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Fourier Diffraction Theorem
and Application in Tomography

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Abstract

Diffraction tomography is a powerful method to obtain a three-dimensional visualisation of a light penetrable object from a number of optical two-dimensional images. It is used in many different fields of science, such as medicine, radiology, physics and biology. Diffraction is a physical effect describing scattering in weakly scattering objects. Our particular studies are motivated from tomographical optical imaging where the probe is moved contact-free with tweezers. The advantage is that the probe can be imaged in a “more” natural enviroment. The downside is that the induced motion of the probe is rather irregular. Thus, tomography with irregular movement will be treated. A valuable tool for deriving the reconstruction formulas for these applications is the Fourier Diffraction Theorem. An analytical investigation of the theorem as well as the mathematical background and the applications of the theorem in diffraction imaging will be presented.

Abstract

Beugungstomographie ist eine effiziente Methode, um eine drei-dimensionale Visualisierung eines licht-durchdringbaren Objekts aus optischen zwei-dimensionalen Bildern. Sie wird in verschiedenen wissenschaftlichen Fachgebieten wie in der Medizin, Radiologie, Physik und Biologie angewendet. Die Beugung ist ein physikalischer Effekt, der die Streuung eines schwach streuenden Objekt beschreibt. Die Masterarbeit ist von tomographische optische Abbildung, bei denen die Probe kontakt-frei mit Pinzetten bewegt wird, motiviert. Der Vorteil ist, dass die Probe in einer “natürlicherer” Umgebung studiert werden kann. Der Nachteil ist, dass die induzierte Bewegung der Sonde irregulär ist. Deswegen wird die Tomographie mit irregulärer Bewegung betrachtet. Ein wertvolles Werkzeug für die Herleitung von Rekonstruktionsformeln ist der Fourier’sche Beugungssatz. Die analytische Untersuchung des Satzes zusammen mit dem mathematischen Hintergrund und die Anwendung des Satzes sind in der Masterarbeit ausgeführt.

Contents

1 Introduction

We use tomography as a synonym for recontructing a three-dimensional object from a sequence of two-dimensional recordings. In diffraction tomography, the object is illuminated by waves from different directions and transmission or reflection recordings give information about the interior of an object. Diffraction is the phenomenon, which describes propagation of waves in weakly scattering objects. Tomographic inversion of waves produced by weakly scattering objects is called diffraction tomography. For example, the tomography using ultrasound or microwaves falls within the scope of diffraction, while using X-ray does not. When it comes to application, diffraction tomography plays a significant part in medical imaging.

While the Fourier Slice Theorem is a fundamental theorem of X-ray tomography, the Fourier Diffraction Theorem is the most valuable theorem for diffraction tomography. It shows the relation between the two-dimensional spatial Fourier transform of the waves resulting from illumination and the three-dimensional spatial Fourier transform of the object.

In order to study Fourier Diffraction Theorem, some fundamental concepts are required. Two of which are Fourier transform and partial Fourier transform, which will be discussed in Section 2 and 3 together with their basic properties defined on different spaces. In particular, in the experiment described later in Section 4, the one-dimensional Fourier transform is applied to the function with respect to one component, while the other two components are fixed. The reason is that the wave, which we want to measure, is recorded at a fixed plane. Therefore, the measured data are in fact the Fourier transforms of the outgoing waves with respect to the first two components only.

In Section 4, a diffraction experiment, which requires the application of the Fourier Diffraction Theorem, is investigated. An incident wave propagates through the scattering object, which is exposed to irregular motion and can be trapped by instruments called tweezers. The measured data are the Fourier transform of the scattered waves. It is required to reconstruct the images of the object from the measurements. Since the probe is weakly scattering, Born and Rytov approximation describe the scattering reasonably well.

Afterwards, the Fourier Diffraction Theorem and its proof in three-dimensional case are studied thoroughly in Section 5.

The measurements gained from the experiment are only given by the data on a space covered by the phase and spatial frequencies with $k_1^2 + k_2^2 < k_0^2$, where k_0 is the wave number of the incident field. This space is called the k-space coverage. Different k-space coverages of different experimental settings will be observed in Section 6.

Finally, in Section 7, a numerical application and visualisation of the experimental results in a two-dimensional setting are considered.

Some background information ultilised in the previous sections is given in the Appendices.

This thesis is written by two authors: Milica Uzelac worked on Section 2, 3 and Subsection 7.1; Thi Lan Nhi Vu worked on Section 4, 5, 6 and Subsection 7.2.

2 Fourier transform and distribution theory (M.U.)

Fourier transform plays a very important role in the study of the Fourier Diffraction Theorem. In this section, we will study the basic properties of the Fourier transform defined on different spaces. In order to achieve the task, we first need to define some function spaces.

Let $1 \leq p < \infty$ and $(\Omega, \Sigma(\Omega), \mu)$ be a measure space with

$$\mathscr{L}^p(\Omega,\mu) = \left\{ f : \Omega \to \mathbb{C} \;\middle|\; f \text{ measurable}, \int_\Omega \left|f(x)\right|^p \,\mathrm{d}\mu(x) < \infty \right\}.$$

For $f \in \mathscr{L}^p(\Omega,\mu)$, define

$$\|f\|_{L^p} := \left(\int_\Omega \left|f(x)\right|^p \,\mathrm{d}\mu(x) \right)^{\frac{1}{p}}.$$

In this $\mathscr{L}^p(\Omega,\mu)$ space, $\|f\|_{L^p}$ is not a norm, since it does not fulfil

$$\|f\|_{L^p} = 0 \Rightarrow f = 0$$

for $f \in \mathscr{L}^p(\Omega,\mu)$. From $\|f\|_{L^p} = 0$, we can only get that $f = 0$ μ-almost everywhere. Therefore, we define the equivalence relation

$$f \sim g \iff \|f - g\|_{L^p} = 0,$$

i.e., the two functions f and g are called equivalent if they coincide μ-almost everywhere. This leads to the definition of the space

$$L^p(\Omega,\mu) := \mathscr{L}^p(\Omega,\mu)/\sim .$$

We can understand this $L^p(\Omega,\mu)$ space as a set of all equivalence classes of measurable function f, for which $\|f\|_{L^p}$ is finite. In this space, if we have that $f = g$ μ-almost everywhere, they are then considered as equal, i.e. $f \equiv g$. Hence, the property

$$\|f\|_{L^p} = 0 \Rightarrow f = 0$$

is fulfilled and $\|f\|_{L^p}$ is a norm in $L^p(\Omega,\mu)$. Furthermore, $L^p(\Omega,\mu)$ is a complete space, that is, every Cauchy sequence $(f_k)_k \subseteq L^p(\Omega,\mu)$ converges to a function

$f \in L^p(\Omega, \mu)$ (cf. Theorem 1.34 in [19]). Thus, $L^p(\Omega, \mu)$ is a Banach space. A function $f \in L^p(\Omega, \mu)$ is called a p-integrable function.

Analogously, for $p = \infty$, we have

$$L^\infty(\Omega, \mu) = \mathscr{L}^\infty(\Omega, \mu)/\{f \in \mathscr{L}^\infty(\Omega, \mu) \mid \|f\|_{L^\infty} = 0\}$$

with

$$\mathscr{L}^\infty(\Omega, \mu) = \left\{ f : \Omega \to \mathbb{C} \;\middle|\; f \text{ measurable}, \|f\|_{L^\infty} < \infty \right\}$$

and

$$\|f\|_{L^\infty} := \inf\{C > 0, C < \infty \mid |f| \leq C \ \mu - a.e\}.$$

$L^\infty(\Omega, \mu)$ is also a Banach space (Theorem 1.34 in [19]).

If the choice of μ is understandable from the context, we can write $L^p(\Omega)$, for all $p \in [1, \infty]$. So, in this paper, we have the Banach space $L^p(\Omega)$ with finite norm

$$\|f\|_{L^p} := \left(\int_\Omega \left| f(\mathbf{r}) \right|^p \, \mathrm{d}\mathbf{r} \right)^{\frac{1}{p}}$$

for $p \in [1, \infty)$ and

$$\|f\|_{L^\infty} := \operatorname*{esssup}_{\mathbf{r} \in \Omega} \left| f(\mathbf{r}) \right|.$$

For $1 \leq p \leq q$, the space $L^q(\Omega)$ is continuously embedded in $L^p(\Omega)$, if $\Omega \subset \mathbb{R}^n$ is bounded.

We also define the space of p-locally integrable functions

$$L^p_{\text{loc}}(\Omega) := \{f : \Omega \to \mathbb{C} \mid \ f|_K \in L^p(K) \ \forall K \subset \Omega, K \text{ compact}\},$$

where $f|_K$ is the restriction of f to the set K.

Moreover, for $\Omega \subseteq \mathbb{R}^n$, we denote:

(i) $\mathscr{C}(\Omega) := \{f : \Omega \to \mathbb{C} \mid f \text{ continuous}\}$ the space of continuous functions endowed with the $\| \cdot \|_{L^\infty}$ norm.

(ii) $\mathscr{C}_c(\Omega) := \{f : \Omega \to \mathbb{C} \mid f \text{ continuous with } \operatorname{supp}(f) \subset K \subset \mathbb{R}^n, K \text{ compact}\}$, where

$$\operatorname{supp}(f) := \Omega \backslash \{y \in \Omega \mid \exists \text{ neighborhood } U \ni y : f|_U = 0\}$$

the space of continuous functions with compact support.

(iii) $\mathscr{C}^k(\Omega) := \{f : \Omega \to \mathbb{C} \mid \forall \alpha \in N_0^n, |\alpha| \leq k, \partial^\alpha f \in \mathscr{C}(\Omega) \ \}$ the space of $k-$times differentiable functions endowed with the norm

$$\|f\|_{\mathscr{C}^k(\Omega)} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty}$$

(iv) $\mathscr{C}^\infty(\Omega) := \{f : \Omega \to \mathbb{C} \mid f \text{ infinitely differentiable}\}$ the space of smooth functions.
We say that a sequence $(f_k)_k \subseteq \mathscr{C}^\infty(\Omega)$ converges to $f \in \mathscr{C}^\infty(\Omega)$ if and only if for all compact subsets K of Ω, for all $\alpha \in \mathbb{N}_0^n$, we have

$$\sup_{x\in K} |\partial^\alpha f_k - \partial^\alpha f| = 0.$$

(v) $\mathscr{C}_c^\infty(\Omega) := \mathscr{C}_c(\Omega) \cap \mathscr{C}^\infty(\Omega)$ the space of smooth functions with compact support or the space of test functions.
A sequence $(f_k)_k \subseteq \mathscr{C}_c^\infty(\Omega)$ converges to $f \in \mathscr{C}_c^\infty(\Omega)$ if and only if there exists a compact subset K of Ω so that for all $\alpha \in \mathbb{N}_0^n$, we have

$$\sup_{x\in K} |\partial^\alpha f_k - \partial^\alpha f| = 0.$$

In the following subsections, we will first study the classical Fourier transform that is defined on the function space $L^1(\mathbb{R}^n)$. Then, we work with the Fourier transform defined on a subspace that is closed under pointwise multiplication of $L^1(\mathbb{R}^n)$. Afterwards, we extend the definition of the Fourier transform to the only p-integrable function space that is a Hilbert space, namely: $L^2(\mathbb{R}^n)$. The study of the Fourier transform in this space does not belong to the field of classical analysis anymore. Finally, we extend our knowledge of the properties of the Fourier transform to a larger space, in particular, the space of tempered distributions. The theory of distributions will be introduced in Subsection 2.4.

2.1 Fourier transform on $L^1(\mathbb{R}^n)$

Now, we define the Fourier transform for functions in $L^1(\mathbb{R}^n)$. A function that depends on the space variables will be transformed into function that depends on the spatial frequencies.

Definition 2.1. For $f \in L^1(\mathbb{R}^n)$, that is, $f : \mathbb{R}^n \to \mathbb{C}$ integrable,

$$\mathscr{F} f(\mathbf{k}) := \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \,\mathrm{d}\mathbf{r} \quad \forall \mathbf{k} \in \mathbb{R}^n \tag{2.1}$$

is called the Fourier transform of f.

Then, we present some basic properties of the Fourier transform.

Lemma 2.2 (Properties of Fourier transform)**.** *For $f, g \in L^1(\mathbb{R}^n)$, $0 \neq \mathbf{a} \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{C}$, we have:*

(i) $\mathscr{F}(\alpha f + \beta g) = \alpha \mathscr{F}(f) + \beta \mathscr{F}(g)$,

(ii) $\mathscr{F} f(\cdot - \mathbf{a}) = e^{-i\mathbf{a}\cdot} \mathscr{F} f$,

(iii) $\mathscr{F}(e^{-i\mathbf{a}\cdot} f) = \mathscr{F} f(\cdot + \mathbf{a})$,

(iv) $\mathscr{F} f(\frac{\cdot}{\mathbf{a}}) = |\mathbf{a}|^n \mathscr{F} f(\mathbf{a}\mathbf{k})$,

(v) $\mathscr{F}(f * g) = (\sqrt{2\pi})^n (\mathscr{F} f)(\mathscr{F} g)$.

Proof. (i) Follows from the linearity of the integral.

(ii) We first apply the Definition 2.1 above to have

$$\mathscr{F} f(\cdot - \mathbf{a})(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{r} - \mathbf{a}) e^{-i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}\mathbf{r}.$$

Substitution $\mathbf{r} - \mathbf{a} = \mathbf{p}$ yields

$$\mathscr{F} f(\cdot - \mathbf{a})(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{p}) e^{-i\mathbf{k}\cdot(\mathbf{p}+\mathbf{a})} \, \mathrm{d}\mathbf{p}.$$

Now, putting the exponential part, which does not depend on $\mathbf{p}$, in front of integral, we arrive at

$$\mathscr{F} f(\cdot - \mathbf{a})(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{a}} \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{p}) e^{-i\mathbf{k}\cdot\mathbf{p}} \, \mathrm{d}\mathbf{p} = e^{-i\mathbf{k}\cdot\mathbf{a}} \mathscr{F} f(\mathbf{k}).$$

(iii) Here we apply Definition 2.1 on $e^{-i\mathbf{a}\cdot} f$ to get

$$\mathscr{F}(e^{-i\mathbf{a}\cdot} f)(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{-i\mathbf{a}\cdot\mathbf{r}} f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}\mathbf{r}.$$

Rearranging the exponential part and applying again Definition 2.1 on function f leads to the following

$$\mathscr{F}(e^{-i\mathbf{a}\cdot} f)(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{r}) e^{-i(\mathbf{k}+\mathbf{a})\cdot\mathbf{r}} \, \mathrm{d}\mathbf{r} = \mathscr{F} f(\mathbf{k} + \mathbf{a}).$$

(iv) Analogous to (ii) with substitution $\frac{\mathbf{r}}{\mathbf{a}} = \mathbf{p}$.

(v) We know that $f * g \in L^1(\mathbb{R}^n)$, due to the Young's theorem (cf. Theorem B.1, Appendix B). First, utilising Definition 2.1 on $f * g \in L^1(\mathbb{R}^n)$ we have

$$\mathscr{F}(f * g)(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{p}) g(\mathbf{r} - \mathbf{p}) \, \mathrm{d}\mathbf{p} \; e^{-i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}\mathbf{r}.$$

Then, changing the order of integration yields

$$\mathscr{F}(f * g)(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{p}) \int_{\mathbb{R}^n} g(\mathbf{r} - \mathbf{p}) e^{-i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}\mathbf{r} \, \mathrm{d}\mathbf{p}.$$

Multiplying and dividing with $e^{i\mathbf{k}\cdot\mathbf{p}}$ gives

$$\mathscr{F}(f * g)(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{p}) \int_{\mathbb{R}^n} g(\mathbf{r} - \mathbf{p}) e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{p})} \, \mathrm{d}\mathbf{r} \; e^{-i\mathbf{k}\cdot\mathbf{p}} \, \mathrm{d}\mathbf{p}.$$

Finally, applying Definition 2.1 once for g and once for f, we arrive at the assertion, namely,

$$\mathscr{F}(f * g)(\mathbf{k}) = \int_{\mathbb{R}^n} f(\mathbf{p})\mathscr{F}g(\mathbf{k})\ e^{-i\mathbf{k}\cdot\mathbf{p}}\ \mathrm{d}\mathbf{p} = (\sqrt{2\pi})^n \mathscr{F}f(\mathbf{k})\mathscr{F}g(\mathbf{k}).$$

□

Furthermore, one of the most important properties of the Fourier transform is that it is a bounded operator that maps L^1-functions to continuous functions:

Lemma 2.3. *The Fourier transform* $\mathscr{F} : L^1(\mathbb{R}^n) \to \mathscr{C}(\mathbb{R}^n)$ *is a linear and bounded operator.*

Proof. We prove first the continuity of $\mathscr{F}f$. If the sequence $\mathbf{k_n}$ converges to some $\mathbf{k}$, then we also have the following pointwise convergence

$$f(\mathbf{r})e^{-i\mathbf{k_n}\cdot\mathbf{r}} \longrightarrow f(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}} \qquad \forall \mathbf{r} \in \mathbb{R}^n.$$

Since $f \in L^1(\mathbb{R}^n)$ and

$$\left| f(\mathbf{r})e^{-i\mathbf{k_n}\cdot\mathbf{r}} \right| \leq \left| f(\mathbf{r}) \right| \qquad \forall \mathbf{r} \in \mathbb{R}^n,$$

we can conclude, using Lebesgue dominated convergence theorem (cf. Theorem B4, Appendix B), that

$$\frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{r})e^{-i\mathbf{k_n}\cdot\mathbf{r}}\ \mathrm{d}\mathbf{r} \longrightarrow \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}\ \mathrm{d}\mathbf{r},$$

or equivalently,

$$\mathscr{F}f(\mathbf{k_n}) \longrightarrow \mathscr{F}f(\mathbf{k}).$$

Therefore, we have that $\mathscr{F}f \in \mathscr{C}(\mathbb{R}^n)$.

Moreover, the space of continuous functions $\mathscr{C}(\mathbb{R}^n)$ endowed with the norm $\|\cdot\|_{L^\infty}$ is a Banach space (Theorem 1.28 in [19]). Therefore, $\mathscr{F}$ is an operator between Banach spaces. The linearity is shown in Lemma 2.2.

Now, to show the boundedness of $\mathscr{F}$, we consider the operator norm of $\mathscr{F}$, which is given by $\|\mathscr{F}\|_{L^1 \to \mathscr{C}} = \frac{1}{(\sqrt{2\pi})^n} < \infty$. Indeed, we can show this in two steps. First, from

$$\left|\mathscr{F}f(\mathbf{k})\right| \leq \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \left|f(\mathbf{r})\right| |e^{-i\mathbf{k}\cdot\mathbf{r}}|\ \mathrm{d}\mathbf{r} = \frac{1}{(\sqrt{2\pi})^n} \|f\|_1$$

for all $f \in L^1(\mathbb{R}^n)$ and all $\mathbf{k} \in \mathbb{R}^n$, we get $\|\mathscr{F}f\|_{L^\infty} \leq \frac{1}{(\sqrt{2\pi})^n}$. Then, choosing

$$f(\mathbf{r}) = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{\|\mathbf{r}\|^2}{2}},$$

we have $\|\mathscr{F}f\|_{L^\infty} = \frac{1}{(\sqrt{2\pi})^n}$. Therefore, $\mathscr{F}$ is bounded. □

We have another important property of the Fourier transform of integrable functions:

Lemma 2.4 (Riemann-Lebesgue)**.** *For $f \in L^1(\mathbb{R}^n)$, we have $\left|\mathscr{F} f(\mathbf{k})\right| \to 0$ as $|\mathbf{k}| \to \infty$.*

Proof. Proposition 2.2.17 in [2]. □

Remark 2.5. From the lemma above, we know that for an integrable function f, $\mathscr{F} f \in \mathscr{C}(\mathbb{R}^n)$. Unfortunately, in general, it does not hold that $\mathscr{F} f \in L^1(\mathbb{R}^n)$. For example, choose the 1-dimensional function

$$f(r) = \begin{cases} \sqrt{\dfrac{\pi}{2}} & \text{for } |r| \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

From

$$\int_{\mathbb{R}} |f(r)| \, \mathrm{d}r = \int_{-1}^{1} \sqrt{\frac{\pi}{2}} \, \mathrm{d}r = \sqrt{2\pi} < \infty,$$

we can conclude that $f \in L^1(\mathbb{R})$. The Fourier transform of f,

$$\mathscr{F} f(k) = \frac{\sin(x)}{x}$$

(formula 17.23.22 in [3]), is not an integrable function (cf. [25], pg. 73). Therefore, in general, we cannot write the inverse transform as a Lebesgue integral, that is, the Fourier operator $\mathscr{F}$ defined on $L^1(\mathbb{R}^n)$ is not an isomorphism. Thus, we want to define the Fourier transform on some other spaces, where $\mathscr{F}$ is guaranteed to be an isomorphism.

We define a new function space:

Definition 2.6. The function space

$$\mathscr{S}(\mathbb{R}^n) := \left\{\psi \in \mathscr{C}^\infty(\mathbb{R}^n) \mid \ p_{\alpha,\beta}(\psi) < \infty \ \forall \alpha, \beta \in \mathbb{N}_0^n\right\}$$

where

$$p_{\alpha,\beta}(\psi) := \sup_{\mathbf{r} \in \mathbb{R}^n} \left|\mathbf{r}^\alpha \partial^\beta \psi(\mathbf{r})\right|, \tag{2.2}$$

is called Schwartz space or the space of rapidly decreasing functions.

Lemma 2.7. *The maps $p_{\alpha,\beta} : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ are for all $\alpha, \beta \in \mathbb{N}^n$ semi-norms. Hence, $\mathscr{S}(\mathbb{R}^n)$ is a locally convex space.*

Proof. By definition, the following properties hold for all $\psi \in \mathscr{S}(\mathbb{R}^n)$:

- $p_{\alpha,\beta}(\psi) \geq 0$,
- $p_{\alpha,\beta}(\lambda\psi) = |\lambda| p_{\alpha,\beta}(\psi)$ for all $\lambda \in \mathbb{C}$,

- $p_{\alpha,\beta}(\psi_1 + \psi_2) \leq p_{\alpha,\beta}(\psi_1) + p_{\alpha,\beta}(\psi_2)$.

Hence, $\{p_{\alpha,\beta} : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C} \mid \alpha, \beta \in \mathbb{N}^n\}$ is a family of seminorms on $\mathscr{S}(\mathbb{R}^n)$ (cf. Definition A.1, Appendix A). From Definition A.2, we can conclude that $\mathscr{S}(\mathbb{R}^n)$ is a locally convex space. $\square$

From the lemma above, we can define the topology of $\mathscr{S}(\mathbb{R}^n)$ as follows:

Remark 2.8. Define the metric $d : \mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \to \mathbb{R}$,

$$d(\psi, \phi) := \sum_{\alpha,\beta \in \mathbb{N}_0^n} \frac{1}{2^{|\alpha|+|\beta|}} \frac{p_{\alpha,\beta}(\psi - \phi)}{1 + p_{\alpha,\beta}(\psi - \phi)} \quad \text{for } \psi, \phi \in \mathscr{S}(\mathbb{R}^n),$$

where $d(\psi_n, \psi) \to 0$ if and only if $p_{\alpha,\beta}(\psi_n - \psi) \to 0$ for all $\alpha, \beta \in \mathbb{N}_0^n$. With respect to d, $\mathscr{S}(\mathbb{R}^n)$ is a complete metric space (see [24], pg. 133).

We have an important property of Schwartz space, which is essential for defining the Fourier transform on the space of L^2-functions:

Theorem 2.9. *For $p \in [1, \infty)$, $\mathscr{S}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ is a dense and continuous embedding.*

Proof. For density: From definition, we know that $\mathscr{C}_c^\infty(\mathbb{R}^n) \subseteq \mathscr{S}(\mathbb{R}^n)$. For $p \in [1, \infty)$, we need to prove that $\mathscr{C}_c^\infty(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$, so that we can conclude that $\mathscr{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ dense. So, to show is: $\mathscr{C}_c^\infty(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ dense, for $1 \leq p < \infty$. Since $\mathscr{C}_c(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ dense (see Corollary 0.2 in [26]), it remains to prove that $\mathscr{C}_c^\infty(\mathbb{R}^n) \subseteq \mathscr{C}_c(\mathbb{R}^n)$ dense. To do that, we take $f \in \mathscr{C}_c(\mathbb{R}^n)$ and a sequence in $\mathscr{C}_c^\infty(\mathbb{R}^n)$ and show that this sequence converges to $f \in \mathscr{C}_c(\mathbb{R}^n)$:

Let $u \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ with $\int u = 1$ so that $u_\epsilon \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ defined as follow is an approximate identity (Definition A.4, Appendix A) with

$$u_\epsilon(\mathbf{r}) = \frac{1}{\epsilon^n} u\left(\frac{\mathbf{r}}{\epsilon}\right).$$

Then, $u_\epsilon * f \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, and by Theorem B.1 in Appendix B, we have $u_\epsilon * f \to f$ in $L^p(\mathbb{R}^n)$ for $\epsilon \to \infty$. So, $\mathscr{C}_c^\infty(\mathbb{R}^n) \subseteq \mathscr{C}_c(\mathbb{R}^n)$ dense with respect to L^p-topology, and therefore, we have $\mathscr{C}_c^\infty(\mathbb{R}^n) \subseteq \mathscr{C}_c(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ dense.

For continuity: see Lemma 5.2 in [1]. $\square$

To make the definition of the Schwartz functions more understandable, we provide a concrete example:

Example 2.10. (i) $f(r) := e^{-r^2} \in \mathscr{S}(\mathbb{R})$.

We have that $f^{(l)}(r) = P_l(r)e^{-r^2}$ for a polynomial P_l. Now, we consider two cases. First, for $k, l \in \mathbb{N}$ and $|r| \geq 1$, we have the estimation

$$|r|^k |P_l(r)| \leq C_l |r|^d \quad \forall r : |r| \geq 1,$$

where $d \geq \deg(P_l) + k$, $C_l \in \mathbb{R}$. Choose $d \in \mathbb{N}$ even.

Since $e^{r^2} = \displaystyle\sum_{k=0}^{\infty} \frac{(r^2)^k}{k!}$, we then get the estimation

$$e^{r^2} \geq 1 + \frac{(r^2)^{\frac{d}{2}}}{\left(\frac{d}{2}\right)!} \quad \forall r : |r| \geq 1.$$

Hence,

$$|e^{-r^2}| \leq \left| \frac{1}{1 + \frac{r^d}{(\frac{d}{2})!}} \right| \quad \forall r : |r| \geq 1.$$

Finally, we get

$$|r^k f^{(l)}(r)| \leq C_l \left| \frac{|r|^d}{1 + \frac{r^d}{(\frac{d}{2})!}} \right| \leq C_l \frac{|r|^d}{\frac{|r^d|}{(\frac{d}{2})!}} \leq C_l \left(\frac{d}{2}\right)! < \infty \; \forall r : |r| \geq 1.$$

Then, consider the case $|r| < 1$. It is clear that $|r|^k |f^{(l)}(r)|$ is bounded for all $k, l \in \mathbb{N}$. Therefore, $f \in \mathscr{S}(\mathbb{R})$.

(ii) $g(r) := e^{-r^2} \sin(e^{-r^2})$ decays faster than any polynomial, i.e., $|r^k g(r)|$ is bounded for all $k \in \mathbb{N}$, but its derivatives are not. Hence $g \notin \mathscr{S}(\mathbb{R})$. Indeed, consider

$$g'(r) = -2re^{-r^2} \sin(e^{-r^2}) + 2r \cos(e^{-r^2}).$$

The first addend is bounded due to the first item, but the second addend is not.

As we introduced before, $\mathscr{S}(\mathbb{R}^n)$ is a subspace of $L^1(\mathbb{R}^n)$ that is closed under pointwise multiplication. Furthermore, the derivatives and convolution of functions in $\mathscr{S}(\mathbb{R}^n)$ are again Schwartz function. Indeed, we have the following proposition:

Proposition 2.11. *Let $\phi, \psi \in \mathscr{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, the followings hold:*

- $\partial^\alpha \phi \in \mathscr{S}(\mathbb{R}^n)$.
- $\phi\psi \in \mathscr{S}(\mathbb{R}^n)$.
- $\phi * \psi \in \mathscr{S}(\mathbb{R}^n)$.

Proof. See Proposition 3.1.6 in [17]. □

Since $\mathscr{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$ (see Theorem 2.9), for a Schwartz function f, we can calculate the Fourier transform of f using Definition 2.1. In the next subsection, we study the properties of the Fourier transform on $\mathscr{S}(\mathbb{R}^n)$.

2.2 Fourier transform on $\mathscr{S}(\mathbb{R}^n)$

Proposition 2.12 (Properties of Fourier transform on $\mathscr{S}(\mathbb{R}^n)$)**.** *For $\psi, \phi \in \mathscr{S}(\mathbb{R}^n)$, we have*

(i) $\mathscr{F}(\alpha\psi + \beta\phi) = \alpha\mathscr{F}(\psi) + \beta\mathscr{F}(\phi)$, *for all* $\alpha, \beta \in \mathbb{C}$.

(ii) For all $\mathbf{k} \in \mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}_0^n$, *we have*
$\partial^\alpha \mathscr{F}\psi(\mathbf{k}) = (-i)^{|\alpha|}\mathscr{F}(\mathbf{r} \mapsto \mathbf{r}^\alpha \psi(\mathbf{r}))(\mathbf{k})$,
$\mathscr{F}(\partial^\alpha \psi)(\mathbf{k}) = i^{|\alpha|}\mathbf{k}^\alpha \mathscr{F}\psi(\mathbf{k})$.

(iii) $\mathscr{F}(\psi * \phi)(\mathbf{k}) = (\sqrt{2\pi})^n \mathscr{F}\psi(\mathbf{k})\mathscr{F}\phi(\mathbf{k})$.

(iv) $\mathscr{F}\phi(\cdot - \mathbf{a}) = e^{-i\mathbf{a}\cdot}\mathscr{F}\phi$, *for all* $\mathbf{a} \in \mathbb{R}^n$.

(v) $\mathscr{F}(e^{-i\mathbf{a}\cdot}\phi) = \mathscr{F}\phi(\cdot + \mathbf{a})$, *for all* $\mathbf{a} \in \mathbb{R}^n$.

Proof. (i) Follows from the linearity of the integral.

(ii) First, we note that

$$|\psi(\mathbf{r})\partial_{\mathbf{k}}^\alpha e^{-i\mathbf{k}\cdot\mathbf{r}}| = |(-i)^{|\alpha|}\mathbf{r}^\alpha \psi(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}| = |\mathbf{r}^\alpha \psi(\mathbf{r})| \in L^1(\mathbb{R}^n) \quad \forall \mathbf{k} \in \mathbb{R}^n.$$

Then, applying Definition 2.1 and using the fact that $|\psi(\mathbf{r})\partial_{\mathbf{k}}^\alpha e^{-i\mathbf{k}\cdot\mathbf{r}}|$ is bounded by the L^1-function $|\mathbf{r}^\alpha\psi(\mathbf{r})|$ for all $\mathbf{k} \in \mathbb{R}^n$, we can apply the Lebesgue dominated convergence theorem and get

$$\partial^\alpha \mathscr{F}\psi(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n}\partial_{\mathbf{k}}^\alpha \int_{\mathbb{R}^n} \psi(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}\mathbf{r} = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} \psi(\mathbf{r})\partial_{\mathbf{k}}^\alpha e^{-i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}\mathbf{r}.$$

Calculating $\partial_{\mathbf{k}}^\alpha e^{-i\mathbf{k}\cdot\mathbf{r}}$, we obtain

$$\partial^\alpha \mathscr{F}\psi(\mathbf{k}) = (-i)^{|\alpha|}\frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} \mathbf{r}^\alpha \psi(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}\mathbf{r},$$

with $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

For the second part of (ii), after writing

$$\mathscr{F}(\partial^\alpha\psi)(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} \left(\partial^\alpha\psi(\mathbf{r})\right) e^{-i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}\mathbf{r},$$

we utilise integration by parts where the first addend vanishes, since $\psi \in \mathscr{S}(\mathbb{R}^n)$ and all boundary terms vanish. This yields

$$\mathscr{F}(\partial^\alpha\psi)(\mathbf{k}) = (-1)^{|\alpha|}\frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} \psi(\mathbf{r})\partial_{\mathbf{r}}^\alpha e^{-i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}\mathbf{r}.$$

Calculating the derivative in the integrand gives

$$\mathscr{F}(\partial^\alpha\psi)(\mathbf{k}) = i^{|\alpha|}\mathbf{k}^\alpha\frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} \psi(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}\mathbf{r} = i^{|\alpha|}\mathbf{k}^\alpha \mathscr{F}\psi(\mathbf{k}).$$

(iii) Using the definition of convolution and the fact that the convolution of two Schwartz functions is again a Schwartz function (cf. Proposition 2.11), we have

$$\mathscr{F}(\psi * \phi)(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(\mathbf{r} - \mathbf{p})\phi(\mathbf{p}) e^{-i\mathbf{k}\cdot\mathbf{r}} \ \mathrm{d}\mathbf{p} \ \mathrm{d}\mathbf{r}$$

and

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \left| \psi(\mathbf{r} - \mathbf{p})\phi(\mathbf{p}) e^{-i\mathbf{k}\cdot\mathbf{r}} \right| \ \mathrm{d}(\mathbf{r}, \mathbf{p}) < \infty$$

because $\phi * \psi \in S(\mathbb{R}^n)$, hence also in $L^1(\mathbb{R}^n)$. Thus,

$$(\mathbf{r}, \mathbf{p}) \mapsto \psi(\mathbf{r} - \mathbf{p})\phi(\mathbf{p}) e^{-i\mathbf{k}\cdot\mathbf{r}} \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$$

and we can apply the Fubini-Tonelli theorem (Theorem B.3, Appendix B). Now, writing $e^{-i\mathbf{k}\cdot\mathbf{r}} = e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{p})} e^{-i\mathbf{k}\cdot\mathbf{p}}$, making substitution $\mathbf{r} - \mathbf{p} = \mathbf{q}$ and applying Fubini-Tonelli theorem to get

$$\mathscr{F}(\psi * \phi)(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \psi(\mathbf{q}) e^{-i\mathbf{k}\cdot\mathbf{q}} \ \mathrm{d}\mathbf{q} \int_{\mathbb{R}^n} \phi(\mathbf{p}) e^{-i\mathbf{k}\cdot\mathbf{p}} \ \mathrm{d}\mathbf{p},$$

which then gives

$$\mathscr{F}(\psi * \phi)(\mathbf{k}) = (\sqrt{2\pi})^n \mathscr{F}\psi(\mathbf{k}) \mathscr{F}\phi(\mathbf{k}),$$

that completes the proof.

□

In contrast to the $L^1(\mathbb{R}^n)$ case (cf. Remark 2.5), the Fourier transform maps $\mathscr{S}(\mathbb{R}^n)$ onto itself. Therefore, the inverse Fourier transform is well-defined. Indeed, we have this following Proposition:

Proposition 2.13. *The Fourier transform $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is a topological isomorphism with continuous inverse $\mathscr{F}^{-1} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ given by*

$$\mathscr{F}^{-1}\psi(\mathbf{r}) := \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \psi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \ \mathrm{d}\mathbf{k} \qquad \forall \psi \in \mathscr{S}(\mathbb{R}^n), \mathbf{r} \in \mathbb{R}^n. \tag{2.3}$$

Proof. Corollary 2.2.15 in [2]. □

Remark 2.14. We have some other useful properties of the Fourier transform on $\mathscr{S}(\mathbb{R}^n)$, namely,

(i) For $\psi \in \mathscr{S}(\mathbb{R}^n)$, $\mathscr{F}(\mathscr{F}\psi)(\mathbf{r}) = \psi(-\mathbf{r})$.

(ii) For $\psi, \phi \in \mathscr{S}(\mathbb{R}^n)$, we have the Plancherel's identity

$$\langle \psi, \phi \rangle_{L^2} = \frac{1}{(2\pi)^n} \langle \mathscr{F}\psi, \mathscr{F}\phi \rangle_{L^2},$$

and also the Parseval's identity as the direct consequence

$$\|\psi\|_{L^2} = \frac{1}{(\sqrt{2\pi})^n} \|\mathscr{F}\psi\|_{L^2}.$$

(iii) For all $\psi, \phi \in \mathscr{S}(\mathbb{R}^n)$, we have that

$$\int_{\mathbb{R}^n} \psi(\mathbf{r})\mathscr{F}\phi(\mathbf{r})\ \mathrm{d}\mathbf{r} = \int_{\mathbb{R}^n} \mathscr{F}\psi(\mathbf{r})\phi(\mathbf{r})\ \mathrm{d}\mathbf{r}. \tag{2.4}$$

For proof see Theorem 2.2.14 in [2].

Now, we want to study the Fourier transform on the space of L^2-functions.

2.3 Fourier transform on $L^2(\mathbb{R}^n)$

Since the dual space of $L^2(\mathbb{R}^n)$ is again $L^2(\mathbb{R}^n)$, and it is the only Hilbert space among all $L^p(\mathbb{R}^n)$ for $p \in [1, \infty]$, it is a convenient setting for utilizing the Fourier transform.

