**MASTERARBEIT/MASTER'S THESIS**

Titel der Masterarbeit/Title of the Master's Thesis

# "Aspects of Vaught's Conjecture"

verfasst von/submitted by

Michael Zechner BSc

angestrebter akademischer Grad/in partial fulfillment of the requirements for the degree of

Master of Science (MSc)

Wien, 2021/Vienna, 2021

| | |
|---|---|
| Studienkennzahl lt. Studienblatt/degree programme code as it appears on the student record sheet: | UA 066 821 |
| Studienrichtung lt. Studienblatt/degree programme as it appears on the student record sheet: | Masterstudium Mathematik |
| Betreut von/Supervisor: | Univ.-Prof. Dr. Sy-David Friedman |

# Declaration of Authorship

I, Michael Zechner, declare that this thesis titled *"Aspects of Vaught's Conjecture"* and the work presented in it are my own. I confirm that:

- This work was done wholly while in candidature for a master's degree at the University of Vienna.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where this thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed.

**Date:**____________________ **Signed:**____________________

*Deo Gratias*

# Aknowledgements

I owe the deepest gratitude to my parents for their love, patience and support. This thesis is dedicated to them.

Special thanks goes to my advisor professor Sy-David Friedman for his patience, his advice and in particular for contacting professor David Marker who provided us with unpublished notes by Martin Ziegler and Elisabeth Bouscaren. I thank professor Marker, professor Ziegler and professor Michael Laskowski for their assistance.

# Abstract

This thesis is a short survey of some major results concerning an open problem of model theory known as Vaught's conjecture (VC).

In its original form VC states that the number of isomorphism types of contable models of a complete first order theory in a countable vocabulary is either at most $\aleph_0$ or else $\mathfrak{c}$, the cardinality of the continuum.

After providing a short historical background and a motivation in section 1, we introduce the infinitary language $\mathcal{L}_{\omega_1,\omega}$ in section 2. Following [12], we define the notions $\alpha$ equivalence ($\sim_\alpha$), infinitary equivalence ($\equiv_{\infty,\omega}$) and study consistency properties as well as end extensions. The most important results in this section are Scott's Isomorphism Theorem, the Omitting Types Theorem, the Model Existence Theorem and a sufficient criterion in order to determine if an infinitary sentence has a small uncountable model.

Still following [12] in section 3 we show how countable models can be coded as elements of standard Borel spaces. We define the notion of a scattered infinitary sentence and present a characterisation of it. We also provide two different proofs of a result by Morley which states that every sentence of $\mathcal{L}_{\omega_1,\omega}$ either has at most $\aleph_1$ many isomorphism types of its countable models or else continuum many.

In section 4 we focus on three major results: 1. A theorem by Harnik and Makkai stating that every counter example to VC has a model of cardinality $\aleph_1$ which is not $\equiv_{\infty,\omega}$ equivalent to any countable model. 2. A model theoretic proof of a theorem by Hjorth which states that if VC is false, then there is a counterexample which has only models of cardinality $\aleph_0$ or $\aleph_1$. 3. A theorem by Harrington stating that the Scott ranks of models of a VC counterexample are unbounded below $\aleph_2$.

Section 5 is based on chapter 5 of [9] and some unpublished notes by Martin Ziegler and Elisabeth Bouscaren. We show that it is enough to prove VC for theories (or infinitary sentences) in the language of graphs or bounded lattices.

In section 6 we mostly follow [2] and [10] in order to see how results of descriptive set theory are used to study model theoretic questions. We present Hjorth's original proof of his theorem which was already discussed in section 4. Then we move on to the more general problem of the topological Vaught conjecture and show that VC for infinitary sentences is equivalent to TVC($S_\infty$), the topological Vaught conjecture with respect to the group $S_\infty$, as well as to TVC($H(C)$), where $C$ is the Cantor space and $H(C)$ is the group of homeomorphisms of $C$. Finally, we present the proof that the general topological Vaught conjecture is equivalent to TVC($H(I^{\mathbb{N}})$), where $I^{\mathbb{N}}$ is the Hilbert cube.

# Zusammenfassung

Diese Masterarbeit behandelt einige wichtige Resultate aus der Forschung an einem offenen Problem der Modelltheorie, bekannt als Vaughts Vermutung, abgekürzt mit VC für Vaught's Conjecture.

VC wurde erstmals im Jahr 1959 formuliert. In ihrer ursprünglichen Form besagt sie, dass eine vollständige Theorie der Prädikatenlogik, die ein abzählbares Vokabular verwendet, entweder höchstens $\aleph_0$ viele Isomorphietypen von abzählbaren Modellen hat oder andernfalls Kontinuum viele. Es werden im Lauf der Arbeit Verallgemeinerungen von VC vorgestellt.

Die Arbeit ist in 6 Abschnitte unterteilt:

Nach einem kurzem geschichtlichen Hintergrund und einer Motivation in Sektion 1 wenden wir uns in Sektion 2 infinitären Sprachen der Form $\mathcal{L}_{\omega_1,\omega}(\tau)$ zu. Es werden unter anderem die Begriffe Consistency property, $\alpha$-Äquivalenz ($\sim_\alpha$), Scott Rang, infinitäre Äquivalenz ($\equiv_{\infty,\omega}$) und End extension eingeführt und untersucht. Die wichtigsten und später benötigten Ergebnisse in diesem Abschnitt sind Scott's Isomorphismus Theorem, das Omitting Types Theorem, das Model Existence Theorem sowie ein hinreichendes Kriterium um festzustellen, ob ein infinitärer Satz ein überabzählbares Modell hat, das nur abzählbar viele Typen realisiert. Wir halten uns hierbei an [12].

Im dritten Abschnitt folgen wir weiter [12] und zeigen, wie abzählbare Modelle als Elemente eines Polnischen Raumes kodiert werden können. Der Begriff scattered sentence wird eingeführt und charakterisiert. Weiters werden zwei verschiedene Beweise eines Theorems von Morley präsentiert, welches besagt, dass ein $\mathcal{L}_{\omega_1,\omega}$ Satz entweder höchstens $\aleph_1$ viele abzählbare Isomorphietypen hat oder andernfalls Kontinuum viele.

Sektion 4 behandelt 3 wesentliche Resultate: 1. Ein Theorem von Harnik und Makkai, welches besagt, dass jedes Gegenbeispiel zu VC ein Modell der Kardinalität $\aleph_1$ hat, das nicht $\equiv_{\infty,\omega}$ äquivalent zu einem abzählbaren Modell ist. 2. Ein modelltheoretischer Beweis eines Theorems von Hjorth, welches besagt, dass wenn VC falsch ist, dann gibt es ein Gegenbeispiel, das nur Modelle der Kardinalität $\aleph_0$ und $\aleph_1$ hat. 3. Ein Theorem von Harrington, welches besagt, dass die Scott Ränge von Modellen eines VC Gegenbeispiels unbeschränkt unter $\aleph_2$ sind.

Der fünfte Abschnitt basiert auf Kapitel 5 von [9] und auf unpublizierten Notizen von Martin Ziegler und Elisabeth Bouscaren. Es wird gezeigt, dass es für einen Beweis von VC ausreicht, nur Theorien (infinitäre Sätze) zu betrachten, die in der Sprache von Graphen oder Verbänden formuliert sind.

Sektion 6 hält sich überwiegend an [2] und [10] um relevante Zusammenhänge zwischen Modelltheorie und deskriptiver Mengentheorie aufzuzeigen. Wir stellen den ursprünglichen Beweis von Hjorths Theorem vor, das schon im 4. Abschnitt studiert wurde. Dann wenden wir uns der allgemeineren topologischen Vaught Vermutung zu - abgekürzt mit TVC für Topological Vaught Conjecture - und zeigen, dass VC für infinitäre Sätze äquivalent ist zu sowohl $TVC(S_\infty)$, der topologischen Vaught Vermutung für die Gruppe $S_\infty$, als auch zu $TVC(H(C))$, wobei $C$ den Cantor-Raum bezeichnet und $H(C)$ die Homöomorphismengruppe von $C$. Schließlich wird gezeigt, dass die allgemeine TVC äquivalent zu $TVC(H(I^{\mathbb{N}}))$ ist, wobei mit $I^{\mathbb{N}}$ der Hilbertwürfel gemeint ist.

# Contents

# 1 Introduction

## 1.1 Preliminaries and Motivation

It is assumed that the reader is familiar with the following basic notions of first order logic:

- A language of first order logic. We call the set of constant, relation and function symbols a vocabulary. Familiarity with free and bound variables, terms, atomic formulas, formulas, subformulas, sentences and theories is also assumed.

- A model $\mathcal{M}$, also called a structure, for a given language $\mathcal{L}$. If $\mathcal{M}$ is a model, $k$ a natural number greater 0, $\phi(v_1, \ldots, v_k)$ a $\mathcal{L}$-formula with free variables among $v_1, \ldots, v_k$ and $a_1, \ldots, a_k$ are in $\mathcal{M}$, then well known expressions like $\mathcal{M} \models \phi(a_1, \ldots, a_k)$ or $\mathcal{M} \models T$, where $T$ is a $\mathcal{L}$-theory, will be used without explanation.

- A Submodel $\mathfrak{M}_1$ and an elementary submodel $\mathfrak{M}_2$ of a given model $\mathfrak{N}$. The former will be notated with $\mathfrak{M}_1 \subset \mathfrak{N}$, the latter with $\mathfrak{M}_2 \prec \mathfrak{N}$.

- A homomorphism, an elementary embedding and an isomorphism of models.

- The semantic consequence. If a set of formulas $\Gamma$ of a given language $\mathcal{L}$ implies another set of formulas $\Sigma$ then this is notated with $\Gamma \models \Sigma$. In case $\Sigma = \{\phi\}$ the notation $\Gamma \models \phi$ is used.

Definitions and examples can be found for example in chapters 1 - 3 of [5]. All languages in this section will be first order.

Throughout this thesis results from set theory are used which require the Axiom of choice. We therefore assume ZFC.

Let $\mathcal{L}$ be a language, $\kappa$ a cardinal and $T$ a complete theory, i.e. $T \models \sigma$ or $T \models \neg\sigma$, for all sentences $\sigma$ of $\mathcal{L}$. A natural question to ask is how many models of cardinality $\kappa$ up to isomorphism does $T$ have. Clearly, every model of cardinality $\kappa$ is isomorphic to one which has the set $\kappa$ as its universe, so if we let $\lambda := |\mathcal{L}|$ and $\mu := max\{\kappa, \lambda\}$ then it is easy to see that there are at most $2^\mu$ many isomorphism classes of models of $T$ of cardinality $\kappa$. This observation leads to the following

**Definition.**
- (The spectrum function) The function $I$ which maps each tuple $(T, \kappa)$, where $T$ is a theory and $\kappa$ a cardinal, onto the cardinal number of isomorphism classes of models of $T$ of cardinality $\kappa$ is called the spectrum function.

- An isomorphism class is also called an isomorphism type.

- If $I(T, \kappa) = 1$ then T is called $\kappa$-categorical.

**Notation.**
- A $k$-tuple $(u_1, \ldots, u_k)$ or a sequence $(u_i : i \in \omega)$ of elements of a set $U$ is often written as $\overline{u}$ if misunderstandings can be ruled out. If $W$ is a set and $g : U \longrightarrow W$ is a function, then $g(\overline{u})$ stands for $(g(u_1), \ldots, g(u_k))$.

- Sometimes expressions like

$$\bigwedge_{i \in I} \phi_i \text{ or } \bigvee_{i \in I} \phi_i$$

are used. Here, $I$ is a finite index set and each $\phi_i$ is a formula. These expressions describe a finite conjunction (or disjunction) in an arbitrary order of the reader's liking, such that for each $i \in I$, $\phi_i$ occurs exactly once.

- We define $\omega_+ := \omega \setminus \{0\}$ and $\mathfrak{c} := 2^{\aleph_0}$, the cardinality of the reals.

We assume that the reader is familiar with basic forcing arguments as some of them will be used later in this thesis. A good introduction to the theory of forcing can be found in [11].

Unless specifically stated otherwise, all vocabularies considered in this thesis are countable, all first order theories are complete and have infinite models. Complete theories with finite models are not interesting for us because of the following

**Proposition 1.1.1.** *(Folklore) Let $T$ be a complete theory of an arbitrary, not necessarily countable, language $\mathcal{L}$. If $T$ has a finite model, then all models of $T$ are isomorphic.*

*Proof.* If $\mathfrak{M} =< \{a_1, \ldots, a_k\}, ... >$ is a model of $T$ and $k := |M| \in \omega_+$, then $T \models$ "There are exactly k elements".

Suppose there is $\mathfrak{N} \models T$ not isomorphic to $\mathfrak{M}$. Let $N$ be the universe of $\mathfrak{N}$ and $S$ the set of all bijections from $M$ onto $N$. Clearly, $|S| = k!$.

For all $f \in S$, there is a quantifier free formula $\phi_f(v_1, \ldots, v_k)$ such that

$$\mathfrak{M} \models \phi_f(\overline{a}) \text{ and } \mathfrak{N} \models \neg\phi_f(f(\overline{a})).$$

By defining

$$\Psi := \exists v_1 \ldots \exists v_k[(\bigwedge_{1 \leq i < j \leq k} (v_i \neq v_j)) \wedge (\bigwedge_{f \in S} \phi_f(\overline{v}))]$$

we get $\mathfrak{M} \models \Psi$ and $\mathfrak{N} \models \neg\Psi$, contradicting the assumption that $T$ is complete. □

$\aleph_0$-categorical theories have been studied thoroughly in the last century. Characterisations were given by Erwin Engeler, Czesław Ryll-Nardzewski and Lars Svenonius, see for example theorem 7.3.1 of [9]. A well known example for such a theory is $DLO$, the theory of dense linear orders without endpoints.

Section 3.3 of [5] shows that there is a complete $T$ with $I(T, \aleph_0) = k$, if $k \in (\omega + 1) \setminus \{0, 2\} \cup \{\mathfrak{c}\}$.

"Vaught's Never Two" theorem - see theorem 6.1 of [21] and also theorem 3.3.48 of [5] - is a surprising result by Robert Vaught stating that $I(T, \aleph_0) = 2$ is impossible. At the end of [21] the author asks the following question:

> " *Can it be proved, without the use of the continuum hypothesis, that there exists a complete theory having exactly $\aleph_1$ non-isomorphic denumerable models?* "

This problem has not been solved yet. Since many mathematicans believe the answer to this question to be negative, it was reformulated as a conjecture, known as Vaught's Conjecture, which is abbreviated by $VC_1$ in this thesis:

$VC_1$: If $T$ is a complete theory of a countable language, then either $I(T, \aleph_0) \leq \aleph_0$ or $I(T, \aleph_0) = \mathfrak{c}$.

This statement is trivially true if we assume the continuum hypothesis (CH), i.e. $2^{\aleph_0} = \aleph_1$, which is undecidable in $ZFC$.

The study of $VC_1$ has led to new questions which can be seen as more general versions of the original problem and are independent of the value of $\mathfrak{c}$. We will see proofs of results related to $VC_1$ using methods of descriptive set theory and model theory. A first new perspective gives us infinitary logic.

# 2 The Language $\mathcal{L}_{\omega_1,\omega}$

## 2.1 Basics

Definitions and results of this subsection can be found in chapter 1 of [12].

Let $\tau$ be a vocabulary and $V := \{v_\alpha | \alpha < \omega_1\}$ a set of distinct variables. The language $\mathcal{L}_{\omega_1,\omega}$ $(\tau)$ is defined analogously to a first order language with the exception that countable conjunctions and disjunctions are allowed. That means:

- All variables of terms and formulas are in $V$.
- Atomic formulas are atomic first order formulas.
- The rules for negation, finite conjuntion, finite disjunction and quantification are the same as for first order logic.
- New rules: If $F$ is a countable set of $\mathcal{L}_{\omega_1,\omega}(\tau)$ formulas, then

$$\bigvee_{\phi \in F} \phi \quad \text{and} \quad \bigwedge_{\phi \in F} \phi$$

  are $\mathcal{L}_{\omega_1,\omega}(\tau)$ formulas.

For this thesis, the author has chosen to expand the standard definition of $\mathcal{L}_{\omega_1,\omega}$: A new symbol $\top$, called verum, is added and defined as an atomic sentence with no variables or constnats.

Hence forth, when considering a specific vocabulary $\tau$, the notation "$\mathcal{L}_{\omega_1,\omega}$" will be used instead of "$\mathcal{L}_{\omega_1,\omega}(\tau)$" if misunderstandings can be ruled out.The notation is chosen to indicate that countable conjunctions and disjunctions are allowed but only finite blocks of quantifiers.

The definitions of subformulas as well as of free and bound variables of a $\mathcal{L}_{\omega_1,\omega}$ formula are defined analogously to first order formulas and extended in an obvious way. The following facts are easily checked via induction on formula complexity:

**Fact 2.1.1.** Every $\mathcal{L}_{\omega_1,\omega}$ formula has at most $\aleph_0$ many variables and subformulas.

**Fact 2.1.2.** If a $\mathcal{L}_{\omega_1,\omega}$ formula has only finitely many free variables, then the same holds for each of its subformulas.

For a $\tau$ model $\mathfrak{M} := \langle M, \dots \rangle$, a $\mathcal{L}_{\omega_1,\omega}$-formula $\phi(\overline{v})$ and a map $\pi : V \longrightarrow M$, one defines the notion $\mathfrak{M} \models \phi(\pi(\overline{v}))$ inductively just as for first order formulas with the exception that if $F$ is a countable set of $\mathcal{L}_{\omega_1,\omega}$ forumlas then

$$\mathfrak{M} \models \bigwedge_{\psi \in F} \psi(\pi(\overline{v})) \text{ iff } \mathfrak{M} \models \psi(\pi(\overline{v})), \text{ for all } \psi \in F,$$

and

$$\mathfrak{M} \models \bigvee_{\psi \in F} \psi(\pi(\overline{v})) \text{ iff } \mathfrak{M} \models \psi(\pi(\overline{v})), \text{ for some } \psi \in F.$$

Furthermore, we define $\mathfrak{M} \models \top$ for all $\tau$ structures.

**Notation.** From now on, as in texts on first order logic, the expression

$$\mathfrak{M} \models \phi(\overline{m})$$

is used, where $\overline{m} := \pi(\overline{v})$.

Infinitary logic enables us to characterize classes of certain structures which cannot be axiomatized in first order logic.

**Example 2.1.3.** Let $\tau := \{e, \circ\}$, the vocabulary of groups. The class of torsion groups is not axiomatizable by a first order theory. This can be shown using the compactness theorem - see theorem 2.1.8 of [5]. However, we can characterize these structures by a single $\mathcal{L}_{\omega_1,\omega}$-sentence, namely

$$\sigma_G \wedge \forall x (\bigvee_{n \in \omega_+} \underbrace{x \circ \cdots \circ x}_{\text{n times}} = e),$$

where $\sigma_G$ is the conjunction of the group axioms.

On the other hand, some useful results of first order logic do not hold in infinitary logic:

**Example 2.1.4.** The compactness theorem. For $n \in \omega_+$, let $\phi_n$ be defined as

$$\exists x_1 \dots \exists x_n \forall y (\bigvee_{i=1}^{n} y = x_i)$$

and for $n > 1$, let

$$\psi_n := \exists x_1 \dots \exists x_n (\bigwedge_{1 \le i < j \le n} x_i \neq x_j).$$

Consider the set $\Sigma := \{\sigma\} \cup \{\psi_n | n \in \omega, \ n > 1\}$, where

$$\sigma := \bigvee_{n \in \omega_+} \phi_n.$$

Clearly, $\Sigma$ is not satisfiable, even though each of its finite subsets is.

**Example 2.1.5.** The upward Löwenheim-Skolem theorem. Let $\tau := \{c_i | i < \omega\}$ be a vocabulary of constant symbols. Define

$$\sigma := [\bigwedge_{0 \leq i < j < \omega} (c_i \neq c_j)] \wedge [\forall x (\bigvee_{i < \omega} x = c_i)].$$

This $\mathcal{L}_{\omega_1,\omega}$-sentence has an infinite model, but not in every cardinality. In fact, every model of $\sigma$ is countable.

**Definition.** We define the function $\sim: \mathcal{L}_{\omega_1,\omega} \longrightarrow \mathcal{L}_{\omega_1,\omega}$ inductively:

- $\sim(\phi) := \neg\phi$, if $\phi$ is atomic.
- $\sim(\neg\phi) := \phi$.
- For $F \subseteq \mathcal{L}_{\omega_1,\omega}$ and $|F| \leq \aleph_0$,

$$\sim(\bigwedge_{\phi \in F} \phi) := \bigvee_{\phi \in F} \sim(\phi) \text{ and } \sim(\bigvee_{\phi \in F} \phi) := \bigwedge_{\phi \in F} \sim(\phi).$$

- If $v$ is a variable, then $\sim(\forall v \phi) := \exists v \sim(\phi)$ and $\sim(\exists v \phi) := \forall v \sim(\phi)$.

The next result can be easily verified via induction on formula complexity:

**Proposition 2.1.6.** *Let $\mathfrak{M}$ be a $\tau$ structure and $\phi$ a $\mathcal{L}_{\omega_1,\omega}$-formula. Then*

$$\mathfrak{M} \models \sim\phi \text{ iff } \mathfrak{M} \models \neg\phi.$$

Every first order theory of a countable language can be written as a $\mathcal{L}_{\omega_1,\omega}$-sentence. By expanding the definition of the spectrum function to include infinitary sentences we get a generalized version of $VC_1$:

$VC_2$: For every $\mathcal{L}_{\omega_1,\omega}$-sentence $\sigma$ of a countable vocabulary, either $I(\sigma, \aleph_0) \leq \aleph_0$ or $I(\sigma, \aleph_0) = \mathfrak{c}$.

Clearly, $VC_2 \Rightarrow VC_1$.

**Definition.** A fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$ is a set $\mathbb{A}$ of $\mathcal{L}_{\omega_1,\omega}(\tau)$ formulas such that there is an infinite set $W$ of variables with the following properties:

- All variables of $\mathbb{A}$ are in $W$.
- All closed atomic formulas and those with variables of $W$ are in $\mathbb{A}$.
- If $\phi(x, \dots) \in \mathbb{A}$ and $t$ is a term, either closed or with variables of $W$, then $\phi(t, \dots) \in \mathbb{A}$.
- If $\phi \in \mathbb{A}$, $v$ is a variable of $\phi$ and $u \in W$, then $\phi' \in \mathbb{A}$, where $\phi'$ is gained from $\phi$ by replacing each occurrence of $v$ with $u$.
- $\mathbb{A}$ is closed under the function $\sim$ as well as under negation ($\neg$), finite conjunctions, finite disjunctions and quantification over variables of $W$.
- $\mathbb{A}$ is closed with respect to subformulas.

**Definition.** Let $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}$ be a fragment and $\mathfrak{M} := \langle M, \dots \rangle$, $\mathfrak{N} := \langle N, \dots \rangle$ be $\tau$ structures such that $M \subseteq N$. Then $\mathfrak{M}$ is called an $\mathbb{A}$ elementary submodel of $\mathfrak{N}$, notated by $\mathfrak{M} \prec_{\mathbb{A}} \mathfrak{N}$, if for all $n < \omega$, all $\phi(v_1, \dots, v_n) \in \mathbb{A}$ and all $\overline{m} = (m_1, \dots, m_n) \in M^n$, we have

$$\mathfrak{M} \models \phi(\overline{m}) \text{ iff } \mathfrak{N} \models \phi(\overline{m}).$$

We can now state the infinitary version of the Tarski-Vaught criterion, a well known result of first order logic:

**Proposition 2.1.7.** *Let $\mathfrak{M} := \langle M, \dots \rangle$, $\mathfrak{N} := \langle N, \dots \rangle$ be $\tau$ structures such that $\mathfrak{M} \subset \mathfrak{N}$ and $\mathbb{A}$ be a fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$. Then $\mathfrak{M} \prec_{\mathbb{A}} \mathfrak{N}$ if and only if for all $\phi(\overline{v}, y) \in \mathbb{A}$ and all $\overline{m}$ of $M$, $\mathfrak{N} \models \exists y \phi(\overline{m}, y)$ implies $\mathfrak{N} \models \phi(\overline{m}, a)$, for some $a \in M$.*

The proof can be done via induction on formula complexity.

There is also a downward Löwenheim-Skolem theorem for $\mathcal{L}_{\omega_1,\omega}$:

**Lemma 2.1.8.** *Suppose $\mathbb{A}$ is a countable fragment in which all formulas have only finitely many free variables, $\mathfrak{M} := \langle M, \dots \rangle$ is a $\tau$ structure and $T \subseteq M$. Then there is $\mathcal{B} := \langle B, \dots \rangle$ such that $\mathcal{B} \prec_{\mathbb{A}} \mathfrak{M}$, $T \subseteq B$ and $|B| \leq max\{\aleph_0, |T|\}$.*

The proof is similar to that of the first order version, using Skolem functions.

There are also other infinitary languages. One of particular interest for us is $\mathcal{L}_{\infty,\omega}$.

**Definition.** The language $\mathcal{L}_{\infty,\omega}$ is the class of all formulas which can be built by the rules for formulas of first order logic with the additional rule:

- If $F$ is a arbitrary set of $\mathcal{L}_{\infty,\omega}$ formulas, then

$$\bigwedge_{\phi \in F} \phi \text{ and } \bigvee_{\phi \in F} \phi$$

  are $\mathcal{L}_{\infty,\omega}$ formulas.

Clearly, $\mathcal{L}_{\omega_1,\omega} \subset \mathcal{L}_{\infty,\omega}$.

In model theory, one is often interested, if two models agree on sentences of a certain complexity which is determined by quantifiers.

**Definition.** We inductively define the quantifier rank of a $\mathcal{L}_{\infty,\omega}$ formula $\phi$, notated by $qr(\phi)$:

- $qr(\phi) := 0$, if $\phi$ is atomic.
- $qr(\neg\phi) := qr(\phi)$.
- If $F$ is a set of $\mathcal{L}_{\infty,\omega}$ formulas, then

$$qr(\bigwedge_{\phi \in F} \phi) = qr(\bigvee_{\phi \in F} \phi) := sup\{qr(\phi) | \phi \in F\}$$

- $qr(\exists v \phi) := qr(\phi) + 1$

One can verify by induction on formula complexity that $qr(\phi)$ is an ordinal number, for every $\phi \in \mathcal{L}_{\infty,\omega}$, and if $\phi \in \mathcal{L}_{\omega_1,\omega}$, then $qr(\phi) < \omega_1$.

## 2.2 Scott Rank and Scott's Isomorphism Theorem

Unless otherwise stated, definitions and results are from section 2.2 of [12]

**Definition.** Via induction on $\alpha \in \mathbf{ON}$, for a given vocabulary $\tau$, the equivalence relation $\sim_\alpha$ is defined on the class of all tuples $(\mathfrak{M}; \overline{a})$, where $\mathfrak{M} := \langle M, \dots \rangle$ is a $\tau$ structure and $\overline{a} \in M^{<\omega}$.

- If $(\mathfrak{M}; \overline{a}) \sim_\alpha (\mathfrak{N}; \overline{b})$, then $length(\overline{a}) = length(\overline{b})$.
- $(\mathfrak{M}; \overline{a}) \sim_0 (\mathfrak{N}; \overline{b})$ iff for all atomic $\phi(\overline{v})$, $\mathfrak{M} \models \phi(\overline{a}) \Leftrightarrow \mathfrak{N} \models \phi(\overline{b})$.
- $(\mathfrak{M}; \overline{a}) \sim_{\alpha+1} (\mathfrak{N}; \overline{b})$ iff for all $c \in M$ there is $d \in N$ such that

  $$(\mathfrak{M}; \overline{a}^\frown c) \sim_\alpha (\mathfrak{N}; \overline{b}^\frown d),$$

  and for all $d \in N$ there is $c \in M$ such that

  $$(\mathfrak{M}; \overline{a}^\frown c) \sim_\alpha (\mathfrak{N}; \overline{b}^\frown d).$$

  For a limit ordinal $\alpha > 0$, $(\mathfrak{M}; \overline{a}) \sim_\alpha (\mathfrak{N}; \overline{b})$ iff for all $\beta < \alpha$

  $$(\mathfrak{M}; \overline{a}) \sim_\beta (\mathfrak{N}; \overline{b}).$$

By induction on $\alpha \in \mathbf{ON}$ one checks:

**Fact 2.2.1.** If $(\mathfrak{M}; \overline{a}) \sim_\alpha (\mathfrak{N}; \overline{b})$ and $\beta < \alpha$, then $(\mathfrak{M}; \overline{a}) \sim_\beta (\mathfrak{N}; \overline{b})$.

We will see that each $\sim_\alpha$ equivalence class can be described by a single infinitary foumula.

**Definition.** For every $\tau$ structure $\mathfrak{M} := \langle M, \dots \rangle$ and every $\overline{a} \in M^{<\omega}$, we inductively define the formula $\chi^{\mathfrak{M}}_{\overline{a},\alpha}(\overline{v})$, where $length(\overline{v}) = length(\overline{a})$:

- $\chi^{\mathfrak{M}}_{\overline{a},0}(\overline{v})$ is the conjunction of all atomic and negated atomic formulas $\psi(\overline{v})$ such that $\mathfrak{M} \models \psi(\overline{a})$.
- $$\chi^{\mathfrak{M}}_{\overline{a},\alpha+1}(\overline{v}) := [\bigwedge_{c \in M} \exists w \chi^{\mathfrak{M}}_{\overline{a}^\frown c,\alpha}(\overline{v}, w)] \wedge [\forall w (\bigvee_{c \in M} \chi^{\mathfrak{M}}_{\overline{a}^\frown c,\alpha}(\overline{v}, w))].$$
- If $\alpha > 0$ is a limit ordinal, then

  $$\chi^{\mathfrak{M}}_{\overline{a},\alpha}(\overline{v}) := \bigwedge_{\beta < \alpha} \chi^{\mathfrak{M}}_{\overline{a},\beta}(\overline{v}).$$

**Remark.** It is because of this definition that we added the closed atomic sentence $\top$ to the language. If $\tau$ is a vocabulary without constant symbols and $\overline{a} = \emptyset$, then $\chi^{\mathfrak{M}}_{\overline{a},0} = \top$.

We can make the following immediate observations:

**Proposition 2.2.2.** *Let $\mathfrak{M} := \langle M, \dots \rangle$ be a $\tau$ structure, $k < \omega$ and $\overline{a} \in M^k$. Then for every ordinal $\alpha$:*

1. *$\chi^{\mathfrak{M}}_{\overline{a},\alpha}(\overline{v}) \in \mathcal{L}_{\infty,\omega}$ and has quantifier rank $\alpha$.*

2. $\mathfrak{M} \models \chi^{\mathfrak{M}}_{\overline{a},\alpha}(\overline{a})$.

3. *If* $\mathfrak{M}$ *is countable and* $\alpha < \omega_1$*, then* $\chi^{\mathfrak{M}}_{\overline{a},\alpha}(\overline{v}) \in \mathcal{L}_{\omega_1,\omega}$.

*Proof.* Straight forward induction on $\alpha \in \mathbf{ON}$. For the third statement, keep in mind that we are only considering countable vocabularies. □

The next Lemma is also easy to prove and will turn out useful later in this thesis.

**Lemma 2.2.3.** *Let* $\mathfrak{M} := \langle M, \dots \rangle$, $\mathfrak{N} := \langle N, \dots \rangle$ *be* $\tau$ *structures,* $k \in \omega$, $\overline{a} \in M^k$ *and* $\overline{b} \in N^k$*. The following are equivalent for* $\alpha \in$ ***ON****:*

1. $(\mathfrak{M}; \overline{a}) \sim_\alpha (\mathfrak{N}; \overline{b})$.

2. $\chi^{\mathfrak{M}}_{\overline{a},\alpha}(\overline{v}) = \chi^{\mathfrak{N}}_{\overline{b},\alpha}(\overline{v})$.

3. *For all k formulas* $\psi(\overline{v}) \in \mathcal{L}_{\infty,\omega}$ *with quantifier rank* $\leq \alpha$*,*

$$\mathfrak{M} \models \psi(\overline{a}) \text{ iff } \mathfrak{N} \models \psi(\overline{b}).$$

*Proof.* The equivalence of all three statements can be proved by induction on $\alpha \in \mathbf{ON}$. □

**Proposition 2.2.4.** *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be infinite* $\tau$ *structures and* $\kappa := max\{|M|, |N|\}$*. There is* $\gamma < \kappa^+$ *such that for all* $k < \omega$*,* $\overline{a} \in M^k$ *and* $\overline{b} \in N^k$ *the following holds:*

$$\text{If } (\mathfrak{M}; \overline{a}) \sim_\gamma (\mathfrak{N}; \overline{b}), \text{ then } (\mathfrak{M}; \overline{a}) \sim_\alpha (\mathfrak{N}; \overline{b}), \text{ for all } \alpha \in \mathbf{ON}.$$

*Proof.* We define the length of a tuple $\overline{a}$ as $lg(\overline{a})$ and for every $\alpha \in \mathbf{ON}$,

$$\Gamma_\alpha := \{(\overline{a}, \overline{b}) \in M^{<\omega} \times N^{<\omega} |\ lg(\overline{a}) = lg(\overline{b}) \text{ and } (\mathfrak{M}; \overline{a}) \not\sim_\alpha (\mathfrak{N}; \overline{b})\}.$$

By fact 2.2.1, $\Gamma_\alpha \subseteq \Gamma_\beta$, for $\alpha \leq \beta$, and via induction on $\alpha$ one can show that if $\Gamma_\alpha = \Gamma_{\alpha+1}$, then $\Gamma_\alpha = \Gamma_\beta$, for all $\beta \geq \alpha$. Since $|M^{<\omega} \times N^{<\omega}| = \kappa$, there is $\gamma < \kappa^+$ such that $\Gamma_\gamma = \Gamma_{\gamma+1}$. Clearly, $\gamma$ has the proposed property. □

**Definition.**

- We write $\mathfrak{M} \equiv_\alpha \mathfrak{N}$ if $(\mathfrak{M}; \emptyset) \sim_\alpha (\mathfrak{N}; \emptyset)$, that is if $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the same $\mathcal{L}_{\infty,\omega}$ sentences of quantifier rank $\leq \alpha$. Two structures $\mathfrak{M}$ and $\mathfrak{N}$ are called infinitarily equivalent, notated by $\mathfrak{M} \equiv_{\infty,\omega} \mathfrak{N}$ or in this thesis simply by $\mathfrak{M} \equiv_\infty \mathfrak{N}$, if $\mathfrak{M} \equiv_\alpha \mathfrak{N}$, for all $\alpha \in \mathbf{ON}$.
- For every $\tau$ structure $\mathfrak{M}$, the least $\gamma < |M|^+$ such that for all $k < \omega$ and all $k$-tuples $\overline{a}$, $\overline{b}$ of $M$,

  $$(\mathfrak{M}; \overline{a}) \sim_\gamma (\mathfrak{M}; \overline{b}) \text{ implies } (\mathfrak{M}; \overline{a}) \sim_{\gamma+1} (\mathfrak{M}; \overline{b}),$$

  is called the Scot rank or Scott height of $\mathfrak{M}$, denoted by $sr(\mathfrak{M})$.

Combining Lemma 2.2.3 with proposition 2.2.4, we see that the Scott rank of a model can be described by an infinitary sentence. Given a $\tau$ structure $\mathfrak{M}$ with Scott rank $\gamma$, we call the formula

$$\Psi_{\mathfrak{M}} := \chi^{\mathfrak{M}}_{\emptyset,\gamma} \wedge \bigwedge_{k<\omega} [\bigwedge_{\overline{a} \in M^k} \forall v_1 \dots \forall v_k (\chi^{\mathfrak{M}}_{\overline{a},\gamma}(\overline{v}) \rightarrow \chi^{\mathfrak{M}}_{\overline{a},\gamma+1}(\overline{v}))]$$

the ***Scott sentence*** of $\mathfrak{M}$.

The next result gives a characterisation of infinitary equivalence and of the isomorphism relation on countable structures.

**Theorem 2.2.5.** (**Scott's Isomorphism Theorem**) *Let $\mathfrak{M}$, $\mathfrak{N}$ be two models. Then*

1. $\mathfrak{M} \equiv_\infty \mathfrak{N}$ *iff* $\mathfrak{M} \models \Psi_\mathfrak{N}$ *iff* $\mathfrak{N} \models \Psi_\mathfrak{M}$.

2. *If $\mathfrak{M}$ and $\mathfrak{N}$ are countable, then $\mathfrak{M} \cong \mathfrak{N}$ iff $\mathfrak{N} \models \Psi_\mathfrak{M}$.*

*Proof.* 1. We only check $\mathfrak{M} \models \Psi_\mathfrak{N} \Rightarrow \mathfrak{M} \equiv_\infty \mathfrak{N}$: Let $\delta := sr(\mathfrak{N})$ and as in the proof of proposition 2.2.4 for $\alpha \in \mathbf{ON}$, define

$$\Gamma_\alpha := \{(\bar{a}, \bar{b}) \in M^{<\omega} \times \; N^{<\omega} |\; lg(\bar{a}) = lg(\bar{b}) \text{ and } (\mathfrak{M}; \bar{a}) \not\sim_\alpha (\mathfrak{N}; \bar{b})\}.$$

It follows form the definition of $\Psi_\mathfrak{N}$ and Lemma 2.2.3, that $\Gamma_\delta = \Gamma_{\delta+1}$ and then by induction $\Gamma_\delta = \Gamma_\beta$, for all $\beta \geq \delta$.

$\mathfrak{M} \models \chi^\mathfrak{N}_{\emptyset,\delta}$ means that $(\emptyset, \emptyset) \notin \Gamma_\delta$ and therefore $\mathfrak{M} \equiv_\infty \mathfrak{N}$.

2. The direction ($\Rightarrow$) is clear.

The proof of ($\Leftarrow$) can be done via a back and forth argument:

Let $(a_n : n < \omega)$ be an enumeration of $M$, $(b_n : n < \omega)$ be one of $N$ and $\delta := sr(\mathfrak{M})$. Since $\mathfrak{N} \models \Psi_\mathfrak{M}$, it follows - using Lemma 2.2.3 - $\mathfrak{M} \equiv_{\delta+1} \mathfrak{N}$, so for $a_0 \in M$ we can choose $i \in \omega$ minimal such that $(\mathfrak{M}; a_0) \sim_\delta (\mathfrak{N}; b_i)$. Suppose $k < \omega_+$, $\bar{a} \in M^k$, $\bar{b} \in N^k$ and $(\mathfrak{M}; \bar{a}) \sim_\delta (\mathfrak{N}; \bar{b})$. In particular, $\bar{a} \mapsto \bar{b}$ is a finite partial embedding from $\mathfrak{M}$ into $\mathfrak{N}$. Choose $j \in \omega$ minimal such that $b_j$ does not occur in $\bar{b}$. As in the first step, we have $(\mathfrak{M}; \bar{a}) \sim_{\delta+1} (\mathfrak{N}; \bar{b})$, hence we can choose $r \in \omega$ minimal such that $(\mathfrak{M}; \bar{a}^\frown a_r) \sim_\delta (\mathfrak{N}; \bar{b}^\frown b_j)$. In the following step we choose $j \in \omega$ minimal such that $a_j$ does not occur in $\bar{a}$ and then use the same argument to extend the finite partial embedding. Since both enumerations are countable, the union of these finite embeddings is an isomorphism from $\mathfrak{M}$ onto $\mathfrak{N}$.

