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"Planar Ising model – Color representation with random currents"

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Abstract

We use random currents to study the Ising model on planar graphs. In particular, building on the recent work of Aizenman et al.[2] on the stochastic geometry of random currents, we use a version of the Switching lemma to prove that the dual Ising model has a color representation in terms of random currents. This leads to a new formula for the truncated two-point function in terms of random currents. Moreover, we show that on the complete graph on three vertices, this representation is distinct from the random-cluster representation, even on the level of partitions. We also revisit some well-known expansions of the Ising model.

Zusammenfassung

Wir verwenden zufällige Ströme (random currents), um das Ising-Modell in der Ebene zu untersuchen. Aufbauend auf den neuen Resultaten über die stochastische Geometrie der zufälligen Ströme von Aizenman et al.[2] benutzen wir eine Version des Switching Lemmas, um eine neue Darstellung für das duale Ising-Modell herzuleiten. Dies führt zu einer neuen Formel für die Zwei-Punkt-Funktion (truncated two-point function) mit zufälligen Strömen. Wir zeigen weiter, dass sich diese Darstellung von der random-cluster-Darstellung sogar auf dem Niveau der Partitionen unterscheidet. Im ersten Teil der Arbeit werden bekannte graphische Darstellungen für das Ising-Modell vorgestellt.

Contents

1 Introduction

In his doctoral thesis in 1895, Pierre Curie observed a phase transition between ferromagnetic[1] and paramagnetic[2] behaviour in some materials such as iron, cobalt, or nickel. When the temperature increases above a certain point (Curie's critical point), the material loses its permanent magnetic properties. In 1920, the German physicist W. Lenz[19] suggested a mathematical model with an attempt to capture the sudden qualitative change in behaviour and assigned it as a thesis topic to his PhD student E. Ising [17]. Ising proved the model exhibits no phase transition in one dimension on $\mathbb{Z}^d, d = 1$, and conjectured the same for all higher dimensions $d \geq 2$. The model was abandoned for many years until 1936 when R. Peierls [25] showed a phase transition occurs for $d = 2$, thus disproving Ising's false conjecture. The now called Ising model became one of the most studied models in statistical mechanics for physicists and mathematicians up to this day.

1.1 Structure of the thesis

The first part of the thesis is mainly expository, revisiting some well-known graphical expansions of the Ising model. We present the high-temperature expansion and the low-temperature expansion. These are of great historical significance and were utilized in the famous Peierls argument on the existence of a phase transition on $\mathbb{Z}^2$. We then explore the relationship between the two expansions, known as the Kramers–Wannier duality [18]. The random-cluster expansion, introduced by K. Fortuin and P. Kasteleyn in 1972 [11] and its connection to the Ising model via the Edwards–Sokal coupling [9], provide the prototypical example of the so-called ‘divide and color’ model. The well-known ideas presented in this chapter are of great significance in statistical mechanics. For a detailed modern exposition of these topics, we refer the reader to [6, 15, 12, 4].

In the second part, we discuss the random current expansion and the related Switching lemma; ideas introduced by Griffiths, Hurst, and Sherman [14]. Random currents became of great interest to mathematicians ever since and were used to show continuity of the phase transition of the Ising model on various graphs, including $\mathbb{Z}^3$ [1]. Recently, a new distributional connection between random currents and the random-cluster model was established by W. Werner and T. Lupu [23]. We apply a modified version of the Switching lemma and related ideas developed in the paper by M. Aizenman et al.[2] to show the dual planar Ising model has a color representation in terms of the dual double random current measure (Theorem 3.7). We then use it to derive a new expression

[1] material which retains its magnetization even in the absence of an external magnetic field

[2] material with no magnetization, but with the ability to gain magnetization when immersed in an external magnetic field

for the truncated two-point function (Theorem 3.13), which yields an explicit bound in terms of connection probabilities in the Ising model. Lastly, we compare the new color representation with currents to the FK-Ising representation and show they have distinct distributions on the level of partitions. This supports the recent result of M. Forsström [10] on the non-uniqueness of the color representation for the Ising model.

1.2 Ising model

The model can be defined explicitely on any finite graph $G = (V(G), E(G)) = (V, G)$. For each vertex $x \in V(G)$, we consider an associated *spin variable* σ_x taking values in $\{+1, -1\}$ and we call the collection of these spin variables $\sigma = (\sigma_x : x \in V(G)) \in \{\pm 1\}^{V(G)}$ a *spin configuration*. The *Ising model* on G with coupling constants $(J_{xy})_{xy \in E(G)} \geq 0$ [3], *inverse temperature* β and *external field* h is the probability measure, which assigns to each spin configuration σ a probability proportional to $\exp(-\beta \mathbf{H}(\sigma))$, where

$$\mathbf{H}(\sigma) := -\sum_{xy \in E(G)} J_{xy}\sigma_x\sigma_y - h\sum_{x \in V(G)} \sigma_x$$

is the *energy* of the spin configuration. Since we will be working with no external magnetic field (h=0) and a constant temperature, it is convenient to implicitly include β in the coupling constants J_{xy}. With this, the Ising measure μ (= $\mu_{G,J_{xy}}$) with free boundary conditions is defined as

$$\mu(\sigma) := \frac{\exp(-\mathbf{H}(\sigma))}{Z(G, J_{xy})},$$

$$Z(G, J_{xy}) := \sum_{\sigma \in \{\pm 1\}^{V(G)}} \exp(-\mathbf{H}(\sigma)),$$

where $Z(G, J_{xy})$ is the normalizing constant called the *partition function*.
Given a function $f : \{\pm 1\}^{V(G)} \to \mathbb{R}$, we will be interested in the expected value of f under μ, which we denote by $\langle f \rangle$, or $\mu(f)$. We will also use the notation

$$Z(f) := \sum_{\sigma \in \{\pm 1\}^{V(G)}} f(\sigma)\exp(-\mathbf{H}(\sigma)).$$

Remark 1.1. Throughout the thesis, G will denote an embedding of a finite planar graph in $\mathbb{R}^2$, meaning that the edges will be depicted by bounded simple arcs, which do not cross except at the endpoints. The *faces* are defined as the connected components of the plane without the edges.

[3] The positive coupling $J_{xy} \geq 0$ makes the model *ferromagnetic*, which means the measure assigns a higher probability to configurations with aligned spins. This is in contrast with the *ferrimagnetic* behaviour, which favors anti-parallel alignment of spins.

1.3 Infinite-volume limit

Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be an infinite, but locally finite planar graph and write $\Omega := \{-1, +1\}^{\mathbb{V}}$ for the set of all spin configurations. Given a finite subgraph G of $\mathbb{G}$, we define the boundary of G as $\partial G := \{x \in G : \exists y \in \mathbb{G} \setminus G \text{ with } xy \in \mathbb{E}\}$. For a fixed spin configuration $\tau \in \Omega$, consider the finite set $\Omega_G^\tau := \{\sigma \in \Omega : \sigma|_{\mathbb{V} \setminus V(G)} = \tau\}$ of spin configurations on $\mathbb{G}$, which coincide with τ outside of G. The Ising model on G with boundary conditions τ is a probability measure on Ω_G^τ given by

$$\mu_G^\tau(\sigma) := \frac{\exp(-\mathbf{H}_G^\tau(\sigma))}{\sum_{\sigma \in \Omega_G^\tau} \exp(-\mathbf{H}_G^\tau(\sigma))}, \quad \text{where}$$

$$\mathbf{H}_G^\tau(\sigma) := -\sum_{xy \in E(G)} J_{xy}\sigma_x\sigma_y - \sum_{x \in \partial G, y \notin V(G)} J_{xy}\sigma_x\sigma_y = -\sum_{\{x,y\} \cap V(G) \neq \emptyset} J_{xy}\sigma_x\sigma_y.$$

Remark 1.2. We will only consider *nearest neighbor* models, meaning that $J_{xy} = 0$ if $xy \notin \mathbb{E}$, hence the measure μ_G^τ is influenced only by the neighboring vertices outside of G. Of particular interest are the $+$, and $-$ boundary conditions, where $\tau = +1$ resp. $\tau = -1$ for all vertices in $\mathbb{V}$. In this case, we adopt the standard notation μ_G^+ resp. μ_G^-.

We now state two fundamental properties of the Ising model; the Spatial Markov property (also known as the Dobrushin-Landford-Ruelle property) and the FKG-Inequality. These also hold for many other models in statistical mechanics, such as the random-cluster model as well. They are also crucial for the construction of the infinite-volume Ising measures on $\mathbb{G}$. For details, we refer to [4].

Proposition 1.3 (Spatial Markov Property). *Let $G \subset F$ be finite subgraphs of $\mathbb{G}$ and $\tau \in \Omega$, $\eta \in \Omega_F^\tau$. Then*

$$\mu_F^\tau(\,\cdot \mid \sigma_x = \eta_x \text{ on } V(F) \setminus V(G)) = \mu_G^\eta(\,\cdot\,).$$

Note that the only relevant 'outer' spins that determine the measure μ_G^τ, are the neighbouring vertices outside of G. In particular, the Ising measure with $+$ boundary conditions can be interpreted as the free model on a bigger graph, where we condition on the 'outer' spins being $+$.

The second fundamental property relies on the partial order on the spin configurations, where $\sigma \le \sigma'$, whenever $\sigma_x \le \sigma'_x$ for all vertices x. We call an event $A \subset \Omega$ increasing, if $\sigma \le \sigma'$ and $\sigma \in A$ imply $\sigma' \in A$. Increasing events are exactly those events, which are closed under the operation of flipping $-$ spins to $+$ spins. The FKG-inequality asserts that increasing events are positively correlated.

Proposition 1.4 (FKG-inequality). *Let $G \subset \mathbb{G}$ be a finite subgraph, $\tau \in \Omega$ and A, B two increasing events. Then*

$$\mu_G^\tau(A \cap B) \ge \mu_G^\tau(A)\mu_G^\tau(B).$$

To define the Ising measure on the infinite graph $\mathbb{G}$, we proceed indirectly using a limiting procedure, since the partition function on an infinite graph is not well-defined. The space $\Omega = \{-1, 1\}^{\mathbb{V}}$ can be naturally equipped with a sigma-algebra $\mathcal{F} = \sigma(\mathcal{A})$ generated by the cylinder sets (or local events). Moreover, endowed with the product topology, Ω becomes a Cantor space. In particular Ω is compact (by Tychonoff's theorem), metrizable, separable (cylinder events form a countable basis), and as such Ω is a Polish space. We also consider the space $\mathcal{PM}(\Omega)$ of probability measures on Ω, equipped with the topology of weak convergence. Note that μ_n converges to μ in $\mathcal{PM}(\Omega)$ (or weakly), if the convergence is on every local event.

We say that an increasing sequence of subgraphs $(G_n)_{n\in\mathbb{N}} \subseteq \mathbb{G}$ *exhausts* $\mathbb{G}$, if for every finite $F \subseteq \mathbb{G}$ there exists a sufficiently large N such that $F \subseteq G_n$ for all $n \geq N$, and write $G_n \uparrow \mathbb{G}$.

Theorem 1.5 (Thermodynamic limit). *Let $(J_e)_{e\in\mathbb{E}} \geq 0$ be a family of coupling constants and $(G_n)_{n\in\mathbb{N}}$ a sequence of finite subgraphs with $G_n \uparrow \mathbb{G}$. There exists a probability measure μ^+ on $(\Omega, \mathcal{F})$ such that $\mu^+_{G_n} \to \mu^+$ in $\mathcal{PM}(\Omega)$ as $n \to \infty$. Moreover, the limit is independent of the sequence $G_n \uparrow \mathbb{G}$.*

Proof. By metrizability and compactness of Ω, the family of measures $\{\mu^+_{G_n} | n \in \mathbb{N}\}$ is automatically tight. Therefore, by Prokhorov's theorem, the family is relatively compact in $\mathcal{PM}(\Omega)$, and hence it contains an accumulation point, which we denote μ^+. This proves existence of a limit on a subsequence.
Let $A \in \mathcal{F}$ be any increasing local event and $G \subseteq F \subseteq \mathbb{G}$ two finite subgraphs. From the Spatial Markov Property and the FKG-inequality we obtain

$$\mu^+_G(A) = \mu^+_F(A \mid \sigma_x = 1 \text{ on } V(F)\setminus V(G)) = \frac{\mu^+_F(A \cap \sigma_x = 1 \text{ on } V(F)\setminus V(G))}{\mu^+_F(\sigma_x = 1 \text{ on } V(F)\setminus V(G))} \geq \mu^+_F(A).$$

Consequently, $\mu^+_{G_n}(A) \to \mu^+(A)$ for any sequence $G_n \uparrow \mathbb{G}$. Since every cylinder event can be expressed in terms of increasing events using the inclusion-exclusion principle, it follows that increasing events form a convergence-determining class [3]. This proves the limit does not depend on the specific sequence. □

2 Expansions of the Ising model

There are various ways to rewrite the partition function of the Ising model in terms of other combinatorial objects. Here, we consider the graphical expansions; the high-temperature, the low-temperature, and the FK-random-cluster expansion. This will allow us to change the perspective from the Ising measure on spin configurations to related probability measures on subsets of edges. Throughout, G will denote a finite planar graph.

2.1 High-temperature expansion

For any subset of egdes $\eta \subseteq E(G)$, let $\Delta(\eta, x)$ be the number of edges in η, which are incident to x and $\partial\eta := \{x \in V(G) : \Delta(\eta, x) \equiv 1 \pmod 2\}$ denote the set of all x in $V(G)$, which have an odd number of incident edges in η. We will call such vertices *sources* of η. We call a subset $\eta \subseteq E(G)$ *sourceless* if $\partial\eta = \emptyset$. Observe that $\partial\eta$ contains an even number of vertices.