We consider a function $f \in L^2(\mathbb{R}^n)$. The integral

$$\frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}\ \mathrm{d}\mathbf{r}$$

does not converge absolutely. For example, consider the 1-dimensional function

$$f(r) := \begin{cases} \frac{1}{r} & \text{for } r \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$

From

$$\int_{\mathbb{R}} |f(r)|^2\ \mathrm{d}r = \int_1^\infty \left|\frac{1}{r}\right|^2\ \mathrm{d}r = \left.\frac{-1}{x}\right|_1^\infty = 1$$

and

$$\int_{\mathbb{R}} |f(r)|\ \mathrm{d}r = \int_1^\infty \left|\frac{1}{r}\right|\ \mathrm{d}r = \left.\frac{x\ln(|x|)}{|x|}\right|_1^\infty = \infty,$$

we can conclude that $f \in L^2(\mathbb{R})$ but $\notin L^1(\mathbb{R})$. For $k \in \mathbb{R}$, the integral

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(r)e^{-ikr}\ \mathrm{d}r = \frac{1}{\sqrt{2\pi}} \int_1^\infty \frac{e^{-ikr}}{r}\ \mathrm{d}r$$

does not converge absolutely, since

$$\int_{\mathbb{R}} \left|f(r)e^{-ikr}\right|\ \mathrm{d}r = \|f\|_{L^1(\mathbb{R})} = \infty.$$

In consequence, we need to define the Fourier transform for $f \in L^2(\mathbb{R}^n)$ differently. We find an extenstion of $\mathscr{F}$ to $L^2(\mathbb{R}^n)$ as follows:

Since we know that $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ (c.f. Theorem 2.9), for every $f \in L^2(\mathbb{R}^n)$, there exists a sequence of functions f_k in $\mathscr{S}(\mathbb{R}^n)$ that converges to f. Moreover, the Fourier transform of f_k is actually a Cauchy sequence. Indeed,

$$\|\mathscr{F}f_k - \mathscr{F}f_l\|_{L^2} = \|\mathscr{F}(f_k - f_l)\|_{L^2} = \|f_k - f_l\|_{L^2} \xrightarrow{k,l\to\infty} 0,$$

using the property in Remark 2.14, (ii). Therefore, the limit of $\mathscr{F} f_k$ exists. Now, let us define

$$\mathscr{F} f := \lim_{k\to\infty} \mathscr{F} f_k.$$

This definition of Fourier transform of $f \in L^2(\mathbb{R}^n)$ is well-defined, i.e., it does not depend on the choice of the sequence f_k. We can see that by the following: First, we take another sequence $g_k \in \mathscr{S}(\mathbb{R}^n)$ that converges to f. From the triangle inequality, we have that

$$\|\mathscr{F} f - \mathscr{F} g_k\|_{L^2} \leq \|\mathscr{F} f - \mathscr{F} f_k\|_{L^2} + \|\mathscr{F} f_k - \mathscr{F} g_k\|_{L^2}.$$

From the linearity of the Fourier transform, we can rewrite the inequality above as

$$\|\mathscr{F} f - \mathscr{F} g_k\|_{L^2} \leq \|\mathscr{F} f - \mathscr{F} f_k\|_{L^2} + \|\mathscr{F} (f_k - g_k)\|_{L^2}.$$

The Parseval's identity in Remark 2.14 gives $\|\mathscr{F}(f_k - g_k)\|_{L^2} = \|f_k - g_k\|_{L^2}$, which yields

$$\|\mathscr{F} f - \mathscr{F} g_k\|_{L^2} \leq \|\mathscr{F} f - \mathscr{F} f_k\|_{L^2} + \|f_k - g_k\|_{L^2}.$$

Using the triangle inequality again, we obtain

$$\|\mathscr{F} f - \mathscr{F} g_k\|_{L^2} \leq |\mathscr{F} f - \mathscr{F} f_k\|_{L^2} + \|f_k - f\|_{L^2} + \|f - g_k\|_{L^2}.$$

Since we have that $\|\mathscr{F} f - \mathscr{F} f_k\|_{L^2} \xrightarrow{k\to\infty} 0$ due to the fact that $\mathscr{F} f_k$ is a Cauchy sequence, $\|f_k - f\|_{L^2} \xrightarrow{k\to\infty} 0$ and $\|f - g_k\|_{L^2} \xrightarrow{k\to\infty} 0$ according to the assumptions, we get that

$$\lim_{k\to\infty} \mathscr{F} f_k =: \mathscr{F} f = \lim_{k\to\infty} \mathscr{F} g_k.$$

Hence, we have an extension of the Fourier transform from $\mathscr{S}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, i.e., the operator $\mathscr{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Similarly, we get $\mathscr{F}^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, which means that the extension of the Fourier transform on $L^2(\mathbb{R}^n)$ is also a topological isomorphism.

Remark 2.15. For $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, the definition on Fourier transform in Definition 2.1 and $\mathscr{F} f := \lim\limits_{k\to\infty} \mathscr{F} f_k$ coincide pointwise a.e..

To see this, first we denote the Fourier transform in Definition 2.1 as $\mathscr{F}_1$, i.e.,

$$\mathscr{F}_1 f(\mathbf{k}) := \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}\mathbf{r} \quad \forall \mathbf{k} \in \mathbb{R}^n,$$

and the extension of the Fourier transform from $\mathscr{S}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ as $\mathscr{F}_2$, i.e.,

$$\mathscr{F}_2 f := \lim_{k\to\infty} \mathscr{F} f_k,$$

where $(f_k)_k$ is a sequence in $\mathscr{S}(\mathbb{R}^n)$ that converges to f in $L^2(\mathbb{R}^n)$. Then, for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, take a sequence $(f_k)_k \subseteq \mathscr{S}(\mathbb{R}^n)$ so that f_k converges to f in both $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. Now, from

$$\|\mathscr{F}_1 f_k - \mathscr{F}_1 f\|_\infty \leq \|f_k - f\|_1$$

and the assumption that f_k converges to f in $L^1(\mathbb{R}^n)$, we can conclude that $\mathscr{F}_1 f_k$ converges uniformly to $\mathscr{F}_1 f$. But by definition, we have that $\mathscr{F}_1 f_k$ converges to $\mathscr{F}_2 f$ in $L^2(\mathbb{R}^n)$ and it follows that we can find a subsequence of $\mathscr{F}_1(f_k)$ that converges pointwise a.e. to $\mathscr{F}_2 f_k$. Since $\mathscr{F}_1 f_k$ converges uniformly to $\mathscr{F}_1 f$, we have that the entire sequence converges pointwise to $\mathscr{F}_1 f$. Thus, $\mathscr{F}_1 f = \mathscr{F}_2 f$ a.e..

Similar to the case of Schwartz functions, the properties of the Fourier transform written in Remark 2.14 also hold for $L^2(\mathbb{R}^n)$. We can prove the Parseval's identity for functions in $L^2(\mathbb{R}^n)$ as in the next theorem. The other two properties can be proved analogously.

Theorem 2.16. *For $f \in L^2(\mathbb{R}^n)$, we have that*

$$\|\mathscr{F} f\|_{L^2} = (\sqrt{2\pi})^n \|f\|_{L^2}.$$

Proof. Choose a sequence $f_k \in \mathscr{S}(\mathbb{R}^n)$ that converges to $f \in L^2(\mathbb{R}^n)$. Consider

$$\|\mathscr{F} f\|_{L^2} = \|\lim_{k\to\infty} \mathscr{F} f_k\|_{L^2} = \lim_{k\to\infty} \|\mathscr{F} f_k\|_{L^2},$$

using the continuity of norm. Then, due to Remark 2.14, (ii), we get

$$\|\mathscr{F} f\|_{L^2} = \lim_{k\to\infty} \|\mathscr{F} f_k\|_{L^2} = \lim_{k\to\infty} \|f_k\|_{L^2} = \|f\|_{L^2}$$

and arrive at the assertion. □

We have already studied the definition and fundamental properties of the Fourier transform on both $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ spaces. However, the case $L^p(\mathbb{R}^n)$, for $p \in (1, 2)$, has not been examined yet. In this situation, we have the subsequent remark.

Remark 2.17. Let $f \in L^p(\mathbb{R}^n)$, $p \in (1, 2)$. The Fourier transform is defined as

$$\mathscr{F} f := \mathscr{F} f_1 + \mathscr{F} f_2, \tag{2.5}$$

with $f = f_1 + f_2$, $f_1 \in L^1(\mathbb{R}^n)$, $f_2 \in L^2(\mathbb{R}^n)$.
We can always find such functions f_1 and f_2. For example, take $f_1 = f \mathbb{1}_{|f|>1}$ and $f_2 = f \mathbb{1}_{|f|\leq 1}$, where

$$\mathbb{1}_A(r) = \begin{cases} 1 & \text{for } \ r \in A, \\ 0 & \text{otherwise} \end{cases}$$

is the characteristic function of the set A.

The definition in Equation (2.5) is well-defined. In fact, let $f_1 + f_2 = g_1 + g_2$ be two different decompositions of f, with $f_1, g_1 \in L^1(\mathbb{R}^n)$, $f_2, g_2 \in L^2(\mathbb{R}^n)$. Thus, $f_1 - g_1 = g_2 - f_2 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, which yields

$$\mathscr{F}(f_1 - g_1) = \mathscr{F}(g_2 - f_2),$$

which is equivalent to

$$\mathscr{F} f_1 - \mathscr{F} g_1 = \mathscr{F} g_2 - \mathscr{F} f_2.$$

This gives

$$\mathscr{F} f_1 + \mathscr{F} f_2 = \mathscr{F} g_1 + \mathscr{F} g_2,$$

or equivalently,

$$\mathscr{F}(f_1 + f_2) = \mathscr{F}(g_1 + g_2).$$

Hence, $\mathscr{F} f$ does not depend on the choice of decomposition components, i.e., f_1, f_2.

Additionally, in $L^2(\mathbb{R}^n)$, there are some special functions, called the eigenfunctions of the Fourier operator $\mathscr{F}$. They are functions that have the same form as their own Fourier transforms. A properly scaled Gaussian curve is one prominent example of eigenfunctions of the Fourier operator. Indeed, we have

Proposition 2.18. *For*

$$\begin{aligned} f : \mathbb{R}^n &\to \mathbb{R} \\ \mathbf{r} &\mapsto e^{-\frac{1}{2}\|\mathbf{r}\|^2}, \end{aligned}$$

it holds $\mathscr{F} f = f$.

Proof. Before we prove this, we first need to show that the following statement holds true: For $\alpha > 0$, $\beta \in \mathbb{C}^n$

$$\int_{\mathbb{R}^n} e^{\beta \cdot \mathbf{r} - \frac{\|\mathbf{r}\|^2}{\alpha}} \, \mathrm{d}\mathbf{r} = (\pi\alpha)^{\frac{n}{2}} e^{\frac{\alpha}{4} \sum_{j=1}^n \beta_j^2} \tag{2.6}$$

(see Example 2.2.9 in [2]).

For $n = 1$, $\beta = 0$, we have

$$\left(\int_{\mathbb{R}} e^{-\frac{r^2}{\alpha}} \, \mathrm{d}r \right)^2 = \left(\int_{\mathbb{R}} e^{-\frac{r^2}{\alpha}} \, \mathrm{d}r \right) \left(\int_{\mathbb{R}} e^{-\frac{p^2}{\alpha}} \, \mathrm{d}p \right) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\frac{r^2 + p^2}{\alpha}} \, \mathrm{d}r \, \mathrm{d}p.$$

Using polar coordienates $\rho > 0, \varphi \in [0, 2\pi)$ to have $r = \rho \cos\varphi$ and $p = \rho \sin\varphi$, we arrive at

$$\left(\int_{\mathbb{R}} e^{-\frac{r^2}{\alpha}} \, \mathrm{d}r \right)^2 = \int_0^\infty \int_0^{2\pi} e^{-\frac{\rho^2}{\alpha}} \rho \, \mathrm{d}\varphi \, \mathrm{d}\rho = 2\pi \int_0^\infty e^{-\frac{\rho^2}{\alpha}} \rho \, \mathrm{d}\rho.$$

Now, we make use of the substitution $\tau = -\frac{\rho^2}{\alpha}$ to calculate the indefinite integral

$$2\pi \int e^{-\frac{\rho^2}{\alpha}} \rho \, \mathrm{d}\rho = -2\pi \int \frac{\alpha}{2} e^{\tau} \, \mathrm{d}\tau = -\pi\alpha e^{-\frac{\rho^2}{\alpha}}.$$

Therefore, we get

$$\left(\int_{\mathbb{R}} e^{-\frac{r^2}{\alpha}} \, \mathrm{d}r \right)^2 = -\pi\alpha e^{-\frac{\rho^2}{\alpha}} \Big|_0^{\infty} = \pi\alpha.$$

Thus, utilising substitution $p = r + \frac{\alpha\beta}{2}$, we obtain

$$\sqrt{\pi\alpha} = \int_{\mathbb{R}} e^{-\frac{r^2}{\alpha}} \, \mathrm{d}r = \int_{\mathbb{R}} e^{-\frac{\left(p - \frac{\alpha\beta}{2}\right)^2}{\alpha}} \, \mathrm{d}p = e^{-\frac{\alpha\beta^2}{4}} \int_{\mathbb{R}} e^{-\frac{p^2}{\alpha} + p\beta} \, \mathrm{d}p. \tag{2.7}$$

For high-dimensional case with $\beta \in \mathbb{R}^n$, since $r_j \mapsto e^{\beta_j \cdot r_j - \frac{r_j^2}{\alpha}} \in L^1(\mathbb{R})$ for all $j = 1, ..., n$, we can conclude that $e^{\beta \cdot \mathbf{r} - \frac{\|\mathbf{r}\|^2}{\alpha}} \in L^1(\mathbb{R}^n)$ using the second item of Theorem B.3. Applying the first item of Fubini-Tonelli theorem, together with Equation (2.7), we obtain

$$\int_{\mathbb{R}^n} e^{\beta \cdot \mathbf{r} - \frac{\|\mathbf{r}\|^2}{\alpha}} \, \mathrm{d}\mathbf{r} = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{\beta_1 r_1 - \frac{r_1{}^2}{\alpha}} \, \mathrm{d}r_1 \cdots e^{\beta_n r_n - \frac{r_n^2}{\alpha}} \, \mathrm{d}r_n = (\sqrt{\pi\alpha})^n e^{\frac{\alpha\beta_1^2}{4}} \cdots e^{\frac{\alpha\beta_n^2}{4}}.$$

For the case $\beta \in \mathbb{C}^n$, consider analytic continuation.

Now, taking the Fourier transform of $f(\mathbf{r}) = e^{-\frac{1}{2}\|\mathbf{r}\|^2}$ gives

$$(\mathscr{F} f)(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{-i\mathbf{k} \cdot \mathbf{r} - \frac{1}{2}\|\mathbf{r}\|^2} \, \mathrm{d}\mathbf{r}$$

and applying Equation (2.6) by taking $\alpha = 2$ and $\beta = -i\mathbf{k}$ yields

$$(\mathscr{F} f)(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} (2\pi)^{\frac{n}{2}} e^{\frac{2}{4} \sum_{j=1}^n (-i\mathbf{k}_j)^2} = e^{-\frac{1}{2}\|\mathbf{k}\|^2} = f(\mathbf{k}),$$

which completes the proof. □

We are left with the task of determining all eigenvalues and their possible corresponding eigenfunctions. We do this in four steps:

1. First, we show that a eigenvalue of $\mathscr{F}$ must be an element of $\{\pm 1, \pm i\}$.
2. ± 1 and $\pm i$ are indeed eigenvalues of $\mathscr{F}$.
3. We specify the corresponding eigenfunctions in 1-dimensional case.
4. We extend the specified eigenfunctions to n-dimensional case.

According to the first item of Remark 2.14, that is, $\mathscr{F}^2 f = f(-\cdot)$, we have that the Fourier transform operator applied four times gives us the identity operator $\mathscr{I}$. For $f \in L^2(\mathbb{R}^n)$ eigenfunction of $\mathscr{F}$, i.e., $\mathscr{F} f = \lambda f$, $\lambda \in \mathbb{R}$, we have $\mathscr{F}^n f = \lambda^n f$, $\lambda \in \mathbb{R}$, $n > 0$, which holds true because the Fourier transform is linear operator. Taking $n = 4$, we have the following:

$$\mathscr{F}^4 f = \lambda^4 f \Leftrightarrow \mathscr{I} f = \lambda^4 f \Leftrightarrow f = \lambda^4 f.$$

So, eigenvalues of Fourier transform can only be the 4th roots of unity, i.e., $1, -1, i, -i$.

Now, we want to show that $1, -1, i, -i$ are indeed eigenvalues of $\mathscr{F}$ and the corresponding eigenfunctions are indeed the Hermite functions defined as follows:

Definition 2.19. For $l \in \mathbb{N}$, the polynomials $P_l : \mathbb{R} \to \mathbb{R}$ with

$$P_l(r) := (-1)^l e^{r^2} \partial^l e^{-r^2},$$

are called the Hermite polynomials of degree l. The Schwartz functions $p_l : \mathbb{R} \to \mathbb{R}$ with

$$p_l(r) := (2^l l! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{r^2}{2}} P_l(r),$$

are called Hermite functions.

These Hermite functions build the orthonormal basis of $L^2(\mathbb{R})$. In fact, we have the following proposition:

Proposition 2.20. *The orthonormal basis for* $L^2(\mathbb{R})$ *is* $\{p_l \mid l \in \mathbb{N}\}$.

Proof. Before we start, we will show a statement, which will help us prove this proposition: Namely, for $r, s \in \mathbb{R}$, it holds

$$\sum_{l \in \mathbb{N}} \frac{P_l(r)}{l!} s^l = e^{2rs - s^2}. \tag{2.8}$$

In order to prove this, we first define a function $f(q) = e^{-q^2}$, for $q \in \mathbb{R}$ and then, for $r, s \in \mathbb{R}$, using Taylor series expansion and the definition of Hermite polynomials, we get the following:

$$f(q) = \sum_{l \in \mathbb{N}} \frac{\partial^l f(r)}{l!} (q - r)^l = \sum_{l \in \mathbb{N}} (-1)^l e^{-r^2} \frac{P_l(r)}{l!} (q - r)^l.$$

Applying this gives

$$\sum_{l \in \mathbb{N}} \frac{P_l(r)}{l!} s^l = e^{r^2} \sum_{l \in \mathbb{N}} (-1)^l e^{-r^2} \frac{P_l(r)}{l!} (-s)^l = e^{r^2} f(r - s).$$

Plugging in $f(r - s) = e^{-(r-s)^2}$, we arrive at

$$\sum_{l \in \mathbb{N}} \frac{P_l(r)}{l!} s^l = e^{2rs - s^2}.$$

Now, we concentrate on proving the proposition. First, we show the orthonormality, i.e.,

$$\langle p_l, p_m \rangle_{L^2(\mathbb{R})} = \delta_{l,m}. \tag{2.9}$$

We observe the Hermite polynomials and define a space $L^2(\mathbb{R}, \tau \mathrm{d}r)$ with

$$\tau(r) = \frac{1}{\sqrt{\pi}} e^{-r^2}, \tag{2.10}$$

endowed with the inner product

$$\langle P_l, P_m \rangle_{L^2(\mathbb{R}, \tau \mathrm{d}r)} := \int_{\mathbb{R}} P_l(r) P_m(r) \tau(r) \; \mathrm{d}r,$$

that is,

$$L^2(\mathbb{R}, \tau \mathrm{d}r) := \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; \|f\|_{L^2(\mathbb{R}, \tau \mathrm{d}r)} < \infty \right\} \Big/ \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; \|f\|_{L^2(\mathbb{R}, \tau \mathrm{d}r)} = 0 \right\}.$$

On the one hand, from Equation (2.6), for $\gamma, \delta \in \mathbb{R}$, we have

$$e^{2\gamma\delta} = e^{2\gamma\delta} \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(r-(\gamma+\delta))^2} \; \mathrm{d}r,$$

and after some calculations in the exponential part we get

$$e^{2\gamma\delta} = \int_{\mathbb{R}} \frac{1}{\sqrt{\pi}} e^{-r^2} e^{2r\gamma - \gamma^2 + 2r\delta - \delta^2} \; \mathrm{d}r.$$

Applying both Equation (2.8) and the definition of τ yields

$$e^{2\gamma\delta} = \int_{\mathbb{R}} \tau(r) \sum_{l \in \mathbb{N}} \sum_{m \in \mathbb{N}} \frac{\gamma^l}{l!} \frac{\delta^m}{m!} P_l(r) P_m(r) \; \mathrm{d}r.$$

Now, interchanging the sums and the integral and using the definition of $L^2(\mathbb{R}, \tau \mathrm{d}r)$-inner product leads to

$$e^{2\gamma\delta} = \sum_{l \in \mathbb{N}} \sum_{m \in \mathbb{N}} \frac{\gamma^l}{l!} \frac{\delta^m}{m!} \int_{\mathbb{R}} \tau(r) P_l(r) P_m(r) \; \mathrm{d}r = \sum_{l \in \mathbb{N}} \sum_{m \in \mathbb{N}} \frac{\gamma^l}{l!} \frac{\delta^m}{m!} \langle P_l, P_m \rangle_{L^2(\mathbb{R}, \tau \mathrm{d}r)}.$$

Finally, we utilise the definition of Hermite functions to obtain

$$e^{2\gamma\delta} = \sum_{l \in \mathbb{N}} \sum_{m \in \mathbb{N}} \frac{\gamma^l}{l!} \frac{\delta^m}{m!} \sqrt{2^l l!} \sqrt{2^m m} \langle p_l, p_m \rangle_{L^2(\mathbb{R})}. \tag{2.11}$$

On the other hand, we have that the exponential function can be defined by the power series, i.e.,

$$e^{2\gamma\delta} = \sum_{l \in \mathbb{N}} \frac{2^l}{l!} \gamma^l \delta^l. \tag{2.12}$$

Now, comparing the coefficients in (2.11) and (2.12), we arrive at (2.9).

What is left to show is that $\operatorname{span}\{p_l \mid l \in \mathbb{N}\}$ is dense in $L^2(\mathbb{R})$. We know that $\operatorname{span}\{P_l \mid l \leq m\}$ coincides with all polynomials of degree at most m and therefore, by applying Stone-Weierstrass theorem (cf. Appendix B8), we have that $\operatorname{span}\{P_l \mid l \in \mathbb{N}\}$ is dense in $L^2(\mathbb{R}, \tau \mathrm{d}r)$. Thus, from

$$\langle f, P_l \rangle_{L^2(\mathbb{R}, \tau \mathrm{d}r)} = 0 \quad \forall\, l \in \mathbb{N}, \tag{2.13}$$

where $f \in L^2(\mathbb{R}, \tau \mathrm{d}r)$, it follows that $f = 0$. Assume that we have

$$\langle f, p_l \rangle_{L^2(\mathbb{R})} = 0 \quad \forall\, l \in \mathbb{N}, \tag{2.14}$$

where $f \in L^2(\mathbb{R})$. Applying the definition of Hermite functions gives

$$0 = \langle f, p_l \rangle_{L^2(\mathbb{R})} = \langle e^{r^2} f, P_l \rangle_{L^2(\mathbb{R}, \tau \mathrm{d}r)} \quad \forall\, l \in \mathbb{N}, \tag{2.15}$$

from which we conclude that $e^{r^2} f = 0$ and therefore $f = 0$. So, $\operatorname{span}\{p_l \mid l \in \mathbb{N}\}$ is dense in $L^2(\mathbb{R})$. $\square$

Now, we show that the Hermite functions are the eigenfunctions of the Fourier transform on $L^2(\mathbb{R})$.

Theorem 2.21. *For $n = 1$, $l \in \mathbb{N}$, the following holds true:*

$$\mathscr{F} p_l = (-i)^l p_l.$$

Proof. From Proposition 2.12, we have for all $k, r \in \mathbb{R}$,

$$\mathscr{F}(\partial \psi)(k) = ik \mathscr{F} \psi(k) \tag{2.16}$$

and

$$\partial \mathscr{F} \psi(k) = -i \mathscr{F} \left[r \mapsto r\psi(r) \right](k). \tag{2.17}$$

Setting $\mathscr{E}\psi := \left[r \mapsto r\psi(r) - \partial\psi(r) \right]$, we have

$$\mathscr{F}(\mathscr{E}\psi)(k) = \mathscr{F}\left[r \mapsto r\psi(r) - \partial\psi(r) \right](k) = \mathscr{F}\left[r \mapsto r\psi(r) \right](k) - \mathscr{F}\left[r \mapsto \partial\psi(r) \right](k)$$

and applying Equations (2.16) and (2.17), we get

$$\mathscr{F}(\mathscr{E}\psi)(k) = i\partial \mathscr{F}\psi(k) - ik\mathscr{F}\psi(k) = -i(k\mathscr{F}\psi - \partial\mathscr{F}\psi)(k) = -i\mathscr{E}(\mathscr{F}\psi)(k).$$

Thus,

$$\mathscr{F}\mathscr{E}^l = (-i\mathscr{E})^l \mathscr{F}. \tag{2.18}$$

Consider the following calculations, which will help us prove this theorem. Applying the setting above together with the definition of Hermite functions yields

$$\mathscr{E}p_l(r) = rp_l(r) - \partial p_l(r) = r(2^l l! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{r^2}{2}} P_l(r) - \partial (2^l l! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{r^2}{2}} P_l(r).$$

Plugging in the definition of Hermite polynomials leads us to

$$\mathscr{E}p_l(r) = (2^l l!\sqrt{\pi})^{-\frac{1}{2}}(-1)^l\big[re^{\frac{r^2}{2}}\partial^l e^{-r^2} - \partial(e^{\frac{r^2}{2}}\partial^l e^{-r^2})\big].$$

After differentiating the product, we arrive at

$$\mathscr{E}p_l(r) = -(2^l l!\sqrt{\pi})^{-\frac{1}{2}}(-1)^l e^{\frac{r^2}{2}}\partial^{l+1}e^{-r^2} = \frac{1}{c_l}p_{l+1}(r),$$

with a properly chosen constant $c_l \in \mathbb{R}$. Iteratively, we get

$$p_l(r) = c_{l-1}\mathscr{E}p_{l-1}(r) = c_{l-1}\mathscr{E}c_{l-2}\mathscr{E}p_{l-2} = \cdots = \underbrace{c_{l-1}\cdots c_0}_{=:C_l}\mathscr{E}^l p_0. \tag{2.19}$$

Applying Proposition 2.18 for $l = 0$, $p_0(r) = (\sqrt{\pi})^{-\frac{1}{2}}e^{-\frac{r^2}{2}}$ gives $\mathscr{F}p_0 = p_0$. Eventually, using Equation (2.18), it holds:

$$\mathscr{F}p_l = \mathscr{F}(C_l\mathscr{E}^l p_0) = C_l\mathscr{F}\mathscr{E}^l p_0 = C_l(-i\mathscr{E})^l\mathscr{F}p_0.$$

Equation (2.19) yields

$$\mathscr{F}p_l = (-i)^l C_l\mathscr{E}^l p_0 = (-i)^l p_l$$

and we arrive at the assertion. □

Remark 2.22. According to the theorem above, we note that the Fourier transform of p_l is a complex-valued function, despite the fact that p_l is real-valued.

Now, using the fact that $\{p_{l_1}\otimes\cdots\otimes p_{l_n} \mid l \in \mathbb{N}^n\}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ (cf. Appendices) for high dimensional case $n > 1$, we finally obtain the following result for the eigenfunctions of the high-dimensional Fourier transform:

Corollary 2.23. *For $n \in \mathbb{N}$, $l \in \mathbb{N}^n$, it holds:*

$$\mathscr{F}(p_{l_1}\otimes\cdots\otimes p_{l_n}) = (-i)^{|l|}p_{l_1}\otimes\cdots\otimes p_{l_n}.$$

Proof. We first have

$$\mathscr{F}(p_{l_1}\otimes\cdots\otimes p_{l_n})(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} p_{l_1}(r_1)\cdots p_{l_n}(r_n)e^{-i\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}\mathbf{r}$$

and rewriting the exponential part yields

$$\mathscr{F}(p_{l_1}\otimes\cdots\otimes p_{l_n})(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} p_{l_1}(r_1)e^{-ik_1r_1}\cdots p_{l_n}(r_n)e^{-ik_nr_n}\,\mathrm{d}\mathbf{r}.$$

Since $\mathbf{r}\mapsto p_{l_1}(r_1)...p_{l_n}(r_n)e^{-i\mathbf{k}\cdot\mathbf{r}} \in L^1(\mathbb{R}^n)$, we can apply Fubini-Tonelli theorem to get

$$\mathscr{F}(p_{l_1}\otimes\cdots\otimes p_{l_n}) = \mathscr{F}(p_{l_1})\otimes\cdots\otimes\mathscr{F}(p_{l_n}).$$

Utilising Theorem 2.21 gives

$$\mathscr{F}(p_{l_1})\otimes\cdots\otimes\mathscr{F}(p_{l_n}) = (-i)^{|l|}p_{l_1}\otimes\cdots\otimes p_{l_n}.$$

So, the assertion follows. □

Despite having many important and useful properties, the space of $L^2(\mathbb{R}^n)$-functions still has a shortcoming as described in the remark below:

Remark 2.24. Consider the Helmholtz equation

$$-(\Delta + k_0^2)u = g \quad \text{on } \mathbb{R}^n \tag{2.20}$$

with $k_0 \in \mathbb{R}$. To handle this equation appropriately, we need to find a suitable space so that the Δ-operator is well-defined. We keep in mind that Δ is not well-defined on $L^2(\mathbb{R}^n)$. It is well-defined on $\mathscr{S}(\mathbb{R}^n)$, that is,

$$\Delta : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n).$$

So, in a $\mathscr{S}(\mathbb{R}^n)$ setting, taking the Fourier transform of both sides of Equation (2.20), we obtain

$$(-\|\mathbf{k}\|^2 + k_0^2)\mathscr{F}u(\mathbf{k}) = \mathscr{F}g(\mathbf{k}).$$

Unfortunately, in $\mathscr{S}(\mathbb{R}^n)$, this is not equivalent to

$$\mathscr{F}u(\mathbf{k}) = \frac{1}{(-\|\mathbf{k}\|^2 + k_0^2)}\mathscr{F}g(\mathbf{k}),$$

since in general, we do not have that

$$\frac{1}{k_0^2 - \|\cdot\|^2}\mathscr{F}g(\cdot) \in \mathscr{S}(\mathbb{R}^n),$$

because of the fact that $\dfrac{1}{k_0^2 - \|\cdot\|^2}$ is not guaranteed to be in $\mathscr{S}(\mathbb{R}^n)$. Indeed, for example, take the 1-dimensional function $f(r) := \dfrac{1}{k_0^2 - |r|^2}$. From

$$r^3 f(r) = r\frac{r^2}{k_0^2 - r^2} = r\left(1 + \frac{k_0^2}{k_0^2 - r^2}\right) \xrightarrow{r\to\infty} \infty,$$

we have that $f \notin \mathscr{S}(\mathbb{R})$. Thus, in order to solve the Helmholtz equation using the Fourier transform, we need another concept, namely, distributions, which will be studied in the next subsection.

2.4 Fourier transform on $\mathscr{S}'(\mathbb{R}^n)$

Let $\mathscr{D}(\mathbb{R}^n) := \mathscr{C}_c^\infty(\mathbb{R}^n)$ be the set of test functions. First, we define the space of distributions:

Definition 2.25. A functional $T : \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ is called $\mathbb{C}$-linear, if

$$T(\lambda_1\psi_1 + \lambda_2\psi_2) = \overline{\lambda_1}T(\psi_1) + \overline{\lambda_2}T(\psi_2). \tag{2.21}$$

Definition 2.26. A functional $T : \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ is called a distribution, if it is $\mathbb{C}$-linear and if for every compact set $K \subseteq \mathbb{R}^n$, there exist constants $C > 0$ and $k \in \mathbb{N}$, such that for all $\psi \in \mathscr{D}(\mathbb{R}^n)$ with $\operatorname{supp}(\psi) \subseteq K$, the following holds:

$$\left|T(\psi)\right| \leq C \sum_{|\alpha| \leq k} \sup_{x \in K} \left|\partial^\alpha \psi(x)\right| . \tag{2.22}$$

The set of all distributions is denoted by $\mathscr{D}'(\mathbb{R}^n)$.

We have some note on the set of distributions:

Remark 2.27. (i) The set

$$\left\{ \mathscr{D}(\mathbb{R}^n) \ni \psi \mapsto \sum_{|\alpha| \leq k} \sup_{x \in K} \left|\partial^\alpha \psi(x)\right| \;\middle|\; k, l \in \mathbb{N} \right\}$$

is a family of semi-norms and hence, $\mathscr{D}(\mathbb{R}^n)$ is a locally convex space.

(ii) The space of distributions $\mathscr{D}'(\mathbb{R}^n)$ is the dual space of $\mathscr{D}(\mathbb{R}^n)$, i.e., for all $T \in \mathscr{D}'(\mathbb{R}^n)$ the following holds

- $T(\alpha\psi + \beta\phi) = \alpha T(\psi) + \beta T(\phi) \quad \forall \alpha, \beta \in \mathbb{C}, \psi, \phi \in \mathscr{D}(\mathbb{R}^n)$
- $T(\psi_l) \overset{l \to \infty}{\longrightarrow} T(\psi)$, for $\psi_l \overset{l \to \infty}{\longrightarrow} \psi$ in $\mathscr{D}(\mathbb{R}^n)$.

For proof see Theorem 1.26 in [23].

(iii) The convergence in $\mathscr{D}'(\mathbb{R}^n)$ can be defined as follows: For T and a sequence $(T_k)_k$ in $\mathscr{D}(\mathbb{R}^n)$, we define

$$T_k \xrightarrow{\mathscr{D}'(\mathbb{R}^n)} T :\Longleftrightarrow \; T_k(\psi) \xrightarrow{\mathscr{D}(\mathbb{R}^n)} T(\psi) \quad \forall \psi \in \mathscr{D}(\mathbb{R}^n).$$

In the following, we introduce some familiar examples of distributions:

Example 2.28. (i) If f is locally integrable, then $T_f : \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ given by

$$T_f(\psi) = \int_{\mathbb{R}^n} f(\mathbf{r})\overline{\psi(\mathbf{r})} \, \mathrm{d}\mathbf{r} \tag{2.23}$$

is called regular distribution. This functional is clearly linear. The boundedness follows by the following consideration:

$$\left|T_f(\psi)\right| = \left|\int_{\mathbb{R}^n} f(\mathbf{r})\overline{\psi(\mathbf{r})} \, \mathrm{d}\mathbf{r}\right| \leq \int_{\mathbb{R}^n} \left|f(\mathbf{r})\overline{\psi(\mathbf{r})}\right| \mathrm{d}\mathbf{r}. \tag{2.24}$$

Now, let $K \subseteq \mathbb{R}^n$ be a compact subset such that $\operatorname{supp}(\psi) \subseteq K$. We have

$$\left|T_f(\psi)\right| \leq \int_K \left|f(\mathbf{r})\overline{\psi(\mathbf{r})}\right| \mathrm{d}\mathbf{r} \leq \sup_{\mathbf{r} \in K} \left|\psi(\mathbf{r})\right| \underbrace{\int_K |f(\mathbf{r})| \, \mathrm{d}\mathbf{r}}_{=:\mathrm{C}}, \tag{2.25}$$

which holds for all $\psi \in \mathscr{D}(\mathbb{R}^n)$. Therefore, if zero is the limit of the sequence $(\psi_l)_l \subseteq \mathscr{D}(\mathbb{R}^n)$, then it is also the limit of $(T_f(\psi_l))_l \subseteq \mathscr{D}'(\mathbb{R}^n)$.

For $f, \psi \in L^2(\mathbb{R}^n)$, it holds that

$$T_f(\psi) = \langle f, \psi \rangle_{L^2(\mathbb{R}^n)}.$$

Therefore, some authors utilise the notation $\langle f, \psi \rangle$ for $T_f(\psi)$, although f, ψ are not elements of $L^2(\mathbb{R}^n)$. Here, $\langle f, \psi \rangle$ is considered as a distribution, not an inner product.

Furthermore, the distribution T_f is uniquely determined by function f. Indeed, the identical symbol is usually being used for the function and the related distribution. The reason is: We have that, if $T_f = T_g$, then $f = g$. To verify that, observe the following: If $T_f = T_g$, then

$$\int_{\mathbb{R}^n} f(\mathbf{r})\overline{\psi(\mathbf{r})}\ \mathrm{d}\mathbf{r} = \int_{\mathbb{R}^n} g(\mathbf{r})\overline{\psi(\mathbf{r})}\ \mathrm{d}\mathbf{r} \quad \forall \psi \in \mathscr{D}(\mathbb{R}^n).$$

Hence,

$$\int_{\mathbb{R}^n} (f(\mathbf{r}) - g(\mathbf{r}))\overline{\psi(\mathbf{r})}\ \mathrm{d}\mathbf{r} = 0 \quad \forall \psi \in \mathscr{D}(\mathbb{R}^n).$$

Therefore, $f = g$.

(ii) For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the operator defined as

$$\psi \mapsto \int_{\mathbb{R}^n} f(\mathbf{r})\overline{\partial^\alpha \psi(\mathbf{r})}\ \mathrm{d}\mathbf{r}\ , \psi \in \mathscr{D}(\mathbb{R}^n),$$

is a distribution as well.

We give another well-known example of distribution:

Definition 2.29. For $\mathbf{r}_0 \in \mathbb{R}^n$, we define the Dirac-delta distribution $\delta_{\mathbf{r}_0} \in \mathscr{D}'(\mathbb{R}^n)$ as

$$\delta_{\mathbf{r}_0}(\psi) := \psi(\mathbf{r}_0),$$

for $\psi \in \mathscr{D}(\mathbb{R}^n)$. For $r_0 = 0$, we write δ_0 as δ.