□

Let us look at some examples:

**Example 2.2.6.** Define $\mathfrak{M} := (\mathbb{Q}, <)$ with the standard linear order. Then $sr(\mathfrak{M}) = 0$, since if $k < \omega$ and $\bar{a}, \bar{b} \in \mathbb{Q}^k$ satisfy the same atomic formulas, there is an automorphism of $\mathfrak{M}$ mapping $\bar{a}$ onto $\bar{b}$.

**Example 2.2.7.** Consider the models $\mathfrak{M} := (\aleph_1 \times \mathbb{Q}, <_1)$ and $\mathfrak{N} := (\aleph_1 \times \mathbb{Q}, <_2)$. In $\mathfrak{M}$ we define $(\alpha, q) <_1 (\beta, r)$ if either $\beta < \alpha$ or $\alpha = \beta$ and $q < r$, and in $\mathfrak{N}$ $(\alpha, q) <_2 (\beta, r)$ iff $\alpha \in \beta$ or $\alpha = \beta$ and $q < r$ in $\mathbb{Q}$.

Using the downward Löwenheim Skolem theorem, one can show

$$\mathfrak{M} \equiv_\infty (\mathbb{Q}, <) \equiv_\infty \mathfrak{N},$$

hence $sr(\mathfrak{M}) = sr(\mathfrak{N}) = 0$. Clearly, $\mathfrak{M} \not\cong \mathfrak{N}$, since every element of $\mathfrak{M}$ has $\aleph_1$ many predecessors, which is not the case in $\mathfrak{N}$.

**Remark.** This example shows that infinitary equivalence of two models does not imply that they are isomorphic. However, two structures are $\equiv_{\infty,\omega}$ equivalent if and only if they are isomorphic in some generic extension of the universe:

(⇒) Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are two infinitarily equivalent models and let $\kappa := max\{|M|, |N|\}$. Consider the forcing notion $P := (Fn(\omega, \kappa), <)$, the set of all finite functions from $\omega$ into $\kappa$, where for all $p, q \in P$:

$$p \leq q :\Leftrightarrow p \supseteq q.$$

If $G$ is a generic filter, then since the relation $\sim_\alpha$ is absolute for transitive models of ZFC, $\mathfrak{M}$ and $\mathfrak{N}$ are $\equiv_{\infty,\omega}$ equivalent and countable in $V[G]$, hence by Scott's theorem the models are isomorphic.

The direction (⇐) follows from the fact that the statement

$$"\mathfrak{M} \models \phi(\overline{a})"$$

is absolute for transitive models of $ZFC$.

**Example 2.2.8.** Consider $\mathfrak{M} := (\mathbb{Z}, \dot{0}, \dot{+})$, where $\dot{0}^{\mathbb{Z}} = 0$ and $\dot{+}$ is interpreted with the standard addition. We check that $sr(\mathfrak{M}) = 1$: Suppose $k \in \omega_+$, $\overline{a} = (a_1, \ldots, a_k), \overline{b} = (b_1, \ldots, b_k) \in \mathbb{Z}^k$ and $(\mathfrak{M}; \overline{a}) \sim_1 (\mathfrak{M}; \overline{b})$, then for $1 \leq j \leq k$, we have $|a_j| = |b_j|$. This follows as for every divider $d > 0$ of $a_j$, the formula

$$\exists y(\underbrace{y \dot{+} \ldots \dot{+} y}_{\text{d times}} = x)$$

has quantifier rank 1 and is therefore satisfied by both $a_j$ and $b_j$. With a similar argument one can show that either $\overline{a} = \overline{b}$ or $\overline{a} = -1 \cdot \overline{b}$. Obviously, the identity map and the map $a \mapsto -a$ are automorphisms of $\mathfrak{M}$, hence $sr(\mathfrak{M}) \leq 1$.

On the other hand, it is easy to check that $(\mathfrak{M}, 1) \sim_0 (\mathfrak{M}, 2)$, but

$$(\mathfrak{M}, 1) \not\sim_1 (\mathfrak{M}, 2),$$

which means $sr(\mathfrak{M}) > 0$.

## 2.3 Model Existence and Omitting Types

Even though the compactness theorem does not hold in $\mathcal{L}_{\omega_1,\omega}$, there is another way to check whether a set of formulas has a model. The idea is due to Michael Makkai and adapts a Henkin argument for infinitary logic. One important notion we will need several times in this thesis is that of a consistency property. We assume that our countable vocabulary contains an infinite set $C$ of constant symbols. Also, recall the definition of $\sim \phi$, for $\phi \in \mathcal{L}_{\infty,\omega}$.

**Definition.** A consistency property is a set $\Sigma$ of countable sets of $\mathcal{L}_{\omega_1,\omega}$ sentences such that for all $\sigma \in \Sigma$ the following conditions hold:

$(C_0)$ If $\mu \subseteq \sigma$, then $\mu \in \Sigma$.

$(C_1)$ If $\phi \in \sigma$, then $\neg\phi \notin \sigma$.

$(C_2)$ If $\neg\phi \in \sigma$, then there is $\mu \in \Sigma$ such that $\sigma \cup \{\sim \phi\} \subseteq \mu$.

$(C_3)$ If

$$\bigwedge_{\phi \in F} \phi \in \sigma,$$

then for all $\phi \in F$ there is $\mu \in \Sigma$ such that $\sigma \cup \{\phi\} \subseteq \mu$.

$(C_4)$ If

$$\bigvee_{\phi \in F} \in \sigma,$$

then there is $\phi \in F$ and $\mu \in \Sigma$ such that $\sigma \cup \{\phi\} \subseteq \mu$.

$(C_5)$ If $\forall v \phi(v) \in \sigma$, then for all $c \in C$ there is $\mu \in \Sigma$ such that $\sigma \cup \{\phi(c)\} \subseteq \mu$.

$(C_6)$ If $\exists v \phi(v) \in \sigma$, then there is $c \in C$ and $\mu \in \Sigma$ such that $\sigma \cup \{\phi(c)\} \subseteq \mu$.

$(C_7)$ Let $t$ be an arbitrary closed term, i.e. a term with no variables, and $c, d \in C$.

a.) If $(c = d) \in \sigma$, then there is $\mu \in \Sigma$ such that $\sigma \cup \{d = c\} \subseteq \mu$.

b.) If $c = t, \phi(t) \in \sigma$, then there is $\mu \in \Sigma$ such that $\sigma \cup \{\phi(c)\} \subseteq \mu$.

c.) There is $e \in C$ and $\mu \in \Sigma$ such that $\sigma \cup \{e = t\} \subseteq \mu$.

The next two facts are easy to verify:

**Fact 2.3.1.** If $\Sigma_0$ is a set of countable sets of $\mathcal{L}_{\omega_1,\omega}$-sentences satisfying $C_1 - C_7$, then $\Sigma := \{\delta | \exists \sigma \in \Sigma_0,\ \delta \subseteq \sigma\}$ is a consistency property.

**Fact 2.3.2.** If $\Sigma_0$ is a consistency property, then so are $\Sigma_1$ and $\Sigma_2$, where

$$\Sigma_1 := \{\sigma \in \Sigma_0 |\ \sigma \text{ is finite}\}$$

and $\Sigma_2$ is the set of all $\sigma \in \Sigma_0$, in which only finitely many constant symbols of $C$ occur.

**Theorem 2.3.3.** ***(Model Existence Theorem)*** *If $\Sigma$ is a consistency property and $\sigma \in \Sigma$, then there is a countable model $\mathfrak{M} \models \sigma$.*

*Proof.* (Sketch) Let $\mathbb{A}$ be the smallest fragment conatining $\sigma$ which is closed under $\sim$. Clearly, $\mathbb{A}$ is countable.

Using an enumeration of $\mathbb{A}$ which lists every formula infinitely many times and the definition of a consistency property, one can inductively build a set $\Gamma := \bigcup\{\sigma_n | \, n < \omega\} \subseteq \mathbb{A}$, where $\sigma_0 := \sigma$ and $\sigma_n \in \Sigma$, for all $n < \omega$. $\Gamma$ has the following properties:

- For every closed term $t$ there is a constant $c \in C$ such that $(c = t) \in \Gamma$.
- If

$$\bigvee_{\psi \in F} \psi$$

  is in $\Gamma$, then some $\psi \in F$ is also in $\Gamma$.
- If $\neg\psi \in \Gamma$, then $\sim \psi \in \Gamma$.
- If $\exists x \psi(x)$ is in $\Gamma$, then for some constant symbol $c$, $\psi(c) \in \Gamma$.

Then consider the equivalence relation on the set of constant symbols $C$ defined by $c \sim d :\Leftrightarrow\ (c = d) \in \Gamma$. The model $\mathfrak{M}$ has the set of equivalence classes $C/\sim$ as its universe. All symbols of the vocabulary can be interpreted in an obvious way such that every element of the universe is the interpretation of a constant symbol and for all $\psi \in \Gamma$, $\mathfrak{M} \models \psi$. See the proof of theorem 4.1.6 in [12] for the details. □

By a slight modification of the previous proof one can show:

**Theorem 2.3.4.** *(Extended Model Existence Theorem) Let $\Sigma$ be a consistency property and $T$ a countable set of $\mathcal{L}_{\omega_1,\omega}$-sentences (in the vocabulary $\tau$) such that for all $\sigma \in \Sigma$ and all $\psi \in T$ $\sigma \cup \{\psi\} \in \Sigma$. Then for all $\sigma \in \Sigma$, $T \cup \sigma$ has a countable model.*

As in the proof of the previous theorem, a countable model is constructed in which every element is the interpretation of a constant symbol.

One important application of the Model Existence Theorem is the infinitary and generalized version of the Omitting Types Theorem:

**Theorem 2.3.5.** *Let $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}$ be a countable fragment, $T \subseteq \mathbb{A}$ a satisfiable theory and for every $n \in \omega$ $\Theta_n(v_1, \ldots, v_{k_n})$ a set of $\mathbb{A}$ formulas. Suppose that for all $n$ and all $\mathbb{A}$ formulas $\psi(v_1, \ldots, v_{k_n})$ such that $T + \exists \overline{v}\psi(\overline{v})$ is satisfiable there is $\theta \in \Theta_n$ such that $T + \exists \overline{v}(\psi(\overline{v}) \wedge \theta(\overline{v}))$ is satisfiable. Then*

$$T + \bigwedge_{n<\omega} (\forall \overline{v} \bigvee_{\theta \in \Theta_n} \theta(\overline{v}))$$

*is satisfiable.*

*Proof.* Add a new set of constant symbols $C$ to the given vocabulary. Let $\mathbb{A}'$ be the set of all formulas of $\mathbb{A}$ with only finitely many free variables. Then $\mathbb{A}^*$ is defined as the smallest fragment containing $\{\phi(\overline{c}) | \phi(\overline{v}) \in \mathbb{A}',\ \overline{c} \in C^{<\omega}\}$ and

$$\Delta := \{ \bigvee_{\theta \in \Theta_n} \theta(\overline{c}) |\ n < \omega,\ \overline{c} \in C^{k_n}\}.$$

Now consider the set $\Sigma$ of all elements of the form $\sigma_0 \cup T \cup \Delta$, where $\sigma_0 \subseteq \mathbb{A}^*$ is a finite set of sentences and $T \cup \sigma_0$ is satisfiable. It is not difficult to show that $\Sigma$ is a consistency property. The only interesting case is $(C_4)$ when

$$\bigvee_{\phi \in F} \phi \in \Delta.$$

Let $\mu := \sigma_0 \cup T \cup \Delta$ be an element of $\Sigma$. Since $\sigma_0$ is finite, there are only finitely many constant symbols of $C$, $c_1, \ldots, c_l$, occurring in it, and therefore

$$\psi(c_1, \ldots, c_l) := \bigwedge_{\phi \in \sigma_0} \phi \in \mathbb{A}^*.$$

Without loss of generality assume that $l = k_n$. Then by assumption for some $\theta \in \Theta_n$

$$T + \exists \overline{v}(\psi(\overline{v}) \wedge \theta(\overline{v}))$$

is satisfiable, hence $\mu \cup \{\theta(\overline{c})\} \in \Sigma$.

Since $T$ is satisfiable, we have $\mu_0 := T \cup \Delta \in \Sigma$ and so there is a countable model of $\mu_0$ in which every element is the interpretation of a constant symbol of $C$. □

The result is called omitting types theorem because it gives us a sufficient criterion for the existence of a countable model which does not realise a countable set of given types: Suppose we have a countable fragment $\mathbb{A}$ a theory $T \subseteq \mathbb{A}$, and a set

$$\{t_n(v_1, \dots, v_{i_n}) : n < \omega\}$$

of types such that for every $n < \omega$ and every $\phi(v_1, \dots, v_{i_n})$ satisfiable with $T$, there $\gamma(\overline{v}) \in t_n$ such that $T + \exists \overline{v}(\phi(\overline{v}) \wedge \neg\gamma(\overline{v}))$ is satisfiable, then by defining $\Theta_n := \{\neg\gamma : \gamma \in t_n\}$, we can apply the omitting types theorem an get a model which omits every $t_n$.

In first order logic one can use the compactness theorem to prove that the class of all countable well orders cannot be axiomatized. With the help of theorem 2.3.3 we can show that this class cannot be characterized by a $\mathcal{L}_{\omega_1,\omega}$-sentence either.

**Lemma 2.3.6.** *Suppose the countable vocabulary $\tau$ has a binary relation symbol $\dot{<}$ and $\phi$ is a $\mathcal{L}_{\omega_1,\omega}$-sentence such for all $\alpha < \omega_1$ there is a model $\mathfrak{M}$ of $\phi$ containing $(\alpha, <)$ as a submodel. Then there is $\mathfrak{N} \models \phi$ such that $(\mathbb{Q}, <)$ embeds into it as a submodel.*

*Proof.* First, we consider a new vocabulary $\tau^*$ by adding two countably infinite sets $C$ and $D := \{d_s | s \in \mathbb{Q}\}$ of new constant symbols. Let

$$\Delta := \{d_r \dot{<} d_s |\ r, s \in \mathbb{Q},\ r < s\} \subseteq \mathcal{L}_{\omega_1,\omega}(\tau^*)$$

and $\Sigma$ the set of all elements $\mu$ with the following properties:

$(P1)$ $\mu = \sigma_0 \cup \{\phi\} \cup \Delta$, where $\sigma_0$ is a finite set of $\tau^*$ sentences in which only finitely many new constant symbols occur and $\sigma_0 \cup \{\phi\}$ is satisfiable.

$(P2)$ If no symbols of $D$ occur in $\sigma_0$, then for all $\alpha < \omega_1$, there is a model of $\sigma_0 \cup \{\phi\}$ such that $(\alpha, <)$ can be embedded into it, and if $d_{s_1}, \dots, d_{s_k}$ are all symbols of $D$ occurring in $\sigma_0$ and $s_1 < \cdots < s_k$ in $\mathbb{Q}$, then for all $\alpha < \omega_1$, there is a model $\mathfrak{M} \models \sigma_0 \cup \{\phi\}$ such that $d_{s_1}^{\mathfrak{M}} < \cdots < d_{s_k}^{\mathfrak{M}}$ in $\mathfrak{M}$ and for $1 \leq j < k$, $(\alpha, <)$ can be embedded into the sets $(<, d_{s_1}^{\mathfrak{M}}) := \{a \in M |\ a <^{\mathfrak{M}} d_{s_1}^{\mathfrak{M}}\}$,

$$(d_{s_j}^{\mathfrak{M}}, d_{s_{j+1}}^{\mathfrak{M}}) := \{a \in M | d_{s_j}^{\mathfrak{M}} <^{\mathfrak{M}} a <^{\mathfrak{M}} d_{s_{j+1}}^{\mathfrak{M}}\}$$

and $(d_{s_k}^{\mathfrak{M}}, <) := \{a \in M |\ d_{s_k}^{\mathfrak{M}} <^{\mathfrak{M}} a\}$.

Once we have shown that $\Sigma$ is a consistency property, the proof is complete, since clearly $\{\phi\} \cup \Delta \in \Sigma$. This is a routine exercise and there are only two interesting cases.

$C_4$: Suppose $\mu = \sigma_0 \cup \{\phi\} \cup \Delta \in \Sigma$ and without loss of generality for some countable set of $\tau^*$ formulas $F$,

$$\bigvee_{\psi \in F} \psi$$

is in $\sigma_0$. Then using the regularity of $\omega_1$, it follows that for some $\psi \in F$, $\mu \cup \{\psi\} \in \Sigma$.

$(C_7) - b.)$: We only check the case when $\phi(t) = (d_r \dot{<} d_q) \in \Delta$, for some $r < s \in \mathbb{Q}$ and $t \in \{d_r, d_q\}$. Assume without loss of generality that $t = d_r$, $\mu = \sigma_0 \cup \{\phi\} \cup \Delta \in \Sigma$, $c$ is a constant symbol $\notin D$, $s_1 < \cdots < r < \cdots < s_k$ a finite sequence in $\mathbb{Q}$ such that all constant symbols of $D$ occurring in $\sigma_0$

are among $d_{s_1}, \dots, d_r, \dots, d_{s_k}$ and $\{(d_r \dot{<} d_q), (c = d_r)\} \subseteq \mu$. We will see that $\mu_2 := \mu \cup \{c \dot{<} d_q\} \in \Sigma$. If $\alpha < \omega_1$ and $\beta := \alpha + \alpha$, then there is a model $\mathfrak{M}$ of $\sigma_0 \cup \{\phi\}$ such that $(\beta, <)$ can be emebbed into it according to $(P2)$. If $d_s$ occurs in $\sigma_0$, then $\mathfrak{M}$ witnesses both $(P1)$ and $(P2)$ for $\mu_2$. Otherwise, suppose $r \leq s_i < q < s_{i+1}$. We can get a model $\mathfrak{M}'$ of $\mu_2$ by interpreting $d_q$ in $M$ such that $(\alpha, <)$ can be embedded into $(d_{s_i}^{\mathfrak{M}'}, d_q^{\mathfrak{M}'})$. This is possible, since by assumption $(\beta, <)$ can be embedded into $(d_{s_i}^{\mathfrak{M}}, d_{s_{i+1}}^{\mathfrak{M}})$, hence both $(P1)$ and $(P2)$ hold for $\mu_2$. □

**Remark.** The assumption that every $\alpha < \omega_1$ can be embedded into some model of $\phi$ is necessary, since otherwise the argument of the proof cannot simultaneously guarantee that $\Sigma$ satisfies $C_4$ and that $\{\phi\} \cup \Delta \in \Sigma$.

As an immediate consequence we have

**Corollary 2.3.7.** *Let $\phi$ be a $\mathcal{L}_{\omega_1,\omega}$-sentence in a vocabulary with a binary relation symbol $\dot{<}$ such that in every model $\mathfrak{M}$ of $\phi$ $\dot{<}^{\mathfrak{M}}$ is a well order on $M$. Then the set of order types of models of $\phi$ is bounded below $\omega_1$.*

**Remark.** (See also page 85 of [2]) It is possible to characterize the class of countable well orders with a $\mathcal{L}_{\omega_1,\omega}$-theory: Let $\tau$ be a vocabulary with a single binary relation symbol $\dot{<}$. Via induction on $\alpha < \omega_1$ one can define a $\mathcal{L}_{\omega_1,\omega}$-formula $\phi_\alpha(v)$ such that for every model $\mathfrak{M}$ and all $a \in M$: $\mathfrak{M} \models \phi_\alpha(a)$ iff $\dot{<}^{\mathfrak{M}}$ is a linear order and $((<, a), \dot{<}^{\mathfrak{M}})$ is a well order of order type $\alpha$.

Let

$$\sigma_\alpha := \forall x(\bigvee_{\beta<\alpha} \phi_\beta(x)) \vee \exists x(\phi_\alpha(x)),$$

and $T := \{\sigma_\alpha | \ \alpha < \omega_1\}$. Then a countable structure is a model of $T$ if and only if it is a well order.

## 2.4 End Extensions and Small Uncountable Models

Even though $VC_2$ focuses on isomorphism types of countable models, we will also take a look at what can be said about uncountable models of presumed counterexamples.

**Definition.** (1) Let $(\mathfrak{M}_\alpha : \alpha < \mu)$ be a sequence of models such that for all $\alpha < \beta < \mu$, $\mathfrak{M}_\alpha \subset \mathfrak{M}_\beta$. Define the model $\mathfrak{N}$ as follows:

- The universe of $\mathfrak{N}$ is $N := \bigcup_{\alpha<\mu} M_\alpha$.
- $c^{\mathfrak{N}} := c^{\mathfrak{M}_0}$, for every constant symbol $c$ of $\tau$.
- For every relation symbol $R$ of $\tau$,

$$R^{\mathfrak{N}} := \bigcup_{\alpha<\mu} R^{\mathfrak{M}_\alpha}.$$

- For every function symbol $f$ of $\tau$,

$$f^{\mathfrak{N}} := \bigcup_{\alpha<\mu} f^{\mathfrak{M}_\alpha}.$$

Then $\mathfrak{N}$ is called the limit of this sequence, also notated as

$$lim_{\alpha\to\mu}\mathfrak{M}_\alpha.$$

(2) Let $\mu \in \mathbf{ON}$ and $(\mathfrak{M}_\alpha := \langle M_\alpha, \dots\rangle : \alpha < \mu)$ be a sequence of models such that the following hold:

- $\mathfrak{M}_\alpha \subseteq \mathfrak{M}_\beta$, for $\alpha < \beta$.
- For every limit ordinal $\lambda < \mu$, $\mathfrak{M}_\lambda = lim_{\alpha\to\lambda}\mathfrak{M}_\alpha$.

Then $(\mathfrak{M}_\alpha : \alpha < \mu)$ is called a chain of models. If $\mathbb{A} \subseteq \mathcal{L}_{\infty,\omega}$ is a fragment and in addition $\mathfrak{M}_\alpha \prec_\mathbb{A} \mathfrak{M}_\beta$, for $\alpha < \beta < \mu$, then $(\mathfrak{M}_\alpha : \alpha < \mu)$ is called an $\mathbb{A}$-elementary chain of models.

**Proposition 2.4.1.** *(i) Let $\mu$ be an arbitrary ordinal, $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}$ is a countable fragment and $(\mathfrak{M}_\alpha : \alpha < \mu)$ is an $\mathbb{A}$ elementary chain with limit $\mathfrak{N}$, then $\mathfrak{M}_\alpha \prec_\mathbb{A} \mathfrak{N}$, for all $\alpha < \mu$.*

*(ii) If $\mu$ is a regular uncountable cardinal, $(\mathfrak{M}_\alpha : \alpha < \mu)$ be a chain of infinite models with limit $\mathfrak{N}$ and $\sigma \in \mathcal{L}_{\omega_1,\omega}$ a sentence. Then $\mathfrak{N} \models \sigma$ if and only if the set*

$$\{\alpha < \mu : \mathfrak{M}_\alpha \models \sigma\}$$

*contains a club set.*

*Proof.* (i) is easy to check.

(ii) is an application of the downward Löwenheim-Skolem theorem and uses the fact that $\mathfrak{M}_\lambda$ is the limit of the chain restricted to $\lambda$, for every nonempty limit $\lambda < \mu$. E.g. for the direction ($\Rightarrow$) consider the fragment $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}$ generated by $\sigma$. Clearly, $\mathbb{A}$ is countable and using the regularity of $\mu$, one can easily check that

$$\{\alpha < \mu : \mathfrak{M}_\alpha \prec_\mathbb{A} \mathfrak{N}\}$$

is as desired. □

In first order logic every theory with an uncountable model has a model in every cardinality. This can be shown with the help of the compactness theorem. As we have seen this result is not true in infinitary logic.

However, if $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}$ is a countable fragment, $\sigma \in \mathbb{A}$ is a sentence with an infinite model and every countable model of $\sigma$ has a proper $\mathbb{A}$-elementary extension, then there is an $\mathbb{A}$-elementary chain $(\mathfrak{M}_\alpha : \alpha < \omega_1)$ of countable models of $\sigma$ such that $\mathfrak{M}_\alpha \precneqq_\mathbb{A} \mathfrak{M}_\beta$, for $\alpha < \beta$. Clearly, the limit of this chain is also a model of $\sigma$ and has cardinality $\aleph_1$. We present a sufficient criterion for the existence of a proper elementary chain.

**Definition.** (1) Let $\mathcal{R}$ be a binary relation symbol of a given vocabulary $\tau$ and $\mathfrak{M} := \langle M, \dots\rangle \subseteq \mathfrak{N} := \langle N, \dots\rangle$ $\tau$-structures. $\mathfrak{N}$ is called an end extension of $\mathfrak{M}$ if for all $a, b \in N$ $aRb$ and $b \in M$ implies $a \in M$, where $R = \mathcal{R}^\mathfrak{N}$.

$\mathfrak{N}$ is called a strong end extension of $\mathfrak{M}$ if $\mathfrak{N}$ is an end extension of $\mathfrak{M}$, $\mathfrak{M} \subsetneqq \mathfrak{N}$, and for all $b \in N \setminus M$ and $a \in M$ $aRb$.

(2) Let $\psi(x,\overline{w}) \in \mathcal{L}_{\infty,\omega}$. The expression $\exists^* x\psi(x,\overline{w})$ stands for the formula $\forall y\exists x(y\mathcal{R}x \wedge \psi(x,\overline{w}))$.
$\forall^* x\psi(x,\overline{w})$ stands for the formula $\exists y\forall x(y\mathcal{R}x \to \psi(x,\overline{w}))$. In both formulas $y$ is a variable not occurring in $(x,\overline{w})$.

(3) Let $\mathbb{A} \subseteq \mathcal{L}_{\infty,\omega}$ be a fragment. We say $\mathbb{A}$ has property $(*)$, if it has the following closure properties:

$(*)_1$ If
$$\psi, \bigvee_{\phi\in F} \phi \in \mathbb{A},$$
then
$$\bigvee_{\phi\in F} (\phi \wedge \psi), \bigvee_{\phi\in F} (\psi \wedge \phi) \in \mathbb{A}.$$

$(*)_2$ If
$$\exists\overline{v} \bigvee_{\phi\in F} \phi(\overline{v}) \in \mathbb{A}, \text{ then } \bigvee_{\phi\in F} \exists\overline{v}\phi(\overline{v}) \in \mathbb{A},$$
where $F \subseteq \mathbb{A}$ and all formulas of $F$ have their free variables among $\overline{v}$.

With respect to linear orders, strong end extensions are just proper end extensions, and we only consider such end extensions in this thesis.

**Theorem 2.4.2.** *Let $\tau$ be a countable vocabulary with a binary relation symbol $\dot{<}$ and $\mathfrak{M}$ a countable model in which $<:= \dot{<}^{\mathfrak{M}}$ is a linear order. If $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}(\tau)$ is a countable fragment with property $(*)$, then $\mathfrak{M}$ has a countable $\mathbb{A}$-elementary strong end extension if and only if the following holds in $\mathfrak{M}$:*

*(i) For all $a \in M$ there is $b \in M$ such that $a < b$.*

*(ii) If $\psi(x,y,\overline{w}) \in \mathbb{A}$, then*
$$\mathfrak{M} \models \forall\overline{w}[\exists^* x\exists y\psi(x,y,\overline{w}) \to (\exists^* y\exists x\psi(x,y,\overline{w}) \vee \exists y\exists^* x\psi(x,y,\overline{w}))].$$

*(iii) If*
$$\bigvee_{\psi\in F} \psi(x,\overline{w}) \in \mathbb{A},$$
*then*
$$\mathfrak{M} \models \forall\overline{w}(\exists^* x \bigvee_{\psi\in F} \psi(x,\overline{w}) \to \bigvee_{\psi\in F} \exists^* x\psi(x,\overline{w})).$$

*Proof.* $(\Rightarrow)$: Let $\mathfrak{N}$ be a strong $\mathbb{A}$-elementary extension of $\mathfrak{M}$ and $b \in N\backslash M$. We will check that $(ii)$ holds, a similar argument works for the other two properties.

Let $\overline{u} \in M$ and assume - using $\mathfrak{M} \prec_{\mathbb{A}} \mathfrak{N}$ - $\mathfrak{N} \models \exists y\psi(b,y,\overline{u})$. If $\mathfrak{N} \models \psi(b,m,\overline{u})$, for some $m \in M$, then since $a < b$ for all $a \in M$, we have
$$\mathfrak{M} \models \exists y\exists^* x\psi(x,y,\overline{u}).$$
Otherwise, $\mathfrak{N} \models \psi(b,n,\overline{u})$, where $n \in N \setminus M$, and consequently
$$\mathfrak{M} \models \exists^* y\exists x\psi(x,y,\overline{u}).$$

($\Leftarrow$): Let $\mathfrak{M} := \langle M, \dots \rangle$ be countable and satisfy $(i) - (iii)$. Enlarge the given vocabulary $\tau$ to $\tau'$, by adding new constants

$$\{d\}\dot{\cup}C,$$

where $C := \{c_m | m \in M\}$ .If $\overline{c} = (c_{m_1}, \dots, c_{m_k})$ is a $k$-tuple of $C$, then $m(\overline{c})$ denotes $(m_1, \dots, m_k)$. Define

$$T := \{\psi(d, \overline{c}) | \psi(x, \overline{v}) \in \mathbb{A},\ \overline{c} \in C^{<\omega} \text{ and } \mathfrak{M} \models \forall^* x \psi(x, m(\overline{c}))\}$$

and for $a \in M$,

$$\theta_a(y) := (\bigvee_{m \in M} y = c_m) \vee c_a \dot{<} y.$$

Note that there is a strong end extension of $\mathfrak{M}$ iff $T \cup \{\forall y \theta_a(y) : a \in M\}$ has a model. This oservation uses property $(i)$. We will show the existence of such a model by using the omitting types theorem (2.3.5).

<u>*Claim* 1 :</u> $T$ is satisfiable.

Extend $\tau'$ to $\tau''$ by adding a countable set $U$ of new constants and let $\Sigma$ be the set of all $\sigma$ with the following properties:

- $\sigma$ is a finite set of $\mathcal{L}_{\omega_1,\omega}(\tau'')$ sentences all of which are of the form

  $$\psi(d, \overline{c}, \overline{u}),$$

  for some $\psi(x, \overline{v}, \overline{w}) \in \mathbb{A}$ with only finitely many free variables, $\overline{c} \in C^{<\omega}$ and $\overline{u} \in U^{<\omega}$.

- For

  $$\chi_\sigma(d, \overline{c}, \overline{u}) := \bigwedge_{\psi \in \sigma} \psi(d, \overline{c}, \overline{u}),$$

  we have

  $$\mathfrak{M} \models \exists^* x \exists \overline{w} \chi_\sigma(x, m(c), \overline{w}).$$

Then $\Sigma$ is a consistency property. The proof is routine and we will only check $(C_4)$ for which the property $(*)$ is used: Suppose $\sigma \in \Sigma$ and

$$\bigvee_{\psi \in F} \psi(d, \overline{c}, \overline{u}) \in \sigma.$$

It follows that

$$\mathfrak{M} \models \exists^* x \bigvee_{\psi \in F} \exists \overline{w} [\psi \wedge \chi_\sigma](x, m(c), \overline{w}).$$

Note that by the definition of the $(*)$ property, this formula is in $\mathbb{A}$, therefore we can apply $(iii)$ and conclude $\sigma \cup \{\psi\} \in \Sigma$, for some $\psi \in F$.

Using property $(i)$, it is easy to check that for every $\phi \in T$ and every $\sigma \in \Sigma$, $\sigma \cup \{\phi\} \in \Sigma$, hence the extended model existence theorem (2.3.4) guarantees that $T$ is satisfiable.(q.e.d.-*Claim* 1)

<u>*Claim* 2 :</u> If $\psi(x, \overline{v}) \in \mathbb{A}$ and $\overline{c} \in C^{<\omega}$, then $T + \psi(d, \overline{c})$ is satisfiable if and only if $\mathfrak{M} \models \exists^* x \psi(x, m(\overline{c}))$.

($\rightarrow$): If $\mathfrak{M} \not\models \exists^* x \psi(x, m(\overline{c}))$, then $\mathfrak{M} \models \forall^* x \neg\psi(x, m(\overline{c}))$, hence $\neg\psi(d, \overline{c})$ is in $T$ and $T + \psi(d, \overline{c})$ is not satisfiable.

($\leftarrow$): If $\mathfrak{M} \models \exists^* x\psi(x, m(\bar{c}))$, then $\{\psi(d, \bar{c})\} \in \Sigma$ and by the extended model existence theorem $T + \psi(d, \bar{c})$ is satisfiable. (q.e.d.-*Claim* 2)

In order to apply the omitting types theorem, we check that for every, $\psi(x, y, \bar{v}) \in \mathbb{A}$ and $\bar{c} \in C^{<\omega}$ such that $T + \exists y\psi(d, y, \bar{c})$ is satisfiable, there is a model for

$$T + \exists y[\psi(d, y, \bar{c}) \wedge \theta_a(y)],$$

for every $a \in M$.

By claim 2 and the fact that $<$ is a linear order, we have

$$\mathfrak{M} \models \exists^* x\exists y[(\psi(x, y, m(\bar{c})) \wedge a < y) \vee (\psi(x, y, m(\bar{c}) \wedge y \leq a)].$$

Since *(iii)* holds in $\mathfrak{M}$, there are two possibilities:

(1) $\mathfrak{M} \models \exists^* x\exists y(\psi(x, y, m(\bar{c}) \wedge a < y)$: In this case we are done by claim 2 and the fact that $c_a \dot{<} y \models \theta_a(y)$.

(2) $\mathfrak{M} \models \exists^* x\exists y(\psi(x, y, m(\bar{c})) \wedge y \leq a)$: Apply *(ii)* and since clearly

$$\mathfrak{M} \not\models \exists^* y\exists x(\psi(x, y, m(\bar{c})) \wedge y \leq a),$$

there is $n \in M$ such that

$$\mathfrak{M} \models \exists^* x(\psi(x, n, m(\bar{c})) \wedge n \leq a),$$

which by claim 2 implies that

$$T + \exists y[\psi(d, y, \bar{c}) \wedge \bigvee_{m \in M} y = c_m]$$

and therefore $T + \exists y[\psi(d, y, \bar{c}) \wedge \theta_a(y)]$ is satisfiable.

The omitting types theorem now guarantees the existence of a strong end extension of $\mathfrak{M}$. □

**Corollary 2.4.3.** *Let $<$ be a binary relation symbol of the vocabulary $\tau$, $\mathbb{A}$ a countable fragment with property $(*)$ and $\mathfrak{M}$ a countable model such that $<^{\mathfrak{M}}$ is a linear order. If $\mathfrak{M}$ has a strong $\mathbb{A}$-elementary end extension, then it has one with cardinality $\aleph_1$.*

*Proof.* If $\mathfrak{M}_1$ is a countable strong end extension of $\mathfrak{M}$, then by the previous theorem properties $(i) - (iii)$ hold in $\mathfrak{M}$. Since $\mathfrak{M} \prec_{\mathbb{A}} \mathfrak{M}_1$, this is also true in $\mathfrak{M}_1$, hence $\mathfrak{M}_1$ has a countable strong end extension $\mathfrak{M}_2$. Clearly, we can build an $\mathbb{A}$ elementary chain of length $\aleph_1$ of strong end extensions of $\mathfrak{M}$. Then the limit of this chain has cardinality $\aleph_1$ and is also a strong end extension of $\mathfrak{M}$. □

**Corollary 2.4.4.** *Let $\tau$ be a vocabulary with a binary relation symbol $<$ and $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$ a sentence. Suppose $\mathfrak{M}$ is a model of $\phi$ and $<^{\mathfrak{M}}$ is a well order of order type $\omega_1$. Then there is a model $\mathfrak{N}$ of $\phi$ with cardinality $\aleph_1$ such that $(\mathbb{Q}, <)$ embeds into it.*

*Proof.* Let $\mathbb{A}$ be the smallest fragment containing $\phi$ with property $(*)$.

Add a new unary relation symbol $P$ and let $\sigma$ be the sentence in the new vocabulary $\tau'$ stating:

(1) $\exists x \neg Px$.

(2) $<$ is a linear order and $\forall x \forall y[(x < y \wedge Py) \to Px]$.

(3) $P$ is an $\mathbb{A}$-elementary submodel.

(3) can easily be expressed as an infinite conjunction using the Tarski-Vaught criterion. Note that since $\mathbb{A}$ is countable, $\sigma \in \mathcal{L}_{\omega_1,\omega}(\tau')$.

For every $\alpha < \omega_1$, there is a countable model $(\mathcal{M}_\alpha, P_\alpha)$ of $\phi \wedge \sigma$ such that $(\alpha, <)$ embeds into it and $P_\alpha \prec_{\mathbb{A}} \mathcal{M}_\alpha \prec_{\mathbb{A}} \mathfrak{M}$: This is a straight forward application of the downward Löwenheim-Skolem theorem. By the undefinability of well orders (see Lemma 2.3.6) there is a countable model $(\mathcal{B}, \mathcal{P})$ of $\phi \wedge \sigma$, such that $(\mathbb{Q}, <)$ embeds into it. But then clearly $\mathcal{B}$ is a strong $\mathbb{A}$-elementary end extension of $\mathcal{P}$, seen as $\tau$-structures. Thus properties $(i) - (iii)$ of theorem 2.4.2 hold in $\mathcal{B}$. By the previous corrolary $\mathcal{B}$ has a strong end extension $\mathfrak{N}$ of cardinality $\aleph_1$. □

Now we have the necessary means to present a sufficient criterion for a $\mathcal{L}_{\omega_1,\omega}$-sentence to have an uncountable model that realizes only countably many types.

**Definition.** Let $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}$ be a fragment. A model $\mathfrak{M}$ is called $\mathbb{A}$-small if for all $k < \omega$, $\mathfrak{M}$ realizes $\leq \aleph_0$ many $k$-types $\subseteq \mathbb{A}$. $\mathfrak{M}$ is called small if it is $\mathcal{L}_{\omega_1,\omega}$-small.

**Proposition 2.4.5.** *A model $\mathfrak{N}$ is small iff there is a countable model $\mathfrak{M}$ such that $\mathfrak{M} \equiv_{\infty,\omega} \mathfrak{N}$.*

*Proof.* $(\Rightarrow)$: Recall the definitions of $\sim_\alpha$, for $\alpha \in \mathbf{ON}$, $\chi^{\mathfrak{N}}_{\overline{a},\alpha}(\overline{v})$, where $\overline{a} \in N^{<\omega}$, and the characterisation given in Lemma 2.2.3.

