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Abstract

According to Felix Klein's Erlangen program geometry is the study of invariants under a certain group of transformations. For example for Euclidean geometry this is the well-known Euclidean group. Analogously the less-known Laguerre geometry - which this thesis is dedicated to - is the study of invariants under the group of Laguerre transformations. Their characterizing property is that they preserve oriented hyperspheres, oriented hyperplanes and their oriented contact. Depending on the space on which those transformations operate, we will deal with hyperbolic, elliptic and Euclidean Laguerre geometry. To do this, we use quadric models, where we can identify oriented hyperplanes with points on and oriented hyperspheres with planar sections of a given quadric, the so-called Laguerre quadric. Laguerre transformations then can be lifted to projective transformations which preserve this quadric.

For this purpose it is necessary to review some basics on projective geometry and quadrics, as well as introduce some tools for the work in quadric models. This will be done in a preliminary chapter.

For hyperbolic and elliptic Laguerre geometry the quadric models are very similar, since the corresponding Laguerre quadrics are both non-degenerate. Thus we will treat those non-Euclidean Laguerre geometries in a common chapter.

For Euclidean Laguerre geometry the quadric is degenerate, making this model slightly more complicated. For this reason, we treat this case in a separate chapter. Here, we will also deal with another model of Euclidean Laguerre geometry, namely the cyclographic model, which has already been treated by earlier generations (see e.g., [1]). Particularly for the more thoroughly studied planar case, circles and lines of the plane are identified with points and planes of the space, respectively.

Another perspective on Laguerre geometry will be given as subgeometry of Lie geometry, which can also be treated in a quadric model. The points of the so-called Lie quadric represent oriented hyperspheres, oriented hyperplanes and points in the space that the Lie geometry "lives" in. The Laguerre quadric can then be recovered as a hyperplanar section of the Lie quadric, containing all the points of it that correspond to oriented hyperplanes. In this sense Laguerre transformations are special Lie transformations, preserving oriented hyperplanes. Therefore we will show how to embed the Laguerre quadric into the Lie quadric in Chapter 5.

Finally, we shall give some chosen applications to show the benefits of working with

Laguerre geometry. With this we want to show the potential of research on this geometry, without going into detail since the main goal of this thesis shall remain to give an overview of various models of Laguerre geometries.

Zusammenfassung

Nach Felix Kleins Erlanger Programm ist eine Geometrie die Invariantentheorie einer bestimmten Transformationsgruppe. So untersucht beispielsweise die wohlbekannte euklidische Geometrie Invarianten unter der Gruppe der Euklidischen Bewegungen. Analog dazu befasst sich die weniger bekannte Laguerre-Geometrie - der die vorliegende Arbeit gewidmet ist - mit Invarianten der Gruppe der sogenannten Laguerre-Transformationen. Charakterisierend für diese ist, dass sie orientierte Hyperebenen, orientierte Hyperkugeln und deren orientierten Kontakt erhalten. Je nachdem in welchem Raum diese Transformationen operieren, unterscheiden wir zwischen hyperbolischer, elliptischer und euklidischer Laguerre-Geometrie. Diese werden wir hauptsächlich in Quadrikenmodellen behandeln, in denen orientierte Hyperebenen bzw. Hyperkugeln mit Punkten auf bzw. innerhalb einer ausgezeichneten Quadrik, der sogenannten Laguerre-Quadrik, identifiziert werden können. In diesem Sinne können Laguerre-Transformationen zu projektiven Transformationen "gehoben" werden, die diese Quadrik erhalten.

Hierfür ist es offensichtlich notwendig, einige Grundlagen zur projektiven Geometrie und insbesondere zu Quadriken zu wiederholen, sowie einige neue Werkzeuge zum Arbeiten in Quadrikenmodellen einzuführen. Dies wird in einem vorbereitenden Kapitel vorgenommen.

Für hyperbolische und elliptische Laguerre-Geometrie sind die zugehörigen Quadrikenmodelle ähnlich, da die entsprechenden Laguerre-Quadriken nicht degeneriert sind. Daher werden wir diese nicht-Euklidischen Geometrien in einem gemeinsamen Kapitel behandeln.

Für euklidische Laguerre-Geometrie ist die Quadrik degeneriert, was das zugehörige Modell etwas komplizierter macht. Daher behandeln wir diesen Fall in einem separaten Kapitel. Hier werden wir zusätzlich auf ein weiteres Modell der euklidischen Laguerre-Geometrie eingehen, nämlich auf das zyklographische Modell, welches bereits von früheren Generationen studiert worden ist (siehe z.B. [1]). Insbesondere im gründlicher erforschten ebenen Fall werden hier Kreise bzw. Geraden der Ebene mit Punkten bzw. Ebenen des Raumes identifiziert.

Eine weitere Möglichkeit, Laguerre-Geometrie zu betrachten, ist als Teilgeometrie der Lie-Geometrie, welche ebenfalls in einem Quadrikenmodell behandelt werden kann. Die Punkte der sogenannten Lie-Quadrik repräsentieren orientierte Hyperkugeln, orientierte Hyperebenen und Punkte des Raumes, in dem die Lie-Geometrie "lebt". Die Laguerre-Quadrik ist damit in der Lie-Quadrik als Menge aller Punkte, die orientierten Hyperebe-

nen entsprechen, enthalten und kann als Schnitt dieser mit einer Hyperebene dargestellt werden. In diesem Sinne sind Laguerre-Transformationen spezielle Lie-Transformationen, nämlich jene, die orientierte Hyperebenen erhalten. Daher werden wir in Kapitel 5 zeigen, wie man die Laguerre-Quadrik in die Lie-Quadrik einbettet.

Abschließend behandeln wir einige ausgewählte Anwendungen, um die Vorteile des Arbeitens mit Laguerre-Geometrie zu demonstrieren. Dadurch soll das Forschungspotential dieser Geometrie aufgezeigt werden, ohne ins Detail zu gehen, da es das Hauptanliegen der vorliegenden Arbeit ist, einen Überblick über die Theorie der Laguerre-Geometrie zu geben.

Contents

1 Introduction

Laguerre geometry is often referred to as the geometry of oriented hyperplanes and hyperspheres. This is due to the fact that Laguerre transformations are defined as follows:

Definition 1.1. *Laguerre transformations* are bijective transformations that preserve

- oriented hyperplanes,
- oriented hyperspheres and
- oriented contact and non-contact between them.

The orientation of the fundamental objects can be defined by their normal vectors or by the direction in which they are traversed. For hyperspheres one can also choose a signed radius. The basic relation of oriented contact means that the objects do not only need to be tangent, but their normal vectors have to face in the same direction as well (see Figure 1.1) [12, p. 165–166]. Since in Laguerre geometry we are only interested in oriented objects and their oriented contact we will sometimes omit the word "oriented" in this thesis where no misunderstandings can arise. Also we will sometimes omit the prefix "hyper" where the dimension is clear.

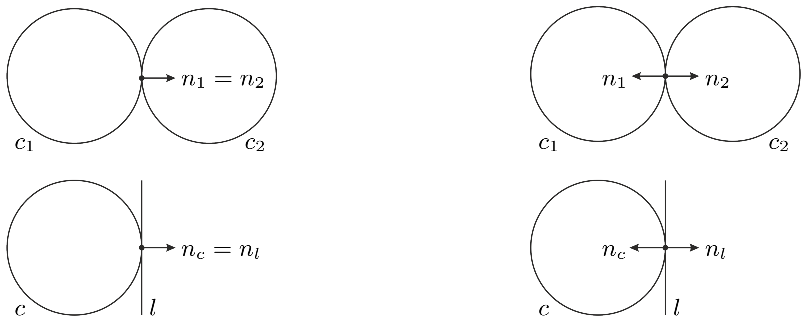


Figure 1.1: Circles and lines in oriented contact (left) and non-contact (right)

Note that in Laguerre geometry the notion of "point" does not exist. What we intuitively consider a point is a hypersphere of radius zero. Thus, under Laguerre transformations points are in general not preserved, i.e., can be mapped to "any other sphere" (we will see an example of this at the end of Chapter 4). For a point P being in oriented contact with a hyperplane p or sphere s means the usual incidence relation of the point lying on the plane or sphere, i.e., $P \in p$, or $P \in s$, respectively.

Sometimes it will be useful to consider a hypersphere as envelope of its (oriented) tangent hyperplanes [12, p. 166]. Analogously a point can be interpreted as the envelope of all lines passing through it.

In this thesis we want to study Laguerre geometry mainly with the aid of quadric models based on A. I. Bobenko et al.'s *Non-Euclidean Laguerre geometry and incircular nets* [2]. The basic idea of a quadric model is to embed the space in which the Laguerre geometry "lives", i.e., hyperbolic, elliptic or Euclidean space, in a projective space of one (or, for Lie geometry, two) dimension(s) higher and identify a certain quadric in this projective space. Hyperbolic/elliptic/Euclidean hyperplanes can then be identified with points on this given quadric and hyperspheres with hyperplanar sections of it. Therefore Laguerre transformations are induced by projective transformations which preserve this quadric.

Clearly, it is necessary to review some basics on the projective space and quadrics. We will provide these as well as introduce some tools for the work in quadric models in the following chapter, based on [2], [6] and [11].

2 Preliminaries

Definition 2.1. Let $\sim$ be an equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ with

$$x \sim y \Leftrightarrow \exists \lambda \in \mathbb{R} : x = \lambda y.$$

We call

$$\mathbb{P}^n := P(\mathbb{R}^{n+1}) := (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$$

the *n-dimensional real projective space* and P the *projectivization operator*.

Denoting the equivalence class, i.e., the span of a vector $x \in \mathbb{R}^{n+1}$ by $[x]$, we can say that points $[x]$ of the projective space $\mathbb{P}^n$ correspond to lines of $\mathbb{R}^{n+1}$ through the origin. Thus, two vectors x, y with the same direction represent the same equivalence class, i.e., the same point $[x] = [y]$ of $\mathbb{P}^n$. Choosing an arbitrary representative $x \in \mathbb{R}^{n+1} \setminus \{0\}$ we denote by

$$\boldsymbol{x} := [x] = [x_1, \ldots, x_{n+1}]$$

the *homogeneous coordinates* of the corresponding point.

Definition 2.2. Points $\boldsymbol{x}$ with $x_{n+1} = 0$ are called *points at infinity*.

The affine space $\mathbb{R}^n$ can be extended to the projective space $\mathbb{P}^n$ by adding a point at infinity to each line of $\mathbb{R}^n$. Then parallel lines share the same point at infinity, i.e., intersect in it. The set of all points at infinity is called the *hyperplane at infinity* with equation $x_{n+1} = 0$.

Definition 2.3. The projectivization $\boldsymbol{U} := P(U) \subset \mathbb{P}^n$ of a linear subspace $U \subset \mathbb{R}^{n+1}$ is a *projective subspace*. It has one dimension less than the corresponding linear subspace, i.e., $dim(\boldsymbol{U}) = dim(U) - 1$.

We denote the projective subspace spanned by two projective points $\boldsymbol{x}, \boldsymbol{y}$ by

$$\boldsymbol{x} \vee \boldsymbol{y} := [x \vee y] := P(\operatorname{span}\{x, y\})$$

and analogously for an arbitrary number of points of $\mathbb{P}^n$ [2, p. 13].

Definition 2.4. Let f be a linear automorphism of $\mathbb{R}^{n+1}$, i.e.,

$$\begin{aligned} f : \mathbb{R}^{n+1} &\to \mathbb{R}^{n+1} \\ x &\mapsto f(x) := Ax \end{aligned}$$

for an invertible matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$. Then we call a map

$$\begin{aligned} t : \mathbb{P}^n &\to \mathbb{P}^n \\ \boldsymbol{x} &\mapsto t(\boldsymbol{x}) = t([x]) := [Ax] \end{aligned}$$

which is induced by f, a *projective transformation* [11]. The group of projective transformations of $\mathbb{P}^n$ is denoted by $\mathrm{PGL}(n+1)$.

Lemma 2.5. *Projective transformations map projective subspaces to projective subspaces while preserving their dimension and incidences [2, p. 13].*

If we interpret the homogeneous coordinates $[y_1, y_2, y_3]$ of a point $\boldsymbol{y} \in \mathbb{P}^2$ as line coordinates, i.e., as coefficients in the equation $y_1x_1 + y_2x_2 + y_3x_3 = 0$ representing a line, we get the *dual projective plane* $\mathbb{P}^{2*}$. All theorems that hold for $\mathbb{P}^2$ also hold for $\mathbb{P}^{2*}$ upon exchanging the terms "points" and "lines" while preserving incidences. In analogy to the 2-dimensional case, the n-dimensional projective space $\mathbb{P}^n$ also has a dual projective space $\mathbb{P}^{n*}$, where the dual counterpart of a k-dimensional projective subspace $\boldsymbol{U}$ of $\mathbb{P}^n$ is an $(n-k-1)$-dimensional projective subspace $\boldsymbol{U}^*$ of $\mathbb{P}^{n*}$.

Definition 2.6. A map from a projective space to its dual projective space that preserves incidences is called a *duality* [11].

Definition 2.7. Let $\langle \cdot, \cdot \rangle$ be a non-zero symmetric bilinear form on $\mathbb{R}^{n+1}$. A vector $x \in \mathbb{R}^{n+1}$ is called

- *spacelike* if $\langle x, x \rangle > 0$,
- *timelike* if $\langle x, x \rangle < 0$,
- *lightlike* or *isotropic* if $\langle x, x \rangle = 0$.

The triple (r, s, t) consisting of the numbers r, s, t of spacelike, timelike and lightlike vectors in an arbitrary orthogonal basis of $\mathbb{R}^{n+1}$ w.r.t. the bilinear form $\langle \cdot, \cdot \rangle$ is called the *signature* of $\langle \cdot, \cdot \rangle$. If $t = 0$, the bilinear form $\langle \cdot, \cdot \rangle$ is called *non-degenerate* and t is omitted in the signature [2, p. 14].

Definition 2.8. The subgroup of projective transformations whose corresponding linear transformations of $\mathbb{R}^{n+1}$ preserve a bilinear form with signature (r, s, t) is called the *projective orthogonal group* $\mathrm{PO}(r, s, t)$.

Definition 2.9. Let $\langle \cdot, \cdot \rangle$ be a non-zero symmetric bilinear form on $\mathbb{R}^{n+1}$ with signature (r, s, t). Then

$$Q := \{\boldsymbol{x} \in \mathbb{P}^n \mid \langle x, x \rangle = 0\}$$

is a *quadric* in $\mathbb{P}^n$ (*conic* for $n = 2$). Its signature is well-defined as the signature of the corresponding bilinear form $\langle \cdot, \cdot \rangle$ up to interchanging r and s, which is equivalent to multiplying the equation of the quadric by -1. A quadric is called degenerate if $\langle \cdot, \cdot \rangle$ is degenerate, i.e., if $t > 0$ [2, p. 14].

Remark 2.10. A quadric is the projectivization of the set of all lightlike vectors in $\mathbb{R}^{n+1}$ w.r.t. $\langle\cdot,\cdot\rangle$. Thus a non-zero scalar multiple of $\langle\cdot,\cdot\rangle$ defines the same quadric.

Definition 2.11. For a quadric Q the set of points dual to its tangent hyperplanes is also a quadric which we call the *dual quadric* Q^* to Q. For non-degenerate Q it has the same signature as Q [2, p. 83].

Definition 2.12. A point $\boldsymbol{v}$ on a quadric Q is called a *vertex* of Q if

$$\langle v, x\rangle = 0 \quad \forall x \in \mathbb{R}^{n+1}.$$

A non-degenerate quadric contains no vertices. For a degenerate quadric its set of vertices is a projective subspace of dimension $t-1$.

Remark 2.13. A non-empty quadric that does not consist of only vertices uniquely determines its corresponding bilinear form up to a non-zero scalar multiple and vice versa [2, p. 14].

The signature of a projective subspace is defined by the signature (r,s,t) of the bilinear form restricted to the subspace. Thus choosing a signature for the bilinear form already determines the signature for every projective subspace. Since the signature of a quadric is determined by the signature of the corresponding bilinear form up to interchanging r and s, from now on (w.l.o.g.) we assume $r \geq s$ throughout this thesis.

Definition 2.14. A projective subspace which is contained in a quadric is called an *isotropic subspace.*

Lemma 2.15. *For a quadric $Q \subset \mathbb{P}^n$ of signature (r,s,t) the maximal dimension of an isotropic subspace of Q is $n-r = s+t-1$ (see, e.g. [6, p. 57]).*

Lemma 2.16. *A line which is not an isotropic line intersects a quadric in either 0,1 or 2 points.*

Definition 2.17. For a quadric Q with corresponding bilinear form $\langle\cdot,\cdot\rangle$ we call

$$Q^+ := \{\boldsymbol{x} \in \mathbb{P}^n \mid \langle x, x\rangle > 0\}$$
$$Q^- := \{\boldsymbol{x} \in \mathbb{P}^n \mid \langle x, x\rangle < 0\}$$

the *outside* and the *inside* of the quadric Q, respectively [2, p. 14].

Remark 2.18. For a quadric Q of signature (r,s,t) with

- $r = 0$ the outside Q^+ is empty.
- $s = 0$ the inside Q^- is empty.

Lemma 2.19. *For $r \neq s$ and $rs \neq 0$ the subgroup of $PGL(n+1)$ that preserves a quadric Q of signature (r,s,t) is exactly the projective orthogonal group $PO(r,s,t)$. For $r = s$ the projective orthogonal transformations and additionally all projective transformations that interchange the two sides Q^+, Q^- of the quadric preserve the quadric Q.*

Definition 2.20. Let $\boldsymbol{U} \subset \mathbb{P}^n$ be a projective subspace. We call

$$\boldsymbol{U}^\perp := \{\boldsymbol{x} \in \mathbb{P}^n \mid \langle x, y\rangle = 0 \quad \forall y \in U\}$$

the *polar subspace* of $\boldsymbol{U}$ w.r.t. the quadric Q corresponding to the bilinear form $\langle\cdot,\cdot\rangle$. If Q is non-degenerate, we have:

$$\dim(\boldsymbol{U}) + \dim(\boldsymbol{U}^\perp) = n - 1$$

[2, p. 15-16].

From the definition we see that points which lie in their own polar subspace are exactly the points of the quadric. The polar subspace of a point $\boldsymbol{x} \in Q$ is the tangent hyperplane of Q in $\boldsymbol{x}$.

Definition 2.21. Let $\boldsymbol{x} \in \mathbb{P}^n \setminus Q$ be a point not on a quadric Q. Then

$$C_Q(\boldsymbol{x}) := \bigcup_{\boldsymbol{y} \in \boldsymbol{x}^\perp \cap Q} \boldsymbol{x} \vee \boldsymbol{y}$$

is called the *tangent cone* or the *cone of contact* to Q from $\boldsymbol{x}$ [2, p. 17].

For a non-degenerate quadric Q the intersection with the polar subspace $\boldsymbol{x}^\perp$ of a point $\boldsymbol{x} \in \mathbb{P}^n \setminus Q$ is a quadric and the rulings of the tangent cone are exactly the tangents to Q from $\boldsymbol{x}$, touching Q along the quadric $\boldsymbol{x}^\perp \cap Q$ (see Figure 2.1).

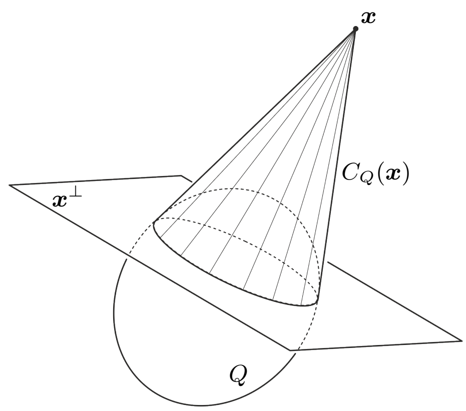


Figure 2.1: Tangent cone to a quadric in $\mathbb{P}^3$

Let us now consider a fixed quadric Q of signature (r, s, t) with corresponding bilinear form $\langle\cdot,\cdot\rangle$.

Definition 2.22. A point $\boldsymbol{p} \in \mathbb{P}^n \setminus Q$ determines three maps:

$$\begin{aligned} \sigma_{\boldsymbol{p}} : \mathbb{P}^n &\to \mathbb{P}^n \\ \boldsymbol{x} &\mapsto \sigma(\boldsymbol{x}) := [x - 2\frac{\langle x, p\rangle}{\langle p, p\rangle} p] \end{aligned} \tag{2.1}$$

which is called the *involution associated with* $\boldsymbol{p}$ or the *reflection in the hyperplane* $\boldsymbol{p}^\perp$,

$$\begin{aligned} \pi_{\boldsymbol{p}} : \mathbb{P}^n \setminus \{\boldsymbol{p}\} &\to \mathbb{P}^n \\ \boldsymbol{x} &\mapsto \pi_{\boldsymbol{p}}(\boldsymbol{x}) := [x - \frac{\langle x, p\rangle}{\langle p, p\rangle} p] = (\boldsymbol{x} \vee \boldsymbol{p}) \cap \boldsymbol{p}^\perp \end{aligned} \tag{2.2}$$

which is the *central projection associated with* $\boldsymbol{p}$ or *central projection onto* $\boldsymbol{p}^\perp$ [2, p. 25], and

$$\begin{aligned} \pi_{\boldsymbol{p}}^* : \mathbb{P}^n \setminus \{\boldsymbol{p}\} &\to \mathbb{P}^n \\ \boldsymbol{x} &\mapsto \pi_{\boldsymbol{p}}^*(\boldsymbol{x}) := \boldsymbol{x}^\perp \cap \boldsymbol{p}^\perp = (\boldsymbol{x} \vee \boldsymbol{p})^\perp \end{aligned} \tag{2.3}$$

which is the *polar projection associated with* $\boldsymbol{p}$ or *polar projection onto* $\boldsymbol{p}^\perp$ [2, p. 38].

Proposition 2.23. *The map $\sigma_{\boldsymbol{p}}$ has the following properties:*

a) $\sigma_{\boldsymbol{p}}$ fixes $\boldsymbol{p}$ and its polar hyperplane $\boldsymbol{p}^\perp$.

b) For a line through $\boldsymbol{p}$ that intersects Q in two points, $\sigma_{\boldsymbol{p}}$ interchanges the intersection points.

c) $\sigma_{\boldsymbol{p}}$ is a projective involution, i.e., $\sigma_{\boldsymbol{p}} \circ \sigma_{\boldsymbol{p}} = \mathrm{id}$ (see Figure 2.2).

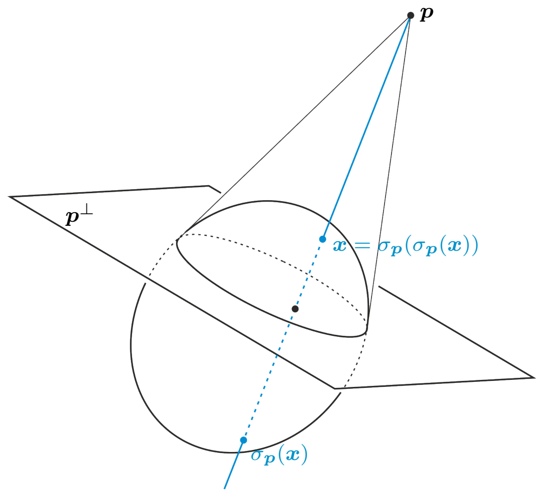


Figure 2.2: $\sigma_{\boldsymbol{p}}$ acting on a quadric in $\mathbb{P}^3$

Before describing also the properties of the other two maps we first consider the intersection of Q and $\boldsymbol{p}^\perp$ which we denote by $\tilde{Q}$, i.e.,

$$\tilde{Q} := Q \cap \boldsymbol{p}^\perp.$$

Lemma 2.24. *Let $\boldsymbol{p} \in \mathbb{P}^n \setminus Q$ be a point not on the quadric Q.*

a) If $\boldsymbol{p} \in Q^+$, then $\tilde{Q}$ is a quadric of signature $(r-1, s, t)$ and Q projects onto $\tilde{Q}^- \cup \tilde{Q}$ under $\pi_{\boldsymbol{p}}$, i.e., $\pi_{\boldsymbol{p}}(Q) = \tilde{Q}^- \cup \tilde{Q}$.

b) If $\boldsymbol{p} \in Q^-$, then $\tilde{Q}$ is a quadric of signature $(r, s-1, t)$ and Q projects onto $\tilde{Q}^+ \cup \tilde{Q}$ under $\pi_{\boldsymbol{p}}$, i.e., $\pi_{\boldsymbol{p}}(Q) = \tilde{Q}^+ \cup \tilde{Q}$ [2, p. 25–26].

This means, depending on whether $\boldsymbol{p}$ lies outside or inside of Q, the projection $\pi_{\boldsymbol{p}}$ restricted to $Q \setminus \tilde{Q}$ projects either onto the inside or the outside of the quadric $\tilde{Q}$. Since $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ for a point $\boldsymbol{x}$ can also be described as $(\boldsymbol{x} \vee \boldsymbol{p}) \cap \boldsymbol{p}^\perp$, all points on the line $\boldsymbol{x} \vee \boldsymbol{p}$ project down to the same point in $\boldsymbol{p}^\perp$. And since the line $\boldsymbol{q} \vee \boldsymbol{p}$ for a point $\boldsymbol{q} \in Q \setminus \tilde{Q}$ intersects Q in exactly a second point $\tilde{\boldsymbol{q}} \neq \boldsymbol{q}$ (according to Lemma 2.16), a point of $\tilde{Q}^\pm$ has exactly two pre-images in $Q \setminus \tilde{Q}$ under $\pi_{\boldsymbol{p}}$ which can be interchanged by $\sigma_{\boldsymbol{p}}$ (according to Proposition 2.23b). Thus, we have:

Proposition 2.25. *The map $\pi_{\boldsymbol{p}}$ has the following properties:*

a) *It fixes $\boldsymbol{p}^\perp$.*

b) *For $\boldsymbol{p} \in Q^\pm$ the restriction of $\pi_{\boldsymbol{p}}$ to $Q \setminus \tilde{Q}$ doubly covers $\tilde{Q}^\mp$.*

c) *Two points that project to the same point under $\pi_{\boldsymbol{p}}$ can be interchanged via $\sigma_{\boldsymbol{p}}$ (see Figure 2.3), thus:*

$$\pi_{\boldsymbol{p}}(Q) \simeq Q/\sigma_{\boldsymbol{p}}$$

[2, p. 26].