Remark 2.30. (i) It is clear that $\delta_{\mathbf{r}_0}$ is $\mathbb{C}$-linear. From

$$|\delta_{\mathbf{r}_0}(\psi)| = |\psi(\mathbf{r}_0)| \le \sup_{x \in K} \left|\psi(x)\right|,$$

the equation (2.22) is satisfied with $k = 0$ and $C = 1$. Hence, $\delta_{\mathbf{r}_0}$ is indeed a distribution.

(ii) It is not a regular distribution. We will prove it by contradiction. Suppose that there exists a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $\delta_{\mathbf{r}_0} = T_f$. Let $\psi \in \mathscr{D}(\mathbb{R}^n)$ with $\text{supp}(\psi) \subseteq \overline{\mathscr{B}} \subseteq \mathbb{R}^n$, where $\mathscr{B}$ is the unit ball, and $\psi(0) = 1$. Then, for $k \in \mathbb{N}$, define

$$\psi_k(x) := \psi(k(\mathbf{r} - \mathbf{r}_0)).$$

From definition, we have $\text{supp}(\psi_k) \subseteq \overline{\mathscr{B}_k}$ where $\mathscr{B}_k$ is the open ball with radius $\frac{1}{k}$ centered at $\mathbf{r}_0$, and $\psi_k(\mathbf{r}_0) = 1$. Then, we have

$$|\delta_{\mathbf{r}_0}(\psi_k)| = |T_f(\psi_k)| \leq \int_{\overline{\mathscr{B}_k}} |f(\mathbf{r})||\psi(k(\mathbf{r} - \mathbf{r}_0))| \, \mathrm{d}\mathbf{r} \leq \int_{\overline{\mathscr{B}_k}} |f(\mathbf{r})| \|\psi\|_{L^\infty} \, \mathrm{d}\mathbf{r}.$$

Substitute $\mathbf{u} := k(\mathbf{r} - \mathbf{r}_0)$, we get

$$\int_{\overline{\mathscr{B}_k}} |f(\mathbf{r})| \|\psi\|_{L^\infty} \, \mathrm{d}\mathbf{r} = \|\psi\|_{L^\infty} \int_{\overline{\mathscr{B}}} \frac{|f(\frac{\mathbf{u}+k\mathbf{r}_0}{k})|}{k} \, \mathrm{d}\mathbf{u} \leq \frac{1}{k} \int_{\overline{\mathscr{B}}} \left| f\left(\frac{\mathbf{u} + k\mathbf{r}_0}{k}\right)\right| \, \mathrm{d}\mathbf{u},$$

which converges to 0 as $k \to \infty$, since $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Thus, we have that

$$|\delta_{\mathbf{r}_0}(\psi_k)| \xrightarrow{k\to\infty} 0,$$

which is a contradiction to the fact that

$$|\delta_{\mathbf{r}_0}(\psi_k)| = \psi_k(\mathbf{r}_0) = 1.$$

Hence, $\delta_{\mathbf{r}_0}$ is not a regular distribution and also not in $L^1_{\text{loc}}(\mathbb{R}^n)$.

From these examples, we get the observation: $\mathscr{D}'(\mathbb{R}^n)$ is a larger space than $\mathscr{D}(\mathbb{R}^n)$. Moreover, we know that each locally integrable function f uniquely determines a regular distribution T_f. Besides, since δ is an element of $\mathscr{D}'(\mathbb{R}^n)$ and is not locally integrable, the space of distributions is a strict extension of $L^1_{\text{loc}}(\mathbb{R}^n)$. Thus, we also have the other name for distribution, viz., generalised function.

Remark 2.31. Note that Fourier transform of a compactly supported function is not guaranteed to have a compact support. Thus, we can say that $\mathscr{F}(\mathscr{D}'(\mathbb{R}^n)) \neq \mathscr{D}'(\mathbb{R}^n)$.

Indeed, for a function $f \in \mathscr{D}(\mathbb{R}^n)$, $\mathscr{F}f$ can only have compact support if $f = 0$. We can show that as follows: Consider $f \in \mathscr{D}(\mathbb{R}^n)$. We know that $\mathscr{F}f$ can be extended to a holomorphic function on $\mathbb{C}^n$ (see Theorem 5.30 in [23]). Therefore, we have that if $\mathscr{F}f$ is a compactly supported function, it is zero on a open subset of $\mathbb{R}^n$ and hence will be zero on $\mathbb{C}^n$. But because of the fact that the Fourier operator is injective, we can conclude that $f = 0$.

As before, we need a space so that the Fourier transform defined on this space is again a isomorphism. We have the following definition:

Definition 2.32. A functional $T : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ is called a tempered distribution, if it is $\mathbb{C}$-linear and there exist constants $C > 0$ and $k, l \in \mathbb{N}$ such that for all $\psi \in \mathscr{S}(\mathbb{R}^n)$, the following holds:

$$\big|T(\psi)\big| \leq C \sum_{|\alpha|\leq k,\ |\beta|\leq l} \underbrace{\sup_{\mathbf{r}\in\mathbb{R}^n} \Big|\mathbf{r}^{\alpha}\partial^{\beta}\psi(\mathbf{r})\Big|}_{=p_{\alpha,\beta}(\psi)}. \tag{2.26}$$

The set of all tempered distributions is denoted by $\mathscr{S}'(\mathbb{R}^n)$.

Similar to the space of distributions $\mathscr{D}'(\mathbb{R}^n)$ (cf. Remark 2.27), for the set of tempered distributions, we note the followings:

Remark 2.33. (i) The set

$$\left\{ \sum_{|\alpha|\leq k,\ |\beta|\leq l} p_{\alpha,\beta} \;\middle|\; k, l \in \mathbb{N} \right\}$$

is a family of semi-norms and hence, $\mathscr{S}'(\mathbb{R}^n)$ is a locally convex space.

(ii) $\mathscr{S}'(\mathbb{R}^n)$ is the topological dual of $\mathscr{S}(\mathbb{R}^n)$ (see Theorem 5.17 in [23]).

(iii) The convergence in $\mathscr{S}'(\mathbb{R}^n)$ can be defined as follows: For T and a sequence $(T_k)_k$ in $\mathscr{S}(\mathbb{R}^n)$, we define

$$T_k \xrightarrow{\mathscr{S}'(\mathbb{R}^n)} T :\Longleftrightarrow\ T_k(\psi) \xrightarrow{\mathscr{S}(\mathbb{R}^n)} T(\psi) \quad \forall \psi \in \mathscr{S}(\mathbb{R}^n).$$

(iv) We have $\mathscr{S}'(\mathbb{R}^n) \subseteq \mathscr{D}'(\mathbb{R}^n)$, see Remark 5.20 in [23].

The following lemma is fundamental for determining whether the distribution T_f, where f is a locally integrable function, is tempered.

Lemma 2.34. *If the function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and there exists $N \in \mathbb{N}$ such that*

$$\lim_{\|\mathbf{r}\|\to\infty} \frac{\big|f(\mathbf{r})\big|}{\|\mathbf{r}\|^N} = 0,$$

then $T_f \in \mathscr{S}'(\mathbb{R})$. We simply write $f \in \mathscr{S}'(\mathbb{R})$, with

$$\psi \mapsto \int_{\mathbb{R}^n} f(\mathbf{r})\psi(\mathbf{r})\ \mathrm{d}\mathbf{r}.$$

Proof. Example 5.12 in [1]. $\square$

For a better understanding of the space of tempered distributions, we give some basic examples.

Example 2.35. (i) We know from Example 2.28 that $T_f : \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ given by

$$T_f(\psi) = \int_{\mathbb{R}^n} f(\mathbf{r})\overline{\psi(\mathbf{r})}\ \mathrm{d}\mathbf{r}$$

is a distribution. However, it is not a tempered distribution. To see this, we take the 1-dimensional case and the function $f(r) := e^{r^2}$ and $\psi(r) := e^{-\frac{r^2}{2}} \in \mathscr{S}(\mathbb{R})$. So we have

$$\int_{\mathbb{R}} e^{\frac{r^2}{2}}\ \mathrm{d}r = \infty.$$

(ii) As in Definition 2.29, that is, $\delta_{\mathbf{r}_0}(\psi) := \psi(\mathbf{r}_0)$ for $\psi \in \mathscr{S}(\mathbb{R}^n)$, the Dirac-delta distribution is an element of $\mathscr{S}'(\mathbb{R}^n)$. Indeed, for $\psi \in \mathscr{S}(\mathbb{R}^n)$, we have

$$|\delta_{\mathbf{r}_0}(\psi)| = |\psi(\mathbf{r}_0)| \leq \sup_{x\in K}\left|\psi(x)\right|,$$

Equation (2.26) is satisfied with $k, l = 0$ and $C = 1$.

(iii) For all $p \in [1, \infty)$, $L^p(\mathbb{R}^n) \subseteq \mathscr{S}'(\mathbb{R}^n)$. For proof see Remark 5.20 in [23].

In the following, we can find three examples of functions that are elements of $\mathscr{S}'(\mathbb{R}^n)$, which hold true, due to the fact that these functions are locally integrable and polynomially bounded at infinity as stated in Lemma 2.34.

(iv) $\mathbb{1} \in \mathscr{S}'(\mathbb{R}^n)$.

(v) The characteristic function $\mathbb{1}_{[0,\infty)} \in \mathscr{S}'(\mathbb{R}^n)$.

(vi) The function $f : \mathbb{R}^n \to \mathbb{C}$, $\mathbf{r} \mapsto \|\mathbf{r}\|^N$, $N \in \mathbb{N}$ is also an element of $\mathscr{S}'(\mathbb{R}^n)$.

Now, we want to define the Fourier transform on $\mathscr{S}'(\mathbb{R}^n)$.

Definition 2.36. For $T \in \mathscr{S}'(\mathbb{R}^n)$, the Fourier transform is defined by

$$\mathscr{F}T(\psi) := T(\mathscr{F}\psi) \tag{2.27}$$

for all $\psi \in \mathscr{S}(\mathbb{R}^n)$.

This definition of the Fourier transform is well-defined because of the fact that for $T \in \mathscr{S}'(\mathbb{R}^n)$, the map $\mathscr{S}(\mathbb{R}^n) \ni \psi \mapsto T(\mathscr{F}\psi) \in \mathbb{C}$ is also a tempered distribution, together with the equation (2.4) by third item of Remark 2.14.

As an example, we calculate the Fourier transform of the identity function:

Example 2.37. Applying the Fourier transform to the constant function gives us the Dirac-delta distribution. Indeed, for $\phi \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\mathscr{F}1(\phi) := 1(\mathscr{F}\phi) = \int_{\mathbb{R}^n} 1(\mathscr{F}\phi)(\mathbf{r})\ \mathrm{d}\mathbf{r} = \frac{(\sqrt{2\pi})^n}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} (\mathscr{F}\phi)(\mathbf{r})e^{i\mathbf{r}\cdot 0}\ \mathrm{d}\mathbf{r}.$$

Using the definition of the inverse Fourier transform and Definition 2.29, we obtain

$$\mathscr{F}1(\phi) = (\sqrt{2\pi})^n \mathscr{F}^{-1}(\mathscr{F}\phi)(0) = (\sqrt{2\pi})^n \phi(0) = (\sqrt{2\pi})^n \delta_0.$$

Similarly to the case of $\mathscr{S}(\mathbb{R}^n)$, we also have $\mathscr{F}\left(\mathscr{S}'(\mathbb{R}^n)\right) = \mathscr{S}'(\mathbb{R}^n)$ and hence, the inverse Fourier transform can also be defined uniquely. Indeed, we have the following important result:

Proposition 2.38. *The Fourier transform $\mathscr{F} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is topological isomorphism with continuous inverse $\mathscr{F}^{-1} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$, given by*

$$\mathscr{F}^{-1}T(\psi) := T(\mathscr{F}^{-1}\psi) \tag{2.28}$$

for $T \in \mathscr{S}^{'}(\mathbb{R}^n)$, $\psi \in \mathscr{S}(\mathbb{R}^n)$.

Proof. Theorem 5.17 in [1]. □

Remark 2.39. The Fourier operator $\mathscr{F}$ defined on $\mathscr{S}'(\mathbb{R}^n)$ is the extension of $\mathscr{F}$ on $\mathscr{S}(\mathbb{R}^n)$ (see Remark 5.15 in [1]). Therefore, the properties (i), (ii), (iv), (v) in Proposition 2.12 also hold for the Fourier transform defined on the space of tempered distribution.

Subsequently, we want to learn about one of the crucial properties of the Fourier transform, namely, the convolution theorem. To accomplish this, we first take into account the following facts.

First, we define the multiplication of a tempered distribution and a Schwartz function and observe its property:

Remark 2.40. Consider the multiplication operation

$$\begin{aligned} \mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) &\to \mathscr{S}(\mathbb{R}^n) \\ (\psi, \phi) \qquad &\mapsto \psi(\mathbf{r})\phi(\mathbf{r}), \end{aligned}$$

which turns $\mathscr{S}(\mathbb{R}^n)$ into a commutative algebra without identity (cf. Proposition 2.11).

For $T \in \mathscr{S}'(\mathbb{R}^n)$ and $\psi \in \mathscr{S}(\mathbb{R}^n)$, the tempered distribution $T\psi \in \mathscr{S}'(\mathbb{R}^n)$ is defined by

$$(T\psi)(\phi) := T(\psi\phi), \quad \phi \in \mathscr{S}(\mathbb{R}^n).$$

For $f, \psi, \phi \in \mathscr{S}(\mathbb{R}^n)$, we have $T_f\psi = T_{f\psi}$, because due to Example 2.28, it holds

$$(T_f\psi)(\phi) = T_f(\psi\phi) = \int_{\mathbb{R}^n} f(\mathbf{r})(\psi\phi)(\mathbf{r})\ \mathrm{d}\mathbf{r}$$

and commutativity property of pointwise multiplication gives

$$(T_f\psi)(\phi) = \int_{\mathbb{R}^n} f(\mathbf{r})\psi(\mathbf{r})\phi(\mathbf{r})\ \mathrm{d}\mathbf{r} = \int_{\mathbb{R}^n} (f\psi)(\mathbf{r})\phi(\mathbf{r})\ \mathrm{d}\mathbf{r} = T_{f\psi}(\phi).$$

This extends the multiplication of functions from $\mathscr{S}(\mathbb{R}^n)$ to the multiplication of a tempered distribution with a Schwartz function on $\mathscr{S}'(\mathbb{R}^n)$.

Then, we define the convolution of a tempered distribution and a Schwartz function as follows:

Definition 2.41. For $T \in \mathscr{S}'(\mathbb{R}^n)$ and $\psi \in \mathscr{S}(\mathbb{R}^n)$, the convolution is defined as

$$(T * \psi)(\phi) := T\left(\psi(-\cdot) * \phi\right) \tag{2.29}$$

for all $\phi \in \mathscr{S}(\mathbb{R}^n)$, yielding $T * \psi \in \mathscr{S}'(\mathbb{R}^n)$ for $\mathbf{r} \in \mathbb{R}^n$.

An example of convolving a tempered distribution with a Schwartz function is given in the lemma below:

Lemma 2.42. *For $\psi \in \mathscr{S}(\mathbb{R}^n)$, it holds that $\delta_0 * \psi = T_\psi$. Hence, $\delta_0 * \psi = \psi$ in $\mathscr{S}'(\mathbb{R}^n)$.*

Proof. For $\phi \in \mathscr{S}(\mathbb{R}^n)$, due to Definitions 2.41 and 2.29, we have that

$$(\delta_0 * \psi)(\phi) = \delta_0(\psi(-\cdot) * \phi) = (\psi(-\cdot) * \phi)(0).$$

Using the definition of convolution in $\mathscr{S}(\mathbb{R}^n)$, we arrive at

$$(\delta_0 * \psi)(\phi) = \int_{\mathbb{R}^n} \psi(-\mathbf{r})\phi(0 - \mathbf{r})\ \mathrm{d}\mathbf{r} = \int_{\mathbb{R}^n} \psi(\mathbf{r})\phi(\mathbf{r})\ \mathrm{d}\mathbf{r} = T_\psi(\phi).$$

□

Analogous to the pointwise multiplication described above, from Definition 2.41, we can also extend the convolution of functions from $\mathscr{S}(\mathbb{R}^n)$ to the convolution of a tempered distribution with a Schwartz function on $\mathscr{S}'(\mathbb{R}^n)$ as follows:

Remark 2.43. For $f, \psi, \phi \in \mathscr{S}(\mathbb{R}^n)$, applying the known definitions from above yields

$$(T_f * \psi)(\phi) = T_f(\psi^- * \phi) = \int_{\mathbb{R}^n} f(\mathbf{r}) \int_{\mathbb{R}^n} \psi(-\mathbf{p})\phi(\mathbf{r} - \mathbf{p})\ \mathrm{d}\mathbf{p}\ \mathrm{d}\mathbf{r}.$$

Now, utilising the substitution $\mathbf{r} - \mathbf{p} = \mathbf{q}$ gives

$$(T_f * \psi)(\phi) = \int_{\mathbb{R}^n} f(\mathbf{r}) \int_{\mathbb{R}^n} \psi(\mathbf{q} - \mathbf{r})\phi(\mathbf{q})\ \mathrm{d}\mathbf{q}\ \mathrm{d}\mathbf{r}.$$

Again, since $f, \phi, \psi \in \mathscr{S}(\mathbb{R}^n)$, from Proposition 2.11 and Theorem B.3(ii), we have that

$$(\mathbf{r}, \mathbf{q}) \mapsto f(\mathbf{r})\psi(\mathbf{q} - \mathbf{r})\phi(\mathbf{q}) \in L^1(\mathbb{R}^n \times \mathbb{R}^n).$$

Hence, we can apply the Fubini-Tonelli theorem to interchange the order of integration, namely,

$$(T_f * \psi)(\phi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{r})\psi(\mathbf{q} - \mathbf{r})\mathrm{d}\mathbf{r}\ \phi(\mathbf{q})\ \mathrm{d}\mathbf{q} = \int_{\mathbb{R}^n} (f * \psi)(\mathbf{q})\phi(\mathbf{q})\ \mathrm{d}\mathbf{q}.$$

So, we arrive at

$$(T_f * \psi)(\phi) = T_{f*\psi}(\phi).$$

From the two remarks above, we can finally state the convolution theorem as follows:

Theorem 2.44 (Convolution theorem)**.** *For $T \in \mathscr{S}'(\mathbb{R}^n)$ and $\psi \in \mathscr{S}(\mathbb{R}^n)$, it holds:*
(i)
$$\mathscr{F}(T * \psi) = (\sqrt{2\pi})^n \mathscr{F}T\mathscr{F}\psi, \tag{2.30}$$
(ii)
$$\mathscr{F}T * \mathscr{F}\psi = (\sqrt{2\pi})^n \mathscr{F}(T\psi). \tag{2.31}$$
The same equalities holds for $\mathscr{F}^{-1}$.

Proof. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$.

(i) Due to the definition of Fourier transform on the space of tempered distributions and Definition 2.41, we have:

$$\mathscr{F}(T * \psi)(\phi) = (T * \psi)\mathscr{F}\phi = T\left(\psi(-\cdot) * \mathscr{F}\phi\right)$$

Now, applying again the definition of $\mathscr{F}$ on $\mathscr{S}'(\mathbb{R}^n)$, we obtain

$$\mathscr{F}(T * \psi)(\phi) = T\left(\mathscr{F}\mathscr{F}^{-1}\left(\psi(-\cdot) * \mathscr{F}\phi\right)\right) = \mathscr{F}T\left(\mathscr{F}^{-1}\left(\psi(-\cdot) * \mathscr{F}\phi\right)\right).$$

Finally, utilising Proposition 2.12(iii) yields

$$\mathscr{F}(T * \psi)(\phi) = \mathscr{F}T\left((\sqrt{2\pi})^n(\mathscr{F}\psi)\phi\right) = (\sqrt{2\pi})^n(\mathscr{F}T\mathscr{F}\psi)(\phi).$$

(ii) Definition 2.41, definition and the property $(\mathscr{F}\psi)(-\cdot) = \mathscr{F}^{-1}$ of the Fourier transform on the space of tempered distributions give

$$(\mathscr{F}T * \mathscr{F}\psi)(\phi) = \mathscr{F}T\left((\mathscr{F}\psi)(-\cdot) * \phi\right) = T\left(\mathscr{F}\left(\mathscr{F}^{-1}\psi * \phi\right)\right).$$

At last, Proposition 2.12(iii) and the definition of the Fourier transform on $\mathscr{S}'(\mathbb{R}^n)$ lead us to

$$(\mathscr{F}T * \mathscr{F}\psi)(\phi) = T\left((\sqrt{2\pi})^n\psi\mathscr{F}\phi\right) = (\sqrt{2\pi})^n(T\psi)(\mathscr{F}\phi),$$

which is equivalent to

$$(\mathscr{F}T * \mathscr{F}\psi)(\phi) = (\sqrt{2\pi})^n(\mathscr{F}(T\psi))(\phi).$$

□

At last, we show that the space of tempered distributions overcomes the shortcoming of the two function spaces $\mathscr{S}(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, which is stated in the previous subsection. Compared to Remark 2.24, which shows that despite $g \in \mathscr{S}(\mathbb{R}^n)$, solving the Helmholtz equation $(\Delta + k_0)u = g$ on $\mathbb{R}^n$ for $k_0 \in \mathbb{R}$ does not give a $\mathscr{S}(\mathbb{R}^n)$ solution. The following shows that if $g \in \mathscr{S}'(\mathbb{R}^n)$, the solution of the Helmholtz equation is again a tempered distribution. Indeed, let $g \in \mathscr{S}'(\mathbb{R}^n)$. First, we have

$$\int_{\mathscr{B}} \frac{1}{k_0 - \|\mathbf{r}\|^2}\ \mathrm{d}\mathbf{r} = \int_{S_{n-1}} \int_0^1 \frac{\rho^{n-1}}{k_0^2 - \rho^2}\ \mathrm{d}\rho\ \mathrm{d}S = \omega(S_{n-1}) \int_0^1 \frac{\rho^{n-1}}{k_0^2 - \rho^2}\ \mathrm{d}\rho,$$

with $\mathrm{d}S$ is the surface element, $\mathscr{B} \subseteq \mathbb{R}^n$ is the unit ball, $\omega(S_{n-1})$ denotes the surface area of the $(n-1)$-sphere S_{n-1}. Now, we consider two cases:

- For $n \in \mathbb{N}$ even, we use the substitution $k_0^2 - \rho^2 =: u$ to get

  $$\int \frac{\rho^{n-1}}{k_0^2 - \rho^2}\ \mathrm{d}\rho = \int \frac{(u - k_0^2)^{\frac{n-2}{2}}}{u}\ \mathrm{d}u.$$

  From binomial expansion, we get

  $$\int \frac{(u - k_0^2)^{\frac{n-2}{2}}}{u}\ \mathrm{d}u = \sum_{k=0}^{\frac{n-2}{2}} \binom{\frac{n-2}{2}}{k} (-k_0)^{\frac{n-2}{2} - k} \int u^{k-1}\ \mathrm{d}u.$$

  Integrating $\int u^{k-1}\ \mathrm{d}u$ for each $k \in \{0, 1, ..., \frac{n-2}{2}\}$ gives

  $$\int \frac{(u - k_0^2)^{\frac{n-2}{2}}}{u}\ \mathrm{d}u = (-k_0)^{\frac{n-2}{2}} \ln(|u|) + \sum_{k=1}^{\frac{n-2}{2}} \binom{\frac{n-2}{2}}{k} (-k_0)^{\frac{n-2}{2} - k} \frac{u^k}{k}.$$

  Since

  $$(-k_0)^{\frac{n-2}{2}} \ln(|k_0^2 - \rho^2|) + \sum_{k=1}^{\frac{n-2}{2}} \binom{\frac{n-2}{2}}{k} (-k_0)^{\frac{n-2}{2} - k} \frac{(k_0^2 - \rho^2)^k}{k} \Bigg|_0^1 < \infty,$$

  we can conclude that

  $$\int_0^1 \frac{\rho^{n-1}}{k_0^2 - \rho^2}\ \mathrm{d}\rho < \infty.$$

- For $n \in \mathbb{N}$ odd, we have

  $$\int_0^1 \frac{\rho^{n-1}}{k_0^2 - \rho^2} = -\int_0^1 \sum_{k=0}^{n-3} \rho^{n-1-k} k_0^{k-2} - \frac{k_0^{n-1}}{\rho^2 - k_0^2} = -\frac{\rho^{n-k}}{n-k} - \frac{\operatorname{arctanh}\left(\frac{\rho}{k_0}\right)}{k_0} \Bigg|_0^1$$
  $$< \infty.$$

Hence, we can conclude that $\dfrac{1}{k_0^2 - \|\cdot\|^2} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. It is also clear that

$$\lim_{\|\mathbf{r}\| \to \infty} \frac{1}{k_0^2 - \|\mathbf{r}\|^2} = 0.$$

Hence, we can apply Lemma 2.34 and obtain

$$\mathbf{k} \mapsto \frac{1}{k_0^2 - \|\mathbf{k}\|^2} \in \mathscr{S}'(\mathbb{R}^n).$$

Thus, the equation

$$-(\Delta + k_0)u = g \Leftrightarrow (k_0^2 - \|\mathbf{k}\|^2)\mathscr{F}u(\mathbf{k}) = \mathscr{F}g(\mathbf{k})$$

is now equivalent to

$$\mathscr{F}u(\mathbf{k}) = \frac{1}{k_0^2 - \|\mathbf{k}\|^2}\mathscr{F}g(\mathbf{k}) \in \mathscr{S}'(\mathbb{R}^n).$$

Therefore, from Definition 2.41, we obtain

$$u(\mathbf{r}) = \mathscr{F}^{-1}\left(\frac{1}{k_0^2 - \|\cdot\|^2}\mathscr{F}g\right)(\mathbf{r}) = \mathscr{F}^{-1}\left(\frac{1}{k_0^2 - \|\cdot\|^2}\right) * g \in \mathscr{S}'(\mathbb{R}^n).$$

To summarise this section, we first give the following relation between the studied spaces:

$$\mathscr{C}_c^\infty \overset{\text{Def}}{\subseteq} \mathscr{D}(\mathbb{R}^n) \overset{2.6}{\subseteq} \mathscr{S}(\mathbb{R}^n) \overset{2.9}{\subseteq} L^p(\mathbb{R}^n) \overset{2.35}{\subseteq} \mathscr{S}'(\mathbb{R}^n) \overset{2.33}{\subseteq} \mathscr{D}'(\mathbb{R}^n).$$

Then, we provide a table comparing the properties of the Fourier transform defined on these spaces:

$L^1(\mathbb{R}^n)$	$\mathscr{F}(L^1(\mathbb{R}^n)) \neq L^1(\mathbb{R}^n)$, $\mathscr{F} : L^1(\mathbb{R}^n) \to \mathscr{C}(\mathbb{R}^n)$ is not an isomorphism, Solution of $-(\Delta + k_0)u = g$ is not in $L^1(\mathbb{R}^n)$.
$\mathscr{S}(\mathbb{R}^n)$, $L^2(\mathbb{R}^n)$	$\mathscr{F}(\mathscr{S}(\mathbb{R}^n)) = \mathscr{S}(\mathbb{R}^n)$, $\mathscr{F}(L^2(\mathbb{R}^n)) = L^2(\mathbb{R}^n)$, $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$, $\mathscr{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are isomorphisms, Solution of $-(\Delta + k_0)u = g$ is not in $\mathscr{S}(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$.
$\mathscr{D}'(\mathbb{R}^n)$	$\mathscr{F}(\mathscr{D}'(\mathbb{R}^n)) \neq \mathscr{D}'(\mathbb{R}^n)$, $\mathscr{F}$ is not an isomorphism, Solution of $-(\Delta + k_0)u = g$ is not in $\mathscr{D}'(\mathbb{R}^n)$.
$\mathscr{S}'(\mathbb{R}^n)$	$\mathscr{F}(\mathscr{S}'(\mathbb{R}^n)) = \mathscr{S}'(\mathbb{R}^n)$, $\mathscr{F} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is an isomorphism, Solution of $-(\Delta + k_0)u = g$ is in $\mathscr{S}'(\mathbb{R}^n)$.

Among all of these spaces, the space of tempered distributions $\mathscr{S}'(\mathbb{R}^n)$ is the best for our use, because the Fourier transform is a topological isomorphism on $\mathscr{S}'(\mathbb{R}^n)$ and many PDE problems that do not have a function solution, have solutions in $\mathscr{S}'(\mathbb{R}^n)$, e.g., the Helmholtz equation, which will be studied thoroughly in Section 5.

3 Partial Fourier transforms (M.U.)

As the Fourier transform is defined on $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$, we define the partial Fourier transform on the same spaces. For Schwartz functions, we have the definition:

Definition 3.1. For $j \in \{1, 2, ..., n\}$ and $\psi \in \mathscr{S}(\mathbb{R}^n)$, the partial Fourier transform $\mathscr{F}_j$ is defined by

$$\mathscr{F}_j\psi(r_1, ..., r_{j-1}, k_j, r_{j+1}, ..., r_n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(r_1, ..., r_{j-1}, s, r_{j+1}, ..., r_n)e^{-ik_j s}\, \mathrm{d}s. \tag{3.1}$$

For $I = \{j_1, ..., j_m\} \subset \{1, 2, ..., n\}$ and $\psi \in \mathscr{S}(\mathbb{R}^n)$, the partial Fourier transform $\mathscr{F}_I$ is given by

$$\mathscr{F}_I\psi := \mathscr{F}_{j_m} \cdots \mathscr{F}_{j_1}\psi. \tag{3.2}$$

The order of the partial Fourier transform is not important due to the Fubini-Tonelli theorem. Note that $\mathscr{F}_{1,...,n} = \mathscr{F}$.

Remark 3.2. The map $r_j \mapsto \psi(\mathbf{r})$ with fixed $r_1, ..., r_{j-1}, r_{j+1}, ..., r_n$ is an element of $\mathscr{S}(\mathbb{R})$ and therefore in $L^1(\mathbb{R})$ with respect to r_j, too. Hence, the partial Fourier transform in Definition 3.1 is well-defined.

The partial Fourier transform has the same properties as the n-dimensional Fourier transform:

Proposition 3.3. *The partial Fourier transforms $\mathscr{F}_I : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ are topological isomorphisms with continuous inverses $\mathscr{F}_I^{-1} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ given by*

$$\mathscr{F}_I^{-1}\psi := \mathscr{F}_{j_m}^{-1} \cdots \mathscr{F}_{j_1}^{-1}\psi, \tag{3.3}$$

where

$$\mathscr{F}_j^{-1}\psi(r_1, ..., r_{j-1}, k_j, r_{j+1}, ..., r_n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(r_1, ..., r_{j-1}, s, r_{j+1}, ..., r_n)e^{ik_j s}\mathrm{d}s. \tag{3.4}$$

Proof. Due to Definition 3.1 and Equation (3.2), we know that

$$\|\mathscr{F}_I\psi\|_{L^\infty} < C,$$

for a $C > 0$. $\mathscr{F}_I\psi \in \mathscr{C}^\infty(\mathbb{R}^n)$ and $\mathscr{F}_I$ interchanges differentiation and multiplication with polynomials. Suppose $I = \{1, ..., m\}$, we find

$$(\mathbf{k}, \mathbf{r})^\alpha \partial^\beta \mathscr{F}_I\psi = (-i)^{|\alpha_1|+|\beta_1|}\mathscr{F}_I(\partial_{\mathbf{s}}^{\alpha_1}\mathbf{r}^{\alpha_2}\mathbf{s}^{\beta_1}\partial_{\mathbf{r}}^{\beta_2}\psi) \tag{3.5}$$

for all $\mathbf{k} \in \mathbb{R}^m, \mathbf{r} \in \mathbb{R}^{n-m}$ and all $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{N}^m \times \mathbb{N}^{n-m}$. Indeed, consider

$$\partial_{\mathbf{k}}^\beta \mathscr{F}_I\psi(\mathbf{k}, \mathbf{r}) = (-i)^{|\beta|}\mathscr{F}_I(\mathbf{s} \mapsto \mathbf{s}^\beta\psi(\mathbf{s}))(\mathbf{k}, \mathbf{r}) \tag{3.6}$$

and

$$\mathscr{F}_I(\partial^\beta\psi)(\mathbf{k},\mathbf{r}) = i^{|\beta|}\mathbf{k}^\beta\mathscr{F}_I\psi(\mathbf{k},\mathbf{r}). \tag{3.7}$$

First, we have

$$(\mathbf{k},\mathbf{r})^\alpha\partial^\beta\mathscr{F}_I\psi = \mathbf{k}^{\alpha_1}\mathbf{r}^{\alpha_2}\partial_{\mathbf{k}}^{\beta_1}\partial_{\mathbf{r}}^{\beta_2}\mathscr{F}_I\psi = \mathbf{k}^{\alpha_1}\mathbf{r}^{\alpha_2}\partial_{\mathbf{k}}^{\beta_1}\mathscr{F}_I(\partial_{\mathbf{r}}^{\beta_2}\psi).$$

Then, applying Equation (3.6) leads to

$$(\mathbf{k},\mathbf{r})^\alpha\partial^\beta\mathscr{F}_I\psi = \mathbf{k}^{\alpha_1}\mathbf{r}^{\alpha_2}(-i)^{|\beta_1|}\mathscr{F}_I(\mathbf{s}\mapsto\mathbf{s}^{\beta_1}\partial_{\mathbf{r}}^{\beta_2}\psi)$$

and additionally making some rearrangements with i gives

$$(\mathbf{k},\mathbf{r})^\alpha\partial^\beta\mathscr{F}_I\psi = \mathbf{k}^{\alpha_1}\mathbf{r}^{\alpha_2}(-i)^{|\alpha_1|+|\beta_1|}i^{|\alpha_1|}\mathscr{F}_I(\mathbf{s}^{\beta_1}\partial_{\mathbf{r}}^{\beta_2}\psi).$$

After that, we use Equation (3.7) to obtain

$$(\mathbf{k},\mathbf{r})^\alpha\partial^\beta\mathscr{F}_I\psi = (-i)^{|\alpha_1|+|\beta_1|}\mathscr{F}_I(\partial_{\mathbf{s}}^{\alpha_1}\mathbf{r}^{\alpha_2}\mathbf{s}^{\beta_1}\partial_{\mathbf{r}}^{\beta_2}\psi).$$

So, since $\mathscr{F}_I\psi$ is continuous, $\|\mathscr{F}_I\psi\|_{L^\infty} < C$, $C > 0$ and Equation (3.5) yields $\left\|(\mathbf{k},\mathbf{r})^\alpha\partial^\beta\mathscr{F}_I\psi\right\|_{L^\infty} < C_2$, $C_2 > 0$, we have $\mathscr{F}_I\psi \in \mathscr{S}(\mathbb{R}^n)$.

