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„Blow-up Phenomena on Friedman-Lemaître-Robertson-Walker spacetimes"

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Contents

1. Introduction

Understanding the properties of solutions the Einstein equations

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathbf{R}g_{\mu\nu} = 8\pi T_{\mu\nu}$$

is the backbone of Mathematical Relativity and thus of gaining a mathematically rigorous understanding of our universe. Due to the complexity of these equations, a common strategy to gain more insight into them is to consider how the equations behave for certain simple, well-known models. From a cosmological perspective, one often uses Friedman-Lemaître-Robertson-Walker (FLRW) spacetimes

$$M = I \times \overline{M},\ g = -dt^2 + a(t)^2\bar{g}$$

to describe the universe as a whole as a spacetime filled with an expanding homogeneous ideal fluid with energy density ρ and pressure p, where I is an open subinterval of $\mathbb{R}$, $a \in C^\infty(I)$ is governed by the Friedman equations and $\left(\overline{M}, \bar{g}\right)$ is a Riemannian manifold of constant sectional curvature κ. Further, to slowly approach more phenomenological matter models encoded by the energy-momentum tensor T on such geometric backgrounds, it is a reasonable first step to couple it with simplest matter model conceivable, namely by a scalar field associated with

$$T_{\mu\nu}[\varphi] = \nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}\nabla^\eta\varphi\nabla_\eta\varphi\,.$$

Since this FLRW model is, of course, a considerable simplification of the known universe, the question arises of whether and which properties of FLRW spacetimes are conserved when the structure of the spacetime only slightly deviates from this large-scale model – in particular whether near-FLRW solutions to the Einstein Scalar-Field equations also exhibit a Big Bang in the sense that the Riemann curvature tensor blows up toward the left-hand boundary of I. While this past nonlinear stability of FLRW spacetimes has been proven when the sectional curvature κ of $\left(\overline{M}, \bar{g}\right)$ is zero or positive (see [14, 15, 16]), the problem remains open for $\kappa = -1$.

To approach this mathematically rather involved analysis, the aim of this thesis will be to formulate a rigorous understanding on how solutions to the wave equation

$$\Box_g\psi = g^{\mu\nu}\nabla_\mu\nabla_\nu\psi = 0$$

blow up toward the Big Bang singularity on a fixed FLRW background (M, g) as above, where $\left(\overline{M}, \bar{g}\right)$ is of zero or negative sectional curvature, i.e. whether they also diverge along with the geometric quantities and if so, at what rate and which rough asymptotic form they take. Waves are of interest here since they arise as the natural form scalar fields take and since they can be understood as a toy model for the linearized Einstein equations. To be slightly more precise, it will become apparent that the geometric properties of $\left(\overline{M}, \bar{g}\right)$ are only relevant in so far as they influence the scale factor via the Friedman equations. Thus, a more general class of *warped product* spacetimes will be analyzed where any additional geometric assumptions on the closed Riemannian manifold $\left(\overline{M}, \bar{g}\right)$ are dropped, but the expansion rate within this spacetime is the one obtained from the Friedman equations for $\kappa = 0$ and $\kappa = -1$ respectively. In this sense, this thesis is a vast generalization of the arguments and results in [1] that provided an asymptotic blow-up analysis of waves for "pure" FLRW spacetimes with $\kappa = 0$ (along with generalizing the analysis to waves to the stiff-fluid background $\rho = p$ as far as possible), even though certain structural ideas will follow along similar lines. To be more precise, the fact that, unlike in flat spatial geometry, coordinate derivatives no longer commute will severly complicate proving regularity statements toward the Big Bang.

After covering some basic notation in Chapter 2, Chapter 3 will provide a more comprehensive overview on how our mathematical question is naturally motivated from physical considerations, in particular with regards to how FLRW spacetimes and the Friedman equations arise in cosmology and as to how waves and the scalar-field energy-momentum tensor are connected. The more general mathematical setup can then be established in Chapter 4 along with many useful preliminary properties of the scale factor a and of waves, after which Chapter 5 provides a necessary mathematical excursion for our analysis, generalizing the concepts of Sobolev spaces and ellipticity from the well-known Euclidean case to the more general Riemannian setting.
With all of these tools in hand, it will be shown Chapter 6 that certain energies of waves remain bounded toward the Big Bang singularity at $t = 0$, and similarly for waves rescaled by their suspected leading asymptotic order, allowing to extract pointwise upper bounds for the rescaled variables that extend to $t = 0$. With this in hand, it will be proven in Section 7.1 that, for $p = (\gamma - 1)\rho,\ \gamma \in (2/3, 2)$, any smooth wave ψ takes the asymptotic form

$$\begin{aligned} \psi(t,x) &= A(x) t^{1-\frac{2}{\gamma}} + o\left(t^{1-\frac{2}{\gamma}}\right), \quad \text{respectively} \\ \psi(t,x) &= A(x) \int_t^\infty a(s)^{-3}\, ds + o\left(\int_t^\infty a(s)^{-3}\, ds\right) \end{aligned}$$

for some $A \in C^\infty\left(\overline{M}\right)$ as $t \to 0$ on warped product spacetimes associated with $\kappa = 0$, respectively $\kappa = -1$. Furthermore, rather general sufficient conditions on the initial data will be provided in Section 7.2 that ensure that A does not vanish. and hence guarantee blow-up of highest possible order. More precisely, these conditions essentially state that if the initial data of derivatives of ψ on some close enough initial hypersurface $\overline{M}_{t_0}$ is dominated in a sufficiently strong L^2-sense by velocity terms $\|\partial_t \psi(t_0, \cdot)\|_{L^2(\overline{M})}$ (i.e. spatial inhomogeneities of the wave are comparatively small), then this is preserved when developing toward the Big Bang hypersurface, ensuring that the highest order in t does not vanish. The asymptotic results will be extended to the stiff case as far as possible in Section 7.3, before closing out the main line analysis in Chapter 8 by connecting the obtained results to the more general question of nonlinear stability of FLRW spacetimes as in [14, 15, 16] that was posed earlier.

2. Notation

Throughout this thesis and unless specified otherwise, Greek indices $(\mu, \nu, \dots)$ are numbered starting from 0, while Latin indices $(i, j, k, \dots)$ start from 1, and the Einstein summation convention will be in force unless explictly stated otherwise.

When using local coordinates $(x_\mu)_{\mu=0,1,2,3}$ on a semi-Riemannian manifold of the form $(M = I \times \overline{M}, g)$, where I is an open interval and $\overline{M}$ is a three dimensional Riemannian manifold $\left(\overline{M}, \bar{g}\right)$, we tacitly assume local coordinates on a suitable open coordinate neighbourhood $J \times U$ take the form

$$x_0(t,p) = t,\ x_i(t,p) = \chi_i(p),$$

where (χ_i) is a corresponding local chart on U. For the sake of notational simplicity, we will occasionally use x_i to refer to both the coordinates on some submanifold $\overline{M}_t = \{t\} \times \overline{M}$ and to the corresponding χ-chart on $\overline{M}$ where this nuance of embedding isn't relevant.
Accordingly, Γ^c_{ab} and ∂_a (resp. $\bar{\Gamma}^k_{ij}$ and $\bar{\partial}_i$) denote Christoffel symbols and coordinate derivatives for local charts on M (resp. $\overline{M}$). Further, ∇ (resp. $\overline{\nabla}$)[1] are the Levi-Civita-connections and $\Box_g$ (resp. Δ) the Laplace-Beltrami operators on (M, g) (resp. $\left(\overline{M}, \bar{g}\right)$).

In particular, if φ is a smooth function on $(M = I \times \overline{M}, g)$, where $\iota_t : \overline{M} \longrightarrow \overline{M}_t$ is the standard smooth embedding, we will use the notation $\varphi(t, \cdot) := \varphi \circ \iota_t$, similarly meaning $\overline{\nabla}\varphi(t, \cdot)$ to be $\overline{\nabla}(\varphi \circ \iota_t)$ and so on for higher order spatial derivatives.
Additionally, vol_N denotes the volume form on the Riemannian manifold (N, h), and unless stated otherwise, $\psi : M \to \mathbb{R}$ is a smooth wave on the semi-Riemannian manifold (M, g), i.e. $\Box_g \psi \equiv 0$.

For two non-negative functions f, g, $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leq Cg$. If C is dependent on a set of variables $v_1, \dots, v_n$, we write $f \lesssim_{v_1,\dots,v_n} g$. Furthermore, $f \simeq g$ means that $f \lesssim g$ and $g \lesssim f$ hold, and $f \simeq_{v_1,\dots,v_n} g$ is defined analogously.

Finally, our convention for the natural numbers is $\mathbb{N} = \{0, 1, 2, \dots\}$.

[1] It should be stressed that this notation is **not** consistent with [1], one of the main works this thesis will build upon, since they use this symbol to mean the induced covariant derivative on corresponding embedded hypersurfaces $\overline{M}_t$, which differs from our usage by a t-dependent scaling.

3. Physical background

3.1. FLRW spacetimes and cosmology

To start off, this section will (re-)introduce FLRW spacetimes as they arise naturally in cosmology, as well as collect some geometric formula for later usage. This serves mainly as motivation and backdrop to the more general warped product setting introduced in Section 4.1, and will mostly follow [9, Ch. 12.4].

Consider a four dimensional manifold $M = I \times \overline{M}$, where I is some open subinterval of $\mathbb{R}$ and $\overline{M}$ is a connected three-dimensional manifold. Further, we define ∂_t to be the vector field corresponding to classical derivative on I after lifting to M. The goal is to construct a cosmological model in the sense of finding a Lorentzian metric g on M that can approximately match the behaviour of the observable universe considered as a single object, with internal complexities severely simplified. In this model, I and $\overline{M}$ should encode time and space respectively. Thus, the following additional constraints have to be imposed:

1. Due to the absence of any additional influence, a galaxy simplifed in the point p should move along worldlines $\gamma_p : t \mapsto (t, p)$, i.e. these must be affinely parametrized geodesics such that $g(\dot{\gamma}_p, \dot{\gamma}_p) = g(\partial_t, \partial_t)$ is a negative constant. For the sake of convention, one chooses the scaling $g(\partial_t, \partial_t) = -1$. From the perspective of the 3+1-formalism (see Chapter 8), this corresponds to lapse function $n = 1$ and Gaussian shift vector $X = 0$.

2. On a large scale and on average, galaxies do not seem to move much relative to one another. Hence, for any $t \in I$, we assume that ∂_t is normal on any spatial slice $\overline{M}_t = \{t\} \times \overline{M} \subseteq M$ since any point must be influenced by time equally when developing along worldlines. It immediately follows[2] that $\left(\overline{M}_t, g|_{\overline{M}_t}\right)$ must be a Riemannian manifold for any $t \in I$, and because $\overline{M}$ should be isometric to these spatial slices to properly model them, $\overline{M}$ must also be Riemannian with regards to some metric $\bar{g}$.

3. Even more precisely, no galaxy supercluster seems to be special or in a special place compared to any other one. Instead of viewing every single galaxy supercluster as an individual structure, this is modelled by saying that the universe looks the same in all directions from every point. More formally, this isotropy should mean

[2]From here on out, $g|_{\overline{M}_t}$ is meant to denote the map $\overline{M}_t \ni (t,p) \mapsto g_{(t,p)}|_{T_{(t,p)}\overline{M}_t \times T_{(t,p)}\overline{M}_t}$.

that, for any $(t,p) \in M$, there exists an open neigbourhood such that, for any unit vectors $x, y \in T_{(t,p)}M$ tangent to $\overline{M}_t$, there exists an isometry φ on M that takes the form $\mathrm{id} \times \overline{\varphi}$ near (t,p) and satisfies $d\varphi(x) = y$.

The last condition in particular forces a very rigid structure onto $\overline{M}$:

Proposition 3.1.1 ([9, p. 342f.])**.** *Given the assumptions stated above, any $\left(\overline{M}_t, g|_{\overline{M}_t}\right)$ (and hence also $\left(\overline{M}, \bar{g}\right)$) is of constant sectional curvature $C(t)$ (resp. κ) (see Example A.1.7). Additionally, for any $s,t \in I$, the diffeomorphism*

$$\mu \equiv \mu_{st} : \overline{M}_s \to \overline{M}_t,\ \ \mu(s,p) = (t,p)$$

satisfies $\mu^*\left(g|_{\overline{M}_t}\right) = g|_{\overline{M}_s}$. The equivalent statement holds for $M \hookrightarrow \overline{M}_t$, and more precisely, there exists a smooth function $a : I \to \mathbb{R}^+$ such that $a(t)^2\bar{g} = \iota_t^*\left(g|_{\overline{M}_t}\right)$ with regards to the standard embedding $\iota_t : \overline{M} \hookrightarrow \overline{M}_t$. Finally, one can choose $\left(\overline{M}, \bar{g}\right)$ such that $\kappa \in \{-1, 0, 1\}$.

Proof. Regarding the former, it suffices by Schur's Lemma (see Lemma A.1.10) to prove that, at any $(t,p) \in \overline{M}_t$, the sectional curvature is constant for nondegenerate planes $\Pi_1, \Pi_2 \subset T_{(t,p)}\overline{M}_t$. Since they must intersect in at least a one-dimensional subspace, they are either the same or one can find unit vectors x, y, z tangent to $\overline{M}_t$ such that (x, z) is an orthonormal basis of Π_1 and (y, z) is one of Π_2. In either case, by the assumption on local isotropy, we can hence find a local isometry $\varphi = \mathrm{id} \times \overline{\varphi}$ such that $d\varphi(x) = y$ holds, and rotate $\overline{\varphi}$ to ensure that $d\varphi(z) = z$ is satisfied additionally. In particular, one then has $d\varphi\left(\Pi_1\right) = \Pi_2$. Since the sectional curvature is uniquely determined by the curvature tensor (see Definition A.1.6) which is preserved under local isometries, κ is also preserved by φ and must thus stay the same on any nondegenerate plane, so the statement now follows.

Regarding the second point, we refer to the proof of [9, p. 342f., Prop. 12.6b)] for details – the essential idea is that since all hypersurfaces have constant sectional curvature and are diffeomorphic to one another via μ, the metrics $\left(g|_{\overline{M}_s}\right)\Big|_{(s,p)}$ and $\left(g|_{\overline{M}_t}\right)\Big|_{(t,p)}$ can only differ by a scale factor $h \equiv h(s,t,p)$, i.e. $h(s,t,p)^2C(t) = C(s)$. Using the spatial isotropy, one can then show that it is independent of p, which gives the statement. Note that h in particular also satisfies $h(s,t)^2 g|_{\overline{M}_t} = g|_{\overline{M}_s}$, and that since μ is a diffeomorphism, h is nonvanishing, so the sign of C is fixed.

Further, for some fixed $s \in I$, we set $\kappa = C(s)a(s)^2$, where $\kappa \in \{-1, 0, 1\}$ encodes the sign of C and we choose $a(s) > 0$ arbitrarily if $\kappa = 0$. Then, we assign $\overline{M}$ with the metric $\bar{g}$ such that ι_s is an isometry with $a(s)^2\bar{g} = \iota_s^*\left(g|_{\overline{M}_s}\right)$. From this, it already follows from a similar argument as in the second step that $\left(\overline{M}, \bar{g}\right)$ has sectional curvature κ, and that $a(t) = {}^{a(s)}/_{h(s,t)}$ satisfies our desired property since, for any consistent family of embeddings $\iota_t : \overline{M} \hookrightarrow \overline{M}_t$, $\iota_t = \mu_{ts} \circ \iota_s$ must hold, and thus

$$\iota_t^* \left(g|_{\overline{M}_t} \right) = \iota_s^* \left(h(t,s)^2 g|_{\overline{M}_s} \right) = h(s,t)^{-2} \iota_s^* \left(g|_{\overline{M}_s} \right) = \frac{a(s)^2}{h(s,t)^2} \bar{g} = a(t)^2 \bar{g} \,.$$

In particular, h and thus also a can always be chosen to be positive without loss of generality. □

Thus, any cosmological model is already constrained to be within the following class of spacetimes:

Definition 3.1.2. A **Friedman-Lemaître-Robertson-Walker** (FLRW) **spacetime** is a four-dimensional Lorentzian manifold (M, g) such that $M = I \times \overline{M}$, where $\left(\overline{M}, \bar{g}\right)$ is a *closed* (three dimensional) Riemannian manifold of constant sectional curvature $\kappa \in \{-1, 0, 1\}$ and I is an open subinterval of $\mathbb{R}$, and[3]

$$g = g_{\mathrm{FLRW}} = -dt^2 + a(t)^2 \bar{g}$$

holds for a function $a \in C^\infty(I, \mathbb{R}^+)$.

Remark 3.1.3. The only condition that does not naturally arize from above discussion is the closedness of $(\overline{M}, \bar{g})$. While this is not always used in the definition for FLRW-spacetimes, it is a rather natural additional assumption to impose upon a cosmological model:
On the one hand, one does not experience the universe to have an "edge" at any point in time and if such an edge were to exist, since we assume no point in the universe to have any special role, it would have to be equally visible and in particular equally far away from any observer – so it at best could exist at infinite distance from any particle in the universe, at which point it is natural to discard it and to assume $\partial \overline{M} = \varnothing$. On the other hand, there are no singularities on the macro scale of galaxy superclusters, so $\left(\overline{M}, \bar{g}\right)$ can reasonably be chosen to be geodesically complete. Further, for the sake of simplicity, assuming the universe to be bounded is not too much of a restriction since only a finite section of the universe could have ever had influence on our observable universe, so by the Hopf-Rinow theorem, it is also closed under that assumption.

Before moving on, we quickly collect some formulas for geometric objects in FLRW spacetimes:

Remark 3.1.4. Using the standard formula

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\frac{\partial g_{\sigma\mu}}{\partial x^\nu} + \frac{\partial g_{\sigma\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right),$$

the Christoffel symbols associated to the FLRW metric as in Definition 3.1.2 are as follows:

[3]dropping the obvious pullbacks along standard embeddings in the notation from here on out

$$
\begin{aligned}
\Gamma^0_{0\mu} &= \Gamma^0_{\mu 0} = \Gamma^\mu_{00} = 0 \\
\Gamma^j_{0i} &= \Gamma^j_{i0} = \frac{1}{2} g^{j\sigma} \frac{\partial g_{\sigma i}}{\partial t} = \frac{1}{2} a(t)^{-2} \bar{g}^{jk} \bar{g}_{ki} \cdot 2a(t)\dot{a}(t) = \frac{\dot{a}(t)}{a(t)} \delta^j_i \\
\Gamma^0_{ij} &= -\frac{1}{2}\left(-\frac{\partial g_{ij}}{\partial t}\right) = a(t)\dot{a}(t) \cdot \bar{g}_{ij} \\
\Gamma^k_{ij} &= \frac{1}{2} a(t)^{-2} \bar{g}^{kl} \cdot a(t)^2 \left(\frac{\partial \bar{g}_{li}}{\partial x^j} + \frac{\partial \bar{g}_{lj}}{\partial x^i} - \frac{\partial \bar{g}_{ij}}{\partial x^l}\right) = \bar{\Gamma}^k_{ij}
\end{aligned}
$$

We deduce the following local identities for covariant derivatives for any vector fields X, Y on M, with $\overline{X}, \overline{Y}$ denoting the induced vector fields on $\overline{M}_t$ (see Definition A.1.12 for the relationship between ∇ and $\overline{\nabla}$):

$$
\begin{aligned}
\nabla_0 X_0 &= \partial_t X_0 \\
\nabla_0 X_i &= \partial_t X_i - \frac{\dot{a}(t)}{a(t)} X_i \\
\nabla_i X_0 &= \partial_i X_0 - \frac{\dot{a}(t)}{a(t)} X_i \\
\nabla_i X_j &= \partial_i X_j - \bar{\Gamma}^k_{ij} X_k - a(t) \cdot \dot{a}(t) \bar{g}_{ij} X_0 = \overline{\nabla}_i \overline{X}_j - a(t)\dot{a}(t) \cdot \bar{g}_{ij} X_0 \\
\nabla_0 Y^0 &= \partial_t Y^0 \\
\nabla_0 Y^k &= \partial_t Y^k + \frac{\dot{a}(t)}{a(t)} Y^k \\
\nabla_i Y^0 &= \partial_i Y^0 + a(t)\dot{a}(t) \cdot \bar{g}_{ij} Y^j \\
\nabla_i Y^k &= \partial_i Y^k + \bar{\Gamma}^k_{ij} Y^j + \frac{\dot{a}(t)}{a(t)} \delta^k_i Y^0 = \overline{\nabla}_i \overline{Y}^k + \frac{\dot{a}(t)}{a(t)} \delta^k_i Y^0
\end{aligned}
$$

Lemma 3.1.5 ([9, p. 354, Corollary 12.10])**.** *If $\left(\overline{M}, \bar{g}\right)$ is of constant sectional curvature κ, the Ricci curvature* Ric *of the associated FLRW spacetime is given as follows:*

$$
\begin{aligned}
\mathrm{Ric}(\partial_t, \partial_t) &= -3\frac{\ddot{a}}{a}, \quad \mathrm{Ric}(\partial_t, X) = 0 \qquad \forall\, X \perp \partial_t \\
\mathrm{Ric}(X, Y) &= \left(2\frac{\dot{a}^2 + \kappa}{a^2} + \frac{\ddot{a}}{a}\right) g(X, Y) \qquad \forall\, X, Y \perp \partial_t
\end{aligned}
$$

Its scalar curvature is

$$
\boldsymbol{R} = 6\left(\frac{\dot{a}^2 + \kappa}{a^2} + \frac{\ddot{a}}{a}\right).
$$

3.2. Perfect fluids, the Friedman equations and Big Bang singularities

In the last section and in particular in Lemma 3.1.5, the shape of the purely geometric components of the Einstein equations for our cosmological model has been determined, so what remains to be done is to formulate a reasonable framework for the matter components. In the initial simplification, we dropped all internal complexities of the universe and viewed any point as equivalent to any other. So, one essentially views the universe as one single homogeneous substance. As such, the closest physical approximation is a perfect fluid with some energy density ρ and pressure p that moves through spacetime with four-velocity u. The energy-stress tensor for such a perfect fluid takes the form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$$

(see [9, p.337f.] for a heuristic derivation). Further, since our model has to remain spatially homogeneous in how energy and pressure are distributed, we can assume p and ρ to be covariantly constant, and because we assumed the worldlines of any "molecule" in our fluid to flow along ∂_t, we simply have $(u^\mu) = (-1, 0, 0, 0)^T$. For any tangent vectors X, Y orthogonal to ∂_t, the tensor thus reduces to:

$$T(\partial_t, \partial_t) = (\rho + p) - p = \rho, \quad T(\partial_t, X) = 0, \quad T(X, Y) = pg(X, Y) \tag{1}$$

However, since the energy stress tensor must be divergence free because the Einstein tensor is, some further constraints have to be imposed on how ρ and p may be chosen:

Lemma 3.2.1. *Consider an FLRW spacetime (M, g) with scale factor a. The energy-momentum tensor given by* (1)*, where p and ρ are only dependent on t, is divergence-free if and only if the following* ***continuity equation*** *is satisfied:*

$$\partial_t \rho = -3\frac{\dot{a}}{a}(\rho + p) \tag{2}$$

Proof. By [10, p.28, (2.66)], $\nabla_\mu T^{\mu\nu} = 0$ simply reduces[4] to

$$\partial_t \rho - tr(k)\rho - k_{ij}T^{ij} = 0,$$

where k is the second fundamental form of the spatial hypersurfaces $\overline{M}_t$ (see Definition A.1.13) and $tr(k)$ is its trace with regards to g. With Remark 3.1.4, we compute:

$$k_{ij} = -g\left(\nabla_{\bar{\partial}_i}(\partial_t), \bar{\partial}_j\right) = -\Gamma^c_{i0}g_{cj} = -\frac{\dot{a}(t)}{a(t)}a(t)^2\bar{g}_{ij} = -\dot{a}(t)a(t)\bar{g}_{ij}$$

In particular, $tr(k) = -\dot{a}(t)a(t)\bar{g}_{ij}g^{ij} = -3\frac{\dot{a}(t)}{a(t)}$ holds, and since $k_{ij}T^{ij} = p \cdot tr(k)$ by (1), the statement now follows. □

Hence, the Einstein equations for our cosmological model take the following form:

Proposition 3.2.2. *An FLRW spacetime (M, g) of the form $M = I \times \overline{M}$, with scale factor a and where $\left(\overline{M}, \bar{g}\right)$ is of constant sectional curvature κ, solves the Einstein*

[4] In general, this also contains an evolution condition on the shear stress components T_{0i}, but these are zero for our stress tensor and hence this is trivially satisfied.

equations when coupled with the divergence-free energy-momentum tensor as in (1) *if and only if, along with the continuity equation, the* ***Friedman equations*** *are satisfied:*

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{\kappa}{a^2} \tag{3}$$

$$\frac{\ddot a}{a} = -\frac{4\pi}{3}\left(\rho + 3p\right) . \tag{4}$$

Proof. Again, it obviously suffices to show this equivalence when inserting any combination of ∂_t and arbitrary vectors X, Y orthogonal to ∂_t into either sides of the Einstein equations

$$\mathrm{Ric} - \frac{1}{2}\mathbf{R}g = 8\pi T.$$

With Lemma 3.1.5, one computes that both sides always vanish when inserting ∂_t and X, and can rephrase the remaining equations as

$$8\pi\rho = -3\frac{\ddot a}{a} + 3\left(\left(\frac{\dot a + \kappa}{a^2}\right)^2 + \frac{\ddot a}{a}\right) = 3\frac{\dot a^2 + \kappa}{a^2} .$$

$$\begin{aligned} 8\pi p \cdot g(X,Y) &= \left(2\frac{\dot a^2 + \kappa}{a^2} + \frac{\ddot a}{a}\right) g(X,Y) - 3\left(\frac{\dot a^2 + \kappa}{a^2} + \frac{\ddot a}{a}\right) g(X,Y) \\ &= \left[-\frac{\dot a^2 + \kappa}{a^2} - 2\frac{\ddot a}{a}\right] g(X,Y) . \end{aligned}$$

The first equation is already (3). Inserting this into the second equation and dropping $g(X,Y)$, we equivalently obtain

$$8\pi p = -\frac{1}{3}\cdot 8\pi\rho - 2\frac{\ddot a}{a}$$

and thus (4) after rearranging. □

Remark 3.2.3. At this point, it should be stressed that one ultimately, the Friedman equations being satisfied for some fixed $\kappa \in \{-1, 0, 1\}$ is the only additional constraint that is going to be imposed on warped product spacetimes $\left(I \times \overline{M}, g\right)$ with metrics of the form

$$g = -dt^2 + a(t)^2 \bar g$$

later on: $\left(\overline{M}, \bar g\right)$ will be considered as a general closed Riemannian manifold that is not necessarily of constant sectional curvature (see Section 4.1 for more on this point and the general setting of our statements). However, it is still instructive to see how these equations arise from geometric properties that follow from intuitive restrictions a simple cosmological model should satisfy. Furthermore, while the main physical application of our results still lies within “true” FLRW spacetimes, the relevance of being able to consider these more general warped products will be further explored in Chapter 8.