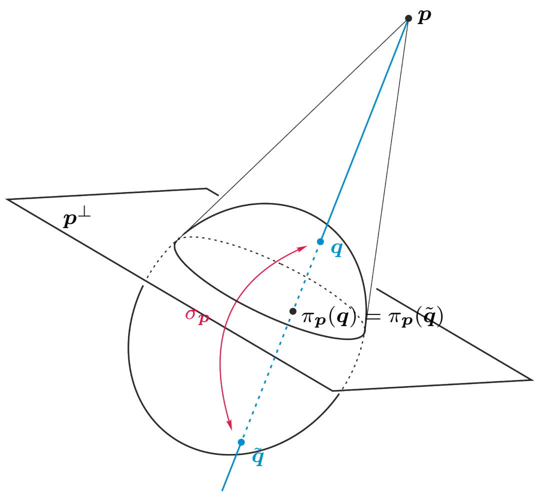


Figure 2.3: $\pi_{\boldsymbol{p}}$ acting on a quadric in $\mathbb{P}^3$

Finally, the map $\pi^*_{\boldsymbol{p}}$ maps each point $\boldsymbol{x}$ of $\mathbb{P}^n \setminus \{\boldsymbol{p}\}$ to the polar subspace of $\boldsymbol{x} \vee \boldsymbol{p}$ w.r.t. Q. Since this is exactly the polar hyperplane of the (central) projected point $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ w.r.t. $\tilde{Q}$, we have:

Proposition 2.26. *The map $\pi^*_{\boldsymbol{p}}$ has the following properties:*

a) *The polar projection $\pi^*_{\boldsymbol{p}}(\boldsymbol{x})$ is the polar hyperplane of the point $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ w.r.t. $\tilde{Q}$.*

b) *Applying $\pi^*_{\boldsymbol{p}}$ to a point $\boldsymbol{x}$ is equivalent to first applying $\pi_{\boldsymbol{p}}$ to $\boldsymbol{x}$ and then polarity w.r.t. $\tilde{Q}$ to the projected point $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ (see Figure 2.4).*

c) *For $\boldsymbol{p} \in Q^\pm$ the restriction of $\pi^*_{\boldsymbol{p}}$ to $Q \setminus \tilde{Q}$ doubly covers the set of all hyperplanes of $\boldsymbol{p}^\perp$ that are polar to points in $\tilde{Q}^\mp$.*

d) *Each two points of Q that have the same image line under $\pi^*_{\boldsymbol{p}}$ can be interchanged via $\sigma_{\boldsymbol{p}}$ [2, p. 39].*

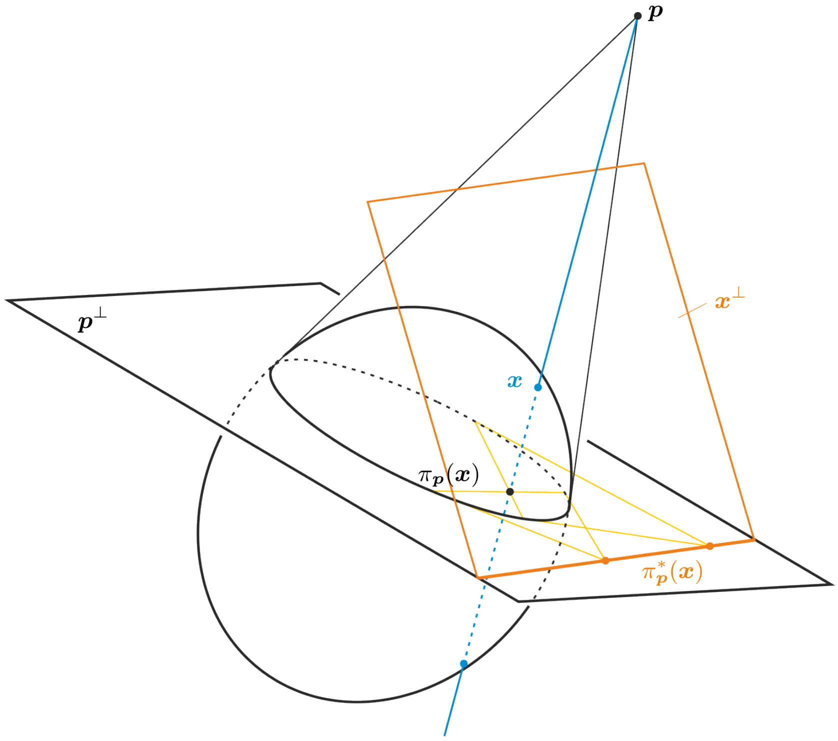


Figure 2.4: $\pi^*_{\boldsymbol{p}}$ acting on a quadric in $\mathbb{P}^3$

3 Non-Euclidean Laguerre geometry

Now that we have all the tools we need, we can finally start treating Laguerre geometry from the perspective of quadric models. As mentioned in the introduction, the basic idea of quadric models is to embed the space in which one studies Laguerre geometry, i.e., hyperbolic, elliptic or Euclidean space, into the real projective space of one dimension higher and identify oriented hyperplanes and spheres with points or hyperplanar sections of a certain quadric in this projective space. For each space form a fitting quadric has to be chosen. For the hyperbolic and elliptic case the correspondence between elements of the quadric and hyperplanes and spheres of the studied space can be established by a polar projection, or equivalently the composition of a central projection and a polarity. Due to degeneracy of the quadric in the Euclidean case this is not possible and one has to work in the dual projective space instead of applying polarity. For this reason, we will treat Non-Euclidean and Euclidean Laguerre Geometry separately, starting with the 2-dimensional hyperbolic case to give the reader a geometric intuition. We then generalize the idea to arbitrary dimensions, before treating the elliptic case analogously. This study in quadric models is based on [2].

3.1 Hyperbolic Laguerre geometry

3.1.1 2-dimensional hyperbolic Laguerre geometry

In this chapter we want to embed the hyperbolic plane $\mathcal{H}$ into 3-dimensional projective space $\mathbb{P}^3$ and find a correspondence between points on a quadric $Q \subset \mathbb{P}^3$ and oriented lines of $\mathcal{H}$, as well as between planar sections of Q and oriented circles of $\mathcal{H}$. After that we will study Laguerre transformations of $\mathcal{H}$, including the subgroup of projective transformations that induces them and how transformations of this group can be decomposed, as well as introduce an invariant of them. For that we mainly use [2, Sections 2.2 and 6.2].

For the hyperbolic plane $\mathcal{H}$ we use the Cayley-Klein model where the inside of the unit circle, called the *absolute circle*, is identified with $\mathcal{H}$ and the points on the circle, so-called *ideal points*, take the role of points at infinity. Hyperbolic lines are segments inside of the absolute circle with two ideal endpoints. Amongst hyperbolic circles we distinguish between ordinary hyperbolic circles, hyperbolic curves of constant distance to a hyperbolic line and horocycles (the differences will be discussed later in this section). The group of hyperbolic motions consists of the transformations that preserve the absolute circle [4].

To embed $\mathcal{H}$ into $\mathbb{P}^3$ we write the absolute (unit) circle as

$$\tilde{Q}:\begin{cases} x_1^2+x_2^2-x_4^2=0 \\ x_3=0 \end{cases} \tag{3.1}$$

which is

$$\tilde{Q}:\begin{cases} x^2+y^2=1 \\ z=0 \end{cases} \tag{3.2}$$

in inhomogeneous coordinates. It is the smallest circle lying on the rotational hyperboloid

$$Q: x^2+y^2-z^2=1 \tag{3.3}$$

(see Figure 3.1) [2, p. 8]. The latter one is a quadric with homogeneous equation

$$Q: x_1^2+x_2^2-x_3^2-x_4^2=0, \tag{3.4}$$

i.e., a quadric with signature $(2,2)$, containing $\tilde{Q}$ as quadric of signature $(2,1)$. Thus, we found a quadric in $\mathbb{P}^3$ containing the absolute circle of $\mathcal{H}$. Now we show how to identify points and planar sections of Q with oriented lines and circles of $\mathcal{H}$.

Correspondence between points/planar sections of Q and oriented hyperbolic lines/circles

As mentioned above, the correspondence between elements of the quadric Q and of $\mathcal{H}$ can be established by a polar projection associated with a point $\boldsymbol{p}\in\mathbb{P}^3\setminus Q$, which raises the necessity for choosing a fitting point $\boldsymbol{p}$ (the choice of $\boldsymbol{p}$ as well as the choice of the quadric Q will be explained later). Let $\boldsymbol{p}$ be the point at infinity of the z-axis, i.e.,

$$\boldsymbol{p} := [0,0,1,0] \tag{3.5}$$

in homogeneous coordinates. Then the plane $\boldsymbol{p}^\perp$ carries the absolute circle $\tilde{Q}$, i.e., $\tilde{Q}=Q\cap\boldsymbol{p}^\perp$ (see Figure 3.1).

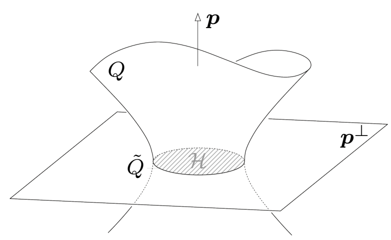


Figure 3.1: Embedding of $\mathcal{H}$ into the hyperbolic Laguerre quadric

Proposition 3.1. *The restriction of the polar projection π_p^*, defined as in (2.3) with $n=3$, to $Q\setminus\tilde{Q}$ yields a double cover of the set of all hyperbolic lines [2, p. 39].*

Proof. This follows from Proposition 2.26c. □

Geometrically, this can be explained as follows (see Figure 3.2): For an arbitrary hyperbolic line the pole w.r.t. $\tilde{Q}$ is a point outside of $\tilde{Q}$, i.e., in $\tilde{Q}^+$. Each point of

$\tilde{Q}^+$ has exactly two pre-images under $\pi_{\boldsymbol{p}}$ because the corresponding projection line is a vertical line through the point of $\tilde{Q}^+$ which has exactly two intersection points with Q, one above $\boldsymbol{p}^\perp$ and one below. So the set of poles of hyperbolic lines (i.e., Q^+) is doubly covered by $\pi_{\boldsymbol{p}}|_{Q\setminus\tilde{Q}}$. Thus, since any polar projection can be decomposed into a central projection and a polarity w.r.t. $\tilde{Q}$, the set of all hyperbolic lines is doubly covered by $\pi^*_{\boldsymbol{p}}|_{Q\setminus\tilde{Q}}$.

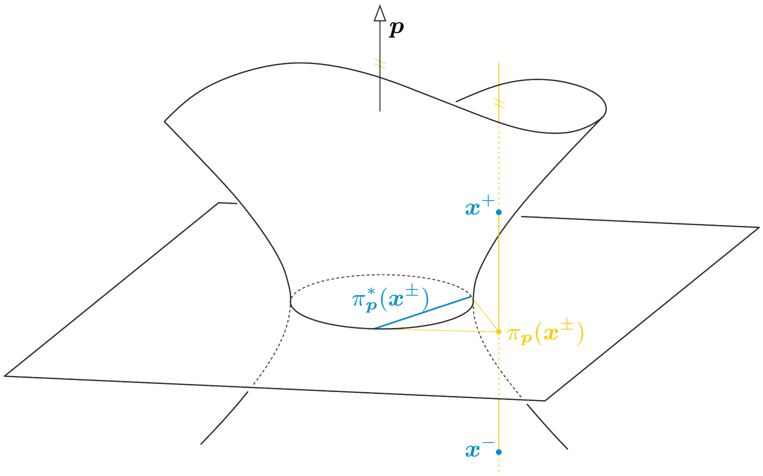


Figure 3.2: Double cover of the hyperbolic lines

Each two points of the quadric that have the same image line under $\pi^*_{\boldsymbol{p}}$ differ only by the sign of their x_3-components upon a fitting normalization, e.g. by setting one of the other components equal to 1, for all points of $\mathbb{P}^3$. From now on, we will assume such a normalization each time we distinguish orientations by the sign of a component in this thesis. Furthermore, the two pre-image points can be interchanged via the reflection $\sigma_{\boldsymbol{p}}$ in $\boldsymbol{p}^\perp$ (defined as in (2.3) with $n = 3$). Thus $(Q \setminus \tilde{Q})/\sigma_{\boldsymbol{p}}$ corresponds bijectively to the set of all hyperbolic lines. Since we are interested in oriented lines, we consider a map $\overrightarrow{\pi^*_{\boldsymbol{p}}}$ that maps each point $\boldsymbol{x}$ of $Q \setminus \tilde{Q}$ to a hyperbolic line in the same way as $\pi^*_{\boldsymbol{p}}$ but additionally endows the image line $\overrightarrow{\pi^*_{\boldsymbol{p}}}(\boldsymbol{x})$ with an orientation determined by the sign of the last component of $\boldsymbol{x}$. That is, we treat each hyperbolic line - which is the image line of exactly two points $\boldsymbol{x}^\pm = [x_1, x_2, \pm x_3, 1]$ of $Q \setminus \tilde{Q}$ under $\pi^*_{\boldsymbol{p}}$ - as consisting of two oppositely oriented lines, where $\overrightarrow{\pi^*_{\boldsymbol{p}}}$ maps the point above $\boldsymbol{p}^\perp$ (positive x_3-component) to the positively oriented line, and the point under $\boldsymbol{p}^\perp$ (negative x_3-component) to the negatively oriented line. Thus, $\overrightarrow{\pi^*_{\boldsymbol{p}}}$ yields a bijective correspondence between points on $Q \setminus \tilde{Q}$ and the set of all oriented hyperbolic lines, where $\sigma_{\boldsymbol{p}}$ acts orientation reversing.

To describe $\overrightarrow{\pi_{\boldsymbol{p}}^*}$ analytically, one can distinguish the points $\boldsymbol{x}^\pm$ as the intersection points of two different pairs of rulings from the two families of rulings on Q:

Let $R^\pm$ be the two families of rulings on Q generated by rotation of the rulings $r^\pm : x = 1, y = \pm z$ around the z-axis (see Figure 3.3). Let l be an arbitrary positively oriented hyperbolic line and denote by l_1, l_2 the ideal points (on $\tilde{Q}$) of l, in the order in which they are traversed. Let $r^\pm_{l_{1,2}}$ denote the rulings from $R^\pm$ containing $l_{1,2}$. Then the point $\boldsymbol{x}^+$ on Q corresponding to l can be written as

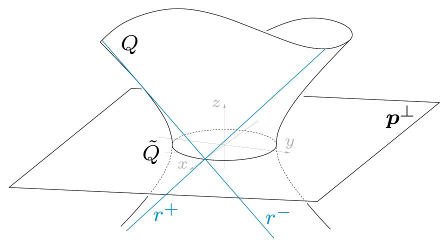


Figure 3.3: Rulings on Q generating R^+ and R^-

$$\boldsymbol{x}^+ = r^+_{l_1} \cap r^-_{l_2}. \tag{3.6}$$

For the negatively oriented line $\tilde{l}$ the corresponding point x^- is

$$\boldsymbol{x}^- = r^+_{\tilde{l}_1} \cap r^-_{\tilde{l}_2} = r^+_{l_2} \cap r^-_{l_1} \tag{3.7}$$

(see Figure 3.4) [2, p. 8]. Thus $\overrightarrow{\pi_{\boldsymbol{p}}^*}$ maps $\boldsymbol{x}^+$ to the positively oriented line and $\boldsymbol{x}^-$ to the negatively oriented one. Therefore we have:

Proposition 3.2. *The set $Q \setminus \tilde{Q}$ bijectively corresponds to the set $\vec{\mathcal{L}}$ of oriented hyperbolic lines via the map $\overrightarrow{\pi_{\boldsymbol{p}}^*} : Q \setminus \tilde{Q} \to \vec{\mathcal{L}}$, and the involution $\sigma_{\boldsymbol{p}}$ acts orientation reversing on $\vec{\mathcal{L}}$.*

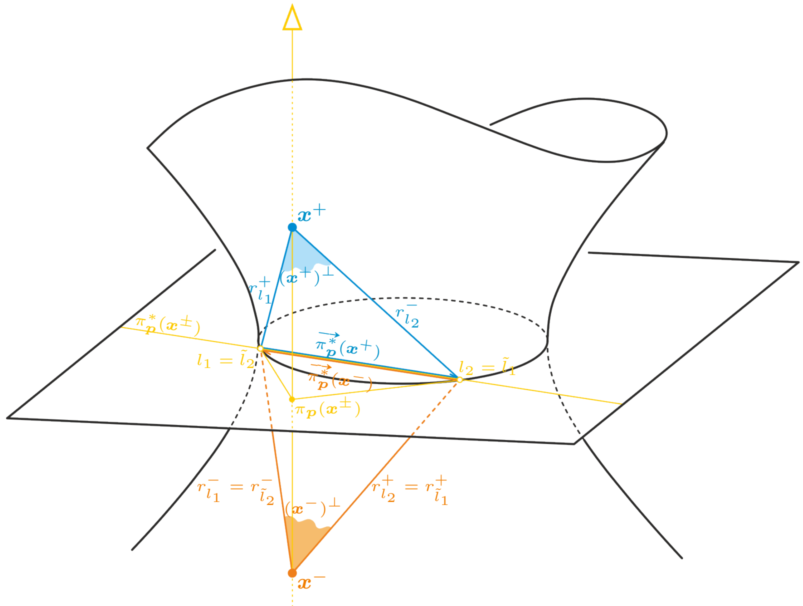


Figure 3.4: Bijective correspondence between points of Q and oriented lines of $\mathcal{H}$

Remark 3.3. Particularly points on isotropic lines of Q correspond to pencils of parallel lines of $\mathcal{H}$, since the lines on Q are exactly its rulings from which each intersects $\tilde{Q}$ in exactly one point yielding the point at infinity of these parallel lines (see Figure 3.5).

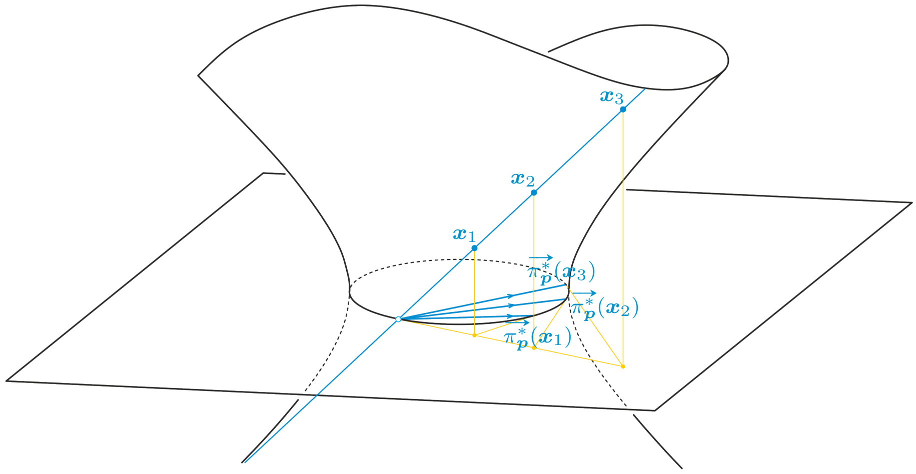


Figure 3.5: Pencil of parallel oriented lines in $\mathcal{H}$

We now want to find a correspondence between planar sections of Q and oriented hyperbolic circles. As we already know how to retrieve a point on the quadric from an oriented hyperbolic line, we might consider an oriented circle as envelope of its (oriented) tangents and lift those to their corresponding points on Q (by applying $\overrightarrow{\pi}^{*-1}_{\boldsymbol{p}}$). Geometrically, this means to find the pole of each tangent t w.r.t. $\tilde{Q}$ and lift it up or down to the quadric along its connecting line with $\boldsymbol{p}$ (which is the polar line of t w.r.t Q), depending on the orientation of t. Points corresponding to positive orientation lie on the "upper half" of Q, i.e., above $\boldsymbol{p}^\perp$, those corresponding to negative orientation lie below $\boldsymbol{p}^\perp$. The poles of tangents of a hyperbolic circle c w.r.t. $\tilde{Q}$ lie on a concentric circle in $\tilde{Q}^+$. We define:

Definition 3.4. We call the circle consisting of the poles of the tangents of a hyperbolic circle c w.r.t. $\tilde{Q}$ the *polar circle* $c^\perp$ of c [2, p. 21].

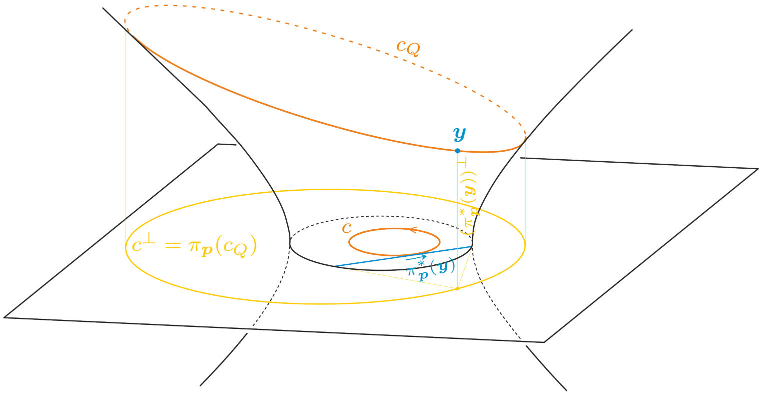
Lifting all tangents of a hyperbolic circle c to Q is therefore equivalent to intersecting the cylinder through the polar circle $c^\perp$ of c (which consists of the polar lines of the tangents of c w.r.t. Q) with the "corresponding half" of Q, yielding a conic, i.e., a planar section of Q (see Figure 3.6a). Thus, every oriented circle of $\mathcal{H}$ (bijectively) corresponds to a planar section c_Q of Q (where $\pi_{\boldsymbol{p}}(c_Q) = c^\perp$).

Remark 3.5. As points are circles (of radius zero), they must also correspond to planar sections of Q. Considering a point P as the intersection of all hyperbolic lines through it, the poles to these lines lie on a common line $l \subset \boldsymbol{p}^\perp$ that is polar to P w.r.t $\tilde{Q}$. Lifting l to Q means intersecting the vertical lines though l with Q. Thus, hyperbolic points correspond to conics on Q in z-parallel planes, i.e., hyperbolas [2, p. 11].

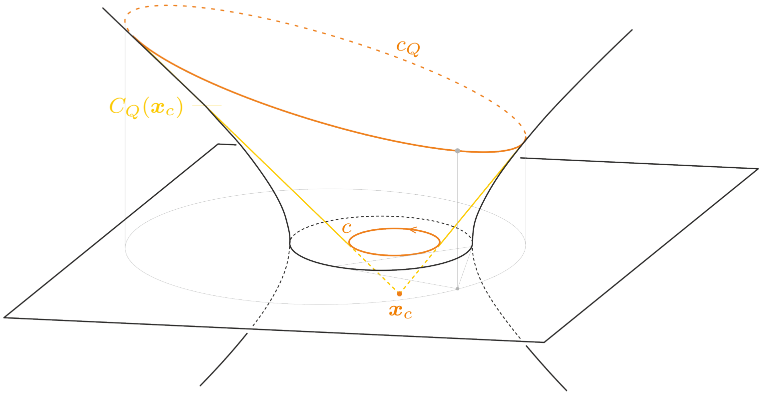
We lifted circles of $\mathcal{H}$ to Q by identifying them with their tangents and by applying $\overrightarrow{\pi_{\boldsymbol{p}}^{*}}^{-1}$ to them. In other words: applying $\overrightarrow{\pi_{\boldsymbol{p}}^{*}}$ to a planar section of Q yields the tangents of a hyperbolic circle, not its points. To get a map that actually gives us the points, we identify a planar section c_Q with the pole $\boldsymbol{x}_c$ of the plane carrying c w.r.t. Q, which is exactly the vertex of the cone of contact $C_Q(\boldsymbol{x}_c)$ touching Q along c_Q (see Figure 3.6b). As $C_Q(\boldsymbol{x}_c)$ is exactly the envelope of the tangent planes to Q along c_Q, and $\pi_{\boldsymbol{p}}^{*}$ maps each point y of c_Q to a tangent of the corresponding hyperbolic circle by intersecting the tangent plane $y^{\perp}$ in y with $\boldsymbol{p}^{\perp}$, the cone $C_Q(\boldsymbol{x}_c)$ corresponding to x_c intersects $\boldsymbol{p}^{\perp}$ exactly in the hyperbolic circle corresponding to c_Q (see Figure 3.6c). Finally, due to the fact that the vertex of a cone of contact intersecting $\boldsymbol{p}^{\perp}$ in a hyperbolic circle, i.e., a circle inside $\tilde{Q}$, must lie inside Q, we conclude: The map

$$\begin{aligned} \pi_{\boldsymbol{p}}^{*c} : Q^{-} &\to \mathcal{C} \\ \boldsymbol{x} &\mapsto \pi_{\boldsymbol{p}}^{*c}(\boldsymbol{x}) := C_Q(\boldsymbol{x}) \cap \boldsymbol{p}^{\perp}, \end{aligned}$$

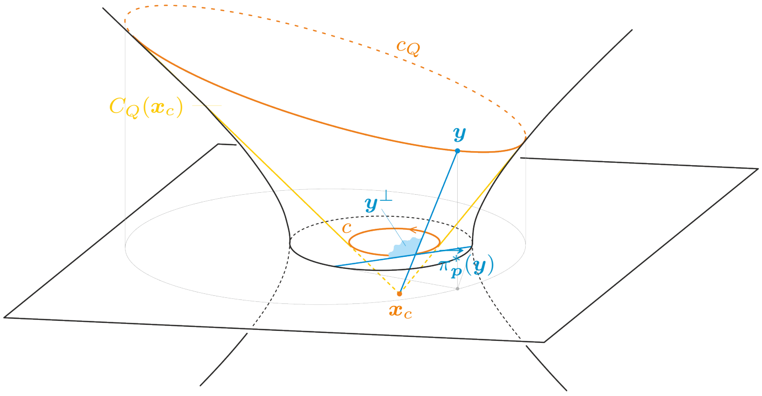
where $\mathcal{C}$ is the set of hyperbolic circles, maps each point $\boldsymbol{x}$ of Q^{-} to a circle in $\mathcal{H}$. The center of such a hyperbolic circle $\pi_{\boldsymbol{p}}^{*c}(\boldsymbol{x})$ is $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ and the polar line of $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ w.r.t. $\tilde{Q}$ is the intersection line of $\boldsymbol{p}^{\perp}$ and the plane $\boldsymbol{x}^{\perp}$ (carrying the conic corresponding to c) [2, p. 9–10, 38–41]. Just as for $\pi_{\boldsymbol{p}}^{*}$ and $\overrightarrow{\pi_{\boldsymbol{p}}^{*}}$ we want to consider the "oriented version" $\overrightarrow{\pi_{\boldsymbol{p}}^{*c}}$ of the map $\pi_{\boldsymbol{p}}^{*c}$, where the orientation of a circle $\pi_{\boldsymbol{p}}^{*c}(\boldsymbol{x})$ is again encoded in the x_3-component of $\boldsymbol{x}$ (in contrast to $\overrightarrow{\pi_{\boldsymbol{p}}^{*}}$, $\overrightarrow{\pi_{\boldsymbol{p}}^{*c}}$ maps points with negative x_3-component to positively oriented circles, since the corresponding planar sections lie above $\boldsymbol{p}^{\perp}$). The map $\overrightarrow{\pi_{\boldsymbol{p}}^{*c}}$ then yields a bijective correspondence between the points of Q^{-} and the set $\overrightarrow{\mathcal{C}}$ of oriented circles in $\mathcal{H}$.