Now, to show is: For all $\alpha,\beta \in \mathbb{N}^m \times \mathbb{N}^{n-m}$, there exist $\tilde{\alpha},\tilde{\beta}$ and $C > 0$ such that $p_{\alpha,\beta}(\mathscr{F}_I\psi) \leq Cp_{\tilde{\alpha},\tilde{\beta}}(\psi)$. First, we apply Equation (3.5) to get

$$p_{\alpha,\beta}(\mathscr{F}_I\psi) = \sup_{(\mathbf{k},\mathbf{r})\in\mathbb{R}^m\times\mathbb{R}^{n-m}}\left|\mathscr{F}_I(\partial_{\mathbf{s}}^{\alpha_1}\mathbf{r}^{\alpha_2}\mathbf{s}^{\beta_1}\partial_{\mathbf{r}}^{\beta_2}\psi)\right|.$$

We use the definition of the Fourier transform to have

$$\begin{aligned}&p_{\alpha,\beta}(\mathscr{F}_I\psi)\\ &= \sup_{(\mathbf{k},\mathbf{r})\in\mathbb{R}^m\times\mathbb{R}^{n-m}}\frac{1}{(2\pi)^n}\left|\int_{\mathbb{R}^m}\int_{\mathbb{R}^{n-m}}\partial_{\mathbf{s}}^{\alpha_1}(\mathbf{r}^{\alpha_2}\mathbf{s}^{\beta_1}\partial_{\mathbf{r}}^{\beta_2}\psi)e^{-i\mathbf{k_1}\cdot\mathbf{s_1}-i\mathbf{k_2}\cdot\mathbf{s_2}}\mathrm{d}\mathbf{s_1}\mathrm{d}\mathbf{s_2}\right|\end{aligned}$$

Now, we use the triangle inequality to have

$$p_{\alpha,\beta}(\mathscr{F}_I\psi) \leq \frac{1}{(2\pi)^n}\sup_{(\mathbf{k_1},\mathbf{k_2})\in\mathbb{R}^m\times\mathbb{R}^{n-m}}\int_{\mathbb{R}^m}\int_{\mathbb{R}^{n-m}}\left|\partial_{\mathbf{s}}^{\alpha_1}(\mathbf{r}^{\alpha_2}\mathbf{s}^{\beta_1}\partial_{\mathbf{r}}^{\beta_2}\psi)\right|\mathrm{d}\mathbf{s_1}\mathrm{d}\mathbf{s_2}.$$

Note that $\mathbf{s_1} = (s_{11},...,s_{1m}) \in \mathbb{R}^m$ and $\mathbf{s_2} = (s_{21},...,s_{2,(n-m)}) \in \mathbb{R}^{(n-m)}$. Multiplying the integrand with

$$\prod_{j=1}^{m}\frac{1+s_{1j}^2}{1+s_{1j}^2}\prod_{k=1}^{n-m}\frac{1+s_{2k}^2}{1+s_{2k}^2}$$

yields

$$\begin{aligned}p_{\alpha,\beta}(\mathscr{F}_I\psi) \leq \frac{1}{(2\pi)^n}\sup_{\mathbf{s}\in\mathbb{R}^n}&\left|\prod_{j=1}^{m}\prod_{k=1}^{n-m}(1+s_{1j}^2)(1+s_{2k}^2)\partial_{\mathbf{s}}^{\alpha_1}(\mathbf{r}^{\alpha_2}\mathbf{s}^{\beta_1}\partial_{\mathbf{r}}^{\beta_2}\psi)\right|\\ &\underbrace{\prod_{j=1}^{m}\prod_{k=1}^{n-m}\left(\int_{\mathbb{R}}\frac{1}{1+s_{1j}^2}\mathrm{d}s_{1j}\right)\left(\int_{\mathbb{R}}\frac{1}{1+s_{2k}^2}\mathrm{d}s_{2k}\right)}_{=\pi^n}.\end{aligned}$$

So, we have the following:

$$p_{\alpha,\beta}(\mathscr{F}_I\psi) = \frac{\pi^{n-1}}{2^n} \sup_{\mathbf{s}\in\mathbb{R}^n} \left| \prod_{j=1}^{m} \prod_{k=1}^{n-m} (1+s_{1j}^2)(1+s_{2k}^2)\partial_{\mathbf{s}}^{\alpha_1}(\mathbf{r}^{\alpha_2}\mathbf{s}^{\beta_1}\partial_{\mathbf{r}}^{\beta_2}\psi) \right|$$

Using the Leibniz formula (Theorem B.5, Appendix B), we obtain

$$p_{\alpha,\beta}(\mathscr{F}_I\psi) = \frac{\pi^{n-1}}{2^n}$$
$$\sup_{\mathbf{s}\in\mathbb{R}^n} \left| \prod_{j=1}^{m} \prod_{k=1}^{n-m} (1+s_{1j}^2)(1+s_{2k}^2) \sum_{\gamma\leq\alpha_1} \binom{\alpha_1}{\gamma} \partial_{\mathbf{s}}^{\gamma}\mathbf{r}^{\alpha_2}\mathbf{s}^{\beta_1}\partial_{\mathbf{s}}^{\alpha_1-\gamma}\partial_{\mathbf{r}}^{\beta_2}\psi(\mathbf{s_1},\mathbf{s_2},\mathbf{r}) \right|.$$

Applying triangle inequality and differentiating yield

$$p_{\alpha,\beta}(\mathscr{F}_I\psi) \leq \frac{\pi^{n-1}}{2^n} \sum_{\gamma=0}^{\min\{\alpha_1,\beta_1\}}$$
$$\sup_{\mathbf{s}\in\mathbb{R}^n} \left| \prod_{j=1}^{m} \prod_{k=1}^{n-m} (1+s_{1j}^2)(1+s_{2k}^2) \binom{\alpha_1}{\gamma} \frac{\beta_1!}{(\beta_1-\gamma)!} \mathbf{r}^{\alpha_2}\mathbf{s}^{\beta_1-\gamma}\partial_{\mathbf{s}}^{\alpha_1-\gamma}\partial_{\mathbf{r}}^{\beta_2}\psi(\mathbf{s_1},\mathbf{s_2},\mathbf{r}) \right|$$

and finally, we arrive at

$$p_{\alpha,\beta}(\mathscr{F}_I\psi) \leq \frac{\pi}{2} \sum_{\gamma=0}^{\min\{\alpha_1,\beta_1\}} \binom{\alpha_1}{\gamma} \frac{\beta_1!}{(\beta_1-\gamma)!}$$
$$\prod_{j=1}^{m} \prod_{k=1}^{n-m} (1+s_{1j}^2)(1+s_{2k}^2)(\mathbf{s},\mathbf{r})^{(\beta_1-\gamma,\alpha_2)}\partial^{(\alpha_1-\gamma,\beta_2)}\psi(\mathbf{s},\mathbf{r}) < \infty.$$

Therefore, $p_{\alpha,\beta}(\mathscr{F}_I\psi) \leq Cp_{\tilde{\alpha},\tilde{\beta}}(\psi)$ and hence $\mathscr{F}_I$ is continuous on $\mathscr{S}(\mathbb{R}^n)$.

Since $\mathbf{k} \mapsto \mathscr{F}_I(\psi)(\mathbf{k},\mathbf{r})$ is the partial Fourier transform of $\mathbf{s} \mapsto \psi(\mathbf{s},\mathbf{r}) \in \mathscr{S}(\mathbb{R}^n)$, $\mathscr{F}_I$ has an inverse, which is also continuous on $\mathscr{S}(\mathbb{R}^n)$. $\square$

Remark 3.4. Let $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ be a linear and bounded operator and assume that there exists an operator $B : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$, which is also linear and bounded with

$$\int_{\mathbb{R}^n} (A\psi)\phi(\mathbf{r})\ \mathrm{d}\mathbf{r} = \int_{\mathbb{R}^n} \psi(B\phi)(\mathbf{r})\ \mathrm{d}\mathbf{r} \quad \forall\psi,\phi \in \mathscr{S}(\mathbb{R}^n). \tag{3.8}$$

Then, A can be uniquely extended to a bounded operator on $\mathscr{S}'(\mathbb{R}^n)$ by applying $Au(\psi) = u(B\psi)$ for $u \in \mathscr{S}'(\mathbb{R}^n)$.

In the same way, we can make the extension of the partial Fourier transform to the space of tempered distributions as below:

Definition 3.5. For $T \in \mathscr{S}'(\mathbb{R}^n)$, we define $\mathscr{F}_I T \in \mathscr{S}'(\mathbb{R}^n)$ by

$$\mathscr{F}_I T(\psi) := T(\mathscr{F}_I \psi) \quad \forall \psi \in \mathscr{S}(\mathbb{R}^n). \tag{3.9}$$

Similar to the Fourier transform on $\mathscr{S}'(\mathbb{R}^n)$, this property also holds for the partial Fourier transform:

Proposition 3.6. *The partial Fourier transforms $\mathscr{F}_I : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ are topological isomorphisms with continuous inverses $\mathscr{F}_I^{-1} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ given by*

$$\mathscr{F}_I^{-1} T(\psi) := T(\mathscr{F}_I^{-1} \psi) \qquad \forall T \in \mathscr{S}'(\mathbb{R}^n), \psi \in \mathscr{S}(\mathbb{R}^n). \tag{3.10}$$

Proof. Since $\mathscr{F}_I : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is bounded, due to Remark 3.4, $\mathscr{F}_I^{-1} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is also bounded. We use Definition 2.36 twice to obtain the following:

$$\mathscr{F}_I \mathscr{F}_I^{-1} T(\psi) = \mathscr{F}_I T(\mathscr{F}_I^{-1}\psi) = T(\mathscr{F}_I \mathscr{F}_I^{-1}\psi) = T(\psi) = \mathscr{F}_I^{-1} \mathscr{F}_I T(\psi). \tag{3.11}$$

Therefore, we can conclude that $\mathscr{F}_I^{-1}$ is the inverse of $\mathscr{F}_I$. $\square$

Remark 3.7. For $\psi \in \mathscr{S}(\mathbb{R}^n)$, we have that $\mathscr{F}_I \psi \in \mathscr{S}(\mathbb{R}^n)$. So, from Fubini-Tonelli theorem, it follows that for the partial Fourier transform, it holds:

$$\mathscr{F}_J \mathscr{F}_I = \mathscr{F}_I \mathscr{F}_J \psi = \mathscr{F}_{I \cup J} \psi \quad \forall \psi \in \mathscr{S}(\mathbb{R}^n),$$

with disjoint $I, J \subseteq \{1, ..., n\}$. Thereby, for all $T \in \mathscr{S}'(\mathbb{R}^n)$ and $\psi \in \mathscr{S}(\mathbb{R}^n)$, using Definition 2.36, we get

$$\mathscr{F}_I \mathscr{F}_J T(\psi) = \mathscr{F}_I T(\mathscr{F}_J \psi) = T(\mathscr{F}_I \mathscr{F}_J \psi) = T(\mathscr{F}_{I \cup J} \psi) = \mathscr{F}_{I \cup J} T(\psi).$$

Therefore, applying n-dimensional Fourier transform is in fact successively applying the partial Fourier transform n times, i.e. $\mathscr{F} = \mathscr{F}_1 \mathscr{F}_2 ... \mathscr{F}_n$.

We will later use in Section 5 the special case of $\mathscr{F} = \mathscr{F}_1 \mathscr{F}_2 ... \mathscr{F}_n$ for $n = 3$, that is, $\mathscr{F}_3 \mathscr{F}_{1,2} = \mathscr{F}$, which leads to

$$\mathscr{F}_{1,2} = \mathscr{F}_3^{-1} \mathscr{F}, \tag{3.12}$$

where $\mathscr{F}_{1,2}$ and $\mathscr{F}$ stand for the 2-dimensional Fourier transform with respect to the first two components and the 3-dimensional Fourier transform, respectively.

Now, we introduce some other definitions, namely, the partial convolution of two Schwartz functions and the partial convolution of a Schwartz function with a tempered distribution, which are necessary for the next sections.

Definition 3.8. For $\psi, \phi \in \mathscr{S}(\mathbb{R}^n)$ and $j \in \{1, ..., n\}$, the convolution along j-th coordinate is defined by

$$(\psi \overset{j}{*} \phi)(\mathbf{r}) := \int_{\mathbb{R}} \psi(r_1, ..., r_{j-1}, s, r_{j+1}, ..., r_n) \phi(r_1, ..., r_{j-1}, r_j - s, r_{j+1}, ..., r_n) \mathrm{d}s. \tag{3.13}$$

The partial convolution possesses the following property:

Proposition 3.9. *Let $j \in \{1, ..., n\}$ and $\psi \in \mathscr{S}(\mathbb{R}^n)$. Then, for all $\phi \in \mathscr{S}(\mathbb{R}^n)$, we have*

$$(\psi * \cdot) : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$$
$$\phi \mapsto \psi \overset{j}{*} \phi.$$

Proof. We know that $\psi \overset{j}{*} \phi \in \mathscr{C}^\infty(\mathbb{R}^n)$, for all $\psi, \phi \in \mathscr{S}(\mathbb{R}^n)$, due to the interchange of differential and integral sign. Now, for every $\alpha \in \mathbb{N}_0^m$, we consider

$$\partial^\alpha(\psi \overset{j}{*} \phi)(\mathbf{r}) = \partial^\alpha \int_{\mathbb{R}} \psi(s)\phi(r_j - s)\ \mathrm{d}s = \int_{\mathbb{R}} \partial^\alpha \psi(s)\phi(r_j - s)\ \mathrm{d}s.$$

Applying the Leibniz formula and Lebesgue dominated convergence theorem, since $\partial^\alpha\psi\partial^{\alpha-\beta}\phi \in \mathscr{S}(\mathbb{R}^n)$ hence also in $L^1(\mathbb{R}^n)$, we obtain

$$\partial^\alpha(\psi \overset{j}{*} \phi)(\mathbf{r}) = \int_{\mathbb{R}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta\psi\partial^{\alpha-\beta}\phi\ \mathrm{d}s = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}} \partial^\beta\psi\partial^{\alpha-\beta}\phi\ \mathrm{d}s.$$

This, together with $\|\mathbf{r}^\beta(\chi \overset{j}{*} \phi)\|_{L^\infty} < C_1$ for a constant $C_1 > 0$, for all $\beta \in \mathbb{N}^{n-m}$ and $\chi, \phi \in \mathscr{S}(\mathbb{R}^n)$, implies that $\|\mathbf{r}^\beta\partial^\alpha(\psi \overset{j}{*} \phi)\|_{L^\infty} < C_2$ for a constant $C_2 > 0$. From the proof of Proposition 3.3, for $\tilde{\alpha}, \tilde{\beta} \in \mathbb{N}^n$, we can find $C_\psi > 0$ such that

$$\mathbf{r}^\beta\partial^\alpha(\psi \overset{j}{*} \phi) \leq C_\psi p_{\tilde{\alpha},\tilde{\beta}}(\phi) < \infty.$$

$\square$

Now, we extend the convolution operators from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$ (cf. Remark 3.4):

Definition 3.10. For $T \in \mathscr{S}'(\mathbb{R}^n)$ and $\psi \in \mathscr{S}(\mathbb{R}^n)$, the partial convolution along the j-the coordinate of ψ and T is defined as

$$(T \overset{j}{*} \psi)(\phi) := T(\psi_j^- \overset{j}{*} \phi) \qquad \forall \phi \in \mathscr{S}(\mathbb{R}^n), \tag{3.14}$$

yielding $T \overset{j}{*} \psi \in \mathscr{S}'(\mathbb{R}^n)$, where $\psi_j^-(\mathbf{r}) := \psi(r_1, ..., r_{j-1}, -r_j, r_{j+1}, ..., r_n)$.

Applying the Fourier transform to a partial convolution of a Schwartz function with a tempered distribution leads to the following results:

Theorem 3.11 (Partial convolution theorem)**.** *For $T \in \mathscr{S}'(\mathbb{R}^n)$ and $\psi \in \mathscr{S}(\mathbb{R}^n)$, it holds:*
(i)

$$\mathscr{F}_j(T \overset{j}{*} \psi) = \sqrt{2\pi}\mathscr{F}_j T \mathscr{F}_j \psi,$$

(ii)

$$\mathscr{F}_j T \overset{j}{*} \mathscr{F}_j \psi = \sqrt{2\pi}\mathscr{F}_j(T\psi).$$

The same holds for $\mathscr{F}_j^{-1}$.

Proof. For $\psi, \phi \in \mathscr{S}(\mathbb{R}^n)$,

$$\mathscr{F}_j(\psi \overset{j}{*} \phi) = \sqrt{2\pi}\mathscr{F}_j\psi\mathscr{F}_j\phi, \tag{3.15}$$

and

$$\mathscr{F}_j\psi \overset{j}{*} \mathscr{F}_j\phi = \sqrt{2\pi}\mathscr{F}_j(\psi\phi), \tag{3.16}$$

hold true, also for $\mathscr{F}_j^{-1}$, as the properties of the 1-dimensional Fourier transform. Here, r_i are fixed for $i \neq j$, and $r_j \mapsto \psi(\mathbf{r})$, $r_j \mapsto \phi(\mathbf{r})$ are 1-dimensional complex-valued functions.

(i) We can use Definition 3.5 and Definition 3.10 to get

$$\mathscr{F}_j(T \overset{j}{*} \psi)(\phi) = (T \overset{j}{*} \psi)\mathscr{F}_j\phi = T(\psi_j^- \overset{j}{*} \mathscr{F}_j\phi).$$

In order to obtain the desired result, we first take $\mathscr{F}_j\mathscr{F}_j^{-1}$ to have

$$\mathscr{F}_j(T \overset{j}{*} \psi)(\phi) = T(\mathscr{F}_j\mathscr{F}_j^{-1}(\psi_j^- \overset{j}{*} \mathscr{F}_j\phi)) = \mathscr{F}_jT(\mathscr{F}_j^{-1}(\psi_j^- \overset{j}{*} \mathscr{F}_j\phi))$$

and then apply Equation (3.15) yields

$$\mathscr{F}_j(T \overset{j}{*} \psi)(\phi) = \mathscr{F}_jT(\sqrt{2\pi}\mathscr{F}_j\psi \cdot \phi) = \sqrt{2\pi}(\mathscr{F}_jT\mathscr{F}_j\psi)(\phi). \tag{3.17}$$

(ii) Analogous to (i).

□

4 Conceptual experiment (N.V.)

In this section, we describe a diffraction experiment and its mathematical setting.

As it can be seen in the following figures, a plane wave u^{inc} spreads in the direction $e_3 = (0, 0, 1)^T$, i.e.,

$$u^{\text{inc}}(\mathbf{r}) = e^{ik_0r_3}, \mathbf{r} \in \mathbb{R}^3,$$

where k_0 is the wave number, and illuminates the object. The wavelength λ of u^{inc} is inversely propotional to the wave number k_0,

$$\lambda = \frac{2\pi}{k_0}.$$

Let the object be surrounded by the open ball $\mathscr{B}_{r_s}(\mathbf{0}) \subseteq \mathbb{R}^3$ centered at $\mathbf{0}$ with radius r_s. Due to the rotation of the object around $\mathbf{0}$, we obtain numerous illuminations of the object. For transmission imaging, the incident wave u^{inc} gives rise to a scattered waves u^{scat}. At the plane $r_3 = r_M > r_s > 0$, the real and imaginary parts of the scattered waves are recorded. In practice, it means that we measure the waves using interferometry. For reflection imaging, at $r_3 = -r_M < -r_s < 0$, the same holds.

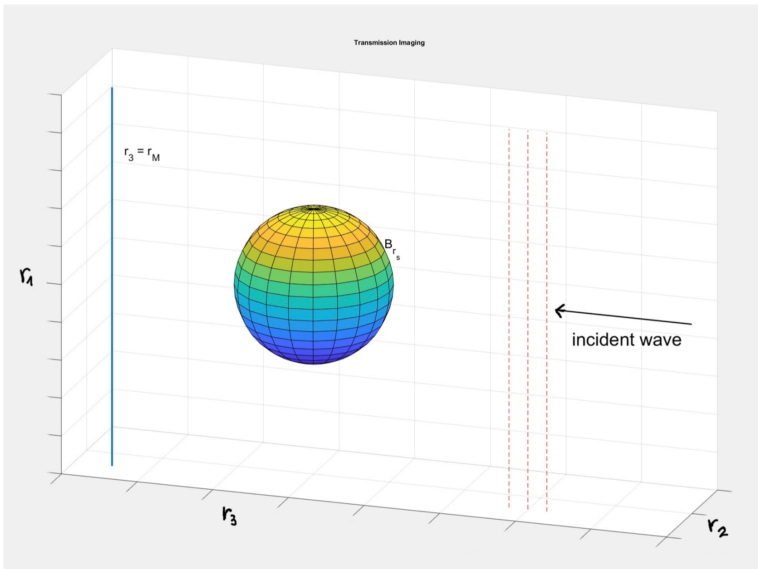

Figure 1: Setting of transmission imaging experiment

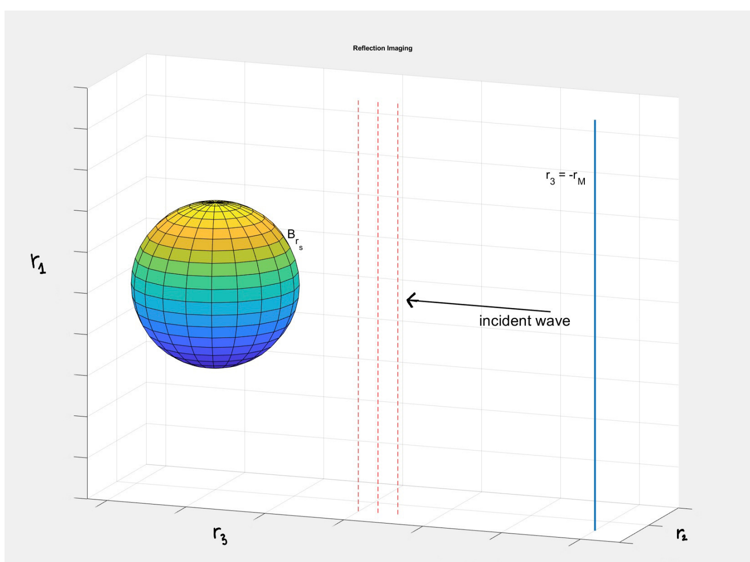

Figure 2: Setting of reflection imaging experiment

An adequate function used for characterisation of the scattering property of our observed object is

$$k(\mathbf{r}) = k_0 \frac{n(\mathbf{r})}{n_0},$$

where $\mathbf{r} \in \mathbb{R}^3$, $n(\mathbf{r}) \in \mathbb{R}$ is the refractive index of the object and n_0 is the constant refractive index of the background. We have that $k(\mathbf{r}) \neq k_0^2$ if and only if $\mathbf{r} \in \mathscr{B}_{r_s}(\mathbf{0})$.

Our goal is to reconstruct the scattering potential, given by a real-valued function

$$f(\mathbf{r}) := k^2(\mathbf{r}) - k_0^2,$$

from the scattered data. By assumptions, we have $\operatorname{supp}(f) \subseteq \mathscr{B}_{r_s}(\mathbf{0}) \subseteq (-r_s, r_s)^3$ for $r_s < r_M$.

We can describe wave by a complex-valued function $p(\mathbf{r}, t) \in \mathbb{C}$, which depends on the position $\mathbf{r} \in \mathbb{R}^n$ and time $t > 0$. The progression of the wavefunction p is characterised by a partial differential equation, namely: the inhomogeneous wave equation.

Remark 4.1. Consider the inhomogeneous wave equation

$$\partial_{tt} p(\mathbf{r}, t) - k^2(\mathbf{r}) \Delta p(\mathbf{r}, t) = 0, \qquad (\mathbf{r}, t) \in \Omega \times (0, \infty) \subseteq \mathbb{R}^3 \times \mathbb{R}. \tag{4.1}$$

If we assume that $p(\mathbf{r}, t) := u^{\text{tot}}(\mathbf{r}) e^{-i\omega t}$, the complex harmonic wave with frequency ω, solves the wave equation (4.1), then the total field u^{tot} must solve the reduced wave equation

$$\left(\Delta + \frac{\omega^2}{k^2(\mathbf{r})} \right) u^{\text{tot}}(\mathbf{r}) = 0. \tag{4.2}$$

Indeed, we have

$$\begin{aligned} & \partial_{tt} p(\mathbf{r}, t) - k^2(\mathbf{r}) \Delta p(\mathbf{r}, t) = 0 \\ \Leftrightarrow \;\; & \partial_{tt}(u^{\text{tot}}(\mathbf{r}) e^{-i\omega t}) - k^2(\mathbf{r}) \Delta (u^{\text{tot}}(\mathbf{r}) e^{-i\omega t}) = 0. \end{aligned}$$

Calculating the derivatives yields

$$u^{\text{tot}}(\mathbf{r})(-\omega^2) e^{-i\omega t} - e^{-i\omega t} k^2(\mathbf{r}) \Delta u^{\text{tot}}(\mathbf{r}) = 0.$$

Divide both sides by $-e^{-i\omega t}$ gives

$$\omega^2 u^{\text{tot}}(\mathbf{r}) + k^2(\mathbf{r}) \Delta u^{\text{tot}}(\mathbf{r}) = 0.$$

Rewriting the equation gives

$$(\omega^2 + k^2(\mathbf{r}) \Delta) u^{\text{tot}}(\mathbf{r}) = 0$$

and therefore, we arrive at

$$\left(\frac{\omega^2}{k^2(\mathbf{r})} + \Delta\right) u^{\mathrm{tot}}(\mathbf{r}) = 0.$$

Now, we denote $\dfrac{\omega^2}{k^2(\mathbf{r})}$ by our $k^2(\mathbf{r})$ as in the experimental setting above and get

$$(\Delta + k^2(\mathbf{r}))u^{\mathrm{tot}}(\mathbf{r}) = 0, \tag{4.3}$$

where

$$u^{\mathrm{tot}}(\mathbf{r}) := u^{\mathrm{inc}}(\mathbf{r}) + u^{\mathrm{scat}}(\mathbf{r})$$

represents the total field.

Moreover, we let u^{inc} satisfy the homogeneous Helmholtz equation

$$(\Delta + k_0^2)u^{\mathrm{inc}}(\mathbf{r}) = 0. \tag{4.4}$$

Our inverse problem of determining $f(\mathbf{r})$ from the measurements of u^{tot} is non-linear. Rather than solving directly, a simpler way to solve this non-linear problem is linearising it first. The two frequently used ways for linearisation are based on Born and Rytov approximation.

4.1 Born approximation

Recall the equation

$$\Delta u^{\mathrm{tot}}(\mathbf{r}) + k^2(\mathbf{r})u^{\mathrm{tot}}(\mathbf{r}) = 0.$$

Adding the term $k_0^2 u^{\mathrm{tot}}(\mathbf{r})$ to both sides, we obtain

$$\Delta u^{\mathrm{tot}}(\mathbf{r}) + (k_0^2 + k^2(\mathbf{r}))u^{\mathrm{tot}}(\mathbf{r}) = k_0^2 u^{\mathrm{tot}}(\mathbf{r}).$$

Now, setting $u^{\mathrm{tot}}(\mathbf{r}) = u^{\mathrm{inc}}(\mathbf{r}) + u^{\mathrm{scat}}(\mathbf{r})$ yields

$$\Delta(u^{\mathrm{inc}} + u^{\mathrm{scat}})(\mathbf{r}) + k_0^2(u^{\mathrm{inc}}(\mathbf{r}) + u^{\mathrm{scat}}(\mathbf{r})) = (k_0^2 - k^2(\mathbf{r}))(u^{\mathrm{inc}}(\mathbf{r}) + u^{\mathrm{scat}}(\mathbf{r})),$$

which is equivalent to

$$(\Delta + k_0^2)u^{\mathrm{scat}}(\mathbf{r}) + (\Delta + k_0^2)u^{\mathrm{inc}}(\mathbf{r}) = (k_0^2 - k^2(\mathbf{r}))(u^{\mathrm{inc}}(\mathbf{r}) + u^{\mathrm{scat}}(\mathbf{r})).$$

Since by assumption, u^{inc} solves $(\Delta + k_0^2)u^{\mathrm{inc}}(\mathbf{r}) = 0$, we finally arrive at

$$(\Delta + k_0^2)u^{\mathrm{scat}}(\mathbf{r}) = (k_0^2 - k^2(\mathbf{r}))(u^{\mathrm{inc}}(\mathbf{r}) + u^{\mathrm{scat}}(\mathbf{r})). \tag{4.5}$$

By letting the higher order term $(k_0^2 - k^2(\mathbf{r}))u^{\mathrm{scat}}(\mathbf{r}) = 0$, under the assumptions that u^{scat} is relatively small with respect to u^{inc} and $k_0^2 - k^2(\mathbf{r})$ is relatively small with respect to k_0, we get the Born approximation

$$-(\Delta + k_0^2)u^{\mathrm{B}}(\mathbf{r}) = f(\mathbf{r})u^{\mathrm{inc}}(\mathbf{r}) =: g(\mathbf{r}). \tag{4.6}$$

The explicit solution of the Born approximation will be discussed in the next section.

4.2 Rytov approximation

Rytov approximation uses the same idea as the Born approximation, with the difference that the complex logarithms of the incident and scattered field are considered.

Now, we define

$$\begin{aligned}
u^{\text{tot}}(\mathbf{r}) &= u^{\text{inc}}(\mathbf{r}) + u^{\text{scat}}(\mathbf{r}) =: e^{\varphi^{\text{tot}}(\mathbf{r})} \\
u^{\text{inc}}(\mathbf{r}) &= e^{ik_0 r_3} =: e^{\varphi^{\text{inc}}(\mathbf{r})} \\
\varphi^{\text{tot}}(\mathbf{r}) &= \varphi^{\text{inc}}(\mathbf{r}) + \varphi^{\text{scat}}(\mathbf{r}).
\end{aligned} \tag{4.7}$$

We keep in mind that all the calculations below are only formal because we do not specify the branch of φ^{tot}.

From the definition in Equations (4.7), it follows that

$$u^{\text{scat}}(\mathbf{r}) = u^{\text{tot}}(\mathbf{r}) - u^{\text{inc}}(\mathbf{r}) = e^{\varphi^{\text{tot}}(\mathbf{r})} - e^{\varphi^{\text{inc}}(\mathbf{r})} = e^{\varphi^{\text{inc}}(\mathbf{r})}(e^{\varphi^{\text{scat}}(\mathbf{r})} - 1).$$

Since $u^{\text{tot}} = e^{\varphi^{\text{tot}}}$ solves Equation (4.3), we obtain

$$\Delta e^{\varphi^{\text{tot}}(\mathbf{r})} + k_0^2 e^{\varphi^{\text{tot}}(\mathbf{r})} = (k_0^2 - k^2(\mathbf{r}))e^{\varphi^{\text{tot}}(\mathbf{r})}.$$

Using chain rule and product rule, we get

$$\Delta e^{\varphi^{\text{tot}}} = \nabla \cdot (e^{\varphi^{\text{tot}}} \nabla \varphi^{\text{tot}}) = e^{\varphi^{\text{tot}}} \Delta \varphi^{\text{tot}} + e^{\varphi^{\text{tot}}} \nabla \varphi^{\text{tot}} \cdot \nabla \varphi^{\text{tot}}.$$

So,

$$\Delta e^{\varphi^{\text{tot}}} = e^{\varphi^{\text{tot}}} \Big(\Delta \varphi^{\text{tot}} + (\nabla \varphi^{\text{tot}})^2 \Big),$$

where $(\nabla \varphi^{\text{tot}})^2 = \displaystyle\sum_{i=1}^{3} \left(\frac{\partial \varphi^{\text{tot}}}{\partial r_i} \right)^2$.

Similarly, for $u^{\text{inc}} = e^{\varphi^{\text{inc}}}$, we have

$$\Delta e^{\varphi^{\text{inc}}} = e^{\varphi^{\text{inc}}} \Big(\Delta \varphi^{\text{inc}} + (\nabla \varphi^{\text{inc}})^2 \Big).$$

Plugging these results in the Helmholtz equation (4.4), we get

$$\Delta e^{\varphi^{\text{inc}}(\mathbf{r})} + k_0^2 e^{\varphi^{\text{inc}}(\mathbf{r})} = 0 \Leftrightarrow e^{\varphi^{\text{inc}}}(\mathbf{r}) \Big(\Delta \varphi^{\text{inc}}(\mathbf{r}) + (\nabla \varphi^{\text{inc}}(\mathbf{r}))^2 \Big) = -k_0^2 e^{\varphi^{\text{inc}}(\mathbf{r})},$$

which is equivalent to

$$-k_0^2 = \Delta \varphi^{\text{inc}}(\mathbf{r}) + (\nabla \varphi^{\text{inc}}(\mathbf{r}))^2. \tag{4.8}$$

Analogously, plugging in the reduced wave equation (4.3), we get

$$-k^2(\mathbf{r}) = \Delta \varphi^{\text{tot}}(\mathbf{r}) + (\nabla \varphi^{\text{tot}}(\mathbf{r}))^2,$$

where we can apply $\varphi^{\text{tot}}(\mathbf{r}) = \varphi^{\text{inc}}(\mathbf{r}) + \varphi^{\text{scat}}(\mathbf{r})$ to obtain

$$-k^2(\mathbf{r}) = \Delta\varphi^{\text{inc}}(\mathbf{r}) + \Delta\varphi^{\text{scat}}(\mathbf{r}) + (\nabla\varphi^{\text{inc}}(\mathbf{r}) + \nabla\varphi^{\text{scat}}(\mathbf{r}))^2.$$

Finally, we arrive at

$$\begin{aligned}-k^2(\mathbf{r}) = \Delta\varphi^{\text{inc}}(\mathbf{r}) + (\nabla\varphi^{\text{inc}}(\mathbf{r}))^2 + \Delta\varphi^{\text{scat}}(\mathbf{r}) + (\nabla\varphi^{\text{scat}}(\mathbf{r}))^2 \\ + 2\nabla\varphi^{\text{inc}}(\mathbf{r}) \cdot \nabla\varphi^{\text{scat}}(\mathbf{r}).\end{aligned} \tag{4.9}$$

Taking the difference of Equations (4.8) and (4.9), we get

$$-f(\mathbf{r}) = k_0^2 - k^2(\mathbf{r}) = \Delta\varphi^{\text{scat}}(\mathbf{r}) + (\nabla\varphi^{\text{scat}}(\mathbf{r}))^2 + 2\nabla\varphi^{\text{inc}}(\mathbf{r}) \cdot \nabla\varphi^{\text{scat}}(\mathbf{r}). \tag{4.10}$$

Now, we want to simplify the term $\Delta\varphi^{\text{scat}}(\mathbf{r}) + 2\nabla\varphi^{\text{inc}}(\mathbf{r}) \cdot \nabla\varphi^{\text{scat}}(\mathbf{r})$. Note that,

$$\begin{aligned}\Delta(u^{\text{inc}}\varphi^{\text{scat}}) &= \nabla\big(u^{\text{inc}}\nabla\varphi^{\text{scat}} + (\nabla u^{\text{inc}})\varphi^{\text{scat}}\big) \\ &= u^{\text{inc}}\Delta\varphi^{\text{scat}} + 2\nabla u^{\text{inc}} \cdot \nabla\varphi^{\text{scat}} + (\Delta u^{\text{inc}})\varphi^{\text{scat}}.\end{aligned}$$

Since

$$\nabla u^{\text{inc}} = \nabla e^{\varphi^{\text{inc}}} = e^{\varphi^{\text{inc}}}\nabla\varphi^{\text{inc}} = u^{\text{inc}}\nabla\varphi^{\text{inc}}$$

and

$$\Delta u^{\text{inc}} = u^{\text{inc}}\Big(\Delta\varphi^{\text{inc}} + (\nabla\varphi^{\text{inc}})^2\Big) = -u^{\text{inc}}k_0^2,$$

we arrive at the equation

$$\Delta(u^{\text{inc}}\varphi^{\text{scat}}) = u^{\text{inc}}\Delta\varphi^{\text{scat}} + 2(u^{\text{inc}}\nabla\varphi^{\text{inc}}) \cdot \nabla\varphi^{\text{scat}} - k_0^2 u^{\text{inc}}\varphi^{\text{scat}}.$$

Therefore, putting $-k_0^2 u^{\text{inc}}\varphi^{\text{scat}}$ on the other side of the equation gives

$$(\Delta + k_0^2)(u^{\text{inc}}\varphi^{\text{scat}}) = u^{\text{inc}}\Delta\varphi^{\text{scat}} + 2(u^{\text{inc}}\nabla\varphi^{\text{inc}}) \cdot \nabla\varphi^{\text{scat}}$$

and applying Equation 4.10 yields

$$(\Delta + k_0^2)(u^{\text{inc}}\varphi^{\text{scat}}) = u^{\text{inc}}\big(-f - (\nabla\varphi^{\text{scat}})^2\big),$$

or equivalently, we have

$$-(\Delta + k_0^2)(u^{\text{inc}}\varphi^{\text{scat}})(\mathbf{r}) = \big(f(\mathbf{r}) + (\nabla\varphi^{\text{scat}}(\mathbf{r}))^2\big)u^{\text{inc}}(\mathbf{r}).$$

If the phase gradient $\nabla\varphi^{\text{scat}}(\mathbf{r})$ is small compared to the scattering potential $f(\mathbf{r})$, we neglect the higher order term $(\nabla\varphi^{\text{scat}})^2$ and get the Rytov approximation φ^{R} of the phase gradient:

$$-(\Delta + k_0^2)(u^{\text{inc}}\varphi^{\text{R}})(\mathbf{r}) = f(\mathbf{r})u^{\text{inc}}(\mathbf{r}) = g(\mathbf{r}). \tag{4.11}$$

Then, together with the Born approximation (4.6), we obtain the relation

$$u^{\text{B}}(\mathbf{r}) = u^{\text{inc}}(\mathbf{r})\varphi^{\text{R}}(\mathbf{r}). \tag{4.12}$$

So far, we have only studied the Rytov approximation of the phase gradient φ^{R}. In the next remark, we will examine the Rytov approximation u^{R} of the scattered wave.

Remark 4.2. From the definition of u^{scat} in Equation (4.7) and the relation (4.12), the Rytov approximation of the scattered wave u^{scat} is

$$u^{\mathrm{R}}(\mathbf{r}) = u^{\text{inc}}(\mathbf{r})(e^{\varphi^{\mathrm{R}}(\mathbf{r})} - 1) = u^{\text{inc}}(\mathbf{r}) \left(\exp\left(\frac{u^{\mathrm{B}}(\mathbf{r})}{u^{\text{inc}}(\mathbf{r})} \right) - 1 \right).$$

The equation above yields

$$\frac{u^{\mathrm{R}}(\mathbf{r})}{u^{\text{inc}}(\mathbf{r})} + 1 = \exp\left(\frac{u^{\mathrm{B}}(\mathbf{r})}{u^{\text{inc}}(\mathbf{r})} \right).$$

Taking logarithm on both sides, we have

$$u^{\mathrm{B}}(\mathbf{r}) = u^{\text{inc}} \log\left(\frac{u^{\mathrm{R}}(\mathbf{r})}{u^{\text{inc}}(\mathbf{r})} + 1 \right). \tag{4.13}$$

Here, the complex logarithm $\log(z)$, which is the inverse function of the exponential function e^z, is given by the multi-valued function

$$\log(re^{i\theta}) = \log(r) + i\theta + 2\pi k i$$

for $z = re^{i\theta} = re^{i\theta + 2\pi k i} \in \mathbb{C}$ for all $k \in \mathbb{Z}$ (i.e. $r = |z| \in \mathbb{R}$ is the modulus and $\theta = \arg(z) \in (-\pi, \pi]$ is the principle argument of z). Each $k \in \mathbb{Z}$ corresponds to a branch of logarithm. The k-th branch satisfies

$$\Im(\log(z)) \in (\pi(2k-1), \pi(2k+1)].$$

Thus, in Equation (4.13), we can choose, for example, the principle branch ($k = 0$):

$$\mathrm{Log}(z) = \log(|z|) + i \arg(z),$$

or typically, in physics, one uses phase unwrapping in order to suppress the 2π-discontinuity of the complex logarithms. It can be done by adding or substracting multiplies of 2π (see [28]).

In practice, the measurements are either assumed to be the Born approximation u^{B} or the Rytov approximation u^{R}. For the latter case, we then calculate u^{B} from u^{R} using (4.13). Hence, in the following sections, we will only take into account the Born approximation u^{B}. For simplicity, we write u instead of u^{B}.

