{6.2.37a}$$

$$C_6 := L\sqrt{2} + \mu, \tag{6.2.37b}$$

$$C_7 := 1 + \frac{1}{\sigma\beta} + \left(\frac{2}{\sigma} - 1\right)\|A\| + 4\left(\sigma\tau + \|A\|\right)C_0\|A\|. \tag{6.2.37c}$$

Proof. Let $k \geqslant 1$ be fixed. Applying the calculus rules of the limiting subdifferential we get

$$\begin{aligned} \nabla_x\Psi\left(\mathbf{X}_k\right) = {} & \nabla_x h\left(x_k, y_k\right) + A^* u_k + \beta A^*\left(Ax_k - z_k\right) + 2C_1\left(x_k - x_{k-1}\right) \\ & + 2\sigma C_0 B^T\left(A^*\left(u_k - u_{k-1}\right) + \sigma B\left(x_k - x_{k-1}\right)\right), \end{aligned} \tag{6.2.38a}$$

$$\partial_y\Psi\left(\mathbf{X}_k\right) = \partial g\left(y_k\right) + \nabla_y h\left(x_k, y_k\right), \tag{6.2.38b}$$

$$\partial_z\Psi\left(\mathbf{X}_k\right) = \partial f\left(z_k\right) - u_k - \beta\left(Ax_k - z_k\right), \tag{6.2.38c}$$

$$\nabla_u\Psi\left(\mathbf{X}_k\right) = Ax_k - z_k + 2C_0 A\left(A^*\left(u_k - u_{k-1}\right) + \sigma B\left(x_k - x_{k-1}\right)\right), \tag{6.2.38d}$$

$$\nabla_{x'}\Psi\left(\mathbf{X}_k\right) = -2\sigma C_0 B^T\left(A^*\left(u_k - u_{k-1}\right) + \sigma B\left(x_k - x_{k-1}\right)\right) - 2C_1\left(x_k - x_{k-1}\right), \tag{6.2.38e}$$

$$\nabla_{u'}\Psi\left(\mathbf{X}_k\right) = -2C_0 A\left(A^*\left(u_k - u_{k-1}\right) + \sigma B\left(x_k - x_{k-1}\right)\right). \tag{6.2.38f}$$

Then (6.2.35a) and (6.2.35d) - (6.2.35f) follow directly from (6.2.38a) and (6.2.38d) - (6.2.38f), respectively. By combining (6.2.38b) with the optimality criterion for (6.2.1a)

$$0 \in \partial g\left(y_k\right) + \nabla_y h\left(x_{k-1}, y_{k-1}\right) + \mu\left(y_k - y_{k-1}\right),$$

we obtain (6.2.35b). Similarly, by combining (6.2.38c) with the optimality criterion for (6.2.1b)

$$0 \in \partial f\left(z_k\right) - u_{k-1} - \beta\left(Ax_{k-1} - z_k\right),$$

we get (6.2.35c).

In the following we will derive the upper estimates for the components of the limiting subgradient. From (6.2.28) it follows

$$\begin{aligned} \left\|d_x^k\right\| \leqslant {} & \left\|\nabla_x h\left(x_k, y_k\right) + A^* u_k\right\| + \beta\|A\|\left\|Ax_k - z_k\right\| + 2\left(C_1 + \sigma^2\tau^2 C_0\right)\left\|x_k - x_{k-1}\right\| \\ & + 2\sigma\tau C_0\|A\|\left\|u_k - u_{k-1}\right\| \\ \leqslant {} & \left(L\sqrt{2} + \tau + 2C_1 + 2\sigma^2\tau^2 C_0\right)\left\|x_k - x_{k-1}\right\| + \left(\frac{2}{\sigma} - 1 + 2\sigma\tau C_0\right)\|A\|\left\|u_k - u_{k-1}\right\|. \end{aligned}$$

In addition, we have

$$\begin{aligned}
\left\|d_y^k\right\| &\leqslant L\sqrt{2}\left\|x_k - x_{k-1}\right\| + \left(L\sqrt{2} + \mu\right)\left\|y_k - y_{k-1}\right\|, \\
\left\|d_z^k\right\| &\leqslant \beta\left\|A\right\|\left\|x_k - x_{k-1}\right\| + \left\|u_k - u_{k-1}\right\|, \\
\left\|d_u^k\right\| &\leqslant 2\sigma\tau C_0\left\|A\right\|\left\|x_k - x_{k-1}\right\| + \left(\frac{1}{\sigma\beta} + 2C_0\left\|A\right\|^2\right)\left\|u_k - u_{k-1}\right\|, \\
\left\|d_{x'}^k\right\| &\leqslant 2\left(\sigma^2\tau^2 C_0 + C_1\right)\left\|x_k - x_{k-1}\right\| + 2\sigma\tau C_0\left\|A\right\|\left\|u_k - u_{k-1}\right\|, \\
\left\|d_{u'}^k\right\| &\leqslant 2\sigma\tau C_0\left\|A\right\|\left\|x_k - x_{k-1}\right\| + 2C_0\left\|A\right\|^2\left\|u_k - u_{k-1}\right\|.
\end{aligned}$$

The inequality (6.2.36) follows by combining the above relations with (5.1.2). □

We denote by $\Omega := \Omega\left(\{\mathbf{X}_k\}_{k\geqslant 1}\right)$ the set of cluster points of the sequence $\{\mathbf{X}_k\}_{k\geqslant 1} \subseteq \mathbf{R}$, which is nonempty thanks to the boundedness of $\{\mathbf{X}_k\}_{k\geqslant 1}$. The main result of this section follows.