(a) Cylinder through $c^\perp$ intersecting Q in the conic $c_Q \subset Q$ corresponding to the oriented circle $c \in \mathcal{H}$



(b) Identification of an oriented circle of $\mathcal{H}$ with the pole of the plane carrying the corresponding planar section of Q



(c) Tangent cone $C_Q(\boldsymbol{x}_c)$ intersecting $\boldsymbol{p}^\perp$ in c, $\boldsymbol{y}^\perp$ intersecting in the tangent to c corresponding to $y \in c_Q$

Figure 3.6: Bijective correspondence between planar sections of Q and oriented circles of $\mathcal{H}$

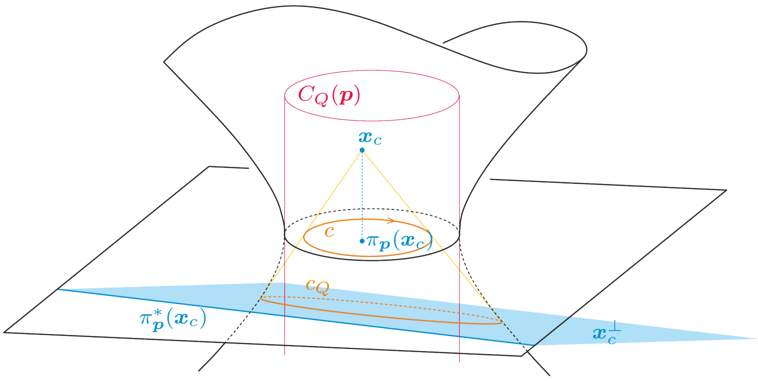
Having identified hyperbolic circles with points inside of Q let us distinguish for which points we get which type of hyperbolic circle:

As mentioned in the introduction of this section, in hyperbolic geometry we distinguish between ordinary hyperbolic circles, hyperbolic curves of constant distance to a hyperbolic line and horocycles:

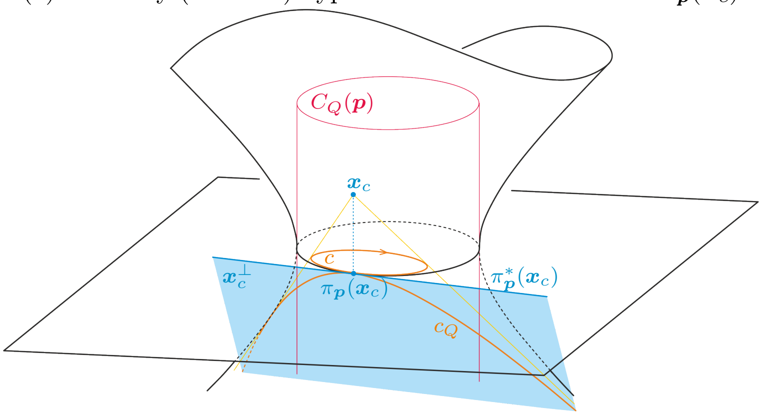
- A hyperbolic curve c of constant distance to a hyperbolic line l touches the absolute circle $\tilde{Q}$ in two points, the ideal endpoints of the line l. Therefore, the pole z of this line lies outside of $\tilde{Q}$ and we call z the center of c.
- A horocycle c has third order contact with $\tilde{Q}$ at its center z, whose polar line is the common tangent of $\tilde{Q}$ and c in z.
- (Ordinary) hyperbolic circles do not have (real) intersection points with $\tilde{Q}$. As the center of an ordinary circle lies inside of it, it also lies inside of $\tilde{Q}$, which means that its polar line does not intersect $\tilde{Q}$.

Since the center of each type of a hyperbolic circle c is obtained by applying $\pi_{\boldsymbol{p}}$ to the corresponding point $\boldsymbol{x}_c := \overrightarrow{\pi^{*}_{\boldsymbol{p}} c}^{-1}(c)$ of Q^-, the type of c is determined by:

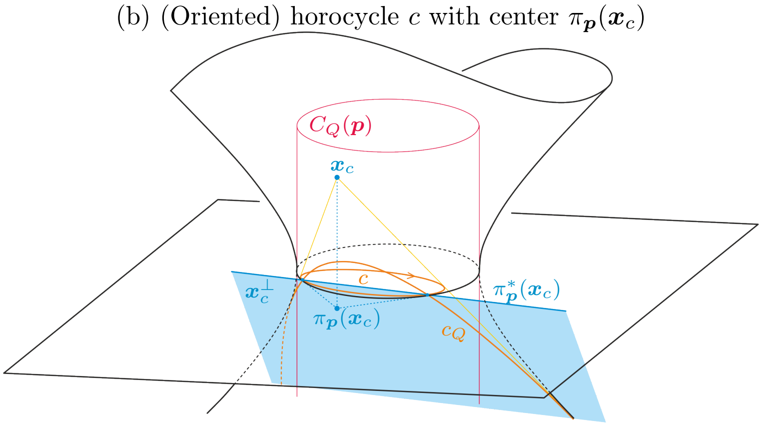
- the location of $\pi_{\boldsymbol{p}}(\boldsymbol{x}_c)$ (inside, outside or on $\tilde{Q}$), i.e., by the value of $\langle \pi_{\boldsymbol{p}}(\boldsymbol{x}_c), \pi_{\boldsymbol{p}}(\boldsymbol{x}_c) \rangle$ (smaller, greater or equal to zero),
- whether the polar plane of $\boldsymbol{x}_c$ w.r.t. Q (=plane carrying the conic corresponding to c) intersects $\boldsymbol{p}^{\perp}$ in a line (=polar line of the center of c) outside, inside or in a tangent of $\tilde{Q}$, or by
- whether $\boldsymbol{x}_c \in Q^-$ lies inside, outside or on the cylinder $C_Q(\boldsymbol{p}) : x^2 + y^2 = 1$ (see Figure 3.7) [2, p. 8–10, 41].



(a) Ordinary (oriented) hyperbolic circle c with center $\pi_{\boldsymbol{p}}(\boldsymbol{x}_c)$



(b) (Oriented) horocycle c with center $\pi_{\boldsymbol{p}}(\boldsymbol{x}_c)$



(c) (Oriented) hyperbolic curve c of constant distance to hyperbolic line $\pi^*_{\boldsymbol{p}}(\boldsymbol{x}_c)$

Figure 3.7: Types of hyperbolic circles c depending on location of $\boldsymbol{x}_c$

Going back to bijection $\overrightarrow{\pi^{*c}_{\boldsymbol{p}}}$, we notice that it has two major advantages over $\overrightarrow{\pi^*_{\boldsymbol{p}}}$:

- We get a point model also for circles, not just for lines, which for computational reasons is more convenient than to identify the circles with planar sections.
- If we allow $\overrightarrow{\pi^{*c}_{\boldsymbol{p}}}$ to operate on $Q \setminus \tilde{Q}$, we notice that it also yields a bijective correspondence for the lines, because: The cone of contact for a point $\boldsymbol{x}$ on the quadric is its tangent plane. By the definition of $\pi^*_{\boldsymbol{p}}$, intersecting this tangent plane with $\boldsymbol{p}^\perp$ while preserving the orientation, we get exactly $\overrightarrow{\pi^*_{\boldsymbol{p}}}(\boldsymbol{x})$.
  For this reason we expand the domain of $\overrightarrow{\pi^{*c}_{\boldsymbol{p}}}$ to $(Q^- \cup Q) \setminus \tilde{Q}$ and name the expanded map b (indicating the bijective correspondence for oriented circles as well as lines).

Proposition 3.6. *The map $b : (Q^- \cup Q) \setminus \tilde{Q} \to \vec{\mathcal{C}} \cup \vec{\mathcal{L}}$ is bijective, where $\vec{\mathcal{C}} \cup \vec{\mathcal{L}}$ are oriented hyperbolic circles and lines. In particular:*

- *For $\boldsymbol{x} \in Q \setminus \tilde{Q}$, $b(\boldsymbol{x})$ is a hyperbolic line with pole $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ w.r.t. $\tilde{Q}$.*
- *For $\boldsymbol{x} \in C_Q(\boldsymbol{p})^-$, i.e., $\langle \pi_{\boldsymbol{p}}(x), \pi_{\boldsymbol{p}}(x) \rangle < 0$, $b(\boldsymbol{x})$ is an (ordinary) hyperbolic circle with center $\pi_{\boldsymbol{p}}(\boldsymbol{x})$.*
- *For $\boldsymbol{x} \in Q^- \cap C_Q(\boldsymbol{p})^+$, i.e., $\langle \pi_{\boldsymbol{p}}(x), \pi_{\boldsymbol{p}}(x) \rangle > 0$, $b(\boldsymbol{x})$ is a hyperbolic curve of constant distance to a hyperbolic line with pole $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ w.r.t. Q.*
- *For $\boldsymbol{x} \in C_Q(\boldsymbol{p}) \setminus \tilde{Q}$, i.e., $\langle \pi_{\boldsymbol{p}}(x), \pi_{\boldsymbol{p}}(x) \rangle = 0$, $b(\boldsymbol{x})$ is a horocycle with center $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ [2, p. 41].*

Let us summarize what we have so far: We chose a quadric Q and a point $\boldsymbol{p} \in \mathbb{P}^3$ such that the inside of the quadric $\tilde{Q} = Q \cap \boldsymbol{p}^\perp$ could be identified with the hyperbolic plane, in the sense of points and planar sections of Q corresponding to oriented hyperbolic lines and circles, respectively. Now it also becomes clear, why we chose Q and $\boldsymbol{p}$ the way we did: For the hyperbolic plane we need the inside of the unit circle $\tilde{Q}$ (or any projectively equivalent model) which is a quadric of signature $(2, 1)$. For Q we need a quadric of one dimension higher which contains $\tilde{Q}$, i.e., Q is either a quadric of signature $(2, 2)$ or $(3, 1)$. For the latter case, according to Lemma 2.24, $\boldsymbol{p}$ would have to lie outside of Q and Q would project down to $\tilde{Q}^- \cup \tilde{Q}$ under $\pi_{\boldsymbol{p}}$. Since the projected points of Q are supposed to be the poles of hyperbolic lines w.r.t. $\tilde{Q}$, they cannot lie inside the hyperbolic plane. Thus, Q needs to have signature $(2, 2)$, i.e., it is projectively equivalent to the rotational hyperboloid (3.3). In this case, $\boldsymbol{p}$ has to lie on the inside of Q, in order for $\tilde{Q}$ to be a quadric of signature $(2, 1)$. In particular, it needs to be the pole of the plane carrying $\tilde{Q}$. Since $\tilde{Q}$ is the unit circle lying on the rotational hyperboloid (3.3) as its smallest circle, $\boldsymbol{p}$ must be the point at infinity of the z-axis.

Hyperbolic Laguerre transformations

Now we turn our attention to Laguerre transformations of the hyperbolic plane and how they can be characterized in our quadric model. Since those transformations preserve oriented hyperbolic circles and lines, and since b yields a bijective correspondence between certain points of the projective space (in particular: points of $(Q^- \cup Q) \setminus \tilde{Q} \subset \mathbb{P}^3$) and hyperbolic circles and lines, a Laguerre transformation can be lifted to a transformation of $\mathbb{P}^3$, i.e.:

$$\alpha = b \circ t \circ b^{-1}$$

for any hyperbolic Laguerre transformation α and a certain transformation t (see Figure 3.8). Our goal is to find out of which type of transformation t is.

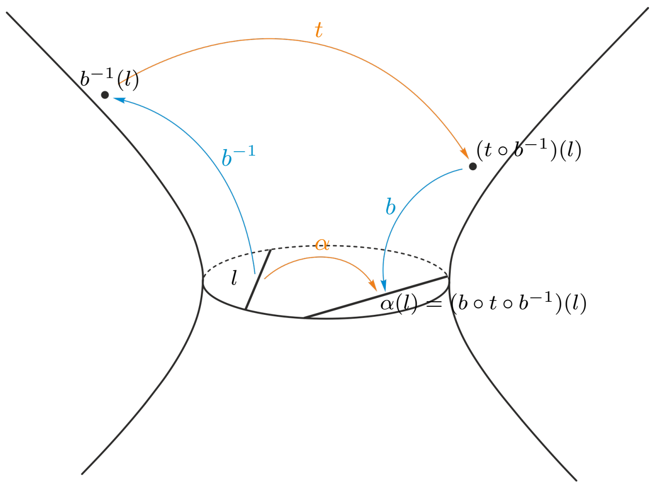


Figure 3.8: Lifting hyperbolic Laguerre transformations to projective transformations

By their definition, Laguerre transformations have to preserve a) oriented circles and lines, and b) their oriented contact. What does this mean for the transformation t of $\mathbb{P}^3$ that induces an arbitrary Laguerre transformation α?

a) Since α preserves oriented circles and lines, t must preserve Q and must not interchange the quadric's sides, as oriented lines correspond to points on Q, and oriented circles to points inside of Q.

b) The Laguerre transformation α also preserves the oriented contact of circles and lines of $\mathcal{H}$. Now the question arises: How do points of $Q^- \cup Q$ have to lie in $\mathbb{P}^3$ in order for the corresponding oriented circles and lines to be in oriented contact?

   Let us consider two circles c_1, c_2 in oriented contact with a common line l. Let $\boldsymbol{x}_{c_1} := b^{-1}(c_1), \boldsymbol{x}_{c_2} := b^{-1}(c_2), \boldsymbol{x}_l := b^{-1}(l)$ be their corresponding points in $Q^- \cup Q$. Then the tangent cones $C_Q(\boldsymbol{x}_{c_1}), C_Q(\boldsymbol{x}_{c_2})$ have a common ruling r through the contact point of c_1, c_2 and l, which is a tangent to Q in a point $\boldsymbol{q}$. The common tangent plane T_r to the tangent cones along r, which is also the tangent plane to Q in $\boldsymbol{q}$ (i.e., $T_r = \boldsymbol{q}^\perp$), intersects $\boldsymbol{p}^\perp$ in l, yielding $\boldsymbol{q} = \boldsymbol{x}_l$ (see Figure 3.9). Thus, since the ruling r carries the vertices $\boldsymbol{x}_{c_1}, \boldsymbol{x}_{c_2}$ of the tangent cones corresponding to c_1, c_2 and also carries $\boldsymbol{x}_l$, circles and lines of $\mathcal{H}$ are in oriented contact if and only if they lie on a tangent to Q. For the special case of lines being in oriented contact at a common point at infinity, we already have seen that the corresponding points on Q lie on a common ruling of Q.

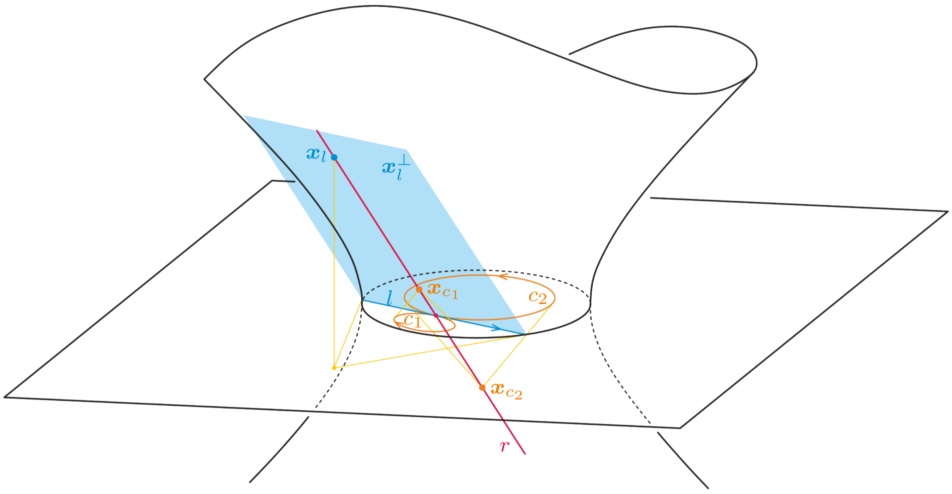


Figure 3.9: Lifting hyperbolic Laguerre transformations to projective transformations

From b) we see that the transformation t has to preserve lines of $\mathbb{P}^3$, i.e., t must be a projective transformation. From a) we know that t has to preserve Q without interchanging its sides. Hence, due to Lemma 2.19, the group of transformations that t belongs to is exactly the projective orthogonal group $\mathrm{PO}(2,2)$. In other words: Hyperbolic Laguerre transformations are induced by $\mathrm{PO}(2,2)$ [2, p. 42].

Proposition 3.7. *Every Laguerre transformation α of the hyperbolic plane can be written in the form:*

$$\alpha = b \circ t \circ b^{-1}$$

for some $t \in PO(2,2)$.

Therefore, it is of interest how $\mathrm{PO}(2,2)$ can be generated.
Let us denote by $\mathrm{PO}(2,2)_{\boldsymbol{p}}$ the subgroup of transformations of $\mathrm{PO}(2,2)$ which fix the point $\boldsymbol{p}$ [2, p. 13]. We want to show that every transformation t of $\mathrm{PO}(2,2)$ can be decomposed into two transformations of the subgroup $\mathrm{PO}(2,2)_{\boldsymbol{p}}$ and a so-called "scaling along a pencil of concentric circles". To understand this we first study which elements of $\mathbb{P}^3$ correspond to pencils of concentric circles.

Let $(c_i)_{i\in I}$ be a pencil of concentric circles in $\mathcal{H}$. For an arbitrary circle c_i of the pencil, the center is given by $\pi_{\boldsymbol{p}}(\boldsymbol{x}_i)$ for $\boldsymbol{x}_i := b^{-1}(c_i)$. The point $\boldsymbol{x}_i$ lies on the projection line $\pi_{\boldsymbol{p}}(\boldsymbol{x}_i) \vee \boldsymbol{p}$ which is vertical. Thus, the centers of all circles of the pencil must lie on this line. Since b^{-1} lifts circles of $\mathcal{H}$ to points inside of Q, every pencil of concentric hyperbolic circles corresponds to points of Q^- that lie on a vertical line of $\mathbb{P}^3$. Hence the planes carrying the conics on Q corresponding to the circles of the pencil intersect $\boldsymbol{p}^\perp$ in a common line, the polar line $\pi^*_{\boldsymbol{p}}(\boldsymbol{x}_i)$ of the center of the circles c_i (see Figure 3.10).

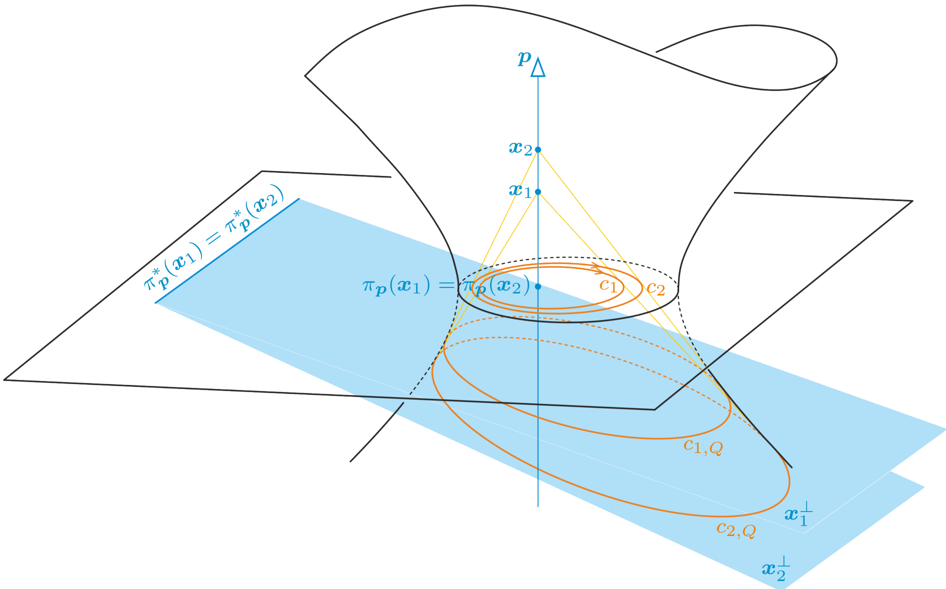


Figure 3.10: Pencil of concentric (ordinary) hyperbolic circles

Definition 3.8. Let c_1, c_2 be two circles of a pencil $(c_i)_{i\in I}$ of concentric hyperbolic circles, and let $\boldsymbol{x}_1 := b^{-1}(c_1), \boldsymbol{x}_2 := b^{-1}(c_2)$ be their corresponding points in Q^-. Then we call the unique transformation $T_{\boldsymbol{x}_1,\boldsymbol{x}_2} \in \mathrm{PO}(2,2)_{\boldsymbol{p}}$ that maps $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ and preserves each plane through the line $\boldsymbol{x}_1 \vee \boldsymbol{x}_2$ (containing $\boldsymbol{p}$) a *scaling along the pencil* $(c_i)_{i\in I}$ *of concentric hyperbolic circles* [2, p. 32–33].

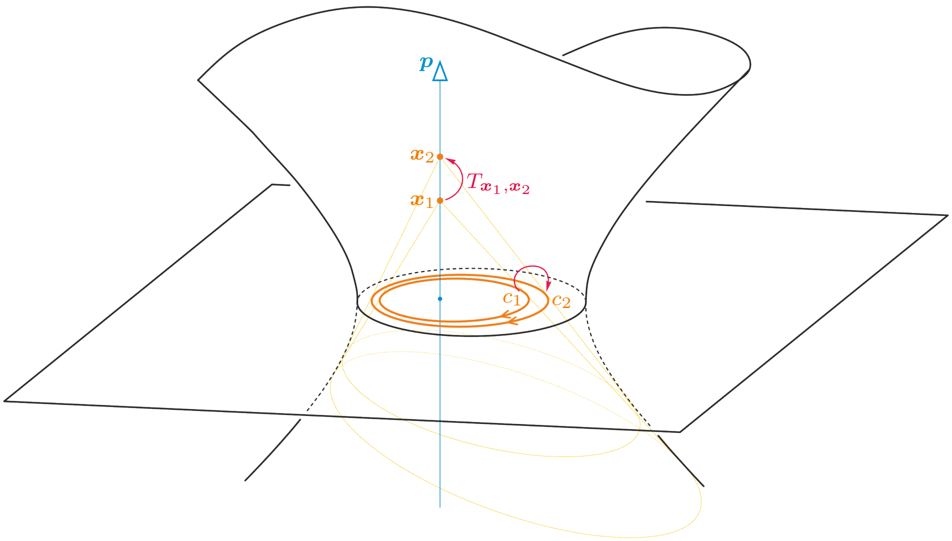


Figure 3.11: Scaling along a pencil of concentric (ordinary) hyperbolic circles

There are three types of scalings depending on the type of hyperbolic circles of the pencil. A one-parameter family of scalings along a pencil of concentric (ordinary) hyperbolic circles (see Figure 3.11)/hyperbolic curves of constant distance to a common hyperbolic line/concentric horocycles (with center on $\tilde{Q}$) can be represented by the matrices

$$T_u^c := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sin(u) & \cos(u) \\ 0 & 0 & \cos(u) & -\sin(u) \end{bmatrix},$$

$$T_u^l := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh(u) & \sinh(u) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \sinh(u) & \cosh(u) & 0 \end{bmatrix},$$

$$T_u^h := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\frac{u^2}{2} & u & \frac{u^2}{2} \\ 0 & -\frac{u^2}{2} & -u & 1-\frac{u^2}{2} \\ 0 & u & 1 & u \end{bmatrix}$$

for $u \in \mathbb{R}$, respectively [2, p. 43].

Proposition 3.9. *Every transformation $t \in \mathrm{PO}(2,2)$ can be written in the form:*

$$t = \Phi \circ T_{u_t} \circ \Psi \tag{3.8}$$

where $\Phi, \Psi \in \mathrm{PO}(2,2)_{\boldsymbol{p}}$ and $T_{u_t} \in \{T_{u_t}^c, T_{u_t}^l, T_{u_t}^h\}$ is a scaling for some $u_t \in \mathbb{R}$ [2, p. 44].

Remark 3.10. The group $\mathrm{PO}(2,2)_{\boldsymbol{p}}$ doubly covers the group of hyperbolic motions $\mathrm{PO}(2,1)$ [2, p. 42]. In particular:

$$\mathrm{PO}(2,1) \simeq \mathrm{PO}(2,2)_{\boldsymbol{p}}/\sigma_{\boldsymbol{p}} \tag{3.9}$$

[2, p. 27].

Knowing that hyperbolic Laguerre transformations are induced by transformations of $\mathrm{PO}(2,2)$, invariants of this group can bring insights into invariants of Laguerre transformations.

Definition 3.11. For a quadric $Q \subset \mathbb{P}^n$ with corresponding bilinear form $\langle \cdot, \cdot \rangle$ and two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{P}^n \setminus Q$ we call

$$K_Q(\boldsymbol{x}, \boldsymbol{y}) := \frac{\langle x, y \rangle^2}{\langle x, x \rangle \langle y, y \rangle} \tag{3.10}$$

the *Cayley-Klein distance* (short: *CK-distance*) between $\boldsymbol{x}$ and $\boldsymbol{y}$.

Lemma 3.12. *The CK-distance K_Q associated with the quadric Q is invariant under projective transformations that preserve Q [2, p. 18].*

The CK-distance associated with the absolute circle $\tilde{Q}$ induces the hyperbolic distance as follows:

Lemma 3.13. *For $\tilde{Q}$, defined as in (3.2), the equation*

$$K_{\tilde{Q}}(\boldsymbol{x}, \boldsymbol{y}) = \cosh^2 d(\boldsymbol{x}, \boldsymbol{y}) \tag{3.11}$$

defines the hyperbolic distance $d(\boldsymbol{x}, \boldsymbol{y})$ between two points $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$ [2, p. 21].

Remark 3.14. Lemma 3.12 implies that $K_{\tilde{Q}}$ is preserved under $\mathrm{PO}(2,1)$ and because of Lemma 3.13 the hyperbolic distance is preserved as well - which is obvious since $\mathrm{PO}(2,1)$ is exactly the group of hyperbolic motions.

