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„TQFTs with additional structure"

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Abstract

Topological quantum field theories (TQFTs) are quantum field theories where only the global structure of spacetime plays a role. In this thesis we study TQFTs in the presence of (topological) defects, and tangential structures, such as orientations or spin structures. After a review of the relevant mathematical notions, and some simpler variants of TQFTs, we propose a definition for the appropriate bordism category underlying such field theories. As an important example we then focus on two dimensional defect TQFTs with spin structures. For these we construct a 2-category. For this 2-category we find a pivotal structure like in the oriented case. In addition we further find a 2-endofunctor coming from deck transformations of the underlying spin bundles.

Zusammenfassung

Topologische Quantenfeldtheorien (TQFTs) sind Quantenfeldtheorien in denen nur
die globale Struktur der zugrunde liegenden Raumzeit eine Rolle spielt. In dieser Ar-
beit untersuchen wir TQFTs mit (topologischen) Defekten und tangentialen Struk-
turen, wie Orientierungen oder Spinstrukturen. Nachdem wir die relevanten math-
ematischen Begriffe einführen und einen Überblick über simplere Varienten von
TQFTs geben, schlagen wir eine mögliche Definition der Bordismen Kategorien, die
solchen Theorien zugrunde liegen, vor. Anschließend konzentrieren wir uns auf zwei
dimensionale defekt TQFTs mit Spinstrukturen, welche ein wichtiges Beispiel der
allgemeinen Theorien bilden. Für diese TQFTs konstruieren wir eine 2-Kategorie
mit zusätzlicher Struktur. Insbesondere finden wir eine pivotale Struktur wie im
orientierten Fall. Zusätzlich finden wir einen 2-Endofunktor welcher durch Deck-
transformationen auf den zugrunde liegenden Spinbündeln induziert wird.

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Contents

Introduction

Quantum field theories (QFTs) are an indispensable tool in modern theoretical physics with applications ranging from particle physics over cosmology to condensed matter systems and quantum optics. However even though QFTs are such a widely accepted tool in physics a precise mathematical understanding of them remains elusive. Finding a rigorous definition and constructing non-trivial examples of QFTs is thus one of the most prominent questions in modern mathematical physics. There are various ways to work towards such a definition and a complete list of these would be beyond the scope of this thesis. We will follow the categorical approach called *functorial field theory*, this approach to QFTs goes back to Segal's axiomatic definition of 2-dimensional conformal field theories [Seg04], and Atiyah's and Witten's work on topological quantum field theories [Ati88; Wit88] in the 1980s. To motivate the functorial definition one usually makes a heuristic argument using path integrals. In the following we will sketch such an argument along the lines of [CR18, Section 2.1] and [Bar05, Chapter 1]:

Path integral motivation

First we need to choose a *spacetime*, this means a (compact) Lorentzian manifold M with metric g.[1] Next we need to choose a set of *fields* Φ, for example a single scalar field would be modeled by a $\phi \in C^\infty(M, \mathbb{R})$.[2] Furthermore we need a *action functional* $S[\Phi]$, usually given as

$$S[\Phi] = \int_M \mathcal{L}(\Phi, \nabla\Phi) \mathrm{dvol}_g \tag{0.0.1}$$

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[1] For simplicity let us assume that M has empty boundary, otherwise we would need to specify boundary conditions for the fields.

[2] For more complicated theories like gauge theories, such as the standard model of particle physics, we would need further choices such as a compact Lie group G and a principal G-bundle P over M.

with dvol_g the volume form on M and $\mathcal{L}(\Phi, \nabla\Phi)$ the *Lagrangian density* which depends on Φ and its first derivative $\nabla\Phi$. For a free massless scalar field the Lagrangian density would be

$$\mathcal{L}(\phi, \nabla\phi) = \frac{1}{2}g(\nabla\phi, \nabla\phi). \tag{0.0.2}$$

Up to now everything is classical and can be formulated completely rigorously. The *path integral* or *partition function* is given as

$$\mathcal{Z} = \int D\Phi \, e^{iS[\Phi]} \tag{0.0.3}$$

where $\int D\Phi$ should mean "the integral over all fields Φ". We can use the path integral to compute the amplitude that a state $|\Phi_1\rangle$ at time t_1 evolves into another state $\langle\Phi_2|$ at time t_2 as

$$\langle\Phi_2|U|\Phi_1\rangle = \int_{\Phi|_{\Sigma_1}=\Phi_1}^{\Phi|_{\Sigma_2}=\Phi_2} D\Phi \, e^{iS[\Phi]} \tag{0.0.4}$$

where Σ_1 and Σ_2 are the spatial hypersurfaces of M at time t_1 and t_2, respectively. This prescription reads as follows: "Sum over all field configurations Φ which restrict to Φ_1 and Φ_2 on Σ_1 and Σ_2, respectively, and weight each contribution by $e^{iS[\Phi]}$." From this formula we can see two things:

1. A spatial hypersurface Σ corresponds to an Hilbert space of states $\mathcal{H}_\Sigma$.

2. A submanifold N of M "between" two spatial hypersurfaces Σ_1 and Σ_2 as above corresponds to a *time evolution operator*

$$U(t_2, t_1) \colon \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}. \tag{0.0.5}$$

The submanifold N is called a *bordism* from Σ_1 to Σ_2, a precise definition will be given in Section 1.4. The picture behind this heuristic description should be the

following:

(0.0.6)
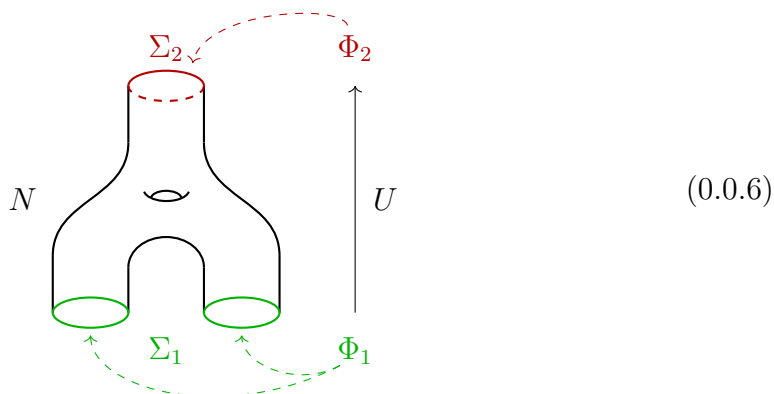
where the dashed arrows signify that the Φ_i are fields localised on the Σ_i. From this heuristic argument we expect a QFT to be something like a map:

$$\text{Spacetimes} \longrightarrow \text{Algebra}$$
$$\text{spatial hypersurface } \Sigma \longmapsto \text{space of states } \mathcal{H}_\Sigma \qquad (0.0.7)$$
$$\text{bordism } N \longmapsto \text{time evolution operator } U,$$

which preserves certain structures. Let us call this map $\mathcal{Z}$, in reminiscence to the path integral from above. To understand the nature of $\mathcal{Z}$ let us consider some properties it should satisfy:

Firstly we want that evolving from Φ_1 to Φ_2 along N and then further evolving to Φ_3 along N' should be the same as evolving straight from Φ_1 to Φ_3 along the "glued bordism" $N' \circ N$. For $\mathcal{Z}$ this means that

$$\mathcal{Z}(N_2 \circ N_1) = \mathcal{Z}(N_2) \circ \mathcal{Z}(N_1) \qquad (0.0.8)$$

should hold. This equation is precisely *functoriality* of $\mathcal{Z}$.

For the second property suppose that the spatial hypersurface Σ decomposes into a disjoint union $\Sigma = \Sigma_1 \sqcup \Sigma_2$ then the fields Φ should also decompose into fields Φ_1 and Φ_2 on Σ_1 and Σ_2, respectively. This means the space of states should satisfy

$$\mathcal{Z}(\Sigma) = \mathcal{Z}(\Sigma_1) \otimes_\mathbb{C} \mathcal{Z}(\Sigma_2). \qquad (0.0.9)$$

Furthermore if we assume $N = N_1 \sqcup N_2$, then we expect the time evolution to decompose, from this we find that

$$\mathcal{Z}(N_1 \sqcup N_2) = U_1 \otimes_\mathbb{C} U_2, \qquad (0.0.10)$$

3

here U_1 and U_2 are the time evolution operators of Φ_1 and Φ_2 along N_1 and N_2, respectively. Equations (0.0.9) and (0.0.10) mean that $\mathcal{Z}$ should be *monoidal*. This captures the quantum nature of the theory, as in quantum theories the state space of a composite system is the tensor product of the state spaces of the components. Moreover $\Sigma_1 \sqcup \Sigma_2 \cong \Sigma_2 \sqcup \Sigma_1$ and $\mathcal{Z}$ should respect this symmetry property, i.e. we expect an isomorphism

$$\mathcal{Z}(\Sigma_1) \otimes_{\mathbb{C}} \mathcal{Z}(\Sigma_2) \cong \mathcal{Z}(\Sigma_2) \otimes_{\mathbb{C}} \mathcal{Z}(\Sigma_1). \tag{0.0.11}$$

The precise choice of this isomorphism depends on whether the fields are bosonic or fermionic. This property states that $\mathcal{Z}$ is *symmetric*.

Finally we expect an inner product on the state spaces and the corresponding notion of adjoint operators. These algebraic structures are related to orientation reversal on the geometric side and are more carefully motivated in [CR18, Section 2.1].

From this motivation we expect that $\mathcal{Z}$ should be a *symmetric monoidal functor* from a geometric bordism category to a category of vector spaces with extra structure, e.g. Hilbert spaces. Such a $\mathcal{Z}$ is called a *functorial field theory*.

Topological quantum field theories

Topological quantum field theories (TQFTs) are QFTs where only the topological, or global, structure of the spacetime plays a role. In physics this is often expressed as the requirement that all correlation functions are independent of the metric, such theories are said to be of *Schwarz type*. A related notion is that of *cohomological* or *Witten type* TQFTs where the action and the stress energy tensor are zero in some certain cohomology, see [Bar05, Chapter 1.3] for more details. For example Chern-Simons theory and BF theory are TQFTs of Schwarz type. In the functorial approach to QFTs a TQFT is a symmetric monoidal functor

$$\mathcal{Z} \colon \mathrm{Bord} \to \mathrm{Vect}_{\mathbb{C}} \tag{0.0.12}$$

where the manifolds in the bordism category Bord only contain topological structure such as an orientation or spin structure, and no geometric structure such as a metric or a connection. This "simplicity" on the geometric side is a main reason TQFTs can be defined and studied rigorously. Despite this they are still non-trivial and arise naturally in many areas of both physics and pure mathematics:

- In theoretical high energy physics as twists of certain supersymmetric conformal field theories, these TQFTs are of Witten-type [Hor+03, Chapter 16];

4

- in condensed matter physics in the classification of topological phases of matter [FH19; FH21];

- in topological quantum computing, where the worldlines of anyonic quasiparticles appear as quantum gates. The topological nature of their effective interaction are modelled by specific TQFTs [DFN06];

- in algebraic topology as topological invariants of manifolds per definition. Moreover some classes of TQFTs, such as Chern-Simons theory, are related to knot invariants [Wit89];

- as relations between the traditionally separate fields of algebra, geometry, topology and number theory, more specifically those connected to "homological mirror symmetry" or the "geometric Langlands program" [Hor+03].

In general one can differentiate between types of TQFTs depending on the precise details of the bordism category for which they are defined, more specifically there are different (topological) bordism categories depending on the following "variables":

- the *dimension* of bordisms;

- allowed *tangential structures* on bordisms;

- allowed types of *boundaries* or *stratifications*;

- *categorical degree*;

We will only very briefly describe how these variables enter the bordism category and which effect varying them has for the algebraic description of the TQFTs, full definitions will be given throughout the main text.

How the dimension $n \in \mathbb{Z}_+$ enters is straightforward to see. In the algebraic description the dimension is in parts responsible for the "categorical degree" of the algebraic structure. Very roughly the connection is as follows: The higher the dimension of the bordism category, the richer the algebraic structure of the TQFTs. We will come back to this in the third point.

Tangential structures are a type of extra topological structure on the manifold and can be thought of as a generalisation of orientations and spin structures. Further down we will give a very brief motivation to study spin structures. Tangential structures determine some of the algebraic properties of the TQFTs. In Chapter 2 we will make this more precise and see how spin TQFTs differ from oriented TQFTs.

The third point refers to the type of manifolds underlying the bordism category. In principle these are either manifolds with *boundaries*, *corners*, or *stratifications*.

In the heuristic description above we tacitly assumed that the spatial hypersurfaces Σ_i are *closed*, i.e. compact and without boundaries, this lead to the bordism N being a compact manifold with boundaries. The corresponding bordism categories are called *closed*. The TQFT defined on closed bordism categories are the simplest in the algebraic description, for example in two dimensions they are described by certain algebras, see for example [Koc03] for oriented TQFTs.

If we allow the Σ_i to also have boundaries, we would get that N needs to be a manifold with corners. In this case we should also specify *boundary conditions* for the boundaries of the Σ_i. The corresponding bordism categories are called *open* or *open-closed* depending on whether all objects have boundaries or not. Open 2-dimensional TQFTs are described by certain categories, see [MS09; LP08].

Finally stratified manifolds are needed if we want to describe *defects* in field theories. We will give more motivation for this in the next section and make a small excursion to defects in general QFTs. The algebraic description of defect TQFTs is the richest, it is expected that n-dimensional defect TQFTs are described by n-*categories* [CRS19]. A n-category is a "higher dimensional" version of a category in the sense that there will not only be objects and morphisms, which can be seen as 0- and 1-dimensional things, but objects, 1-morphisms between objects, 2-morphisms between 1-morphisms, and so on until n-morphisms between $(n-1)$-morphisms.

The relation between higher categories and defect TQFTs was made precise in [DKR11] and [CMS20] where for any 2- and 3-dimensional oriented defect TQFT it was shown how to extract a 2- or 3-category, respectively. In Chapter 5 we will construct a 2-category from any 2-dimensional defect spin TQFT. Here we already see how higher categories naturally appear while studying TQFTs.

Higher categories can enter further on a more direct level if we allow the bordism category itself to be a higher category, consequently also the target category of vector spaces needs to be replaced by a suitable higher category and the symmetric monoidal functor by a higher functor. Such TQFTs are called *extended*. Although we will not be concerned with such theories in this thesis, a brief description of these fascinating and rich theories should not be left out.

Defects in QFTs

A physical *defect* is a lower-dimensional region of spacetime which behaves differently then its surroundings. The resulting theory is "defective" in the sense that two regions of spacetime disjoint by a defect could have vastly different physical properties. We call a defect *topological* if geometric details are not necessary to characterise them. Such a loose description of topological defects already captures basic prop-

erties of more complicated physical defects such as domain walls in ferromagnets or cosmic strings in cosmology.

Furthermore for topological defects it is possible to formulate them completely rigorous. In such a rigorous framework they can be used on a more conceptual level, to relate and compare different QFTs. For example dualities (such as mirror symmetry) between different theories are special cases of topological defects [Car18].

To incorporate defects in the functorial approach to TQFTs the bordism category needs to have more "local" structure. More precisely the manifolds should come with embedded submanifolds called *strata*. In analogy to open and open-closed bordisms one further needs to assign labels to these strata. The physical interpretation of these labels depends on the dimension of the strata:

- *domains* or *phases* of the field theory for n-dimensional strata;

- *domain walls* for $(n-1)$-dimensional strata;

- *junctions* for $(n-2)$-dimensional to 1-dimensional strata;

- *local operators* or *junction points* for 0-strata;

with n being the dimension of the whole manifold. For example a stratified manifold without labels could look something like this:

$$(0.0.13)$$
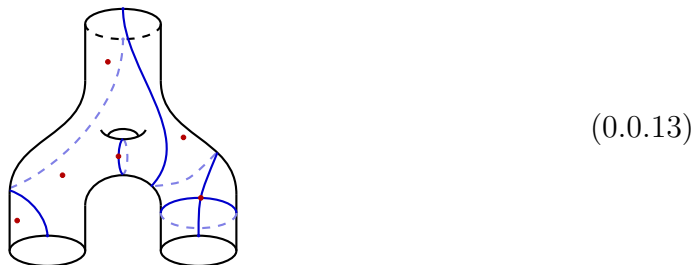
Defect TQFTs are the most general of the TQFTs we described here in the sense that closed, open, and open-closed TQFTs are special cases of them:

7
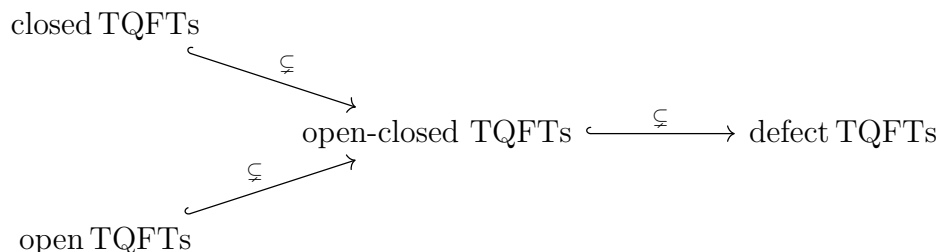
To see that closed TQFTs are special cases of defect TQFTs observe that we can simply "forget" the stratification. For open and open-closed TQFTs we need to assume that there is a "trivial phase", the boundary conditions are then the domain walls between a fixed phase and the trivial phase, see [Car18, Section 3.3] for a detailed explanation.

Spin structures

In this thesis we are mainly interested in TQFTs with *spin structures*. These structures are needed in physics to describe fermionic fields on manifolds through *spinors*. Very roughly one needs a spin structure in order to define the *Dirac operator* on a manifold, see [Wal84, Chapter 13] or [Ham17, Chapter 6] for a thorough motivation.

Spinors are mathematical objects which transform under a full rotation by a change of sign, in terms of the rotation group $SO(n)$ this means that not a 2π rotation but a 4π rotation acts as the identity on spinors. More precisely a *spinorial representation* of $SO(n)$ is a representation of its double covering group $Spin(n)$ the *spin group*, and spinors are vectors in the representation space.

Spin structures also play an important role in mathematics, for example the *Atiyah-Singer index theorem* is a statement that relates analytic properties of the Dirac operator to a topological invariant of the manifold, see for example [LM89, Chapter III] or [Nak03, Chapter 12].

Outline of this thesis

In Chapter 1 we will introduce and review some of the algebraic and topological background in great detail. To this end we briefly introduce the relevant algebraic notions such as *monoidal categories, string diagrams, and bicategories*. We will then focus on the geometric side where we review the standard construction of (un)oriented closed *bordism categories* in more detail. To illustrate the general theme of how a classification result for TQFTs works, we will discuss the classification result of 2-dimensional oriented TQFTs in terms of *commutative Frobenius algebras*.

Chapter 2 is mostly focused on the notion of *spin structures* on manifolds. We begin with a quick review of the double covering of the special orthogonal groups $SO(n)$, the so-called *spin groups* $Spin(n)$, and discuss their physical origin. After this we will define the notion of a *spin structure* on a fixed manifold as a class of $Spin(n)$-bundles where the action of $Spin(n)$ is compatible with the oriented frame bundle of the manifold. To answer questions about existence and uniqueness of such structures, we will introduce some tools from algebraic topology. These tools will first be used to answer the same questions for orientations, where the computations are

more straightforward. After this general discussion of spin structures we will focus on 1- and 2-dimensional manifolds. There we show that any oriented 2-dimensional manifold posses at least one spin structure. As an important example we will then explicitly construct the two possible spin structures on a circle and relate them to physical terminology. Finally we will define general *tangential structures* using the language of classifying spaces. This notion encompasses both orientations and spin structures as special cases, more generally any reduction of the frame bundle along a Lie group homomorphism gives rise to a type of tangential structure. In this setting we will then define a bordism category where all manifolds are equipped with a fixed type of structure, and the gluing of such manifolds along boundaries is compatible with this structure.

In Chapter 3 we discuss a 2-dimensional bordism category appropriate to study spin TQFTs with *boundary conditions*. More precisely we will define the *open-closed spin bordism category*. After this we will introduce the corresponding algebraic notions and review the classification results of *open-closed spin TQFTs* by [SS20]. We will then slightly extend this result to incorporate different boundary conditions and operators between them. Finally we will reformulate these results categorically in analogy to the oriented case.

In Chapter 4 we will propose a possible definition of *stratified spin manifolds* and the corresponding bordism categories for any dimension. For this we will modify the definitions of stratified oriented manifold given in [CMS20, Section 2] to account for non-trivial spin structures on the strata. We will then discuss a special case of stratified spin manifolds where the spin structures on all strata are induced by the global spin structure of the whole manifold, see Section 4.1 for the precise statement. After this we will discuss the relations between the stratified spin bordism category and the closed spin bordism category. After this we will focus solely on the 2-dimensional case and describe how to consistently label a stratified spin bordism. These labels will be interpreted as either 1-dimensional topological defects, so-called *line defects*, which can be seen as generalisations of boundary conditions as discussed in Chapter 3, or phases of the TQFT.

In Chapter 5 we will finally define 2-dimensional *defect spin TQFTs*. Following the ideas of [DKR11] for oriented defect TQFTs we will construct a 2-category for a fixed defect spin TQFT. In this 2-category the objects will correspond to the phases of the TQFT while the 1- and 2-morphisms will be interpreted as line defects and local operators, respectively. We will then study the extra structure of this 2-category, where we will first show that it is pivotal, in analogy to the oriented case. After this we will describe a completion procedure which makes the physical intuition of fusing defect lines precise, however this will only work for the before mentioned

9

special case of the bordism category where all spin structures on the strata are induced. Both the pivotal structure and the completion procedure are present for oriented defect TQFTs, in the final section we will discuss genuine new structures coming from the non-trivial spin bundles. More precisely we will use the non-trivial deck transformation to find a 2-endofunctor and discuss how certain cylinders in the bordism category can be used to define a map between different sectors of the TQFT.

Chapter 1

Preliminaries

In this chapter we will introduce, and review some of the necessary mathematical foundations we will use throughout this thesis. In Section 1.1 we will give a very broad overview of the assumed background in algebra, geometry, and topology. We will then state the conventions and notations for these topics which we will use throughout the rest of this thesis.

In Section 1.2 we will give a brief review of *monoidal categories* and their graphical calculus through *string diagrams*. Using the graphical calculus we will then generalise familiar notions from vector spaces such as *duals* or *algebras*.

In Section 1.3 we will very briefly review the basic definitions of 2-*categories* and discuss how monoidal categories can be interpreted in this setting. In particular we will see how the before introduced calculus with string diagrams descends from the graphical calculus of 2-categories.

After this we will review the relevant topological foundations in Section 1.4. More precisely we will define *bordisms* and describe the *closed n-dimensional bordism category* in detail.

In the final Section 1.5 of this chapter we will define *closed TQFTs*. After studying some of their basic properties we will focus on the simplest non-trivial case namely 2-*dimensional closed oriented TQFTs* and discuss their algebraic classification in terms of *commutative Frobenius algebras*.

1.1 Prerequisites, notations and conventions

1.1.1 Prerequisites

We suppose the reader is comfortable with basic notions of category theory such as categories, functors, and natural transformations. Familiarity with universal constructions, such as colimits, will also be helpful to understand the more abstract concepts. A gentle introduction to category theory and its relation to physics, computer science, and logic is given in [BS10]. For a complete introduction see [Lei14].

Furthermore the reader is assumed to have a working knowledge of topology and differential geometry, including smooth manifolds, tangent bundles, orientations, Lie groups, and basics about Riemannian metrics. The required definitions and facts about (principal) fiber bundles are collected in Appendix A. For a gentle introduction to these topics see [Nak03], a book on general relativity such as [Wal84] will also suffice, for a mathematical minded reader we recommend [Wal16], [Bau14], and [Ham17]. Further references are given in the relevant sections of the main text.

1.1.2 Notation and conventions

Algebra

Throughout this thesis let $\Bbbk$ be a fixed field, for example $\Bbbk = \mathbb{C}$. In the following table we give a list of our notation for standard categories the reader should be familiar with:

Set	category of sets
$\mathrm{Vect}_{\Bbbk}$	category of vector spaces over $\Bbbk$
$\mathrm{vect}_{\Bbbk}$	category of finite-dimensional vector spaces over $\Bbbk$
Top	category of topological spaces and continuous maps

Recall the notion of a *super vector space* as a $\mathbb{Z}_2$-graded vector space, i.e. a $V \in \mathrm{Vect}_{\Bbbk}$ together with a decomposition $V = V_0 \oplus V_1$. The elements of V_0 are said to have *degree* 0 and will be called *even* or *bosonic*; elements of V_1 are said to have degree 1 and will be called *odd* or *fermionic*. For $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ super vector spaces we say a linear map $f \colon V \to W$ is of *degree j* if $f(V_i) \subseteq W_{i+j}$ for $i, j \in \mathbb{Z}_2$. The category of super vector spaces $\mathcal{S}\mathrm{Vect}_{\Bbbk}$ has super vector spaces as objects and linear maps of degree 0, also called *even*, as morphisms. The full subcategory of finite-dimensional super vector spaces is denoted $\mathcal{S}\mathrm{vect}_{\Bbbk}$.

Geometry

Throughout this thesis we will work with manifolds, where we always mean smooth manifolds, which are second countable, and Hausdorff as topological space. Furthermore we will assume all manifolds to be compact.

1.2   Symmetric monoidal categories

We begin with an introduction to (symmetric) monoidal categories and their graphical calculus. Monoidal categories are categories with extra structure which allow to also "compose" objects with each other. It turns out that these categories allow for a graphical calculus quite analogously to Feynman diagrams in particle physics. These *string diagrams*[1] will be indispensable for the rest of this thesis. As an illustration of this, we will use the graphical calculus to generalise the notion of $\Bbbk$-algebras and modules, to algebras and modules *internal* to a given monoidal category. This allows us to define so-called *Frobenius algebras* and study their basic properties. Frobenius algebras turn out to be indispensable for the algebraic description of 2-dimensional TQFTs.

Definition 1.2.1. A *monoidal category* consists of:

- a category $\mathcal{C}$;

- a functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product* (or *monoidal product*), where we write $\otimes(A, B) = A \otimes B$ and $\otimes(f, g) = f \otimes g$ for objects $A, B \in \mathcal{C}$ and morphisms f and g in $\mathcal{C}$;

- an object called the *identity object* $\mathbb{1} \in \mathcal{C}$;

- natural isomorphisms called the *associator*:

$$\alpha_{A,B,C} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

the *left unit law*:

$$\lambda_A \colon \mathbb{1} \otimes A \to A,$$

and the *right unit law*:

$$\rho_A \colon A \otimes \mathbb{1} \to A;$$

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[1]The name does not come from string theory, but is motivated by using 'strings' to represent the objects of the category [BS10].

such that the following diagrams commute for all $A, B, C, D \in \mathcal{C}$:

- the *pentagon equation*:

$$
(A \otimes B) \otimes (C \otimes D)
$$

$$
\alpha_{A \otimes B, C, D} \qquad \alpha_{A, B, C \otimes D}
$$

$$
((A \otimes B) \otimes C) \otimes D \qquad A \otimes (B \otimes (C \otimes D))
$$

$$
\alpha_{A, B, C} \otimes 1_D \qquad 1_A \otimes \alpha_{B, C, D}
$$

$$
(A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A, B \otimes C, D}} A \otimes ((B \otimes C) \otimes D)
$$

governing the associator;

- the *triangle equations*:

$$
(A \otimes \mathbb{1}) \otimes B \xrightarrow{\alpha_{A, \mathbb{1}, B}} A \otimes (\mathbb{1} \otimes B)
$$

$$
\rho_A \otimes 1_B \qquad 1_A \otimes \lambda_B
$$

$$
A \otimes B
$$

governing the left and right unit laws.

A monoidal category in which the associator and the left and right unitors are given by the identity morphisms is called *strict*.

Examples 1.2.2. Here we will list a few standard examples of monoidal categories, some of which are particularly important for TQFTs. Verifying the axioms is straightforward and will not be discussed in detail.

1. Let $\mathcal{C}$ be any category, then the category of endofunctors $\operatorname{End}(\mathcal{C})$ together with composition of functors is a strict monoidal category.

2. The category Set is a monoidal category together with the Cartesian product $\times$ of sets and any one element set as unit.[2] The structure morphisms α, λ, ρ are the obvious ones.

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[2]Note here that the monoidal structure of Set is not unique because the unit is not unique as any one element set would work.

3. The category $\mathrm{Vect}_{\Bbbk}$ of vector spaces and its full subcategory $\mathrm{vect}_{\Bbbk}$ of finite-dimensional vector spaces are monoidal categories together with $\otimes_{\Bbbk}$, the tensor product over $\Bbbk$.

4. The category $\mathcal{S}\mathrm{Vect}_{\Bbbk}$ of super vector spaces and its full subcategory of finite-dimensional super vector spaces $\mathcal{S}\mathrm{vect}_{\Bbbk}$ are monoidal categories together with $\otimes_{\Bbbk}$ the tensor product over $\Bbbk$.

We will often abbreviate the data of a monoidal category $(\mathcal{C}, \otimes, \alpha, \rho, \lambda)$ to $(\mathcal{C}, \otimes)$ or only $\mathcal{C}$. One way to justify these abbrevations is given by the *Mac Lane strictness theorem*, see Theorem 1.2.11.

Definition 1.2.3. A *braided monoidal category* consists of:

- a monoidal category $\mathcal{C}$;

- a natural isomorphism called the *braiding*:

$$\beta_{A,B} \colon A \otimes B \to B \otimes A,$$

such that the following two diagrams, called *hexagon equations*, commute:

$$
\begin{array}{ccccc}
A \otimes (B \otimes C) & \xrightarrow{\alpha^{-1}_{A,B,C}} & (A \otimes B) \otimes C & \xrightarrow{\beta_{A,B} \otimes 1_C} & (B \otimes A) \otimes C \\
\Big\downarrow{\scriptstyle \beta_{A,B\otimes C}} & & & & \Big\downarrow{\scriptstyle \alpha_{B,A,C}} \\
(B \otimes C) \otimes A & \xleftarrow{\alpha^{-1}_{B,C,A}} & B \otimes (C \otimes A) & \xleftarrow{1_B \otimes \beta_{A,C}} & B \otimes (A \otimes C)
\end{array}
$$

$$
\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{1_A \otimes \beta_{B,C}} & A \otimes (C \otimes B) \\
\Big\downarrow{\scriptstyle \beta_{A\otimes B,C}} & & & & \Big\downarrow{\scriptstyle \alpha^{-1}_{A,C,B}} \\
C \otimes (A \otimes B) & \xleftarrow{\alpha_{C,A,B}} & (C \otimes A) \otimes B & \xleftarrow{\beta_{A,C} \otimes 1_B} & (A \otimes C) \otimes B
\end{array}
$$

Definition 1.2.4. A *symmetric monoidal category* is a braided monoidal category $\mathcal{C}$ for which the braiding satisfies $\beta_{B,A} = \beta^{-1}_{A,B}$ for all $A, B \in \mathcal{C}$.

A braided monoidal category can be thought of as a generalization of commutativity where $X \otimes Y$ only needs to be isomorphic to $Y \otimes X$ and not equal. For a symmetric monoidal category this condition is strengthened in the sense that commuting twice is equal to the identity.

15

Example 1.2.5. The category of vector spaces $\mathrm{Vect}_{\Bbbk}$ is symmetric with braiding

$$\beta_{V,W} \colon V \otimes_{\Bbbk} W \to W \otimes_{\Bbbk} V \tag{1.2.1}$$

$$v \otimes_{\Bbbk} w \mapsto w \otimes_{\Bbbk} v. \tag{1.2.2}$$

Example 1.2.6. The category of super vector spaces $\mathcal{S}\mathrm{Vect}_{\Bbbk}$ has two different symmetric structures: The first one is the one induce by $\mathrm{Vect}_{\Bbbk}$ from the previous example. The second one is given by

$$\beta_{V,W} \colon V \otimes_{\Bbbk} W \to W \otimes_{\Bbbk} V \tag{1.2.3}$$

$$v \otimes_{\Bbbk} w \mapsto (-1)^{|v| \cdot |w|} w \otimes_{\Bbbk} v, \tag{1.2.4}$$

where v and w are of degree $|v|$ and $|w|$, respectively, and extend linearly otherwise.

If we want functors between monoidal categories to respect the monoidal structures on their source and target categories, we will need some extra data to guarantee compatibility.

Definition 1.2.7. A functor $F \colon \mathcal{C} \to \mathcal{D}$ between monoidal categories is called *monoidal* if it is equipped with:

- a natural isomorphism $\Phi_{A,B} \colon F(A) \otimes F(B) \to F(A \otimes B)$;

- an isomorphism $\phi \colon \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$,

such that

- the diagram

$$
\begin{array}{ccccc}
(F(A) \otimes F(B)) \otimes F(C) & \xrightarrow{\Phi_{A,B} \otimes 1_{F(C)}} & F(A \otimes B) \otimes F(C) & \xrightarrow{\Phi_{A \otimes B, C}} & F((A \otimes B) \otimes C) \\
\Big\downarrow \alpha_{F(A),F(B),F(C)} & & & & \Big\downarrow F(\alpha_{A,B,C}) \\
F(A) \otimes (F(B) \otimes F(C)) & \xrightarrow[1_{F(A)} \otimes \Phi_{B,C}]{} & F(A) \otimes F(B \otimes C) & \xrightarrow[\Phi_{A, B \otimes C}]{} & F(A \otimes (B \otimes C))
\end{array}
$$

  commutes for all $A, B, C \in \mathcal{C}$;

- the diagrams

$$
\begin{array}{ccc}
\mathbb{1}_{\mathcal{D}} \otimes F(A) & \xrightarrow{\lambda_{F(A)}} & F(A) \\
\Big\downarrow \phi \otimes 1_{F(A)} & & \Big\uparrow F(\lambda_A) \\
F(\mathbb{1}_{\mathcal{C}}) \otimes F(A) & \xrightarrow[\Phi_{\mathbb{1},A}]{} & F(\mathbb{1}_{\mathcal{C}} \otimes A)
\end{array}
$$

16

$$
\begin{array}{ccc}
F(A) \otimes \mathbb{1}_{\mathcal{D}} & \xrightarrow{\ \rho_{F(A)}\ } & F(A) \\
{\scriptstyle 1_{F(A)}\otimes\phi}\big\downarrow & & \big\uparrow{\scriptstyle F(\rho_A)} \\
F(A) \otimes F(\mathbb{1}_{\mathcal{C}}) & \xrightarrow[\ \Phi_{A,\mathbb{1}}\ ]{} & F(A \otimes \mathbb{1}_{\mathcal{C}})
\end{array}
$$

commute for any $A \in \mathcal{C}$.

A monoidal functor correspondes is a triple (F, Φ, ϕ) which we will often abbreviate to F. A monoidal functor (F, Φ, ϕ) is called *strict* if Φ and ϕ are identity morphisms.

Definition 1.2.8. A monoidal functor $F \colon \mathcal{C} \to \mathcal{B}$ is called *braided monoidal* if for for all $A, B \in \mathcal{C}$ the following diagram commutes:

$$
\begin{array}{ccc}
F(A) \otimes F(B) & \xrightarrow{\ \beta_{F(A),F(B)}\ } & F(B) \otimes F(A) \\
{\scriptstyle \Phi_{A,B}}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_{B,A}} \\
F(A \otimes B) & \xrightarrow[\ F(\beta_{A,B})\ ]{} & F(B \otimes A)
\end{array}
$$

In case of symmetric monoidal categories, a braided monoidal functor automatically respects the symmetric monoidal structure and is therefore called *symmetric monoidal*.

Definition 1.2.9. Let (F, Φ, ϕ) and (G, Γ, γ) be monoidal functors. A *monoidal natural transformation* is a natural transformation $\eta \colon F \Rightarrow G$, such that the diagrams

$$
\begin{array}{ccc}
F(A) \otimes F(B) & \xrightarrow{\ \eta_A \otimes \eta_B\ } & G(A) \otimes G(B) \\
{\scriptstyle \Phi_{A,B}}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma_{A,B}} \\
F(A \otimes B) & \xrightarrow[\ \eta_{A\otimes B}\ ]{} & G(A \otimes B)
\end{array}
$$

and

$$
\begin{array}{ccc}
& \mathbb{1} & \\
{\scriptstyle \phi}\swarrow & & \searrow{\scriptstyle \gamma} \\
F(\mathbb{1}) & \xrightarrow[\ \eta_{\mathbb{1}}\ ]{} & G(\mathbb{1})
\end{array}
$$

commute for all $A, B \in \mathcal{C}$.

Remark 1.2.10. For braided and symmetric monoidal categories and functors, monoidal natural transformations automatically respect the braided (resp. symmetric) monoidal structure.