Theorem 6.2.8. *Let Assumption 6.2.3 be satisfied and $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 6.2.1. The following statements are true:*

(i) if $\{(x_{k_n}, y_{k_n}, z_{k_n}, u_{k_n})\}_{k\geqslant 0}$ is a subsequence of $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ which converges to the point (x_*, y_*, z_*, u_*) as $k \to +\infty$, then

$$\lim_{n\to+\infty} \Psi_{k_n} = \Psi\left(x_*, y_*, z_*, u_*, x_*, u_*\right);$$

(ii) it holds

$$\begin{aligned}
\Omega \subseteq \operatorname{crit}(\Psi) \subseteq \{\mathbf{X}_* \in \mathbf{R} : -A^* u_* &= \nabla_x h\left(x_*, y_*\right), \\
0 \in \partial g\left(y_*\right) + \nabla_y h\left(x_*, y_*\right), u_* &\in \partial f\left(z_*\right), z_* = A x_*\},
\end{aligned} \tag{6.2.39}$$

where $\mathbf{X}_* := (x_*, y_*, z_*, u_*, x_*, u_*)$;

(iii) it holds $\lim_{k\to+\infty} \operatorname{dist}(\mathbf{X}_k, \Omega) = 0$;

(iv) the set Ω is nonempty, connected and compact;

(v) the function Ψ takes on Ω the value

$$\Psi_* = \lim_{k\to+\infty} \Psi_k = \lim_{k\to+\infty} \left\{f\left(z_k\right) + g\left(y_k\right) + h\left(x_k, y_k\right)\right\}.$$

Proof. Let $(x_*, y_*, z_*, u_*) \in \mathcal{H} \times \mathcal{K} \times \mathcal{G} \times \mathcal{G}$ be such that the subsequence

$$\{\mathbf{X}_{k_n} := (x_{k_n}, y_{k_n}, z_{k_n}, u_{k_n}, x_{k_n-1}, u_{k_n-1})\}_{k\geqslant 1}$$

of $\{\mathbf{X}_k\}_{n\geqslant 1}$ converges to $\mathbf{X}_* := (x_*, y_*, z_*, u_*, x_*, u_*)$.

(i) From (6.2.1a) and (6.2.1b) we have for any $k \geqslant 1$

$$\begin{aligned}
& g\left(y_{k_n}\right) + \left\langle \nabla_y h\left(x_{k_n-1}, y_{k_n-1}\right), y_{k_n} - y_{k_n-1}\right\rangle + \frac{\mu}{2}\left\|y_{k_n} - y_{k_n-1}\right\|^2 \\
\leqslant\ & g\left(y_*\right) + \left\langle \nabla_y h\left(x_{k_n-1}, y_{k_n-1}\right), y_* - y_{k_n-1}\right\rangle + \frac{\mu}{2}\left\|y_* - y_{k_n-1}\right\|^2
\end{aligned}$$

and

$$\begin{aligned} & f\left(z_{k_n}\right)+\left\langle u_{k_n-1}, A x_{k_n-1}-z_{k_n}\right\rangle+\frac{\beta}{2}\left\|A x_{k_n-1}-z_{k_n}\right\|^2 \\ \leqslant & f\left(z_*\right)+\left\langle u_{k_n-1}, A x_{k_n-1}-z_*\right\rangle+\frac{\beta}{2}\left\|A x_{k_{k-1}}-z_*\right\|^2, \end{aligned}$$

respectively. From (6.2.1d) and Theorem 6.2.5 follows $Ax^* = z^*$. Taking the limit superior as $n \to +\infty$ on both sides of the above inequalities, we get

$$\limsup_{k \to +\infty} f\left(z_{k_n}\right) \leqslant f\left(z_*\right) \qquad \text{and} \qquad \limsup_{k \to +\infty} g\left(y_{k_n}\right) \leqslant g\left(y_*\right)$$

which, combined with the lower semicontinuity of f and g, lead to

$$\lim_{k \to +\infty} f\left(z_{k_n}\right) = f\left(z_*\right) \qquad \text{and} \qquad \lim_{k \to +\infty} g\left(y_{k_n}\right) = g\left(y_*\right).$$

The desired statement follows thanks to the continuity of h.