Lemma 3.15. *Let c_1, c_2 be two hyperbolic circles with a common tangent touching the circles in $\boldsymbol{y}_1, \boldsymbol{y}_2$ and let $\boldsymbol{x}_1, \boldsymbol{x}_2$ be the points of $\mathbb{P}^3$ corresponding to c_1, c_2 (via b^{-1}), i.e., $\boldsymbol{x}_1 := b^{-1}(c_1)$, $\boldsymbol{x}_2 := b^{-1}(c_2)$ (see Figure 3.12). Then*

$$K_Q(\boldsymbol{x}_1, \boldsymbol{x}_2) = K_{\tilde{Q}}(\boldsymbol{y}_1, \boldsymbol{y}_2) \tag{3.12}$$

[2, p. 39].

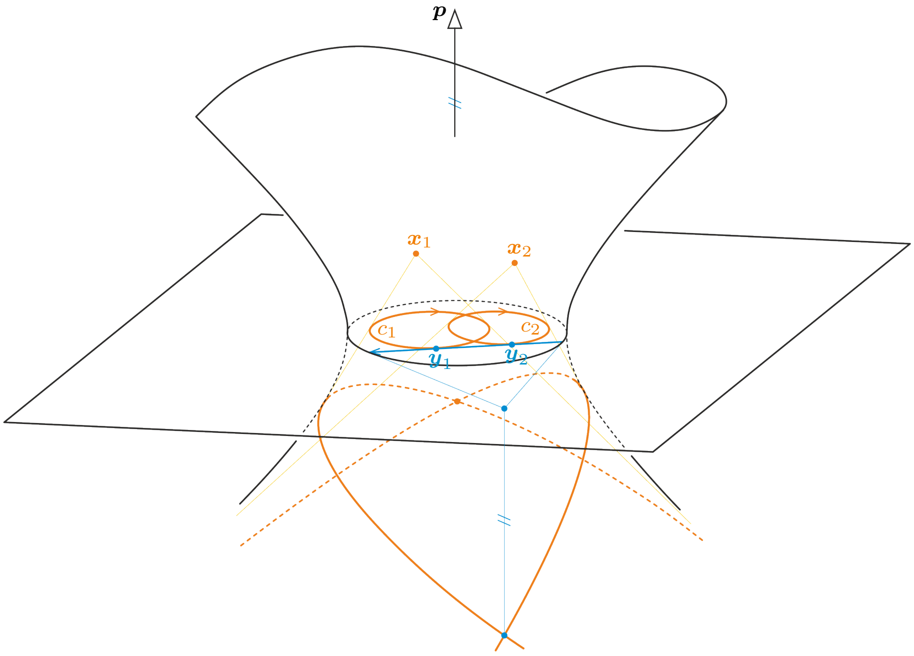


Figure 3.12: Two hyperbolic circles with a common (oriented) tangent

A proof of this Lemma can be found in [2, p. 40]. It implies:

Proposition 3.16. *The tangential distance between two (oriented) hyperbolic circles is invariant under hyperbolic Laguerre transformations.*

Proof. The tangential distance between two hyperbolic circles c_1, c_2 is the hyperbolic distance $d(\boldsymbol{y}_1, \boldsymbol{y}_2)$ between the touching points of a common oriented tangent. The hyperbolic distance is induced by $K_{\tilde{Q}}(\boldsymbol{y}_1, \boldsymbol{y}_2)$, which is equal to the CK-distance $K_Q(\boldsymbol{x}_1, \boldsymbol{x}_2)$ of the points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{P}^3$ corresponding to the hyperbolic circles c_1, c_2, according to Lemma 3.15. Since $K_Q(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is the CK-distance associated with the quadric Q, due to Lemma 3.12 it is invariant under the group $\mathrm{PO}(2,2)$ which induces Laguerre transformations of $\mathcal{H}$. Thus the tangential distance is preserved under hyperbolic Laguerre transformations. $\square$

3.1.2 n-dimensional hyperbolic Laguerre geometry

In this chapter we generalize the quadric model of hyperbolic Laguerre geometry for arbitrary dimensions bigger than 2 following [2, Section 6.2]. In analogy to the 2-dimensional case, we choose a quadric Q and a point $\boldsymbol{p}$ in $\mathbb{P}^{n+1}$ such that the inside of the quadric $\tilde{Q} = Q \cap \boldsymbol{p}^\perp$ can be identified with the n-dimensional hyperbolic space $\mathcal{H}$. As Bobenko et al. demonstrate in [2, Section 4.4] this can be done with the inside of a quadric $\tilde{Q}$ with signature $(n, 1)$. For the signature of Q we therefore have two possibilities, $(n+1, 1)$ or $(n, 2)$. In analogy to our argumentation in the 2-dimensional case (on p. 20), we need the points of Q to project onto the outside of $\tilde{Q}$ under $\pi_{\boldsymbol{p}}$, in order for the polar hyperplanes of the projected points to be hyperbolic hyperplanes (i.e., hyperplanes inside of $\tilde{Q}$). Hence, due to Lemma 2.24, we can only choose $(n, 2)$. Thus we have

$$Q : \langle x, x\rangle_Q := x_1^2 + ... + x_n^2 - x_{n+1}^2 - x_{n+2}^2 = 0$$

which we call the *hyperbolic Laguerre quadric*, and

$$\tilde{Q} : \begin{cases} \langle x, x\rangle_{\tilde{Q}} := x_1^2 + ... + x_n^2 - x_{n+1}^2 = 0 \\ \qquad\qquad\qquad\qquad\quad x_{n+2} = 0. \end{cases} \tag{3.13}$$

The point $\boldsymbol{p}$ thus has to lie inside of Q, i.e., $\langle p, p\rangle_Q < 0$. W.l.o.g we choose

$$\boldsymbol{p} := [0, ..., 0, 1]$$

so $\tilde{Q}$ is the intersection $Q \cap \boldsymbol{p}^\perp$, in analogy to the absolute circle in the 2-dimensional case. The involution and the central projection associated with $\boldsymbol{p}$ then take the form:

$$\sigma_{\boldsymbol{p}} : [x_1, ..., x_{n+1}, x_{n+2}] \mapsto [x_1, ..., x_{n+1}, -x_{n+2}], \tag{3.14}$$
$$\pi_{\boldsymbol{p}} : [x_1, ..., x_{n+1}, x_{n+2}] \mapsto [x_1, ..., x_{n+1}, 0], \tag{3.15}$$

[2, p. 40].

Correspondence between points/hyperplanar sections of Q and oriented hyperbolic hyperplanes/spheres

In analogy to Proposition 3.1 for the 2-dimensional case, for dimension n we have:

Proposition 3.17. *The restriction of the polar projection $\pi^*_{\boldsymbol{p}}$ (defined as in (2.3)) to $Q \setminus \tilde{Q}$ yields a double cover of the set of all hyperbolic hyperplanes [2, p. 39].*

Proof. See Proposition 2.26c. □

Each two points $\boldsymbol{x}^\pm$ of $\mathbb{P}^{n+1}$ that project onto the same hyperbolic hyperplane p under $\pi^*_{\boldsymbol{p}}$ can be interchanged via $\sigma_{\boldsymbol{p}}$ which yields reversing the orientation of the corresponding oriented hyperplanes that p carries (the orientation is encoded in the last component of $\boldsymbol{x}^\pm$, as we see in 3.14) [2, p. 41]. Thus, by defining a map $\overrightarrow{\pi^*_{\boldsymbol{p}}}$ that maps points of $Q \setminus \tilde{Q}$ to the same hyperplanes as $\pi^*_{\boldsymbol{p}}$ but additionally endows them with an orientation determined by the x_{n+2}-component of the corresponding point, we get a bijective correspondence between $Q \setminus \tilde{Q}$ and the set of oriented hyperbolic hyperplanes.

Proposition 3.18. *The set $Q \setminus \tilde{Q}$ bijectively corresponds to the set $\overrightarrow{\mathcal{P}}$ of oriented hyperbolic hyperplanes via the map $\overrightarrow{\pi^*_{\boldsymbol{p}}} : Q \setminus \tilde{Q} \to \overrightarrow{\mathcal{P}}$, and the involution $\sigma_{\boldsymbol{p}}$ acts orientation reversing on $\overrightarrow{\mathcal{P}}$.*

Moving on to hyperbolic hyperspheres, since $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ and $\pi^*_{\boldsymbol{p}}(\boldsymbol{x})$ for any point $\boldsymbol{x} \in \mathbb{P}^{n+1}$ are polar w.r.t. $\tilde{Q}$ (according to Proposition 2.26a), we define (in analogy to Definition 3.4):

Definition 3.19. The set of poles of all tangents of a hyperbolic hypersphere s is called the *polar hypersphere* of s [2, p. 21].

Lifting an oriented hyperbolic hypersphere s to the quadric Q by applying $\overrightarrow{\pi_{\boldsymbol{p}}^{*}}^{-1}$ to its tangent hyperplanes can geometrically be described as follows: First we find the poles of the tangent hyperplanes w.r.t. $\tilde{Q}$, which lie on the polar hypersphere $s^{\perp}$ of s. Then we intersect the connecting lines of points of $s^{\perp}$ and $\boldsymbol{p}$ with Q, yielding two hyperplanar sections of Q. One of those corresponds to s, the other one to the hyperbolic hypersphere with opposite orientation. Thus, oriented hyperbolic hyperspheres correspond to hyperplanar sections of Q.

To get a point model of the hyperspheres we identify each hyperplanar section s_Q of Q corresponding to a hyperbolic hypersphere s with the pole $\boldsymbol{x}_s$ (w.r.t. Q) of the hyperplane carrying s_Q. In analogy to the 2-dimensional case, the pole $\boldsymbol{x}_s$ is the vertex of the cone of contact $C_Q(\boldsymbol{x}_s)$ touching Q along s_Q, i.e., $s_Q = \boldsymbol{x}_s^{\perp} \cap Q$. The tangent hyperplanes to $C_Q(\boldsymbol{x}_s)$ and Q in points of s_Q coincide, and therefore those tangent hyperplanes intersect $\boldsymbol{p}^{\perp}$ in tangent hyperplanes of s and the cone itself intersects in s. Furthermore, $\boldsymbol{x}_s$ has to lie inside Q in order for the hypersphere s to lie inside $\tilde{Q}$, i.e., to be hyperbolic. Thus the map

$$\begin{aligned} \pi_{\boldsymbol{p}}^{*\mathcal{S}} &: Q^{-} \to \mathcal{S} \\ \boldsymbol{x} &\mapsto \pi_{\boldsymbol{p}}^{*\mathcal{S}}(\boldsymbol{x}) := C_Q(\boldsymbol{x}) \cap \boldsymbol{p}^{\perp}, \end{aligned}$$

where $\mathcal{S}$ is the set of all hyperbolic hyperspheres, takes each point of Q^{-} to a hypersphere of $\mathcal{H}$ [2, p. 39]. Its "oriented version" $\overrightarrow{\pi_{\boldsymbol{p}}^{*\mathcal{S}}}$ (which endows the image circle of a point $\boldsymbol{x} \in Q^{-}$ with an orientation corresponding to the sign of the x_{n+2}-component of $\boldsymbol{x}$) yields a bijective correspondence between Q^{-} and the set of all oriented hyperbolic hyperspheres. Just as in the 2-dimensional case, amongst those hyperspheres we distinguish between ordinary hyperbolic hyperspheres, hyperbolic surfaces of constant distance to a hyperbolic hyperplane and horospheres depending on the location of $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ w.r.t. $\tilde{Q}$, i.e., the value of $\langle \pi_{\boldsymbol{p}}(x), \pi_{\boldsymbol{p}}(x) \rangle$. Besides the spheres, $\overrightarrow{\pi_{\boldsymbol{p}}^{*\mathcal{S}}}$ also yields the hyperbolic hyperplanes when applied to points of $Q \setminus \tilde{Q}$. Hence we expand its domain to $Q^{-} \cup Q \setminus \tilde{Q}$ and name the expanded map b (in analogy to the approach in the 2-dimensional case).

Proposition 3.20. *The map $b : Q^{-} \cup Q \setminus \tilde{Q} \to \vec{\mathcal{S}} \cup \vec{\mathcal{P}}$ is bijective, where $\vec{\mathcal{S}} \cup \vec{\mathcal{P}}$ are oriented hyperbolic hyperspheres and planes. In particular:*

- *For $\boldsymbol{x} \in Q \setminus \tilde{Q}$, $b(\boldsymbol{x})$ is a hyperbolic hyperplane with pole $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ w.r.t. $\tilde{Q}$.*
- *For $\boldsymbol{x} \in Q^{-} \cap C_Q(\boldsymbol{p})^{-}$, i.e., $\langle \pi_{\boldsymbol{p}}(x), \pi_{\boldsymbol{p}}(x) \rangle < 0$, $b(\boldsymbol{x})$ is an (ordinary) hyperbolic hypersphere with center $\pi_{\boldsymbol{p}}(\boldsymbol{x})$.*
- *For $\boldsymbol{x} \in Q^{-} \cap C_Q(\boldsymbol{p})^{+}$, i.e., $\langle \pi_{\boldsymbol{p}}(x), \pi_{\boldsymbol{p}}(x) \rangle > 0$, $b(\boldsymbol{x})$ is a hyperbolic hypersurface of constant distance to a hyperbolic hyperplane with pole $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ w.r.t. $\tilde{Q}$.*
- *For $\boldsymbol{x} \in Q^{-} \cap C_Q(\boldsymbol{p}) \setminus \tilde{Q}$, i.e., $\langle \pi_{\boldsymbol{p}}(x), \pi_{\boldsymbol{p}}(x) \rangle = 0$, $b(\boldsymbol{x})$ is a horosphere with center $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ [2, p. 41].*

Hyperbolic Laguerre transformations

Since points and planar sections of Q correspond to oriented hyperplanes and spheres of $\mathcal{H}$ respectively (via b) and Laguerre transformations preserve those, we can lift hyperbolic Laguerre transformations to projective transformations that preserve the quadric Q. Since the signature of the hyperbolic Laguerre quadric for dimensions $n > 2$ is non-neutral, this is exactly the projective orthogonal group $\mathrm{PO}(n, 2)$ (c.f. Lemma 2.19) [2, p. 42].

Proposition 3.21. *Every Laguerre transformation α of the n-dimensional hyperbolic space can be written in the form:*

$$\alpha = b \circ t \circ b^{-1}$$

for some $t \in \mathrm{PO}(n, 2)$.

In analogy to the 2-dimensional case, transformations of the group $\mathrm{PO}(n, 2)$ can be decomposed into transformations of the subgroup $\mathrm{PO}(n, 2)_{\boldsymbol{p}}$, which besides preserving Q also fixes $\boldsymbol{p}$, and a "scaling along a pencil of concentric hyperspheres" (that belongs to one of three one-parameter families of scalings). A pencil of concentric hyperbolic hyperspheres with center $\boldsymbol{z}$ corresponds to points of Q^- that lie on the line $\boldsymbol{z} \vee \boldsymbol{p} \subset \mathbb{P}^{n+1}$.

Definition 3.22. Let s_1, s_2 be two hyperspheres of a pencil $(s_i)_{i \in I}$ of concentric hyperbolic hyperspheres, and let $\boldsymbol{x}_1 := b^{-1}(s_1), \boldsymbol{x}_2 := b^{-1}(s_2)$ be their corresponding points in Q^-. Then we call the unique transformation $T_{\boldsymbol{x}_1, \boldsymbol{x}_2} \in \mathrm{PO}(n, 2)$ that maps $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ and preserves each hyperplane through the line $\boldsymbol{x}_1 \vee \boldsymbol{x}_2$ (containing $\boldsymbol{p}$) a *scaling along the pencil $(s_i)_{i \in I}$ of concentric hyperbolic hyperspheres* [2, p. 32–33].

Depending on the type of the hyperspheres of the pencil, i.e., whether they are hyperbolic hyperspheres, hyperbolic hypersufaces of constant distance to a hyperbolic hyperplane or horospheres, we have three types of scalings represented by:

$$T_u^s := \left[\begin{array}{ccc|cc} & & & 0 & 0 \\ & I_n & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \dots & 0 & \cos(u) & \sin(u) \\ 0 & \dots & 0 & -\sin(u) & \cos(u) \end{array}\right],$$

$$T_u^p := \left[\begin{array}{ccc|ccc} & & & 0 & 0 & 0 \\ & I_{n-1} & & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 \\ \hline 0 & \dots & 0 & \cosh(u) & 0 & \sinh(u) \\ 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & \dots & 0 & \sinh(u) & 0 & \cosh(u) \end{array}\right],$$

$$T_u^h := \left[\begin{array}{ccc|cc|c} & & & 0 & 0 & 0 \\ & I_{n-1} & & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 \\ \hline 0 & \dots & 0 & 1+\frac{u^2}{2} & \frac{u^2}{2} & u \\ 0 & \dots & 0 & -\frac{u^2}{2} & 1-\frac{u^2}{2} & -u \\ \hline 0 & \dots & 0 & u & u & 1 \end{array}\right]$$

where I_n is the identity matrix of size n and $u \in \mathbb{R}$ [2, p. 43].

Proposition 3.23. *Every transformation $t \in \mathrm{PO}(n,2)$ can be written in the form:*

$$t = \Phi \circ T_{u_t} \circ \Psi \tag{3.16}$$

where $\Phi, \Psi \in \mathrm{PO}(n,2)_{\boldsymbol{p}}$ and $T_{u_t} \in \{T_u^s, T_u^p, T_u^h\}$ is a scaling for some $u_t \in \mathbb{R}$ [2, p. 44].

Remark 3.24. The group $\mathrm{PO}(n,2)_{\boldsymbol{p}}$ doubly covers the group of hyperbolic motions $\mathrm{PO}(n,1)$ [2, p. 42]. In particular:

$$\mathrm{PO}(n,1) \simeq \mathrm{PO}(n,2)_{\boldsymbol{p}}/\sigma_{\boldsymbol{p}} \tag{3.17}$$

[2, p. 27].

In Section 3.1.1 we already defined the CK-distance associated to a quadric in a projective space of arbitrary dimension. Associated with $\tilde{Q}$ as in (3.13) it induces the hyperbolic distance on the n-dimensional hyperbolic space:

Lemma 3.25. *For $\tilde{Q}$, defined as in (3.13), the equation*

$$K_{\tilde{Q}}(\boldsymbol{x}, \boldsymbol{y}) = \cosh^2 d(\boldsymbol{x}, \boldsymbol{y}) \tag{3.18}$$

defines the hyperbolic distance $d(\boldsymbol{x}, \boldsymbol{y})$ between two points $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$ [2, p. 21].

Remark 3.26. Analogous to Remark 3.14, from the previous Lemma we see that the group of hyperbolic motions $\mathrm{PO}(n,1)$ preserves the hyperbolic distance, since it is induced by $K_{\tilde{Q}}$ which is invariant under projective transformations that preserve $\tilde{Q}$ (according to Lemma 3.12), i.e., transformations from $\mathrm{PO}(n,1)$.

According to [2] Lemma 3.15 holds true for arbitrary dimension, that is:

Lemma 3.27. *Let s_1, s_2 be two hyperbolic hyperspheres with a common tangent hyperplane touching the hyperspheres in $\boldsymbol{y}_1, \boldsymbol{y}_2$ and $\boldsymbol{x}_1, \boldsymbol{x}_2$ the points of $\mathbb{P}^{n+1}$ corresponding to s_1, s_2 (via b^{-1}), i.e., $\boldsymbol{x}_1 := b^{-1}(s_1)$, $\boldsymbol{x}_2 := b^{-1}(s_2)$. Then*

$$K_Q(\boldsymbol{x}_1, \boldsymbol{x}_2) = K_{\tilde{Q}}(\boldsymbol{y}_1, \boldsymbol{y}_2) \tag{3.19}$$

[2, p. 39].

Thus the tangential distance is also an invariant of the n-dimensional hyperbolic Laguerre geometry.

Proposition 3.28. *The tangential distance between two (oriented) hyperbolic hyperspheres is invariant under hyperbolic Laguerre transformations.*

Proof. The proof works analogously to the proof of Proposition 3.16 using Lemmas 3.12, 3.25 and 3.27. □

3.2 Elliptic Laguerre geometry

In analogy to hyperbolic Laguerre geometry we want to embed the elliptic space into the projective space of one dimension higher and find a correspondence between points/hyperplanar sections of a chosen quadric in this projective space and oriented elliptic hyperplanes/spheres. The foundation for this is provided by [2, Sections 2.1 and 6.2].

3.2.1 2-dimensional elliptic Laguerre geometry

Bobenko et al. have shown in [2, Section 4.5] that the n-dimensional elliptic space can be identified with the outside of a quadric with signature $(n+1,0)$, whose inside is empty while the outside is the entire projective space containing the quadric. Thus, for 2-dimensional elliptic Laguerre geometry, similarly to the hyperbolic case, we again choose a quadric Q and a point $\boldsymbol{p}$ in $\mathbb{P}^3$, but this time identify the outside of the quadric $\tilde{Q} = Q \cap \boldsymbol{p}^\perp$ (with signature $(3,0)$) with the elliptic plane $\mathcal{E}$. Since the inside of $\tilde{Q}$ must be empty (according to Remark 2.18), we want the quadric Q, containing $\tilde{Q}$, to project to $\tilde{Q}^+$ under the central projection $\pi_{\boldsymbol{p}}$ that we will associate with $\boldsymbol{p}$. Hence, due to Lemma 2.24, for the *elliptic Laguerre quadric* Q we choose a quadric of signature $(3,1)$, namely

$$Q : \langle x, x\rangle_Q := x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0,$$

which is the unit sphere in $\mathbb{R}^3$.

Since the real part of $\tilde{Q}$ should be empty (i.e., $\boldsymbol{p}^\perp$ should not intersect Q in real points), $\boldsymbol{p}$ must lie inside of Q. W.l.o.g. we choose

$$\boldsymbol{p} := [0,0,0,1],$$

making $\boldsymbol{p}^\perp$ the plane at infinity of $\mathbb{P}^3$. The maps $\sigma_{\boldsymbol{p}}$ and $\pi_{\boldsymbol{p}}$, defined as in (2.1) and (2.2), then take the forms

$$\begin{aligned}
\sigma_{\boldsymbol{p}} &: [x_1, x_2, x_3, x_4] \mapsto [x_1, x_2, x_3, -x_4],\\
\pi_{\boldsymbol{p}} &: [x_1, x_2, x_3, x_4] \mapsto [x_1, x_2, x_3, 0].
\end{aligned}$$

In particular, $\sigma_{\boldsymbol{p}}$ interchanges antipodal points of the sphere Q, and $\pi_{\boldsymbol{p}}$ takes each point of $\mathbb{P}^3$ to the point at infinity of its connecting line with $\boldsymbol{p}$ (see Figure 3.13). For $\tilde{Q} = \boldsymbol{p}^\perp \cap Q$ we get

$$\tilde{Q} : \begin{cases} \langle x, x\rangle_{\tilde{Q}} := x_1^2 + x_2^2 + x_3^2 = 0 \\ \qquad\qquad\qquad\qquad x_4 = 0. \end{cases} \tag{3.20}$$

As mentioned above, we identify the elliptic plane

$$\mathcal{E} = \tilde{Q}^+ = \boldsymbol{p}^\perp$$

with the outside of $\tilde{Q}$ which is the plane at infinity [2, p. 44].

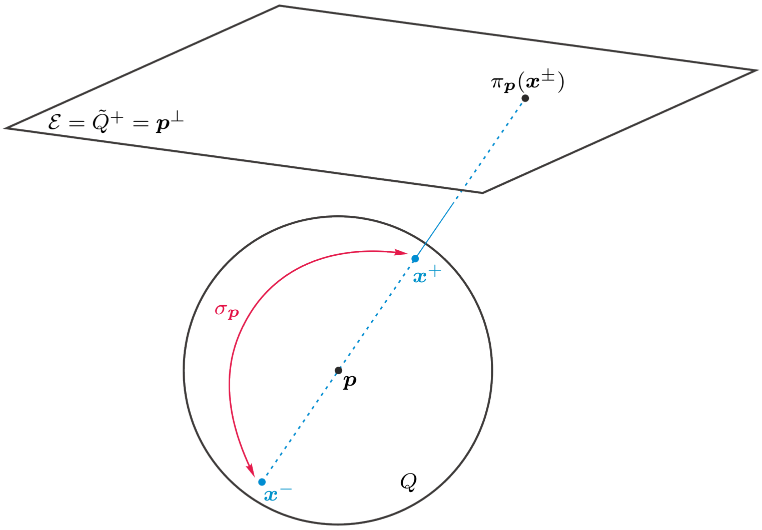


Figure 3.13: Elliptic Laguerre quadric Q and $\mathcal{E}$ embedded into $\mathbb{P}^3$

Correspondence between points/planar sections of Q and oriented elliptic lines/circles

Just as for hyperbolic Laguerre geometry, we establish a correspondence with the help of the polar projection $\pi^*_{\boldsymbol{p}}$ associated with $\boldsymbol{p}$.

Proposition 3.29. *The restriction of the polar projection $\pi^*_{\boldsymbol{p}}$, defined as in (2.3) with $n = 3$, to Q yields a double cover of the set of all elliptic lines [2, p. 39].*

Proof. See Proposition 2.26 c. □

Figure 3.13 illustrates how the elliptic lines are doubly covered, since the projection line under $\pi_{\boldsymbol{p}}$ for each point $\boldsymbol{x}^{+}$ of Q contains a second point of Q, namely its antipodal point $\boldsymbol{x}^{-}$. Applying polarity w.r.t. $\tilde{Q}$ to $\pi_{\boldsymbol{p}}(\boldsymbol{x}^{\pm})$ yields the line $\pi^*_{\boldsymbol{p}}(\boldsymbol{x}^{\pm})$ in $\mathcal{E}$. By the definition of the polar projection (see (2.3)), we geometrically get $\pi^*_{\boldsymbol{p}}(\boldsymbol{x}^{\pm})$ by intersecting the tangent plane in $\boldsymbol{x}^{+}$ or $\boldsymbol{x}^{-}$ with $\boldsymbol{p}^{\perp}$, i.e., $\pi^*_{\boldsymbol{p}}(\boldsymbol{x}^{\pm})$ is the line at infinity of these (parallel) tangent planes (see Figure 3.14).