5 Fourier Diffraction Theorem (N.V.)

The Fourier Diffraction Theorem gives us the relation between the 2-dimensional Fourier transform of the measured data u at the measurement plane and the 3-dimensional Fourier transform of the scattering potential f. In this section, we first state the Fourier Diffraction Theorem in 3-dimensional case, then apply it in our particular experimental setting. Afterwards, the theorem will be proved in Subsection 5.3. Finally, a generalised version of Fourier Diffraction Theorem in a n-dimensional setting will be introduced in Subsection 5.4.

Now, consider again the Born approximation u that satisfies the inhomogenous Helmholtz equation (4.6), that is,

$$-(\Delta + k_0^2)u(\mathbf{r}) = f(\mathbf{r})u^{\mathrm{inc}}(\mathbf{r}) = g(\mathbf{r}) \quad \text{on } \mathbb{R}^3,$$

and assume that u fulfils the Sommerfeld radiation condition

$$\lim_{s\to\infty} \max_{\|\mathbf{r}\|=s} \|\mathbf{r}\| \left| \frac{\partial}{\partial \mathbf{r}} u(\mathbf{r}) - ik_0 u(\mathbf{r}) \right| = 0, \tag{5.1}$$

where $\dfrac{\partial}{\partial \mathbf{r}}$ denotes the directional derivative. Here, we assume that the complex harmonic wave is $p(\mathbf{r},t) := u^{\mathrm{tot}}(\mathbf{r})e^{-i\omega t}$ (cf. Remark 4.1). If $p(\mathbf{r},t) := u^{\mathrm{tot}}(\mathbf{r})e^{i\omega t}$, we replace $-i$ with $+i$ in the equation (5.1) (see [10], pg. 328).

The assumption that the Born approximation u has to satisfy the Sommerfeld radiation condition is worthwhile. We will enquire into the reason in the first subsection.

5.1 Sommerfeld radiation condition

In order to get a unique solution of some partial differential equations, it is necessary to have the Sommerfeld radiation condition, which describes the asymptotic behaviour of the solutions. These problems describe waves that propagate from the object.

The Born approximation (4.6), that is,

$$-(\Delta + k_0^2)u(\mathbf{r}) = f(\mathbf{r})u^{\mathrm{inc}}(\mathbf{r}) = g(\mathbf{r}) \;\; \text{on } \mathbb{R}^3$$

is one example of mathematical formulation of these problems.

Solving the Born approximation mathematically might give us not only the outgoing wave obtained by the wave scattering, but also the incoming waves coming from the infinity to the object, which is physically not possible. Arnold Sommerfeld characterised the radiation condition: "the source must be sources, not sinks of energy. The energy which is radiated from the sources must scatter to infinity; no energy may be radiated from infinity into the field [4]". Therefore, we need the Sommerfeld radiation condition to prevent the latter case.

In the case of a 3-dimensional problem, the outgoing wave u must fulfil Equation (5.1)

$$\lim_{s\to\infty} \max_{\|\mathbf{r}\|=s} \|\mathbf{r}\| \left| \frac{\partial}{\partial \mathbf{r}} u(\mathbf{r}) - ik_0 u(\mathbf{r}) \right| = 0$$

uniformly, i.e., the limit must be 0 for all directions going to infinity.

For an incoming wave u, we have the absorption condition

$$\lim_{s\to\infty} \max_{\|\mathbf{r}\|=s} \|\mathbf{r}\| \left| \frac{\partial}{\partial \mathbf{r}} u(\mathbf{r}) + ik_0 u(\mathbf{r}) \right| = 0$$

(see [13], pg. 394).

The inhomogenous Helmholtz equation (4.6) has infinite number of solutions, that is,

$$u(\mathbf{r}) = \int_{\mathbb{R}^3} \left(C_1 G_+(\mathbf{r}-\mathbf{r}') + C_2 G_-(\mathbf{r}-\mathbf{r}') \right) f(\mathbf{r}') u^{\mathrm{inc}}(\mathbf{r}')\ \mathrm{d}\mathbf{r}',$$

where $C_1, C_2 \in \mathbb{C}$,

$$G_+(\mathbf{r}) = \frac{e^{ik_0\|\mathbf{r}\|}}{4\pi\|\mathbf{r}\|}$$

and

$$G_-(\mathbf{r}) = \frac{e^{-ik_0\|\mathbf{r}\|}}{4\pi\|\mathbf{r}\|}.$$

Indeed, let $G := C_1 G_+ + C_2 G_-$. We have

$$-(\Delta + k_0^2) u = g \Leftrightarrow -(\Delta + k_0^2) \left[\int_{\mathbb{R}^3} G(\mathbf{r}-\mathbf{r}') g(\mathbf{r}')\ \mathrm{d}\mathbf{r}' \right] = g(\mathbf{r}),$$

which is equivalent to

$$\int_{\mathbb{R}^3} -(\Delta + k_0^2) G(\mathbf{r}-\mathbf{r}') g(\mathbf{r}')\ \mathrm{d}\mathbf{r}' = g(\mathbf{r}).$$

Moreover, the following equality also holds true, as a property of the 3-dimensional Dirac-delta distribution:

$$g(\mathbf{r}) = \int_{\mathbb{R}^3} \delta(\mathbf{r}-\mathbf{r}') g(\mathbf{r}')\ \mathrm{d}\mathbf{r}'.$$

Thus, G has to solve the equation

$$-(\Delta + k_0^2) G(\mathbf{r}-\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'),$$

or in other words, G must be the Green's function of the Helmholtz operator $-(\Delta + k_0^2)$. The expression of G will be shown in Subsection 5.3.

However, only one solution satisfies the Sommerfeld radiation condition, namely:

$$u_+(\mathbf{r}) = \int_{\mathbb{R}^3} G_+(\mathbf{r}-\mathbf{r}') f(\mathbf{r}') u^{\mathrm{inc}}(\mathbf{r}')\ \mathrm{d}\mathbf{r}' = (G_+ * g)(\mathbf{r}).$$

This will also be proved in Subsection 5.3. Thus, we call G_+ the outgoing Green's function and G_- the incoming Green's function of the Helmholtz operator.

5.2 Fourier Diffraction Theorem

To express Fourier Diffraction Theorem, we first define

$$\kappa = \kappa(k_1, k_2) := \begin{cases} \sqrt{k_0^2 - k_1^2 - k_2^2} & \text{if } \ k_1^2 + k_2^2 \leq k_0^2 \\ i\sqrt{k_1^2 + k_2^2 - k_0^2} & \text{if } \ k_1^2 + k_2^2 > k_0^2 \end{cases}.$$

Then, we define the Heaviside function

$$\chi_{r_3}(s_1, s_2, s_3) := \begin{cases} 0 & \text{if } \ s_3 < r_3 \\ 1 & \text{otherwise} \end{cases}.$$

The Heaviside function χ_{r_3} is piecewise continuous and therefore a regular distribution.

Now, we can state the Fourier Diffraction Theorem as follows:

Theorem 5.1 (**Fourier Diffraction Theorem**). *Let $k_0 > 0$ and $0 \neq g \in L^p(\mathbb{R}^3)$, $p > 1$ with supp$(g) \subseteq \mathscr{B}_{r_s}(\mathbf{0})$ for some $r_s > 0$. Assume that u solves*

$$-(\Delta + k_0^2)u = g$$

and satisfies the Sommerfeld radiation condition. Then, we have

$$\mathscr{F}_{1,2}u(k_1, k_2, r_3) = \sqrt{\frac{\pi}{2}}\frac{i}{\kappa}\Big[e^{i\kappa r_3}\mathscr{F}\big((1 - \chi_{r_3})g\big)(k_1, k_2, \kappa) + e^{-i\kappa r_3}\mathscr{F}\big(\chi_{r_3}g\big)(k_1, k_2, -\kappa)\Big] \tag{5.2}$$

almost everywhere.

We have some notices on the relation (5.2):

Remark 5.2. (i) The following Lemma is necessary for proving the theorem: **du Bois - Reymond's lemma** (Lemma 3.2 in [1])
Let $f \in L^1_{\text{loc}}(\Omega)$, which satisfies

$$\int_\Omega f(x)\varphi(x) \, \mathrm{d}x = 0$$

for all test functions $\varphi \in \mathscr{D}(\Omega)$. Then $f = 0$.

(ii) We will prove the Theorem 5.1 assuming that $\kappa > 0$, since physically, if the object is not rotated, the measurements are only given by the data of the spatial frequencies with $k_1^2 + k_2^2 < k_0^2$, that is, we only get the measurements on

$$\left\{(k_1, k_2, \pm\kappa - k_0)^{\mathrm{T}} \in \mathbb{R}^3 \mid k_1^2 + k_2^2 < k_0^2\right\},$$

because if κ is complex, then $e^{i\kappa r_3}$ is asymptotically decaying and we can not measure such waves. This set is a hemisphere (depicted in Subsection 5.4) and is called the k-space coverage of the experiment. Some other authors call it frequency coverage.

Now, to get the Fourier Diffraction Theorem for our particular experiment described in Section 4, we first summarise the mathematical setting of the experiment, namely:

Assumption 5.3. *(i) f is the scattering potential of the object,*

(ii) $u^{\mathrm{inc}}(\mathbf{r}) = e^{ik_0 r_3}$ is the incident field,

(iii) the scattered field u fulfils the Born approximation (4.6) *with the Sommerfeld radiation condition* (5.1) *and*

(iv) the measured data of the scattered field are $u(k_1, k_2, r_M)$ for transmission imaging and $u(k_1, k_2, -r_M)$ for reflection imaging.

Then, we can apply the Fourier Diffraction Theorem 5.1 to our setting to attain

Corollary 5.4. *Assume that Assumption 5.3 holds. Then, for $f \in L^p(\mathbb{R}^3)$, $p > 1$, with supp$(f) \subseteq \mathscr{B}_{r_s}(\mathbf{0})$ for some $0 < r_s < r_M$, it holds:*

$$\mathscr{F}_{1,2}u(k_1, k_2, \pm r_M) = \sqrt{\frac{\pi}{2}} \frac{ie^{i\kappa r_M}}{\kappa} \mathscr{F} f(k_1, k_2, \pm\kappa - k_0) \tag{5.3}$$

for $k_1, k_2 \in \mathbb{R}$, $k_1^2 + k_2^2 < k_0^2$.

Proof. From Theorem 5.1, with $g = f u^{\mathrm{inc}}$, we have

$$\begin{aligned}\mathscr{F}_{1,2}u(k_1, k_2, r_3) = \sqrt{\frac{\pi}{2}} \frac{i}{\kappa} \Big[& e^{i\kappa r_3} \mathscr{F}\big((1 - \chi_{r_3}) f u^{\mathrm{inc}}\big)(k_1, k_2, \kappa) \\ & + e^{-i\kappa r_3} \mathscr{F}\big(\chi_{r_3} f u^{\mathrm{inc}}\big)(k_1, k_2, -\kappa)\Big].\end{aligned}$$

Since $u^{\mathrm{inc}}(\mathbf{r}) = e^{ik_0 r_3}$, we can use the property $\mathscr{F}[\mathbf{r} \mapsto e^{i\mathbf{b}\cdot\mathbf{r}} f(\mathbf{r})](\mathbf{k}) = \mathscr{F} f(\mathbf{k} - \mathbf{b})$ of the Fourier transform with $\mathbf{r} = (r_1, r_2, r_3)^{\mathrm{T}}$ and $\mathbf{b} = (0, 0, k_0)^{\mathrm{T}}$ to get

$$\begin{aligned}\mathscr{F}_{1,2}u(k_1, k_2, r_3) = \sqrt{\frac{\pi}{2}} \frac{i}{\kappa} \Big[& e^{i\kappa r_3} \mathscr{F}\big((1 - \chi_{r_3}) f\big)(k_1, k_2, \kappa - k_0) \\ & + e^{-i\kappa r_3} \mathscr{F}\big(\chi_{r_3} f\big)(k_1, k_2, -\kappa - k_0)\Big].\end{aligned}$$

If $r_3 = r_M$, since $r_s < r_M$, we have $\chi_{r_M} f = 0$ and $(1 - \chi_{r_M}) f = f$. Analogously, if $r_3 = -r_M$, since $-r_M < -r_s$, we have $\chi_{-r_M} f = f$ and $(1 - \chi_{-r_M}) f = 0$, which yields the desired result. □

Remark 5.5. The function

$$r_3 \mapsto \frac{ie^{i\kappa |r_3|}}{2\kappa},$$

which appears in the equation (5.3), is the outgoing Green's function for the one-dimensional Helmholtz equation with wave number κ (cf. Corollary B.15, Appendix B).

5.3 Proof of the Fourier Diffraction Theorem

To prove the Fourier Diffraction Theorem 5.1, we need to show some results first.

Definition 5.6. A function $f : \mathbb{R}^n \to \mathbb{C}$ is called radial if for all $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^n$ with $\|\mathbf{r}\| = \|\mathbf{r}'\|$, we have that $f(\mathbf{r}) = f(\mathbf{r}')$.
A radial function can be written in the form

$$f(\mathbf{r}) = F(\|\mathbf{r}\|),$$

where $F : [0, \infty) \to \mathbb{C}$.

Theorem 5.7. *If $f \in L^1(\mathbb{R}^n)$ is a radial function of the form $f(\mathbf{r}) = F(r)$, $r = \|\mathbf{r}\|$, then $\mathscr{F} f$ is also a radial function and*

$$\mathscr{F} f(\mathbf{k}) = \|\mathbf{k}\|^{1-\frac{n}{2}} \int_0^\infty F(r) r^{\frac{n}{2}} J_{\frac{n}{2}-1}(r\|\mathbf{k}\|) \, \mathrm{d}r, \tag{5.4}$$

where $\mathbf{k} \in \mathbb{R}^n \backslash \{0\}$ and

$$J_\nu(t) = \frac{t^\nu}{(2\pi)^{\nu+1}} \omega(S_{2\nu}) \int_0^\pi e^{-it\cos(\theta)} \sin^{2\nu}(\theta) \, \mathrm{d}\theta \tag{5.5}$$

is the Bessel function of the first kind of order $\nu \geq 0$ (cf. Theorem B.10, Appendix B). Here, $\omega(S_{d-1})$ is the surface area of the unit sphere $S_{d-1} = \{\mathbf{r} \in \mathbb{R}^d \; : \; \|\mathbf{r}\| = 1\} \subseteq \mathbb{R}^d$,

$$\omega(S_{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}.$$

In the case $n = 3$, we have that

$$\mathscr{F} f(\mathbf{k}) = \sqrt{\frac{2}{\pi}} \frac{1}{\|\mathbf{k}\|} \int_0^\infty F(r) r \sin(r\|\mathbf{k}\|) \, \mathrm{d}r. \tag{5.6}$$

Proof. Since $f \in L^1(\mathbb{R}^n)$ is a radial function, we have $f(\mathbf{r}) = F(r)$, $r = \|\mathbf{r}\|$.

Now, consider the definition of the Fourier transform and the Bessel function (5.5), we have that

$$\mathscr{F} f(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}\mathbf{r}.$$

Then, we use the substitution $\mathbf{r} = r\mathbf{n}$ with $\mathbf{n}$ is the unit vector in S_{n-1} to get

$$\mathscr{F} f(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \int_0^\infty F(r) \int_{S_{n-1}} e^{-ir\mathbf{k}\cdot\mathbf{n}} \, \mathrm{d}S \; r^{n-1} \, \mathrm{d}r.$$

Now, use the polar coordinates with the $z-$axis along $\mathbf{k}$, so that $\mathbf{k}\cdot\mathbf{n} = \|\mathbf{k}\| \cos(\theta)$, for $\theta \in [0, \pi]$, we have

$$\int_{S_{n-1}} e^{-ir\mathbf{k}\cdot\mathbf{n}} \, \mathrm{d}S = \int_0^\pi e^{-i\|\mathbf{k}\| r\cos(\theta)} \omega(S_{n-2}) \sin^{n-2}(\theta) \, \mathrm{d}\theta$$

Thus, the following holds:

$$\mathscr{F}f(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \int_0^\infty \int_0^\pi F(r) e^{-i\|\mathbf{k}\| r \cos(\theta)} \omega(S_{n-2}) \sin^{n-2}(\theta) r^{n-1} \, \mathrm{d}\theta \mathrm{d}r.$$

In order to use the Poisson's integral representation of the Bessel function of the first kind of order $\frac{n}{2} - 1$ in Equation (5.5), we first multiply the integrand with

$$1 = (r\|\mathbf{k}\|)^{1-\frac{n}{2}} (r\|\mathbf{k}\|)^{\frac{n}{2}-1}$$

to obtain

$$\begin{aligned} &\mathscr{F}f(\mathbf{k}) \\ &= \int_0^\infty F(r) r^{\frac{n}{2}} \|\mathbf{k}\|^{1-\frac{n}{2}} \left[\frac{(r\|\mathbf{k}\|)^{\frac{n}{2}-1}}{(\sqrt{2\pi})^n} \int_0^\pi e^{-i\|\mathbf{k}\| r \cos(\theta)} \sin^{n-2}(\theta) \omega(S_{n-2}) \mathrm{d}\theta \right] \mathrm{d}r, \end{aligned}$$

which yields

$$\mathscr{F}f(\mathbf{k}) = \|\mathbf{k}\|^{1-\frac{n}{2}} \int_0^\infty F(r) r^{\frac{n}{2}} J_{\frac{n}{2}-1}(r\|\mathbf{k}\|) \mathrm{d}r.$$

The formula of the surface area of the $(d-1)$-unit sphere

$$\omega(S_{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$$

is proved in Theorem 1 in [9].

Now, for the case $n = 3$, setting $d = 2$ in the formula above, we get $\omega_1 = 2\pi$ and hence,

$$J_{\frac{1}{2}}(t) = \frac{t^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}+1}} 2\pi \int_0^\pi e^{-it\cos(\theta)} \sin(\theta) \, \mathrm{d}\theta = \frac{t^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \frac{2\sin(t)}{t} = \frac{\sqrt{2}}{\sqrt{\pi}} t^{-\frac{1}{2}} \sin(t). \tag{5.7}$$

Therefore,

$$\mathscr{F}f(\mathbf{k}) = \|\mathbf{k}\|^{-\frac{1}{2}} \int_0^\infty F(r) r^{\frac{3}{2}} J_{\frac{1}{2}}(r\|\mathbf{k}\|) \, \mathrm{d}r.$$

Applying Equation 5.7, we obtain

$$\mathscr{F}f(\mathbf{k}) = \sqrt{\frac{2}{\pi}} \frac{1}{\|\mathbf{k}\|} \int_0^\infty F(r) r \sin(r\|\mathbf{k}\|) \, \mathrm{d}r.$$

□

Corollary 5.8. *The function*

$$G(\mathbf{r})_+ := \frac{e^{ik_0\|\mathbf{r}\|}}{4\pi\|\mathbf{r}\|}$$

is the Green's function of the Helmholtz operator $-(\Delta + k_0^2)$ that satisfies the Sommerfeld radiation condition, whereas

$$G(\mathbf{r})_- := \frac{e^{-ik_0\|\mathbf{r}\|}}{4\pi\|\mathbf{r}\|}$$

is the Green's function of the Helmholtz operator that satisfies the absorption condition.

Proof. By definition, the Green's function of the Helmholtz operator $-(\Delta + k_0^2)$ must solve the Helmholtz equation

$$-(\Delta + k_0^2)G(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$

where δ is the 3-dimensional Dirac-delta distribution.

Since this partial differential equation is translationally invariant, we can introduce a new coordinate

$$\rho = \mathbf{r} - \mathbf{r}'$$

so that we have

$$-(\Delta + k_0^2)G(\rho) = \delta(\rho). \tag{5.8}$$

Because δ is a radial function, $G(\rho)$ is a function that depends only on $\|\rho\|$, that is, $G(\rho) = G(\|\rho\|)$. Hence, for the sake of simplicity, we write ρ instead of $\|\rho\|$.

Now we want to write Equation (5.8) in polar coordinates. First, let

$$g(\rho, \theta) := D\Phi(\rho, \theta)^T D\Phi(\rho, \theta),$$

where $\Phi(\rho, \theta) = (\rho\cos(\theta), \rho\sin(\theta))$. Then,

$$\begin{aligned} \Delta f(\rho\cos(\theta), \rho\sin(\theta)) &= \frac{1}{\sqrt{\det g}} \operatorname{div}\big(\sqrt{\det g} g^{-1} \nabla_{\rho,\theta} \tilde{f}(\rho, \theta)\big) \\ &= \frac{1}{\rho}\partial_\rho(\rho\partial_\rho \tilde{f})(\rho, \theta) + \frac{1}{\rho}\partial_\theta\left(\rho\frac{1}{\rho^2}\partial_\theta \tilde{f}\right)(\rho, \theta), \end{aligned}$$

with $\tilde{f}(\rho, \theta) := f(\rho\cos(\theta), \rho\sin(\theta))$.

Since $G(\rho)$ does not depend on the angles, we get from Equation (5.8) the equation

$$-\frac{1}{\rho}\partial_\rho(\rho\partial_\rho G)(\rho) - k_0^2 G(\rho) = 0,$$

which is equivalent to the ordinary differential equation

$$(\rho G)'' + k_0^2(\rho G) = 0, \quad \rho \neq 0.$$

Solving this equation gives us the solution

$$\rho G(\rho) = C_1 e^{ik_0\rho} + C_2 e^{-ik_0\rho},$$

i.e.,

$$G(\rho) = C_1 \frac{e^{ik_0\rho}}{\rho} + C_2 \frac{e^{-ik_0\rho}}{\rho}$$

for $C_1, C_2 \in \mathbb{C}$. Now, we want to find the solution that satisfies the Sommerfeld radiation condition. First, consider

$$G_+(\rho) = C_1 \frac{e^{ik_0\rho}}{\rho}.$$

Then,

$$\frac{\partial}{\partial\rho} G_+(\rho) = C_1 \frac{e^{ik_0\rho}}{\rho} \left(ik_0 - \frac{1}{\rho} \right) = G_+(\rho) \left(ik_0 - \frac{1}{\rho} \right),$$

which gives

$$\left| \frac{\partial}{\partial\rho} G_+(\rho) - ik_0 G_+(\rho) \right| = C_1 \frac{1}{\rho^2},$$

which converges to 0 as $\rho \to \infty$. Therefore, $G_+(\rho)$ satisfies the Sommerfeld radiation condition and hence is an outgoing wave. Analogously, we can consider

$$G_-(\rho) = C_1 \frac{e^{-ik_0\rho}}{\rho},$$

which does not satisfy the Sommerfeld radiation condition. Instead, it satisfies the absorption condition

$$\left| \frac{\partial}{\partial\rho} G_-(\rho) + ik_0 G_-(\rho) \right| = C_2 \frac{1}{\rho^2}.$$

Therefore, G_- is an incoming wave.

Finally, to determine the constant C_1 and C_2, we consider the behavior of the equation as $\rho \to 0$. Assume that C_1, C_2 are independent of k_0. As $k_0 \to 0$, the Helmholtz equation turns into the Laplace equation $-\Delta G(\rho) = \delta(\rho)$. Since it is well-known that the Green's function of the Laplace operator Δ is given by $-\frac{1}{4\pi\rho}$, we get

$$C_1 = C_2 = \frac{1}{4\pi}.$$

Hence, we find that

$$G_+(\mathbf{r}) := \frac{e^{ik_0\|\mathbf{r}\|}}{4\pi\|\mathbf{r}\|}$$

is the Green's function of the Helmholtz operator $-(\Delta + k_0^2)$ that satisfies the Sommerfeld radiation condition, whereas $G_-(\mathbf{r})$ is the solution that satisfies the absorption condition. $\square$

From now on, we only consider the outgoing solution of the Helmholtz equation, because it is the only physically possible solution. The reason is that in reality, waves decay. Therefore, for simplicity, in this section, we denote G_+ as G.

We have some properties of the Green's function of the Helmholtz operator:

Lemma 5.9. *For $p \in [1, \infty]$, we have that $G \notin L^p(\mathbb{R}^3)$. Therefore, $G \notin \mathscr{S}(\mathbb{R}^3)$.*

Proof. First, consider for $p \in [1, \infty)$:

$$\int_{\mathbb{R}^3} \left|G(\mathbf{r})\right|^p \, \mathrm{d}\mathbf{r} = \int_{\mathbb{R}^3} \left|\frac{e^{ik_0\|\mathbf{r}\|}}{4\pi\|\mathbf{r}\|}\right|^p \, \mathrm{d}\mathbf{r} = \frac{1}{(4\pi)^p} \int_{\mathbb{R}^3} \frac{1}{\|\mathbf{r}\|^p} \, \mathrm{d}\mathbf{r}.$$

Then, making the substitution with the unit vector $\mathbf{n} \in S_2$, we obtain

$$\int_{\mathbb{R}^3} \left|G(\mathbf{r})\right|^p \, \mathrm{d}\mathbf{r} = \frac{1}{(4\pi)^p} \int_0^\infty \int_{S_2} \frac{1}{\|\rho \cdot \mathbf{n}\|^p} \, \mathrm{d}S \; \rho^2 \, \mathrm{d}\rho,$$

which leads to

$$\int_{\mathbb{R}^3} \left|G(\mathbf{r})\right|^p \, \mathrm{d}\mathbf{r} = \frac{1}{(4\pi)^p} \omega(S_2) \int_0^\infty \rho^{2-p} \, \mathrm{d}\rho = \infty,$$

where $w(S_2)$ denotes the surface area of the unit 2-sphere $S_2 \subseteq \mathbb{R}^3$.

For $p = \infty$, we have the following:

$$\left|G(\mathbf{r})\right| = \frac{1}{4\pi\|\mathbf{r}\|}.$$

Letting $\|\mathbf{r}\| \longrightarrow 0$, we get $\left|G(\mathbf{r})\right| \longrightarrow \infty$, so $G \notin L^\infty(\mathbb{R}^3)$, and 0 is the singularity of G. $\square$

Lemma 5.10. *The function*

$$G(\mathbf{r}) = \frac{e^{ik_0\|\mathbf{r}\|}}{4\pi\|\mathbf{r}\|},$$

with $\mathbf{r} \in \mathbb{R}^3$, $k_0 \in \mathbb{R}$, is an element of $\mathscr{S}'(\mathbb{R}^3)$.

Proof. We need to prove that $G \in L^1_{\text{loc}}(\mathbb{R}^3)$ and

$$\lim_{\|\mathbf{r}\| \to \infty} \frac{\left|G(\mathbf{r})\right|}{\|\mathbf{r}\|^N} = 0$$

for a $N \in \mathbb{N}$ to apply Lemma 2.34 so that we can conclude that $G \in \mathscr{S}'(\mathbb{R}^3)$.

Let $\mathscr{B} \subseteq \mathbb{R}^3$ denote the unit ball and consider

$$\int_{\mathscr{B}} \left|G(\mathbf{r})\right| \, \mathrm{d}\mathbf{r} = \int_{\mathscr{B}} \left| \frac{e^{ik_0\|\mathbf{r}\|}}{4\pi\|\mathbf{r}\|} \right| \, \mathrm{d}\mathbf{r} = \frac{1}{4\pi} \int_{\mathscr{B}} \frac{1}{\|\mathbf{r}\|} \, \mathrm{d}\mathbf{r}.$$

As in the previous lemma, we make the substitution to get

$$\int_{\mathscr{B}} \left|G(\mathbf{r})\right| \, \mathrm{d}\mathbf{r} = \frac{1}{4\pi} \omega(S_2) \int_0^1 \rho \, \mathrm{d}\rho < \infty,$$

which yields $G \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. It is clear that

$$\lim_{\|\mathbf{r}\| \to \infty} \frac{\left|G(\mathbf{r})\right|}{\|\mathbf{r}\|^N} = 0$$

for a $N \in \mathbb{N}$. Hence, the proof is complete. $\square$

Remark 5.11. Similar to the proof of the two lemmas above, if we fixed $r_3 \in \mathbb{R}$, we can also prove that $G(\cdot, \cdot, r_3) \notin L^1(\mathbb{R}^2)$ and therefore, $\mathscr{F}_{1,2}G$ can not be defined on a $L^1(\mathbb{R}^2)$ or $\mathscr{C}(\mathbb{R}^2)$ setting, but we can calculate $\mathscr{F}_{1,2}G$ in distributional sense.

Now, the idea of proving the Fourier Diffraction Theorem 5.1 is as follows:

- First, we define a function G_ϵ, which is the Green's function of the viscosity equation
$$-\left(\Delta + (k_0 + i\epsilon)^2\right) u = g$$
that satisfies the Sommerfeld radiation condition. We have G_ϵ converges to G in $\mathscr{S}'(\mathbb{R}^3)$ for $\epsilon \to 0$. Then we calculate $\mathscr{F}_{1,2}G_\epsilon$ and take the limit to get $\mathscr{F}_{1,2}G$.

- Second, we take a sequence g_n that converges to g and consider the equation $-(\Delta + k_0^2)u_n = g_n$ with the unique solution $u_n = G * g_n$ that satisfies the Sommerfeld radiation condition and calculate $\mathscr{F}_{1,2}u_n$.

- Finally, from $\mathscr{F}_{1,2}u_n$ we find $\mathscr{F}_{1,2}u$.

For the first step, define

$$G_\epsilon(\mathbf{r}) := e^{-\epsilon\|\mathbf{r}\|} G(\mathbf{r}), \quad \epsilon > 0. \tag{5.9}$$

G_ϵ solves the viscosity equation $-\left(\Delta + (k_0 + i\epsilon)^2\right) u = g$ (see proof of Corollary B.15 in Appendix B). Note that G and G_ϵ are both in $\mathscr{S}'(\mathbb{R}^3)$. Indeed, we have

Lemma 5.12. *For*

$$G_\epsilon(\mathbf{r}) := e^{-\epsilon\|\mathbf{r}\|} G(\mathbf{r}), \quad \epsilon > 0,$$

we have that $G_\epsilon \in \mathscr{S}'(\mathbb{R}^3)$.

Proof. From

$$\int_{\mathscr{B}} \left| e^{-\epsilon\|\mathbf{r}\|} G(\mathbf{r}) \right| \, \mathrm{d}\mathbf{r} = \int_{\mathscr{B}} \left| e^{-\epsilon\|\mathbf{r}\|} \frac{e^{ik_0\|\mathbf{r}\|}}{4\pi\|\mathbf{r}\|} \right| \, \mathrm{d}\mathbf{r} = \int_{\mathscr{B}} e^{-\epsilon\|\mathbf{r}\|} \frac{1}{4\pi\|\mathbf{r}\|} \, \mathrm{d}\mathbf{r},$$

we use again the substitution as in the proof of Lemma 5.10 to get

$$\frac{1}{4\pi}\omega(S_2) \int_0^1 \frac{1}{\rho} e^{-\epsilon\rho} \rho^2 \, \mathrm{d}\rho = \frac{1}{4\pi}\omega(S_2) \int_0^1 \rho e^{-\epsilon\rho} \, \mathrm{d}\rho.$$

Therefore,

$$\int_{\mathscr{B}} \left| e^{-\epsilon\|\mathbf{r}\|} G(\mathbf{r}) \right| \, \mathrm{d}\mathbf{r} = \frac{1}{4\pi}\omega(S_2) \left[\frac{e^{-\epsilon}}{-\epsilon} + \frac{1}{\epsilon} \left(\frac{e^{-\epsilon}}{-\epsilon} + \frac{1}{\epsilon} \right) \right] < \infty$$

for $\epsilon > 0$, and we can conclude that $G_\epsilon \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. Moreover, from

$$\lim_{\|\mathbf{r}\|\to\infty} \frac{\left|G(\mathbf{r})\right|}{\mathbf{r}^N} = 0,$$

with a $N \in \mathbb{N}$, (see proof of Lemma 5.10), together with

$$\lim_{\|\mathbf{r}\|\to\infty} e^{-\epsilon\|\mathbf{r}\|} = 0,$$

we obtain

$$\lim_{\|\mathbf{r}\|\to\infty} \frac{\left|G_\epsilon(\mathbf{r})\right|}{\mathbf{r}^N} = \lim_{\|\mathbf{r}\|\to\infty} e^{-\epsilon\|\mathbf{r}\|} \frac{\left|G(\mathbf{r})\right|}{\mathbf{r}^N} = 0.$$

Therefore, we can apply Lemma 2.34 to arrive at the assertion that $G_\epsilon \in \mathscr{S}'(\mathbb{R}^3)$. ☐

Then, we want to show that G_ϵ converges to G in $\mathscr{S}'(\mathbb{R}^3)$ as ϵ goes to 0, as in the following lemma:

Lemma 5.13. *For G and G_ϵ defined above, $G_\epsilon \xrightarrow{\epsilon\to 0} G$ in $\mathscr{S}'(\mathbb{R}^3)$.*

Proof. First, using the definition of G_ϵ, we get

$$\left| \mathbf{r}^\alpha \partial^\beta \left[G(\mathbf{r}) - G_\epsilon(\mathbf{r}) \right] \right| = \left| \mathbf{r}^\alpha \partial^\beta \left[(1 - e^{-\epsilon\|\mathbf{r}\|}) G(\mathbf{r}) \right] \right|.$$

Then, using the Leibniz's formula to calculate $\partial^\beta \left[(1 - e^{-\epsilon\|\mathbf{r}\|}) G(\mathbf{r}) \right]$ yields

$$\left| \mathbf{r}^\alpha \partial^\beta [G(\mathbf{r}) - G_\epsilon(\mathbf{r})] \right| = \left| \mathbf{r}^\alpha \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \left[\partial^\gamma (1 - e^{-\epsilon\|\mathbf{r}\|}) \right] \left[\partial^{\beta-\gamma} G(\mathbf{r}) \right] \right|.$$

Calculating $\partial^\gamma(1-e^{-\epsilon\|\mathbf{r}\|})$ and $\partial^{\beta-\gamma}G(\mathbf{r})$ gives

$$\left|\mathbf{r}^\alpha \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\left[\left(-\epsilon\frac{1}{2}\|\mathbf{r}\|^{-\frac{1}{2}}\right)^\gamma e^{-\epsilon\|\mathbf{r}\|}\right]\left[\partial^{\beta-\gamma}G(\mathbf{r})\right]\right|.$$

Utilising the triangle inequality, we obtain

$$\left|\mathbf{r}^\alpha\partial^\beta[G(\mathbf{r})-G_\epsilon(\mathbf{r})]\right| \leq \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\left(\frac{\epsilon}{2\|\mathbf{r}\|^{\frac{1}{2}}}\right)^\gamma e^{-\epsilon\|\mathbf{r}\|}\left|\mathbf{r}^\alpha\partial^{\beta-\gamma}G(\mathbf{r})\right|.$$

Since the sum is finite,

$$\left(\frac{\epsilon}{2\|\mathbf{r}\|^{\frac{3}{2}}}\right)^\gamma e^{-\epsilon\|\mathbf{r}\|} \xrightarrow{\epsilon\to 0} 0$$

and

$$\|\mathbf{r}\|\left|\mathbf{r}^\alpha\partial^{\beta-\gamma}G(\mathbf{r})\right| < \infty$$

for all $\mathbf{r}\in\mathbb{R}^n$ and all $\alpha,\beta\in\mathbb{N}_0^n$ and $\gamma\leq\beta$, we can conclude that

$$\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\left(\frac{\epsilon}{2\|\mathbf{r}\|^{\frac{1}{2}}}\right)^\gamma e^{-\epsilon\|\mathbf{r}\|}\left|\mathbf{r}^\alpha\partial^{\beta-\gamma}G(\mathbf{r})\right| \xrightarrow{\epsilon\to 0} 0.$$

Hence, we have that

$$p_{\alpha,\beta}(G-G_\epsilon) \xrightarrow{\epsilon\to 0} 0,$$

or equivalently, $G_\epsilon \to G$ as $\epsilon\to 0$ in $\mathscr{S}'(\mathbb{R}^3)$.

□

Next, we want to calculate $\mathscr{F}_{1,2}G_\epsilon$ and $\mathscr{F}_{1,2}G$ to complete the first step. To do this, we need to define, for $\epsilon>0$,

$$\kappa_\epsilon := \sqrt{(k_0+i\epsilon)^2-k_1^2-k_2^2} \tag{5.10}$$

the root with positive imaginary part. With the definition of κ_ϵ, we can now compute the partial Fourier transform of G_ϵ and G.

Lemma 5.14. *The partial Fourier transform $\mathscr{F}_{1,2}G\in\mathscr{S}'(\mathbb{R}^3)$ is given by*

$$\mathscr{F}_{1,2}G(\phi) = \lim_{\epsilon\to 0}\mathscr{F}_{1,2}G_\epsilon(\phi) \tag{5.11}$$

$$= \lim_{\epsilon\to 0}\int_{\mathbb{R}^3}\frac{ie^{i\kappa_\epsilon|r_3|}}{4\pi\kappa_\epsilon}\phi(k_1,k_2,r_3)\ \mathrm{d}(k_1,k_2,r_3) \tag{5.12}$$

for all $\phi\in\mathscr{S}(\mathbb{R}^3)$.

Proof. First, from Proposition 3.6, we have that $\mathscr{F}_{1,2}$ is a bounded operator on $\mathscr{S}'(\mathbb{R}^3)$. Therefore, $\mathscr{F}_{1,2}G_\epsilon \xrightarrow{\epsilon\to 0} \mathscr{F}_{1,2}G$ in $\mathscr{S}'(\mathbb{R}^3)$ as $G_\epsilon \xrightarrow{\epsilon\to 0} G$ and we get Equation (5.11).