1.2.1 Graphical calculus

It turns out to be hugely beneficial to introduce a graphical calculus for monoidal categories. We will now describe how this works in detail: Let $(\mathcal{C}, \otimes)$ be a monoidal category. An object $X \in \mathcal{C}$ is drawn as a *string* labeled with X:

$$\Big| \ X \ ; \tag{1.2.5}$$

a morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is drawn as a vertex that connects an X and Y labeled strand:

$$\begin{matrix} Y \\ \bullet \phi \\ X \end{matrix} \ \equiv \phi \in \mathrm{Hom}_{\mathcal{C}}(X, Y). \tag{1.2.6}$$

The unit morphism of an objects will be "drawn" as an invisible vertex, i.e. just a string as for the object. Note that we read such an diagram from bottom to top. This is compatible with the rule that for composable morphisms $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\psi \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$. the composition will be drawn by gluing the strands vertically, i.e.

$$\begin{matrix} Z \\ \bullet \psi \\ \bullet \phi \\ X \end{matrix} \ \equiv \ \begin{matrix} Z \\ \bullet \psi \circ \phi \\ X \end{matrix} \ . \tag{1.2.7}$$

This composition rule is compatible with the rule for drawing the identity morphisms as just a string labeled by an object. We also have a monoidal structure on $\mathcal{C}$ and can therefore "compose" objects and morphisms differently using the monoidal product

$\otimes$, this will be denoted by horizontal composing the strings, i.e.

$$
\begin{array}{ccc}
X \otimes Y & & X \quad Y \\
\big| & \equiv & \big| \quad \big| \\
X \otimes Y & & X \quad Y
\end{array}
\qquad (1.2.8)
$$

The monoidal product of morphisms $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\phi' \in \mathrm{Hom}_{\mathcal{C}}(X', Y')$ will be drawn as

$$
\begin{array}{ccccc}
Y \otimes Y' & & Y \quad Y' & & Y \quad Y' \\
{\scriptstyle\bullet}\phi \otimes \phi' & \equiv & {\times}\phi \otimes \phi' & \equiv & \phi{\scriptstyle\bullet} \quad {\scriptstyle\bullet}\phi' \\
X \otimes X' & & X \quad X' & & X \quad X'
\end{array}
\qquad (1.2.9)
$$

For example, the diagram

$$
\begin{array}{c}
X' \quad Y' \\
{\times}\psi \\
X \ Y \ Z
\end{array}
\qquad (1.2.10)
$$

corresponds to the morphism $\psi \in \mathrm{Hom}_{\mathcal{C}}((X \otimes Y) \otimes Z, X' \otimes Y')$. To make the analogy with Feynman diagrams more explicit we can interpret vertical composition with time evolution. In this interpretation diagram (1.2.10) corresponds to the evolution of three "particles" X, Y, Z into two "particles" X', Y' through the interaction ψ.

Note that in the graphical calculus we did not distinguish between the isomorphic objects $(X \otimes Y) \otimes Z$ and $X \otimes Y (\otimes Z)$, which means we treat the associator of $\mathcal{C}$ as an identity. Similarly, the unit object is usually "drawn" as an invisible string, which amounts to the unitors also not being displayed in the graphical calculus. These observations suggest the graphical calculus we introduced only makes sense for strict monoidal categories. The reason it works for any monoidal category is the before-mentioned *Mac Lane strictification theorem*.

19

Theorem 1.2.11 (Mac Lane). Every monoidal category $\mathcal{C}$ is monoidally equivalent to a strict monoidal category $\mathcal{C}^{\mathrm{str}}$. Moreover, for every monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$, there is a strict monoidal functor $F^{\mathrm{str}}\colon \mathcal{C}^{\mathrm{str}} \to \mathcal{D}^{\mathrm{str}}$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\;F\;} & \mathcal{D} \\
\cong \downarrow & & \downarrow \cong \\
\mathcal{C}^{\mathrm{str}} & \xrightarrow[F^{\mathrm{str}}]{} & \mathcal{D}^{\mathrm{str}}
\end{array}
$$

commutes up to a monoidal natural transformation.

A detailed proof is given in [Eti+16, Section 2.8]. There are similar statements for braided and symmetric monoidal categories.

An important consequence of the strictification theorem are the so-called coherence theorems, see [Eti+16, Section 2.9] for more details. We will not delve further into the specifics of these important results and only note that they allow us to turn the heuristic description of the graphical calculus into a concrete definition.

Remark 1.2.12. It should be noted here that monoidal categories can equivalently be thought of as a special case of 2-*categories*. The concrete definition of a 2-category will be given in Section 1.3. Roughly, they can be thought of as a 2-dimensional version of categories with not only objects and morphisms, but objects, 1-morphisms, and 2-morphisms between 1-morphisms. A standard example is the 2-category of (small) categories, functors and natural transformations.

For a braided monoidal category $\mathcal{C}$ we will draw the braiding and its inverse as

$$
\beta_{X,Y} = \;\;\raisebox{-1em}{\begin{array}{c} Y \quad X \\ \diagdown\!\!\!\diagup \\ X \quad Y \end{array}}\;\;, \qquad
\beta_{X,Y}^{-1} = \;\;\raisebox{-1em}{\begin{array}{c} X \quad Y \\ \diagup\!\!\!\diagdown \\ Y \quad X \end{array}}\;\;. \tag{1.2.11}
$$

If the braiding is symmetric we will draw it simply as

$$
\beta_{X,Y} = \;\;\raisebox{-1em}{\begin{array}{c} Y \quad X \\ \times \\ X \quad Y \end{array}}\;\;. \tag{1.2.12}
$$

The notion of *duality* is incredibly important in both mathematics as well as physics. It turns out that monoidal categories provide a framework for a general notion of *dual*, for which the dual of finite-dimensional vector spaces is an example.

Definition 1.2.13. Let $\mathcal{C}$ be a monoidal category. A *left dual* of $X \in \mathcal{C}$ is an object $^\vee X \in \mathcal{C}$ together with morphisms

$$
\begin{array}{cc}
& \\
{}^\vee X \quad X & = \mathrm{ev}_X \colon {}^\vee X \otimes X \to \mathbb{1},
\end{array}
\tag{1.2.13}
$$

$$
\begin{array}{cc}
X \quad {}^\vee X & \\
& = \mathrm{coev}_X \colon \mathbb{1} \to X \otimes {}^\vee X,
\end{array}
\tag{1.2.14}
$$

called the *left evaluation* and *coevaluation* morphisms fulfilling the so-called *Zorro identities* (or *Zorro moves*):

$$
\begin{array}{cccc}
X & X & {}^\vee X & {}^\vee X \\
& & & \\
& = & & = \\
X & X & {}^\vee X & {}^\vee X
\end{array}
\tag{1.2.15}
$$

Analogously, one defines a *right dual* $X \in \mathcal{C}$ as an object $X^\vee \in \mathcal{C}$ together with morphisms

$$
\begin{array}{cc}
& \\
X \quad X^\vee & = \widetilde{\mathrm{ev}}_X \colon X \otimes X^\vee \to \mathbb{1},
\end{array}
\tag{1.2.16}
$$

$$
\begin{array}{cc}
X^\vee \quad X & \\
& = \widetilde{\mathrm{coev}}_X \colon \mathbb{1} \to X^\vee \otimes X,
\end{array}
\tag{1.2.17}
$$

called the *right evaluation* and *coevaluation* morphisms, again fulfilling the Zorro identities:

$$
\begin{array}{cccc}
X & X & X^\vee & X^\vee \\
& & & \\
& = & & = \\
X & X & X^\vee & X^\vee
\end{array}
\tag{1.2.18}
$$

A monoidal category in which every object has left or right duality data is called left resp. right *rigid*, if it is both left and right rigid it is called *rigid*. For a braided monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \beta)$, left or right rigidity imply rigidity. To see this note

that for $(^\vee X, \mathrm{ev}_X, \mathrm{coev}_x)$ left duality data of $X \in \mathcal{C}$, we have the canonical right duality data $(^\vee X, \mathrm{ev}_X \circ \beta_{X,^\vee X}, \beta_{^\vee X,X}^{-1} \circ \mathrm{coev}_X)$, in string diagrams:

$$\overset{\frown}{\underset{X \quad ^\vee X}{}} := \overset{\bigcirc}{\underset{X \quad ^\vee X}{\times}} \quad , \quad \underset{X^\vee \quad X}{\smile} := \overset{X^\vee \quad X}{\underset{}{\bigtimes}} . \tag{1.2.19}$$

The Zorro moves follow from a straightforward computation using naturality of the braiding and the Zorro moves for ev_X and coev_X. For left dualisable objects in braided and symmetric monoidal categories we will always choose the above right duality data.

Proposition 1.2.14. Let $\mathcal{C}$ be a monoidal category. If $X \in \mathcal{C}$ has a left/right dual then it is unique up to unique isomorphism.

See [Eti+16, Proposition 2.10.5] for the proof.

Definition 1.2.15. Let $\mathcal{C}$ be a rigid monoidal category. The *left dual* of the morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is the morphism

$$^\vee \phi := \overset{^\vee X}{\underset{^\vee X}{\curvearrowright \bullet \phi \curvearrowleft}} \tag{1.2.20}$$

Definition 1.2.16. Let $\mathcal{C}$ be a left rigid monoidal category. A *pivotal structure* on $\mathcal{C}$ is a monoidal natural isomorphism $\delta \colon 1_{\mathcal{C}} \to {}^{\vee\vee}(-)$; a left rigid monoidal category with pivotal structure is called *pivotal*.

Proposition 1.2.17. A pivotal category $(\mathcal{C}, \delta)$ is rigid.

Proof. A pivotal category is left rigid by definition, thus we need to show that $\mathcal{C}$ is also right rigid. For this let $X \in \mathcal{C}$ with left duality data $(^\vee X, \mathrm{ev}_X, \mathrm{coev}_X)$. We define right duality data by setting $X^\vee := {}^\vee X$, $\widetilde{\mathrm{ev}}_X := \mathrm{ev}_X \circ (\delta_X \otimes 1_{X^\vee})$, and $\widetilde{\mathrm{coev}}_X := (1_{X^\vee} \otimes \delta_X^{-1}) \circ \mathrm{coev}_X$. The Zorro moves can be directly verified through the corresponding string diagrams. $\square$

An important example of a pivotal category is the category $\mathrm{vect}_{\Bbbk}$.

1.2.2 Algebras in monoidal categories

Before we define what an algebra in a monoidal category should be first recall the notion of an $\Bbbk$-*algebra* over a field $\Bbbk$. A $\Bbbk$-algebra is a vector space $A \in \mathrm{Vect}_\Bbbk$ together with a linear map $\mu \colon A \otimes_\Bbbk A \to A$, called the *multiplication* and a distinguished element $e \in A$, called the *unit*, such that μ is associative, and unital with unit e, i.e.

$$\mu \circ (\mu \otimes 1_A) = \mu \circ (1_A \otimes \mu) \qquad \text{(associativity)},$$
$$\mu(e \otimes a) = a = \mu(a \otimes e) \qquad \text{(unitality)}$$

for any $a \in A$. The associativity equation is already suitable to be interpreted in any monoidal categories. To properly interpret unitality in a categorical setting we will use the notion of *generalised elements* of an object in a monoidal category. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category and $X \in \mathcal{C}$, the set of *generlised elements of X* is given by $\mathrm{Hom}_\mathcal{C}(\mathbb{1}, X)$. Why this definition? Let us look at the special case of $\mathcal{C} = \mathrm{Vect}_\Bbbk$, in this setting we can explicitly construct an isomorphism $V \cong \mathrm{Hom}_\Bbbk(\Bbbk, V)$ by sending an element $v \in V$ to the unique(!) linear map $f \colon \Bbbk \to V$ with $f(1) = v$.[3] With this we can equivalently consider the morphism $\eta \in \mathrm{Hom}_\Bbbk(\Bbbk, A)$, corresponding to e, as the unit of our algebra A. In this formulation unitality is given by

$$\mu \circ (\eta \otimes 1_A) = 1_A = \mu \circ (1_a \otimes \eta).$$

With this preparation we can now finally define the notion of algebra internal to a monoidal category using the graphical calculus.

Definition 1.2.18. Let $\mathcal{C}$ be a monoidal category.

1. An *algebra*[4] in $\mathcal{C}$ is a triple (A, μ, η) where

$$A \in \mathcal{C}, \quad \text{\Large$\curlywedge$} = \mu \colon A \otimes A \to A, \quad \text{\Large$|$} = \eta \colon \mathbb{1} \to A \qquad (1.2.21)$$

   with μ and η called *multiplication* and *unit*, respectively, such that the following identities hold

$$\qquad \qquad \text{(associativity)}, \qquad \qquad (1.2.22)$$

$$\qquad \qquad \text{(unitality)}. \qquad \qquad (1.2.23)$$

---

[3]This prescription fully determines f due to linearity.
[4]Sometimes also called a *monoid*.

An *algebra map* or *algebra morphism* $(A, \mu, \eta) \to (A', \mu', \eta')$ is a morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, A')$ which preserves the multiplication and unit in the sense that

$$\phi \bigwedge \phi = \bigwedge^{\phi}, \qquad \phi \; \Big| = \Big|. \tag{1.2.24}$$

The *category of algebras* $\mathrm{Alg}(\mathcal{C})$ has algebras in $\mathcal{C}$ as objects, algebra maps as morphisms, and composition and units induced from $\mathcal{C}$.

If $\mathcal{C}$ also has a braiding β, we call an algebra (A, μ, η) *commutative* if

$$\bigotimes = \bigwedge . \tag{1.2.25}$$

The full subcategory of *commutative algebras* will be denoted by $\mathrm{ComAlg}(\mathcal{C})$.

2. A *coalgebra*[5] in $\mathcal{C}$ is a triple (A, Δ, ϵ) where

$$A \in \mathcal{C}, \quad \bigvee = \Delta \colon A \to A \otimes A, \quad \Big| = \epsilon \colon A \to \mathbb{1} \tag{1.2.26}$$

with Δ and ϵ called *comultiplication* and *counit*, respectively, such that the following identities hold

$$\bigvee\!\!\!\bigvee = \bigvee\!\!\!\bigvee \qquad \text{(coassociativity)}, \tag{1.2.27}$$

$$\bigvee = \Big| = \bigvee \qquad \text{(counitality)}. \tag{1.2.28}$$

A *coalgebra map* or *coalgebra morphism* $(A, \Delta, \epsilon) \to (A', \Delta', \epsilon')$ is a morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, A')$ which preserves the comultiplication and counit in the sense that

$$\bigvee_{\phi} = {}^{\phi}\bigvee^{\phi}, \qquad \phi \; \Big| = \Big|. \tag{1.2.29}$$

---

[5]Sometimes also called a *comonoid*.

24

The *category of coalgebras* coAlg($\mathcal{C}$) has coalgebras in $\mathcal{C}$ as objects, coalgebra maps as morphisms, and composition and units induced from $\mathcal{C}$. Note that CoAlg($\mathcal{C}$) = Alg($\mathcal{C}^{\mathrm{op}}$).

If $\mathcal{C}$ also has a braiding β, we call a coalgebra (A, Δ, ϵ) *cocommutative* if

$$\text{(diagram)} = \text{(diagram)}. \tag{1.2.30}$$

The full subcategory of *cocommutative coalgebras* will be denoted by CoComAlg($\mathcal{C}$).

3. A *Frobenius algebra* in $\mathcal{C}$ is an object A together with an algebra and coalgebra structure such that

$$\text{(diagram)} = \text{(diagram)} = \text{(diagram)} \qquad \text{(Frobenius relations)}. \tag{1.2.31}$$

A *map of Frobenius algebras* is simultaneously a map of algebras and coalgebras. The *category of Frobenius algebras* will be denoted by Frob($\mathcal{C}$), it can be shown that this is a groupoid, i.e. every morphism is invertible [REF]. A Frobenius algebra is called *commutative* if its underlying algebra is commutative, the full subcategory of commutative Frobenius algebras will be denoted by ComFrob($\mathcal{C}$).

There are several equivalent descriptions of Frobenius algebras, we will mostly only be interested in the one given above and the following.

Proposition 1.2.19. Let $\mathcal{C}$ be a monoidal category. The following are equivalent:

1. $(A, \mu, \eta, \Delta, \epsilon) \in \mathrm{Frob}(\mathcal{C})$.

2. $(A, \mu, \eta) \in \mathrm{Alg}(\mathcal{C})$ together with a non-degnerate pairing[6]

$$\text{(diagram)} = \kappa \colon A \otimes A \to \mathbb{1} \tag{1.2.32}$$

which is compatible with μ in the sense that

$$\text{(diagram)} = \text{(diagram)}. \tag{1.2.33}$$

___

[6]The notion of non-degenerate pairings in a monoidal category is a generalization of the non-degenerate pairings in $\mathrm{Vect}_{\Bbbk}$, see [FS08, Definition 3]. Precise knowledge of this general will not be relevant as we will only use this characterisation for $\mathcal{C} = \mathrm{Vect}_{\Bbbk}$ or $\mathcal{C} = \mathcal{S}\mathrm{Vect}_{\Bbbk}$.

See [FS08, Section 3] for a proof. One useful fact from this proof is that instead of specifying a Frobenius pairing κ, one could also use a *Frobenius trace* $\epsilon\colon A \to \mathbb{1}$ which is related to the pairing by $\kappa = \epsilon \circ \mu$.

Lemma 1.2.20. Let $\mathcal{C}$ be a monoidal category and $(A, \mu, \eta, \Delta, \epsilon) \in \mathrm{Frob}(\mathcal{C})$. Then A is dualisable with $^{\vee}A \cong A$.

Proof. We set

$$\mathrm{ev}_A := \epsilon \circ \mu = \quad , \quad \mathrm{coev}_A := \Delta \circ \eta = \qquad (1.2.34)$$

the Zorro identities then directly follow from the Frobenius relation together with unitality and counitality. For example

$$= \quad = \qquad (1.2.35)$$

where the first equation is the Frobenius relation and the second on unitality and counitality. $\qquad\square$

Lemma 1.2.21. Let $\mathcal{C}$ be a monoidal category. The category of Frobenius algebras in $\mathcal{C}$ is a groupoid.

Proof. Let $(A, \mu, \eta, \Delta, \epsilon), (A', \mu', \eta', \Delta', \epsilon') \in \mathrm{Frob}(\mathcal{C})$ and $\phi\colon A \to A'$ be a map of Frobenius algebras. We set $\phi^{-1} := {}^{\vee}\phi$ for the duality data of A and A' from the

previous proof. Then

$$\phi^{-1} \circ \phi = \quad (1.2.36)$$

$$= \quad (1.2.37)$$

$$= \quad = 1_A \quad (1.2.38)$$

where in the first equation we used the definition of ϕ^{-1} and the duality data, in the second step we used that ϕ is an Algebra morphism and in the final step the Zorro move. Analogously one shows that $\phi \circ \phi' = 1_{A'}$. $\qquad\square$

A Frobenius algebra A in a symmetric monoidal category with Frobenius pairing κ is called *symmetric* if

$$= \quad . \quad (1.2.39)$$

More generally one defines the *Nakayama automorphism* N_A of A by

$$= N_A \quad . \quad (1.2.40)$$

Then A is symmetric if and only if $N_A = 1_A$. It can be shown that in our conventions for Frobenius algebras the Nakayma automorphism is given by

$$N_A = \quad . \quad (1.2.41)$$

27

Definition 1.2.22. Let $\mathcal{C}$ be a monoidal category and let $A, A' \in \mathrm{Alg}(\mathcal{C})$. An A'-A-*bimodule* is an object $X \in \mathcal{C}$ together with morphisms

$$
\begin{array}{ll}
\text{(diagram)} & = \rho^l_X \in \mathrm{Hom}_{\mathcal{C}}(X \otimes A, X), \\
\text{(diagram)} & = \rho^r_X \in \mathrm{Hom}_{\mathcal{C}}(A' \otimes X, X)
\end{array}
$$

$$(1.2.42)$$

such that

$$(1.2.43)$$

and

$$(1.2.44)$$

A *left A-module* is a A-$\mathbb{1}$-bimodule X with $\rho^r_X = \rho_X$, analogously a *right A-module* is a $\mathbb{1}$-A-bimodule X with $\rho^l_X = \lambda_X$.

A *bimodule map* between two A'-A-bimodules X, Y is a morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ such that

$$(1.2.45)$$

There is a category of A'-A-bimodules ${}_{A'}\mathrm{Mod}_A$ which has A'-A-bimodules as objects and bimodule maps as morphisms. Analogously there is a category of left A-modules denoted by ${}_A\mathrm{Mod}$ or $\mathrm{Mod}(A')$ as well as a category Mod_A of right A-modules.

Analogously one can define *bicomodules* over a coalgebra by considering the above string diagrams read from top to bottom instead. We will not give the precise definition.

1.3 Short introduction to 2-categories

From the previous section we know how to interpret the one dimensional lines and zero dimensional points of string diagrams as objects and morphisms in a monoidal category, respectively. What about the two dimensional regions? Do these also correspond to some algebraic objects? The answer is yes! However to understand this we need to leave the world of ordinary categories and go one dimension higher to so-called 2-*categories*, sometimes also called *bicategories*. Very roughly, a 2-category consists of objects, morphisms between these objects, and "higher" morphisms between the morphisms, such that certain coherence axioms hold. This is only the first step into the realm of *higher category theory*, where even higher morphisms between morphisms exist. We will however not delve further into the theory of higher categories and focus our attention on 2-categories.

Giving a full introduction to 2-categories would go beyond the scope of this thesis, therefore we will only consider the very basics of 2-categories, and focus on their graphical calculus without worrying about details. We will follow [JY20] for the basic definitions, and [Car18; Lau11] for the graphical calculus.

Definition 1.3.1. A 2-*category* $\mathcal{B}$ consists of:

- a collection of *objects* $\mathrm{Ob}(\mathcal{B})$, where we will often times write $a, b \in \mathcal{B}$ instead of $a, b \in \mathrm{Ob}(\mathcal{B})$;

- for every pair of objects $a, b \in \mathcal{B}$, a category $\mathcal{B}(a, b)$, called the *Hom category*; we call the objects $X \colon a \to b$ of $\mathcal{B}(a, b)$ 1-*morphisms* of $\mathcal{B}$; a morphism $\phi \in \mathrm{Hom}_{\mathcal{B}(a,b)}(X, Y)$ is called 2-*morphism* of $\mathcal{B}$ and will be denoted as

$$
b \quad \overset{Y}{\underset{X}{\Longleftarrow \varphi \Uparrow \Longleftarrow}} \quad a
$$

  or simply $\phi \colon X \implies Y$; composition and identity morphisms in $\mathcal{B}(a, b)$ are called *vertical composition* and *identity* 2-*morphisms*, respectively; we will denote vertical composition with $\cdot$ and display it as

$$
b \longleftarrow \overset{\overset{Z}{\psi \Uparrow}}{\underset{\phi \Uparrow}{Y}} \longrightarrow a
$$

  for $\phi \in \mathrm{Hom}_{\mathcal{B}(a,b)}(X, Y)$ and $\psi \in \mathrm{Hom}_{\mathcal{B}(a,b)}(Y, Z)$;

- for every triple of objects $a, b, c \in \mathcal{B}$ a functor

$$\otimes_{c,b,a} \colon \mathcal{B}(b,c) \times \mathcal{B}(a,b) \to \mathcal{B}(a,c)$$
$$(Y, X) \mapsto Y \otimes X$$
$$(\psi, \phi) \mapsto \psi \otimes \phi$$

  called the *horizontal composition*;[7]

- for each object $a \in \mathcal{B}$ a functor

$$1_a \colon \mathbf{1} \to \mathcal{B}(a,a) \tag{1.3.1}$$

  where $\mathbf{1}$ denotes the category with one object and exactly one morphism; note that this just gives a 1-morphism, which we will also denote by 1_a, and its identity 2-morphism 1_{1_a}; the functor 1_a will be called the *identity* 1-morphism of a;

- for objects $a, b, c, d \in \mathcal{B}$, a natural isomorphism

$$\alpha_{d,c,b,a} \colon \ \otimes_{d,b,a} \circ (\otimes_{d,c,b} \times \mathrm{id}_{\mathcal{B}(a,b)}) \to \otimes_{d,c,a} \circ (\mathrm{id}_{\mathcal{B}(c,d)} \times \otimes_{c,b,a})$$

  called the *associator*;

- for each pair of objects $a, b \in \mathcal{B}$, natural isomorphisms

$$\otimes_{b,b,a} \circ \left(1_b \times \mathrm{id}_{\mathcal{B}(a,b)} \right) \xrightarrow{\ \lambda_{a,b}\ } \mathrm{id}_{\mathcal{B}(a,b)} \xleftarrow{\ \rho_{a,b}\ } \otimes_{b,a,a} \circ \left(\mathrm{id}_{\mathcal{B}(a,b)} \times 1_a \right)$$

  called the *left unitor* and the *right unitor*, respectively;

We will often times omit the labeling subscripts of $\otimes, \alpha, \lambda$, and ρ. Instead we will use subscripts to denote the components of the natural isomorphisms, e.g. for $a, b, c, d \in \mathcal{B}$, $X \in \mathcal{B}(a,b)$, $Y \in \mathcal{B}(b,c)$, and $Z \in \mathcal{B}(c,d)$ we write $\alpha_{Z,Y,X}$ instead of $(\alpha_{d,c,b,a})_{Z,Y,X}$. All of this data needs to satisfy coherence axioms in the sense that the following diagrams commute for all $W \in \mathcal{B}(a,b)$, $X \in \mathcal{B}(b,c)$, $Y \in \mathcal{B}(c,d)$, and $Z \in \mathcal{B}(d,e)$:

---

[7]The notation is on purpose reminiscent to the one of monoidal products, as we will become clear soon.

- The *pentagon axiom*:

$$
\begin{array}{ccc}
& (Z \otimes Y) \otimes (X \otimes W) & \\
\end{array}
$$

$$
\alpha_{Z \otimes Y, X, W} \nearrow \qquad \searrow \alpha_{Z, Y, X \otimes W}
$$

$$
((Z \otimes Y) \otimes X) \otimes W \qquad\qquad Z \otimes (Y \otimes (X \otimes W))
$$

$$
\alpha_{Z,Y,X} \otimes 1_W \Big\downarrow \qquad\qquad \Big\uparrow 1_Z \otimes \alpha_{Y,X,W}
$$

$$
(Z \otimes (Y \otimes X)) \otimes W \xrightarrow{\ \alpha_{Z, Y \otimes X, W}\ } Z \otimes ((Y \otimes X) \otimes W)
$$

- The *unit axiom*:

$$
(X \otimes 1_b) \otimes W \xrightarrow{\ \alpha_{W, 1_b, X}\ } X \otimes (1_b \otimes W)
$$

$$
\rho_W \otimes 1_X \searrow \qquad\qquad \swarrow 1_X \otimes \lambda_W
$$

$$
X \otimes W
$$

We will often abbreviate the data of a 2-catgory $(\mathcal{B}, \otimes, 1, \alpha, \lambda, \rho)$ to $\mathcal{B}$.

Example 1.3.2. Let $\mathcal{C}$ be a monoidal category, the *delooping* $\mathrm{B}\mathcal{C}$ of $\mathcal{C}$, is defined as the 2-category with a single object $*$, and Hom category $\mathrm{B}\mathcal{C}(*, *) := \mathcal{C}$. It is straightforward to check that the pentagon and unit axioms correspond directly to the pentagon and triangle equations of $\mathcal{C}$, respectively. Conversely, in any 2-category $\mathcal{B}$ and object $a \in \mathcal{B}$, the Hom category $\mathcal{B}(a, a)$ is canonically a monoidal category. This explains the choice of notation for horizontal composition.

This example allows us to interpret 2-catgories as "many monoidal categories together". This interpretation suggests to ask if there is also a strict version of 2-categories, the answer is yes.

Definition 1.3.3. A 2-category $(\mathcal{B}, \otimes, 1, \alpha, \lambda, \rho)$ is called *strict* if the associatior α, and the unitors λ and ρ are identities. Note that some people call a strict 2-category a 2-category.

Example 1.3.4. A standard example of a strict 2-category is given by Cat, where objects, 1-morphisms, and 2-morphisms are (small) categories, functors, and natural transformations, respectively.

We introduced strict 2-categories as 2-categories with special properties, however there are also different ways to look at them either directly in terms of their data, or through enriched category theory. Each of these views is equivalent to the others and useful in different circumstances. Our choice is purely to embed our previous discussion of monoidal categories into the larger world of 2-categories. For more on the other views see [JY20, Section 2.3].

In analogy to Theorem 1.2.11 there is also a strictification result for 2-categories to strict 2-categories, which states that every 2-category is 2-equivalent[8] to a strict 2-category [JY20, Theorem 8.4.1]. This theorem allows us to develop a graphical calculus without worrying too much about diagrams being only defined up to isomorphism, e.g. associativity of horizontal composition.

Before getting to string diagrams, we will now shortly discuss *pasting diagrams*. Classically these diagrams are often used in ordinary category theory to illustrate natural transformations. However from Example 1.3.4 we know that categories, functors, and natural transformations form a (strict) 2-category. This suggests to develop a calculus of pasting diagrams for any 2-category. Note that we already began to do this in the definition of a 2-category we denoted 2-morphisms and their vertical composition through pasting diagrams. For the general case let $\mathcal{B}$ be a 2-category, which we can always assume by the coherence theorem of 2-categories. For $a, b, c, d \in \mathcal{B}$, $X, Y \in \mathcal{B}(a,b)$, $X', Y' \in \mathcal{B}(b,c)$, $X'', Y'' \in \mathcal{B}(c,d)$, $\phi \in \mathrm{Hom}_{\mathcal{B}(a,b)}(X,Y)$, $\phi' \in \mathrm{Hom}_{\mathcal{B}(b,c)}(X',Y')$, and $\phi' \in \mathrm{Hom}_{\mathcal{B}(b,c)}(X',Y')$. We denote the horizontal composition of ϕ and ϕ' by

$$
c \xleftarrow{\ \ \phi' \ \ } b \xleftarrow{\ \ \phi \ \ } a \quad = \quad c \xleftarrow{\ \ \phi' \otimes \phi \ \ } a
$$

Note that we really need strict associativity of horizontal composition for this prescription to make sense because

$$
d \xleftarrow{\ \ \phi'' \ \ } c \xleftarrow{\ \ \phi' \ \ } b \xleftarrow{\ \ \phi \ \ } a
$$

could mean either $(\phi'' \otimes \phi') \otimes \phi$ or $\phi'' \otimes (\phi' \otimes \phi)$ otherwise. The prescription of diagrams for horizontal and vertical composition is already enough to start working

---

[8]See [JY20, Definition 6.2.8] for the definition of 2-equivalences, called biequivalence there.

with these diagrams, the rest of the rules can be inferred directly from the definition of a 2-category, for example the unitality axioms amount to the following equality of diagrams:

$$b \overset{1_b}{\underset{1_b}{\rightleftarrows}} 1_{1_b} \; b \overset{Y}{\underset{X}{\rightleftarrows}} \phi \; a \;\; = \;\; b \overset{Y}{\underset{X}{\rightleftarrows}} \phi \; a \;\; = \;\; b \overset{Y}{\underset{X}{\rightleftarrows}} \phi \; a \overset{1_a}{\underset{1_a}{\rightleftarrows}} 1_{1_a} \; a$$

However these pasting diagrams are not the graphical calculus in which we are ultimately interested, instead we will work with their *Poincaré duals*. In Chapter 5 we will construct a 2-category from a 2-dimensional defect spin TQFT, for this 2-category the graphical calculus discussed below can be interpreted directly through the defects of the TQFT. For a full introduction to pasting diagrams and the rules by which they are governed see [JY20, Chapter 3] and [Lau11, Section 2.1]. To turn a pasting diagram into their Poincaré dual, a *string diagram*, we work with the following rules:

- A zero dimensional arrow, corresponding to an object, gets replaced by a two dimensional region.

- A one dimensional arrow, labeled with a 1-morphism, gets replaced with a one dimensional string which goes into the orthogonal direction.

- A two dimensional arrow, labeled with a 2-morphism, gets replaced with a zero dimensional point.

The so obtained string diagrams are exactly the generalisation of string diagrams of monoidal categories we are looking for. To illustrate the rules of turning a pasting diagram into a string diagram, consider this simple example;

$$b \overset{Y}{\underset{X}{\rightleftarrows}} \phi \; a \quad \longrightarrow \quad b \; \underset{X}{\overset{Y}{\Big|}} \bullet \phi \quad a \; .$$

It is evident that this is exactly the type of diagram we sought out to find! Furthermore it allows us to understand how the string diagrams of monoidal categories are

really just a special case of string diagrams for 2-categories. More precisely: view a monoidal category $\mathcal{C}$ as its corresponding 2-category $\mathrm{B}\mathcal{C}$, the string diagrams for $\mathcal{C}$ are obtained by the ones of $\mathrm{B}\mathcal{C}$ by omitting the label of the single object of $\mathrm{B}\mathcal{C}$.

It is now straightforward to generalise the structures we discussed for monoidal categories, such as rigidity, to 2-categories however we will not delve further into this for now and refer the interested reader to either [Lau11, Section 2.3] or [Car18, Section 2.2] for more details.

1.4 Closed bordism category

With the necessary algebraic preliminaries understood, we can now turn to the geometric side. The goal of this section is to introduce the notion of *bordisms* and their (ordinary) categories for both the oriented and unoriented case. Very roughly, a bordism between two closed $(n-1)$-dimensional manifolds Σ_1, Σ_2, is an n-dimensional manifold M that "connects" Σ_1 to Σ_2. As explained in the Introduction such a bordism corresponds to the "global evolution" from Σ_1 to Σ_2.

Definition 1.4.1. Let Σ_1, Σ_2 be closed $(n-1)$-dimensional manifolds.

1. A *bordism*[9] $\Sigma_1 \to \Sigma_2$ consists of

   - a n-dimensional manifold with boundary M,
   - a decomposition $\partial M = (\partial M)_1 \sqcup (\partial M)_2$ into the *in-going boundary* $(\partial M)_1$ and *out-going boundary* $(\partial M)_2$,
   - germs[10] (in $\epsilon > 0$) of embeddings (called *collars*)

$$\theta_1 \colon [0, \epsilon) \times \Sigma_1 \to M \tag{1.4.1}$$
$$\theta_2 \colon (-\epsilon, 0] \times \Sigma_2 \to M \tag{1.4.2}$$

   such that $\theta_i(\{0\} \times \Sigma_i) = (\partial M)_i$ for $i \in \{1,2\}$, and $\mathrm{Im}(\theta_1) \cap \mathrm{Im}(\theta_2) = \varnothing$.

2. Let (M, θ_1, θ_2) and $(M', \theta_1', \theta_2')$ be bordisms $\Sigma_1 \to \Sigma_2$. A *diffeomorphism* $(M, \theta_1, \theta_2) \to (M', \theta_1', \theta_2')$ is a diffeomorphism $f \colon M \to M'$ such that $f((\partial M)_i) = (\partial M')_i$, and $f \circ \theta_i = \theta_i'$ hold for $i \in \{1,2\}$.

---

[9]Some people prefer the term *cobordism*.

[10]Germs are useful for technical reasons concerning the "composition" of two bordisms. More on this before Definition 1.4.4.

We will often times abbreviate a bordism $(M, \theta_1, \theta_2) \colon \Sigma_1 \to \Sigma_2$ to $M \colon \Sigma_1 \to \Sigma_2$ or just M. It is not hard to see that being diffeomorphic is an equivalence relation for bordisms, we will denote the diffeomorphism class of a bordism by $[M]$ or M. The definition of a bordism is illustrated with the following picture:

$$(1.4.3)$$
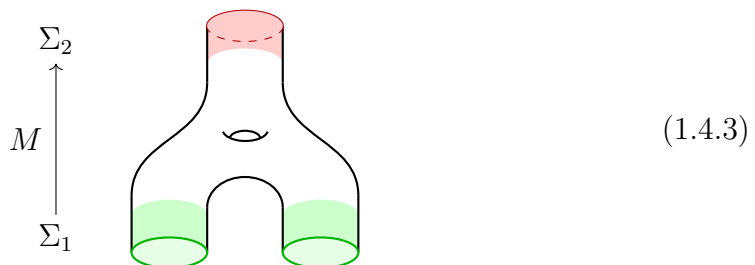
Here the two green circles on the bottom illustrate the source Σ_1 of the bordism M, while the red circle at the top illustrates the target Σ_1. The shaded regions indicate the collar neighbourhood of Σ_1 and Σ_2 in M, given through the collars θ_1, respectively θ_2.

Examples 1.4.2. Two simple, yet important examples of bordisms are given by *cylinders* over an $(n-1)$-dimensional closed manifold.

1. The cylinder $I \times \Sigma$, with $I = [0, 1]$ the closed unit interval, becomes a bordism $\Sigma \to \Sigma$ by setting $\left(\partial (I \times \Sigma)\right)_1 := \Sigma =: \left(\partial (I \times \Sigma)\right)_2$ and

$$\theta_1 \colon [0, \epsilon) \times \Sigma \to I \times \Sigma$$

$$(t, p) \mapsto \left(\frac{t}{2}, p\right),$$

$$\theta_2 \colon (-\epsilon, 0] \times \Sigma \to I \times \Sigma$$

$$(t, p) \mapsto \left(\frac{t}{2} + 1, p\right).$$

2. For any diffeomorphism $f \colon \Sigma \to \Sigma$, the cylinder $I \times \Sigma$ becomes a bordism $\Sigma \to \Sigma$, called the *mapping cylinder* and denoted as C_f, by setting $\left(\partial C_\Sigma^f\right)_1 :=$

$$\Sigma =: \left(\partial C_\Sigma^f \right)_2 \text{ and }$$

$$\theta_1 \colon [0, \epsilon) \times \Sigma \to C_\Sigma^f \tag{1.4.4}$$

$$(t, p) \mapsto \left(\frac{t}{2}, p \right), \tag{1.4.5}$$

$$\theta_2 \colon (-\epsilon, 0] \times \Sigma \to C_\Sigma^f \tag{1.4.6}$$

$$(t, p) \mapsto \left(\frac{t}{2} + 1, f(p) \right). \tag{1.4.7}$$

The first example is a special case of the second example with $f = \mathrm{id}_\Sigma$.