(ii) For the sequence $\{D_k\}_{n \geqslant 0}$ defined in (6.2.34) - (6.2.35), we have that $D_{k_n} \in \partial\Psi\left(\mathbf{X}_{k_n}\right)$ for any $k \geqslant 1$ and $D_{k_n} \to 0$ as $n \to +\infty$, while $\mathbf{X}_{k_n} \to \mathbf{X}_*$ and $\Psi_{k_n} \to \Psi(\mathbf{X}_*)$ as $n \to +\infty$. The closedness criterion of the limiting subdifferential guarantees that $0 \in \partial\Psi(\mathbf{X}_*)$ or, in other words, $\mathbf{X}_* \in \operatorname{crit}(\Psi)$.

Choosing now an element $\mathbf{X}_* \in \operatorname{crit}(\Psi)$, it holds

$$\begin{cases} 0 & = \nabla_x h\left(x_*, y_*\right) + A^* u_* + \beta A^*\left(A x_* - z_*\right), \\ 0 & \in \partial g\left(y_*\right) + \nabla_y h\left(x_*, y_*\right), \\ 0 & \in \partial f\left(z_*\right) - u_* - \beta\left(A x_* - z_*\right), \\ 0 & = A x_* - z_*, \end{cases}$$

which is further equivalent to (6.2.39).

(iii)-(iv) The proof follows in the lines of the proof of Theorem 5 (ii)-(iii) in [36], also by taking into consideration [36, Remark 5], according to which the properties in (iii) and (iv) are generic for sequences satisfying $\mathbf{X}_k - \mathbf{X}_{k-1} \to 0$ as $k \to +\infty$, which is indeed the case due to (6.2.25).

(v) The sequences $\{f(z_k) + g(y_k) + h(x_k, y_k)\}_{k \geqslant 0}$ and $\{\Psi_k\}_{k \geqslant 0}$ have the same limit due to (6.2.25) and the fact that $\{u_k\}_{k \geqslant 0}$ is bounded

$$\Psi_* = \lim_{k \to +\infty} \Psi_k = \lim_{k \to +\infty} \left\{f\left(z_k\right) + g\left(y_k\right) + h\left(x_k, y_k\right)\right\}.$$

The conclusion follows by taking into consideration the first two statements of this theorem. □

Remark 6.2.6. An element (x_*, y_*, z_*, u_*) fulfilling (6.2.39) is a so-called KKT point of the optimization problem (6.1.1). Such a KKT point obviously fulfils

$$0 \in A^* \partial f\left(A x_*\right) + \nabla_x h\left(x_*, y_*\right), \qquad 0 \in \partial g\left(y_*\right) + \nabla_y h\left(x_*, y_*\right). \tag{6.2.40}$$

If A is injective, then this system of inclusions is further equivalent to

$$\begin{aligned} & 0 \in \partial\left(f \circ A\right)\left(x_*\right) + \nabla_x h\left(x_*, y_*\right) = \partial_x\left(f \circ A + H\right), \\ & 0 \in \partial g\left(y_*\right) + \nabla_y h\left(x_*, y_*\right) = \partial_y\left(G + H\right), \end{aligned} \tag{6.2.41}$$

in other words, (x_*, y_*) is a critical point of the optimization problem (6.1.1). On the other hand, if the functions f, g and h are convex, then, even without asking A to be injective, (6.2.40) and (6.2.41) are equivalent, which means that (x_*, y_*) is a global minimum of the optimization problem (6.1.1).

6.3 Global convergence and rates

In this section we will prove global convergence for the sequence $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ generated by Algorithm 6.2.1 in the context of the Kurdyka-Łojasiewicz property and provide convergence rates for it in the context of the Łojasiewicz property.

6.3.1 Global convergence under Kurdyka-Łojasiewicz assumptions

From now on we will use the following notations

$$C_8 := \frac{1}{\min\{C_2, C_3, C_4\}}, \qquad C_9 := \max\{C_5, C_6, C_7\} \qquad \text{and} \qquad \mathcal{E}_k := \Psi_k - \Psi_* \ \forall k \geqslant 1,$$

where $\Psi_* = \lim_{k\to+\infty} \Psi_k$.

The next result shows that if Ψ is a KŁ function, then the sequence $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ converges to a KKT point of the optimization problem (6.1.1). This hypothesis is fulfilled if, for instance, f, g and h are semi-algebraic functions.

Theorem 6.3.1. *Let Assumption 6.2.3 be satisfied and $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 6.2.1. If Ψ is a KŁ function, then the following statements are true:*

(i) the sequence $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ has finite length, namely,

$$\sum_{k\geqslant 0} \|x_{k+1} - x_k\| < +\infty, \sum_{k\geqslant 0} \|y_{k+1} - y_k\| < +\infty,$$
$$\sum_{k\geqslant 0} \|z_{k+1} - z_k\| < +\infty, \sum_{k\geqslant 0} \|u_{k+1} - u_k\| < +\infty; \tag{6.3.1}$$

(ii) the sequence$\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ converges to a KKT point of the optimization problem (6.1.1).