Remark 3.2.4. Finally, we can use these equations to give some intuition to the formal definition of a Big Bang singularity for warped products as just described:
The universe is currently observed to be expanding (see [9, p.347]), so $\frac{\dot{a}}{a}$ is currently positive, hence so is $\dot{a}$ by our sign convention. As long as one has $\rho + 3p > 0$, $\ddot{a}$ is strictly decreasing, hence $\dot{a}$ must always have been positive, i.e. a must have been strictly increasing. In particular, this means that if we want to consider the time variable t on the maximal interval of existence $I = (t_{min}, t_{max})$ for a, a must (heuristically) converge to 0 approaching t_{min} since this would be the first point at which the spacetime metric would fail to be Lorentzian. This would correspond to matter being contracted to a singular point from a physical perspective, i.e. ρ should diverge toward ∞. To safely ensure this, it suffices to assume, after multiplying both sides of (3) by a^2 and taking the limit, that $\dot{a}^2$ (and hence $\dot{a} > 0$) diverges toward ∞ approaching t_{min}. Note that precisely if all of these requirements are met, Lemma 3.1.5 implies that the Ricci tensor (for an FLRW spacetime) also diverges, so the breakdown of our spacetime is encoded directly within the Einstein equations via the curvature.

Altogether, this means that we say an FLRW spacetime, or more generally a warped product spacetime of the type outlined in Remark 3.2.3, forms a **Big Bang singularity** when $a \to 0$ and $\dot{a} \to \infty$ hold approaching t_{min} (see [9, p.348, Def. 12.16]). In Section 4.2, we will more rigorously show that a Big Bang actually forms in our setting, along with the singularities being "physical" in the sense that $\rho \to \infty$ holds toward t_{min}.

3.3. Waves and the scalar-field matter model

In addition to the now essentially complete set-up for the substratum of our universe, it is now of interest to consider first ways of introducing matter beyond just the substratum into this model and to analyse its behaviour. In this short section following [10, p. 30-35], we will briefly recap how the wave equation arises from a very simple matter model and then verify the continuity equation for the corresponding energy-momentum tensor.

As mentioned previously, one of the most basic approaches that we could conceivably use is modelling matter by a smooth scalar function $\varphi : M \to \mathbb{R}$ that is not influenced by any potential. Hence, in analogy to the classical Lagrangian of a free particle, the relativistic Lagrangian associated with such matter would be

$$\mathcal{L} = -\frac{1}{2}\nabla^\mu\varphi\nabla_\mu\varphi = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi\,. \tag{5}$$

In analogy to the non-relativistic Euler-Lagrange equations, the corresponding equation of motion for a Lagrangian $\mathcal{L}$ are given by

$$\nabla^\mu\left(\frac{\delta\mathcal{L}}{\delta\left(\nabla^\mu\varphi\right)}\right) = \frac{\delta L}{\delta\varphi}\,,$$

while the energy-momentum tensor (arising from minimizing the action functional corresponding to the Lagrangian) reads

$$T_{\mu\nu}[\varphi] = -2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} + \mathcal{L}g_{\mu\nu}$$

(see [10, p.30, (3.1)-(3.2)]). For our simple matter model arising from (5), one now easily calculates that the energy-momentum tensor reads

$$\begin{aligned} T_{\mu\nu}[\varphi] &= -2 \cdot \left(-\frac{1}{2}\nabla_\mu\varphi\nabla_\nu\varphi\right) - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\varphi\nabla_\alpha\varphi \\ &= \nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\varphi\nabla_\alpha\varphi \end{aligned} \tag{6}$$

and that the equation of motion simply becomes

$$\nabla^\mu\left(-\frac{1}{2}\nabla_\mu\varphi\right) = 0 \;\Leftrightarrow \Box_g\varphi = \nabla^\mu\nabla_\mu\varphi = 0\,.$$

To fully introduce scalar field matter into the gravitational system, one would have to add (6) to the energy-momentum tensor of the perfect fluid substratum within our Einstein equations, or at the very least just couple the Einstein tensor with the scalar-field energy-momentum for a fixed scalar field, either of which would create co-dependencies which would greatly complicate our analysis. As a first step toward such problems, we will hence fix the geometric FLRW background as in Definition 3.1.2 with a scale factor a arising from the Friedman equations and consider the wave equation on it. This decouples the geometry from the influence of the scalar field while one can still reasonably expect similar asymptotics to the coupled case since many perfect fluid solutions to the Einstein equations as discussed in Section 3.2 can arise from scalar field solutions, as discussed in [3, Ch. 1]

It should be checked whether we need to impose further conditions on the wave for our energy momentum tensor to be divergence free. However, this is not the case:

Lemma 3.3.1. *Let ψ be a smooth wave on a Lorentzian manifold (M, g) and let $T_{\mu\nu} \equiv T_{\mu\nu}[\psi]$ be the corresponding energy-momentum tensor given by (6). Then, the continuity equations $\nabla^\mu T_{\mu\nu} = 0$ are satisfied.*

Proof. Using that, for the Levi-Civita connection, $\nabla^\mu g_{\mu\nu} = 0$ and $\nabla_\nu\nabla^\mu\psi = \nabla^\mu\nabla_\nu\psi$ hold, one calculates:

$$\begin{aligned} \nabla^\mu T_{\mu\nu}[\psi] &= \nabla^\mu\nabla_\mu\psi \cdot \nabla_\nu\psi + \nabla^\mu\nabla_\nu\psi \cdot \nabla_\mu\psi - \frac{1}{2}\left(\nabla^\mu\left(\nabla^\alpha\psi \cdot \nabla_\alpha\psi\right)\right)g_{\mu\nu} \\ &= \Box_g\psi \cdot \partial_\nu\psi + \nabla_\nu\nabla^\mu\psi\nabla_\mu\psi - \frac{1}{2}\left(\nabla_\nu\left(\nabla^\alpha\psi \cdot \nabla_\alpha\psi\right)\right) \\ &= 0 + \nabla^\mu\nabla_\nu\psi \cdot \nabla_\mu\psi - \frac{1}{2}(\nabla_\nu\nabla^\alpha\psi \cdot \nabla_\alpha\psi + \nabla^\alpha\psi \cdot \nabla_\nu\nabla_\alpha\psi) \\ &= 0 \end{aligned}$$

□

As a final remark, it should be noted that the initial value problem

$$\Box_g \psi = 0,\ \ \psi(t_0, x) = u_0(x),\ \ \partial_t \psi(t_0, x) = u_1(x)$$

admits unique smooth solutions in a vast class of (maximally developed globally hyperbolic) spacetimes, including FLRW spacetimes and the slightly more general notion of warped product spacetimes which will be formally introduced shortly (see [13, p.144, Thm. 12.6]), so “only” considering smooth waves in this thesis isn’t a restriction in any meaningful way.

4. Mathematical preparation

4.1. Setting

To summarize the framework established in the previous chapter, the cosmological model at the basis of our analysis is an FLRW spacetime (Definition 3.1.2), in which the expansion rate is coupled with the energy density ρ and pressure p of the substratum via the Friedman equations (3),(4) as well as the continuity equation in Lemma 3.2.1. As a toy case for introducing scalar field matter, we consider a scalar field ψ that satisfies the wave equation $\square_g \psi = 0$. Within this thesis, we will further amend and simplify this framework as follows:

- It will turn out that *the precise type of the Riemannian spatial geometry is irrelevant beyond its influence on the scale factor via the Friedman equations.* In particular, we will call a spacetime (M, g) a **warped product spacetime** (or simply warped product) if it satisfies all conditions of Definition 3.1.2 except that $\left(\overline{M}, \bar{g}\right)$ **need not have constant sectional curvature**, as briefly introduced in Remark 3.2.3. In the following, (M, g) will always be a warped product unless stated otherwise.
Note again that, from a purely physical perspective, only FLRW spacetimes themselves are of direct interest due to the arguments in section 3.1, but allowing these toy considerations of spatial inhomogeneities is an interesting indicator for further nonlinear stability analysis of FLRW spacetimes and more that goes beyond sheer mathematical generality for its own sake. Furthermore, it is easily seen that all formulas in Remark 3.1.4 also apply to warped products.

- We will restrict ourselves to considering spatial geometries of zero and negative sectional curvature, i.e. $\kappa = 0$ and $\kappa = -1$ by Proposition 3.1.1. In particular, we call (M, g) of **type 0** (resp. **type −1**) if a satisfies the Friedman equations (3),(4) and the continuity equation (2) for $\kappa = 0$ (resp. $\kappa = -1$). The former case is considered mainly as a generalization of [1] and since it will provide slightly easier proofs that make the arguments associated with $\kappa = -1$ a little more accessible. This latter case is relevant since this is where the question of nonlinear stability of the FLRW Big Bang singularity in presence of scalar field matter is still open. (See Chapter 8 for more on how the results we will have achieved lead into that more general problem.)

- As will be shown in Section 4.2, we can (and thus will) consider $M = \mathbb{R}^+ \times \overline{M}$ for all warped product spacetimes in this thesis, setting the Big Bang singularity at $t = 0$, i.e. $a(0) = 0$ (see Remark 3.2.4).

- Again to simplify our calculations, we want to keep p and ρ as simple as possible while still having some physical relevance. However, simply setting $\rho = p = 0$ would amount to a being constant in type 0 and the Milne universe with $a(t) = t$ in type -1, neither of which would admit a Big Bang singularity even in their true FLRW analogues (see [12, p.360f.] for details on the latter, i.e. the Milne universe). Thus, the next best way to connect them is via linear dependency, i.e. $p = (\gamma - 1)\rho$, along with $\rho > 0$, which will also always be assumed for the scale factor of a warped product spacetime unless stated otherwise.

  We will mostly constrain ourselves to $\gamma \in (2/3, 2)$, additionally considering $\gamma = 2$ whenever possible. The latter is called the **stiff** case since $\gamma - 1$ corresponds to the square of the speed of sound c_s within the material (when the former is nonnegative) which can at most be equal to $c^2 = 1$. While this interpretation of $\gamma - 1$ is of course only reasonable when $\gamma \geq 1$ is satisfied, extending to $\gamma > 2/3$ (or equivalently to all $\gamma < 2$ with $\rho + 3p > 0$) allows considerations of all possible values of γ where a Big Bang as defined in Remark 3.2.4 actually forms, since precisely then, $\ddot{a}$ remains strictly negative by the second Friedman equation (4). For $\gamma \leq 2/3$, since ρ should be strictly positive from a physical perspective, $\ddot{a}$ would be constant or strictly increasing, which obviously prevents $\dot{a}(0) = \infty$ and thus a Big Bang singularity as defined in Remark 3.2.4. Finally, it should be noted that a dust filled universe is associated to $\gamma = 1$, while a radiation filled universe corresponds to $\gamma = 4/3$ (see [4, Chapters 6.4.5, 6.4.6]).

The goal will be to analyze how waves blow up on type 0 and type -1 warped products, with scale factors arising from ρ and p given by the Friedman equations and choice of γ, towards the Big Bang singularity. As will be mentioned in the following analysis, some of the core ideas (especially in type 0) are to an extent generalisations of the work done in [1]. However, these generalisations not only manage to extend some of the ideas to $\gamma = 2$, but also show that assumptions on the spatial geometry are mostly not necessary, while [1] heavily used flatness of the spatial geometry within their analysis by using the fact that coordinate derivatives commute. Furthermore, and a little more obviously, we additionally extend the analysis to a different class of scale factors that require significantly more care, as will be seen in the next section.

4.2. Analysis of the scale factor

We have to quickly collect how the linear relation between pressure and energy density influences the scale factor and verify that a Big Bang singularity then actually manifests:

The continuity equation from Lemma 3.2.1 now reads

$$\dot{\rho} = -3\frac{\dot{a}}{a}\gamma\rho \tag{7}$$

or after rearranging (since we assume ρ to be non-vanishing)

$$\frac{\dot{\rho}}{\rho} = -3\gamma\frac{\dot{a}}{a}\,. \tag{8}$$

Recall from Definition 3.1.2 that (without loss of generality), one considers a to be positive on $I = \mathbb{R}^+$. After integrating using $\rho > 0$ and $a > 0$, it follows there is some $B > 0$ such that

$$\log(\rho(t)) = -3\gamma\log(a(t)) + \log(B)$$

and hence

$$\rho(t) = B \cdot a(t)^{-3\gamma} > 0\,. \tag{9}$$

The first Friedman equation (3) now reduces to

$$\dot{a} = \sqrt{\frac{8\pi B}{3}a^{2-3\gamma} - \kappa}\;, \tag{10}$$

again choosing the positive sign since this is what is physically observed (see Remark 3.2.4). In particular, one immediately sees $\lim_{t\to 0}\dot{a}(t) = \infty$ when requiring $a(0) = 0$ since $2-3\gamma < 2$ holds for $\gamma > \frac{2}{3}$, so any solution to this equation exhibits a Big Bang singularity which is then also physical by (9). The reformulated first Friedman equation (10) now also immediately implies the second one, i.e. (4), by derivating and using $p = (\gamma - 1)\rho$ in $(*)$:

$$\begin{aligned}\ddot{a} &= \frac{1}{2\dot{a}}\frac{8\pi B}{3}(2-3\gamma)a^{1-3\gamma}\dot{a}\\ &\overset{(9)}{=} \frac{4\pi}{3}(2-3\gamma)\rho a\\ &= -\frac{4\pi}{3}(1+3(\gamma-1))\rho a\\ &\overset{(*)}{=} -\frac{4\pi}{3}(\rho+3p)a\end{aligned} \tag{11}$$

Hence, one only needs to analyze the solutions to (10). For $\kappa = 0$, this simplifies to

$$\dot{a} = \sqrt{\frac{8\pi B}{3}}\cdot a^{1-\frac{3\gamma}{2}},$$

or after rearranging (since $1 - \frac{3\gamma}{2} < 0$ already rules out that a vanishes in the interior of any solution interval)

$$\dot{a}a^{\frac{3\gamma}{2}-1} = \sqrt{\frac{8\pi B}{3}}\,.$$

After integrating on $[0, t]$ and substituting the integrand, it follows with $a(0) = 0$ that

$$\frac{2}{3\gamma}a(t)^{\frac{3\gamma}{2}} = \sqrt{\frac{8\pi B}{3}}t$$

or equivalently

$$a(t) = \left(\frac{3\gamma}{2} \sqrt{\frac{8\pi B}{3}} \right)^{\frac{2}{3\gamma}} t^{\frac{2}{3\gamma}} .$$

holds for all $t > 0$. Since we can absorb all time-independent constants into the metric without loss of generality (both strictly within FLRW spacetimes and in warped products), it is completely sufficient for the warped products of type 0 to analyse those endowed with the scale factor $a(t) = t^{\frac{2}{3\gamma}}$ for $2/3 < \gamma \leq 2$.

For $\kappa = -1$, the situation is roughly similar, but the analysis is significantly more involved:

Lemma 4.2.1. *Consider the initial value problem*

$$\dot{a} = f(a) := \sqrt{\frac{8\pi B}{3} a^{2-3\gamma} + 1}, \quad a(0) = 0 \tag{12}$$

for $a : \mathbb{R}_0^+ \longrightarrow \mathbb{R}, B > 0, \gamma \in (2/3, 2]$. *This problem has a unique solution* a *with the following properties:*

- *a is strictly increasing.*
- $a(t) \geq t$ *for all* $t \geq 0$, *with equality only at* $t = 0$.
- $a \in C\left([0,\infty), \mathbb{R}_0^+\right) \cap C^\omega\left((0,\infty), \mathbb{R}^+\right)$
- $a(t) \simeq t^{\frac{2}{3\gamma}}$ *as* $t \to 0$
- $\int_t^\infty a(s)^{-3} ds < \infty$ *for all* $t > 0$
- *For* $t_0 > 0$ *small enough,* $0 < t < t_0$,

$$\int_t^{t_0} a(s)^{-3} ds \simeq \begin{cases} t^{1-\frac{2}{\gamma}} - t_0^{1-\frac{2}{\gamma}} & \gamma < 2 \\ \log(t_0) - \log(t) & \gamma = 2 \end{cases}$$

Proof. To start off, the first point is immediate for any positive[5] solution of $\dot{a} = f(a)$ on its maximal open interval of existence since $\dot{a} > 1$, and so is the second when additionally considering the initial value.
Now, we move to the shifted initial value problem

$$\dot{a} = f(a), \quad a(t_0) = a_0 > 0 \tag{13}$$

[5] Any real solution of (12) must be nonnegative, even if we weren't requiring this w.l.og. : If it were to become negative, then the differential equation can only be satisfied as long as $a(t) < -\left(\frac{3}{8\pi B}\right)^{3\gamma-2} < 0$, so no continuous extension to $a(0) = 0$ is possible.

at $t_0 > 0$. Since $f : (0, \infty) \longrightarrow \mathbb{R}$ is smooth, a unique smooth real-valued solution exists on some maximal interval of existence $I = (t_{min}, t_{max})$. Since $2 - 3\gamma < 0$, f is monotonously decreasing and in particular one has

$$a(t) \leq \sqrt{1 + \frac{8\pi B}{3} a_0^{2-3\gamma}} \cdot (t - t_0) + a_0 \,.$$

In particular, assuming one were to have $t_{max} \in \mathbb{R}$, $\lim_{t \uparrow t_{max}} a(t)$ would exist in $\mathbb{R}^+$ since a is increasing and would be bounded, and then the Picard-Lindelöf Theorem would admit a local solution of $\dot{a} = f(a)$ at $a(t_{max})$ that, by uniqueness, would be an extension of a beyond I, contradicting maximality. Thus, $I = (t_{min}, \infty)$.

Furthermore, because a is strictly increasing and positive on I, a converges approaching t_{min}. It follows that $a(t) \to 0$ as $t \downarrow t_{min}$ – else, one could again find a local continuation at $a(t_{min})$ by the Picard-Lindelöf Theorem, contradicting maximality. Finally, $t \in (0, \infty) \mapsto a(t + t_{min})$ now solves (12) – or equivalently, we can assume $t_{min} = 0$ without loss of generality for a solution of (13) since no solution can be extended past 0.

Additionally, f extends to a holomorphic function on the simply connected set $V := \mathbb{C} \backslash \{z \in \mathbb{C} | \mathrm{Im}(z) \geq 0\}$ by appropriate choice of logarithm, thus (13) has a unique holomorphic local solution around any $t_0 \in I$ with initial condition $a(t_0) \in \mathbb{R}$ by the Cauchy-Kovalevskaya Theorem [6, p.46f.]. Hence, this local uniqueness yields a real analytic solution on $I = (0, \infty)$ to the real-valued differential equation that must agree with any real solution on an open subinterval of I, and in particular any real solution on I must be analytic.

On the other hand, assume there were two different (maximally extended) solutions a_1, a_2 to (12), then some $\tilde{a} > 0$ has to exist such that $a_1(t_1) = \tilde{a} = a_2(t_2)$ for some $0 < t_1 < t_2$. However, both a_1 and $t \mapsto a_2(t + t_2 - t_1)$ locally solve the initial value problem

$$\dot{\varphi}(t) = f(a), \;\; \varphi(t_1) = \tilde{a} \,.$$

Its solutions are locally unique and (as argued before) analytic on their open existence intervals, hence any two local solutions are extendible to a common maximal solution since analytic functions are uniquely determined by their values on any open set. In particular, it follows that $a_2(t_2 - t_1) = a_1(0) = 0$. Since a_2 is strictly increasing, $t_2 - t_1 = 0$ would have to hold, which is a contradiction. Hence, (12) has a unique continuous solution on $[0, \infty)$ which must then also be analytic on $(0, \infty)$.

To prove the asymptotic behaviour of a, consider $b(t) := a(t)^{\frac{3\gamma}{2}}$ which satisfies

$$\dot{b} = \frac{3\gamma}{2} a^{\frac{3\gamma}{2} - 1} \dot{a} = \frac{3\gamma}{2} \sqrt{a^{3\gamma - 2} + \frac{8\pi B}{3}}.$$

Hence, using $a(0) = 0$ and $3\gamma - 2 > 0$, we obtain $\lim_{t\to 0} \dot{b}(t) = \frac{3\gamma}{2}\sqrt{\frac{8\pi B}{3}} > 0$. By the l'Hospital rule, it now follows that

$$\lim_{t\to 0} \frac{a(t)}{t^{\frac{2}{3\gamma}}} = \left(\lim_{t\to 0} \frac{b(t)}{t}\right)^{\frac{2}{3\gamma}} = \lim_{t\to 0} \left(\dot{b}(t)\right)^{\frac{2}{3\gamma}} > 0\,,$$

which shows the fourth bullet point.
Finally, with all the previous results, one calculates for any $t > 0$ that

$$0 < \int_t^\infty a(s)^{-3} ds < \int_t^\infty s^{-3} ds = \frac{1}{2t^2} < \infty,$$

and respectively for t_0 small enough and $0 < t < t_0$ that

$$\begin{aligned}\int_t^{t_0} a(s)^{-3} ds &\simeq \int_t^{t_0} s^{-\frac{2}{\gamma}} ds \\ &= \begin{cases} \frac{1}{\frac{2}{\gamma}-1}\left(t^{1-\frac{2}{\gamma}} - {t_0}^{1-\frac{2}{\gamma}}\right) & \gamma < 2 \\ \log(t_0) - \log(t) & \gamma = 2 \end{cases}.\end{aligned}$$

□

4.3. Some central formulas

Next, we need to relate the Laplace-Beltrami-operators of (M, g) and $\left(\overline{M}, \bar{g}\right)$ in warped product spacetimes:

Lemma 4.3.1. *For any $\varphi \in C^\infty(M)$,*

$$(\Box_g \varphi)(t,\cdot) = -\left(\partial_t^2 \varphi\right)(t,\cdot) + a(t)^{-2}\Delta\varphi(t,\cdot) - 3\frac{\dot{a}(t)}{a(t)}(\partial_t\varphi)(t,\cdot).$$

Proof.

$$\begin{aligned}\Box_g \varphi =& g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi \\ \overset{(*)}{=}& -\partial_t(\partial_t\varphi) + g^{0l}\left(\partial_t(\nabla_l\varphi) - \frac{\dot{a}(t)}{a(t)}\nabla_l\varphi\right) \\ &+ a(t)^{-2}\bar{g}^{kl}\left(\overline{\nabla}_k\overline{\nabla}_l\varphi(t,\cdot) - \frac{\dot{a}(t)}{a(t)}a(t)^2\bar{g}_{kl}\partial_t\varphi\right) \\ =& -\partial_t^2\varphi + a(t)^{-2}\Delta\varphi(t,\cdot) - 3\frac{\dot{a}(t)}{a(t)}\partial_t\varphi,\end{aligned}$$

with $(*)$ due to Remark 3.1.4 and the last step with $g^{0l} = 0$, $\bar{g}^{kl}\bar{g}_{kl} = \bar{g}^{lk}\bar{g}_{kl} = \delta^l{}_l = 3$. □

Remark 4.3.2. In particular, in warped products of type 0, one has $a(t) = t^{\frac{2}{3\gamma}}$ w.l.o.g. (see the discussion preceding Lemma 4.2.1) and thus obtains

$$(\square_g \varphi)(t, \cdot) = -\left(\partial_t^2 \varphi\right)(t, \cdot) + t^{-\frac{4}{3\gamma}} \Delta\varphi(t, \cdot) - \frac{2}{\gamma t}(\partial_t \varphi)(t, \cdot).$$

Corollary 4.3.3. *For any smooth wave ψ and any $t > 0$, it holds that*

$$\left(\partial_t^2 \psi\right)(t, \cdot) = a(t)^{-2} \Delta\psi(t, \cdot) - 3\frac{\dot{a}(t)}{a(t)}(\partial_t \psi)(t, \cdot).$$

Furthermore, for any $N \in \mathbb{N}_0$, $\Delta^N \psi : (t, x) \mapsto \left(\Delta^N \psi(t, \cdot)\right)(x)$ is also a smooth wave.

Proof. The former is immediate from Lemma 4.3.1, which we also use to prove the latter, along with the fact that Δ and ∂_t commute when acting on smooth functions:

$$\begin{aligned}
\left(\square_g \Delta^N \psi\right)(t, \cdot) &= -\partial_t^2 \Delta^N \psi(t, \cdot) + a(t)^{-2} \Delta^{N+1} \psi(t, \cdot) - 3\frac{\dot{a}(t)}{a(t)} \partial_t \Delta^N \psi(t, \cdot) \\
&= \left[\Delta^N \left(-\partial_t^2 \psi + a^{-2} \Delta\psi - 3\frac{\dot{a}}{a}\partial_t \psi\right)(t, \cdot)\right] \\
&= \Delta^N \left[\square_g \psi(t, \cdot)\right] = 0.
\end{aligned}$$

□

It should be noted that the fairly unassuming fact that Δ and $\square_g$ commute irrespective of spatial geometry, as essentially just shown, will become central in extending many of the ideas of [1] to our analysis – there, energy estimates could simply be extended to spatial coordinate derivatives of arbitrary high order since they all commute with the Laplacian in flat spatial geometry, which makes it comparatively easy to extract regularity statements on waves towards the Big Bang. Along with the ellipticity properties that will be collected in Section 5.2, $[\Delta, \square_g] = 0$ will bridge the gap left by moving from this to different and more general spatial geometries, in particular in the proof of Theorem 7.1.1.