Now, we compute $\mathscr{F}_{1,2}G_\epsilon$ using $\mathscr{F}_{1,2} = \mathscr{F}_3^{-1}\mathscr{F}$. Since G_ϵ is a radial function, using Theorem 5.7, we have

$$\mathscr{F}G_\epsilon(\mathbf{k}) = \sqrt{\frac{2}{\pi}}\frac{1}{\|\mathbf{k}\|}\int_0^\infty G_\epsilon(r) r \sin(r\|\mathbf{k}\|)\ \mathrm{d}r.$$

Applying the definition of G_ϵ, i.e., $G_\epsilon(r) = e^{-\epsilon r}\frac{e^{ik_0 r}}{4\pi r}$, $\epsilon > 0$ yields

$$\mathscr{F}G_\epsilon(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^3}\frac{1}{\|\mathbf{k}\|}\int_0^\infty e^{r(-\epsilon+ik_0)}\sin(r\|\mathbf{k}\|)\ \mathrm{d}r. \tag{5.13}$$

Now, we want to integrate the term on the right-hand side, but before doing that, we first use the interpretation of the sin function with exponential functions, namely,

$$\sin(r\|\mathbf{k}\|) = \frac{e^{ir\|\mathbf{k}\|} - e^{-ir\|\mathbf{k}\|}}{2i},$$

which then gives

$$\int_0^\infty e^{r(-\epsilon+ik_0)}\sin(r\|\mathbf{k}\|)\ \mathrm{d}r = \frac{1}{2i}\left[\int_0^\infty e^{r(-\epsilon+ik_0+i\|\mathbf{k}\|)} - e^{r(-\epsilon+ik_0-i\|\mathbf{k}\|)}\ \mathrm{d}r\right].$$

So, integrating yields

$$\begin{aligned}&\int_0^\infty e^{s(-\epsilon+ik_0+i\|\mathbf{k}\|)} - e^{s(-\epsilon+ik_0-i\|\mathbf{k}\|)}\ \mathrm{d}r\\ &= \frac{-1}{-\epsilon+i(k_0+\|\mathbf{k}\|)} + \frac{1}{-\epsilon+i(k_0-\|\mathbf{k}\|)}\end{aligned}$$

After finding a common denominator and simplifying the fraction, we obtain the following:

$$\frac{-1}{-\epsilon+i(k_0+\|\mathbf{k}\|)} + \frac{1}{-\epsilon+i(k_0-\|\mathbf{k}\|)} = \frac{2i\|\mathbf{k}\|}{(\epsilon^2 - 2i\epsilon k_0 - k_0^2) + \|\mathbf{k}\|^2}.$$

Therefore,

$$\int_0^\infty e^{r(-\epsilon+ik_0)}\sin(r\|\mathbf{k}\|)\ \mathrm{d}r = \frac{1}{2i}\frac{2i\|\mathbf{k}\|}{(\epsilon - ik_0)^2 + \|\mathbf{k}\|^2}.$$

Hence, plugging this in Equation (5.13) gives

$$\mathscr{F}G_\epsilon(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^3}\frac{1}{(\epsilon - ik_0)^2 + \|\mathbf{k}\|^2} = \frac{1}{(\sqrt{2\pi})^3}\frac{1}{\|\mathbf{k}\|^2 - i^2(\epsilon - ik_0)^2}$$

and finally, we arrive at

$$\mathscr{F}G_\epsilon(\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^3}\frac{1}{\|\mathbf{k}\|^2 - (i\epsilon + k_0)^2} = \frac{1}{(\sqrt{2\pi})^3}\frac{1}{k_3^2 - \kappa_\epsilon^2}.$$

Because of the property $\mathscr{F}^{-1}f = \mathscr{F}[x \mapsto f(-x)]$ of the 1-dimensional Fourier transform and the formula 17.23.14 in [3], we have that

$$\mathscr{F}^{-1}\left[k_3 \mapsto \frac{1}{k_3^2 + a^2}\right] = \mathscr{F}\left[k_3 \mapsto \frac{1}{k_3^2 + a^2}\right] = \sqrt{\frac{\pi}{2}}\frac{e^{-a|r_3|}}{a}$$

for $a \in \mathbb{C}, Re(a) > 0$. Now, choose $a = -i\kappa_\epsilon$. Since $Im(\kappa_\epsilon) > 0$, the condition $Re(a) > 0$ is satisfied. So we arrive at

$$\mathscr{F}_{1,2}G_\epsilon(\mathbf{k}) = \mathscr{F}_3^{-1}\mathscr{F}G_\epsilon(\mathbf{k}) = \sqrt{\frac{\pi}{2}}\frac{1}{(\sqrt{2\pi})^3}\frac{e^{i\kappa_\epsilon|r_3|}}{-i\kappa_\epsilon} = -\frac{1}{4\pi}\frac{e^{i\kappa_\epsilon|r_3|}}{i\kappa_\epsilon} = \frac{1}{4\pi}\frac{ie^{i\kappa_\epsilon|r_3|}}{\kappa_\epsilon}$$

and we have shown Equation (5.12).

□

Now, we do the second part of the proof of the Fourier Diffraction Theorem.

Let $g \in L^p(\mathbb{R}^3), p > 1$ with $\operatorname{supp}(g) \subseteq \mathscr{B}_r(0)$, $r > 0$. Since for $1 \le q \le p$, $L^p(\Omega) \hookrightarrow L^q(\Omega)$ for any $\Omega \subseteq \mathbb{R}^3$ bounded and $\mathscr{D}(\mathbb{R}^3)$ is dense in $L^p(\mathbb{R}^3)$ for $p \in [0, \infty)$, we can find a sequence of functions $g_n \in \mathscr{D}(\mathbb{R}^3)$ with $\operatorname{supp}(g_n) \subseteq \mathscr{B}_r(0)$ so that $g_n \xrightarrow{n\to\infty} g$ in $L^q(\mathbb{R}^3)$ for

$$\begin{cases} q \in [1, p] & \text{if } p \in (1, \infty) \\ q \in [1, \infty) & \text{if } p = \infty \end{cases}.$$

Now, consider $-(\Delta + k_0^2)u_n = g_n$. The unique solution that satisfies the Sommerfeld radiation condition is $u_n = g_n * G$ (cf. Subsection 5.1). We have this following result of the partial Fourier transform of u_n:

Lemma 5.15. *For the outgoing solution u_n of the equation $-(\Delta + k_0^2)u_n = g_n$, we have*

$$\mathscr{F}_{1,2}u_n(\phi) = \frac{i\sqrt{\pi}}{\sqrt{2}}\int_{\mathbb{R}^3}\frac{\phi}{\kappa}\Big(e^{i\kappa r_3}\mathscr{F}\left[(1-\chi_{r_3})g_n\right](k_1,k_2,\kappa) \\ + e^{-i\kappa r_3}\mathscr{F}\left[\chi_{r_3}g_n\right](k_1,k_2,-\kappa)\Big)\ \mathrm{d}(k_1,k_2,r_3).$$

Proof. From $\mathscr{F}_{1,2} = \mathscr{F}_3^{-1}\mathscr{F}$ and the two convolution theorems, i.e., Theorem 2.44 for Fourier transform and Theorem 3.11 for partial Fourier transform, we have

$$\mathscr{F}_{1,2}u_n = \mathscr{F}_{1,2}(g_n * G) = \mathscr{F}_3^{-1}\mathscr{F}(g_n * G) \overset{2.44}{=} \mathscr{F}_3^{-1}\left[(\sqrt{2\pi})^3(\mathscr{F}g_n)(\mathscr{F}G)\right],$$

which equals

$$(\sqrt{2\pi})^3 \mathscr{F}_3^{-1}\left[(\mathscr{F}g_n)(\mathscr{F}G)\right] \overset{3.11}{=} (\sqrt{2\pi})^3\sqrt{2\pi}(\mathscr{F}_3^{-1}\mathscr{F}g_n) \overset{3}{*} (\mathscr{F}_3^{-1}\mathscr{F}G).$$

Hence, we get

$$\mathscr{F}_{1,2}u_n = 2\pi\mathscr{F}_{1,2}g_n \overset{3}{*} \mathscr{F}_{1,2}G.$$

Now, for every $\phi \in \mathscr{S}(\mathbb{R}^3)$, since partial convolutions are continuous on $\mathscr{S}'(\mathbb{R}^3)$, we have that,

$$\mathscr{F}_{1,2}u_n(\phi) = 2\pi(\mathscr{F}_{1,2}g_n \overset{3}{*} \mathscr{F}_{1,2}G)(\phi) = 2\pi \lim_{\epsilon\to 0}(\mathscr{F}_{1,2}g_n \overset{3}{*} \mathscr{F}_{1,2}G_\epsilon)(\phi).$$

Due to Definition 3.10, we get

$$(\mathscr{F}_{1,2}g_n \overset{3}{*} \mathscr{F}_{1,2}G_\epsilon)(\phi) = \mathscr{F}_{1,2}G_\epsilon\left[(\mathscr{F}_{1,2}g_n)_3^- \overset{3}{*} \phi\right]$$

and applying Lemma 5.14 leads to

$$\mathscr{F}_{1,2}G_\epsilon\left[(\mathscr{F}_{1,2}g_n)_3^- \overset{3}{*} \phi\right] = \frac{i}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i\kappa_\epsilon|r_3|}}{\kappa_\epsilon}\left[(\mathscr{F}_{1,2}g_n)_3^- \overset{3}{*} \phi\right](k_1,k_2,r_3)\ \mathrm{d}(k_1,k_2,r_3).$$

Now, the definition of convolution yields

$$\begin{aligned}&(\mathscr{F}_{1,2}g_n \overset{3}{*} \mathscr{F}_{1,2}G_\epsilon)(\phi)\\ = \ &\frac{i}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i\kappa_\epsilon|r_3|}}{\kappa_\epsilon}\int_{\mathbb{R}}\mathscr{F}_{1,2}g_n(k_1,k_2,-x)\phi(k_1,k_2,r_3-x)\ \mathrm{d}x\ \mathrm{d}(k_1,k_2,r_3).\end{aligned}$$

Then, utilising the substitution, we obtain

$$\begin{aligned}&(\mathscr{F}_{1,2}g_n \overset{3}{*} \mathscr{F}_{1,2}G_\epsilon)(\phi)\\ = \ &\frac{i}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i\kappa_\epsilon|r_3|}}{\kappa_\epsilon}\int_{\mathbb{R}}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\phi(k_1,k_2,r_3+x)\ \mathrm{d}x\ \mathrm{d}(k_1,k_2,r_3),\end{aligned}$$

which then equals

$$\frac{i}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i\kappa_\epsilon|r_3-x|}}{\kappa_\epsilon}\int_{\mathbb{R}}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\phi(k_1,k_2,r_3)\ \mathrm{d}x\ \mathrm{d}(k_1,k_2,r_3).$$

Since $\phi \in \mathscr{S}(\mathbb{R}^3)$ with respect to (k_1,k_2,r_3) and $\mathscr{F}_{1,2}(g_n) \in \mathscr{S}(\mathbb{R}^3)$ with respect to (k_1,k_2,x), we can conclude that

$$(k_1,k_2,r_3,x) \mapsto e^{i\kappa_\epsilon|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\phi(k_1,k_2,r_3)$$

is Lebesgue integrable. Hence, we can apply Fubini-Tonelli theorem to get

$$\mathscr{F}_{1,2}u_n(\phi) = \frac{i}{2}\lim_{\epsilon\to 0}\int_{\mathbb{R}^3}\frac{\phi}{\kappa_\epsilon}\left(\int_{\mathbb{R}}e^{i\kappa_\epsilon|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x\right)\mathrm{d}(k_1,k_2,r_3). \tag{5.14}$$

Now, for $\phi \in \mathscr{D}(\mathbb{R}^3)$, we want to interchange the limit and the integration, using Lebesgue dominated convergence theorem twice. First, since $\mathscr{F}_{1,2}g_n(k_1,k_2,\cdot) \in L^1(\mathbb{R})$, we can use Lebesgue's theorem to get

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}} e^{i\kappa_\epsilon|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x = \int_{\mathbb{R}} e^{i\kappa|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x$$

for all $(k_1,k_2,r_3) \in \mathbb{R}^3$. Then,

$$\begin{aligned}&\lim_{\epsilon\to 0}\frac{\phi(k_1,k_2,r_3)}{\kappa_\epsilon(k_1,k_2)}\int_{\mathbb{R}} e^{i\kappa_\epsilon|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x\\ &= \frac{\phi(k_1,k_2,r_3)}{\kappa(k_1,k_2)}\int_{\mathbb{R}} e^{i\kappa|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x\end{aligned} \tag{5.15}$$

for $k_1^2+k_2^2 \neq k_0^2$ (or $\kappa \neq 0$).

Next, in order to apply the Lebesgue's dominated convergence theorem to interchange the limit and the integral over $\mathbb{R}^3$ in equation (5.14), we need to show that $\dfrac{1}{|\kappa_\epsilon|} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ with respect to (k_1,k_2) in order to show that

$$(k_1,k_2) \mapsto \left|\frac{\phi}{\kappa_\epsilon}\int_{\mathbb{R}} e^{i\kappa_\epsilon|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x\right|$$

is bounded by a L^1-function.

For $\mathscr{B}_c(0) \subseteq \mathbb{R}^2$ with $c > k_0$, we have:

$$\int_{\mathscr{B}_c(0)}\frac{1}{|\kappa_\epsilon|}\ \mathrm{d}(k_1,k_2) = \int_{\mathscr{B}_c(0)}\left|k_0^2-k_1^2-k_2^2\right|^{-\frac{1}{2}}\ \mathrm{d}(k_1,k_2),$$

which equals

$$\int_0^{2\pi}\int_0^{k_0^2}\left|k_0^2-\rho^2\right|^{-\frac{1}{2}}\rho\ \mathrm{d}\rho\ \mathrm{d}\varphi,$$

using polar coordinate $(k_1,k_2) \mapsto (\rho\cos\varphi,\rho\sin\varphi)$. Since $\left|k_0^2-\rho^2\right|^{-\frac{1}{2}}\rho$ is integrable with respect to ρ and does not depend on φ, we have that

$$(\rho,\varphi) \mapsto \left|k_0^2-\rho^2\right|^{-\frac{1}{2}}\rho$$

is Lebesgue integrable. Hence, we can apply Fubini-Tonelli theorem to get

$$\int_{\mathscr{B}_c(0)}\frac{1}{|\kappa_\epsilon|}\ \mathrm{d}(k_1,k_2) = \left(\int_0^{2\pi}1\ \mathrm{d}\varphi\right)\left(\int_0^{k_0^2}\left|k_0^2-\rho^2\right|^{-\frac{1}{2}}\rho\ \mathrm{d}\rho\right).$$

So,

$$\int_{\mathscr{B}_c(0)} \frac{1}{|\kappa_\epsilon|}\ \mathrm{d}(k_1,k_2) = 2\pi\ \frac{(\rho^2-k_0^2)^{\frac{3}{2}}}{\left|\rho^2-k_0^2\right|}\Bigg|_0^{k_0^2} = 2\pi k_0 < \infty,$$

since

$$\frac{(\rho^2-k_0^2)^{\frac{3}{2}}}{\left|\rho^2-k_0^2\right|} = \begin{cases} -(\rho^2-k_0^2)^{\frac{1}{2}} & \text{for } |k_0| > |\rho|\,, \\ (\rho^2-k_0^2)^{\frac{1}{2}} & \text{otherwise.} \end{cases}$$

Therefore, $\dfrac{1}{|\kappa_\epsilon|} \in L^1(\mathscr{B}_c(0))$ for all $c > k_0$, i.e., $\dfrac{1}{|\kappa_\epsilon|} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$.

Now, consider the integrand on the right-hand side of Equation 5.14 and use the triangle inequality to get

$$\begin{aligned} &\left|\frac{\phi}{\kappa_\epsilon}\int_{\mathbb{R}} e^{i\kappa_\epsilon|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x\right| \\ \leq\ &\frac{|\phi|}{|\kappa_\epsilon|}\int_{\mathbb{R}}\left|\mathscr{F}_{1,2}g_n(k_1,k_2,x)\right|\ \mathrm{d}x \end{aligned}$$

Then, we apply the 2-dimensional Fourier transform with respect to the first two components to have

$$\begin{aligned} &\frac{|\phi|}{|\kappa_\epsilon|}\int_{\mathbb{R}}\left|\mathscr{F}_{1,2}g_n(k_1,k_2,x)\right|\ \mathrm{d}x \\ =\ &\frac{|\phi|}{|\kappa_\epsilon|}\int_{\mathbb{R}}\left|\frac{1}{2\pi}\int_{\mathbb{R}^2} g_n(s_1,s_2,x)e^{-i(k_1s_1+k_2s_2)}\ \mathrm{d}(s_1,s_2)\right|\mathrm{d}x. \end{aligned}$$

Applying the triangle inequality again, we obtain

$$\frac{|\phi|}{|\kappa_\epsilon|}\int_{\mathbb{R}}\left|\mathscr{F}_{1,2}g_n(k_1,k_2,x)\right|\ \mathrm{d}x \leq \frac{|\phi|}{|\kappa_\epsilon|}\int_{\mathbb{R}}\frac{1}{2\pi}\int_{\mathbb{R}^2}\left|g_n(s_1,s_2,x)\right|\ \mathrm{d}(s_1,s_2)\mathrm{d}x.$$

The right-hand side of the above inequality equals

$$\frac{|\phi|}{|\kappa_\epsilon|}\frac{1}{2\pi}\int_{\mathbb{R}^3}\left|g_n(s_1,s_2,x)\right|\ \mathrm{d}(s_1,s_2,x).$$

So, since $\phi \in \mathscr{S}'(\mathbb{R}^3)$ and $\dfrac{1}{|\kappa_\epsilon|} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, we have

$$\left|\frac{\phi}{\kappa_\epsilon}\int_{\mathbb{R}} e^{i\kappa_\epsilon|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x\right| \leq \frac{1}{2\pi}\frac{|\phi|}{|\kappa_\epsilon|}\|g_n\|_{L^1(\mathbb{R}^3)} \in L^1(\mathbb{R}^3). \tag{5.16}$$

Now, we can apply the Lebesgue's dominated convergence theorem with Equations (5.15) and (5.16) to Equation (5.14) to get

$$\mathscr{F}_{1,2}u_n(\phi) = \frac{i}{2}\int_{\mathbb{R}^3}\frac{\phi}{\kappa}\left(\int_{\mathbb{R}} e^{i\kappa|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x\right)\mathrm{d}(k_1,k_2,r_3).$$

We then calculate

$$
\begin{aligned}
& \int_{\mathbb{R}} e^{i\kappa|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x \\
= \ & \int_{-\infty}^{r_3} e^{i\kappa(r_3-x)}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x + \int_{r_3}^{\infty} e^{i\kappa(x-r_3)}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x,
\end{aligned}
$$

which equals

$$
e^{i\kappa r_3}\int_{-\infty}^{r_3} e^{-i\kappa x}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x + e^{-i\kappa r_3}\int_{r_3}^{\infty} e^{i\kappa x}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x.
$$

Now, we rewrite this for the integral over $\mathbb{R}$ that yields

$$
\begin{aligned}
& \int_{\mathbb{R}} e^{i\kappa|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x \\
= \ & e^{i\kappa r_3}\int_{\mathbb{R}} \big(1-\chi_{r_3}(k_1,k_2,x)\big)e^{-i\kappa x}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x \\
+ \ & e^{-i\kappa r_3}\int_{\mathbb{R}} e^{i\kappa x}\chi_{r_3}(k_1,k_2,x)\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x.
\end{aligned}
$$

So, we have that

$$
\begin{aligned}
& \int_{\mathbb{R}} e^{i\kappa|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x \\
= \ & e^{i\kappa r_3}\int_{\mathbb{R}} e^{-i\kappa x}\mathscr{F}_{1,2}\left[(1-\chi_{r_3})g_n\right](k_1,k_2,x)\ \mathrm{d}x \\
+ \ & e^{-i\kappa r_3}\int_{\mathbb{R}} e^{i\kappa x}\mathscr{F}_{1,2}\left[\chi_{r_3}g_n\right](k_1,k_2,x)\ \mathrm{d}x.
\end{aligned}
$$

Then, the definition of the 1-dimensional Fourier transform gives

$$
\begin{aligned}
& \int_{\mathbb{R}} e^{i\kappa|r_3-x|}\mathscr{F}_{1,2}g_n(k_1,k_2,x)\ \mathrm{d}x \\
= \ & e^{i\kappa r_3}\sqrt{2\pi}\mathscr{F}_3\mathscr{F}_{1,2}\left[(1-\chi_{r_3})g_n\right](k_1,k_2,\kappa) \\
+ \ & e^{-i\kappa r_3}\sqrt{2\pi}\mathscr{F}_3\mathscr{F}_{1,2}\left[\chi_{r_3}g_n\right](k_1,k_2,-\kappa),
\end{aligned}
\tag{5.17}
$$

which equals

$$
\begin{aligned}
\sqrt{2\pi}\big(& e^{i\kappa r_3}\mathscr{F}\left[(1-\chi_{r_3})g_n\right](k_1,k_2,\kappa) \\
& + e^{-i\kappa r_3}\mathscr{F}\left[\chi_{r_3}g_n\right](k_1,k_2,-\kappa)\big)
\end{aligned}
$$

due to the fact that $\mathscr{F}_3\mathscr{F}_{1,2} = \mathscr{F}$ holds true. Thus,

$$
\begin{aligned}
\mathscr{F}_{1,2}u_n(\phi) = \frac{i\sqrt{\pi}}{\sqrt{2}}\int_{\mathbb{R}^3}\frac{\phi}{\kappa}\Big(& e^{i\kappa r_3}\mathscr{F}\left[(1-\chi_{r_3})g_n\right](k_1,k_2,\kappa) \\
& + e^{-i\kappa r_3}\mathscr{F}\left[\chi_{r_3}g_n\right](k_1,k_2,-\kappa)\Big)\ \mathrm{d}(k_1,k_2,r_3).
\end{aligned}
\tag{5.18}
$$

□

Lastly, we find $\mathscr{F}_{1,2}u$ from $\mathscr{F}_{1,2}u_n$. We do this in two steps:

- First, find the limit of the right-hand side of (5.18) as $n \to \infty$.
- Then, consider the limit of the left-hand side of (5.18) as $n \to \infty$.

What we get from these two limits will be $\mathscr{F}_{1,2}u(\phi)$.

For the first step, since $g_n \to g$ in $L^1(\mathbb{R}^3)$ and $\mathscr{F} : L^1(\mathbb{R}^3) \to \mathscr{C}(\mathbb{R}^3)$, we obtain the pointwise limit:

$$\begin{aligned}&\lim_{n\to\infty} \frac{\phi}{\kappa} \Big(e^{i\kappa r_3} \mathscr{F}_3 \mathscr{F}_{1,2} \big[(1-\chi_{r_3})g_n\big] (k_1,k_2,\kappa) + e^{-i\kappa r_3} \mathscr{F}_3\mathscr{F}_{1,2}\left[\chi_{r_3} g_n\right](k_1,k_2,-\kappa)\Big) \\ &= \frac{\phi}{\kappa} \Big(e^{i\kappa r_3} \mathscr{F}_3 \mathscr{F}_{1,2} \big[(1-\chi_{r_3})g\big] (k_1,k_2,\kappa) + e^{-i\kappa r_3} \mathscr{F}_3\mathscr{F}_{1,2}\left[\chi_{r_3} g\right](k_1,k_2,-\kappa)\Big)\end{aligned} \tag{5.19}$$

for $k_1^2 + k_2^2 \neq k_0^2$. Moreover, from Equations (5.16) and (5.17), by applying the triangle inequality, we have

$$\begin{aligned}&\left| \frac{\phi}{\kappa} \Big(e^{i\kappa r_3} \mathscr{F} \big[(1-\chi_{r_3})g_n\big] + e^{-i\kappa r_3} \mathscr{F}\left[\chi_{r_3} g_n\right]\Big)\right| \\ \leq &\left|\frac{\phi}{\kappa}\right| \Big(\left|\mathscr{F}\big[(1-\chi_{r_3})g_n\big]\right| + \left|\mathscr{F}\left[\chi_{r_3} g_n\right]\right|\Big)\end{aligned}$$

Then, utilising the definition of the Fourier transform, the right-hand side of the above inequality equals

$$\begin{aligned}&\left|\frac{\phi}{\kappa}\right| \left(\left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_3} e^{-i\kappa x} \mathscr{F}_{1,2} g_n \,\mathrm{d}x \right| + \left| \frac{1}{\sqrt{2\pi}} \int_{r_3}^{\infty} e^{-i\kappa x} \mathscr{F}_{1,2} g_n \,\mathrm{d}x \right| \right) \\ \leq &\frac{1}{\sqrt{2\pi}} \left|\frac{\phi}{\kappa}\right| \left(\int_{-\infty}^{r_3} \left|\mathscr{F}_{1,2} g_n\right| \mathrm{d}x + \int_{r_3}^{\infty} \left|\mathscr{F}_{1,2} g_n\right| \mathrm{d}x \right) = \frac{1}{\sqrt{2\pi}} \frac{|\phi|}{|\kappa|} \int_{\mathbb{R}} \left|\mathscr{F}_{1,2} g_n\right| \mathrm{d}x.\end{aligned}$$

Finally, we arrive at

$$\left| \frac{\phi}{\kappa} \Big(e^{i\kappa r_3} \mathscr{F} \big[(1-\chi_{r_3})g_n\big] + e^{-i\kappa r_3} \mathscr{F}\left[\chi_{r_3} g_n\right]\Big)\right| \leq \frac{1}{\sqrt{(2\pi)^3}} \frac{|\phi|}{\kappa} \|g_n\|_{L^1(\mathbb{R}^3)}.$$

Since $g_n \to g$ in $L^1(\mathbb{R}^3)$, we can find a $C > 0$ and $N \in \mathbb{N}$ such that

$$\frac{1}{\sqrt{(2\pi)^3}} \frac{|\phi|}{\kappa} \|g_n\|_{L^1(\mathbb{R}^3)} \leq C \frac{|\phi|}{\kappa} \|g\|_{L^1(\mathbb{R}^3)} \in L^1(\mathbb{R}^3),$$

for all $n \geq N$ and almost every $(k_1, k_2, r_3) \in \mathbb{R}^3$. Hence,

$$\left| \frac{\phi}{\kappa} \Big(e^{i\kappa r_3} \mathscr{F} \big[(1-\chi_{r_3})g_n\big] + e^{-i\kappa r_3} \mathscr{F}\left[\chi_{r_3} g_n\right]\Big)\right| \leq C \frac{|\phi|}{\kappa} \|g\|_{L^1(\mathbb{R}^3)}. \tag{5.20}$$

Now, from Equations (5.19) and (5.20), we can apply Lebesgue's dominated convergence theorem to get

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^3}\frac{\phi}{\kappa}\left(e^{i\kappa r_3}\mathscr{F}\left[(1-\chi_{r_3})g_n\right]+e^{-i\kappa r_3}\mathscr{F}\left[\chi_{r_3}g_n\right]\right)\,\mathrm{d}(k_1,k_2,r_3)\\
=&\int_{\mathbb{R}^3}\frac{\phi}{\kappa}\left(e^{i\kappa r_3}\mathscr{F}\left[(1-\chi_{r_3})g\right]+e^{-i\kappa r_3}\mathscr{F}\left[\chi_{r_3}g\right]\right)\,\mathrm{d}(k_1,k_2,r_3). \qquad (5.21)
\end{aligned}$$

Then, for the second step, we consider the limit of the left-hand side of equation (5.18) as $n\to\infty$. To do this, we first need a remark:

Remark 5.16. (Theorem 6, Remark 1 in [14])
For the unique solution u that satisfies the Sommerfeld radiation condition u of $-(\Delta+k_0^2)u=g$, where $g\in L^{q_1}(\mathbb{R}^3)$, it holds:

$$\|u\|_{L^{q_2}}\le C(k_0)\|g\|_{L^{q_1}} \qquad (5.22)$$

for

$$q_1<\frac{3}{2},\; q_2>3,\; \frac{1}{q_1}-\frac{1}{q_2}\in\left[\frac{1}{2},\frac{2}{3}\right]. \qquad (5.23)$$

If we take

$$q_2=3+\epsilon,\; q_1\in\left[\frac{9+3\epsilon}{9+2\epsilon},\;\frac{6+2\epsilon}{5+\epsilon}\right]$$

for ϵ sufficiently small, q_1 and q_2 will satisfy the conditions in (5.23). By choosing ϵ small enough, q_1 also satisfies the condition $1<q_1<p$ for any $p>1$.

Now, consider $g\in L^p(\mathbb{R}^3), p>1$ with $\operatorname{supp}(g)\subseteq\mathscr{B}_r(0)$. We know that for $1\le q_1\le p$, $L^p(\mathbb{R}^3)\hookrightarrow L^{q_1}(\mathbb{R}^3)$. Therefore, $g\in L^{q_1}(\mathbb{R}^3)$. That yields $g_n\xrightarrow{n\to\infty}g$ in $L^{q_1}(\mathbb{R}^3)$ and because of Equation (5.22), we get $u_n\xrightarrow{n\to\infty}u$ in $L^{q_2}(\mathbb{R}^3)$ and hence $u_n\xrightarrow{n\to\infty}u$ in $\mathscr{S}'(\mathbb{R}^3)$. Due to the continuity of the Fourier transform, we get

$$\mathscr{F}_{1,2}u_n(\phi)=\mathscr{F}_{1,2}u(\phi) \qquad (5.24)$$

for all $\phi\in\mathscr{S}(\mathbb{R}^3)$.

Finally, from Equations (5.18), (5.21) and (5.24), we obtain the equality

$$\begin{aligned}
\mathscr{F}_{1,2}u(\phi)=\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^3}\phi\frac{i}{\kappa}&\left(e^{i\kappa r_3}\mathscr{F}\left[(1-\chi_{r_3})g_n\right](k_1,k_2,\kappa)\right.\\
&\left.+e^{-i\kappa r_3}\mathscr{F}\left[\chi_{r_3}g_n\right](k_1,k_2,-\kappa)\right)\,\mathrm{d}(k_1,k_2,r_3)
\end{aligned}$$

for all test functions $\phi \in \mathscr{D}(\mathbb{R}^3)$. Using du Bois-Reymond's lemma (see first item of Remark 5.2), we arrive at the assertion that the equation

$$\mathscr{F}_{1,2}u(k_1, k_2, r_3) = \sqrt{\frac{\pi}{2}}\frac{i}{\kappa}\Big(e^{i\kappa r_3}\mathscr{F}\left[(1-\chi_{r_3})g_n\right](k_1, k_2, \kappa)$$
$$+e^{-i\kappa r_3}\mathscr{F}\left[\chi_{r_3}g_n\right](k_1, k_2, -\kappa)\Big)$$

holds almost everywhere.

5.4 Generalisation of Fourier Diffraction Theorem in $\mathbb{R}^n$

In this subsection, we introduce a more general case of the conceptual experiment described in Section 4, where the scattering potential f is now a real-valued function of n variables, $n \geq 2$, i.e.,

$$f : \mathbb{R}^n \to \mathbb{R},$$

and the incident field u^{inc} is now defined as

$$u^{\mathrm{inc}}(\mathbf{r}) = e^{ik_0\mathbf{d_0}\cdot\mathbf{r}}, \qquad \mathbf{r} \in \mathbb{R}^n.$$

Here, $\mathbf{d_0} \in \mathbb{R}^n$ is the unit vector of the direction, in which u^{inc} propagates, and k_0 is the wave number.

We summarise the n-dimensional experimental setting in the assumption below:

Assumption 5.17. *(i) f is the scattering potential of the object,*

(ii) $u^{\mathrm{inc}}(\mathbf{r}) = e^{ik_0\mathbf{d_0}\cdot\mathbf{r}}$ is the incident field,

(iii) the scattered field u fulfils the n-dimensional Born approximation with the Sommerfeld radiation condition and

(iv) the measured data of the scattered field are $u(k_1, ..., k_{n-1}, r_M)$ for transmission imaging and $u(k_1, ..., k_{n-1}, -r_M)$ for reflection imaging.

Remark 5.18. In Section 4 and Subsection 5.2, the special case, where $n = 3$ and $\mathbf{d_0} = e_3 = (0, 0, 1)^{\mathrm{T}}$ is considered.

For the n-dimensional case, we also define κ as

$$\kappa = \kappa(k_1, ..., k_{n-1}) := \begin{cases} \sqrt{k_0^2 - \sum\limits_{j=1}^{n-1} k_j^2} & \text{if } \sum\limits_{j=1}^{n-1} k_j^2 \leq k_0^2 \\ i\sqrt{\sum\limits_{j=1}^{n-1} k_j^2 - k_0^2} & \text{if } \sum\limits_{j=1}^{n-1} k_j^2 > k_0^2 \end{cases}.$$

Then, we can state the Fourier Diffracion Theorem for the n-dimensional setting:

Theorem 5.19. *Let $k_0 > 0$ and $g \in L^p(\mathbb{R}^n)$, $n \geq 2$, $p > 1$ with supp$(g) \subseteq \mathscr{B}_{r_s}(0) \subseteq \mathbb{R}^n$ for some $r_s > 0$. Assume that u solves*

$$-(\Delta + k_0^2)u = g$$

and satisfies the Sommerfeld radiation condition. Then, we have

$$\mathscr{F}_{1,...,n-1}u(k_1, ..., k_{n-1}, r_n) = \sqrt{\frac{\pi}{2}}\frac{i}{\kappa}\Big[e^{i\kappa r_n}\mathscr{F}\big((1-\chi_{r_n})g\big)(k_1, ..., k_{n-1}, \kappa) + e^{-i\kappa r_n}\mathscr{F}\big(\chi_{r_n}g\big)(k_1, ...k_{n-1}, -\kappa)\Big] \tag{5.25}$$

almost everywhere.

Proof. see Theorem 2.3 in [12]. □

Analogous to what we did in Subsection 5.2, we get the Fourier Diffraction Theorem for our particular n-dimensional experimental setting:

Corollary 5.20. *Assume that Assumption 5.17 holds. Then, for $f \in L^p(\mathbb{R}^n)$, $p > 1$, $n \geq 2$, with supp$(f) \subseteq \mathscr{B}_{r_s}(\mathbf{0})$ for some $0 < r_s < r_M$, the following holds:*

$$\mathscr{F}_{1,...,n-1}u(k_1, ..., k_{n-1}, \pm r_M) = \sqrt{\frac{\pi}{2}}\frac{ie^{i\kappa r_M}}{\kappa}\mathscr{F}f\left((k_1, ..., k_{n-1}, \pm\kappa)^{\mathrm{T}} - k_0\mathbf{d}_0\right) \tag{5.26}$$

for $k_1, k_2 \in \mathbb{R}, k_1^2 + k_2^2 \leq k_0^2$.

Proof. Analogous to the proof of Corollary 5.4. □

Remark 5.21. If the object is not rotated, the measurements are only given by the data of the spatial frequencies with $\sum_{j=1}^{n-1} k_j^2 \leq k_0^2$, i.e. we only get the measurements on the k-space coverage, which is a $(n-1)$-hemisphere:

$$\left\{(k_1, ..., k_{n-1}, \pm\kappa)^{\mathrm{T}} - k_0\mathbf{d_0} \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^{n-1} k_j^2 \leq k_0^2\right\}.$$

We will discuss the k-space coverage in more detail in the next section.

The following figures visualise the k-space coverage for the case $n = 3$ and $\mathbf{d_0} = e_3 = (0, 0, 1)^{\mathrm{T}}$:

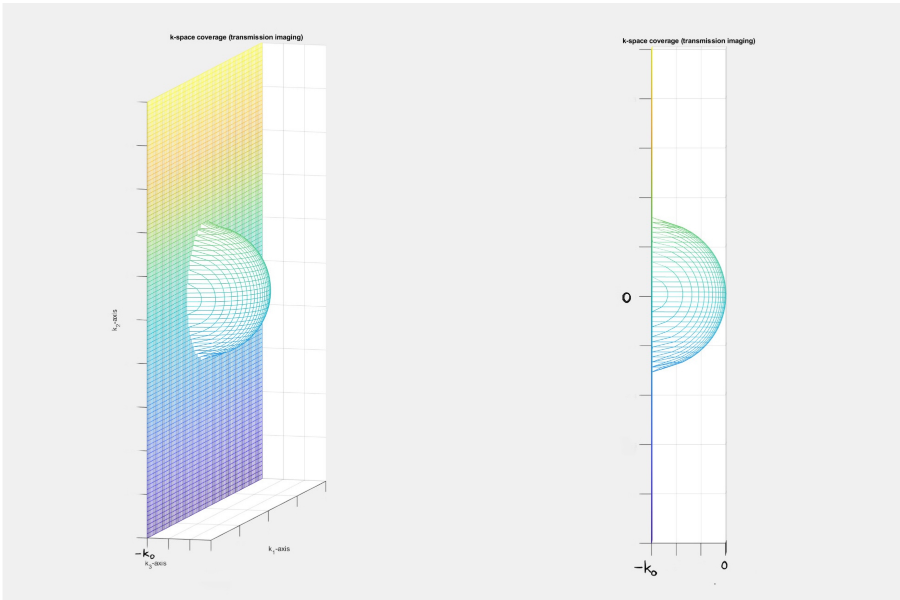

Figure 3: k-space coverage in the case of transmission imaging

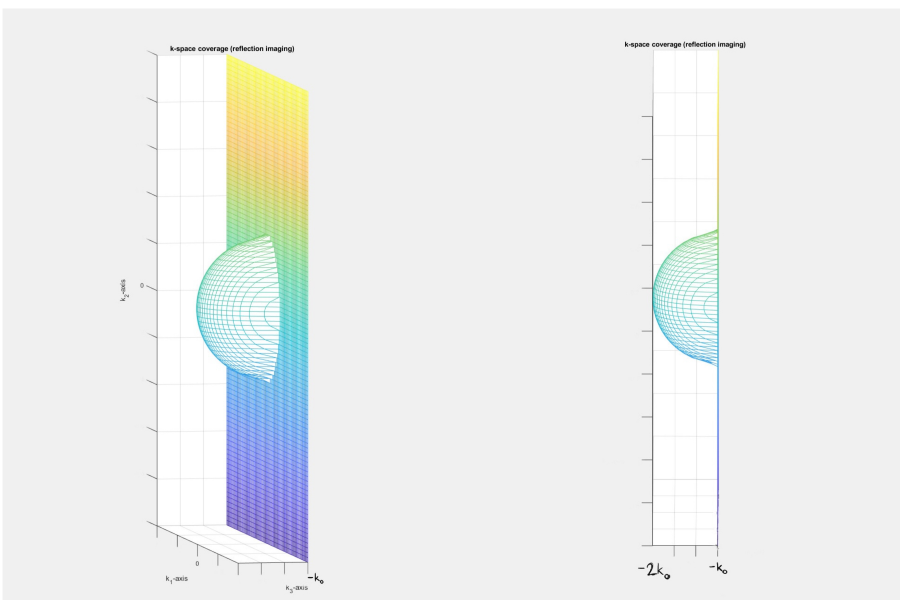

Figure 4: k-space coverage in the case of reflection imaging

6 k-space coverage (N.V.)

