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„Normalizing conditions of Chern-Moser-Beloshapka Normalforms in $\mathbb{C}^{2\times 2}$"

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Abstract

This thesis aims to introduce Chern-Moser-Beloshapka-Normalforms in $\mathbb{C}^{2\times 2}$. First a general overview of the topic has been made, covering the Fischer inner product, the group of isomorphisms in $\mathbb{C}^{2\times 2}$ as well as a general introduction on the area of normalforms. Furthermore, an overview of the results of the paper *Convergence of the Chern–Moser–Beloshapka normal forms*, by Bernhard Lamel and Laurent Stolovitch has been made, on which the research of this thesis is based. For the main part of this thesis the program Wolfram Mathematica has been used to solve the appearing normalizing conditions. These are explicitly stated for the elliptic case in $\mathbb{C}^{2\times 2}$, and a simple adaptation of the code leads to results for the parabolic and hyperbolic cases.

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Contents

1 Introduction

This chapter aims to cover the basic themes and concepts needed to construct Chern-Moser-Beloshapka Normalforms, which are the basis of this thesis. Normalforms for hypersurfaces, the Fischer inner product and Levi-nondegenerate manifolds will be introduced.

1.1 Normalforms

In this subsection Normalforms for Hypersurfaces will be introduced following the paper *Real Hypersurfaces in Complex Manifolds* by S.S. Chern and J.K. Moser [4]. Furthermore the Chern-Moser normalizing conditions will be stated, based on *Normal forms in Cauchy-Riemann geometry* by Martin Kolar, Ilya G. Kossovskiy, Dmitri Zaitsev [7] as well as *Convergence of the Chern–Moser–Beloshapka normal forms* by B. Lamel and L. Stolovitch [8]. These notions are fundamental for the further concepts of this thesis.

Take $z_1, \ldots, z_{n+1}$ to be the coordinates in $\mathbb{C}^{n+1}$. Let M be a real hypersurface in $\mathbb{C}^{n+1}$ at the origin, which can be described by the following equation

$$r(z_1, \ldots, z_{n+1}, \overline{z_1}, \ldots, \overline{z_{n+1}}) = 0$$

with r being a real analytic function, where at the origin its first derivatives are not all equal to zero.

Applying transformation on M, which are holomorphic near the origin, can help us find a simple normal form. We can define the individual variables z_{n+1} and $\overline{z_{n+1}}$ as

$$z_{n+1} = w = u + iv, \quad \overline{z_{n+1}} = u - iv \tag{1.1}$$

and further assume that

$$\begin{aligned} r_{z_\alpha} &= 0, \quad \alpha = 1, \ldots, n \\ r_w &= - r_{\overline{w}} \neq 0 \end{aligned}$$

holds at the origin. By using a linear transformation this step can be accomplished. Solving equation 1.1 for the variable v yields

$$v = F(z, \overline{z}, u) \tag{1.2}$$

where the function F maintains the analytic property in the $2n + 1$ variables $z, \overline{z}, u$ and vanishes at the origin along with its first derivatives. The hypersurface M uniquely determines the function F.

We can apply a holomorphic transformation to the hypersurface M of the form

$$z^* = f(z, w), \quad w^* = g(z, w) \tag{1.3}$$

with f being an n-vector valued holomorphic function and g a holomorphic scalar. Furthermore, we require f and g to vanish at the origin and would like the complex tangent space 1.2 at the origin ($w = 0$) to be preserved. This results in the following conditions

$$f = 0, \quad g = 0, \quad \frac{\partial g}{\partial z} = 0 \quad \text{at } z = w = 0. \tag{1.4}$$

Thus the new hypersurface M^* can be written as

$$v^* = F^*(z^*, \overline{z^*}, u^*).$$

Choosing 1.3 wisely will reduce the representation of M^* to a simpler form. We can replace the assumption that F is real analytic and instead consider it as a formal power series in the variables $z_1, \dots, z_n, \overline{z_1}, \dots, \overline{z_n}$ and u with $\overline{F(z, \overline{z}, u)} = F(\overline{z}, z, u)$ as its reality condition. Furthermore we will assume that F does not contain any constant or linear terms. The space of these formal power series $\mathcal{F}$ is linear. The transformations in 1.3 can also be described by two formal power series for f and g in $z_1, \dots, z_n, w$ where no constant terms appear and for g no linear terms appear based on the last equation in 1.4. Formal transformations of this type create a group $\mathcal{G}$ under composition. Elements inside $\mathcal{F}$ can be decomposed into quasihomogeneous parts, such as $F \in \mathcal{F}$, $F = \sum_{\nu}^{\infty} F_\nu(z, \overline{z}, u)$, with $F_\nu(tz, t\overline{z}, t^2 u) = t^\nu F_\nu(z, \overline{z}, u)$ for any $t > 0$. Therefore u will be given the weight 2 and z and $\overline{z}$ will be assigned weight 1. The terms of weight $\nu = 2$ do not contain any terms with u, since F does not contain any linear terms. This means that

$$F_2 = Q(z) + \overline{Q(z)} + H(z, z)$$

with Q being a quadric form of z and H a Hermitian form.