Remark 4.3.4. The basic structure of the wave operator just established allows to compute the spatially homogeneous waves that will serve as the main points of comparison for the general blow-up behaviour: A smooth homogeneous wave $\psi(t, x) \equiv \psi(t)$ satisfies the second order differential equation

$$\begin{aligned}
-\partial_t^2 \psi - 3\frac{\dot{a}}{a}\partial_t \psi = 0 \;&\Leftrightarrow\; a^3 \partial_t \dot{\psi} + 3a^2 \dot{a}\dot{\psi} = 0 \\
&\Leftrightarrow\; \partial_t \left(a^3 \dot{\psi}\right) = 0 \\
&\Leftrightarrow\; \dot{\psi} = C_1 a^{-3} \text{ for } C_1 \in \mathbb{R}
\end{aligned}$$

In type 0, homogeneous waves thus take the explicit form

$$\dot{\psi}(t) = \begin{cases} C_1 t^{1-\frac{2}{\gamma}} + C_2 & \gamma \in (2/3, 2) \\ C_1 \log(t) + C_2 & \gamma = 2 \end{cases} \tag{14}$$

while one can use the second-to-last point in Lemma 4.2.1 for type -1 to write the homogeneous waves as

$$\psi(t) = C_1 \int_t^\infty a(s)^{-3}\, ds + C_2 \text{ for } C_1, C_2 \in \mathbb{R}\,. \tag{15}$$

In particular, in either setting, **we thus expect waves to behave like $t^{1-\frac{2}{\gamma}}$ towards the Big Bang singularity in warped product spacetimes that don't arise from stiff fluids, and like $\log(t)$ in the stiff case**. In the following, when referring to homogeneous waves, it will always be assumed that they vanish in the far field where this is possible ($C_2 = 0$) and are not constant ($C_1 \neq 0$).

Finally, before moving on, we quickly collect what will in the following always be referred to as "integration by parts" and will be used ad nauseum:

Lemma 4.3.5. *For any $\varphi, \tilde{\varphi} \in C^\infty\left(\overline{M}\right)$, the following holds:*

$$\int_{\overline{M}} \bar{g}\left(\overline{\nabla}\varphi, \overline{\nabla}\tilde{\varphi}\right)\, \mathrm{vol}_{\overline{M}} = -\int_{\overline{M}} \varphi \Delta \tilde{\varphi}\, \mathrm{vol}_{\overline{M}}$$

Proof. By Stokes' Theorem on closed manifolds, one has

$$\int_{\overline{M}} \mathrm{div}\left(\varphi \overline{\nabla}\tilde{\varphi}\right)\, \mathrm{vol}_{\overline{M}} = 0\,.$$

The statement now follows with $\mathrm{div}\left(\varphi\overline{\nabla}\tilde{\varphi}\right) = \bar{g}\left(\overline{\nabla}\varphi, \overline{\nabla}\tilde{\varphi}\right) + \varphi\Delta\tilde{\varphi}$. □

5. Mathematical background

A key ingredient in analyzing and controlling the blow-up behaviour of waves will be energy estimates that are closely related to integrals over derivatives that structurally resemble what, in Euclidean space, one would call Sobolev norms. Thus, before moving on, it needs to be (re-)established how (and to which extent) the theory of Sobolev spaces can be extended to the Riemannian setting, including the theory of elliptic operators and their relation to these new Sobolev spaces:

5.1. Sobolev Spaces and Sobolev Estimates

Let V be a bounded open subset of $\mathbb{R}^n$, and $k \in \mathbb{N}$. We define

$$\|f\|_{H^k_{eucl}(V)} := \sqrt{\sum_{|\alpha|\leq k} \int_V |D^\alpha f|^2},$$

as the standard L^2-Sobolev norm of order k on V, where one sums over all multiindices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = \sum_{i=1}^n \alpha_i \leq k$. $H^k_{eucl}(V)$ then denotes the completion of $C^\infty\left(\overline{V}\right)$ with regards to this norm. In the following, we will assume that V is also bounded and has a smooth boundary (for example a ball), since in this case, one can find bounded extension operators $E_k : H^k_{eucl}(V) \longrightarrow H^k_{eucl}(\mathbb{R}^n)$ for any $k \in \mathbb{N}$ such that, for any $u \in H^k(V)$, $E_k u|_V = u$ holds almost everywhere and the support of $E_k u$ is contained in some bounded open set W with $V \subset\subset W$ (see [5, p. 254, Thm. 5.4.1]).
This allows us to deduce the following Sobolev inequality for any $k > \frac{n}{2}$ and any $u \in C^\infty(V)$:

$$\|u\|_{L^\infty(V)} = \|E_k u\|_{L^\infty(V)} \leq \|E_k u\|_{L^\infty(\mathbb{R}^n)} \lesssim \|E_k u\|_{H^k_{eucl}(\mathbb{R}^n)} \lesssim \|u\|_{H^k_{eucl}(V)} \tag{16}$$

Here, we used Morrey's inequality on $\mathbb{R}^n$ for $p = 2$ for the penultimate step, i.e.

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{H^k(\mathbb{R}^n)} \quad \text{for all } f \in C^\infty\left(\mathbb{R}^n\right)$$

for $k > \frac{n}{2}$ (see [5, p. 266, Thm 5.6.4]), and the boundedness of E_k in the final step.
Next, we define (L^2–)Sobolev norms on closed Riemannian manifolds as follows for any open subset $U \subseteq \overline{M}$ and $k \in \mathbb{N}$ (see [2, p. 457, (3)]):

$$\|f\|_{H^k(U)} = \sqrt{\int_U f^2 \mathrm{vol}_{\overline{M}} + \sum_{1\leq m\leq k} \int_U \left|\overline{\nabla}^m f\right|^2_{\bar{g}} \mathrm{vol}_{\overline{M}}}$$

$$\left|\overline{\nabla}^m f\right|_{\bar{g}}^2 = \bar{g}^{i_1 j_1}\bar{g}^{i_2 j_2}\dots\bar{g}^{i_m j_m}\left(\overline{\nabla}_{i_1}\overline{\nabla}_{i_2}\dots\overline{\nabla}_{i_m} f\right)\left(\overline{\nabla}_{j_1}\overline{\nabla}_{j_2}\dots\overline{\nabla}_{j_m} f\right)$$

Again, $H^k(U)$ then denotes the completion of $C^\infty\left(\overline{U}\right)$ with regards to the respective Sobolev norms. In particular, for $k = 1$, the norm takes the form

$$\|f\|_{H^1(U)} = \sqrt{\int_U f^2 \mathrm{vol}_{\overline{M}} + \int_U \left|\overline{\nabla} f\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}}}$$

which then also induces a Hilbert space structure on $H^1(U)$ with regards to the scalar product

$$\left\langle f, \tilde{f} \right\rangle_{H^1(U)} := \int_U f\tilde{f}\, \mathrm{vol}_{\overline{M}} + \int_U \bar{g}\left(\overline{\nabla} f, \overline{\nabla}\tilde{f}\right) \mathrm{vol}_{\overline{M}} .$$

One can show the following lemma, using normal neighbourhoods and that any open neighbourhood in $\overline{M}$ is relatively compact:

Lemma 5.1.1. *[7, Lemma 2.2.1] For any chart (U, x) on a closed connected Riemannian manifold $\left(\overline{M}, \bar{g}\right)$, $\|\cdot\|_{H^1(U)} \simeq_{\bar{g}} \|(\cdot)\circ x^{-1}\|_{H^1_{eucl}(x(U))}$.*

More central to our analysis will be an inequality proved along very similar lines:

Lemma 5.1.2. *For any small enough normal chart (U, x) centered around $p \in U$ on a closed connected three-dimensional Riemannian manifold $\left(\overline{M}, \bar{g}\right)$, one has*

$$\|u\|_{L^\infty(U)} \lesssim_{\bar{g}} \|u\|_{H^2(U)} .$$

Covering $\overline{M}$ with finitely many of these neighbourhoods, it follows by a standard approximation argument that, for all $u \in H^2\left(\overline{M}\right)$,

$$\|u\|_{C(\overline{M})} \lesssim_{\bar{g}} \|u\|_{H^2(\overline{M})}. \tag{17}$$

Proof. Note that for any $\varepsilon > 0$ (w.l.o.g. $\varepsilon < 1$), there exists a small enough normal neighbourhood U such that, in the respective normal coordinates, one has

$$\left|\bar{g}^{ij} - \delta^{ij}\right| < \varepsilon,\ \left|\overline{\Gamma}^k_{ij}\right| < \varepsilon$$

(see Corollary A.1.5) and hence

$$\begin{aligned}\left|\bar{g}^{i_1 j_1}\bar{g}^{i_2 j_2} - \delta^{i_1 j_1}\delta^{i_2 j_2}\right| &\le \left|\bar{g}^{i_1 j_1}\right|\left|\bar{g}^{i_2 j_2} - \delta^{i_2 j_2}\right| + \delta^{i_2 j_2}\left|\bar{g}^{i_1 j_1} - \delta^{i_1 j_1}\right| \\ &< (1+\varepsilon)\varepsilon + \varepsilon = (2+\varepsilon)\varepsilon .\end{aligned}$$

Setting $C > 0$ such that

$$\left(\sum_{k=1}^9 a_k\right)^2 \le C \sum_{k=1}^9 a_k^2$$

is satisfied for any $(a_k)_{k=1,\dots,9} \subseteq \left(\mathbb{R}_0^+\right)^9$, the following thus holds for all $\varphi \in C^\infty\left(\overline{M}\right)$:

$$\begin{aligned}
&\left|\left|\overline{\nabla}^2\varphi\right|_{\bar{g}}^2 - \delta^{i_1j_1}\delta^{i_2j_2}\partial_{i_1}\partial_{i_2}\varphi\partial_{j_1}\partial_{j_2}\varphi\right| \\
&\quad\le \left|\bar{g}^{i_1j_1}\bar{g}^{i_2j_2} - \delta^{i_1j_1}\delta^{i_2j_2}\right|\left|\overline{\nabla}_{i_1}\overline{\nabla}_{j_1}\varphi\overline{\nabla}_{i_2}\overline{\nabla}_{j_2}\varphi\right| + \\
&\quad\quad + \delta^{i_1j_1}\delta^{i_2j_2}\left|\overline{\nabla}_{i_1}\overline{\nabla}_{i_2}\varphi\overline{\nabla}_{j_1}\overline{\nabla}_{j_2}\varphi - \partial_{i_1}\partial_{j_1}\varphi\partial_{i_2}\partial_{j_2}\varphi\right| \\
&\quad\le (2+\varepsilon)\varepsilon\left(\sum_{i,j=1}^{3}\left|\overline{\nabla}_i\overline{\nabla}_j\varphi\right|\right)^2 + \sum_{i,j=1}^{3}\left|\left(\overline{\nabla}_i\overline{\nabla}_j\varphi\right)^2 - (\partial_i\partial_j\varphi)^2\right| \\
&\quad\le [C(2+\varepsilon)+1]\varepsilon\left(\sum_{i,j=1}^{3}\left|\overline{\nabla}_i\overline{\nabla}_j\varphi\right|^2\right) + \sum_{i,j=1}^{3}|\partial_i\partial_j\varphi|^2 \\
&\quad\le [2C(2+\varepsilon)+2+1]\varepsilon\left(\sum_{i,j=1}^{3}|\partial_i\partial_j\varphi|^2\right) + [2C(2+\varepsilon)+2]\varepsilon\left(\sum_{k=1}^{3}\left|\bar{\Gamma}_{ij}^k\right||\partial_k\varphi|\right)^2 \\
&\quad\le [2C(2+\varepsilon)+2+1]\varepsilon\left(\sum_{i,j=1}^{3}|\partial_i\partial_j\varphi|^2\right) + 4[2C(2+\varepsilon)+2]\varepsilon\left(\sum_{k=1}^{3}\varepsilon^2|\partial_k\varphi|^2\right)
\end{aligned}$$

After integration, using $\varepsilon < 1$ and updating C, one obtains

$$\begin{aligned}
&\left|\int_U\left|\overline{\nabla}^2\varphi\right|_{\bar{g}}^2 \mathrm{vol}_U - \int_U \delta^{i_1j_1}\delta^{i_2j_2}\partial_{i_1}\partial_{i_2}\varphi\partial_{j_1}\partial_{j_2}\varphi\,\mathrm{vol}_U\right| \\
\le& C\varepsilon\left[\sum_{i,j=1}^{3}\int_U|\partial_i\partial_j\varphi|^2\,\mathrm{vol}_U + \sum_{k=1}^{3}\int_U|\partial_k\varphi|^2\,\mathrm{vol}_U\right]
\end{aligned}$$

and after re-arranging that

$$(1-C\varepsilon)\sum_{i,j=1}^{3}\int_U|\partial_i\partial_j\varphi|^2\,\mathrm{vol}_U - C\varepsilon\sum_{k=1}^{3}\int_U|\partial_k\varphi|^2\mathrm{vol}_U \le \int_U\left|\overline{\nabla}^2\varphi\right|_{\bar{g}}^2\,\mathrm{vol}_U\,.$$

The terms on the left hand side correspond to the terms in the Euclidean Sobolev norms after coordinate transformation to $x(U)$. Putting this together with the other contributions to the Sobolev norm over U and using Lemma 5.1.1, we deduce

$$\begin{aligned}
&(1-C\varepsilon)\left\|\varphi\circ x^{-1}\right\|_{H^2_{eucl}(x(U))}^2 \\
\le& \left\|\varphi\circ x^{-1}\right\|_{L^2(x(U))}^2 + (1-C\varepsilon)\sum_{k=1}^{3}\int_U|\partial_k\varphi|^2\,\mathrm{vol}_U + (1-C\varepsilon)\sum_{i,j=1}^{3}\int_U|\partial_i\partial_j\varphi|^2\,\mathrm{vol}_U
\end{aligned}$$

$$= \left\| \varphi \circ x^{-1} \right\|^2_{H^1_{eucl}(x(U))} + (1 - C\varepsilon) \sum_{i,j=1}^{3} \int_U |\partial_i \partial_j \varphi|^2 \, \mathrm{vol}_U - C\varepsilon \sum_{k=1}^{3} \int_U |\partial_k \varphi|^2 \, \mathrm{vol}_U$$

$$\lesssim \|\varphi\|^2_{H^1(U)} + \int_U \left| \overline{\nabla}^2 \varphi \right|^2_{\bar{g}} \mathrm{vol}_U = \|\varphi\|^2_{H^2(U)} \,,$$

In particular, the statement now follows with (16) when choosing $\varepsilon > 0$ small enough such that $1 - C\varepsilon > 0$ holds, since one then has

$$\|\varphi\|_{L^\infty(U)} = \left\| \varphi \circ x^{-1} \right\|_{L^\infty(x(U))} \lesssim \left\| \varphi \circ x^{-1} \right\|_{H^2_{eucl}(x(U))} \lesssim \|\varphi\|_{H^2(U)} \,.$$

□

Remark 5.1.3. The proof of Lemma 5.1.2 also shows one half of the norm equivalence between $H^2(U)$ and $H^2_{eucl}(x(U))$ on small enough normal charts (even in arbitrary dimensions, since $\dim\left(\overline{M}\right) = 3$ was only used in the final set of inequalities), where the other half can be argued completely analogously and is thus omitted here, but then of course also extends to $\overline{M}$. For higher order Sobolev spaces, the situation is more intricate a priori since curvature terms enter into the analysis.

Further we note that, as in the Euclidean case, notions of weak derivative and integration by parts naturally extend from smooth functions to Sobolev spaces by the same density arguments.

5.2. Elliptic Differential Operators and Elliptic Regularity

In the light of Corollary 4.3.3, we need to better understand Laplace-Beltrami operator Δ on $\left(\overline{M}, \bar{g}\right)$ for our analysis, in particular due to the ellipticity properties of the standard Laplacian being useful tools in Euclidean space that we would like to have at our disposal. However, the metric dependencies again necessitate some caution in trying to extend even the notion of elliptic operators to manifolds. In this short section, we will quickly collect some results in this vein from [2, p. 459-467].

In the following, $P : C^\infty\left(\overline{M}\right) \to C^\infty\left(\overline{M}\right)$ is a linear differential operator of order $k \in \mathbb{N}\backslash\{0\}$ and

$$Pu = \sum_{|\alpha| \leq k} a_\alpha \overline{\partial}^\alpha u$$

holds in local coordinates for any $u \in C^\infty\left(\overline{M}\right)$ and with summing over multi-indices α, where a_α are smooth.

Definition 5.2.1 (see [2, p. 460f.])**.** For $\xi \in T^*_x\overline{M}$, the **principal symbol** $\sigma_\xi(P, \cdot) : \overline{M} \to \mathbb{R}$ is defined in local coordinates around $x \in \overline{M}$ as

$$\sigma_\xi(P, x) = \frac{i^k}{k!} \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha$$

$\left(\text{with } \xi^\alpha = \prod_{i=1}^{\dim(\overline{M})} \xi^{\alpha_i}\right)$. For an equivalent coordinate-invariant definition, let $x \in \overline{M}$ and $\varphi \in C^\infty\left(\overline{M}\right)$ with $\varphi(x) = 0$, $d\varphi(x) = \xi$. Then,

$$\sigma_\xi(P; x) = \frac{i^k}{k!} P\left(\varphi^k\right)\Big|_x .$$

Morally, the principal symbol locally isolates the highest order derivatives and combines this information into a global object. The appearance of i is an admittedly slightly inconvenient remnant of trying to preserve consistency with taking adjoints – however, we are only going to apply this to second order differential operators (i.e. $k = 2$) where everything remains real-valued.

Definition 5.2.2 ([2, p. 461f.])**.** A differential operator $P : C^\infty\left(\overline{M}\right) \to C^\infty\left(\overline{M}\right)$ is called **elliptic at** $x \in \overline{M}$ if $\sigma_\xi(P; x) \neq 0$ holds for all (real) $\xi \in T_x^*\overline{M}\backslash\{0\}$. An **elliptic operator** is elliptic at every $x \in \overline{M}$.

Example 5.2.3. Note that this is consistent with ellipticity of a differential operator P of order 2 in an open subset of $\mathbb{R}^n$, i.e.

$$P = \sum_{i,j=1}^n a_{ij}\partial_i\partial_j + \sum_{k=1}^n b_k\partial_k + c, \quad \sum_{i,j=1}^n a_{ij}\xi^i\xi^j \neq 0 \quad \left(\text{resp. } \sum_{i,j=1}^n a_{ij}\xi^i\xi^j \geq \theta > 0\right) \quad \forall \xi \in \mathbb{R}^n .$$

Example 5.2.4. In local coordinates, we can write the Laplace-Beltrami operator $\Delta : C^\infty\left(\overline{M}\right) \to C^\infty\left(\overline{M}\right)$ as

$$\Delta = \bar{g}^{ij}\overline{\nabla}_i\overline{\nabla}_j = \bar{g}^{ij}\left(\overline{\partial}_i\overline{\partial}_j + \overline{\Gamma}_{ij}^k\overline{\partial}_k\right).$$

Thus, the principal symbol reads

$$\sigma_\xi(\Delta; p) = -\frac{1}{2}\bar{g}^{ij}(p)\xi_i\xi_j .$$

Since $\bar{g}$ is a Riemannian metric, $\bar{g}^{ij}(p)$ is negative definite, so the princial symbol is strictly negative for $\xi \neq 0$. Thus, Δ is an elliptic operator.

With this necessary nomenclature established, one can now also formulate the following estimates that we will apply to the Laplace Beltrami operator Δ throughout the rest of the thesis:

Proposition 5.2.5 ([2, p.463, Thm. 27])**.** *For any smooth function $u \in C^\infty\left(\overline{M}\right)$, where $\overline{M}$ is a closed Riemannian manifold, $k, l, m \in \mathbb{N}$, $k \neq 0$ and an elliptic differential operator P of order k, one has that*

$$\|u\|_{H^{k+l}(\overline{M})} \lesssim \|Pu\|_{H^l(\overline{M})} + \|u\|_{L^1(\overline{M})} \lesssim \|u\|_{H^{k+l}(\overline{M})} \text{ and}$$

$$\|u\|_{C^{k+m}(\overline{M})} \lesssim \|Pu\|_{C^m(\overline{M})} + \|u\|_{C(\overline{M})} \lesssim \|u\|_{C^{k+m}(\overline{M})}$$

These estimates extend to corresponding Sobolev spaces and weak derivatives by the standard density argument.

Note that closedness is essential for this to hold, since the lack of boundary allows to translate statements on *interior* regularity in Euclidean space to the entire manifold by covering the compact manifold $\overline{M}$ with finitely many suitable coordinate neighbourhoods. Additionally, this statement on elliptic regularity yields the following crucial estimate:

Corollary 5.2.6. *For any smooth function u on a closed Riemannian manifold $\left(\overline{M}, \bar{g}\right)$, one has*

$$\|u\|_{C(\overline{M})} \leq C \left(\|u\|_{L^2(\overline{M})} + \|\Delta u\|_{L^2(\overline{M})} \right) . \tag{18}$$

for a $\bar{g}$-dependent constant $C > 0$

Proof. With (17) and then applying the first inequality of Proposition 5.2.5 with $k = 2, l = 0$ for the elliptic operator Δ of order 2 (see Example 5.2.4), we collect the following estimates for any $u \in C^\infty\left(\overline{M}\right)$:

$$\|u\|_{C(\overline{M})} \lesssim_{\bar{g}} \|u\|_{H^2(\overline{M})} \lesssim \|u\|_{L^1(\overline{M})} + \|\Delta u\|_{L^2(\overline{M})}.$$

In particular, we can estimate first summand on the right hand side with the Hölder inequality applied to $1 \cdot u$, since $\overline{M}$ is of finite volume, and the statement now follows. ☐

6. Energy estimates

With all of the necessary tools now introduced, this chapter will establish the necessary energy estimates that allow for some a priori control of (rescaled) waves toward the Big Bang singularity, which will be essential in proving the asymptotic results.

For a smooth function $\varphi : M \to \mathbb{R}$, consider the following energies:

$$E(t,\varphi) = E(\varphi(t,\cdot)) = \int_{\overline{M}} |\partial_t\varphi(t,\cdot)|^2 + a(t)^{-2}\left|\overline{\nabla}\varphi(t,\cdot)\right|^2_{\bar{g}} \operatorname{vol}_{\overline{M}} \tag{19}$$

$$E_N(t,\varphi) = E\left(\Delta^N\varphi(t,\cdot)\right) \tag{20}$$

Since φ and a are smooth and $\overline{M}$ is closed, the derivative of the integrand of $E_N(t,\varphi)$ is uniformly bounded on any compact subset of $\mathbb{R}^+$, so the Dominated Convergence Theorem allows to pull the differentiation in time past the integral, which we will use without additional remarks from here on out. It follows analogously that $t \mapsto E(t,\varphi)$ is smooth for $\varphi \in C^\infty(M)$. Similarly, we will use without further notice that ∂_t commutes with any differential operator on $\overline{M}$, in particular $\partial_t\overline{\nabla}\varphi = \overline{\nabla}\partial_t\varphi$ holds for any $\varphi \in C^\infty(M)$.

6.1. Wave energy estimates

The behaviour of homogeneous waves established in Remark 4.3.4 already indicates that general waves should roughly behave like $t^{1-\frac{2}{\gamma}}$ (resp. $\int_t^\infty a(s)^{-3}\,ds$) towards the Big Bang singularity at $t = 0$ in type 0 (resp. -1) warped product spacetimes, at the very least when the spatial inhomogeneities entering via $\Delta\psi$ are comparatively small. For homogeneous waves, the energy of order $N = 0$ is easily seen to take form

$$E(t,\psi_{hom}) \simeq \left|t^{-\frac{2}{\gamma}}\right|^2 = t^{-\frac{4}{\gamma}} = a(t)^{-6}$$

for type 0 by the calculation preceding Lemma 4.2.1, and

$$E(t,\psi_{hom}) \simeq \left|-a(t)^{-3}\right|^2 = a(t)^{-6}$$

for type -1. The next proposition thus formalizes this intuition to general waves:

Proposition 6.1.1. *For any $N \in \mathbb{N}$ and $0 < t < t_0$, the following estimate holds on any warped product spacetime (M,g) with scale factor a such that $\dot{a} > 0$ holds and for arbitrary p,ρ such that the Friedman and continuity equations are satisfied:*

$$a(t)^6 E_N(t,\psi) \le a(t_0)^6 E_N(t_0,\psi)\,.$$

In warped product spacetimes of type 0 *with* $p = (\gamma - 1)\rho$, $\gamma \in (2/3, 2]$, *i.e.* $a(t) = t^{\frac{2}{3\gamma}}$, *this reads*

$$t^{\frac{4}{\gamma}} E_N(t, \psi) \le t_0{}^{\frac{4}{\gamma}} E_N(t_0, \psi)\,.$$

Note that by the rephrased Friedman equation (10) for $\kappa = 0$ and $\kappa = -1$, our "standard" warped product spacetimes of type 0 and -1 with linear equation of phase $p = (\gamma - 1)\rho$ are actually included in this proposition.
Two different proofs of this statement will be provided: The former is a little shorter and relies solely on the form of the wave operator via Corollary 4.3.3. The latter arises more or less directly from the relativistic framework established in Section 3.3 that was also utilized in [1] (which we will follow closely in that proof) as well as for nonlinear stability analysis in [15, 16]. It will also become important in Section 7.2 because makes this inequality a little more precise. For both proofs, we quickly note that it suffices to prove the estimate for $N = 0$ as it then immediately extends to the waves $\Delta^N \psi$ for any $N \in \mathbb{N}$ by Corollary 4.3.3.