Since the vocabulary $\tau$ is countable and $\mathfrak{N}$ realizes only countably many complete types $\subseteq \mathcal{L}_{\omega_1,\omega}$, it is easy to show inductively that for $\alpha < \omega_1$ the following conditions hold:

- $\chi^{\mathfrak{N}}_{\overline{a},\alpha}(\overline{v}) \in \mathcal{L}_{\omega_1,\omega}$, for all $\overline{a} \in N^{<\omega}$.
- $|\{\chi^{\mathfrak{N}}_{\overline{a},\alpha}(\overline{v}) | \ \overline{a} \in N^{<\omega}\}| \leq \aleph_0$.

Using this and the regularity of $\omega_1$ it follows that there is $\alpha < \omega_1$ such that for all $n < \omega$ and $\overline{a}, \overline{b} \in N^n$:

$$(\mathfrak{N}; \overline{a}) \sim_\alpha (\mathfrak{N}; \overline{b}) \text{ implies } (\mathfrak{N}; \overline{a}) \sim_{\alpha+1} (\mathfrak{N}; \overline{b}).$$

Thus $sr(\mathfrak{N}) < \omega_1$ and $\Psi_{\mathfrak{N}} \in \mathcal{L}_{\omega_1,\omega}$, where $\Psi_{\mathfrak{N}}$ is the Scott sentence of $\mathfrak{N}$. By the downward Löwenheim-Skolem theorem there is a countable model $\mathfrak{M} \models \Psi_{\mathfrak{N}}$, which by Scott's isomorphism theorem means $\mathfrak{N} \equiv_{\infty,\omega} \mathfrak{M}$.

$(\Leftarrow)$: Let $\mathfrak{M}$ be countable, $\mathfrak{N} \equiv_{\infty,\omega} \mathfrak{M}$, $k < \omega$ and $\overline{a} \in N^k$. We will see that the $\mathcal{L}_{\omega_1,\omega}$-type of $\overline{a}$ is realized in $\mathfrak{M}$.

Suppose this is not so. Then for every $\overline{b} \in M^k$ there is $\phi_{\overline{b}}(\overline{v}) \in \mathcal{L}_{\omega_1,\omega}$ such that

$$\mathfrak{N} \models \phi_{\overline{b}}(\overline{a}) \text{ and } \mathfrak{M} \models \neg\phi_{\overline{b}}(\overline{b}).$$

Hence

$$\mathfrak{N} \models \exists \overline{v} \bigwedge_{\overline{b} \in M^k} \phi_{\overline{b}}(\overline{v}), \text{ but } \mathfrak{M} \not\models \exists \overline{v} \bigwedge_{\overline{b} \in M^k} \phi_{\overline{b}}(\overline{v}),$$

a contradiction. Therefore $\mathfrak{N}$ realizes at most as many complete $\mathcal{L}_{\omega_1,\omega}$-types as $\mathfrak{M}$ does, i.e. $\leq \aleph_0$ many. □

**Definition.** Let $(X,<)$ be a linear order and $p \in X$. An element $q \in X$ is called successor of $p$ if $p < q$ and there is no $r \in X$ such that $p < r < q$.

$p$ is called a limit point if $p$ is not minimal and for all $r < p$ there is $s \in X$ such that $r < s < p$.

**Theorem 2.4.6.** *If a sentence $\phi \in \mathcal{L}_{\omega_1,\omega}$ has an uncountable model which is $\mathbb{A}$-small, for every countable fragment $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}$, then $\phi$ has a small uncountable model.*

*Proof.* Let $\mathfrak{M}$ be an uncountable model of $\phi$ which is $\mathbb{A}$-small for every countable fragment $\mathbb{A}$. Since $\phi$ is a sentence, we can by the downward Löwenheim-Skolem theorem assume that $M$, the universe of $\mathfrak{M}$, is $\omega_1$.

As in the proof of proposition 2.4.5 one can show via induction on $\alpha < \omega_1$ that for all $n < \omega$ the equivalence relation $E_{n,\alpha}$ on $M^n$ defined by

$$\overline{a}\ E_{n,\alpha}\ \overline{b} :\Leftrightarrow (\mathfrak{M};\overline{a}) \sim_\alpha (\mathfrak{M};\overline{b})$$

has only countably many equivalence classes. Here, the conditions of the theorem and the characterisation of Lemma 2.2.3 are explicitly used.

Now we extend the vocabulary $\tau$ to $\tau'$ by adding the following symbols:

- A binary relation symbol $<$.
- For every $n < \omega$ a $2n+1$ ary relation symbol $\mathcal{E}_n$.
- For every $n < \omega$ a $n+1$ ary function symbol $f_n$.

Then extend $\mathfrak{M}$ to $\mathfrak{M}'$ by interpreting the new symbols as follows:

- $<^{\mathfrak{M}'}$ is the element relation on $\omega_1$.
- $\mathcal{E}_n^{\mathfrak{M}'}(\alpha, a_1, \ldots, a_n, b_1, \ldots, b_n)$ iff $\overline{a}E_{n,\alpha}\overline{b}$.
- Choose $f_n^{\mathfrak{M}'} : M^{n+1} \mapsto \omega$ such that for all $\alpha < \omega_1$ and $\overline{a}, \overline{b} \in M^n$:

  $$f_n^{\mathfrak{M}'}(\alpha, \overline{a}) = f_n^{\mathfrak{M}'}(\alpha, \overline{b}) \Leftrightarrow \overline{a}E_{n,\alpha}\overline{b}.$$

  We can do this, since $E_{n,\alpha}$ has only countably many equivalence classes.

Now let $\sigma \in \mathcal{L}_{\omega_1,\omega}(\tau')$ be the sentence stating:

$(S1)$ $<$ is a linear order, there is a minimal element and every element has a successor.

$(S2)$ For every $n < \omega$: If $y$ is the $<$-minimal element, then for all $\overline{v}, \overline{w}$:

$$\mathcal{E}_n(y, \overline{v}, \overline{w}) \leftrightarrow \bigwedge_{\psi \in F} [\psi(\overline{v}) \leftrightarrow \psi(\overline{w})],$$

where $F$ is the set of all atomic $\tau$-formulas.

$(S3)$ For every $n < \omega$ and all $y$: If $z$ is the successor of $y$ then for all $\overline{v}, \overline{w}$:

$$\mathcal{E}_n(z, \overline{v}, \overline{w}) \leftrightarrow (\forall r \exists t[\mathcal{E}_{n+1}(y, \overline{v}^\frown r, \overline{w}^\frown t)] \wedge \forall t \exists r[\mathcal{E}_{n+1}(y, \overline{v}^\frown r, \overline{w}^\frown t)]).$$

$(S4)$ For all $n < \omega$ and all $y$: If $y$ is a limit point then

$$\mathcal{E}_n(y, \overline{v}, \overline{w}) \leftrightarrow \forall x < y(\mathcal{E}_n(x, \overline{v}, \overline{w})).$$

$(S5)$ For all $n < \omega$ and all $\overline{v}, \overline{w}$:

$$\forall y \forall x[(\mathcal{E}_n(y, \overline{v}, \overline{w}) \wedge x < y) \rightarrow \mathcal{E}_n(x, \overline{v}, \overline{w})].$$

$(S6)$ For all $n < \omega$: There is an initial segment $I$ with respect to $<$ of order type $\omega$ such that $im(f_n) \subseteq I$ and:

$$\forall y \forall \overline{v} \forall \overline{w}[f_n(y, \overline{v}) = f_n(y, \overline{w}) \leftrightarrow \mathcal{E}_n(y, \overline{v}, \overline{w})].$$

Clearly, $\mathfrak{M}' \models \phi \wedge \sigma$, and $(M, <^{\mathfrak{M}'})$ is a well order of order type $\omega_1$. Therefore we can apply corrolary 2.4.4. Let $\mathfrak{N} := \langle N, \dots \rangle$ be a model of cardinality $\aleph_1$ such that $\mathfrak{N} \models \phi \wedge \sigma$ and $(\mathbb{Q}, <) \subset (\mathfrak{N}, <^{\mathfrak{N}})$.

It follows from $(S5)$ and $(S6)$ that for all $k < \omega$, we have an equivalence relation $E_n$ on $N^n$ defined by

$$\overline{a} E_n \overline{b} :\Leftrightarrow \exists q \in \mathbb{Q}\ (\mathcal{E}_n^{\mathfrak{N}}(q, \overline{a}, \overline{b})).$$

Via induction on $\alpha < \mathbf{ON}$ one can easily show that for all $n < \omega$ and $\overline{a}, \overline{b} \in N^n$, $\overline{a} E_n \overline{b}$ implies $(\mathfrak{N} \restriction \tau; \overline{a}) \sim_\alpha (\mathfrak{N} \restriction \tau; \overline{b})$, hence $\overline{a}$ and $\overline{b}$ realize the same $\mathcal{L}_{\omega_1,\omega}(\tau)$-type in $\mathfrak{N} \restriction \tau$. $E_n$ has only countably many equivalence classes, thus $\mathfrak{N} \restriction \tau$ is small. □

## 2.5 Atomic and Prime Models

In Robert Vaught's studies about isomorphism types of theories and especially in the proof of his Never Two Theorem the notions of atomic and saturated models played a crucial role. We can generalize these ideas for $\mathcal{L}_{\infty,\omega}$ theories. Only atomic models are important for this thesis.

**Notation.** If $T \subseteq \mathcal{L}_{\infty,\omega}$ is a theory and $\Sigma, \Delta$ are sets of $\mathcal{L}_{\infty,\omega}$ formulas, then

$$\Sigma \models_T \Delta$$

stands for $T \cup \Sigma \models \Delta$. In case $\Sigma = \{\phi\}$, we write $\phi \models_T \psi$.

**Definition.** Let $\mathbb{A} \subseteq \mathcal{L}_{\infty,\omega}$ be a fragment and $T \subseteq \mathbb{A}$ a satisfiable theory.

- If $k \in \omega$, then a formula is called a $k$ formula, if it has at most $k$ free variables.
- $T$ is $\mathbb{A}$-complete if for all sentences $\sigma \in \mathbb{A}$ either $T \models \sigma$ or $T \models \neg\sigma$.
- Let $k < \omega$ and $\phi(v_1, \dots, v_k) \in \mathbb{A}$ a $k$-formula. We say $\phi(\overline{v})$ is $k$- complete in $\mathbb{A}$ over a theory $T \subseteq \mathbb{A}$ if for all $k$-formulas $\psi(\overline{v}) \in \mathbb{A}$ either $\phi \models_T \psi$ or $\phi \models_T \neg\psi$, but not both. If the parameters $k, T, \mathbb{A}$ are clear from the context, then we simply call $\phi(\overline{v})$ complete. A $k$ formula $\psi(\overline{v}) \in \mathbb{A}$ is called completable (over $T$) if for some complete $\phi(\overline{v})$, we have $\phi \models_T \psi$.

- Let $k < \omega$. A $k$-type of $\mathbb{A}$ is a set of formulas $\mathfrak{t} \subseteq \mathbb{A}$ such that for some variables $v_1, \ldots, v_k$, every formula of $\mathfrak{t}$ has its free variables among $v_1, \ldots, v_k$ and for some model $\mathfrak{M}$ and some $\overline{a} \in M^k$, $\mathfrak{M} \models \psi(\overline{a})$, for all $\psi(\overline{v}) \in \mathfrak{t}$. If for all $k$-formulas $\psi \in \mathbb{A}$ either $\psi \in \mathfrak{t}$ or $\neg\psi \in \mathfrak{t}$, then $\mathfrak{t}$ is called complete in $\mathbb{A}$. Given a model $\mathfrak{M}$ and $\overline{a} \in M^{<\omega}$, $\mathfrak{tp}^{\mathbb{A}}_{\mathfrak{M}}(\overline{a})$ is defined as the set of all $lg(\overline{a})$-formulas $\psi(\overline{v}) \in \mathbb{A}$ such that $\mathfrak{M} \models \psi(\overline{a})$.

- If $\mathfrak{M}$ is a model $Th_{\mathbb{A}}(\mathfrak{M})$ is the set of all sentences $\in \mathbb{A}$ which are true in $\mathfrak{M}$.

- A model $\mathfrak{M}$ is called $\mathbb{A}$-atomic if for all $k < \omega$ and all $\overline{a} \in M^k$, $\mathfrak{tp}^{\mathbb{A}}_{\mathfrak{M}}(\overline{a})$ contains a complete formula over $Th_{\mathbb{A}}(\mathfrak{M})$. We simply call a model atomic, if $\mathbb{A}$ is clear from the context.

- A model $\mathfrak{M}$ is called $\mathbb{A}$-prime, if for all models $\mathfrak{N}$ of $Th_{\mathbb{A}}(\mathfrak{M})$ there is an $\mathbb{A}$-elementary embedding $\mathfrak{M} \hookrightarrow_{\mathbb{A}} \mathfrak{N}$

- A complete theory $T$ is called atomic if it has an atomic model.

If $\mathfrak{t}(\overline{v})$ is a complete type of $\mathbb{A}$ containing the complete theory $T$ and $\psi(\overline{v}) \in \mathfrak{t}$ is complete over $T$, then $\mathfrak{t}$ is called isolated over $T$.

**Remark.** For every $n < \omega$, one can define a topology on the set of all complete $n$-types. This topological space is called a Stone space with the Stone topology. A $n$-type $\mathfrak{t}$ is isolated if and only if it is an isolated point in the corresponding Stone space.

In this case we also have, that for some complete formula $\phi(\overline{v}) \in \mathfrak{t}$,

$$\mathfrak{t} = \{\psi(\overline{v}) \in \mathbb{A} : \phi \models_T \psi\}.$$

From here on, unless stated otherwise, $\mathbb{A}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$.

It turns out that important results about atomic models in first order logic also hold in infinitary logic.

**Lemma 2.5.1.** *Let $T$ be an $\mathbb{A}$-complete theory. Then $T$ has an atomic model iff for every $k < \omega$ and every $k$-formula $\psi(\overline{v}) \in \mathbb{A}$ satisfiable with $T$ there is a $k$-formula $\phi(\overline{v})$ which is complete over $T$ such that $\phi \models_T \psi$.*

*Proof.* The directon ($\Rightarrow$) is immediate.

The direction ($\Leftarrow$) can be shown by using the omitting types theorem (2.3.5) and defining $\Theta_n$ as the set of all $n$-complete formulas over $T$. □

**Lemma 2.5.2.** *If $\mathfrak{M}$ is a countable model, then $\mathfrak{M}$ is $\mathbb{A}$-atomic iff it is $\mathbb{A}$-prime.*

*Proof.* ($\Rightarrow$): Let $M := \{a_k | k < \omega\}$, $T := Th_{\mathbb{A}}(\mathfrak{M})$ and $\mathfrak{N}$ an arbitrary model of $T$.

Since $\mathfrak{M}$ is atomic and $T$ complete, there is $\psi_1(v_1)$ complete over $T$ such that $\mathfrak{M} \models \psi_1(a_1)$, hence $\mathfrak{N} \models \psi_1(b_1)$, for some $b_1 \in N$.

Then $\mathfrak{M} \models \psi_2(a_1, a_2)$, for some complete 2-formula $\psi_2(v_1, v_2) \in \mathbb{A}$. It follows $\psi_1(v_1) \models_T \exists v_2 \psi_2(v_1, v_2)$ and so $\mathfrak{N} \models \psi_2(b_1, b_2)$, for some $b_2 \in N$. By continuing this process we get a function from $M$ into $N$. The elementarity follows from the fact that for every $n < \omega$ $\overline{a}$ and $\overline{b}$ satisfy the same $n$-complete formula.

($\Leftarrow$): Suppose $\mathfrak{M}$ is prime, $k < \omega$ and $\overline{a} \in M^k$ such that $S := \mathfrak{tp}^{\mathbb{A}}_{\mathfrak{M}}(\overline{a})$ does not contain a $k$-complete formula. Define $\Theta(\overline{v}) := \{\neg\psi(\overline{v}) |\ \psi(\overline{v}) \in S\}$. It follows that $\Theta$ and $T$ satisfy the conditions of the omitting types theorem, hence there is a countable model $\mathfrak{N}$ of $T$ in which $S$ is omitted, therefore $\mathfrak{M}$ cannot be embedded $\mathbb{A}$-elementarily into $\mathfrak{N}$, a contradiction. □

**Remark.** The argument for the direction ($\Rightarrow$) can also be used in a back and forth manner, in order to show that two countable atomic models of the same complete theory are isomorphic.

**Lemma 2.5.3.** *Suppose $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}(\tau)$ is a countable fragment, $T \subseteq \mathbb{A}$ is $\mathbb{A}$-complete and for every $k < \omega$, there are at most $\aleph_0$ many complete $k$-types $\mathfrak{t} \subseteq \mathbb{A}$ such that $T \subseteq \mathfrak{t}$. Then $T$ has a countable $\mathbb{A}$-atomic model.*

*Proof.* Suppose this is not so, let $k < \omega$ and $\psi(\overline{v})$ be a $k$-formula satisfiable with $T$ which is not completable in $\mathbb{A}$.

By assumption, there is an enumeration $(\mathfrak{t}_n : n < \omega)$ of all complete $k$-types of $\mathbb{A}$, which contain $T \cup \{\psi\}$ as a subset. If none of them is isolated, then for every $k$-formula $\gamma(\overline{v})$ which is satisfiable with $T$ and for every $n < \omega$, there is $\theta_n(\overline{v}) \in \mathfrak{t}_n$ such that $T + \exists\overline{v}(\gamma(\overline{v}) \wedge \neg\theta_n(\overline{v}))$ is satisfiable. Then by the omiting types theorem, there is a countable model $\mathfrak{M}$ of $T$ which omitts $\mathfrak{t}_n$, for all $n < \omega$. But $T$ is complete, hence $\exists\overline{v}\psi(\overline{v}) \in T$ and since every complete type of $\mathbb{A}$ containing $\psi$ is listed, one of them is realized in $\mathfrak{M}$, a contradiction. □

The previous three lemmas are also true in first order logic.We now have the tools to prove

**Lemma 2.5.4.** *Suppose $\phi \in \mathcal{L}_{\omega_1,\omega}$ is a complete sentence and $\mathfrak{M}$ is a model of $\phi$. Then $sr(\mathfrak{M}) \leq qr(\phi) + \omega$.*

*Proof.* It is not difficult to show that a $\mathcal{L}_{\omega_1,\omega}$-sentence is complete if and only if it is $\omega$-categorical. By Scott's isomorphism theorem and the downward Löwenheim-Skolem theorem, we can without loss of generality assume that $\mathfrak{M}$ is countable.

Let $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}$ be the smallest fragment containing $\phi$ and $\alpha := qr(\phi) + \omega$. Clearly, $qr(\psi) \leq \alpha$, for all $\psi \in \mathbb{A}$.

Since $\phi$ is $\omega$-categorical, $\mathfrak{M}$ is $\mathbb{A}$-prime for models of $\phi$ and therefore atomic.

Now suppose $k < \omega$, $\overline{a}, \overline{b} \in M^k$ and $(\mathfrak{M}; \overline{a}) \sim_\alpha (\mathfrak{M}; \overline{b})$. Then $\overline{a}$ and $\overline{b}$ satisfy the same $\mathbb{A}$-complete $k$-formulas in $\mathfrak{M}$. We can then use a back and forth argument to construct an automorphism of $\mathfrak{M}$ mapping $\overline{a}$ onto $\overline{b}$. Thus, $(\mathfrak{M}; \overline{a}) \sim_{\alpha+1} (\mathfrak{M}; \overline{b})$ and by the definition of the Scott rank, we have $sr(\mathfrak{M})$ is less or equal $\alpha$. □

# 3 Connections to Descriptive Set Theory

Unless stated otherwise, results of this section can be found in or are based on chapters from [12].

## 3.1 Important Results for this Thesis

Familiarity with the following notions is required:

- A topological $T_2$( metric, complete, seperable, product, analytic) space.

- A Borel (analytic) function and a homeomorphism between topological spaces.
- A $\sigma$-algebra, a Borel (analytic($\Sigma^1_1$), coanalytic ($\Pi^1_1$)) subset of a topological space.

**Definition.** (i) A topological space $X$ is completely metrizable if it admits a compatible metric $d$ such that $(X, d)$ is complete. A seperable completely metrizable space is called a Polish space.

(ii) A standard Borel space is a measurable space $(X, S)$, such that for some Polish topology $\mathcal{O}$ on $X$, $S$ is the $\sigma$-algebra of Borel ssets generated by $\mathcal{O}$.

$\mathcal{N}$ denotes the Baire space which is Polish.

Polish spaces have been mathamatical objects of great interest and it is an important observation that $\mathcal{L}_{\omega_1,\omega}$-formulas and every model with universe $\omega$ can be interpreted as elements of Polish spaces. This enables us to study VC in the framework of descriptive set theory.

We can interpret countable trees and relations on $\omega$ as elements of $\mathcal{N}$. This will be useful in later proofs and for coding $\mathcal{L}_{\omega_1,\omega}$-formulas.

The following four well known results will be needed, proofs of them can be found for example in [10].

**Lemma 3.1.1.** *Let $X, Y$ be analytic $T_2$ spaces and $f : X \mapsto Y$ be a function. The following are equivalent:*

*(i) $f$ is Borel.*

*(ii) The graph of $f$ is a Borel subset of $X \times Y$.*

*(iii) The graph of $f$ is an analytic set.*

*(iv) $f$ is analytic.*

**Theorem 3.1.2.** *(Souslin's Perfect Set Theorem for Analytic Sets) Let $X$ be a Polish space and $A \subseteq X$ be analytic. Then $A$ is either countable or else contains a perfect set.*

**Definition.** An $\omega$-tree is a tree $\subseteq \omega^{<\omega}$. A tree is well founded if it has no infinite branch.

**Lemma 3.1.3.** *$WF$, the set of codes of well founded $\omega$-trees, is $\Pi^1_1$ but not analytic.*

**Corollary 3.1.4.** *(i) The set of codes of well founded relations on $\omega$ is $\Pi^1_1$ but not analytic.*

*(ii) ($\Sigma^1_1$-bounding of well founded relations) If $A \subseteq \mathcal{N}$ is an analytic subset of codes of well founded relations on $\omega$, then*

$$sup\{\rho(R) : R \in A\} < \omega_1,$$

*where $\rho(R) \in \boldsymbol{ON}$ is the rank of $R$.*

Next, we look at a coding method for $\mathcal{L}_{\omega_1,\omega}$-formulas with finitely many free variables: Suppose we have a countable set of variables $V$, coded by an infinite subset of $\omega$, a coding of the symbols "$\neg$", "$\wedge$", "$\bigwedge$", "$\exists y$", where $y \in V$ and a coding of first order formulas as Gödel numbers. For a first order formula $\psi$ with variables of $V$, let $\ulcorner \psi \urcorner$ denote the Gödel number of $\psi$.

**Definition.** A labeled tree is a triple $(T, l, v) \in \mathcal{N}^3$, where $T$ codes an $\omega$-tree and $l, v$ code functions from $T$ into $\omega$ such that for all $s \in T$, one of the following holds:

(1) $s \in T$ is a terminal node, $l(s)$ is the Gödel number of an atomic formula $\psi$ and $v(s)$ is the set (code) of the variables of $\psi$.

(2) $l(s) = \ulcorner \neg \urcorner$, $s^\frown 0$ is the only successor in $T$ and $v(s) = v(s^\frown 0)$.

(3) $l(s) = \ulcorner \exists y \urcorner$, for some $y \in V$, $s^\frown 0$ is the only successor of $s$ in $T$ and $v(s) = v(s^\frown 0) \setminus \{y\}$.

(4) $l(s) = \ulcorner \wedge \urcorner$ and

$$v(s) = \bigcup_{\substack{i<\omega \\ s^\frown i \in T}} v(s^\frown i)$$

is finite.

A $\mathcal{L}_{\omega_1,\omega}$ code, i.e. a code for a $\mathcal{L}_{\omega_1,\omega}$ formula with finitely many free variables, is a labeled tree $(T, l, v)$, where $T$ is well founded. A sentence code is a $\mathcal{L}_{\omega_1,\omega}$ code with $v(\emptyset) = \emptyset$.

So an atomic formula $\psi$ is coded by $(\{\emptyset\}, l, v)$, where $l(\emptyset) = \ulcorner \psi \urcorner$ and $v(\emptyset)$ codes the variables of $\psi$.

If $F \subset \omega$ and $\{\psi_j : j \in F\}$ is set of $\mathcal{L}_{\omega_1,\omega}$ formulas all of which have free variables among $v_1, \ldots, v_k$ and $\psi_j$ is coded by the labeled tree $(T_j, l_j, v_j)$, for $j \in F$, then $\bigwedge_{j \in F} \psi_j$ is coded by the labeled tree $(T, l, v)$, where

$$T := \{\emptyset\} \cup \{j^\frown s : j \in F, s \in T_j\},$$

$l(\emptyset) := \ulcorner \bigwedge \urcorner$ and for$j \in F$ and $s \neq \emptyset$, $l(s) := l_j(t)$ if and only if $s = j^\frown t$ and $t \in T_j$. The value $v(\emptyset)$ is defined as in (4) of the definition and for $s = j^\frown t$, where $j \in F$ and $t \in T_j$, $v(s) := v_j(t)$.

Analogously, we can define labeled trees when $\psi$ is of the form $\exists y \phi$ or $\neg \phi$. The value $l(\emptyset)$ always tells us if we are dealing with an atomic formula, a negation, an existential formula or a conjunction.

Since $WF$ is $\Pi^1_1$, we have that the set of formula codes and the set of sentence codes is $\Pi^1_1$.

## 3.2 The Space of Countable Models

**Definition.** Let $\omega$ and $2 = \{0, 1\}$ be equipped with the discrete topology and the vocabulary $\tau = C \cup R \cup F$, where $C$ is the set of constant symbols, $R$ the set of relation symbols and $F$ the set of function symbols. For each $r \in R$ and $f \in F$, let $n_r$, $(n_f)$be the arity of $r$, respectively $f$, $M_r := \omega^{n_r} \times \{r\}$ and $M_f := \omega^{n_f} \times \{f\}$. Define the index set

$$J := C \cup \bigcup \{M_r : r \in R\} \cup \bigcup \{M_f : f \in F\}$$

and the topological spaces $(X_j, O_j)$, for $j \in J$ as follows: $O_j$ is the discrete topology on $X_j$, i.e $P(X_j)$, and

$$X_j := \begin{cases} 2, & j \in \omega^{n_r} \times \{r\}, \text{ for some } r \in R \\ \omega, & else \end{cases}$$

Then

$$\mathcal{X}_\tau := \prod_{j \in J} X_j$$

equipped with the product topology is called the space of $\tau$-structures.

Each element $x \in \mathcal{X}_\tau$ codes a $\tau$ structure $\mathfrak{M}_x$ with universe $\omega$ in a canonical way:

- $c^{\mathfrak{M}_x} := x(c)$, for $c \in C$.
- If $r \in R$, then $r^{\mathfrak{M}_x}(a_1, \ldots, a_{n_r}) :\Leftrightarrow x((a_1, \ldots, a_{n_r}, r)) = 1$
- If $f \in F$, then $f^{\mathfrak{M}_x} : \omega^{n_f} \mapsto \omega$ is defined by

$$(a_1, \ldots, a_{n_f}) \to x((a_1, \ldots, a_{n_f}, f)).$$

Conversely, for every $\tau$ structure $\mathfrak{M}$ with universe $\omega$ there is a unique $x \in \mathcal{X}_\tau$ such that $\mathfrak{M}_x = \mathfrak{M}$.

Since $\tau$ is countable, $\mathcal{X}_\tau$ is a Polish space. For a first order quantifier free formula $\psi(\overline{v})$ and $\overline{a} \in \omega^{lg(\overline{v})}$ let $\mathcal{B}_{\psi(\overline{a})} := \{x \in \mathcal{X}_\tau : \mathfrak{M}_x \models \psi(\overline{a})\}$. Then

$$\{\mathcal{B}_{\psi(\overline{a})} : \psi(\overline{v}) \text{ quantifier free and first order, } \overline{a} \in \omega^{lg(\overline{v})}\}$$

is a basis for the topology on $\mathcal{X}_\tau$. The next proposition can be proven via straight forward induction on formula complexity.

**Proposition 3.2.1.** *Let $\psi(v_1, \ldots, v_k)$ be a $\mathcal{L}_{\omega_1,\omega}$-formula and $a_1, \ldots, a_k \in \omega$. Then $\{x \in \mathcal{X}_\tau : \mathfrak{M}_x \models \psi(\overline{a})\}$ is Borel.*

**Definition.** For a sentence $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$, $Mod(\phi)$ is defined as

$$\{x \in \mathcal{X}_\tau : \mathfrak{M}_x \models \phi\}.$$

Hence, $Mod(\phi)$ with the inherited $\sigma$-algebra is a standard Borel space. We identify $Mod(\phi)$ with $\{\mathfrak{M}_x : x \in Mod(\phi)\}$.

**Definition.** Let $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}$ be a fragment, $n < \omega$ and $T \subseteq \mathbb{A}$ a theory. $S_n(\mathbb{A}, T)$ denotes the set of all complete $n$-types $\mathfrak{s}(\overline{v}) \subseteq \mathbb{A}$ such that $T \cup \mathfrak{s}(\overline{v})$ is satisfiable.

Suppose that $\mathbb{A}$ is countable, $\phi \in \mathbb{A}$ is a sentence and $n < \omega$. If we take the index set $J$ of all $n$-formulas of $\mathbb{A}$, then $Y :=^J 2$ equipped with the product topology is homeomorphic to the Cantor space. We identify it with $P(J)$ by identifying each subset of $J$ with its characteristic function.

Now consider the function $g : Mod(\phi) \times \omega^n \mapsto Y$ defined by

$$(x, \overline{a}) \mapsto \mathfrak{tp}^{\mathbb{A}}_{\mathfrak{M}_x}(\overline{a}).$$

With the help of proposition 3.2.1 we see that this function is Borel. Since both the domain and the codomain of $g$ are Polish spaces it follows, that the image $im(g)$ is analytic. Clearly, $im(g) = S_n(\mathbb{A}, \phi)$. Thus, for all $n < \omega$, either $S_n(\mathbb{A}, \phi)$ is countable, or else it contains a perfect set. In particular:

$$|S_n(\mathbb{A}, \phi)| \leq \aleph_0 \text{ or } |S_n(\mathbb{A}, \phi)| = \mathfrak{c}.$$

**Proposition 3.2.2.** *(i) For every $n < \omega$ and every $\alpha < \omega_1$, the equivalence relation $E_{\alpha,n}$ on $\mathcal{X}_\tau \times \omega^n$ defined by*

$$(x, \overline{a})E_{\alpha,n}(y, \overline{b}) :\Leftrightarrow (\mathfrak{M}_x; \overline{a}) \sim_\alpha (\mathfrak{M}_y; \overline{b})$$

*is Borel.*

*(ii) For every $\alpha < \omega_1$, the set of all $x \in \mathcal{X}_\tau$ with $sr(\mathfrak{M}_x) \leq \alpha$ is Borel.*

*Proof.* (i) The proof can be done via induction on $\alpha < \omega_1$. For the case $\alpha = 0$, we use poposition 3.2.1 and the fact that the vocabulary $\tau$ is countable.

(ii) is an easy consequence of (i).

□

The isomorphism relation $\cong$ on $\mathcal{X}_\tau$ is analytic, since it is the projection of the Borel set of all $(x, y, f) \in \mathcal{X}_\tau^2 \times \mathcal{N}$ such that $f$ is an isomorphism from $\mathfrak{M}_x$ onto $\mathfrak{M}_y$ but it is not necessarily Borel. In fact, we have

**Lemma 3.2.3.** *Let $\phi \in \mathcal{L}_{\omega_1,\omega}$ be a sentence. Then $\cong$ is Borel on $Mod(\phi)$ iff there is a countable bound on the set of Scott ranks of elements of $Mod(\phi)$.*

*Proof.* ($\Rightarrow$):We define a set $W$ of 4-tuples $(\mathfrak{M}, \mathfrak{N}, R, z) \subseteq Mod(\phi)^2 \times \mathcal{N} \times \mathcal{N}$, where $R$ codes a linear order on $\omega$ and $z$ a subset of $\omega \times (\cup_{n<\omega}(\omega^n \times \omega^n))$.The linear order $R$ has the following properties:

- 0 is $R$-minimal.
- Every element $a \in \omega$ that is not $R$-maximal has a successor, denoted by $s_R(a)$.

For all $n < \omega$, $z$ satisfies:

(1) For all $\overline{a}, \overline{b} \in \omega^n$, $(0, \overline{a}, \overline{b}) \in z$ iff $(\mathfrak{M}; \overline{a}) \sim_0 (\mathfrak{N}; \overline{b})$.

(2) If $(m, \overline{a}, \overline{b}) \in z$ and $xRm$, then $(x, \overline{a}, \overline{b}) \in z$.

(3) For all $m \in \omega$ and $\overline{a}, \overline{b} \in \omega^n$, $(s_R(m), \overline{a}, \overline{b}) \in z$ iff for all $c \in \omega$ there is $d \in \omega$ such that $(m, \overline{a}^\frown c, \overline{b}^\frown d) \in z$ and for all $d \in \omega$ there is $c \in \omega$ such that $(m, \overline{a}^\frown c, \overline{b}^\frown d) \in z$.

(4) If $m \in \omega$ is a $R$-limit point $\neq 0$ and $\overline{a}, \overline{b} \in \omega^n$, then $(m, \overline{a}, \overline{b}) \in z$ iff for all $xRm$, $(x, \overline{a}, \overline{b}) \in z$.

One can show that $W$ is a Borel subset and for all $\alpha < \omega_1$ and $\mathfrak{M}, \mathfrak{N} \in Mod(\phi)$, there is $(\mathfrak{M}, \mathfrak{N}, R, z) \in W$, where $R$ is a wellorder of order type $\alpha$, and

$$(\mathfrak{M}; \overline{a}) \sim_{\beta_x} (\mathfrak{N}; \overline{b}) \Leftrightarrow (x, \overline{a}, \overline{b}) \in z,$$

where $\beta_x := ot(\{y \in \omega : yRx\})$.

If $\cong$ is Borel on $Mod(\phi)$, then the set $A$, defined as

$$\{R \in \mathcal{N} : (\mathfrak{M}, \mathfrak{N}, R, z) \in W, \text{ for some } z, (x, \emptyset, \emptyset) \in z, \text{ for all } x \in \omega, \text{ and } \mathfrak{M} \ncong \mathfrak{N}\},$$

is analytic.

**Claim:** *If $(\mathfrak{M}, \mathfrak{N}, R, z) \in W$, $(x_k : k < \omega)$ is a sequence in $\omega$ such that for all $k$, $x_{k+1} R x_k$ and for some $k, n < \omega$, $\overline{a}, \overline{b} \in \omega^n$, we have $(x_k, \overline{a}, \overline{b}) \in z$, then there is an isomorphism from $\mathfrak{M}$ onto $\mathfrak{N}$ mapping $\overline{a}$ onto $\overline{b}$.*

This is easily proven via a back and forth argument using (1)-(3).

As a consequence of the claim we get that each $R \in A$ is a wellorder. By $\Sigma^1_1$-bounding there is $\alpha < \omega_1$ such that $ot(R) < \alpha$, for all $R \in A$. But this means that for all $\mathfrak{M}, \mathfrak{N} \in Mod(\phi)$, $\mathfrak{M} \equiv_\alpha \mathfrak{N}$ is equivalent to $\mathfrak{M} \cong \mathfrak{N}$ or in other words, $\chi^{\mathfrak{M}}_{\emptyset,\alpha}$ is $\omega$-categorical, for all $\mathfrak{M} \in Mod(\phi)$.

Now we apply Lemma 2.5.4 and conclude that the Scott ranks of $Mod(\phi)$ are bounded by $\alpha + \omega$.

($\Leftarrow$): Let $\gamma < \omega_1$ such that $sr(\mathfrak{M}) < \gamma$, for all $\mathfrak{M} \in Mod(\phi)$.

If we define $\Psi_{\mathfrak{M}}$ as the Scott sentence of $\mathfrak{M}$, then by the definition of the Scott-sentence it follows that $qr(\Psi_{\mathfrak{M}}) \leq \gamma + \omega$, for all $\mathfrak{M} \in Mod(\phi)$. Therefore, if $\mathfrak{M}, \mathfrak{N} \in Mod(\phi)$ and $\mathfrak{M} \sim_{\gamma+\omega} \mathfrak{N}$, then $\mathfrak{M} \cong \mathfrak{N}$, hence the $\cong$-relation is equal to the $\sim_{\gamma+\omega}$ relation which is Borel. □

The following results are presented without proof.

**Theorem 3.2.4.** *(i) (Silver [19]) If $E$ is a $\Pi^1_1$-equivalence relation on a standard Borel space $X$ with uncountably many equivalence classes, then there is a perfect set of $E$-inequivalent elements $\subseteq X$.*

*(ii) (Burgess [3]) If $E$ is a $\Sigma^1_1$-equivalence relation on a standard Borel space $X$ with $\geq \aleph_2$ equivalence classes, then there is a perfect set of $E$-inequivalent elements $\subseteq X$.*

**Corollary 3.2.5.** *(Morley) For every sentence $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$, either there are at most $\aleph_1$ many countable isomorphism types of $\phi$ or else there is a perfect set of pairwise non isomorphic countable models $\subseteq Mod(\phi)$.*

*Proof.* Since $\cong$ is $\Sigma^1_1$ and $Mod(\phi)$ is a standard Borel space, the statement immediately follows from $(ii)$, but it can also be proven using $(i)$:

Suppose $I(\phi, \aleph_0) \geq \aleph_2$. Then for some $\alpha < \omega_1$,

$$\mathcal{A}_\alpha := \{x \in Mod(\phi) : sr(\mathfrak{M}_x) = \alpha\}$$

contains $\aleph_2$ many pairwise non isomorphic models. As we have seen before, $\mathcal{A}_\alpha$ is a standard Borel space and by the definition of the Scott sentence and Lemma 2.2.3 it follows that the isomorphism relation restricted to $\mathcal{A}_\alpha$ is identical to the $\equiv_{\alpha+\omega}$-relation which is Borel and therefore $\Pi^1_1$. Now $(i)$ implies the existence of a perfect set of pairwise non isomorphic models of $\phi$. □

**Definition.** Let $T \subseteq \mathcal{L}_{\omega_1,\omega}(\tau)$ be a theory. We say $T$ has perfectly many models if there is a perfect set of pairwise non isomorphic models in $Mod(T)$.

Currently, no example of a $\mathcal{L}_{\omega_1,\omega}$-sentence $\phi$ with $I(\phi, \aleph_0) = \aleph_1$ is known. Since every known example for $I(\phi, \aleph_0) = \mathfrak{c}$ has perfectly many models, we introduce a new version of $VC$.

$VC_3$: For every $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence $\phi$ of a countable vocabulary $\tau$, either $I(\phi, \aleph_0) \leq \aleph_0$ or else $\phi$ has perfectly many models.