Definition 1.4.3. Let Σ_1, Σ_2 be closed $(n-1)$-dimensional manifolds and $(M, \theta_1, \theta_2) \colon \Sigma_1 \to \Sigma_2$ a bordism. The *reversed bordism* $(M^{\mathrm{rev}}, \theta_1^{\mathrm{rev}}, \theta_2^{\mathrm{rev}})$ from Σ_2 to Σ_1 is given $M^{\mathrm{rev}} = M$, $(\partial M^{\mathrm{rev}})_1 = \Sigma_2$, $(\partial M^{\mathrm{rev}})_2 = \Sigma_1$, and

$$\theta_1^{\mathrm{rev}}(t, p) := \theta_2(-t, p) \quad \text{for} \quad t \in [0, \epsilon), \quad p \in \Sigma_2, \tag{1.4.8}$$
$$\theta_2^{\mathrm{rev}}(t, p) := \theta_1(-t, p) \quad \text{for} \quad t \in (-\epsilon, 0], \quad p \in \Sigma_1. \tag{1.4.9}$$

We can informally picture the reversed bordism as the original bordism "turned upside down".

Ultimately we want to construct a category with closed $(n-1)$-dimensional manifolds as objects and bordism classes as morphisms, for this we need to find a way to "compose" or *glue* bordisms. More precisely, suppose we are given three closed $(n-1)$-dimensional manifolds $\Sigma_1, \Sigma_2, \Sigma_3$ and two bordisms $M \colon \Sigma_1 \to \Sigma_2$ and $M' \colon \Sigma_2 \to \Sigma_3$, can we glue M and M' along their common boundary Σ_2? The following discussion is a combination of the arguments given in [Koc03, Section 1.3], [Wal16, Section 2.7], and [Die08, Section 15.1]. For more details we refer to these sources.

First we recall the notion of a *pushout* of topological spaces. Formally a pushout in Top is defined as the colimit of two morphisms $\phi_1 \colon X \to Y$, $\phi_2 \colon X \to Y'$. This means it is given by a topological space $Y \sqcup_X Y'$ together with two morphisms $Y \to Y \sqcup_X Y' \leftarrow Y'$ making the square with ϕ_1 and ϕ_2 commute, and such that for every other such triple (Z, f, f') there exists a unique morphism $Y \sqcup_X Y' \to Z$

making the following diagram commute

An explicit construction of the pushout is given as the quotient $Y \sqcup_X Y' := (Y \sqcup Y') / \sim$, where two elements $y, y' \in Y \sqcup Y'$ are equivalent iff $y \in Y$, $y' \in Y'$ and there exists a $x \in X$ such that $\phi_1(x) = y$ and $\phi_2(x) = y'$. With the quotient topology, i.e. the finest topology such that the canonical surjection $\pi \colon M \sqcup M' \to M \sqcup_{\Sigma_2} M'$ is surjective.

We now want to apply this construction to our bordisms M and M'. For this we take as maps $\phi_1 := \theta_2|_{\{0\} \times \Sigma_2}$ and $\phi_2 := \theta_1'|_{\{0\} \times \Sigma_2}$, where θ_2 and θ_1' are representatives of the germs of collars θ_2 and θ_1', respectively. To see that the construction of $M \sqcup_{\Sigma_2} M'$ is independent of the choice of representatives of the germs note that by definition of a germ all representatives are equal when restricted to the boundary.

Next we need to endow the topological space $M \sqcup_{\Sigma_2} M'$ with the structure of a smooth manifold. To do this first note that $M \sqcup M'$ can naturally be endowed with the structure of a smooth manifold, such that the canonical maps $\iota \colon M \to M \sqcup M' \leftarrow M' \colon \iota'$ are smooth. We now glue the collars $\theta_2 \colon (-\epsilon, 0] \times \Sigma_2 \to M$ and $\theta_1' \colon [0, \epsilon) \times \Sigma_2 \to M'$, for fixed $\epsilon > 0$, to a map

$$\theta \colon (-\epsilon, \epsilon) \times \Sigma_2 \to M \sqcup_{\Sigma_2} M' \tag{1.4.10}$$

$$(t, p) \mapsto \begin{cases} (\pi \circ \iota \circ \theta_2)(t, p), & \text{if} \quad t \leqslant 0, \\ (\pi \circ \iota' \circ \theta_1')(t, p), & \text{if} \quad t \geqslant 0. \end{cases} \tag{1.4.11}$$

This is well-defined by definition of $M \sqcup_{\Sigma_2} M'$, and can be shown to be an embedding. Using this map we can now give $M \sqcup_{\Sigma_2} M'$ the structure of a smooth manifold: We define a real valued function f on $M \sqcup_{\Sigma_2} M'$ to be smooth if $f \circ \pi$ is a smooth function on $M \sqcup M'$ and $f \circ \theta$ is a smooth function on $(-\epsilon, \epsilon) \times \Sigma_2$. This gives $M \sqcup_{\Sigma_2} M'$ the structure of a smooth manifold because coordinate neighbourhoods in $M \sqcup M'$ and $(-\epsilon, \epsilon) \times \Sigma_2$ give coordinate neighbourhoods on $M \sqcup_{\Sigma_2} M'$ which agree on their overlaps.

The last remaining question concerns the uniqueness of this construction. The only arbitrary element was the fixing of ϵ and corresponding choice of collars. Fortunately, collars are unique up to diffeotopy [Wal16, Section 2.5], therefore our construction is unique up to diffeomorphism.

Definition 1.4.4. Let $\Sigma_1, \Sigma_2, \Sigma_3$ be closed $(n-1)$-dimensional manifolds, and let $M \colon \Sigma_1 \to \Sigma_2$ and $M' \colon \Sigma_2 \to \Sigma_3$ be bordisms. Then the bordism $(M \sqcup_{\Sigma_2} M', \theta_1, \theta_2')$ with $(\partial(M \sqcup_{\Sigma_2} M'))_1 = \Sigma_1$ and $(\partial(M \sqcup_{\Sigma_2} M'))_2 = \Sigma_3$ is called a *gluing* of M and M' along Σ_2. The gluing is only unique up to diffeomorphism.

The idea behind this whole construction is illustrated in the following picture:

$$(1.4.12)$$
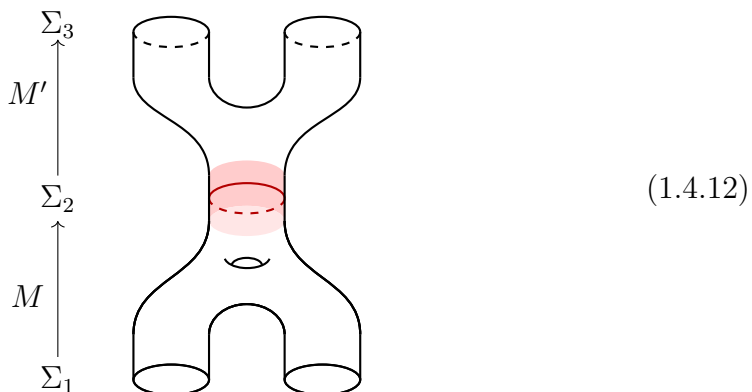
where the red shaded area indicates the neighbourhood $(-\epsilon, \epsilon) \times \Sigma_2$ used in the construction.

We have just seen, using gluing as composition of bordisms is not well-defined because it is not unique. To circumvent this problem in the definition of our category we will instead consider diffeomorphism classes of bordisms.

Definition 1.4.5. The *unoriented closed bordism category* $\mathrm{Bord}_{n,n-1}$ in dimension n is defined as:

- objects are given by closed $(n-1)$-dimensional manifolds,

- morphisms are given by diffeomorphism classes of bordisms,

- unit morphisms are given by the class of cylinders, i.e. $1_\Sigma := [C_\Sigma]$ for $\Sigma \in \mathrm{Bord}_{n,n-1}$,

- composition is induced by gluing of bordisms, i.e. $[M] \circ [M'] = [M \sqcup_{\Sigma_2} M']$ for $M \colon \Sigma_1 \to \Sigma_2$ and $M' \colon \Sigma_2 \to \Sigma_3$;

The composition is unital with respect to 1_Σ. To show associativity one uses the universal property of the pushout, together with well-definedness of the smooth structure on the gluing up to diffeomorphism.

Proposition 1.4.6. The category $\text{Bord}_{n,n-1}$ has a symmetric monoidal structure given by:

- the monoidal product is disjoint union $\sqcup$,

- the unit object is the empty set $\varnothing$ viewed as an $(n-1)$-dimensional manifold,

- the symmetric braiding with components $\beta_{\Sigma,\Sigma'} := [C_{\text{twist}_{\Sigma,\Sigma'}}]$, where

$$\text{twist}_{\Sigma,\Sigma'} : \Sigma \sqcup \Sigma' \to \Sigma' \sqcup \Sigma \tag{1.4.13}$$
$$(x, x')) \mapsto (x', x) \tag{1.4.14}$$

with $x \in \Sigma$ and $x' \in \Sigma'$ is the diffeomorphism which "interchanges" Σ and Σ'.[11]

Proof. The associator and unitors are induced by the universal property of disjoint union. That β is a symmetric braiding follows from the definition of cylinders and properties of the disjoint union, see [Koc03, Lemma 1.3.28]. □

Examples 1.4.7. In low dimensions the bordism categories can be described quite explicitly:

1. Every object in $\text{Bord}_{1,0}$ is a disjoint union of finitely many points such as

$$\bullet \quad \bullet \quad \bullet \quad \bullet \tag{1.4.15}$$

   Every morphism decomposes, under disjoint union and composition, into a disjoint union of finitely many closed intervals and circles [Mil65, Appendix], for example

(1.4.16)

   is a morphism from the disjoint union of three points to one.

2. Every object in $\text{Bord}_{2,1}$ is the disjoint union of finitely many circles, such as

$$\bigcirc \quad \bigcirc \quad \bigcirc \tag{1.4.17}$$

   Every closed, connected two dimensional manifold is diffeomorphic to one of the following [Wal16, Section 5.7]:

---

[11]We are a bit sloppy here with the disjoint union and use the $'$ as "index".

- a sphere S^2

(1.4.18)

- a torus $T^2 = S^1 \times S^1$
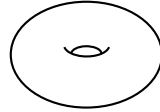
(1.4.19)

- a connected sum[12] of g tori $T^2 \# T^2 \# \ldots \# T^2$
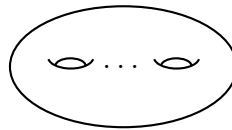
(1.4.20)

  for any $g \in \mathbb{N}_{\geqslant 2}$, giving a surface of genus g
- a finite connected sum of real projective planes $\mathbb{RP}^2 \cong S^2/\mathbb{Z}_2$:

$$\mathbb{RP}^2 \# \mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2 \qquad (1.4.21)$$

Note that spheres and the connected sum of tori are orientable manifolds while connected sums of real projective planes are unorientable.

Every morphism in $\mathrm{Bord}_{2,1}$ is represented by a finite disjoint union of closed two dimensional manifolds as above, with a finite number (possibly zero) of disks B^2 removed and the boundary components S^1 endowed with a *choice* of collars.[13] This follows from the classification of compact two dimensional manifolds, see [Wal16, Section 5.7] for a detailed discussion using handle decompositions. For example
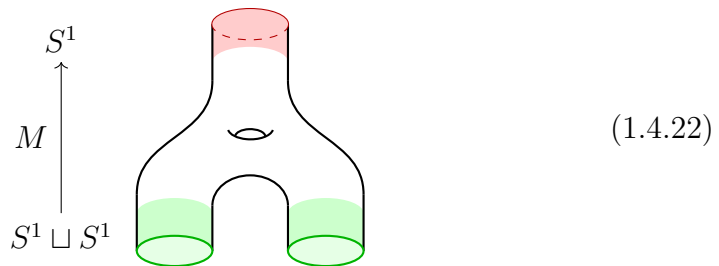
(1.4.22)

---

[12]A connected sum of two manifolds is defined by cutting out a ball on each and gluing the resulting spheres together.

[13]Different choices of collars can lead to distinct bordisms, however such choices can always be absorbed into composition with a mapping cylinder C_f.

is obtained from cutting three disks B^2 out of the torus T^2.

Before we get to the notion of topological quantum field theories, we will now consider a different variant of a bordism category, one for which all manifolds are oriented. This can be seen as an instance of a bordism category with extra structure. In general this "extra structure" could be of topological nature, such as orientation, or geometric, such as a metric. We will only be interested in bordisms with extra structure of topological nature, more precisely with so-called *tangential structures*, more on these and their bordism categories in Section 2.3.1. For a detailed introduction of orientations on a manifold see for example [Fre12, Chapter 2].

Definition 1.4.8. The symmetric monoidal, *oriented, closed bordism category* $\mathrm{Bord}^{\mathrm{or}}_{n,n-1}$ in dimension n is defined as:

- objects are given by closed, oriented $(n-1)$-dimensional manifolds,

- morphisms are given by classes of bordisms (M, θ_1, θ_2) such that M is oriented, θ_1 is orientation reversing, θ_2 is orientation preserving, and diffeomorphisms between bordisms are orientation preserving,

- unit morphisms, composition, and the symmetric monoidal structure are the ones induced by $\mathrm{Bord}_{n,n-1}$.

Examples 1.4.9.   1. Every object in $\mathrm{Bord}^{\mathrm{or}}_{1,0}$ is a disjoint union of finitely many oriented points such as
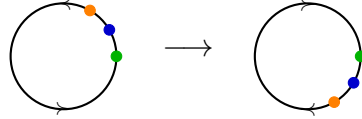
$$\bullet \quad \bullet \quad \bullet \quad \bullet \qquad (1.4.23)$$
$$+ \quad - \quad - \quad +$$

Every morphism can be obtained from disjoint union and gluing, from the following six oriented bordisms:

$$(1.4.24)$$
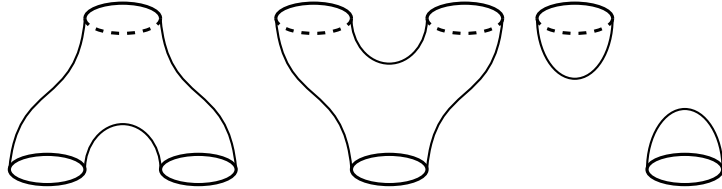
and the symmetric braiding.

2. Every object in $\mathrm{Bord}^{\mathrm{or}}_{2,1}$ is the disjoint union of oriented circles. However since the following diffeomorphism between the differently oriented circles is orientation preserving

$$(1.4.25)$$

it induces an isomorphism in $\mathrm{Bord}^{\mathrm{or}}_{2,1}$. We will therefore identify both circles and denote both of them by S^1.

Every morphism of $\mathrm{Bord}^{\mathrm{or}}_{2,1}$ is obtained from composition and disjoint union through the following four generators

$$(1.4.26)$$

referred to as the *pair of pants*, the *co pair of pants*, the *cup*, and the *cap*. This result follows almost immediately from the classification result of 2-dimensional compact manifolds cited above. One different way to proof this decomposition is through *Morse theory*, this is worked out for example in [Koc03, Chapter 1]. This result plays a crucial role in the classification of TQFTs "living" on compact two dimensional oriented manifolds, see Theorem 1.5.4 below.

A property of bordism categories which turns out to be extremely important for both the theory and constructions of TQFTs is their rigidity:

Proposition 1.4.10. The monoidal categories $\mathrm{Bord}^{\mathrm{or}}_{n,n-1}$ and $\mathrm{Bord}_{n,n-1}$ are rigid for any $n \in \mathbb{Z}_+$.

Proof. We will only prove the oriented case, the unoriented case can be treated analogously by ignoring the orientations. For $\Sigma \in \mathrm{Bord}^{\mathrm{or}}_{n,n-1}$, we set ${}^{\vee}\Sigma = \Sigma^{\vee} := -\Sigma$, where $-\Sigma$ is the same manifold but with opposite orientation.[14] Choosing the standard orientation on the unit interval I, the oriented cylinder $I \times \Sigma$ can be viewed as both an evaluation map $\Sigma \sqcup \Sigma^{\vee} \to \varnothing$ and a coevaluation map $\varnothing \to \Sigma^{\vee} \sqcup \Sigma$ in $\mathrm{Bord}^{\mathrm{or}}_{n,n-1}$. These maps satisfy the Zorro identities because the associated bordisms are in the same oriented diffeomorphism class. $\square$

We will always consider our bordism categories to be rigid with duality data chosen as in the proof above.

---

[14]See [Fre12, Definition 2.14] for the definition of the opposite orientation.

1.5 Two dimensional closed oriented TQFTs

In this last section of the preliminaries we are now finally able to give a rigorous definition of *topological quantum field theories* in the spirit of [Ati88]. After discussing a few general properties we will focus on the two dimensional oriented case and discuss the classification result in Theorem 1.5.4. We will only sketch a proof of this result because a full proof would lead us to far outside the scope of this thesis. Nonetheless we think it is important to at least illustrate the proof in some detail as it serves as the archetypical example of a classification result for TQFTs.

Definition 1.5.1. Let $\mathcal{C}$ be a symmetric monoidal category. A *closed n-dimensional topological quantum field theory* (*TQFT*) with values in $\mathcal{C}$ is a symmetric monoidal functor

$$\mathcal{Z}\colon \mathrm{Bord}_{n,n-1} \to \mathcal{C}. \tag{1.5.1}$$

A *morphism* between closed n-dimensional TQFTs $\mathcal{Z}$ and $\mathcal{Z}'$ is a monoidal natural transformation $\eta\colon \mathcal{Z} \to \mathcal{Z}'$. The *category of closed n-dimensional TQFTs* is the category $\mathrm{Fun}^{\otimes,\beta}\left(\mathrm{Bord}_{n,n-1}, \mathcal{C}\right)$ of braided monoidal functors together with monoidal natural transformations.

We will often times abbreviate closed n-dimensional TQFT to only closed TQFT when the context is clear. We can analogously define *closed oriented TQFTs*, their morphisms, and their category by substituting the unoriented bordism category $\mathrm{Bord}_{n,n-1}$ with the oriented bordism category $\mathrm{Bord}^{\mathrm{or}}_{n,n-1}$. More generally there is a notion of *closed $\mathcal{X}$-TQFT* for any tangential structure $\mathcal{X}$, see Section 2.3.2. We will also be interested in variations of $\mathrm{Bord}_{n,n-1}$ where the underlying manifolds have corners or stratifications, the rule of thumb is as long as the bordism category contains only topological data, we will call a symmetric monoidal functor out of it a *TQFT*, with appropriate prefixes discussed in the following chapters. If it is clear from the context we will drop the prefixes and simply call it a TQFT.

For $\mathcal{C} = \mathrm{Vect}_{\Bbbk}$ or $\mathcal{C} = \mathcal{S}\mathrm{Vect}_{\Bbbk}$ and a TQFT $\mathcal{Z}\colon \mathrm{Bord}_{n,n-1} \to \mathcal{C}$, we call $\mathcal{Z}(\Sigma)$ the *state space* of $\Sigma \in \mathrm{Bord}_{n,n-1}$. One immediate consequence of the definition of TQFTs and Proposition 1.4.10 is that the state spaces are finite-dimensional:

Proposition 1.5.2. Let $\mathcal{Z}\colon \mathrm{Bord}_{n,n-1} \to \mathcal{C}$ be a TQFT. Then $\mathcal{Z}(\Sigma)$ is dualisable for any $\Sigma \in \mathrm{Bord}_{n,n-1}$.

Proof. Every object $\Sigma \in \mathrm{Bord}_{n,n-1}$ is dualisable by Proposition 1.4.10, furthermore $\mathcal{Z}$ is a symmetric monoidal functor and monoidal functors preserve duality [Eti+16, Section 2.10]. $\square$

In the case of $\mathcal{C} = \mathrm{Vect}_{\Bbbk}$ (or $\mathcal{S}\mathrm{Vect}_{\Bbbk}$), the dualisable objects are exactly the finite-dimensional vector spaces (super vector spaces). One further direct consequence of Proposition 1.4.10 is the following:

Proposition 1.5.3. Let $\mathcal{C}$ be a symmetric monoidal category. Then $\mathrm{Fun}^{\otimes,\beta}\left(\mathrm{Bord}_{n,n-1}, \mathcal{C}\right)$ is a groupoid, i.e. morphisms between TQFTs are always isomorphisms.

See [CR18, Appendix A.2].

Both of the above propositions have a corresponding version for TQFTs with tangential structure. For more on general properties of TQFTs see for example [CR18, Section 2.4.].

A natural question to consider when studying TQFTs (or really any QFT) is the following: What algebraic structure is common to all TQFTs in a given fixed dimension $n \in \mathbb{Z}_+$? In our functorial formulation of TQFTs this amounts to finding a groupoid (with algebraic interpretation/origin) which is equivalent to $\mathrm{Fun}^{\otimes,\beta}\left(\mathrm{Bord}_{n,n-1}, \mathcal{C}\right)$. Finding such a classification of TQFTs in terms of algebraic data is in general a very hard problem, however the basic idea to tackle this problem is actually quite simple. Very roughly one works in essentially three steps:

1. Find a description of the bordism category, e.g. $\mathrm{Bord}_{n,n-1}$, in terms of *generators and relations*[15], and observe an algebraic structure, expressed through the relations, on the generators;

2. Construct a functor from the category of TQFTs, e.g. $\mathrm{Fun}^{\otimes,\beta}\left(\mathrm{Bord}_{n,n-1}, \mathcal{C}\right)$, to the category, corresponding to the found algebraic structure;

3. Show that the functor is an equivalence;

The second and the third step are always quite similar for any type of TQFT and usually straightforward, modulo some verifications of course. The first part usually the most difficult because finding a presentation of $\mathrm{Bord}_{n,n-1}$ is equivalent to classifying all compact n-manifolds up to diffeomorphism. For exactly this reason there are currently no classification results of closed TQFTs in dimension $n > 3$. For the rest of this chapter, and a large part of this thesis, we will focus on $n = 2$. To illustrate the above procedure we will now discuss the classification of closed two dimensional oriented TQFTs in terms of commutative Frobenius algebras.

Theorem 1.5.4. Let $(\mathcal{C}, \otimes, \mathbb{1}, \beta)$ be a symmetric monoidal category. There is an equivalence of groupoids $\mathrm{Fun}^{\otimes,\beta}(\mathrm{Bord}_{2,1}^{\mathrm{or}}, \mathcal{C}) \cong \mathrm{ComFrob}(\mathcal{C})$.

---

[15]See [CR18, Section 3.2.] for a detailed introduction and a precise definition of what it means to be generated as a symmetric monoidal category by generators and relations.

Proof sketch. According to the general idea outlined above, we begin our proof with finding a presentation of the bordism category $\mathrm{Bord}_{2,1}^{\mathrm{or}}$ through *generators and relations*. Fortunately in dimension $n = 2$ such a presentation is known, we already used it in the Examples 1.4.7 and 1.4.9. The closed two dimensional oriented bordism category $\mathrm{Bord}_{2,1}^{\mathrm{or}}$ is generated as a symmetric monoidal category by the following data:

$$\mathcal{G}_0 = \{\,\bigcirc\,\} \equiv \{S^1\}$$

$$\mathcal{G}_1 = \left\{\,\text{(pair of pants)},\ \text{(copants)},\ \text{(cap)},\ \text{(cup)}\,\right\}$$

$$\mathcal{G}_2 = \left\{\ \begin{array}{cc} \text{(assoc. left)} = \text{(assoc. right)}, & \text{(coassoc. left)} = \text{(coassoc. right)}, \\[2mm] \text{(unit left)} = \text{(identity)} = \text{(unit right)}, & \text{(counit left)} = \text{(identity)} = \text{(counit right)}, \\[2mm] \text{(Frobenius left)} = \text{(identity)} = \text{(Frobenius right)}, & \text{(commutativity)} = \text{(mult.)} \end{array}\ \right\}$$

With $\mathcal{G}_0$ and $\mathcal{G}_1$ being the *generators*, and the set $\mathcal{G}_2$ the *relations*. The "interchange bordism" on the left hand side of the last relation is not in the set of generators $\mathcal{G}_1$ because we get it "by definition" of $\mathrm{Bord}_{2,1}^{\mathrm{or}}$ generated as a *symmetric* monoidal category. To show that $\{\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2\}$ really present $\mathrm{Bord}_{2,1}^{\mathrm{or}}$ one usually uses Morse theory [Koc03, Section 1.4.]. The relations $\mathcal{G}_2$ are reminiscent of the string diagrams used to define commutative Frobenius algebras. This reminiscence motivates us to define the functor

$$\mathcal{E} \colon \mathrm{Fun}^{\otimes, \beta}\left(\mathrm{Bord}_{2,1}^{\mathrm{or}}, \mathcal{C}\right) \to \mathrm{ComFrob}(\mathcal{C})$$

$$\mathcal{Z} \mapsto \left(\mathcal{Z}\left(\bigcirc\right), \mathcal{Z}\left(\text{(pants)}\right), \mathcal{Z}\left(\text{(copants)}\right), \mathcal{Z}\left(\text{(cap)}\right), \mathcal{Z}\left(\text{(cup)}\right)\right),$$

$$\eta \mapsto \eta_{S^1}.$$

From the relations $\mathcal{G}_2$ it is clear that the functor is well-defined on objects. To see that it is well-defined on morphism let us assume $\mathcal{Z}$ is strict, then $\eta_{S^1 \sqcup S^1} = \eta_{S^1} \otimes \eta_{S^1}$ and naturality of η imply that η_{S^1} is a morphism of Frobenius algebras in $\mathcal{C}$. For

example commutativity of the diagram

$$
\begin{array}{ccc}
\mathcal{Z}(S^1 \sqcup S^1) & \xrightarrow{\;\mathcal{Z}\left(\vcenter{\hbox{\includegraphics{pants}}}\right)\;} & \mathcal{Z}(S^1) \\
\Big\downarrow{\scriptstyle \eta_{S^1 \sqcup S^1} = \eta_{S^1} \otimes \eta_{S^1}} & & \Big\downarrow{\scriptstyle \eta_{S^1}} \\
\widetilde{\mathcal{Z}}(S^1 \sqcup S^1) & \xrightarrow{\;\widetilde{\mathcal{Z}}\left(\vcenter{\hbox{\includegraphics{pants}}}\right)\;} & \widetilde{\mathcal{Z}}(S^1)
\end{array}
$$

corresponds to η_{S^1} being compatible with the mutliplications. The other compatibility conditions can be shown in a similar way. If $\mathcal{Z}$ is not strict one needs to be a bit more careful and use the isomorphism $\eta_{S^1 \sqcup S^1} \cong \eta_{S^1} \otimes \eta_{S^1}$ and its inverse.

Finally we want to show that $\mathcal{E}$ is an equivalence. First note that η is fully determined by η_{S^1}, thus $\mathcal{E}$ is fully faithful. To show that $\mathcal{E}$ is essentially surjective, let $(A, \mu, \delta, \Delta, \epsilon) \in \mathrm{ComFrob}(\mathcal{C})$. We use this data to define a TQFT $\mathcal{Z}_A$ by setting

$$
\begin{aligned}
\mathcal{Z}_A \left(\bigcirc\right) &= A \\
\mathcal{Z}_A \left(\vcenter{\hbox{\includegraphics{pants}}}\right) &= \mu \\
\mathcal{Z}_A \left(\vcenter{\hbox{\includegraphics{cap}}}\right) &= \delta \\
\mathcal{Z}_A \left(\vcenter{\hbox{\includegraphics{copants}}}\right) &= \Delta \\
\mathcal{Z}_A \left(\vcenter{\hbox{\includegraphics{disk}}}\right) &= \eta
\end{aligned}
\tag{1.5.2}
$$

This fully determines a symmetric monoidal functor $\mathcal{Z}_A \colon \mathrm{Bord}_{2,1}^{\mathrm{or}} \to \mathcal{C}$ because $\mathrm{Bord}_{2,1}^{\mathrm{or}}$ is freely generated as a symmetric monoidal category by $\{\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2\}$: A symmetric monoidal functor from a freely generated symmetric monoidal category is fully determined by its action on the generators $\mathcal{G}_0$ and $\mathcal{G}_1$ as long as the relations $\mathcal{G}_2$ are satisfied, see [CR18, Section 3.2] or [Koc03, Chapter 3]. This means we only need to check if the relations are satisfied, for example we need to check that

$$
\mathcal{Z}_A \left(\vcenter{\hbox{\includegraphics{assoc1}}}\right) = \mathcal{Z}_A \left(\vcenter{\hbox{\includegraphics{assoc2}}}\right)
$$

holds, which it indeed does by associativity of μ. Similarly the other relations are satisfied exactly because A is a commutative Frobenius algebra. $\qquad\square$

To appreciate this result, we will give a few examples of two dimensional closed oriented TQFTs and their physical origin. For this we will heavily use the characterisation of Frobenius algebras in $\mathrm{Vect}_{\Bbbk}$ given by Proposition 1.2.19. We will be

very brief with our discussion of these, as a proper treatment of each of the examples would need familiarity with further concepts from algebra and geometry than we presuppose.

Examples 1.5.5. 1. Let G be a finite abelian group with unit element e. The free vector space $\Bbbk G$ is naturally a $\Bbbk$-algebra with multiplication induced by the multiplication of G, furthermore it is a Frobenius algebra with non-degenerate pairing defined on basis elements $g, h \in G$ by

$$\langle g, h \rangle = \delta_{g,h^{-1}}. \tag{1.5.3}$$

This example is related to *Dijkgraaf–Witten* theory, which is a discrete model of gauge theory, see [Bar05, Chapter 4] for more details.

2. Let X be an oriented closed n-dimensional manifold. The *de Rahm cohomology* $H_{\mathrm{dR}}^\bullet(X) = \bigoplus_{k=0}^n H_{\mathrm{dR}}^k(X)$ of X is an $\mathbb{R}$-algebra together with multiplication induced by the wedge product $\wedge$ of differential forms. Note that this multiplication is only graded commutative, i.e. commutative in $\mathcal{S}\mathrm{Vect}_\mathbb{R}$. The pairing

$$\langle [\alpha], [\beta] \rangle = \int_X \alpha \wedge \beta \tag{1.5.4}$$

is well-defined by Stokes' theorem and non-degenerate by Poincaré duality. Due to the graded commutativity of $\wedge$, $H_{\mathrm{dR}}^\bullet(X)$ is a Frobenius algebra in $\mathcal{S}\mathrm{Vect}_\mathbb{R}$ and not $\mathrm{Vect}_\mathbb{R}$. If X is a Kähler manifold, this commutative Frobenius algebra is related to the *A-twisted sigma model* with target X, see [Hor+03, Section 16.4.].

3. Let $n \in \mathbb{Z}_+$, and let $W \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial in n variables such that its *Jacobi algebra*

$$J_W = \mathbb{C}[x_1, \ldots, x_n]/(\partial_{x_1} W, \ldots, \partial_{x_n} W) \tag{1.5.5}$$

is finite-dimensional. Here $(\partial_{x_1} W, \ldots, \partial_{x_n} W)$ denotes the ideal generated by all partial derivatives of W, i.e.

$$(\partial_{x_1} W, \ldots, \partial_{x_n} W) = \left\{ \sum_{k=1}^n p_k \cdot \partial_{x_k} W \ \middle| \ p_1, \ldots, p_n \in \mathbb{C}[x_1, \ldots, x_n] \right\} \tag{1.5.6}$$

47

as an infinite-dimensional vector space over $\mathbb{C}$. The Jacobi algebra inherits its multiplication from $\mathbb{C}[x_1, \ldots, x_n]$, furthermore it can be shown that

$$\langle [\phi], [\psi] \rangle = \mathrm{Res}_{\{0\}} \left[\frac{\psi(x) \cdot \phi(x) \mathrm{d}x}{\partial_{x_1} W, \ldots, \partial_{x_n} W} \right] \tag{1.5.7}$$

is a non-degenerate pairing on J_W, see for example [GH94, Chapter 5]. Where $\mathrm{Res}_{\{0\}} \left[\frac{\psi(x) \cdot \phi(x) \mathrm{d}x}{\partial_{x_1} W, \ldots, \partial_{x_n} W} \right]$ is a generalization of the residue from complex analysis. This type of Frobenius algebra is associated to so-called *B-twisted affine Landau–Ginzburg models*, see [CM16] for details and further references.

Chapter 2

Spin and other tangential structures

The goal of this chapter is to give a thorough understanding of spin and more general tangential structures. For this the reader is assumed to have knowledge of fiber bundles, especially reductions of principal fiber bundles, associated vector bundles, and classifying spaces. The relevant background is given in Appendix A and the literature cited therein. Furthermore some basic knowledge about (co)homology will be needed for the existence and uniqueness of spin structures for a fixed manifold.

In Section 2.1 we begin by briefly reviewing the construction of the frame bundle associated to the tangent bundle of a chosen manifold and its relation to the special orthogonal group $SO(n)$. After this we will define a spin structure as a reduction of this frame bundle to a certain principal fiber bundle. Before we can do this, we will need to define what our structure group should be, this group will be the spin group. We will then briefly comment on the use of the spin group in physics and in particular its *spinor representation*. The question of existence and uniqueness of spin structures on manifolds will be answered using tools from algebraic topology, which we will first employ to answer the same question for orientations.

Afterwards we focus on one and two dimensional compact manifolds, where our question on the existence and uniqueness can be answered directly through elementary results in algebraic topology. Furthermore we will describe the group $Spin(2)$ in detail, and briefly discuss an equivalent definition of spin structures on surfaces. We then focus on describing the spin structures of an oriented circle in great detail, this example will be crucial for closed two dimensional spin TQFTs.

Finally we will consider general *tangential structures*, discuss a few examples thereof, and generalize the bordism categories from Section 1.4 to one for arbitrary

tangential structures, including a somewhat heuristic description of the spin bordism category.

2.1 Spin structures on manifolds

A spin structure can in general be defined on vector bundles E where a certain characteristic class vanishes. We will solely focus on the case where $E = TM$ for a given Riemannian manifold M and define the spin structure directly as a special principal fiber bundle over M.

Recall that for every n-dimensional manifold M the *frame bundle* $\mathrm{GL}(M)$ is a principal $\mathrm{GL}(n)$-bundle associated to the tangent bundle TM. Let

$$\mathrm{GL}(M)_x := \{\nu_x = (\nu_1, \ldots, \nu_n) \subset T_x M \,\big|\, \nu_x \text{ is a basis of } T_x M\},$$

then we define the set

$$\mathrm{GL}(M) = \bigsqcup_{x \in M} \mathrm{GL}(M)_x.$$

To construct a principal $\mathrm{GL}(n)$-bundle from this set we define a $\mathrm{GL}(n)$ right action by multiplication of a basis with a matrix from the right. Furthermore $\mathrm{GL}(M)$ can be equipped with the structure of a smooth manifold using the atlas of M [Ham17, Theorem 4.4.1]. For the projection map $\pi\colon \mathrm{GL}(M) \to M$ set $\pi(\nu_x) := x$. It can be shown that $\mathrm{GL}(M)$ can always be reduced to an $\mathrm{O}(n)$-bundle of *orthonormal frames* denoted by $\mathrm{O}(M)$, because $\mathrm{O}(n)$ is a maximally compact subgroup of $\mathrm{GL}(n)$. In the language of classifying spaces this statement corresponds to the fact that the inclusion $\iota\colon \mathrm{O}(n) \hookrightarrow \mathrm{GL}(n)$ induces a homotopy equivalence $B\iota\colon B\mathrm{O}(n) \to B\mathrm{GL}(n)$ [Die08, Proposition 14.4.13]. On the manifold level this reduction corresponds to the choice of a Riemannian metric g on M, see [Bau14, Beispiel 2.12] for details. Thus from now on we will only consider Riemannian manifolds. Recall further that a manifold is called *orientable* if there is a reduction of $\mathrm{O}(M)$ to an $\mathrm{SO}(n)$-bundle. In Section 2.3.2 we will see a different reason why working with $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ instead of $\mathrm{GL}(n)$ and $\mathrm{GL}^+(n)$ will lead to the same notion of spin structures. The corresponding bundle of oriented, orthonormal frames is denoted by $\mathrm{SO}(M)$. This statement tells us that we can choose all cocycles of the oriented frame bundle as maps into $\mathrm{SO}(n)$.

To motivate the following definition recall that in physics fermionic particles are described by so called *spinors* which transform under a full rotation by a change of sign. In terms of the rotation group this means that not a 2π rotation but a 4π rotation acts as the identity on spinors. To be more precise a *spinorial representation*

of the rotation group is a representation of the double cover of the rotation group and spinors are the elements in the representation space [SU01, Chapter 7].

Definition 2.1.1. Let $n \in \mathbb{N}$. The n-dimensional *spin group* $\mathrm{Spin}(n)$ to be the double cover of the special orthogonal group $\mathrm{SO}(n)$. This means there exists a group homomorphism $\lambda \colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ which is also a topological double cover.

Remarks 2.1.2. The following is a collection of further properties concerning spin groups, a generalisation for pseudo-orthogonal groups, as well as a few low dimensional examples and are mostly taken from either [Ham17, Chapter 6.5.1] or [LM89, Chapter 8].