Proof 1 of Proposition 6.1.1. Since $t \mapsto E(t, \psi)$ is smooth, one calculates using Corollary 4.3.3 to replace $\partial_t^2 \psi$ as well as integration by parts (see Lemma 4.3.5):

$$\begin{aligned}
\partial_t E(t, \psi) = &\int_{\overline{M}} \Big[2\partial_t^2 \psi(t, \cdot)\partial_t \psi(t, \cdot) + 2a(t)^{-2} \cdot \bar{g}\left(\partial_t \overline{\nabla}\psi(t, \cdot), \overline{\nabla}\psi(t, \cdot)\right) \\
&-2\frac{\dot{a}(t)}{a(t)^3}\left|\overline{\nabla}\psi(t, \cdot)\right|_{\bar{g}}^2 \Big] \operatorname{vol}_{\overline{M}} \\
= &\int_{\overline{M}} \Big[2\left(a(t)^{-2}\Delta\psi(t, \cdot) - 3\frac{\dot{a}(t)}{a(t)}\partial_t\psi(t, \cdot)\right)\partial_t\psi(t, \cdot) \\
&-2a(t)^{-2}\partial_t\psi(t, \cdot)\Delta\psi(t, \cdot) - 2\frac{\dot{a}(t)}{a(t)}a(t)^{-2}\left|\overline{\nabla}\psi(t, \cdot)\right|_{\bar{g}}^2 \Big] \operatorname{vol}_{\overline{M}} \\
= &\int_{\overline{M}} - \left[6\frac{\dot{a}(t)}{a(t)}\left|\partial_t\psi(t, \cdot)\right|^2 + 2\frac{\dot{a}(t)}{a(t)}a(t)^{-2}\left|\overline{\nabla}\psi(t, \cdot)\right|_{\bar{g}}^2 \right] \operatorname{vol}_{\overline{M}} \\
\ge & -6\frac{\dot{a}(t)}{a(t)}E(t, \psi)
\end{aligned}$$

By integration on $[t, t_0]$, we obtain

$$E(t, \psi) \le E(t_0, \psi) + \int_t^{t_0} 6\frac{\dot{a}(s)}{a(s)}E(s, \psi)\, ds\,.$$

From the Gronwall lemma, it now follows that

$$\begin{aligned}
E(t, \psi) &\le E(t_0, \psi)\exp\left(\int_t^{t_0} 6\frac{\dot{a}(s)}{a(s)}\, ds\right) \\
&= E(t_0, \psi)\exp\left(\int_{a(t)}^{a(t_0)} \frac{6}{x}\, dx\right) \\
&= E(t_0, \psi)\left(\frac{a(t_0)}{a(t)}\right)^6
\end{aligned}$$

and thus the statement holds. $\square$

Before moving on to the second proof, we need to establish energy fluxes in this context:

Definition 6.1.2. The **energy flux** $J^X[\varphi]$ is the covector field defined by the projection of the energy-momentum tensor of scalar field matter (see (6)) along the vector field $X \in \mathcal{X}(M)$, i.e. one defines

$$J_a^X[\varphi] = X^b T_{ab}[\varphi] = X^b \left(\nabla_a \varphi \nabla_b \varphi - \frac{1}{2} g_{ab} \nabla^c \varphi \nabla_c \varphi \right) .$$

for a smooth function $\varphi : M \to \mathbb{R}$. Note that

$$J_0^{\partial_t}[\varphi] = T_{00}[\varphi] = \frac{1}{2} \left(|\partial_t \varphi|^2 + a(t)^{-2} \left| \overline{\nabla} \varphi \right|_{\bar{g}}^2 \right)$$

holds, so in a certain sense, the energies E_N naturally arise from how scalar-field matter is encoded in an energy-momentum tensor.

Proof 2 of Proposition 6.1.1. Set $X = a(t)^3 \partial_t$. Then, one computes

$$\begin{aligned} \nabla^\mu X^\nu &= g^{\mu\sigma} \left[\left(\partial_\sigma a^3 \right) \partial_t^\nu + a^3 \nabla_\sigma \partial_t^\nu \right] \\ &= \begin{cases} -3a^2 \dot{a} & \mu = \nu = 0 \\ g^{\mu\sigma} a^3 \Gamma^\nu_{0\sigma} = g^{\mu\nu} \dot{a} a^2 = \bar{g}^{\mu\nu} \dot{a} & \mu, \nu \neq 0 \\ 0 & \text{else} \end{cases} \end{aligned}$$

with Remark 3.1.4. Thus, recalling that, since ψ is a wave, the divergence of T vanishes (see Lemma 3.3.1), one sees:

$$\begin{aligned} \nabla^\mu \left(J_\mu^X[\psi] \right) &= \left(\nabla^\mu X^\nu \right) T_{\mu\nu}[\psi] + X^\nu \left(\nabla^\mu T_{\mu\nu}[\psi] \right) \\ &= -3a^2 \dot{a} T_{00}[\psi] + \dot{a} \bar{g}^{ij} T_{ij}[\psi] \\ &= a^2 \dot{a} \left[-\frac{3}{2} \left(|\partial_t \psi|^2 + a^{-2} \left| \overline{\nabla} \psi \right|_{\bar{g}}^2 \right) + a^{-2} \bar{g}^{ij} \overline{\nabla}_i \psi \overline{\nabla}_j \psi \right. \\ &\qquad \left. - \frac{1}{2} \left(a^{-2} \bar{g}^{ij} \right) \left(a^2 \bar{g}_{ij} \nabla^c \psi \nabla_c \psi \right) \right] \\ &= a^2 \dot{a} \left[-\frac{3}{2} \left(|\partial_t \psi|^2 + a^{-2} \left| \overline{\nabla} \psi \right|_{\bar{g}}^2 \right) + a^{-2} \left| \overline{\nabla} \psi \right|_{\bar{g}}^2 \right. \\ &\qquad \left. + \frac{3}{2} |\partial_t \psi|^2 - \frac{3}{2} a^{-2} \left| \overline{\nabla} \psi \right|_g^2 \right] \\ &= -2\dot{a} \left| \overline{\nabla} \psi \right|_{\bar{g}}^2 \leq 0 \end{aligned} \tag{21}$$

since $\dot{a} > 0$ by assumption. The induced volume form $\mathrm{vol}_{\overline{M}_s}$ on $\overline{M}_s = \{s\} \times \overline{M}$ is given by

$$\mathrm{vol}_{\overline{M}_s} = \sqrt{\frac{\det(g|_{\overline{M}_s})}{\det(\bar{g})}} \mathrm{vol}_{\overline{M}} = \sqrt{\frac{\left(a(s)^2\right)^3 \det(\bar{g})}{\det(\bar{g})}} \mathrm{vol}_{\overline{M}} = a(s)^3 \mathrm{vol}_{\overline{M}}$$

by the Jacobi transformation law. Now, we choose the orientation on M such that $(-\partial_t, \mathcal{B})$ is positively oriented for any positively oriented local basis $\mathcal{B}$ on $T\overline{M}$. Using the divergence theorem by integration over the volume form vol_M associated with said orientation yields

$$\begin{aligned}
-\int_t^{t_0}\int_{\overline{M}_s} \mathrm{div}\left(J^X[\psi]\right)\mathrm{vol}_{\overline{M}_s} ds &= \int_{[t,t_0]\times\overline{M}} \mathrm{div}\left(J^X[\psi]\right)\mathrm{vol}_M \\
&= \int_{\overline{M}_{t_0}} J_0^X[\psi]\mathrm{vol}_{\overline{M}_{t_0}} - \int_{\overline{M}_t} J_0^X[\psi]\mathrm{vol}_{\overline{M}_t} \\
&= \frac{1}{2}a(t_0)^6\int_{\overline{M}}\left[|\partial_t\psi(t_0,\cdot)|^2 + a(t_0)^{-2}\left|\overline{\nabla}\psi(t_0,\cdot)\right|_{\bar{g}}^2\right]\mathrm{vol}_{\overline{M}} \\
&\quad - \frac{1}{2}a(t)^6\int_{\overline{M}}\left[|\partial_t\psi(t,\cdot)|^2 + a(t)^{-2}\left|\overline{\nabla}\psi(t,\cdot)\right|_{\bar{g}}^2\right]\mathrm{vol}_{\overline{M}} \\
&= \frac{1}{2}a(t_0)^6E(t_0,\psi) - \frac{1}{2}a(t)^6E(t,\psi),
\end{aligned}$$

which can be rearranged to

$$a(t)^6E(t,\psi) = a(t_0)^6E(t_0,\psi) + 2\int_t^{t_0}\int_{\overline{M}_s}\mathrm{div}\left(J^X[\psi]\right)\mathrm{vol}_{\overline{M}_s} ds\,. \tag{22}$$

Since the divergence term is nonpositive by (21), the statement now follows. □

Along with some of the Sobolev machinery established in Chapter 5, this already allows for rather precise pointwise control of waves towards $t = 0$:

Corollary 6.1.3. *In the setting of Proposition 6.1.1, with $(t,x) \in M$, $0 < t < t_0$, the following estimate holds for any smooth wave ψ:*

$$\left|\Delta^N\psi(t,x)\right| \leq Ca(t_0)^3\left(\int_t^{t_0} a(s)^{-3}\,ds\right)\left(\sqrt{E_N(t_0,\psi)} + \sqrt{E_{N+1}(t_0,\psi)}\right) + \left|\Delta^N\psi(t_0,x)\right| \tag{23}$$

where $C > 0$ is a $\bar{g}$-dependent constant. In particular, in the type 0 warped products with $a(t) = t^{\frac{2}{3\gamma}}$ with $p = (\gamma-1)\rho$, it follows that

$$\begin{aligned}
\left|\Delta^N\psi(t,x)\right| \leq{}& Ct_0^{\frac{2}{\gamma}}\left(\sqrt{E_N(t_0,\psi)} + \sqrt{E_{N+1}(t_0,\psi)}\right)\begin{cases}\frac{t^{1-\frac{2}{\gamma}} - t_0^{1-\frac{2}{\gamma}}}{\frac{2}{\gamma}-1} & \frac{2}{3} < \gamma < 2 \\ \log(t_0) - \log(t) & \gamma = 2\end{cases} \\
&+ \left|\Delta^N\psi(t_0,x)\right|
\end{aligned} \tag{24}$$

and this extends to warped products of type -1 with the same equations of phase for p and ρ, choosing small enough $t_0 > 0$ and updating $C \equiv C(\bar{g}, t_0, \rho(t_0))$.

Proof. Applying Corollary 5.2.6 in the third line, one computes

$$\begin{aligned}
\left|\Delta^N \psi(t,\cdot)\right| &\leq \left|\int_t^{t_0} \partial_t \Delta^N \psi(s,x) ds\right| + \left|\Delta^N \psi(t_0,x)\right| \\
&\leq \int_t^{t_0} \left\|\partial_t \Delta^N \psi(s,\cdot)\right\|_{L^\infty(\overline{M})} ds \,+\, \left|\Delta^N \psi(t_0,x)\right| \\
&\leq C\cdot \int_t^{t_0} \left(\left\|\partial_t \Delta^N \psi(s,\cdot)\right\|_{L^2(\overline{M})} + \left\|\partial_t \Delta^{N+1} \psi(s,\cdot)\right\|_{L^2(\overline{M})}\right) ds \,+\, \left|\Delta^N \psi(t_0,x)\right| \\
&\leq C\cdot \int_t^{t_0} \left(\sqrt{E_N(s,\psi)} + \sqrt{E_{N+1}(s,\psi)}\right) ds \,+\, \left|\Delta^N \psi(t_0,x)\right| \\
&\overset{(*)}{\leq} C\cdot \left(\sqrt{E_N(t_0,\psi)} + \sqrt{E_{N+1}(t_0,\psi)}\right) \int_t^{t_0} \frac{a(t_0)^3}{a(s)^3} ds \,+\, \left|\Delta^N \psi(t_0,x)\right| ,
\end{aligned}$$

where $(*)$ follows from Proposition 6.1.1.
In type 0, (24) is simply obtained by computing the integral. Moving on to type -1, by the last point in Lemma 4.2.1, one has

$$\int_t^{t_0} a(s)^{-3}\, ds \lesssim_{t_0,\rho(t_0)} \int_t^{t_0} \left(s^{-\frac{2}{3\gamma}}\right)^3 ds = \int_t^{t_0} s^{\frac{2}{\gamma}}\, ds$$

for $t_0 > 0$ small enough, and thus the final claim follows. □

6.2. Rescaled energy estimates

To derive a more precise asymptotic behaviour, it is now intuitive to consider the analogous energies for waves rescaled by the leading order suggested by Proposition 6.1.1 and Corollary 6.1.3. We start with type 0 warped products:

Proposition 6.2.1. *Let $2/3 < \gamma < 2$ and set*

$$\Gamma = \max\left(\frac{4}{3\gamma}, 4 - \frac{4}{\gamma}\right) .$$

For a smooth wave ψ in a warped product spacetime (M,g) of type 0, we set $\hat{\psi}(t,x) = \psi(t,x)/t^{1-2/\gamma}$. Then, for any $N \in \mathbb{N}$ and $0 < t < t_0$, the following estimates hold for a $\bar{g}$-dependent constant $C > 0$:

$$t^\Gamma E_N\left(t,\hat{\psi}\right) \leq t_0^\Gamma E_N\left(t_0,\hat{\psi}\right) ,$$

$$\left|\Delta^N \hat{\psi}(t,\cdot)\right| \leq \frac{C t_0^{\frac{\Gamma}{2}}}{1-\frac{\Gamma}{2}} \left(t_0^{1-\frac{\Gamma}{2}} - t^{1-\frac{\Gamma}{2}}\right) \left(\sqrt{E_N\left(t_0,\hat{\psi}\right)} + \sqrt{E_{N+1}\left(t_0,\hat{\psi}\right)}\right) + \left|\Delta^N \hat{\psi}(t_0,\cdot)\right| \tag{25}$$

Proof. Again, it suffices to just prove the case $N = 0$. First, we calculate[6] with the explicit form of $\Box_g$ given by Remark 4.3.2:

$$\begin{aligned}\Box_g\left(\hat{\psi}\right) =& -\partial_t^2\left(\frac{\psi}{t^{1-\frac{2}{\gamma}}}\right) + t^{-\frac{4}{3\gamma}}\Delta\left(\frac{\psi}{t^{1-\frac{2}{\gamma}}}\right) - \frac{2}{\gamma t}\partial_t\left(\frac{\psi}{t^{1-\frac{2}{\gamma}}}\right)\\ =&\left(\frac{-\partial_t^2\psi + t^{-\frac{4}{3\gamma}}\Delta\psi - \frac{2}{\gamma t}\partial_t\psi}{t^{1-\frac{2}{\gamma}}}\right) - 2\left(\frac{2}{\gamma}-1\right)\frac{\partial_t\psi}{t^{2-\frac{2}{\gamma}}}\\ &-\left(\frac{2}{\gamma}-1\right)\left(\frac{2}{\gamma}-2\right)\frac{\psi}{t^{3-\frac{2}{\gamma}}} - \frac{2}{\gamma t}\left(\frac{2}{\gamma}-1\right)\frac{\psi}{t^{2-\frac{2}{\gamma}}}\\ =&\frac{\Box_g\psi}{t^{1-\frac{2}{\gamma}}} - \frac{2}{t}\left(\frac{2}{\gamma}-1\right)\left[\frac{\partial_t\psi}{t^{1-\frac{2}{\gamma}}} + \left(\frac{2}{\gamma}-1\right)\frac{\psi}{t^{2-\frac{2}{\gamma}}}\right]\\ =&-\frac{2}{t}\left(\frac{2}{\gamma}-1\right)\partial_t\hat{\psi}\end{aligned}$$

In particular, after rearranging, one has:

$$\partial_t^2\hat{\psi} = t^{-\frac{4}{3\gamma}}\Delta\hat{\psi} + \frac{2}{t}\left(\frac{1}{\gamma}-1\right)\partial_t\hat{\psi}$$

Now, one can perform a similar computation to the first proof of Proposition 6.1.1:

$$\begin{aligned}\partial_t E\left(t,\hat{\psi}\right) =& \int_{\overline{M}}\left[2\partial_t^2\hat{\psi}\cdot\partial_t\hat{\psi} + 2t^{-\frac{4}{3\gamma}}\cdot\bar{g}\left(\partial_t\nabla\hat{\psi},\nabla\hat{\psi}\right) - \frac{4}{3\gamma}t^{-\frac{4}{3\gamma}-1}\left|\nabla\hat{\psi}\right|_{\bar{g}}^2\right]\mathrm{vol}_{\overline{M}}\\ =&\int_{\overline{M}}\left[2\left(t^{-\frac{4}{3\gamma}}\Delta\hat{\psi} + \frac{2}{t}\left(\frac{1}{\gamma}-1\right)\partial_t\hat{\psi}\right)\partial_t\hat{\psi}\right.\\ &\left.-2t^{-\frac{4}{3\gamma}}\partial_t\hat{\psi}\cdot\Delta\hat{\psi} - \frac{4}{3\gamma t}t^{-\frac{4}{3\gamma}}\left|\nabla\hat{\psi}\right|_{\bar{g}}^2\right]\mathrm{vol}_{\overline{M}}\\ =&\int_{\overline{M}}\left[\frac{1}{t}\left(\frac{4}{\gamma}-4\right)\left|\partial_t\hat{\psi}\right|^2 - \frac{4}{3\gamma t}t^{-\frac{4}{3\gamma}}\left|\nabla\hat{\psi}\right|_{\bar{g}}^2\right]\mathrm{vol}_{\overline{M}}\\ \geq& -\frac{1}{t}\max\left(4-\frac{4}{\gamma},\frac{4}{3\gamma}\right)\int_{\overline{M}}\left[\left|\partial_t\hat{\psi}\right|^2 + t^{-\frac{4}{3\gamma}}\left|\nabla\hat{\psi}(t,\cdot)\right|_{\bar{g}}^2\right]\mathrm{vol}_{\overline{M}}\\ =&-\frac{\Gamma}{t}E\left(t,\hat{\psi}\right)\end{aligned}$$

From here, we can deduce the first estimate precisely as in Proposition 6.1.1, replacing $\frac{4}{\gamma}$ with Γ. The pointwise estimate also follows analogously to Corollary 6.1.3, with

$$\left\|\partial_t\hat{\psi}(t,\cdot)\right\|_{L^\infty(\overline{M})} \leq C\left(\frac{t_0}{t}\right)^{\frac{\Gamma}{2}}\left(\sqrt{E\left(t_0,\hat{\psi}\right)} + \sqrt{E_1\left(t_0,\hat{\psi}\right)}\right) \tag{26}$$

for any $0 < t < t_0$, $x \in \overline{M}$ (and similarly for $N > 0$). □

[6] notationally surpressing t-dependency of ψ and its derivatives here and in the rest of the proof

Remark 6.2.2. Note that

$$\Gamma = \begin{cases} \frac{4}{3\gamma} & \frac{2}{3} < \gamma \leq \frac{4}{3} \\ 4 - \frac{4}{\gamma} & \frac{4}{3} \leq \gamma < 2 \end{cases}$$

holds since the former is strictly decreasing in γ, the latter strictly increasing and both agree for $\gamma = \frac{4}{3}$. Thus, one has

$$0 < 1 - \frac{\Gamma}{2} = \begin{cases} 1 - \frac{2}{3\gamma} & \frac{2}{3} < \gamma \leq \frac{4}{3} \\ \frac{2}{\gamma} - 1 & \frac{4}{3} \leq \gamma < 2 \end{cases}$$

and Proposition 6.2.1 also proves that

$$(t, x) \mapsto \frac{\Delta^N \psi(t, x)}{t^{1-\frac{2}{\gamma}}}$$

is uniformly bounded, since the first summand in (25) is independent of x and the second is uniformly bounded in x since ψ is continuous on the compact submanifold $\overline{M}_{t_0}$. With $1 - \Gamma/2 > 0$, the inequailty (26) even shows that, for any $x \in \overline{M}$, $t \mapsto \left(\frac{\Delta^N \psi(t,x)}{t^{1-\frac{2}{\gamma}}}\right)$ is absolutely continuous on $[0, t_0]$.

Expanding this to scale factors associated with negatively curved space works along similar lines because the same asymptotic behaviour is exhibited in principle. However, because this similarity truly only holds as t approaches 0, the statements become a little more technically involved:

Proposition 6.2.3. *Let ψ be a smooth wave on a warped product spacetime (M, g) of type -1 with $\gamma \in (2/3, 2]$. We define $\hat{\psi}(t, x) := \psi(t,x)/h(t)$, $h(t) = \int_t^\infty a(s)^{-3}\, ds$. Then, for any $\varepsilon > 0$, there exists $t_0 > 0$ small enough such that, for*

$$\Gamma_\varepsilon = \max(6(\gamma - 1) + \varepsilon, 2),$$
$$a(t)^{\Gamma_\varepsilon} E\left(t, \hat{\psi}\right) \leq a(t_0)^{\Gamma_\varepsilon} E\left(t_0, \hat{\psi}\right)$$

holds for any $0 < t < t_0$. Additionally, and strictly improving the case $\gamma = 2$, the following estimate is satisfied for arbitrary $t_0 > 0$ and again any $0 < t < t_0$:

$$a(t)^6 E\left(t, \hat{\psi}\right) \leq a(t_0)^6 E\left(t_0, \hat{\psi}\right)$$

Proof. Step 1: Once again, we first need to understand how the wave operator acts on $\hat{\psi}$, using $\partial_t h = -a^{-3}$:

$$\begin{aligned} \square_g \hat{\psi} &= -\partial_t^2 \left(\frac{\psi}{h}\right) + a^{-2} \Delta \frac{\psi}{h} - 3\frac{\dot{a}}{a} \partial_t \left(\frac{\psi}{h}\right) \\ &= -\frac{\partial_t^2 \psi}{h} - 2\partial_t \psi \left(-\frac{-a^{-3}}{h^2}\right) - \psi \left(2\frac{a^{-6}}{h^3} - 3\frac{\dot{a}a^{-4}}{h^2}\right) \\ &\quad + \frac{1}{h} a^{-2} \Delta \psi - 3\frac{\dot{a}}{a} \left(\frac{\partial_t \psi}{h} - \psi \frac{-a^{-3}}{h^2}\right) \end{aligned}$$

$$= \frac{\Box_g \psi}{h} - 2\frac{a^{-3}}{h}\left[\frac{\partial_t \psi}{h} - \psi \frac{-a^{-3}}{h^2}\right]$$

$$= 2\frac{\dot{h}}{h}\partial_t \hat{\psi}$$

Hence, again after re-arranging,

$$\partial_t^2 \hat{\psi} = a(t)^{-2}\Delta\hat{\psi} - \left(3\frac{\dot{a}}{a} + 2\frac{\dot{h}}{h}\right)\partial_t \hat{\psi}$$

follows. In trying to analogize the proof of Proposition 6.2.1 as much as possible, we will need to compare ${}^{\dot{h}}\!/\!{}_h$ to ${}^{\dot{a}}\!/\!{}_a$ for small times:

Step 2: We will show

$$\lim_{t\to 0}\frac{\dot{h}/h}{\dot{a}/a}(t) = \frac{3\gamma}{2} - 3$$

for any $\gamma \in (2/3, 2]$. First, we simplify this expression:

$$\frac{\dot{h}/h}{\dot{a}/a} = \frac{-a^{-3}a}{\dot{a}h} = -\frac{\left(a^2\dot{a}\right)^{-1}}{h}$$

As $t \to 0$, the denominator diverges toward ∞ as shown in Lemma 4.2.1. Regarding the numerator, the rephrased Friedman equation (10) with $\kappa = -1$ gives

$$a^2\dot{a} = a^2\sqrt{1 + \frac{8\pi B}{3}a^{2-3\gamma}} = \sqrt{a^4 + \frac{8\pi B}{3}a^{6-3\gamma}}.$$

With $a(0) = 0$ and $6 - 3\gamma > 0$ for $\gamma < 2$, this gives

$$\lim_{t\to 0}\left(a(t)^2\dot{a}(t)\right)^{-1} = \begin{cases}\infty & \gamma < 2\\ \sqrt{\frac{3}{8\pi B}} & \gamma = 2\end{cases}.$$

Thus, the claim already follows for $\gamma = 2$. Else, we can apply the l'Hospital's rule:

$$\begin{aligned}\lim_{t\to 0}\frac{\dot{h}/h}{\dot{a}/a}(t) &= -\lim_{t\to 0}\frac{a(t)^{-2}\dot{a}(t)^{-1}}{h(t)}\\ &= -\lim_{t\to 0}\frac{-2a(t)^{-3}\dot{a}(t)\dot{a}(t)^{-1} - a(t)^{-2}\dot{a}(t)^{-2}\ddot{a}(t)}{-a(t)^{-3}}\\ &= -2 - \lim_{t\to 0}\frac{a(t)\ddot{a}(t)}{\dot{a}(t)^2}\\ &\overset{(*)}{=} -2 - \lim_{t\to 0}\frac{a(t)\left(-\frac{4\pi}{3}(1+3(\gamma-1))\rho\right)}{1+\frac{8\pi B}{3}a(t)^{2-3\gamma}}\\ &\overset{(**)}{=} -2 - \lim_{t\to 0}\frac{a(t)^2\left(-\frac{4\pi}{3}(3\gamma-2)Ba(t)^{-3\gamma}\right)}{1+\frac{8\pi B}{3}a(t)^{2-3\gamma}}\end{aligned}$$

$$= -2 + \lim_{t\to 0} \frac{\frac{4\pi B}{3}(3\gamma-2)}{a(t)^{3\gamma-2} + \frac{8\pi B}{3}}$$
$$= -2 + \frac{1}{2}(3\gamma-2) = \frac{3\gamma}{2} - 3,$$

where we used (11) for $(*)$ to substitute $\ddot{a}$ in the numerator, and (10) with $\kappa = -1$ to replace $\dot{a}$ in the denominator, as well as $p = (\gamma-1)\rho$ and (9) to replace ρ in $(**)$. For the final limit, we recall that $3\gamma - 2$ is positive for $\gamma > 2/3$, that $a(0) = 0$ holds and that B isn't zero.