The $+/-$ symmetry of the spins leads to the useful observation that for any exponents λ_x, $x \in V(G)$, we have

$$\sum_{\sigma \in \{\pm 1\}^{V(G)}} \prod_{x \in V(G)} \sigma_x^{\lambda_x} = \begin{cases} 2^{|V(G)|}, & \text{if } \lambda_x \equiv 0 \pmod 2 \ \forall x \in V(G), \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Indeed, if λ_x is even for all $x \in V(G)$, then $\prod_{x \in V(G)} \sigma_x^{\lambda_x} = 1$ and there are $2^{|V(G)|}$ such terms. If there exists $y \in V(G)$ with λ_y odd, then

$$\sum_{\sigma \in \{\pm 1\}^{V(G)}} \prod_{x \in V(G)} \sigma_x^{\lambda_x} = \sum_{\sigma:\sigma_y = 1} \sigma_y \prod_{x \in V(G) \setminus y} \sigma_x^{\lambda_x} + \sum_{\sigma:\sigma_y = -1} \sigma_y \prod_{x \in V(G) \setminus y} \sigma_x^{\lambda_x} = 0.$$

The high-temperature expansion of the partition function goes back to van der Waerden[27] and utilizes the identity $\exp(J_{xy}\sigma_x\sigma_y) = \cosh(J_{xy})(1 + \tanh(J_{xy})\sigma_x\sigma_y)$. We shall use the standard notation $\sigma_A = \prod_{x \in A} \sigma_x$ and follow the calculation in [6].

$$
\begin{aligned}
Z(\sigma_A) &= \sum_{\sigma\in\{-1,1\}^{V(G)}} \sigma_A \exp(-H(\sigma)) = \sum_{\sigma\in\{-1,1\}^{V(G)}} \sigma_A \prod_{xy\in E(G)} \exp(J_{xy}\sigma_x\sigma_y) \\
&= c_0 \sum_{\sigma\in\{-1,1\}^{V}} \sigma_A \prod_{xy\in E(G)} (1+\tanh(J_{xy})\sigma_x\sigma_y) = c_0 \sum_{\sigma\in\{-1,1\}^{V}} \sigma_A \sum_{\eta\subseteq E(G)} \prod_{xy\in\eta} \tanh(J_{xy})\sigma_x\sigma_y \\
&= c_0 \sum_{\eta\subseteq E(G)} x(\eta) \sum_{\sigma\in\{-1,1\}^{V(G)}} \sigma_A \prod_{xy\in\eta} \sigma_x\sigma_y = c_0 \sum_{\eta\subseteq E(G)} x(\eta) \sum_{\sigma\in\{-1,1\}^{V(G)}} \prod_{x\in V(G)} \sigma_x^{\deg(\eta,x)+\mathbb{I}[x\in A]} \\
&= c_0\, 2^{|V(G)|} \sum_{\eta:\partial\eta=A} x(\eta),
\end{aligned}
$$

where the last equality follows from (2.1), $c_0 = \prod_{e\in E(G)} \cosh(J_e)$ and $x(\eta) = \prod_{e\in\eta} \tanh(J_e)$. In particular, we obtain an expansion of the partition function

$$
Z(G, J_{xy}) = Z(1) = c_0\, 2^{|V(G)|} \sum_{\eta:\partial\eta=\emptyset} x(\eta).
$$

This motivates the definition of the high-temperature measure.

Definition 2.1. For any $A\subseteq V(G)$ the high-temperature measure $\nu^A = \nu^A_{G,J_e}$ on subsets of edges is defined as

$$
\nu^A(\eta_0) = \begin{cases} \dfrac{x(\eta_0)}{\sum_{\partial\eta=A} x(\eta)}, & \text{if } \partial\eta_0 = A, \\ 0, & \text{if } \partial\eta_0 \neq A. \end{cases}
$$

We will be mostly interested in the case $A=\emptyset$. The high-temperature measure $\nu^\emptyset$ is supported on *sourceless* configurations $\partial\eta=\emptyset$, which coincide with even subgraphs of G.

2.2 Low-temperature expansion

The dual G^* of G is the planar graph (multigraph) whose vertices V^* are the faces of G (including the outer face) and whose edges E^* are connecting neighbouring faces.[1] By construction, E and E^* are in bijective correspondence and the faces of G^* correspond to the vertices $V(G)$. We note that the dual of the dual planar graph is isomorphic to the primal graph.[2] This allows us to freely change perspectives between G and G^*. (see [24])

The low-temperature is the oldest expansion and it expresses the partition function in terms of even subgraphs on the dual graph G^*. It is motivated by the observation that the information of a spin configuration σ is captured in the $+/-$ interface, separating the $+$ spins from the $-$ spins. (We only lose the one bit information because of the $+/-$ symmetry.) For an edge e with endpoints x and y let $I_\sigma(e) = \sigma_x\sigma_y$. We define the $+/-$ interface $C(\sigma) := \{e^*\in E^* : I_\sigma(e) = -1\}$. However, we shall mostly work in

[1] Bridges in G correspond to loops in G^* and a boundary between two faces in G made of multiple edges corresponds to a multiedge in G^*.

[2] Different embeddings of the same planar graph in $\mathbb{R}^2$ might result in non-isomorphic dual graphs.

the context of the dual Ising measure, where spins are assigned to faces of G and have $C(\sigma) := \{e \in E : I_\sigma(e^*) = -1\}$. This is depicted in Figure 2.1b, which illuminates the correspondence between spin configurations on faces and even subgraphs. We include a combinatorial proof of the statement to illustrate the switching principle, which will play an important role later.

Lemma 2.2. *Let G be a connected planar graph. The map $C : \sigma \mapsto C(\sigma)$ is a two-to-one correspondence from the set of spin configurations on faces $\{\pm 1\}^{F(G)}$ to the set of all even subgraphs $\{\eta \subseteq E : \partial\eta = \emptyset\}$. An equivalent statement holds for C from $\{\pm 1\}^{V(G)}$ to $\{\eta \subseteq E^* : \partial\eta = \emptyset\}$.*

Proof. For any vertex $x \in V(G)$ the sign of σ on the incident faces changes an even number of times, hence $C(\sigma)$ is an even subgraph. It suffices to determine the cardinality of the set of even subgraphs $\{\eta \subseteq E : \partial\eta = \emptyset\}$.

For two subsets $\eta, \xi \subseteq E$, let $\eta \triangle \xi := (\eta \setminus \xi) \cup (\xi \setminus \eta)$ denote the symmetric difference. In this context, the symmetric difference corresponds to the group addition in $\mathbb{Z}_2^E$ and it is compatible with the operator ∂ in the sense that $\partial(\eta \triangle \xi) = \partial\eta \triangle \partial\xi$. Since G is connected, for any $x, y \in V$ there exists $\xi \subseteq E$ with $\partial\xi = \{x, y\}$. Given any $A \subseteq V$ with $|A|$ even, we can write $A = \{x_1, y_1, \ldots x_n, y_n\}$. Taking $\xi_k \subseteq E$ such that $\partial\xi_k = \{x_k, y_k\}$ for $k = 1, .., n$ and setting $\xi := \xi_1 \triangle \ldots \triangle \xi_n$, gives $\xi \subseteq E$ with $\partial\xi = A$. With this, the map $\eta \mapsto \xi \triangle \eta$ defines an involution between $\{\eta \subseteq E : \partial\eta = \emptyset\}$ and $\{\eta \subseteq E : \partial\eta = A\}$. In particular, this shows that the cardinality of $\{\eta \subseteq E : \partial\eta = A\}$ does not depend on A and thus

$$2^{|E|} = \sum_{A \subseteq V} |\{\eta \subseteq E : \partial\eta = A\}| = \underbrace{|\{A \subseteq V : |A| \text{ is even}\}|}_{2^{|V|-1}} \, |\{\eta \subseteq E : \partial\eta = \emptyset\}|$$

And by Euler's formula

$$|\{\eta \subseteq E : \partial\eta = \emptyset\}| = 2^{|E|-|V|+1} = 2^{|F|-1},$$

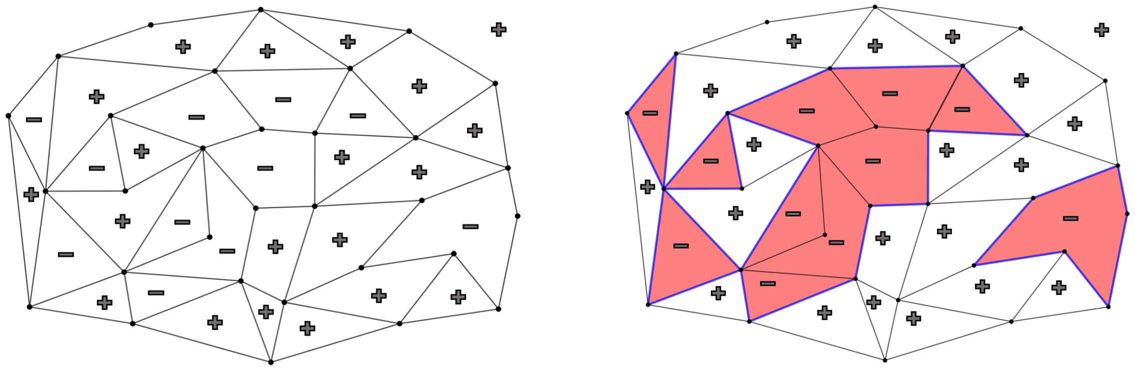
which shows there is a two-to-one correspondence between spin configurations on the faces of G and even subgraphs of G. □

More on even graphs can be found in [16]. We shall expand the switching principle used in the above argument when discussing the Switching lemma in the section on random currents.

As a consequence of the correspondence between spin configurations σ and sourceless subsets $\eta \subseteq E$ we get the low-temperature expansion

$$\begin{aligned} Z(G, J_{xy}) &= \sum_{\sigma \in \{\pm 1\}^{V(G)}} \prod_{xy \in E(G)} \exp(J_{xy}\sigma_x\sigma_y) = c_1 \sum_{\sigma \in \{\pm 1\}^{V(G)}} \prod_{e \in C(\sigma)} \exp(-2J_e) \\ &= 2c_1 \sum_{\substack{\eta \subseteq E \\ \partial\eta = \emptyset}} \prod_{e \in \eta} \exp(-2J_e), \end{aligned}$$

where $c_1 = \prod_{xy \in E(G)} \exp(J_{xy})$.



(a) $\sigma \in \{\pm 1\}^{V}$ (b) blue contour $C(\sigma) \subseteq E(G)$

Figure 2.1: low-temperature expansion

The dual Ising model assigns spins σ_u to the faces of G. It is defined by

$$\mu^*(\sigma) = \frac{1}{Z}\exp\left(\sum_{uv\in E} J_{uv}\sigma_u\sigma_v\right),$$

where

$$\boxed{\exp(-2J_e^*) = \tanh(J_e)} \tag{2.2}$$

and where $Z^* = Z(G^*, J_e^*)$ denotes the normalizing constant. One can check that the function $J_e \mapsto J_e^*$ induced by the coupling defines an involution. Therefore, the primal and the dual model enjoy a symmetry, which allows for a change in perspective between the two models. Introducing boundary conditions breaks this symmetry. The dual Ising model μ_+ with + boundary conditions is obtained by conditioning on the outer face g to be +1, i.e. $\mu_+(\sigma) = \mu(\sigma|\sigma_{\mathrm{g}=+1})$. Furthermore, the restriction $C_+ = C|_{\{\sigma:\sigma_{\mathrm{g}}=+1\}}$ becomes a bijection between spin configurations on the dual graph G^* and even subgraphs in G. Any $\eta \subseteq E(G)$ with $\partial\eta = \emptyset$ uniquely determines $\sigma = C_+^{-1}(\eta)$. The sourceless configuration $C(\sigma)$ is depicted as the blue contour in Fig.2.1b. The low-temperature expansion of the dual Ising model can be expressed in terms of the high-temperature expansion using the coupling 2.2.

$$Z(G^*, J_e^*) = 2c_1 \sum_{\substack{\eta\subseteq E(G)\\ \partial\eta=\emptyset}} \prod_{e\in\eta} \exp(-2J_e^*) = 2c_1 \sum_{\substack{\eta\subseteq E(G)\\ \partial\eta=\emptyset}} x(\eta) = \frac{2c_1}{c_0 2^{|V(G)|}} Z(G, J_e), \tag{2.3}$$

where $c_1 = \prod_{e\in E} \exp(J_e^*)$.

With this we obtain a distributional identity between the dual Ising measure μ^* and the high-temperature measure $\nu^{\emptyset}$.

Proposition 2.3. *Sampling σ according to μ^* has the same distribution as sampling $\eta \subseteq E(G)$ according to $\nu^{\emptyset}$ and then taking $\sigma = C_+^{-1}(\eta)$ or $\sigma = C_-^{-1}(\eta)$ with equal probability. In particular, $\mu_+ = C_+^{-1}\nu^{\emptyset}$ and $\nu^{\emptyset} = C\mu_+ = C\mu^*$.*

Proof. We want to show $\mu(\sigma) = 1/2\,\nu^{\emptyset}(C(\sigma))$. Using (2.3), the relation $\tanh(J_e) = \exp(-2J_e)$ and the definition of $C(\sigma)$, we obtain

$$\frac{\nu^{\emptyset}(C(\sigma))}{2} = \frac{1}{2}\frac{x(C(\sigma))}{\sum\limits_{\substack{\eta \subset E(G)\\ \partial\eta=\emptyset}} x(\eta)} = \frac{c_1 \prod\limits_{e\in C(\sigma)} \tanh(J_e)}{Z(G, J_e)} = \frac{\prod\limits_{e\notin C(\sigma)} \exp(J_e) \prod\limits_{e\in C(\sigma)} \exp(-J_e)}{Z(G, J_e)} = \mu(\sigma).$$

This shows that the two different ways of sampling a spin configuration σ have the same law. Conditioning on $\sigma_g = 1$ yields $\mu_+ = C_+^{-1}\nu^{\emptyset}$. □

The relationship between the low and high-temperature expansion is known as the Kramers–Wannier duality [18]. In the second part of the thesis, we will be interested in the correlation functions $\langle\sigma_u\sigma_v\rangle_\mu$ for faces u, v of G. We also note that Proposition 2.3 can be used to define the high-temperature measure $\nu^{\emptyset}_{\mathrm{G}}$ in the infinite-volume limit. Let $G_n \uparrow$ G a sequence of finite planar subgraphs, which exhausts G. From the weak convergence of μ_{G_n} to μ_{G} and continuity of the map C between spin configurations on faces of G and edge-configurations on G, it follows that $\nu^{\emptyset}_{G_n} = C\,\mu_{G_n}$ converges weakly to $\nu^{\emptyset}_{\mathrm{G}} := C\,\mu_{\mathrm{G}}$.

2.3 FK-random-cluster expansion

The random-cluster model was invented by Fortuin and Kasteleyn [11]. It generalizes to other models in statistical mechanics and unifies Bernoulli percolation, Ising, and Potts models into one framework. We will only consider its applications to the Ising model. A detailed analysis of the random-cluster model can be found in [15].

Any subset $\omega \subset E(G)$ induces a spanning subgraph $G(\omega) \subset G$, where $V(\omega) = V(G)$ and $E(\omega) = \omega$.

Definition 2.4. Let A be a subset of vertices in $V(G)$. We denote by F_A the set of all $\omega \subset E(G)$ such that each connected component of $G(\omega)$ contains an even number of points in A. (An isolated point is counted as a component of $G(\omega)$ as well.)

In particular, for $A = \{x, y\}$ we have $\omega \in F_A$ if and only if there exists a path from x to y using only edges in ω. Besides, if A has an odd number of points, F_A must be empty since at least one component of ω contains an odd number of points in A. Henceforth, we shall assume A has even cardinality. Complementary to the set $C(\sigma)$, we define the set of edges $E(\sigma) := \{xy \in E(G) : \sigma_x = \sigma_y\}$.