Clearly, $VC_3$ is the strongest version introduced so far. Also note that it is independent of the value of $\mathfrak{c}$. From now on our focus will be on this problem.

We have considered Vaught's conjecture for infinitary sentences. One might think about a generalisation to theories of infinitary sentences but this question can be answered easily:

First, note that a satisfiable theory $T \subseteq \mathcal{L}_{\omega_1,\omega}(\tau)$ which is complete in $\mathcal{L}_{\omega_1,\omega}(\tau)$ is $\omega$ categorical: If $\mathfrak{M}$ is a countable model of $T$, then $T \models \Psi_{\mathfrak{M}}$, the Scott sentence of $\mathfrak{M}$. It follows that $VC_3$ is trivially true for complete theories of $\mathcal{L}_{\omega_1,\omega}$.

If we consider arbirtary theories of $\mathcal{L}_{\omega_1,\omega}$, then we have a counterexample.

**Example 3.2.6.** (also see page 85 of [2]) Let $\tau := \{<\}$, where $<$ is binary. Via induction on $\alpha < \omega_1$ we can define a sentence $\sigma_\alpha \in \mathcal{L}_{\omega_1,\omega}(\tau)$ which characterizes the structure $(\alpha, <)$ up to isomorphism. Let $\sigma_0 := \forall x(x \neq x)$.

Suppose we have defined $\sigma_\alpha$. Then let $\sigma_{\alpha+1}$ state that $<$ is a linear order, there is a largest element $x$ and the set of all $y < x$ satisfies $\sigma_\alpha$.

At limit stages $\alpha > 0$, $\sigma_\alpha$ states that $<$ is a linear order, for all $\beta < \alpha$, there is $x$ such that the set of all $y < x$ satisfies $\sigma_\beta$, and for all $x$, the set of all $y < x$ satisfies $\sigma_\beta$, for some $\beta < \alpha$. This can be expressed with a countable conjunction and a countable disjunction, hence $\sigma_\alpha \in \mathcal{L}_{\omega_1,\omega}(\tau)$.

Now, for $\alpha < \omega_1$, the sentence $\phi_\alpha \in \mathcal{L}_{\omega_1,\omega}(\tau)$ states that $<$ is a linear order and either $\sigma_\beta$, for some $\beta \leq \alpha$, holds, or there is $x$ such that the set of all $y < x$ satisfies $\sigma_\alpha$.

Consider the theory $T := \{\phi_\alpha : \alpha < \omega_1\}$. It is easy to show that a countable $\tau$-structure $\mathfrak{M}$ satisfies $T$ if and only if it is a well order. Thus, $T$ has uncountably many isomorphism types.

Clearly, $T$ cannot have perfectly many models, because that would give us an analytic set of well orders whose order types are unbounded below $\aleph_1$, a contradiction to $\Sigma^1_1$ bounding of well founded relations.

## 3.3 Scattered Sentences

In the previous section we saw that for every countable fragment $\mathbb{A}$ and every $n < \omega$, the set of complete $n$-types $\subseteq \mathbb{A}$ can be seen as an analytic subset of the Cantor space. Using Silver's theorem, one can easily show that if for some $\mathcal{L}_{\omega_1,\omega}$-sentence $\phi$, some $\mathbb{A}$ and some $n < \omega$, there is a perfect set of complete and satisfiable $n$-types $\subseteq \mathbb{A}$ containing $\phi$, then there is a perfect set of pairwise non isomorphic countable models $\subseteq Mod(\phi)$. In that case $\phi$ trivially satisfies $VC_3$. Therefore, we are interested in sentences which do not have this property.

**Definition.** A $\mathcal{L}_{\omega_1,\omega}$-sentence $\phi$ is called scattered, if for every countable fragment $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}$ and every $n < \omega$, $|S_n(\mathbb{A}, \phi)| \leq \aleph_0$.

Clearly, if $I(\phi, \aleph_0) \leq \aleph_0$, then $\phi$ must be scattered.

For a scattered sentence $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$, let $(\mathbb{A}_{\phi,\alpha} : \alpha < \omega_1)$ be a sequence of fragments defined as follows:

- $\mathbb{A}_{\phi,0}$ is the smallest fragment containing $\phi$.
- $\mathbb{A}_{\phi,\alpha+1}$ is the smallest fragment containing all formulas of the form

$$\bigwedge_{\psi \in \mathfrak{t}} \psi(\overline{v}),$$

  where $\mathfrak{t}(v_1, \ldots, v_n) \subseteq \mathbb{A}_\alpha$ is a complete and satisfiable $n$-type such that $\phi \in \mathfrak{t}$, and $n < \omega$.

- For every limit ordinal $\lambda \in (0, \omega_1)$,

$$\mathbb{A}_{\phi,\lambda} := \bigcup_{\alpha<\lambda} \mathbb{A}_{\phi,\alpha}.$$

**Proposition 3.3.1.** *Let $\phi$ be scattered.*

*(i) For all $\alpha < \omega_1$, $\mathbb{A}_{\phi,\alpha}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$ in which all formulas have only finitely many free variables and if $\alpha \leq \beta < \omega_1$, then $\mathbb{A}_{\phi,\alpha} \subseteq \mathbb{A}_{\phi,\beta}$.*

*(ii) If $\mathfrak{M}, \mathfrak{N} \in Mod(\phi)$, $n < \omega$ and $\overline{a}, \overline{b} \in \omega^n$, then for all $\alpha < \omega_1$,*

$$\mathfrak{tp}_{\mathfrak{M}}^{\mathbb{A}_{\phi,\alpha}}(\overline{a}) = \mathfrak{tp}_{\mathfrak{N}}^{\mathbb{A}_{\phi,\alpha}}(\overline{b}) \textit{ implies } (\mathfrak{M}; \overline{a}) \sim_\alpha (\mathfrak{N}; \overline{b}).$$

*Proof.* $(i)$ can easily be shown via induction on $\alpha < \omega_1$.

We check $(ii)$ inductively:

- The case for $\alpha = 0$ is clear,since every atomic formula is in $\mathbb{A}_{\phi,0}$.
- $\alpha \to \alpha + 1$: Let $n < \omega$, $(\mathfrak{M}, \overline{a}), (\mathfrak{N}, \overline{b}) \in Mod(\phi) \times \omega^n$, such that

$$\mathfrak{tp}_{\mathfrak{M}}^{\mathbb{A}_{\phi,\alpha+1}}(\overline{a}) = \mathfrak{tp}_{\mathfrak{N}}^{\mathbb{A}_{\phi,\alpha+1}}(\overline{b}),$$

  and $c \in \omega$. Then

$$\gamma(v_1, \ldots, v_{n+1}) := \bigwedge_{\psi \in \mathfrak{t}} \psi(v_1, \ldots, v_{n+1})$$

  is in $\mathbb{A}_{\phi,\alpha+1}$, where $\mathfrak{t} := \mathfrak{tp}_{\mathfrak{M}}^{\mathbb{A}_{\phi,\alpha}}(\overline{a}^\frown c)$.

  By assumption, it follows that $\mathfrak{N} \models \exists v_{n+1} \gamma(\overline{b})$, which by induction hypothesis implies $(\mathfrak{M}; \overline{a}^\frown c) \sim_\alpha (\mathfrak{N}; \overline{b}^\frown d)$, for some $b \in \omega$. For symmetry reasons this means $(\mathfrak{M}; \overline{a}) \sim_{\alpha+1} (\mathfrak{N}; \overline{b})$.

- The case for limit ordinals $\in (0, \omega_1)$ follows, since $(\mathbb{A}_{\phi,\alpha} : \alpha < \omega_1)$ is a chain of fragments.

□

**Remark.** Regarding $(ii)$: In general, $(\mathfrak{M}; \bar{a}) \sim_\alpha (\mathfrak{N}; \bar{b})$ does not imply

$$\mathfrak{tp}_{\mathfrak{M}}^{\mathbb{A}_{\phi,\alpha}}(\bar{a}) = \mathfrak{tp}_{\mathfrak{N}}^{\mathbb{A}_{\phi,\alpha}}(\bar{b}).$$

E.g.let $\mathfrak{M} := (\mathbb{Z}, <)$ $a_1 := 1$, $a_2 := 2$, $b_1 := 1$, $b_2 := 3$ and $\phi$ the Scott sentence of $\mathfrak{M}$.

We have $sr(\mathfrak{M}) = \omega$, $(\mathfrak{M}; \bar{a}) \sim_0 (\mathfrak{M}; \bar{b})$ but $\bar{a}$ and $\bar{b}$ do not satisfy the same first order formulas in $\mathfrak{M}$, and these are contained in $\mathbb{A}_{\phi,0}$.

We can now give a first characterisation for a sentence to be scattered.

**Lemma 3.3.2.** *Let $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$ be a sentence. The following are equivalent:*

*(i) $\phi$ is scattered.*

*(ii) For every $\alpha < \omega_1$, there are only countably many $\equiv_\alpha$-equivalence classes on $Mod(\phi)$.*

*(iii) $\phi$ does not have perfectly many models.*

*Proof.* $(i) \Rightarrow (ii)$: If $\phi$ is scattered and for some $\alpha < \omega_1$ there are uncountably many $\equiv_\alpha$-equivalence classes on $Mod(\phi)$, then by the previous proposition the countable fragment $\mathbb{A}_{\phi,\alpha}$ has uncountably many satisfiable complete types containing $\phi$, a contradiction.

$(ii) \Rightarrow (i)$: If $\phi$ is not scattered, then for some countable fragment $\mathbb{A}$ of $\mathcal{L}_{\omega_1,\omega}(\tau)$ and some $n < \omega$, uncountably many complete $n$-types $\subseteq \mathbb{A}$ are realized in $Mod(\phi)$.

Since $\mathbb{A}$ is countable, there is $\gamma < \omega_1$ such that $qr(\psi) < \gamma$, for all $\psi \in \mathbb{A}$. Let $\alpha := \gamma + \omega$.

For every $\mathfrak{M} \in Mod(\phi)$ there is a $\mathcal{L}_{\omega_1,\omega}(\tau)$ sentence $\sigma_{\mathfrak{M}}$ of quantifier rank $\alpha + n$ describing the set of $n$-types of $\mathbb{A}$ which are realized in $\mathfrak{M}$. Clearly, for $\mathfrak{M}, \mathfrak{N} \in Mod(\phi)$, we have

$$\mathfrak{M} \text{ realizes the same } n\text{-types in } \mathbb{A} \text{ as } \mathfrak{N} \text{ iff } \mathfrak{M} \models \sigma_{\mathfrak{N}} \text{ iff } \sigma_{\mathfrak{M}} = \sigma_{\mathfrak{N}}.$$

Since there are continuum many complete $n$-types in $\mathbb{A}$, it follows that there is an uncountable set $S \subseteq Mod(\phi)$, such that if $\mathfrak{M}, \mathfrak{N} \in S$ and $\mathfrak{M} \neq \mathfrak{N}$ then $\sigma_{\mathfrak{M}} \neq \sigma_{\mathfrak{N}}$, hence there are uncountably many $\equiv_{\alpha+n}$ classes in $Mod(\phi)$.

(iii) $\Rightarrow$ (i): If $\phi$ is not scattered, then for some $\alpha < \omega_1$, there are uncountably many $\equiv_\alpha$ classes on $Mod(\phi)$, and since $\equiv_\alpha$ is a Borel equivalence relation, it follows from Silver's theorem, that $\phi$ has perfectly many models.

(i) $\Rightarrow$ (iii): Suppose $S \subseteq Mod(\phi)$ is a perfect set of pairwise nonisomorphic models. We proceed similarly to the proof of Lemma 3.2.3.

Let $W \subseteq Mod(\phi)^2 \times \mathcal{N} \times \mathcal{N}$ be the set of all tuples $(\mathfrak{M}, \mathfrak{N}, R, z)$, where $z$ and the linear order $R$ are as in the specified proof.

Consider the set of all $R$ such that for some $\mathfrak{M}, \mathfrak{N} \in S$ and some $z \in \mathcal{N}$, $(\mathfrak{M}, \mathfrak{N}, R, z)$ is in $W$, where $\mathfrak{M} \neq \mathfrak{N}$ and $(x, \emptyset, \emptyset) \in z$, for all $x \in \omega$. As in 3.2.3 we argue that it is an analytic set of well orders and hence, by $\Sigma^1_1$ bounding is bounded by some $\alpha < \omega_1$. It follows that there are uncountably many $\equiv_{\alpha+1}$ classes in $Mod(\phi)$ and thus, by (ii) $\phi$ is not scattered. □

**Remark.** With a slight modification of the proof one can show that $\phi$ is scattered if and only if for all $n < \omega$ and all $\alpha < \omega_1$, there are at most $\aleph_0$ many $\sim_\alpha$ classes in $Mod(\phi) \times \omega^n$.

We also have the necessary knowledge to give a second proof of Morley's theorem: If $\phi$ is scattered, then for every $\alpha < \omega_1$ there are $\leq \aleph_0$ many isomorphism classes of models of $\phi$ with Scott rank $\leq \alpha$, since we have already seen that for these models the isomorphism relation is identical to the $\sim_{\alpha+\omega}$-relation. Therefore, $I(\phi, \aleph_0) \leq \aleph_1 \cdot \aleph_0 = \aleph_1$.

**Definition.** Let $T \cup \{\phi(\overline{v})\}$ be a set of first order formulas of an arbitrary vocabulary, where $T$ is a theory.

- $\phi$ is called upwards absolute for models of $T$ if for all models $\mathfrak{M}, \mathfrak{N}$ of $T$ such that $\mathfrak{M} \subset \mathfrak{N}$ we have

$$\mathfrak{M} \models \phi(\overline{a}) \text{ implies } \mathfrak{N} \models \phi(\overline{a}),$$

  for all $\overline{a}$ in $\mathfrak{M}$.

- $\phi$ is called downwards absolute if for all models $\mathfrak{M}, \mathfrak{N}$ of $T$ such that $\mathfrak{M} \subset \mathfrak{N}$ and all $\overline{a}$ in $M$, we have

$$\mathfrak{N} \models \phi(\overline{a}) \text{ implies } \mathfrak{M} \models \phi(\overline{a}).$$

- $\phi$ is called absolute for models of $T$ if it is both downwards and upwards absolute.

**Definition.** In the language of set theory the set of $\Delta_0$ formulas is the smallest set $S$ of first order formulas such that

(1) $S$ contains all atomic formulas.

(2) $S$ is closed under negation and conjunction.

(3) If $\psi \in S$ and $x, y$ are variables, then $\forall x(x \in y \to \psi)$ and $\exists x(x \in y \wedge \psi)$ are in $S$.

For $n < \omega$, we can now recursively define a hierachy of first order formulas:

- A formula $\phi$ is $\Sigma_0$ or $\Pi_0$ if it is $\Delta_0$.
- $\phi$ is $\Sigma_{n+1}$ if $\phi$ is of the form $\exists \overline{v} \psi(\overline{v})$, where $\psi(\overline{v})$ is $\Pi_n$.
- $\phi$ is $\Pi_{n+1}$ if it is of the form $\forall \overline{v} \psi(\overline{v})$, where $\psi(\overline{v})$ is $\Sigma_n$.

A formula $\phi$ is called $\Delta_n$ if modulo $ZFC$ it is equivalent to both a $\Sigma_n$ and a $\Pi_n$ formula.

One important result which will not be proved here is

**Theorem 3.3.3.** *(Shoenfield's Absoluteness Theorem) If $M \subseteq N$ are transitive models of $ZFC$, $(\omega_1)^N \subseteq M$ and $P \in M$ is a Polish space of the form $\mathcal{N}^y \times \omega^l$, where $y \in \omega + 1$ and $l \in \omega$, then every $\Sigma^1_2$ relation on $P$ is absolute between $M$ and $N$.*

This implies that if $M \subseteq N$ are transitive models of $ZFC$, $(\omega_1)^N \subseteq M$, $\phi(\overline{v})$ is a $\Sigma_2$ formula and $\overline{a} \in P^{<\omega}$, where $P \in M$ is Polish as in the theorem, such that $\phi(\overline{a})$ defines a $\Sigma^1_2$ relation on $P$ and all unbounded quantifiers range over elements of $P$ only, then

$$M \models \phi(\overline{a}) \Leftrightarrow N \models \phi(\overline{a}).$$

An important consequence of this result is that the property of being a scattered sentence of $\mathcal{L}_{\omega_1,\omega}$ is absolute for the transitive models of $ZFC$ which are considered in this thesis, as it can be expressed by a $\Sigma_2$ formula $\phi(v)$ stating

"$v$ is a code for a $\mathcal{L}_{\omega_1,\omega}$ sentence and there is a code $T$ for a binary tree such that every distinct branches $f_1, f_2$ of $T$ are codes for nonisomorphic elements of $Mod(v)$."

**Lemma 3.3.4.** *If $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$ is not scattered, then there is a perfect set of pairwise nonisomorphic models of $\phi$ all of which have the same Scott rank.*

*Proof.* It suffices to show that the lemma holds in countable transitive models of $ZFC$.

Let $M$ be such a model and $\phi \in M$ be a non scattered sentence of $\mathcal{L}_{\omega_1,\omega}(\tau)$.

Choose a forcing notion $P \in M$ which preserves cardinals such that for a $P$ generic $G \subseteq P$

$$M[G] \models \mathfrak{c} > \aleph_1,$$

where $M[G]$ is the generic extension of $M$.

By Shoenfield's absoluteness theorem, $\phi$ is not scattered in $M[G]$, hence there is a perfect set $S$ of pairwise nonisomorphic models in $Mod(\phi)^{M[G]}$.

Recall that for every $\alpha < \omega_1$ the set of all $A \in Mod(\phi)$ with $sr(A) = \alpha$ is Borel.

In $M[G]$ we have $|S| > \aleph_1$ thus by the perfect set theorem for some $\alpha < \omega_1$ there is a perfect set of pairwise nonisomorphic models with Scott rank $\alpha$. This can be expressed with a $\Sigma_2$ formula and some real $f \in \mathcal{N}^M$ which codes a well order on $\omega$ of order type $\alpha$. Here we are using the fact that the $\sim_\alpha$ relation is Borel.

Since $\omega_1^M = \omega_1^{M[G]}$ and by absoluteness, $\phi$ has perfectly many models of Scott rank $\alpha$ in $M$. ☐

It is for absoluteness reasons that we prefer $VC_3$. However, we have

**Proposition 3.3.5.** *If $ZFC \vdash VC_2$ then $ZFC \vdash VC_3$.*

*Proof.* We show that if $ZFC$ proves $VC_2$, then $VC_3$ holds in all countable transitive models of $ZFC$.

Suppose this is not so and let $M$ be a countable transitive model of $ZFC$ such that for some $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau) \cap M$

$$M \models I(\phi, \aleph_0) > \aleph_0,$$

but $\phi$ does not have perfectly many models in $M$. It follows that $\phi$ is scattered and by Morley's theorem

$$M \models I(\phi, \aleph_0) = \aleph_1.$$

Let $P \in M$ be a forcing notion which preserves cardinals such that for a $P$ generic filter $G \subseteq P$

$$M[G] \models \mathfrak{c} > \aleph_1.$$

By absoluteness, $\phi$ is scattered in $M[G]$ and has uncountably many isomorphism types.

Since $M[G] \models ZFC$, it follows that $VC_2$ holds in the generic extension and therefore

$$M[G] \models I(\phi, \aleph_0) = \mathfrak{c}.$$

But then by Morley's theorem, $\phi$ has perfectly many models and by absoluteness $\phi$ is not scattered in $M$, a contradiction. □

# 4 Uncountable Models of Vaught Counter Examples

Unless stated otherwise, the proofs presented in this section are based on [12].

Since $VC_3$ is still an open problem, we can ask what consequences can be derived if it fails.

**Definition.** A counterexample to $VC_3$, or simply a VCE, is a scattered sentence $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$ with $I(\phi, \aleph_0) = \aleph_1$.

Let us first check that every VCE has an uncountable model.

## 4.1 Minimal Vaught Counter Examples

**Definition.** Let $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$ be a VCE.

(1) We call $\phi$ a minimal VCE if for all sentences $\sigma \in \mathcal{L}_{\omega_1,\omega}(\tau)$ either

$$\phi \wedge \sigma, \text{ or } \phi \wedge \neg\sigma$$

has uncountably many countable models but not both.

(2) Let $n < \omega$ and $\psi(v_1, \ldots, v_n) \in \mathcal{L}_{\omega_1,\omega}(\tau)$. We say $\psi(\overline{v})$ is $\phi$-large if

$$\phi \wedge \exists\overline{v}\psi(\overline{v})$$

has uncountably many countable models. If for all $n$-formulas $\chi(\overline{v})$ of $\mathcal{L}_{\omega_1,\omega}(\tau)$, either $\psi \wedge \chi$ or $\psi \wedge \neg\chi$ is $\phi$-large but not both, then $\psi(\overline{v})$ is called minimal $\phi$-large.

**Lemma 4.1.1.** *Suppose $\phi_1 \in \mathcal{L}_{\omega_1,\omega}$ is a VCE. Then there is a minimal VCE $\phi_0$ such that $\phi_0 \models \phi_1$.*

*Proof.* Assume $\phi_1$ is not implied by a minimal VCE. We construct a countable fragment $\mathbb{A}$ with uncountably many types containing $\phi_1$.

Let $C$ be a countable set of new constants, $\tau' := \tau \cup C$ and define $\Sigma$ as the set of all $\sigma$ with the following properties:

(i) $\sigma$ is a finite set of $\tau'$-sentences in which only finitely many new constants occur.

(ii) $\sigma \cup \{\phi_1\}$ has uncontably many countable models.

It is not difficult to check that $\Sigma$ is a consistency property, the case $(C_4)$ uses the regularity of $\omega_1$.

Next, we choose an enumeration of $C$, a well order of $\mathcal{L}_{\omega_1,\omega}$ and construct a binary tree $T := \{\sigma_s : s \in 2^{<\omega}\}$ and a sequence $(\mathbb{A}_n : n < \omega)$ of countable fragments. Each $\sigma_s$ will be a finite set of $\tau'$-sentences containing $\phi_1$.

Let $\mathcal{V}$ be a countable set of variables containing all variables of $\phi_1$, choose an enumeration $(t_n : n < \omega)$ of all $\tau'$-terms with variables in $\mathcal{V}$ and a function $h : \omega \mapsto \omega \times \omega$ , such that $h^{-1}(m, n)$ is infinite, for every $(m, n) \in \omega \times \omega$.

For every countable fragment $\mathbb{A} \in \mathcal{L}_{\omega_1,\omega}(\tau')$ with variables in $\mathcal{V}$, let $f_{\mathbb{A}}$ be a function from $\omega$ into the set of all sentences of $\mathbb{A}$, such that every sentence $\psi \in \mathbb{A}$ has an infinite preimage. $T$ is defined recursively on every level:

- $\sigma_\emptyset :=< \phi >$ and let $\mathbb{A}_0$ be the smallest fragment containing $\phi$.
- Suppose $n < \omega$, $\sigma_s$ is defined for $s \in^n 2$ and $h(n) = (k_1, k_2)$. Let

$$\chi_n := \begin{cases} f_{\mathbb{A}_{k_1}}(k_2), & k_1 \leq n \\ f_{\mathbb{A}_n}(k_2), & else \end{cases}$$

Now extend $\sigma_s$ to $\sigma'_s$ as follows: If $\sigma_s + \chi_n \in \Sigma$, then $\chi_n \in \sigma'_s$, otherwise $\neg\chi_n \in \sigma'_s$. Furthermore, if $\sigma_s + \chi_n \in \Sigma$ and

.) $\chi_n = \neg\psi$, then $\sim \psi \in \sigma'_s$.

.) $\chi_n = \bigwedge_{\psi \in F} \psi$, then the smallest $\psi \in F \setminus \sigma_s$ is in $\sigma'_s$.

.) $\chi_n = \bigvee_{\psi \in F}$, then $\psi \in \sigma'_s$, where $\psi \in F$ is minimal such that $\sigma_s + \psi \in \Sigma$.

.) $\chi_n = \forall x \psi(v)$, then $\psi(c) \in \sigma'_s$, where $c$ is the minimal constant of $\tau'$ such that $\psi(c) \notin \sigma_s$.

.) $\chi_n = \exists v \psi(v)$, then $\psi(c) \in \sigma'_s$, where $c$ is the minimal new constant not occurring in $\sigma_s$.

Then add $t_n = c$ to $\sigma'_s$, where $c$ is the minimal new constant not occurring in $\sigma'_s$. By this construction, $\sigma'_s \in \Sigma$, and if we define

$$\gamma(c_1, \ldots, c_k) := \bigwedge_{\psi \in \sigma'_s} \psi,$$

where $c_1, \ldots, c_k$ are all new constants occurring in $\sigma'_s$, then $\overline{\gamma} := \exists \overline{v} \gamma(\overline{v})$ is a $\phi_1$-large sentence $\in \mathcal{L}_{\omega_1,\omega}(\tau)$ and since $\overline{\gamma} \models \phi_1$, $\overline{\gamma}$ is a VCE, which by our assumption is not minimal. Hence there is a $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence $\rho$ with variables in $\mathcal{V}$, such that $\overline{\gamma} \wedge \rho$ and $\overline{\gamma} \wedge \neg\rho$ are $\phi_1$-large. Then define

$$\sigma_{s^\frown 0} := \sigma'_s \cup \{\overline{\gamma} \wedge \rho\}, \text{ and } \sigma_{s^\frown 1} := \sigma'_s \cup \{\overline{\gamma} \wedge \neg\rho\}.$$

- Let $\mathbb{A}_{n+1}$ be the smallest fragment of $\mathcal{L}_{\omega_1,\omega}(\tau')$ containing

$$\{\sigma_s : s \in^{n+1} 2\}.$$

This completes the definition of $T$. Let $f \in^\omega 2$. Then by our construction, the set

$$P(\bigcup_{n<\omega} \sigma_{f \restriction n})$$

is a consistency property, hence $B_f := \bigcup_{n<\omega} \sigma_{f\restriction n}$ is satisfiable.

Now consider the countable fragment

$$\mathbb{A} := \bigcup_{n<\omega} \mathbb{A}_n \cap \mathcal{L}_{\omega_1,\omega}(\tau).$$

If $f, g \in^\omega 2$ are distinct, then $B_f \cap \mathbb{A}, B_g \cap \mathbb{A}$ are distinct, complete and satisfiable types containing $\phi_1$, a contradiction to $\phi_1$ being scattered. $\square$

**Corollary 4.1.2.** *If $\phi$ is a VCE and $\psi(v_1, \ldots, v_k)$ is a $\phi$-large $k$-formula, then there is a minimal $\phi$-large formula $\psi_2(\overline{v})$ such that $\psi_2 \models \psi$.*

*Proof.* Add $k$ new constants $c_1, \ldots, c_k$, thereby getting a new vocabulary $\tau'$. Then $\phi \wedge \psi(c_1, \ldots, c_k)$ is a VCE in $\mathcal{L}_{\omega_1,\omega}(\tau')$. By the previous Lemma, there is a minimal VCE $\gamma(\overline{c})$ sucht that $\gamma(\overline{c}) \models \phi \wedge \psi(\overline{c})$. Clearly, $\psi_2 := \gamma(v_1, \ldots, v_k)$ is minimal $\phi$-large. $\square$

Now we have the necessary knowledge to show that every VCE has an uncountable model with many types.

**Theorem 4.1.3.** *If $\phi \in \mathcal{L}_{\omega_1,\omega}$ is a VCE, then there is a model $\mathfrak{N}$ of $\phi$ which has cardinality $\aleph_1$ and is not $\equiv_{\infty,\omega}$ equivalent to any countable model.*

*Proof.* By Lemma 4.1.1, we can assume that $\phi$ is a minimal VCE. Our goal is to define a chain of special countable fragments $(\mathbb{A}_\alpha : \alpha < \omega_1)$ and a chain of countable models of $\phi$ $(\mathfrak{M}_\alpha : \alpha < \omega_1)$ such that for $\alpha < \beta$, $\mathfrak{M}_\alpha$ is a proper $\mathbb{A}_\alpha$-elementary submodel of $\mathfrak{M}_\beta$. The limit of this model chain will be the desired model.

Let $\mathbb{A}_0$ be a countable fragment containig $\phi$ such that for all $n < \omega$ and all $\phi$-large formulas $\psi(v_1, \ldots, v_n)$, there is a minimal $\phi$-large formula $\psi'(\overline{v}) \in \mathbb{A}_0$ implying $\psi$, and let $T_0$ be the set of all $\phi$-large sentences of $\mathbb{A}_0$. Because $\phi$ is a minimal VCE and $\aleph_1$ is regular, $T_0$ is a satisfiable $\mathbb{A}_0$-complete theory.

If $\psi(\overline{v}) \in \mathbb{A}_0$ is satisfiable with $T_0$, then for some minimal $\phi$-large $\psi'(\overline{v}) \in \mathbb{A}$, we have $\forall \overline{v}(\psi'(\overline{v}) \rightarrow \psi(\overline{v})) \in T_0$, hence $\psi'(\overline{v}) \models_{T_0} \psi(\overline{v})$.

If $\psi'(\overline{v}) \in \mathbb{A}_0$ is a minimal $\phi$-large $n$-formula and $\psi(\overline{v}) \in \mathbb{A}$ is an arbitrary $n$-formula such that $\psi' \wedge \psi$ is $\phi$-large, then $\neg\exists\overline{v}(\psi' \wedge \neg\psi) \in T_0$, thus $\psi' \models_{T_0} \psi$. This means that for all $n < \omega$, the minimal $\phi$-large $n$-formulas are exactly the $n$-complete formulas over $T_0$ and every $n$-formula in $\mathbb{A}_0$ which satisfiable with $T_0$ is implied by a $n$-complete formulas. By Lemma 2.5.1, $T_0$ has a countable $\mathbb{A}_0$-prime model $\mathfrak{M}_0$.

Given $\mathbb{A}_\alpha$ countable, $T_\alpha \subseteq \mathbb{A}_\alpha$ as the set of all $\phi$-large sentences and $\mathfrak{M}_\alpha$ $\mathbb{A}_\alpha$-prime, we let $\mathbb{A}_{\alpha+1}$ be a countable fragment such that

- $\mathbb{A}_\alpha \cup \{\Psi_{\mathfrak{M}_\alpha}\} \subseteq \mathbb{A}_{\alpha+1}$, where $\Psi_{\mathfrak{M}_\alpha}$ is the Scott sentence of $\mathfrak{M}_\alpha$.
- For all $n < \omega$, every $\phi$-large $n$-formula is implied by a minimal $\phi$-large $n$-formula in $\mathbb{A}_{\alpha+1}$.

Then let $T_{\alpha+1}$ be the set of all $\phi$-large sentences of $\mathbb{A}_{\alpha+1}$.

As in the case for $\alpha = 0$, one can show that $T_{\alpha+1}$ has a countable $\mathbb{A}_{\alpha+1}$-prime model, and since $T_\alpha \subseteq T_{\alpha+1}$, we can assume

$$\mathfrak{M}_\alpha \prec_{\mathbb{A}_\alpha} \mathfrak{M}_{\alpha+1}.$$

Furthermore, since $\neg\Psi_{\mathfrak{M}_\alpha} \in T_{\alpha+1}$, $\mathfrak{M}_{\alpha+1}$ is a proper extension.

At limit stage $\alpha$ let $\mathbb{A}_\alpha := \bigcup_{\beta<\alpha} \mathbb{A}_\alpha$, $T_\alpha := \bigcup_{\beta<\alpha} T_\beta$ and $\mathfrak{M}_\alpha := \bigcup_{\beta<\alpha} \mathfrak{M}_\beta$. Clearly, $\mathfrak{M}_\alpha$ is countable and $\mathfrak{M}_\alpha \models T_\alpha$. If $n < \omega$, and $\overline{a} \in M_\alpha^n$, then for some $\beta < \alpha$, $\overline{a} \in M_\beta^n$. Since $\mathfrak{M}_\beta$ is $\mathbb{A}_\beta$-atomic, there is a minimal $\phi$-large formula $\psi(\overline{v})$, such that $\mathfrak{M}_\alpha \models \psi(\overline{a})$. Notice that $\psi(\overline{v})$ is also $n$-complete over $T_\alpha$, hence $\mathfrak{M}_\alpha$ is $\mathbb{A}_\alpha$-atomic and therefore prime.

Now let $\mathfrak{N} := \bigcup_{\alpha<\omega_1} \mathfrak{M}_\alpha$ and $T := \bigcup_{\alpha<\omega_1} T_\alpha$. Then $\phi \in T$, $\mathfrak{N} \models T$ and $|N| = \aleph_1$.

Suppose there is a countable model $\mathfrak{M}$, such that $\mathfrak{N} \equiv_{\infty,\omega} \mathfrak{M}$. Then by the downward Löwenheim-Skolem theorem, there is $\alpha < \omega_1$ such that $\mathfrak{M}_\alpha \equiv_{\infty,\omega} \mathfrak{M}$, hence by Scott's isomorphism theorem, we have $\mathfrak{N} \models \Psi_{\mathfrak{M}_\alpha}$, a contradiction, since $\neg\Psi_{\mathfrak{M}_\alpha} \in T$. □

Every uncountable model of a VCE $\phi$ is $\mathbb{A}$-small, for every countable fragment $\mathbb{A} \subseteq \mathcal{L}_{\omega_1,\omega}$, because $\phi$ is scattered. Thus, using theorem 2.4.6, we get

**Corollary 4.1.4.** *Every VCE has at least $\aleph_1$ many small models of cardinality $\aleph_1$.*

*Proof.* Let $\phi$ be a VCE. The existence of a small uncountable model is clear.

Since $\phi$ is scattered, it has only countably many isomorphism types of Scott rank $\alpha$, for every $\alpha < \omega_1$. Hence for every $\alpha < \omega_1$, there is a countable set $F_\alpha$ of Scott sentences such that every countable model of $\phi$ with Scott rank $\leq \alpha$ satisfies exactly one sentence of $F_\alpha$.

Now let

$$\rho_\alpha := \phi \wedge \neg(\bigvee_{\psi \in F_\alpha} \psi).$$

Then $\rho_\alpha$ is also a VCE, which has a small model $\mathfrak{N}$ of cardinality $\aleph_1$. Clearly, every model of $\rho_\alpha$ is also a model of $\phi$ and by definition has Scott rank $> \alpha$. □

## 4.2 Hjorth's Theorem

We know that an infinitary sentence with an infinite model does not need to have a model in every infinite cardinality. Gerg Hjorth showed in[6] that if $VC_3$ fails, then there must be a VCE with no models in any cardinality greater than $\aleph_1$. For this he used a descriptive set theoretic approach but there is also a model theoretic one which will be presented here.

**Definition.** Let $\mathcal{K}$ be a class of finite $\tau$-structures. We say

(Em) $\mathcal{K}$ has the joint embedding property if for all $A_1, A_2 \in \mathcal{K}$, there is $B \in \mathcal{K}$ and embeddings $g_1 : A_1 \mapsto B$, $g_2 : A_2 \mapsto B$.

(Am) $\mathcal{K}$ has the amalgamation property if for all $A, B_1, B_2 \in \mathcal{K}$ with embeddings $f_i : A \mapsto B_i$, $1 \leq i \leq 2$, there is $C \in \mathcal{K}$ with embeddings $g_i : B_i \mapsto C$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

(Dam) $\mathcal{K}$ has the disjoint or strong amalgamation property if it satisfies (Am), and for the embeddings $g_i$, $1 \leq i \leq 2$, we have

$$g_1(B_1 \setminus im(f_1)) \cap g_2(B_2 \setminus im(f_2)) = \emptyset.$$

**Example 4.2.1.** Let $\mathcal{K}$ be the class of all finite linear orders. Then $\mathcal{K}$ satisfies (Em) and (Dam).

**Example 4.2.2.** The class $\mathcal{K}$ of finite groups satisfies (Em,Dam):

(Em): If $A, B \in \mathcal{K}$, then the direct product $C := A \times B$ is also in $\mathcal{K}$ and clearly, there are monomorphisms $A \hookrightarrow C$ and $B \hookrightarrow C$.

(Dam): Let $(A, \circ), (B_1, \circ), (B_2, *) \in \mathcal{K}$, $A \subset B_i$, for $1 \leq i \leq 2$, and $A = B_1 \cap B_2$. Consider the group $C$ given by the presentation $< S|R >$, where $S := B_1 \cup B_2$ is the set of generators and $R$ the set of relations defined by:

- $xyz^{-1} = 1$, for $x, y, z \in B_1$ and $x \circ y = z$ or $x, y, z \in B_2$ and $x * y = z$.
- $xyx^{-1}y^{-1} = 1$, for $x \in B_1 \setminus A$ and $y \in B_2 \setminus A$.

It is easy to show that $C \in \mathcal{K}$ and that it is isomorphic to some $C' \in \mathcal{K}$ which contains both $B_1$ and $B_2$ as subgroups.

**Definition.** Let $\mathcal{K}$ be a class of finite structures. A model $\mathfrak{M} := \langle M, \dots \rangle$ is called $\mathcal{K}$-generic if

(G1) For all finite $F \subseteq M$, there is $A \in \mathcal{K}$ such that $F \subseteq A \subset \mathfrak{M}$.

(G2) For all $A \in \mathcal{K}$, there is an embedding from $A$ into $\mathfrak{M}$.

(G3) If $A, B \in \mathcal{K}$, $A \subset B$ and $A \subset \mathfrak{M}$, then there is $B' \in \mathcal{K}$ and an isomorphism $f : B \mapsto B'$ such that $B' \subset \mathfrak{M}$ and $f \upharpoonright A = id_A$.

**Proposition 4.2.3.** *Suppose $\mathcal{K}$ is a class of finite $\tau$-structures which satisfies (Em,Am), is closed under isomorphism and has at most $\aleph_0$ many isomorphism types. Then there is a countable $\mathcal{K}$-generic model which is unique up to isomorphism.*

*Proof.* Let $(\mathcal{B}_n : n < \omega)$ be an enumeration of representatives of all isomorphism types of $\mathcal{K}$ and $\mathcal{K}'$ be the set of all models $\mathcal{A}$ such that $A \subset \omega$ and $\mathcal{A}$ is isomorphic to $\mathcal{B}_k$, for some $k < \omega$. Clearly, $\mathcal{K}'$ is countable, so let $(\mathcal{A}_n : n < \omega)$ be an enumeration of $\mathcal{K}'$.

Let $i : \omega \mapsto \omega \times \omega$, $n \mapsto (i_0(n), i_1(n))$ be a map such that for all $m, n \in \omega$, $(m, n)$ has an infinite preimage.

Using (Em), we can choose $\mathcal{C}_0 \in \mathcal{K}$ such that $C_0 \subset \omega$ and $\mathcal{A}_{i_0(0)}, \mathcal{A}_{i_1(0)}$ can be embedded into it.