- $\mathrm{Spin}(n)$ exists and is unique up to isomorphism. It can be explicitly constructed as a subgroup of the Clifford algebra $\mathrm{Cl}(n)$. It can furthermore be shown that $\mathrm{Spin}(n)$ is a Lie group and λ a Lie group homomorphism.

- For $n > 2$, $\mathrm{Spin}(n)$ is simply connected and thus the universal covering group of $\mathrm{SO}(n)$.

- Considering the pseudo-orthogonal group $\mathrm{SO}(s,t)$ of signature (s,t) instead leads to further spin groups. Especially $\mathrm{Spin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})$, the double cover of the special orthochronous Lorentz group $\mathrm{SO}^+(1,3)$, plays an important role in physics.

- In low dimensions the spin groups are given by:

  - $\mathrm{Spin}(1) \cong \mathbb{Z}_2$
  - $\mathrm{Spin}(2) \cong \mathrm{U}(1)$
  - $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$
  - $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$

With this preparation we can now define a *spin structure* on a manifold. This additional structure is needed to be able to describe fermionic fields in a consistent way on a manifold, see [Wal84, Chapter 13] for a more thorough motivation.

Definition 2.1.3. Let (M, g) be an oriented, n-dimensional Riemannian manifold. A *spin structure* on M is a $\mathrm{Spin}(n)$-principal fiber bundle

$$\pi_{\mathrm{Spin}} \colon \mathrm{Spin}(M) \to M$$

together with a smooth double covering

$$\Lambda \colon \mathrm{Spin}(M) \to \mathrm{SO}(M)$$

such that

$$
\begin{array}{ccc}
\mathrm{Spin}(M) \times \mathrm{Spin}(n) & \xrightarrow{\;\dot{\;}\;} & \mathrm{Spin}(M) \\
\downarrow{\scriptstyle \Lambda \times \lambda} & & \downarrow{\scriptstyle \Lambda} \quad\searrow{\scriptstyle \pi_{\mathrm{Spin}}} \\
& & \qquad\qquad M \\
\downarrow & & \downarrow \quad\nearrow{\scriptstyle \pi_{SO}} \\
\mathrm{SO}(M) \times \mathrm{SO}(n) & \xrightarrow{\;\dot{\;}\;} & \mathrm{SO}(M)
\end{array}
\qquad (2.1.1)
$$

commutes. Here $\lambda \colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the double covering of Lie groups and the $\cdot$ on the horizontal arrows indicates right action of the respective structure group. We will often abbreviate the data of a spin structure on a manifold to the tuple (M, Λ) and call this a *manifold with spin structure*.

A spin structure on M is more than a $\mathrm{Spin}(n)$-principal fiber bundle: it needs to be compatible with the natural $\mathrm{SO}(n)$-bundle of M in the sense that the triangle on the right of Diagram (2.1.1) commutes. In the language of principal fiber bundles this means a spin structure is a λ-equivariant bundle morphism $\Lambda \colon \mathrm{Spin}(M) \to \mathrm{SO}(M)$, i.e. a λ-reduction of $\mathrm{SO}(M)$. Furthermore $\Lambda \colon \mathrm{Spin}(M) \to \mathrm{SO}(M)$ is a $\mathbb{Z}_2$-principal fiber bundle, i.e. a double covering of $\mathrm{SO}(M)$.

We can generalize the definition of spin structures to the pseudo-Riemannian case by replacing all groups with their pseudo-Riemannian correspondences, see for example [Ham17, Chapter 6]. Even more generally we can define a spin structure without a (pseudo) Riemannian metric by using the group $\mathrm{GL}^+(n)$ of orientation preserving automorphisms of $\mathbb{R}^n$ and its double cover $\widetilde{\mathrm{GL}}^+(n)$. This approach will not be considered for now as we first want to gain more intuition in this topic. Also working with $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$ will allow us to simplify the discussion of existence and uniqueness of spin structures. We also think that the relation to the physical motivation is clearer using the chosen definition. In later chapters we will introduce the concept of *tangential structures* using classifying spaces. Orientations and spin structures are special cases of such tangential structures. As briefly mentioned in the discussion of the frame bundle, the inclusion of compact subgroups induces a homotopy equivalence of the corresponding classifying spaces. This means that using using $\mathrm{GL}^+(n)$ and $\widetilde{GL}^+(n)$ will lead us to an equivalent notion of tangential structure.

Even though the physical motivation for spin structures lies not in the spin group, but the associated spinors. We will only briefly recall the definition of the *spinor*

representation of Spin(n), as this would lead us to far in to the theory of Clifford algebras, see [Ham17, Chapter 6] for a thorough discussion. From the structure theorem of complex Clifford algebras [Ham17, Theorem 6.3.21] we get an algebra representation $\rho \colon \mathbb{C}l(n) \to \mathrm{End}(\mathbb{C}^N)$ with $N = n/2$ if n is even and $N = (n-1)/2$ if n is odd. We will call this the *spinor representation* of $\mathbb{C}l(n)$ and the space $\mathbb{C}^N$ the space of *spinors*. As noted before the spin group is a subgroup of the real Clifford algebra $Cl(n)$. We further know that $Cl(n) \otimes \mathbb{C} \cong \mathbb{C}l(n)$. Putting this together means we get a complex representation $\kappa \colon \mathrm{Spin}(n) \to \mathrm{GL}(N, \mathbb{C})$ by composing the inclusion $\mathrm{Spin}(n) \subset Cl(n)$ with the spinor representation. We will call this representation the *spinor representation* of $\mathrm{Spin}(n)$.

Definitions 2.1.4. Let (M, g) be an oriented, n-dimensional Riemannian manifold with spin structure $\Lambda \colon \mathrm{Spin}(M) \to \mathrm{SO}(M)$, and $\kappa \colon \mathrm{Spin}(n) \to \mathrm{GL}(N, \mathbb{C})$ the spinor representation of $\mathrm{Spin}(n)$. The associated vector bundle

$$\mathcal{S} := \mathrm{Spin}(M) \times_\kappa \mathbb{C}^N$$

is called the *spinor bundle* of M, and its sections are called *spinors* or *spinor fields*.

Before we can talk about existence and uniqueness of spin structures, we need a notion of morphism between spin structures. For this the notion of a principal bundle morphism from Definition A.0.5 is a bit too weak, instead we will employ the following.

Definition 2.1.5. Let (M, Λ) and $(\widetilde{M}, \widetilde{\Lambda})$ be manifolds with spin structure. A *morphism between manifolds with spin structure* is given by a bundle morphism $F \colon \mathrm{Spin}(M) \to \mathrm{Spin}(\widetilde{M})$ such that

$$
\begin{array}{ccc}
\mathrm{Spin}(M) & \xrightarrow{\ F\ } & \mathrm{Spin}(\widetilde{M}) \\
\Lambda \downarrow & & \downarrow \widetilde{\Lambda} \\
\mathrm{SO}(M) & \xrightarrow{\ Tf_*\ } & \mathrm{SO}(\widetilde{M}) \\
\pi_{SO} \downarrow & & \downarrow \widetilde{\pi}_{SO} \\
M & \xrightarrow{\ f\ } & \widetilde{M}
\end{array}
$$

commutes, where f is the induced map on base spaces, and Tf_* the bundle morphism induced by the derivative of f. An *isomorphism of spin structures* on a fixed manifold M is a morphism of manifolds with spin structure F as above where $M = \widetilde{M}$,

$F\colon \mathrm{Spin}(M) \to \widetilde{\mathrm{Spin}}(M)$ is a bundle isomorphism, and the induced map is $f = \mathrm{id}_M$, i.e. we have the following commutative triangle

$$
\begin{array}{ccc}
\mathrm{Spin}(M) & \xrightarrow{\quad F \quad} & \widetilde{\mathrm{Spin}}(M) \\
& \Lambda \searrow \quad \swarrow \tilde{\Lambda} & \\
& \mathrm{SO}(M) &
\end{array}
\quad .
$$

This means F is also a bundle isomorphism between $\mathrm{Spin}(M)$ and $\widetilde{\mathrm{Spin}}(M)$ viewed as $\mathbb{Z}_2$-bundles over $\mathrm{SO}(M)$.

Before stating a result for the existence of spin structures on a given manifold we will review the simpler question if a given manifold can be oriented. This allows us to introduce the relevant concepts from algebraic topology in a simpler setting. To this end we will now outline the definition of *Čech cohomology* for a manifold following [Nak03, Chapter 11.6]. We will only remark on subtleties and choices when they arise and refer the reader to a more thorough introduction given in [Wer19, Chapter 2.3.2], also [LM89, Appendix A] for a more general discussion on the relation between Čech cohomology and principal fiber bundles.

Let M be a manifold, $\{U_i\}_{i \in I}$ a good open cover of M^1, and $r \in \mathbb{N}$. A *Čech r-cochain* is a locally constant function $f\colon U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_r} \to \mathbb{Z}_2$ for $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_r} \neq \varnothing$, which is totally symmetric. We will employ the notation $f(i_0, \ldots, i_r)$ for such a function, total symmetry can now be expressed as

$$
f(i_{\sigma(0)}, \ldots, i_{\sigma(r)}) = f(i_0, \ldots, i_r)
$$

for an arbitrary permutation $\sigma \in S_{r+1}$. The space of these maps will be denoted by $\check{C}^r(M, \mathbb{Z}_2)$, this space can be endowed with the structure of an abelian group stemming from $\mathbb{Z}_2$. We will use multiplicative notation for $\mathbb{Z}_2$, therefore the unit will be denoted by 1. We further define the *coboundary operator* $\delta_r\colon \check{C}^r(M, \mathbb{Z}_2) \to \check{C}^{r+1}(M, \mathbb{Z}_2)$ by

$$
(\delta_r f)(i_0, \ldots, i_{r+1}) := \prod_{j=0}^{r+1} f(i_0, \ldots, \hat{i}_j, \ldots, i_{r+1})
$$

where the variable with the hat "$\hat{\ }$" is omitted. By a direct computation it can be shown that $\delta_r \circ \delta_{r-1} = 1$. Therefore we can define a δ-cohomology, the *r*th-*Čech cohomology group* $\check{H}^r(M, \mathbb{Z}_2) := \ker(\delta_r) / \mathrm{im}(\delta_{r-1})$.

---

[1]Good means that all the sets U_i are contractible, and all finite intersections are either empty or contractible, this always exists for a manifold as we assume it to be paracompact.

Usually one defines the Čech cohomology ring $\check{H}^*(\mathcal{U}, R)$ with coefficients in a ring R only for a cover $\mathcal{U} := \{U_i\}_{i \in I}$, however there is a theorem [Wer19, Theorem 2.3.7] which states that if $\mathcal{U}$ is a good cover of M then $\check{H}^*(\mathcal{U}; R) \cong H^*_{\text{sing}}(M, R)$, where $H^*_{sing}(M, R)$ is the singular cohomology of M. Which in particular implies that our construction is independent of the choice of cover $\mathcal{U}$ up to isomorphism.

For a given Riemannian manifold (M, g) we will now construct an element of $\check{H}^1(M, \mathbb{Z}_2)$ using the orthonormal frame bundle $\text{SO}(M)$. Let $\{U_i\}_{i \in I}$ be a good covering of M and let $\{U_i, \phi_i\}_{i \in I}$ be a bundle atlas for $\text{SO}(M)$. Then the cocycle $\{\phi_{i,j}\}_{i,j \in I}$, see Appendix A, can be used to define a Čech 1-cochain by setting $f(i, j) := \det(\phi_{ij}(x)) = \pm 1$ for any $x \in U_i \cap U_j \neq \varnothing$. This is independent of x because $\{U_i\}_{i \in I}$ is a good cover, thus we will drop the x dependence from now on. In particular the so defined function is locally constant and furthermore $f(i, j) = f(j, i)$ so it is indeed a Čech 1-cochain. Acting with the coboundary operator δ on f shows that it is δ-closed:

$$\begin{aligned}
\delta f(i, j, k) &= \det(\phi_{ij})\det(\phi_{jk})\det(\phi_{ki}) \\
&= \det(\phi_{ij}\phi_{jk}\phi_{ki}) \\
&= \det(\phi_{ik}\phi_{ki}) \\
&= \det(\text{id}_n) \\
&= 1
\end{aligned}$$

by the cocycle conditions. We define the first *first Stiefel–Whitney class* $w_1(M)$ of M as the cohomology element represented by f: $w_1(M) := [f] \in \check{H}^1(M, \mathbb{Z}_2)$. It can be shown that $w_1(M)$ is independent of the choice of bundle maps $\{\phi_i\}_{i \in I}$, see [Wer19, Chapter 2.3.2] for more details. The significance of the first Stiefel–Whitney class lies in the following theorem.

Theorem 2.1.6. Let (M, g) be an Riemannian manifold.

1. M is orientable if and only if its first Stiefel–Whitney class is trivial.

2. If M is orientable, then there is a bijection between the set of isomorphism classes of orientations on M and the Čech cohomology group $\check{H}^0(M, \mathbb{Z}_2)$.

Proof. " $\implies$ " If M is orientable, the frame bundle can be reduced to a $\text{SO}(n)$-bundle and for every bundle atlas $\{U_i, \phi_i\}_{i \in I}$ we have $\phi_{ij} \in \text{SO}(n)$ and therefore $f(i, j) = \det(\phi_{ij}) = 1$, so $[f] = w_1(M)$ is trivial.
" $\impliedby$ " If $w_1(M)$ is trivial we have an $f_0 \in \check{C}^0(M, \mathbb{Z}_2)$ such that $\delta f_0 = f$. Since $f_0(i) = \pm 1$ we always find $h_i \in \text{O}(n)$ such that $\det(h_i) = f_0(i)$, using these $\{h_i\}_{i \in I}$

we can define a new chart $\{\bar{\phi}_i\}_{i \in I}$ by setting $\bar{\phi}_i := h_i \cdot \phi_i$ such that for any pair of indices (i, j) with $\det(\phi_{ij}) = -1$ we set $f_0(i) = 1$ and $f_0(j) = -1$ then $\det(\bar{\phi}_{ij}) = \det(h_i \cdot \phi_{ij} \cdot h_j^{-1}) = -\det(\phi_{ij}) = 1$. So M is orientable.

For a proof of the second statement see [LM89, Theorem 1.2]. $\qquad\square$

This theorem shows that the first Stiefel–Whitney class is an obstruction to the orientability of M, in other words $w_1(M)$ is a characteristic class for $\mathrm{O}(n)$-bundles.

As explained in Appendix A we should be able to obtain the first Stiefel–Whitney on more general grounds. For this recall that orientability of M is equivalent to the existence of a continuous map $g \colon M \to B\mathrm{SO}(n)$ lifting the classifying map τ_M of the frame bundle $\mathrm{O}(M)$, such that the following diagram commutes up to homotopy

$$
\begin{array}{ccc}
 & & B\mathrm{SO}(n) \\
 & \nearrow^{g} & \downarrow \\
M & \xrightarrow{\tau_M} & B\mathrm{O}(n).
\end{array}
\tag{2.1.2}
$$

Further consider the short exact sequence of groups

$$
0 \longrightarrow \mathrm{SO}(n) \longrightarrow \mathrm{O}(n) \longrightarrow \mathbb{Z}_2 \longrightarrow 0
$$

given by the determinant homomorphism, this sequence induces the homotopy fibration

$$
B\mathrm{SO}(n) \longrightarrow B\mathrm{O}(n) \longrightarrow B\mathbb{Z}_2.
$$

We can now combine this fibration with Diagram (2.1.2) to obtain

$$
\begin{array}{cccc}
 & & B\mathrm{SO}(n) \\
 & \nearrow^{g} & \downarrow \\
M & \xrightarrow{\tau_M} & B\mathrm{O}(n) & \xrightarrow{w_1} & B\mathbb{Z}_2.
\end{array}
\tag{2.1.3}
$$

This diagram is exactly of the form of diagram (A.0.2), because $B\mathbb{Z}_2$ is a $K(\mathbb{Z}_2, 1)$ space [Die08, Example 14.4.8]. The cohomology element $w_1 \in H^1(B\mathrm{O}(n), \mathbb{Z}_2)$ corresponds to the first Stiefel–Whitney class from above in the sense that $\tau_M^*(w_1) = w_1(M)$. Indeed using more general methods, it is possible to show that $w_1(M)$ corresponds to an element $w_1 \in H^*(B\mathrm{O}(n); \mathbb{Z}_2)$ [LM89, Chapter II].

Continuing in a similar way we will now consider the question if a manifold admits a spin structure. Let (M, g) be an oriented n-dimensional Riemannian manifold, $\{U_i\}_{i \in I}$ a good covering of M, and $\{U_i, \phi_i\}_{i \in I}$ a bundle atlas for $\mathrm{SO}(M)$. Using the

group homomorphism $\lambda \colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ we consider a lifting $\tilde{\phi}_{ij}$ of the cocycle ϕ_{ij}, i.e. $\lambda(\tilde{\phi}_{ij}) = \phi_{ij}$. Note that for the lift of ϕ_{ij} a choice between $\tilde{\phi}_{ij}$ and $-\tilde{\phi}_{ij}$ can be made. Since

$$\lambda\left(\tilde{\phi}_{ij}\tilde{\phi}_{jk}\tilde{\phi}_{ki}\right) = \phi_{ij}\phi_{jk}\phi_{ki} = \mathrm{id}_n$$

we have $\tilde{\phi}_{ij}\tilde{\phi}_{jk}\tilde{\phi}_{ki} \in \ker(\lambda) = \{-\mathrm{I}, \mathrm{I}\}$, where I denotes the unit element of $\mathrm{Spin}(n)$. In order to define a $\mathrm{Spin}(n)$-bundle the $\tilde{\phi}_{ij}$ need to satisfy the cocycle condition $\tilde{\phi}_{ij}\tilde{\phi}_{jk}\tilde{\phi}_{ki} = \mathrm{I}$. This motivates the definition of a Čech 2-cochain $f(i, j, k)$ by

$$f(i, j, k)\mathrm{I} := \tilde{\phi}_{ij}\tilde{\phi}_{jk}\tilde{\phi}_{ki}.$$

It can be checked that f is symmetric and closed. Furthermore the cohomology element $[f]$ is independent of any choices we made. Therefore we have an element $w_2(M) := [f] \in \check{H}^2(M, \mathbb{Z}_2)$ called the *second Stiefel–Whitney class* of M.

Theorem 2.1.7. Let (M, g) be an oriented, Riemannian manifold.

1. M admits a spin structure if and only if the second Stiefel–Whitney class $w_2(M) \in \check{H}^2(M, \mathbb{Z}_2)$ is trivial, i.e. $w_2(M) = 1$.

2. If M admits a spin structure, then there is a bijection between the set of isomorphism classes of spin structures on M and the Čech cohomology group $\check{H}^1(M, \mathbb{Z}_2)$.

Proof sketch. The proof of the first statement is similar to the proof of the statement about orientability and the first Stiefel–Whitney class:
" $\Longrightarrow$ " If M has a spin structue, then we have a $\mathrm{Spin}(n)$-bundle with transition functions $\{\tilde{\phi}_{ij}\}_{i,j\in I}$ which satisfy the cocycle condition, $\tilde{\phi}_{ij}\tilde{\phi}_{jk}\tilde{\phi}_{ki} = \mathrm{I}$, therefore $w_2(M)$ is trivial.
" $\Longleftarrow$ " Assume $w_2(M)$ is trivial and consider a lift $\{\tilde{\phi}_{ij}\}_{i,j\in I}$ of the transition functions $\{\phi_{ij}\}_{i,j\in I}$ of $\mathrm{SO}(M)$ through λ, remember there is a choice to pick $\pm\tilde{\phi}_{ij}$. Let $f \in w_2(M)$ be a representative of $w_2(M)$, there is a Čech 1-cochain f_1, such that $f(i, j, k) = \delta f_1(i, j, k)$. We consider the 1-cochain $f_1(i, j)$ to be defined as the sign of the chosen lift $\tilde{\phi}_{ij}$. If we define new transition functions by $\tilde{\phi}'_{ij} = f_1(i, j)\tilde{\phi}_{ij}$, this gives us $\tilde{\phi}'_{ij}\tilde{\phi}'_{jk}\tilde{\phi}'_{ki} = \left(\delta f_1(i, j, k)\right)^2 \mathrm{I} = \mathrm{I}$. Hence the $\{\tilde{\phi}'_{ij}\}_{i,j\in I}$ define a $\mathrm{Spin}(n)$-bundle over M, which gives a spin structure because the transition functions are explicitly constructed from the oriented orthonormal frame bundle [Wer19, Chapter 2.3].
For a proof of the second statement see [LM89, Corollary 1.5]. $\qquad\square$

In analogy to the first Stiefel–Whitney class also the second defines a universal characteristic class as explained in [LM89, Appendix A].

2.2 Spin surfaces and circles

We will now focus on spin structures on surfaces, where a *surface* is understood as an oriented, connected, compact two dimensional manifold, possibly with boundary. Recall from Section 1.4, the classical result from differential topology, which states that surfaces are classified, up to diffeomorphism, by their genus g and the number of oriented boundary components h. We will denote a surface of genus g and h oriented boundary components by $\Sigma_{g,h}$ and simply Σ_g for a closed surface of genus g. For our purposes only oriented surfaces will be of interest so in the following smooth maps between them are understood as orientation preserving, furthermore the oriented orthonormal frame bundle $\mathrm{SO}(\Sigma)$ of the surface Σ will be simply referred to as the frame bundle of Σ.

Definitions 2.2.1. A *spin surface* is a tuple given by a surface Σ together with a spin structure $\Lambda \colon \mathrm{Spin}(\Sigma) \to \mathrm{SO}(\Sigma)$.

According to Theorem 2.1.7 the classification and existence of spin structures on a surface Σ is tied to the Čech cohomology groups $\check{H}^1(\Sigma, \mathbb{Z}_2)$ and $\check{H}^2(\Sigma, \mathbb{Z}_2)$ which are, as noted in the previous chapter, isomorphic to the singular cohomology groups $H^1(\Sigma, \mathbb{Z}_2)$ and $H^2(\Sigma; \mathbb{Z}_2)$. For a closed surface Σ_g it is well known from algebraic topology that $H^1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ and $H^2(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}$ [Nak03, Example 3.12], thus $H^2(\Sigma_g, \mathbb{Z}_2) = 0$, by the universial coefficient theorem [Hat01, Theorem 3.2]. This means that every closed surface can be equipped with a spin structure. Furthermore $|H^1(\Sigma_g, \mathbb{Z}_2)| = 2^{2g}$, so there are 2^{2g} inequivalent spin structures.

In two dimensions the spin group can be constructed using quite elementary means, this will be sketched in what follows, see [Ham17, Chapter 6] for the general approach based on Clifford algebras. For the rest of this chapter we will tacitly identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ and use a complex coordinate z.[2] Furthermore by identifying $z \in \mathbb{C}$ with the real 2×2 matrix $\begin{pmatrix} \mathrm{Re}(z) & -\mathrm{Im}(z) \\ \mathrm{Im}(z) & \mathrm{Re}(z) \end{pmatrix}$ gives us isomorphisms $S^1 \cong \mathrm{SO}(2) \cong \mathrm{U}(1)$ of Lie groups.

We define a map

$$\mathrm{U}(1) \to \mathrm{U}(1)$$
$$z \mapsto z^2.$$

It can be checked that composition of this map with the before mentioned ismorphisms defines a smooth double covering homomorphism $\lambda \colon \mathrm{U}(1) \to \mathrm{SO}(2)$. Thus we find $\mathrm{Spin}(2) \cong \mathrm{U}(1)$ and $\ker(\lambda) \cong \mathbb{Z}_2$.

---

[2]We still consider our manifolds and maps to be real and smooth, and not complex and holomorphic.

For a given spin surface $\mathrm{Spin}(\Sigma)$ acting from the right with the non-trivial element in the kernel of λ induces an involution $w\colon \mathrm{Spin}(\Sigma) \to \mathrm{Spin}(\Sigma)$ sometimes called the *spin flip*. This involution is precisely a *deck transformation* of $\mathrm{Spin}(\Sigma)$ viewed as a $\mathbb{Z}_2$-bundle over $\mathrm{SO}(\Sigma)$.[3]

Remark 2.2.2. It should be noted here that in two dimensions there is the possibility to consider the r-fold cover $\mathrm{Spin}^r(2)$ of $\mathrm{SO}(2)$ instead of the double cover and define a corresponding *r-spin structure* on a surface analogous to Definition 2.1.3 for any $r \in \mathbb{Z}_+$, see [RS21; SS20] for more details. In this context what we call a spin structure would be a 2-spin structure and an orientation a 1-spin structure.

Closely related to a surface in our context is the notion of a compact Riemann surface, i.e. a complex one dimensional manifold. For such a compact Riemann surface $\widetilde{\Sigma}$ there is another way to think of spin structures: The spinor bundle $\mathcal{S}$ associated to a spin structure is a square root bundle ($\mathcal{S} \otimes \mathcal{S} \cong K$) of the canonical line bundle $K = \Lambda^1 T^* \widetilde{\Sigma}$. The isomorphism is induced by the double covering of principal fiber bundles,[4] for a more detailed account on this point of view see [Ati71].

An important topological invariant for a manifold with spin structure is given by the *Atiyah invariant*, this is defined as the index of a differential operator (the Dirac operator) associated to the canonical spinor bundle of the manifold. For surfaces there is an equivalent algebraic invariant, given by the Arf invariant of a quadratic form on $H_1(\Sigma, \mathbb{Z}_2)$ of symplectic type which is related to the choice of spin structure on the surface [Joh80].

Example 2.2.3. As our motivation comes from topological field theory in two dimensions we will need to describe a category for bordisms with spin structure. To this end we will now discuss spin structures for the oriented circle S^1 and the associated spinor bundles. This will also allow us to relate our discussion to the use of different spinor fields in physics. Note that for $n = 1$ the spinor space from Defintion 2.1.4 is $\mathbb{C}$.

According to Theorem 2.1.7 we expect two non-isomorphic spin structures on S^1, because $H^1(S^1, \mathbb{Z}_2) \cong \mathbb{Z}_2$ [Nak03, Example 3.8]. To construct them note that S^1 is a Lie group and therefore its frame bundle is trivial, i.e. $\mathrm{SO}(S^1) = S^1 \times \mathrm{SO}(1)$. Note further that this space is isomorphic to S^1 itself because $\mathrm{SO}(1) \cong \{1\}$, the trivial group. In more geometric terms this means that there is exactly one oriented unit basis vector for the tangent space at each point. From Remark 2.1.2 we know

---

[3]The involution w exists for any manifold with spin structure, not just 2-dimensional ones.

[4]Technically this gives a spin structure with respect to the cotangent bundle and not the tangent bundle, but these two structures are "dual" to one another, see [Ebe06, Chapter 3.2].

that $\mathrm{Spin}(1) \cong \mathbb{Z}_2$, thus a spin structure on S^1 corresponds to a double cover of S^1. We will therefore use the terms double cover and spin structure interchangeably. As noted above there are two non-isomorphic spin structures of the circle. We will now directly construct the spaces and double covering maps. A different way to construct them is by lifting a cocycle of $\mathrm{SO}(S^1)$ to a $\mathrm{Spin}(1)$ cocycle for a good cover of S^1, see [Wer19, Example 2.3.17] for this approach.

The first double cover is the trivial, disconnected one with total space $\mathrm{Spin}^{\mathrm{R}}(S^1) = S^1 \times \mathbb{Z}_2 \cong S^1 \sqcup S^1$, and covering map given by

$$\pi^{\mathrm{R}} \colon S^1 \sqcup S^1 \to S^1$$
$$(z_1, z_2) \mapsto z_1.$$

We will call this spin structure the *Ramond structure* on S^1. The terminology is chosen to be in accordance to the one used in physics, as will be shown below. As an associated vector bundle of a trivial bundle, the spinor bundle is also trivial $\mathcal{S}^{\mathrm{R}} = S^1 \times \mathbb{C}$ [Nak03, Corollary 9.2]. We can therefore view spinor fields as maps from S^1 to $\mathbb{C}$, i.e. spinor fields of the Ramond structure are complex valued 2π periodic functions.

The second spin structure is given by the connected double cover $\mathrm{Spin}^{\mathrm{NS}}(S^1) \cong S^1$. The map

$$\pi^{\mathrm{NS}} \colon S^1 \to S^1$$
$$z \mapsto z^2$$

provides a double cover of S^1 because every $z \in S^1$ has exactly two square roots $\pm\sqrt{z}$. We will call this the *Neveu–Schwarz* structure on S^1. Embedded in $\mathbb{R}^3$ this space can be thought of as the boundary of the Möbius strip, see Figure (2.1b).

The associated spinor bundle is given by a "Möbius like" complex vector bundle $\mathcal{S}^{\mathrm{NS}} = (S^1 \times \mathbb{C})/\sim$ where we identify $(z_1, z_2) \sim (z_1, -z_2)$. The spinors from this bundle are 2π anti-periodic. This can be seen by considering $\mathcal{S}^{\mathrm{NS}}$ as the quotient space of $\mathbb{R} \times \mathbb{C}$ by the equivalence relation $(x, z) \sim (x + 2\pi, -z)$. As indicated above the name convention we have chosen for the spin bundles is in accordance to the convention used in physics for periodic and anti-periodic spinors on a circle [Pol98, Chapter 10.2].

Interestingly from the point of view of bordism theory, the NS-structure is the "trivial" one, which means that the NS-structure is the one induced by the unique (up to isomorphism) spin structure of the unit disk $B^2 \subset \mathbb{R}^2$. To see this first note that B^2 is contractible and therefore the fundamental group of B^2 is trivial, this implies the first cohomology group of B^2 is trivial as well, now according to Theorem 2.1.7

60
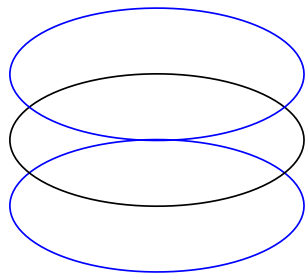
(a) Disconnected or "Ramond" double cover.
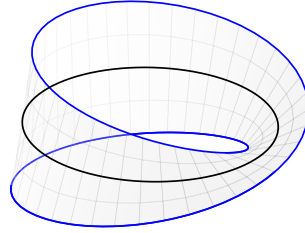
(b) Connected or "Neveu–Schwarz" double cover with shaded mesh to indicate the "Möbius band like" structure.

Figure 2.1: The two non isomorphic double covers of the circle in blue with base circle in black and shaded mesh to indicate the "Möbius band like" structure for the connected case.

there is a unique spin structure on B^2. Furthermore contractability implies triviality of the frame bundle $\mathrm{SO}(B^2) = B^2 \times \mathrm{SO}(2) \cong B^2 \times S^1$ [Ham17, Corollary 4.2.9]. As S^1 and B^2 are both embedded submanifolds of $\mathbb{R}^2$ we will use the standard coordinates (x, y) of $\mathbb{R}^2$, with these we get a global section, which gives in particular a global trivialisation of $\mathrm{SO}(B^2)$ which assigns the standard orthonormal basis (e_x, e_y) to a point. In this picture the right action of $\mathrm{SO}(2)$ is given by counterclockwise rotation of this basis. A natural choice for the spin structure is now given by the trivial $\mathrm{Spin}(2)$-bundle $\mathrm{Spin}(B^2) = B^2 \times \mathrm{Spin}(2) \cong B^2 \times S^1$ with projection to $\mathrm{SO}(B^2)$ given by the map

$$\Lambda \colon \mathrm{Spin}(B^2) \to \mathrm{SO}(B^2)$$
$$(x, g) \mapsto (x, \lambda(g)),$$

i.e. the identity on the B^2 factor and the doubling map on the S^1 factor. A direct computation shows that this indeed defines a spin structure on B^2. The inclusion of $\mathrm{SO}(S^1)$ into $\mathrm{SO}(B^2)$ in the trivialization chosen above is given by

$$\iota \colon \mathrm{SO}(S^1) \to \mathrm{SO}(B^2)$$
$$e_\theta \mapsto ((\cos(\theta), \sin(\theta)), \theta),$$

where e_θ denotes the unit tangent vector at $\theta \in S^1$. Using this inclusion we can pullback the double cover $\mathrm{Spin}(B^2)$ of $\mathrm{SO}(B^2)$ to a double cover of $\mathrm{SO}(S^1) \cong S^1$ and therefore obtain a $\mathrm{Spin}(1) \cong \mathbb{Z}_2$ bundle over S^1 which we will denote by $\mathrm{Spin}^{B^2}(S^1)$.

A natural question now is which of the two double covers described above have we obtained. From the definition of the pullback bundle in Theorem A.0.8 we can see that $\text{Spin}^{B^2}(S^1) = \Lambda^{-1}\left(\iota(\text{SO}(S^1))\right)$, which is connected, therefore we indeed find $\text{Spin}^{B^2}(S^1) \cong \text{Spin}^{\text{NS}}(S^1)$.

2.3 Bordisms with extra structure

2.3.1 Tangential structures

Following [Fre12, Chapter 9] and [Sch18, Section 6] we will now define the notion of *tangential structures*, which generalizes the previous idea of defining extra structure on a manifold by a reduction of the frame bundle. First we will define the general notion of a class of structure, afterwards we will define what it means for a manifold M to be equipped with this structure.

Most of this section will not be directly relevant for the rest of this thesis and serves only as an outlook on how to embed the previous ideas into a more general framework. This means we will not explain all of the details and further suppose the reader has familiarity with concepts from algebraic topology such as fibrations and direct limits as developed in [Die08] or [DK01].

First recall, for example from [LM89, Appendix B] the construction of the classifying space $BO(n)$ of the orthogonal group in n-dimensions as a direct limit of finite dimensional Grassmanians, i.e.

$$BO(n) = \operatorname*{colim}_{m \to \infty} Gr_n\left(\mathbb{R}^{m+n}\right)$$

from the inclusions $Gr_n\left(\mathbb{R}^{m+n}\right) \subset Gr_n\left(\mathbb{R}^{m+n+1}\right)$, induced by the inclusions $\mathbb{R}^{m+n} \subset \mathbb{R}^{m+n+1}$. Furthermore we also have inclusions $Gr_n\left(\mathbb{R}^{m+n}\right) \subset Gr_{n+1}\left(\mathbb{R}^{m+n+1}\right)$ which induce maps $BO(n) \to BO(n+1)$, using these we define

$$BO = \operatorname*{colim}_{n \to \infty} BO(n).$$

This is a classifying space for the stable orthogonal group $O = \operatorname{colim}_{n\to\infty} O(n)$ [Fre12, Definition 9.45].

Definitions 2.3.1.   1. (a) An *n-dimensional tangential structure* is a topological space $\mathcal{X}(n)$ together with a fibration $\xi_n \colon \mathcal{X}(n) \to BO(n)$. A *stable tangential structure* is a topological space $\mathcal{X}$ and a fibration $\xi \colon \mathcal{X} \to BO$. A stable tangential structure defines a n-dimensional tangential structure

for every $n \in \mathbb{Z}_+$ by defining $\xi_n \colon \mathcal{X}(n) \to BO(n)$ to be given as the pullback

$$
\begin{array}{ccc}
\mathcal{X}(n) & \dashrightarrow & \mathcal{X} \\
{\scriptstyle \xi_n} \downarrow & & \downarrow {\scriptstyle \xi} \\
BO(n) & \longrightarrow & BO
\end{array}
$$

Note that $\xi_n \colon \mathcal{X}(n) \to BO(n)$ is indeed a fibration. Analogously, a similar pullback diagram defines a m-dimensional tangential structure coming from a n-dimensional tangential structure for any $m < n$.

(b) Two n-dimensional tangential structures $(\mathcal{X}(n), \xi_n)$ and $(\mathcal{X}'(n), \xi_n')$ are called *equivalent* if there is a homotopy equivalence $\mathcal{X}(n) \simeq \mathcal{X}'(n)$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{X}(n) & \xrightarrow{\ \ \simeq\ \ } & \mathcal{X}'(n) \\
 & \searrow {\scriptstyle \xi_n} \qquad {\scriptstyle \xi_n'} \swarrow & \\
 & BO(n) &
\end{array}
$$

commutes up to homotopy. Analogously, two stable tangential structures $(\mathcal{X}, \xi)$ and $(\mathcal{X}', \xi')$ are called *equivalent* if there is a homotopy equivalence $\mathcal{X} \simeq \mathcal{X}'$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\ \ \simeq\ \ } & \mathcal{X}' \\
 & \searrow {\scriptstyle \xi} \qquad {\scriptstyle \xi'} \swarrow & \\
 & BO &
\end{array}
$$

commutes up to homotopy.

2. (a) An $\mathcal{X}(n)$-*structure* on an m-dimensional manifold M, for $m \leqslant n$, is given by the homotopy class of a lift

$$
\begin{array}{ccc}
 & & \mathcal{X}(n) \\
 & {\scriptstyle \theta_M} \nearrow & \downarrow {\scriptstyle \xi_n} \\
M & \xrightarrow[\ \tau_M^{(m,n)}\ ]{} & BO(n)
\end{array}
$$

where $\tau_M^{(m,n)}$ denotes a classifying map of the bundle $TM \oplus \underline{\mathbb{R}^{n-m}}$, i.e. the tangent bundle stabilized to rank n, where $\underline{\mathbb{R}^{n-m}}$ denotes the trivial

$\mathbb{R}^{n-m}$ vector bundle over M. Note that the classifying map $\tau_M^{(m,n)}$ is only specified up to homotopy, therefore a $\mathcal{X}(n)$-structure on M is really a homotopy class of lifts coming from the class of $\tau_M^{(m,n)}$.