Taking the following transformation

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z \\ w - 2iQ(z) \end{pmatrix}$$

will remove the quadric forms, and leaves us with a Hermitian form for F_2 of the shape $F_2 = H(z, z)$. This specific form is referred to as the Levi form, which we will denote by $\langle z, z \rangle = F_2$ and require to be nondegenerate. $\langle z, z \rangle$ denotes the corresponding bilinear form, which fulfills $\langle \lambda z_1, \mu z_2 \rangle = \lambda \overline{\mu} \langle z_1, z_2 \rangle$.

Applying these concepts to our hypersurface, M can be described by

$$v = \langle z, z \rangle + F \tag{1.5}$$

with $F = \sum_{\nu=3}^{\infty} F_\nu$, containing only those terms which have weight ≥ 3. We will further restrict the transformation 1.3 with the condition that $\partial^2 g / \partial z^\alpha \partial z^\beta = 0$ at the origin of the manifold.
To achieve a normal form for M we will need to find a proper formal transformation in $\mathcal{G}$ and will begin by writing it in the form

$$z^* = z + \sum_{\nu=2}^{\infty} f_\nu, \quad w^* = z + \sum_{\nu=3}^{\infty} g_\nu \tag{1.6}$$

where $f_\nu(tz, t^2 w) = t^\nu f_\nu(z, w)$, $g_\nu(tz, t^2 w) = t^\nu g_\nu(z, w)$, with ν being the weight of the polynomials f_ν, g_ν.
These new forms 1.6 can be inserted into the equation $v^* = \langle z^*, z^* \rangle + F^*$. The variables z and w can be restricted to the hypersurface 1.5 to yield us the transformation equations, in which the variables z, $\overline{z}$ and u are independent. In this relation we can gather all terms of the same weight μ and get

$$F_\mu + \operatorname{Im} g_\mu(z, u + i\langle z, z \rangle) = 2 \operatorname{Re}\langle f_{\mu-1}, z \rangle + F_\mu^* + \ldots$$

where the dots represent terms which depend on $f_{\nu-1}, g_\nu, F_\nu, F_\nu^*$ where $\nu < \mu$. The arguments in F_μ are z and $w = u + i\langle z, z \rangle$. Furthermore we can define a linear operator L, referred to as the $Chern - Moseroperator$ which maps $h = (f, g)$ into

$$Lh = \operatorname{Re}\{2\langle z, f \rangle + ig\}_{w=u+i\langle z,z \rangle} \tag{1.7}$$

which turns the relation above into

$$Lh = F_\mu - F_\mu^* + \ldots \quad \text{for } h = (f_{\mu-1}, g_\mu) \tag{1.8}$$

noting that L maps $f_{\mu-1}$,g_μ into terms which have the weight μ. Simplification of the power series F_μ^* is closely related to finding the complement of the range of the operator L.
The goal is to establish a linear subspace $\mathcal{N}$ of $\mathcal{F}$ such that $\mathcal{N}$ and the range of the operator L span $\mathcal{F}$. This means that if $\mathcal{O}$ is the space of $h = (f, g)$ with $f = \sum_{\nu=2}^{\infty} f_\nu$ and $g = \sum_{\nu=3}^{\infty} g_\nu$, then

$$\mathcal{F} = L\,\mathcal{O} + \mathcal{N} \text{ and } \mathcal{N} \cap L\,\mathcal{O} = \{0\} \tag{1.9}$$

Therefore $\mathcal{N}$ is indeed the complement of the range of L. In equation 1.8 we can directly see that we can choose F_μ^* to be in $\mathcal{N}$ and solve the rest of the equation for h. By induction we can see that equation 1.6 can be solved such that the function F^* lies inside $\mathcal{N}$. Hypersurfaces M^* where $F* \in \mathcal{N}$ are said to be in $normalform$.