Step 3: With this information in hand, we can now treat the energy as usual: Using Step 1 to replace $\partial_t^2\hat{\psi}$, we calculate

$$\begin{aligned}\partial_t E\left(t,\hat{\psi}\right) &= \int_{\overline{M}} \left(2\partial_t^2\hat{\psi}\cdot\partial_t\hat{\psi} - 2\partial_t\hat{\psi}\cdot a^{-2}\Delta\hat{\psi} - 2\frac{\dot{a}}{a^3}\left|\overline{\nabla}\hat{\psi}\right|_{\bar{g}}^2\right)\mathrm{vol}_{\overline{M}}\\ &= \int_{\overline{M}}\left(-\left(6\frac{\dot{a}}{a}+4\frac{\dot{h}}{h}\right)\left|\partial_t\hat{\psi}\right|^2 - 2\frac{\dot{a}}{a}a^{-2}\left|\overline{\nabla}\hat{\psi}\right|_{\bar{g}}^2\right)\mathrm{vol}_{\overline{M}}\\ &\geq -\max\left(6\frac{\dot{a}}{a}+4\frac{\dot{h}}{h}, 2\frac{\dot{a}}{a}\right)E(t,\hat{\psi})\end{aligned}$$

Now, it follows from Step 2 that, for any $\varepsilon > 0$, there exists some small enough $t_0 > 0$ such that, for all $0 < t < t_0$,

$$\frac{\dot{h}(t)}{h(t)} \leq \left(\frac{3\gamma}{2} - 3 + \frac{\varepsilon}{4}\right)\frac{\dot{a}(t)}{a(t)}$$

(since both a and $\dot{a}$ are positive) and hence

$$\begin{aligned}\partial_t E\left(t,\hat{\psi}\right) &\geq -\max\left(6 + 4\cdot\left(\frac{3\gamma}{2} - 3 + \frac{\varepsilon}{4}\right), 2\right)\frac{\dot{a}(t)}{a(t)}E\left(t,\hat{\psi}\right)\\ &= -\Gamma_\varepsilon\frac{\dot{a}(t)}{a(t)}E\left(t,\hat{\psi}\right).\end{aligned}$$

The stated energy estimate is now just the usual Gronwall argument. The argument for the second inequality works analogously, simply estimating

$$\partial_t E(t,\psi) \geq -\max\left(6\frac{\dot{a}}{a}+4\frac{\dot{h}}{h}, 2\frac{\dot{a}}{a}\right)E(t,\hat{\psi}) \geq -6\frac{\dot{a}}{a}E(t,\hat{\psi}),$$

since $\dot{h} = -a^{-3} < 0$, $h > 0$ and $\dot{a}/a > 0$, and then continuing as usual. □

In particular, we can derive the following pointwise estimate along similar lines as before:

Corollary 6.2.4. *For (M, g), $\hat{\psi}$ and Γ_ε as in Proposition 6.2.3 and $2/3 < \gamma < 2$, there exists $t_0 > 0$ small enough for any $\varepsilon > 0$ such that, for any $0 < t < t_0$, the following pointwise estimate holds:*

$$\left|\hat{\psi}(t,\cdot)\right| \leq \left|\hat{\psi}(t_0,\cdot)\right| + Ca(t_0)^{\frac{\Gamma_\varepsilon}{2}}\left(\sqrt{E_N(t_0,\hat{\psi})} + \sqrt{E_{N+1}(t_0,\hat{\psi})}\right)\frac{t_0^{1-\Gamma_\varepsilon/3\gamma} - t^{1-\Gamma_\varepsilon/3\gamma}}{1-\Gamma_\varepsilon/3\gamma}$$

For the stiff case ($\gamma = 2$), one analogously obtains, again not requiring $t_0 > 0$ to be small,

$$\left|\hat{\psi}(t,\cdot)\right| \leq |\hat{\psi}(t_0,\cdot)| + Ca(t_0)^3 \left(\sqrt{E_N\left(t_0,\hat{\psi}\right)} + \sqrt{E_{N+1}\left(t_0,\hat{\psi}\right)}\right)(\log(t_0) - \log(t))$$

Proof. We quickly sketch the ansatz for the first case (w.l.o.g. for $N = 0$) so far as it differs from the proof of the corresponding estimate in Proposition 6.1.1 (possibly updating C along the way):

$$\begin{aligned}\left|\hat{\psi}(t,\cdot)\right| &\leq \left|\hat{\psi}(t_0,\cdot)\right| + \int_t^{t_0} \left\|\partial_t\hat{\psi}(s,\cdot)\right\|_{L^\infty(\overline{M})} ds \\ &\leq \left|\hat{\psi}(t_0,\cdot)\right| + C\int_t^{t_0}\left(\sqrt{E\left(s,\hat{\psi}\right)} + \sqrt{E_1\left(s,\hat{\psi}\right)}\right) ds \\ &\leq \left|\hat{\psi}(t_0,\cdot)\right| + C\left(\sqrt{E\left(t_0,\hat{\psi}\right)} + \sqrt{E_1\left(t_0,\hat{\psi}\right)}\right) a(t_0)^{\Gamma_\varepsilon/2}\int_t^{t_0} a(s)^{-\Gamma_\varepsilon/2}\, ds\end{aligned}$$

Further, by Lemma 4.2.1, there exists some $K > 0$ for $t_0 > 0$ small enough such that, for all $0 < s < t_0$,

$$a(s)^{-\Gamma_\varepsilon/2} \leq K \cdot \left(s^{\frac{2}{3\gamma}}\right)^{-\Gamma_\varepsilon/2} = Ks^{\frac{-\Gamma_\varepsilon}{3\gamma}}.$$

Hence,

$$\int_t^{t_0} a(s)^{-\Gamma_\varepsilon/2}\, ds \leq K \cdot \frac{t_0^{1-\Gamma_\varepsilon/3\gamma} - t^{1-\Gamma_\varepsilon/3\gamma}}{1 - \Gamma_\varepsilon/3\gamma}$$

holds and the estimate follows after updating C.
The statement for $\gamma = 2$ follows by precisely the same argument, using the respective energy estimate in Proposition 6.2.3. □

Remark 6.2.5. Again, we turn to the question of whether the rescaled wave is absolutely continuous toward the Big Bang, which will help us answer whether we can extend it to the Big Bang hypersurface: If $\Gamma_\varepsilon = 2$, one has

$$1 - \frac{\Gamma_\varepsilon}{3\gamma} = 1 - \frac{2}{3\gamma} > 0,$$

and else

$$1 - \frac{\Gamma_\varepsilon}{3\gamma} = 1 - \left(2\frac{\gamma-1}{\gamma} + \frac{\varepsilon}{3\gamma}\right) = \frac{2}{\gamma} - 1 - \frac{\varepsilon}{3\gamma}$$

is positive for small enough $\varepsilon > 0$ since $\frac{2}{\gamma} - 1 > 0$ for $\gamma < 2$. Hence, the proof once again even shows that $\hat{\psi}$ is absolutely continuous close to $t = 0$, so $\lim_{t\to 0}\hat{\psi}(t,x)$ exists for any $x \in \overline{M}$.
Furthermore, it should be noted that this does not work for the stiff case since the

upper estimate just obtained still diverges toward ∞ logarithmically when approaching $t = 0$. This is also related to why we need to work with the slightly unwieldy Γ_ε in type -1: Just using the secondary estimate in Proposition 6.2.3 doesn't yield an improved behaviour of the rescaled wave compared to the wave itself a priori, while on the other hand the asymptotic nature of our analysis necessitates some smallness assumptions on $t_0 > 0$ to obtain a sufficiently strong control of the energy in powers of the scale factor.

These estimates will prove to be completely sufficient for the upcoming analysis on warped product spacetimes of types 0 and -1. However, when restricting back to "true" FLRW spacetimes with flat and hyperbolic spatial geometry, additional energies can provide more precise control of relevant Sobolev norms, as outlined in Section A.2.

7. Global blow-up of waves

In this chapter, all results will be combined to rigorously prove what the energy estimates already suggest – that smooth waves exhibit blow-up towards the Big Bang singularity of at most the same rate as homogenous waves. Further, sufficient conditions to verify whether a wave blows up at precisely this rate will be provided. However, as the estimates in Section 6.2 suggest, the stiff case will have to be treated somewhat separately.

7.1. Asymptotics outside of the stiff case

To start off, the rescaled pointwise estimates can be directly utilized to extract the highest possible leading order term:

Theorem 7.1.1. *Let ψ be a smooth wave on a warped product spacetime (M, g) of type 0 or -1, with scale factor a associated to $\gamma \in (2/3, 2)$. Then,*

$$A(x) := \lim_{t\to 0} \frac{\psi(t,x)}{t^{1-\frac{2}{\gamma}}}\,, \text{ respectively } A(x) := \lim_{t\to 0} \frac{\psi(t,x)}{\int_t^\infty a(s)^{-3}\,ds}\,,$$

exists and defines a smooth function on $\overline{M}$.

Proof. First, let's turn to type 0: Since, by Remark 6.2.2,

$$t \in (0, t_0] \mapsto \frac{\Delta^N \psi(t,x)}{t^{1-\frac{2}{\gamma}}}$$

is absolutely continuous for any fixed $x \in \overline{M}$, with a time derivative that is integrable on $[0, t_0]$, $A_N(x) := \lim_{t\to 0} \frac{\Delta^N \psi(t,x)}{t^{1-\frac{2}{\gamma}}}$ exists for any $N \in \mathbb{N}$, $x \in \overline{M}$.

To prove smoothness, we will prove that, for any $N \in \mathbb{N}$, this pointwise convergence also extends to $H^2\left(\overline{M}\right)$, and thus $A_N \in H^2\left(\overline{M}\right)$, as well as that $\Delta A_N = A_{N+1}$ is satsified almost surely in $L^2\left(\overline{M}\right)$ (in the sense of weak derivatives). To see why this is sufficient, choose an arbitrary decreasing sequence $(t_n)_{n\in\mathbb{N}}$ with $0 < t_n \le t_0$ for all $n \in \mathbb{N}$ and $t_n \to 0$ as $n \to \infty$. Further, define

$$f_{N,n}(x) := \frac{\Delta^N \psi(t_n,x)}{t_n^{1-\frac{2}{\gamma}}}\,,$$

so $(f_{N,n})_{n\in\mathbb{N}}$ converges to A_N pointwise for any $N \in \mathbb{N}$ and these subsequences are *consistent* in the sense that $\Delta f_{N,n} = f_{N+1,n}$ holds for all $n, N \in \mathbb{N}$. If the convergence also holds in $H^2\left(\overline{M}\right)$ for all $N \in \mathbb{N}$, then one in particular has that $(f_{N,n})_{n\in\mathbb{N}}$ is a Cauchy sequence with regards to the norm

$$\|\Delta(\cdot)\|_{H^2(\overline{M})} + \|\cdot\|_{H^2(\overline{M})}$$

for any $N \in \mathbb{N}$ *since we chose the subsequences to be consistent.* By using Lemma 5.1.2 on each of these terms, it is then also a Cauchy sequence with regards to

$$\|\Delta(\cdot)\|_{C(\overline{M})} + \|\cdot\|_{C(\overline{M})}$$

for any $N \in \mathbb{N}$ and thus a Cauchy sequence in $C^2\left(\overline{M}\right)$ equipped with the standard norm, by the second set of inequalities in Proposition 5.2.5 (with $k = 0$, $m = 2$). Since the latter is a Banach space and any limit in $C^2\left(\overline{M}\right)$ must coincide with the pointwise limit, it follows that $A_N \in C^2\left(\overline{M}\right)$ must hold for any $N \in \mathbb{N}$. If $\Delta A_N = A_{N+1}$ holds almost surely in $L^2\left(\overline{M}\right)$, it must now also hold classically. Now using Proposition 5.2.5 for $k = 2, m = 2$, it now follows by the same approximation argument that $A_N \in C^4\left(\overline{M}\right)$ is satisfied for any N, and by iterating this argument that $A_N \in C^\infty\left(\overline{M}\right)$ must hold for any $N \in \mathbb{N}$. In particular, this shows that A is smooth and $\Delta^N A_0 = A_N$.

By Remark 6.2.2, $(t, x) \mapsto \left|\frac{\Delta^N\psi(t,x)}{t^{1-\frac{2}{\gamma}}}\right|^2$ is uniformly bounded on $[0, t_0] \times \overline{M}$ for any $N \in \mathbb{N}$. In particular, since $\overline{M}$ is closed and thus has finite volume, $x \mapsto \left|\frac{\Delta^N\psi(t,x)}{t^{1-\frac{2}{\gamma}}}\right|^2$ is bounded from above by an integrable function. Hence, we can use the Dominated Convergence Theorem for t approaching 0 to deduce that A_N is square integrable. More precisely, choosing $(f_{N,n})_{n\in\mathbb{N}}$ as before, this shows that $(f_{N,n})_{n\in\mathbb{N}}$ converges to A_N in $L^2\left(\overline{M}\right)$ for any $N \in \mathbb{N}$ as $n \to \infty$ after rearranging. Again by the consistency property $\Delta f_{N,n} = f_{N+1,n}$, it follows that this sequence must be a Cauchy sequence with regards to

$$\|\Delta(\cdot)\|_{L^2(\overline{M})} + \|\cdot\|_{L^2(\overline{M})},$$

so also with regards to $\|\cdot\|_{H^2(\overline{M})}$ by Proposition 5.2.5. Thus, $f_{N,n}$ converges in $H^2\left(\overline{M}\right)$, and this limit must obviously agree with A_N almost everywhere, so $A_N \in H^2\left(\overline{M}\right)$. Furthermore, by the consistency property and uniqueness of limits, ΔA_N and A_{N+1} must represent the same element of $L^2\left(\overline{M}\right)$. Smoothness of A now follows as argued above.

For type -1, Remark 6.2.5 yields existence along the same lines and fulfills the role of Remark 6.2.2 in the rest of the proof as well. Besides replacing $t^{1-\frac{2}{\gamma}}$ by $\int_t^\infty a(s)^{-3}\,ds$, everything else now follows identically since no (other) properties of the scale factor were used at any point. □

This statement already shows that one has, as $t \to 0$,

$$\psi(t,\cdot) - A \cdot t^{1-\frac{2}{\gamma}} = o\left(t^{1-\frac{2}{\gamma}}\right), \text{ resp.}$$
$$\psi(t,\cdot) - A \cdot \int_t^\infty a(s)^{-3}\,ds = o\left(\int_t^\infty a(s)^{-3}\,ds\right) = o\left(t^{1-\frac{2}{\gamma}}\right)$$

in warped product spacetimes of type 0 (resp. type -1) outside of the stiff setting. However, this behaviour can be shown to be more uniform in the sense of convergence with regards to our energies, which will be proven along similar lines as in [1]:

Theorem 7.1.2. *Under the conditions of Theorem 7.1.1 in type* 0 *warped products, the following holds:*

$$\lim_{t\to 0} t^{\frac{4}{\gamma}} E_N\left(t, \psi(t,\cdot) - A\cdot t^{1-\frac{2}{\gamma}}\right) = 0$$

Proof. For $t > 0$, one calculates (using the Cauchy-Schwarz and Young inequalities in the first estimate and Proposition 6.2.1 in the second):

$$\begin{aligned}
&t^{\frac{4}{\gamma}} E\left(t, \psi(t,\cdot) - A\cdot t^{1-\frac{2}{\gamma}}\right)\\
=\,&t^{\frac{4}{\gamma}} \int_{\overline{M}} \left[\left|\partial_t\psi(t,\cdot) - \left(1-\frac{2}{\gamma}\right)t^{-\frac{2}{\gamma}}A\right|^2 + t^{-\frac{4}{3\gamma}}\left|\overline{\nabla}\psi(t,\cdot) - t^{1-\frac{2}{\gamma}}\overline{\nabla}A\right|_{\bar{g}}^2\right]\mathrm{vol}_{\overline{M}}\\
=\,&t^{\frac{4}{\gamma}} \int_{\overline{M}} \left[\left|t^{1-\frac{2}{\gamma}}\partial_t\left(\frac{\psi(t,\cdot)}{t^{1-\frac{2}{\gamma}}}\right) + \left(1-\frac{2}{\gamma}\right)t^{-\frac{2}{\gamma}}\left(\frac{\psi(t,\cdot)}{t^{1-\frac{2}{\gamma}}} - A\right)\right|^2 + \right.\\
&\qquad\qquad\left. + t^{2-\frac{4}{\gamma}-\frac{4}{3\gamma}}\left|\overline{\nabla}\left(\frac{\psi(t,\cdot)}{t^{1-\frac{2}{\gamma}}}\right) - \overline{\nabla}A\right|_{\bar{g}}^2\right]\mathrm{vol}_{\overline{M}}\\
\leq\,&2t^{\frac{4}{\gamma}} \int_{\overline{M}} \left[t^{2-\frac{4}{\gamma}}\left|\partial_t\hat{\psi}\right|^2 + t^{-\frac{4}{\gamma}}\left(1-\frac{2}{\gamma}\right)^2\left|\hat{\psi} - A\right|^2 + \right.\\
&\qquad\qquad\left. + t^{2-\frac{4}{\gamma}-\frac{4}{3\gamma}}\left(\left|\overline{\nabla}\hat{\psi}\right|_{\bar{g}}^2 + \left|\overline{\nabla}A\right|_{\bar{g}}^2\right)\right]\mathrm{vol}_{\overline{M}}\\
=\,&2\left[t^2 E\left(t,\hat{\psi}\right) + \left(1-\frac{2}{\gamma}\right)^2\left(\int_{\overline{M}}\left|\hat{\psi} - A\right|^2\mathrm{vol}_{\overline{M}}\right) + t^{2-\frac{4}{3\gamma}}\int_{\overline{M}}\left|\overline{\nabla}A\right|_{\bar{g}}^2\mathrm{vol}_{\overline{M}}\right]\\
\leq\,&2\left[t^{2-\Gamma}t_0^\Gamma E\left(t_0,\hat{\psi}\right) + \left(1-\frac{2}{\gamma}\right)^2\left(\int_{\overline{M}}\left|\hat{\psi}(t,\cdot) - A\right|^2\mathrm{vol}_{\overline{M}}\right) + t^{2-\frac{4}{3\gamma}}\int_{\overline{M}}\left|\overline{\nabla}A\right|_{\bar{g}}^2\mathrm{vol}_{\overline{M}}\right]
\end{aligned}$$

As $t \to 0$, the first summand vanishes since $\Gamma < 2$ (see Remark 6.2.2), the second utilizing the Dominated Convergence Theorem and the third because A is smooth (so the integral is finite) and $\gamma > \frac{2}{3}$ forces the prefactor to vanish. Altogether, the statement follows for $N = 0$ and extends to $N \in \mathbb{N}$ as usual by Corollary 4.3.3. □

Theorem 7.1.3. *Denoting* $h(t) = \int_t^\infty a(s)^{-3}\,ds$*, the following holds setting of Theorem 7.1.1 for type* -1*:*

$$a(t)^6 E_N(t, \psi - Ah) \to 0 \text{ as } t \to 0\,.$$

Proof. Similar to before, one calculates:

$$\begin{aligned}
&a(t)^6 E\left(t, \psi - Ah\right)\\
=&a(t)^6 \int_{\overline{M}} \left[\left|\partial_t \psi(t,\cdot) + a(t)^{-3} A\right|^2 + a(t)^{-2}\left|\overline{\nabla}\psi(t,\cdot) - h(t)\overline{\nabla} A\right|_{\bar{g}}^2\right] \mathrm{vol}_{\overline{M}}\\
=&a(t)^6 \int_{\overline{M}} \left[\left|h(t)\partial_t\hat{\psi}(t,\cdot) - \frac{a(t)^{-3}}{h(t)}\psi(t,\cdot) + a(t)^{-3}A\right|^2 + \right.\\
&\left. + a(t)^{-2}h(t)^2\left|\overline{\nabla}\hat{\psi}(t,\cdot) - \overline{\nabla}A\right|_{\bar{g}}^2\right] \mathrm{vol}_{\overline{M}}\\
\leq& 2a(t)^6 \int_{\overline{M}} \left[h(t)^2\left|\partial_t\hat{\psi}(t,\cdot)\right|^2 + a(t)^{-6}\left|\hat{\psi}(t,\cdot) - A\right|^2 + \right.\\
&\left. + a(t)^{-2}h(t)^2\left(\left|\overline{\nabla}\hat{\psi}(t,\cdot)\right|_{\bar{g}}^2 + \left|\overline{\nabla}A\right|_{\bar{g}}^2\right)\right] \mathrm{vol}_{\overline{M}}\\
=&2a(t)^6 h(t)^2 E\left(t,\hat{\psi}\right) + 2\int_{\overline{M}}\left|\hat{\psi}(t,\cdot) - A\right|^2 \mathrm{vol}_{\overline{M}} + 2h(t)^2 a(t)^4 \int_{\overline{M}}\left|\overline{\nabla}A\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}} \qquad (27)
\end{aligned}$$

Now, we analyse all three terms as $t \to 0$:

- Regarding the first term, we have shown in Lemma 4.2.1 that $a(t) = \mathcal{O}\left(t^{\frac{2}{3\gamma}}\right)$ and $h(t) = \mathcal{O}\left(t^{1-\frac{2}{\gamma}}\right)$. Thus, $a(t)^6 h(t)^2 = \mathcal{O}\left(t^2\right)$. On the other hand, combining Proposition 6.2.3 and again Lemma 4.2.1 yields for arbitrarily small $\varepsilon > 0$ as long $t_0 > t > 0$ small enough:

$$E\left(t,\hat{\psi}\right) \leq E\left(t_0,\hat{\psi}\right)\left(\frac{a(t_0)}{a(t)}\right)^{\Gamma_\varepsilon} \leq E\left(t_0,\hat{\psi}\right) a(t_0)^{\Gamma_\varepsilon} \cdot C t^{-2/3\gamma \Gamma_\varepsilon}$$

  If $\Gamma_\varepsilon = 2$, one has $-2\Gamma_\varepsilon/3\gamma = -4/3\gamma > -2$. Else, one has

$$-\frac{2\Gamma_\varepsilon}{3\gamma} = -\frac{2}{3\gamma}\left(6(\gamma-1)+\varepsilon\right) = \frac{4}{\gamma} - 2 - \frac{\varepsilon}{3\gamma}$$

  For $0 < \varepsilon < 3\gamma\left(4/\gamma - 2\right) = 12 - 6\gamma$, one can ensure that this is positive (recalling $\gamma < 2$). Hence, one deduces that $E(t,\psi) = \mathcal{O}\left(t^{-2+\delta}\right)$ holds for some $\delta > 0$ in any case and thus the first summand vanishes.

- The second term simply vanishes by the Dominated Convergence Theorem as before.

- Regarding the final term, one has by Lemma 4.2.1 that

$$h(t)^2 a(t)^4 = \mathcal{O}\left(t^{2-\frac{4}{\gamma}+\frac{8}{3\gamma}}\right) = \mathcal{O}\left(t^{2-\frac{4}{3\gamma}}\right),$$

  so this factor converges to 0 as $t \to 0$ since $\gamma > 2/3$. Since A is smooth, the integral is finite and this term as a whole converges to 0.