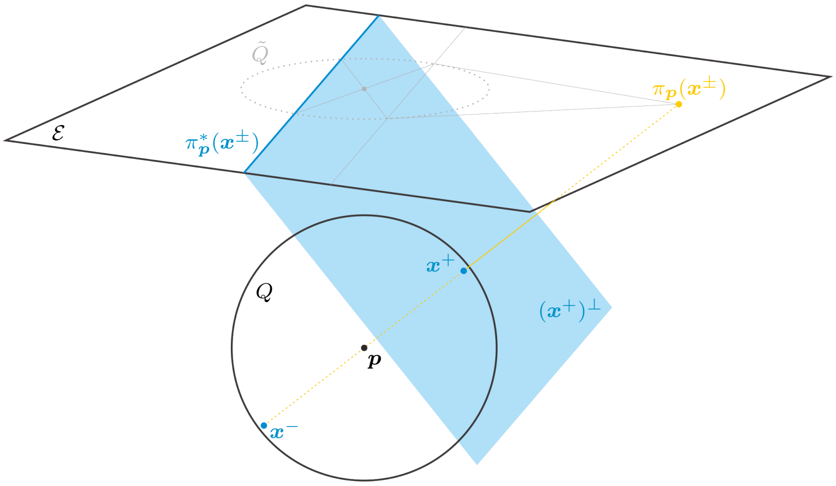


Figure 3.14: Double cover of the elliptic lines

Since $\sigma_{\boldsymbol{p}}$ interchanges the points $\boldsymbol{x}^+$ and $\boldsymbol{x}^-$, it reverses the orientation of the corresponding oriented lines of $\mathcal{E}$ [2, p. 39]. Thus, just as for the hyperbolic case, we consider the map $\overrightarrow{\pi^*_{\boldsymbol{p}}}$ which takes $\boldsymbol{x}^+$ to the positively oriented elliptic line and $\boldsymbol{x}^-$ to the negatively oriented one.

Proposition 3.30. *The quadric Q bijectively corresponds to the set $\vec{\mathcal{L}}$ of oriented hyperbolic lines via the map $\overrightarrow{\pi^*_{\boldsymbol{p}}} : Q \to \vec{\mathcal{L}}$, and the involution $\sigma_{\boldsymbol{p}}$ acts orientation reversing on $\vec{\mathcal{L}}$.*

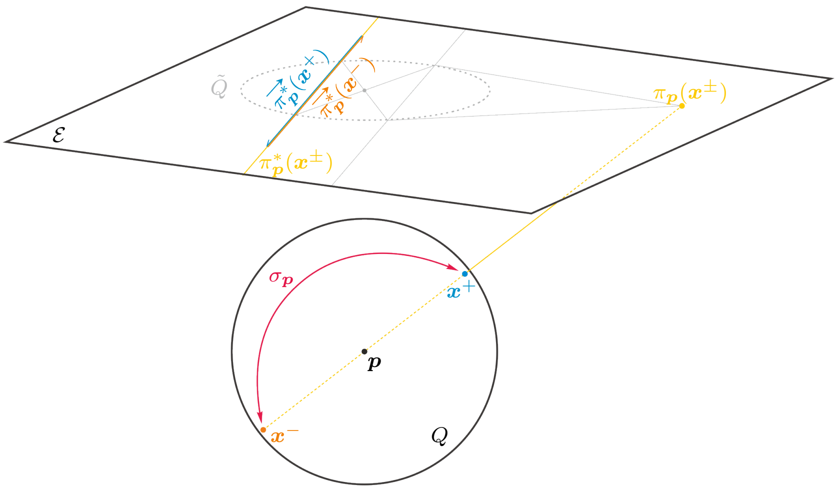
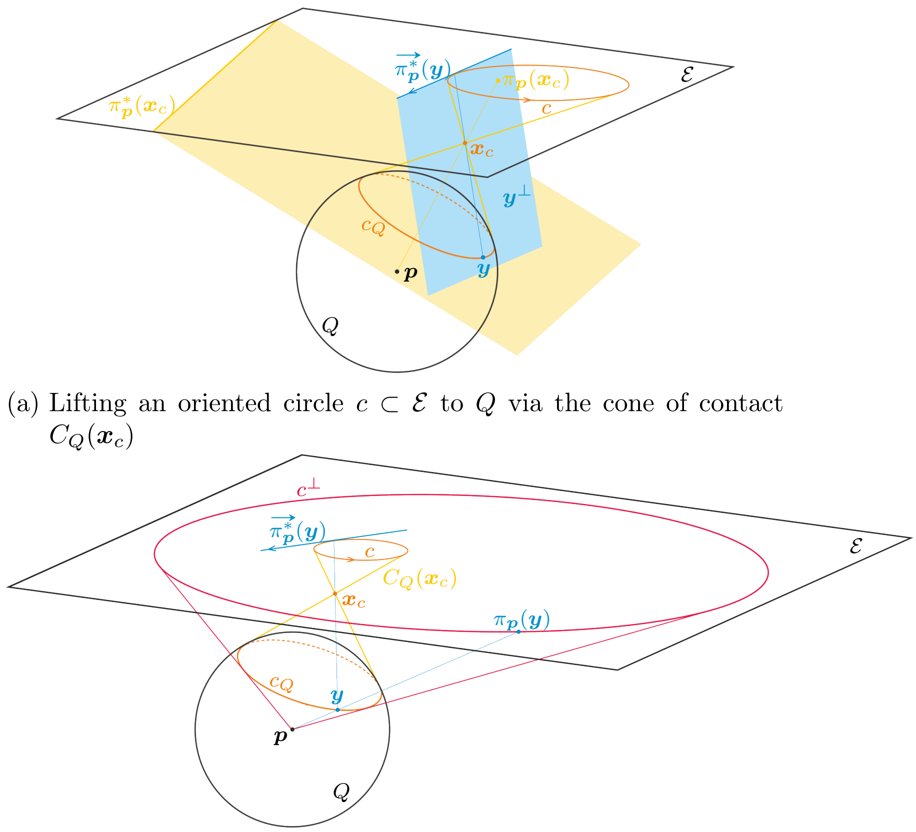


Figure 3.15: Bijective correspondence between points on Q and oriented lines of $\mathcal{E}$

Let us now proceed to oriented elliptic circles. We will show that they can be identified with planar sections of Q, analogously to the quadric model of hyperbolic Laguerre geometry. Interpreting an oriented circle c in $\mathcal{E}$ as envelope of its (oriented) tangents, we can lift each tangent t to Q by applying $\overrightarrow{\pi_{\boldsymbol{p}}^{*}}^{-1}$. Geometrically, we construct the tangent plane to Q with t as its line at infinity and get the corresponding point on Q as contact point of the tangent plane (note that there are actually two such tangent planes, but we choose the one that yields the contact point corresponding to the orientation of t). Repeating this for every tangent of c, those tangent planes envelope a cone touching Q along a circle c_Q, i.e., a planar section of Q. The plane carrying c_Q has $\pi_{\boldsymbol{p}}^{*}(\boldsymbol{x}_c)$ as line at infinity, where $\boldsymbol{x}_c$ is the vertex of the cone and $\pi_{\boldsymbol{p}}(\boldsymbol{x}_c)$ is the center of c (see Figure 3.16a).



(a) Lifting an oriented circle $c \subset \mathcal{E}$ to Q via the cone of contact $C_Q(\boldsymbol{x}_c)$

(b) Lifting an oriented circle $c \subset \mathcal{E}$ to Q via the cone with vertex $\boldsymbol{p}$ containing $c^{\perp}$

Figure 3.16: Bijective correspondence between planar sections of Q and oriented circles of $\mathcal{E}$

Since the polar projection $\pi^*_{\boldsymbol{p}}$ can be decomposed into the central projection $\pi_{\boldsymbol{p}}$ and the polarity w.r.t. $\tilde{Q}$, alternatively to lifting an oriented circle to Q with the aid of a a cone of contact, we could consider:

Definition 3.31. The poles of tangents of an elliptic circle c w.r.t. $\tilde{Q}$ lie on a concentric elliptic circle. We call it the *polar circle* $c^\perp$ of c [2, p. 21].

Thus, by applying polarity w.r.t. $\tilde{Q}$ to the tangents of a circle c in $\mathcal{E}$, we get the points of its polar circle $c^\perp$ in $\mathcal{E}$. Projecting its points to Q is equivalent to intersecting Q with the cone with vertex $\boldsymbol{p}$ that contains $c^\perp$. This yields two circles on Q from which we choose the one that corresponds to the orientation of the tangents of c (see Figure 3.16b).

Remark 3.32. Points in $\mathcal{E}$, as circles with radius zero, also correspond to planar sections of Q, namely great circles [2, p. 8]. This is clear from the paragraph above, since the elliptic lines passing through a point are polar to points on a common elliptic line. Projecting this line back to Q yields the intersection of Q with a plane through $\boldsymbol{p}$ which is the center of the sphere Q.

Finally, we can also identify the oriented elliptic circles with points in the projective space instead of planar sections. For this purpose we identify each planar section c_Q of Q with the vertex $\boldsymbol{x}_c$ of the cone of contact $C_Q(\boldsymbol{x}_c)$ touching Q along c_Q, i.e., with the pole of the plane carrying c_Q. While for the hyperbolic Laguerre quadric the intersection with the polar plane of every point inside of the quadric is non-empty and thus corresponds to a hyperbolic circle, for the elliptic Laguerre quadric Q this is the case for every point outside of Q. Thus the map

$$\begin{aligned} \pi^{*c}_{\boldsymbol{p}} &: Q^+ \to \mathcal{C} \\ &\boldsymbol{x} \mapsto \pi^{*c}_{\boldsymbol{p}}(\boldsymbol{x}) := C_Q(\boldsymbol{x}) \cap \boldsymbol{p}^\perp, \end{aligned}$$

where $\mathcal{C}$ is the set of elliptic circles, maps each point $\boldsymbol{x}$ of Q^+ to a circle in $\mathcal{E}$ (with center $\pi_{\boldsymbol{p}}(\boldsymbol{x})$) [2, p. 45]. Since each elliptic circle c has two pre-image points $\boldsymbol{x}_c^+, \boldsymbol{x}_c^-$ under $\pi^{*c}_{\boldsymbol{p}}$ (differing only by their x_4-component), we denote the map, that distinguishes those points as corresponding to different orientations of c, by $\overrightarrow{\pi^{*c}_{\boldsymbol{p}}}$. In other words: $\overrightarrow{\pi^{*c}_{\boldsymbol{p}}}$ maps $\boldsymbol{x}_c^+, \boldsymbol{x}_c^-$ to the two oppositely oriented circles that c carries.

We see that, just as in the hyperbolic case, $\overrightarrow{\pi^{*c}_{\boldsymbol{p}}}$ also yields the oriented elliptic lines when applied to the points on Q, since the cone of contact to Q with vertex $\boldsymbol{x} \in Q$ is the tangent plane to Q in $\boldsymbol{x}$. Hence, we expand the domain of $\overrightarrow{\pi^{*c}_{\boldsymbol{p}}}$ to $Q^+ \cup Q$ and name the expanded map b. In analogy to Proposition 3.6, we get:

Proposition 3.33. *The map $b : Q^+ \cup Q \to \vec{\mathcal{C}} \cup \vec{\mathcal{L}}$ is bijective, where $\vec{\mathcal{C}} \cup \vec{\mathcal{L}}$ are oriented elliptic circles and lines. In particular:*

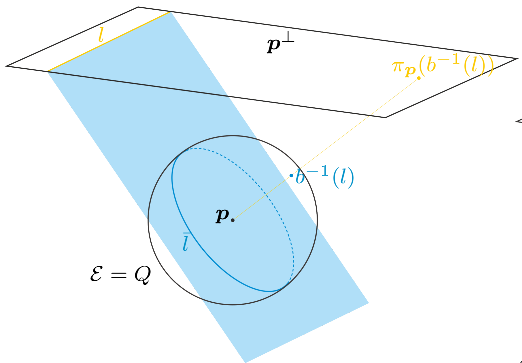
- *For $\boldsymbol{x} \in Q$, $b(\boldsymbol{x})$ is an elliptic line with pole $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ w.r.t. $\tilde{Q}$.*

- *For $\boldsymbol{x} \in Q^+$, $b(\boldsymbol{x})$ is an elliptic circle with center $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ [2, p. 45].*

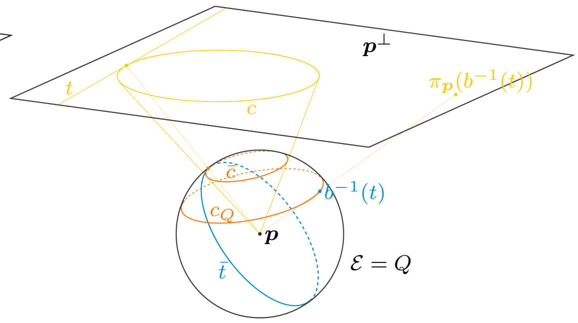
Before turning to Laguerre transformations of the elliptic plane, we want to consider our results in the sphere model of elliptic geometry as well. In this model, the elliptic geometry "takes place" on the unit sphere. Two antipodal points represent the same "point" of $\mathcal{E}$, elliptic lines are great circles, and circles of $\mathcal{E}$ are small circles, i.e., planar sections that do not contain the center of the sphere. The orientation of lines and circles is determined by the direction in which they are traversed [2, p. 7].

Now, if we want to transfer the "place of action" for elliptic geometry from the plane at infinity $\boldsymbol{p}^\perp$ to the unit sphere (which is exactly the elliptic Laguerre quadric Q), we proceed as follows:

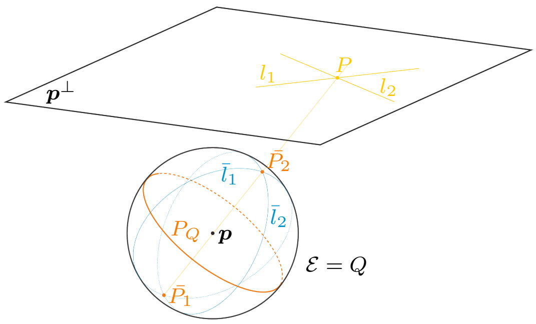
- We take an arbitrary oriented line l in $\boldsymbol{p}^\perp$ to the great circle $\bar{l}$, which is the intersection of $Q = \mathcal{E}$ with the plane containing $\boldsymbol{p}$ and the line at infinity l, and has the same orientation as l. Since elliptic geometry now "takes place" on the surface of the elliptic Laguerre quadric Q, it is interesting to consider the pre-image of l under b. The point $b^{-1}(l)$ on Q is exactly the spherical center of the great circle $\bar{l}$ (i.e., the intersection point of Q with the axis of $\bar{l}$ which lies on the left side of $\bar{l}$ w.r.t. its direction of traversion), because: The point at infinity $\pi_{\boldsymbol{p}}(b^{-1}(l))$ is polar to l w.r.t. $\tilde{Q}$, making $\boldsymbol{p} \vee b^{-1}(l)$ the axis of $\bar{l}$ (see Figure 3.17a).
- With the oriented elliptic lines, it is also determined how to take an oriented circle of $\boldsymbol{p}^\perp$ to the sphere model, i.e., to a small circle of Q. Taking each tangent of a circle c in $\boldsymbol{p}^\perp$ to $\mathcal{E} = Q$, we intersect Q with planes through $\boldsymbol{p}$, yielding great circles in oriented contact with the elliptic circle $\bar{c}$ that we are looking for (see Figure 3.17b). Again, we would like to study the relation between $\bar{c}$ and the corresponding planar section of the elliptic Laguerre quadric Q. For an elliptic line $\bar{l}$, i.e., a great circle, its spherical center is the corresponding point on Q. Thus, for an elliptic circle $\bar{c}$, i.e., a small circle, the corresponding planar section c_Q of Q consists of the spherical centers of all great circles in oriented contact with $\bar{c}$. Geometrically, we get $\bar{c}$ and c_Q (from $c \subset \boldsymbol{p}^\perp$) by intersecting Q with the cone with vertex $\boldsymbol{p}$ containing c or $c^\perp$ respectively, while considering the orientation. The circle c_Q can alternatively be retrieved by intersecting Q with the cone of contact $C_Q(b^{-1}(c))$.
- Finally, each point P of $\boldsymbol{p}^\perp$ is transferred to the sphere model by intersecting the great circles of $Q = \mathcal{E}$, corresponding to the lines through P, yielding the pair $(\bar{P}_1, \bar{P}_2)$ of antipodal points corresponding to P. This pair lies on the connecting line of P and $\boldsymbol{p}$. As points are also circles in Laguerre geometry, there must be a planar section of the elliptic Laguerre quadric Q corresponding to $(\bar{P}_1, \bar{P}_2)$ (or, equivalently, to P), namely a great circle, according to Remark 3.32. This is exactly the great circle that has $\bar{P}_1 \vee \bar{P}_2$ as its axis, because it is contained in the plane that has the polar line of P as its line at infinity (see Figure 3.17c) [2, p. 7–8].



(a) Taking an oriented line of $\boldsymbol{p}^{\perp}$ to a great circle of Q



(b) Taking an oriented circle of $\boldsymbol{p}^{\perp}$ to a small circle of Q



(c) Taking a point of $\boldsymbol{p}^{\perp}$ to a pair of antipodal points on Q

Figure 3.17: Transferring oriented elliptic lines and circles to the sphere model

Since with the sphere model oriented elliptic lines/circles and the corresponding points/ planar sections of the elliptic Laguerre quadric Q are located in the same space, namely the unit sphere Q, one can easily lose track of which space we are currently working in, Q as elliptic plane or Q as quadric in $\mathbb{P}^3$. To help with keeping track of that, the following table is provided, together with a figure below, separating the elliptic space Q and the elliptic Laguerre quadric Q into two different pictures.

	sphere model $\mathcal{E} = Q$	**elliptic Laguerre quadric $Q \subset \mathbb{P}^3$**
elliptic line	great circle	point
elliptic circle	small circle	small circle
elliptic point	pair of antipodal points	great circle

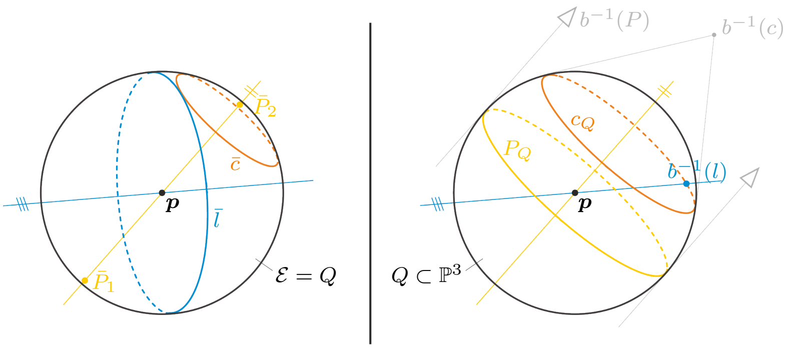


Figure 3.18: Unit sphere Q as elliptic plane (left) and as elliptic Laguerre quadric in $\mathbb{P}^3$ (right)

Elliptic Laguerre transformations

Analogous to hyperbolic Laguerre transformations, elliptic Laguerre transformations of the plane can be lifted to certain transformations of $\mathbb{P}^3$. Just as in the hyperbolic case, those are exactly the projective transformations that preserve the elliptic Laguerre quadric Q, i.e., transformations of the group $\mathrm{PO}(3,1)$, due to the bijective correspondence between planar points/planar sections of Q and oriented lines/circles of $\mathcal{E}$ [2, p. 45].

Proposition 3.34. *Every Laguerre transformation α of the elliptic plane can be written in the form:*

$$\alpha = b \circ t \circ b^{-1}$$

for some $t \in PO(3,1)$.

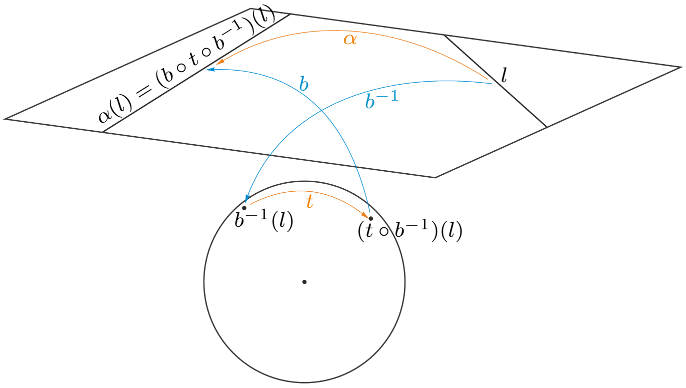


Figure 3.19: Lifting elliptic Laguerre transformations to projective transformations

Just as the group $\mathrm{PO}(2,2)$, which induces Laguerre transformations of the hyperbolic plane, $\mathrm{PO}(3,1)$ can also be generated by scalings along pencils of concentric circles and transformations from the subgroup $\mathrm{PO}(3,1)_{\boldsymbol{p}}$ which preserve $\boldsymbol{p}$. Analogous to our argumentation for pencils of concentric hyperbolic circles (see p. 23), a pencil of elliptic circles corresponds to points of Q^+ that lie on a line through $\boldsymbol{p}$. The planar sections of Q, that correspond to those circles with common center $\boldsymbol{z}$, lie in planes that intersect $\boldsymbol{p}^\perp$ in a common line, namely $\pi^*_{\boldsymbol{p}}(\boldsymbol{z})$. Since $\pi^*_{\boldsymbol{p}}(\boldsymbol{z})$ is a line at infinity, the planes are parallel (see Figure 3.20). With this, we can define a scaling along a pencil of concentric elliptic circles as follows:

Definition 3.35. Let c_1, c_2 be two circles of a pencil $(c_i)_{i\in I}$ of concentric elliptic circles, and let $\boldsymbol{x}_1 := b^{-1}(c_1), \boldsymbol{x}_2 := b^{-1}(c_2)$ be their corresponding points in Q^+. Then we call the unique transformation $T_{\boldsymbol{x}_1,\boldsymbol{x}_2} \in \mathrm{PO}(3,1)$ that maps $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ and preserves each plane through the line $\boldsymbol{x}_1 \vee \boldsymbol{x}_2$ (containing $\boldsymbol{p}$) a *scaling along the pencil* $(c_i)_{i\in I}$ *of concentric elliptic circles* [2, p. 32–33].

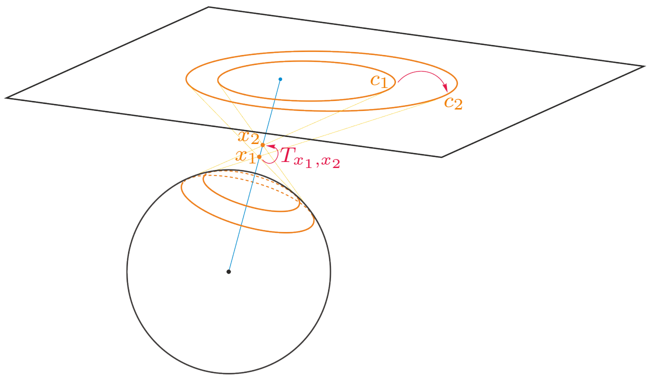


Figure 3.20: Scaling along a pencil of concentric elliptic circles

As mentioned above, with the group $\mathrm{PO}(3,1)_{\boldsymbol{p}}$ and by choosing one specific one-parameter family of scalings, one can generate the entire group $\mathrm{PO}(3,1)$.

Proposition 3.36. *Let*

$$T_u := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(u) & \sinh(u) \\ 0 & 0 & \sinh(u) & \cosh(u) \end{bmatrix}, \qquad u \in \mathbb{R},$$

be a one-parameter family of scalings along concentric elliptic circles. Then every transformation $t \in \mathrm{PO}(3,1)$ *can be written in the form*

$$t = \Phi \circ T_{u_t} \circ \Psi,$$

where $\Phi, \Psi \in \mathrm{PO}(3,1)_{\boldsymbol{p}}$ *and* $u_t \in \mathbb{R}$ *[2, p. 46].*

Remark 3.37. The group $\mathrm{PO}(3,1)_{\boldsymbol{p}}$ doubly covers the group of elliptic motions $\mathrm{PO}(3,0)$ [2, p. 46]. In particular:

$$\mathrm{PO}(3,0) \simeq \mathrm{PO}(3,1)_{\boldsymbol{p}}/\sigma_{\boldsymbol{p}} \tag{3.21}$$

[2, p. 27].

Finally, we want to consider an invariant of $\mathrm{PO}(3,1)$ to find an invariant of elliptic Laguerre transformations. According to [2], the CK-distance $K_{\tilde{Q}}$ associated with $\tilde{Q}$, defined as in (3.20), induces the elliptic distance via the equation

$$K_{\tilde{Q}}(\boldsymbol{x},\boldsymbol{y}) = \cos^2 d(\boldsymbol{x},\boldsymbol{y})$$

[2, p. 23–24], and Lemma 3.15 holds true upon exchanging the term "hyperbolic" with "elliptic" [2, p. 39]. Thus, analogously to Proposition 3.16, we have:

Proposition 3.38. *The tangential distance between two (oriented) elliptic circles is invariant under elliptic Laguerre transformations.*

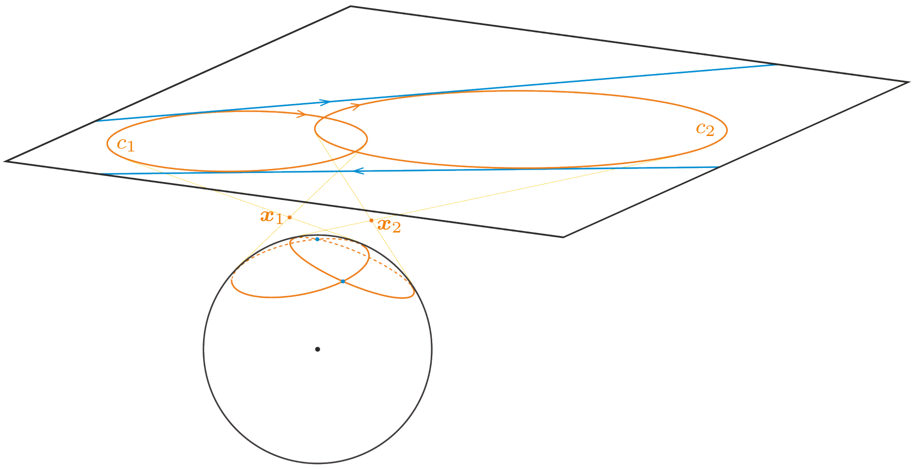


Figure 3.21: Two elliptic circles with common (oriented) tangents

3.2.2 n-dimensional elliptic Laguerre geometry

Just like hyperbolic Laguerre geometry, elliptic Laguerre geometry can also be generalized to arbitrary dimensions. We identify the n-dimensional elliptic space with the outside of a quadric $\tilde{Q}$ of signature $(n+1,0)$ which is contained in the quadric Q with signature $(n+1,1)$ as intersection with the polar hyperplane $\boldsymbol{p}^{\perp}$ (w.r.t. Q) of a point $\boldsymbol{p}$ inside of Q, i.e.: The quadric

$$Q : \langle x,x\rangle_Q := x_1^2 + \ldots + x_{n+1}^2 - x_{n+2}^2$$

is the *elliptic Laguerre quadric*, the polar hyperplane $\boldsymbol{p}^{\perp}$ of the point

$$\boldsymbol{p} := [0,\ldots,0,1]$$

w.r.t. Q intersects the elliptic Laguerre quadric Q in

$$\tilde{Q} : \begin{cases} \langle x, x \rangle_{\tilde{Q}} := x_1^2 + ... + x_n^2 + x_{n+1}^2 = 0 \\ x_{n+2} = 0 \end{cases}$$

whose non-empty side

$$\tilde{Q}^+ = \mathcal{E}$$

is identified with the n-dimensional elliptic space. The involution and central projection associated with $\boldsymbol{p}$ take the same form as in (3.14) and (3.15). Hence, $\pi_{\boldsymbol{p}}$ again maps two points of $\mathbb{P}^{n+1}$, that differ only by their x_{n+2}-component, to the same point in $\boldsymbol{p}^{\perp}$, while $\sigma_{\boldsymbol{p}}$ interchanges them [2, p. 44].