Suppose we have $\mathcal{C}_n$. There are two possibilities:

(i) $\mathcal{A}_{i_0(n+1)} \subset \mathcal{C}_n$ and $\mathcal{A}_{i_0(n+1)} \subset \mathcal{A}_{i_1(n+1)}$. Then by (Am) there is $\mathcal{C}_{n+1} \in \mathcal{K}$ such that $C_{n+1} \subset \omega$, $\mathcal{C}_n \subset \mathcal{C}_{n+1}$ and there is an embedding from $\mathcal{A}_{i_1(n+1)}$ into $\mathcal{C}_{n+1}$ which fixes $\mathcal{A}_{i_0(n+1)}$.

(ii) If we are not in case (i), then use (Em) and choose $\mathcal{C}_{n+1} \in \mathcal{K}$ such that $C_{n+1} \subset \omega$, $\mathcal{C}_n \subset \mathcal{C}_{n+1}$ and $\mathcal{A}_{i_0(n+1)}, \mathcal{A}_{i_1(n+1)}$ can be embedded into $\mathcal{C}_{n+1}$.

Let $\mathfrak{M} := \bigcup_{n<\omega} \mathcal{C}_n$. It is easy to check that $\mathfrak{M}$ is countable and satisfies (G1-G3), hence it is $\mathcal{K}$-generic.

The proof that two countable $\mathcal{K}$-generic models are isomorphic uses a straight forward back and forth argument. For example the extension of a given finite embedding uses (G3). □

**Remark.** Under the conditions of the previous proposition we can express (G1-G3) with a single sentence $\phi_{\mathcal{K}} \in \mathcal{L}_{\omega_1,\omega}(\tau)$, thus a model $\mathfrak{N}$ satisfies $\phi_{\mathcal{K}}$ if and only if it is $\mathcal{K}$-generic. We call $\phi_{\mathcal{K}}$ the generic sentence of $\mathcal{K}$. In particular, $\phi_{\mathcal{K}}$ is $\omega$-categorical.

For the rest of this subsection, the vocabulary $\tau$ is countable and has no constant symbols. Every class $\mathcal{K}$ of finite $\tau$-structures is closed under isomorphism and has exactly $\aleph_0$ many isomorphism types.

**Lemma 4.2.4.** *Suppose the class $\mathcal{K}$ satisfies (Dam) and has the following strong version of the joint embedding property.*

*(Em*): For all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, there is $\mathcal{C} \in \mathcal{K}$ and embeddings*

$$f : \mathcal{A} \mapsto \mathcal{C},\ g : \mathcal{B} \mapsto \mathcal{C}$$

*such that $im(f) \cap im(g) = \emptyset$. Then there is a $\mathcal{K}$-generic model of cardinality $\aleph_1$.*

*Proof.* We expand the vocabulary to $\tau'$ by adding a new unary function symbol $F$ and define $\mathcal{K}'$ as the class of all finite $\tau'$-models $A$ such that $A \restriction \tau \in \mathcal{K}$. Then it follows easily that $\mathcal{K}'$ has only countably many isomorphism types and satisfies (Em,Am). By proposition 4.2.3, there is a countable $\mathcal{K}'$-generic model $\mathfrak{M}'$ and it follows from the previous proof that $|\mathfrak{M}'| = \aleph_0$.

Then $\mathfrak{M} := \mathfrak{M}' \restriction \tau$ is $\mathcal{K}$-generic. Let us check for example (G3): Suppose $A, B \in \mathcal{K}$, $A \subset B$ and $A \subset \mathfrak{M}$. By (G1) for $\mathfrak{M}'$, there is $A' \subset \mathfrak{M}'$ containing $A$ as a $\tau$-submodel. Define $A^* := A' \restriction \tau$

$\mathcal{K}$ satisfies (Dam), hence there is $C \in \mathcal{K}$, such that $B, A^* \subset C$ and

$$(B \setminus A) \cap (A^* \setminus A) = \emptyset.$$

$C$ can easily be extended to $C' \in \mathcal{K}'$ such that $A' \subset C'$. Now apply (G3) for $\mathfrak{M}'$ and then restrict to $\tau$, keeping in mind that $\mathcal{K}$ is closed under isomorphism.

Now we construct a proper embedding from $\mathfrak{M}$ into itself. Let $f := F^{\mathfrak{M}'}$.

Start with an arbitrary $A \in \mathcal{K}$ such that $A \subset \mathfrak{M}$. By interpreting the new function symbol as the $id_A$ we get an element $A' \in \mathcal{K}'$ which by (G2) can be embedded into $\mathfrak{M}'$. Using (G2,G3), one can extend this function to a $\tau$-embedding from $\mathfrak{M}$ into itself. The image of this embedding is a subset of

$$\{x \in M : f(x) = x\}$$

which can easily be shown to be a proper subset.

Thus there is a chain $(\mathfrak{M}_\alpha : \alpha < \omega_1)$ of countable $\mathcal{K}$-generic models with

$$\mathfrak{M}_\alpha \subsetneqq \mathfrak{M}_\beta,$$

whenever $\alpha < \beta < \omega_1$. If we define $\mathfrak{N} := \bigcup_{\alpha<\omega_1} \mathfrak{M}_\alpha$ and $\phi_{\mathcal{K}}$ as the $\tau$-sentence describing (G1-G3), then it is easy to show that $|\mathfrak{N}| = \aleph_1$ and $\mathfrak{N} \models \phi_{\mathcal{K}}$, hence $\mathfrak{N}$ is also $\mathcal{K}$-generic. □

With this theoretic foundation let us consider the following vocabulary $\tau_0$: For every $k < \omega$, $\tau_0$ has a binary relation symbol $S_k$ and a $(k+2)$-ary relation symbol $R_k$. We define $\mathcal{K}_0$ as the class of finite $\tau_0$-structures satisfying the following sentences:

(A1) $\forall x \forall y[(\bigvee_{i<\omega} S_i xy) \wedge \bigwedge_{i<\omega}(S_i xy \rightarrow \bigwedge_{j\neq i} \neg S_j xy)]$.

(A2) For every $k < \omega$, the sentence

$$\forall a_0 \forall a_1 \forall b_1 \dots \forall b_k [R_k a_0 a_1 \bar{b} \to (a_0 \neq a_1 \land \bigwedge_{i \neq j} (b_i \neq b_j))].$$

(A3) For every $k < \omega$ and every permutation $\sigma$ on $\{0, \dots k-1\}$, the sentence

$$\forall a_0 \forall a_1 \forall \bar{b} (R_k a_0 a_1 \bar{b} \leftrightarrow R_k a_0 a_1 \sigma(\bar{b})).$$

(A4) For every $k < \omega$, the sentence

$$\forall a_0 \forall a_1 \forall \bar{b} [R_k a_0 a_1 \bar{b} \to \bigwedge_{i < \omega} (\bigwedge_{j=1}^{k} (S_i a_0 b_j \leftrightarrow S_i a_1 b_j))].$$

(A5) For every $k < \omega$, the sentence

$$\forall a_0 \forall a_1 \forall \bar{b} \forall c [(R_k a_0 a_1 \bar{b} \land \bigwedge_{j=1}^{k} (c \neq b_j)) \to \bigwedge_{i < \omega} (S_i a_0 c \to \neg S_i a_1 c)].$$

(A6) $\forall a_0 \forall a_1 [(a_0 \neq a_1) \to \bigvee_{k < \omega} (\exists b_1 \dots \exists b_k R_k a_0 a_1 \bar{b})]$.

This means that every element of $\mathcal{K}_0$ is a finite, complete, colored and directed graph in which two distinct elements are connected by exactly two directed edges.

If $\mathcal{A} \in \mathcal{K}_0$ and $a_0 \neq a_1$ are in $A$, then the set of all points $b_1, \dots b_k$ in $A$ which are connected to $a_0$ and $a_1$ with the same color is finite. For these elements, we have $R_k^{\mathcal{A}} a_0 a_1 \bar{b}$.

**Proposition 4.2.5.** *$\mathcal{K}_0$ is closed under isomorphism, it has $\aleph_0$ many isomorphism types and satisfies (Em*) and (Dam).*

*Proof.* Clearly, $\mathcal{K}_0$ is closed under isomorphism, and since there are exactly $\aleph_0$ many finite models in $\mathcal{K}_0$ with universe contained in $\omega$, we have $\aleph_0$ many isomorphism types.

Let us check (Dam) for example: Suppose $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \in \mathcal{K}_0$ and without loss of generality $A = B_1 \cap B_2$ and $\mathcal{A} \subset \mathcal{B}_1, \mathcal{B}_2$. Define $C := B_1 \cup B_2$ and $S$ as the set of all $i < \omega$, such that for some $(u, v) \in B_1^2 \cup B_2^2$, we have $S_i^{\mathcal{B}_1}(uv)$ or $S_i^{\mathcal{B}_2}(uv)$. Then let $W := (B_1 \setminus B_2) \times (B_2 \setminus B_1) \cup (B_2 \setminus B_1) \times (B_1 \setminus B_2)$ and $H : W \mapsto \omega \setminus S$ be injective.

We interpret the relation symbols on $\mathcal{C}$ as follows:

- For $i < \omega$ and $(u, v) \in C^2$,

$$S_i^{\mathcal{C}}(u, v) :\Leftrightarrow \begin{cases} (u, v) \in B_1^2 \text{ and } S_i^{\mathcal{B}_1}(u, v) \\ (u, v) \in B_2^2 \text{ and } S_i^{\mathcal{B}_2}(u, v) \\ (u, v) \in W \text{ and } i = H(u, v). \end{cases}$$

- For $k < \omega$ $u, v, \in C$, where $u \neq v$, and $b_1, \dots, b_k \in C$ are distinct, let $R_k^{\mathcal{C}}(uv\bar{b})$ if $\{b_1, \dots, b_k\}$ is the set of all $b \in C$, such that for all $i < \omega$

$$S_i^{\mathcal{C}}(ub) \Leftrightarrow S_i^{\mathcal{C}}(vb).$$

Clearly, $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{C}$ and $\mathcal{C} \in \mathcal{K}_0$.

The argument for (Em*) is similar. □

**Lemma 4.2.6.** *Let $\phi_{\mathcal{K}_0}$ be the generic sentence of $\mathcal{K}_0$. Then $\phi_{\mathcal{K}_0}$ has a model of cardinality $\aleph_1$ but not of any greater cardinality.*

*Proof.* By the previous proposition and Lemma 4.2.4, $\phi_{\mathcal{K}_0}$ has a model of cardinality $\aleph_1$.

Suppose there is a model $\mathfrak{N} := \langle N, \dots \rangle$ of $\phi_{\mathcal{K}_0}$ with cardinality greater than $\aleph_1$. By the downward Löwenheim-Skolem theorem there is a $\mathcal{K}_0$-generic submodel $\mathfrak{M} := \langle M, \dots \rangle$ of $\mathfrak{N}$ with cardinality $\aleph_1$.

Let $c \in N \setminus M$. Then the function $H : M \mapsto \omega$, defined by $a \mapsto i$, where $S_i^{\mathfrak{N}}(ac)$, is injective, which is a contradiction, since $M$ is uncountable. □

Next, we expand the vocabulary $\tau_0$ to $\tau_1$ by adding two unary predicate symbols $P, Q$ and a binary relation symbol $F$.

Let $\mathcal{K}_1$ be the set of all finite $\tau_1$- structures $A$ with the following properties:

- The universe of $A$ is the disjont union of $P^A$ and $Q^A$.
- The symbols of $\tau_0$ are interpreted as relations on $P^A$ only.
- $P^A \restriction \tau_0$ is either empty or an element of $\mathcal{K}_0$.
- $F^A$ is a function from $P^A$ into $Q^A$.

**Proposition 4.2.7.**

*(i) $\mathcal{K}_1$ has $\aleph_0$ many isomorphism types, is closed under isomorphism and satisfies (Em*,Dam). Hence there is a $\mathcal{K}_1$-generic model of cardinality $\aleph_1$.*

*(ii) If $\mathfrak{M}$ is a $\mathcal{K}_1$-generic model, then $|Q^{\mathfrak{M}}| \geq \aleph_0$, $F^{\mathfrak{M}}$ is a surjective function from $P^{\mathfrak{M}}$ onto $Q^{\mathfrak{M}}$ and $P^{\mathfrak{M}} \restriction \tau_0$ is $\mathcal{K}_0$-generic.*

*Proof.* The proof of $(i)$ is similar to that of proposition 4.2.5.

$(ii)$: Let $\mathfrak{M} := \langle M, \dots \rangle$ be $\mathcal{K}_1$-generic. $Q^{\mathfrak{M}}$ is infinite, since for every $k < \omega$, there is a structure $A \in \mathcal{K}_1$ such that $P^A = \emptyset$ and $|Q^A| = k$ which by (G2) can be embedded into $\mathfrak{M}$.

For the surjectivity of $f^{\mathfrak{M}}$ we use the generic property (G3): Let $c \in Q^{\mathfrak{M}}$. Then the model $A$ with universe $\{c\}$, $Q^A := \{c\}$ and all other relation symbols interpreted as $\emptyset$ is a submodel of $\mathfrak{M}$.

Let $b$ be an element not in $M$, and define the model $B \in \mathcal{K}_1$ as follows:

- $P^B := \{b\}$ and $Q^B := \{c\}$.
- $F^B := (b, c)$
- $S_1^B(b, b)$, and all other relation symbols are interpreted as $\emptyset$.

Then $A \subset B$, and by applying (G3) we see that $c \in im(F^{\mathfrak{M}})$.

The proof that $P^{\mathfrak{M}} \restriction \tau_0$ is $\mathcal{K}_0$-generic is a straight forward check of (G1-G3) and can be done by expanding structures of $\mathcal{K}_0$ to ones in $\mathcal{K}_1$ and then use the genericity of $\mathfrak{M}$. □

**Remark.** By slightly modifying the construction of the proper embedding of a countable $\mathcal{K}_1$-generic model into itself - shown in the proof of Lemma 4.2.4 -, one can show that there is a $\mathcal{K}_1$-generic model $\mathfrak{N}$ of cardinality $\aleph_1$ with $|Q^{\mathfrak{N}}| = \aleph_0$. One simply adds the new unary function symbol $G$ and considers the class $\mathcal{K}_1'$ of all finite models $A$ in the new vocabulary such that $A \upharpoonright \tau_0 \in \mathcal{K}_0$ or empty and $G^A \upharpoonright Q^A = id_{Q^A}$. It is then easy to construct the embedding $i$ of a countable $\mathcal{K}_1$-generic model $\mathfrak{M}$ into itself such that $i(P^{\mathfrak{M}}) \subsetneq P^{\mathfrak{M}}$ and $i(Q^{\mathfrak{M}}) = Q^{\mathfrak{M}}$.

**Corollary 4.2.8.** *The $\mathcal{K}_1$-generic sentence $\phi_{\mathcal{K}_1}$ has no models of cardinality greater than $\aleph_1$.*

*Proof.* If $\mathfrak{N}$ is a model of $\phi_{\mathcal{K}_1}$ of cardinality greater than $\aleph_1$, then by $(ii)$ of the previous proposition, $|P^{\mathfrak{N}}| > \aleph_1$ and therefore $\phi_{\mathcal{K}_0}$ has a model of cardinality greater than $\aleph_1$, a contradiction. □

**Definition.** Let $\mathfrak{M} := \langle M, \dots \rangle$ be a $\tau$-structure and $A \subseteq M$. $A$ is called a set of absolute indiscernibles if every permutation of $A$ can be extended to an automorphism of $\mathfrak{M}$.

**Example 4.2.9.** Let $K := (0, 1, +, *)$ be a field and the vocabulary $\tau$ consist of a constant symbol, a binary function symbol and for every $a \in K$, a unary function symbol $f_a$.

We can see every $K$-vector space as a $\tau$-model $\mathcal{V} := \langle V, \dots \rangle$, where the unary function symbols are interpreted as the scalar multiplications.

Then every linearly independent subset $X \subseteq V$ is a set of absolute indiscernibles.

**Proposition 4.2.10.** *If $\mathfrak{M}$ is a countable $\mathcal{K}_1$-generic model, then $Q^{\mathfrak{M}}$ is a set of absolute indiscernibles.*

*Proof.* This can be shown with a back and forth argument. We only present one direction.

Suppose $\pi : Q^{\mathfrak{M}} \mapsto Q^{\mathfrak{M}}$ is bijective, $A \in \mathcal{K}_1$ is a submodel of $\mathfrak{M}$ and $i : A \mapsto \mathfrak{M}$ is an embedding such that

$$i \upharpoonright Q^A = \pi \upharpoonright Q^A.$$

Let $A' := im_i(A)$, $a \in P^{\mathfrak{M}} \setminus A'$ and $q \in Q^{\mathfrak{M}} \setminus A'$. By (G1) there is a $B \in \mathcal{K}_1$ with $A' \subset B \subset \mathfrak{M}$ and $a, q \in B$. Using (G3) it is easy to find a $B_2 \in \mathcal{K}_1$ such that $A \subset B_2 \subset \mathfrak{M}$, $Q^{B_2} = \pi^{-1}(Q^B)$ and there is an isomorphism from $B_2$ onto $B$ which extends $i$ and agrees with $\pi$ on $Q^{B_2}$. □

We can now prove

**Theorem 4.2.11.** *(Hjorth) If $VC_3$ fails, then there is a VCE which has only models of cardinality $\aleph_0$ and $\aleph_1$.*

*Proof.* Let $\mu$ be a countable vocabulary disjoint from $\tau_1$ and $\sigma \in \mathcal{L}_{\omega_1,\omega}(\mu)$ be a VCE. Define $\tau_2 := \tau_1 \cup \mu$ and consider the $\tau_2$-sentence $\psi$ stating the following:

- $\phi_{\mathcal{K}_1}$, the $\mathcal{K}_1$-generic sentence.
- For every constant symbol $c \in \mu$, the sentence $Qc$.

- For every $k < \omega_+$ and every $k$-ary relation symbol $R \in \mu$, the sentence

$$\forall \overline{v}[R(\overline{v}) \to \bigwedge_{i=1}^{k}(Q(v_i))].$$

- For every $k < \omega_+$ and every $k$-ary function symbol $G$ of $\mu$,

$$\forall \overline{v}[(\bigwedge_{i=1}^{k}(Pv_i \to G(\overline{v}) = v_1)) \wedge ([\bigwedge_{i=1}^{k} Qv_i] \to Q(G\overline{v}))].$$

- $\sigma^Q$, which is the sentence $\sigma$ relativised to the predicate $Q$, or in other words, the set defined by the formula $Qv$ is a $\mu$-model of $\sigma$.

So in essence every model of $\psi$ is $\mathcal{K}_1$-generic when restricted to $\tau_1$ and the $\mu$-structure is only interesting on the $Q$-predicate. Furthermore, the $Q$-predicate is a model of $\sigma$, seen as a $\mu$-structure.

Since we know that there is a countable $\mathcal{K}_1$-generic model $\mathfrak{M}$ and $|Q^{\mathfrak{M}}| = \aleph_0$, we can easily interpred every countable model of $\sigma$ in $Q^{\mathfrak{M}}$, thereby getting a model of $\psi$.

*Claim*: If $\mathfrak{M}_1, \mathfrak{M}_2 \in Mod(\psi)$, then $\mathfrak{M}_1 \cong \mathfrak{M}_2$ if and only if

$$Q^{\mathfrak{M}_1} \restriction \mu \cong Q^{\mathfrak{M}_2} \restriction \mu.$$

*(Proof of the claim.)* The direction ($\Rightarrow$) is clear.

The direction ($\Leftarrow$) follows from the fact that if $\mathfrak{M} \in Mod(\psi)$ then $\mathfrak{M} \restriction \tau_1$ is a model of $\phi_{\mathcal{K}_1}$ and $Q^{\mathfrak{M}}$ is a set of absolute indiscernibles. *(q.e.d.-Claim)*

Since $I(\sigma, \aleph_0) = \aleph_1$, we have $I(\psi, \aleph_0) = \aleph_1$.

Furthermore, $\psi$ does not have a perfect set of pairwise non isomorphic countable models, because otherwise $\sigma$ would have one too. Thus $\psi$ is a VCE.

Using the results of the previous subsection or the remark to proposition 4.2.7, we see that $\psi$ has a model of cardinality $\aleph_1$ but of no greater cardinality, since $\psi \models \phi_{\mathcal{K}_1}$. □

## 4.3 Harrington's Theorem

We now take a look at what can be said about the Scott ranks of uncountable models of a VCE. Leo Harrington was the first to prove that the Scott ranks of such models are unbounded below $\aleph_2$ but the proof presented here is based on [1].

Recall from section 3.3 that for a scattered sentence $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$ we defined a chain of countable fragments $(\mathbb{A}_{\phi,\alpha} : \alpha < \omega_1)$. Now suppose that $\phi$ is a VCE. We define a tree of $\mathcal{L}_{\omega_1,\omega}(\tau)$-theories $(\mathcal{T}, <)$, where $t_1 < t_2$ if $t_1 \subsetneq t_2$, for $t_1, t_2 \in \mathcal{T}$. $\mathcal{T}$ is the set of all theories $t$ which satisfy the following conditions:

(1) There is $\alpha < \omega_1$ and $\mathfrak{M} \in Mod(\phi)$ such that $t = Th_{\mathbb{A}_{\phi,\alpha}}(\mathfrak{M})$, the set of all sentences of $\mathbb{A}_{\phi,\alpha}$ which are true in $\mathfrak{M}$.

(2) If $\alpha < \omega_1$ and $\mathfrak{M} \in Mod(\phi)$ such that $t = Th_{\mathbb{A}_{\phi,\alpha}}(\mathfrak{M})$, then for all $\beta < \alpha$, $t \cap \mathbb{A}_{\phi,\beta}$ is not $\omega$-categorical.

$\mathcal{T}$ is easily seen to be a tree and is called the Morley tree of $\phi$.

One can easily check that if $t \in \mathcal{T}$, $\alpha < \omega_1$ and $\mathfrak{M} \in Mod(\phi)$ such that $t = Th_{\mathbb{A}_{\phi,\alpha}}(\mathfrak{M})$, then for all $\beta < \alpha$, $t \cap \mathbb{A}_{\phi,\beta} \in \mathcal{T}$. In this case $\alpha$ is called the height of $t$, notated by $hgt(t)$. We define $\mathcal{T}_\alpha$ as the set of all $t \in \mathcal{T}$ with height $\alpha$.

First, we present some easy results about the Morley tree.

**Proposition 4.3.1.** *For every $\alpha < \omega_1$,*

*(i)* $\mathcal{T}_\alpha \neq \emptyset$.

*(ii) If $t \in \mathcal{T}_\alpha$, then $t$ is $\mathbb{A}_{\phi,\alpha}$-atomic, that is it has a countable $\mathbb{A}_{\phi,\alpha}$-prime model.*

*Proof.* $(i)$: Let $\alpha < \omega_1$. Since $\phi$ is a VCE, it is scattered, hence there is a complete theory $t \subseteq \mathbb{A}_{\phi,\alpha}$ and an uncountable set $D \subseteq Mod(\phi)$ of pairwise non isomorphic models such that for all $\mathfrak{M}, \mathfrak{N} \in D$, $Th_{\mathbb{A}_{\phi,\alpha}}(\mathfrak{M}) = t = Th_{\mathbb{A}_{\phi,\alpha}}(\mathfrak{N})$, which means that $t$ is not $\omega$-categorical.

$(ii)$: This follows immediately from lemma 2.5.3 and the fact that $\phi$ is scattered. □

**Corollary 4.3.2.** *If $\lambda < \omega_1$ is a limit ordinal and $(t_\beta : \beta < \lambda)$ is a chain of theories, where $t_\beta \in \mathcal{T}_\beta$, for $\beta < \lambda$, then $\bigcup_{\beta<\lambda} t_\beta \in \mathcal{T}_\lambda$.*

*Proof.* Every $t_\beta$ is $\mathbb{A}_{\phi,\beta}$-atomic and $\lambda$ is countable, hence there is a sequence $(\beta_n : n < \omega)$ of ordinals in $\lambda$ which is cofinal in $\lambda$ and a sequence $(\mathfrak{M}_n : n < \omega)$ in $Mod(\phi)$ such that for all $n < \omega$, $\mathfrak{M}_n \models t_{\beta_n}$ and

$$\mathfrak{M}_m \prec_{\mathbb{A}_{\phi,\beta_m}} \mathfrak{M}_n,$$

for $m < n < \omega$. If we set

$$\mathfrak{N} := \bigcup_{n<\omega} \mathfrak{M}_n, \text{ and } t_\lambda := \bigcup_{n<\omega} t_{\beta_n},$$

then $t_\lambda = \bigcup_{\beta<\lambda} t_\beta$, $\mathfrak{N} \models t_\lambda$ and since no $t_\beta$ is $\omega$-categorical, we have $t_\lambda \in \mathcal{T}_\lambda$. □

**Lemma 4.3.3.** *(a) If $\mathfrak{M} \in Mod(\phi)$, then there is a terminal node $t \in \mathcal{T}$ such that $\mathfrak{M} \models t$.*

*(b) Let $\lambda < \omega_1$ be a limit ordinlal $\neq \emptyset$. There is $\beta < \lambda$ and $t \in \mathcal{T}_\beta$ such that $t$ has a unique extension in $\mathcal{T}_\lambda$.*

*Proof.* $(a)$: Let $\alpha := sr(\mathfrak{M}) + \omega$. By proposition 3.3.1, we know that for every $\mathfrak{N} \in Mod(\phi)$,

$$Th_{\mathbb{A}_{\phi,\alpha}}(\mathfrak{M}) = Th_{\mathbb{A}_{\phi,\alpha}}(\mathfrak{N}) \text{ implies } \mathfrak{M} \equiv_\alpha \mathfrak{N},$$

which is equivalent to $\mathfrak{M} \cong \mathfrak{N}$. Hence $Th_{\mathbb{A}_{\phi,\alpha}}(\mathfrak{M})$ is $\omega$-categorical. If we choose $\gamma < \omega_1$ minimal with that property, we have the desired terminal node in $\mathcal{T}_\gamma$.

$(b)$: If this is not so, then by using the previous corollary, we can easily build a binary tree $S$ of elements of $\mathcal{T}$ with the following properties:

- Every element of $S$ has height $< \lambda$.
- The union of every branch of $S$ is in $\mathcal{T}_\lambda$.

But then there are $\mathfrak{c}$ many complete $\mathbb{A}_{\phi,\lambda}$-types containing $\phi$, a contradiction. □

**Corollary 4.3.4.** *There is $t \in \mathcal{T}$ and a club set $S \subseteq \omega_1$, such that $t$ has a unique extension in $\mathcal{T}_\alpha$, for every $\alpha \in S$.*

*Proof.* Let $L$ be the set of all limit ordinals $\neq \emptyset$ in $\omega_1$. By the previous lemma, there is a regressive function $f$ on $L$ such that for every $\lambda \in L$, there is $t \in \mathcal{T}_{f(\lambda)}$ with a unique extension in $\mathcal{T}_\lambda$.

Using Fodor's theorem we find that there is a stationary set $L' \subseteq L$ on which $f$ is constant with some value $\beta < \omega_1$.

Since $\phi$ is scattered and $\mathbb{A}_{\phi,\beta}$ is countable, there is some $t \in \mathcal{T}_\beta$ which has a unique extension in $\mathcal{T}_\alpha$ for uncountably many $\alpha \in L'$. Let $S'$ be the set of those $\alpha$. Then $S$ defined as the closure of $S'$ in $\omega_1$ is club, and one easily checks that $t$ has a unique extension in $\mathcal{T}_\alpha$, for every $\alpha \in S$. □

**Lemma 4.3.5.** *If $\lambda < \omega_1$ is a limit ordinal $\neq \emptyset$, $t \in \mathcal{T}_\lambda$ and $\mathfrak{M} \in Mod(\phi)$ such that $\mathfrak{M} \models t$, then $sr(\mathfrak{M}) \geq \lambda$.*

*Proof.* Let $\mathfrak{M} = \langle M, \dots \rangle$ and suppose $sr(\mathfrak{M}) = \beta < \lambda$. We will derive a contradiction by showing that $t_{\beta+2} := t \cap \mathbb{A}_{\phi,\beta+2}$ is $\omega$-categorical.

First, for an arbitrary model $\mathcal{A} := \langle A, \dots \rangle$, $k < \omega$, $\overline{a} \in A^k$ and $\alpha < \omega_1$, we define

$$\Delta(\mathcal{A}, \overline{a}, \alpha) := \mathfrak{tp}^{\mathcal{A}}_{\mathbb{A}_{\phi,\alpha}}(\overline{a}).$$

Using proposition 3.3.1 and the definition of the Scott rank, we conclude that for every $k < \omega$ and every $\overline{a} \in M^k$, the following sentence of $\mathbb{A}_{\phi,\beta+2}$ is true in $\mathfrak{M}$ and therefore in $t$:

$$\sigma_{\overline{a},\beta} := \forall v_1 \dots \forall v_k [\bigwedge_{\psi \in \Delta(\mathfrak{M},\overline{a},\beta)} \psi(\overline{v}) \to \bigwedge_{\psi \in \Delta(\mathfrak{M},\overline{a},\beta+1)} \psi(\overline{v})].$$

Now let $\mathfrak{N} := \langle N, \dots \rangle$ be a model of $t_{\beta+2}$, $k < \omega$, $\overline{a} \in M^k$, $\overline{b} \in N^k$ and suppose $(\mathfrak{M}; \overline{a}) \sim_\beta (\mathfrak{N}; \overline{b})$. Then the following sentence is true in $\mathfrak{M}$:

$$\exists \overline{v} [\bigwedge_{\psi \in \Delta(\mathfrak{N},\overline{b},\beta)} \psi(\overline{v})].$$

By proposition 3.3.1, any $k$-tuple of $\mathfrak{M}$ which satisfies $\Delta(\mathfrak{N}, \overline{b}, \beta)$ is $\sim_\beta$-equivalent to $\overline{b}$ and therefore to $\overline{a}$. Hence $\Delta(\mathfrak{M}, \overline{a}, \beta) = \Delta(\mathfrak{N}, \overline{b}, \beta)$ and since $\mathfrak{N} \models \sigma_{\overline{a},\beta}$, we have $\Delta(\mathfrak{M}, \overline{a}, \beta+1) = \Delta(\mathfrak{N}, \overline{b}, \beta+1)$ which implies $(\mathfrak{M}; \overline{a}) \sim_{\beta+1} (\mathfrak{N}; \overline{b})$.

Clearly, $\mathfrak{N} \models t_\beta$, thus $\mathfrak{M} \equiv_\beta \mathfrak{N}$ and we can in a back and forth manner construct an isomorphism from $\mathfrak{M}$ onto $\mathfrak{N}$. □

With these results we can now present a different proof of the fact that every VCE has an uncountable model which is not small (recall theorem 4.1.3): By corollary 4.3.4, the Morley tree $\mathcal{T}$ has a branch $\mathcal{S}$ of lenght $\omega_1$. For $\alpha < \omega_1$, let $t_\alpha := \mathcal{S} \cap \mathcal{T}_\alpha$.

It is not difficult to show that there is a cofinal sequence of limit ordinals $(\beta_i : i < \omega_1)$ in $\omega_1$ and a chain $(\mathfrak{M}_{\beta_i} : i < \omega_1)$ of countable models with the following properties:

- $\mathfrak{M}_{\beta_i}$ is a $\mathbb{A}_{\phi,\beta_i}$-prime model of $t_{\beta_i}$.

- If $i < j < \omega_1$, then $\mathfrak{M}_{\beta_i}$ is a porper $\mathbb{A}_{\phi,\beta_i}$-elementary submodel of $\mathfrak{M}_{\beta_j}$.

The limit $\mathfrak{N}$ of this chain has cardinality $\aleph_1$ and is a model of $\bigcup \mathcal{S}$.

If $\mathfrak{N}$ is small, then by the downward Löwenheim-Skolem theorem, for every $i < \omega_1$, there is $j$ such that $i < j < \omega_1$ and $\mathfrak{N} \equiv_{\infty,\omega} \mathfrak{M}_{\beta_j}$. This is a contradiction to the previous lemma, since then there is $i < j < \omega_1$ such that

$$\mathfrak{M}_{\beta_i} \equiv_{\infty,\omega} \mathfrak{N} \equiv_{\infty,\omega} \mathfrak{M}_{\beta_j}$$

and $\beta_j > sr(\mathfrak{M}_{\beta_i})$.

**Proposition 4.3.6.** *If $\phi$ is a minimal VCE, then there is a club set $C \subseteq \omega_1$ such that whenever $\alpha \in C$, $\mathfrak{M}, \mathfrak{N} \in Mod(\phi)$ and $sr(\mathfrak{M}), sr(\mathfrak{N}) \geq \alpha$, then*

$$\mathfrak{M} \equiv_\alpha \mathfrak{N}.$$

*Proof.* If $\phi$ is a minimal VCE, then for every $\alpha < \omega_1$, there is a unique $t_\alpha \in \mathcal{T}_\alpha$ with $I(t_\alpha, \aleph_0) = \aleph_1$. Now define $C$ as the set of all non empty limit ordinals $\alpha < \omega_1$ such that every model $\mathfrak{M} \in Mod(\phi)$ with $\mathfrak{M} \not\models t_\alpha$ has Scott rank less than $\alpha$. One can easily check that $C$ is club.

If $\alpha \in C$ and $\mathfrak{M}, \mathfrak{N} \in Mod(\phi)$ with Scott rank $\geq \alpha$, then clearly, $\mathfrak{M}$ and $\mathfrak{N}$ are models of $t_\alpha$ which is a complete type of sentences in $\mathbb{A}_{\phi,\alpha}$. Using proposition 3.3.1 we see that $\mathfrak{M} \equiv_\alpha \mathfrak{N}$. □

Before we look at the proof of the main result we need to make a small detour to a model theoretic construction called a twisted direct limit.

**Definition.** Let $\tau, \tau'$ be vocabularies and $\mathbb{A}, \mathbb{A}'$ be fragments of $\mathcal{L}_{\infty,\omega}(\tau)$ and $\mathcal{L}_{\infty,\omega}(\tau')$ respectively. A fragment embedding from $\mathbb{A}$ into $\mathbb{A}'$ is an injective function $\pi : \mathbb{A} \mapsto \mathbb{A}'$ with the following properties:

(1) $\pi$ is the identity on atomic $\tau$-formulas.

(2) $\pi$ respects finite conjunctions and disjunctions.

(3) $\pi(\forall y \psi) = \forall y \pi(\psi)$ and $\pi(\exists y \psi) = \exists y \pi(\psi)$.

(4) If

$$\phi = \bigwedge_{\psi \in F} \psi \text{ (or } \phi = \bigvee_{\psi \in F} \psi)$$

is an infinite conjunction (disjunction), then $\pi(\phi)$ is an infinite conjunction (disjunction) and for all $\rho \in \mathbb{A}$, $\rho \in F$ if and only if $\pi(\rho)$ is a conjunct (disjunct) of $\pi(\phi)$.

(5) For all $\psi \in \mathbb{A}$, $\pi(\psi)$ has the same free variables as $\psi$.

Now let $\mu$ be a limit ordinal and $(\tau_\alpha : \alpha < \mu)$ be a chain of countable fragments, i.e.

- For $\alpha < \beta < \mu$, $\tau_\alpha \subseteq \tau_\beta$.
- If $\lambda$ is a nonempty limit ordinal in $\mu$, then $\tau_\lambda = \bigcup_{\alpha<\lambda} \tau_\alpha$.

Suppose that for each $\alpha < \mu$, we have a fragment $\mathbb{A}_\alpha \subseteq \mathcal{L}_{\infty,\omega}(\tau_\alpha)$.

**Definition.** (a) For all $\alpha \leq \beta < \mu$, let $\pi_{\alpha,\beta} : \mathbb{A}_\alpha \mapsto \mathbb{A}_\beta$ be a fragment embedding such that $\pi_{\alpha,\alpha} = id_{\mathbb{A}_\alpha}$ and for $\alpha \leq \beta \leq \gamma < \mu$,

$$\pi_{\alpha,\gamma} = \pi_{\beta,\gamma} \circ \pi_{\alpha,\beta}.$$

Then $(\pi_{\alpha,\beta} : \alpha \leq \beta < \mu)$ is called a directed system of fragment embeddings.

(b) Let $\tau^* := \bigcup_{\alpha<\mu} \tau_\alpha$ and $\mathbb{A}^* \subseteq \mathcal{L}_{\infty,\omega}(\tau^*)$ be a fragment. If in addition to the directed system we have for every $\alpha < \mu$ a fragment embedding $\pi_{\alpha,*} : \mathbb{A}_\alpha \mapsto \mathbb{A}^*$ such that $\mathbb{A}^* = \bigcup_{\alpha<\mu} im(\pi_{\alpha,*})$ and for $\alpha \leq \beta < \mu$,

$$\pi_{\alpha,*} = \pi_{\beta,*} \circ \pi_{\alpha,\beta},$$

then $(\mathbb{A}^*, \pi_{\alpha,*} : \alpha < \mu)$ is called the direct limit of $(\pi_{\alpha,\beta} : \alpha \leq \beta < \mu)$.

(c) Suppose that in addition to a directed system of fragment embeddings we have a sequence of models $(\mathfrak{M}_\alpha : \alpha < \mu)$, for all $\alpha \leq \beta < \mu$ a map

$$\rho_{\alpha,\beta} : \mathfrak{M}_\alpha \mapsto \mathfrak{M}_\beta$$

such that $\mathfrak{M}_\alpha$ is a $\tau_\alpha$-structure, $\rho_{\alpha,\alpha} = id_{\mathfrak{M}_\alpha}$, for $\alpha \leq \beta \leq \gamma < \mu$,

$$\rho_{\alpha,\gamma} = \rho_{\beta,\gamma} \circ \rho_{\alpha,\beta},$$

and for all $\psi(\overline{v}) \in \mathbb{A}_\alpha$,

$$\mathfrak{M}_\alpha \models \psi(\overline{a}) \text{ iff } \mathfrak{M}_\beta \models \pi_{\alpha,\beta}(\psi)(\rho_{\alpha,\beta}(\overline{a})).$$

Then $(\mathfrak{M}_\alpha, \pi_{\alpha,\beta}, \rho_{\alpha,\beta} : \alpha \leq \beta < \mu)$ is called a twisted elementary system.

**Lemma 4.3.7.** *Let $(\pi_{\alpha,\beta} : \alpha \leq \beta < \mu)$ be a directed system of fragment embeddings with limit $(\mathbb{A}^*, \pi_{\alpha,*} : \alpha < \mu)$ and $(\mathfrak{M}_\alpha, \pi_{\alpha,\beta}, \rho_{\alpha,\beta} : \alpha \leq \beta < \mu)$ be a twisted elementary system. Then there is a $\tau^*$-structure $\mathfrak{M}_*$ and maps $(\rho_{\alpha,*} : \alpha < \mu)$, where $\rho_{\alpha,*} : \mathfrak{M}_\alpha \mapsto \mathfrak{M}_*$ with the following properties:*

*(i)* $\rho_{\alpha,*} = \rho_{\beta,*} \circ \rho_{\alpha,\beta}$*, for* $\alpha \leq \beta < \mu$.