Altogether, the entire right hand side of (27) now vanishes in the limit, proving the statement. □

7.2. Sufficient conditions for leading order blow-up in the non-stiff case

The statements so far only show that waves blow-up at most with the same rate as homogeneous waves, but given the data of a wave on some spatial hypersuface $\overline{M}_{t_0}$, the previous results don't make a statement on whether this is actually attained since A could just vanish entirely. However, one would heuristically expect that this shouldn't happen for almost homogeneous waves, and this intuition will be made more precise in this section, extending the similar analysis for flat FLRW spacetimes performed in [1]. To this end, some technical lemmata need to be established:

Lemma 7.2.1. *For any smooth wave ψ on a warped product spacetime as in Proposition 6.1.1 and any $0 < t < t_0$, the following holds:*

$$\sqrt{\int_{\overline{M}} \left|\overline{\nabla}\psi(t,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}}} \leq \sqrt{\int_{\overline{M}} \left|\overline{\nabla}\psi(t_0,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}}} + \sqrt{2}\sqrt{E(t_0,\psi)+E_1(t_0,\psi)} \int_t^{t_0} \frac{a(t_0)^3}{a(s)^3}\, ds$$

Proof. For the sake of convenience, we denote $F(t,\psi) := \sqrt{\int_{\overline{M}} \left|\overline{\nabla}\psi(t,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}}}$. One calculates:

$$\begin{aligned}
-\frac{1}{2}\left(\partial_t\left(F(\cdot,\psi)^2\right)\right)(s) &= -\frac{1}{2}\left(\partial_t\left(\int_{\overline{M}}\left|\overline{\nabla}\psi\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}}\right)\right)(s)\\
&= \int_{\overline{M}} -\bar{g}\left(\overline{\nabla}\psi(s,\cdot), \partial_t\overline{\nabla}\psi(s,\cdot)\right) \mathrm{vol}_{\overline{M}}\\
&\leq \int_{\overline{M}} \left|\overline{\nabla}\psi(s,\cdot)\right|_{\bar{g}} \cdot \left|\partial_t\overline{\nabla}\psi(s,\cdot)\right|_{\bar{g}} \mathrm{vol}_{\overline{M}}\\
&\leq \sqrt{\int_{\overline{M}} \left|\overline{\nabla}\psi(s,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}}}\sqrt{\int_{\overline{M}} \left|\overline{\nabla}\partial_t\psi(s,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}}}\\
&\leq F(s,\psi)\sqrt{\int_{\overline{M}} \left|\partial_t\psi(s,\cdot)\cdot\partial_t\Delta\psi(s,\cdot)\right| \mathrm{vol}_{\overline{M}}}\\
&\leq F(s,\psi)\sqrt{\frac{1}{2}\int_{\overline{M}} \left|\partial_t\psi(s,\cdot)\right|^2 + \left|\partial_t\Delta\psi(s,\cdot)\right|^2 \mathrm{vol}_{\overline{M}}}\\
&\leq F(s,\psi)\sqrt{\frac{E(s,\psi)+E_1(s,\psi)}{2}}\\
&\leq F(s,\psi)\sqrt{\frac{E(t_0,\psi)+E_1(t_0,\psi)}{2}\frac{a(t_0)^3}{a(s)^3}}
\end{aligned}$$

On the other hand, one has $\frac{1}{2}\left(\partial_t\left(F(\cdot,\psi)^2\right)\right)(s) = F(t,\psi)\cdot\partial_t F(s,\psi)$. Hence,

$$-\partial_t F(s,\psi) \leq \sqrt{2}\sqrt{E(t_0,\psi)+E_1(t_0,\psi)}\frac{a(t_0)^3}{a(s)^3}$$

and thus the statement follows from integration on $s\in[t,t_0]$. ☐

Next, we need to be able relate A to our energies so that we can then control its size by that of initial data. Even though the arguments in both types are very similar, they will be treated separately for the sake of readability:

Lemma 7.2.2. *In the setting of type* 0 *in Theorem 7.1.1, one has*

$$\lim_{t\downarrow 0} a(t)^6 E(t,\psi) = \lim_{t\downarrow 0} t^{\frac{4}{\gamma}} E(t,\psi) = \left(1-\frac{2}{\gamma}\right)^2 \int_{\overline{M}} |A|^2 \operatorname{vol}_{\overline{M}}$$

Proof. One calculates for arbitrary $t>0$:

$$\begin{aligned}
t^{\frac{4}{\gamma}} E(t,\psi) &= t^{\frac{4}{\gamma}} \int_{\overline{M}} \left[|\partial_t\psi(t,\cdot)|^2 + t^{-\frac{4}{3\gamma}} \left|\overline{\nabla}\psi(t,\cdot)\right|^2_{\bar g}\right] \operatorname{vol}_{\overline{M}} \\
&= t^{\frac{4}{\gamma}} \int_{\overline{M}} \left[\left|\left(1-\frac{2}{\gamma}\right) t^{-\frac{2}{\gamma}} A + \partial_t\left(\psi - At^{1-\frac{2}{\gamma}}\right)\right|^2 + \right. \\
&\qquad + t^{-\frac{4}{3\gamma}} \bar g\left(\overline{\nabla}\left(\psi - At^{1-\frac{2}{\gamma}}\right), \overline{\nabla}\left(\psi(t,\cdot) - At^{1-\frac{2}{\gamma}}\right)\right) + \\
&\qquad \left. + 2t^{1-\frac{2}{\gamma}-\frac{4}{3\gamma}} \bar g\left(\overline{\nabla}\psi(t,\cdot), \overline{\nabla} A\right) - t^{2-\frac{4}{\gamma}-\frac{4}{3\gamma}} \left|\overline{\nabla} A\right|^2_{\bar g}\right] \operatorname{vol}_{\overline{M}} \\
&= \int_{\overline{M}} \left(1-\frac{2}{\gamma}\right)^2 |A|^2 \operatorname{vol}_{\overline{M}} + 2t^{\frac{2}{\gamma}} \int_{\overline{M}} A\cdot\partial_t\left(\psi - At^{1-\frac{2}{\gamma}}\right)\operatorname{vol}_{\overline{M}} \\
&\quad + t^{\frac{4}{\gamma}} E\left(t,\psi - At^{1-\frac{2}{\gamma}}\right) - t^{2-\frac{4}{3\gamma}} \int_{\overline{M}} \left[2\Delta A\cdot\frac{\psi}{t^{1-\frac{2}{\gamma}}} + \left|\overline{\nabla} A\right|^2_{\bar g}\right] \operatorname{vol}_{\overline{M}}
\end{aligned}$$

Taking $t\to 0$, the third summand vanishes by Theorem 7.1.2, and so does the second one since, using Hölder's inequality, its absolute value is bounded by

$$2t^{\frac{2}{\gamma}}\|A\|_{L^2(\overline{M})}\sqrt{\int_{\overline{M}}\left|\partial_t\left(\psi - At^{1-\frac{2}{\gamma}}\right)\right|^2 \operatorname{vol}_{\overline{M}}} \leq 2\|A\|_{L^2(\overline{M})}\sqrt{t^{\frac{4}{\gamma}} E\left(t,\psi - At^{1-\frac{2}{\gamma}}\right)}.$$

As shown in Remark 6.2.2, $(t,x)\mapsto \psi(t,x)/t^{1-\frac{2}{\gamma}}$ is uniformly bounded on $[0,t_0]\times\overline{M}$. Since A is smooth by Theorem 7.1.1, all other terms in the final integral are well-defined and bounded, so the Dominated Convergence Theorem can be applied to the first summand in the final integral, in particular showing that the integral converges as t approaches 0. Because $2-\frac{4}{3\gamma}>0$ holds, the final term also vanishes, proving the lemma. ☐

Lemma 7.2.3. *In the type* -1 *setting of Theorem 7.1.1, the following holds:*

$$\lim_{t\to 0} a(t)^6 E(t,\psi) = \int_{\overline{M}} |A|^2 \operatorname{vol}_{\overline{M}}$$

Proof. As earlier, denote $h(t) = \int_t^\infty a(s)^{-3}\, ds$ and then calculate:

$$\begin{aligned}
a(t)^6 E(t,\psi) =& a(t)^6 \int_{\overline{M}} |\partial_t \psi(t,\cdot)|^2 + a(t)^{-2} \left|\overline{\nabla}\psi(t,\cdot)\right|^2_{\bar{g}} \mathrm{vol}_{\overline{M}} \\
=& a(t)^6 \int_{\overline{M}} \left[\left| \partial_t (\psi - Ah) - a(t)^{-3} \cdot A \right|^2 + a(t)^{-2} \left| \overline{\nabla} (\psi - Ah) \right|^2_{\bar{g}} \right. \\
& \left. + 2a(t)^{-2} h(t) \bar{g}\left(\overline{\nabla}\psi, \overline{\nabla} A\right) - a(t)^{-2} h(t)^2 \left|\overline{\nabla} A\right|^2_{\bar{g}} \right] \mathrm{vol}_{\overline{M}} \\
=& a(t)^6 E(t, \psi - Ah) - 2a(t)^3 \int_{\overline{M}} A \cdot \partial_t (\psi - Ah)\, \mathrm{vol}_{\overline{M}} + \int_{\overline{M}} |A|^2 \mathrm{vol}_{\overline{M}} \\
& - a(t)^4 h(t)^2 \int_{\overline{M}} \left[\frac{\psi}{h} \cdot \Delta A + \left|\overline{\nabla} A\right|^2_{\bar{g}} \right] \mathrm{vol}_{\overline{M}}
\end{aligned} \tag{28}$$

The first term vanishes by Theorem 7.1.3, and so does the second one by the same Hölder-argument as in the previous proof. Regarding the final term, $\psi/h = \hat{\psi}$ converges to A pointwise by definition and is uniformly bounded by Remark 6.2.5, so the integral remains finite in the limit by the Dominated Convergence Theorem. Furthermore, by Lemma 4.2.1, as t approaches 0, $a(t)^4 = \mathcal{O}\left(t^{\frac{8}{3\gamma}}\right)$ and $h(t)^2 = \mathcal{O}\left(t^{2-\frac{4}{\gamma}}\right)$. Hence, the prefactor asymptotically behaves like $t^{2-\frac{4}{3\gamma}}$ and in particular converges to zero, so the entire summand does as well. Since all terms beside $\|A\|^2_{L^2(\overline{M})}$ now vanish in the limit, the statement follows. □

With these lemmata now in hand, we can use the previous energy estimates to construct sufficient conditions that A does not vanish. Again, for the sake of simplicity, we first start out with type 0:

Theorem 7.2.4. *Suppose that, under the assumptions of Theorem 7.1.1 for type* 0 *warped products, for sufficiently small* $t_0 > 0$, $\partial_t \psi(t_0,\cdot)$ *is not identically zero and there exists some* $\varepsilon \in (0,1)$ *such that*

$$\varepsilon \left[1 - G t_0^{2-\frac{4}{3\gamma}}\right] \int_{\overline{M}} |\partial_t \psi(t_0,\cdot)|^2 \mathrm{vol}_{\overline{M}} > G t_0^{2-\frac{4}{3\gamma}} \int_{\overline{M}} |\partial_t \Delta \psi(t_0,\cdot)|^2 \mathrm{vol}_{\overline{M}} \tag{29}$$

and

$$\begin{aligned}
(1-\varepsilon) \left[1 - G t_0^{2-\frac{4}{3\gamma}}\right] t_0^{\frac{4}{3\gamma}} \int_{\overline{M}} |\partial_t \psi(t_0,\cdot)|^2 \mathrm{vol}_{\overline{M}} > \\
> \left(1 + G t_0^{2-\frac{4}{3\gamma}}\right) \int_{\overline{M}} \left|\overline{\nabla}\psi(t_0,\cdot)\right|^2_{\bar{g}} \mathrm{vol}_{\overline{M}} + G t_0^{2-\frac{4}{3\gamma}} \int_{\overline{M}} \left|\overline{\nabla}\Delta\psi(t_0,\cdot)\right|^2_{\bar{g}} \mathrm{vol}_{\overline{M}}
\end{aligned} \tag{30}$$

hold, where $G := \frac{32}{3\gamma\left(1-\frac{2}{\gamma}\right)^2} \left(\frac{3\gamma}{8} - \frac{2}{1+\frac{2}{3\gamma}} + \frac{1}{2-\frac{4}{3\gamma}}\right) = \frac{4}{1-\left(\frac{2}{3\gamma}\right)^2} > 0$. *Then* $\|A\|_{L^2(\overline{M})} > 0$.

Proof. Applying the results (21) and (22) from the energy-flux approach to the original energy estimates, it follows that

$$a(t)^6 E(t,\psi) = a(t_0)^6 E(t_0,\psi) - 4 \int_t^{t_0} \int_{\overline{M}_s} \dot{a}(s) \left|\overline{\nabla}\psi(s,\cdot)\right|^2_{\bar{g}} \mathrm{vol}_{\overline{M}_s} ds$$

Thus, recalling $\mathrm{vol}_{\overline{M}_s} = a(s)^3 \mathrm{vol}_{\overline{M}}$ and using Lemma 7.2.1 for a lower bound, these estimates follow:

$$\begin{aligned}
a(t)^6 E(t,\psi) &= a(t_0)^6 E(t_0,\psi) - \int_t^{t_0} 4\dot{a}(s)a(s)^3 \int_{\overline{M}} \left|\overline{\nabla}\psi(s,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}}\, ds \\
&\geq a(t_0)^6 E(t_0,\psi) - \int_t^{t_0} 4\dot{a}(s)a(s)^3 \left(\sqrt{\int_{\overline{M}} \left|\overline{\nabla}\psi(t_0,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}}} + \right. \\
&\quad \left. + \sqrt{2}\sqrt{E(t_0,\psi)+E_1(t_0,\psi)} \cdot \int_s^{t_0} \frac{a(t_0)^3}{a(r)^3}\, dr \right)^2 ds \\
&\geq a(t_0)^6 E(t_0,\psi) - 8 \left(\int_{\overline{M}} \left|\overline{\nabla}\psi(t_0,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}} \right) \int_t^{t_0} \dot{a}(s)a(s)^3\, ds \\
&\quad - 16 a(t_0)^6 [E(t_0,\psi)+E_1(t_0,\psi)] \int_t^{t_0} \dot{a}(s)a(s)^3 \left(\int_s^{t_0} a(r)^{-3}\, dr \right)^2 ds \\
&= a(t_0)^6 E(t_0,\psi) - 2 \left(\int_{\overline{M}} \left|\overline{\nabla}\psi(t_0,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}} \right) \left(a(t_0)^4 - a(t)^4 \right) \\
&\quad - 16 a(t_0)^6 (E(t_0,\psi)+E_1(t_0,\psi)) \int_t^{t_0} \dot{a}(s)a(s)^3 \left(\int_s^{t_0} a(r)^{-3}\, dr \right)^2 ds
\end{aligned} \tag{31}$$

By Lemma 7.2.2, the left hand side converges to $\left(1-\frac{2}{\gamma}\right)^2 \|A\|^2_{L^2(\overline{M})}$, so it only needs to be shown that the right hand side is strictly greater than zero as $t \to 0$. One quickly collects

$$\left(\int_s^{t_0} a(r)^{-3}\, dr \right)^2 = \left(\int_s^{t_0} r^{\frac{2}{\gamma}}\, dr \right)^2 = \left(\frac{t_0^{1-\frac{2}{\gamma}} - s^{1-\frac{2}{\gamma}}}{1 - 2/\gamma} \right)^2$$

and

$$\begin{aligned}
\dot{a}(s)a(s)^3 \left(\int_s^{t_0} a(r)^{-3}\, dr \right)^2 &= \frac{2}{3\gamma} s^{\left(\frac{2}{3\gamma}-1\right)+\frac{2}{\gamma}} \left(\frac{t_0^{1-\frac{2}{\gamma}} - s^{1-\frac{2}{\gamma}}}{1-\frac{2}{\gamma}} \right)^2 \\
&= \frac{2}{3\gamma\left(1-\frac{2}{\gamma}\right)^2} \left(s^{\frac{8}{3\gamma}-1} t_0^{2-\frac{4}{\gamma}} - 2 s^{\frac{2}{3\gamma}} t_0^{1-\frac{2}{\gamma}} + s^{1-\frac{4}{3\gamma}} \right).
\end{aligned}$$

Thus, when taking the integral over $[0,t_0]$, the respective summands are as follows:

$$\begin{aligned}
\int_0^{t_0} s^{\frac{8}{3\gamma}-1} t_0^{2-\frac{4}{\gamma}}\, ds &= \frac{3\gamma}{8} t_0^{\frac{8}{3\gamma}} t_0^{2-\frac{4}{\gamma}} = \frac{3\gamma}{8} t_0^{2-\frac{4}{3\gamma}} \\
\int_0^{t_0} s^{\frac{2}{3\gamma}} t_0^{1-\frac{2}{\gamma}}\, ds &= \frac{1}{1+\frac{2}{3\gamma}} t_0^{1+\frac{2}{3\gamma}} t_0^{1-\frac{2}{3\gamma}} = \frac{1}{1+\frac{2}{3\gamma}} t_0^{2-\frac{4}{3\gamma}} \\
\int_0^{t_0} s^{1-\frac{4}{3\gamma}}\, ds &= \frac{1}{2-\frac{4}{3\gamma}} t_0^{2-\frac{4}{3\gamma}}
\end{aligned}$$

After taking the limit $t \to 0$, the right hand side of (31) now becomes

$$\begin{aligned}
&t_0^{\frac{4}{\gamma}} E(t_0,\psi) - 2t_0^{\frac{8}{3\gamma}} \int_{\overline{M}} \left|\overline{\nabla}\psi(t_0,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}} \\
&- \frac{32}{3\gamma} \frac{t_0^{\frac{4}{\gamma}} (E(t_0,\psi) + E_1(t_0,\psi))}{\left(1-\frac{2}{\gamma}\right)^2} \left(\frac{3\gamma}{8} - \frac{2}{1+\frac{2}{3\gamma}} + \frac{1}{2-\frac{4}{3\gamma}}\right) t_0^{2-\frac{4}{3\gamma}} \\
=& t_0^{\frac{4}{\gamma}} E(t_0,\psi) - 2t_0^{\frac{8}{3\gamma}} \int_{\overline{M}} \left|\overline{\nabla}\psi(t_0,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}} - G(E(t_0,\psi) + E_1(t_0,\psi)) t_0^{2+\frac{8}{3\gamma}} \\
=& t_0^{\frac{4}{\gamma}} \left(1 - Gt_0^{2-\frac{4}{3\gamma}}\right) \int_{\overline{M}} |\partial_t\psi(t_0,\cdot)|^2 \mathrm{vol}_{\overline{M}} \\
&- Gt_0^{\frac{4}{\gamma}} t_0^{2-\frac{4}{3\gamma}} \int_{\overline{M}} |\partial_t\Delta\psi(t_0,\cdot)|^2 \mathrm{vol}_{\overline{M}} \\
&- t_0^{\frac{8}{3\gamma}} \left(1 + Gt_0^{2-\frac{4}{3\gamma}}\right) \int_{\overline{M}} \left|\overline{\nabla}\psi(t_0,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}} - Gt_0^{\frac{8}{3\gamma}} t_0^{2-\frac{4}{3\gamma}} \int_{\overline{M}} \left|\overline{\nabla}\Delta\psi(t_0,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}}
\end{aligned}$$

One now easily checks that if the conditions (29) and (30) are satisfied, this is positive, i.e. $\|A\|_{L^2(\overline{M})} > 0$.
Finally, one should check that G can be simplified as stated:

$$\begin{aligned}
G &= \frac{32}{3\gamma\left(1-\frac{2}{\gamma}\right)^2} \left(\frac{3\gamma}{8} - \frac{2}{1+\frac{2}{3\gamma}} + \frac{1}{2-\frac{4}{3\gamma}}\right) \\
&= \frac{32}{\left(1-\frac{2}{\gamma}\right)^2} \left(\frac{1}{8} - \frac{2}{3\gamma+2} + \frac{1}{2(3\gamma-2)}\right) \\
&= 32 \left(\frac{(3\gamma+2)(3\gamma-2) - 16(3\gamma-2) + 4(3\gamma+2)}{8(3\gamma+2)(3\gamma-2)\left(1-\frac{2}{\gamma}\right)^2}\right) \\
&= 4 \left(\frac{9\gamma^2 - 4 - 48\gamma + 32 + 12\gamma + 8}{(9\gamma^2-4)\left(1-\frac{2}{\gamma}\right)^2}\right) \\
&= 4 \frac{\gamma^2 - 4\gamma + 4}{\left(\gamma^2 - \frac{4}{9}\right)\left(1-\frac{2}{\gamma}\right)^2} = \frac{4(\gamma-2)^2}{\left(\gamma^2 - \frac{4}{9}\right)\left(1-\frac{2}{\gamma}\right)^2} = \frac{4}{1-\left(\frac{2}{3\gamma}\right)^2}
\end{aligned}$$

□

It should be noted that this criterion is marginally weaker than the corresponding Theorem 1.1.2 in [1] when applying it to flat FLRW spacetimes due to the smallness of t_0 not only being necessary to keep the weights of spatial terms comparatively small, but additionally to ensure $1 - Gt_0^{2-\frac{4}{3\gamma}} > 0$. However, it is significantly stronger in general in the sense that is does not rely on the spatial geometry in any way, while [1] needed this to then leverage energy estimates on coordinate derivatives of ψ. Furthermore, since we essentially only used the energy estimates as well as some arithmetic manipulations on the scale factor, this already indicates how we can extend such a criterion to the framework associated with $\kappa = -1$:

Theorem 7.2.5. *Suppose that, under the assumptions of Theorem 7.1.1 for type -1, for sufficiently small $t_0 > 0$, $\partial_t\psi(t_0,\cdot)$ is not identically zero, that there exist (γ-dependent) constants $k_1, k_2, k_3 > 0$ such that the scale factor a satisfies*

$$k_1 t^{\frac{2}{3\gamma}} \leq a(t) \leq k_2 t^{\frac{2}{3\gamma}} \text{ and} \tag{32}$$

$$0 \leq \dot{a}(t) \leq k_3 \frac{2}{3\gamma} t^{\frac{2}{3\gamma}-1} \tag{33}$$

for all $t \in (0, t_0]$ and that there is some $\varepsilon \in (0,1)$ such that

$$\varepsilon \left[1 - \tilde{G} t_0^{2-\frac{4}{3\gamma}}\right] \int_{\overline{M}} |\partial_t\psi(t_0,\cdot)|^2 \,\mathrm{vol}_{\overline{M}} > \tilde{G} t_0^{2-\frac{4}{3\gamma}} \int_{\overline{M}} |\partial_t\Delta\psi(t_0,\cdot)|^2 \,\mathrm{vol}_{\overline{M}} \tag{34}$$

and

$$\begin{aligned}(1-\varepsilon)&\left[1 - \tilde{G} t_0^{2-\frac{4}{3\gamma}}\right] a(t_0)^2 \int_{\overline{M}} |\partial_t\psi(t_0,\cdot)|^2 \,\mathrm{vol}_{\overline{M}} > \\ &> \left(1 + \tilde{G} t_0^{2-\frac{4}{3\gamma}}\right) \int_{\overline{M}} \left|\overline{\nabla}\psi(t_0,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}} + \tilde{G} t_0^{2-\frac{4}{3\gamma}} \int_{\overline{M}} \left|\overline{\nabla}\Delta\psi(t_0,\cdot)\right|_{\bar{g}}^2 \mathrm{vol}_{\overline{M}}\end{aligned} \tag{35}$$

hold, where $\tilde{G} := \frac{4}{1-\left(\frac{2}{3\gamma}\right)^2}\frac{k_3 k_2^3}{k_1^6} > 0$. Then $\|A\|_{L^2(\overline{M})} > 0$.

Remark 7.2.6. It should be noted that, by Lemma 4.2.1, suitable k_1 and k_2 exist for $t_0 > 0$ small enough due to the asymptotic behaviour near the Big Bang, and an appropriate $k_3 > 0$ then also exists since, by (12), the following holds for small enough $t \in (0, t_0)$:

$$\dot{a}(t) = \sqrt{\frac{8\pi B}{3} a(t)^{2-3\gamma} + 1} \lesssim \sqrt{\frac{8\pi B}{3} t^{\frac{4}{3\gamma}-2} + 1} \lesssim t^{\frac{2}{3\gamma}-1}$$

Proof of Theorem 7.2.5. Up to (31), the proof is identical to the type 0 setting, where the limit of the left hand side even converges precisely to $\|A\|^2_{L^2(\overline{M})}$ by Lemma 7.2.3, and the only thing that needs to be done is to track how the constants incurred from (32) and (33) affect when the right hand side of (31) is positive. Hence, one checks:

$$\int_t^{t_0} \dot{a}(s) a(s)^3 \left(\int_s^{t_0} a(r)^{-3}\,dr\right)^2 ds \leq \frac{k_3 k_2^3}{k_1^6} \int_t^{t_0} \frac{2}{3\gamma} s^{\frac{2}{3\gamma}-1} s^{\frac{2}{\gamma}} \left(\int_s^{t_0} r^{-\frac{2}{\gamma}}\,dr\right)^2 ds$$

One now performs precisely the same calculations as in type 0 and the right hand side of (31) becomes

$$\begin{aligned}
&a(t_0)^6 E(t_0,\psi) - 2a(t_0)^4 \int_{\overline{M}} \left|\overline{\nabla}\psi(t_0,\cdot)\right|^2_{\bar g} \mathrm{vol}_{\overline{M}} \\
&- \frac{32}{3\gamma}\frac{a(t_0)^6(E(t_0,\psi)+E_1(t_0,\psi))}{\left(1-\frac{2}{\gamma}\right)^2}\left(\frac{3\gamma}{8} - \frac{2}{1+\frac{2}{3\gamma}} + \frac{1}{2-\frac{4}{3\gamma}}\right)\frac{k_3k_2^3}{k_1^6} t_0^{2-\frac{4}{3\gamma}} \\
=\,& a(t_0)^6 E(t_0,\psi) - 2a(t_0)^4 \int_{\overline{M}} \left|\overline{\nabla}\psi(t_0,\cdot)\right|^2_{\bar g} \mathrm{vol}_{\overline{M}} - \tilde G a(t_0)^6 t_0^{2-\frac{4}{3\gamma}}\left(E(t_0,\psi)+E_1(t_0,\psi)\right) \\
=\,& a(t_0)^6\left(1-\tilde G t_0^{2-\frac{4}{3\gamma}}\right)\int_{\overline{M}} |\partial_t\psi(t_0,\cdot)|^2 \mathrm{vol}_{\overline{M}} \\
&- \tilde G t_0^{2-\frac{4}{3\gamma}} a(t_0)^6 \int_{\overline{M}} |\partial_t\Delta\psi(t_0,\cdot)|^2 \mathrm{vol}_{\overline{M}} \\
&- a(t_0)^4\left(1+\tilde G t_0^{2-\frac{4}{3\gamma}}\right)\int_{\overline{M}} \left|\overline{\nabla}\psi(t_0,\cdot)\right|^2_{\bar g} \mathrm{vol}_{\overline{M}} - a(t_0)^4 \tilde G t_0^{2-\frac{4}{3\gamma}} \int_{\overline{M}} \left|\overline{\nabla}\Delta\psi(t_0,\cdot)\right|^2_{\bar g} \mathrm{vol}_{\overline{M}}
\end{aligned}$$

Again, one now just checks that the conditions (34) and (35) ensure that this is strictly larger than zero, proving the statement. □

Remark 7.2.7. In slightly less technical terms, Theorems 7.2.4 and 7.2.5 show that, if the L^2-norm of the first order time derivative sufficiently dominates L^2-norms of spatial derivatives of up to third order on some close enough initial data hypersurface $\overline{M}_{t_0}$ – i.e. if the wave is "velocity term dominated (VTD)" near the Big Bang –, then the wave exhibits the maximal possible blow-up. Put even more simply, **small spatial inhomogeneities close to the Big Bang don't significantly influence the asymptotic behaviour of the wave** compared to homogeneous waves. Note that one easily sees from the form of G and $\tilde G$ that this criterion becomes harder to satisfy towards "weaker" Big Bang singularities as $\gamma \downarrow \frac{2}{3}$.