Proof. Let be $\mathbf{X}_* \in \Omega$, thus $\Psi(\mathbf{X}_*) = \Psi_*$. Recall that $\{\mathcal{E}_k\}_{k\geqslant 1}$ is monotonically decreasing and converges to 0 as $k \to +\infty$. We consider two cases.

Case 1. Assume that there exists an integer $k' \geqslant 1$ such that $\mathcal{E}_{k'} = 0$ or, equivalently, $\Psi_{k'} = \Psi_*$. Due to the monotonicity of $\{\mathcal{E}_k\}_{k\geqslant 1}$, it follows that $\mathcal{E}_k = 0$ or, equivalently, $\Psi_k = \Psi_*$ for any $k \geqslant k'$. The inequality (6.2.33) yields for any $k \geqslant k' + 1$

$$x_{k+1} - x_k = 0, \ y_{k+1} - y_k = 0 \text{ and } u_{k+1} - u_k = 0.$$

The inequality (6.2.26) gives us further $z_{k+1} - z_k = 0$ for any $k \geqslant k' + 2$. This proves (6.3.1).

Case 2. Consider now the case when $\mathcal{E}_k > 0$ or, equivalently, $\Psi_k > \Psi_*$ for any $k \geqslant 1$. According to Lemma 2.3.1, there exist $\varepsilon > 0$, $\eta > 0$ and a desingularization function φ such that for any element $\mathbf{X}$ in the intersection

$$\{\mathbf{Z} \in \mathbf{R} \colon \operatorname{dist}(\mathbf{Z}, \Omega) < \varepsilon\} \cap \{\mathbf{Z} \in \mathbf{R} \colon \Psi_* < \Psi(\mathbf{Z}) < \Psi_* + \eta\} \tag{6.3.2}$$

it holds

$$\varphi'(\Psi(\mathbf{X}) - \Psi_*) \cdot \operatorname{dist}(0, \partial\Psi(\mathbf{X})) \geqslant 1.$$

Let be $k_1 \geqslant 1$ such that for any $k \geqslant k_1$

$$\Psi_* < \Psi_k < \Psi_* + \eta.$$

Since $\lim_{k\to+\infty} \operatorname{dist}(\mathbf{X}_k, \Omega) = 0$ (see Lemma 6.2.8 (iii)), there exists $k_2 \geqslant 1$ such that for any $k \geqslant k_2$

$$\operatorname{dist}(\mathbf{X}_k, \Omega) < \varepsilon.$$

Consequently, $\mathbf{X}_k = (x_k, y_k, z_k, u_k, x_{k-1}, u_{k-1})$ belongs to the intersection in (6.3.2) for any $k \geqslant k_0 := \max\{k_1, k_2\}$, which further implies

$$\varphi'(\Psi_k - \Psi_*) \cdot \operatorname{dist}(0, \partial\Psi(\mathbf{X}_k)) = \varphi'(\mathcal{E}_k) \cdot \operatorname{dist}(0, \partial\Psi(\mathbf{X}_k)) \geqslant 1. \tag{6.3.3}$$

Define for two arbitrary nonnegative integers i and j

$$\Delta_{i,j} := \varphi(\Psi_i - \Psi_*) - \varphi(\Psi_j - \Psi_*) = \varphi(\mathcal{E}_i) - \varphi(\mathcal{E}_j).$$

The monotonicity of the sequence $\{\Psi_k\}_{k\geqslant 0}$ and of the function φ implies that $\Delta_{i,j} \geqslant 0$ for any $1 \leqslant i \leqslant j$. In addition, for any $k \geqslant k_0 \geqslant 1$ it holds

$$\sum_{k=k_0}^{N} \Delta_{k,k+1} = \Delta_{k_0,N+1} = \varphi(\mathcal{E}_{k_0}) - \varphi(\mathcal{E}_{k+1}) \leqslant \varphi(\mathcal{E}_{k_0}),$$

from which we get $\sum_{k\geqslant 1} \Delta_{k,k+1} < +\infty$.

By combining Lemma 6.2.3 with the concavity of φ we obtain for any $k \geqslant 1$

$$\begin{aligned}\Delta_{k,k+1} &= \varphi(\mathcal{E}_k) - \varphi(\mathcal{E}_{k+1}) \geqslant \varphi'(\mathcal{E}_k)(\mathcal{E}_k - \mathcal{E}_{k+1}) = \varphi'(\mathcal{E}_k)(\Psi_k - \Psi_{n+1}) \\ &\geqslant \min\{C_2, C_3, C_4\} \varphi'(\mathcal{E}_k) \left(\|x_{k+1} - x_k\|^2 + \|y_{k+1} - y_k\|^2 + \|u_{k+1} - u_k\|^2 \right).\end{aligned}$$