The transformation into normal form directly correlates to the null space of the operator L. Therefore finding a transformation of M into normal form is reduced to finding a complement of the range $\mathcal{N}$ and the null space of the operator L, as described by S.S. Chern and J.K. Moser.
Take $\Phi(z,\bar{z},u) = \sum_{j,k} \Phi_{j,k}(z,\bar{z},u)$ to be a power series in the variables $z, \bar{z}, u$ with the degrees j in z and k in $\bar{z}$ respectively, $\Phi_{j,k}$ is then referred to be of *type* (j,k). Furthermore the condition that $\Phi_{j,k}(tz, s\bar{z}, u) = t^j s^k \Phi_{j,k}(z,\bar{z},u)$ is satisfied. The trace operator, a second order linear differential operator is defined as

$$\mathcal{T} := \left(\frac{\partial}{\partial \bar{z}}\right)^t J \left(\frac{\partial}{\partial z}\right) \tag{1.10}$$

S.S. Chern and J.K. Moser proved that $\ker \mathcal{L} = \{0\}$ and that $\mathcal{N}$ is the linear space containing power series of the form

$$\Phi_{j,0} = \Phi_{0,j} = 0, \quad j \geq 0, \tag{1.11}$$

$$\Phi_{j,1} = \Phi_{1,j} = 0, \quad j \geq 1, \tag{1.12}$$

$$\mathcal{T}\,\Phi_{2,2} = \mathcal{T}^2\,\Phi_{2,3} = \mathcal{T}^3\,\Phi_{3,3} = 0 \tag{1.13}$$

Every real-analytic Levi-nondegenerate hypersurface $M \in \mathbb{C}^{n+1}, n \geq 1$ and any point $p \in M$ can be transformed by formal power series from (M,p) into the normal form

$$v = \langle z, \bar{z} \rangle + \Phi(z,\bar{z},u), \qquad \text{with } \Phi \in \mathcal{N}$$

where $\mathcal{N}$ is the linear space containing the power series fulfilling 1.11.

Furthermore the transformation into normal form is unique only up to holomorphic mappings that preserve the hyperquadric as well as the origin.

1.2 The Fischer inner product

In this subsection the properties of the Fischer inner product will be outlined based on the paper *Über die Differentiationsprozesse der Algebra* by E. Fischer [6] and the paper *Convergence of the Chern–Moser–Beloshapka normal forms* by B. Lamel and L. Stolovitch, as well as *Invariant normal forms of formal series*, by G.R. Belitskii [2]. Further input on this topic can also be found in *Convergence of the Chern–Moser–Beloshapka normal forms* by B. Lamel and L. Stolovitch.
The Fischer inner product sets a basis for the construction of linear operators with formal adjoints, which will be needed for stating the normalization conditions.

Take V to be a finite dimensional vector space over $\mathbb{C}$ or $\mathbb{R}$, with an inner product denoted by $\langle \cdot, \cdot \rangle$. Let $u = (u_1, \ldots, u_d)$ be a formal variable and $V[\![u]\!]$ symbolize the space of all formal power series in u with corresponding coefficient values in V. The elements $f \in V[\![u]\!]$ will be written in the following way

$$f(u) = \sum_{\alpha \in \mathbb{N}^d} f_\alpha u^\alpha, \quad f_\alpha \in V$$

We can extend the inner product defined on V to an inner product defined on $V[\![u]\!]$ by defining

$$\langle f_\alpha u^\alpha, g_\beta u^\beta \rangle = \begin{cases} \alpha! \, \langle f_\alpha, g_\alpha \rangle, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$$

This inner product $\langle f, g \rangle$, referred to as the Fischer inner product, is only defined whenever at most finitely many of the products $\langle f_\alpha, g_\alpha \rangle$ are nonzero. The product $\langle f, g \rangle$ is defined as soon as $g \in F[u]$.
Take $T : F_1[\![u]\!] \to F_2[\![u]\!]$ to be a linear map, then it has a formal adjoint if we can find a map $T^* : F_2[\![u]\!] \to F_1[\![u]\!]$ such that

$$\langle Tf, g \rangle_2 = \langle f, T^* g \rangle_1$$

where both sides have to be defined.
For a linear map T, as above, a formal adjoint exists, if $T(F_1[u]) \subset F_2[u]$, where $F_j[u]$ are spaces of polynomials in u with values in $F_j, j = 1, 2$. For a proof see [8].