Lemma 2.5. *Let A be a set of vertices in $V(G)$ and let ω_0 be a fixed subset in $E(G)$. Then*

$$\sum_{\sigma:\omega_0 \subset E(\sigma)} \sigma_A = \begin{cases} 2^{k(\omega_0)}, & \text{if } \omega_0 \in F_A \\ 0, & \text{if } \omega_0 \notin F_A \end{cases}$$

Proof. Let σ be a spin configuration and ω_0 an edge configuration such that $\omega_0 \subset E(\sigma)$. This means the sign of σ is constant on connected components of $G(\omega_0)$. First assume

$\omega_0 \in F_A$. Then each of the components in $G(\omega_0)$ contains an even number of points in A. It follows $\sigma_A = 1$ and we get

$$\sum_{\sigma:\omega_0 \subseteq E(\sigma)} \sigma_A = |\{\sigma : \omega_0 \subseteq E(\sigma)\}| = 2^{k(\omega_0)}$$

Now let $\omega_0 \notin F_A$. Then there is a connected component ϖ_0 of $G(\omega_0)$ (possibly a single point), which contains an odd number of points from A. Therefore, by changing the sign of σ on ϖ_0, the sign of σ_A also changes. This leads to an involution between

$$S_1 = \{\sigma : \omega_0 \subseteq E(\sigma), \sigma = +1 \text{ on } \varpi_0\} \to \{\sigma : \omega_0 \subseteq E(\sigma), \sigma = -1 \text{ on } \varpi_0\} = S_2$$

Hence

$$\sum_{\sigma:\omega_0 \subseteq E(\sigma)} \sigma_A = \sum_{\sigma \in S_1} \sigma_A + \sum_{\sigma \in S_2} \sigma_A = 0.$$

□

Given probabilities $p_{xy} \in [0,1]$ we define a measure ϕ_{G,p_e} on subsets of $E(G)$ by assigning weights

$$r(\omega) = 2^{k(\omega)} \prod_{xy \in \omega} p_{xy} \prod_{xy \notin \omega} (1 - p_{xy})$$

to each $\omega \subseteq E(G)$. This model, known as the FK-random-cluster model, is directly connected with the Ising model via the so called Edwards–Sokal coupling[9], where the probabilities p_{xy} are specified as $p_{xy} := 1 - \exp(-2J_{xy})$. Indeed, with this coupling we have the identity $\exp(J_{xy}\sigma_x\sigma_y) = \exp(J_{xy})(p_{xy}\mathrm{I}[\sigma_x = \sigma_y] + 1 - p_{xy})$ and one obtains

$$\begin{aligned}
Z(\sigma_A) &= c_1 \sum_{\sigma \in \{-1,1\}^{V(G)}} \sigma_A \prod_{xy \in E(G)} ((p_{xy}\mathrm{I}[\sigma_x = \sigma_y]) + (1 - p_{xy})) \\
&= c_1 \sum_{\sigma \in \{-1,1\}^{V(G)}} \sigma_A \sum_{\substack{\omega \subseteq E(G) \\ \omega \subseteq E(\sigma)}} \prod_{xy \in \omega} p_{xy} \prod_{xy \in \omega^c} (1 - p_{xy}) \\
&= c_1 \sum_{\omega \subseteq E(G)} \prod_{xy \in \omega} p_{xy} \prod_{xy \in \omega^c} (1 - p_{xy}) \sum_{\sigma:\omega \subseteq E(\sigma)} \sigma_A \\
&= c_1 \sum_{\omega \in F_A} 2^{k(\omega)} \prod_{xy \in \omega} p_{xy} \prod_{xy \in \omega^c} (1 - p_{xy}) = c_1 \sum_{\omega \in F_A} r(\omega),
\end{aligned}$$

where we have used that the condition $\omega \subseteq E(\sigma)$ is satisfied if and only if ω has a constant sign on its connected components and the Lemma 2.5.
In particular, we obtain the FK-expansion of the partition function:

$$Z = c_1 \sum_{\omega \subseteq E(G)} r(\omega).$$

Definition 2.6. The coupling $p_e = 1 - \exp(-2J_e)$ defines the FK-Ising measure $\phi = \phi_{G,p_e}$ on $E(G)$ by

$$\phi(\omega) := \frac{r(\omega)}{\sum_{\omega \subseteq E(G)} r(\omega)}.$$

As an immediate consequence of the random-cluster expansion, we can express the correlation function as a probability in terms of the FK-Ising measure. We have

$$\langle\sigma_A\rangle = \frac{Z(\sigma_A)}{Z} = \frac{\sum_{\omega\in F_A} r(\omega)}{\sum_{\omega\subseteq E(G)} r(\omega)} = \phi(F_A). \tag{2.4}$$

2.4 Divide and Color

Sampling $\omega\subseteq E(G)$ according to ϕ and then 'coloring' each connected component of $G(\omega)$ blue or red (+ or $-$) independently with probability $1/2$, we obtain a spin configuration σ. We denote the measure on spin configurations arising from this procedure by ϕ'. We claim that ϕ' has the same distribution as the Ising measure μ. This has several useful consequences, especially the fact that Ising correlations have a representation as connectivity probabilities in the random-cluster model. A direct way to verify that the described color process gives the Ising model is to express:

$$\phi'(\sigma) = \sum_{\omega:\omega\subseteq E(\sigma)} \phi(\omega)\frac{1}{2^{k(\omega)}} = \frac{c_1}{Z}\sum_{\omega:\omega\subseteq E(\sigma)}\prod_{xy\in\omega} p_{xy}\prod_{xy\notin\omega}(1-p_{xy}) = \frac{c_1}{Z}\prod_{xy\notin E(\sigma)}(1-p_{xy}) = \mu(\sigma),$$

where the third equality follows from

$$\sum_{\omega:\omega\subseteq E(\sigma)}\frac{\prod_{xy\in\omega} p_{xy}\prod_{xy\notin\omega}(1-p_{xy})}{\prod_{xy\notin E(\sigma)}(1-p_{xy})} = \sum_{\omega:\omega\subseteq E(\sigma)}\prod_{e\in\omega}p_e\prod_{e\in E(\sigma)\setminus\omega}(1-p_e) = \prod_{e\in E(\sigma)}(p_e+(1-p_e)) = 1$$

and the last equality follows from

$$\mu(\sigma) = \frac{\prod_{xy\in E(\sigma)}\exp(J_{xy})\prod_{xy\notin E(\sigma)}\exp(-J_{xy})}{\prod_{xy\in E(G)}\exp(J_{xy})\sum_{\omega\subseteq E(G)} r(\omega)} = \frac{\prod_{xy\notin E(\sigma)}\exp(-2J_{xy})}{\sum_{\omega\subseteq E(G)} r(\omega)}.$$

However, there is another, more general way to verify that two measures on $\{-1,1\}^{V(G)}$ coincide. It suffices to check the expectations of correlation functions coincide for both measures. Since we will use this result later, we want to formulate it as a lemma. The ideas used in the proof appear in [12].

Lemma 2.7. *Let μ_1 and μ_2 be two measures on $\{-1,1\}^{V(G)}$. If all correlation functions σ_A have the same expectation under both measures, i.e. if $\langle\sigma_A\rangle_{\mu_1} = \langle\sigma_A\rangle_{\mu_2}$ $\forall A\subseteq V(G)$, then $\mu_1=\mu_2$.*

Proof. Given $\sigma,\sigma'\in\{-1,1\}^{V(G)}$ we have

$$\sum_{A\subseteq V(G)}\sigma_A\sigma'_A = \begin{cases} 2^{|V(G)|}, & \text{if } \sigma=\sigma',\\ 0, & \text{if } \sigma\neq\sigma'.\end{cases}$$

Indeed, if $\sigma \neq \sigma'$, then either $|\{x \in V(G) : \sigma_x \neq \sigma'_x\}|$ is odd, in which case for any $A \subseteq V(G)$ one of the sets A, or $V(G) \setminus A$ contains an even number of x with $\sigma_x \neq \sigma'_x$ and the other set contains an odd number of such x. Therefore, $A \mapsto V(G) \setminus A$ is an involution between $\{A \subseteq V(G) : \sigma_A\sigma'_A = 1\}$ and $\{A \subseteq V(G) : \sigma_A\sigma'_A = -1\}$ and the above sum cancels out. If, on the other hand, $|\{x \in V(G) : \sigma_x \neq \sigma'_x\}|$ is even, we can fix $x_0 \in V(G)$ such that $\sigma_{x_0} \neq \sigma'_{x_0}$ and then use the same reasoning on $V(G) \setminus \{x_0\}$ to see the sum vanishes as well. The case $\sigma = \sigma'$ is trivial.
Now for any function $f : \{\pm 1\}^{V(G)} \to \mathbb{R}$, the above observation allows us to rewrite

$$\begin{aligned} f(\sigma) &= \sum_{\sigma' \in \{\pm 1\}^{V(G)}} f(\sigma') 1_{\{\sigma=\sigma'\}}(\sigma) \\ &= \sum_{\sigma' \in \{\pm 1\}^{V(G)}} f(\sigma') 2^{-|V(G)|} \sum_{A \subseteq V(G)} \sigma_A \sigma'_A \\ &= \sum_{A \subseteq V(G)} \underbrace{2^{-|V(G)|} \sum_{\sigma' \in \{\pm 1\}^{V(G)}} f(\sigma')\sigma'_A}_{\lambda_A} \sigma_A = \sum_{A \subseteq V(G)} \lambda_A \sigma_A. \end{aligned}$$

By linearity, it follows $\langle f \rangle_{\mu_1} = \langle f \rangle_{\mu_2}$ for every f. Taking $f = 1_A$ gives the result. ☐

Corollary 2.8. *Sampling $\omega \sim \phi_{G,p_e}$ and then assigning $+1$ or -1 to each component of ω independently with probability $1/2$ has the same distribution as the Ising measure μ_{G,J_e} i.e. $\phi'_{G,p_e} = \mu_{G,J_e}$.*

Proof. Let $A \subseteq V(G)$. We notice that if $\omega \in F_A$, then after coloring $G(\omega)$ on it's components, one necessarily gets $\sigma_A = 1$. If, on the other hand, $\omega \notin F_A$, then $\sigma_A = \pm 1$ with equal probability. It follows $\langle \sigma_A \rangle_{\phi'} = \phi(F_A)$ and using 2.4 we get

$$\langle \sigma_A \rangle_{\phi'} = \phi(F_A) = \langle \sigma_A \rangle_{\mu}. \tag{2.5}$$

Lemma 2.7 concludes the proof. ☐

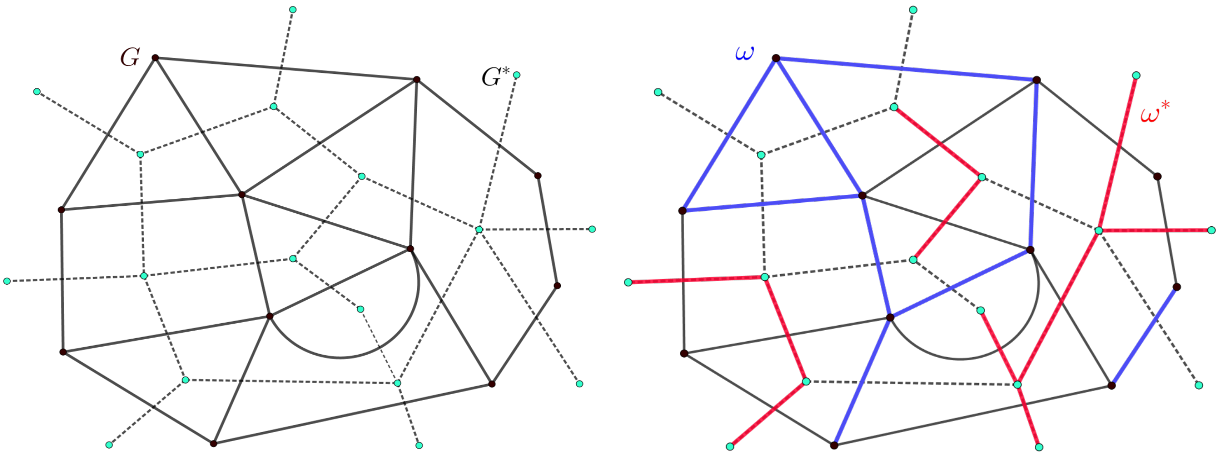
In particular, for any two vertices $x, y \in V$ we have $\langle \sigma_x \sigma_y \rangle = \phi_{G,p_e}(x \overset{\omega}{\leftrightarrow} y)$.

Remark 2.9. We remark that the argument, which gives the first equality in 2.5 remains valid for an arbitrary measure on subsets of edges. In fact, following the coloring procedure, any measure ψ on $\omega \subseteq E(G)$ induces a measure ψ' on spin configurations, satisfying the geometric representation $\langle \sigma_A \rangle_{\psi'} = \psi(F_A)$.

Later, in the section on partitions, we will introduce a more rigorous notation, which formally describes and generalizes the divide and color model. For now, we note that in a general setting the measure ψ' is called the *color process* and is denoted by $\Phi_{1/2}(\psi)$. If $\mu = \Phi_{1/2}(\psi)$ for some measure ψ, we say μ has a *color representation*. We have just shown the random-cluster model is one such color representation of the Ising measure, i.e. $\Phi_{1/2}(\phi_{G,p_e}) = \mu_{G,J_e}$ for $p_e = 1 - e^{-2J_e}$. The question of uniqueness/non-uniqueness of the color representation for the Ising model is discussed in [10], where it is shown that if G is not a tree and $|V(G)| \geq 3$, then there exist at least two distinct color representations $\Phi_{1/2}(\psi_1) = \Phi_{1/2}(\psi_2) = \mu$ of the Ising model on G. We formulate the precise statement in the last section on partitions.

In the second part of the thesis, we use random currents to explicitly describe such a distinct color representation for the dual Ising measure. We show it differs from the random-cluster model on the complete graph on three vertices K_3.

2.5 Duality



(a) graph G with its dual G^* (b) $\omega \subset E$ and its dual config. $\omega^* \subset E^*$

Figure 2.2

Let $d : \omega \mapsto \omega^*$ be the *dual map*, which takes $\omega \subset E(G)$ to its *dual configuration* $\omega^* \subset E^*$, given by $e^* \in \omega^* \iff e \notin \omega$. (Figure 2.2b) (The vertices of G^* in the unbounded component represent the same point at infinity.)