Thus, (6.3.3) implies for any $k \geqslant k_0$

$$\begin{aligned}&\|x_{k+1} - x_k\|^2 + \|y_{k+1} - y_k\|^2 + \|u_{k+1} - u_k\|^2 \\ &\leqslant \operatorname{dist}(0, \partial\Psi(\mathbf{X}_k)) \cdot \varphi'(\mathcal{E}_k) \left(\|x_{k+1} - x_k\|^2 + \|y_{k+1} - y_k\|^2 + \|u_{k+1} - u_k\|^2 \right) \\ &\leqslant C_8 \cdot \operatorname{dist}(0, \partial\Psi(\mathbf{X}_k)) \cdot \Delta_{k,k+1}.\end{aligned}$$

By the Cauchy-Schwarz inequality, the arithmetic mean-geometric mean inequality and Lemma 6.2.7, we have that for any $k \geqslant k_0$ and every $\alpha > 0$

$$\begin{aligned}&\|x_{k+1} - x_k\| + \|y_{k+1} - y_k\| + \|u_{k+1} - u_k\| \\ &\leqslant \sqrt{3} \cdot \sqrt{\|x_{k+1} - x_k\|^2 + \|y_{k+1} - y_k\|^2 + \|u_{k+1} - u_k\|^2} \\ &\leqslant \sqrt{3C_8} \cdot \sqrt{\operatorname{dist}(0, \partial\Psi(\mathbf{X}_k)) \cdot \Delta_{k,k+1}} \\ &\leqslant \alpha \cdot \operatorname{dist}(0, \partial\Psi(\mathbf{X}_k)) + \frac{3C_8}{4\alpha}\Delta_{k,k+1} \\ &\leqslant \alpha C_9 (\|x_k - x_{k-1}\| + \|y_k - y_{k-1}\| + \|u_k - u_{k-1}\|) + \frac{3C_8}{4\alpha}\Delta_{k,k+1}.\end{aligned} \tag{6.3.4}$$

If we denote for any $k \geqslant 0$

$$a_k := \|x_k - x_{k-1}\| + \|y_k - y_{k-1}\| + \|u_k - u_{k-1}\| \qquad \text{and} \qquad d_k := \frac{3C_8}{4\alpha}\Delta_{k,k+1}, \tag{6.3.5}$$

then the above inequality is nothing else than (2.4.6) with

$$\chi_0 := \alpha C_9 \qquad \text{and} \qquad \chi_1 := 0.$$

Since $\sum_{k\geqslant 1} d_n < +\infty$, by choosing $\alpha < 1/C_9$, we can apply Lemma 2.4.4 to conclude that

$$\sum_{k\geqslant 0} \left(\|x_{k+1} - x_k\| + \|y_{k+1} - y_k\| + \|u_{k+1} - u_k\| \right) < +\infty.$$

The proof of (6.3.1) is completed by taking into account once again (6.2.26).

From (i) it follows that the sequence $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ is Cauchy, thus it converges to an element (x_*, y_*, z_*, u_*) which is, according to Lemmas 6.2.8, a KKT point of the optimization problem (6.1.1). □

6.3.2 Convergence rates

In this section we derive convergence rates for the sequence $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ generated by Algorithm 6.2.1 as well as for $\{\Psi_k\}_{k\geqslant 0}$, if the regularized augmented Lagrangian Ψ satisfies the Łojasiewicz property.

If Assumption 6.2.3 is fulfilled and $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ is the sequence generated by Algorithm 6.2.1, then, according to Theorem 6.2.8, the set of cluster points Ω is nonempty, compact and connected and Ψ takes on Ω the value Ψ_*; in addition, $\Omega \subseteq \operatorname{crit}(\Psi)$.

Then there exist $C_L > 0$, $\theta \in [0, 1)$ and $\varepsilon > 0$ such that for any $\mathbf{X} \in \mathbb{B}(\mathbf{X}_*, \varepsilon)$

$$|\Psi(\mathbf{X}) - \Psi_*|^\theta \leqslant C_L \cdot \operatorname{dist}(0, \partial\Psi(\mathbf{X})). \tag{6.3.6}$$

In this case, Ψ is said to satisfy the Łojasiewicz property with Łojasiewicz constant $C_L > 0$ and Łojasiewicz exponent $\theta \in [0, 1)$.

We prove a recurrence inequality for the sequence $\{\mathcal{E}_k\}_{k\geqslant 0}$.