The map $D_\alpha : F[\![u]\!] \to F[\![u]\!]$ defined as

$$D_\gamma f(u) = \frac{\partial^{|\gamma|} f}{\partial u^\gamma} = \sum_\alpha \binom{\alpha!}{\gamma!} \gamma! \, f_\alpha u^{\alpha - \gamma}$$

has the formal adjoint

$$M_\gamma g(u) = u^\gamma g(u),$$

since

$$\langle D_\gamma f_\alpha u^\alpha, g_\beta u^\beta \rangle = \begin{cases} \binom{\alpha}{\beta} \gamma! \, (\alpha - \gamma)! \, \langle f_\alpha, g_{\alpha-\gamma} \rangle = \langle f_\alpha u^\alpha, g_\beta u^{\beta+\gamma} \rangle, & \beta = \alpha - \gamma \\ 0, & \beta \neq \alpha - \gamma. \end{cases}$$

Let $L : F_1 \to F_2$ be a linear operator, then we can define the induced operator $T_L : F_1[\![u]\!] \to F_2[\![u]\!]$ by

$$T_L \left(\sum_\alpha f_\alpha u^\alpha \right) = \sum_\alpha L f_\alpha u^\alpha.$$

This induced operator has the formal adjoint $T_L^* = T_{L^*}$, since

$$\langle T_L f_\alpha u^\alpha, g_\beta u^\beta \rangle_2 = \begin{cases} \alpha! \, \langle L f_\alpha, g_\beta \rangle_2 = \alpha! \, \langle f_\alpha, L^* g_\beta \rangle_1 = \langle f_\alpha u^\alpha, T_{L^*} g_\beta u^\beta \rangle, & \alpha = \beta \\ 0, & else. \end{cases}$$

This concept can be extended to multiple operators. Take $L_j : F[\![u]\!] \to F_j[\![u]\!]$ to be linear operators where $j = 1, \ldots, n$, and each operator has a corresponding formal adjoint L_j^*. Then the operator defined as

$$L = (L_1, \ldots, L_n) : F[\![u]\!] \to \bigoplus_j F_j[\![u]\!],$$

where $\bigoplus_j F_j$ stands for the orthogonal sum, has the formal adjoint $L^* = \sum_j L_j^*$. This concept can also be applied to differential operators. Take the map $D_k : F[\![u]\!] \to Sym^k F$, with $Sym^k F$ being the space of symmetric k-tensors on $\mathbb{C}^d$ (or $\mathbb{R}^d$, respectfully) with values in F, defined by

$$D_k f(u) = (D_\alpha f(u))_{\alpha \in \mathbb{N}^d,\ |\alpha|=k}.$$

This map has the formal adjoint $D_k^* = M_k$ defined by

$$M_k g(u) = \sum_{\gamma \in \mathbb{N}^d,\ |\gamma|=k} g_\gamma(u) u^\gamma, \quad \text{where } g(u) = (g_\gamma(u))_{|\gamma|=k}.$$

The space $Sym^k F$ can be realized as the space of homogeneous polynomials of degree k in d different variables $(u_1, \ldots, u_d)$, i.e.

$$Sym^k F = \bigoplus_{j=1}^{\binom{k+d-1}{d-1}} F,$$

with the induced norm as an orthogonal sum.

If $L_1 : F[\![u]\!] \to F_1[\![u]\!]$ and $L_2 : F[\![u]\!] \to F_2[\![u]\!]$ are two linear maps which each possess a formal adjoint, then $L = L_2 \circ L_1$ has the formal adjoint $L^* = L_1^* \circ L_2^*$.

The *normalized* Fischer inner product, defined by

$$\langle f_\alpha u^\alpha, g_\beta u^\beta \rangle = \begin{cases} \frac{\alpha!}{|\alpha|!} \langle f_\alpha, g_\alpha \rangle, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$$

can be convenient to use in certain cases. The adjoints with respect to the normalized and the standard Fischer inner product only differ by constant factors for terms which have the same homogeneity, but the existence of their adjoints and their corresponding kernels agree. For the sake of looking at the kernels of adjoints it is not necessary to distinguish between the normalized and the standard Fischer inner product.

The coefficient spaces F_1 and F_2 of our linear operator are going to be spaces of polynomials in z and $\bar{z}$ of specific homogeneities, which are equipped with the Fischer norm. Take $\mathcal{H}_{n,m}$ to be the space of homogeneous polynomials of degree m in $z \in \mathbb{C}^n$. Applying the Fischer inner product on monomials gives us

$$\langle z^\alpha, z^\beta \rangle = \begin{cases} \frac{\alpha!}{|\alpha|!}, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$$

.