The dual FK(2)-Ising measure ϕ_{G^*,p_e^*} on configurations ω^* on the dual graph G^* is given by the relation $p_e^* := 1 - e^{-2J_e^*}$

$$\phi_{G^*,p_e^*}(\omega^*) \propto 2^{k(\omega^*)} \prod_{e^* \in \omega^*} p_e^* \prod_{e^* \notin \omega^*} (1 - p_e^*) \propto 2^{k(\omega^*)} \prod_{e^* \in \omega^*} \left(\frac{p_e^*}{1 - p_e^*} \right). \tag{2.6}$$

And one can check that from the relation $\tanh(J_e) = e^{-2J_e^*}$ it follows that

$$\frac{p_e^*}{1 - p_e^*} = \frac{2(1 - p_e)}{p_e}. \tag{2.7}$$

This implies the pushforward of the FK-Ising measure $d_* \phi_{G,p_e}$ coincides with the dual FK-Ising measure ϕ_{G^*,p_e^*}, i.e.

$$\phi_{G^*,p_e^*}(\omega^*) = \phi_{G,p_e}(\omega).$$

To verify this equality, let $f(\omega)$ and $k(\omega)$ denote the number of faces and connected components of $G(\omega)$ respectively.[3] (An isolated vertex counts as one component.) Using the Euler's formula $k(\omega) = |V| - |\omega| + f(\omega) - 1$ for the dual configuration and the fact that $k(\omega) = f(\omega^*)$, the equations 2.6 and 2.7 yield

$$\begin{aligned} \phi_{G,p_e}(\omega) &\propto 2^{k(\omega)} \prod_{e \in \omega} p_e \prod_{e \notin \omega} (1 - p_e) \propto 2^{k(\omega)} \prod_{e \notin \omega} \left(\frac{1 - p_e}{p_e} \right) \propto 2^{f(\omega^*)} \prod_{e^* \in \omega^*} \left(\frac{1 - p_e}{p_e} \right) \\ &\propto 2^{k(\omega^*) + |\omega^*|} \prod_{e^* \in \omega^*} \left(\frac{1 - p_e}{p_e} \right) \propto 2^{k(\omega^*)} \prod_{e^* \in \omega^*} \left(\frac{2(1 - p_e)}{p_e} \right) \propto \phi_{G^*,p_e^*}(\omega^*). \end{aligned}$$

Therefore, the probability measures are the same. For more, see [15].

[3] In 2.2b we have: $k(\omega) = 4, f(\omega) = 3, |\omega| = 9, |V| = 11$ and $k(\omega^*) = 3, f(\omega^*) = 4, |\omega^*| = 10, |V^*| = 10$

2.6 Thermodynamic-limit

The Domain Markov Property (1.3) and the FKG-inequality (1.4) remain valid in the context of the random-cluster model. They are also used to define the infinite-volume random-cluster measures ϕ^0 and ϕ^1. For detailed proofs we refer to [15, 20].

First, we introduce boundary conditions. Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ denote an infinite graph. Let $\Omega_{\mathbb{E}} = \{0, 1\}^{\mathbb{E}}$ be the set of edge-configurations on $\mathbb{G}$. For a finite subgraph $G \subset \mathbb{G}$ and $\xi \in \Omega_{\mathbb{E}}$, let Ω^{ξ}_G be the set of edge-configurations on Ω, which coincide with ξ outside of G. The random-cluster model on G with *boundary conditions* ξ and probability parameters $(p_e)_{e \in E(G)}$ is a probability measure on Ω^{ξ}_G given by

$$\phi^{\xi}_{G,p_e}(\omega) \propto 2^{k(\omega,\xi)} \prod_{e \in \omega} p_e \prod_{e \in E(G) \setminus \omega} (1 - p_e),$$

where the influence of the boundary conditions manifests itself in the number of connected components $k(\omega, \xi)$ intersecting G. The more open edges outside of G, the smaller the number of components $k(\omega, \xi)$. In particular, we consider the *free* and *wired* boundary conditions denoted by ϕ^0_{G,p_e} and ϕ^1_{G,p_e}, where all the edges outside of G are closed, or open respectively. In the wired case there exists one unique infinite cluster.

Proposition 2.10 (Domain Markov Property). *Let $G \subset F \subset \mathbb{G}$ be two finite subgraphs, $\xi \in \Omega_{\mathbb{E}}$ and $\psi \in \Omega^{\xi}_F$. Then*

$$\phi^{\xi}_{F,p_e}(\cdot \mid \omega_e = \psi_e \text{ on } E(F) \setminus E(G)) = \phi^{\psi}_{G,p_e}(\cdot)$$

Proposition 2.11 (FKG-inequality). *Let $G \subset \mathbb{G}$ be a finite subgraph, $\xi \in \Omega_{\mathbb{E}}$ and A, B two increasing events. Then*

$$\phi^{\xi}_{G,p_e}(A \cap B) \geq \phi^{\xi}_{G,p_e}(A)\phi^{\xi}_{G,p_e}(B)$$

Equipping the set $\Omega_{\mathbb{E}}$ with the sigma algebra $\mathcal{F}_{\mathbb{E}}$ generated by the cylinder sets and endowing it with the product topology, we can repeat the argument in the construction of the infinite-volume Ising measure in Theorem 1.5.

Theorem 2.12 (Thermodynamic-limit). *Let $(p_e)_{e \in \mathbb{E}}$ be a family of probability parameters and $(G_n)_{n \in \mathbb{N}}$ a sequence of finite subgraphs with $G_n \uparrow \mathbb{G}$. There exist probability measures ϕ^0 and ϕ^1 on $(\Omega_{\mathbb{E}}, \mathcal{F}_{\mathbb{E}})$ such that $\phi^0_{G_n,p_e} \to \phi^0$ and $\phi^1_{G_n,p_e} \to \phi^1$ in $\mathcal{PM}(\Omega_{\mathbb{E}})$ as $n \to \infty$. Moreover, the limit is independent of the sequence $G_n \uparrow \mathbb{G}$.*

Theorem 2.13 (Edwards–Sokal coupling). *Assume the defining parameters of the infinite-volume Ising model $(J_e)_{e \in \mathbb{E}}$ and the infinite-volume random-cluster model $(p_e)_{e \in \mathbb{E}}$ satisfy the Edwards–Sokal coupling $p_e = 1 - e^{-2J_e}$. Then*

1. *Sampling $\omega \in \Omega_{\mathbb{E}}$ according to ϕ^0 and assigning $+1$ or -1 to each component of ω independently with probability $1/2$ has the same distribution as the Ising measure μ with free boundary conditions on $\mathbb{G}$.*

2. *Sampling $\omega \in \Omega_{\mathbb{E}}$ according to ϕ^1, assigning $+1$ to the infinite component and then $+1$ or -1 to all finite components independently with probability $1/2$ has the same distribution as the Ising measure μ^+ with $+$ boundary conditions on $\mathbb{G}$.*

The Edwards–Sokal coupling provides a unifying probability space for the random-cluster model and the Ising model. We refer the reader to the full proof to Theorem 4.91 [15]. Here, we briefly discuss the powerful method of coupling used in the proof. For any two measures ϕ, ϕ' on $\Omega_{\mathbb{E}}$, we say that ϕ *stochastically dominates* ϕ', and write $\phi \geq_{st.} \phi'$, if $\phi(X) \geq \phi'(X)$ for any continuous[4] increasing random variable $X : \Omega_{\mathbb{E}} \to \mathbb{R}$. A *monotone coupling* between ϕ, ϕ' is a probability measure $\widehat{\mathbb{P}}$ on the product space $\Omega_{\mathbb{E}} \times \Omega_{\mathbb{E}}$, such that the marginals of $\widehat{\mathbb{P}}$ are (ϕ, ϕ') and $\widehat{\mathbb{P}}[(\omega, \omega') : \omega \geq \omega'] = 1$. A useful result on stochastic domination, known as the Strassen's theorem, states that $\phi \geq_{st.} \phi'$ if and only if there exists a monotone coupling between ϕ and ϕ'.

For the random-cluster model, a consequence of the FKG-inequality is the stochastic ordering $\phi^0_{G_n} \leq_{st.} \phi^0_{G_{n+1}}$ and $\phi^1_{G_n} \geq_{st.} \phi^1_{G_{n+1}}$ as $G_n \uparrow \mathbb{G}$. Combining the monotone coupling for the sequence with Corollary 2.8 constitutes the backbone of the proof. We also remark that stochastic domination is preserved under weak limits: If $\phi_n \to \phi$, $\phi'_n \to \phi'$ converge weakly as n goes to infinity, and $\phi_n \geq_{st.} \phi'_n$ for all n, then $\phi \geq_{st.} \phi'$. For more on the coupling method, see [21]. We will use some of these ideas later in the context of the three-way coupling between the high-temperature expansion, random current expansion and the random-cluster model. An important consequence of the Edwards–Sokal coupling is the following geometric representation

$$\langle\sigma_0\rangle^+ = \langle\sigma_0 | 0 \overset{\omega}{\leftrightarrow} \infty\rangle^+ \phi^1(0 \overset{\omega}{\leftrightarrow} \infty) + \langle\sigma_0 | 0 \overset{\omega}{\not\leftrightarrow} \infty\rangle^+ \phi^1(0 \overset{\omega}{\not\leftrightarrow} \infty) = \phi^1(0 \overset{\omega}{\leftrightarrow} \infty) \tag{2.8}$$

2.7 Tail-triviality

Another useful property of the infinite-volume measures is tail-triviality. For a finite subgraph $\Lambda \subset \mathbb{G}$ let $\mathcal{F}_\Lambda$ denote the sigma-algebra generated by the events depending only on the vertices in Λ. Let $\mathcal{T}_\Lambda = \mathcal{F}_{\Lambda^c}$ be the sigma-algebra depending on the vertices outside of Λ and $\mathcal{T} = \bigcap_\Lambda \mathcal{T}_\Lambda$ the tail sigma-algebra consisting of tail events. In other words, A is a tail event if it does not depend on any finite number of spins. A measure ψ on $\{0,1\}^{\mathbb{V}}$ is called *tail-trivial* if for any event $A \in \mathcal{T}$ it follows $\psi(A) \in \{0,1\}$. We use analogous definitions when Λ is a subset of edges and ψ a measure on $\{0,1\}^{\mathbb{E}}$ and use the notation interchangeably.

The following result regards tail-triviality for the extremal Ising measures and random-cluster measures. The proof relies on strong positive association (which is equivalent to the FKG lattice condition) and Markov domain property, and can be found in Theorem 4.19 in [15]. For a more general result about tail-triviality of extremal Gibbs measures we refer to Theorem 7.7 in [13].

Proposition 2.14. *The infinite-volume Ising measures μ^+, μ^-, and the infinite-volume random-cluster measures ϕ^0 and ϕ^1 are tail-trivial.*

Proof. Let $G \subset \Lambda$ be finite subgraphs, $A \in \mathcal{F}_G$ an increasing event and $B \in \mathcal{F}_{\Lambda \setminus G}$. By strong positive association and the Markov property, we get

$$\mu^+(A|B) \geq \mu^+(A| \sigma_{\Lambda \setminus G} = +) = \mu^+_G(A)$$

From this we get $\mu^+(A \cap B) \geq \mu^+_G(A)\mu^+(B)$. Taking $\Lambda \uparrow \mathbb{G}$ and then $G \uparrow \mathbb{G}$ gives $\mu^+(A \cap B) \geq \mu^+(A)\mu^+(B)$ for all $A \in \mathcal{F}$ increasing and for all $B \in \mathcal{T}$. But then, if

[4] Note that by compactness of $\Omega_{\mathbb{E}}$, all continuous functions are bounded.

$B \in \mathcal{T}$, the same inequality must hold for $B^c \in \mathcal{T}$ and adding the two inequalities with B and B^c gives $\mu^+(A \cap B) = \mu^+(A)\mu^+(B)$ for all $A \in \mathcal{F}$ increasing and for all $B \in \mathcal{T}$. Since increasing events generate $\mathcal{F}$ the equality holds for $A = B$, hence $\mu^+(B) \in \{0, 1\}$ for any tail event B in $\mathcal{T}$. The proof for the other measures is analogous. □

Corollary 2.15. *Assuming the Edwards–Sokal coupling we have $\langle\sigma_0\rangle^+ > 0$ if and only if there exists an infinite FK-cluster ϕ^1-a.s.*

Proof. Let C_∞ denote the event that there exists an infinite FK-cluster. This is a tail event, since it is invariant under any local (finite) changes. If $0 < \langle\sigma_0\rangle^+ = \phi^1(0 \overset{\omega}{\leftrightarrow} \infty)$, then $\phi^1(C_\infty) > 0$, therefore by tail-triviality $\phi^1(C_\infty) = 1$. On the other hand, if $0 = \langle\sigma_0\rangle^+ = \phi^1(0 \overset{\omega}{\leftrightarrow} \infty)$, then for any vertex x the FKG-inequality yields

$$0 = \phi^1(0 \overset{\omega}{\leftrightarrow} \infty) \geq \phi^1(0 \overset{\omega}{\leftrightarrow} x, x \overset{\omega}{\leftrightarrow} \infty) \geq \phi^1(0 \overset{\omega}{\leftrightarrow} x)\phi^1(x \overset{\omega}{\leftrightarrow} \infty)$$

Because every local event has a strictly positive probability, we have $\phi^1(0 \overset{\omega}{\leftrightarrow} x) > 0$. The above equation then implies $\phi^1(x \overset{\omega}{\leftrightarrow} \infty) = 0$ for every x. From this we obtain

$$\phi^1(C_\infty) = \phi^1\left(\bigcup_x \{x \overset{\omega}{\leftrightarrow} \infty\}\right) \leq \sum_x \phi^1(x \overset{\omega}{\leftrightarrow} \infty) = 0$$

□

Let C_+ and C_- denote the events on $\{-1, 1\}^V$ that there exists an infinite σ-cluster of + spins and − spins respectively. We use the Edwards–Sokal coupling to prove the following Lemma.

Lemma 2.16. *If $\mu^+(C_-) > 0$, then $\mu^+(C_- \cup C_+) = 1$.*

Proof. First, observe that C_- and C_+ are tail events, which implies $\mu^+(C_-) = 1$. There are two possible cases:

If $\langle\sigma_0\rangle^+ > 0$, then by Corollary 2.15 there exists an infinite FK-cluster ϕ^1-almost surely. This infinite FK-cluster is assigned +1 in the coloring procedure described in Theorem 2.13, and hence $\mu^+(C_+) = 1$. This means $\mu^+(C_- \cup C_+) = 1$.

On the other hand, if $\langle\sigma_0\rangle^+ = 0$, then there exists no infinite FK-cluster ϕ^1-almost surely. This means there is a symmetry between +1 and −1 during the coloring procedure in Theorem 2.13 and we have $\mu^+(C_+) = \mu^+(C_-) = 1$. Hence $\mu^+(C_- \cup C_+) = 1$.