*(ii)* $\mathfrak{M}_* = \bigcup_{\alpha<\mu} im(\rho_{\alpha,*})$.

*(iii) For all $\alpha < \mu$, all $k < \omega$, all k-formulas $\psi(\overline{v}) \in \mathbb{A}_\alpha$ and all $\overline{a} \in M_\alpha^k$,*

$$\mathfrak{M}_\alpha \models \psi(\overline{a}) \textit{ iff } \mathfrak{M}_* \models \pi_{\alpha,*}(\psi)(\rho_{\alpha,*}(\overline{a})).$$

**Remark.** Under the conditions of this Lemma the model $\mathfrak{M}_*$ is called the limit of the twisted elementary system.

*Proof.* Let $\mathcal{F}$ be the set of all functions $f$ such that

- $dom(f) = [\alpha, \mu)$, for some $\alpha < \mu$.
- $f(\alpha) \in M_\alpha$ and for all $\beta \in (\alpha, \mu)$, $f(\beta) = \rho_{\alpha,\beta}(f(\alpha))$.

Clearly, $im(f) \subseteq \bigcup_{\alpha<\mu} M_\alpha$, for every $f \in \mathcal{F}$, and every element of $\mathcal{F}$ is uniquely determined by its value on the minimum of the its domain.

Furthermore, if $f, g \in \mathcal{F}$ and for some $\beta \in dom(f) \cap dom(g)$, $f(\beta) = g(\beta)$, then $f \subseteq g$ or $g \subseteq f$. This follows from the fact that each $\rho_{\alpha,\beta}$ is injective and the commuting property $(c)$ of the definition.

This implies that we have an equivalence relation on $\mathcal{F}$ defined by $f \sim g$ if $f \subseteq g$ or $g \subseteq f$.

The universe $M^*$ of the proposed $\tau^*$-structure is the quotient space $\mathcal{F}/\sim$. For $f \in \mathcal{F}$, let $[f]$ denote the $\sim$-equivalence class of $f$ and $m(f) := min(dom(f))$.

Let $\alpha < \mu$. If $c \in \tau_\alpha$ is a constant symbol, then $c^{\mathfrak{M}_*} := [f_c]$, where $m(f) = \alpha$ and $f_c(\alpha) = c^{\mathfrak{M}_\alpha}$.

If $R \in \tau_\alpha$ is a $k$-ary relation symbol and $f_1, \ldots, f_k \in \mathcal{F}$, then

$$R^{\mathfrak{M}_*}([f_1] \ldots [f_k]) :\Leftrightarrow R^{\mathfrak{M}_\gamma}(f_1(\gamma), \ldots f_k(\gamma)),$$

where $\gamma := max(\{m(f_1), \ldots, m(f_k), \alpha\})$.

If $G \in \tau_\alpha$ is a $k$-ary function symbol and $f_1, \ldots, f_k \in \mathcal{F}$, then again let $\gamma := max(\{m(f_1), \ldots, m(f_k), \alpha\})$ and

$$G^{\mathfrak{M}_*}([f_1], \ldots, [f_k]) := [h],$$

where $m(h) = \gamma$ and $h(\gamma) = G^{\mathfrak{M}_\gamma}(f_1(\gamma), \ldots, f_k(\gamma))$.

Using the fact that fragment embeddings fix atomic formulas and the commuting property, one can easily show that these interpretations are well defined.

For $\alpha < \mu$, let $\rho_{\alpha,*} : M_\alpha \mapsto M_*$ be defined by $\rho_{\alpha,*}(a) := [f]$, where $m(f) = \alpha$ and $f(\alpha) = a$. Then properties $(i)$ and $(ii)$ follow immediately.

$(iii)$ is proven via induction on formula complexity. The case for atomic formulas uses the fact, that atomic formulas are fixed by fragment embeddings. We only check the direction $(\Leftarrow)$ if $\psi$ is of the form $\exists y \phi(y, \overline{v})$.

Suppose $\alpha < \mu$, $\psi \in \mathbb{A}_\alpha$, $\overline{a}$ in $M_\alpha$ and $g \in \mathcal{F}$ such that

$$\mathfrak{M}_* \models \pi_{\alpha,*}(\phi)([g], \rho_{\alpha,*}(\overline{a})).$$

Let $\beta := m(g)$. If $\beta \leq \alpha$, then using the induction hypothesis, we have

$$\mathfrak{M}_\alpha \models \phi(g(\alpha), \overline{a}).$$

If $\alpha < \beta$, then using the commuting property and the induction hypothesis, we have

$$\mathfrak{M}_\beta \models \pi_{\alpha,\beta}(\phi)(g(\beta), \rho_{\alpha,\beta}(\overline{a})).$$

Since fragment embeddings respect quantifiers and by the definition of a twisted elementary system, it follows that

$$\mathfrak{M}_\alpha \models \exists y \phi(y, \overline{a}).$$

□

**Definition.** A twisted elementary system $(\mathfrak{M}_\alpha, \rho_{\alpha,\beta} : \alpha \leq \beta < \mu)$ is called atomic if each $\mathfrak{M}_\alpha$ is $\mathbb{A}_\alpha$-atomic and for all $\alpha \leq \beta < \mu$ and all $\psi(\overline{v}) \in \mathbb{A}_\alpha$,

$$\psi \text{ is complete over } Th(\mathfrak{M}_\alpha) \text{ iff } \pi_{\alpha,\beta}(\psi) \text{ is complete over } Th(\mathfrak{M}_\beta).$$

**Lemma 4.3.8.** *If $\mu < \omega_1$ and $(\mathfrak{M}_\alpha, \rho_{\alpha,\beta}, \alpha \leq \beta < \mu)$ is a twisted elementary atomic system, where all vocabularies, models and fragments are countable, then its limit $\mathfrak{M}_*$ is $\mathbb{A}^*$-atomic. Furthermore, for every $\alpha < \mu$ and every formula $\psi(\overline{v}) \in \mathbb{A}_\alpha$, we have $\psi(\overline{v})$ is complete over $Th_{\mathbb{A}_\alpha}(\mathfrak{M}_\alpha)$ iff $\pi_{\alpha,*}(\psi)(\overline{v})$ is complete over $Th_{\mathbb{A}^*}(\mathfrak{M}_*)$.*

*Proof.* Clearly, $\mathfrak{M}_*$ is countable. Let $\psi(\overline{v}) \in \mathbb{A}_\alpha$ be complete, for some $\alpha < \mu$, $\beta < \mu$ and $\chi(\overline{v}) \in \mathbb{A}_\beta$.

If $\beta \leq \alpha$ and $\mathfrak{M}_\alpha \models \forall \overline{v}(\psi(\overline{v}) \to \pi_{\beta,\alpha}(\chi)(\overline{v}))$, then by Lemma 4.3.7 and the commuting property, $\mathfrak{M}_* \models \forall \overline{v}(\pi_{\alpha,*}(\psi)(\overline{v}) \to \pi_{\beta,*}(\chi)(\overline{v}))$.

If $\beta > \alpha$, then $\pi_{\alpha,\beta}(\psi)$ is $\mathbb{A}_\beta$-complete. Suppose

$$\mathfrak{M}_\beta \models \forall \overline{v}(\pi_{\alpha,\beta}(\psi)(\overline{v}) \to \chi(\overline{v})).$$

Then as in the previous case, $\mathfrak{M}_* \models \forall \overline{v}(\pi_{\alpha,*}(\psi)(\overline{v}) \to \pi_{\beta,*}(\chi)(\overline{v}))$. Hence, $\pi_{\alpha,*}(\psi)(\overline{v})$ is $\mathbb{A}^*$-complete.

If for some $\alpha < \mu$ and $\psi \in \mathbb{A}_\alpha$, $\pi_{\alpha,*}(\psi)$ is complete in $\mathbb{A}^*$, then by the definition of a fragment embedding and Lemma 4.3.7, it easily follows that $\psi$ is $\mathbb{A}_\alpha$-complete.

It also follows from the previous Lemma, that every $k$-tuple of $M_*$ satisfies a $\mathbb{A}^*$-complete formula. □

Now we turn our attention back to the proof of Harrington's theorem. The main idea is to use a forcing notion $P$ which collapses $\aleph_1$ of the ground model. With the help of the forcing relation we will define a new tree of theories $\mathcal{T}_*$, called the generic Morley tree. This tree will have height $\omega_2$ in the ground model but in any $P$-generic extension $\mathcal{T}_*$ will be the Morley tree. Therefore, every theory of $\mathcal{T}_*$ is satisfiable in the gerneric extension but it turns out that it is also satisfiable in the ground model.

The forcing notion $P$ is the set of all finite functions $f$ with $dom(f) \subseteq \omega$ and $im(f) \subseteq \omega_1$, also notated by $Fn(\omega, \omega_1)$. $P$ is ordered by reverse inclusion, i.e. $p \preceq q :\Leftrightarrow q \subseteq p$.

$P$ has the $\omega_2$-c.c. that is all antichains $\subsetneqq P$ have cardinality less than $\aleph_2$, hence it preserves cardinalities greater or equal $\aleph_2$. It follows that if $\mathbb{V}$ is a transitive ground model of $ZFC$ and $G$ is a $P$-generic filter, then in the generic extension $\mathbb{V}[G]$ we have $\aleph_2^{\mathbb{V}} = \aleph_1^{\mathbb{V}[G]}$. For the rest of this subsection we assume $\phi$ to be a VCE of $\mathcal{L}_{\omega_1,\omega}(\tau)$.

**Definition.** Let $(P, R_1), (Q, R_2)$ be partial orders.

- A map $f : P \mapsto Q$ is called order preserving if for all $p, s \in P$, $pR_1s$ implies $f(p)R_2f(s)$. We say $f$ is an isomorphism of orders if it is bijective and both $f$ and $f^{-1}$ are order preserving. An (order) automorphism on $P$ is an isomorphism from $P$ onto itself.
- $(P, \prec)$ is called almost homogeneous if for all $p, q \in P$, there is an automorphism $f : P \mapsto P$ such that $f(p)$ and $q$ are compatible.

The following facts are due to [11] and are easy to prove.

**Fact 4.3.9.** $Fn(\omega, \omega_1)$ is almost homogeneous.

**Fact 4.3.10.** Let $(P, \prec, \mathbb{1})$ be an almost homogeneous partial order, $p \in P$, $\psi(v_1, \dots, v_k)$ be a formula and $a_1, \dots, a_k$ elements of the ground model $\mathbb{V}$. Then either $\mathbb{1} \Vdash \psi(\check{a}_1, \dots, \check{a}_k)$ or $\mathbb{1} \Vdash \neg\psi(\check{a}_1, \dots, \check{a}_k)$.

This enables us to prove

**Lemma 4.3.11.** *In the ground model $\mathbb{V}$, let $\mathbb{A}$ be a fragment of $\mathcal{L}_{\omega_2,\omega}(\tau)$ with cardinality less or equal $\aleph_1$, $\phi \in \mathbb{A}$ and $\dot{t}$ be a $P$-name .*

*(1) If $p \in P$ such that*

$$p \Vdash \text{"}\dot{t} \text{ is an } \check{\mathbb{A}}\text{-complete and satisfiable theory containing } \check{\phi}\text{"},$$

*then there is $q \preceq p$ in $P$ such that for all sentences $\psi \in \mathbb{A}$, either $q \Vdash \check{\psi} \in \dot{t}$ or $q \Vdash \check{\neg\psi} \in \dot{t}$. If $G \subseteq P$ is a $P$-generic filter and $p \in G$, then $\dot{t}_G$ is in $\mathbb{V}$.*

*(2) If $p \in P$ and $n < \omega$ such that*

$$p \Vdash \text{"}\dot{t} \text{ is a complete } n\text{-type of } \check{\mathbb{A}} \text{ containig } \check{\phi}\text{"},$$

*then there is $q \preceq p$ in $P$ such that for all $n$-formulas $\psi(\overline{v}) \in \mathbb{A}$, either $q \Vdash \check{\psi} \in \dot{t}$, or $q \Vdash \check{\neg\psi} \in \dot{t}$.If $G \subseteq P$ is a $P$-generic filter and $p \in G$, then $\dot{t}_G \in \mathbb{V}$.*

*Proof.* (1): The argument takes place in the ground model $\mathbb{V}$.

Suppose that no such $q \in P$ exists. There is a transitive model $\mathfrak{N} := \langle N, \dots \rangle$ of a large enough finite fragment of ZFC such that $|N| = \aleph_1$, $\mathbb{A}, P, \dot{t} \in N$ and the following holds in $\mathfrak{N}$:

- $p \Vdash$ "$\dot{t}$ is an $\check{\mathbb{A}}$-complete and satisfiable theory containing $\check{\phi}$".
- For all $q \preceq p$ in $P$, there is a sentence $\psi \in \mathbb{A}$ and $r \preceq q$ such that

$$r \not\Vdash \check{\psi} \in \dot{t} \text{ and } r \not\Vdash \check{\neg\psi} \in \dot{t}.$$

By the downward Löwenheim-Skolem theorem, there is a countable elementary submodel $\mathfrak{N}'$ of $\mathfrak{N}$, containing $\mathbb{A}, p, P, \dot{t}$ as elements. Now let $\mathcal{B}$ be the Mostowski collapse of $\mathfrak{N}'$ and $P', \mathbb{A}'$ be the image of $P$ and $\mathbb{A}$ under the collapsing isomorphism.

We have an enumeration $(D_n : n < \omega)$ of all dense subsets of $P'$ in $\mathbb{V}$. There is a binary tree $\mathcal{T}_p := \{p_s : s \in 2^{<\omega}\} \subseteq P$ such that

- $p_\emptyset \in D_0$ and $p_\emptyset \preceq p$.
- If $s \in^n 2$, then $p_{s^\frown 0}, p_{s^\frown 1} \in D_n$ and $\preceq p_s$, and there is a sentence $\psi_s \in \mathbb{A}'$ such that

$$p_{s^\frown 0} \Vdash \check{\neg\psi} \in \dot{t} \text{ and } p_{s^\frown 1} \Vdash \check{\psi} \in \dot{t}.$$

For every $f \in^n 2$, there is a $P'$-generic $G_f \subseteq P'$ containing $\{p_{f\upharpoonright n} : n < \omega\}$. Then $t_f := \dot{t}_{G_f}$ is an $\mathbb{A}'$-complete theory $\phi$, and there is $\mathfrak{M}_f \in \mathcal{B}[G_f]$ satisfying $t_f$. Since the satisfaction relation is absolute for transitive models of ZFC, $\mathfrak{M}_f$ is a model of $t_f$ in $\mathbb{V}$.

It easily follows from the construction of $\mathcal{T}_p$ that $\mathfrak{M}_f \ncong \mathfrak{M}_g$, for $f \neq g$ in ${}^n 2$, but then $\phi$ is not scattered, a contradiction.

Hence the set of all $q \in P$ such that $q \Vdash \check{\psi} \in \dot{t}$ or $q \Vdash \check{\neg\psi} \in \dot{t}$ is dense below $p$. If $G$ is $P$-generic and $p \in G$, then for some $q \in G$,

$$\dot{t}_G = \{\psi \in \mathbb{A} : \psi \text{ is a sentence and } q \Vdash \dot{\psi} \in \dot{\mathbb{A}}\},$$

which is in $\mathbb{V}$, since the forcing relation is definable in the ground model.

The argument for (2) is very similar. □

Now consider the chain of $\mathcal{L}_{\omega_2,\omega}(\tau)$-fragments $(\mathbb{A}_{*,\alpha} : \alpha < \omega_2)$ in $\mathbb{V}$ defined as follows:

- $\mathbb{A}_{*,0}$ is the smallest fragment containing $\phi$.
- If $\mathbb{A}_{*,\alpha} \subsetneqq \mathcal{L}_{\omega_2,\omega}$ is a fragment of cardinality less or equal $\aleph_1$, then $\mathbb{A}_{*,\alpha+1}$ is the smallest fragment containing $\mathbb{A}_\alpha$ and all formulas of the form

$$\bigwedge_{\psi \in t} \psi,$$

where for some $n < \omega$, $t \subseteq \mathbb{A}_{*,\alpha}$ is a set of $n$-formulas and for some $p \in P$

$$p \Vdash \text{"}\check{t} \text{ is a complete } \check{\mathbb{A}}_{*,\alpha}\text{-type containing } \check{\phi}\text{"}.$$

- At limit stages $\alpha > 0$, $\mathbb{A}_{*,\alpha} := \bigcup_{\beta<\alpha} \mathbb{A}_{*,\beta}$.

Notice that if $\alpha < \omega_2$ and $|\mathbb{A}_{*,\alpha}| \leq \aleph_1$, then $|\mathbb{A}_{*,\alpha+1}| \leq \aleph_1$. Otherwise there is $p \in P$ and a countable transitive model $\mathfrak{M}$ of ZFC believing that $\phi$ is scattered and for some $n < \omega$ and some fragment $\mathbb{A}$ of cardinality less or equal $\aleph_1$, there are at least $\aleph_2$ many sets $t$ of $n$-formulas of $\mathbb{A}$ such that

$$p \Vdash \text{"}\check{\phi} \text{ is scattered and } \check{t} \text{ is a complete } \check{\mathbb{A}}\text{-type containing } \check{\phi}\text{"}.$$

If $G \subseteq P$ is generic and $p \in G$, then in $\mathfrak{M}[G]$ there are at least $\aleph_1$ many complete $\mathbb{A}$-types containing $\phi$. This is a contradiction, since $\mathbb{A}$ is countable in $\mathfrak{M}[G]$ and by absoluteness, $\phi$ is scattered.

For $\alpha < \omega_2$, let $\mathcal{T}_{*,\alpha}$ be the set of all theories $t \subseteq \mathbb{A}_{*,\alpha}$ such that for some $p \in P$, $p$ forces the statement

$$\text{"}\check{t} \text{ is a satisfiable, complete theory} \subseteq \check{\mathbb{A}}_{*,\alpha},\ \check{\phi} \in \check{t} \text{ and for all } \beta < \check{\alpha},\ \check{t} \cap \mathbb{A}_{*,\beta} \text{ is not } \omega\text{-categorical."}$$

In this case we say $p$ witnesses $t \in \mathcal{T}_{*,\alpha}$.

It follows from the arguments above that for all $\alpha < \omega_2$, $\mathcal{T}_{*,\alpha} \neq \emptyset$, $|\mathcal{T}_{*,\alpha}| \leq \aleph_1$, and using fact 4.3.10 we get

**Proposition 4.3.12.** *For all $\alpha < \omega_2$, if $t \in \mathcal{T}_{*,\alpha}$, then this is witnessed by $\mathbb{1}$.*

In the ground model, the generic Morley tree contains the Morley tree.

**Proposition 4.3.13.** *If $\alpha < \omega_1$, then $\mathbb{A}_{*,\alpha} = \mathbb{A}_{\phi,\alpha}$ and $\mathcal{T}_{*,\alpha} = \mathcal{T}_\alpha$.*

*Proof.* Via induction on $\alpha < \omega_1$: $\mathbb{A}_{*,0} = \mathbb{A}_{\phi,0}$, by definition.

The case for limit stages is clear.

If $t \subseteq \mathbb{A}_{*,\alpha}$ and $p \in P$ forces $t$ to be a complete type, then this is also forced by $\mathbb{1}$. Since $\mathbb{A}_{*,\alpha}$ is countable, there is a countable transitive model $\mathcal{B} := \langle B, \dots \rangle$

of ZFC such that $\mathbb{A}_{*,\alpha}, t \in B$ and $\mathbb{1}$ forces $t$ to be a complete type of $\mathbb{A}_{*,\alpha}$ in $\mathcal{B}$. There is a $(Fn(\omega,\omega_1))^{\mathcal{B}}$- generic $G$ in $\mathbb{V}$ and in $\mathcal{B}[G]$ $t$ is a complete type of $\mathbb{A}_{*,\alpha}$. By absoluteness of the satisfaction relation, this is also true in the ground model, hence

$$\bigwedge_{\psi \in t} \psi \in \mathbb{A}_{\phi,\alpha+1}.$$

If $t \subseteq \mathbb{A}_{\phi,\alpha}$ is a complete type, then in $\mathbb{V}$ there is a countable transitive model of ZFC, where this is also true and therefore forced by $\mathbb{1}$. Hence,

$$\bigwedge_{\psi \in t} \psi \in \mathbb{A}_{*,\alpha}.$$

With an analogous arument one can show that for $\alpha < \omega_1$, $\mathcal{T}_{*,\alpha} = \mathcal{T}_\alpha$. □

We can now prove the crucial result for Harringtons theorem.

**Lemma 4.3.14.** *For all $\alpha < \omega_2$ and all $t \in \mathcal{T}_{*,\alpha}$, $t$ is satisfiable in $\mathbb{V}$.*

*Proof.* By proposition 4.3.13 we can assume $\alpha \geq \omega_1$. For the rest of the proof we denote the fragment simply as $\mathbb{A}$, because the index plays no role.

There is a transitive model $\mathcal{B} := \langle B, \dots \rangle$ of ZFC which has cardinality $\aleph_1$ and $P, t, \mathbb{A} \in B$. $\mathcal{B}$ is the limit of an elementary chain $(B_\beta : \beta < \omega_1)$ of countable submodels such that $P, t, \mathbb{A} \in B_0$. We define $C_\beta$ as the Mostowsky collapse of $B_\beta$, $g_\beta : B_\beta \mapsto C_\beta$ as the collapsing isomorphism, $t_\beta := g_\beta(t)$, $\mathfrak{A}_\beta := g_\beta(\mathbb{A})$ and for $\beta \leq \gamma < \omega_1$, $\pi_{\beta,\gamma} := (g_\gamma \circ g_\beta^{-1}) \restriction \mathfrak{A}_\beta$. Clearly, $im(\pi_{\beta,\gamma}) \subseteq \mathfrak{A}_\gamma$ and since the vocabulary $\tau$ is countable, we can assume that $\pi_{\beta,\gamma}$ is the identity on atomic formulas.

It is easy to check that $(\pi_{\beta,\gamma} : \beta \leq \gamma < \omega_1)$ is a commuting system of fragment embeddings with direct limit $(\mathbb{A}, \pi_{\beta,*} : \beta < \omega_1)$, where

$$\pi_{\beta,*} := g_\beta^{-1} \restriction \mathfrak{A}_\beta.$$

***Claim 1:*** For every $\beta < \omega_1$, $t_\beta$ is satisfiable.
(Proof of the claim): $C_\beta$ is countable and transitive, hence there is a generic $G_\beta \in \mathbb{V}$ such that $t_\beta$ is satisfiable in $C_\beta[G_\beta]$, and then by absoluteness, $t_\beta$ is satisfiable. (q.e.d.-Claim 1)

***Claim 2:*** For $\beta \leq \gamma < \omega_1$, $\psi(\overline{v}) \in \mathfrak{A}_\beta$ is complete over $t_\beta$ iff $\pi_{\beta,\gamma}(\psi)(\overline{v})$ is complete over $t_\gamma$.
(Proof of the claim): This follows from the fact that $B_\beta$ is an elementary submodel of $B_\gamma$: Let $k < \omega$ and $U$ be the set of all $k$-formulas of $\mathfrak{A}_\beta$. Then $U \in C_\beta$ and $\pi_{\beta,\gamma}(U)$ is the set of all $k$-formulas of $\mathfrak{A}_\gamma$. $\psi(\overline{v}) \in U$ is complete over $t_\beta$ if and only if for all $\delta(\overline{v}) \in U$, either

$$\forall \overline{v}[\psi(\overline{v}) \to \delta(\overline{v})]$$

is in $t_\beta$ or

$$\forall \overline{v}[\psi(\overline{v}) \to \neg\delta(\overline{v})]$$

is in $t_\beta$. Using $B_\beta \prec B_\gamma$, we see that this is true in $C_\beta$ if and only if for all $\delta(\overline{v}) \in \pi_{\beta,\gamma}(U)$, either

$$\forall \overline{v}[\pi_{\beta,\gamma}(\overline{v}) \to \delta(\overline{v})]$$

or

$$\forall \overline{v}[\pi_{\beta,\gamma}(\overline{v}) \to \neg\delta(\overline{v})]$$

is in $t_\gamma$.(q.e.d.-Claim 2)

In order to get a model for $t$ we inductively construct an atomic twisted elementary sysetm $(M_\beta, \rho_{\beta,\gamma}, \beta \leq \gamma < \omega_1)$: Suppose $\mu < \omega_1$ and we have an atomic twisted elementary system $(M_\beta, \rho_{\beta,\gamma} : \beta \leq \gamma \leq \mu)$, where each $M_\beta$ is a countable $\mathfrak{A}_\beta$-atomic model of $t_\beta$.

Let $M_{\mu+1}$ be a countable $\mathfrak{A}_{\mu+1}$-atomic model of $t_{\mu+1}$. Such a model exists, since $\phi$ is scattered, $\mathfrak{A}_\beta$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$ and $t_{\mu+1}$ is a complete theory containing $\phi$. Using claim 2, we can find a map $\rho_{\mu,\mu+1}$ from $M_\mu$ into $M_{\mu+1}$ analogously to the proof that atomic models are prime. Then for $\beta < \mu$, we define $\rho_{\beta,\mu+1} := \rho_{\mu,\mu+1} \circ \rho_{\beta,\mu}$, thereby extending the atomic twisted elementary system to $\mu + 1$.

If $\mu < \omega_1$ is a nonempty limit ordinal and $(M_\beta, \rho_{\beta,\gamma} : \beta \leq \gamma < \mu)$ is atomic, then we can simply use lemma 4.3.8 and extend the system to $\mu$.

Hence, we get an atomic twisted elementary system

$$(M_\beta, \rho_{\beta,\gamma} : \beta \leq \gamma < \omega_1).$$

Let $\mathfrak{M}_*$ the limit of this system and $\sigma \in t \cap B_\beta$, for some $\beta < \omega_1$. Then $M_\beta \models g_\beta(\sigma)$, hence

$$\mathfrak{M}_* \models \pi_{\beta,*}(g_\beta(\sigma))$$

but $\pi_{\beta,*}(g_\beta(\sigma)) = \sigma$. We have found a model of $t$ in $\mathbb{V}$. □

**Proposition 4.3.15.** *If $\alpha < \omega_2$ is a nonempty limit ordinal, $t \in \mathcal{T}_{*,\alpha}$ and $\mathfrak{M} \models t$, then $sr(\mathfrak{M}) \geq \alpha$.*

*Proof.* If not, then there is a countable transitive model $M$ of ZFC in which the proposition is false.

In $M$ there is a limit ordinal $\alpha < [\omega_2]^M$, $t \in [\mathcal{T}_{*,\alpha}]^M$ and a model $B$ of $t$ with $sr(B) < \alpha$. But by absoluteness, for every $P$-generic $G$, we have

$$M[G] \models \text{"}B \text{ is a model of } t\text{"}$$

and $sr(B)^M = sr(B)^{M[G]}$. Since the generic Morley tree of $M$ is the Morley tree in $M[G]$ and $\alpha < \omega_1^{M[G]}$, we get a contradiction to lemma 4.3.5. □

**Remark.** If $\phi$ is a VCE, then one could continue the construction of the generic Morley tree beyond $\aleph_2$ in hope of showing the existence of a model of cardinality $\aleph_2$ but then it is unclear whether claim 1 of lemma 4.3.14 would still hold. If $\beta < \omega_2$, then $C_\beta$ can be uncountable in the ground model and we can no longer guarantee the existence of a generic $G_\beta \in \mathbb{V}$.

This completes the proof of Harrington's theorem. An immediate corollary is that every VCE has at least $\aleph_2$ many isomorphism types of cardinality $\aleph_1$. It is not yet known whether $I(\phi, \aleph_1) = 2^{\aleph_1}$, for a VCE $\phi$.

**Proposition 4.3.16.** *If $\phi$ is a VCE, then there are $\aleph_2$ many $\equiv_{\infty,\omega}$-classes of models of cardinality $\aleph_1$.*

*Proof.* Two models with different Scott rank cannot be infinitarily equivalent, hence by Harrington's result, for every VCE $\phi$, there are at least $\aleph_2$ many $\equiv_{\infty,\omega}$-classes of models of cardinality $\aleph_1$.

If for some $\alpha < \omega_2$, there are $\aleph_3$ many classes, then this is true in some countable transitive model $M$ of ZFC.

Let $G$ be $P$-generic. By the absoluteness of the satisfaction relation and since $\phi$ is still scattered in $M[G]$, this gives us some $\beta < \omega_1^{M[G]}$ such that there are uncountably many $\equiv_{\beta,\omega}$ equivalence classes of countable models of $\phi$, a contradiction. □

# 5 Structure Interpretations

In this section we show that it is enough to show $VC_3$ for certain languages.

Subsection 0 and subsection 1 are based on chapter 5.5 of [9].

**Definition.** (1) Let $\mathfrak{M} := \langle M, \dots \rangle$ be a $\tau$-structure and $A \subseteq M$. A subset $X$ of $M^k$ is called (first order) definable over $A$ if for some formula $\psi(v_1, \dots, v_k, w_1, \dots, w_l)$ of $\mathcal{L}(\tau)$ and some $\overline{a} \in A^l$,

$$X = \{\overline{m} \in M^k : \mathfrak{M} \models \psi(\overline{m}, \overline{a})\}.$$

If $A = \emptyset$, then $X$ is simply called definable.

(2) Let $\tau_1, \tau_2$ be vocabularies and $\mathfrak{N} := \langle N, \dots \rangle$ a $\tau_2$ model. Suppose that for some $k \in \omega_+$, we have a set $I$ consisting of the following first order $\tau_2$-formulas:

(I1) One $k$-formula $\psi_D(v_1, \dots, v_k)$ such that $D$, the subset of $N^k$ defined by this formula, is non empty.

(I2) One $2k$-formula $\psi_E(v_1, \dots, v_{2k})$ defining an equivalence relation on $D$.

(I3) For every constant symbol $d \in \tau_1$, a $k$-formula $\psi_d(v_1, \dots, v_k)$ defining an equivalence class of $D/E$.

(I4) For every $l$-ary relation symbol $R \in \tau_1$, a $kl$-formula $\psi_R(\overline{w}_1, \dots, \overline{w}_l)$ defining an $E$ invariant relation on $D^l$.

(I5) For every $l$-ary function symbol $F \in \tau_1$, a $k(l+1)$-formula $\psi_F(\overline{w}_1, \dots, \overline{w}_{l+1})$ defining an $E$ invariant relation on $D^{l+1}$ such that

$$\{([\overline{a}_1], \dots, [\overline{a}_{l+1}]) \in (D/E)^{l+1} : \mathfrak{N} \models \psi_F(\overline{a}_1, \dots, \overline{a}_{l+1})\}$$

is a function from $(D/E)^l$ into $D/E$.

Then we call $I$ a set of $\tau_1$-interpretation formulas and define the $\tau_1$-structure $\mathfrak{N}_I$ as follows:

- The universe is $D/E$.
- A constant symbol $d \in \tau_1$ is interpreted with the $E$ equivalence class defined by $\psi_d$.
- A $l$-ary relation symbol $R \in \tau_1$ is interpreted as

$$\{([\overline{a}_1], \dots, [\overline{a}_l]) \in (D/E)^l : \mathfrak{N} \models \psi_R(\overline{a}_1, \dots, \overline{a}_l)\}.$$

– A $l$-ary function symbol $F \in \tau_1$ is interpreted with the function described in (I5).

If a $\tau_1$ model $\mathfrak{M} := \langle M, \dots \rangle$ is isomorphic to $\mathfrak{N}_I$, then $\mathfrak{N}_I$ is called a $k$ dimensional interpretation of $\mathfrak{M}$ in $\mathfrak{N}$.

**Remark.** If the equivalence relation is the identity, then we identify $D/E$ with $D$.

**Proposition 5.0.1.** *Let $\mathfrak{N}$ be a $\tau_2$ model and $\mathfrak{N}_I$ be a $k$ dimensional $\tau_1$-interpretation. Then there is a map $\pi : \mathcal{L}_{\infty,\omega}(\tau_1) \mapsto \mathcal{L}_{\infty,\omega}(\tau_2)$ such that if $l < \omega$ and $\psi \in \mathcal{L}_{\infty,\omega}(\tau_1)$ is a $l$-formula, then $\pi(\psi)$ is a $kl$-formula, and for all $[\overline{a}_1], \dots, [\overline{a}_l] \in D/E$,*

$$\mathfrak{N}_I \models \psi([\overline{a}_1], \dots, [\overline{a_l}]) \textit{ iff } \mathfrak{N} \models \pi(\psi)(\overline{a}_1, \dots, \overline{a}_l).$$

*Proof.* (Sketch) Using the interpretation formulas of $I$ one can define a map $H$ from all $\tau_1$ terms into the set of all $\tau_2$ first order formulas, such that:

- If $t$ has $l$ variables, then $H(t)$ has $k(l+1)$ free variables.
- If $t(v_1, \dots, v_l)$ is a $\tau_1$-term, then for all $\overline{a}_1, \dots, \overline{a}_l, \overline{b} \in D$,

$$t^{\mathfrak{N}_I}([\overline{a}_1], \dots, [\overline{a}_l]) = [\overline{b}] \text{ iff } \mathfrak{N} \models H(t)(\overline{a}_1, \dots, \overline{a}_l, \overline{b}).$$

If $d \in \tau_1$ is a constant symbol, then $H(d) := \psi_d(\overline{v})$, for a variable $y$,

$$H(y) := \psi_E(\overline{v}_1, \overline{v}_2),$$

and if $f \in \tau_1$ is a $l$-ary function symbol and $t_1, \dots, t_l$ are $\tau_1$-terms with variables among $v_1, \dots, v_m$, then

$$H(f(t_1, \dots, t_l)) := \exists \overline{u}_1, \dots, \exists \overline{u}_l [(\bigwedge_{i=1}^{l} H(t_i)(\overline{w}, \overline{u}_i)) \wedge (\psi_f(\overline{u}_1, \dots, \overline{u}_l, \overline{s}))],$$

where $\overline{w} = (w_1, \dots, w_{km})$. Using $H$, one then defines $\pi$ via induction on formula complexity and proves the proposition simultaneously. □

**Remark.** The map $\pi$ of the previous proof does not depend on the models $\mathfrak{N}_I$ and $\mathfrak{N}$ but only on the $\tau_1$-interpretation formulas. Furthermore, it maps $\tau_1$-first order formulas onto $\tau_2$-first order formulas.

## 5.1 In Infinitary Logic

For a countable vocabulary $\tau$, recall the definitions of the space of countable $\tau$ models, $\mathcal{X}_\tau$, and of $Mod(\phi)$, where $\phi$ is a sentence of $\mathcal{L}_{\omega_1,\omega}(\tau)$ (See subsection 3.2).

Now suppose that for some $k \in \omega_+$, we have a set $I$ of $\tau_2$ formulas of the form $(I1) - (I5)$. If $\tau_1$ is countable, there is a $\mathcal{L}_{\omega_1,\omega}(\tau_2)$-sentence $\sigma_I$ stating:

- $\exists \overline{v} \psi_D(\overline{v})$.
- $\psi_E$ defines an equivalence relation $E$ on $D$, the set defined by $\psi_D$.

- For every constant symbol $d \in \tau_1$ $\psi_d(\overline{v})$ defines an $E$ equivalence class in $D$.
- For every $l$-ary relation or function symbol $s \in \tau_1$, $\psi_s$ is $E$-invariant, and if $s$ is a function symbol, then $\psi_s$ induces a function on $D/E$.

For all models $\mathfrak{N}$ of $\sigma_I$, $\mathfrak{N}_I$ is a $k$ dimensional $\tau_1$-interpretation.

Suppose that for every model $\mathfrak{M} \in \mathcal{X}_{\tau_1}$ there is $\mathfrak{N} \in Mod(\sigma_I)$ such that $\mathfrak{M} \cong \mathfrak{N}_I$. Then by proposition 5.0.1 for every sentence $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau_1)$ there is a sentence $\phi_* \in \mathcal{L}_{\omega_1,\omega}(\tau_2)$ such that

$$Mod(\phi_*) = \{\mathfrak{N} \in Mod(\sigma_I) : \mathfrak{N}_I \in Mod(\phi)\}.$$

Since for all $\mathcal{A}, \mathcal{B} \in Mod(\sigma_I)$, $\mathcal{A} \cong \mathcal{B}$ implies $\mathcal{A}_I \cong \mathcal{B}_I$, we have

$$I(\phi, \aleph_0) \leq I(\phi_*, \aleph_0).$$

It can happen that $I(\phi, \aleph_0) < I(\phi_*, \aleph_0)$, for example let $\tau_1$ be a vocabulary consisting of $\aleph_0$ many constant symbols and $\tau_2 := \tau_1 \cup \{R\}$, where $R$ is an unary relation symbol. For the set $I$ of $\tau_1$-interpretation formulas, we simply choose $D$ to be the entire uiverse $E$ the identity and for every constant symbol $c \in \tau_1$, $\psi_c(v) := (v = c)$. Clearly, every $\tau_1$-model is the reduct of a $\tau_2$-model and therefore isomorphic to a 1-dimensional $\tau_1$-interpretation $\mathfrak{N}_I$, where $\mathfrak{N}$ is a model of $\sigma_I$.

Let

$$\phi := \bigwedge_{i \neq j} (c_i \neq c_j).$$

Then one can show $I(\phi, \aleph_0) = \aleph_0$, but $I(\phi_*, \aleph_0) = \mathfrak{c}$. If instead of $Mod(\sigma_I)$ we consider $Mod(\rho)$, where $\rho := \forall x R(x)$, then we can still interpret every elment of $Mod(\phi)$ in some element of $Mod(\rho)$ but now we have for all $\mathcal{A}, \mathcal{B} \in Mod(\rho)$

$$\mathcal{A} \cong \mathcal{B} \text{ iff } \mathcal{A}_I \cong \mathcal{B}_I,$$

thus, if we define $\phi' := \rho \wedge \phi_*$, then $I(\phi, \aleph_0) = I(\phi', \aleph_0)$.

This illustrates the main focus of this section: We show that if $\tau_2$ is one of certain finite vocabularies, then every $\mathcal{L}_{\omega_1,\omega}(\tau_1)$-sentence of an arbitrary countable vocabulary $\tau_1$ can be interpreted in the language $\mathcal{L}_{\omega_1,\omega}(\tau_2)$ in the sense that there is a $\mathcal{L}_{\omega_1,\omega}(\tau_2)$-sentence $\sigma_2$ with following properties:

(A) For every $\mathfrak{M} \in Mod(\sigma_1)$, there is $\mathfrak{N} \in Mod(\sigma_2)$ in which $\mathfrak{M}$ can be interpreted.

(B) For every $\mathfrak{N} \in Mod(\sigma_2)$, there is $\mathfrak{M} \in Mod(\sigma_1)$ which can be interpreted in $\mathfrak{N}$.

(C) $I(\sigma_1, \aleph_0) = I(\sigma_2, \aleph_0)$.

If we can show this, then clearly $VC_3$ for $\mathcal{L}_{\omega_1,\omega}(\tau_2)$-sentences implies the general version.