The inner product on $(\mathcal{H}_{n,m})^l$ is induced by the Fischer inner product by declaring the components to be orthogonal with each other if $f = (f^1, \dots, f^l) \in (\mathcal{H}_{n,m})^l$, then $\langle f, g \rangle = \sum_{j=1}^{l} \langle f^j, g^j \rangle$

Take $\mathcal{R}_{m,k}$ to be the space of polynomials in z and $\bar{z}$, valued in $\mathbb{C}^d$, that are homogeneous of degree m (respectively k) in z (respectively $\bar{z}$). This space can also be equipped with the Fischer inner product $\langle ., . \rangle_{d,k}$, where the components are also declared to be orthogonal. This means that the inner product of a polynomial $P = (P_1, \dots, P_d)^t \in \mathcal{R}_{m,k}$ with another polynomial $Q = (Q_1, \dots, Q_d)^t \in \mathcal{R}_{m,k}$ is defined by $\langle P, Q \rangle = \sum_j \langle P_l, Q_l \rangle$, where the latter inner products within the sum are given on the basis monomials by

$$\langle z^{\alpha_1} \bar{z}^{\alpha_2}, z^{\beta_1} \bar{z}^{\beta_2} \rangle = \begin{cases} \frac{\alpha_1! \alpha_2!}{(|\alpha_1| + |\alpha_2|)!}, & \alpha_1 = \beta_1, \alpha_2 = \beta_2, \\ 0, & \alpha_1 \neq \beta_1 \text{ or } \alpha_2 \neq \beta_2. \end{cases}$$

The Fischer inner product and its properties prove themselves to very useful for defining the operators used to specify the normalizing conditions.

1.3 Levi-nondegenerate manifolds in $\mathbb{C}_z^2 \times \mathbb{C}_w^2$

This section is based on the paper *Convergence of the Chern–Moser–Beloshapka normal forms* by B. Lamel and L. Stolovitch and will also cover results of the paper *Holomorphic automorphisms of quadrics* by Vladimir Ežov and Gerd Schmalz, Gerd [5].

The objects of study are real-analytic, Levi-nondegenerate manifolds of $\mathbb{C}^N$. Take $M \subset \mathbb{C}^N$ to be a real submanifold. At a point $p \in M$, given the suitable coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d = \mathbb{C}^N$, this manifold can be described using a defining equation of the form

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w)$$

where $\varphi : \mathbb{C}^n \times \mathbb{R}^d \to \mathbb{R}^d$ is a germ of a real analytic map which satisfies

$$\varphi(0,0,0) = 0 \quad \textit{and} \quad \nabla\varphi(0,0,0) = 0$$

The second order invariant of this is its Levi form $\mathcal{L}_p$, which is a natural Hermitian vector-valued form defined on $\mathcal{V}_p = \mathbb{C}T_pM \cap \mathbb{C}T_p^{(0,1)}\mathbb{C}^N$ as

$$\mathcal{L}_p(X_p, Y_p) = [X_p, Y_p] \mod \mathcal{V}_p \oplus \bar{\mathcal{V}_p} \in (\mathbb{C}T_pM)/(\mathcal{V}_p \oplus \bar{\mathcal{V}_p}).$$

The manifold M is said to be Levi-nondegenerate (at p) if the Levi-form $\mathcal{L}_p$ is a nondegenerate, vectorvalued Hermitian form and it is of full rank.
The Levi-nondegenerate condition on the Levi-form $\mathcal{L}_p$ means that if $\mathcal{L}_p(X_p, Y_p) = 0$ for all $Y_p \in \mathcal{V}_p$ then $X_p = 0$.
$\mathcal{L}_p$ is of full rank if $\theta(\mathcal{L}_p(X_p, Y_p)) = 0$ for all $X_p, Y_p \in \mathcal{V}_p$ and $\theta \in T^0_p M = \mathcal{V}_p^\perp \cap \bar{\mathcal{V}}_p^\perp$ implies that $\theta = 0$.
A *hyperquadric* is the typical model for this case, where the corresponding equation to the manifold takes the form

$$\operatorname{Im} w = Q(z,\bar{z}) = \begin{pmatrix} Q_1(z,\bar{z}) \\ \vdots \\ Q_d(z,\bar{z}) \end{pmatrix} = \begin{pmatrix} \bar{z}^t J_1 z \\ \vdots \\ \bar{z}^t J_d z \end{pmatrix},$$

where the J_k are Hermitian $n \times n$ matrices.
In this situation the conditions of nondegeneracy and full rank can be expressed by

$$\bigcap_{k=1}^{d} \ker J_k = \{0\}, \quad \sum_{k=1}^{d} \lambda_k J_k = 0 \implies \lambda_k = 0, \ k = 1, \ldots, d. \tag{1.14}$$

Endowing z with weight 1 and w with weight 2 the defining equation of the hyperquadric becomes of degree 1, which we will assume to be the case from now on.