Finally, all criteria so far could only make statements on whether A is globally identical to zero or not, fundamentally relying on the closedness of $\overline{M}$ to utilize (among other things) the divergence theorem and thus relying on global information. However, one can also formulate a ($\left(\overline{M},\bar g\right)$-dependent) criterion on whether A is pointwise non-vanishing:

Theorem 7.2.8. *Consider the setup of Theorem 7.1.1. Let* $K>0$ *be such that* $\|\varphi\|^2_{C(\overline{M})} \le K^2\left(\|\varphi\|^2_{L^2(\overline{M})} + \|\Delta\varphi\|^2_{L^2(\overline{M})}\right)$ *for all* $\varphi \in C^\infty\left(\overline{M}\right)$. *Further, let* $\varepsilon>0$, $|C| > \frac{K}{1-\frac{2}{\gamma}}\varepsilon$ *(resp.* $|C|>K\varepsilon$*) for type* 0 *(resp. type* -1*) case and* $\psi_{hom}(t,x) := C\cdot t^{1-\frac{2}{\gamma}}$ *(resp.* $\psi_{hom}(t,x) = C\cdot\int_t^\infty a(s)^{-3}\,ds$*) be homogeneous waves (see Remark 4.3.4). Then, if*

$$a(t_0)^6\left[E\left(t_0,\psi-\psi_{hom}\right) + E\left(t_0,\Delta\left(\psi-\psi_{hom}\right)\right)\right] \le \varepsilon^2$$

holds for some $t_0>0$, *A is non-vanishing.*

Proof. Only type 0 will be proven since type -1 follows identically, exchanging $t^{1-\frac{2}{\gamma}}$ with h and adapting for the differences in scaling that causes.
First, note that a suitable $K > 0$ exists by Corollary 5.2.6. Further, ψ_h and hence also $\psi - \psi_h$ are smooth waves. In particular, we obtain

$$\left(1 - \frac{2}{\gamma}\right)^2 \|A - C\|^2_{L^2(\overline{M})} = \lim_{t \to 0} a(t)^6 E(t, \psi - \psi_{hom}) \leq a(t_0)^6 E(t_0, \psi - \psi_{hom}).$$

As shown in the proof of Theorem 7.1.1,

$$\Delta\left(\frac{\psi - \overline{\psi}}{t^{1-\frac{2}{\gamma}}}\right) \to \Delta(A - C) = \Delta A$$

holds as $t \to 0$ since $\psi - \psi_h$ converges to $A - C$ in $C^2\left(\overline{M}\right)$, so energy estimates equally apply here and we obtain with Proposition 6.1.1:

$$\begin{aligned}\left(1 - \frac{2}{\gamma}\right)^2 &\left(\|A - C\|^2_{L^2(\overline{M})} + \|\Delta(A - C)\|^2_{L^2(\overline{M})}\right) \\ \leq a(t_0)^6 &\left[E\left(t_0, \psi - \psi_{hom}\right) + E\left(t_0, \Delta\left(\psi - \psi_{hom}\right)\right)\right] \leq \varepsilon^2\end{aligned}$$

By definition of K, it now follows that

$$|A(x) - C|^2 \leq K^2 \left(\|A - C\|^2_{L^2(\overline{M})} + \|\Delta\left(A - C\right)\|^2_{L^2(\overline{M})}\right) \leq \frac{K^2}{\left(1 - \frac{2}{\gamma}\right)^2}\varepsilon^2$$

and thus by assumption

$$|A(x)| \geq |C| - \frac{K}{1 - \frac{2}{\gamma}}\varepsilon > 0\,.$$

□

In short, this again verifies the intuition that, **if a wave is almost homogeneous on an initial data hypersuface, it remains so approaching the Big Bang** and hence has almost the same leading order asymptotics. This criterion is however less precise than the global criteria since it explicitly needs the rather crude embedding constant K that also depends on $\bar{g}$, and as the VTD-behaviour is far less explicit.

7.3. Asymptotics in the stiff case

For scale factors associated with stiff fluids, the previous strategy doesn't quite work because, as mentioned in Remark 6.2.5, one can a priori at most achieve a logarithmic pointwise bound on the L^1-norm of the time derivative of the rescaled waves, so the absolute continuity argument that was used to prove the existence of A in Theorem 7.1.1 fails a priori. However, we can employ a more explicit calculation to at least show the asymptotic behaviour qualitatively:

Theorem 7.3.1. *If ψ is a smooth wave over a warped product spacetime (M, g) of type 0 or -1, with scale factor a associated to $\gamma = 2$, then there exist $A \in C^\infty\left(\overline{M}\right)$, $r \in C^\infty(M)$ such that, for type 0,*

$$\psi(t, x) = A(x)\log(t) + r(t, x)$$

holds with $r(t, x) = o(log(|t|))$ approaching 0, and for type -1

$$\psi(t, x) = A(x) \int_t^\infty a(s)^{-3}\, ds + r(t, x),$$

is satisfied, where $r(t, x) = o\left(\int_t^{t_0} a(s)^{-3}\, ds\right)$ as $t \to 0$. It then also immediately follows (using Lemma 4.2.1 in type -1 warped products) that

$$\psi(t, x) = \mathcal{O}\left(|\log(t)|\right) \;\; (t \to 0)$$

Proof. From the re-arranged wave equation in Corollary 4.3.3, we have (since $a(t) > 0$ is satisfied for all $t > 0$)

$$\begin{aligned} \partial_t^2\psi + 3\frac{\dot{a}}{a}\partial_t\psi &= a^{-2}\Delta\psi \\ \Leftrightarrow\; a^3\partial_t\dot{\psi} + 3\dot{a}a^2\dot{\psi} &= a\Delta\psi \\ \Leftrightarrow\; \partial_t\left(a^3\dot{\psi}\right) &= a\Delta\psi \end{aligned}$$

By integration, we obtain

$$\dot{\psi}(t, x) = a(t_0)^3\dot{\psi}(t_0, x)a(t)^{-3} - a(t)^{-3}\int_t^{t_0} a(s)\Delta\psi(s, x)\, ds$$

for some $t_0 > 0$. Set $L = t_0$ for type 0 and $L = \infty$ for type -1. Then, again by integration and re-arranging (at first only formally), one obtains

$$\begin{aligned} \psi(t, x) =& \psi(t_0, x) - a(t_0)^3\partial_t\psi(t_0, x)\int_t^{t_0} a(s)^{-3}\, ds + \int_t^{t_0} a(s)^{-3}\left(\int_s^{t_0} a(r)\Delta\psi(r, x)\, dr\right)\, ds \\ =& \left(\int_t^L a(s)^{-3}\, ds\right)\left(-a(t_0)^3\partial_t\psi(t_0, x) + \int_0^{t_0} a(r)\Delta\psi(r, x)\, dr\right) \\ &- \left(\int_{t_0}^L a(s)^{-3}\, ds\right)\left(-a(t_0)^3\partial_t\psi(t_0, x) + \int_0^{t_0} a(r)\Delta\psi(r, x)\, dr\right) + \psi(t_0, x) \\ &- \int_t^{t_0}\int_0^s a(s)^{-3}a(r)\Delta\psi(r, x)\, dr\, ds \end{aligned} \tag{36}$$

Of course, this rearrangement is only allowed if $r \mapsto a(r)\Delta\psi(r, x)$ is integrable on $(0, t_0]$, which we will now check: By Corollary 6.1.3 for $N = 1$, one knows that

$$|\Delta\psi(r, x)| \le |\Delta\psi(t_0, x)| + Ca(t_0)^3\left(\int_r^{t_0} a(s)^{-3}\, ds\right)\left(\sqrt{E_1(t_0, \psi)} + \sqrt{E_2(t_0, \psi)}\right)$$

is satisfied for some $\bar{g}$-dependent constant C. For the sake of this argument, this information is simplified by possibly updating C and working with the estimate

$$|\Delta\psi(r,x)| \leq C\left(1 + \int_r^{t_0} a(s)^{-3}\, ds\right).$$

By Lemma 4.2.1 with $\gamma = 2$, one has $a(t) \simeq t^{\frac{1}{3}}$ and hence $\int_t^{t_0} a(s)^{-3} ds = \mathcal{O}\left(|\log(t)|\right)$ for type -1 as $t \to 0$, and in type 0 one even has $a(t) = t^{\frac{1}{3}}$ for all $t > 0$ and thus $\int_t^{t_0} a(s)^{-3} ds = \log(t_0) - \log(t)$. Hence, one obtains the following (for w.l.o.g. small enough $t_0 > 0$ in type -1, and possibly updating C):

$$\begin{aligned}
\int_s^{t_0} |a(r)\Delta\psi(r,x)|\, dr &\leq C \int_s^{t_0} r^{\frac{1}{3}}(1 + |\log(r)|)\, dr \\
&\leq C\left[\frac{3}{4}\left(t_0^{\frac{4}{3}} - s^{\frac{4}{3}} + t_0^{\frac{4}{3}}|\log(t_0)| + s^{\frac{4}{3}}|\log(s)|\right) + \int_s^{t_0} \frac{3}{4} r^{\frac{1}{3}}\, dr\right] \\
&= C\left[\frac{3}{4}\left(t_0^{\frac{4}{3}}(1 + |\log(t_0)|) + s^{\frac{4}{3}}(-1 + |\log(s)|)\right) + \frac{9}{16}\left(t_0^{\frac{4}{3}} - s^{\frac{4}{3}}\right)\right]
\end{aligned}$$

As s approaches 0, this remains bounded since $s^{\alpha}|\log(s)| \to 0$ as $s \to 0$ for any $\alpha > 0$, so all our above calculations were justified. (Note that, for type -1, $L = \infty$ is allowed by Lemma 4.2.1.)
First, we now finish type -1: As already implied by (36), we set A and r as follows:

$$\begin{aligned}
A(x) &:= -a(t_0)^3 \partial_t \psi(t_0, x) + \int_0^{t_0} a(r)\Delta\psi(r,x)\, dr \\
r(t,x) &:= \psi(t_0,x) - \left(\int_{t_0}^{L} a(s)^{-3}\, ds\right)\left(-a(t_0)^3\partial_t\psi(t_0,x) + \int_0^{t_0} a(q)\Delta\psi(q,x)\, dq\right) \\
&\quad - \int_t^{t_0}\int_0^s a(s)^{-3} a(q)\Delta\psi(q,x)\, dq\, ds
\end{aligned}$$

Since ψ and a are smooth, so are A and r. To prove the statement, it only needs to be shown that r is of strictly lesser order than $\int_t^{\infty} a(s)^{-3}\, ds$, i.e. strictly lesser order than $|\log(t)|$. Obviously, this only needs to be verified for the only non-constant term in the second line. We check, along similar lines to before, w.l.o.g. for $t_0 > 0$ small enough:

$$\begin{aligned}
\left|\int_t^{t_0} a(s)^{-3}\int_0^s a(r)\Delta\psi(q,x)\, dq\, ds\right| &\leq C\int_t^{t_0}\frac{1}{s}\int_0^s q^{\frac{1}{3}}\left(1 + |\log(q)|\right)\, dq\, ds \\
&\leq C\int_t^{t_0}\frac{1}{s}\left[\frac{3}{4}s^{\frac{4}{3}} - 0 + \frac{3}{4}s^{\frac{4}{3}}|\log(s)| - 0 + \frac{3}{4}\int_0^s q^{\frac{1}{3}}\, dq\right] ds \\
&= C\int_t^{t_0}\frac{1}{s}\left[\frac{3}{2}s^{\frac{4}{3}} + \frac{9}{16}s^{\frac{4}{3}} + 0\right] ds \\
&= \frac{33}{16}C\int_t^{t_0} s^{\frac{1}{3}}\, ds \\
&\lesssim \left(t_0^{\frac{4}{3}} - t^{\frac{4}{3}}\right)
\end{aligned}$$

Thus, r remains bounded as $t \to 0$, in particular $r(t,x) = o(|\log(t)|)$ as $t \to 0$ and the statement follows.

For type -1, note that since $L = t_0$, the first summand in the second line of (36) vanishes, and one has

$$\int_t^L a(s)^{-3}\, ds = \int_t^{t_0} \frac{1}{s}\, ds = \log(t_0) - \log(t)\,.$$

Thus, (36) becomes

$$\begin{aligned}\psi(t,x) = &-\log(t)\left(-a(t_0)^3\partial_t\psi(t_0,x) + \int_0^{t_0} a(r)\Delta\psi(r,x)\, dr\right)\\ &+\log(t_0)\left(-a(t_0)^3\partial_t\psi(t_0,x) + \int_0^{t_0} a(r)\Delta\psi(r,x)\, dr\right) + \psi(t_0,x)\\ &-\int_t^{t_0}\int_0^s a(s)^{-3}a(r)\Delta\psi(r,x)\, dr\, ds\end{aligned}$$

and we analogously set

$$\begin{aligned}A(x) &:= a(t_0)^3\partial_t\psi(t_0,x) - \int_0^{t_0}\int_s^{t_0} a(r)\Delta\psi(r,x)\, dr\, ds\\ r(t,x) &:= \psi(t_0,x) + \log(t_0)\left(-a(t_0)^3\partial_t\psi(t_0,x) + \int_0^{t_0}\int_s^{t_0} a(q)\Delta\psi(q,x)\, dq\, ds\right)\\ &\quad - \int_t^{t_0}\int_0^s a(s)^{-3}a(q)\Delta\psi(q,x)\, dq\, ds\end{aligned}$$

The argument now follows identically since the only term that isn't obviously of order $o(|\log(t)|)$ approaching 0 is the same one as in type -1, where all terms also have the same asymptotic behaviour. □

Remark 7.3.2. In principle, Theorem 7.1.1 could also have been proven this way – to be more precise, considering warped product spacetimes of type 0 for the sake of simplicity and since they contain the essential asymptotic information, one has from Corollary 6.1.3 that

$$\Delta\psi(r,x) \lesssim_{\bar{g},t_0,\psi(t_0,\cdot)} 1 + t^{1-\frac{2}{\gamma}}$$

holds for all $0 < t < t_0$. In particular, since $a(t) = t^{\frac{2}{3\gamma}}$, it would follow along similar lines to above that

$$\left|\int_t^{t_0} a(q)\Delta\psi(q,x)\, dq\right| \lesssim \int_t^{t_0}\left(q^{\frac{2}{3\gamma}} + q^{1-\frac{4}{3\gamma}}\right) dq \lesssim_{t_0,\gamma} 1 - t^{1+\frac{1}{3\gamma}} + \frac{1}{2-\frac{4}{3\gamma}}\left(t_0^{2-\frac{4}{3\gamma}} - t^{2-\frac{4}{3\gamma}}\right)$$

which would remain bounded as $t \to 0$. However, the approach in Section 7.1 has the conceptual advantage that it essentially only relies on the energy estimates which (as the energy-flux approach for Proposition 6.1.1 indicates) still allow for explicit "wiggle room" that is explicitly controllable, as we calculated in section 7.2. This could then become essential for nonlinear stability analysis of FLRW spacetimes within solutions to the Einstein equations, since in particular there may be a shift away from the warped product metric structure

$$g = -dt^2 + a(t)^2 \bar{g}$$

which was fundamental to Lemma 4.3.1 and thus the initial re-arrangement that needed to be a precise equality for the argument in this section to work.

Remark 7.3.3. Sadly, one cannot reach blow-up criteria for the stiff case with the same methods as in the non-stiff setting, which we will now quickly illustrate for the framework associated with $\kappa = -1$ (similar issues occur in the setting associated with flat space): Referring to (27) and looking at the first term on the right hand side, one sees that the energy convergence now no longer holds since, by Proposition 6.2.3, one can only obtain

$$a(t)^6 h(t)^2 E\left(t, \hat{\psi}\right) \leq a(t_0)^6 E\left(t_0, \hat{\psi}\right) h(t)^2$$

which would diverge since $h(t)$ diverges logarithmically approaching $t = 0$. Thus, one would need to rescale the energy by some function approaching 0 toward the Big Bang faster than $a(t)^6$ to obtain any type of energy convergence. This rescaling would then have to be carried over the proof of Lemma 7.2.3, or more precisely (28), killing both the entire left hand side since we know that term to be bounded by Proposition 6.1.1 and also the $\|A\|^2_{L^2(\overline{M})}$-term on the right hand side used to relate the energies with A. Thus, this lemma and with it the entire approach to our global and pointwise blow-up conditions as in Theorem 7.2.5 sadly fail.

8. A brief glimpse into nonlinear stability analysis

As indicated in Chapter 1, the results from Chapter 7 aren't just of interest in terms of understanding the behaviour of waves on a fixed FLRW-background, but also in so far as they serve as a toy case for a more general (non-)linear stability analysis of Big Bang formation around FLRW-spacetimes within the Einstein Scalar-Field and Stiff-Fluid equations. To further illustrate this connection, it will be briefly illustrated how many of the arguments and results run in parallel to the nonlinear stability results obtained by J. Speck in spatial sectional $\kappa = 1$ (see [16]):

The core additional consideration that needs to be taken is having to control perturbations of the metric. However, the Einstein equations can be reformulated within the so-called 3+1-formalism in such a way that there still is a regular time-function $\hat{t}$ that has level sets $\overline{M}_t$ and with regards to which the equations can be seen as a system of evolution equations within the following geometric objects: the metric $g|_{\overline{M}_t}$ on the hypersurfaces, the second fundamental form k on $\overline{M}_t$, the lapse function $N = |g\left(\nabla\hat{t}, \nabla\hat{t}\right)|^{-\frac{1}{2}}$ and the shift function $X = \partial_{\hat{t}} - N \cdot n_{\overline{M}_t}$. These are then reformulated for rescaled versions of these variables in such a way that the suspected leading order is eliminated. From here, the arguments for these metric components work along the same conceptual lines as the energy estimates in Propositions 6.1.1, 6.2.1 and 6.2.3 – one defines a suitable energy flux adapted to the geometric framework that produces energies which, once inserting the evolution equations, can be shown to be at the very least only barely divergent a priori. This is achieved by developing the resulting integrals in terms of the scale factor that was analyzed similar to Lemma 4.2.1. In fact, one sees in [16, p.924, (9.7a)][7] that essentially the same energy flux is used for the scalar-field component as was introduced in Definition 6.1.2, up to using a rescaled metric object for the spatial component instead of the fixed background metric in our approach.

Speck then improved these estimates by commuting the equations with a global system of Killing vector fields on $\mathbb{S}^3$ in the same way that we used the fact that $\square_g$ and Δ commute to induce energy estimates of higher order, and in particular also the pointwise controls in Corollary 6.1.3, Proposition 6.2.1 and Corollary 6.2.4, as well as in the absolute continuity arguments in extracting a footprint state A. While no fully global analogue to this vector field system specifically can exist for $\kappa = -1$, the fact that the Laplace-Beltrami operator was completely sufficient for our purposes indicates that this

[7] after decoding the rescaled variables and the different convention in scale factor

may not be necessary to obtain similar statements. In particular, Speck's approach allows for a more immediate comparison of the energies with Sobolev norms by more or less counting orders of derivation, which was circumvented in the above analysis by using the ellipticity of Δ, and had to be replaced again due to the lack of global Killing fields. However, at least in "true" FLRW spacetimes, one could have added further energies that bridge this gap and allow direct control of arbitrarily high Sobolev norms, as outlined in Section A.2. This can then, in turn, yield pointwise control as per our argument in Lemma 5.1.2. However, the fact that spatial geometry was truly irrelevant for our argumentation using elliptic operators does indicate this to be approach that may be better adapted to dealing with the general types of nonlinear perturbations that need to be controlled. Returning to the approximate energy identities themselvs, a priori divergent estimates can then be improved by a bootstrapping argument to obtain sufficiently strong asymptotic control, which was not yet necessary for our toy case.

In comparing the general results of [16] to what was achieved in this thesis, while obviously the statements on stability regarding any geometric information and the actual Big Bang formation itself would still need to be shown, the waves in negative spatial curvature seem to behave very similarly, at least for this toy case, to the scalar fields in [16]: There, the rescaled variables converge to "footprint states" which they can be shown to be close to for small initial data. The latter in particular matches up well with staying pointwise close to homogeneous waves for close enough initial data in Theorem 7.2.8. Furthermore, the solutions in [16] are velocity term dominated in the sense that the footprint states arise from considering only velocity terms in the corresponding evolution equations, which is precisely how the homogeneous waves as points of comparison were obtained in Remark 4.3.4. Again, this behaviour essentially means that time derivative terms are the primary cause of blow-up, which is in accordance with the very explicit L^2 velocity term dominated behaviour in the sufficient blow-up conditions formulated in Theorems 7.2.4 and 7.2.5. The fact that the stiff case causes some slight technical issues in formulation of a sufficient blow-up condition, while not ideal, is also reflected in the results of [16], namely that specifically the rescaled time derivative only converges pointwise in terms of spatial derivatives of at least order 1 (see [16, p. 974, (19.2e)]) – the logarithmic blow-up behaviour seems to be rather ill-behaved under simple rescaling.

In short, the results of this thesis are promising indicators that a more general nonlinear stability analysis of Big Bang formation for near-FLRW spacetimes in negative sectional curvature should not only be possible, but yield equally strong results to [16] where one had $\kappa = 1$ – or, for that matter, [15] with $\kappa = 0$, which utilized a very similar approach to similar results.

A. Appendix

A.1. Basic differential geometry

For all following statements, let (N, h) be a semi-Riemannian manifold with Riemann curvature tensor R unless specified otherwise.

Normal coordinates

Definition A.1.1. An open neighbourhood U of a point $p \in N$ is called **normal** if the exponential map $\exp_p : V \to U$ is a diffeomorphism from a star-shaped neighbourhood $V \subseteq T_pM$ of 0_p onto U, and **totally normal** (often also (geodesically) convex) if it is a normal neighbourhood of all of its points. Any point in N has a totally normal neighbourhood (see [9, p. 130, Prop. 5.7]).

Lemma A.1.2 ([7, p. 21, Thm. 1.4.4])**.** *In normal coordinates centred at p, one has*

$$g_{ij}(p) = \delta_{ij},\, \Gamma^k_{ij}(0) = 0 \text{ for all } i, j, k$$

Remark A.1.3. Since $q \mapsto h_{ij}(q)$ is smooth in any coordinate system, so is $q \mapsto h^{ij}(q)$. In particular, in any normal neighbourhood U centred around p and for any $\varepsilon > 0$, there exists some open neighbourhood $U_\varepsilon \subseteq U$ of p such that, for all $q \in U_\varepsilon$,

$$\max\left(|h_{ij}(q) - \delta_{ij}|\,, \left|h^{ij}(q) - \delta^{ij}\right|\right) < \varepsilon \tag{37}$$

Remark A.1.4 ([8, p. 85, Proof of Thm. 1.9.10])**.** Let $U = \exp_p^{-1}(V)$ be an open normal neighbourhood centred in p. Consider $E = (\pi_{TM}, \exp)$, $Z = E(V)$ and $W \subseteq U$ such that $\overline{W} \times \overline{W} \subset\subset Z$. Then, denoting by $\Gamma^k_{ij}(r_1, r_2)$ the Christoffel symbols of normal coordinates centered around r_1 in r_2, it holds that, for any $\varepsilon > 0$, there exists a neighbourhood $W_\varepsilon \subseteq W$ of p such that one has, for all $q, q' \in W_\varepsilon$,

$$\left|\Gamma^k_{ij}(q, q')\right| < \varepsilon\,. \tag{38}$$

Corollary A.1.5. *For any $\varepsilon > 0$, a compact manifold (N, h) can be covered by finitely many totally normal neighbourhoods $U_1, \ldots, U_n$ such that* (37) *and* (38) *hold on any $U_i, i = 1, \ldots, n$.*

Sectional curvature

Definition A.1.6 ([9, p. 77, Lemma 3.39])**.** Let $p \in N$ and $\Pi \subseteq T_pN$ be a nondegenerate plane spanned by $v, w \in \Pi$. Then, the **sectional curvature** of the plane Π is defined by

$$\kappa(\Pi) \equiv \kappa(v,w) = \frac{h\left(R_{vw}v, w\right)}{h(v,v)h(w,w) - h(v,w)^2}$$

and is independent of choice of basis. Further (see [9, p. 78, Lemma 3.41]), κ uniquely determines R.

Example A.1.7 ([9, p. 83, Corollary 3.43])**.** (N,h) is of **constant sectional curvature** if $\kappa(\Pi) = \kappa \in \mathbb{R}$ is satisfied for any $p \in N$ and all nondegenerate planes $\Pi \subseteq T_pN$. The curvature tensor R and the Ricci tensor Ric then take the form

$$R(u,v)w = \kappa\left(h(w,u)v - h(w,v)u\right) .$$

$$\mathrm{Ric}(u,v) = \kappa(\dim(N) - 1)h(u,v)$$

Example A.1.8 ([9, p.228, Corollary 8.25])**.** The complete and simply connected Riemannian manifolds (N,h) of dimension $n \geq 2$ and of constant curvature κ are isometric to the sphere $\mathbb{S}^n(r)$ for $\kappa = \frac{1}{r^2}$, Euclidean space $\mathbb{R}^n$ for $\kappa = 0$ and hyperbolic space $H^n(r)$ for $\kappa = -\frac{1}{r^2}$.