□

3 Color representations with random currents

3.1 Random currents expansion

This expansion in integer valued functions on edges goes back to Griffiths, Hurst and Sherman [14] and has become an important tool in studying the Ising model ever since. A *random current* $\mathbf{n} = (\mathbf{n}_{xy} : xy \in E(G)) \in \mathbb{N}^{E(G)}$ is an integer valued function $\mathbf{n} : E(G) \mapsto \mathbb{N}$ defined on the edges of the graph. Setting $X(\mathbf{n}, x) := \sum_y \mathbf{n}_{xy}$ to be the sum of the current values around x, we can define the *sources* $\partial\mathbf{n} := \{x \in V(G) : X(\mathbf{n}, x) \equiv 1 \pmod 2\}$. We also define the *weight* of a current by

$$w(\mathbf{n}) := \prod_{xy \in E(G)} \frac{(J_{xy})^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!}.$$

Using the Taylor expansion for the exponential we can write [6]

$$\begin{aligned}
Z(\sigma_A) &= \sum_{\sigma \in \{\pm 1\}^{V(G)}} \sigma_A \prod_{xy \in E(G)} \sum_{\mathbf{n}_{xy}=0}^{\infty} \frac{(J_{xy}\sigma_x\sigma_y)^{\mathbf{n}_{xy}}}{\mathbf{n}_{xy}!} \\
&= \sum_{\sigma \in \{\pm 1\}^{V(G)}} \sigma_A \sum_{\mathbf{n} \in \mathbb{N}^{E(G)}} \prod_{xy \in E(G)} (\sigma_x\sigma_y)^{\mathbf{n}_{xy}} w(\mathbf{n}) \\
&= \sum_{\mathbf{n} \in \mathbb{N}^{E(G)}} w(\mathbf{n}) \sum_{\sigma \in \{\pm 1\}^{V(G)}} \prod_{x \in V(G)} \sigma_x^{X(\mathbf{n},x) + \mathbf{I}[x \in A]} \\
&= 2^{|V(G)|} \sum_{\substack{\mathbf{n} \in \mathbb{N}^{E(G)} \\ \partial\mathbf{n} = A}} w(\mathbf{n}).
\end{aligned}$$

where we have used 2.1 in the last equality. With this we get the expansion of the partition function in terms of currents

$$Z = 2^{|V(G)|} \sum_{\partial\mathbf{n} = \emptyset} w(\mathbf{n}) \text{ and } \langle \sigma_A \rangle = \frac{\sum_{\partial\mathbf{n}=A} w(\mathbf{n})}{\sum_{\partial\mathbf{n}=\emptyset} w(\mathbf{n})}.$$

Similarly to the high-temperature measure ν^A, we can define a measure on currents.

Definition 3.1. For $A, B \subseteq V(G)$ the measure $\mathrm{P}^A = \mathrm{P}^A_{G,J_e}$ on currents is given by

$$\mathrm{P}^A(\mathbf{n}_0) = \begin{cases} \dfrac{w(\mathbf{n}_0)}{\sum\limits_{\partial\mathbf{n}=A} w(\mathbf{n})}, & \text{if } \partial\mathbf{n}_0 = A, \\ 0, & \text{if } \partial\mathbf{n}_0 \neq A. \end{cases}$$

The double random current measure $\mathsf{P}^A \otimes \mathsf{P}^B$ is given by the product measure and is supported on $\{\partial\mathbf{n} = A\} \otimes \{\partial\mathbf{n} = B\}$.

Comparing the random current expansion with the high-temperature expansion gives

$$c_0 \sum_{\partial\eta=\emptyset} x(\eta) = \sum_{\partial\mathbf{n}=\emptyset} w(\mathbf{n}).$$

Let $\widehat{\mathbf{n}} := \{e \in E(G) : \mathbf{n}_e > 0\}$ denote the set of edges on which the current $\mathbf{n}$ has a strictly positive value. The main information carried by a current is the parity of its values. Forgetting the specific value of the current $\mathbf{n}$ and only considering its parity at each edge, we can view a random current as an edge-configuration $\widehat{\mathbf{n}}$ partitioned into two distinguished sets $\widehat{\mathbf{n}} = \mathbf{n}_{odd} \sqcup \mathbf{n}_{ev}$, where $\mathbf{n}_{odd} = \{e \in \widehat{\mathbf{n}} : \mathbf{n}_e \equiv 1(2)\}$ and $\mathbf{n}_{ev} = \{e \in \widehat{\mathbf{n}} : \mathbf{n}_e \equiv 0(2)\}$ [8]. We note that $\partial\mathbf{n} = \partial\mathbf{n}_{odd}$. Let $\widehat{\mathsf{P}}^A = \widehat{\mathsf{P}}^A_{G,J_e}$ denote the pushforward of P^A_{G,J_e} under the map $\mathbf{n} \mapsto \widehat{\mathbf{n}}$. The described division of a random current into its *odd* and *even* edges yields a direct connection between the random current expansion and the high-temperature expansion.

Proposition 3.2.

1. *If $\mathbf{n} \sim \mathsf{P}^A$, then $\mathbf{n}_{odd} \sim \nu^A$. In other words, sampling a random current $\mathbf{n}$ according to P^A and then looking at its odd edges $\mathbf{n}_{odd}$ has the same distribution as sampling η according to the measure ν^A.*

2. *Sampling $\widehat{\mathbf{n}} \sim \widehat{\mathsf{P}}^A$ has the same distribution as sampling η according to ν^A and adding each edge independently with probability $1 - 1/cosh(J_e)$.*

Proof. Let $\eta_0 \subset E(G)$ such that $\partial\eta_0 = A$. We express the probability $\mathsf{P}^A(\{\mathbf{n} : \mathbf{n}_{odd} = \eta_0\})$ as

$$\mathsf{P}^A\left(\left\{\begin{array}{ll}\mathbf{n}_e \equiv 1(2) \text{ on } \eta_0, \\ \mathbf{n}_e \equiv 0(2) \text{ on } \eta_0^c\end{array}\right\}\right) = \frac{\prod_{e\in\eta_0}\sinh(J_e)\prod_{e\in\eta_0^c}\cosh(J_e)}{\sum_{\partial\mathbf{n}=A} w(\mathbf{n})} = \frac{c_0\prod_{e\in\eta_0}\tanh(J_e)}{c_0\sum_{\partial\eta=A} x(\eta)} = \nu^A(\eta_0).$$

Let $\omega = \omega_{odd} \sqcup \omega_{ev} \subset E(G)$ be a disjoint union of two sets of edges, where $\partial\omega_{odd} = A$. Using the notation $q_e = 1 - 1/\cosh(J_e)$ and $x_e = \tanh(J_e)$, by a similar computation we have

$$\mathsf{P}^A\left(\left\{\begin{array}{l}\mathbf{n}_{odd} = \omega_{odd}, \\ \mathbf{n}_{ev} = \omega_{ev}\end{array}\right\}\right) = \frac{\prod_{e\in\omega_{odd}} x_e \prod_{e\in\omega_{ev}} q_e \prod_{e\in E\setminus\omega}(1-q_e)}{\sum_{\partial\eta=A} x(\eta)} = \nu^A(\omega_{odd})\prod_{e\in\omega_{ev}} q_e \prod_{e\in E\setminus\omega}(1-q_e). \tag{3.1}$$

□

Remark 3.3 (The three-way coupling). Informally, the second assertion of the statement says that 'high-temperature' + 'Bernoulli percolation' = 'random current'. More recently, a new distributional relation between random currents and random-clusters was established in [23]. It asserts that 'random current'+'Bernoulli percolation'='random-cluster', where the probabilities are given by $p_e = 1 - e^{-J_e}$ and $A = \emptyset$. In particular, from the three-way coupling and the Strassen's theorem, we obtain a stochastic ordering between the three measures

$$\nu^{\emptyset}_{G,J_e} \preceq_{st.} \widehat{\mathsf{P}}^{\emptyset}_{G,J_e} \preceq_{st.} \phi^{0}_{G,p_e}.$$

Moreover, combining the two consecutive couplings gives a monotone coupling between the high-temperature expansion and the random-cluster model, where the latter arises from sampling $\eta \sim \nu^{\emptyset}_{G,J_e}$ and then independently opening edges with probabilities $p_e = 1 - e^{-J_e}/\cosh(J_e)$. This coupling remains valid in the limit on the infinite planar graph $\mathbb{G}$. Indeed, let $G_n \nearrow \mathbb{G}$, $\eta_n \sim \nu^{\emptyset}_{G_n}$ and $B_n \sim \xi_n$ be the Bernoulli percolation on G_n with probabilities p_e and independent from η_n. Also, let $B \sim \xi_{\mathbb{G}}$ denote the Bernoulli percolation with probabilities p_e on $\mathbb{G}$. We consider $\omega_n := \eta_n + B_n \sim \phi^0_{G_n} = h_*(\nu^{\emptyset}_{G_n} \otimes \xi_n)$, where $h(x,y) = x + y$ is addition on edge-configurations. From the weak convergence of $\nu^{\emptyset}_{G_n} \to \nu^{\emptyset}_{\mathbb{G}}$, $\xi_n \to \xi_{\mathbb{G}}$ and continuity of h, we obtain that $\phi^0_{G_n} = h_*(\nu^{\emptyset}_{G_n} \otimes \xi_n) \to h_*(\nu^{\emptyset}_{\mathbb{G}} \otimes \xi_{\mathbb{G}})$ (see [3]). Therefore, sampling $\omega \sim \phi^0_{\mathbb{G}}$ has the same distribution as sampling $\eta \sim \nu^{\emptyset}_{\mathbb{G}}$ and then independently opening edges $B \sim \xi_{\mathbb{G}}$.

3.2 Multigraphs and Switching lemma

A random current $\mathbf{n}$ can be thought of as a multigraph $\mathcal{N}$, where the value of the current $\mathbf{n}_{xy}$ at an edge determines the number of edges of $\mathcal{N}$ between x and y. We also set $w(\mathcal{N}) := w(\mathbf{n})$ and note that $\partial\mathcal{N} = \partial\mathbf{n}$.
The following result, called the Switching lemma, is a useful tool insofar as it allows to change the perspective from the correlation function to a probability of some connectivity event in terms of the random current measure. More on this topic can be found in [14, 6, 2, 5].

Lemma 3.4 (Switching lemma). *For $A, B \subseteq V(G)$ and a function $F : \mathbb{N}^{E(G)} \to \mathbb{R}$*

$$\sum_{\substack{\partial\mathbf{n}_1 = A\\ \partial\mathbf{n}_2 = B}} F(\mathbf{n}_1 + \mathbf{n}_2) w(\mathbf{n}_1) w(\mathbf{n}_2) = \sum_{\substack{\partial\mathbf{n}_1 = \emptyset\\ \partial\mathbf{n}_2 = A \Delta B}} F(\mathbf{n}_1 + \mathbf{n}_2) w(\mathbf{n}_1) w(\mathbf{n}_2) \mathbb{I}[\widehat{\mathbf{n}_1 + \mathbf{n}_2} \in \mathcal{F}_A]. \quad (3.2)$$

Proof. Setting $\mathbf{n} := \mathbf{n}_1$ and $\mathbf{m} := \mathbf{n}_1 + \mathbf{n}_2$, we have $\left\{\begin{smallmatrix}\partial\mathbf{n}_1 = A\\ \partial\mathbf{n}_2 = B\end{smallmatrix}\right\} \iff \left\{\begin{smallmatrix}\partial\mathbf{m} = A \Delta B\\ \partial\mathbf{n} = A\end{smallmatrix}\right\}$ as well as

$$w(\mathbf{n}_1) w(\mathbf{n}_2) = w(\mathbf{m}) \prod_e \binom{\mathbf{m}_e}{\mathbf{n}_e} \overset{\text{not.}}{=} w(\mathbf{m}) \binom{\mathbf{m}}{\mathbf{n}}.$$

Hence both sides of (3.2) can be written in terms of $\mathbf{m}$ and $\mathbf{n}$ as

$$\sum_{\partial\mathbf{m} = A \Delta B} F(\mathbf{m}) w(\mathbf{m}) \sum_{\substack{\mathbf{n} \le \mathbf{m}\\ \partial\mathbf{n} = A}} \binom{\mathbf{m}}{\mathbf{n}} = \sum_{\partial\mathbf{m} = A \Delta B} F(\mathbf{m}) w(\mathbf{m}) \mathbb{I}[\widehat{\mathbf{m}} \in \mathcal{F}_A] \sum_{\substack{\mathbf{n} \le \mathbf{m}\\ \partial\mathbf{n} = \emptyset}} \binom{\mathbf{m}}{\mathbf{n}}.$$

Therefore, it suffices to show that for a fixed current $\mathbf{m}$ we have the combinatorial identity

$$\sum_{\substack{\mathbf{n} \le \mathbf{m}\\ \partial\mathbf{n} = A}} \binom{\mathbf{m}}{\mathbf{n}} = \mathbb{I}[\widehat{\mathbf{m}} \in \mathcal{F}_A] \sum_{\substack{\mathbf{n} \le \mathbf{m}\\ \partial\mathbf{n} = \emptyset}} \binom{\mathbf{m}}{\mathbf{n}}. \quad (3.3)$$

First, we notice $\partial\mathbf{n} = A \implies \widehat{\mathbf{n}} \in \mathcal{F}_A$, and since $\mathcal{F}_A$ is an increasing event and $\widehat{\mathbf{n}} \le \widehat{\mathbf{m}}$, we have $\hat{\mathbf{n}} \in F_A \implies \hat{\mathbf{m}} \in F_A$. Consequently, if $\hat{\mathbf{m}} \notin F_A$, the LHS of (3.3) vanishes as well. On the other hand, if $\hat{\mathbf{m}} \in F_A$, then the equation (3.3) can be formulated in terms of multigraphs as

$$|\{N \subseteq M : \partial N = A\}| = |\{N \subseteq M : \partial N = \emptyset\}|.$$

But since $\hat{\mathbf{m}} \in F_A$, there exists a submultigraph $K \subseteq M$ with $\partial K = A$. With this the mapping $N \mapsto N \Delta K$ becomes an involution between the above sets, which proves the lemma.

$\square$

As a useful consequence of the Switching lemma, one gets a relation between correlation functions and double random currents.