Therefore, at each point a Levi-nondegenerate manifold can be interpreted as a "higher order deformation" of such a hyperquadric. Thus their defining function can be written as

$$\operatorname{Im} w = Q(z,\bar{z}) + \Phi_{\geq 3}(z,\bar{z},\operatorname{Re} w) \tag{1.15}$$

where $\Phi_{\geq 3}$ contains only quasihomogeneous term of order at least 3.

This thesis aims to study Levi-nondegenerate manifolds in $\mathbb{C}^2_z \times \mathbb{C}^2_w$ which can be written in the way above

$$\operatorname{Im} w = Q(z,\bar{z}) + \Phi_{\geq 3}(z,\bar{z},\operatorname{Re} w),$$

where

$$Q(z,\bar{z}) = \left(\bar{z}^t J_1 z, \bar{z}^t J_2 z\right).$$

We will assume that Q is a Hermitian form on $\mathbb{C}^2$ defined by two Hermitian 2×2 matrices J_k. In particular, we note that

$$\overline{Q(a,\bar{b})} = Q(b,\bar{a})$$

for $a, b \in \mathbb{C}^n$. The higher-order deformation $\Phi_{\geq 3}(z,\bar{z},\operatorname{Re} w)$ is an analytic map germ at 0 and is, based on construction, of quasi-order ≥ 3 and can thus be written as

$$\Phi_{\geq 3}(z,\bar{z},\operatorname{Re} w) = \sum_{p \geq 3} \Phi_p(z,\bar{z},\operatorname{Re} w).$$

The individual terms $\Phi_p(z, \bar{z}, \operatorname{Re} w)$ are homogeneous of degree p.

We have the following three quadrics in the case $n = d = 2$.

Elliptic:

$$\begin{aligned} Q_1 : v_1 &= |z_1|^2 + |z_2|^2 \\ v_2 &= z_1\bar{z}_2 + z_2\bar{z}_1 \end{aligned} \tag{1.16}$$

Hyperbolic:

$$\begin{aligned} Q_{-1} : v_1 &= |z_1|^2 - |z_2|^2 \\ v_2 &= z_1\bar{z}_2 + z_2\bar{z}_1 \end{aligned} \tag{1.17}$$

Parabolic:

$$\begin{aligned} Q_0 : v_1 &= |z_1|^2 \\ v_2 &= z_1\bar{z}_2 + z_2\bar{z}_1 \end{aligned} \tag{1.18}$$

All possible Levi-nondegenerate quadric models of codimension 2 in $\mathbb{C}^4$ are isomorphic to one of the these three aforementioned quadrics by using the following action of the group $G^{2,2} = GL(2, \mathbb{C}) \times GL(2, \mathbb{R})$:

$$(C, \rho)(\langle z, z \rangle) = \rho \langle C^{-1}z, C^{-1}z \rangle.$$

These quadrics are referred to as *elliptic*, *parabolic* or *hyperbolic*, since the corresponding characteristic polynomial $P(t) = det(H^1 + tH^2)$ has either two real, two complex conjugate or one real root, respectively, as introduced by Ezov and Schmalz. These can be written in the following matrix notation.

$$\begin{aligned} Q_1 : v_1 &= \begin{pmatrix} \bar{z_1} & \bar{z_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ v_2 &= \begin{pmatrix} \bar{z_1} & \bar{z_2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{aligned} \tag{1.19}$$

$$\begin{aligned} Q_{-1} : v_1 &= \begin{pmatrix} \bar{z_1} & \bar{z_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ v_2 &= \begin{pmatrix} \bar{z_1} & \bar{z_2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{aligned} \tag{1.20}$$

$$\begin{aligned} Q_0 : v_1 &= \begin{pmatrix} \bar{z_1} & \bar{z_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ v_2 &= \begin{pmatrix} \bar{z_1} & \bar{z_2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{aligned} \tag{1.21}$$

Where the corresponding 2×2 matrices J_1 and J_2 can easily be detected.

2 General case

In this section the results of the paper *Convergence of the Chern-Moser-Beloshapka normal forms* by B. Lamel and L. Stolovitch [8] will be described. Furthermore the paper *Construction of the normal form of the equation of a surface of high codimension* by V. Beloshapka, [3], acts as basis for defining the operators within this chapter.
First the invidivial operators will be introduced, which will help us define the normalizing conditions for the power series in the second section. In the final section of this chapter the main results of the paper will be outlined.

2.1 Operators

The Fischer inner product sets the basis for the construction of the operators and their corresponding adjoints needed to further describe the normalizing conditions.