A *stable $\mathcal{X}$-structure* on an m-dimensional manifold M is a family of coherent $\mathcal{X}(n)$-structures on M for n sufficiently large.[5]

(b) Two n-dimensional tangential structures on M, θ_M and θ'_M are called *equivalent* if there is an isotopy, this means a homotopy over $BO(n)$, between them. If such an isotopy exists we will denote this by $\theta_M \simeq \theta'_M$.

(c) A *manifold with $\mathcal{X}(n)$-structure* is a tuple $(M, [\theta_M])$, where M is a k-dimensional manifold and $[\theta_M]$ is a $\mathcal{X}(n)$-structure on M as above.

Examples 2.3.2. We will now give a list of n-dimensional tangential structures, and indicate if they arise from a stable tangential structure:

- $\mathcal{X}(n) = BO(n)$ with ξ_n the identity is again the trivial tangential structure, and corresponds to the stable tangential structure $\mathcal{X} = BO$.

- $\mathcal{X}(n) = BSO(n)$ with ξ_n induced from the inclusion $SO(n) \to O(n)$ is, as we have seen before, orientation. The corresponding stable tangential structure is $\mathcal{X} = BSO$.

- $\mathcal{X}(n) = BSpin(n)$ with $\xi_n = B\lambda \circ B\iota$ induced from the covering map $\lambda \colon Spin(n) \to SO(n)$ composed with the inclusion $SO(n) \subset O(n)$ is exactly a spin structure, the corresponding stable structure is $\mathcal{X} = BSpin$. Note that a spin structure on M *induces* an orientation on M by post-composing the lift of τ_M with $B\lambda$.

- For $r \in \mathbb{Z}_+$, $\mathcal{X}(2) = BSpin^r(2)$ with ξ_n induced from the r-fold covering map $Spin^r(2) \to SO(2)$ is called an *r-spin structure*. Orientations and spin structures on two dimensional manifolds are examples of this for $r = 1$ and $r = 2$, respectively. Note that r-spin is not stable for $r \notin \{1, 2\}$.

- $\mathcal{X}(n) = BG$ for some topological group G together with a homomorphism $G \to O(n)$ inducing the fibration ξ_n is a reduction of the structure group to G, often called a *G-structure*. Orientations and spin structures are special cases of this.

- $\mathcal{X}(n) = pt$ with ξ_n the inclusion is called an *n-framing*. This is a special case of a G-structure for $G = \{1\}$, the trivial group. From this point of view the

---

[5]For a precise definition see [Sto68, Chapter II].

64

geometric interpretation of an n-framing on an n-dimensional manifold is a trivialization of the tangent bundle.

- $\mathcal{X}(n) = BG \times BO(n)$ for some topological group G with ξ_n the projection corresponds to principal G-bundles.

- More generally $\mathcal{X}(n) = Y \times BO(n)$ for some topological space Y with ξ_n the projection, corresponds to homotopy classes of maps to Y.

Let us now briefly discuss how homotopic groups $G \simeq G'$ lead to equivalent tangential structures. For this recall, for example from the discussion after Definition A.0.9, that the classifying spaces of G and G' are also homotopic $BG \simeq BG'$. Now any group homomorphism $\phi \colon G \to O(n)$ induces a G-structure, moreover the induced group homomorphism $\phi' \colon G' \to O(n)$ induces a G'-structure which is by definition equivalent to the G-structure induced by ϕ. This shows that taking $O(n)$ instead of $GL(n)$ leads to an equivalent theory without the use of an auxiliary metric.

Let $(\mathcal{X}(n), \xi_n)$ be an n-dimensional tangential structure and let $(M, [\theta_M])$ be an n-dimensional manifold with $\mathcal{X}(n)$-structure such that M has non-empty boundary ∂M. Then there is an induced $\mathcal{X}(n)$-structure on the boundary ∂M. To define an $\mathcal{X}(n)$-structure on the boundary ∂M we need a classifying map of the stabilized tangent bundle of the boundary $\widetilde{T(\partial M)} := T(\partial M) \oplus \underline{\mathbb{R}}$. To get this recall the short exact sequence of vector bundles

$$0 \longrightarrow T(\partial M) \longrightarrow \iota^* TM \longrightarrow N(\partial M) \longrightarrow 0$$

where $\iota \colon \partial M \to M$ is the canonical inclusion of the boundary which defines the *normal bundle* $N(\partial M)$ of the boundary. It can be shown that this sequence always splits [Fre12, Section 5.3], i.e. there is a vector bundle isomorphism

$$\iota^* TM \cong T(\partial M) \oplus N(\partial M).$$

Furthermore the bundle $\iota^* TM$ is by definition classified by the map $\tau_M \circ \iota$. Now to use $\tau_M \circ \iota$ as a classifying map of the stabilized tangent bundle of the boundary the normal bundle needs to be trivialized. A trivialization of the normal bundle corresponds to the choice of an outward or an inward pointing normal vector, thus there are, up to isomorphism, two possibilities. The standard convention is to use the outward normal. With this we obtained an $\mathcal{X}(n)$-structure on the boundary ∂M by $[\theta_M \circ \iota]$. The tangential structure on the boundary corresponding to using the inward normal will be called the *opposite* or *reversed* tangential structure $-[\theta_M \circ \iota]$. Technically the $\mathcal{X}(n)$-structure on ∂M is really a $\mathcal{X}(n-1)$-structure.

2.3.2 Bordism category

In order to define a bordism category we need to be able to glue two manifolds with tangential structure along a common boundary. To see how this works let $(\mathcal{X}(n), \xi_n)$ be an n-dimensional tangential structure, and let $(M, [\theta_M])$ and $(N, [\theta_N])$ be n-dimensional manifolds such that $\partial M \cong \partial N =: \Sigma$. Instead of working with the homotopy classes of maps $[\theta_M]$ and $[\theta_N]$ we will fix a representative θ_M of $[\theta_M]$ and θ_N of $[\theta_N]$, and ignore most of the subtleties coming from this to make the argument below clearer. Note that we can define a $\mathcal{X}(n-1)$-structure on Σ by either $\theta_\Sigma = \theta_M \circ \iota_M$ or $\theta_\Sigma = \theta_N \circ \iota_N$, in order to have any chance of gluing M and N along Σ in a way compatible with the tangential structures we need to require that these two maps coincide.[6] This means we have the following commutative diagram

$$
\begin{array}{ccc}
\Sigma & \longhookrightarrow & N \\
\downarrow & & \downarrow{\scriptstyle\theta_N} \\
M & \xrightarrow{\theta_M} & \mathcal{X}(n) \\
\end{array}
$$

(with τ_N, τ_M, ξ_n to $BO(n)$)

Recall now, for example from Section 1.4, the universal property of the pushout $M \sqcup_\Sigma N$, along the inclusions, gives us two maps

$$\tau_{M \sqcup_\Sigma N} \colon M \sqcup_\Sigma N \to BO(n), \tag{2.3.1}$$

$$\theta_{M \sqcup_\Sigma N} \colon M \sqcup_\Sigma N \to \mathcal{X}(n). \tag{2.3.2}$$

Moreover $\theta_{M \sqcup_\Sigma N} \circ \xi_n = \tau_{M \sqcup_\Sigma N}$ by the uniqueness of the map from the universal property, therefore $\theta_{M \sqcup_\Sigma N}$ gives a $\mathcal{X}(n)$-structure on $M \sqcup_\Sigma N$.

With this we can now define closed bordisms with arbitrary tangential structure and the resulting category.

Definition 2.3.3. Let $\mathcal{X}(n) \equiv (\mathcal{X}(n), \xi_n)$ be an n-dimensional tangential structure. The symmetric monoidal category of *closed bordisms with $\mathcal{X}(n)$-structure* $\mathrm{Bord}^{\mathcal{X}}_{n,n-1}$ is defined as follows:

- Objects are given by closed $(n-1)$-dimensional manifolds with $\mathcal{X}(n)$-structure $(\Sigma, [\theta_\Sigma])$.

---

[6]Being homotopic would suffice.

- Morphisms are bordism classes where the underlying compact manifolds are equipped with a $\mathcal{X}(n)$-structure, such that the restriction is equivalent to the ones on the objects.

- Composition is given by gluing as described above.

- The symmetric monoidal structure is given by disjoint union.

Note here that it is equivalent to take cylinders over $(n-1)$-dimensional manifolds together with a $\mathcal{X}(n)$-structure as objects.

Chapter 3

Open-closed spin TQFTs

In this chapter we will give a short review two dimensional open-closed spin TQFTs and their algebraic classification in terms of knowledgebale Λ_2-Frobenius algebras by Stern and Szegedy [SS20]. In addition to this review we will slightly extend their results to allow for decorations on the free boundaries of the bordisms. This will lead us to an algebraic notion which we term "Λ_2-Calabi–Yau category" in analogy to the oriented case. The purpose of this chapter is to acquaint the reader with two dimensional spin TQFTs and can thus be seen as a "warm-up" before we consider two dimensional defect spin TQFTs.

We begin with a very brief geometric description of the open-closed spin bordism category. After this we will turn to the generators and relations description of this category. However instead of giving the relations directly, we will make a slight detour and discuss the corresponding algebraic structures first. This will allow us to shorten the presentation of the bordism category drastically. From this presentation we can then directly review the classification result of open-closed TQFTs.

After this review we will slightly modify the open sector of the bordism category to allow for different decorations on the "free boundaries". These decorations can be interpreted as "boundary conditions" [MS09]. In order to classify TQFTs on this enlarged bordism category we will then introduce the notion of a certain type of category and finally use the results of [SS20] to give a classification result for open-closed spin TQFTs with boundary conditions.

Throughout this chapter, let $(\mathcal{C}, \otimes, \mathbb{1}, \beta)$ be a symmetric monoidal category, which we will often abbreviate to $\mathcal{C}$.

## 3.1 The open-closed spin bordism category

In Section 2.3.2 we gave a definition for closed bordism categories with arbitrary tangential structure, including spin structures. In this section we are interested in a category with not only closed spin bordisms but also open ones. However before we come to the spin case, we will first very briefly describe what *open* and *open-closed bordism* means. First the prefix *open* in open-closed bordism comes from *open strings* in string theory, and has nothing to do with open in point set topology.

The rough idea is to allow the objects of the category to also have non-trivial boundary, in two dimensions this means the objects are no longer only finite disjoint unions of circles S^1 ("closed strings"), but finite disjoint unions of circles and closed intervals $I = [0, 1]$ ("open strings"). Due to this the bordisms can no longer be represented as just compact manifolds with parameterised boundaries[1] but rather compact manifolds with corners.[2]. Moreover we now need to differentiate between *parameterised boundaries* and *free boundaries*. A free boundary is "free" in the sense that it does not come with a boundary parameterisation and therefore does not correspond to an incoming or outgoing boundary.

A bordism from $I \sqcup S^1$ to itself could for example be represented by the following manifold with corners

$$(3.1.1)$$
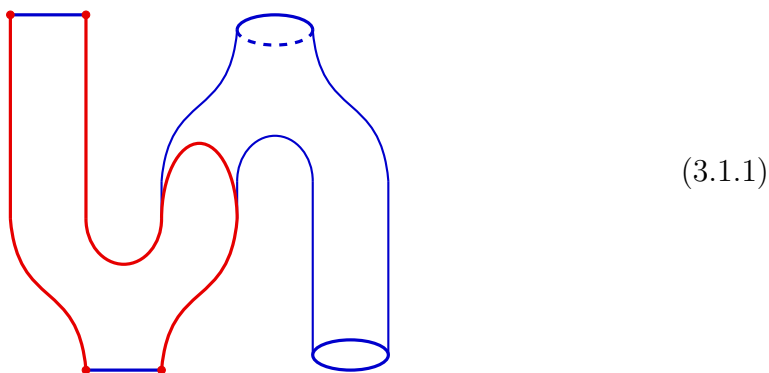
where the thick red line indicates the free boundary, while the thick blue ones correspond to the parameterised boundary. The middle part

---
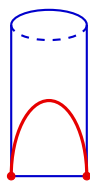
[1]In the sense of coming with germs of collars, see Section 1.4.
[2]This roughly means they are not modelled on the half plane $\mathbb{R} \times \mathbb{R}_{\geqslant 0}$ but the quadrant $\mathbb{R}^2_{\geqslant 0}$.

is called a *whistle bordism*. The underlying manifold of this bordism is a cylinder over the circle, however the bordism is non-trivial because only "half" of the "incoming circle" is parameterised. More precisely the free boundary of this cylinder is an interval (in red) while the parameterised boundary is the disjoint union of an interval and a circle (in blue). This bordism shows that the closed and the open sector are not disjoint from one another. In string theory such a bordism can be interpreted as the worldsheet of an open string which evolves into a closed string over time. Gluing of open-closed bordisms can be defined analogously to the case of closed bordism discussed in Section 1.4. We will not discuss the details of the oriented open-closed bordism category $\mathrm{Bord}_{2,1}^{\mathrm{oc,\,or}}$, for this see [Laz01; MS09; LP08].

The *open-closed spin bordism category* $\mathrm{Bord}_{2,1}^{\mathrm{oc,\,Spin}}$ is defined analogously by replacing oriented manifolds with spin manifolds. This means the objects are finite disjoint unions of intervals with spin structures and circles with spin structures. The interval has only has one spin structure, the trivial one, because it is contractible while the cirlce has two, as explained in Section 2.2. Morphisms are oriented open-closed bordisms together with a spin structure which restricts to the spin structures of the source and target objects. We will now turn to the generators and relation description of the open-closed spin bordism category, for the full definition see [SS20, Section 2].

We already saw that the set of generators $\mathcal{G}_0$ of $\mathrm{Bord}_{2,1}^{\mathrm{oc,\,Spin}}$ consists of the interval I with trivial spin structure, the Neveu–Schwarz circle S^{NS}, and the Ramond circle S^{R}. As explained in [SS20, Section 5] a set of generators $\mathcal{G}_1$ for the open-closed spin bordism category $\mathrm{Bord}_{2,1}^{\mathrm{oc,\,Spin}}$ can be obtained by considering all possible spin structures on the generators of the oriented open-closed bordism category $\mathrm{Bord}_{2,1}^{\mathrm{oc,\,or}}$. However it actually turns out to be sufficient to consider these generators with a single fixed spin structure as well as cylinders with different spin structures. In [SS20] the spin structures on the generators of $\mathrm{Bord}_{2,1}^{\mathrm{oc,\,or}}$ were fixed using a combinatorial model. We will not further discuss this combinatorial model and instead tacitly assume the fixed spin structures we consider are always those, see there for more details.

The set $\mathcal{G}_1$ consists of the following:
the closed sector:

$$\tag{3.1.2}$$

70

the open sector:

$$, \qquad , \qquad , \qquad , \qquad ; \qquad (3.1.3)$$

and finally the "whistle" bordisms of the open-closed sector:

$$\text{and} \qquad \qquad . \qquad (3.1.4)$$

Here the $\circlearrowleft$ inside the cylinders indicates that they are mapping cylinders of the deck transformation mentioned before Remark 2.2.2. These cylinders are not directly part of the generators, however they will be useful later on.

As mentioned above we will not state the relations between these generators now, instead we will first introduce the relevant algebraic structure, and then describe the relations through this structure in a compact form.

3.2 Knowledgeable Λ_2-Frobenius algebras

In this section we will describe the algebraic structure underlying open-closed spin bordisms. We begin with a special type of Frobenius algebra which will turn out to describe open spin TQFTs. After this we will describe the structure corresponding to closed spin TQFTs and the algebraic incarnation of the whistle bordisms. Finally we will state the algebraic classification of open-closed spin TQFTs by [SS20].

Definition 3.2.1. Let $(A, \mu, \Delta, \eta, \epsilon)$ be a Frobenius algebra in $\mathcal{C}$. We call $(A, \mu, \Delta, \eta, \epsilon)$ a Λ_2-*Frobenius algebra* if its Nakayama automorphism N_A satisfies $N_A^2 = id_A$. We will oftentimes only write A to mean the whole data of an Λ_2-Frobenius algebra $(A, \mu, \Delta, \eta, \epsilon)$. Recall that in the graphical calculus we denote the structure maps by:

$$\mu = \qquad , \qquad \Delta = \qquad , \qquad \eta = \qquad , \qquad \epsilon = \qquad ; \qquad (3.2.1)$$

71

Recall further that in our conventions of Frobenius algebras the Nakayama automorphism is given by

$$N_A = \qquad\qquad (3.2.2)$$

A Λ_2-Frobenius algebra is not necessarily symmetric however its pairing satisfies

$$= \; N_A \quad = \quad N_A \; . \qquad\qquad (3.2.3)$$

For $\mathcal{C} = \mathrm{Vect}_{\Bbbk}$ a Λ_2-Frobenius algebra is $\mathbb{Z}_2$-graded Frobenius algebra where the grading is given by decomposition into eigenspaces of the Nakayama automorphism. Furthermore we can recover the open sector of the Frobenius algebras discussed in [MS09, Section 3.4] if we consider $\mathcal{C} = \mathcal{S}\mathrm{Vect}_{\Bbbk}$ and assume that the above grading into eigenspaces of the Nakayama automorphism and the super grading coincide.

Definitions 3.2.2. A *closed Λ_2-Frobenius algebra* in $\mathcal{C}$ is given by a pair of objects (C_0, C_1) of objects in $\mathcal{C}$ together with morphisms

$$
\begin{aligned}
\mu_{x,y} &\colon C_x \otimes C_y \to C_{x+y-1}, &\qquad \eta_1 &\colon \mathbb{1} \to C_1 \\
\Delta_{x,y} &\colon C_{x+y+1} \to C_x \otimes C_y, &\qquad \epsilon_1 &\colon C_1 \to \mathbb{1} \\
N_x &\colon C_x \to C_x
\end{aligned}
$$

where $x, y \in \mathbb{Z}_2$. We will draw these structure morphisms as

$$
\mu_{x,y} = \boxed{x,y} \quad , \quad \Delta_{x,y} = \boxed{x,y} \quad , \quad \eta_1 = \Big| \quad , \quad \epsilon_1 = \Big| \quad , \quad N_x^k = \bigcirc_{k} \; . \qquad (3.2.4)
$$

These morphisms are required to satisfy the following relations for $x, y, z, w \in \mathbb{Z}_2$

72

with $x + y - 2 = z + w$:

$$\text{(associativity)}, \qquad (3.2.5)$$

$$\text{(coassociativity)}, \qquad (3.2.6)$$

$$\text{(unitality)}, \qquad \text{(counitality)},$$

$$(3.2.7)$$

$$\text{(Frobenius)}, \qquad (3.2.8)$$

$$\text{(commutativity)}, \qquad (3.2.9)$$

$$\text{(twist)}, \qquad (3.2.10)$$

and finally

$$\text{(twist}').\qquad (3.2.11)$$

We will often abbreviate the data of a closed Λ_2-Frobenius algebra to just (C_0, C_1).

A *map of closed Λ_2-Frobenius algebras* $\phi \colon (C_0, C_1) \to (D_0, D_1)$ is a collection of morphisms $\phi_x \colon C_x \to D_x$ preserving the structure morphisms in the sense that

$$(3.2.12)$$

$$(3.2.13)$$

The following lemma follows directly from the relations above and provides an explanation of the relationship between C_0 and C_1.

Lemma 3.2.3. Let (C_0, C_1) be a closed Λ_2-Frobenius algebra with structure maps as above. Then the following hold:

1. $(C_1, \mu_{1,1}, \Delta_{1,1}, \eta_1, \epsilon_1)$ is a commutative Frobenius algebra in $\mathcal{C}$.

2. $(C_0, \mu_{0,1}, \mu_{1,0})$ is a C_1-C_1-bimodule.

3. $(C_0, \Delta_{0,1}, \Delta_{1,0})$ is a C_1-C_1-bicomodule.

4. The bimodule and bicomodule structures of C_0 are compatible in the sense that the (co-)actions are (co-)module maps.

5. The composition $\epsilon_1 \circ \mu_{0,0}$ is a non-degenerate pairing with copairing $\Delta_{0,0} \circ \eta_1$, making C_0 self dual.

From this Lemma it follows that maps of closed Λ_2-Frobenius algebras are, analogously to the maps of regular Frobenius algebras, always isomorphisms. This is because maps of closed Λ_2-Frobenius algebras respect the duality data of C_0 and C_1. See the argument for regular Frobenius algebras in Section 1.2 for more details.

Remark 3.2.4. In the case where $\mathcal{C} = \mathrm{Vect}_{\Bbbk}$ the data of a closed Λ_2-Frobenius algebra (C_0, C_1) can be shortened using the direct product of vector spaces. More precisely for $C := C_0 \oplus C_1$ we obtain the linear maps

$$
\begin{aligned}
\mu &\colon C \otimes C \to C, & \eta &\colon \mathbb{1} \to C, \\
\Delta &\colon C \to C \otimes C, & \epsilon &\colon C \to \mathbb{1}, \\
N &\colon C \to C;
\end{aligned}
$$

from the structure morphisms of the closed Λ_2-Frobenius algebra (C_0, C_1) by using the canonical injections $C_x \to C$ and projections $C \to C_x$ for $x \in \mathbb{Z}_2$, and the universal properties of the direct sum.[3] More generally this reformulation works if $\mathcal{C}$ is a additive category, see for example [Eti+16, Chapter 1] for a definition of additive categories, the direct sum is then a *biproduct*, i.e. an object which is a product and a coproduct in a compatible way.

Example 3.2.5. In the case where $\mathcal{C} = \mathcal{S}\mathrm{Vect}_{\Bbbk}$ and the decomposition of C_0 and C_1 coincides with the decomposition given by decomposition into eigenspaces of N_x, we recover the algebras described in [MS09, Section 2.6]. In this case C_1 is purely even while C_0 can have even and odd components.

We now have the open and the closed sector, thus we still need the algebraic incarnation of the whistle bordisms and how the open and closed sector interact through them.

---

[3]Recall, for example from [Lei14, Chapter 5], that the direct sum of vector spaces is both a product and coproduct.

Definition 3.2.6. A *knowledgeable Λ_2-Frobenius algebra* in $\mathcal{C}$ is given by:

- a closed Λ_2-Frobenius algebra (C_0, C_1) in $\mathcal{C}$,

- a Λ_2-Frobenius algebra A in $\mathcal{C}$,

- two morphisms $\iota_x \in \mathrm{Hom}_{\mathcal{C}}(C_x, A)$ and $\pi_x \in \mathrm{Hom}_{\mathcal{C}}(A, C_x)$ for each $x \in \mathbb{Z}_2$, which will be written as

$$\iota_x = \quad\raisebox{-1em}{\text{[diagram]}}\quad, \qquad \pi_x = \quad\raisebox{-1em}{\text{[diagram]}} \tag{3.2.14}$$

such that the following equations are fulfilled for every $x \in \mathbb{Z}_2$:

$$\raisebox{-2em}{\text{[diagram]}} \quad = \quad \raisebox{-2em}{\text{[diagram]}} \qquad \text{(knowledge)}, \tag{3.2.15}$$

$$\raisebox{-2em}{\text{[diagram]}} \quad = \quad \raisebox{-2em}{\text{[diagram]}} \qquad \text{(duality)}, \tag{3.2.16}$$

$$\raisebox{-2em}{\text{[diagram]}} \quad = \quad \raisebox{-2em}{\text{[diagram]}} \qquad \text{(Cardy condition)}. \tag{3.2.17}$$

76

A *map of knowledgeable Λ_2-Frobenius algebras*

$$\Phi\colon ((C_0, C_1), A, \iota_x, \pi_x) \to ((D_0, D_1), B, \iota'_x, \pi'_x) \tag{3.2.18}$$

consists of

- a map of closed Λ_2-Frobenius algebras $\phi\colon (C_0, C_1) \to (D_0, D_1)$;

- a morphism of Frobenius algebras $\psi\colon A \to B$;

such that for any $x \in \mathbb{Z}_2$

$$\tag{3.2.19}$$

The category with knowledgeable Λ_2-Frobenius algebras in $\mathcal{C}$ as objects and maps of knowledgeable Λ_2-Frobenius algebras as morphisms will be denoted by $\Lambda_2 - \mathrm{KnFrob}(\mathcal{C})$.

Note that the category of knowledgeable Λ_2-Frobenius algebras in $\mathcal{C}$ is a groupoid.

Example 3.2.7. Let us assume $\mathcal{C}$ is idempotent complete. For a Λ_2-Frobenius algebras A in $\mathcal{C}$ with $\mu \circ \Delta$ invertible there is a notion of $\mathbb{Z}_2$-graded center $Z^2(A)$ which can be defined using an idempotent build from the structure morphisms of A in a similar way to the usual center of an algebra, see [SS20, Section 4.2] for details. The Λ_2-Frobenius algebras A together with its $\mathbb{Z}_2$-graded center $Z^2(A)$ and the canonical projection and inclusion maps form a knowledgeable Λ_2-Frobenius algebras in $\mathcal{C}$.

With this preparation we can now finally complete the description of $\mathrm{Bord}_{2,1}^{\mathrm{oc,\, Spin}}$ through generators and relations.

Proposition 3.2.8 ([SS20, Proposition 5.1.2])**.** The objects $S^{\mathrm{R}}, S^{\mathrm{NS}}, I \in \mathcal{G}_0$ together with the bordisms $\mathcal{G}_1$ form a knowledgeable Λ_2-Frobenius algebra with $S^{\mathrm{R}} \equiv C_0$, $S^{\mathrm{NS}} \equiv C_1$, and $I \equiv A$, in the notation used above.

Theorem 3.2.9 ([SS20, Theorem 5.2.1])**.** The *open-closed spin bordism category* $\mathrm{Bord}_{2,1}^{\mathrm{oc,\, Spin}}$ is generated as a symmetric monoidal category by the knowledgeable Λ_2-Frobenius algebra $(\mathcal{G}_0, \mathcal{G}_1)$ from above.

From this the classification result of open-closed spin TQFTs follows immediatly:

Corollary 3.2.10 ([SS20, Corollary 5.2.2])**.** The groupoids $\mathrm{Fun}^{\otimes,\mathrm{sym}}(\mathrm{Bord}_{2,1}^{\mathrm{oc},\mathrm{Spin}}, \mathcal{C})$ and $\mathrm{KnFrob}^2(\mathcal{C})$ are equivalent.

The proof of this statement is conceptually analogous to our proof sketch of Theorem 1.5.4.

We will now briefly sketch an intuitive argument on why Proposition 3.2.8 is true in the style of [MS09, Section 2.6], for the closed sector, the open sector can be argued analogously.

We want to find all possible spin structures on the generators of the oriented closed bordism category. For $\mathcal{G}_0$ as in Theorem 1.5.4 we already know that we have two possibilities, the Neveu–Schwarz and the Ramond circle. Now for the generators $\mathcal{G}_1$ of the oriented closed bordism category, we see that we have two types of underlying manifolds: pairs of pants and disks. Thus we now want to find all possible spin structures on pairs of pants and disks, including different spin structures on the boundaries.

From Example 2.2.3 we already know that there is only one spin structure on the disk and it induces an NS-type spin structure on its boundary circle.

To find the possible spin structures on the pair of pants, let us first consider the cylinders over S^{R} and S^{NS}, respectively. To obtain pairs of pants from these cylinders we "cut" a disk out of each cylinder. Now because the disk only has one possible spin structure, the boundary resulting from this cutting process also needs to have the same induced spin structure. Therefore we have two types of spin structure on a pair of pants: the first induces three NS-type boundary circles and the second two R-type boundaries and one of NS-type. This is the geometric origin of the algebraic description of the R-sector as a bimodule over the NS-algebra.

Furthermore for S^{R} there is non-trivial cylinder bordism given by the mapping cylinder induced by the non-trivial deck transformation w discussed before Remark 2.2.2. For S^{NS} the deck transformation cylinder is equivalent to the trivial cylinder through a Dehn twist [MS09, Section 2.6]. This is the geometric origin of the N_x morphisms.

3.3 Boundary conditions

We will generalise the above results to allow decorations or "boundary conditions" on the free boundaries in the sense of [MS09]. Working this out in detail will lead us to the notion of what we term a $\mathcal{C}$-enriched Λ_2-*Calabi–Yau category*, in analogy to

Calabi–Yau categories which are relevant for the oriented version [LP08]. The whole discussion will be quite analogous to the one in [Car18, Section 3.2], and serves as a warm-up for defect spin TQFTs. In this section and Chapter 5 we will need the notion of enriched category, for an introduction to this field see [Kel05].

Let B be any non-empty set, we enlarge the open spin bordism category $\mathrm{Bord}_{2,1}^{\mathrm{o,\,Spin}}$ such that free boundary components of bordisms are labeled with elements in B, and the labels at the endpoints of the objects coincide with the ones of the adjacent free boundaries. More precisely the category of *open spin bordisms with boundary conditions* in the set of *boundary labels* B, is the symmetric monoidal category $\mathrm{Bord}_{2,1}^{\mathrm{o,\,Spin}}(B)$ with generators

$$\mathcal{G}_0 = \left\{\; b \bullet\!\!\!-\!\!\!\!\bullet a \;\equiv I_{b,a} \mid a,b \in B \right\}$$

$$\mathcal{G}_1 = \left\{ \cdots \;\middle|\; a,b,c \in B \right\}$$

and relations analogous to the ones of $\mathrm{Bord}_{2,1}^{\mathrm{o,\,Spin}}$. In analogy to open oriented TQFTs we can use any *open spin TQFT*

$$\mathcal{Z}\colon \mathrm{Bord}_{2,1}^{\mathrm{o,\,Spin}}(B) \to \mathcal{C}$$

to construct a $\mathcal{C}$-enriched category with B as set of objects and morphisms built from the TQFT, see [Car18, Section 3.2.2] for a review of the oriented version:

Construction 3.3.1. Let B be any set of boundary labels. For the open spin TQFT

$$\mathcal{Z}\colon \mathrm{Bord}_{2,1}^{\mathrm{o,\,Spin}}(B) \to \mathcal{C}$$

we define the *category of boundary conditions* $\mathcal{O}_{\mathcal{Z}}$ by:

- For the set of *objects* we set $\mathrm{Ob}\,(\mathcal{O}_{\mathcal{Z}}) = B$.

- For $a,b \in \mathrm{Ob}\,(\mathcal{O}_{\mathcal{Z}})$ we define the set of *morphisms* by

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{Z}}}\,(a,b) := A_{b,a} = \mathcal{Z}\left(b \bullet\!\!\!-\!\!\!\!\bullet a \right) \in \mathcal{C}.$$

79

- Composition $\mu_{c,b,a} \colon A_{c,b} \otimes A_{b,a} \to A_{c,a}$ is defined as

$$\mu_{c,b,a} := \mathcal{Z} \left(\begin{array}{c} \end{array} \right).$$

If we assume $\mathcal{Z}$ to be strict, then this composition is associative due to functoriality of $\mathcal{Z}$, and the relations, see Proposition (3.2.8) for comparison. If $\mathcal{Z}$ is not strict we would need to incorporate the natural ismorphisms governing the monoidality of $\mathcal{Z}$ in a similar way as outlined in the proof of Theorem 1.5.4.

- The unit morphisms are given by

$$1_a := \mathcal{Z} \left(\begin{array}{c} \end{array} \right).$$

Before we discuss the special properties of categories of boundary conditions, we will first define the properties we will encounter in the general setting.

Definition 3.3.2. Let $\mathcal{C}$ be a symmetric monoidal category. A $\mathcal{C}$-enriched Λ_2-*Calabi–Yau category*, consists of the following:

- A $\mathcal{C}$-enriched category $\mathcal{O}$;

- a morphism for all $a \in \mathcal{O}$

$$\epsilon_a \colon \mathrm{End}_{\mathcal{B}}(a) \to \mathbb{1} \tag{3.3.1}$$

in $\mathcal{C}$, called the *trace*;

- a morphism for every pair of objects $a, b \in \mathcal{O}$

$$\gamma_{b,a} \colon \mathrm{Hom}_{\mathcal{B}}(a, b) \equiv H_{b,a} \to H_{b,a} \tag{3.3.2}$$

in $\mathcal{C}$;

such that

- the morphism

$$\kappa_{b,a} := \epsilon_a \circ \mu_{a,b,a} \colon H_{a,b} \otimes H_{b,a} \to \mathbb{1}, \tag{3.3.3}$$

  with μ the composition in $\mathcal{O}$, is a non-degenerate pairing in $\mathcal{C}$;

- the pairing is not necessarily symmetric, but satisfies

$$\begin{aligned}
\kappa_{b,a} \circ \beta_{H_{b,a}, H_{a,b}} &= \kappa_{a,b} \circ (\gamma_{b,a} \otimes 1_{H_{a,b}}) \\
&= \kappa_{a,b} \circ (1_{H_{b,a}} \otimes \gamma_{a,b});
\end{aligned} \tag{3.3.4}$$

- $\gamma_{b,a}^2 = 1_{H_{b,a}}$.

Remark 3.3.3. $\mathcal{C}$-enriched Λ_2-Calabi–Yau categories are a "categorification" of Λ_2-Frobenius algebras in $\mathcal{C}$ in the sense that for every Λ_2-Calabi–Yau category $\mathcal{O}$, the endomorphisms of any object a, i.e. $H_{a,a}$ form a Λ_2-Frobenius algebra in $\mathcal{C}$ with Nakayama automorphism $\gamma_{a,a}$. Furthermore $H_{b,a}$ is a $H_{b,b}$-$H_{a,a}$-bimodule and a $H_{b,b}$-$H_{a,a}$-bicomodule, such that the module and comodule structures commute. In this sense we can think of Λ_2-Calabi–Yau categories as "many" Λ_2-Frobenius algebras together.

Lemma 3.3.4. Let B be any set of boundary conditions, and let $\mathcal{Z} \colon \mathrm{Bord}_2^{\mathrm{o, Spin}}(B) \to \mathcal{C}$ be an open spin TQFT. Then the category of boundary conditions $\mathcal{O}_{\mathcal{Z}}$ of $\mathcal{Z}$ is a $\mathcal{C}$-enriched Λ_2-Calabi–Yau category.

Proof. First we define the data of an Λ_2-Calabi–Yau category, for this let

- the trace morphism

$$\epsilon_a := \mathcal{Z}\left(\begin{array}{c} \bigcap \\ a \quad a \end{array} \right) \colon A_{a,a} \to \mathbb{1} \tag{3.3.5}$$

  for $a \in \mathcal{O}$;

- the "Nakayama" morphism

$$\gamma_{b,a} := \mathcal{Z}\left(\begin{array}{c} b \quad a \\ \circlearrowleft \\ b \quad a \end{array} \right) \tag{3.3.6}$$

  for $a, b \in \mathcal{O}$, $A_{b,a}$;

81

With this data the claim follows by adapting the proof of Theorem 3.2.10 to incorporate labels on the free boundaries. □

Before we come to the classification result, we will first need to define morphisms between $\mathcal{C}$-enriched Λ_2-Calabi–Yau categories. However we only need the ones between two $\mathcal{C}$-enriched Λ_2-Calabi–Yau categories with the same set of objects. In this case the definition can be simplified to the following:

Definition 3.3.5. Let $\mathcal{O}$, $\mathcal{O}'$ be $\mathcal{C}$-enriched Λ_2-Calabi–Yau categories such that $\mathrm{Ob}(\mathcal{O}) = \mathrm{Ob}(\mathcal{O}')$. A *morphism* $F \colon \mathcal{O} \to \mathcal{O}'$ is given by a family of morphisms $F_{b,a} \colon H_{b,a} \to H'_{b,a}$ in $\mathcal{C}$ such that:

$$
\tag{3.3.7}
$$

We call the resulting category of $\mathcal{C}$-enriched Λ_2-Calabi–Yau categories and morphisms the category of *B-colored*, $\mathcal{C}$-enriched Λ_2-Calabi–Yau categories $\Lambda_2\text{-}\mathrm{CY}^B(\mathcal{C})$, where B is the set of objects.

Remark 3.3.6. It can be shown that $\Lambda_2\text{-}\mathrm{CY}^B(\mathcal{C})$ is a groupoid. The proof is again analogous to the one for $\mathrm{FrobAlg}(\mathcal{C})$.

With this preperation we can now state the classification result of open spin TQFTs with boundary conditions.

Theorem 3.3.7. Construction 3.3.1 gives an equivalence of the category of open spin TQFTs with boundary conditions B and the category of B-colored, $\mathcal{C}$-enriched Λ_2-Calabi–Yau categories.