Lemma 6.3.2. *Let Assumption 6.2.3 be satisfied and $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 6.2.1. If Ψ satisfies the Łojasiewicz property with Łojasiewicz constant $C_L > 0$ and Łojasiewicz exponent $\theta \in [0, 1)$, then there exists $k_0 \geqslant 1$ such that the following estimate holds for any $k \geqslant k_0$*

$$\mathcal{E}_{k-1} - \mathcal{E}_k \geqslant C_{10}\mathcal{E}_k^{2\theta}, \qquad \text{where} \quad C_{10} := \frac{C_8}{3(C_L \cdot C_9)^2}. \tag{6.3.7}$$

Proof. For every $n \geqslant 2$ we obtain from Lemma 6.2.3

$$\begin{aligned}
\mathcal{E}_{k-1} - \mathcal{E}_k &= \Psi_{n-1} - \Psi_k \\
&\geqslant C_8\left(\|x_k - x_{k-1}\|^2 + \|y_k - y_{k-1}\|^2 + \|u_k - u_{k-1}\|^2\right) \\
&\geqslant \frac{1}{3}C_8\left(\|x_k - x_{k-1}\| + \|y_k - y_{k-1}\| + \|u_k - u_{k-1}\|\right)^2 \\
&\geqslant C_{10}C_L^2 |||D_k|||^2,
\end{aligned}$$

where $D_k \in \partial\Psi(\mathbf{X}_k)$. Let $\varepsilon > 0$ be such that (6.3.6) is fulfilled and choose $k_0 \geqslant 1$ with the property that for any $k \geqslant k_0$, $\mathbf{X}_k$ belongs to $\mathbb{B}(\mathbf{X}_*, \varepsilon)$. Relation (6.3.6) implies (6.3.7) for any $k \geqslant k_0$. □

The following result follows by combining Lemma 2.4.5 with Lemma 6.3.2.

Theorem 6.3.3. *Let Assumption 6.2.3 be satisfied and $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 6.2.1. If Ψ satisfies the Łojasiewicz property with Łojasiewicz constant $C_L > 0$ and Łojasiewicz exponent $\theta \in [0, 1)$, then the following statements are true:*

(i) if $\theta = 0$, then $\{\Psi_k\}_{k\geqslant 1}$ converges in finite time;

(ii) if $\theta \in (0, 1/2]$, then there exist $k_0 \geqslant 1$, $\widehat{C}_0 > 0$ and $Q \in [0, 1)$ such that for any $k \geqslant k_0$

$$0 \leqslant \Psi_k - \Psi_* \leqslant \widehat{C}_0 Q^k;$$

(iii) if $\theta \in (1/2, 1)$, then there exist $k_0 \geqslant 1$ and $\widehat{C}_1 > 0$ such that for any $k \geqslant k_0 + 1$

$$0 \leqslant \Psi_k - \Psi_* \leqslant \widehat{C}_1 k^{-\frac{1}{2\theta-1}}.$$

The next lemma will play an important role when transferring the convergence rates for $\{\Psi_k\}_{k\geqslant 0}$ to the sequence of iterates $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$.

Lemma 6.3.4. *Let Assumption 6.2.3 be satisfied and $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ be a sequence generated by Algorithm 6.2.1. Let (x_*, y_*, z_*, u_*) be the KKT point of the optimization problem (6.1.1) to which $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ converges as $k \to +\infty$. Then there exists $k_0 \geqslant 1$ such that the following estimates hold for any $k \geqslant k_0$*

$$\begin{aligned}\|x_k - x_*\| &\leqslant C_{11} \max\left\{\sqrt{\mathcal{E}_k}, \varphi(\mathcal{E}_k)\right\}, \quad \|y_k - y_*\| \leqslant C_{11} \max\left\{\sqrt{\mathcal{E}_k}, \varphi(\mathcal{E}_k)\right\}, \\ \|z_k - z_*\| &\leqslant C_{12} \max\left\{\sqrt{\mathcal{E}_k}, \varphi(\mathcal{E}_k)\right\}, \quad \|u_k - u_*\| \leqslant C_{11} \max\left\{\sqrt{\mathcal{E}_k}, \varphi(\mathcal{E}_k)\right\}, \end{aligned} \tag{6.3.8}$$

where

$$C_{11} := 2\sqrt{3C_8} + 3C_8C_9 \qquad \text{and} \qquad C_{12} := \left(\|A\| + \frac{2}{\sigma\beta}\right) C_{11}.$$

Proof. We assume that $\mathcal{E}_k > 0$ for any $k \geqslant 0$. Otherwise, the sequence $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ becomes identical to (x_*, y_*, z_*, u_*) beginning with a given index and the conclusion follows automatically (see the proof of Theorem 6.3.1).

Let $\varepsilon > 0$ be such that (6.3.6) is fulfilled and $k_0 \geqslant 2$ be such that x_k belongs to $\mathbb{B}(\mathbf{X}_*, \varepsilon)$ for any $k \geqslant k_0$.