Correspondence between points/hyperplanar sections of Q and oriented elliptic hyperplanes/spheres

Proposition 3.39. *The restriction of the polar projection $\pi_{\boldsymbol{p}}^*$ (defined as in (2.3)) to $Q \setminus \tilde{Q}$ yields a double cover of the set of all elliptic hyperplanes [2, p. 39].*

Proof. See Proposition 2.26 c. □

Each two points that have the same image-point under $\pi_{\boldsymbol{p}}$ have the same image-hyperplane under $\pi_{\boldsymbol{p}}^*$, and thus are interchangable via the involution $\sigma_{\boldsymbol{p}}$ [2, p. 39]. Therefore we consider a map $\overrightarrow{\pi_{\boldsymbol{p}}^*}$ that takes the point with positive x_{n+2}-component to the positively oriented hyperplane and the point with negative x_{n+2}-component to the negatively oriented hyperplane, while applying $\sigma_{\boldsymbol{p}}$ reverses their orientations.

Proposition 3.40. *The set Q bijectively corresponds to the set $\overrightarrow{\mathcal{P}}$ of oriented elliptic hyperplanes via the map $\overrightarrow{\pi_{\boldsymbol{p}}^*} : Q \to \overrightarrow{\mathcal{P}}$, and the involution $\sigma_{\boldsymbol{p}}$ acts orientation reversing on $\overrightarrow{\mathcal{P}}$.*

To lift elliptic hyperspheres to Q, we identify them with their tangent planes and lift those up to Q via $\overrightarrow{\pi_{\boldsymbol{p}}^*}^{-1}$. For that purpose, we first find their poles w.r.t. $\tilde{Q}$, then project those back to Q by intersecting Q with their connecting lines with $\boldsymbol{p}$ (while considering the orientation of the tangents or, equivalently, of the sphere). For an elliptic hypersphere s the poles of its tangents lie on its polar hypersphere (which is defined analogously to the polar sphere of a hyperbolic sphere, i.e., by exchanging the word "hyperbolic" with "elliptic" in Definition 3.19). Thus, to get the points on Q corresponding to the tangents of s, we intersect the cone with vertex $\boldsymbol{p}$ containing $s^{\perp}$ with Q. This yields two hyperplanar sections of Q, from which each corresponds to a different orientation of the sphere s. Hence, upon identifying the sections with the poles (w.r.t. Q) of the planes carrying them, the set of elliptic hyperspheres $\mathcal{S}$ corresponds to hyperplanar sections of Q via the map

$$\begin{aligned} \pi_{\boldsymbol{p}}^{*\mathcal{S}} : Q^+ &\to \mathcal{S} \\ \boldsymbol{x} &\mapsto \pi_{\boldsymbol{p}}^{*\mathcal{S}}(\boldsymbol{x}) := C_Q(\boldsymbol{x}) \cap \boldsymbol{p}^{\perp} \end{aligned}$$

[2, p. 45]. Denoting the map that encodes their orientation as well by $\overrightarrow{\pi_{\boldsymbol{p}}^{*S}}$, we get a bijective correspondence for the oriented elliptic spheres. Since the cone of contact for a point on Q is the tangent hyperplane in this point, $\overrightarrow{\pi_{\boldsymbol{p}}^{*S}}$ also yields the oriented elliptic hyperplanes. Thus, we expand the domain of $\overrightarrow{\pi_{\boldsymbol{p}}^{*S}}$ to $Q^+ \cup Q$ and denote the expanded map by b. We summarize:

Proposition 3.41. *The map $b : Q^+ \cup Q \mapsto \vec{\mathcal{S}} \cup \vec{\mathcal{P}}$ is bijective, where $\vec{\mathcal{S}} \cup \vec{\mathcal{P}}$ are oriented elliptic hyperspheres and planes. In particular:*

- *For $\boldsymbol{x} \in Q$, $b(\boldsymbol{x})$ is an elliptic hyperplane with pole $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ w.r.t. $\tilde{Q}$.*
- *For $\boldsymbol{x} \in Q^+$, $b(\boldsymbol{x})$ is an elliptic hypersphere with center $\pi_{\boldsymbol{p}}(\boldsymbol{x})$ [2, p. 45].*

Elliptic Laguerre transformations

Analogous to Laguerre transformations of the elliptic plane, the Laguerre transformations of the n-dimensional elliptic space can be lifted to transformations of the group $\mathrm{PO}(n+1,1)$ since those preserve the elliptic Laguerre quadric Q (which carries the points and planar sections corresponding to oriented elliptic hyperplanes and spheres) [2, p. 45].

Proposition 3.42. *Every Laguerre transformation α of the n-dimensional elliptic space can be written in the form:*

$$\alpha = b \circ t \circ b^{-1}$$

for some $t \in \mathrm{PO}(n+1,1)$.

Analogous to dimension 2, pencils of concentric elliptic hyperspheres correspond to points of Q^+ lying on lines through $\boldsymbol{p}$. A scaling along such a pencil is then defined analogously to Definition 3.22 by exchanging the word "hyperbolic" with "elliptic", and $\mathrm{PO}(n,2)$ with $\mathrm{PO}(n+1,1)$ [2, p. 32–33]. After choosing one specific one-parameter family of scalings, with this and the subgroup $\mathrm{PO}(n+1,1)_{\boldsymbol{p}}$, which fixes $\boldsymbol{p}$, we can generate the entire group $\mathrm{PO}(n+1,1)$:

Proposition 3.43. *Let*

$$T_u := \left[\begin{array}{ccc|cc} & & & 0 & 0 \\ & I_n & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \dots & 0 & \cosh(u) & \sinh(u) \\ 0 & \dots & 0 & \sinh(u) & \cosh(u) \end{array}\right], \qquad u \in \mathbb{R},$$

be a one-parameter family of scalings along concentric elliptic hyperspheres. Then every transformation $t \in \mathrm{PO}(n+1,1)$ can be written in the form

$$t = \Phi \circ T_{u_t} \circ \Psi,$$

where $\Phi, \Psi \in \mathrm{PO}(n+1,1)_{\boldsymbol{p}}$ and $u_t \in \mathbb{R}$ [2, p. 46].

Remark 3.44. The group $\mathrm{PO}(n+1,1)_{\boldsymbol{p}}$ doubly covers the group of elliptic motions $\mathrm{PO}(n+1,0)$ [2, p. 46]. In particular:

$$\mathrm{PO}(n+1,0) \simeq \mathrm{PO}(n+1,1)_{\boldsymbol{p}}/\sigma_{\boldsymbol{p}} \tag{3.22}$$

[2, p. 27].

Just as for 2-dimensional Laguerre geometry, the tangential distance between two (oriented) elliptic hyperspheres is invariant under Laguerre transformations of the n-dimensional elliptic space. The argumentation works analogously to the one in the planar case and can be found in [2].

4 Euclidean Laguerre geometry

Euclidean Laguerre geometry can also be studied in a quadric model although the approach slightly differs from the one for non-Euclidean Laguerre geometry due to degeneracy of the Euclidean Laguerre quadric. Analogous to the previous chapter we will first introduce the quadric model for the 2-dimensional case and generalize it for arbitrary dimensions based on [2]. Since 2-dimensional Euclidean Laguerre geometry has already been studied more thoroughly than higher-dimensional and non-Euclidean versions in earlier years, for example by W. Blaschke, we will additionally treat the more known cyclographic model for planar Euclidean Laguerre geometry based on [1] and [9].

4.1 Quadric model of Euclidean Laguerre geometry

The basic idea is again to choose a certain quadric Q and a point $\boldsymbol{p}$ in a projective space such that $\pi_{\boldsymbol{p}}$ would map Q onto the outside of the quadric $\tilde{Q} = Q \cap \boldsymbol{p}^{\perp}$, just like for hyperbolic Laguerre geometry where applying polarity w.r.t. $\tilde{Q}$ to the projected points would give us the hyperbolic hyperplanes (see Chapter 3.1). In the Euclidean case, we cannot use polarity because Q is chosen as a degenerate quadric. Instead, the outside of $\tilde{Q}$ can be identified with the dual Euclidean space E^* and we get the Euclidean hyperplanes by applying duality to the points of E^* [2, Appendix A.2 and A.4].

4.1.1 2-dimensional Euclidean Laguerre geometry

We consider a quadric $\tilde{Q}$ with signature $(2,0,1)$ which is an imaginary cone with its vertex $[0,0,1]$ as only real point. Its inside is empty, $\tilde{Q}^- = \varnothing$, while its outside can be identified with the space E^* of all Euclidean lines, i.e.:

$$\tilde{Q}^+ = \mathbb{P}^2 \setminus \{[0,0,1]\} =: E^*$$

(that is, each equivalence class $[a,b,c] \neq [0,0,1]$ yields the homogeneous line coordinates of the Euclidean line with equation $ax_1 + bx_2 + cx_3 = 0$). Dualization of E^* yields

$$(E^*)^* = (\mathbb{P}^2 \setminus \{[0,0,1]\})^* = (\mathbb{P}^{2*}) \setminus ([0,0,1])^* \simeq \mathbb{R}^2,$$

i.e., the projective plane $\mathbb{P}^2$ without its line at infinity $x_3 = 0$, and thus can be identified with the Euclidean plane E [2, p. 83–84].

Embedding $\tilde{Q}$ into a quadric Q of signature $(2,1,1)$ we have:

$$Q : \langle x, x \rangle_Q := x_1^2 + x_2^2 - x_3^2 = 0 \tag{4.1}$$

is a cone with vertex $\boldsymbol{o} := [0,0,0,1]$ and we call it the *Euclidean Laguerre quadric.*

Remark 4.1. In the literature on planar Euclidean Laguerre geometry one will probably come across the *Blaschke model*, where oriented Euclidean lines and circles are identified with points and planar sections of the so-called *Blaschke cylinder*

$$B : x^2 + y^2 = 1$$

(and Laguerre transformations can be lifted to affine transformations that preserve B). As our goal is to identify points and planar sections of the cone Q with oriented Euclidean lines and circles and as a cone and a cylinder are projectively equivalent, one can already guess the connection between Q and B: The Euclidean Laguerre quadric Q is the projectivization of the Blaschke cylinder B, i.e.,

$$Q = P(B).$$

For $\boldsymbol{p}$ we choose a point inside of Q, w.l.o.g.

$$\boldsymbol{p} := [0, 0, 1, 0]$$

such that (according to Lemma 2.24) $\pi_{\boldsymbol{p}}$ would project to the outside of a quadric with signature $(2, 0, 1)$, namely $\tilde{Q} = \boldsymbol{p}^{\perp} \cap Q$ with equation

$$\tilde{Q} : \begin{cases} \langle x, x \rangle_{\tilde{Q}} := x_1^2 + x_2^2 = 0 \\ \qquad\qquad\qquad\;\; x_3 = 0. \end{cases}$$

The vertex of this imaginary cone $\tilde{Q}$ coincides with the vertex $\boldsymbol{o}$ of Q (see Figure 4.1). As mentioned above we identify

$$\tilde{Q}^{+} = E^{*}$$

with the dual Euclidean plane [2, p. 87].

Correspondence between points/planar sections of Q and oriented Euclidean lines/circles

Proposition 2.25b implies:

Proposition 4.2. *The restriction of the central projection $\pi_{\boldsymbol{p}}$, defined as in (2.2) with $n = 3$, to $Q \setminus \tilde{Q}$ yields a double cover of E^{*}.*

Each two points of $Q \setminus \tilde{Q}$ that project onto the same point of E^{*} can be interchanged via $\sigma_{\boldsymbol{p}}$ (defined as in (2.1)), yielding the reversion of the orientation of the corresponding Euclidean lines [2, p. 87].

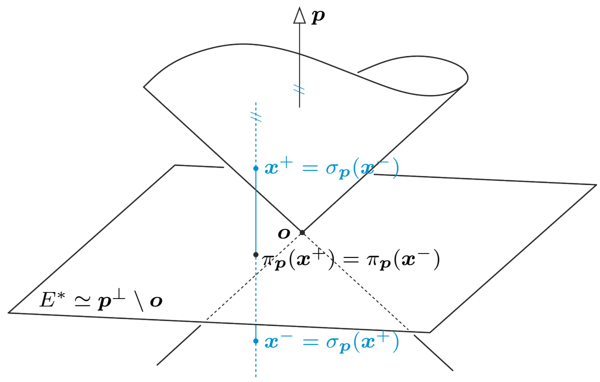


Figure 4.1: Double cover of E^*

Considering a map $\overrightarrow{\pi_{\boldsymbol{p}}^*}$, that first maps a point on $Q \setminus \tilde{Q}$ to a point in E^* while preserving its orientation (that is encoded in the sign of the x_3-component), then takes it to a line in E with this orientation, we get a bijective correspondence between points of $Q \setminus \tilde{Q}$ and oriented Euclidean lines.

Proposition 4.3. *The set $Q \setminus \tilde{Q}$ bijectively corresponds to the set $\vec{\mathcal{L}}$ of oriented Euclidean lines via the map $\overrightarrow{\pi_{\boldsymbol{p}}^*} : Q \setminus \tilde{Q} \to \vec{\mathcal{L}}$, and the involution $\sigma_{\boldsymbol{p}}$ acts orientation reversing on $\vec{\mathcal{L}}$.*

With the correspondence between points of $Q \setminus \tilde{Q}$ and oriented Euclidean lines at hand, we also have a correspondence between planar sections of Q (=conics) and the tangents of oriented Euclidean circles, because: Under $\pi_{\boldsymbol{p}}$ conics on Q project onto circles in E^* since the projection lines (which are orthogonal to $\boldsymbol{p}^\perp$) form a cylinder that we intersect with $\boldsymbol{p}^\perp$. Applying duality to the points of this circle in E^* yields the tangents to the dual circle in E (see Figure 4.2) [2, p. 87].

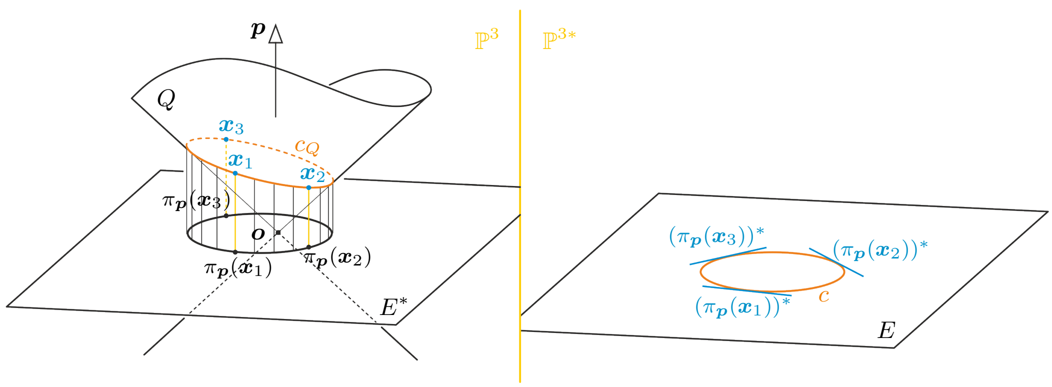


Figure 4.2: Correspondence between planar sections of Q and Euclidean circles

Unlike for non-Euclidean Laguerre geometry, we cannot identify oriented Euclidean circles with the poles (w.r.t. Q, in $\mathbb{P}^3$) of the planes carrying the corresponding conics on Q due to the degeneracy of Q. Thus we cannot find a map b analogously to the one for hyperbolic and elliptic Laguerre geometry respectively (see Proposition 3.6 and 3.33). Instead we can identify them with certain points in the dual projective space $\mathbb{P}^{3*}$: In the dual projective space we have a quadric Q^* which is dual to the cone Q. Since for a cone all tangent (hyper-)planes pass through its vertex, its dual quadric must be contained in the (hyper)plane dual to the vertex. Thus the dual quadric Q^* to Q has signature $(2,1)$ and is contained in the dual plane of the vertex $\boldsymbol{o}$ of Q, i.e., in the plane at infinity of $\mathbb{P}^{3*}$. A point $\boldsymbol{x}$ on a conic $c_Q \subset Q$, corresponding to an (oriented) Euclidean circle c, corresponds to a tangent t of c and is dual to a plane touching Q^* and intersecting E in t. Considering those planes for every $\boldsymbol{x} \in c_Q$, they envelope a cone intersecting the plane at infinity in Q^* and E in c (see Figure 4.3). Identifying c with the vertex $\boldsymbol{x}_c$ of this cone $\Gamma(\boldsymbol{x}_c)$ corresponding to c_Q by duality, we get the so-called "cyclographic model" of planar Euclidean Laguerre geometry (which we will treat in detail in Section 4.2) [2, p. 87].

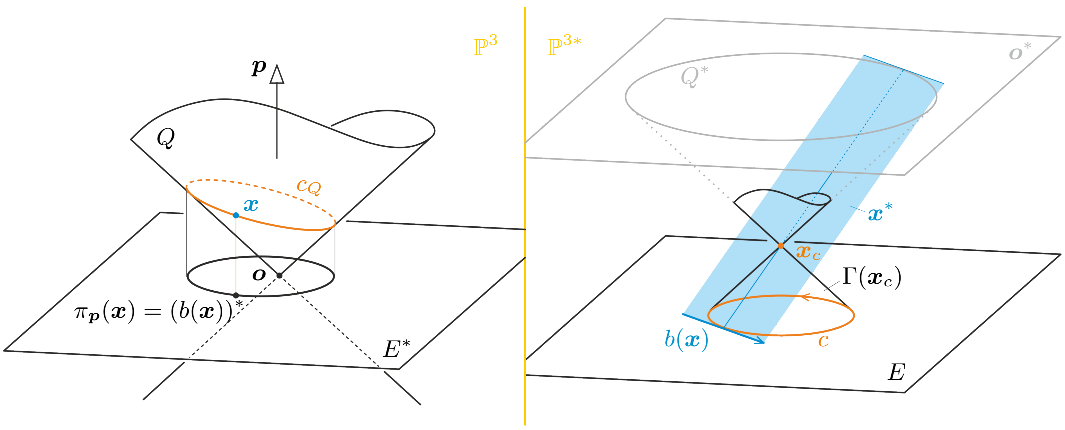


Figure 4.3: Identification of Euclidean circles with points of $\mathbb{P}^{3*}$

Remark 4.4. With the dual quadric Q^* at hand, we get an analogy to the hyperbolic Laguerre geometry (compare Remark 3.3), namely the correspondence between points on isotropic lines of Q and pencils of parallel lines of E, because: An isotropic line l on Q by duality corresponds to a tangent l^* of Q^*, and the points on l correspond to parallel planes with l^* as their plane at infinity. Then these planes intersect E in parallel lines, which correspond to the points on l via b (see Figure 4.4).

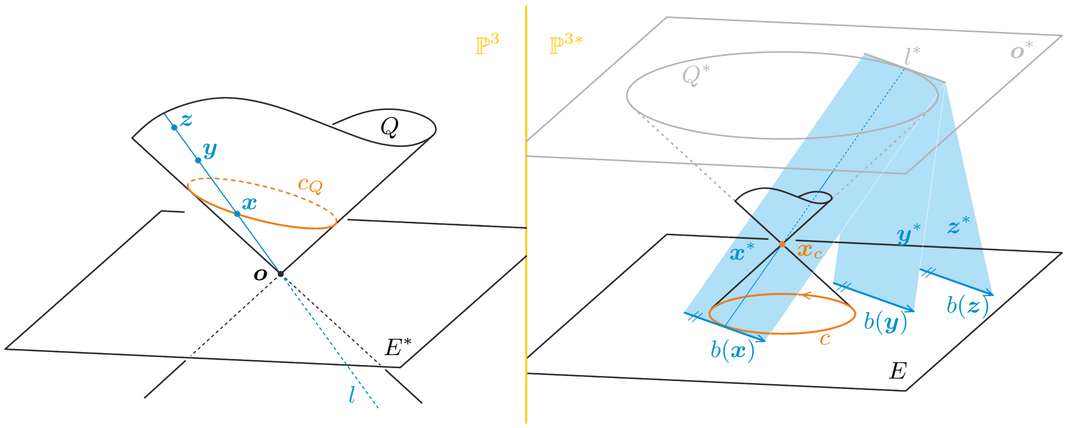


Figure 4.4: Pencil of parallel (oriented) lines in E

Euclidean Laguerre transformations

Just as for non-Euclidean Laguerre geometry, due to the correspondence of elements of the quadric Q and oriented Euclidean lines and circles, the projective transformations inducing Euclidean Laguerre transformations have to preserve Q. These are exactly the transformations of the group $\mathrm{PO}(2,1,1)$ according to Lemma 2.19 [2, p. 87]. Since we cannot identify lines and circles with points in the same projective space (i.e., as mentioned before, there is no map $\overrightarrow{\pi_{\boldsymbol{p}}^{*}\mathcal{S}}$ as for hyperbolic or elliptic Laguerre geometry), we cannot lift Euclidean Laguerre transformations to projective transformations in the same way as for hyperbolic and elliptic Laguerre transformations (i.e., by decomposing them into the form $b \circ t \circ b^{-1}$ for some projective transformation t). But we can interpret each circle as envelope of its tangents which bijectively correspond to points on a planar section of Q via $\overrightarrow{\pi_{\boldsymbol{p}}^{*}}$ (see Proposition 4.3). Then $\overrightarrow{\pi_{\boldsymbol{p}}^{*}}$ yields a bijective correspondence for lines as well as for circles and thus we can decompose each Euclidean Laguerre transformation as follows:

Proposition 4.5. *Every Laguerre transformation α of the Euclidean plane can be written in the form:*

$$\alpha = \overrightarrow{\pi_{\boldsymbol{p}}^{*}} \circ t \circ \overrightarrow{\pi_{\boldsymbol{p}}^{*}}^{-1}$$

for some $t \in \mathrm{PO}(2,1,1)$.

Remark 4.6. The subgroup $\mathrm{PO}(2,1,1)_{\boldsymbol{p}} \subset \mathrm{PO}(2,1,1)$ doubly covers the group of dual Euclidean similarity transformations $\mathrm{PO}(2,0,1)$. In particular:

$$\mathrm{PO}(2,0,1) \simeq \mathrm{PO}(2,1,1)_{\boldsymbol{p}}/\sigma_{\boldsymbol{p}} \tag{4.2}$$

[2, p. 87]. The dual transformations of $\mathrm{PO}(2,0,1)$, i.e., the transformations of the group $\mathrm{PO}(2,0,1)^*$, are the Euclidean similarity transformations (=motions and scalings) [2, p. 85].

If we do not want to identify circles with their tangents, alternatively to using $\overrightarrow{\pi_{\boldsymbol{p}}^*}$, we can "split" the correspondence into two maps, acting on the set of oriented Euclidean circles and lines respectively. This approach is taken for the cyclographic model, as we will see later.

4.1.2 n-dimensional Euclidean Laguerre geometry

Euclidean Laguerre geometry for arbitrary dimensions works analogously to the 2-dimensional case: The n-dimensional dual Euclidean space E^* can be identified with the outside of a quadric $\tilde{Q}$ with signature $(n, 0, 1)$, which we embed into a quadric Q with signature $(n, 1, 1)$ as intersection of Q with the polar hyperplane $\boldsymbol{p}^\perp$ of

$$\boldsymbol{p} := [0, \ldots, 0, 1, 0]$$

w.r.t. Q [2, p. 87].

Correspondence between points/planar sections of Q and oriented Euclidean lines/circles

Proposition 4.7. *The restriction of the central projection $\pi_{\boldsymbol{p}}$ (defined as in (2.2)) to $Q \setminus \tilde{Q}$ yields a double cover of E^*.*

Each two points of $Q \setminus \tilde{Q}$ that project onto the same point of E^* can be interchanged via $\sigma_{\boldsymbol{p}}$ (defined as in (2.1)), which implies reversing orientation of the corresponding Euclidean hyperplanes [2, p. 87]. Considering the related map $\overrightarrow{\pi_{\boldsymbol{p}}^*}$ that preserves the orientation, we get:

Proposition 4.8. *The set $Q \setminus \tilde{Q}$ bijectively corresponds to the set $\overrightarrow{\mathcal{P}}$ of oriented Euclidean hyperplanes via the map $\overrightarrow{\pi_{\boldsymbol{p}}^*} : Q \setminus \tilde{Q} \to \overrightarrow{\mathcal{P}}$, and the involution $\sigma_{\boldsymbol{p}}$ acts orientation reversing on $\overrightarrow{\mathcal{P}}$.*

Turning to the hyperspheres, we again identify them with their tangent hyperplanes. For each hypersphere those are dual to the points of a hypersphere in E^*, which is the central projection of a hyperplanar section of Q. Thus oriented hyperspheres bijectively correspond to hyperplanar sections of Q, where $\sigma_{\boldsymbol{p}}$ again acts orientation reversing. Just as in the 2-dimensional case, we cannot identify hyperspheres with the poles (w.r.t. Q) of the hyperplanes carrying the corresponding planar sections of Q (because Q is degenerate). Instead, we identify them with points in the dual projective space (in particular, we identify each hypersphere s with the vertex $\boldsymbol{x}_s$ of the cone $\Gamma(\boldsymbol{x}_s)$ intersecting E^* in s, and the hyperplane at infinity in the dual quadric Q^*), yielding the cyclographic model for dimension n [2, p. 87].

Euclidean Laguerre transformations

Finally, considering the hyperspheres as envelopes of their tangent hyperplanes, $\overrightarrow{\pi_{\boldsymbol{p}}^*}$ yields a bijective correspondence between points/hyperplanar sections of Q and hyperplanes/spheres of E. Since the transformations of $\mathrm{PO}(n,1,1)$ preserve Q, we get:

Proposition 4.9. *Every Laguerre transformation α of the n-dimensional Euclidean space can be written in the form:*

$$\alpha = \overrightarrow{\pi_{\boldsymbol{p}}^*} \circ t \circ \overrightarrow{\pi_{\boldsymbol{p}}^*}^{-1}$$

for some $t \in \mathrm{PO}(n,1,1)$.