**Proposition 5.1.1.** *Let $\tau_2$ be a countable relational vocabulary with symbols of unbounded arity. Then for every countable vocabulary $\tau_1$, there is a $\mathcal{L}_{\omega_1,\omega}(\tau_2)$ sentence $\sigma$ and a homeomorphism*

$$H : \mathcal{X}_{\tau_1} \mapsto Mod(\sigma)$$

*such that for all $\mathcal{A}, \mathcal{B} \in \mathcal{X}_{\tau_2}$, $\mathcal{A} \cong \mathcal{B}$ iff $H(\mathcal{A}) \cong H(\mathcal{B})$.*

*Proof.* Let $s \mapsto R^s$ be an injective map from $\tau_1$ into $\tau_2$ such that $R^s$ has at least the same arity as $s$ and if $s$ is a $l$-ary function symbol, then $R^s$ is at least $l+1$-ary.

Given an arbitrary $\tau_1$-model $\mathfrak{M}$, it is not difficult to find a $\tau_2$-model $\mathfrak{N}$ in which it can be interpreted:

- The universe of $\mathfrak{N}$ is $M$.
- If $d \in \tau_1$ is a constant symbol and $R^d$ is $n$-ary, then $R^d$ is interpreted as $\{(d^{\mathfrak{M}}, \ldots, d^{\mathfrak{M}})\} \subseteq M^n$.
- If $s \in \tau_1$ is a $l$-ary relation or function symbol and $R^s$ is $n$-ary, then $R^s$ is interpreted such that the projection on $M^l$ or $M^{l+1}$ is identical with $s^{\mathfrak{M}}$.

This observation motivates the choice of the following $\mathcal{L}_{\omega_1,\omega}(\tau_2)$-sentences:

- If $c \in \tau_1$ is a constant symbol and $R^c$ is $n$-ary, then $\sigma_c$ states that $R^c$ has exactly one element which is in the diagonal of the universe, i.e. all of its coordinates are equal.
- If $r \in \tau_1$ is a $l$-ary relation symbol and $R^r$ is $n$ ary, then $\sigma_r$ states that if some $n$-tuple $\overline{v}$ is in $R^r$, then all $n$-tuples with the same first $l$ coordinates as $\overline{v}$ are in $R^r$.
- If $f \in \tau_1$ is a $l$-ary function symbol and $R^f$ is $n$-ary, then

  $$\phi_1 := \forall v_1, \ldots, \forall v_l \exists v_{l+1}(R^f(v_1, \ldots, v_l, v_{l+1}, \ldots, v_{l+1})),$$

  and the sentence $\phi_2$ states that for all $l$-tuples $\overline{v}$, there is at most one $(n-l)$-tuple $\overline{w}$ such that $(\overline{v}, \overline{w})$ is in $R^f$. Then set $\sigma_f := \phi_1 \wedge \phi_2$.
- Let $T$ be the set of all symbols $R \in \tau_2$ which are not of the form $R^s$, for some $s \in \tau_1$. Then for $R \in T$ set $\sigma_R := \forall \overline{v} R(\overline{v})$.

Now let $\psi_D(v_1) := (v_1 = v_1)$ and $\psi_E(v_1, v_2) := (v_1 = v_2)$. For a constant symbol $c \in \tau_1$, we choose $\psi_c(v_1) := R^c(v_1, \ldots, v_1)$, for a $l$-ary relation symbol$r \in \tau_1$, we define $\psi_r(v_1, \ldots, v_l) := R^r(v_1, \ldots, v_l, v_l, \ldots, v_l)$, and if $f \in \tau_1$ is a $l$-ary function symbol set $\psi_f(v_1, \ldots, v_{l+1}) := R^f(v_1, \ldots, v_{l+1}, v_{l+1}, \ldots, v_{l+1})$. This gives us the set $I$ of $\tau_1$-interpretation formulas.

Then for

$$\sigma := (\bigwedge_{s \in \tau_1} \sigma_s) \wedge (\bigwedge_{R \in T} \sigma_R),$$

we see that every $\mathcal{A} \in \mathcal{X}_{\tau_1}$ is of the form $\mathfrak{N}_I$, for a uniquely determined $\mathfrak{N}$ in $Mod(\sigma)$. It is easily checked that the map $H : \mathcal{X}_{\tau_1} \mapsto Mod(\sigma)$,

$$\mathcal{A} \mapsto H(\mathcal{A})$$

such that $H(\mathcal{A})_I = \mathcal{A}$ is a homeomorphism and respects the isomorphism relation in both directions. □

**Corollary 5.1.2.** *Let $\tau_2$ be a relational vocabulary with symbols of unbounded arity. Then $VC_3$ is true if and only if $VC_3$ is true for $\mathcal{L}_{\omega_1,\omega}(\tau_2)$-sentences.*

*Proof.* The direction (⇒) is clear.

(⇐): If $\tau_1$ is an arbitrary countable vocabulary and $\phi$ a $\mathcal{L}_{\omega_1,\omega}(\tau_1)$-sentence, then by the proof of the previous result, every $\mathcal{A} \in Mod(\phi)$ can be interpreted in $H(\mathcal{A})$.

Using proposition 5.0.1, we see, that the image of $H \upharpoonright Mod(\phi)$ is of the form $Mod(\sigma_*)$, for some $\sigma_* \in \mathcal{L}_{\omega_1,\omega}(\tau_2)$. Thus, if $\sigma_*$ has perfectly many models, then so does $\phi$. □

**Definition.**

- Let $\tau_g$ be the vocabulary with a single binary relation symbol $\dot{E}$. An undirected graph is a $\tau_g$-structure $\mathcal{V} := (V, \dot{E}^{\mathcal{V}})$ in which $\dot{E}$ is interpreted as a symmetric binary relation.
- If $(V, E)$ is an undirected graph and $x \in V$, then the valency of $x$ is defined as
$$val(x) := |\{b \in V : xEb\}|.$$
If $\kappa = val(x)$, then we call $x$ a $\kappa$-point.
- Let $(V, E)$ be an undirected graph, $n \in \omega_+$ and $p \in V$. We say $p$ has a tail of length $n$, if there is an injective sequence $(x_i : 1 \leq i \leq n)$ in $V \setminus \{p\}$ such that $pEx_1$, $x_iEx_{i+1}$, for $1 \leq i < n$, $val(x_i) = 2$, for $1 \leq i < n$, and $val(x_n) = 1$. We say $p$ has an infinite tail, if there is sequence $(x_i : i \in \omega_+)$ as above with the exception that no point of this sequence has valency 1.

All graphs considered in this thesis are undirected.

**Theorem 5.1.3.** *Let $\tau$ be an arbitrary countable vocabulary and $\phi$ a sentence of $\mathcal{L}_{\omega_1,\omega}(\tau)$. There is a $\mathcal{L}_{\omega_1,\omega}(\tau_g)$-sentence $\sigma$ such that:*

*(i) Every model of $\sigma$ is a graph and every model of $\phi$ can be interpreted in a model of $\sigma$.*

*(ii) $I(\phi, \aleph_0) = I(\sigma, \aleph_0)$.*

*Proof.* By proposition 5.0.1, it suffices to show this for the case that $\tau$ is a relational vocabulary with exactly one $k$-ary relation symbol, for every $k \in \omega_+$.

Let $(p_k : k \in \omega_+)$ be an enumeration of all prime numbers $\geq 5$. We code a $\tau$-structure $\mathfrak{M} := \langle M, \dots \rangle$ in a graph $G$ as follows:

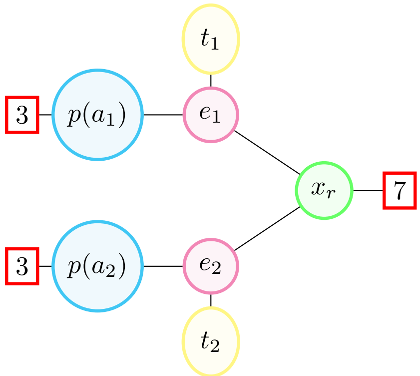
Every element $a \in M$ is identified with a point $p(a) \in G$, which has exactly 3 points of valency 1 connected to it. We call the subgraph of $G$ defined by these four points a $D$-code, because this will be expressed by the interpretation formula $\psi_D$.

Suppose $R_k \in \tau$ is $k$-ary and $\overline{a} \in R_k^{\mathfrak{M}}$. This will be coded by a finite subgraph $G_{\overline{a}} \subseteq G$, called a $R_k$-code:

- The $D$-code containing $p(a_i)$ is a subgraph of $G_{\overline{a}}$, for $1 \leq i \leq k$.
- There is exactly one point $x_r \in G_{\overline{a}}$, which is connected to exactly $p_k$ many points with valency 1 which also belong to $G_{\overline{a}}$.
- If $1 \leq i \leq k$, then $p(a_i)$ is connected to a point $e_i \in G_{\overline{a}}$ of valency 3 which is in turn connected to $x_r$ and has a tail $t(e_i) \subseteq G_{\overline{a}}$ of length $i + 1$. For $i \neq j$, $e_i \neq e_j$ and $t(e_i)$, $t(e_j)$ are disjoint.

- Apart from $p_k$ many points of valency 1, $x_r$ is only connected to $e_i$, for $1 \leq i \leq k$, hence $val(x_r) = k + p_k$.
- For $1 \leq i \leq j \leq k$, $p(a_i)$ is not connected to $p(a_j)$.
- $G_{\overline{a}}$ contains no other points than the ones described above.

Let us look at one illustration of such a code graph: Suppose $r$ is a binary relation symbol, $a_1, a_2 \in M$ and $(a_1, a_2) \in r^{\mathfrak{M}}$. Then $G_{(a_1,a_2)}$ looks like this:




Here, the red squares with number 3 symbolise 3 points of valency 1 which are connected to $p(a_1)$ and $p(a_2)$.

Seven points of valency 1 are connected to $x_r$, which is symbolised by the red square with number 7.

$t_1$ symbolises a tail of length 2 connected to $e_1$ and $t_2$ a tail of length 3 connected to $e_2$.

We can easily define such a code graph with a first order $\tau_g$-formula.

The $\tau_g$-formula $\psi_D(v_1)$ says that $v_1$ is connected to 3 distinct points of valency 1. This formula defines the set $D$.

Let $k \in \omega_+$ and $\psi_k(v_1, \ldots, v_k)$ be the $\tau_g$-formula saying that $v_i \in D$, for $1 \leq i \leq k$, and $v_1, \ldots, v_k$ belong to a $R_k$-code, where $v_i E e_i$.

The equivalence relation on $D$ is the identity. We now have the set $I$ of $\tau$-interpretation formulas.

The sentence $\sigma_0 \in \mathcal{L}_{\omega_1,\omega}(\tau_g)$ states the following:

(1) $E$ is a symmetric relation and for all $(x, y) \in D \times D$, $\neg(xEy)$.

(2) Every element belongs to a $D$-code or to a $R_k$-code, for some $k < \omega_+$.

(3) For all $k < \omega_+$ and for all $(v_1, \ldots, v_k) \in D^k$, there is at most one $R_k$-code of the form $G_{(v_1,\ldots,v_k)}$.

Notice that, since we are working in infinitary logic, there is such a sentence.

It is easy to see that for every $\mathfrak{M} \in \mathcal{X}_\tau$, there is a $\mathfrak{N} \in Mod(\sigma_0)$ such that $\mathfrak{M} \cong \mathfrak{N}_I$. Furthermore, if $\mathcal{A}, \mathcal{B} \in Mod(\sigma_0)$ and $\mathcal{A}_I \cong \mathcal{B}_I$, then $\mathcal{A} \cong \mathcal{B}$.

Now we can use proposition 5.0.1 and conclude that there is a $\tau_g$-sentence $\sigma$ such that $\sigma \models \sigma_0$, every $\mathfrak{M} \in Mod(\phi)$ is isomorphic to some $\mathfrak{N}_I$, where $\mathfrak{N} \in Mod(\sigma)$, and for all $\mathfrak{N} \in Mod(\sigma)$, $\mathfrak{N}_I \cong \mathfrak{M}$, for some $\mathfrak{M} \in Mod(\phi)$. This completes the proof. □

**Corollary 5.1.4.** *$VC_3$ is true if and only if it holds for every sentence of $\mathcal{L}_{\omega_1,\omega}(\tau_g)$.*

## 5.2 In First Order Logic

This subsection is based on [23]. For a first order theory $T$ in the countable vocabulary $\tau$, we write $Mod(T)$ for $Mod(\bigwedge_{\sigma \in T} \sigma)$.

We return to Vaught's conjecture for first order theories. Since we only considererd first order interpretation formulas, the definition of an interpretation and the notion $\mathfrak{N}_I$, where $\mathfrak{N}$ is a $\tau_2$-model and $I$ a set of $\tau_1$ interpretation formulas, are the same as for infinitary logic.

Proposition 5.0.1 also holds for first order logic as does proposition 5.1.1 with the difference that instead of an infinitary sentence we get a first order theory in the relational language with the specific properties.

We consider the following stronger version of $VC_1$:

$VC_3$(FO): If $T$ is a complete first order theory of a countable vocabulary with infinite models then either $I(T, \aleph_0) \leq \aleph_0$ or else there is a perfect set of pairwise nonisomorphic models in $Mod(T)$.

We also have

**Proposition 5.2.1.** *If $\tau_2$ is a relational vocabulary with symbols of unbounded arity then $VC_3$(FO) is true if and only if it is true for $\tau_2$ theories.*

The proof is similar to that of corollary 5.1.2. If $\tau_1$ is an arbitrary countable vocabulary, then a complete first order $\tau_1$ theory $T$ can be interpreted in a complete first order $\tau_2$ theory $T^*$ such that there is a homeomorphism from $Mod(T)$ onto $Mod(T^*)$.

Next, we want to show that $VC_3$(FO) for theories of graphs is equivalent to the general version. This turns out to be a bit more complicated, because we cannot simply use the coding method used in the proof of theorem 5.1.3. The main problem with this approach is that in general we cannot express in a first order theory that every element of the graph either belongs to a $D$-code or to a $R$-code, where $R$ is a relation symbol. In infinitary logic we can do this by using infinite conjunctions and disjunctions.

Given an arbitrary countable relational vocabulary $\tau$ and a complete $\tau$ theory $T$, we could use the set $I$ of $\tau$ intrepretation formulas from 5.1.3 and get a theory $T^*$ of the graph vocabulary $\tau_g$ such that every element of $Mod(T)$ is isomorphic to some $\mathfrak{M}_I$, where $\mathfrak{M} \in Mod(T^*)$ but we cannot guarantee that $\mathcal{A}_I \cong \mathcal{B}_I$ implies $\mathcal{A} \cong \mathcal{B}$, for $\mathcal{A}, \mathcal{B} \in Mod(T^*)$. It can be the case that $I(T, \aleph_0) \leq \aleph_0$ but $I(T^*, \aleph_0) = \mathfrak{c}$. $T^*$ in general will not be complete but how can we find a complete extension of it without increasing the number of its countable isomorphism types?

Therefore, we choose a slightly different technique of coding which gives us the following

**Theorem 5.2.2.** *Let $\tau_g$ be the vocabulary of graphs and $\tau$ be an arbirtary countable vocabulary. There is a $\tau_g$-theory $T_*$ and a set of $\tau$ interpretation formulas $I$ such that the following hold:*

*(i) Every $\mathfrak{M} \in \mathcal{X}_\tau$ is isomorphic to $\mathfrak{N}_I$, for some $\mathfrak{N} \in Mod(T_*)$.*

*(ii) For every $\mathfrak{N} \in Mod(T_*)$, the set*

$$\{\mathcal{A} \in Mod(T_*) : \mathcal{A}_I \cong \mathfrak{N}_I\}$$

*has $\aleph_0$ many isomorphism types.*

*Proof.* Let $R$ be the binary relation symbol of $\tau_g$. Again, by proposition 5.1.1, it suffices to show the theorem for the case that $\tau$ is a relational vocabulary with exactly one $k$-ary relation symbol, for every $k \in \omega_+$.

If $G$ is a graph and $x \in G$, then we call $x$ a P-point if $val(x) \geq 4$.

Fix a function $t : \omega_+ \mapsto \bigcup_{k\in\omega_+} \omega^k$, $k \mapsto t_k$, with the following properties:

- $t_k = (a_1, \ldots, a_{k+1}) \in \omega^{k+1}$ and $2 < a_i < a_j$, for $1 \leq i < j \leq k+1$.
- If $k_1 < k_2$, then no element of $\omega$ occurs both in $t_{k_1}$ and in $t_{k_2}$.
- Every natural number $> 2$ occurs in some $t_k$.

First, we give a general idea of how we code a $\tau$-model $\mathfrak{M}$ in a graph $G$:

- Every element $x$ of $\mathfrak{M}$ is identified with a P-point $p_x \in G$ which has exactly one tail with length 2.
- If $S_k$ is the $k$-ary relation symbol of $\tau$, $t_k = (a_1, \ldots, a_{k+1})$ and

  $$(x_1, \ldots, x_k) \in S_k^{\mathfrak{M}},$$

  then there is a P-point $u \in G$ which has exactly one tail of length $a_{k+1}$ and for $1 \leq i \leq k$, $p_{x_i}$ is connected with a 3-point $e_i$ which has a tail of length $a_i$ and is also connected to $u$. We call the subgraph consisting of $\{p_{x_i} : 1 \leq i \leq k\} \cup \{e_i : 1 \leq i \leq k\} \cup \{u\}$ together with the tails mentioned above a $S_k$-code.

We now make this idea precise by formulating the theory $T_*$ which states:

(1) There is no isolated point and $\forall x \neg(xRx)$.

(2) Every 1-point is connected to a 2-point.

(3) Every 3-point is connected to two P-points and one 2-point.

(4) For every natural number $n \geq 5$, $T_*$ contains the sentence stating

  " Every P-point has valency $\geq n$ and there are at least $n$ P-points with a tail of length 2".

(5) Every P-point is connected to exactly one 2-point and otherwise only 3-points.

(6) If $k \in \omega_+$ and $t_k = (a_1, \ldots, a_{k+1})$, then $T_*$ has the sentence stating

  "For all P-points $x_1, \ldots, x_k$ with a tail of length 2 , there is at most one $S_k$-code containing $x_1, \ldots, x_k$ such that for $1 \leq i \leq k$, $x_i$ is conneceted to the 3-point $e_i$ which has a tail of length $a_i$."

(7) If $k, t_k$ are as in (6), then $T_*$ contains the sentence stating

  "For every P-point $u$ with a tail of length $a_{k+1}$, there is a unique $S_k$-code containing $u$ and every 3-point with a tail of length $a_i$ belongs to a unique $S_k$ code."

(8) For $k \in \omega_+$, $t_k = (a_1, \ldots, a_{k+1})$ and $1 \leq i \leq k$,

"There is no P-point with a tail of length $a_i$ and there is no 3-point with a tail of length $a_{k+1}$ or of length 2".

(9) For every natural number $n > 1$, $T_*$ states that if $x, y$ are two points of valency $\geq 3$ or if both have valency 1, then there is no path of length $n$ from $x$ to $y$ consisting of 2-points.

(10) For every $n > 2$, there are no circles consisting of $n$ 2-points.

(11) For every $n \in \omega_+$, $T_*$ states:

"If $x, y$ are two P-points then there are $n$ 3-points with a tail of length at least $n$ each of which is connected to both $x$ and $y$".

Let $\psi_E(v_1, v_2) := (v_1 = v_2)$, $\psi_D(v_1)$ be the $\tau_g$ formula stating that $v_1$ is a P-point with a tail of length 2 and for $k \in \omega_+$ with $t_k = (a_1, \ldots, a_{k+1})$, let $\psi_k(v_1, \ldots, v_k)$ state that $v_1, \ldots, v_k$ are P-points with a tail of length 2 and belong to a $S_k$-code such that for $1 \leq i \leq k$, $v_i$ is connected to the 3-point $e_i$ which has a tail of length $a_i$. This defines the set $I$ of $\tau$-interpretation formulas.

It is easily checked that $T_*$ is satifiable and for every $\mathcal{A} \in Mod(T_*)$, $\mathcal{A}_I \in \mathcal{X}_\tau$. Furthermore, for every $\mathfrak{M} \in \mathcal{X}_\tau$, there is $\mathcal{A} \in Mod(T_*)$ such that $\mathcal{A}_I \cong \mathfrak{M}$, which completes the proof of (i).

Suppose $\mathcal{A} \in Mod(T_*)$ and $g \in \omega$. If $val(g) \leq 2$, then $g$ belongs to a tail which is either finite or infinite. If the tail is finite then it leads to a 3- or a P-point.

If the tail is infinite, then it either belongs to a point of valency $\geq 3$ or it is a tail with an initial 1-point but not leading to a 3- or P-point or the tail is a connected subgraph consisting of 2-points without initial points.

Otherwise $g$ is either a 3- or a P-point with a finite or infinite tail.

Define $Y(\mathcal{A})$ as the set of all P-points of $\mathcal{A}$ with an infinite tail, $H_1(\mathcal{A})$ as the set of all inifinite tails in $\mathcal{A}$ with an initial 1-point and $H_2(\mathcal{A})$ as the set of all infinite connected subgraphs of $\mathcal{A}$ consisting of 2-points. If $Y(\mathcal{A}), H_1(\mathcal{A}), H_2(\mathcal{A})$ are empty, then we call $\mathcal{A}$ a root model of $T_*$.

Let $M(\mathcal{A})$ be the submodel of $\mathcal{A}$ consisting of all elements $x$ with one of the following properties:

(A) $x$ is a P-point with a finite tail or $x$ is a 3-point connected to two P-points with a finite tail.

(B) $x$ belongs to a tail of some $y$ which satisfies (A).

Then $M(\mathcal{A})$ is a root model of $T_*$ and $M(\mathcal{A})_I = \mathcal{A}_I$. We call $M(\mathcal{A})$ the root model of $\mathcal{A}$. One can now prove without difficulty the following

***Claim:*** If $\mathcal{A}, \mathcal{B} \in Mod(T_*)$, then $\mathcal{A} \cong \mathcal{B}$ if and only if $\mathcal{A}_I \cong \mathcal{B}_I$ and

$$|Y(\mathcal{A})| = |Y(\mathcal{B})|,\ |H_1(\mathcal{A})| = |H_1(\mathcal{B})| \text{ and } |H_2(\mathcal{A})| = |H_2(\mathcal{B})|.$$

Thus, for any given $\mathcal{A} \in Mod(T_*)$, there are $\aleph_0$ many isomorphism types in $Mod(T_*)$ whose root models are isomorphic to $M(\mathcal{A})$. □

**Corollary 5.2.3.** *$VC_3$(FO) is true if and only if it holds for theories of graphs in the vocabulary $\tau_g$.*

*Proof.* We only show ($\Leftarrow$) and assume without loss of generality that $T$ is a complete theory with infinite models in the relational vocabulary $\tau$ which has exactly one $k$-ary relation symbol for every $k \in \omega_+$.

Suppose $I(T, \aleph_0) > \aleph_0$ and let the vocabulary $\tau_g$, the $\tau_g$ theory $T_*$ and the set $I$ of interpretation formulas be as in the previous proof. By interpreting $T$ in $\tau_g$, we get a $\tau_g$ theory $T_g$ which contains $T_*$ as a subset.For $\mathcal{A} \in Mod(T_g)$, we define $Y(\mathcal{A}), H_1(\mathcal{A}), H_2(\mathcal{A})$ as in the previous proof.

$T_g$ does not have to be complete but there are either at most coutably many complete and satisfiable extensions of $T_g$ or else continuum many.

In the former case there must be a complete extension of $T_g$ with uncountably many isomorphism types. This can be seen by using the main claim of the previous proof. Thus, by assumption there is a perfect set $S \subset Mod(T_g)$ of pairwise nonisomorphic models. For all triples $(k_1, k_2, k_3)$ with $k_1, k_2, k_3 \leq \aleph_0$, the set

$$\{A \in Mod(T_g) : |Y(A)| = k_1,\ |H_1(A)| = k_2,\ |H_2(A)| = k_3\}$$

is Borel, since it is of the form $Mod(\sigma)$, where $\sigma$ is a sentence of $\mathcal{L}_{\omega_1,\omega}(\tau_g)$. Clearly, there are $\aleph_0$ many of such triples $(k_1, k_2, k_3)$, hence we can use the claim of the previous proof and the perfect set theorem and assume without loss of generality that for all distinct $A, B \in S$, $A_I \ncong B_I$.

Now consider the following map $\rho : S \mapsto Mod(T)$: If $A \in S$ then $\rho(A)$ is the element of $Mod(T)$ which is isomorphic to $A_I$ by enumerating the elements of the set $D$ which is defined by $\psi_D(v_1)$.

It is easy to show that $\rho$ is Borel and injective, hence $im(\rho)$ contains a perfect set.

If there is a perfect set of complete extensions of $T_g$, then since the elementary equivalence relation on $\mathcal{X}_{\tau_g}$ is Borel we can use Silver's theorem to see that there is a perfect set of pairwise nonisomorphic models of $T_g$ and then use the same argument as before. $\square$

We now consider another different finite vocabulary $\tau_L := \{\wedge, \vee\}$, where $\wedge, \vee$ are binary function symbols.

**Definition.** A lattice is a $\tau_L$-structure $\mathfrak{M} := \langle M, \dots \rangle$, in which the following sentences are true:

(1) (Associativity) For all $a, b, c \in M$,

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c; \text{ and } a \vee (b \vee c) = (a \vee b) \vee c.$$

(2) (Absorption) For all $a, b \in M$

$$a \wedge (a \vee b) = a = a \vee (a \wedge b).$$

(3) (Commutativity) For all $a, b \in M$,

$$a \wedge b = b \wedge a \text{ and } a \vee b = b \vee a.$$

(4) (Idempotence) For all $a \in M$, $a \wedge a = a = a \vee a$.

Lattices are often studied in universal algebra.

**Example 5.2.4.** If $(M, <)$ is a linear order, then we can define a lattice $L$ with $M$ as its universe simply by setting for all $a, b \in M$,

- $a \wedge b := min\{a, b\}$ and
- $a \vee b := max\{a, b\}$.

**Definition.** Let $\tau_l := \tau_L \cup \{0_*, 1_*\}$, where $0_*, 1_*$ are two constant symbols. A bounded lattice is a $\tau_l$-structure $\mathfrak{M}$ such that $\mathfrak{M} \upharpoonright \tau_L$ is a lattice and the following holds

- (Identity) For all $a \in M$, $a \wedge 1_* = a = a \vee 0_*$.

**Example 5.2.5.** If $(B, +, *, -, 0, 1)$ is a Boolean algebra, then $(B, +, *, 0, 1)$ is a bounded lattice.

If $L$ is a lattice we can define a partial reflexive order via

$$a \leq b :\Leftrightarrow a \wedge b = a.$$

One can easily check that $a \leq b$ if and only if $a \vee b = b$. Furthermore, it follows that for every $a, b \in L$,

$$a \wedge b = inf\{c \in L : c \leq a,\ c \leq b\}$$

and

$$a \vee b = sup\{c \in L : a \leq c,\ b \leq c\}$$

On the other hand, suppose $(L, \leq)$ is a reflexive partial order such that for all $a, b \in L$, $inf\{a, b\}$ and $sup\{a, b\}$ exists then $L$ can be made into a lattice by defining $a \wedge b := inf\{a, b\}$ and $a \vee b := sup\{a, b\}$.

Given a lattice $L$ and $x \in L$, we define $L_x := \{a \in L : a \leq x\}$ and the height of $x$ as

$$h(x) := sup\{|A| : A \subseteq L_x,\ \text{A is linearly ordered by } \leq\}.$$

The height of a lattice $L$ is defined as

$$sup\{h(x) : x \in L\}.$$

We now show how graphs can be interpreted in lattices.

**Theorem 5.2.6.** *Let $\tau_g$ be the vocabulary of graphs. There is a set $I$ of $\tau_g$ interpretation formulas in the vocabulary $\tau_l$ and a satisfiable $\tau_l$ theory $T_*$ with the following properties:*

*(i) Every $\mathfrak{N} \in Mod(T_*)$ is a bounded lattice of height $\leq 4$.*

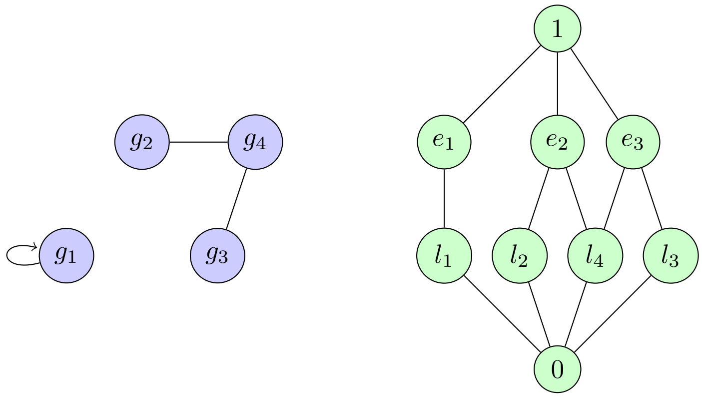
*(ii) There is a Borel surjection $\rho$ from $Mod(T_*)$ onto the set of all graphs in $\mathcal{X}_{\tau_g}$ such that for all $A, B \in Mod(T_*)$*

$$A \cong B \Leftrightarrow \rho(A) \cong \rho(B).$$

*Proof.* We will identify lattices in this proof with their corresponding partial orders.

The idea is to identify a vertex $g$ of a graph $G$ with an element $p(g)$ of height 2 in a lattice $L$. If $a, b \in G$ are connected, then the supremum of $p(a)$ and $p(b)$ will have height 3, otherwise it is 1.

Here is an illistration:




Graph G

Lattice L(G)

We see how the graph $G$ is coded by a lattice $L(G)$ of height 4. The elements $l_1, \ldots, l_4$ in $L(G)$ have height 2 and correspond to the vertices $g_1, \ldots, g_4$ of $G$. The elements $e_1, e_2, e_3$ indicate which vertices belong to an edge.

Apart from the axioms for bounded lattices $T_*$ states:

(1) The maximal element 1 has height at most 4.

(2) For every $n \in \omega_+$:

" There are at least $n$ elements of height 2."

(3) Every element of height 3 has at most 2 predecessors of height 2.

$T_*$ is clearly consistent and satisfies (i).

Let $\psi_E(v_1, v_2) := (v_1 = v_2)$ and $\psi_D(v_1)$ define the set of all elements of height 2. The formula $\psi_R(v_1, v_2)$ states the following:

"$v_1$ and $v_2$ have height 2 and either $v_1 = v_2$ and there is one element $c$ of height 3 such that $v_1 \leq c$ and for all $u \neq v_1$ of height 2, $u \not\leq c$, or $v_1 \neq v_2$ and there is an element $c$ of height 3 such that $v_1 \leq c$ and $v_2 \leq c$."

This gives us the set $I$ of $\tau_g$ interpretation formulas.

It is clear that for every graph $G$ of cardinality $\aleph_0$ there is $\mathfrak{M} \in Mod(T_*)$ which is unique up to isomorphism such that $\mathfrak{M}_I \cong G$.

For $L \in Mod(T_*)$, consider the order isomorphism $H$ from $\omega$ onto

$$D = \{a \in L : L \models \psi_D(a)\}.$$

Then we define $\rho(L)$ as the graph with universe $\omega$ where $k, m \in \omega$ are connected if and only if

$$L \models \psi_R(H(k), H(m)).$$

That is $\rho(L)$ is the unique element of $\mathcal{X}_{\tau_g}$ which is isomorphic to $L_I$ via $H$.

It is a routine exercise to show that $\rho$ satisfies (ii). □

**Corollary 5.2.7.** *(i) $VC_3$(FO) is true if and only if it holds for theories of bounded lattices of height $\leq 4$.*

*(ii) $VC_3$ is true if and only if it holds for every infinitary sentence all of whose models are bounded lattices of height $\leq 4$.*

*Proof.* (i): By corollary 5.2.3 it suffices to show that $VC_3$(FO) for bounded lattices of height $\leq 4$ implies $VC_3$(FO) for graphs. We can use an analogous argument for this.

Let the set $I$ of $\tau_g$ interpretation formulas, the $\tau_l$ theory $T_*$ and the map $\rho$ be defined as in the previous proof.

If $T$ is a complete theory of graphs with infinite models and $I(T, \aleph_0) > \aleph_0$, then there is a theory of bounded lattices containing $T_*$ with a perfect set $S \subseteq Mod(T_*)$ of pairwise nonisomorphic models such that $\rho[S] \subseteq Mod(T)$. This gives us an uncountable Borel set of pairwise nonisomorphic models of $T$ and by the perfect set theorem, $T$ has perfectly many models.

(ii): If $VC_3$ is true for every sentence of $\mathcal{L}_{\omega_1,\omega}(\tau_l)$ all of whose models are bounded lattices of height $\leq 4$, then by the previous proof $VC_3$ holds for $\mathcal{L}_{\omega_1,\omega}(\tau_g)$ sentences, which as we already know implies the general conjecture. □

$VC_1$(FO) has been proven for several special cases, e.g. in [14] for theories with one unary function symbol, for $\omega$-stable theories in [18] and for $o$-minimal theories in [13].

In [20] John Steel proved $VC_3$ for tree like orders, i.e. for every infinitary sentence using only a binary relation symbol each of whose model is a partial order $(A, <)$ such that for all $a \in A$, $\{y \in A : y < a\}$ is linearly ordered, thereby genealising Matatyahu Rubin's proof of $VC_1$(FO) for linear orders (see [17]).

It is an open question whether Vaught's conjecture for first order theories implies the infinitary version.

# 6 The Topological Vaught Conjecture

The definitions of this section can be found in [10] and in [2], unless stated otherwise, the results can be found in [2].

We turn our attention to a further generalisation of Vaught's conjecture.

## 6.1 Topological Group Actions

**Definition.** (1) A topological group is a group $(G, \circ, e)$ together with a topology $\mathcal{O}$ on $G$ such that the function $H : G \times G \mapsto G$ defined by

$$(x, y) \mapsto x \circ y^{-1}$$

is continuous. The group is called Polish if $\mathcal{O}$ is Polish.

(2) Let $X$ be a set and $(G, \circ, e)$ a group. A (left)group action of $G$ on $X$ is a map

$$* : G \times X \mapsto X$$

such that for all $x \in X$ and $g, h \in G$,

$$e * x = x \text{ and } (g \circ h) * x = g * (h * x).$$

(3) If the topological group $G$ acts on the set $X$ via $*$, then the binary relation $\sim$ on $X$ defined by

$$x \sim y :\Leftrightarrow \exists g \in G(g * x = y)$$

is an equivalence relation called the orbit equivalence relation. For every $x \in X$, the equivalence class of $x$ is called the orbit of $x$. If $x \sim y$, then $x, y$ are called orbit or $G$ equivalent.

(4) Let $X$ be a set, $A \subseteq X$ and $R$ an equivalence relation on $X$. $A$ is called $R$-invariant,if for all $x \in X$ and all $a \in A$, $aRx$ implies $x \in A$.

(5) If $G$ is a Polish group, $X$ is a standard Borel space and the action $*$ is Borel, then $X$ is called a Borel $G$-space. In case the action is continuous and $X$ is Polish, then $X$ is called a Polish $G$ space.

(6) A map $\rho : X \mapsto Y$ between Borel $G$ spaces is called a homomorphism if it respects the group action. It is called an embedding (isomorphism) if it is injective (bijective). A homomorphism is Borel if it is Borel measurable.

In this section we only consider invariant sets with respect to the orbit equivalence relation.

**Lemma 6.1.1.** *If $X, Y$ are Borel $G$ spaces and $\rho : X \mapsto Y$, $\pi : Y \mapsto X$ are Borel embeddings, then there are invariant Borel subsets $A \subseteq X$ and $B \subseteq Y$ such that $\rho(A) = Y \setminus B$ and $\pi(B) = X \setminus A$. In particular $X$ and $Y$ are Borel isomorphic.*

*Proof.* Let $A_0 := \emptyset$, $S_0 := Y$ and for $n \in \omega$,

$$A_{n+1} := X \setminus \pi(S_n) \text{ and } S_{n+1} := Y \setminus \rho(A_{n+1}).$$

Then $A := \bigcup_{n<\omega} A_n$ and $B := \bigcap_{n\in\omega} S_n$ are as claimed. □

**Theorem 6.1.2.** *Let $H$ be a Polish group and $G \subseteq H$ be a closed subgroup. If $X$ is a Borel $G$ space with action $*_1$, then there is a Borel $H$ space $Y$ with action $*_2$such that*

*(i) $X$ is a Borel subset of $Y$.*

*(ii) $g *_2 x = g *_1 x$, for all $g \in G$ and $x \in X$, and every $H$ orbit of $Y$ contains exactly one $G$ orbit of $X$.*

*(iii) If $X$ is a Polish $G$ space, then $Y$ can be chosen to be a Polish $H$ space such that $X$ is a closed subset of $Y$.*

*Proof.* (Sketch, for the details see theorem 2.3.5 of [2]) $G$ acts on $X \times H$ via

$$g \circ (x, h) := (g *_1 x, gh),$$

where $gh$ is the group product of $g, h$ in $H$.

Consider the quotient space $Y := (X \times H/G, S)$, where $S$ is the quotient $\sigma$ algebra with respect to the Borel algebra of $X \times H$. One can show that this is a standard Borel space. For $x \in X$ and $h \in H$, the orbit equivalence class of $(x, h)$ is denoted by $[x, h]$.

Now the action $*_2$ of $H$ on $Y$ is given by

$$h *_2 [x, h'] := [x, h'h^{-1}].$$

One can easily show that this action is well defined and Borel.

The map $x \mapsto [x, e]$ is a Borel injection from $X$ into $Y$, hence $X$ can be seen as a Borel subset of $Y$. This proves (i).

$*_2$ agrees with $*_1$ on $G \times X$, as $[g *_1 x, e] = [x, g^{-1}] = g *_2 [x, e]$, for all $(g, x) \in G \times X$.

If $[x, h] \in Y$, then $h *_2 [x, y] = [x, e] \in X$, thus every $H$ orbit contains a $G$ orbit of $X$. Suppose $x, y \in X$ and for some $h \in H$, $h *_2 [x, e] = [y, e]$. Then by definition, $y$ is in the $G$ orbit of $x$, hence (ii) is proven.

For (iii) one checks that the quotient topology on $Y$ is Polish. This is done by first proving that it is $T_2$ and second countable and regular. Urysohn's metrization lemma then implies that $Y$ is seperably metrizable. It follows that $Y$ is Polish. □

For a topological space $X$, let $F(X)$ denote the set of all closed subsets of $X$. The Effros space is defined as $(F(X), S)$, where $S$ is the $\sigma$ algebra on $F(X)$ generated by the sets of the form

$$\{F \in F(X) : F \cap U \neq \emptyset\}$$

and $U$ ranges over all nonempty open subsets of $X$. One can show that if $X$ is Polish, then the Effros space is a standard Borel space (see e.g. theorem 12.6 of [10]).

A Borel $G$ space is called universal if every Borel $G$ space can be embedded into it.