Proposition 3.5. *For any $A \subseteq V(G)$ we have:*

$$\langle\sigma_A\rangle^2 = \mathbf{P}^{\emptyset} \otimes \mathbf{P}^{\emptyset}[\widehat{\mathbf{n}_1 + \mathbf{n}_2} \in F_A]$$

Proof. Setting $A = B$ and $F \equiv 1$ in the Switching lemma yields

$$\begin{aligned}
\langle\sigma_A\rangle^2 &= \frac{1}{Z^{\emptyset,\emptyset}} \sum_{\partial\mathbf{n}_1 = \partial\mathbf{n}_2 = A} w(\mathbf{n}_1)w(\mathbf{n}_2) \\
&= \frac{1}{Z^{\emptyset,\emptyset}} \sum_{\partial\mathbf{n}_1 = \partial\mathbf{n}_2 = \emptyset} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbb{I}[\widehat{\mathbf{n}_1 + \mathbf{n}_2} \in F_A] \\
&= \mathbf{P}^{\emptyset} \otimes \mathbf{P}^{\emptyset}[\widehat{\mathbf{n}_1 + \mathbf{n}_2} \in F_A],
\end{aligned}$$

where we abbreviated $Z^{\emptyset,\emptyset} = \sum_{\partial\mathbf{n}_1 = \partial\mathbf{n}_2 = \emptyset} w(\mathbf{n}_1)w(\mathbf{n}_2)$. $\square$

Proposition 3.6. *Let $\tau := \sigma^1\sigma^2$ where $\sigma^1, \sigma^2 \sim$ Ising measure and independent. Then sampling a spin configuration σ according to τ has the same distribution as sampling σ according to the divide and color model $\psi' = {}_{1/2}(\psi)$, where $\psi(\omega) := \mathbf{P}^{\emptyset} \otimes \mathbf{P}^{\emptyset}[\widehat{\mathbf{n}_1 + \mathbf{n}_2} = \omega]$.*

Proof. Let $g : (\sigma^1, \sigma^2) \mapsto \sigma^1\sigma^2$ be the pointwise multiplication of two spin configurations. The distribution of sampling σ according to τ is given by the pushforward of the product Ising measure under g. We have $\tau \sim g_*(\mu \otimes \mu)$ and we want to show $\psi' = g_*(\mu \otimes \mu)$.

For any $A \subseteq E(G)$, using Remark 2.9 and Proposition 3.5 yields

$$\langle\sigma_A\rangle_{\psi'} = \psi(F_A) = \mathbf{P}^{\emptyset} \otimes \mathbf{P}^{\emptyset}[\widehat{\mathbf{n}_1 + \mathbf{n}_2} \in F_A] = \langle\sigma_A\rangle^2 = \langle\sigma^1_A\sigma^2_A\rangle_{\mu\otimes\mu} = \langle\sigma_A\rangle_{g_*(\mu\otimes\mu)}$$

By Lemma 2.7, it follows $\psi' = g_*(\mu \otimes \mu)$, which concludes the proof.

$\square$

3.3 Main result

Let $d_*\psi$ be the pushforward measure on E^* under the dual map $d : \omega \mapsto \omega^*$, meaning that for any $\alpha \subseteq E^*$; $d_*\psi(\alpha) := \psi(d^{-1}(\alpha)) = \mathrm{P}^{\emptyset} \otimes \mathrm{P}^{\emptyset}[\widehat{\mathbf{n_1} + \mathbf{n_2}}^* = \alpha] = \psi(\alpha^*)$.

The measure $d_*\psi$ on subsets of edges in E^* defines a divide and color model on the dual G^*. The natural question is about the distribution of the induced measure $\mathbb{P}_{1/2}(d_*\psi)$ on spin configurations on G^*. The answer to this question might be surprising and constitutes the main result of the thesis.

Theorem 3.7. *Let G be a finite planar graph. The dual Ising measure μ^* has the following color representation in terms of random currents*

$$\boxed{\mathbb{P}_{1/2}(d_*\psi) = \mu^*}$$

The measure $\mathbb{P}_{1/2}(d_*\psi)$ is obtained from sampling $\omega \sim \psi$ on $E(G)$, then taking its dual configuration ω^*, and assigning $+1$ or -1 to the connected components of ω^* independently.

In other words, sampling a spin configuration σ on G^* according to the divide and color model with the dual double random current measure $d_*\psi$ has the same distribution as the dual Ising model μ^* on G^*. The strategy for the proof is the same as in previous results. The goal is to show that for any set of faces $A = \{u_1, ..., u_n\}$ on G^* the correlation functions σ_A have the same expectations under μ^* and $\mathbb{P}_{1/2}(d_*\psi)$, and then apply the Lemma 2.7. Note that for an odd number of faces $u_1, ..., u_n$

$$\langle \sigma_{u_1} ... \sigma_{u_n} \rangle_{\mu^*} = \langle \sigma_{u_1} ... \sigma_{u_n} \rangle_{\mathbb{P}_{1/2}(d_*\psi)} = 0$$

by symmetry, and it suffices to verify the statement for an even number of faces. First, we need some prerequisite ideas, including a modified version of the Switching lemma, developed in the work of Aizenmann et al.[2].

Definition 3.8. Given a multiset of edges E in $E(G)$, let ϵ_{xy} denote the multiplicity of the edge xy in E. For a random current $\mathbf{n}$ we introduce the quantity

$$(\mathbf{n}|E) := \sum_{xy \in E(G)} \mathbf{n}_{xy} \epsilon_{xy}$$

Similarly, we can extend this notation for a multigraph N associated to the random current $\mathbf{n}$ by $(N|E) := (\mathbf{n}|E)$ and for a subset $\omega \subseteq E(G)$ by $(\omega|E) := \sum_{xy \in \omega} \epsilon_{xy}$.

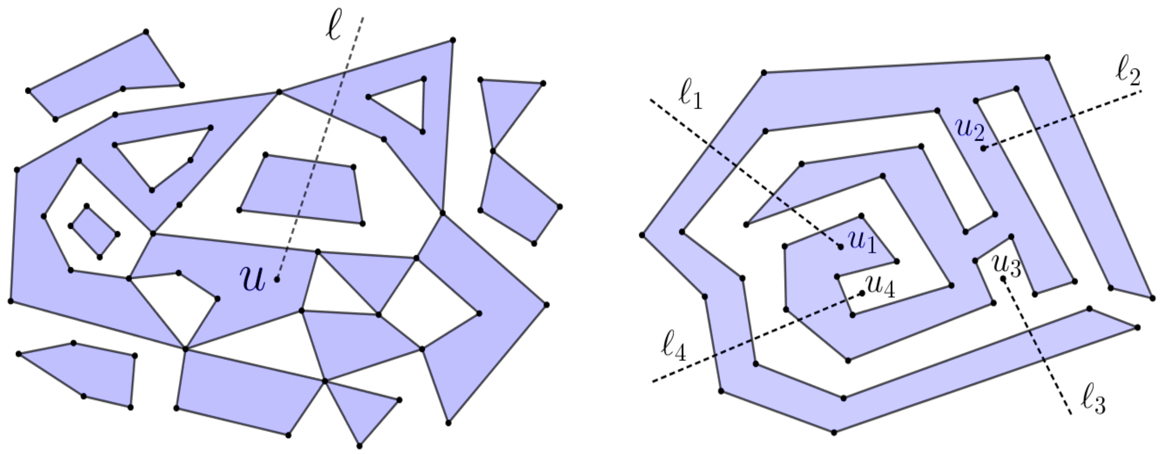
The quantity $(\mathbf{n}|E)$ is additive in both components, i.e. $(\mathbf{n}_1 + \mathbf{n}_2|E) = (\mathbf{n}_1|E) + (\mathbf{n}_2|E)$ and $(\mathbf{n}|E_1 + E_2) = (\mathbf{n}|E_1) + (\mathbf{n}|E_2)$.

Definition 3.9 (The S_E-condition)**.** Given a multiset of edges in $E(G)$, we say a current $\mathbf{n}$ satisfies the S_E-condition and write $\mathbf{n} \in S_E$ if every loop γ supported on the corresponding multigraph N fulfills $(-1)^{(\gamma|E)} = 1$. Similarly, for $\omega \subseteq E(G)$ we write $\omega \in S_E$ if every loop γ on ω satisfies $(-1)^{(\gamma|E)} = 1$.

In other words, $\mathbf{n} \in S_E$ if and only if every loop supported on N intersects the edges of odd multiplicity in E an even number of times [2].

Remark 3.10. We observe that $\mathbf{n} \in S_E \iff \widehat{\mathbf{n}} \in S_E$.
If $(-1)^{(\gamma|E)} = 1$ for every loop γ supported on N, then this must hold for every loop γ on $\widehat{\mathbf{n}}$ as well. Conversely, any loop γ supported on N can be viewed as a current $\gamma \subseteq \mathbf{n}$ and the associated subset $\gamma_{odd} \subseteq \widehat{\gamma} \subseteq \widehat{\mathbf{n}}$ satisfies $\partial\gamma_{odd} = \partial\gamma = \emptyset$. Therefore, $\gamma_{odd} = \gamma_1 \sqcup \ldots \sqcup \gamma_k \subseteq \widehat{\mathbf{n}}$ decomposes into a disjoint union of loops in $\widehat{\mathbf{n}}$ and we obtain $(-1)^{(\gamma|E)} = (-1)^{(\gamma_{odd}|E)} = (-1)^{(\gamma_1 \sqcup \ldots \sqcup \gamma_k|E)} = (-1)^{(\gamma_1|E)} \ldots (-1)^{(\gamma_k|E)} = 1$.

Another noteworthy property is that S_E is a decreasing event, i.e. if $\omega \in S_E$ and $\varpi \leq \omega$, then $\varpi \in S_E$. This stands in contrast with the increasing event F_A. What is more, the event F_A can be fully characterized in terms of the event S_E for a suitable multiset E, which we now specify. Given a graph G and a face $u \in V^*$ of G, we consider a *disorder line* ℓ; a straight line connecting u to the outer face of G without intersecting any of the vertices in G. By ℓ we also denote the set of edges intersected by the disorder line. With this notation, the quantity $(\mathbf{n}|\ell)$ equals the sum of the current values on ℓ. For more faces $u_1, .., u_n$ and disorder lines $\ell_1, .., \ell_n$, one obtains a multiset $L = \ell_1 + \ldots + \ell_n$. (Multiple lines can go through the same edge, hence the need for multisets.) At this point, we make two observations about the disorder lines, depicted in Figure 3.1.



(a) $\eta \subseteq E(G), \partial\eta = \emptyset$ (b) closed loop $\gamma \subseteq E(G)$

Figure 3.1

First, by Lemma 2.2, regions of any $\eta \subseteq E(G)$ with $\partial\eta = \emptyset$ can be 2-colored. (Two faces u_1, u_2 being in the same region if they are connected in η^*). In terms of spins, this is captured by the low-temperature expansion as $\sigma = C_+^{-1}(\eta)$, where we have made the choice of fixing a positive spin on the unbounded face. Let u be any region of η and let ℓ be a disorder line of u as depicted in Figure 3.1a. The spin at u is uniquely determined by the parity of the number of intersections between ℓ and η, which can be expressed as $\sigma_u = C_+^{-1}(\eta)_u = (-1)^{(\eta|\ell)}$. By the above argument, the quantity $(-1)^{(\eta|\ell)}$ is well-defined, since it is independent of the direction of the line ℓ. This naturally extends to multiple faces $u_1, \ldots, u_n$ and their respective disorder lines $\ell_1, .., \ell_n$

$$\sigma_{u_1} \ldots \sigma_{u_n} = (-1)^{(\eta|\ell_1 + \ldots + \ell_n)}.$$

In particular, recall that by Proposition 2.3, for $\eta \sim \nu^{\emptyset}$, one gets $\sigma = C_+^{-1}(\eta) \sim \mu_+$, implying

$$\langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle_{\mu_+} = \langle (-1)^{(\eta|\ell_1 + \ldots + \ell_n)} \rangle_{\nu^{\emptyset}}. \tag{3.4}$$

Moreover, for the boundary conditions we have $\sigma_u = C^{-1}(\eta)_u = (-1)^{(\eta|\ell)+1}$, and hence if the number of faces is even, the low-temperature expansion becomes

$$\langle\sigma_{u_1}...\sigma_{u_n}\rangle_{\mu} = \frac{1}{2}\left(\langle\sigma_{u_1}...\sigma_{u_n}\rangle_{\mu} + \langle\sigma_{u_1}...\sigma_{u_n}\rangle_{\mu_+}\right) = \langle(-1)^{(\eta|\ell_1+...+\ell_n)}\rangle_{\nu_{\emptyset}}. \tag{3.5}$$

Second, by the Jordan Curve Theorem, any closed loop $\gamma \subseteq E(G)$ divides the plane into two connected components, the inside (bounded) component and the outside (unbounded) component. As a consequence, given a set of faces $u_1, ..., u_n$ together with their corresponding disorder lines $\ell_1, .., \ell_n$, the points are partitioned/separated by the curve γ between the inner and outer component (Figure 3.1b). There are two possibilities, either the points are *separated evenly* (both components contain an even number of points), in which case $(-1)^{(\gamma|L)} = 1$, or they are *separated oddly* (both components contain an odd number of points), and $(-1)^{(\gamma|L)} = -1$, where $L = \ell_1 + ... + \ell_n$. It follows that for $\omega \subseteq E(G)$, we have $\omega \in S_L$ if and only if every closed loop γ on ω separates the faces $u_1, ..., u_n$ evenly. We are rewarded with the promised characterization of the S_E-condition.

Lemma 3.11. *Given any set of faces $A = \{u_1, u_2, ...u_n\} \subseteq V$ of the graph G together with the corresponding multiset of disorder lines $L = \ell_1 + ... + \ell_n$ and a subset of edges $\omega \subseteq E(G)$, we have $\omega \in S_L \iff \omega^* \in F_A$.*

Proof. With the second observation and the definition of F_A we have

$$\begin{aligned}\omega \notin S_L &\iff \exists \text{ closed loop } \gamma \text{ on } \omega \text{ with } (-1)^{(\gamma|L)} = -1\\ &\iff \exists \text{ closed loop } \gamma \text{ on } \omega \text{ which separates } u_1, u_2, ..., u_n \text{ oddly.}\\ &\iff \exists \text{ connected component of } \omega^* \text{ containing an odd number of } u_1, ..., u_n.\\ &\iff \omega^* \notin F_A.\end{aligned}$$

□

By Remark 3.10, the statement of the above lemma extends to random currents as $\mathbf{n} \in S_L \iff \widehat{\mathbf{n}} \in F_A$.

The following statement is an analogue of the Switching lemma, but it is expressed in terms of the S_E-condition. The full result can be found in Lemma 6.3 in [2]. For completeness, we include the proof.