The operator $\mathcal{K}$ is the first important operator which will be defined. $\mathcal{K}$ acts on formal power series in z and u and maps these to power series in $z, \bar{z}, u$, that are linear in $\bar{z}$.

$$\mathcal{K} : \mathbb{C}[\![z, u]\!]^d \to (\mathbb{C}[\![z, \bar{z}, u]\!]^d)/((\bar{z}^2))$$

$\mathcal{K}$ is defined as

$$\mathcal{K}(\varphi(z, u)) = Q(\varphi(z, u), \bar{z}) = \begin{pmatrix} \bar{z}^t J_1(\varphi(z, u)) \\ \vdots \\ \bar{z}^t J_d(\varphi(z, u)) \end{pmatrix}$$

where $Q(\cdot, \cdot)$ stands for the Hermitian form of the normal form and consists of Hermitian matrices $J_1, \ldots, J_d$.

The complex conjugate of this operator, $\bar{\mathcal{K}}$, will also be used to define the complex conjugates of the normalizing conditions, which will need to be fulfilled as well.

$$\bar{\mathcal{K}} : \mathbb{C}[\![\bar{z}, u]\!]^d \to (\mathbb{C}[\![z, \bar{z}, u]\!]^d)/((z^2))$$

is defined as

$$\bar{\mathcal{K}}(\varphi(\bar{z}, u)) = Q(z, \varphi(\bar{z}, u)) = \begin{pmatrix} \varphi(\bar{z}, u)^t J_1 z \\ \vdots \\ \varphi(\bar{z}, u) J_d z \end{pmatrix}$$

The corresponding adjoint operator with respect to the Fischer inner product of $\mathcal{K}$, which will be denoted by $\mathcal{K}^*$, maps vice versa power series in $z, \bar{z}, u$, which are linear in $\bar{z}$ to power series in z and u

$$\mathcal{K}^* : (\mathbb{C}[\![z,\bar{z},u]\!]^d)/((\bar{z}^2)) \to \mathbb{C}[\![z,u]\!]^d$$

with

$$\mathcal{K}^* \begin{pmatrix} b_1(z,\bar{z},u) \\ \vdots \\ b_d(z,\bar{z},u) \end{pmatrix} = \sum_{j=1}^{d} \left(J_j \begin{pmatrix} \frac{\partial}{\partial \bar{z}_1}\Big|_0 \\ \vdots \\ \frac{\partial}{\partial \bar{z}_d}\Big|_0 \end{pmatrix} \right) b_j.$$

The complex conjugate of $\mathcal{K}^*$, namely $\bar{\mathcal{K}}^*$ will also be of use when stating the normalizing conditions.

The operator Δ, introduced in by Beloshapka is another key operator and acts on power series in $z, \bar{z}$ and u with values in an arbitrary space. It is defined by

$$(\Delta\varphi)(z,\bar{z},u) = \sum_{j=1}^{d} \varphi_{u_j}(z,\bar{z},u) Q_j(z,\bar{z})$$

with $Q(\cdot,\cdot)$ representing the Hermitian form of the normal form again.
The corresponding adjoint, with respect to the Fischer inner product, Δ^*, is the final operator we will need to define the normalizing conditions. This operator is defined as

$$\Delta^*\varphi = \sum_{j=1}^{d} u_j Q_j \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) \varphi$$

Δ^* has similar properties to the trace operator $\mathcal{T}$, defined in 1.10, and will be used in a similar sense.

These operators will not only be helpful for the specific normalization conditions but also for simplifying the conjugacy equations.

2.2 Normalization conditions

In this section the individual normalization conditions will be stated. In particular, the choice of the third normalization condition is an interesting one. These will later help define the spaces of power series for which the general results of section 2.4 apply to.

The set of all power series in $z, \bar{z}$ and u is denoted by $\mathbb{C}[\![z,\bar{z},u]\!]$. A power series in this set $\Phi(z,\bar{z},u) \in \mathbb{C}[\![z,\bar{z},u]\!]$ can be decomposed into a sum in the following way

$$\Phi(z,\bar{z},u) = \sum_{j,k=0}^{\infty} \Phi_{j,k}(z,\bar{z},u).$$

The first set of normalizing conditions applies to the $(0,p)-$ and $(p,0)-$terms, namely

$$\Phi_{p,0} = \Phi_{0,p} = 0, \quad \text{for } p \geq 0. \tag{2.1}$$

These conditions are equivalent to the definition of "normal" coordinates as Baouendi, Ebenfelt and Rothschild introduced (e.g. [1]). Furthermore this is equivalent to the fact that Φ does not contain any harmonic terms. The set of all power series which fulfill this first condition will be denoted by $\mathcal{N}^0$ and specifically is given by

$$\mathcal{N}^0 := \{\Phi \in \mathbb{C}[\![z, \bar{z}, u]\!] : \Phi(z, 0, u) = \Phi(0, \bar{z}, u) = 0\}.$$