**Theorem 6.1.3.** *For any Polish group G there is a Borel G space $(U_G, *)$ such that for every seperably metrizable topological space $X$ and every Borel group action $\circ$ of $G$ on $X$, there is a Borel embedding from $(X, \circ)$ into $(U_G, *)$.*

*Proof.* (Sketch) $U_G$ is defined as $(F(G))^\omega$, where $F(G)$ is the Effros Borel space of $G$, and $G$ acts on $U_G$ via

$$g * (F_n)_{n\in\omega} := (gF_n)_{n\in\omega}.$$

This action is Borel.

If $X$ is a seperably metrizable space with a Borel $G$ action $\circ$, then choose an enumeration $(S_n : n \in \omega)$ of a basis of the topology.

For $A \subseteq G$ let

$$E(A) := \{g \in G : \text{For all open nbhd V of g}, V \cap A \text{ is not meager}\}.$$

Clearly, $E(A)$ is closed.

One then defines $\pi : X \mapsto U_G$, $x \mapsto (\pi_n(x))_{n\in\omega}$, where

$$\pi_n(x) := (E(\{g \in G : g \circ x \in S_n\}))^{-1}.$$

We skip the details of the proof that this is the desired embedding. $\square$

**Remark.** The result can be sharpened(theorem 2.6.6 of [2]): For every Polish group $G$, there is a compact universal Polish $G$ space.

The following result will also be useful. For a proof see theorem 5.2.1 of [2].

**Theorem 6.1.4.** *Let $X$ be a Borel $G$ space. There is a Polish $G$ space $Y$ which is Borel isomorphic to $X$.*

## 6.2 Applications to Model Theory

The most interesting Polish group for this thesis is $S_\infty$, the group of permutations of $\omega$, where multiplication is the composition of functions and the topology is inherited from $\mathcal{N}$. $S_\infty$ acts on $\mathcal{X}_\tau$ in a canonical way: Let $g \in S_\infty$ and $x \in \mathcal{X}_\tau$ then $g * x$ is the element $y \in \mathcal{X}_\tau$ such that $\mathfrak{M}_x \cong \mathfrak{M}_y$ via $g$. This action is continuous and called the logic action.

**Proposition 6.2.1.** *A subgroup $G$ of $S_\infty$ is closed if and only if there is a countable relational vocabulary $\tau$ and some $\mathfrak{M} \in \mathcal{X}_\tau$ such that*

$$G = Aut(\mathfrak{M}),$$

*where $Aut(\mathfrak{M})$ is the automorphism group of $\mathfrak{M}$.*

*Proof.* ($\Rightarrow$): Suppose $G \leq S_\infty$ is closed and for every $k \in \omega_+$, let $M_k$ be the quotient space of $\omega^k$ of the orbit equivalence relation with respect to $G$, i.e. $\overline{a}, \overline{b} \in \omega^k$ are equivalent if for some $g \in G$, $\overline{b} = g(\overline{a})$.

For every $k \in \omega_+$ and every $s \in M_k$ let $R_s$ be a $k$ ary relation symbol and define $\tau := \{R_s : s \in M_k, \text{ for some } k \in \omega_+\}$.

Consider $\mathfrak{M} \in \mathcal{X}_\tau$ in which $R_s$ is interpreted with $s$.

Using the fact that $G$ is closed, it is easy to show that $G = Aut(\mathfrak{M})$.

($\Leftarrow$): If $G = Aut(\mathfrak{M})$, for some $\mathfrak{M} \in \mathcal{X}_\tau$, where $\tau$ is countable, and $f$ in $S_\infty \setminus G$, then without loss of generality, for some atomic formula $\psi(v_1, \ldots, v_l)$ and some $\overline{a} \in \omega^l$, we have

$$\mathfrak{M} \models \psi(\overline{a}), \text{ and } \mathfrak{M} \not\models \psi(f(\overline{a})).$$

This is also true for the open set of all $h \in S_\infty$ which agree with $f$ on $\overline{a}$. $\square$

In the proof of proposition 5.1.1 we saw that if $\tau_2$ is a relational vocabulary of unbounded arity then for every countable vocabulary $\tau_1$ there is a continuous injection from $\mathcal{X}_{\tau_1}$ into $\mathcal{X}_{\tau_2}$. It is easy to check that this function respects the $S_\infty$ action on the logic spaces, in particular it is a Borel embedding of Polish $S_\infty$ spaces. By lemma 6.1.1 we have, that if $\tau_1, \tau_2$ are relational vocabularies of unbounded arity, then $\mathcal{X}_{\tau_1}$ and $\mathcal{X}_{\tau_2}$ are Borel isomorphic.

**Lemma 6.2.2.** *Let $\tau_1$ be a relational vocabulary such that for some $k \in \omega_+$, all of its symbols have arity less or equal $k$. If $\tau_2$ is a vocabulary consisting of a single $n$ ary relation symbol, where $n > k$, then there is no embedding from $\mathcal{X}_{\tau_2}$ into $\mathcal{X}_{\tau_1}$.*

*Proof.* We can assume without loss of generality that $n = k + 1$.

For a finite injection $b \in \omega^{<\omega}$, we define $S_b := \{g \in S_\infty : b \subseteq g\}$.

Since all symbols of $\tau_1$ are bounded by $k$, every $x \in \mathcal{X}_{\tau_1}$ has the following property

$P_*$: If $g * x \neq x$, then for some injective $b \in \omega^k$, $b \subseteq g$ and $h * x \neq x$, for all $h \in S_b$.

We show that there is a model $M \in \mathcal{X}_{\tau_2}$ which does not satisfy $P_*$, hence there cannot be an embedding from $\mathcal{X}_{\tau_2}$ into $\mathcal{X}_{\tau_1}$, as $\neg P_*$ is preserved under such maps.

Let $Y := \{(a_1, \ldots, a_n) : a_i \neq a_j, 1 \leq i < j \leq n\}$. We identify a subset of $Y$ with its characteristic function which is an element of $2^Y$. Equipped with the product topology $2^Y$ is homeomorphic to the Cantor space.

For $n \leq m < \omega$ and $b \in \omega^m$ injective, define

$$P(m, b) := \{f \in 2^Y : \forall \overline{a} \in dom(b)^n,\ \overline{a} \in f \Leftrightarrow b(\overline{a}) \in f\}.$$

For $n \leq m$ and $l \in \omega$, let $Q_+(m, b, l)$ be the set of all $f \in 2^Y$ such that if $f \in P(m, b)$, then for some injective $c \in \omega^{m+1}$, $b \subseteq c$, $l \in dom(c)$ and $f \in P(m+1, c)$, and $Q_-(m, b, l)$ the set of all $f \in 2^Y$ such that if $f \in P(m, b)$, then for some injective $c \in \omega^{m+1}$, $b \subseteq c$, $l \in range(c)$ and $f \in P(m+1, c)$. It follows that for all $m \geq n$, all $l \in \omega$ and all $b \in \omega^m$, both $Q_+(m, b, l)$ and $Q_-(m, b, l)$ are open and dense in $2^Y$.

For $b \in \omega^k$, we also define $A(k, b)$ as the set of all $f \in 2^Y$ for which there is an injective $c \in \omega^n$ such that $b \subseteq c$ and $f \in P(n, c)$. It is easy to verify that $A(k, b)$ is also open and dense.

Since $2^Y$ is a Baire space, there is some $S$ in

$$\bigcap_{b \in \omega^k} A(k, b) \cap \bigcap_{\substack{l, m \in \omega, m \geq n \\ b \in \omega^m}} (Q_+(m, b, l) \cap Q_-(m, b, l))$$

such that $\emptyset \neq S$ and $Y \neq S$.

Now let $M := (\omega, R)$, where $R \subseteq \omega^n$ is defined by

$$\overline{a} \in R :\Leftrightarrow \overline{a} \in S.$$

Clearly, there is $g \in S_\infty$ such that $g * M \neq M$. If $b \in \omega^k$ and $b \subseteq g$, then by the specific choice of $S$, we can via a back and forth argument construct an automorphism of $M$ extending $b$. Thus, every element of the $S_\infty$ orbit of $M$ does not satisfy $P_*$ and can therefore not be embedded into $\mathcal{X}_{\tau_1}$. □

**Theorem 6.2.3.** *If $\tau$ is a relational vocabulary of unbounded arity, then $\mathcal{X}_\tau$ is a universal Borel $S_\infty$ space.*

*Proof.* It suffices to prove the theorem for the special case that $\tau$ has infinitely many $k$ ary relation symbols, for every $k \in \omega_+$.

First note that $S_\infty$ acts contionusly on the Baire space $\mathcal{N}$ via composition

$$g * f := g \circ f,$$

for $(g, f) \in S_\infty \times \mathcal{N}$, and that for every $g \in S_\infty$ the map

$$\mathcal{N} \ni f \mapsto g * f$$

is a homeomorphism on $\mathcal{N}$.

Thus, we have a Borel action of $S_\infty$ on the Effros Borel space $F(\mathcal{N})$ defined by

$$g *_2 A := gA = \{g * f : f \in A\}.$$

Now let $V := (F(\mathcal{N}))^\omega$ with the componentwise action of $S_\infty$. Clearly, this is also a Borel $S_\infty$ space.

By theorem 6.1.3, we know that $U := (F(S_\infty))^\omega$ is an universal Borel $S_\infty$ space, where $F(S_\infty)$ is the Effros Borel space of $S_\infty$.

It is easy to check that the map

$$U \ni (A_n)_{n\in\omega} \mapsto (\overline{A_n})_{n\in\omega} \in V$$

is a Borel embedding of $S_\infty$ spaces, where $\overline{A}$ is the closure of $A$ in $\mathcal{N}$, for $A \subseteq S_\infty \subseteq \mathcal{N}$, therefore, $V$ is an universal Borel $S_\infty$ space.

If we can find an embedding from $V$ into $\mathcal{X}_\tau$, the theorem follows.

For $(m, n) \in \omega \times \omega_+$, let $R_{(m,n)}$ be a $n$ ary relation symbol of $\tau$.

Every closed subset of $\mathcal{N}$ is the set of branches of a tree $T \subseteq \omega^{<\omega}$.

Suppose $a := (A_m)_{m\in\omega} \in V$ and let $(T_m)_{m\in\omega}$ be the sequence of trees such that $A_m$ is the set of branches of $T_m$. We define a $\tau$ model $M_a$ by interpreting $R_{(m,n)}$ as

$$\{s \in \omega^n : s \in T_m\}.$$

Let $< M_a >$ be the code of $M_a$ in $\mathcal{X}_\tau$. Then the proof that

$$V \ni a \mapsto < M_a > \in \mathcal{X}_\tau$$

is a Borel embedding of $S_\infty$ spaces is routine. □

**Corollary 6.2.4.** *Let $\tau$ be a countable relational vocabulary, $M \in \mathcal{X}_\tau$ with automorphism group $G \leq S_\infty$ and $\mu$ be a relational vocabulary of unbounded arity disjoint from $\tau$. Then the set of all $\mu$ extensions of $M$ is an universal $G$ space.*

*Proof.* Without loss of generality we can assume that $\mu$ is the vocabulary from the previous proof with infinitely many relation symbols in every arity.

$G$ is a closed subgroup of $S_\infty$, hence by theorem 6.1.2 and the previous result, $\mathcal{X}_\mu$ is an universal Borel $G$ space.

If we modify the map $a \mapsto < M_a >$ of the previous proof such that $M_a$ is a $\mu$ extension of $M$, then we get a Borel embedding of $G$ spaces from $(F(\mathcal{N}))^\omega$ into

$$\{B \in \mathcal{X}_{\tau\cup\mu} : B \restriction \tau = M\}.$$

□

We know that by Scott's isomorphism every $x \in \mathcal{X}_\tau$ can be characterized by a single sentence and we have also seen that the set of all models which satisfy a given sentence is Borel. Thus, we have

**Proposition 6.2.5.** *For every $x \in \mathcal{X}_\tau$, the orbit of $x$ under the logic action is Borel.*

**Fact 6.2.6.** Let $X$ be a Polish $G$-space and $A, B \subseteq X$ be analytic, disjoint and $G$-invariant. There is a $G$-invariant Borel subset $C$ such that $A \subseteq C \subseteq X \setminus B$. $C$ is said to seperate $A$ and $B$.

This can be proven using the well known fact that two analytic and disjoint sets in a Polish space can be seperated by a Borel set.

A subset $B$ of a topological space $X$ is called nowhere dense if $(\overline{B})^\circ = \emptyset$. If $B \subseteq X$ is a countable union of nowhere dense subsets of $X$, then $B$ is called meager. A set is comeager if its complement is meager. A topological space is called a Baire space if every nonempty open subset is non meager.

Every Polish space is a Baire space and every open subset of a Polish space equipped with the relative topology is also Baire.

**Notation.** (Category Quantifiers) If $X$ is a topological space and $P$ is a property, then $\forall^* x P(x)$ means that $\{x \in X : P(x)\}$ is comeager.

$\exists^* x P(x)$ means that $\{x \in X : P(x)\}$ is not meager.

**Definition.** (Vaught Transforms) Let $G$ be a Polish group acting on a set $X$ via $*$, $A \subseteq X$ and $U \subseteq G$ open and nonempty. Then

- $A^{\Delta U} := \{x \in X : \exists^* g \in U \text{ such that } g * x \in A\}$.
- $A^{*U} := \{x \in X : \forall^* g \in U (g * x \in A)\}$.

If $U = G$, we simply use $A^*$ and $A^\Delta$.

It easily follows that $A \subseteq X$ is invariant iff $A = A^*$, iff $A = A^\Delta$.

With this notions we can show a theorem by Lopez-Escobar:

**Theorem 6.2.7.** *(Theorem 16.8 of [10]) The invariant Borel sets of $\mathcal{X}_\tau$ are exactly the ones of the form $Mod(\phi)$, where $\phi$ is a $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence.*

*Proof.* (Sketch) For each $k < \omega$, let $U_k$ be the set of all injective $k$-sequences in $\omega^{<\omega}$ and if $u \in U_k$, then let $[u] := \{g \in S_\infty : u \subset g^{-1}\}$. Clearly, $[u]$ is clopen.

Then, for $A \subseteq \mathcal{X}_\tau$, define

$$A_k^* := \{(x, \overline{u}) \in \mathcal{X}_\tau \times \omega^k : \overline{u} \in U_k, x \in A^{*[u]}\}.$$

For every Borel set $A \subseteq \mathcal{X}_\tau$ and every $k < \omega$, $A_k^*$ is of the form

$$A_{\phi,k:=}\{(x, \overline{a}) \in \mathcal{X}_\tau \times \omega^k : \mathfrak{M}_x \models \phi(\overline{a})\},$$

where $\phi(\overline{v})$ is a $k$-formula of $\mathcal{L}_{\omega_1,\omega}(\tau)$ and $\overline{a} \in U_k$. This follows from the fact that the set of all $A \subseteq \mathcal{X}_\tau$, for which $A_k^*$ is of the form $A_{\phi,k}$, contains every open set and is closed under complementation and countable intersections.

Then consider the case for $k = 0$: Clearly, if $\phi \in \mathcal{L}_{\omega_1,\omega}$ is a sentence, then $A_\phi$ is Borel and invariant.

On the other hand, if $A \subseteq \mathcal{X}_\tau$ is Borel and invariant, then $A = A^* = A_0^*$ which is of the form $A_\phi$. □

An immediate consequence of this result is that every countable $\tau$-structure can be characterized up to isomorphism by a $\mathcal{L}_{\omega_1,\omega}$-sentence, since we have pointed out that the orbit of every model is a Borel set which is clearly invariant. This result was already proven in theorem 2.2.5 but notice that here we have only given an existential proof whereas in section 2 we provided an example of a characterizing sentence. We also have:

**Corollary 6.2.8.** *(Interpolation Theorem, see corollary 16.11 of [10]) Let $\tau_1, \tau_2$ be countable vocabularies, $\phi$ a $\mathcal{L}_{\omega_1,\omega}(\tau_1)$-sentence, $\psi$ a $\mathcal{L}_{\omega_1,\omega}(\tau_2)$-sentence and define $\tau' := \tau_1 \cap \tau_2$. Suppose that with respect to $\tau_1 \cup \tau_2$, we have $\phi \models \psi$. Then there is a $\mathcal{L}_{\omega_1,\omega}(\tau')$-sentence $\sigma$ such that $\phi \models \sigma \models \psi$.*

*Proof.* Let $A$ be the set of all $x \in \mathcal{X}_{\tau'}$ for which there is a $y \in \mathcal{X}_{\tau_1}$ such that

$$\mathfrak{M}_y \models \phi, \text{ and } \mathfrak{M}_y \restriction \tau' = \mathfrak{M}_x,$$

and $B$ the set of all $x \in \mathcal{X}_{\tau'}$ for which there is a $y \in \mathcal{X}_{\tau_2}$ such that

$$\mathfrak{M}_y \models \neg\psi, \text{ and } \mathfrak{M}_y \restriction \tau' = \mathfrak{M}_x.$$

Then $A$ and $B$ are analytic, disjoint and invariant, hence there is an invariant Borel set $C \subseteq \mathcal{X}_{\tau'}$ seperating them. By the Lopez-Escobar theorem $C = A_\sigma$ for some sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}(\tau')$. it is easy to check that $\phi \models \sigma \models \psi$. □

We can now present Hjorth's original proof of theorem 4.2.11:

Recall that in section 4 we considered a countable infinite vocabulary $\tau_1$ which has an unary predicate symbol $Q'$. We defined a special sentence $\psi$ of $\mathcal{L}_{\omega_1,\omega}(\tau_1)$, namely the $\mathcal{K}_1$ generic sentence, which has models of cardinality $\aleph_0$ and $\aleph_1$ but of no higher cardinality.

Let $M \in Mod(\psi)$ and $Q := Q'^M$.In proposition 4.2.7 it was shown that $Q^M$ is infinite and later that $Q$ is a set of absolute indiscernibles.

Fix an arbitrary bijection $\rho$ from $Q$ onto $\omega$. It follows that there is a continuous group homomorphism $\pi$ from $Aut(M)$ onto $S_\infty$ given by

$$Aut(M) \ni g \mapsto \rho \circ g \circ \rho^{-1} \in S_\infty.$$

Suppose $\mu$ is an arbitrary contable vocabulary disjoint from $\tau_1$ and $\sigma \in \mathcal{L}_{\omega_1,\omega}(\mu)$ is a counterexample to $VC_3$. Then $Mod(\sigma)$ is a Borel $S_\infty$ space with uncountably many but not perfectly many orbits. It is also a Borel $Aut(M)$ space with the acton $g *_2 x := \pi(g) * x$, for $g \in Aut(M)$ and $x \in Mod(\sigma)$. Note that the orbit equivalence relation generated by $*_2$ is equal to the one generated by $*$.

Let $\tau_1'$ be a relational vocabulary of unbounded arity disjoint from $\tau_1$ and $\tau_2 := \mu \cup \tau_1 \cup \tau_1'$. By corollary 6.2.4, the $\tau_2$ extensions of $M$ form a universal $Aut(M)$ space, denoted by $Y$. Clearly, $Y \subseteq \mathcal{X}_{\tau_2}$.

Thus, there is a Borel embedding

$$F : Mod(\sigma) \mapsto Y.$$

Let $B := im(F)$. Then $B$ is Borel. Now consider

$$C := \{N \in \mathcal{X}_{\tau_2} : N \cong A, \text{ for some } A \in B\}.$$

$C$ is Borel, since it can also be described as the set of all $N \in \mathcal{X}_{\tau_2}$ which satisfy $\Psi_M$, the Scott sentence of $M$, and for which all $N' \in Y$ isomorphic to $N$ are in $B$. Clearly, $C$ is invariant with respect to the standard action of $S_\infty$.

By theorem 6.2.7, there is a sentence $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau_2)$ such that $C = Mod(\phi)$.
We have $\phi \models \Psi_M$, hence $\phi$ has no models in cardinality greater than $\aleph_1$.

It is also obvious that $\phi$ has uncountably but not perfectly many models, as any perfect set of pairwise non isomorphic models of $\phi$ yields a perfect set of nonisomorphic models of $\sigma$. (q.e.d.)

We have seen that $VC_3$ is a problem about the orbit equivalence relation of a Polish group action on an invariant Borel subset of a Polish space. This observation leads to the so called topological Vaught conjecture (TVC).

> **TVC2(G)**: Let $X$ be a Polish $G$-space and $A \subseteq X$ be invariant and Borel. Then either there are $\leq \aleph_0$ many $G$-orbits in $A$ or else $A$ contains a perfect set of pairwise non orbit equivalent elements.

Clearly, $TVC2(S_\infty)$ implies $VC_3$ and is independent of the value of $\mathfrak{c}$. There are also other versions:

> **TVC1(G)**: If $X$ is a Polish $G$ space, then either there are at most $\aleph_0$ many $G$ orbits or else there is a perfect subset of $X$ of pairwise non orbit equivalent elements.

> **TVC3(G)**: If $X$ is a Borel $G$ space, then either there are at most $\aleph_0$ many $G$ orbits or else there is a perfect set of pairwise non orbitequivalent elements in $X$.

Historically, $TVC1(G)$ is the first version and was formulated by D.H. Miller(see [16] p.484). The idea to consider model theoretic problems from a descriptive set theoretic and topological point of view was already discussed by R. Vaught in [22].

**Proposition 6.2.9.** *If G is a Polish group, then all three versions of the topological Vaught conjecture for G are equivalent.*

*Proof.* Clearly, we have $TVC3(G) \Rightarrow TVC2(G) \Rightarrow TVC1(G)$.

$TVC1(G) \Rightarrow TVC3(G)$ follows imediately from the fact that evrey Borel $G$ space is Borel isomorphic to a Polish $G$ space. □

Since we are only considering Polish groups here and in light of the previous proposition, we simply write $TVC(G)$ from now on.

**Proposition 6.2.10.** *Let G be a Polish group. Then $TVC(G)$ holds if and only if $TVC(A)$ is true, for every closed subgroub $A \leq G$.*

*Proof.* (I thank Mathematics Stack Exchange user Edward H for his help with this proof.)

($\Rightarrow$): Let $A \leq G$ be closed and $X$ be a Polish $A$ space with uncountably many orbits.

By theorem 6.1.2 there is a Polish $G$ space $Y$ containing $X$ as a closed subset such that the action $\circ$ of $G$ on $Y$ extends the action of $A$ on $X$ and every $G$ orbit of $Y$ contains exactly one $A$ orbit of $X$.

Assuming $TVC(G)$, it follows that there is a perfect set $W \subseteq Y$ of pairwise $G$ independent elements.

The set

$$S := \{(w, g) \in W \times G : g \circ w \in X\} \subseteq Y \times G$$

is closed and has domain $W$. Thus, by the Jankov, von NeumannUniformization Theorem (see theorem 18.1 of [10]), there is a $\sigma(\Sigma^1_1)$ measurable function

$$\rho : (W, \sigma(\Sigma^1_1)) \mapsto (G, \mathcal{B}(G)),$$

where $\sigma(\Sigma^1_1)$ is the $\sigma$ algebra of $W$ generated by the $\Sigma^1_1$ sets and $\mathcal{B}(G)$ is the Borel algebra of $G$, such that for all $w \in W$, $(w, \rho(w)) \in S$. In particular $\rho$ is Baire measurable, i.e. for every open set $U \subseteq G$, $\rho^{-1}(U)$ has the Baire property. It follows that $\rho$ is continuous on a dense $G_\delta$ subset $B \subseteq W$. Then

$$\{\rho(w) \circ w : w \in B\}$$

is an uncountable Borel subset of pairwise $A$ independent elements of $X$, hence by the perfect set theorem, $X$ has perfectly many $A$ orbits.

The direction $(\Leftarrow)$ is immediate. □

**Theorem 6.2.11.** *The following are equivalent:*

*(1)* $VC_3$

*(2)* $TVC(S_\infty)$.

*Proof.* (2) ⇒ (1) is clear.

(1) ⇒ (2): Suppose $X$ is a Borel $S_\infty$ space.

Let $\tau$ be the relational vocabulary with exactly one $k$ ary symbol for every $k \in \omega_+$. By theorem 6.2.3, $\mathcal{X}_\tau$ is a universal $S_\infty$ space, hence there is a Borel embedding $\pi$ from $X$ into $\mathcal{X}_\tau$. Since $im(\pi)$ is an invariant Borel subset of $\mathcal{X}_\tau$, there is a sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}(\tau)$ such that $im(\pi) = Mod(\sigma)$. If there are uncountably many $S_\infty$ orbits in $X$, then $I(\sigma, \aleph_0) > \aleph_0$ and thus by $VC_3$, $\sigma$ has perfectly many models.

Any perfect set of pairwise non isomorphic models of $\sigma$ easily gives us a perfect set of non orbit equivalent elements in $X$. □

Let $X, Y$ be metrizable spaces, $X$ compact and $C(X, Y)$ be the set of all continuous functions from $X$ into $Y$.

For $K \subseteq X$ closed an $U \subseteq Y$ open, define

$$V(K, U) := \{f \in C(X, Y) : f[K] \subseteq U\}.$$

Then $\{V(K, U) : K \subseteq X \text{ closed}, U \subseteq Y \text{ open}\}$ is a subbasis for a topology on $C(X, Y)$, called the compact open topology of $C(X, Y)$. In this thesis we only deal with this topology when considering $C(X, Y)$.

If $d$ is a compatible metric for $Y$, then

$$d_u(f, g) := sup\{d(f(a), g(a)) : a \in X\}$$

is a compatible metric for the compact open topology of $C(X, Y)$. One can show - see for example theorem 4.19 of [10] - that if $Y$ is Polish then so is $C(X, Y)$.

In the special case where $X = Y$, we consider the set of all homeomorphisms on $X$. Together with the composition operation this set is a group, called the homeomorphism group of $X$ and denoted by $H(X)$. It is not difficult to show that $H(X)$ is a Polish group with the toplogy it inherits as a $G_\delta$ subset of $C(X, X)$. A compatible metric is given by

$$d_*(f, g) := d_u(f, g) + d_u(f^{-1}, g^{-1}).$$

Homeomorphism groups are studied in particular with respect to the connections between the topological properties of a space $X$ and the algebraic properties of its homoemorphism group.

**Lemma 6.2.12.** *Let $C := 2^{\mathbb{N}}$ be the Cantor space. The following are equivalent:*

*(i)* $TVC(S_\infty)$.

*(ii)* $TVC(H(C))$.

*Proof.* $(i) \Rightarrow (ii)$: First, we show that $H(C)$ is isomorphic to a closed subgroup of $S_\infty$.

Let $M$ be the set of all clopen subsets of $C$. Since $C$ is compact and metrizable, it follows that $|C| = \aleph_0$.

$M$ is closed under finite intersections and unions, hence we can see it as a countable model $\mathfrak{M} :=< M, \cap, \cup >$ of a language with two binary function symbols.

Every homeomorphism $\pi \in H(C)$ determines an automorphism $H_\pi \in Aut(\mathfrak{M})$ via

$$H_\pi(A) := \pi[A],$$

for $A \in M$.

Conversly, $H \in Aut(\mathfrak{M})$ gives us a homeomorphism $\pi_H \in H(C)$ via

$$\pi_H(x) := \bigcap\{H(A) : A \in M,\ x \in A\},$$

for $x \in C$. It is a routine exercise to show that $\pi_H$ is well defined and that the map

$$H(C) \ni \pi \mapsto H_\pi \in Aut(\mathfrak{M})$$

is an isomorphism of topological groups.

If we fix a bijection from $M$ onto $\omega$, then by proposition 6.2.1, $H(C)$ is isomorphic to a closed subgroup of $S_\infty$, therefore by proposition 6.2.10 $TVC(S_\infty)$ implies $TVC(H(C))$.

$(ii) \Rightarrow (i)$: Similarily to the previous direction, we show that $S_\infty$ is isomorphic to a closed subgroup of $H(C)$: Given $\sigma \in S_\infty$ and $f \in C$, let $f_\sigma \in C$ be defined via

$$f_\sigma(n) := f(\sigma^{-1}(n)),$$

for $n \in \omega$.

It is not difficult to show that the map $G_\sigma : C \mapsto C,\ f \mapsto f_\sigma$ is a homeomorphism, and that the map $G : S_\infty \mapsto H(C),\ \sigma \mapsto G_\sigma$ is a continuous group monomorphism. Furthermore, $G$ is a homeomorphism between $S_\infty$ and $im(G)$ equipped with the relative topology.

It remains to show that $im(G) = \overline{im(G)}$. Let $n \in \omega$, $g \in \overline{im(G)}$ and define the following clopen sets

$$B_{(n,0)} := \{f \in C : f(n) = 0\} \text{ and } B_{(n,1)} := \{f \in C : f(n) = 1\}.$$

Clearly, $C = B_{(n,0)} \dot\cup B_{(n,1)}$, hence $C = g[B_{(n,0)}] \dot\cup g[B_{(n,1)}]$, and since $g \in \overline{im(G)}$, it follows that for some $m \in \omega$,

$$g[B_{(n,0)}] = B_{(m,0)} \text{ and } g[B_{(n,1)}] = B_{(m,1)}.$$

Using this and the fact that $g^{-1} \in \overline{im(G)}$, it is easy to see that for some $\sigma \in S_\infty$, $g = G_\sigma$.

By proposition 6.2.1, we get $TVC(S_\infty)$. □

**Definition.** Let $I := [0,1] \subseteq \mathbb{R}$ be equipped with its standard topology. The product space $I^{\mathbb{N}}$ is called the Hilbert cube.

The topological Vaught conjecture is in fact a problem about a specific group.

**Lemma 6.2.13.** *The following are equivalent:*

*(i)* $TVC(G)$*, for every Polish group* $G$*.*

*(ii)* $TVC(H(I^{\mathbb{N}}))$*.*

*Proof.* $(i) \Rightarrow (ii)$ is clear.

$(ii) \Rightarrow (i)$ is based on the following result by Uspenskii (for a proof see theorem 9.18. of [10]):

Every Polish group is isomorphic to a closed subgroup of $H(I^{\mathbb{N}})$.

Then using proposition 6.2.10 completes the proof. □

Some obvious attemps to generalise $TVC$ further can be answered:
Let us first look at a version for analytic sets.

> $\mathbf{TVC}(\mathbf{G}, \boldsymbol{\Sigma}^1_1)$: "For every Polish $G$ space $X$ and every invariant analytic subset $A \subseteq X$, there are either at most $\aleph_0$ many $G$ orbits in $A$ or else $A$ contains a perfect set of pairwise non orbit equivalent elements."

Then we have a counterexample:

**Example 6.2.14.** For $\alpha < \omega_1$, let $M_\alpha$ be the set of all sequences $(z_\beta : \beta < \alpha)$, where $z_\beta \in \mathbb{Z}$ and $z_\beta \neq 0$ for only finitely many $\beta$. We define a linear order$<_\alpha$ on $M_\alpha$ as follows: $(x_\beta : \beta < \alpha) <_\alpha (y_\beta : \beta < \alpha)$, if $x_i < y_i$, where $i < \alpha$ is maximal such that $x_i \neq y_i$.

Let $\tau$ be the vocabulary with only one binary relation symbol and $\mathcal{A}$ be the set of all elements in $\mathcal{X}_\tau$ which are isomorphic to some $M_\alpha$ or to $M_\alpha \times (\mathbb{Q}, <)$ with the colexicographic order, for some $\alpha < \omega_1$. Using a $\Sigma^1_1$ bounding argument, one can show that $\mathcal{A}$ has $\aleph_1$ many but not perfectly many isomorphism types.

In [15] it is shown that $\mathcal{A}$ is the set restricions of countable linearly ordered abelian groups with universe $\omega$ to the vocabulary $\tau$, hence there is an analytic set on which the logic action of $S_\infty$ has uncountably but not perfectly many orbits.

**Remark.** It follows from John Steel's article ([20]) that the set $\mathcal{A}$ of the previous example is not Borel: Suppose otherwise. Then since $\mathcal{A}$ is invariant with respect to the $S_\infty$ action, by theorem 6.2.7, $\mathcal{A} = Mod(\phi)$ for some sentence $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$.

Note that all elements of $\mathcal{A}$ are linear and therefore tree like orders, hence by [20], $\phi$ satisfies $VC_3$ and $\mathcal{A}$ has perfectly many orbits, a contradiction.

**Remark.** Sets of models of the form $Mod(\phi)$ are called elementary classes.

More generally, if $\mathcal{K} \subseteq \mathcal{X}_\tau$ and for some countable extension $\tau_*$ of $\tau$ and a sentence $\phi_* \in \mathcal{L}_{\omega_1,\omega}(\tau_*)$, $\mathcal{K}$ is the set of $\tau$ restrictions of $Mod(\phi_*)$, then $\mathcal{K}$ is called a pseudoelementary class.

If $\mathcal{K}$ is pseudoelementary and determined by $\phi_*$, then we can define an equivalence relation on $Mod(\phi_*)$ via

$$A \sim B :\Leftrightarrow A \restriction \tau \cong B \restriction \tau.$$

It is easy to check that this is a $\Sigma^1_1$ relation, therefore by Burgess' theorem there are either at most $\aleph_1$ many equivalence classes or else perfectly many.

The set $\mathcal{A}$ of the previous example shows that the version of $VC_3$ for pseudoelementary classes is false.

$TVC2(S_\infty)$ cannot be generalized to coanalytic sets, as we can simply choose $A$ to be the set of codes of countable wellorders in the language $\tau$.

One might try to generalise $TVC3(S_\infty)$ with respect to the group action. Instead of the Borel algebra we could consider the previously mentioned $\sigma$ algebra $S := \sigma(\Sigma^1_1)$ generated by the analytic sets and demand that the action of $S_\infty$ on the Polish space is $S$ measurable. In this case we get a counterexample as follows: Let $\tau, \mathcal{A}$ be as in the previous example and $C := 2^{\mathbb{N}}$ be the Cantor space.

There is a continuous embedding from $\mathcal{A}$, equipped with the relative topology of $\mathcal{X}_\tau$, into $C$. This follows from the well known fact that every Polish space can be continuously embedded into $C$.

Since $\mathcal{A}$ is uncountable and analytic, there is a continuous embedding from $C$ into $\mathcal{A}$.

With a Schröder-Bernstein argument we can construct a bijection $\pi : \mathcal{A} \mapsto C$ which is $S$ measurable in both directions.

Using $\pi$, we get a $S$ measurable $S_\infty$ action on $C$ with uncountably but not perfectly many orbits.

$TVC(G)$ has been proven for several special cases, e.g.:

- If $G$ is nilpotent or if it admits an invariant metric (see [8]). In particular every abelian Polish group satisfies the $TVC$.
- If $G$ is locally compact. This is a well known and immediate consequence of a result by Edward Effros (see [4]).

**Definition.** Let $G, H$ be topological groups. $G$ is said to divide $H$ if there is a closed subgroup $H' \leq H$ and a continuous surjective group homomorphism

$$\pi : H' \twoheadrightarrow G.$$

We have seen that $TVC(S_\infty, \Sigma^1_1)$ is false. In [7] it is shown that for every Polish group $G$, $TVC(G, \Sigma^1_1)$ fails if and only if $S_\infty$ divides $G$. As a consequence we get that should $TVC(S_\infty)$ be false then $TVC(G)$ is true if and only if $S_\infty$ does not divide $G$.

## References

[1] Baldwin John, Friedman Sy David, Koerwien Martin, Laskowski Michael, "Three Red Herrings around Vaught's Conjecture", Transactions of the American Mathematical Society, vol. 268 (2016), pp. 3673-3694.

[2] Becker Howard, Kechris Alexander, "The Descriptive Set Theory of Polish Group Actions", London Mathematical Society Lecture Notes Series 232, Cambridge:Cambridge University Press (1996).

[3] Burgess John, "Equivalences generated by families of Borel sets", Proceedings of the American Mathematical Society, vol. 69 (1978), pp. 323-326

[4] Effros Edward G.,"Polish Transformation Groups and Classification Problems", General topology and modern analysis, Rao and McAuley (eds.), New York, Academic Press, 1981,pp. 217–227.

[5] Goldstern Martin, Judah Haim, "The Incompleteness Phenomenon", A K Peters/CRC Press; 1 edition (1998).

[6] Hjorth Greg, "A Note to Counterexamples to the Vaught Conjecture", Notre Dame Journal of Formal Logic, vol. 48 (2007), pp. 49-51

[7] Hjorth Greg, "Vaught's Conjecture on Analytic Sets", Journal of the American Mathematical Society vol. 14(2001), pp. 125-143.

[8] Hjorth Greg, Solecki Slawomir, " Vaught's Conjecture and the Glimm-Effros Property for Polish Transformation Groups ", Transactions of the American Mathemathical Society, vol. 351 (1999), number 7, pp. 2623-2643.

[9] Hodges Wilfrid, "Model Theory", Cambridge: Cambridge University Press (1993).

[10] Kechris Alexander, "Classical Descriptive Set Theory", Springer-Verlag (1995).

[11] Kunen Kenneth, "SetTheory: An Introduction to Independence Proofs", Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland, Amsterdam, 1983.

[12] Marker David, "Lectures on Infinitary Model Theory", Lecture Notes in Logic 46, Cambridge: Cambridge University Press (2016).

[13] Mayer Laura L., "Vaught's Conjecture for $o$-minimal Theories", Journal of Symbolic Logic vol. 53 (1988), pp. 146-149.

[14] Miller Arnold W., "Vaught's conjecture for theories of one unary operation", Fundamenta Mathematicae, vol. 111(1981), pp.135-141.

[15] Morel Anne C., "Structure and Order Structure in Abelian Groups", Colloquium Mathematicum, vol. 19 (1968), pp. 199-209.

[16] Rogers C. A. et al., Analytic sets, Academic Press, 1980.

[17] Rubin Matatyahu, "Theories of Linear Order", Israel Journal of Mathematics vol. 17 (1974), pp. 392-443.

[18] Shelah Saharon, Harrington Leo, Makkai Michael, "A Proof of Vaught's Conjecture for $\omega$-stable Theories", Israel Journal of Mathematics vol. 49 (1984), pp.259-280.

[19] Silver Jack, "Counting the number of equivalence classes of Borel and coanalytic equivalence relations", Annals of Mathematical Logic, vol. 18 (1980), pp. 1-28.

[20] Steel John R., "On Vaught's Conjecture", Cabal Seminar 76-77, Lecture Notes in Mathematics vol. 689, Springer Verlag, Berlin, 1978, pp. 193-208.

[21] Vaught Robert, "Denumerable models of complete theories" , Infinitistic Methods (Proc. Symp. Foundations Math., Warsaw, 1959) , Państwowe Wydawnictwo Nauk. Warsaw/Pergamon Press (1961) pp. 303–321.

[22] Vaught Robert, "Invariant Sets in Topology and Logic", Fund. Math. Vol. 82 (1974) pp. 269-293.

[23] Ziegler Martin, Bouscaren Elisabeth, "Interpreting in Graphs", preprint (Paris 1992), http://home.mathematik.uni-freiburg.de/ziegler/preprints/INTERPR.pdf