We fix $k \geqslant k_0$ now. One can easily notice that

$$\|x_k - x_*\| \leqslant \|x_{k+1} - x_k\| + \|x_{k+1} - x_*\| \leqslant \cdots \leqslant \sum_{i\geqslant k} \|x_{k+1} - x_k\|.$$

Similarly, we derive

$$\|y_k - y_*\| \leqslant \sum_{i\geqslant k} \|y_{k+1} - y_k\|, \quad \|z_k - z_*\| \leqslant \sum_{i\geqslant k} \|z_{k+1} - z_k\|, \quad \|u_k - u_*\| \leqslant \sum_{i\geqslant k} \|u_{k+1} - u_k\|.$$

On the other hand, in view of (6.3.5) and by taking $\alpha := \dfrac{1}{2C_9}$ the inequality (6.3.4) can be written as

$$a_{k+1} \leqslant \frac{1}{2} a_k + b_k \ \forall k \geqslant k_0.$$

Let us fix now an integer $N \geqslant k$. Summing up the above inequality for $i = k, ..., N$, we have

$$\begin{aligned}\sum_{i=k}^N a_{i+1} \leqslant \frac{1}{2}\sum_{i=k}^N a_i + \sum_{i=k}^N b_i &= \frac{1}{2}\sum_{i=k}^N a_{i+1} + a_k - a_{N+1} + \sum_{i=k}^N b_i \\ &\leqslant \frac{1}{2}\sum_{i=k}^N a_{i+1} + a_k + \frac{3C_8C_9}{2}\varphi(\mathcal{E}_k).\end{aligned}$$

By passing $N \to +\infty$, we obtain

$$\begin{aligned}\sum_{i\geqslant k} a_{k+1} &= \sum_{i\geqslant k} \left(\|x_{k+1} - x_k\| + \|y_{k+1} - y_k\| + \|u_{k+1} - u_k\|\right) \\ &\leqslant 2\left(\|x_{k+1} - x_k\| + \|y_{k+1} - y_k\| + \|u_{k+1} - u_k\|\right) + 3C_8C_9\varphi(\mathcal{E}_k) \\ &\leqslant 2\sqrt{3} \cdot \sqrt{\|x_{k+1} - x_k\|^2 + \|y_{k+1} - y_k\|^2 + \|u_{k+1} - u_k\|^2} + 3C_8C_9\varphi(\mathcal{E}_k) \\ &\leqslant 2\sqrt{3C_8} \cdot \sqrt{\mathcal{E}_k - \mathcal{E}_{k+1}} + 3C_8C_9\varphi(\mathcal{E}_k),\end{aligned}$$

which gives the desired statement. □

We can now formulate convergence rates for the sequence of generated iterates.

Theorem 6.3.5. *Let Assumption 6.2.3 be satisfied and* $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ *be a sequence generated by Algorithm 6.2.1. Suppose further that* Ψ *satisfies the Łojasiewicz property with Łojasiewicz constant* $C_L > 0$ *and Łojasiewicz exponent* $\theta \in [0,1)$*. Let* (x_*, y_*, z_*, u_*) *be the KKT point of the optimization problem (6.1.1) to which* $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ *converges as* $k \to +\infty$*. Then the following statements are true:*

(i) if $\theta = 0$*, then the algorithm converges in finite time;*

(ii) if $\theta \in (0, 1/2]$*, then there exist* $k_0 \geqslant 1$*,* $\widehat{C}_{0,1}, \widehat{C}_{0,2}, \widehat{C}_{0,3}, \widehat{C}_{0,4} > 0$ *and* $\widehat{Q} \in [0,1)$ *such that for any* $k \geqslant k_0$

$$\|x_k - x_*\| \leqslant \widehat{C}_{0,1}\widehat{Q}^k, \quad \|y_k - y_*\| \leqslant \widehat{C}_{0,2}\widehat{Q}^k, \quad \|z_k - z_*\| \leqslant \widehat{C}_{0,3}\widehat{Q}^k, \quad \|u_k - u_*\| \leqslant \widehat{C}_{0,4}\widehat{Q}^k;$$

(iii) if $\theta \in (1/2, 1)$*, then there exist* $k_0 \geqslant 1$ *and* $\widehat{C}_{1,1}, \widehat{C}_{1,2}, \widehat{C}_{1,3}, \widehat{C}_{1,4} > 0$ *such that for any* $k \geqslant k_0 + 1$

$$\|x_k - x_*\| \leqslant \widehat{C}_{1,1}k^{-\frac{1-\theta}{2\theta-1}}, \quad \|y_k - y_*\| \leqslant \widehat{C}_{1,2}k^{-\frac{1-\theta}{2\theta-1}},$$
$$\|z_k - z_*\| \leqslant \widehat{C}_{1,3}k^{-\frac{1-\theta}{2\theta-1}}, \quad \|u_k - u_*\| \leqslant \widehat{C}_{1,4}k^{-\frac{1-\theta}{2\theta-1}}.$$

Proof. Let

$$\varphi : [0, +\infty) \to [0, +\infty), \quad s \mapsto \frac{1}{1-\theta}C_L s^{1-\theta},$$

be the desingularization function.