Proof sketch. One direction is clear by Lemma 3.3.4. For the other direction let $\mathcal{O}$ be any B-colored, $\mathcal{C}$-enriched Λ_2-Calabi–Yau. We define an open spin TQFT $\mathcal{Z}_{\mathcal{O}} \colon \mathrm{Bord}_{2,1}^{\mathrm{o,\,Spin}}(\mathrm{Ob}(\mathcal{O})) \to \mathcal{C}$ by defining its action on the generators $\mathcal{G}_0$ and $\mathcal{G}_1$ of $\mathrm{Bord}_{2,1}^{\mathrm{o,\,Spin}}(B)$:

- The intervals get mapped to the Hom sets:

$$
\mathcal{Z}_{\mathcal{O}} \left(b \bullet\!\!-\!\!\!-\!\!\bullet a \right) := \mathrm{Hom}_{\mathcal{O}}(a, b).
$$

- The pair of pants gets mapped to composition in $\mathcal{O}$:

$$\mathcal{Z}_O \left(\begin{array}{c} \includegraphics{pants} \end{array} \right) := \mu_{c,b,a} : \mathrm{Hom}_{\mathcal{O}}(b,c) \otimes \mathrm{Hom}_{\mathcal{O}}(a,b) \to \mathrm{Hom}_{\mathcal{O}}(a,c).$$

- The cup gets mapped to the identity morphism:

$$\mathcal{Z}_{\mathcal{O}} \left(\begin{array}{c} \includegraphics{cup} \end{array} \right) := 1_a.$$

- The cap gets mapped to the trace morphism:

$$\mathcal{Z}_{\mathcal{O}} \left(\begin{array}{c} \includegraphics{cap} \end{array} \right) := \epsilon_a.$$

- The upside down pair of pants gets mapped to the "co-composition" in $\mathcal{O}$ obtained from the non-degenerate pairing κ and the composition μ.

- The deck transformation cylinder gets mapped to the "Nakayama" morphism:

$$\mathcal{Z} \left(\begin{array}{c} \includegraphics{cylinder} \end{array} \right) := \gamma_{b,a} \tag{3.3.8}$$

Note that we ignored subtleties leading to a non-strict $\mathcal{Z}_{\mathcal{O}}$ as in the proof of Theorem 1.5.4. $\qquad \square$

3.3.1 Serre functors

For regular Calabi–Yau categories, there is a shorter definition using the notion of a *Serre functor*, see [Car18, Section 3.2] for a few more comments on this. In this subsection we will discuss the question if being a Λ_2-Calabi–Yau category can also be phrased using a Serre functor.

Definitions 3.3.8. Let $\mathcal{O}$ be a $\mathcal{C}$-enriched category.

1. A *Serre functor* on $\mathcal{O}$ is a $\mathcal{C}$-enriched functor $\mathcal{S}\colon \mathcal{O} \to \mathcal{O}$, together with non-degenerate pairing

$$\mathrm{Hom}_{\mathcal{O}}(a,b) \otimes \mathrm{Hom}_{\mathcal{O}}(b, \mathcal{S}(a)) \to \mathbb{1} \tag{3.3.9}$$

   natural in $a, b \in B$.

2. $\mathcal{O}$ is called a *Calabi–Yau* category if it has trivialisable Serre functor, i.e. $\mathcal{S} \cong 1_{\mathcal{O}}$.

Before we can discuss this further we need to make a simplifying assumption on the category $\mathcal{C}$.

Assumption 3.3.9. *The symmetric monoidal category $\mathcal{C}$ is* sovereign, *i.e. $\mathcal{C}$ is rigid and the left and right duality endofunctors are equal [FS08]. This property can be seen as a strict version of pivotality.*[4]

Lemma 3.3.10. Let $\mathcal{C} = \mathrm{vect}_{\Bbbk}$. The category of boundary conditions $\mathcal{O}$ of any open spin TQFT $\mathcal{Z}$ has a Serre functor $\mathcal{S}$ given by

$$\mathcal{S}\colon \mathcal{O} \to \mathcal{O}$$
$$a \mapsto a$$
$$\phi \mapsto \gamma_{b,a}(\phi)$$

for $\phi\colon a \to b$, with *Serre pairing*

$$\widetilde{\kappa}_{b,a} = \epsilon_a \circ \mu_{a,b,a}\colon \mathrm{Hom}_{\mathcal{O}}(b, \mathcal{S}(a)) \otimes \mathrm{Hom}_{\mathcal{O}}(a,b) \to \mathbb{1}.$$

Proof. Functoriality of $\mathcal{S}$ can be proved analogously to the proof that the Nakayama automorphism of a Frobenius algebra in a sovereign category is an algebra morphism [FS08, Proposition 18], note here that for this we need the above assumption.

---

[4]It would already suffice to assume equality of left and right duals for some of the structure morphisms of the Λ_2-CY-categories.

Non-degeneracy of the pairing is clear by Lemma 3.3.4. To prove naturality note that for $\mathcal{C} = \mathrm{vect}_{\Bbbk}$ non-degeneracy of the pairing $\widetilde{\kappa}_{b,a}$ is equivalent to having isomorphisms $\widetilde{\kappa}'_{b,a} \colon \mathrm{Hom}_{\mathcal{O}}(b, \mathcal{S}(a)) \to \mathrm{Hom}_{\mathcal{O}}(a,b)^*$. For naturallity we now need to show that the following diagram, and an analogous one,

$$
\begin{array}{ccc}
\mathrm{Hom}_{\Bbbk}(b, \mathcal{S}(a)) & \xrightarrow{\ \mathcal{S}(f) \circ (-)\ } & \mathrm{Hom}_{\Bbbk}(b, \mathcal{S}(a')) \\
\Big\downarrow{\scriptstyle\widetilde{\kappa}'_{a,b}} & & \Big\downarrow{\scriptstyle\widetilde{\kappa}'_{a',b}} \\
\mathrm{Hom}_{\Bbbk}(a, b)^* & \xrightarrow[\ (-) \circ f\ ]{} & \mathrm{Hom}_{\Bbbk}(a', b)^*
\end{array}
$$

commutes for all $a, b \in \mathcal{O}$, and $f \colon a \to a'$. For this let $\phi \in \mathrm{Hom}_{\Bbbk}(b, a)$, we get the condition

$$
\widetilde{\kappa}_{a',b} \circ \left((\gamma_{a',b}(f) \circ \phi) \otimes (-) \right) = \widetilde{\kappa}_{a',b} \circ \left(\phi \otimes ((-) \circ f) \right).
$$

This condition is fulfilled by the defining property Equation (3.3.4) of $\gamma_{a,b}$. The second naturality diagram yields a similar condition. $\qquad\square$

This Lemma also holds in the case for general $\mathcal{C}$, however the naturality is harder to prove.

Corollary 3.3.11. The square of Serre functor $\mathcal{S}$ of the category of boundary conditions $\mathcal{O}_{\mathcal{Z}}$ of any open spin TQFT $\mathcal{Z}$ is trivialisable, i.e. $\mathcal{S}^2 \cong 1_{\mathcal{O}_{\mathcal{Z}}}$.

Proof. This follows immediatly from the previous lemma and Lemma 3.3.4. $\qquad\square$

With these results the following equivalent characterisation of Λ_2-Calabi–Yau categories is straightforward.

Proposition 3.3.12. Let B be any set, and let $\mathcal{O}$ be a $\mathcal{C}$-enriched category with $\mathrm{Ob}(\mathcal{O}) = B$. The following are equivalent:

1. $\mathcal{O}$ is a B-colored, $\mathcal{C}$-enriched Λ_2-Calabi–Yau category,

2. $\mathcal{O}$ has a Serre functor $\mathcal{S}$ with $\mathcal{S}^2 \cong 1_{\mathcal{O}}$;

An interesting consequence of this whole discussion is the following.

Lemma 3.3.13. Let $\mathcal{Z} \colon \mathrm{Bord}_{2,1}^{\mathrm{o};\,\mathrm{Spin}}(B) \to \mathrm{Vect}_{\Bbbk}$ be an open spin TQFT. Then the category of boundary conditions $\mathcal{O}$ of $\mathcal{Z}$ is a *supercategory*, i.e. it is an $\mathcal{S}\mathrm{Vect}_{\Bbbk}$-enriched category.

Proof. The grading on the Hom spaces $A_{b,a}$ is given by decomposition into eigenspaces of the linear maps $\gamma_{b,a} \colon A_{b,a} \to A_{b,a}$. The composition is an even function with respect to this grading due to functoriality of the Serre functor $\mathcal{S}$. $\qquad\qquad\square$

We are now finally in a place to at least mention a possible example of a Λ_2-Calabi–Yau category, we will be very brief with this as an adequate treatment would need further techniques from homological algebra.

Example 3.3.14. Let a $W \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial in $n \in \mathbb{Z}_+$ variables such that its Jacobi algebra, see Example 1.5.5, is finite-dimensional. A *matrix factorization* of W is a $\mathbb{Z}_2$-graded $\mathbb{C}[x_1, \ldots, x_n]$-module $X = X_0 \oplus X_1$ together with an odd $\mathbb{C}[x_1, \ldots, x_n]$-linear endomorphism $d_X \colon X \to X$ such that $d_X^2 = W 1_X$. For (X, d_X) and (Y, d_Y) matrix factorization of W we define a map

$$
\begin{aligned}
\delta_{XY} \colon \operatorname{Hom}_{\mathbb{C}[x_1, \ldots, x_n]}(X, Y) &\to \operatorname{Hom}_{\mathbb{C}[x_1, \ldots, x_n]}(X, Y) \\
\Phi &\mapsto d_Y \circ \Phi - (-1)^{|\Phi|} \Phi \circ d_X.
\end{aligned}
\tag{3.3.10}
$$

It is straightforward to show that δ_{XY} is a differential, i.e. $\delta_{XY}^2 = 0$. The *homotopy category of matrix factorization* $\operatorname{hmf}(\mathbb{C}[x_1, \ldots, x_n], W)$ of W is defined as:

- objects are matrix factorization (X, d_X) of W;

- morphisms are elements of the δ_{XY}-cohomology, i.e.

$$
\operatorname{Hom}_{\operatorname{hmf}(\mathbb{C}[x_1, \ldots, x_n], W)}((X, d_X), (Y, d_Y)) := H_{\delta_{XY}}(\operatorname{Hom}_{\mathbb{C}[x_1, \ldots, x_n]}(X, Y)). \tag{3.3.11}
$$

The cohomology is naturally $\mathbb{Z}_2$-graded because δ_{XY} is an odd operator. This category corresponds to the open sector of the B-twisted Landau–Ginzburg model, see for example [CM16, Section 2.2] for more details and further references to the original literature. The results of [CS21, Section 4.2] suggest that if the number of variables n of the potential W is odd then $\operatorname{hmf}(\mathbb{C}[x_1, \ldots, x_n], W)$ should correspond to an open spin TQFT, however a more careful analysis of this is still needed.

3.3.2 Open-closed sector

We can now extend the description of open spin TQFTs we just described to open-closed spin TQFTs by combining it with Definition 3.2.6. For this note that the bordisms (3.1.4) will carry only a label on their free boundaries, therefore the maps ι_x and π_x will now be indexed by an element in B. From this idea we get the following final result by slightly modifying Theorem (3.2.10):

Theorem 3.3.15. An open-closed spin TQFT with set of boundary conditions B
$\mathcal{Z}\colon \mathrm{Bord}_{2,1}^{\mathrm{oc},\,\mathrm{Spin}}(B) \to \mathcal{C}$ is equivalent to the following data:

- a closed Λ_2-Frobenius algebra (C_0, C_1);

- a B-colored, $\mathcal{C}$-enriched Λ_2-Calabi–Yau category $\mathcal{O}$;

- two morphisms $\iota_x \in \mathrm{Hom}_{\mathcal{C}}(C_x, A_{a,a})$ and $\pi_x \in \mathrm{Hom}_{\mathcal{C}}(A_{a,a}, C_x)$ for each $x \in \mathbb{Z}_2$, which will be written as

$$\iota_x^a = \;\;,\qquad \pi_x^a = \;\; \tag{3.3.12}$$

such that for any $a, b \in B$ and $x \in \mathbb{Z}_2$ the following equations, of morphisms in $\mathcal{C}$, are fulfilled:

$$= \;\gamma_{a,a}^x \qquad \text{(knowledge)}, \tag{3.3.13}$$

$$= \qquad \text{(duality)}, \tag{3.3.14}$$

87

$$\begin{array}{c} A_{b,b} \\ \big| \\ \bigtriangleup x \\ \big| \\ \bigtriangledown x \\ \big| \\ A_{a,a} \end{array} \quad = \quad \gamma_{b,a}^{x} \qquad \text{(Cardy condition).} \qquad (3.3.15)$$

The data $\left((C_0, C_1), \mathcal{O}, (\iota_x^a)_{x \in \mathbb{Z}_2,\, a \in B}, (\pi_x^a)_{x \in \mathbb{Z}_2,\, a \in B} \right)$ is called a *B-colored, knowledge-able* Λ_2-*Calabi–Yau category* enriched in $\mathcal{C}$.

Chapter 4

Defect spin bordisms

In this chapter we will propose a possible definition for the appropriate bordism category to study spin TQFTs in the presence of *topological defects*. To this end we will modify the definitions of stratified oriented manifolds, and their labeled version, given in [CMS20, Section 2], to account for the presence of non-trivial spin structures on the strata.

In Section 4.1 we begin our discussion with a general definition of what we term *stratified spin manifolds* of dimension $n \in \mathbb{Z}_+$, with or without boundaries, and the corresponding morphisms between such manifolds. We will then define a bordism category for such manifolds. Finally we will explain the relation of our bordism category to the closed spin bordism category, discussed in Section 2.3.2, and the stratified oriented bordism category of [CMS20].

After this in Section 4.2 we will focus solely on 2 dimensions and describe how to consistently label a stratified spin bordism. The interpretations of these labels will be either as *topological defects*, which can be seen as a generalisation of boundary conditions as discussed in Section 3.3, or closed spin TQFTs.

4.1   Stratified spin bordisms

In this section we propose a definition for *stratified spin manifolds*, and the corresponding bordisms in any dimension. To do this we modify the definitions of the oriented case given in [CMS20, Section 2] to account for spin structures on the strata. The content of this section can in principle be generalised to any tangential structure which factorizes through orientations, see Section 2.3.1, in a straightforward manner.

Definition 4.1.1. An n-dimensional *stratified manifold with spin structure* (without

boundary) is an n-dimensional manifold with spin structure (Σ, Λ) (without boundary) together with a *filtration* $(F_j) := (\Sigma = F_n \supset F_{n-1} \supset \cdots \supset F_0 \supset F_{-1} = \varnothing)$ of Σ subject to the following conditions:

1. $\Sigma_j := F_j \backslash F_{j-1}$ is a j-dimensional submanifold of Σ (which may be empty) for all $j \in \{0, 1, \ldots, n\}$; connected components of Σ_j are denoted by Σ_j^α and are referred to as *j-strata*; each j-stratum Σ_j^α is equipped with a *choice* of spin structure λ_j^α on $\Sigma_j^\alpha \times (0,1)^{n-j}$, which may differ from the one induced by Σ unless the filtration is trivial, i.e. $F_j = \varnothing$ for $j \neq n$.[1] For any n-stratum we further impose that the chosen spin structure induces the same orientation as the one induced by Λ.

2. Frontier condition: for all strata $\Sigma_i^\alpha, \Sigma_j^\beta$ with $\Sigma_i^\alpha \cap \overline{\Sigma}_j^\beta \neq \emptyset$, we have $\Sigma_i^\alpha \subset \overline{\Sigma}_j^\beta$, where $\overline{(-)}$ denotes topological closure.

3. Finiteness condition: the total number of strata is finite.

We will often denote a stratified manifold with spin structure $((\Sigma, \Lambda), (F_j))$ by (Σ, Λ) or Σ and will also call it a *stratified spin manifold*. For a fixed stratified spin manifold, the set of all j-strata with spin structure λ will be denoted by $\mathcal{S}_j^\lambda$.

It is important to observe here that this is really well-defined, i.e. any stratum indeed admits at least one spin structure, namely the one induced by the global spin structure Λ through the inclusion map.

Definition 4.1.2. A *morphism* from an n-dimensional stratified manifold with spin structure (Σ, Λ) to an n-dimensional stratified manifold with spin structure $(\widetilde{\Sigma}, \widetilde{\Lambda})$ is given by:

- a continuous map $f \colon \Sigma \to \widetilde{\Sigma}$;[2]

- a family of isomorphisms $F_j^\alpha \colon (\Sigma_j^\alpha, \lambda_j^\alpha) \to (\widetilde{\Sigma}_j^\beta, \widetilde{\lambda}_j^\beta)$ of manifolds with spin structure for every $j \in \{0, 1, \ldots, n\}$ and all α.[3]

such that for f and the induced maps between base spaces $f_j^\alpha \colon \Sigma_j^\alpha \to \widetilde{\Sigma}_j^\beta$ it holds that $f(\Sigma_j) \subseteq \widetilde{\Sigma}_j$, and for the restriction to strata $f|_{\Sigma_j^\alpha} = f_j^\alpha$. The morphism is called an *embedding of stratified spin manifolds* if all the f_j^α are diffeomorphisms.

---

[1]Note that we equip every j-stratum with a spin structure corresponding to the group $\mathrm{Spin}(n)$ and not $\mathrm{Spin}(j)$.

[2]If we would require f to be smooth or a morphism of manifolds with spin structure we would run into the same problem as mentioned in the footnote at [CRS19, Page 11].

[3]Note that specifying α already determines β because continuous functions map connected sets to connected sets.

When comparing with the oriented case, one might feel like our definition of stratified spin manifolds is to "loose", in the sense that the global spin structure Λ only enters in at two places: To ensure, as explained above, the definition makes sense, and to give a global notion of orientation on the n-strata. There are several other places where the global spin structure Λ could enter. One important stricter special case is the following.

Definition 4.1.3. An n-dimensional *induced* stratified spin manifold (without boundary) is an n-dimensional stratified spin manifold (without boundary) (Σ, Λ) such that each j-stratum Σ_j^α is equipped with a spin structure that is equivalent to the one induced by Λ up to a change of underlying orientation. For any n-stratum the spin structure is equivalent to the one induced by Λ.

Induced stratified spin manifolds consist of alot less data then other stratified spin manifolds, they are basically the data of an oriented stratified manifold together with a spin structure on the whole space. We will see how induced stratified spin manifolds are more rigid in the proof of Lemma 5.1.6.

In order to define a bordism category we also need to define stratified spin manifolds with boundary, for this recall from Section 2.3.2 that a spin structure on a manifold with boundary induces a spin structure on its boundary. Furthermore we call a submanifold $N \subset M$ *neat* if $\partial N = N \cap \partial M$ and $N \cap \mathring{M} \neq \emptyset$, where $\mathring{M} := M \backslash \partial M$ denotes the interior of M.

Definition 4.1.4. An n-dimensional *stratified spin manifold with boundary* is an n-dimensional spin manifold with boundary (M, Ξ) together with a filtration $(F_j) := (M = F_n \supset F_{n-1} \supset \cdots \supset F_0 \supset F_{-1} = \emptyset)$ of M subject to the following conditions:

1. The interior $\mathring{M}$ together with the filtration $(\mathring{M} \cap F_j)$ is an n-dimensional stratified manifold with spin structure.

2. $M_j := F_j \backslash F_{j-1}$ is a neat j-dimensional submanifold of M (which may be empty) for all $j \in \{0, 1, \ldots, n\}$; connected components of M_j are denoted by M_j^α and are referred to as j-*strata*; each j-stratum M_j^α is equipped with a choice of spin structure on $M_j^\alpha \times (0,1)^{n-j}$, which induce the choices for the j-strata of $\mathring{M}$.

3. ∂M together with the filtration $(\partial M \cap F_{j+1})$ is an $(n-1)$-dimensional stratified manifold with spin structures, such that the spin structures on the strata are the ones induces from the M_j.

Definition 4.1.5. A *morphism* from an n-dimensional stratified spin manifold with boundary (M, Ξ) to an n-dimensional stratified spin manifold with boundary $(\widetilde{M}, \widetilde{\Xi})$ is given by:

- a continuous map $f\colon M \to \widetilde{M}$ between manifolds with spin structure;

- a family of isomorphisms $F_j^\alpha\colon (M_j^\alpha, \xi_j^\alpha) \to (\widetilde{M}_j^\beta, \widetilde{\xi}_j^\beta)$ of manifolds with spin structure for every $j \in \{0, 1, \ldots, n\}$ and all α;

such that for f and the induced maps between base spaces $f_j^\alpha\colon M_j^\alpha \to \widetilde{M}_j^\beta$ it holds that

- $f(\partial M) \subseteq \partial \widetilde{M}$,

- $f(M_j) \subseteq \widetilde{M}_j$,

- $f|_{M_j^\alpha} = f_j^\alpha$.

In particular $f|_{\mathring{M}}$ and $F_j^\alpha|_{\mathring{M}}$, and $f|_{\partial M}$ and $F_j^\alpha|_{\partial M_j^\alpha}$ are compatible morphisms of stratified spin manifolds without boundary. we will sometimes abbreviate the data to just $f\colon M \to \widetilde{M}$

With this we can now define *stratified spin bordisms*.

Definition 4.1.6. Let $(\Sigma_1, \Lambda_1), ((\Sigma_2, \Lambda_2))$ be closed $(n-1)$-dimensional stratified spin manifolds. A *stratified spin bordism* is given by:

- a n-dimensional stratified spin manifold with boundary (M, Ξ);

- a decomposition $\partial M = (\partial M)_1 \sqcup (\partial M)_2$ into the in-going boundary $(\partial M)_1$ and out-going boundary $(\partial M)_2$;

- germs of embeddings of stratified spin manifolds[4]

$$\theta_1\colon [0, \epsilon) \times \Sigma_1 \to M \tag{4.1.1}$$
$$\theta_2\colon (-\epsilon, 0] \times \Sigma_2 \to M \tag{4.1.2}$$

such that

– the continous map underlying θ_1 is orientation reversing and the one of θ_2 orientation preserving;

– $\theta_i(\{0\} \times \Sigma_i) = (\partial M)_i$ for $i \in \{1, 2\}$;

– $\mathrm{Im}(\theta_1) \cap \mathrm{Im}(\theta_2) = \varnothing$;

---

[4]With spin structures on the stratified cylinder $[0, \epsilon) \times \Sigma_1$ the ones induced by the ones of Σ_1 through the deformation retract $[0, \epsilon) \times \Sigma_1 \to \Sigma_1$ and analogously for $(-\epsilon, 0] \times \Sigma_2$.

For (M, θ_1, θ_2) and $(M', \theta'_1, \theta'_2)$ stratified spin bordisms $\Sigma_1 \to \Sigma_2$. A *diffeomorphism* $(M, \theta_1, \theta_2) \to (M', \theta'_1, \theta'_2)$ is a morphism of stratified spin manifolds $f \colon M \to M'$ such that $f((\partial M)_i) = (\partial M')_i$, and $f \circ \theta_i = \theta'_i$ hold for $i \in \{1, 2\}$.

We may now define a symmetric monoidal category $\mathrm{Bord}^{\mathrm{strat,Spin}}_{n,n-1}$ of stratified spin bordisms in analogy to the closed spin bordism category $\mathrm{Bord}^{\mathrm{Spin}}_{n,n-1}$:

Definition 4.1.7. The category of *stratified spin bordisms* $\mathrm{Bord}^{\mathrm{strat,Spin}}_{n,n-1}$ is defined as follows:

- objects are isomorphism classes of closed $(n-1)$-dimensional stratified spin manifolds;

- morphisms are diffeomorphism classes of stratified spin bordisms;

- composition is given by applying the standard construction of gluing in terms of collars in $\mathrm{Bord}^{\mathrm{Spin}}_{n,n-1}$ to the boundary strata;

- Disjoint union of manifolds gives the standard symmetric monoidal structure;

The composition can be shown to be associate, and unital with respect to the bordism class of the cylinder over (Σ, Λ), see Section 1.4 and Section 2.3.2 for a detailed discussion.

In complete analogy we can define a category of *induced stratified spin bordisms*, which we will denote as $\mathrm{Bord}^{\mathrm{strat,Spin}}_{n,n-1}$, it is clear that this is a non-full subcategory of $\mathrm{Bord}^{\mathrm{strat,Spin}}_{n,n-1}$.

An example of a 2-dimensional stratified spin bordism from two circles with one 0-stratum, and one 1-stratum to a circle with two 0-strata, and two 1-strata could

have

(4.1.3)
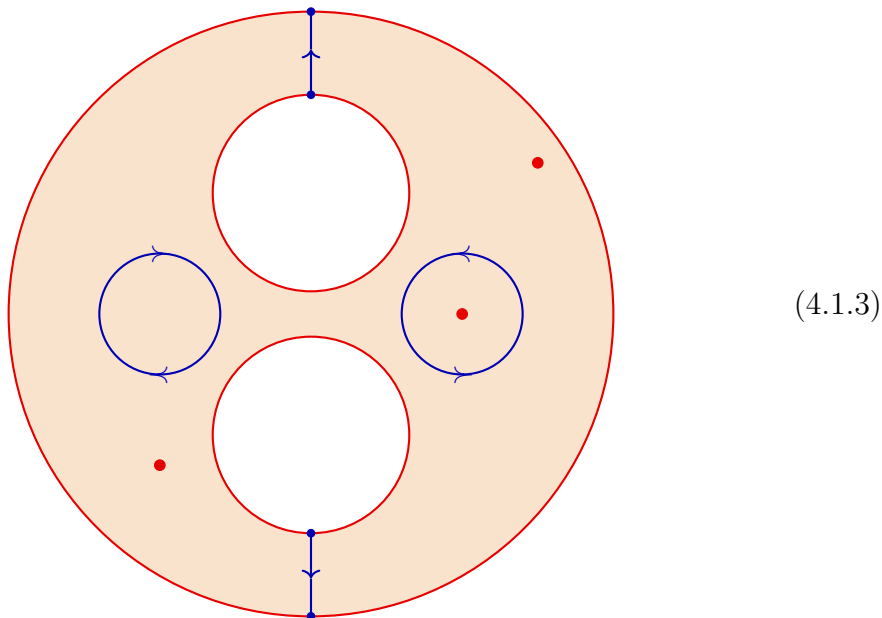
as underlying stratified manifold, where we view the smaller orange circles inside the disk as ingoing boundaries, and the big one as outgoing boundary. The red dots illustrate the 0-strata, with any possible orientation. The blue lines illustrate the 1-strata with the arrows indicating the orientation. The orange patches illustrate the 2-strata. The 0-strata can only have on spin structure per orientation. For the 1-strata we need to differentiate between circles and intervals. The circles could have different spin structures, for example one could be of NS-type and the other one of R-type. The intervals can only have the trivial spin structure because they are contractible, note that the boundaries of these 1-strata, lie completely in the boundary of the whole manifold. The 2-stratum with no 0-stratum inside is also contractible and thus only allows the trivial spin structure. The other two 2-strata are topologically non-trivial and can thus allow non-trivial spin structures.

Remarks 4.1.8. 1. The category of closed spin bordisms $\mathrm{Bord}_{n,n-1}^{\mathrm{Spin}}$ is a subcategory of $\mathrm{Bord}_{n,n-1}^{\mathrm{strat,Spin}}$ because spin manifolds are stratified spin manifolds with trivial stratification. It is a non-full subcategory due to the presence of strata that do not reach the boundary.

2. The oriented category $\mathrm{Bord}_{n,n-1}^{\mathrm{strat},SO}$ is obtainable from $\mathrm{Bord}_{n,n-1}^{\mathrm{strat,Spin}}$ because a spin structure needs an underlying orientation, see Chapter 2, in other words there is a functor

$$\mathrm{Bord}_{n,n-1}^{\mathrm{strat,Spin}} \to \mathrm{Bord}_{n,n-1}^{\mathrm{strat},SO}$$

94

that forgets the spin structure of every stratum and only remembers the underlying orientation: For $n \leqslant 2$ every oriented n-manifold posses at least one spin structure therefore the above forgetful functor is essentially surjective and full, it will however not be faithful because there are already non-isomorphic spin structures on S^1.

4.2 Defect spin bordisms in two dimensions

Similarly to the boundary labels we introduced on the free boundaries of open bordisms in Section 3.3, we want to find a consistent way to label the strata of a stratified spin bordism. To do this we will solely focus on $n = 2$, as we would need to introduce a notion of *standard neighbourhoods* of strata, see [CMS20, Section 2.2]. Furthermore we will not discuss how to label 0-strata in the interior of bordisms for now, because in Chapter 5 we will see how the TQFTs we are going to study will give us the allowed labels automatically.

To decorate a bordism, we follow the "physical" interpretation of the different strata from [DKR11] for the oriented case:

$$\text{labels for 2-strata} \mathrel{\widehat{=}} \text{closed spin TQFTs}$$
$$\text{labels for 1-strata} \mathrel{\widehat{=}} \text{line defects}$$

See Section 3.2 for a review of closed spin TQFTs.

First we need to consider if the labels should depend on the spin structure of the strata. For the 2-strata this can be answered right away through the physical interpretation. The labels should correspond to closed spin TQFTs and a closed spin TQFT can differentiate between different spin structures on the same smooth manifold. Thus the label of a 2-stratum should not depend on the spin structure of the 2-strata.

For the 1-strata the argument is a bit more subtle. First note that a 1-stratum is either homeomorphic to an interval I or to the circle S^1. Any oriented interval only has one spin structure, the trivial one, because it is contractible while the circles have two, as explained in Section 2.2. From this point of view it seems reasonable for us to use three different sets to label 1-strata, one for each type. However this is a fallacy, because we can glue two intervals to obtain a circle. Naively one would expect that we can only get one of the possible spin structures of the circle, however using a mapping cylinder of a non-trivial deck transformation it is possible to obtain both.[5] To understand this it is easier to think about the associated spinor bundles

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[5]This is a similar construction as the clutching construction of vector bundles on spheres, see
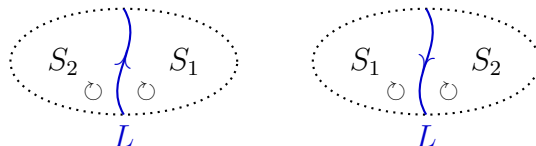
instead of thinking about the spin bundles. Recall from Example 2.2.3 that the spinor bundles of a circle are cylinder like for the R-structure and Möbius strip like for the NS-structure. The spinor bundle of an interval is just a "strip" $I \times \mathbb{C}$. By gluing two of such strips we get a cylinder $S^1 \times \mathbb{C}$, however if we twist one end of one strip we obtain the Möbius strip. The non-trivial deck transformation mentioned above corresponds exactly to twisting of one end of the strip before gluing. Thus the labels of 1-strata should also be independent of the spin structure.

We now need to find a way to label bordisms consistently, for this we will need to understand how 2-strata are allowed to meet at a 1-stratum. To understand this let (Σ, Λ) be a stratified spin bordism, recall that we denote with $\mathcal{S}_j^\lambda$ the set of all j-strata with spin structure λ. However as we just discussed, the labels will not depend on the spin structure, therefore we are interested in $\mathcal{S}_j$. Let L be a 1-stratum. We can find exactly at most 2-strata S_1 and S_2 such that $\overline{L} \subseteq \overline{S_1} \cap \overline{S_2}$, where $\overline{(-)}$ denotes the topological closure. This property can be pictured as:

$$(4.2.1)$$

Note that S_1 and S_2 are not necessarily distinct. In analogy to the oriented case, we have a global notion of orientation, by definition. Using this global orientation of the 2-strata we can employ the same convention as in [CRS19; CMS20] and think of S_1 as the 'source' (and S_2 as the 'target') of L if and only if an arrow from S_1 to S_2 together with a positive frame of L gives a positive frame of Σ. This convention is illustrated as:

$$(4.2.2)$$

Note that we choose the opposite global orientation than the one given in [DKR11; CMS20; CRS19], this is done because we want to read such a picture from right to left, bottom to top, in analogy to string diagrams. This means we have a map

$$m_1 \colon \mathcal{S}_1 \times \{\pm\} \to \mathcal{S}_2 \times \mathcal{S}_2$$
$$(L, +) \mapsto (S_1, S_2). \qquad (4.2.3)$$
$$(L, -) \mapsto (S_2, S_1)$$

[Hat17, Section 1.2]

With this we can now define the allowed labels on bordisms and how to label a bordism.

Definition 4.2.1. By *defect data* $\mathbb{D}$ we mean a choice of:

- two sets D_1 and D_2, called *defect labels*

- maps $\widetilde{s}, \widetilde{t} \colon D_1 \to D_2$, which we extend to maps $s, t \colon D_1 \times \{\pm\} \to D_2$ by setting

$$s(x, +) = \widetilde{s}(x), \qquad s(x, -) = \widetilde{t}(x)$$
$$t(x, +) = \widetilde{t}(x), \qquad t(x, -) = \widetilde{s}(x)$$

  for $x \in D_1$. We call s and t the *source* and *target maps*, respectively.

We can now decorate any stratified spin bordism (M, Ξ) with defect data $\mathbb{D}$ as follows: A $\mathbb{D}$-*decorated spin bordism* or *defect spin bordism* is a stratified spin bordism (M, Ξ) together with *decoration maps*

$$d_1 \colon \mathcal{S}_1 \to D_1, \tag{4.2.4}$$
$$d_2 \colon \mathcal{S}_2 \to D_2 \tag{4.2.5}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{S}_1 \times \{\pm\} & \xrightarrow{\ d_1\ } & D_1 \times \{\pm\} \\ {\scriptstyle m_1}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ \mathcal{S}_2 \times \mathcal{S}_2 & \xrightarrow[\ d_2 \times d_2\ ]{} & D_2 \times D_2 \end{array}$$

We say that a j-stratum Σ_j^α is *decorated* with $x \in D_j$ if $d_j(\Sigma_j^\alpha) = x$. A *morphism* of $\mathbb{D}$-decorated spin bordisms is a morphism $(M, \Xi) \to (\widetilde{M}, \widetilde{\Xi})$ of stratified spin bordisms such that every stratum in M carries the same decoration as the stratum in $\widetilde{M}$ into which it is mapped.

With this we can now finally define the bordism category on which we want to study TQFTs.

Definition 4.2.2. The symmetric monoidal category $\mathrm{Bord}_{2,1}^{\mathrm{def},\,\mathrm{Spin}}(\mathbb{D})$ of 2-dimensional *defect spin bordisms with defect data* $\mathbb{D}$ is given by:

- Objects are isomorphism classes of 1-dimensional stratified spin manifolds $[(\Sigma, \Lambda)] \in \mathrm{Bord}_{2,1}^{\mathrm{strat},\,\mathrm{Spin}}$ together with the structure of a $\mathbb{D}$-decorated spin bordism on the cylinder $(\Sigma \times [0,1], \Lambda)$.[6] Note that on the object Σ itself this means that j-strata are labeled by $(j+1)$-dimensional defect data.

---

[6]With spin structures on the strata of $\Sigma \times [0,1]$ induced analogously to Defintion 4.1.6.

- Morphisms are equivalence classes of $\mathbb{D}$-decorated spin bordisms such that the collars are morphisms of $\mathbb{D}$-decorated spin bordisms.

- Composition is the gluing from $\mathrm{Bord}_{2,1}^{\mathrm{strat,\,Spin}}$ such that the decorations match.

- The symmetric monoidal structure is given by disjoint union of manifolds.

There is again a subcategory with only induced spin structures which will be denoted as $\mathrm{Bord}_{2,1}^{\mathrm{def,\,Spin}}(\mathbb{D})$.

Remark 4.2.3. For any set of defect data $\mathbb{D}$ there is a forgetful functor

$$\mathrm{Bord}_2^{\mathrm{def,\,Spin}}(\mathbb{D}) \to \mathrm{Bord}_2^{\mathrm{strat,\,Spin}} \tag{4.2.6}$$

which sends a defect spin bordism to its underlying stratified spin bordism.

As an example consider the stratified bordism (4.1.3). A possible labeled version of this could be

$$\tag{4.2.7}$$
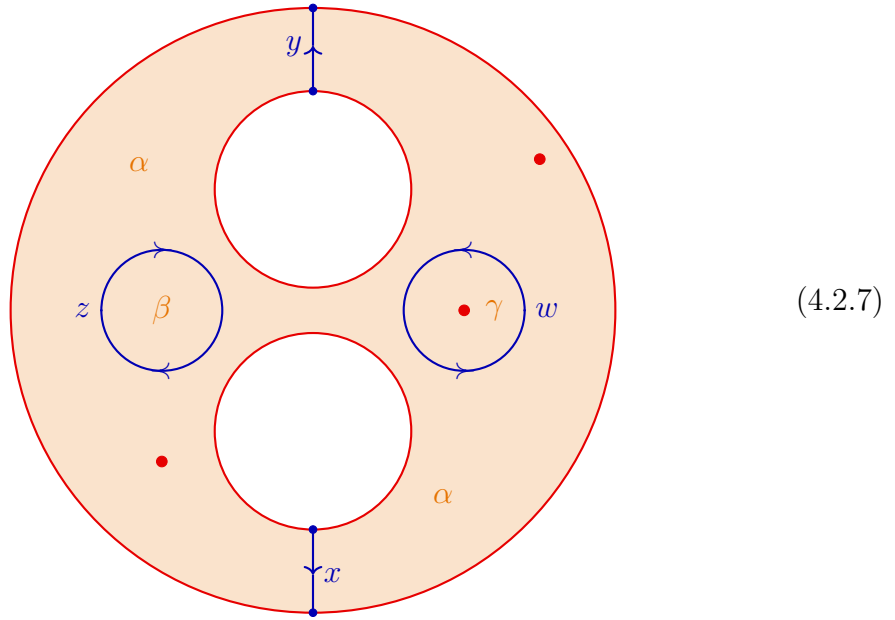
for $w, x, y, z \in D_1$, and $\alpha, \beta, \gamma \in D_2$. If we assume the standard orientation for 2-strata then the source and target of for example x would both be α while for z we have $s(z, -) = \beta$ and $t(z, -) = \alpha$.