Remark 4.10. The subgroup $\mathrm{PO}(n,1,1)_{\boldsymbol{p}} \subset \mathrm{PO}(n,1,1)$ doubly covers the group of dual Euclidean similarity transformations $\mathrm{PO}(n,0,1)$. In particular:

$$\mathrm{PO}(n,0,1) \simeq \mathrm{PO}(n,1,1)_{\boldsymbol{p}}/\sigma_{\boldsymbol{p}} \tag{4.3}$$

[2, p. 87]. The dual transformations of $\mathrm{PO}(n,0,1)$, i.e., the transformations of the group $\mathrm{PO}(n,0,1)^*$, are the Euclidean similarity transformations [2, p. 85].

4.2 Cyclographic model of Euclidean Laguerre geometry

In this section we first explain how to identify circles and lines in the cyclographic model of planar Euclidean Laguerre geometry. Then we show its relations to the quadric model, before proceeding to Euclidean Laguerre transformations in this model. This section is based on [1] and [9].

Correspondence between points/planes of $\mathbb{R}^3$ and oriented Euclidean circles/lines

For the cyclographic model of planar Euclidean Laguerre geometry, we embed the Euclidean plane $E = \mathbb{R}^2$ into the space $\mathbb{R}^3$ as the plane with equation $z = 0$. We then identify each point $X = (X_1, X_2, X_3)$ of $\mathbb{R}^3$ with an oriented circle c of E, namely the circle with the orthogonal projection (X_1, X_2) as center and radius X_3 (in particular, if $X_3 = 0$ we get a point of E, i.e., a Euclidean circle of radius 0). Geometrically, we can get the circle c by intersecting E with the cone $\Gamma(X)$ with vertex X and rulings with isotropic directions. Since the radius of the circle c is the X_3-coordinate of X, its orientation depends on whether X lies above or below E. Thus, we have a bijective correspondence between the points of $\mathbb{R}^3$ and oriented Euclidean circles.

Next, we identify each plane of $\mathbb{R}^3$ that intersects E at an angle of $\frac{\pi}{4}$ (which we call an *isotropic plane* from now on) with its intersection line with E. Since each two planes that can be interchanged by a reflection in E have the same intersection line, they yield different orientations. Thus, we also have a bijective correspondence between isotropic planes of $\mathbb{R}^3$ and oriented Euclidean lines [1, p. 136–137].

Definition 4.11. Consider the map

$$\begin{aligned} z : \mathbb{R}^3 &\to \vec{\mathcal{C}} \\ X = (X_1, X_2, X_3) &\mapsto z(X) := c((X_1, X_2), X_3), \end{aligned}$$

where $c((X_1, X_2), X_3)$ denotes the oriented circle with center (X_1, X_2) and oriented radius X_3 (i.e., $\operatorname{sgn}(X_3)$ encodes the orientation of the circle), and the map

$$\begin{aligned} \tilde{z} : \mathcal{P}_{\text{iso}} &\to \vec{\mathcal{L}} \\ p : p_0 + p_1 X_1 + p_2 X_2 + p_3 X_3 = 0 &\mapsto \tilde{z}(p) := l(n, d), \end{aligned}$$

where $\mathcal{P}_{\text{iso}}$ denotes the set of isotropic planes in $\mathbb{R}^3$ and $l(n, d)$ denotes the oriented line with oriented normal vector

$$n = \begin{cases} (p_1, p_2) & \text{if } p_3 < 0 \\ -(p_1, p_2) & \text{if } p_3 > 0 \end{cases}$$

(i.e., $\operatorname{sgn}(p_3)$ encodes the orientation of the line) and distance $d = \frac{-p_0}{\|n\|}$ to the origin. Then the pair $(z, \tilde{z})$ is called the *cyclographic map* [9, p. 6], [12, p. 167, 170].

Proposition 4.12. *The cyclographic map yields a bijective correspondence between points/isotropic planes of* $\mathbb{R}^3$ *and oriented circles/lines of* E.

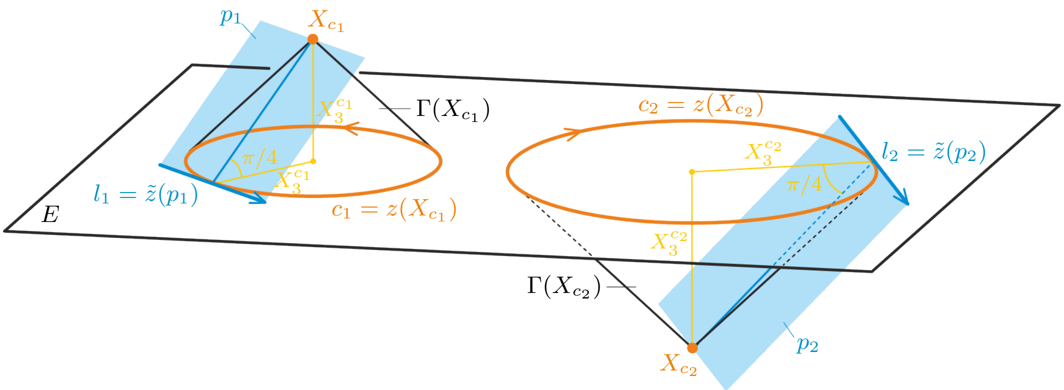


Figure 4.5: The cyclographic map

Now the question arises: Which connection is there between this cyclographic model and the quadric model from Section 4.1?

In Section 4.1.1, we mentioned that every oriented Euclidean circle c can be identified with a point in the (projectively extended) space, namely the vertex $\boldsymbol{x}_c$ of the cone $\Gamma(\boldsymbol{x}_c)$ (see Figure 4.3). The cone $\Gamma(\boldsymbol{x}_c)$ is the envelope of its tangent planes which (by duality) correspond to points of the planar section c_Q corresponding to the circle c. Every such point $\boldsymbol{q} = [q_1, q_2, q_3, q_4]$ of $c_Q \subset Q$ satisfies the equation of the quadric, i.e.,

$$\langle q, q \rangle_Q = q_1^2 + q_2^2 - q_3^2 = 0. \tag{4.4}$$

Since the corresponding tangent plane $\boldsymbol{q}^*$ to $\Gamma(\boldsymbol{x}_c)$ has the homogeneous coordinates $[q_1, q_2, q_3, q_4]$ of $\boldsymbol{q}$ as homogeneous plane coordinates, it satisfies the same equation. And since (4.4) is equivalent to $q_1^2 + q_2^2 = q_3^2$, the Euclidean intersection angle of $\boldsymbol{q}^*$ with E is $\frac{\pi}{4}$ (which explains the term "isotropic" for such a plane, since (4.4) implies that its normal vector is isotropic w.r.t. $\langle \cdot, \cdot \rangle_Q$). Thus, we see that $\Gamma(\boldsymbol{x}_c)$ is exactly the projectivization of the cone $\Gamma(X)$ with $X = z^{-1}(c)$ in the cyclographic model.

Before proceeding to Euclidean Laguerre transformations in the cyclographic model, let us investigate what isotropic lines, i.e., lines with isotropic directions, correspond to. Let l be an isotropic line of $\mathbb{R}^3$ and X an arbitrary point on l. Then l is a ruling of $\Gamma(X)$. Let $Y \neq X$ be another point on l. Then the unique isotropic plane through l corresponds to a common tangent of the circles $z(X)$ and $z(Y)$ (via $\tilde{z}$). Thus, isotropic lines of $\mathbb{R}^3$ correspond to pencils of circles that are all in oriented contact with a common tangent (see Figure 4.6). Hence we have:

Proposition 4.13. *Let c_1, c_2 be two oriented circles of E, and $X_{c_1} := z^{-1}(c_1), X_{c_2} := z^{-1}(c_2)$ their corresponding points of $\mathbb{R}^3$. Let Q be the quadric defined as in (4.1). Then the following statements are equivalent:*

- *The circles c_1 and c_2 are in oriented contact.*

- *The vector* $X_{c_1} - X_{c_2}$ *is isotropic.*
- $\langle X_{c_1} - X_{c_2}, X_{c_1} - X_{c_2} \rangle_Q = 0$ *[9, p. 7–8].*

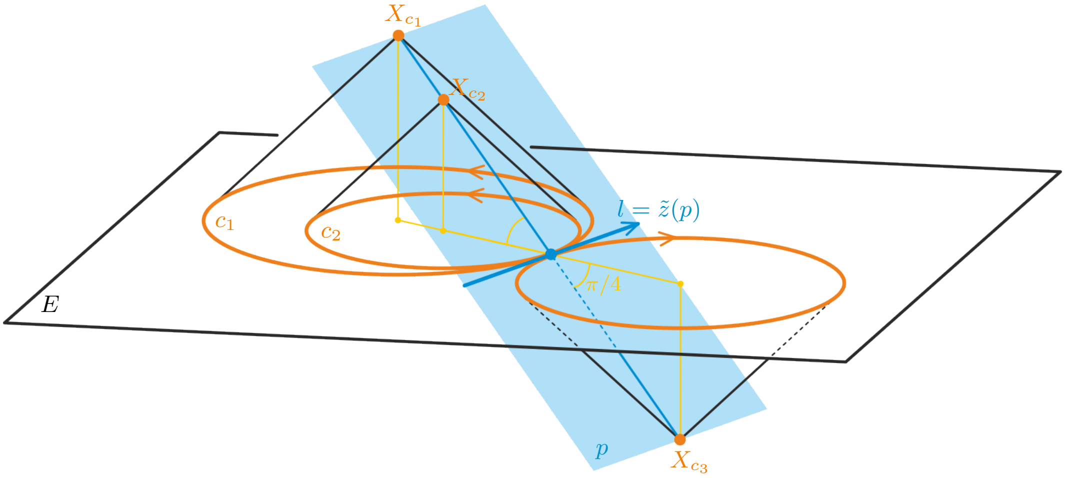


Figure 4.6: Pencil of circles in oriented contact with a common line in E

Euclidean Laguerre transformations

Analogous to the quadric model, we want to lift Euclidean Laguerre transformations to the cyclographic model. In the cyclographic model this is slightly more complicated due to the fact that the correspondence between objects of $\mathbb{R}^3$ and E is split into two maps $z, \tilde{z}$ (cf. Section 4.1.1). By splitting also each Laguerre transformation into two maps, operating on two different domains, it is possible though:

Proposition 4.14. *Let T be the group of transformations t of the form*

$$t(X) = \lambda M \cdot X + b \qquad \lambda \in \mathbb{R}, b \in \mathbb{R}^3, M \in \mathbb{R}^{3\times 3}$$

with M orthogonal w.r.t. the matrix $D := diag(1, 1, -1)$, i.e.,

$$M^T \cdot D \cdot M = D.$$

Let the pair $(\alpha, \tilde{\alpha})$ be a Euclidean Laguerre transformation, where α and $\tilde{\alpha}$ operate on the set of oriented Euclidean circles and lines respectively. Then the Laguerre transformation $(\alpha,\tilde{\alpha})$ can be decomposed as follows:

$$\alpha = z \circ t \circ z^{-1}$$
$$\tilde{\alpha} = \tilde{z} \circ t \circ \tilde{z}^{-1}$$

for some $t \in T$ [9, p. 13–15].

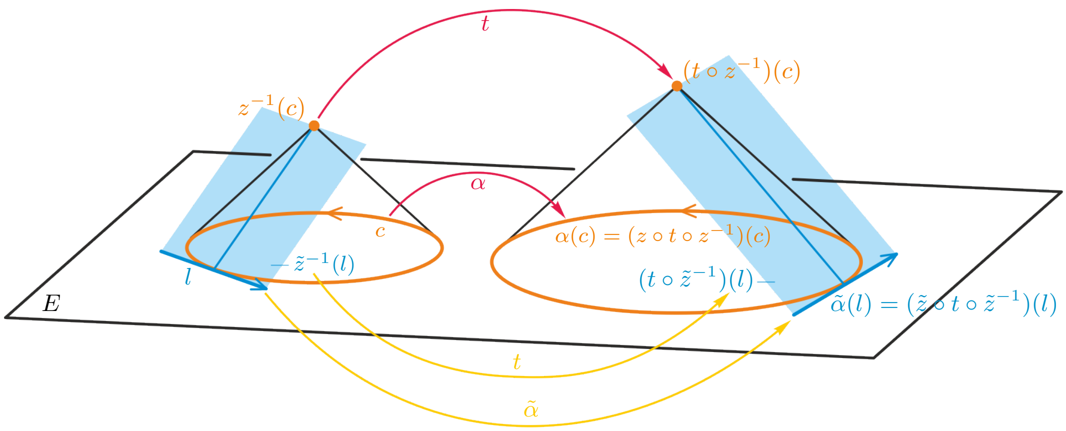


Figure 4.7: Lifting Euclidean Laguerre transformations to $\mathbb{R}^3$

An exact proof of Proposition 4.14 can be found in [9, p. 15-16]. For our purposes we just mention that the matrix D induces the bilinear form

$$\langle x, y \rangle := x^T \cdot D \cdot y = x_1 y_1 + x_2 y_2 - x_3 y_3$$

that corresponds to a quadric with signature $(2, 1)$. This quadric can be interpreted as the dual quadric Q^* of the Euclidean Laguerre quadric Q (defined as in (4.1)). The transformation matrix M of each transformation $t \in T$ being orthogonal w.r.t. D means that Q^* is preserved unter t, because for any $\boldsymbol{x} \in Q^*$ we have:

$$\langle Mx, Mx \rangle = (xM)^T \cdot D \cdot (Mx) = x^T \cdot (M^T \cdot D \cdot M) \cdot x = x^T \cdot D \cdot x = \langle x, x \rangle = 0.$$

Since every isotropic line/plane is the ruling/tangent plane of the cone $\Gamma(X)$ for some $X \in \mathbb{R}^3$ that intersects the plane at infinity in Q^*, isotropic lines and planes touch Q^*. Hence they are preserved under t and thus the cyclographic image of t preserves oriented Euclidean circles, lines and their oriented contact, i.e., is a Laguerre transformation [1, p. 138–140].

Finally, to conclude the chapter on Euclidean Laguerre geometry, we want to treat a few examples of Laguerre transformations of the Euclidean plane:

- Euclidean similarity transformations are (point-preserving) Laguerre transformations since they are induced by $\mathrm{PO}(2,1,1)_{\boldsymbol{p}} \subset \mathrm{PO}(2,1,1)$ (whose transformations induce Laguerre transformations according to Proposition 4.5).
- The cyclographic image of the reflection in the plane $X_3 = 0$ is a Laguerre transformation which fixes all Euclidean circles and lines but reverses their orientation (induced by $\sigma_{\boldsymbol{p}} \in \mathrm{PO}(2,1,1)_{\boldsymbol{p}}$).
- The cyclographic image of a translation in $\mathbb{R}^3$ parallel to the X_3-axis, i.e., the translation by a vector $(0, 0, d)$ for $d \in \mathbb{R}$, is called a *dilatation* and is a Laguerre

transformation that increases or decreases the radius of each circle by d. This means that each point gets "blown up" to a circle of radius d (see Figure 4.8). Thus, dilatations are not point-preserving [9, p. 16].

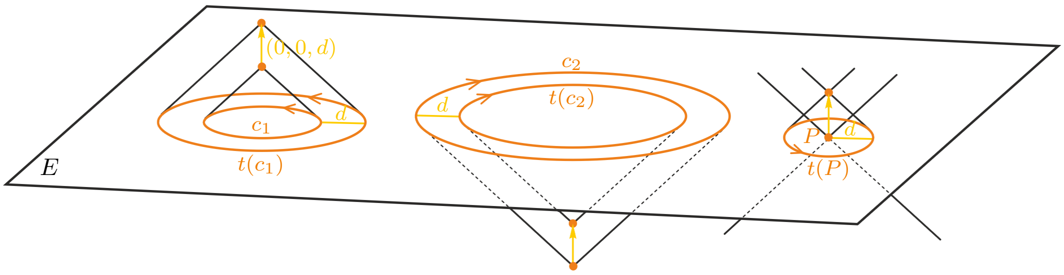


Figure 4.8: Dilatation t induced by the translation by $(0, 0, d)$

Note that the contact point $\bar{P}$ of the images $t(c_1), t(c_2)$ of two circles c_1, c_2, that are in oriented contact, under a dilatation t cannot be the image point of the original contact point P [1, p. 4], since P becomes a circle $t(P)$ of radius d touching $t(c_1)$ and $t(c_2)$. Nevertheless, the new contact point $\bar{P}$ needs to lie on all three circles, $t(c_1), t(c_2)$ and $t(P)$ (see Figure 4.9).

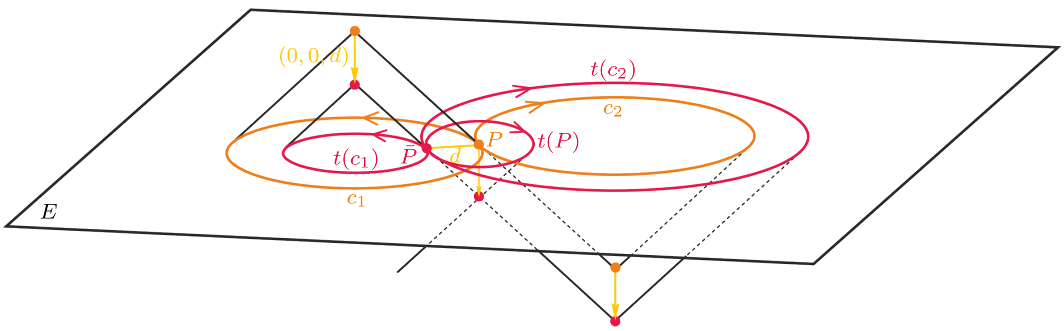


Figure 4.9: Oriented contact of image circles under a dilatation

Remark 4.15. Just like the quadric model, the cyclographic model can also be generalized to arbitrary dimensions by embedding the Euclidean space $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ [12, p. 167, 170].

5 Laguerre geometry from Lie geometry

In this chapter we will treat another perspective on Laguerre geometry, namely as subgeometry of Lie (sphere) geometry. For this purpose we give a short introduction to Lie geometry and work in the quadric model, to embed the Laguerre quadric $Q \subset \mathbb{P}^{n+1}$ (for n-dimensional Laguerre geometry) into the so-called (one dimension higher) "Lie quadric" $Q_L \subset \mathbb{P}^{n+2}$. This is based on [1], [2] and [3].

In analogy to our approach for Laguerre geometry, we explain the basic concept for dimension 2, for reasons of intuition and visual imagination. While in planar Laguerre geometry the fundamental objects are oriented circles and lines, in Lie geometry we are dealing only with oriented circles. Points are considered to be circles of radius 0, just as in Laguerre geometry, and lines are circles of radius ∞. All these types of (oriented) "circles" are referred to as *Lie circles*, while keeping the terms "circles", "lines" and "points" for the objects that we intuitively imagine as such, in order to avoid misunderstandings. Just as for Laguerre geometry, the basic relation is that of oriented contact between Lie circles, and *Lie transformations* preserve it as well as the Lie circles themselves. The "types" however are in general not preserved, i.e., for example an oriented circle can be mapped to a line or a point and vice versa. Here we can see that Laguerre transformations are special Lie transformations, namely those which preserve oriented lines.

Remark 5.1. Lie transformations which preserve points also form a special group of transformations, namely the well-known Möbius transformations [1, p. 3–4, 13–15, 177–178]. In particular, the group of Lie transformations is generated by the union of the Laguerre and Möbius groups [3, p. 29].

We now want to consider a quadric model of Lie geometry. For that purpose, we define

$$Q_L : \langle x, x \rangle_L := x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 = 0,$$

which is a quadric of signature $(3, 2)$, as the *Lie quadric* [2, p. 48]. While points on the Laguerre quadric Q only correspond to oriented lines, points on the Lie quadric Q_L correspond to Lie circles, i.e., oriented circles, lines and points. In particular, the correspondence is bijective (we denote it by b_L from here on) [3, p. 1], and in [2] or [3] we can find tables which show the correspondence between the coordinates of a point on Q_L and "elements" of the Lie circle that it uniquely determines (i.e., center and radius of a circle/unit normal vector and distance from the origin for a line). Especially for points in the plane it is common in literature to choose the quadric model in such a way that they would correspond to points on Q_L whose last component is 0 (representing the radius).

Particularly interesting are isotropic lines on Q_L. Analogous to the quadric models of planar hyperbolic and Euclidean Laguerre geometry, where isotropic lines of the Laguerre quadric Q correspond to pencils of parallel lines with their point at infinity as common point (see Remark 3.3 and 4.4), isotropic lines of the Lie quadric Q_L correspond to pencils of Lie circles with a common contact element. In particular, such a pencil can either consist of parallel oriented lines or of oriented circles with a common tangent [3, p. 22–27]. The latter case reminds us of the cyclographic model of Euclidean Laguerre geometry, where we concluded that isotropic lines correspond to pencils of oriented circles that are in oriented contact with a common line. We even get a similar result to Proposition 4.13, after defining:

Definition 5.2. Two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{P}^4$ which are orthogonal w.r.t. the bilinear form induced by Q_L, i.e., which satisfy

$$\langle x_1, x_2 \rangle_L = 0,$$

are called *Lie orthogonal.* In particular, Lie orthogonal points on Q_L lie on isotropic lines of Q_L [2, p. 48].

Proposition 5.3. *Let c_1, c_2 be two oriented circles and $\boldsymbol{x}_1 := b_L^{-1}(c_1), \boldsymbol{x}_2 := b_L^{-1}(c_2)$ their corresponding points on L. Then the following statements are equivalent:*

- *The circles c_1 and c_2 are in oriented contact.*
- *The line $\boldsymbol{x}_1 \vee \boldsymbol{x}_2$ is isotropic.*
- $\langle x_1, x_2 \rangle_L = 0.$
- *The points $\boldsymbol{x}_1, \boldsymbol{x}_2$ are Lie orthogonal [3, p. 25].*

Let us note: Lie circles correspond to points on Q_L and especially those which are in oriented contact lie on isotropic lines of Q_L. Recall that Lie transformations preserve Lie circles and their oriented contact. Now if we want to lift Lie transformations to transformations in $\mathbb{P}^4$ analogously to Laguerre transformations, we know that the required transformations need to preserve Q_L and need to preserve lines. Thus, due to Lemma 2.19, those are exactly the transformations of the group $\mathrm{PO}(3,2)$ [2, p. 48].

Proposition 5.4. *Every Lie transformation α can be written in the form:*

$$\alpha = b_L \circ t \circ b_L^{-1}$$

for some $t \in PO(3,2)$.

Now that we have an idea of Lie geometry, we want to see how Laguerre geometry comes into play here, or in terms of the quadric model, how we can embed the Laguerre quadric Q into the Lie quadric Q_L. In order to do that, we need the following definition:

Definition 5.5. Let $\boldsymbol{x}$ be a point in $\mathbb{P}^4$. Then the set

$$L \cap \boldsymbol{x}^\perp$$

is called the *sphere complex* determined by the point $\boldsymbol{x}$. The complex is called

- *elliptic* if $\boldsymbol{x} \in Q_L^+$,
- *hyperbolic* if $\boldsymbol{x} \in Q_L^-$,
- *parabolic* if $\boldsymbol{x} \in Q_L$ [2, p. 50–51].

Remark 5.6.

a) Since the points $\boldsymbol{y}$ in the polar hyperplane of a point $\boldsymbol{x}$ w.r.t. Q_L satisfy the equation $\langle x, y\rangle_L = 0$, a sphere complex can also be defined as the set of points $\boldsymbol{y} \in Q_L$ that are Lie orthogonal to the point $\boldsymbol{x} \in \mathbb{P}^4$ that determines the complex [3, p. 46].

b) Any two elliptic/hyperbolic/parabolic sphere complexes are equivalent up to a transformation of $\mathrm{PO}(3,2)$ [2, p. 50].

As mentioned before, it is common in the literature to distinguish the points $\boldsymbol{x}$ on Q_L which correspond to points in the plane, from such which correspond to oriented circles and lines, by their last component being 0, i.e., $x_5 = 0$. For this reason we consider the hyperbolic sphere complex determined by the point

$$\boldsymbol{p}_L := [0, 0, 0, 0, 1].$$

The complex $Q_M := Q_L \cap \boldsymbol{p}_L^\perp$ then consists of all points on Q_L that correspond to points in the plane, since all points of $\boldsymbol{p}_L^\perp$ satisfy $x_5 = 0$. Thus, Lie transformations whose corresponding transformations of $\mathrm{PO}(3,2)$ preserve Q_M are exactly the Möbius transformations since they preserve points. Q_M itself is a quadric of signature $(3,1)$ which can be identified as the so-called "Möbius quadric" [2, p. 49–50], i.e., a quadric that can be used for a model of Möbius geometry that works analogously to the ones for Laguerre and Lie geometry, where the points and planar sections of Q_M can be identified with points and circles, which are the fundamental objects of Möbius geometry [2, Section 5.4]. However, going into details here would go beyond the scope of this thesis. Our focus is Laguerre geometry, the geometry of oriented circles and lines, hence we want to find a sphere complex on Q_L that represents the oriented lines.

We consider the involution $\sigma_{\boldsymbol{p}_L}$ associated with the point $\boldsymbol{p}_L$. It should play the role of reversing the orientation in Lie geometry as well. Thus, the complex determined by a point $\boldsymbol{q} \in \mathbb{P}^4$, that should represent the oriented lines, must be preserved under $\sigma_{\boldsymbol{p}_L}$. This condition is equivalent to $\langle p_L, q\rangle_L = 0$. Hence the complex $\tilde{Q}_L := Q_L \cap \boldsymbol{q}^\perp$ for a point $\boldsymbol{q} \in \mathbb{P}^4$ with $\langle p_L, q\rangle_L = 0$ corresponds to the set of all oriented lines of the plane. Depending on whether the complex is elliptic, hyperbolic or parabolic, we recover different geometries:

- For $\boldsymbol{q} \in Q_L^+$, the elliptic complex $\tilde{Q}_L$ is a quadric of signature $(2,2)$ and can be identified with the hyperbolic Laguerre quadric. In this case, we rename $\boldsymbol{q} =: \boldsymbol{q}_{\text{hyp}}$ and $\tilde{Q}_L =: Q_{\text{hyp}}$.

  The central projection $\pi_{\boldsymbol{q}_{\text{hyp}}}$ associated with the point $\boldsymbol{q}_{\text{hyp}}$ projects the Lie quadric Q_L onto the inside of Q_{hyp}, in particular (Lemma 2.24 implies)

  $$\pi_{\boldsymbol{q}_{\text{hyp}}}(Q_L) = \tilde{Q}_L^- \cup \tilde{Q}_L = Q_{\text{hyp}}^- \cup Q_{\text{hyp}},$$

  and the point $\boldsymbol{p}_{\text{hyp}} := \pi_{\boldsymbol{q}_{\text{hyp}}}(\boldsymbol{p}_L)$ plays the role of the point $\boldsymbol{p}$ from the quadric model of hyperbolic Laguerre geometry (see Section 3.1). Thus, by applying the bijective correspondence b_{hyp} from Proposition 3.6, that is determined by the polar projection from $\boldsymbol{p}_{\text{hyp}}$, to the points of $\pi_{\boldsymbol{q}_{\text{hyp}}}(Q_L)$, we recover hyperbolic Laguerre geometry in the plane $\boldsymbol{p}_L^\perp \cap \boldsymbol{q}_{\text{hyp}}^\perp$. In particular,

  $$\tilde{Q}_{\text{hyp}} := Q_{\text{hyp}} \cap \boldsymbol{p}_{\text{hyp}}^{\perp_{Q_{\text{hyp}}}} = Q_L \cap \boldsymbol{q}_{\text{hyp}}^\perp \cap \boldsymbol{p}_{\text{hyp}}^{\perp_{Q_{\text{hyp}}}} \subset \boldsymbol{p}_L^\perp \cap \boldsymbol{q}_{\text{hyp}}^\perp$$

  (where the upper index $\perp_{Q_{\text{hyp}}}$ implies polarity w.r.t. Q_{hyp}) plays the role of the absolute circle $\tilde{Q}$ from Section 3.1.1.