(i) If $\theta = 0$, then $\{\Psi_k\}_{k\geqslant 1}$ converges in finite time. As seen in the proof of Theorem 6.3.1, the sequence $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ becomes identical to (x_*, y_*, z_*, u_*) starting from a given index. In other words, the sequence $\{(x_k, y_k, z_k, u_k)\}_{k\geqslant 0}$ converges also in finite time and the conclusion follows.

Let be $\theta \neq \frac{1}{2}$ and $k_0' \geqslant 1$ such that for any $k \geqslant k_0'$ the inequalities (6.3.8) in Lemma 6.3.4 and

$$\mathcal{E}_k \leqslant \left(\frac{1}{1-\theta}C_L\right)^{\frac{2}{2\theta-1}}$$

hold.

(ii) If $\theta \in (0, 1/2)$, then $2\theta - 1 < 0$ and thus for any $k \geqslant k_0'$

$$\frac{1}{1-\theta}C_L\mathcal{E}_k^{1-\theta} \leqslant \sqrt{\mathcal{E}_k},$$

which implies that

$$\max\left\{\sqrt{\mathcal{E}_k}, \varphi(\mathcal{E}_k)\right\} = \sqrt{\mathcal{E}_k}.$$

If $\theta = 1/2$, then

$$\varphi(\mathcal{E}_k) = 2C_L\sqrt{\mathcal{E}_k},$$

thus

$$\max\left\{\sqrt{\mathcal{E}_k}, \varphi(\mathcal{E}_k)\right\} = \max\{1, 2C_L\} \cdot \sqrt{\mathcal{E}_k} \ \forall k \geqslant 1.$$

In both cases we have

$$\max\left\{\sqrt{\mathcal{E}_k}, \varphi(\mathcal{E}_k)\right\} \leqslant \max\{1, 2C_L\} \cdot \sqrt{\mathcal{E}_k} \ \forall k \geqslant k_0'.$$

By Theorem 6.3.3, there exist $k_0'' \geqslant 1$, $\widehat{C}_0 > 0$ and $Q \in [0,1)$ such that for $\widehat{Q} := \sqrt{Q}$ and every $k \geqslant k_0''$ it holds

$$\sqrt{\mathcal{E}_k} \leqslant \sqrt{\widehat{C}_0}Q^{k/2} = \sqrt{\widehat{C}_0}\widehat{Q}^k.$$

The conclusion follows from Lemma 6.3.4 for $k_0 := \max\{k_0', k_0''\}$.

(iii) If $\theta \in (1/2, 1)$, then $2\theta - 1 > 0$ and thus for any $k \geqslant k_0'$

$$\sqrt{\mathcal{E}_k} \leqslant \frac{1}{1-\theta} C_L \mathcal{E}_k^{1-\theta},$$

which implies that

$$\max \left\{ \sqrt{\mathcal{E}_k}, \varphi\left(\mathcal{E}_k\right) \right\} = \varphi\left(\mathcal{E}_k\right) = \frac{1}{1-\theta} C_L \mathcal{E}_k^{1-\theta}.$$

By Theorem 6.3.3, there exist $k_0'' \geqslant 1$ and $\widehat{C}_1 > 0$ such that for any $k \geqslant k_0''$

$$\frac{1}{1-\theta} C_L \mathcal{E}_k^{1-\theta} \leqslant \frac{1}{1-\theta} C_L \widehat{C}_1^{1-\theta} \left(k-2\right)^{-\frac{1-\theta}{2\theta-1}}.$$

The conclusion follows again for $k_0 := \max\left\{k_0', k_0''\right\}$ from Lemma 6.3.4. □

6.4 Further perspectives

The following difference of convex optimization model is of huge interest, since it captures many applied problems

$$\min_{x \in \mathcal{H}} \left\{ \psi\left(Ax\right) - \phi\left(Bx\right) + \Theta\left(x\right) \right\}, \tag{6.4.1}$$

where $\psi \colon \mathcal{G} \to \mathbb{R} \cup \{+\infty\}$; $\phi \colon \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ are proper, convex and lower semicontinuous functions with $A \colon \mathcal{H} \to \mathcal{G}; B \colon \mathcal{H} \to \mathcal{K}$ are linear operators and $\Theta \colon \mathcal{H} \to \mathbb{R}$ is a Fréchet differentiable function with L-Lipschitz continuous gradient.

Following the idea of Banert and Boţ in [22], we can rewrite the problem (6.4.1) as

$$\min_{(x,y) \in \mathcal{H} \times \mathcal{K}} \left\{ \psi\left(Ax\right) + \phi^*\left(y\right) - \langle Bx, y \rangle + \Theta\left(x\right) \right\}. \tag{6.4.2}$$

One could use the investigation in this chapter to formulate an algorithm to solve (6.4.2) and provide a setting in which this converges. The numerical validation of the method can be done by considering applications in image processing and machine learning.

On the other hand, recently, Boţ and Kanzler proposed in [55] a continuous time approach for the optimization problem (6.2.2). It would be interesting to also addressing (6.1.1) from the same perspective and to develop corresponding asymptotic analysis.

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