A simple example of defect data, and corresponding bordism category is the following:

Example 4.2.4. We define a set of defect data $\mathbb{D}^*$ by:

- $D_2 = \{*\}$;

- $D_1 = \emptyset$;

- s, t the empty functions;

This defect data can only label bordism with $\mathcal{S}_1 = \emptyset$, i.e. $\mathrm{Bord}_{2,1}^{\mathrm{def},\,\mathrm{Spin}}(\mathbb{D}^*) \cong \mathrm{Bord}_{2,1}^{\mathrm{Spin}}$. In this sense, we see that closed spin TQFTs are special cases of defect TQFTs. Moreover this explains the physical interpretation of defect labels given in the beginning of this chapter. We will see the relations between elements in D_2 and closed spin TQFTs even more clearly in Chapter 5.

Chapter 5

The 2-category of a defect spin TQFT

We are now finally in a position to define and rigorously study *defect spin TQFTs*. In the first part of this chapter we will follow the ideas of [DKR11] and construct something similar to a 2-category from the data of a given defect spin TQFT. For ease of notation we will still call the resulting mathematical object a 2-category and postpone the discussion of the precise 2-categorical structure to Section 5.2. The construction can be thought of as the defect version of Construction 3.3.1. We will then study the 2-category in some detail, especially focusing on the differences to the oriented version. For the necessary background on 2-categories see Section 1.3, and the references therein.

5.1 Defect spin TQFTs and their 2-categories

Throughout this chapter let $(\mathcal{C}, \widetilde{\otimes}, \mathbb{1}, \beta)$ be a symmetric monoidal category. Note that we use $\widetilde{\otimes}$ instead of $\otimes$ for the monoidal product of $\mathcal{C}$, we do this because $\otimes$ will be used to denote horizontal composition later on. Now for any choice of defect data $\mathbb{D}$, we can finally define the central object of this thesis:

Definition 5.1.1. A 2-*dimensional defect spin TQFT with defect data* $\mathbb{D}$ *and values in* $\mathcal{C}$ is a symmetric monoidal functor

$$\mathcal{Z} \colon \mathrm{Bord}_{2,1}^{\mathrm{def, Spin}}(\mathbb{D}) \to \mathcal{C}.$$

From now on we will also fix the defect data $\mathbb{D}$ and abbreviate 2-dimensional defect spin TQFT with defect data $\mathbb{D}$ and values in $\mathcal{C}$ to *defect spin TQFT*. We will

not discuss morphisms and the category of defect spin TQFTs here, as we would first need to discuss morphisms of defect data and the category of defect data, see [CRS19, Section 2.3] for a discussion of these for oriented defect TQFTs. The definitions for the spin version can be obtained analogously. However we will not need these notions for the following.

Following the spirit of [DKR11], which was reviewed in great detail in [Car18], we will now construct a 2-category $\mathcal{B}_{\mathcal{Z}}$ out of a given defect spin TQFT

$$\mathcal{Z} \colon \mathrm{Bord}_{2,1}^{\mathrm{def,\,Spin}}(\mathbb{D}) \to \mathcal{C}$$

which captures as much of the structure of the TQFT as possible. This construction will greatly resemble Construction 3.3.1. Indeed under some assumptions, it should in principle be possible to derive the category of boundary conditions as a subsector of the *defect 2-category*. For the oriented case this is explained in [Car18, Section 3.3].

The data of the 2-category we want to construct should have a physical interpretation in terms of the defect spin TQFT. In analogy to [DKR11] we try to find the following structure:

$$
\begin{aligned}
\text{objects} &\mathrel{\widehat{=}} \text{closed spin TQFTs} \\
\text{1-morphisms} &\mathrel{\widehat{=}} \text{line defects} \\
\text{2-morphisms} &\mathrel{\widehat{=}} \text{``local'' operators} \\
\text{vertical composition} &\mathrel{\widehat{=}} \text{operator product} \\
\text{horizontal composition} &\mathrel{\widehat{=}} \text{fusion product} \\
\text{adjunction} &\mathrel{\widehat{=}} \text{spin structure reversal}
\end{aligned}
$$

This interpretation will in turn allow us to compute correlation functions of the TQFT as string diagrams in the 2-category, which indisputably gives a powerful tool for computations in specific models.

However before we can begin with the construction we need to make one assumption on the target category $\mathcal{C}$.

Assumption 5.1.2. *The monoidal category $\mathcal{C}$ is additive. We will denote the biproduct of $\mathcal{C}$ by $\oplus$.*

This assumption is motivated by the idea that local operators of the TQFT should be generalized elements of the state space, and the bulk of a defect spin TQFT should be described by a closed spin TQFTs. From Remark 3.2.4 we know that if $\mathcal{C}$ is additive, we can form the complete state space as the biproduct of the NS- and R-sector of a closed spin TQFT.

Construction 5.1.3. Given any defect spin TQFT $\mathcal{Z}\colon \mathrm{Bord}_{2,1}^{\mathrm{def,\,Spin}}(\mathbb{D}) \to \mathcal{C}$ we will now construct a 2-category $\mathcal{B}_{\mathcal{Z}}$:
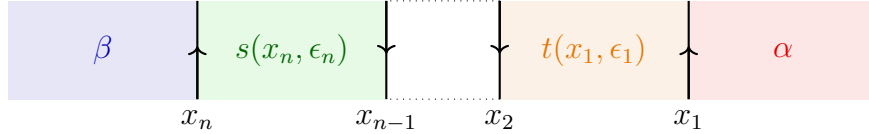
- For the set of *objects* we set $\mathrm{Ob}(\mathcal{B}_{\mathcal{Z}}) = D_2$.

- For $\alpha,\ \beta \in \mathrm{Ob}(\mathcal{B}_{\mathcal{Z}})$ we define a *1-morphism* from α to β to be a list

$$\big((x_n, \epsilon_n), \ldots, (x_1, \epsilon_1)\big) \in \prod_{i=1}^{n}(D_1 \times \{\pm\})$$

  for any $n \in \mathbb{N}$, such that

$$s(x_1, \epsilon_1) = \alpha, \qquad s(x_i, \epsilon_i) = t(x_{i+1}, \epsilon_{i+1}), \qquad t(x_n, \epsilon_n) = \beta.$$

 The reason for this definition comes from the following picture on bordisms:

  By abuse of notation we will denote the set of all 1-morphisms between α and β in the same way as the Hom category, i.e. by $\mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$.
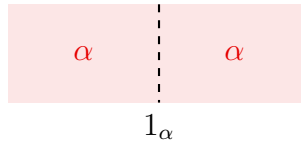
- For $X = \big((x_n, \epsilon_n), \ldots, (x_1, \epsilon_1)\big) \in \mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$ and $Y = \big((y_m, \nu_m), \ldots, (y_1, \nu_1)\big) \in \mathcal{B}_{\mathcal{Z}}(\beta, \gamma)$ we define their *horizontal composition* to be the concatenation of lists:

$$Y \otimes X := \big((y_m, \nu_m), \ldots, (y_1, \nu_1), (x_n, \epsilon_n), \ldots, (x_1, \epsilon_1)\big) \in \mathcal{B}_{\mathcal{Z}}(\alpha, \gamma).$$

  Again this corresponds to the following picture:
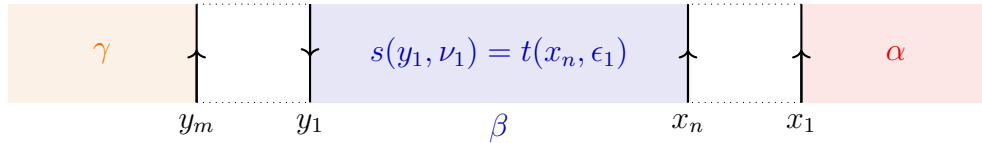
  This composition is strictly associative and unital with respect to the empty sequence, i.e. $1_\alpha = ()$, we will sometimes denote this by

- We want 2-*morphisms* to correspond to "local" operators but such operators should correspond to labels for 0-strata, which we did not discuss until now. We will now use the defect spin TQFT $\mathcal{Z}$ to compute the local operators.

  If we view boundary conditions as a special type of defect lines we expect, according to the classification of open/closed spin TQFTs, three types of possible local operators: *Neveu–Schwarz operators*, and *Ramond operators* in the bulk, and *defect operators* living on junction points of defect lines.

  We begin with the defect operators, for this let α, $\beta \in \mathrm{Ob}(\mathcal{B}_{\mathcal{Z}})$, and non-empty lists $X, Y \in \mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$. Consider the following neighborhood of a 0-stratum
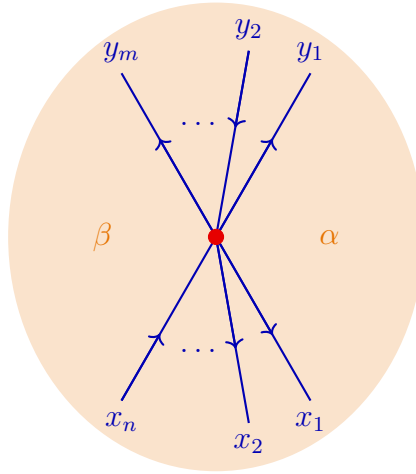
  of some defect bordism, where the orientations of the 1-strata are in way to incorporate the signs in the lists X and Y. For this we employ the convention that the 1-stratum labeled with x_i (resp. y_j) points towards the red dot if $\epsilon_i = -$ (resp. $\nu_j = -$) and away from it otherwise, e.g. in this example $\epsilon_1 = +$ while $\epsilon_n = -$.

  We want to find a way to label the red junction point so that it corresponds to a local operator. To do this we think of a small circle around the unlabeled

0-stratum:
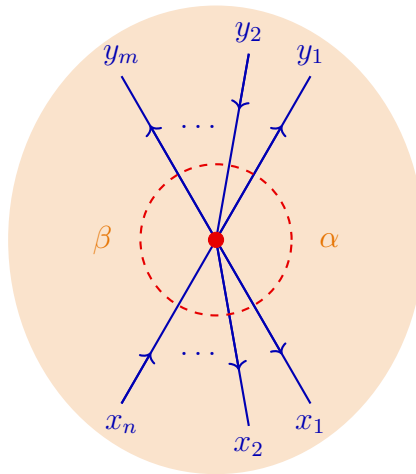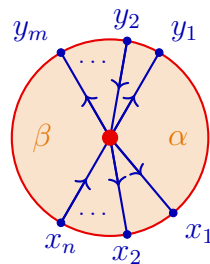

If we now cut out the stratified disk bounded by this circle, and label the new boundary of the bordism and the disk in such a way that the labels on the strata match the ones from the bordism we started with, we obtain
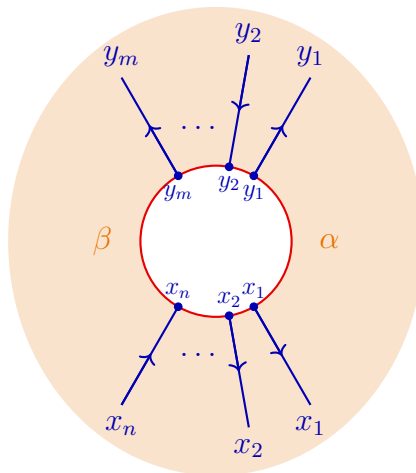
and

The resulting boundary circle will be denoted by $S^1(X, Y)$, i.e.

$$S^1(X, Y) \equiv \qquad\qquad\qquad\qquad\qquad\qquad (5.1.1)$$

for the example above. The labels for 1-strata of $S^1(X, Y)$ are fully determined by X and Y through the source and target maps. Furthermore the spin structures on the strata can only be the trivial ones because every stratum is contractible.

Note that we did not loose any information because we could just glue the disk back in. Moreover this also holds if we apply $\mathcal{Z}$ due to functoriality. Applying $\mathcal{Z}$ on the disk we obtain a generalized element of the state space $\mathcal{Z}\big(S^1(X, Y)\big)$. A *defect operator* is now defined to be exactly a generalized element of the state space $\mathcal{Z}\big(S^1(X, Y)\big)$. We can think of this procedure as "regularizing" the point defect to a disk. However it is important to stress here that not every generalized element of $\mathcal{Z}\big(S^1(X, Y)\big)$ is the image of a defect disk under $\mathcal{Z}$, because $\mathcal{Z}$ might not be full.

With this motivation, we define the set of 2-*morphisms* between X and Y as

$$\mathrm{Hom}_{\mathcal{B}_{\mathcal{Z}}(\alpha, \beta)}(X, Y) := \mathrm{Hom}_{\mathcal{C}}\Big(\mathbb{1}, \mathcal{Z}\big(S^1(X, Y)\big)\Big).$$

We now come to the trickier part, the bulk. Consider again a defect bordism for which 0-strata are not labeled. We are interested in the situation where the 0-stratum lies completely inside a 2-stratum
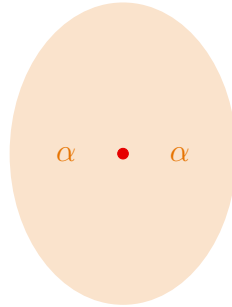
from the above procedure we expect to obtain a defect disk with label α for its single 2-stratum and no label for the 0-stratum
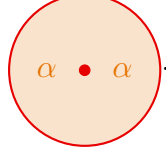
We will use the shorthand notation $S_\alpha^1 \equiv S^1(1_\alpha, 1_\alpha)$ for the boundary circle of this defect disk.

In contrast to the previous case with 1-strata present the spin structure on the 2-stratum is not fixed. This is because even though the disk itself is contractible, the 2-stratum is homotopic to a cylinder and could thus have a spin structure of either NS- or R-type. Therefore we need to incorporate both possibilities into our definition of 2-morphisms.

In general this would not be possible, however due to Assumption 5.1.2 we know that the biproduct $\mathcal{Z}\left(S_\alpha^{\mathrm{NS}}\right) \oplus \mathcal{Z}\left(S_\alpha^{\mathrm{R}}\right)$ exists, where we used the notation S_α^{NS} and S_α^{R} for α labeled NS- and R-circles, respectively. We can now define

$$\mathrm{End}_{\mathcal{B}_\mathcal{Z}(\alpha,\alpha)}(1_\alpha) := \mathrm{Hom}_\mathcal{C}\left(\mathbb{1}, \mathcal{Z}\left(S_\alpha^{\mathrm{NS}}\right) \oplus \mathcal{Z}\left(S_\alpha^{\mathrm{R}}\right)\right). \qquad (5.1.2)$$

Furthermore from the universal properties of the biproduct it follows that generalized elements $\mathcal{Z}\left(S_\alpha^{\mathrm{NS}}\right) \oplus \mathcal{Z}\left(S_\alpha^{\mathrm{R}}\right)$ decompose into elements of $\mathcal{Z}(S_\alpha^{\mathrm{NS}})$ and $\mathcal{Z}(S_\alpha^{\mathrm{R}})$, respectively. The generalized elements of $\mathcal{Z}(S_\alpha^{\mathrm{NS}})$ will be called *Neveu–Schwarz operators* and the generalized elements of $\mathcal{Z}(S_\alpha^{\mathrm{R}})$ *Ramond operators*. Motivated by Section 3.2 we will introduce the notation

$$C_0^\alpha := \mathrm{Hom}_\mathcal{C}\left(\mathbb{1}, \mathcal{Z}\left(S_\alpha^{\mathrm{R}}\right)\right),$$

$$C_1^\alpha := \mathrm{Hom}_\mathcal{C}\left(\mathbb{1}, \mathcal{Z}\left(S_\alpha^{\mathrm{NS}}\right)\right).$$

- For two 2-morphisms $\Phi \in \mathrm{Hom}_{\mathcal{B}_\mathcal{Z}(\alpha,\beta)}(X,Y)$ and $\Psi \in \mathrm{Hom}_{\mathcal{B}_\mathcal{Z}(\alpha,\beta)}(Y,Z)$ we

define their *vertical composition* to be

$$\Psi \cdot \Phi := \mathcal{Z} \left(\begin{array}{c} \text{[diagram]} \end{array} \right) \circ (\Psi \,\tilde{\otimes}\, \Phi)$$

where we view the smaller circles inside the disk as *ingoing*, and the big one as *outgoing*. From now on we will often suppress the orientation on the 1-strata of defect bordisms.

Associativity (even strict) of this composition follows from scaling invariance in the bordism category as well as functoriality of $\mathcal{Z}$. It is unital with respect to the identity 2-morphism 1_X associated to the disk

$$1_X = \mathcal{Z} \left(\begin{array}{c} \text{[diagram]} \end{array} \right)$$

- Finally for two 2-morphisms $\Phi \in \mathrm{Hom}_{\mathcal{B}(\alpha,\beta)}(X,Y)$ and $\tilde{\Phi} \in \mathrm{Hom}_{\mathcal{B}(\beta,\gamma)}(\tilde{X},\tilde{Y})$ we define their *horizontal composition* to be

$$\tilde{\Phi} \otimes \Phi := \mathcal{Z} \left(\begin{array}{c} \end{array} \right) \circ (\widetilde{\tilde{\Phi}\tilde{\otimes}\Phi})$$
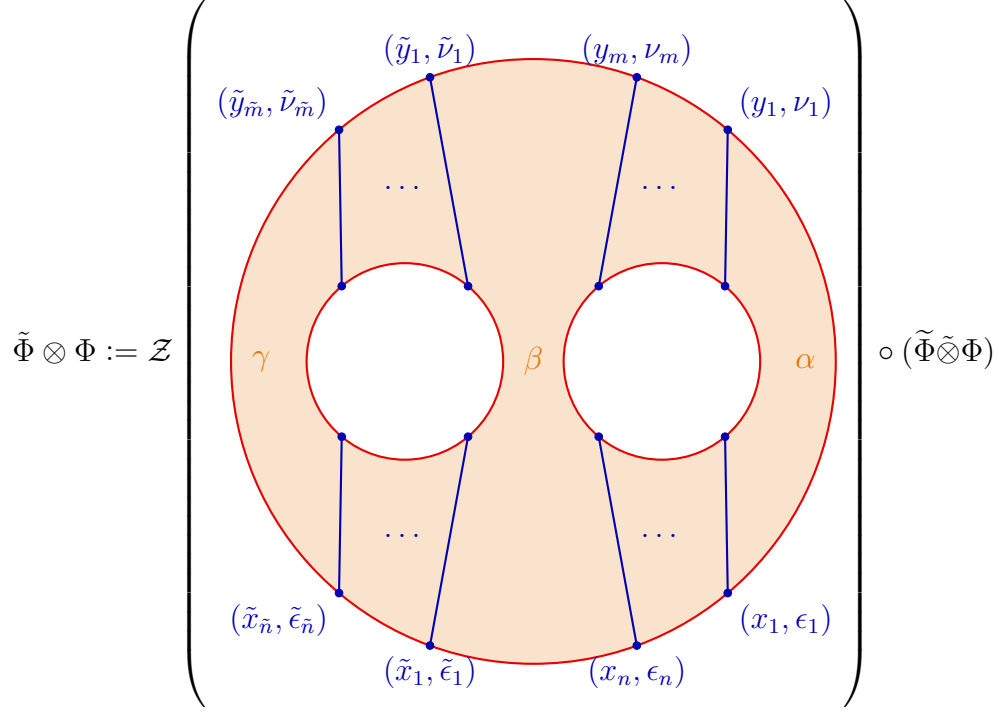
where we again view the smaller circles as ingoing, and the big one as outgoing.

Until now we have been very vague about why the above construction does not define a 2-category in contrast to the oriented case. This will be explained in the following: In any 2-category the 2-endomorphisms of any unit 1-morphism should form a commutative algebra with multiplication given by vertical composition as the endomorphism space of the unit of any monoidal category always is a commutative algebra [Eti+16, Proposition 2.2.10]. This follows from the *interchange law* $(-\otimes -)\cdot(-\otimes -) = (-\cdot -)\otimes(-\cdot -)$ which is a consequence of functoriality of horizontal composition [JY20, Explanation 2.1.6]. However for any $\alpha \in \mathcal{B}_{\mathcal{Z}}$ the endomorphisms $\mathrm{End}_{\mathcal{B}_{\mathcal{Z}}(\alpha,\alpha)}(1_\alpha)$ might not be a commutative algebra because

$$\Psi \cdot \Phi = \Phi \cdot \Psi \qquad (5.1.3)$$

does not need to hold if Ψ and Φ are Ramond operators. This follows from Equation (3.2.9) in Chapter 3 and the observation that horizontal (as well as vertical) composition of 2-endomorphisms is defined through a spin pair of pants bordism with

trivial stratification. Therefore $\mathcal{B}_{\mathcal{Z}}$ might not be a 2-category, but something closely related. We will postpone our discussion of the subtleties relating to the interchange law to Subsection 5.2.1 and will still the 2-categorical structure $\mathcal{B}_{\mathcal{Z}}$ a 2-category for now.

It is important to note here that this construction only uses information of the genus zero sector of the bordism category. The rest of this thesis is focused on studying $\mathcal{B}_{\mathcal{Z}}$ in more detail. In particular we will discuss extra structure, which is not present in the oriented case, in the next section. Before we come to this, we will now first prove an analogous rigidity result as the one for oriented defect TQFTs [DKR11, Remark 2.3].

Lemma 5.1.4. Let $\mathcal{Z}$ be a defect spin TQFT with corresponding 2-category $\mathcal{B}_{\mathcal{Z}}$. Then any 1-morphism $X = ((x_n, \epsilon_n), \ldots, (x_1, \epsilon_1)) \in \mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$ has left and right adjoints given by:

$$^{\dagger}X = X^{\dagger} = ((x_1, -\epsilon_1), \ldots, (x_n, -\epsilon_n)) \in \mathcal{B}_{\mathcal{Z}}(\beta, \alpha) \tag{5.1.4}$$

with adjunction maps

$$\mathrm{ev}_X = \mathcal{Z}\left(\quad \right) : {}^{\dagger}X \otimes X \to 1_{\alpha},$$

$$\mathrm{coev}_X = \mathcal{Z}\left(\quad \right) : 1_{\beta} \to X \otimes {}^{\dagger}X.$$

The maps $\widetilde{\mathrm{ev}}_X$ and $\widetilde{\mathrm{coev}}_X$ are defined analogously, by appropriately reversing the orientations and orders.

Proof. To verify this assertion we will prove one of the Zorro moves:

$$
\beta \, X \, \alpha \; = \; \mathcal{Z}\left(\begin{array}{c} X \\ \beta \quad X^\dagger \quad \alpha \\ X \end{array} \right) \circ \left(\mathrm{coev}_X \widetilde{\otimes} \mathrm{ev}_X\right)
$$

$$
= \; \mathcal{Z}\left(\begin{array}{c} X \\ \beta \quad X^\dagger \quad \alpha \\ X \end{array} \right) \tag{5.1.5}
$$

$$
= \; \mathcal{Z}\left(\begin{array}{c} \beta \quad X \quad \alpha \end{array} \right) \; = \; 1_X
$$

where in the first step we used the definition of horizontal and vertical composition in $\mathcal{B}_{\mathcal{Z}}$, in the second step we used functoriality of $\mathcal{Z}$, and in the last step we used

110

isotopy invariance in $\mathrm{Bord}_{2,1}^{\mathrm{def},\mathrm{Spin}}(\mathbb{D})$. The second Zorro move, and the ones for right adjoints, can be proven analogously. $\qquad\square$

Remark 5.1.5. In complete analogy to the oriented case one can even prove that $\mathcal{B}_{\mathcal{Z}}$ is pivotal, i.e. the right and left adjoints of 2-morphisms are isomorphic. More precisely the components of the pivotal structure are identities by a similar argument as the one above.

Before we come to the differences to the oriented case, we will now discuss a lemma which works for the oriented case, but can only be generalized to the subcategory $\mathrm{Bord}_{2,1}^{\mathrm{def},\underline{\mathrm{Spin}}}(\mathbb{D})$ of $\mathrm{Bord}_{2,1}^{\mathrm{def},\mathrm{Spin}}(\mathbb{D})$ where all spin structures are induced from a global spin structure. However this is not really a problem, as this result is not particularly useful from a practical point of view and will not be used in the rest of the thesis. We still include it as it can be viewed as a justification of the physical intuition of calling the horizontal composition a "fusion product" of defects, and thus serves to give a conceptually clearer picture. For this reason we will skip over some details of the proof, which would require more care from a differential topology viewpoint, e.g. smoothness of gluing.

Lemma 5.1.6. Let $\mathcal{Z}\colon \mathrm{Bord}_{2,1}^{\mathrm{def},\underline{\mathrm{Spin}}}(\mathbb{D}) \to \mathcal{C}$ be a defect spin TQFT and $\mathcal{B}_{\mathcal{Z}}$ its corresponding 2-category. Then there exist defect data $\overline{\mathbb{D}}$ and a defect spin TQFT $\overline{\mathcal{Z}}\colon \mathrm{Bord}_{2,}^{\mathrm{def},\underline{\mathrm{Spin}}}(\overline{\mathbb{D}}) \to \mathcal{C}$ such that $D_1 \subseteq \overline{D}_1$, $D_2 = \overline{D}_2$, $\mathcal{B}_{\mathcal{Z}} \cong \mathcal{B}_{\overline{\mathcal{Z}}}$, and

$$
\begin{array}{ccc}
\mathrm{Bord}_{2,1}^{\mathrm{def},\underline{\mathrm{Spin}}}(\mathbb{D}) & \xrightarrow{\ \mathcal{Z}\ } & \mathcal{C} \\[4pt]
{\scriptstyle \iota_*}\big\downarrow & \nearrow{\scriptstyle \overline{\mathcal{Z}}} & \\[4pt]
\mathrm{Bord}_{2,1}^{\mathrm{def},\underline{\mathrm{Spin}}}(\overline{\mathbb{D}}) & &
\end{array}
$$

commutes, where the functor ι_* induced by the inclusion $D_1 \subseteq \overline{D}_1$ is an equivalence of categories.

Proof sketch. We begin with defining the defect data $\overline{\mathbb{D}}$:

- The labels for 2-strata stay the same:
$$
\overline{D}_2 = D_2.
$$

- The labels for 1-strata are precisely the 1-morphisms in $\mathcal{B}_{\mathcal{Z}}$:
$$
\overline{D}_1 := \bigsqcup_{\alpha,\beta\in D_2} \mathcal{B}_{\mathcal{Z}}(\alpha,\beta).
$$

- For $X \in \mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$ we define source and target maps $\bar{s}_{\alpha,\beta}, \bar{t}_{\alpha,\beta} \colon \mathcal{B}_{\mathcal{Z}}(\alpha, \beta) \times \{\pm\} \to \overline{D}_2$ by

$$\bar{s}_{\alpha,\beta}(X, +) = \alpha, \quad \bar{t}_{\alpha,\beta}(X, +) = \beta \tag{5.1.6}$$
$$\bar{s}_{\alpha,\beta}(X, -) = \beta, \quad \bar{t}_{\alpha,\beta}(X, -) = \alpha \tag{5.1.7}$$

which we extend to

$$\bar{s}, \bar{t} \colon \overline{D}_1 \times \{\pm\} \to \overline{D}_2 \tag{5.1.8}$$

through the universal property of the disjoint union.

This gives rise to the bordism category $\mathrm{Bord}_{2,1}^{\mathrm{def, Spin}}(\overline{\mathbb{D}})$. In this category any 1-stratum between two 2-strata labeled with α and β, respectively, is by definition labeled with an element of $\mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$.
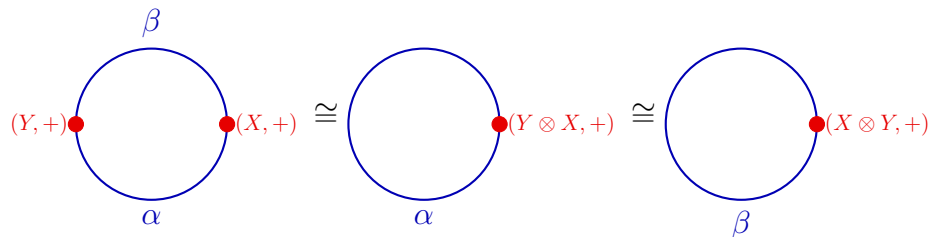
The functor $\iota_* \colon \mathrm{Bord}_{2,1}^{\mathrm{def, Spin}}(\mathbb{D}) \to \mathrm{Bord}_{2,1}^{\mathrm{def, Spin}}(\overline{\mathbb{D}})$ sends any defect spin bordism labeled with $\mathbb{D}$ to the same stratified spin bordism labeled with $\overline{\mathbb{D}}$ such that a 1-stratum with label $x \in D_1$ becomes a 1-stratum with label $(x, \epsilon) \in \overline{D}_1$, where ϵ is the orientation of the 1-stratum. It is straightforward to see that this functor is faithful and symmetric monoidal. In the following we will see that it is also essentially surjective and full, and therefore an equivalence.

Now we extend the TQFT $\mathcal{Z}$ to one defined on the new bordism category. To do this we first define the action of $\overline{\mathcal{Z}}$ on connected objects. For this let $\alpha \in D_2$ and $X = ((x_n, \epsilon_n), \ldots, (x_1, \epsilon_1)) \in \mathcal{B}_{\mathcal{Z}}(\alpha, \alpha)$, we set

$$\tag{5.1.9}$$

$$\tag{5.1.10}$$

112

where the circle in the first line on the right is an object in the original bordism category $\mathrm{Bord}_{2,1}^{\mathrm{def,\,Spin}}(\mathbb{D})$, with induced spin structures on the 0-strata such that the underlying orientation matches the ϵ_i. Note that the labels on the 2-strata are fully determined by X through the original source and target maps. This prescription already fully determines the action of $\overline{\mathcal{Z}}$ on connected objects because every connected object in $\mathrm{Bord}_{2,1}^{\mathrm{def,\,Spin}}(\overline{\mathbb{D}})$ is represented by one of this form through the horizontal composition $\otimes$ in $\mathcal{B}_{\mathcal{Z}}$. For example for $X \in \mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$ and $Y \in \mathcal{B}_{\mathcal{Z}}(\beta, \alpha)$ we have
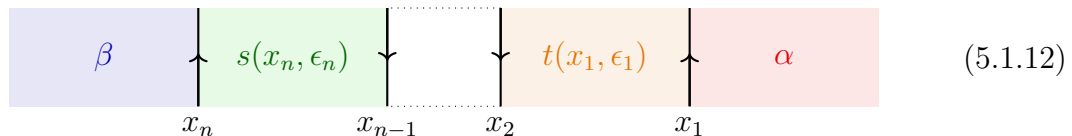
$$
\begin{array}{ccc}
\beta \\
(Y,+) \bullet \qquad \bullet (X,+) & \cong & \bullet (Y \otimes X, +) \cong \bullet (X \otimes Y, +) \\
\alpha & \alpha & \beta
\end{array}
$$

in $\mathrm{Bord}_{2,1}^{\mathrm{def,\,Spin}}(\overline{\mathbb{D}})$. The same reasoning shows that ι_* is essentially surjective.

The action on objects with multiple connected components follows from monoidality of $\mathcal{Z}$.

Using the same idea we can define the action of $\overline{\mathcal{Z}}$ on bordisms: We define the action of $\overline{\mathcal{Z}}$ on a bordism $[M]$ "locally", for this let $X = ((x_n, \epsilon_n), \ldots, (x_1, \epsilon))\in \mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$ be the label of a chosen 1-stratum in M. Locally a neighbourhood of X in M is of the form

$$
\begin{array}{c}
\beta \qquad \uparrow \qquad \alpha \\
X
\end{array}
\tag{5.1.11}
$$
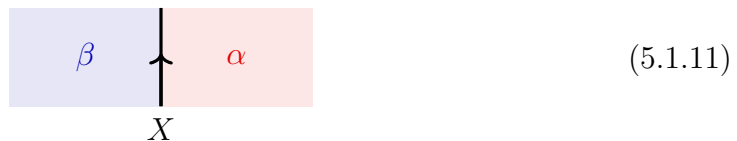
We now define a new defect spin manifold $\widetilde{M}$ by replacing all such neighbourhoods of X with ones of the form

$$
\begin{array}{c}
\beta \quad \uparrow \quad s(x_n, \epsilon_n) \quad \uparrow \qquad \uparrow \quad t(x_1, \epsilon_1) \quad \uparrow \quad \alpha \\
x_n \qquad \qquad x_{n-1} \quad x_2 \qquad \quad x_1
\end{array}
\tag{5.1.12}
$$

where the spin structures on the 1-strata are fixed using the ϵ_i as underlying orientation. By repeating this step for every 1-stratum in M we arrive at a manifold $\overline{M}$ which represents a bordism in $\mathrm{Bord}_{2,1}^{\mathrm{def,Spin}}(\overline{\mathbb{D}})$. We will not make this construction

113

more precise because it would require more care from a differential topology viewpoint and we will not use this result in the rest of the thesis. The things one would need to check are for example smoothness of the gluing and if the resulting bordism $[\overline{M}]$ is independent of the representative M of $[M]$. This argument can be used to show that ι_* is full.

We can now define

$$\overline{\mathcal{Z}}(M) := \mathcal{Z}(\overline{M}). \tag{5.1.13}$$

This indeed defines a functor because the above procedure respects gluing of stratified manifolds and $\mathcal{Z}$ is a functor. Furthermore it is symmetric monoidal because $\mathcal{Z}$ is symmetric monoidal and we can use the isomorphisms giving the symmetric monoidal structure of $\mathcal{Z}$ and the inverse of the functor ι_*. Finally from the definition of $\overline{D}_1$ and the construction of $\overline{\mathcal{Z}}$ it is evident that $\mathcal{B}_{\overline{\mathcal{Z}}} \cong \mathcal{B}_{\mathcal{Z}}$. $\qquad\square$

We call $\overline{\mathcal{Z}}$ the D_1-*completion* of $\mathcal{Z}$.

Remark 5.1.7. It should be stressed here that the above proof fails for $\mathrm{Bord}_{2,1}^{\mathrm{def},\,\mathrm{Spin}}(\mathbb{D})$ because we would loose information. More precisely to construct the TQFT $\overline{\mathcal{Z}}$ we constructed bordisms labeled with $\mathbb{D}$ from ones labeled with $\overline{\mathbb{D}}$. To do this we need to fix a way to choose the spin structures for the strata, otherwise the procedure could never be unique. However as soon as we fix a choice of assigning spin structures, we loose the information of what $\mathcal{Z}$ assigns to the different possibilities. For example let $\alpha, \beta \in \mathcal{B}_{\mathcal{Z}}$ and $X = ((x_n, \epsilon_n), \ldots, (x_1, \epsilon_1)) \in \mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$. There are two possible spin structures on the 1-stratum of the defect spin bordism

114
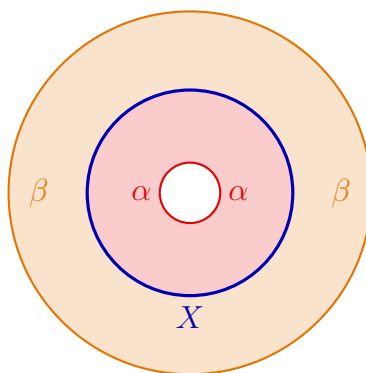
in $\mathrm{Bord}_{2,1}^{\mathrm{def},\,\mathrm{Spin}}(\overline{\mathbb{D}})$. From the construction in the proof we should obtain a bordism
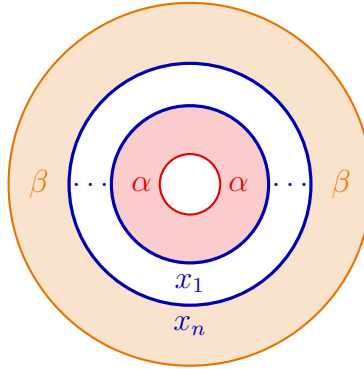
in $\mathrm{Bord}_{2,1}^{\mathrm{def},\,\mathrm{Spin}}(\mathbb{D})$, however there are 2^n possible combinations of spin structures on the 1-strata. In this sense there can never be an equivalence $\mathrm{Bord}_{2,1}^{\mathrm{def},\,\mathrm{Spin}}(\overline{\mathbb{D}}) \cong \mathrm{Bord}_{2,1}^{\mathrm{def},\,\mathrm{Spin}}(\mathbb{D})$.

5.2 Extra structure of $\mathcal{B}_{\mathcal{Z}}$

Up to now no structure on $\mathcal{B}_{\mathcal{Z}}$ was related to spin and could already be found in the oriented case. In this section we are going to discuss the extra structures on $\mathcal{B}_{\mathcal{Z}}$ related to the spin structures on the underlying bordisms. More precisely we will first briefly consider the question what type of 2-categorical structure $\mathcal{B}_{\mathcal{Z}}$ is. Afterwards we will discuss certain 2-morphisms in $\mathcal{B}_{\mathcal{Z}}$ and closely related morphisms $\mathcal{C}$.

5.2.1 Deck transformations

In Chapter 3 we saw that deck transformation bordisms play an important role in the algebraic structure of open-closed spin TQFTs, we thus expect them to also be of great importance for defect spin TQFTs. In this subsection we will incorporate them as a family of functors acting on the Hom categories of $\mathcal{B}_{\mathcal{Z}}$.