- For $\boldsymbol{q} \in Q_L^-$, the hyperbolic complex $\tilde{Q}_L$ is a quadric of signature $(3,1)$ and can be identified with the elliptic Laguerre quadric. In this case, we rename $\boldsymbol{q} =: \boldsymbol{q}_{\text{ell}}$ and $\tilde{Q}_L =: Q_{\text{ell}}$.

  The central projection $\pi_{\boldsymbol{q}_{\text{ell}}}$ associated with the point $\boldsymbol{q}_{\text{ell}}$ projects the Lie quadric Q_L onto the outside of Q_{ell}, in particular (Lemma 2.24 implies)

  $$\pi_{\boldsymbol{q}_{\text{ell}}}(Q_L) = \tilde{Q}_L^+ \cup \tilde{Q}_L = Q_{\text{ell}}^+ \cup Q_{\text{ell}},$$

  and the point $\boldsymbol{p}_{\text{ell}} := \pi_{\boldsymbol{q}_{\text{ell}}}(\boldsymbol{p}_L)$ plays the role of the point $\boldsymbol{p}$ from the quadric model of hyperbolic Laguerre geometry (see Section 3.2). Thus, by applying the bijective correspondence b_{ell} from Proposition 3.33 that is determined by the polar projection from $\boldsymbol{p}_{\text{ell}}$ to the points of $\pi_{\boldsymbol{q}_{\text{ell}}}(Q_L) \setminus Q_{\text{hyp}} = Q_{\text{hyp}}^+$, we recover elliptic Laguerre geometry in the plane $\boldsymbol{p}_L^\perp \cap \boldsymbol{q}_{\text{ell}}^\perp$. In particular,

  $$\tilde{Q}_{\text{ell}} := Q_{\text{ell}} \cap \boldsymbol{p}_{\text{ell}}^{\perp_{Q_{\text{ell}}}} = Q_L \cap \boldsymbol{q}_{\text{ell}}^\perp \cap \boldsymbol{p}_{\text{ell}}^{\perp_{Q_{\text{ell}}}} \subset \boldsymbol{p}_L^\perp \cap \boldsymbol{q}_{\text{ell}}^\perp$$

  (where the upper index $\perp_{Q_{\text{ell}}}$ implies polarity w.r.t. Q_{ell}) plays the role of the imaginary cone $\tilde{Q}$ from Section 3.2.1.

- For $\boldsymbol{q} \in Q_L$, the parabolic complex $\tilde{Q}_L$ is a quadric of signature $(2,1,1)$ and can be identified with the Euclidean Laguerre quadric. In this case, we rename $\boldsymbol{q} =: \boldsymbol{q}_{\text{Euc}}$ and $\tilde{Q}_L =: Q_{\text{Euc}}$.

  Since the Euclidean Laguerre quadric is degenerate and thus we do not use polar projection, we apply central projection from the point $\boldsymbol{p}_{\text{Euc}} := \pi_{\boldsymbol{q}_{\text{Euc}}}(\boldsymbol{p}_L)$ to recover dual Euclidean geometry in $\boldsymbol{p}_{\text{Euc}}^{\perp_{Q_{\text{Euc}}}}$ (where the upper index $\perp_{Q_{\text{Euc}}}$ implies polarity w.r.t. Q_{Euc}), then apply duality to recover Euclidean geometry [2, p. 51–54].

Finally, the subgroup of the transformation group $\mathrm{PO}(3,2)$ (which induces Lie transformations) that fixes the point $\boldsymbol{q}$ (which determines the Laguerre quadric $Q = Q_L \cap \boldsymbol{q}^\perp$), i.e., $\mathrm{PO}(3,2)_{\boldsymbol{q}}$, doubly covers the projective orthogonal group inducing hyperbolic/elliptic/Euclidean Laguerre transformations [2, p. 52]:

$$\mathrm{PO}(3,2)_{\boldsymbol{q}}/\sigma_{\boldsymbol{q}} \simeq \begin{cases} \mathrm{PO}(2,2) & \text{if } \boldsymbol{q} \in Q_L^+ \\ \mathrm{PO}(3,1) & \text{if } \boldsymbol{q} \in Q_L^- \\ \mathrm{PO}(2,1,1) & \text{if } \boldsymbol{q} \in Q_L. \end{cases}$$

Remark 5.7. Since projective transformations that fix $\boldsymbol{p}_L$ and $\boldsymbol{q}$ also fix $\pi_{\boldsymbol{q}}(\boldsymbol{p}_L) \in \boldsymbol{q}^\perp$ (which plays the role of the center of the polar projection used to induce hyperbolic/elliptic/Euclidean Laguerre geometry in $\boldsymbol{p}_L \cap \boldsymbol{q}$) and since $\sigma_{\boldsymbol{p}_L}$ preserves $\boldsymbol{q}^\perp$, we also have:

$$\mathrm{PO}(3,2)_{\{\boldsymbol{p}_L,\boldsymbol{q}\}}/\{\sigma_{\boldsymbol{p}_L},\sigma_{\boldsymbol{q}}\} \simeq \begin{cases} \mathrm{PO}(2,2)_{\boldsymbol{p}_L}/\sigma_{\boldsymbol{p}_L} \simeq \mathrm{PO}(2,1) & \text{if } \boldsymbol{q} \in Q_L^+ \\ \mathrm{PO}(3,1)_{\boldsymbol{p}_L}/\sigma_{\boldsymbol{p}_L} \simeq \mathrm{PO}(3,0) & \text{if } \boldsymbol{q} \in Q_L^- \\ \mathrm{PO}(2,1,1)_{\boldsymbol{p}_L}/\sigma_{\boldsymbol{p}_L} \simeq \mathrm{PO}(2,0,1) & \text{if } \boldsymbol{q} \in Q_L. \end{cases}$$

This means that the subgroup of $\mathrm{PO}(3,2)$ that fixes $\boldsymbol{p}_L$ and $\boldsymbol{q}$, i.e., $\mathrm{PO}(3,2)_{\{\boldsymbol{p}_L,\boldsymbol{q}\}}$, quadruply covers the hyperbolic motions/elliptic motions/group $\mathrm{PO}(2,0,1)$ whose dual transformation group $\mathrm{PO}(2,0,1)^*$ consists of Euclidean motions and scalings [2, p. 53].

Lastly, it should be mentioned that everything can be generalized for arbitrary dimensions, i.e., Lie hyperspheres of the n-dimensional space can be identified with points on the Lie quadric of signature $(n+1,2)$ in $\mathbb{P}^{n+2}$, and the group $\mathrm{PO}(n+1,2)$ induces Lie transformations. The hyperbolic/elliptic/Euclidean Laguerre quadric (signature $(n,2)$, $(n+1,1)$ or $(n,1,1)$, respectively) can be recovered from the Lie quadric as intersection with the polar hyperplane of a point outside/inside/on the Lie quadric [2, Section 7.2].

6 Applications

In this chapter we show how the theory that we have acquired throughout this thesis can be applied in practice. We demonstrate this by solving a few chosen geometrical problems with Laguerre geometry.

6.1 Problem of Apollonius

The problem of Apollonius is posed as follows:

> *Given three circles in the plane, which are not all tangent to each other, find all circles that are tangent to all three of them.*

We will solve this problem for three oriented circles within the cyclographic model of planar Euclidean Laguerre geometry. There, three oriented circles c_1, c_2, c_3 can be lifted to points $X_{c_1}, X_{c_2}, X_{c_3}$ in $\mathbb{R}^3$, which are the vertices of the cones $\Gamma(X_{c_1}), \Gamma(X_{c_2}), \Gamma(X_{c_3})$ that intersect the base plane in c_1, c_2, c_3. Recall that in this model two circles are in oriented contact if and only if the connecting line of their corresponding points in $\mathbb{R}^3$ is isotropic. This means that for a circle c in oriented contact to c_1, $X_c := z^{-1}(c)$ must lie on (a ruling of) the cone $\Gamma(X_{c_1})$. Thus, we are looking for circles c whose corresponding points X_c lie on all three cones $\Gamma(X_{c_1}), \Gamma(X_{c_2})$ and $\Gamma(X_{c_3})$. In other words: We need to find the intersection points of those three cones. Each two of them intersect in a conic, yielding three conics that intersect in at most two points. These points correspond to the circles that we are looking for (via z) [9, p. 9].

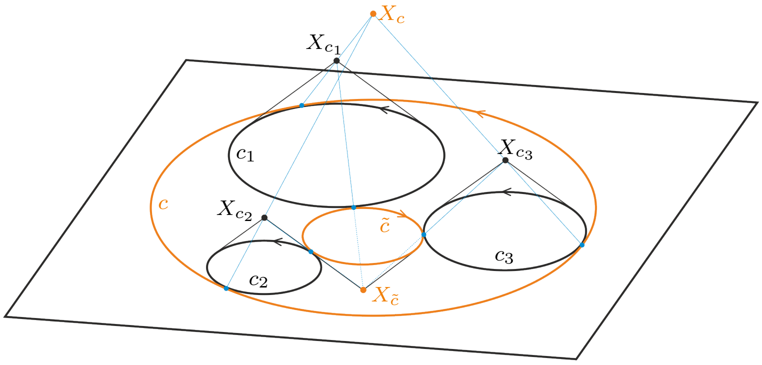


Figure 6.1: Solution circles for the problem of Apollonius (incl. corresponding cones in $\mathbb{R}^3$)

The 3-dimensional problem of Apollonius (where we determine the tangent sphere to 4 given spheres) is used for GPS (Global Positioning System). By measuring the time difference between the time at which a signal is sent by a satellite and when it is received by a vehicle, the GPS calculates the distance of the vehicle to the satellite. This distance is called a *pseudo range* since it is not accurate. It contains an error because the clocks of the satellite and the vehicle are not synchronized. Denoting the position of a satellite by S_i, its pseudo range by p_i and the error by e, the position V of the vehicle must lie on the sphere s_i with center S_i and radius $p_i + e$. Now if the GPS calculates the pseudo-distances to 4 satellites ($i \in \{1, 2, 2, 4\}$), the errors must be equal since the clocks of the satellites are synchronized, so V would be the intersection point of the spheres s_i. Since the value of the error is unknown, we instead find the inscribed sphere s with radius e touching all 4 spheres s_i and determine its center, which differs from s only by a Laguerre transformation (namely a dilatation by e). Thus, the center of s is exactly the position V of the vehicle (see Figure 6.2) [13], [5].

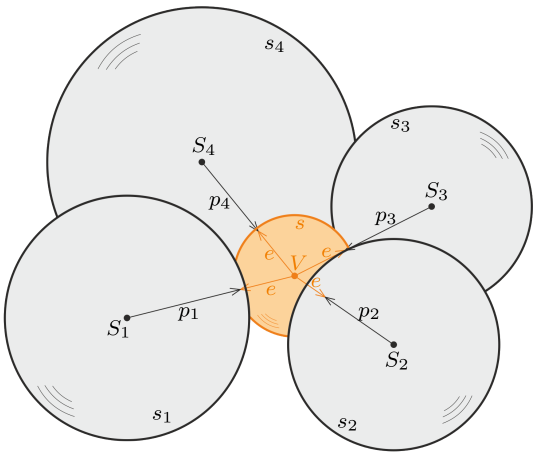


Figure 6.2: The principle of GPS

6.2 Offsets

In this section we show how to determine the offset of a planar curve using the cyclographic model of Laguerre geometry. For this purpose we first define:

Definition 6.1. The offset c^d of a planar curve c at a distance d is the envelope of a one-parameter family of circles with centers on c and radius d [9, p. 31].

Note that if the distance d is zero, we get exactly the original curve c, since the one-parameter family of circles then degenerates to the points of c. Since in the sense of Laguerre geometry, points are circles as well, we get an offset at an arbitrary distance $d \neq 0$ by applying a dilatation by d to the points of c and finding the envelope of the image circles [12, p. 166]. In terms of the cyclographic model (cf. Section 4.2) this dilatation corresponds to a translation t_d of c by the vector $(0, 0, d)$ (considering the plane that carries c as the base plane $z = 0$) (see Figure 6.4a). For each point x of c, the corresponding dilatated circle has a common tangent t with c^d parallel to the tangent of c in x [9, p. 31]. The cyclographic pre-image $\tilde{z}^{-1}(t)$ of this tangent is an isotropic plane containing the isotropic line through $t_d(x)$ and the contact point of t and c^d (see Figure 6.4b). For all points of c this one-parameter family of (isotropic) planes envelopes a developable surface containing $t_d(c)$ with isotropic rulings, intersecting the base plane exactly in c^d [12, p. 172]. In our case $t_d(c)$ is a planar curve, but this developable surface can more generally be found for a space curve from which each tangent is contained in at least one isotropic plane. This is the case only for certain types of directional vectors of the tangents. In Section 4.2 we have seen that the condition $\langle x, x\rangle = 0$ for the directional vector x of an isotropic line means that the line intersects the base plane at an angle of $\frac{\pi}{4}$. Analogously the definition of spacelike and timelike vectors (see Definition 2.7) implies that a spacelike/timelike line intersects at an angle smaller/bigger than $\frac{\pi}{4}$. Thus, through a spacelike tangent there are exactly two isotropic planes, while through a timelike tangent there are none. Therefore, the aforementioned developable surface can be defined for space curves with isotropic or spacelike tangents.

Definition 6.2. Let c be a space curve in $\mathbb{R}^3$ with spacelike or isotropic tangents. We call the developable surface(s), that the isotropic planes through the tangents of c envelope, the *isotropic developable(s)* of c.

Remark 6.3. A curve with only isotropic tangents possesses only one isotropic developable. Curves which have spacelike tangents as well have two isotropic developables (see Figure 6.3) [9, p. 25].

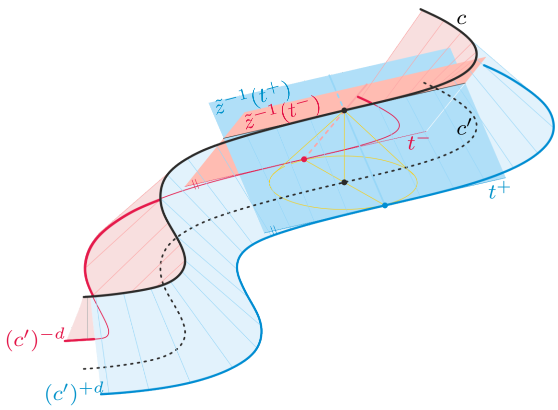
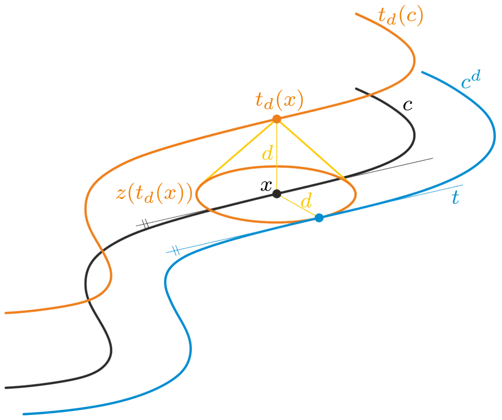
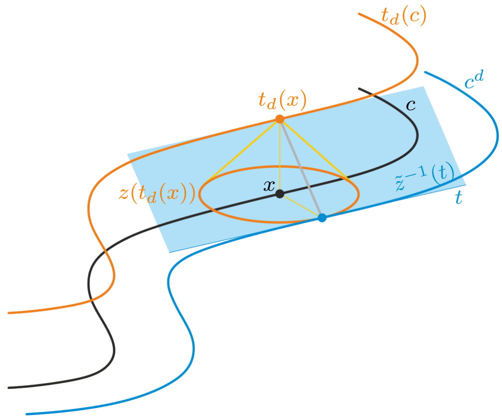


Figure 6.3: Isotropic developables of a curve c with spacelike tangents

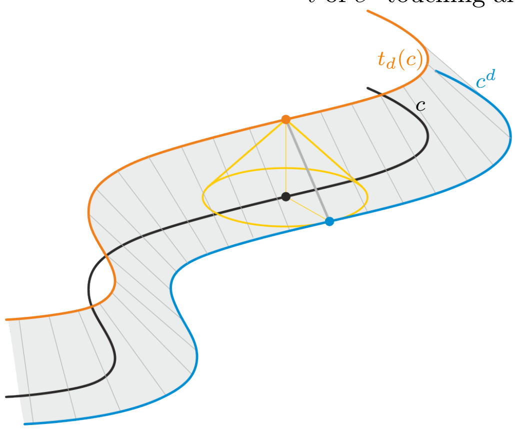
Thus, we get the offset of a curve c at a distance d by first applying translation t_d by $(0,0,d)$ to it, then intersecting the base plane with the isotropic developable of $t_d(c)$, that corresponds to the orientation of (the tangents of) c (see Figure 6.4c).



(a) Lifting a dilatation by d to a translation by $(0,0,d)$



(b) Pre-image plane of an oriented tangent t of c^d touching along ruling of $\Gamma(t_d(x))$



(c) Offset c^d as intersection with an isotropic developable of $t_d(c)$

Figure 6.4: Constructing the offset of a planar curve c within the cyclographic model

Remark 6.4. In the case of $t_d(c)$ having two isotropic developables, the intersection with the second one yields the offset of the oppositely oriented curve at distance d, or the offset of c at distance $-d$. Thus, for a certain orientation of c and a fixed d, the isotropic developable through c^d is uniquely determined. Therefore, to get the same developable surface, one can also "start" from a curve in the base plane and define the unique isotropic developable of it as the cyclographic pre-image of all circles in oriented contact with the planar curve, or equivalently, as the envelope of the pre-image planes of the oriented tangents of the planar curve.

Alternatively, one can also construct the isotropic developable of the original (planar) curve c (in the sense of Remark 6.4), intersect it with a plane $z = -d$ parallel to the base plane and project the intersection curve orthogonally back onto the base plane (which is equivalent to applying translation by $(0, 0, d)$ to the intersection curve) (see Figure 6.5) [12, p. 171–173].

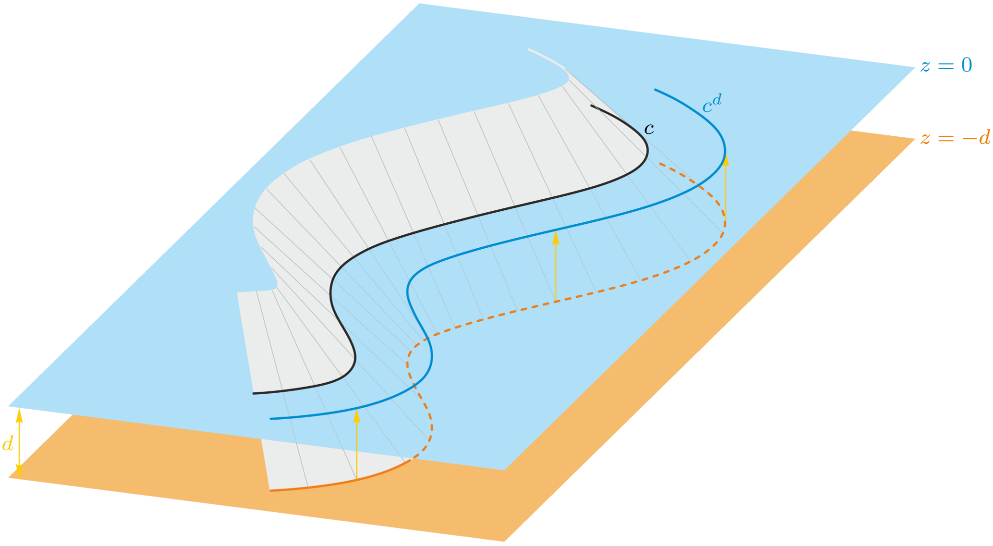


Figure 6.5: Isotropic developable of planar curve c

6.3 Medial axis

In this section we show how to get the medial axis of a polygon (or more general, of a closed curve) using the cyclographic model of Laguerre geometry.

Definition 6.5. The *medial axis* of a polygon or a closed curve is the closure of the set of all points which have the same distance to at least two points on the polygon/curve [7].

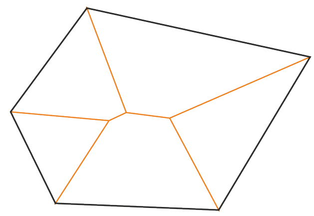


Figure 6.6: Medial axis of a polygon

Let us consider a polygon P whose edges are oriented in such a way that their normal vectors lie on the outside of P. We denote the inside, including the boundary, by I_P. Let x be a point on the medial axis m of P that has the same distance to two points x_1, x_2 of P. This means that there is a circle c_x with center x and radius $r_x := \overline{xx_1} = \overline{xx_2}$ touching P in x_1 and x_2. If we take the plane which P lies in as base plane for Euclidean Laguerre geometry and consider the pre-image of c_x under the cyclographic map z, the point $z^{-1}(c_x)$ lies above x and its z-coordinate is exactly r_x. We get $z^{-1}(c_x)$ by intersecting the rulings of the cone $\Gamma(z^{-1}(c_x))$ (=isotropic lines) through the points p_1 and p_2 (see Figure 6.7).

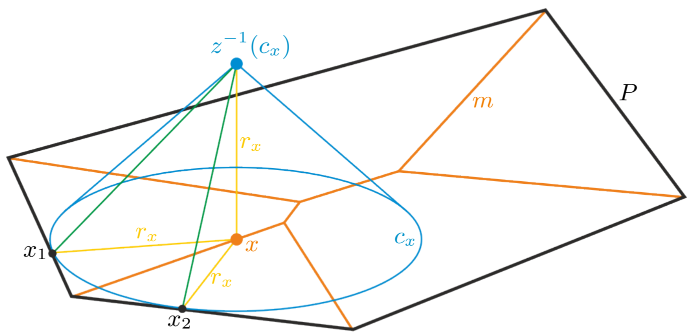


Figure 6.7: Cyclographic pre-image of a maximal circle c_x in I_P with center $x \in m$

Repeating this process for every point of m, i.e., finding the maximal circle with the point on m as its center and determining its pre-image under z, we get a polyline $\tilde{m}$, whose orthogonal projection to the base plane is the medial axis of P. We call $\tilde{m}$ the *medial axis transform* of P. Due to the argumentation above, through every point y of $\tilde{m}$ there are at least two isotropic lines that intersect P in the contact points of the circle $z(y)$. Thus, these lines corresponding to all points on $\tilde{m}$ cover the isotropic developables through the edges of P. In other words: Intersecting the isotropic developables through the (oriented) edges of a polygon and projecting the intersection polyline onto the base plane orthogonally yields the medial axis of the polygon (see Figure 6.8). In particular, the orthogonal projection of the intersection polyline, which we call the *untrimmed medial axis transform*, carries more points than just the medial axis, therefore we call it the *untrimmed medial axis*. To get the actual medial axis one has to remove points y from the untrimmed medial axis transform whose orthogonal projection y' does not lie in I_P and whose cyclographic image circles $z(y)$ are not maximal circles in I_P (i.e., where the radius of $z(y)$ is not the minimal distance of y' to P), yielding the medial axis transform $\tilde{m}$.

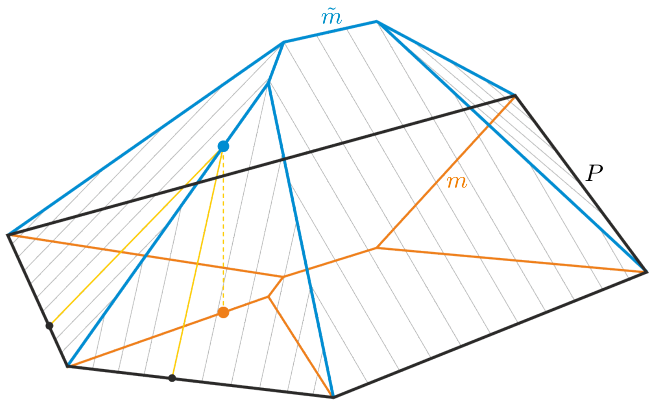


Figure 6.8: Medial axis transform $\tilde{m}$ and medial axis m of the polygon P

In the same way we can find the medial axis of a closed curve: The isotropic developables through the components of a closed curve c intersect in the untrimmed medial axis transform. Removing points y whose orthogonal projection y' does not lie on the inside I_c of the curve and whose cyclographic image circles $z(y)$ are not maximal circles in I_c, we get the medial axis transform whose orthogonal projection is the medial axis of c [12, p. 173–174].

Remark 6.6. Since the edges/components of a polygon/closed curve are the envelopes of the cyclographic image circles of the points of the medial axis transform, the medial axis transform is sometimes used for shape reconstruction. As shown in the previous section, we can also get offsets of the polygon/curve by applying a translation to the medial axis transform [12, p. 175].

7 Conclusion

Since the goal of this thesis was mainly to give an overview over the theory of Laguerre geometry in different classical geometries, we have only seen a few applications of it in the last chapter. Nevertheless, there exist plenty more, e.g., if we proceed further in the area of offsets, Laguerre-geometric methods allow us to obtain any so-called “PH-curve” (or “PN surface”), whose offset is always rational, which is important for CAGD (for details, see [10]). As an example for a higher-dimensional application, a canal surface (=envelope of a one-parameter family of spheres) can be interpreted as envelope of the cyclographic image spheres of a curve in $\mathbb{R}^4$ [12, p. 176–177]. Other Laguerre-geometric objects are “S*-nets” (=quadrilateral meshes where each quad possesses a common tangential sphere with its four adjacent quads) [8, p. 7] and “Laguerre checkerboard incircular nets” (=quadrilateral meshes where the edges of every second quad - arranged in a checkerboard pattern - are in oriented contact with a common circle) [2, p. 58–59], where the latter ones pose an example for an application of non-Euclidean Laguerre geometry as well.

As indicated above (non-)Euclidean Laguerre geometry has many applications in fields like applied geometry, CAGD, incidence geometry, discrete differential geometry, etc. and appears to leave many interesting and useful properties yet to be discovered.

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