At this moment, we examine the effects of modifying some components on the k-space coverage. Precisely, first, we consider the wave number $k(t)$, which changes with respect to $t \in [0, T]$. The other components are in correlation with each other, namely, altering one of them, is the same as changing the others. The three of them are the incident direction, the orientation of the object and the hyperplane where the data are measured.

The following table provides an overview of the incident fields when different modifications are made. These modifications are made on wave number, incidence direction, orientation of an object and the measurement plane.

Modified	Fixed	Incident field
Wave number	Incident direction Object Measurement plane	$u_t^{\text{inc}}(\mathbf{r}) = e^{ik(t)\mathbf{d_0}\cdot\mathbf{r}}$ $t \in [0, T]$
Incidence direction	Wave number Object Measurement plane	$u_t^{\text{inc}}(\mathbf{r}) = e^{ik_0\mathbf{d}(t)\cdot\mathbf{r}}$ $t \in [0, T]$
Object	Wave number Incident direction Measurement plane	$u_t^{\text{inc}}(\mathbf{r}) = e^{ik_0\mathbf{d_0}\cdot\mathbf{r}}$ $t \in [0, T]$
Measurement plane	Wave number Incident direction Object	$u_t^{\text{inc}}(\mathbf{r}) = e^{ik_0\mathbf{d_0}\cdot\mathbf{r}}$ $t \in [0, T]$

6.1 Modifying the wave number

Here, we concentrate ourselves on the effects of changing the wave number on the k-space coverage. We fix the object and consider the incident field

$$u_t^{\text{inc}}(\mathbf{r}) = e^{ik(t)\mathbf{d_0}\cdot\mathbf{r}}, \quad t \in [0, T], \tag{6.1}$$

with wave number $k(t)$, which propagates towards the object and produces the scattered wave, which is represented by u_t. Using Corollary 5.20, we get the following formula:

$$\mathscr{F}_{1,..,n-1} u_t(k_1, .., k_{n-1}, \pm r_M) = \sqrt{\frac{\pi}{2}} \frac{i e^{i\kappa(t) r_M}}{\kappa(t)} \mathscr{F} f\left((k_1, .., k_{n-1}, \pm\kappa)^{\mathrm{T}} - k(t)\mathbf{d_0}\right), \tag{6.2}$$

where

$$\kappa(t) = \begin{cases} \sqrt{k^2(t) - \sum_{j=1}^{n-1} k_j^2} & \text{for } k^2(t) > \sum_{j=1}^{n-1} k_j^2, \\ i\sqrt{k^2(t) - \sum_{j=1}^{n-1} k_j^2} & \text{otherwise.} \end{cases}$$

In this case, the k-space coverage is given by

$$\mathscr{U} = \left\{ (k_1, .., k_{n-1}, \pm\kappa)^{\mathrm{T}} - k(t)\mathbf{d_0} \in \mathbb{R}^n \;\middle|\; k^2(t) > \sum_{j=1}^{n-1} k_j^2,\ t \in [0, T] \right\}. \tag{6.3}$$

We have

$$\mathscr{U} = \bigcup_{t\in[0,T]} \mathscr{H}_t,$$

where $\mathscr{H}_t,\ t \in (0, T]$ is $(n-1)$-hemisphere obtained by translating and scaling

$$\mathscr{H}_0 := \left\{ (k_1, .., k_{n-1}, \pm\kappa)^{\mathrm{T}} - k_0\mathbf{d}_0 \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^{n-1} k_j^2 < k_0^2 \right\}$$

so that $\mathbf{0} \in \mathscr{H}_t$.

Example 6.1. We study the 2-dimensional case, where we choose $\mathbf{d_0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The data are recorded at the plane $r_2 = r_M$ for transmission imaging and $r_2 = -r_M$ for reflection imaging. Furthermore, the wave number $k(t)$ is given by $k(t),\ t \in [0, T]$, so that $[\underline{k}, \bar{k}] = k([0, T])$.

Here, the k-space coverage is

$$\mathscr{U} = \left\{ \left(k_1, \pm\kappa - k(t)\right)^{\mathrm{T}} \in \mathbb{R}^2 \;\middle|\; k^2(t) > k_1^2,\ t \in [0, T] \right\}. \tag{6.4}$$

Since $k_1^2 < k^2(t)$, we must have $|k_1| < k(t)$ for all $t \in [0, T]$. Therefore, $|k_1| < \bar{k}$. Moreover, for $|k_1| < \underline{k}$ we have

$$\sqrt{\underline{k}^2 - k_1^2} - \underline{k}^2 \leq k_2 \leq \sqrt{\bar{k}^2 - k_1^2} - \bar{k}^2$$

and for $|k_1| \geq \underline{k}$, we have

$$-|k_1| \leq k_2 \leq \sqrt{\bar{k}^2 - k_1^2} - \bar{k}^2.$$

The figures in Figure 5 are the visualisations of $\mathscr{U}$. According to these figures, in both cases of transmission and reflection imaging, there are no accessible points in the area close to the origin. This means that the lower frequency data are missing in the reconstruction. In particular, if we measure at both $r_2 = r_M$ and $r_2 = -r_M$, the area mentioned above is a circle with radius $\underline{k}$.

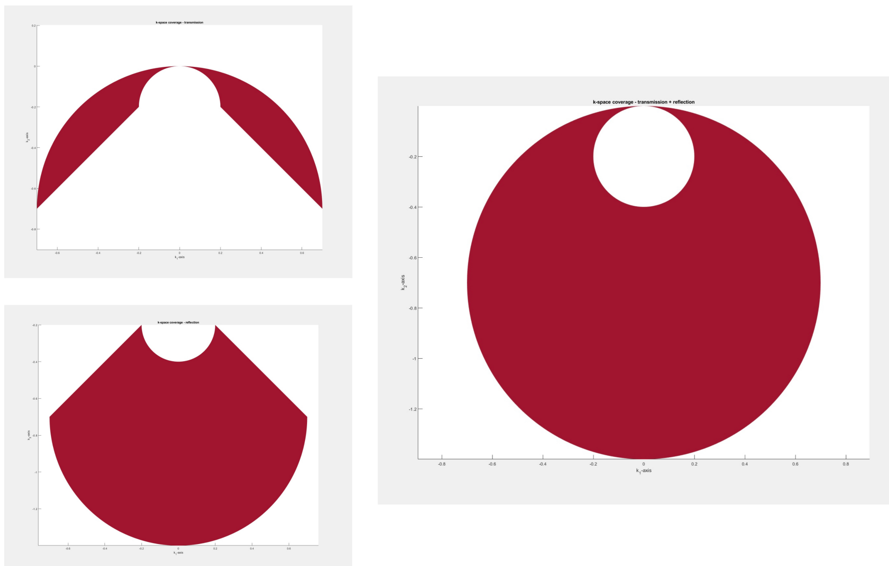

Figure 5: k-space coverage of the 2-dimensional case with $\mathbf{d_0} = (0,1)^{\mathrm{T}}$, $k(t)$ covers the interval $[\underline{k}, \bar{k}] = [0.2, 0.7]$. Upper left: k-space coverage for transmission imaging. Lower left: k-space coverage for reflection imaging. Right: Combine the k-space coverage of transmission imaging and reflection imaging.

Now, we pay attention to modification of incidence direction.

6.2 Modifying the incidence direction

In this subsection, the incidence field is

$$u_t^{\mathrm{inc}}(\mathbf{r}) = e^{ik_0 \mathbf{d}(t)\cdot\mathbf{r}}, \quad t \in [0, T], \tag{6.5}$$

where the incidence direction $\mathbf{d}(t) \in S_{n-1}$ changes with respect to $t \in [0, T]$ so that $\mathbf{d}(0) = \mathbf{d_0}$. The wave number and the object are fixed. From Corollary 5.20, we have the relation

$$\mathscr{F}_{1,..,n-1} u_t(k_1, .., k_{n-1}, \pm r_M) = \sqrt{\frac{\pi}{2}} \frac{ie^{i\kappa r_M}}{\kappa} \mathscr{F} f\left((k_1, .., k_{n-1}, \pm\kappa)^{\mathrm{T}} - k_0 \mathbf{d}(t)\right), \tag{6.6}$$

where u_t is again the scattered wave. In this case, if the incident direction is rotated, the k-space coverage is obtained by translations of the $(n-1)$-hemisphere

$\mathscr{H}_0$, namely,

$$\mathscr{U} = \left\{ (k_1, .., k_{n-1}, \pm\kappa)^{\mathrm{T}} - k_0 \mathbf{d}(t) \in \mathbb{R}^n \;\middle|\; k^2(t) > \sum_{j=1}^{n-1} k_j^2,\ t \in [0, T] \right\}. \tag{6.7}$$

Example 6.2. We choose $n = 2$ and $\mathbf{d}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$. The k-space coverage is

$$\mathscr{U} = \left\{ \big(k_1 - k_0 \cos(t), \pm\kappa - k_0 \sin(t)\big)^{\mathrm{T}} \in \mathbb{R}^2 \;\middle|\; k^2(t) > k_1^2,\ t \in [0, T] \right\}, \tag{6.8}$$

which is illustrated by the following figures.

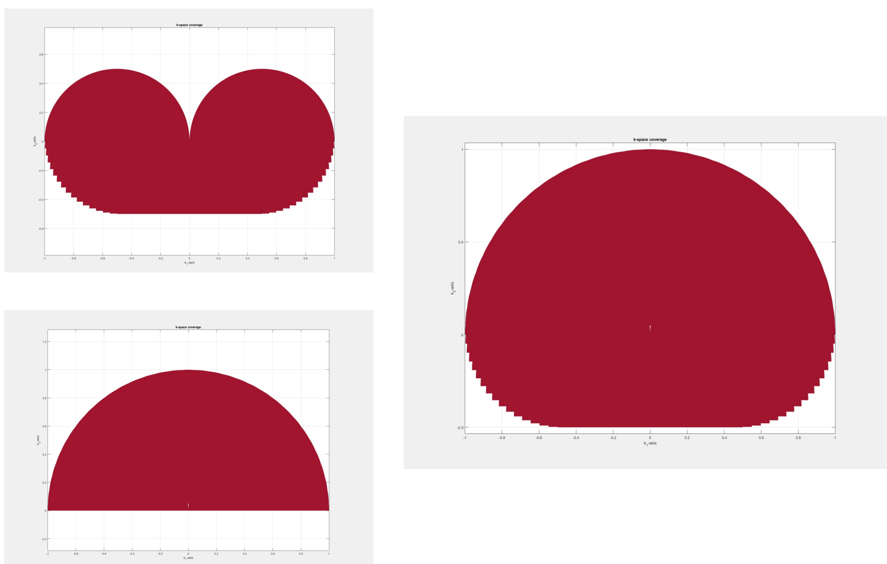

Figure 6: k-space coverage of the 2-dimensional case for transmission imaging with $\mathbf{d}(t) = (\cos(t), \sin(t))^{\mathrm{T}}$, $k_0 = 0.5$. Upper left: Taking $t = \frac{k}{64}\pi$, $k = 0, 1, ..., 64$, i.e. $t \in [0, \pi]$. Lower left: Taking $t = \frac{k}{64}\pi$, $k = 64, ..., 128$, i.e. $t \in [\pi, 2\pi]$. Right: Taking $t = \frac{k}{64}\pi$, $k = 0, ..., 128$, i.e. $t \in [0, 2\pi]$.

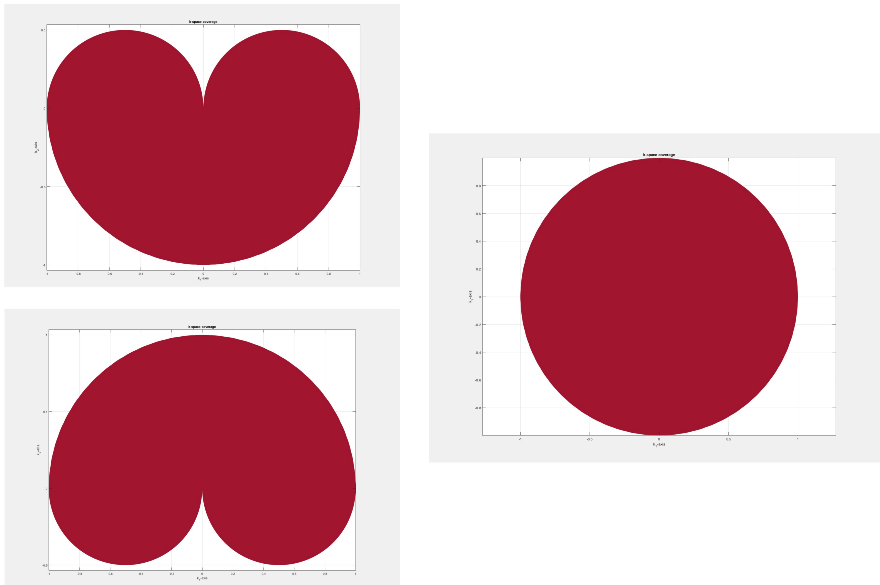

Figure 7: k-space coverage of the 2-dimensional case with data taken at both $r_2 = r_M$ and $r_2 = -r_M$, $\mathbf{d}(t) = (\cos(t), \sin(t))^{\mathrm{T}}$, $k_0 = 0.5$. Upper left: Taking $t = \frac{k}{64}\pi$, $k = 0, 1, ..., 64$, i.e. $t \in [0, \pi]$. Lower left: Taking $t = \frac{k}{64}\pi$, $k = 64, ..., 128$, i.e. $t \in [\pi, 2\pi]$. Right: Taking $t = \frac{k}{64}\pi$, $k = 0, ..., 128$, i.e. $t \in [0, 2\pi]$.

Next, we deal with the k-space coverage in the case of altering the object orientation.

6.3 Modifying the orientation of the object

Here, the object is assumed to be rotated using the matrix $R_t^{\mathrm{T}} \in SO(n)$, $t \in [0, T]$. The rotated scattering potential is $f \circ R_t$. The incident direction and the wave number are fixed. The scattered wave, which is generated by the incident field illuminating the object,

$$u^{\mathrm{inc}}(\mathbf{r}) = e^{ik_0 \mathbf{d}_0 \cdot \mathbf{r}}, \tag{6.9}$$

is denoted by u_t, $t \in [0, T]$. According to Corollary 5.20, we obtain the formula

$$\mathscr{F}_{1,..,n-1} u_t(k_1, .., k_{n-1}, \pm r_M) = \sqrt{\frac{\pi}{2}} \frac{i e^{i\kappa r_M}}{\kappa} \mathscr{F} f \left(R_t \left((k_1, .., k_{n-1}, \pm\kappa)^{\mathrm{T}} - k_0 \mathbf{d}_0 \right) \right). \tag{6.10}$$

The k-space coverage is

$$\mathscr{U} = \left\{ R_t \left((k_1, .., k_{n-1}, \pm\kappa)^{\mathrm{T}} - k_0 \mathbf{d}_0 \right) \in \mathbb{R}^n \;\middle|\; k^2(t) > \sum_{j=1}^{n-1} k_j^2, \; t \in [0, T] \right\}. \tag{6.11}$$

If the object is rotated, we get the k-space coverage by rotations of the $(n-1)$-hemisphere $\mathscr{H}_0$. We can see it in the example below:

Example 6.3. For $n = 2$ and $R_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$:

(i) Let $\mathbf{d_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then,

$$\mathscr{U} = \left\{ \begin{pmatrix} (k_1 - k_0)\cos(t) + \kappa \sin(t) \\ -(k_1 - k_0)\sin(t) + \kappa \cos(t) \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; k_0^2 > k_1^2,\; t \in [0, T] \right\}, \tag{6.12}$$

which is showed in the pictures:

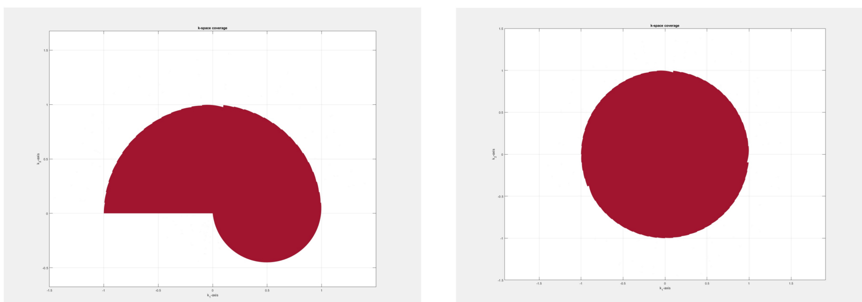

Figure 8: k-space coverage of the 2-dimensional case for transmission imaging with $\mathbf{d_0} = (1, 0)^{\mathrm{T}}$, $k_0 = 0.5$ and the object is rotated using the standard rotation matrix. Left: Taking $t = \frac{k}{64}\pi,\ k = 0, 1, ..., 64$, i.e. half turn $[0, \pi]$. Right: Taking $t = \frac{k}{64}\pi,\ k = 0, ..., 128$, i.e. full turn $[0, 2\pi]$. The rotation of $\mathscr{H}_0$ goes clockwise.

(ii) Let $\mathbf{d_0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Hence,

$$\mathscr{U} = \left\{ \begin{pmatrix} (\kappa - k_0)\sin(t) + k_1 \cos(t) \\ (\kappa - k_0)\cos(t) - k_1 \sin(t) \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; k_0^2 > k_1^2,\; t \in [0, T] \right\}. \tag{6.13}$$

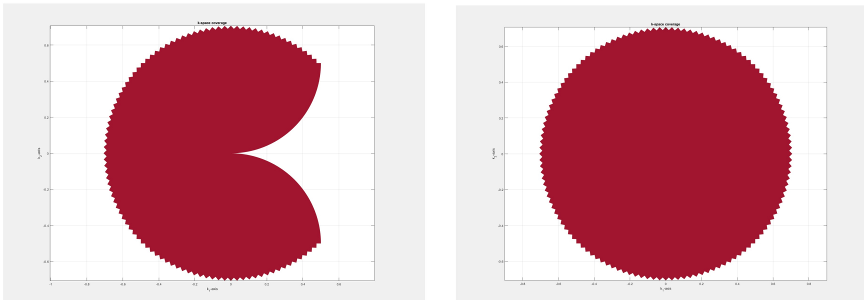

Figure 9: k-space coverage of the 2-dimensional case for transmission imaging with $\mathbf{d_0} = (0,1)^{\mathrm{T}}$, $k_0 = 0.5$ and the object is rotated using the standard rotation matrix. Left: Taking $t = \frac{k}{64}\pi,\ k = 0, 1, ..., 64$, i.e. half turn $[0, \pi]$. Right: Taking $t = \frac{k}{64}\pi,\ k = 0, ..., 128$, i.e. full turn $[0, 2\pi]$. The rotation of $\mathscr{H}_0$ goes clockwise.

Finally, we move to the case where the location of the measurement hyperplane is changed.

6.4 Modifying the measurement hyperplane's location

If we vary the distance between the origin and the measurement hyperplane, such that the hyperplane does not meet the support of the scattering potential, that is, $r_M > r_s$ for transmission imaging and $-r_M < -r_s$ for reflection imaging, while the object and incident field are fixed, the k-space coverage stays unmodified.

Now, assume that the hyperplane is rotated around $\mathbf{0}$ by the rotation matrix $R_t^{\mathrm{T}} \in SO(n)$, $t \in [0, T]$. The formula

$$\begin{aligned} &\mathscr{F}_{1,..,n-1} u_t(k_1, .., k_{n-1}, \pm r_M) \\ &= \sqrt{\frac{\pi}{2}} \frac{i e^{i\kappa r_M}}{\kappa} \mathscr{F} f\left(R_t\left((k_1, .., k_{n-1}, \pm\kappa)^{\mathrm{T}} - k_0 R_t^{\mathrm{T}} \mathbf{d}_0\right)\right) \\ &= \sqrt{\frac{\pi}{2}} \frac{i e^{i\kappa r_M}}{\kappa} \mathscr{F} f\left(R_t(k_1, .., k_{n-1}, \pm\kappa)^{\mathrm{T}} - k_0 \mathbf{d}_0\right), \end{aligned} \tag{6.14}$$

is attained by utilising Equations (6.6) and (6.10), where $\mathbf{d}(t) = R_t^{\mathrm{T}} \mathbf{d}_0$, because rotating the measurement hyperplane is equivalent to rotating the incident direction and the object. The k-space coverage in this case is:

$$\mathscr{U} = \left\{ R_t(k_1, .., k_{n-1}, \pm\kappa)^{\mathrm{T}} - k_0 \mathbf{d}_0 \in \mathbb{R}^n \;\middle|\; k^2(t) > \sum_{j=1}^{n-1} k_j^2,\ t \in [0, T] \right\}. \tag{6.15}$$

We study an example of the 2-dimensional case:

Example 6.4. For $n = 2$ and $R_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$:

(i) Let $\mathbf{d_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus,

$$\mathscr{U} = \left\{ \begin{pmatrix} k_1 \cos(t) + \kappa \sin(t) - k_0 \\ -k_1 \sin(t) + \kappa \cos(t) \end{pmatrix} \in \mathbb{R}^2 \mid k_0^2 > k_1^2 \ , t \in [0, T] \right\}. \tag{6.16}$$

(ii) $\mathbf{d_0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore,

$$\mathscr{U} = \left\{ \begin{pmatrix} k_1 \cos(t) + \kappa \sin(t) \\ -k_1 \sin(t) + \kappa \cos(t) - k_0 \end{pmatrix} \in \mathbb{R}^2 \mid k_0^2 > k_1^2 \ , t \in [0, T] \right\}. \tag{6.17}$$

As we can see, since rotating the measurement hyperplane is equivalent to rotating the object and the incident direction, we obtain more data than in the two previous cases. $\mathscr{U}$s in both cases are represented by the following illustrations:

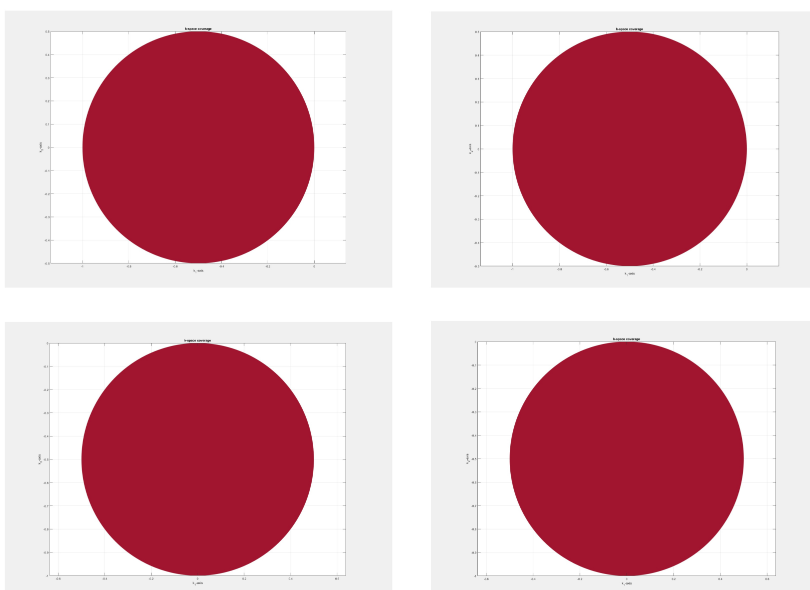

Figure 10: k-space coverage of the 2-dimensional case for transmission imaging with $k_0 = 0.5$ and the measurement hyperplane is rotated using the standard rotation matrix. The incidence direction is chosen to be $\mathbf{d_0} = (1, 0)^{\mathrm{T}}$ (top) and $\mathbf{d_0} = (0, 1)^{\mathrm{T}}$ (below). Left: Taking $t = \frac{k}{64}\pi, \ k = 0, 1, ..., 64$, i.e. half turn $[0, \pi]$. Right: Taking $t = \frac{k}{64}\pi, \ k = 0, ..., 128$, i.e. full turn $[0, 2\pi]$.

In the previous sections, we considered the continuous case, where the scattered wave is assumed to be continuous. In this case, the k-space coverage $\mathscr{H}_0$ is the entire $(n-1)$-hemisphere. Moreover, the modifications discussed in Section 6 are also studied with t covering the entire interval $[0,T]$. Thus, the number of accessible data is infinite. However, in practice, we can only measure a finite number of scattered data. Therefore, the study of the discretised Born approximation is required. For simplicity, in the next section, we will examine the discreteised Born approximation in the 2-dimensional case.

7 Discretisation of the Born approximation and application

7.1 Discretisation of data (M.U.)

Consider the unique solution of the 2-dimensional Born approximation that satisfies the Sommerfeld radiation condition,

$$u(\mathbf{r}) = \int_{\mathbb{R}^2} G(\mathbf{r}-\mathbf{r}')f(\mathbf{r}')u^{\mathrm{inc}}(\mathbf{r}')\ \mathrm{d}\mathbf{r}' =: G * g, \quad \mathbf{r} \in \mathbb{R}^2, \tag{7.1}$$

where $g := fu^{\mathrm{inc}}$ and G is 2-dimensional Green's function of the Helmholtz operator, that is,

$$G(\mathbf{r}) = \frac{i}{4}H_0^{(1)}(k_0\|\mathbf{r}\|), \quad \mathbf{r} \in \mathbb{R}^2 \setminus \{0\},$$

with $H_0^{(1)}$ is the Hankel function of the first kind of order zero (cf. Corollary B.15 in Appendix B).

We can discretise Equation (7.1) on the uniform grid

$$D_N := \frac{2r_s}{N} I_N \subseteq [-r_s, r_s]^2, \tag{7.2}$$

with $I_N := \{-\frac{N}{2} + j; j = 0, ..., n-1\}$, $N \in 2\mathbb{N}$, to get

$$u(\mathbf{r}) \approx u_N(\mathbf{r}) := \left(\frac{2r_s}{N}\right)^2 \sum_{\mathbf{r}' \in D_N} G(\mathbf{r}-\mathbf{r}')g(\mathbf{r}') \quad \mathbf{r} \in \mathbb{R}^2. \tag{7.3}$$

In the experiment described in Section 4, the measured data are assumed to be the Fourier transforms of the discretised Born approximation $u_N(r_1, \pm r_M)$, $N \in 2\mathbb{N}$, with $r_1 \in \frac{2l_M}{q} I_q$ are the equidistant points discretising $[-l_M, l_M]$. Our goal is the reconstruction of the scattering potential f, which is related to the measured data by the Fourier Diffraction Theorem. Therefore, it is desirable to calculate the discrete Fourier transform (DFT) of $u_N(r_1, \pm r_M)$, which approximates

$$\mathscr{F}_1 u(k_1, \pm r_M) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(r_1, \pm r_M) e^{ir_1k_1}\ \mathrm{d}r_1, \quad k_1 \in [-k_0, k_0].$$

The discrete Fourier transform of the Born approximation is given by

$$F_{1,q}u(k_1, \pm r_M) = \frac{1}{\sqrt{2\pi}} \frac{2l_M}{q} \sum_{r_1 \in \frac{2l_M}{q} I_q} u(r_1, \pm r_M) e^{i r_1 k_1} \, \mathrm{d}r_1, \quad k_1 \in k_0 \frac{I_q}{2l_M}, \tag{7.4}$$

with the equidistant points $r_1 \in \frac{2l_M}{q} I_q$ discretising the interval $[-l_M, l_M]$.

In reality, the object moves. Therefore, we consider the case where the object is rotated using the rotation matrix $R_t^{\mathrm{T}} \in SO(2)$. From Equation (6.10) with $n = 2$, i.e.,

$$\mathscr{F}_1 u_t(k_1, \pm r_M) = \sqrt{\frac{\pi}{2}} \frac{i e^{i\kappa r_M}}{\kappa} \mathscr{F} f \left(R_t \left((k_1, \pm\kappa)^{\mathrm{T}} - k_0 \mathbf{d}_0 \right) \right), \tag{7.5}$$

we obtain the relation

$$\mathscr{F} f \left(R_t \left((k_1, \pm\kappa)^{\mathrm{T}} - k_0 \mathbf{d}_0 \right) \right) = -i \sqrt{\frac{2}{\pi}} \kappa e^{-i\kappa r_M} \mathscr{F}_1 u_t(k_1, \pm r_M), \tag{7.6}$$

which holds true if $k_1^2 < k_0^2$. In order to get the reconstruction of the scattering potential, we need to approximate $\mathscr{F} f$ with the discrete Fourier transform on the discretised k-space coverage

$$\mathscr{U}_{q,q_t} = \left\{ R_t \left((k_1, \pm\kappa)^{\mathrm{T}} - k_0 \mathbf{d}_0 \right) \;\middle|\; k_1 \in k_0 \frac{I_q}{2l_M}, \ k_1^2 \le k_0^2, \ t \in \frac{T}{q_t} \{0, 1, ..., q_t\} \right\}, \tag{7.7}$$

where $\frac{T}{q_t}\{0, 1, ..., q_t\}$ are the equidistant samples of $t \in [0, T]$ and $k_1 \in k_0 \frac{I_q}{2l_M}$ are the equidistant points discretising the interval $[-k_0, k_0]$.

Since $\mathscr{U}_{q,q_t}$ is a non-uniform grid, the analysis of the 2-dimensional non-uniform discrete Fourier transform is desired. Accordingly, we define the non-uniform discrete fourier transform elementwise as

$$\mathbf{F}_N \mathbf{f}_N(\mathbf{y}) := \left(\frac{2r_s}{N} \right)^2 \sum_{\mathbf{r} \in D_N} f(\mathbf{r}) e^{i \mathbf{r} \cdot \mathbf{y}} \approx \mathscr{F} f(\mathbf{y}), \quad \mathbf{y} \in \mathscr{U}_{N,N_t}, \tag{7.8}$$

for $\mathbf{f}_N := (f(\mathbf{r}))_{\mathbf{r} \in D_N} \in \mathbb{R}^{N^2}$. To determine $\mathbf{f}_N$, we use the iterative method (Algorithm 7.27, described in Chapter 7.6.2 of [7]). This process is referred to as the inverse non-uniform discrete Fourier transform.

To summarise, we list the steps required for reconstructing the scattering potential:

1. The measured data are the discretised Fourier transform of the Born approximation.

2. Using the relation (7.5), we calculate the approximation of the Fourier transform of the scattering potential.

3. Applying inverse non-uniform discrete Fourier transform to the previously gained $\mathscr{F}f$, we get the scattering potential approximately.

7.2 Data generation (N.V.)

In our concrete example, we assume that the wave number k_0 of the incident field is 2π. Then, we scrutinise the efficacy of the Born approximation in 2-dimensional case by first generating the data, then solving the inverse problem and finally comparing the obtained solutions with the generated data.

For data generation,

1. We choose a test function f.

2. Then, we solve the equation

$$-\left(\Delta + k^2(\mathbf{r})\right)\left(u^{\mathrm{inc}}(\mathbf{r}) + u(\mathbf{r})\right) = 0,$$

   where

$$k^2(\mathbf{r}) = f(\mathbf{r}) + k_0^2,$$

   to get u. The solution u is either assumed to be the Born approximation u^{B}, or we let it be the Rytov approximation u^{R} and calculate the Born approximation using the relation

$$u^{\mathrm{B}}(\mathbf{r}) = u^{\mathrm{inc}} \log\left(\frac{u^{\mathrm{R}}(\mathbf{r})}{u^{\mathrm{inc}}(\mathbf{r})} + 1\right).$$

We solve the inverse problem by performing the following:

1. First, we calculate the discrete Fourier transform of the Born approximation obtained from the previous step.

2. From the discrete Fourier transform of the Born approximation, we determine the approximation of the Fourier transform of function f.

3. Next, we utilise inverse non-uniform discrete Fourier transform to get the approximation of f.

Let us specify the first step of data genarating by taking the indicator function of the disc $B_\rho(\mathbf{0})$ with radius $0 < \rho < r_s$ and centre $\mathbf{0}$, i.e., for $\mathbf{r} \in \mathbb{R}^2$,

$$f(\mathbf{r}) := \mathbb{1}_{B_\rho(\mathbf{0})}(\mathbf{r}) = \begin{cases} 1 & \text{if } \mathbf{r} \in B_\rho(\mathbf{0}), \\ 0 & \text{otherwise.} \end{cases}$$

We compute the Fourier transform of f in the following lemma:

Lemma 7.1. *Let $B_\rho(\mathbf{0})$ be the disc. Then, for $\mathbf{k} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, it holds*

$$\mathscr{F}\mathbb{1}_{B_\rho(\mathbf{0})}(\mathbf{k}) = \frac{\rho}{\|\mathbf{k}\|} J_1(\rho\|\mathbf{k}\|), \tag{7.9}$$

where J_1 denotes the Bessel function of the first kind of order 1. For $\mathbf{k} = 0$, we have

$$\mathscr{F}\mathbb{1}_{B_\rho(\mathbf{0})}(\mathbf{0}) = \frac{\rho^2}{2}. \tag{7.10}$$

Proof. Since $\mathbb{1}_{B_\rho(\mathbf{0})}$ is a radial function, we can apply Theorem 5.7 to obtain

$$\mathscr{F}\mathbb{1}_{B_\rho(\mathbf{0})}(\mathbf{k}) = \int_0^\infty 1_{[0,\rho]}(s) s J_0(s\|\mathbf{k}\|)\ \mathrm{d}s = \int_0^\rho s J_0(s\|\mathbf{k}\|)\ \mathrm{d}s.$$

Now, for $\mathbf{k} \neq \mathbf{0}$, we get

$$\int_0^\rho s J_0(s\|\mathbf{k}\|)\ \mathrm{d}s = \frac{1}{\|\mathbf{k}\|} \int_0^\rho s\|\mathbf{k}\|\ J_0(s\|\mathbf{k}\|)\ \mathrm{d}s.$$

Substituting $t = s\|\mathbf{k}\|$ yields

$$\int_0^\rho s\|\mathbf{k}\|\ J_0(s\|\mathbf{k}\|)\ \mathrm{d}s = \frac{1}{\|\mathbf{k}\|^2} \int_0^{\rho\|\mathbf{k}\|} t J_0(t)\ \mathrm{d}t.$$

Then, using the property

$$t^n J_{n-1}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(t^n J_n(t)),$$

which is proved in Lemma B.12, Appendix B, with $n = 1$, we get

$$\frac{1}{\|\mathbf{k}\|^2} \int_0^{\rho\|\mathbf{k}\|} t J_0(t)\ \mathrm{d}t = \frac{1}{\|\mathbf{k}\|^2} \int_0^{\rho\|\mathbf{k}\|} \frac{\mathrm{d}}{\mathrm{d}t}(t J_1(t))\ \mathrm{d}t = \frac{1}{\|\mathbf{k}\|^2} (t J_1(t)) \Big|_0^{\rho\|\mathbf{k}\|},$$

which equals

$$\frac{\rho}{\|\mathbf{k}\|} J_1(\rho\|\mathbf{k}\|).$$

For $\mathbf{k} = \mathbf{0}$, we have

$$\mathscr{F}\mathbb{1}_{B_\rho(\mathbf{0})}(\mathbf{0}) = \int_0^\rho s J_0(s\|\mathbf{0}\|)\ \mathrm{d}s = \int_0^\rho s J_0(0)\ \mathrm{d}s.$$

Then, using the fact that $J_0(0) = 1$ (Theorem B.13, Appendix B), we get

$$\int_0^\rho s J_0(0)\ \mathrm{d}s = \int_0^\rho s\ \mathrm{d}s = \frac{\rho^2}{2}$$

and arrive at the assertion. $\square$

For the second step, namely, computing the Born aproximation, we first need to discretise the test function f, for which we choose different radii $\rho = 2$ and $\rho = 4.5$. We sample f on the 40×40 grid D_{40}, defined by equation (7.2). Here, the square $[-r_s, r_s]^2$ is $\left[-\frac{N}{4\sqrt{2}}, \frac{N}{4\sqrt{2}}\right]$ with $N = 40$. Then, we evaluate discretised Born approximation from equation (7.3). In [12], the authors calculate the approximation of $u(\cdot, r_M)$ using the full waveform inversion.

Now, we solve the inverse problem. First, by utilising Equation (7.4), we get the discrete Fourier transform of the Born approximation. Then, applying Equation (7.6) leads to discrete Fourier transform of f, which completes the fourth step. Further step requires us to find an approximantion of f by solving Equation (7.8).