Lemma 3.12. *For $A, B \subseteq V(G)$, multisets E_1, E_2 in $E(G)$ and any function f on currents*

$$\begin{aligned}&\sum_{\substack{\partial\mathbf{n}_1 = A\\ \partial\mathbf{n}_2 = B}} f(\mathbf{n_1} + \mathbf{n_2}) w(\mathbf{n}_1)(-1)^{(\mathbf{n}_1|E_1)} w(\mathbf{n}_2)(-1)^{(\mathbf{n}_2|E_2)}\\ &= \sum_{\substack{\partial\mathbf{n}_1 = A\\ \partial\mathbf{n}_2 = B}} f(\mathbf{n_1} + \mathbf{n_2}) w(\mathbf{n}_1)(-1)^{(\mathbf{n}_1|E_1)} w(\mathbf{n}_2)(-1)^{(\mathbf{n}_2|E_2)} \mathbb{I}[\mathbf{n_1} + \mathbf{n_2} \in S_{E_1+E_2}]\end{aligned}$$

Proof.

$$\sum_{\substack{\partial\mathbf{n_1}=A\\ \partial\mathbf{n_2}=B}} f(\mathbf{n_1}+\mathbf{n_2})w(\mathbf{n}_1)(-1)^{(\mathbf{n}_1|E_1)}w(\mathbf{n}_2)(-1)^{(\mathbf{n}_2|E_2)}\mathrm{I}[\mathbf{n_1}+\mathbf{n_2}\notin S_{E_1+E_2}] \tag{3.6}$$

$$= \sum_{\partial\mathbf{m}=A\Delta B} f(\mathbf{m})w(\mathbf{m})\mathrm{I}[\mathbf{m}\notin S_{E_1+E_2}] \sum_{\substack{\mathbf{n}\leq\mathbf{m}\\ \partial\mathbf{n}=A}} \binom{\mathbf{m}}{\mathbf{n}}(-1)^{(\mathbf{n}|E_1)}(-1)^{(\mathbf{m}-\mathbf{n}|E_2)} \tag{3.7}$$

$$= \sum_{\partial M=A\Delta B} f(\mathbf{m})w(M)\mathrm{I}[M\notin S_{E_1+E_2}] \sum_{\substack{N\subseteq M\\ \partial N=A}} (-1)^{(N|E_1)}(-1)^{(M\setminus N|E_2)}. \tag{3.8}$$

Now, given $M\notin S_{E_1+E_2}$, it follows that there exists a loop $\gamma\subseteq M$ with $(-1)^{(\gamma|E_1+E_2)}=-1$. Using $\partial\gamma=\emptyset$, the map $N\mapsto N\Delta\gamma$ provides an involution on $\{N\subseteq M:\partial N=A\}$. By setting $g(N):=(-1)^{(N|E_1)}(-1)^{(M\setminus N|E_2)}$, the involution enables us to express

$$\sum_{\substack{N\subseteq M\\ \partial N=A}} g(N) = \sum_{\substack{N\subseteq M\\ \partial N=A}} g(N\Delta\gamma) = \sum_{\substack{N\subseteq M\\ \partial N=A}} (-1)^{(N\Delta\gamma|E_1)}(-1)^{((M\setminus N)\Delta\gamma|E_2)}$$

$$= \sum_{\substack{N\subseteq M\\ \partial N=A}} (-1)^{(N|E_1)(\gamma|E_1)}(-1)^{(M\setminus N|E_2)(\gamma|E_2)} = -\sum_{\substack{N\subseteq M\\ \partial N=A}} g(N)$$

Therefore, $\sum_{\substack{N\subseteq M\\ \partial N=A}} g(N)=0$, which proves the terms with $\mathbf{n_1}+\mathbf{n_2}\notin S_{E_1+E_2}$ have zero contribution to the sum in (3.6). □

We are finally ready to prove the main result.

Proof. (Theorem 3.7) Let $A=\{u_1,\ldots,u_n\}$ be a set of faces in G, with n even and let $\ell_1,\ldots,\ell_n$ be the corresponding disorder lines with $L=\ell_1+\ldots+\ell_n$. Using the low-temperature expansion as presented in (3.5) and then Proposition 3.2, we can express the correlation function $\langle\sigma_A\rangle_\mu$

$$\langle\sigma_{u_1}\ldots\sigma_{u_n}\rangle_\mu = \left\langle(-1)^{(\eta|\ell_1+\ldots+\ell_n)}\right\rangle_{\nu_\emptyset} = \left\langle(-1)^{(\mathbf{n}_{odd}|L)}\right\rangle_{\mathrm{P}_\emptyset} = \left\langle(-1)^{(\mathbf{n}|L)}\right\rangle_{\mathrm{P}_\emptyset} = \frac{\sum_{\partial\mathbf{n}=\emptyset} w(\mathbf{n})(-1)^{(\mathbf{n}|L)}}{\sum_{\partial\mathbf{n}=\emptyset} w(\mathbf{n})}.$$

We abbreviate

$$Z^{\emptyset,\emptyset} = \sum_{\partial\mathbf{n_1}=\partial\mathbf{n_2}=\emptyset} w(\mathbf{n_1})w(\mathbf{n_2}).$$

Applying the Switching lemma 3.12, where $E_1=L$ and $E_2=\emptyset$, and Lemma 3.11 respectively, yields

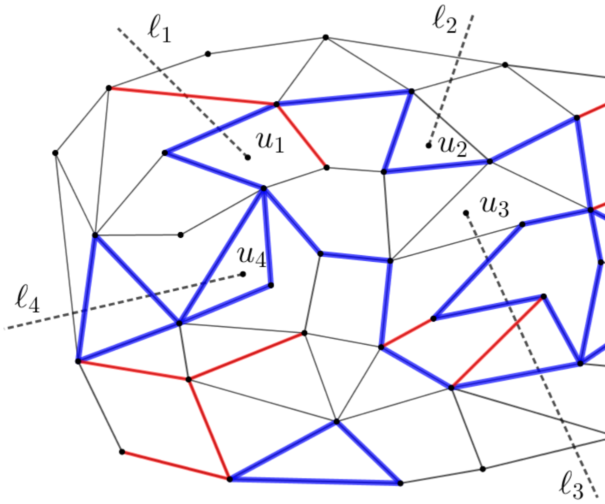


Figure 3.2: random current $\mathbf{n}$ divided into $\mathbf{n}_{odd}$ (blue) and $\mathbf{n}_{ev}$ (red) + disorder lines

$$\frac{\sum_{\partial \mathbf{n}=\emptyset} w(\mathbf{n})(-1)^{(\mathbf{n}|L)}}{\sum_{\partial \mathbf{n}=\emptyset} w(\mathbf{n})} = \frac{1}{Z^{\emptyset,\emptyset}} \sum_{\partial \mathbf{n_1}=\partial \mathbf{n_2}=\emptyset} w(\mathbf{n_1})(-1)^{(\mathbf{n_1}|L)} w(\mathbf{n_2}) \tag{3.9}$$

$$= \frac{1}{Z^{\emptyset,\emptyset}} \sum_{\partial \mathbf{n_1}=\partial \mathbf{n_2}=\emptyset} w(\mathbf{n_1})(-1)^{(\mathbf{n_1}|L)} w(\mathbf{n_2}) \mathbb{I}[\mathbf{n_1}+\mathbf{n_2} \in S_L] \tag{3.10}$$

$$= \frac{1}{Z^{\emptyset,\emptyset}} \sum_{\partial \mathbf{n_1}=\partial \mathbf{n_2}=\emptyset} w(\mathbf{n_1}) w(\mathbf{n_2}) \mathbb{I}[\mathbf{n_1}+\mathbf{n_2} \in S_L] \tag{3.11}$$

$$= \frac{1}{Z^{\emptyset,\emptyset}} \sum_{\partial \mathbf{n_1}=\partial \mathbf{n_2}=\emptyset} w(\mathbf{n_1}) w(\mathbf{n_2}) \mathbb{I}[\mathbf{n_1} \dotplus \mathbf{n_2} \in F_A] \tag{3.12}$$

$$= \mathrm{P}^{\emptyset} \otimes \mathrm{P}^{\emptyset}[\mathbf{n_1} \dotplus \mathbf{n_2} \in F_A] = d_\psi(F_A) = \langle \sigma_A \rangle_{1/2(d\,\psi)}, \tag{3.13}$$

To obtain the line (3.11), we have used that any sourceless current $\mathbf{n}$ can be written as a sum of loops $\mathbf{n} = \gamma_1 + .. + \gamma_k$, and thus for currents $\mathbf{n} \perp \mathbf{m}$ with $\mathbf{m} \in S_E$ and $\partial \mathbf{n} = \emptyset$, we have $(-1)^{(\mathbf{n}|E)} = 1$.

□

3.4 Truncated two-point function

A similar type of argument can be used to express the truncated two-point function

$$\langle \sigma_u; \sigma_v \rangle^+ := \langle \sigma_u \sigma_v \rangle^+ - \langle \sigma_u \rangle^+ \langle \sigma_v \rangle^+$$

in terms of the double random current measure $\mathrm{P}^{\emptyset} \otimes \mathrm{P}^{\emptyset} = \mathrm{P}^{\emptyset,\emptyset}$. Here, we abbreviate $\langle \cdot \rangle^+ = \langle \cdot \rangle^+_G$ for the dual Ising model on a finite planar graph G with + boundary conditions. Furthermore, we slightly abuse the notation by writing μ^+ for both the primal and the dual Ising model, and distinguish between the two based on whether the spins are on faces or vertices of the graph. However, we note that changing between the primal and the dual model is a matter of perspective.

Theorem 3.13. *Let u, v be two faces of a finite planar graph G together with their respective disorder lines ℓ_u and ℓ_v. Then the truncated two-point function*

$$\langle \sigma_u ; \sigma_v \rangle_G^+ = 2\mathbf{P}^{\emptyset,\emptyset}\left[u \overset{\mathbf{n}_1+\mathbf{n}_2}{\nleftrightarrow} v, \tau_u = \tau_v = -1 \right],$$

where $\tau_u = (-1)^{(\mathbf{n}_1+\mathbf{n}_2|\ell_u)}$, has the distribution of the XOR Ising spin with + boundary conditions, obtained by multiplying two independent Ising$^+$ spin configurations σ^1 and σ^2.

Proof. Using the low-temperature expansion in (3.4) and Proposition 3.2 gives

$$\langle \sigma_u \sigma_v \rangle^+ = \frac{\sum_{\partial \mathbf{n}=\emptyset} w(\mathbf{n})(-1)^{(\mathbf{n}|\ell_u+\ell_v)}}{\sum_{\partial \mathbf{n}=\emptyset} w(\mathbf{n})} \quad \text{and} \quad \langle \sigma_u \rangle^+ = \frac{\sum_{\partial \mathbf{n}=\emptyset} w(\mathbf{n})(-1)^{(\mathbf{n}|\ell_u)}}{\sum_{\partial \mathbf{n}=\emptyset} w(\mathbf{n})}.$$

We can write

$$\begin{aligned}
\langle \sigma_u ; \sigma_v \rangle^+ &= \frac{1}{Z^{\emptyset,\emptyset}} \sum_{\partial \mathbf{n}_1=\partial \mathbf{n}_2=\emptyset} w(\mathbf{n}_1)w(\mathbf{n}_2)[(-1)^{(\mathbf{n}_1|\ell_u+\ell_v)} - (-1)^{(\mathbf{n}_1|\ell_u)}(-1)^{(\mathbf{n}_2|\ell_v)}] \\
&= \frac{1}{Z^{\emptyset,\emptyset}} \sum_{\partial \mathbf{n}_1=\partial \mathbf{n}_2=\emptyset} w(\mathbf{n}_1)w(\mathbf{n}_2)(-1)^{(\mathbf{n}_1|\ell_u+\ell_v)} \, 2\mathrm{I}[(-1)^{(\mathbf{n}_1+\mathbf{n}_2|\ell_v)} = -1] \\
&= \frac{2}{Z^{\emptyset,\emptyset}} \sum_{\partial \mathbf{n}_1=\partial \mathbf{n}_2=\emptyset} w(\mathbf{n}_1)w(\mathbf{n}_2)(-1)^{(\mathbf{n}_1|\ell_u+\ell_v)} \, \mathrm{I}[(-1)^{(\mathbf{n}_1+\mathbf{n}_2|\ell_v)} = -1, \mathbf{n}_1+\mathbf{n}_2 \in S_{\ell_u+\ell_v}],
\end{aligned}$$

where in the last line we have used the Switching lemma 3.12 with $E_1 = \ell_u + \ell_v$, $E_2 = \emptyset$ and $f(\mathbf{n}_1 + \mathbf{n}_2) = \mathrm{I}[(-1)^{(\mathbf{n}_1+\mathbf{n}_2|\ell_v)} = -1]$. Furthermore, the condition $\mathbf{n}_1 + \mathbf{n}_2 \in S_{\ell_u+\ell_v}$ implies $(-1)^{(\mathbf{n}_1|\ell_u+\ell_v)} = 1$, since the sourceless current $\mathbf{n}_1$ decomposes into a sum of closed loops, and we can erase this term from the expression. Similarly, it implies $(-1)^{(\mathbf{n}_1+\mathbf{n}_2|\ell_u+\ell_v)} = 1$, which gives $(-1)^{(\mathbf{n}_1+\mathbf{n}_2|\ell_u)} = (-1)^{(\mathbf{n}_1+\mathbf{n}_2|\ell_v)}$. This yields

$$\begin{aligned}
&\frac{2}{Z^{\emptyset,\emptyset}} \sum_{\partial \mathbf{n}_1=\partial \mathbf{n}_2=\emptyset} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathrm{I}[(-1)^{(\mathbf{n}_1+\mathbf{n}_2|\ell_u)} = (-1)^{(\mathbf{n}_1+\mathbf{n}_2|\ell_v)} = -1, \mathbf{n}_1+\mathbf{n}_2 \in S_{\ell_u+\ell_v}] \\
&= \frac{2}{Z^{\emptyset,\emptyset}} \sum_{\partial \mathbf{n}_1=\partial \mathbf{n}_2=\emptyset} w(\mathbf{n}_1)w(\mathbf{n}_2)\mathrm{I}[\tau_u = \tau_v = -1, u \overset{\mathbf{n}_1+\mathbf{n}_2}{\nleftrightarrow} v],
\end{aligned}$$

where the last line follows from Lemma 3.11. □

Corollary 3.14. *Let u, v be two faces of G. Then we have the following bound for the truncated two-point function*

$$\langle \sigma_u ; \sigma_v \rangle^+ \leq 2\mu^+(u \overset{-}{\longleftrightarrow} v),$$

where $u \overset{-}{\longleftrightarrow} v$ is the event of connection between u and v via faces with $-$ spins. Moreover, the bound is valid in the infinite-volume limit on the infinite planar graph G.