Let $\alpha, \beta \in \mathcal{B}_{\mathcal{Z}}$ and $X, Y \in \mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$. Recall the definition of the defect circle $S^1(X, Y)$ from the construction of $\mathcal{B}_{\mathcal{Z}}$, we define the *deck transformation bordism* $C_{S^1(X,Y)}^w$ to be the mapping cylinder over $S^1(X, Y)$ of a morphism w of stratified spin manifolds where the bundle map for any stratum is the non-trivial deck transformation discussed above Remark 2.2.2. More precisely w is an endomorphism of the stratified spin manifold underlying $S^1(X, Y)$ where for any stratum the endomorphism of spin manifolds is the non-trivial deck transformation, i.e. the action of

the non-trivial element in $\mathbb{Z}_2$ on the spin bundles. It is straightforward to see that w is an involution.
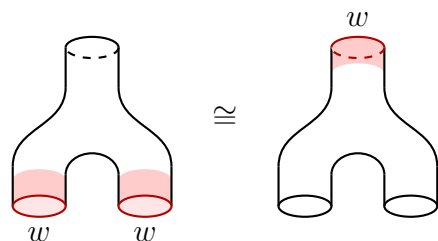
Lemma 5.2.1. There is a family of functors

$$
\begin{aligned}
S_{\beta,\alpha}\colon \mathcal{B}_Z(\alpha,\beta) &\to \mathcal{B}_Z(\alpha,\beta)\\
X &\mapsto X\\
\Phi &\mapsto \mathcal{Z}\left(C^w_{S^1(X,Y)}\right) \circ (\Phi)
\end{aligned}
\tag{5.2.1}
$$

for $\Phi\colon X \to Y$ which lift to a 2-functor

$$
\mathcal{S}\colon \mathcal{B}_{\mathcal{Z}} \to \mathcal{B}_Z.
\tag{5.2.2}
$$

Proof. First to show that $S_{\beta,\alpha}$ really defines a functor we need to check if it respects vertical composition in $\mathcal{B}_{\mathcal{Z}}$. For this recall that vertical composition is defined through a pair of pants bordism. Recall further, for example from Footnote 13, that gluing a bordism M with a mapping cylinder C^f_Σ along Σ, is equivalent to the bordism M^f which has the same underlying manifold as M but the boundary parameterization of Σ is precomposed with f. From this we see that it is sufficient to analyze pairs of pants with incoming boundaries parameterized through a non-trivial deck transformations, and the usual parameterisation for the outgoing boundary. But such a bordism is diffeomorphic to a pair of pants bordism with only the outgoing boundary parameterized by a deck transformation, and the incoming ones trivially parameterized, through a deck transformation on the whole bordism. This is completely analogous to the reasoning in [Koc03, Remark 1.3.22]. Schematically the described equivalence of bordisms can be pictured as:

$$
\tag{5.2.3}
$$

Note that in the case of trivial stratifications this is exactly the statement that the morphisms N_x of a closed Λ_2-Frobenius algebra, from Chapter 3, commute with the rest of the structure morphisms.

Applying this reasoning to the defect spin bordism corresponding to vertical composition, and using functoriality of $\mathcal{Z}$ we indeed find that $S_{\beta,\alpha}$ is functorial on

$\mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$ in the sense that

$$S_{\beta,\alpha}(\Psi \cdot \Phi) = S_{\beta,\alpha}(\Psi) \cdot S_{\beta,\alpha}(\Phi). \tag{5.2.4}$$

Moreover the same argument shows that $S_{\beta,\alpha}(1_X) = 1_X$ for any $X \in \mathcal{B}_{\mathcal{Z}}$, thus $(S_{\beta,\alpha})$ is indeed a functor.

Now to show that the family of functors $(S_{\beta,\alpha})$ lifts to a 2-functor we need to prove that it is compatible with horizontal composition of $\mathcal{B}_{\mathcal{Z}}$ in the sense that for $\gamma \in \mathcal{B}_{\mathcal{Z}}$ we have

$$S_{\gamma,\beta} \otimes S_{\beta,\alpha} = S_{\gamma,\alpha}. \tag{5.2.5}$$

To see that this holds recall that horizontal composition is also defined through a pair of pants bordism. For this pair of pants we can argue analogously as for the one of vertical composition. Thus we indeed find

$$S_{\gamma,\beta} \otimes S_{\beta,\alpha} = S_{\gamma,\alpha}. \tag{5.2.6}$$

$\square$

Corollary 5.2.2. The square of the 2-functor $\mathcal{S} \colon \mathcal{B}_{\mathcal{Z}} \to \mathcal{B}_{\mathcal{Z}}$ is the identity 2-functor, i.e. $\mathcal{S}^2 = 1_{\mathcal{B}_{\mathcal{Z}}}$.

Proof. This follows immediately because w is an involution. $\square$

We can now come back to the issue discussed at the end of Construction 5.1.3. Recall the problem was that for any $\alpha \in \mathcal{B}_{\mathcal{Z}}$ the space of 2-endomorphisms $\mathrm{End}_{\mathcal{B}_{\mathcal{Z}}(\alpha,\alpha)}(1_\alpha)$ should form a commutative algebra. However we already saw that for Ramond operators this is not necessarily the case since $\Psi, \Phi \in C_0^\alpha$ satisfy

$$\Psi \cdot \Phi = \Phi \cdot \mathcal{S}(\Psi) = \mathcal{S}(\Phi) \cdot \Psi \tag{5.2.7}$$

by Equation (3.2.9) and the observation that vertical (as well as horizontal) composition of 2-endomorphisms is defined through a spin pair of pants bordism with trivial stratification.

One possible resolution of this problem would be to check if $\mathcal{B}_{\mathcal{Z}}$ is something similar to a category enriched over the category of Π-*categories*. A Π-category as defined in [BE17, Definition 1.6] is a $\Bbbk$-linear category $\mathcal{A}$ together with a $\Bbbk$-linear endofunctor Π and a natural isomorphism $\Pi^2 \cong 1_{\mathcal{A}}$ together with a mild compatibility condition. There is also a notion of Π-functors between Π-categories which are $\Bbbk$-linear functors together with natural isomorphisms which mediate the compatibility

of with the Π-structures. Verifying if $\mathcal{B}_{\mathcal{Z}}$ has such a structure is still an object for further research.

However before we go on we will very roughly sketch the idea for $\mathcal{C} = \mathrm{Vect}_{\Bbbk}$: First note that any Hom category $\mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$ of $\mathcal{B}_{\mathcal{Z}}$ is a $\Bbbk$-linear category and the component $S_{\beta,\alpha}$ of $\mathcal{S}$ is a $\Bbbk$-linear endofunctor. Moreover from Corollary 5.2.2 we know that $S_{\beta,\alpha}^2 = 1_{\mathcal{B}_{\mathcal{Z}}(\alpha,\beta)}$, this gives $\mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$ the structure of a Π-category. Furthermore because the $S_{\beta,\alpha}$ lift to $\mathcal{S}$ it seems reasonable to expect that horizontal composition is a Π-functor.

It should also be noted here that there is a close relation between super categories and Π-categories. For this one needs the additional notion of Π-*super categories*. A Π-super category is a super category $\mathcal{A}$ together with a super functor $\Pi \colon \mathcal{A} \to \mathcal{A}$ and an odd supernatural isomorphism $\Pi \cong 1_{\mathcal{A}}$. There are functors

$$\mathcal{S}\mathrm{Cat} \xrightarrow{\ \pi\ } \Pi\text{-}\mathcal{S}\mathrm{Cat} \xrightarrow{\ \varepsilon\ } \Pi\text{-}\mathrm{Cat}$$

between the category of super categories $\mathcal{S}\mathrm{Cat}$, the category of Π-super categories Π-$\mathcal{S}\mathrm{Cat}$, and the category of Π-categories Π-Cat. The relation between super categories, Π-super categories, and Π-categories is explained through properties of thes functors. The functor $\pi \colon \mathcal{S}\mathrm{Cat} \to \Pi$-$\mathcal{S}\mathrm{Cat}$ has the property that for any super category $\mathcal{A}$ and any Π-super category $\mathcal{B}$ there is a super equivalence of super functor categories

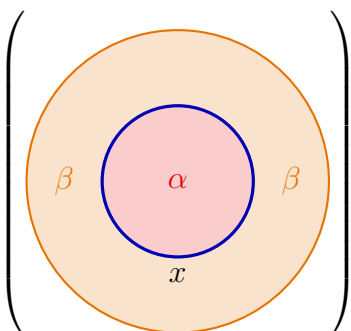$$[\pi\mathcal{A}, \mathcal{B}] \cong [\mathcal{A}, \nu\mathcal{B}]$$

with $\nu \colon \Pi$-$\mathcal{S}\mathrm{Cat} \to \mathcal{S}\mathrm{Cat}$ the forgetful functor.[1] The functor $\varepsilon \colon \Pi$-$\mathcal{S}\mathrm{Cat} \to \Pi$-$\mathrm{Cat}$ is an equivalence of categories. For a more precise statement and details see [BE17, Theorem 1.9].

5.2.2 States from line defects

In this final subsection we are going to focus on special 2-morphisms, which are induced by cup bordisms. We will start with the simplest case with exactly one 1-stratum. For this let $\alpha, \beta, \gamma \in D_2$, $X = (x, \epsilon) \in \mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$, and $Y = (y, \nu) \in \mathcal{B}_{\mathcal{Z}}(\beta, \gamma)$.
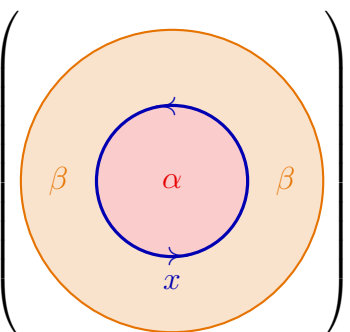
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[1] In [BE17] this super equivalence is called a 2-adjunction.

For $\sigma_\beta, \sigma_X \in \{\mathrm{NS}, \mathrm{R}\}$ we define the bordisms

$$\Phi^{\sigma_\beta}_{X,\sigma_X} := \mathcal{Z} \left(\begin{array}{c} \beta \quad \alpha \quad \beta \\ x \end{array} \right) \in C^\beta_\sigma \qquad (5.2.8)$$

where the boundary circle is outgoing and of R-type if $\sigma_\beta = \mathrm{R}$ and of NS-type if $\sigma_\beta = \mathrm{NS}$. The orientation underlying the spin structure σ_X of the 1-stratum is counterclockwise if $\epsilon = +$ and clockwise for $\epsilon = -$.

Note that $\Phi^\mathrm{R}_{X,\mathrm{R}}$ is the left or right (depending on ϵ) quantum dimension of the 1-morphism X in $\mathcal{B}_\mathcal{Z}$, e.g.

$$\Phi^\mathrm{R}_{(x,+),\mathrm{R}} = \mathcal{Z} \left(\begin{array}{c} \beta \quad \alpha \quad \beta \\ x \end{array} \right) = \dim_l(X). \qquad (5.2.9)$$

This is because gluing of defect spin bordisms respects the spin structure and gluing two spin intervals without twisting results in a R-type circle, see the discussion in the beginning of Section 4.2 for more details.

For the special case where $\alpha = \beta$ and $X = 1_\alpha$, i.e. no 1-stratum, we will employ the notation $\Omega_\alpha := \Phi^\mathrm{NS}_{1_\alpha}$, this notation will become clearer in a moment. Note here that $\Phi^\mathrm{R}_{1_\alpha}$ does not exist because for trivially stratified spin disks there is only one possible spin structure, for this reason we drop the spin structure σ in Ω_α. From now on we condense the label X and spin structure σ_X on any stratum to a single expression $\widehat{X} \equiv (X, \sigma_X)$.

Next we define four types of morphism in $\mathcal{C}$ with which we can "act" on the Φs:

$$F_{\widehat{X}}^{\sigma_\alpha,\sigma_\beta} := \mathcal{Z}\left(\vcenter{\hbox{}} \right) \in \mathrm{Hom}_{\mathcal{C}}\left(C_{\sigma_\alpha}^\alpha, C_{\sigma_\beta}^\beta \right) \qquad (5.2.10)$$

where orientations and spin structures on the 1-stratum are defined analogous to the definition of $\Phi_{\widehat{X}}^x$. The spin structures on the 2-strata are defined through $\sigma_\alpha, \sigma_\beta \in \{\mathrm{NS}, \mathrm{R}\}$ in analogy to above, e.g. $\sigma_\alpha = \mathrm{R}$ and $\sigma_\beta = \mathrm{NS}$ means that the α labeled 2-stratum is of R-type while the β labeled stratum is of NS-type. We thus have four possible versions of these maps, one for each combination of spin structures on the 2-strata.

Lemma 5.2.3. The map $F_{\widehat{X}}^{\mathrm{NS},\sigma_\beta}$ generates the state $\Phi_{\widehat{X}}^{\sigma_\beta}$ in the sense that

$$\Phi_{\widehat{X}}^{\sigma_\beta} = F_{\widehat{X}}^{\mathrm{NS},\sigma_\beta} \circ \Omega_\alpha. \qquad (5.2.11)$$

Proof. A direct computation shows

$$F_{\widehat{X}}^{\mathrm{NS},\sigma_\beta} \circ \Omega_\alpha = \mathcal{Z}\left(\vcenter{\hbox{}} \right) \circ \mathcal{Z}\left(\vcenter{\hbox{}} \right) \qquad (5.2.12)$$

$$= \mathcal{Z}\left(\vcenter{\hbox{}} \right) \qquad (5.2.13)$$

$$= \Phi_{\widehat{X}}^{\sigma_\beta} \qquad (5.2.14)$$

where in the second step we used functoriality of $\mathcal{Z}$. □

The significance of Lemma (5.2.11) is that it allows us to restrict our attention to only the maps $F_{\widehat{X}}^{1,\sigma_\beta}$ because we can recover the $\Phi_{\widehat{X}}^{\sigma_\beta}$ from them. This also explains the notation for Ω_α. In analogy to conventional quantum field theory this equation allows us to interpret Ω_α as the "vacuum state" of the phase α, and we can view the map $F_{\widehat{X}}^{\text{NS},\sigma_\beta}$ as a "creation operators" of the "state" $\Phi_{\widehat{X}}^{\sigma_\beta}$. Moreover the vacuum Ω_α is the unit of the NS algebra. It is important to note here that we are able to get the states in the NS- and R-sector of the phase β through these operators.

We can relax the definition of the F's two allow for bordisms with more then one 1-stratum. The 1-strata are labeled with elements $X = ((x_n, \epsilon_n), \ldots, (x_1, \epsilon_1)) \in \mathcal{B}_{\mathcal{Z}}(\alpha, \beta)$ such that the orientations match with the ϵ, i.e.

$$F_{\widehat{X}}^{\sigma_\alpha,\sigma_\beta} := \mathcal{Z}\left(\begin{array}{c} \includegraphics \end{array} \right) \in \mathrm{Hom}_{\mathcal{C}}\left(C_{\sigma_\alpha}^\alpha, C_{\sigma_\beta}^\beta \right). \qquad (5.2.15)$$

The following lemma can be proved completely analogously to Lemma 5.2.11.

Lemma 5.2.4. The maps $F_{\widehat{X}}^{\sigma_\alpha,\sigma_\beta}$ respect horizontal composition in the sense that

$$F_{\widehat{Y}}^{\sigma_\beta,\sigma_\gamma} \circ F_{\widehat{X}}^{\sigma_\alpha,\sigma_\beta} = F_{\widehat{Y\otimes X}}^{\sigma_\alpha,\sigma_\gamma} \qquad (5.2.16)$$

for composable X and Y.

Remark 5.2.5. The F operators cannot be used to define an 2-functor because they do not respect vertical composition of 2-morphisms as

$$\includegraphics \neq \includegraphics . \qquad (5.2.17)$$

In summary we found how defect circles give rise to special 2-morphisms in both the NS- and R-sector. Furthermore we described how cylinders can be used to define "operators" which act on these 2-morphisms.

Appendix A

Fiber bundles

In this appendix we will review some important notions from the theory of fiber bundles and the closely associated classifying spaces. Regarding the general theory of fiber bundles we will mostly follow [Ham17] and [Bau14], and [Joh80, Appendix B], [Die08] and [Nak03] for characteristic classes and classifying spaces.

We begin by defining the notion of a *fiber bundle* over a smooth manifold with general smooth fiber and introduce standard terminology and notions. After this we will focus on the important cases of principal fiber bundles and vector bundles where the fiber type is assumed to be a Lie group or a vector space respectively. Using the pullback bundle construction we will then introduce the notion of a classifying space of a topological group and the corresponding universal bundle.

The topological spaces we will be most interested in will be (smooth) manifolds which we assume to be second countable.

Definition A.0.1. Let M, E, F be smooth manifolds and let $\pi\colon E \to M$ be a smooth surjection. The tuple (E, π, M, F) is called a *smooth fiber bundle* of fiber type F if for any point $x \in M$ there exists a neighbourhood $U \subset M$ and a diffeomorphism $\phi\colon \pi^{-1}(U) \to U \times F$ such that $\mathrm{pr}_1 \circ \phi = \pi$:

$$
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\ \phi\ } & U \times F \\
{\scriptstyle \pi}\big\downarrow & \swarrow {\scriptstyle \mathrm{pr}_1} & \\
U & &
\end{array}
$$

E is called the *total space*, M the *base space* and F the *fiber type*. We will often abbreviate (E, π, M, F) by E. The pair $(U; \phi_U)$ is called a bundle chart or local trivialization of E over U.

Let $\{U_i\}_{i\in I}$ be a covering of M and let (U_i, ϕ_i) be a bundle chart for each $i \in I$. Then $\{U_i, \phi_i\}_{i\in I}$ is called a *bundle atlas* and the maps

$$\phi_i \circ \phi_k^{-1} \colon (U_i \cap U_k) \times F \to (U_i \cap U_k) \times F$$

are called *transition functions* between the bundle charts (U_i, ϕ_i) and (U_k, ϕ_k). Denoting by $\mathrm{Diff}(F)$ the diffeomorphism group of F we obtain maps

$$\phi_{ik} \colon (U_i \cap U_k) \to \mathrm{Diff}(F)$$
$$x \mapsto \left(\phi_i \circ \phi_k^{-1}(x) \colon F \to F\right)$$

that satisfy the so-called *cocycle conditions*

$$\phi_{ik}(x) \circ \phi_{kj}(x) = \phi_{ij}(x) \quad \text{and} \quad \phi_{ii}(x) = \mathrm{id}_F$$

or equivalently

$$\phi_{ik}(x) \circ \phi_{kj}(x) \circ \phi_{ji}(x) = \mathrm{id}_F$$

The collection of maps $\{\phi_{ik}\}_{i,k\in I}$ is called the *cocycle* of the bundle atlas $\{U_i, \phi_i\}_{i\in I}$.

Definition A.0.2. Two fiber bundles (E, π, M, F) and $(\tilde{E}, \tilde{\pi}, M, \tilde{F})$ over the same base space M are called *isomorphic* if there exists a diffeomorphism $H \colon E \to \tilde{E}$ such that $\tilde{\pi} \circ H = \pi$.

Definition A.0.3. Let G be a Lie group, P, M smooth manifolds and $\pi \colon P \to M$ a smooth map. The tuple (P, π, M, G) is called a *G-principal fiber bundle over M* if:

1. G acts on P from the right as a Lie transformation group, and the action is free and simply transitive on the fibers.

2. There exists a bundle atlas $\{(U_i, \phi_i)\}$ consisting of G-equivariant bundle maps, i.e.:

   - $\phi_i \colon \pi^{-1}(U_i) \to U_i \times G$ is a diffeomorphism.
   - $\mathrm{pr}_1 \circ \phi_i = \pi|_{U_i}$.
   - $\phi_i(p \cdot g) = \phi_i(p) \cdot g$ for all $p \in \pi^{-1}(U_i)$ and $g \in G$, where G acts on $U_i \times G$ via $(x, a) \cdot g = (x, ag)$.

We will often abbreviate the term G-principal fiber bundle simply to G-bundle.

Remark A.0.4. It is also possible to define a G-bundle over a topological space M, using a topological group G and homeomorphisms instead of diffeomorphisms. We will however mostly be working in the smooth category with the exception of $\mathbb{Z}_2$-bundles.

Definition A.0.5. Let (P, π, M, G) and (Q, π', N, H) be principal fiber bundles. A *bundle morphism* $Q \to P$ is a tuple (f, λ) where $\lambda \colon H \to G$ is a Lie group homomorphism and $f \colon Q \to P$ is a smooth map such that f is a λ-equivariant bundle map, i.e. $f(q \cdot h) = f(q) \cdot \lambda(h)$ for all $q \in Q$ and $h \in H$.

We call two principal G-bundles (P, π, M, G) and (P', π', M, G) equivalent (or isomorphic) if there exists a bundle morphism (f, id_G) such that $f \colon P \to P'$ is a diffeomorphism. We denote the set of equivalence classes of G-bundles over a manifold M by $\mathrm{Prin}_G(M)$. If we have a bundle morphism (f, λ) where the induced map on the base manifold is given by id_M, we call Q together with the morphism a λ-*reduction* of P. Furthermore if H is a subgroup of G and λ the inclusion, we call it an H-reduction of P.

Definition A.0.6. A fiber bundle (E, π, M, V) is called a $\Bbbk$-*vector bundle* of *rank n* if:

1. The typical fiber V is a n-dimensional $\Bbbk$-vector space.

2. Every fiber E_x is a $\Bbbk$-vector space.

3. There exists a bundle atlas $\{(U_i, \phi_i)\}$ such that the fiber diffeomorphisms

$$\phi_{ix} \colon E_x \to V$$

are linear isomorphisms.

1-dimensional vector bundles are usually called *line bundles*.

Definition A.0.7. Let (P, π, M, G) be a principal fiber bundle, F a smooth manifold and $\rho \colon G \times F \to F$ a smooth left action of G on F. Then we have a right action of G on $P \times F$ by $(p, v) \cdot g := (p \cdot g, \rho(g^{-1})v)$ we call the quotient by this action

$$P \times_\rho F := (P \times F)/G$$

the *associated fiber bundle* to P and ρ.

A particularly important special case arises when F is a vector space and ρ a representation of G on F. Another construction we will need is the so called *pullback bundle*.

Theorem A.0.8. Let $f\colon N \to M$ be smooth and let (E, π, M, F) be a smooth fiber bundle. Then $(f^*E, \bar{\pi}, N, F)$ with

$$f^*E := \{(x, e) \in N \times E \mid f(x) = \pi(e)\} \subset N \times E$$
$$\bar{\pi}((x, e)) := x$$

is a smooth fiber bundle called the *pullback bundle*.

See [Bau14, Satz 2.2] or [Ham17, Theorem 4.1.17].

The question to classify principal G-bundles over a fixed space relies heavily on the pullback bundle construction and the additional fact that homotopic maps induce isomorphic bundles, see [Die08, Theorem 14.3.3] for a proof of this assertion.

Definition A.0.9. Let G be a topological group. A *classifying space* of G is a connected topological space BG together with a G-bundle $EG \to BG$ called the *universal bundle*, such that the following is true. For any compact space M there is a one-to-one correspondence between $\mathrm{Prin}_G(M)$ and the homotopy classes of maps $M \to BG$. The correspondence is given by associating a homotopy class of a map $f_P\colon M \to BG$ to a fixed bundle P over M such that the pullback bundle f_P^*EG is isomorphic to P. Such a map f_P is called a *classifying map* of the bundle P.

This idea is illustrated in the following diagram:

$$
\begin{array}{ccccc}
P & \xrightarrow{\;\cong\;} & f_P^*EG & \xleftarrow{\;\;f_P^*\;\;} & EG \\
 & \searrow & \downarrow & & \downarrow \\
 & & M & \xrightarrow[f_P]{} & BG
\end{array}
$$

where the vertical and diagonal arrows are bundle projections.

BG is unique up to homotopy and exists for any topological group [Die08, Theorem 14.4.2]. Furthermore a principal G-bundle $E \to B$ is universal if and only if E is contractible [LM89, Theorem B.3] [Die08, Theorem 14.4.12]. This result implies the useful relation $\pi_n(BG) = \pi_{n-1}(G)$ between homotopy groups of G and homotopy groups of BG. To show this consider the long exact sequence of homotopy groups

$$\cdots \to \pi_n(G) \to \pi_n(EG) \to \pi_n(BG) \to \pi_{n-1}(G) \to \ldots$$

of the fibration

$$G \to EG \to BG,$$

see [Hat01, Section 4.2]. Now because EG is contractible we know that all its homotopy groups are trivial, i.e. $\pi_n(EG) = \{1\}$ for all $n \in \mathbb{N}$, thus exactness of the sequence means $\pi_n(BG) = \pi_{n-1}(G)$. From this relation it immediately follows that homotopic groups $G \simeq G'$ have homotopic classifying spaces $BG \simeq BG'$.

Given a Lie group homomorphism $\phi\colon H \to G$, there is a continuous map

$$B\phi\colon BH \to BG$$

which classifies the principal G-bundle $EH \times_\phi G$ over BH associated via ϕ, i.e. we have

$$(B\phi)^*(EG) := EH \times_\phi G.$$

The notation $B\phi$ comes from a functorial construction of classifying spaces. More precisely for G a group one construction of the classifying space BG is as the geometric realization of the nerve of the delooping BG, i.e. $BG \cong |\mathcal{N}(\mathrm{B}G)|$, see for example [Ric20, Section 11.2].

Using the above language, a reduction (or lift) of a principle G-bundle P over M to a principle H-bundle Q over M along the group homomorphism ϕ is equivalent to the existence of a map $f_Q\colon M \to BH$ such that the following diagram commutes up to homotopy:

$$
\begin{array}{ccc}
 & & BH \\
 & {\scriptstyle f_Q}\nearrow & \downarrow{\scriptstyle B\phi} \\
M & \xrightarrow{\ f_P\ } & BG
\end{array}
\tag{A.0.1}
$$

where f_P is a classifying map for the G-bundle P [Fre12, Proposition 9.38].

In general it is quite difficult to find necessary and sufficient conditions to find a lifting f_Q of f_P, in the following we will sketch how to reformulate the problem using *obstruction theory*, this will mostly be based on [DK01, Chapter 7]. Suppose we also have the homotopy fibration

$$BH \xrightarrow{\ B\phi\ } BG \xrightarrow{\ c\ } K(A, k).$$

with $K(A, k)$ the k-th *Eilenberg–Mac Lane space* [DK01, Chapter 7.7] of an abelian group A and $c\colon BG \to K(A, k)$ a continuous map. We now combine this fibration with Diagram (A.0.1) and consider the diagram

$$
\begin{array}{cccc}
 & & BH \\
 & {\scriptstyle f_Q}\nearrow & \downarrow{\scriptstyle B\phi} \\
M & \xrightarrow{\ f_P\ } & BG & \xrightarrow{\ c\ } K(A, k).
\end{array}
\tag{A.0.2}
$$

126

Obstruction theory tells us that the existence of f_Q turns out to be equivalent to the condition that $c \circ f_P$ is nullhomotopic [DK01, Chapter 7.10]. If we further use the important property of Eilenberg–Mac Lane spaces that homotopy classes of maps $BG \to K(A, k)$ are naturally isomorphic to the k-th *singular cohomology* $H^k(BG, A)$ with coefficients in A, i.e. $[BG, K(A, k)] \cong H^k(BG, A)$, we can thus view the map c as an element in $H^k(BG, A)$ [DK01, Theorem 7.22].[1] In this formulation the existence of f_Q is equivalent to the requirement that $f_P^*(c) \in H^k(M, A)$ vanishes.

This motivates us to define a *universal characteristic class c* for principal G-bundles to be a non-zero element in the singular cohomology ring $H^*(BG, A)$ with coefficients in A. For a fixed class $c \in H^k(BG; A)$ and any principal G-bundle $P \to M$. We define the *c-characteristic class* of P to be the class $c(P) = f_P^*(c) \in H^k(M, A)$, where $f_P \colon M \to BG$ is a classifying map of P. $c(P)$ is uniquely defined, as f_P is unique up to homotopy and cohomology is invariant under homotopy. Given a continuous map $F \colon N \to M$, the c-characteristic classes satisfy the naturality condition $c(F^*P) = F^*c(P)$ [LM89, Appendix B].

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[1] This result precisely states that the singular cohomology functor $H^k(-, A)$ is representable.

Bibliography

[Ati71]   M. F. Atiyah. "Riemann surfaces and spin structures". In: *Annales scientifiques de l'École Normale Supérieure* Ser. 4, 4.1 (1971), 47–62.

[Ati88]   Michael F. Atiyah. "Topological quantum field theory". In: *Publications Mathématiques de l'IHÉS* 68 (1988), 175–186.

[Bar05]   B. Bartlett. *Categorical Aspects of Topological Quantum Field Theories.* 2005. arXiv: `0512103 [math.QA]`.

[Bau14]   Helga. Baum. *Eichfeldtheorie.* ger. 2nd ed. 2014. Masterclass. Berlin, Heidelberg: Springer Berlin Heidelberg, 2014. ISBN: 3-642-38539-7.

[BE17]   J. Brundan and A. P. Ellis. "Monoidal supercategories". In: *Communications in Mathematical Physics* 351.3 (2017), pp. 1045–1089. arXiv: `1603.05928 [math.RT]`.

[BS10]   J. Baez and M. Stay. "Physics, Topology, Logic and Computation: A Rosetta Stone". In: *New structures for physics.* Springer, 2010, pp. 95–172. arXiv: `0903.0340 [quant-ph]`.

[Car18]   N. Carqueville. "Lecture notes on 2-dimensional defect TQFT". In: *Banach Center Publications* 114 (2018), pp. 49–84. arXiv: `1607.05747 [math.QA]`.

[CM16]   N. Carqueville and D. Murfet. "Adjunctions and defects in Landau–Ginzburg models". In: *Advances in Mathematics* 289 (2016), 480–566. DOI: `10.1016/j.aim.2015.03.033`.

[CMS20]   N. Carqueville, C. Meusburger, and G. Schaumann. "3-dimensional defect TQFTs and their tricategories". In: *Advances in Mathematics* 364 (2020), p. 107024. arXiv: `1603.01171 [math.QA]`.

[CR18]   N. Carqueville and I. Runkel. "Introductory lectures on topological quantum field theory". In: *Banach Center Publications* 114 (2018), pp. 9–47. arXiv: `1705.05734 [math.QA]`.

[CRS19]   N. Carqueville, I. Runkel, and G. Schaumann. "Orbifolds of n-dimensional defect TQFTs". In: *Geometry & Topology* 23.2 (2019), pp. 781–864. arXiv: `1705.06085 [math.QA]`.

[CS21]   N. Carqueville and L. Szegedy. *Fully extended r-spin TQFTs*. 2021. arXiv: `2107.02046 [math.QA]`.

[DFN06]   S. Das Sarma, M. Freedman, and C. Nayak. "Topological quantum computation". In: *Physics today* 59.7 (2006), pp. 32–38.

[Die08]   T. tom Dieck. *Algebraic Topology*. EMS textbooks in mathematics. European Mathematical Society, 2008. DOI: `10.4171/048`.

[DK01]   J. Davis and P. Kirk. *Lecture Notes in Algebraic Topology*. Vol. 35. Graduate Studies in Mathematics. Providence, R.I.: American Mathematical Society, 2001. ISBN: 978-0-8218-2160-2, 978-1-4704-2088-8. DOI: `10.1090/gsm/035`.

[DKR11]   A. Davydov, L. Kong, and I. Runkel. "Field theories with defects and the centre functor". In: *Mathematical foundations of quantum field theory and perturbative string theory* 83 (2011), pp. 71–128. arXiv: `1107.0495 [math.QA]`.

[Ebe06]   J. F. Ebert. *Characteristic classes of spin surface bundles: Appliations of the Madsen-Weiss theory*. 2006. arXiv: `math/0611612 [math.AT]`.

[Eti+16]   P. Etingof et al. *Tensor Categories*. Mathematical Surveys and Monographs. American Mathematical Society, 2016.

[FH19]   D. S. Freed and M. J. Hopkins. *Invertible phases of matter with spatial symmetry*. 2019. arXiv: `1901.06419 [math-ph]`.

[FH21]   D. S. Freed and M. J. Hopkins. "Reflection positivity and invertible topological phases". In: *Geometry & Topology* 25.3 (2021), pp. 1165–1330.

[Fre12]   D. S. Freed. *Bordism: old and new*. Lecture Notes. University of Texas, Austin, Fall 2012.

[FS08]   J. Fuchs and C. Stigner. "On Frobenius algebras in rigid monoidal categories". In: *The Arabian Journal for Science and Engineering* 33.2C (2008), pp. 175–192. arXiv: `0901.4886 [math.CT]`.

[GH94]   P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley Classics Library. Wiley, 1994.

[Ham17]   J.D. Hamilton Mark. *Mathematical Gauge Theory: With Applications to the Standard Model of Particle Physics*. Universitext. Cham: Springer International Publishing, 2017.

[Hat01]   A. Hatcher. *Algebraic Topology*. 2001.

[Hat17]   A. Hatcher. *Vector Bundles and K-Theory*. 2017.

[Hor+03] K. Hori et al. *Mirror Symmetry*. Vol. 1. Clay mathematics monographs. American Mathematical Society, 2003.

[Joh80]   D. Johnson. "Spin Structures and Quadratic forms on Surfaces". In: *Journal of the London Mathematical Society* s2-22.2 (Oct. 1980), pp. 365–373.

[JY20]   N. Johnson and D. Yau. *2-Dimensional Categories*. 2020. arXiv: 2002. 06055 [math.CT].

[Kel05]   G. M. Kelly. "The Basic Concepts of Enriched Category Theory". In: *Reprints in Theory and Applications of Categories* 2005 (Jan. 2005).

[Koc03]   J. Kock. *Frobenius Algebras and 2-D Topological Quantum Field Theories*. London Mathematical Society Student Texts. Cambridge University Press, 2003.

[Lau11]   A. D. Lauda. *An introduction to diagrammatic algebra and categorified quantum sl(2)*. 2011. arXiv: 1106.2128 [math.QA].

[Laz01]   C. I. Lazaroiu. "On the structure of open–closed topological field theory in two dimensions". In: *Nuclear Physics B* 603.3 (2001), pp. 497–530. ISSN: 0550-3213. DOI: https://doi.org/10.1016/S0550-3213(01)00135-3.

[Lei14]   T. Leinster. *Basic Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2014.

[LM89]   H.B. Lawson and M.L. Michelsohn. *Spin Geometry (PMS-38), Volume 38*. Princeton Mathematical Series. Princeton University Press, 1989.

[LP08]   A. D. Lauda and H. Pfeiffer. "Open–closed strings: Two-dimensional extended TQFTs and Frobenius algebras". In: *Topology and its Applications* 155.7 (2008), pp. 623–666. ISSN: 0166-8641. DOI: https://doi.org/10. 1016/j.topol.2007.11.005.

[Mil65]   J. W. Milnor. *Topology from the differentiable viewpoint*. eng. 1. publ.. Charlottesville, Va.: Univ. Press of Virginia, 1965.

[MS09]   G. W. Moore and G. Segal. "D-branes and K-theory in 2D topological field theory". In: *Dirichlet Branes and Mirror Symmetry*. Ed. by P. S. Aspinwall et al. American Mathematical Society, 2009. arXiv: `hep-th/0609042 [hep-th]`.

[Nak03]   M. Nakahara. *Geometry, Topology and Physics*. Taylor & Francis Ltd, 2003.

[Pol98]   Joseph Polchinski. *String Theory*. Vol. 2. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1998. DOI: `10.1017/CBO9780511618123`.

[Ric20]   B. Richter. *From categories to homotopy theory*. Vol. 188. Cambridge studies in advanced mathematics. Cambridge University Press, 2020. ISBN: 1108479626.

[RS21]   I. Runkel and L. Szegedy. "Topological field theory on r-spin surfaces and the Arf invariant". In: *Journal of Mathematical Physics* 62.10 (2021), p. 102302. arXiv: `1802.09978 [math.QA]`.

[Sch18]   C. Schommer-Pries. *Tori Detect Invertibility of Topological Field Theories*. 2018. arXiv: `1511.01772 [math.AT]`.

[Seg04]   G. Segal. "The definition of conformal field theory". eng. In: *Topology, Geometry and Quantum Field Theory*. Cambridge University Press, 2004, pp. 421–422. ISBN: 0521540496.

[SS20]   W. H. Stern and L. Szegedy. *Topological field theories on open-closed r-spin surfaces*. 2020. arXiv: `2004.14181 [math.QA]`.

[Sto68]   R. E. Stong. *Notes on Cobordism Theory*. eng. Princeton: Princeton University Press, 1968. ISBN: 069108064X.

[SU01]   R. U. Sexl and H. K. Urbantke. *Relativity, Groups, Particles*. Springer-Verlag Wien, 2001. DOI: `10.1007/978-3-7091-6234-7`.

[Wal16]   C. T. C. Wall. *Differential Topology*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2016.

[Wal84]   Robert M. Wald. *General Relativity*. First Edition. University Of Chicago Press, 1984. ISBN: 0226870332.

[Wer19]   K. Wernli. *Lecture Notes on Spin Geometry*. 2019. arXiv: `1911.09766 [math.DG]`.

[Wit88]   E. Witten. "Topological quantum field theory". In: *Communications in mathematical physics* 117.3 (1988), pp. 353–386. ISSN: 0010-3616.

[Wit89]   E. Witten. "Quantum field theory and the Jones polynomial". In: *Communications in Mathematical Physics* 121.3 (1989), pp. 351–399.