Eventually, we rate the approximations of f found in the previous step. Depending on the value of the peak signal-to-noise-ratio (PSNR), we can conclude how high the quality of reconstruction is; the higher the values of PSNR, the finer the reconstruction. The PSNR is defined by

$$\mathrm{PSNR}(\mathbf{f}, \mathbf{g}) := 10 \log_{10} \frac{\max\limits_{\mathbf{r} \in D_N} \left|\mathbf{f}(\mathbf{r})\right|^2}{\frac{1}{N^2} \sum\limits_{\mathbf{r} \in D_N} \left|\mathbf{f}(\mathbf{r}) - \mathbf{g}(\mathbf{r})\right|^2}, \tag{7.11}$$

where $\mathbf{f}$ is the discretisation of the test function f chosen in the first step and $\mathbf{g}$ is the approximation of $\mathbf{f}$.

The following figures give us adequate visualisatons of reconstruction. We describe the dissimilarities between different visualisations depending on frequency, radius of test function, choice of approximation (Born or Rytov) and k-space coverage. The authors of [12] reconstruct f under the setting:

- grid size: $N = 120$,
- length of the measurement plane: $l_M = 10$, i.e.,

$$r_1 \in \left\{ -10 + \frac{j}{10} \;\middle|\; j \in \{0, ..., 200\} \right\},$$

- number of time steps: $q_t = 40$,
- $r_s = \frac{N}{8\sqrt{2}} \approx 10$.

For the radius of test function $\rho = 4.5$ and the frequency $\omega = 1$, we have:

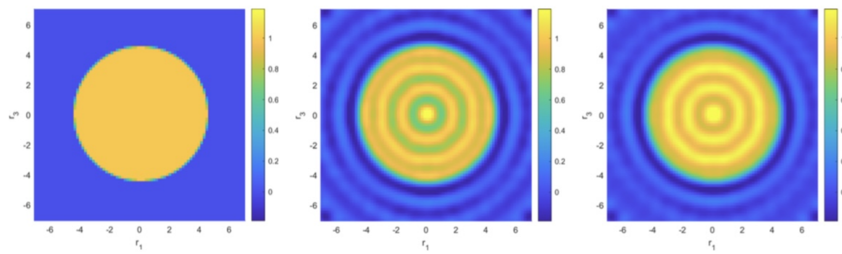

Figure 11: [12] Data generating using full waveform method and reconstruction using non-uniform discrete Fourier transform. The frequency of the incident field is $\omega = 1$. Left: Original function $f(\mathbf{r}) = \mathbb{1}_{B_{4.5}(\mathbf{0})}(\mathbf{r})$. Center: Reconstruction using Born approximation. Right: Reconstruction using Rytov approximation.

Figure 11 depicts the result of applying either Born or Rytov approximation. The visualisations show that using Born approximation gives higher difference between function values than utilising Rytov approximation. This difference is zero for the original function f. Additionally, the PSNR of Born approximation is lower than the one of Rytov. In particular, the PSNR of the Born is 13.26, whereas the PSNR of the Rytov is 17.67.

If we take $\rho = 2$ with the same frequency as above, then we can see that the difference between Born and Rytov approximation is hardly identifiable. Furthermore, PSNRs are almost identical, namely, for Born and Rytov reconstructions, 18.04 and 18.49, respectively. In fact, for objects with sufficiently small radii, Born and Rytov approximation are almost equal to each other (cf. [5], pg. 218). The statement is visualised by the following figures:

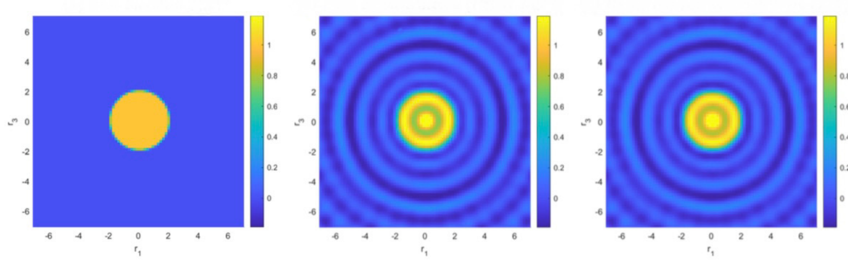

Figure 12: [12] Data generating using full waveform method and reconstruction using non-uniform discrete Fourier transform. The frequency of the incident field is $\omega = 1$. Left: Original function $f(\mathbf{r}) = \mathbb{1}_{B_2(\mathbf{0})}(\mathbf{r})$. Center: Reconstruction using Born approximation. Right: Reconstruction using Rytov approximation.

Taking smaller frequencies, $\omega_1 = 0.7$ and $\omega_2 = 0.4$, the wave number also changes, so $k_1 = 0.14\pi$ or $k_2 = 0.8\pi$. Observing Figure 13, we can conclude that: the lower the frequency, the worse the quality of reconstruction.

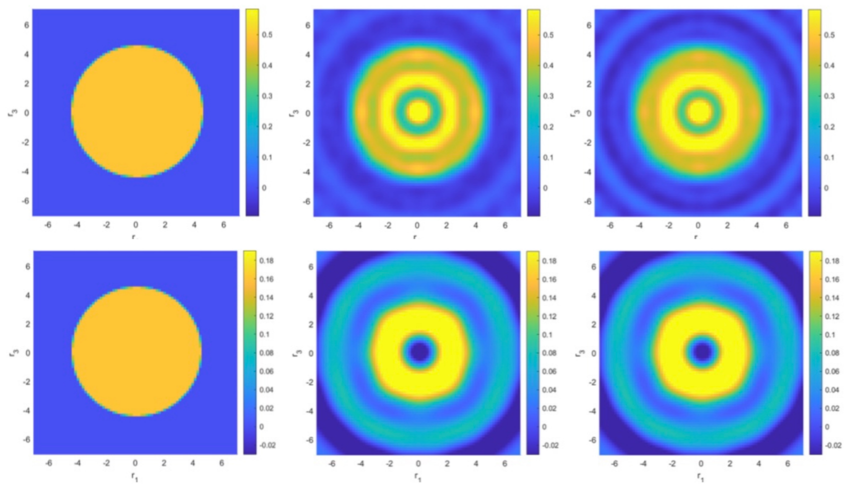

Figure 13: [12] Data generating using full waveform method and reconstruction using non-uniform discrete Fourier transform. The frequency of the incident field is $\omega = 0.7$ (top). and $\omega = 0.4$ (below). Left: Original function $f(\mathbf{r}) = \mathbb{1}_{B_{4.5}(\mathbf{0})}(\mathbf{r})$. Center: Reconstruction using Born approximation. Right: Reconstruction using Rytov approximation.

Keeping the rotation of the object $R_0^{\mathrm{T}} = I$ and the incidence direction $\mathbf{d_0} = (0,1)^{\mathrm{T}}$ fixed, combining the reconstructions with diverse frequencies $\omega \in [0,1]$ ($k(t)$ covers the interval $[0, 2\pi]$, see equation (6.4) in Subsection 6.1), i.e., reconstructions with data on the k-space coverage

$$\mathscr{U} = \left\{ \left(k_1, \pm\kappa - k(t)\right)^{\mathrm{T}} \in \mathbb{R}^2 \;\middle|\; k^2(t) > k_1^2,\ k(t) \in [0, 2\pi] \right\},$$

we get Figure 14.

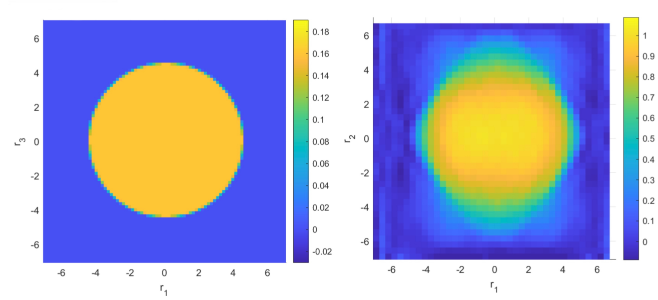

Figure 14: [12] Data generating using full waveform method and reconstruction using non-uniform discrete Fourier transform. The frequency ω of the incident field ranging from 0 to 1. The incidence direction is $\mathbf{d_0} = (0,1)^{\mathrm{T}}$. Left: Original function $f(\mathbf{r}) = \mathbb{1}_{B_{4.5}(\mathbf{0})}(\mathbf{r})$. Right: Reconstruction using Born approximation.

As we can see, if we compare Figure 13 with Figure 14, reconstructions with only one wave number are worse than the reconstruction with multiple frequencies.

Now, we choose the rotation matrix $R_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$ and keep the frequency of the incident field $\omega = 1$ and the incidence direction $\mathbf{d_0} = (0,1)^{\mathrm{T}}$ fixed. In other words, we reconstruct with data on the k-space coverage (6.13):

$$\mathscr{U} = \left\{ \begin{pmatrix} (\kappa - k_0)\sin(t) + k_1\cos(t) \\ (\kappa - k_0)\cos(t) - k_1\sin(t) \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; k_0^2 > k_1^2,\ t \in [0,T] \right\}.$$

The following figures show the reconstruction of the full turn and the half turn rotation of an object (cf. Figure 9 in Subsection 6.3) with the artefacts on some parts of the boundary in the latter case.

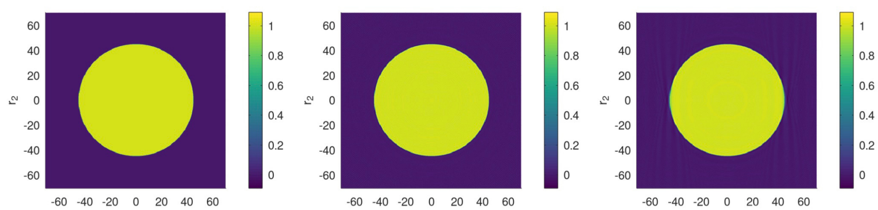

Figure 15: [12] Data generating using full waveform method and reconstruction using non-uniform discrete Fourier transform. The frequency of the incident field is $\omega = 1$. The incidence direction is $\mathbf{d_0} = (0,1)^{\mathrm{T}}$. The object is rotated using the standard rotation matrix. Left: Original function $f(\mathbf{r}) = \mathbb{1}_{B_{45}(\mathbf{0})}(\mathbf{r})$. Center: Reconstruction from full turn rotation ($t \in [0, 2\pi]$). Right: Reconstruction from half turn rotation ($t \in [0, \pi]$).

Appendix A Definitions

Definition A.1 (Semi-norm)**.**
Let X be a vector space over $\mathbb{K}$, where $\mathbb{K}$ is a subfield of $\mathbb{C}$. A semi-norms on X is a map $p : X \to \mathbb{R}$ that satisfies

- $p_{\alpha,\beta}(\psi) \geq 0$,
- $p_{\alpha,\beta}(\lambda\psi) = |\lambda| p_{\alpha,\beta}(\psi)$ for all $\lambda \in \mathbb{C}$,
- $p_{\alpha,\beta}(\psi_1 + \psi_2) \leq p_{\alpha,\beta}(\psi_1) + p_{\alpha,\beta}(\psi_2)$.

Definition A.2 (Locally convex space)**.**
A vector space X over $\mathbb{K}$, where $\mathbb{K}$ is a subfield of $\mathbb{C}$, along with a family $\mathscr{P}$ of semi-norms on X is called a locally convex space.

Definition A.3 (Topological isomorphism)**.**
Let X, Y be topological spaces. We call a continuous map $f : X \to Y$ an isomorphism if there exists a function $g : Y \to X$ such that $f \circ g = id_X$ and $g \circ f = id_Y$.

Definition A.4 (Approximate identity)**.**
An approximate identity is a family $(u_\epsilon)_{\epsilon>0} \subseteq L^1(\mathbb{R}^n)$ such that

(i) $\exists c > 0$ such that $\|u_\epsilon\|_{L^1} \leq c$ for all $\epsilon > 0$

(ii) $\int u_\epsilon = 1$ for all $\epsilon > 0$

(iii) for any neighborhood U of 0,

$$\int_{X \setminus U} |u_\epsilon| \xrightarrow{\epsilon \to 0} 0.$$

Definition A.5 (Tensor products of Hilbert spaces)**.** Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces. For each $\psi_1 \in \mathscr{H}_1$, $\psi_2 \in \mathscr{H}_2$, we define the conjugate bilinear form

$$\begin{aligned} \psi_1 \otimes \psi_2 : \mathscr{H}_1 \times \mathscr{H}_2 &\to \mathbb{R} \\ (\varphi_1, \varphi_2) &\mapsto \langle \varphi_1, \psi_1 \rangle \langle \varphi_2, \psi_2 \rangle. \end{aligned}$$

Let $\mathscr{E} := \operatorname{span}\big(\{\psi_1 \otimes \psi_2 \mid \psi_1 \in \mathscr{H}_1, \psi_2 \in \mathscr{H}_2\}\big)$. The inner product on $\mathscr{E}$, $\langle \cdot, \cdot \rangle_{\mathscr{E}}$, is defined by

$$\langle \eta_1 \otimes \eta_2, \zeta_1 \otimes \zeta_2 \rangle_{\mathscr{E}} := \langle \eta_1, \zeta_1 \rangle \langle \eta_2, \zeta_2 \rangle,$$

where $\eta_1, \zeta_1 \in \mathscr{H}_1$ and $\eta_2, \zeta_2 \in \mathscr{H}_2$. $\langle \cdot, \cdot \rangle_{\mathscr{E}}$ is well-defined and positive definite (cf. Appendix B). Then, the tensor products $\mathscr{H}_1 \otimes \mathscr{H}_2$ of the two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ is defined to be $\mathscr{E}$ endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathscr{E}}$.

Definition A.6 (Bessel functions of the first kind and second kind)**.** For $\nu \geq 0$, we define the Bessel function of the first kind of order ν as

$$J_\nu(t) := \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu + k + 1) k!} \left(\frac{t}{2}\right)^{2k}$$

and the Bessel function of the second kind of order ν as

$$Y_\nu(t) := \frac{J_\nu(t)\cos(\nu\pi) - J_{-\nu}(t)}{\sin(\nu\pi)}.$$

Definition A.7 (Hankel functions)**.** The Hankel functions of first and second kind (or Bessel functions of the third kind) are defined as

$$H_\nu^{(1)}(x) := J_\nu(x) + iY_\nu(x) \quad x > 0,$$
$$H_\nu^{(2)}(x) := J_\nu(x) - iY_\nu(x) \quad x > 0,$$

respectively.

Appendix B Results

Theorem B.1 (Young's convolutional inequality)**.** *For $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, the convolution*

$$(f * g)(\mathbf{r}) := \int_{\mathbb{R}^n} f(\mathbf{r}')g(\mathbf{r} - \mathbf{r}') \, \mathrm{d}\mathbf{r}' \quad \in L^p(\mathbb{R}^n)$$

is well-defined a.e. and the Young's inequality

$$\|f * g\|_{L^p} \leq \|f\|_{L^1}\|g\|_{L^p}$$

holds.

Proof. Theorem 2.2.4 in [17]. □

Theorem B.2 (Convolution with an approximate identity)**.** *Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^n)$. If $(u_\epsilon)_{\epsilon>0}$ is an approximate identity, then*

$$u_\epsilon * f \xrightarrow{\epsilon\to\infty} f$$

in $L^p(\mathbb{R}^n)$.

Proof. Theorem 2.3.4 in [17]. □

Theorem B.3 (Fubini's theorem)**.** *Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ be Lebesgue measurable subsets and $f : X \times Y \to \mathbb{R} \cup \{-\infty, \infty\}$.*

(i) If $f \in L^1(X \times Y)$, then, we have

$$x \mapsto \int_Y f(x,y) \, \mathrm{d}y \ \in L^1(X),$$

$$y \mapsto \int_X f(x,y) \, \mathrm{d}x \ \in L^1(Y),$$

and the following equalities hold:

$$\int_{X\times Y} f(x,y) \, \mathrm{d}(x,y) = \int_X \left(\int_Y f(x,y) \, \mathrm{d}y\right) \, \mathrm{d}x = \int_Y \left(\int_X f(x,y) \, \mathrm{d}x\right) \mathrm{d}y.$$

(ii) If f is Lebesgue measurable on $X \times Y$, then,

$$x \mapsto \int_Y |f(x,y)| \, \mathrm{d}y \quad \text{and} \quad y \mapsto \int_X |f(x,y)| \, \mathrm{d}x$$

are Lebesgue measurable on X and Y, respectively. The following equalities also hold:

$$\int_{X\times Y} |f(x,y)| \mathrm{d}(x,y) = \int_X \left(\int_Y |f(x,y)| \, \mathrm{d}y \right) \mathrm{d}x = \int_Y \left(\int_X |f(x,y)| \mathrm{d}x \right) \mathrm{d}y.$$

If any of the integration above is finite, then $f \in L^1(X \times Y)$.

Proof. Theorem 2.4.11 in [19]. □

Theorem B.4 (Lebesgue's dominated convergence theorem)**.** *Let $(f_k)_k$ be a sequence of functions in $L^1(\Omega)$ such that for almost all $\mathbf{r} \in \Omega$,*

$$\lim_{k\to\infty} f_k(\mathbf{r}) = f(\mathbf{r})$$

for a function $f : \Omega \to \mathbb{C}$ in $L^1(\Omega)$. Suppose that there exists an integrable function $g \in L^1(\Omega)$ such that

$$|f_k| \leq g$$

almost everywhere for all $k \in \mathbb{N}$. Then, $f \in L^1(\Omega)$ and

$$\lim_{k\to\infty} \int_\Omega f_k(\mathbf{r}) \, \mathrm{d}\mathbf{r} = \int_\Omega f(\mathbf{r}) \, \mathrm{d}\mathbf{r}.$$

Proof. Theorem 2.3.14 in [19]. □

Theorem B.5 (Multivariable Leibniz's formula)**.** *Let f, g be elements of $\mathscr{C}^k(\Omega)$ for some open set $\Omega \subseteq \mathbb{R}^n$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}_0^n$ be a multi-index, with $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq k$. We have*

$$\partial^\alpha (fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g).$$

Here,

$$\beta \leq \alpha \;\; :\Leftrightarrow \;\; \alpha_i \leq \beta_i \;\; \forall i = 1, ..., n,$$

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_n}{\beta_n},$$

and

$$\partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}.$$

Proof. Follows by mathematical induction:
For $n = 1$, it is the Leibniz's formula in one variable

$$\partial^m(fg) = \sum_{j=0}^{m} \binom{m}{j} (\partial^j f)(\partial^{m-j} g).$$

For the induction step, suppose that the result is true for $n-1$ variables:

$$\begin{aligned}&\partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}(fg)\\ &= \sum_{\beta_2=0}^{\alpha_2} \cdots \sum_{\beta_n=0}^{\alpha_n} \binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_n}{\beta_n} \left((\partial_2^{\beta_2} \cdots \partial_n^{\beta_n} f)(\partial_2^{\alpha_2-\beta_2} \cdots \partial_n^{\alpha_n-\beta_n} g) \right).\end{aligned}$$

Then, applying $\partial_1^{\alpha_1}$ to both sides gives

$$\begin{aligned}&\partial_1^{\alpha_1}\partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}(fg)\\ &= \sum_{\beta_2=0}^{\alpha_2} \cdots \sum_{\beta_n=0}^{\alpha_n} \binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_n}{\beta_n} \partial_1^{\alpha_1} \left((\partial_2^{\beta_2} \cdots \partial_n^{\beta_n} f)(\partial_2^{\alpha_2-\beta_2} \cdots \partial_n^{\alpha_n-\beta_n} g) \right).\end{aligned} \tag{B.1}$$

Then, we use Leibniz's formula for the case $n = 1$ to get

$$\begin{aligned}&\partial_1^{\alpha_1} \left((\partial_2^{\beta_2} \cdots \partial_n^{\beta_n} f)(\partial_2^{\alpha_2-\beta_2} \cdots \partial_n^{\alpha_n-\beta_n} g) \right)\\ &= \sum_{\beta_1=0}^{\alpha_1} \binom{\alpha_1}{\beta_1} (\partial_1^{\beta_1}\partial_2^{\beta_2} \cdots \partial_n^{\beta_n} f)(\partial_1^{\alpha_1-\beta_2}\partial_2^{\alpha_2-\beta_2} \cdots \partial_n^{\alpha_n-\beta_n} g)\end{aligned}$$

and insert it in equation (B.1). Since all the sums are finite, we can moves $\sum_{\beta_1=0}^{\alpha_1}$ to the far left to get $\sum_{\beta_1=0}^{\alpha_1} \sum_{\beta_2=0}^{\alpha_2} \cdots \sum_{\beta_n=0}^{\alpha_n}$ on the left-hand side of equation (B.1) and arrive at the assertion. □

Proposition B.6 (Inner product of $\mathscr{H}_1 \otimes \mathscr{H}_2$)**.** *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces. The inner product $\langle \cdot, \cdot \rangle$,*

$$\langle \eta_1 \otimes \eta_2, \zeta_1 \otimes \zeta_2 \rangle := \langle \eta_1, \zeta_1 \rangle \langle \eta_2, \zeta_2 \rangle,$$

where $\eta_1, \zeta_1 \in \mathscr{H}_1$ and $\eta_2, \zeta_2 \in \mathscr{H}_2$, is well-defined and positive definite.

Proof. Chapter II, Proposition 1 in [8]. □

Proposition B.7 (Orthonormal basis of $\mathscr{H}_1 \otimes \mathscr{H}_2$)**.** *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and $\{\varphi_k\}_k$ and $\{\psi_l\}_l$ be the orthonormal bases of $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. Then $\{\varphi_k \otimes \psi_l\}_{k,l}$ is an orthonormal basis of $\mathscr{H}_1 \otimes \mathscr{H}_2$.*

Proof. Chapter II, Proposition 2 in [8]. □

Theorem B.8 (Weierstrass Approximation Theorem)**.** *Let $f \in \mathscr{C}([a,b],\mathbb{R})$ be given. Then, for every $\epsilon > 0$, there exists a polynomial $P : [a,b] \to \mathbb{R}$ such that*

$$\|f - p\|_\infty < \epsilon.$$

In the other words, the space of polynomials $\mathscr{P}([a,b],\mathbb{R})$ is dense in $\mathscr{C}([a,b],\mathbb{R})$.

Proof. Theorem 2.1.1 in [21]. □

Theorem B.9 (Solution of Bessel differential equation)**.** *For $\nu \geq 0$, the Bessel functions of the first and second kind of order ν are linear independent solutions of the Bessel differential equation*

$$\left(x^2\partial_x^2 + x\partial_x + (x^2 - \nu^2)\right) f = 0.$$

Hence, the Hankel functions also solve the Bessel differential equation.

Proof. cf. [31]. □

Theorem B.10 (Poisson's integral representation of Bessel function)**.** *For $\nu \in \mathbb{C}$ with $\Re(\nu) > -\dfrac{1}{2}$, we have*

$$J_\nu(t) = \frac{1}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{t}{2}\right)^\nu \int_0^\pi e^{-it\cos(\theta)} \sin^{2\nu}(\theta)\ \mathrm{d}\theta.$$

Proof. Theorem 3 in [31]. □

Theorem B.11 (Integral representation of Hankel functions)**.** *For $\nu \in \mathbb{C}$ with $\Re(\nu) > -\dfrac{1}{2}$, we have*

$$H_\nu^{(1)}(t) = \frac{\Gamma\left(\frac{1}{2} - \nu\right)}{\pi i \Gamma\left(\frac{1}{2}\right)} \left(\frac{t}{2}\right)^\nu \left(1 - e^{-2\left(\nu - \frac{1}{2}\right)\pi i}\right) \int_1^\infty e^{itp}(p^2 - 1)^{\nu - \frac{1}{2}}\ \mathrm{d}p$$

and

$$H_\nu^{(2)}(t) = -\frac{\Gamma\left(\frac{1}{2} - \nu\right)}{\pi i \Gamma\left(\frac{1}{2}\right)} \left(\frac{t}{2}\right)^\nu \left(1 - e^{2\left(\nu - \frac{1}{2}\right)\pi i}\right) \int_1^\infty e^{-itp}(p^2 - 1)^{\nu - \frac{1}{2}}\ \mathrm{d}p.$$

Proof. 6.13 in [9]. □

Lemma B.12 (Property of Bessel function of the first kind)**.** *For $\nu \geq 0$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x).$$

Proof. From the definition of the Bessel function of the first kind of order ν, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^\nu J_\nu(x)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[x^\nu \left(\frac{x}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(\nu + k + 1)k!}\left(\frac{x}{2}\right)^{2k}\right],$$

which equals

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu+k+1)k!}\frac{x^{2\nu+2k}}{2^{\nu+2k}}\right]=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu+k+1)k!}\frac{1}{2^{\nu+2k}}\left[\frac{\mathrm{d}}{\mathrm{d}x}x^{2\nu+2k}\right].$$

Calculating the derivative $\frac{\mathrm{d}}{\mathrm{d}x}x^{2\nu+2k}$ gives

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu+k+1)k!}\frac{(2\nu+2k)x^{2\nu+2k-1}}{2^{\nu+2k}},$$

which equals

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu+k)k!}\frac{x^{2\nu+2k-1}}{2^{\nu+2k-1}}=x^{\nu}\left(\frac{x}{2}\right)^{\nu-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu+k)k!}\left(\frac{x}{2}\right)^{2k}=x^{\nu}J_{\nu-1}(x)$$

and we arrive at the assertion.

□

Theorem B.13 (Representation of Bessel function as Fourier series coefficients)**.** *For $\nu\in\mathbb{Z}$, we have*

$$J_\nu(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i\nu\theta}e^{it\sin(\theta)}\ \mathrm{d}\theta.$$

Therefore,

$$J_\nu(0)=\begin{cases}1 & \textit{for } \nu=0\\ 0 & \textit{otherwise}\end{cases}.$$

Proof. The first equality is proved in [31]. Setting $t=0$ yields

$$J_\nu(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i\nu\theta}\ \mathrm{d}\theta,$$

which is 0 for all integer $\nu\neq 0$ and 1 for $\nu=0$. □

Theorem B.14 (Residue Theorem)**.** *Let $U\subseteq\mathbb{C}$ be a domain, A be closed in U without any accumulation point in U and γ be a cycle, i.e., a sum of closed curves, in $U\backslash A$ that is homologous to zero in U, i.e., $\mathrm{ind}_\gamma(z)=0$ for all $z\in\mathbb{C}\backslash U$, where ind_γ denotes the winding number of $z\notin|\gamma|$ with respect to the cycle γ,*

$$\mathrm{ind}_\gamma(z):=\frac{1}{2\pi i}\int_\gamma\frac{1}{\xi-z}\ \mathrm{d}\xi.$$

Then, for any f holomorphic in $U\backslash A$, which has either a removable singularity or a pole at each point of A, we have that the set $\{a\in A\mid \mathrm{ind}_\gamma(a)\neq 0\}$ is finite and

$$\frac{1}{2\pi i}\int_\gamma f\ \mathrm{d}z=\sum_{a\in A}\mathrm{Res}(f;a)\mathrm{ind}_\gamma(a),$$

where $\operatorname{Res}(f;a)$ *is the residue of* f *at* a*, defined as the coefficient* c_{-1} *of the Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n, \qquad 0 < |z-a| < r$$

of f*, where* $r > 0$ *such that* $\overline{\mathscr{B}_r(a)} \subseteq U$ *and* $\overline{\mathscr{B}_r(a)} \cap A = a$*.*

Proof. Theorem 8.1 in [22]. □

Corollary B.15 (Green's function of Helmholtz operator)**.** *The Green's function of the Helmholtz operator* $-(\Delta + k_0)$ *that satisfies the Sommerfeld radiation condition is*

(i) $G_1(r) = \dfrac{ie^{ik_0|r|}}{2k_0}$ *in the* 1*-dimensional case,*

(ii) $G_2(\mathbf{r}) = \dfrac{i}{4}H_0^{(1)}(k_0\|\mathbf{r}\|)$ *in the* 2*-dimensional case, where* $H_0^{(1)}$ *is the Hankel function of the first kind of order zero, and*

(iii) $G_3(\mathbf{r}) = \dfrac{ie^{ik_0\|\mathbf{r}\|}}{4\pi\|\mathbf{r}\|}$ *in the* 3*-dimensional case.*

Proof. 1. Consider first the one-dimensional case. By definition, the Green's function of the Helmholtz operator must solve the Helmholtz equation

$$-\left(\frac{\mathrm{d}}{\mathrm{d}r^2} + k_0^2\right) G(r-r') = \delta(r-r'),$$

where δ is the Dirac delta function. For simplicity, write $\rho = r - r'$ and get

$$\left(\frac{\mathrm{d}}{\mathrm{d}r^2} + k_0^2\right) G(\rho) = -\delta(\rho).$$

Then, taking the Fourier transform on both sides yields

$$(-k^2 + k_0^2)\mathscr{F}G(k) = -\frac{1}{\sqrt{2\pi}}, \tag{B.2}$$

using the fact that

$$\mathscr{F}(\delta) = \frac{1}{\sqrt{2\pi}}.$$

Indeed, for $\mathscr{F} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$, we have that

$$\mathscr{F}^{-1}1(\phi) := 1(\mathscr{F}^{-1}\phi) = \int_{\mathbb{R}^n} 1(\mathscr{F}^{-1}\phi)(\mathbf{r})\ \mathrm{d}\mathbf{r},$$

which equals

$$\frac{(\sqrt{2\pi})^n}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} (\mathscr{F}^{-1}\phi)(\mathbf{r})e^{-i\mathbf{r}\cdot 0}\ \mathrm{d}\mathbf{r} = (\sqrt{2\pi})^n\mathscr{F}(\mathscr{F}^{-1}\phi)(0) = (\sqrt{2\pi})^n\phi(0).$$

Using definition of the n-dimensional Dirac-delta distribution, we get

$$(\sqrt{2\pi})^n \phi(0) = (\sqrt{2\pi})^n \delta_0 = (\sqrt{2\pi})^n \delta.$$

Then, using $\mathscr{F}^{-1}\mathscr{F} = \mathscr{F}\mathscr{F}^{-1} = \mathscr{I}$, we get

$$\mathscr{F}\mathscr{F}^{-1} 1 = (\sqrt{2\pi})^n \mathscr{F}\delta,$$

which is equivalent to

$$\mathscr{F}\delta = \frac{1}{(\sqrt{2\pi})^n}.$$

In our case, $n = 1$. Thus, from (B.2), we have

$$\mathscr{F}G(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{k^2 - k_0^2}.$$

Taking the inverse Fourier transform of both sides, we get

$$G(\rho) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik\rho}}{k^2 - k_0^2} \, \mathrm{d}k.$$

Since $k \mapsto e^{ik\rho}$ is entire, the map $k \mapsto \dfrac{e^{ik\rho}}{k^2 - k_0^2}$ has only two simple poles at $\pm k_0$, which lie in the real axis. On one hand, one could apply Residue theorem (Theorem B.14) directly to calculate this integral along the contour depicted in the following figure:

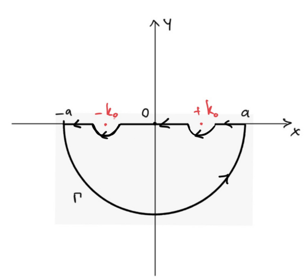

Figure 16: Contour using to calculate the integral directly.

On the other hand, for simplicity, one could consider k_0 as a limit of $k_0 + i\epsilon$ for $\epsilon > 0$ as $\epsilon \to 0$ instead. Thus, $G(\rho)$ becomes

$$G(\rho) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik\rho}}{k^2 - (k_0 + i\epsilon)^2} \, \mathrm{d}k,$$

and the two simple poles are now $\pm(k_0 + i\epsilon)$. Now, to determine the integral

$$\int_{-\infty}^{\infty} \frac{e^{ik\rho}}{k^2 - (k_0 + i\epsilon)^2} \, \mathrm{d}k,$$

we first consider the case $\rho > 0$ and calculate the contour integral

$$\int_C \frac{e^{i\rho z}}{z^2 - (k_0 + i\epsilon)^2} \, \mathrm{d}z,$$

where C is a contour that goes along the real line from $-a$ to a, $a > k$, and then goes counterclockwise along a semicircle centered at 0 from a to $-a$. We note that only $k_0 + i\epsilon$ lies in the region bounded by this contour. The contour is depicted in the following figure:

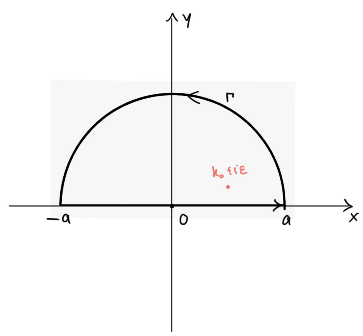

Figure 17: Contour C in the case $\rho > 0$.

The Residue theorem yields

$$\begin{aligned}
\int_C \frac{e^{i\rho z}}{z^2 - (k_0 + i\epsilon)^2} \, \mathrm{d}z &= 2\pi i \mathrm{Res}\left(\frac{e^{i\rho z}}{z^2 - (k_0 + i\epsilon)^2}; k_0 + i\epsilon\right) \\
&= 2\pi i \lim_{z \to k_0 + i\epsilon} \left[z - (k_0 + i\epsilon)\right] \frac{e^{i\rho z}}{z^2 - (k_0 + i\epsilon)^2} \\
&= 2\pi i \frac{e^{i(k_0 + i\epsilon)\rho}}{2(k_0 + i\epsilon)}.
\end{aligned}$$

Now, split the contour C into the interval $[-a, a]$ and the arc Γ (cf. Figure 17), we get

$$\int_{-a}^{a} \frac{e^{i\rho z}}{z^2 - (k_0 + i\epsilon)^2} \, \mathrm{d}z = \int_C \frac{e^{i\rho z}}{z^2 - (k_0 + i\epsilon)^2} \, \mathrm{d}z - \int_\Gamma \frac{e^{i\rho z}}{z^2 - (k_0 + i\epsilon)^2} \, \mathrm{d}z.$$

Since

$$\left| \int_\Gamma \frac{e^{i\rho z}}{z^2 - (k_0 + i\epsilon)^2} \, \mathrm{d}z \right| \leq \pi a \sup_\Gamma \left| \frac{e^{i\rho z}}{z^2 - (k_0 + i\epsilon)^2} \right|$$
$$\leq \pi a \sup_\Gamma \frac{1}{\left| z^2 - (k_0 + i\epsilon)^2 \right|}$$
$$\leq \frac{\pi a}{a^2 - |k_0 + i\epsilon|^2} \xrightarrow{a \to \infty} 0,$$

we obtain

$$\int_{-\infty}^{\infty} \frac{e^{i\rho k}}{k^2 - (k_0 + i\epsilon)^2} \, \mathrm{d}k = 2\pi i \frac{e^{i(k_0 + i\epsilon)\rho}}{2(k_0 + i\epsilon)}.$$

Analogously, for $\rho < 0$, we choose another contour C as in the following figure.

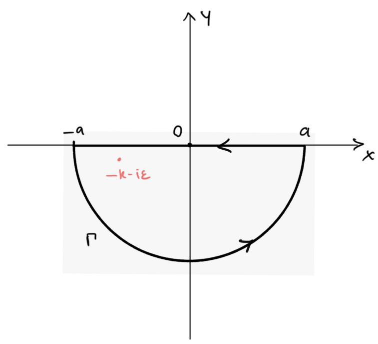

Figure 18: Contour C in the case $\rho < 0$.

From the same argument, we get

$$\int_{-\infty}^{\infty} \frac{e^{i\rho k}}{k^2 - (k_0 + i\epsilon)^2} \, \mathrm{d}k = 2\pi i \frac{e^{-i(k_0 + i\epsilon)\rho}}{2(k_0 + i\epsilon)}.$$

Taking the limit $\epsilon \to 0$, we arrive to the assertion that

$$G(\rho) = \frac{i e^{i k_0 |\rho|}}{2k_0}.$$

Note that

$$G_\epsilon(\rho) = \frac{i e^{i(k_0 + i\epsilon)|\rho|}}{2(k_0 + i\epsilon)}$$

is the Green's function of the viscosity Helmholtz operator $-\left(\Delta + (k_0 + i\epsilon)^2\right)$.

2. Now, consider the 2-dimensional Helmholtz equation

$$-(\Delta + k_0^2)G(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}').$$

Because of the translational invariance property of this partial differential equation, we can introduce the new coordinate $\rho = \mathbf{r} - \mathbf{r}'$. Furthermore, since δ is a radial function, $G(\rho)$ only depends on $\|\rho\|$. Therefore, we can write ρ instead of $\|\rho\|$ for simplicity.

Analogous to the first case, we take the Fourier transforms of both sides of $-(\Delta + k_0)G(\rho) = \delta(\rho)$ to get

$$\mathscr{F}G(\mathbf{k}) = \frac{1}{2\pi} \frac{1}{\|\mathbf{k}\|^2 - k_0^2},$$

then apply the inverse Fourier transform to obtain

$$G(\rho) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{i\mathbf{k}\cdot\rho}}{\|\mathbf{k}\|^2 - k_0^2} \, \mathrm{d}\mathbf{k}.$$

This integral can also be evaluated as part of a contour integral, resulting

$$G(\rho) = \frac{1}{\pi^2} \int_1^\infty \frac{\pi}{2} \frac{e^{ik_0\rho t}}{\sqrt{t^2 - 1}} \, \mathrm{d}t = \frac{i}{4} H_0^{(1)}(k_0\rho),$$

using the integral representation of the first Hankel function of order zero

$$H_0^{(1)}(x) = \frac{-2i}{\pi^2} \int_1^\infty \frac{e^{itx}}{\sqrt{t^2 - 1}} \, \mathrm{d}t,$$

cf. Theorem B.11. The calculation of the contour integral can be found in Section 3.2 of [15].

3. Already proved in Corollary 5.8.

□

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