Corollary A.1.9 ([9, p. 220, Corollary 8.11])**.** *Any space of constant sectional curvature is locally symmetric with regards to the Levi-Civita-connection ∇^N, i.e. $\nabla^N R = 0$.*

Lemma A.1.10 (Schur's Lemma)**.** *Any connected semi-Riemannian manifold (N,h) of dimension $\dim N = n \geq 3$, where the sectional $\kappa(\Pi_p) = \kappa_p$ is constant for any $p \in N$ and any nondegenerate plane $\Pi_p \subseteq T_pN$, is of constant sectional curvature.*

Proof. Starting from the second Bianchi identity (see [9, p. 76, Prop. 3.37])

$$\nabla_e R_{abcd} + \nabla_d R_{abec} + \nabla_c R_{abde} = 0,$$

one obtains after contraction in a and c that

$$\nabla_e \mathrm{Ric}_{bd} - \nabla_d \mathrm{Ric}_{be} + \nabla^a R_{abde} = 0$$

and after contracting again b and d

$$\nabla_e \mathbf{R} - \nabla^b \mathrm{Ric}_{be} - \nabla^a \mathrm{Ric}_{ae} = 0,$$

which becomes the contracted Bianchi identity

$$\frac{1}{2}\nabla \mathbf{R} = \mathrm{div}\left(\mathrm{Ric}\right)$$

after rearranging. By the assumption on the sectional curvature in any point $p \in N$, it follows that the curvature tensor takes the form

$$R_{uv}w = \kappa(p)\left(h(w,u)v - h(w,v)u\right),$$

so one has after contraction

$$\text{Ric} = \kappa(p)(n-1)h, \quad \mathbf{R} = \kappa(p)n(n-1)\,.$$

Inserting this back into the contracted Bianchi identity, it follows that

$$\frac{n(n-1)}{2}\nabla\kappa = (n-1)\nabla\kappa$$

holds since $\nabla h = 0$, which can only be true if $\nabla\kappa = 0$ since $n \neq 1, 2$. Thus, κ must be constant. □

Riemannian submanifolds and the second fundamental form

For these final collected statements, let $\left(\overline{M}, \bar{g}\right)$ be a (semi-)Riemannian submanifold of the semi-Riemannian manifold (M, g), i.e. there is an inclusion map $\iota : \overline{M} \to M$ such that $\iota^* g = \bar{g}$.

Definition A.1.11 ([9, p.98f.])**.** For any $p \in M$, one has the decomposition

$$T_pM = T_p\overline{M} \oplus T_p\overline{M}^{\perp}$$

with associated **normal projection** $\text{nor} : T_pM \to T_p\overline{M}^{\perp}$. Further, the set of **vector fields over** ι is defined as

$$\overline{\mathcal{X}}\left(\overline{M}\right) = \left\{ X \in C^\infty\left(\overline{M}, TM\right) \middle| \pi_{T\overline{M}} \circ X = \iota \right\}\,.$$

In particular, for any $Y \in \mathcal{X}(M)$, $Y \circ \iota$ is a vector field over ι, and one has the analogous orthogonal decomposition

$$\overline{\mathcal{X}}\left(\overline{M}\right) = \mathcal{X}\left(\overline{M}\right) \oplus \mathcal{X}\left(\overline{M}\right)^{\perp}$$

with associated normal projection

$$\text{nor} : \overline{\mathcal{X}}\left(\overline{M}\right) \to \mathcal{X}\left(\overline{M}\right)^{\perp}\,.$$

Definition A.1.12. Let $V, X \in \mathcal{X}(M)$ be smooth extensions of $\overline{V}, \overline{X} \in \mathcal{X}\left(\overline{M}\right)$ (these exist by [9, p.33]). The **induced connection** $\nabla : \mathcal{X}\left(\overline{M}\right) \times \overline{\mathcal{X}}\left(\overline{M}\right) \to \overline{\mathcal{X}}\left(\overline{M}\right)$ is defined by $\nabla_{\overline{V}}\overline{X} \equiv \nabla_{\overline{V}}X = (\nabla_V X) \circ \iota$. (To see that this is well-defined, see [9, p. 99, Lemma 4.1].)

Definition A.1.13. The second fundamental form

$$k : \mathcal{X}\left(\overline{M}\right) \times \mathcal{X}\left(\overline{M}\right) \to \mathcal{X}\left(\overline{M}\right)^{\perp}, \quad k\left(\overline{V}, \overline{W}\right) = \text{nor}\left(\nabla_{\overline{V}}\overline{W}\right)$$

is a symmetric and $C^\infty\left(\overline{M}\right)$-bilinear tensor ([9, p. 100]). If $\left(\overline{M}, \bar{g}\right)$ has the unit normal $n \in \mathcal{X}\left(\overline{M}\right)^{\perp}$ (after embedding along ι), one has (see [11, p.35, (3.19)])

$$k\left(\overline{V}, \overline{W}\right) = -g\left(\iota_*\overline{V}, \nabla_{\overline{W}}n\right)\,.$$

A.2. An alternative approach to energies

The goal of the energy estimates in Chapter 6 was to ultimately gain control on sufficiently high Sobolev norms of $\partial_t\psi$ and $\partial_t\hat{\psi}$, respectively. For our purposes, the energies introduced were completely sufficient even on warped product spacetimes due to the ellipticity of Δ. However, it may be instructive to consider how precise we could have made this control if needed, in particular if reverting back to "true" FLRW-spacetimes could allow for some simplifications when extending our results to nonlinear stability analysis as outlined in Chapter 8.

First, one needs to define energies capable of controlling derivatives of even order in an L^2-sense: In our proof, the "important" term in these energies is the one containing the time derivative, since this is used to extract absolute continuity of ψ and $\hat{\psi}$ respectively. Thus, for this alternative approach, we are interested in how we can find suitable energies that can be utilized to directly control spatial Sobolev norms of $\partial_t\psi(t,\cdot)$ – i.e. whether, given additional geometric assumptions, simply using L^2-norms of $\Delta^N\varphi$ and $\overline{\nabla}\Delta^N\varphi$ can sufficiently control the Sobolev norms of any $\varphi\in C^\infty\left(\overline{M}\right)$ in such a way that we can apply this to an energy conservation calculus. First, we need to construct energies containing gradients of $\partial_t\psi$ that remain bounded toward the Big Bang:

Lemma A.2.1. *We define*

$$\mathcal{E}\left(t,\overline{\nabla}\psi\right)=\int_{\overline{M}}\left|\overline{\nabla}\partial_t\psi(t,\cdot)\right|^2_{\bar{g}}+a(t)^{-2}|\Delta\psi(t,\cdot)|^2\,\mathrm{vol}_{\overline{M}}\,.$$

Then, we obtain for any warped product spacetime (M,g) as in Proposition 6.1.1 and for any wave ψ:

$$a(t)^6\mathcal{E}\left(t,\overline{\nabla}\psi\right)\leq a(t_0)^6\mathcal{E}\left(t_0,\overline{\nabla}\psi\right)$$

As previously, these estimates still hold when replacing $\overline{\nabla}\psi$ by $\overline{\nabla}\Delta^N\psi$.

Proof. As is by now almost routine, we use integration by parts and Corollary 4.3.3 to calculate the following:

$$\begin{aligned}\partial_t\mathcal{E}\left(t,\overline{\nabla}\psi\right)=&\;2\int_{\overline{M}}\Big[\bar{g}\left(\overline{\nabla}\partial_t^2\psi(t,\cdot),\overline{\nabla}\partial_t\psi(t\cdot)\right)+a(t)^{-2}\Delta\partial_t\psi(t,\cdot)\cdot\Delta\psi(t,\cdot)\\&\quad-\frac{\dot{a}(t)}{a(t)}a(t)^{-2}|\Delta\psi(t,\cdot)|^2\Big]\,\mathrm{vol}_{\overline{M}}\\=&\int_{\overline{M}}\Big[-2\partial_t^2\psi(t,\cdot)\cdot\Delta\partial_t\psi(t,\cdot)+2a(t)^{-2}\Delta\partial_t\psi(t,\cdot)\cdot\Delta\psi(t,\cdot)\\&\quad-2\frac{\dot{a}(t)}{a(t)}a(t)^{-2}|\Delta\psi(t,\cdot)|^2\Big]\,\mathrm{vol}_{\overline{M}}\\=&\int_{\overline{M}}\left[6\frac{\dot{a}(t)}{a(t)}\partial_t\psi(t,\cdot)\Delta\partial_t\psi(t,\cdot)-2\frac{\dot{a}(t)}{a(t)}a(t)^{-2}|\Delta\psi(t,\cdot)|^2\right]\,\mathrm{vol}_{\overline{M}}\\\geq&-6\frac{\dot{a}(t)}{a(t)}\mathcal{E}(t,\overline{\nabla}\psi)\end{aligned}$$

The statement now follows from the Gronwall lemma as in Proposition 6.1.1. □

Obviously, to control the H^1-norm of $\partial_t\psi$, no further argument needed now, so we directly turn our attention to the next order, after quickly introducing some notation:

Definition A.2.2. For a Riemannian manifold $\left(\overline{M},\bar{g}\right)$ and $k\in\mathbb{N}$, we call

$$\|\varphi\|_{\dot{H}^k(\overline{M})}=\sqrt{\int_{\overline{M}}\left|\overline{\nabla}^k\varphi\right|_{\bar{g}}^2\operatorname{vol}_{\overline{M}}}$$

the **homogeneous (L^2-)Sobolev norm** of order k of $\varphi\in C^\infty\left(\overline{M}\right)$.

Lemma A.2.3. *Assume that $\varphi\in C^\infty\left(\overline{M}\right)$ and that there exist some $c_1,c_2\in\mathbb{R}$ such that $-c_1\bar{g}\geq\operatorname{Ric}\geq-c_2\bar{g}$. Then, we have*

$$\|\Delta\varphi\|^2_{L^2(\overline{M})}+c_1\|\varphi\|^2_{\dot{H}^1(\overline{M})}\leq\|\varphi\|^2_{\dot{H}^2(\overline{M})}\leq\|\Delta\varphi\|^2_{L^2(\overline{M})}+c_2\|\varphi\|^2_{\dot{H}^1(\overline{M})}\,.$$

In particular, if $\overline{M}$ is a (three dimensional) manifold of constant sectional curvature κ, equality holds with $c_1=c_2=-2\kappa$.

Proof. Recall that, for any tensor $T_{k_1\dots k_m}$ of order m, one has

$$[\nabla_a,\nabla_b]\,T_{k_1\dots k_m}=-R^l_{k_1ab}T_{lk_2\dots k_m}-\dots-R^l_{k_mab}T_{k_1\dots k_{m-1}l}\qquad(39)$$

For the upper estimate, we calculate (using integration by parts and commuting derivatives with according error terms):

$$\begin{aligned}
&\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\overline{\nabla}_{i_1}\overline{\nabla}_{i_2}\varphi\overline{\nabla}_{j_1}\overline{\nabla}_{j_2}\varphi\operatorname{vol}_{\overline{M}}\\
&=-\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\overline{\nabla}_{j_1}\overline{\nabla}_{i_1}\overline{\nabla}_{i_2}\varphi\overline{\nabla}_{j_2}\varphi\operatorname{vol}_{\overline{M}}\\
&=-\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\overline{\nabla}_{j_1}\overline{\nabla}_{i_2}\overline{\nabla}_{i_1}\varphi\overline{\nabla}_{j_2}\varphi\operatorname{vol}_{\overline{M}}\\
&\overset{(39)}{=}-\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\left(\overline{\nabla}_{i_2}\overline{\nabla}_{j_1}\overline{\nabla}_{i_1}\varphi-R^k_{i_1j_1i_2}\overline{\nabla}_k\varphi\right)\overline{\nabla}_{j_2}\varphi\operatorname{vol}_{\overline{M}}\\
&=\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\overline{\nabla}_{j_1}\overline{\nabla}_{i_1}\varphi\overline{\nabla}_{i_2}\overline{\nabla}_{j_2}\varphi\operatorname{vol}_{\overline{M}}\\
&\quad-\int_{\overline{M}}\bar{g}^{i_2j_2}\bar{g}^{kl}\operatorname{Ric}_{ki_2}\overline{\nabla}_l\varphi\overline{\nabla}_{j_2}\varphi\operatorname{vol}_{\overline{M}}\\
&\leq\int_{\overline{M}}|\Delta\varphi|^2\operatorname{vol}_{\overline{M}}+c_2\int_{\overline{M}}\bar{g}^{i_2j_2}\bar{g}^{kl}\bar{g}_{ki_2}\overline{\nabla}_l\varphi\overline{\nabla}_{j_2}\varphi\operatorname{vol}_{\overline{M}}\\
&=\int_{\overline{M}}|\Delta\varphi|^2\operatorname{vol}_{\overline{M}}+c_2\int_{\overline{M}}\bar{g}^{lj_2}\overline{\nabla}_l\varphi\overline{\nabla}_{j_2}\varphi\operatorname{vol}_{\overline{M}}
\end{aligned}$$

The lower estimate obviously follows by inserting the upper Ricci bound instead of the lower one in the penultimate line. The final statement is also immediate since $\operatorname{Ric}=2\kappa\bar{g}$ holds by Example A.1.7. □

For the next order, we will restrict ourselves to constant sectional curvature entirely and only show the necessary estimate for the intended argument:

Lemma A.2.4. *Let $\left(\overline{M},\bar{g}\right)$ be a three-dimensional Riemannian manifold of constant sectional curvature κ. Then, for any $\varphi\in C^\infty\left(\overline{M}\right)$, the following holds:*

$$\|\varphi\|^2_{\dot{H}^3\left(\overline{M}\right)}\leq\left(1+2|\kappa|+4\kappa^2\right)\left(\|\varphi\|^2_{\dot{H}^1\left(\overline{M}\right)}+\|\Delta\varphi\|^2_{H^1\left(\overline{M}\right)}\right)$$

Proof. We note that spaces of constant sectional curvature are locally symmetric (see Corollary A.1.9), so we can simply pull all Riemannian curvature tensors that appear during the calculation past covariant derivative acting upon then. Now, one calculates:

$$\begin{aligned}
&\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}\overline{\nabla}_{i_1}\overline{\nabla}_{i_2}\overline{\nabla}_{i_3}\varphi\overline{\nabla}_{j_1}\overline{\nabla}_{j_2}\overline{\nabla}_{j_3}\varphi\,\mathrm{vol}_{\overline{M}}\\
=&-\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}\overline{\nabla}_{j_1}\overline{\nabla}_{i_1}\overline{\nabla}_{i_2}\overline{\nabla}_{i_3}\varphi\overline{\nabla}_{j_2}\overline{\nabla}_{j_3}\varphi\,\mathrm{vol}_{\overline{M}}\\
\overset{(39)}{=}&-\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}\overline{\nabla}_{j_1}\overline{\nabla}_{i_2}\overline{\nabla}_{i_1}\overline{\nabla}_{i_3}\varphi\overline{\nabla}_{j_2}\overline{\nabla}_{j_3}\varphi\,\mathrm{vol}_{\overline{M}}\\
&+\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}\overline{\nabla}_{j_1}\left(R^l_{i_3i_1i_2}\overline{\nabla}_l\varphi\right)\overline{\nabla}_{j_2}\overline{\nabla}_{j_3}\varphi\,\mathrm{vol}_{\overline{M}}\\
\overset{(39)}{=}&-\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}\overline{\nabla}_{i_2}\overline{\nabla}_{j_1}\overline{\nabla}_{i_1}\overline{\nabla}_{i_3}\varphi\overline{\nabla}_{j_2}\overline{\nabla}_{j_3}\varphi\,\mathrm{vol}_{\overline{M}}\\
&+\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}R^l_{i_1j_1i_2}\overline{\nabla}_l\overline{\nabla}_{i_3}\varphi\overline{\nabla}_{j_2}\overline{\nabla}_{j_3}\varphi\,\mathrm{vol}_{\overline{M}}\\
&+\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}R^l_{i_3j_1i_2}\overline{\nabla}_{i_1}\overline{\nabla}_l\varphi\overline{\nabla}_{j_2}\overline{\nabla}_{j_3}\varphi\,\mathrm{vol}_{\overline{M}}\\
&+\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}R^l_{i_3i_1i_2}\overline{\nabla}_{j_1}\overline{\nabla}_l\varphi\overline{\nabla}_{j_2}\overline{\nabla}_{j_3}\varphi\,\mathrm{vol}_{\overline{M}}\\
=&\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}\overline{\nabla}_{j_1}\overline{\nabla}_{i_1}\overline{\nabla}_{i_3}\varphi\overline{\nabla}_{i_2}\overline{\nabla}_{j_2}\overline{\nabla}_{j_3}\varphi\,\mathrm{vol}_{\overline{M}}\\
&\int_{\overline{M}}\left(-\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}\mathrm{Ric}^l_{i_2}\overline{\nabla}_l\overline{\nabla}_{i_3}\varphi\overline{\nabla}_{j_2}\overline{\nabla}_{j_3}\varphi+2R^{li_1i_2i_3}\overline{\nabla}_{i_1}\overline{\nabla}_l\varphi\overline{\nabla}_{i_2}\overline{\nabla}_{i_3}\varphi\right)\mathrm{vol}_{\overline{M}}
\end{aligned}$$

In particular, by Example A.1.7, we observe $\mathrm{Ric}^k_l=\bar{g}^{km}\cdot 2\kappa\bar{g}_{ml}=2\kappa\delta^k_l$ and estimate the first term as follows:

$$\begin{aligned}
&\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}\overline{\nabla}_{j_1}\overline{\nabla}_{i_1}\overline{\nabla}_{i_3}\varphi\overline{\nabla}_{i_2}\overline{\nabla}_{j_2}\overline{\nabla}_{j_3}\varphi\,\mathrm{vol}_{\overline{M}}\\
=&\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}\overline{\nabla}_{j_1}\overline{\nabla}_{i_3}\overline{\nabla}_{i_1}\varphi\overline{\nabla}_{i_2}\overline{\nabla}_{j_3}\overline{\nabla}_{j_2}\varphi\,\mathrm{vol}_{\overline{M}}\\
=&\int_{\overline{M}}\bar{g}^{i_1j_1}\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}\left(\overline{\nabla}_{i_3}\overline{\nabla}_{j_1}\overline{\nabla}_{i_1}\varphi-R^l_{j_1i_1i_3}\overline{\nabla}_l\varphi\right)\left(\overline{\nabla}_{j_3}\overline{\nabla}_{i_2}\overline{\nabla}_{j_2}\varphi-R^k_{j_2i_2j_3}\overline{\nabla}_k\varphi\right)\mathrm{vol}_{\overline{M}}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\overline{M}} \Big(\left|\overline{\nabla}\Delta\varphi\right|_{\bar{g}}^2 + \bar{g}^{i_3j_3}\mathrm{Ric}^l_{i_3}\overline{\nabla}_l\varphi\overline{\nabla}_{j_3}\Delta\varphi \\
&\qquad + \bar{g}^{i_3j_3}\mathrm{Ric}^k_{j_3}\overline{\nabla}_{i_3}\Delta\varphi\overline{\nabla}_k\varphi + \bar{g}^{i_3j_3}\mathrm{Ric}^l_{i_3}\mathrm{Ric}^k_{j_3}\overline{\nabla}_k\varphi\overline{\nabla}_l\varphi \Big) \,\mathrm{vol}_{\overline{M}} \\
&= \int_{\overline{M}} \Big(\left|\overline{\nabla}\Delta\varphi\right|_{\bar{g}}^2 + 4|\kappa|\bar{g}^{i_3j_3}\overline{\nabla}_{i_3}\Delta\varphi\overline{\nabla}_{j_3}\varphi + 4\kappa^2\bar{g}^{kl}\overline{\nabla}_k\varphi\overline{\nabla}_l\varphi \Big) \,\mathrm{vol}_{\overline{M}} \\
&\leq \int_{\overline{M}} \Big(\left|\overline{\nabla}\Delta\varphi\right|_{\bar{g}}^2 + \Big(2|\kappa| + 4\kappa^2\Big) \left|\overline{\nabla}\varphi\right|_{\bar{g}}^2 + 2|\kappa| \left|\overline{\nabla}\Delta\varphi\right|_{\bar{g}}^2 \Big) \,\mathrm{vol}_{\overline{M}},
\end{aligned}
$$

with the last step by applying the Cauchy-Schwarz inequality to the second and third summand. For the remaining terms, we finally compute

$$
\begin{aligned}
&\int_{\overline{M}} -\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}\mathrm{Ric}^l_{i_2}\overline{\nabla}_l\overline{\nabla}_{i_3}\varphi\overline{\nabla}_{j_2}\overline{\nabla}_{j_3}\varphi \,\mathrm{vol}_{\overline{M}} \\
&= \int_{\overline{M}} 2|\kappa|\bar{g}^{i_2j_2}\bar{g}^{i_3j_3}\overline{\nabla}_{i_2}\overline{\nabla}_{i_3}\varphi\overline{\nabla}_{j_2}\overline{\nabla}_{j_3}\varphi \,\mathrm{vol}_{\overline{M}} = 2|\kappa|\|\Delta\varphi\|^2_{L^2(\overline{M})}
\end{aligned}
$$

and

$$
\int_{\overline{M}} 2R^{li_1i_2i_3}\overline{\nabla}_{i_1}\overline{\nabla}_l\varphi\overline{\nabla}_{i_2}\overline{\nabla}_{i_3}\varphi \,\mathrm{vol}_{\overline{M}} = 0
$$

since $R^{li_1i_2i_3}$ is antisymmetric in l and i_1 while $\overline{\nabla}_{i_1}\overline{\nabla}_l\varphi$ is symmetric in l and i_1 for any $\varphi \in C^\infty\left(\overline{M}\right)$. □

Combining the last two lemmas, this shows that one can control the H^3-norm of $\varphi \in C^\infty\left(\overline{M}\right)$ by only considering L^2-norms of φ, $\overline{\nabla}\varphi$, $\Delta\varphi$ and $\overline{\nabla}\Delta\varphi$. Going back to the start, this shows that $\|\partial_t\psi(t,\cdot)\|^2_{H^3\left(\overline{M}\right)}$ is controlled by $\sum_{N=0,1}\left(E_N(t,\psi) + \mathcal{E}_N(t,\psi)\right)$ (up to κ-dependent constant). One could extend this argumentation to arbitrarily high order tensors, again simply using (39) to rearrange derivatives, then $\overline{\nabla}R = 0$ to pull all curvature terms to the front, and finally treating the resulting terms just like above.

Thus, in "true" FLRW spacetimes, one could also circumvent using ellipticity properties since there exist energies that can control arbitrarily high Sobolev norms of $\partial_t\psi$ to yield arbitrarily high regularity, in particular once considering the limit $t \to 0$ as in Chapter 7.

Abstract

Friedman-Lemaître-Robertson-Walker (FLRW) spacetimes play a central role within cosmology since they are believed to roughly describe the observable universe at large scales. Ultimately, one would like to understand whether the Big Bang formation these spacetimes exhibit is stable when perturbing around them within the Einstein Scalar-Field and Stiff-Fluid equations. As a first step approaching this complicated issue, this thesis is concerned with the blow-up behaviour of scalar waves towards the Big Bang singularity **on a fixed FLRW background** with zero or negative spatial sectional curvature.

To this end, energy estimates adapted to the respective geometries will be developed that, along with some Sobolev space and ellipticity theory on Riemannian manifolds, allow for uniform pointwise bounds on (rescaled) waves that extend to the Big Bang hypersurface. In particular, this will enable us to extract a smooth limit of the wave rescaled by its suspected leading order. Finally, for FLRW spacetimes governed by an ideal fluid that isn't stiff, sufficient conditions will be established such that blow-up of precisely this order can be achieved if the L^2-initial data is sufficiently dominated by velocity terms for a hypersurface close enough to the singularity. The analysis conceptually often parallels the methods and ideas first presented in [1], but significantly generalizes them not only by additionally analyzing the case of negative sectional curvature instead of only $\kappa = 0$ as well as that stiff fluids as far as possible, but also since a wider class of Lorentzian manifolds is considered where any curvature and homogeneity assumptions on the spatial geometry are dropped.

Zusammenfassung

Friedman-Lemaître-Robertson-Walker (FLRW) Raumzeiten spielen eine zentrale Rolle in der Kosmologie, da sie das beobachtbare Universum auf großen Skalen annähernd beschreiben. Schlussendlich würde man gerne verstehen, ob die Art und Weise, wie in diesen Zeiten ein Urknall auftritt, unter Störung innerhalb der Einstein Gleichungen stabil ist in Anwesenheit von Materie, z.B. modelliert durch ein Skalarfeld oder in Form einer steifen Flüssigkeit. Als erster Schritt in Richtung dieser Problematik beschäftigt sich diese Arbeit mit dem Blow-up-Verhalten von skalaren Wellen in Richtung des Urknalls auf einem fixierten FLRW Hintergrund bei nicht-positiver räumlicher sektionaler Krümmung.

Hierfür werden Energieabschätzungen entwickelt, die an die jeweiligen Geometrien angepasst sind. Zusammen mit etwas Sobolev- und Elliptizitätstheorie auf Riemannschen Mannigfaltigkeiten ermöglichen sie es, gleichmäßige punktweise obere Schranken für (reskalierte) Wellen zu finden, die zur Urknall-Hyperfläche fortgesetzt werden können. Insbesondere erlaubt das, einen glatten Grenzwert für die um die (vermutete) führende Ordnung reskalierte Welle zu extrahieren. Zuguterletzt werden für FLRW Raumzeiten, deren Expansion von einer idealen, aber nicht steifen Flüssigkeit angetrieben wird, hinreichende Bedingungen aufgestellt, die Blow-up von genau dieser führenden Ordnung sicherstellen, falls die L^2-Anfangsdaten auf einer raumartigen Hyperfläche nahe der Singularität ausreichend von geschwindigkeitsartigen Termen dominiert werden. Diese Analyse läuft konzeptuell oft parallel zu den Methoden und Ideen, die zuerst in [1] präsentiert wurden, aber verallgemeinert sie nicht nur signifikant durch die zusätzliche Betrachtung von $\kappa = -1$ anstatt nur $\kappa = 0$ sowie die teilweise Ausweitung der Analyse auf steife Flüssigkeiten, sondern auch dadurch, dass eine größere Klasse von Lorentz-Mannigfaltigkeiten betrachtet wird, die zwar ähnlich zu FLRW Raumzeiten sind, aber keine zusätzlichen Bedinungen an die Geometrie der zugrundeliegenden räumlichen Riemannschen Mannigfaltigkeit stellen.

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