Proof. Writing $\tau_u = (-1)^{(\mathbf{n}_1+\mathbf{n}_2|\ell_u)} = \sigma_u^1\sigma_u^2$, Theorem 3.13 and Proposition 3.2 yield

$$
\begin{aligned}
\frac{1}{2}\langle\sigma_u;\sigma_v\rangle_G^+ &= \mathbf{P}_G^{\cdot\,\cdot}\left[u \xleftrightarrow{\mathbf{n}_1+\mathbf{n}_2} v, \tau_u = \tau_v = -1\right] \\
&\quad \mathbf{P}_G^{\cdot\,\cdot}\left[u \xleftrightarrow{\mathbf{n}_1 \flat dd} v, u \xleftrightarrow{\mathbf{n}_2 \flat dd} v, \sigma_u^1\sigma_u^2 = -1\right] \\
&= \mathbf{P}_G^{\cdot}\left[u \xleftrightarrow{\mathbf{n}_1 \flat dd} v, \sigma_u^1 = -1\right] \mathbf{P}_G^{\cdot}\left[u \xleftrightarrow{\mathbf{n}_2 \flat dd} v, \sigma_u^2 = 1\right] \\
&= \mu_G^+(u \leftrightarrow v)\mu_G^+(u \xleftrightarrow{+} v) \\
&\quad \mu_G^+(u \leftrightarrow v)
\end{aligned}
$$

Let $G_n \nearrow \mathrm{G}$ be a sequence of finite planar graphs, which exhausts G and let $\Lambda \subset \mathrm{G}$ be a finite subgraph. The event $\{u \leftrightarrow v$ in $\Lambda\}$ is decreasing, therefore, by the FKG-inequality, $\mu_{G_n}^+(\{u \leftrightarrow v$ in $\Lambda\})$ increases to $\mu_{\mathrm{G}}^+(u \leftrightarrow v$ in $\Lambda)$ as n goes to infinity. Fixing an arbitrary n, and then taking $\Lambda \nearrow \mathrm{G}$, gives $\mu_{G_n}^+(u \leftrightarrow v) \leq \mu_{\mathrm{G}}^+(u \leftrightarrow v)$. In particular, $\langle\sigma_u;\sigma_v\rangle_{G_n}^+ \leq 2\mu_{G_n}^+(u \leftrightarrow v) \leq 2\mu_{\mathrm{G}}^+(u \leftrightarrow v)$ for all n, thus $\langle\sigma_u;\sigma_v\rangle_{\mathrm{G}}^+ \leq 2\mu_{\mathrm{G}}^+(u \leftrightarrow v)$. □

As a consequence, if there is no infinite σ-cluster of faces with $-$ signs, then the truncated two-point function $\langle\sigma_u;\sigma_v\rangle_{\mathrm{G}}^+$ converges to 0, as the graph distance between the faces u and v increases. The following statement gives another sufficient condition for the convergence of the truncated two-point function. Let C_- denote the event of having an infinite σ-cluster of faces with $-$ signs and C_+ be the corresponding event for + spins. Recall that C_∞ is the percolation event of having an infinite cluster of edges and denote C_∞^* the corresponding event on the dual graph. Finally, let ϕ^1 be the dual random-cluster measure on faces of G.

Proposition 3.15. *If either* $\phi^1(C_\infty^*) = 0$ *or* $\phi^0(C_\infty) = 0$*, then* $\langle\sigma_u;\sigma_v\rangle_{\mathrm{G}}^+ \xrightarrow[d(u,v)\to 0]{} 0$

Proof. First, we assume $\phi^1(C_\infty^*) = 0$. Then we have $\langle\sigma_u\rangle^+ = \phi^1(u \xleftrightarrow{\omega} \infty) = 0$ and $\langle\sigma_u\sigma_v\rangle^+ = \phi^1(u \xleftrightarrow{\omega} v) \to 0$ as $d(u,v) \to 0$. Therefore, also $\langle\sigma_u;\sigma_v\rangle_{\mathrm{G}}^+ \to 0$ as $d(u,v) \to 0$.

Second, if $\phi^0(C_\infty) = 0$, it suffices to show that $\mu_{\mathrm{G}}^+(C_-) = 0$. Assume, by contradiction, that $\mu_{\mathrm{G}}^+(C_-) > 0$. Then by Lemma 2.16 $\mu_{\mathrm{G}}^+(C_-, C_+) = 1$, which means there must be an infinite $+/-$ interface $C(\sigma)$, separating the + faces from the $-$ faces μ_{G}^+-almost surely. Using the low-temperature expansion, this is equivalent to saying that $\nu_{\mathrm{G}}^{\cdot}(C_\infty) = 1$. Now, using the fact that $\nu_{\mathrm{G}}^{\cdot} \leq_{st.} \phi^0$ established in Remark 3.3 on the three-way coupling, we conclude that $\phi^0(C_\infty) = 1$, which is a contradiction. □

It is known that for the Ising model with $J_e = \beta\ \forall e \in E$ on the hypercubic lattice $\mathbb{Z}^d, d \geq 1$, the truncated two-point function decays exponentially to zero when $\beta \neq \beta_c$ [7]. For the planar FK-Ising on $\mathbb{Z}^2$ with constant parameter $p_e = p\ \forall e \in E$, and away from criticality (i.e. $p \neq p_c$), coexistence of an infinite FK-cluster C_∞ and an infinite dual FK-cluster C_∞^* is impossible. Indeed, on $\mathbb{Z}^2$, the self-dual parameter equals the critical point: $p_{sd} = p_c = \sqrt{2}/(1+\sqrt{2})$. Thus $\phi^0(C_\infty) = 1 \implies p > p_{sd} \implies p^* < p_{sd} \implies \phi^1(C_\infty^*) = 0$, and by Proposition 3.15, the truncated two-point function converges to zero. This argument works for any planar model with $p_{sd} = p_c$, but the Proposition is much more general.

3.5 Partitions

In Theorem 3.7, we have established that the dual FK-Ising measure ϕ_{G,p_e} and the dual double random current measure $d\psi$ coincide on the connectivity events $\mathcal{F}_A$ for $A \subseteq V$. Now we want to investigate, whether this equality extends to the generated σ-algebra $\sigma(\mathcal{F}_A; A \subseteq V)$.

First, we generalize and formalize the divide and color model (see [26]). Let Part_V denote the set of all partitions of V and $\sigma(\text{Part}_V)$ the discrete σ-algebra. Let $\pi : \omega \mapsto \pi[\omega]$ be the map from subsets of $E(G)$ to Part_V^G given by the equivalence relation $x \sim_\omega y \iff x \overset{\omega}{\leftrightarrow} y$. We denote the image of the map π by $\text{Part}_V^G := \{\pi[\omega] : \omega \subseteq E(G)\}$. Let RER_V denote the set of all probability measures on $(\text{Part}_V, \sigma(\text{Part}_V))$ and let RER_V^G be the subset of all measures in RER_V supported on partitions in Part_V^G. We now consider a mapping Φ_p from RER_V to the set of probability measures on $\{\pm 1\}^V$ defined as follows. Given a measure $\nu \in \text{RER}_V$, we sample a partition $\pi \in \text{Part}_V$ according to ν ('divide'). Then, we assign the number 1 with probability p, and -1 with probability $1-p$, to each element of the partition π independently ('color'). The resulting probability measure $\Phi_p(\nu)$ on $\{\pm 1\}^V$ is called the *color process* or the *Generalized Divide and Color model*. For instance, the FK-Ising measure ϕ induces a measure $\pi_*[\phi] \in \text{RER}_V^G$, which is mapped with $\Phi_{1/2}$ to the Ising model $\Phi_{1/2}(\pi_*[\phi]) = \mu$. In [10], it is shown that if G is not a tree and $|V(G)| \geq 3$, the color representation of the Ising model is not unique, i.e. there exist at least two distinct measures $\nu_1, \nu_2 \in \text{RER}_V^G$ such that $\Phi_{1/2}(\nu_1) = \Phi_{1/2}(\nu_2) = \mu$.

Moreover, the map $\pi : \omega \mapsto \pi[\omega]$ from $2^{E(G)}$ to Part_V^G encodes all the information about the connectivity properties within ω. Knowing $\pi[\omega]$, we know whether any two points are connected in ω, and hence we know if $\mathcal{F}_A$ occurred. (We do, however, lose the information about the specific way the points are connected.) Conversely, knowing whether two points are connected in ω for any pair of points x, y, we can, by definition, deduce the partition $\pi[\omega]$. This means the σ-algebra $\sigma(\pi)$ generated by the map π coincides with the σ-algebra generated by connectivity events $\sigma(\mathcal{F}_A; A \subseteq V)$. It follows that for measures ϕ_1 and ϕ_2 on subsets of $E(G)$, the following three statements are equivalent:

i) $\phi_1 = \phi_2$ on $\sigma(\mathcal{F}_A; A \subseteq V)$,

ii) $\omega \sim \phi_1$ and $\varpi \sim \phi_2 \implies \pi[\omega] \sim \pi[\varpi]$,

iii) $\pi_*[\phi_1] = \pi_*[\phi_2]$ in RER_V^G.

Proposition 3.16. *$\pi_*[\phi_{G,p_e}] \neq \pi_*[d\psi]$ in RER_V^G, where $G = K_3$ is the complete graph on three vertices and $J_e = 1$ for $e = 1, 2, 3$. In other words, for $\omega \sim \phi_{G,p_e}$ and $\varpi \sim d\psi$, $\pi[\omega] \nsim \pi[\varpi]$.*

Proof. Let G be a graph consisting of two vertices $V(G) = \{x, y\}$ and connected by three edges $E(G) = \{e_1, e_2, e_3\}$. Using the viewpoint introduced in (3.1) of a random current as an edge configuration consisting of two disjoint parts $\mathbf{n} = \omega_{odd} \sqcup \omega_{even}$, where ω_{odd} is an even subgraph, we can explicitly express probabilities for the currents (see [22])

$$\mathrm{P}^{\emptyset}(\{\mathbf{n} = \omega_{odd} \sqcup \omega_{ev}\}) = \frac{\prod_{e \in \omega_{odd}} x_e \prod_{e\in\omega_{ev}} q_e \prod_{e \in E(G)\setminus\omega} (1 - q_e)}{\sum_{\partial\eta=\emptyset} x(\eta)},$$

where $\{\mathbf{n} = \omega_{odd} \sqcup \omega_{ev}\} := \{\mathbf{n}_{odd} = \omega_{odd}, \mathbf{n}_{ev} = \omega_{ev}\}$, $q_e = 1 - 1/\cosh(J_e)$ and $x_e = \tanh(J_e)$ We abbreviate x_j for x_{e_j} and p_j for p_{e_j}. For instance, we can compute

$$\begin{aligned}
\mathrm{P}^{\emptyset}(\widehat{\mathbf{n}} = \{e_1, e_2, e_3\}) &= \mathrm{P}^{\emptyset}(\{e_1, e_2\} \sqcup \{e_3\}) + \mathrm{P}^{\emptyset}(\{e_2, e_3\} \sqcup \{e_1\}) \\
&\quad + \mathrm{P}^{\emptyset}(\{e_1, e_3\} \sqcup \{e_2\}) + \mathrm{P}^{\emptyset}(\emptyset \sqcup \{e_1, e_2, e_3\}) \\
&= \frac{x_1x_2q_3 + x_2x_3q_1 + x_1x_3q_2 + q_1q_2q_3}{x_1x_2 + x_2x_3 + x_1x_3 + 1},
\end{aligned}$$

where the denominator equals the normalizing constant $\sum_{\partial\eta=\emptyset} x(\eta)$. In particular, we have

$$\begin{aligned}
\delta_1 &:= \mathrm{P}^{\emptyset}(\{\widehat{\mathbf{n}} = \{e_1\}\}) \propto q_1(1 - q_2)(1 - q_3), \\
\delta_2 &:= \mathrm{P}^{\emptyset}(\{\widehat{\mathbf{n}} = \{e_2\}\}) \propto (1 - q_1)q_2(1 - q_3), \\
\delta_3 &:= \mathrm{P}^{\emptyset}(\{\widehat{\mathbf{n}} = \{e_3\}\}) \propto (1 - q_1)(1 - q_2)q_3, \\
\delta_4 &:= \mathrm{P}^{\emptyset}(\{\widehat{\mathbf{n}} = \emptyset\}) \propto (1 - q_1)(1 - q_2)(1 - q_3).
\end{aligned}$$

We denote the three faces of G by $\{u_1, u_2, u_3\}$. We shall compute the probability that all three faces are in the same partition element after sampling ω with $d\,\psi$.

$$\begin{aligned}
d\,\psi(\pi[\omega] = \{\{u_1, u_2, u_3\}\}) &= d\,\psi\left(u_1 \overset{\omega}{\leftrightarrow} u_2 \overset{\omega}{\leftrightarrow} u_3\right) \\
&= d\,\psi(\omega = \{e_1, e_2, e_3\} \cup \{e_1, e_2\} \cup \{e_2, e_3\} \cup \{e_1, e_3\}) \\
&= \psi(\omega = \emptyset \cup \{e_1\} \cup \{e_2\} \cup \{e_3\}) \\
&= \delta_1\delta_1 + \delta_2\delta_2 + \delta_3\delta_3 + \delta_4\delta_4 + 2\delta_1\delta_4 + 2\delta_2\delta_4 + 2\delta_3\delta_4.
\end{aligned}$$

Setting $J_e = J = 1$, we have $q_j = q = 1 - 1/\cosh(1)$ and $x_j = x = \tanh(1)$ for $j = 1, 2, 3$ and get

$$\begin{aligned}
d\,\psi(\pi[\omega] = \{\{u_1, u_2, u_3\}\}) &= \frac{(1 - q)^6 + 6(1 - q)^5 q + 3(1 - q)^4 q^2}{(3x + 1)^2} \\
&= \frac{e^4(3 - 2e^2 + 3e^4)}{(1 + e^6)^2} \approx 0.0507 \qquad (3.14)
\end{aligned}$$

Secondly, we compute the probability of the same event under the dual FK-Ising measure. We have

$$\phi_{G,p_e}(\omega) \propto 2^{k(\omega)} \prod_{xy\in\omega} p_{xy} \prod_{xy \in \omega^c} (1 - p_{xy}),$$

where $p_e = 1 - e^{-2J_e}$ and the normalizing constant equals $1 + (1-p_1)(1-p_2)(1-p_3)$

$$
\begin{aligned}
&\phi_{G,p_e}(\omega = \{e_1\}) \propto p_1(1-p_2)(1-p_3),\\
&\phi_{G,p_e}(\omega = \{e_2\}) \propto (1-p_1)p_2(1-p_3),\\
&\phi_{G,p_e}(\omega = \{e_3\}) \propto (1-p_1)(1-p_2)p_3,\\
&\phi_{G,p_e}(\omega = \emptyset) \propto 2(1-p_1)(1-p_2)(1-p_3),
\end{aligned}
$$

$$
\begin{aligned}
\phi_{G',p_e}(\pi[\omega'] = \{\{u_1,u_2,u_3\}\}) &= \phi_{G',p_e}\left(u_1 \overset{\omega'}{\leftrightarrow} u_2 \overset{\omega'}{\leftrightarrow} u_3\right)\\
&= \phi_{G',p_e}(\omega' = \{e_1,e_2,e_3\} \cup \{e_1,e_2\} \cup \{e_2,e_3\} \cup \{e_1,e_3\})\\
&= \phi_{G,p_e}(\omega = \emptyset \cup \{e_1\} \cup \{e_2\} \cup \{e_3\}).
\end{aligned}
$$

Setting all $J_e = J = 1$, we have $p_j = p = 1 - e^{-2J_e}$ for $j = 1,2,3$ and get

$$
\phi_{G',p_e}(\pi[\omega'] = \{\{u_1,u_2,u_3\}\}) = \frac{2(1-p)^3 + 3(1-p)^2 p}{(1-p)^3 + 1} = \frac{1+3e^2}{1+e^6} \approx 0.0523,
$$

which is different from (3.14). This completes the proof. □

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