The second set of normalizing conditions apply to the $(p,1)-$ and $(1,p)-$terms for $p \geq 0$. The operators used for these conditions $\mathcal{K}^*$ and $\bar{\mathcal{K}}^*$ are defined on spaces which are linear in $\bar{z}$, respectively z. Since $\Phi_{p,1} \in (\mathbb{C}[\![z, \bar{z}, u]\!]^2)/((\bar{z}^2))$ and $\Phi_{1,p} \in (\mathbb{C}[\![z, \bar{z}, u]\!]^2)/((z^2))$ the computation is possible. These conditions differ from the Chern-Moser conditions and state that

$$\mathcal{K}^* \, \Phi_{p,1} = (\bar{\mathcal{K}})^* \Phi_{1,p} = 0, \quad \text{for } p \geq 0. \tag{2.2}$$

The analogical normal form space is then

$$\mathcal{N}^1_{\leq k} = \left\{\Phi \in \mathbb{C}[\![z, \bar{z}, u]\!] : \mathcal{K}^* \, \Phi_{p,1} = \bar{\mathcal{K}}^* \Phi_{1,p} = 0, 1 < p \leq k\right\}.$$

In the $\Phi_{1,1}$ there is the possibility that terms appear which cannot be removed, so the trace conditions need to in fact remove all invariant parts of $\Phi_{j,j}$ for $j \leq 3$. In general $\Phi_{1,1}$ does not have a polar decomposition, so the choice of which terms should be removed and which terms should be kept is quite difficult. The approach B. Lamel and L. Stolovitch choose is a balanced one where the diagonal terms $(1,1)$, $(2,2)$ and $(3,3)$ are included. The normalization conditions state that

$$-6\Delta^* \Phi_{1,1} + (\Delta^*)^3 \Phi_{3,3} = 0 \tag{2.3}$$

$$\mathcal{K}^*(\Phi_{1,1} - i\Delta^* \Phi_{2,2} - (\Delta^*)^2 \Phi_{3,3}) = 0, \tag{2.4}$$

where the set of all power series $\Phi \in \mathbb{C}[\![z, \bar{z}, u]\!]$, which fulfill these two equations, will be denoted by $\mathcal{N}^d$.

In the final set of normalizing conditions the off-diagonal terms $(2,3)$ and $(3,2)$ will be used, which do not appear in the Chern-Moser approach. These state that

$$\mathcal{K}^*(\Delta^*)^2(\Phi_{2,3} + i\Delta\Phi_{1,2}) = \bar{\mathcal{K}}^*(\Delta^*)^2(\Phi_{3,2} - i\Delta\Phi_{2,1}) = 0 \tag{2.5}$$

$\mathcal{N}^{off}$ shall denote the set of all power series which fulfill these conditions. It is given by

$$\mathcal{N}^{off} = \left\{\Phi \in \mathbb{C}[\![z, \bar{z}, u]\!] : \mathcal{K}^*(\Delta^*)^2(\Phi_{2,3} + i\Delta\Phi_{1,2}) = \bar{\mathcal{K}}^*(\Delta^*)^2(\Phi_{3,2} - i\Delta\Phi_{2,1}) = 0\right\}.$$

These individual normal form spaces help us define two spaces which the general results are based on. Set

$$\hat{\mathcal{N}}_f := \mathcal{N}^0 \cap \mathcal{N}^1_{\leq\infty} \cap \mathcal{N}^d \cap \mathcal{N}^{off} \tag{2.6}$$

and

$$\hat{\mathcal{N}}^w_f := \mathcal{N}^0 \cap \mathcal{N}^1_{\leq\infty} \cap \mathcal{N}^d\,. \tag{2.7}$$

We can see that $\hat{\mathcal{N}}_f \subset \hat{\mathcal{N}}^w_f$.

2.3 Transformation of a perturbation of the intial quadric

This section covers the transformation of a perturbation of the inital quadric based on the construction of *Lamel* and *Stolovitch*. The inital quadric can be rewritten using the defined operators from section 2.1. The conjugacy equation describes these terms. Individual equations based on the found operators and previously discovered equations will be stated which will later help define the $(p,q)-$ terms of the conjugacy equation.

Take $\tilde{M}$ to be the germ of a real analytic manifold at 0 of the space $\mathbb{C}^{n+d}$. This can be described using following equation

$$v' = Q(z', \bar{z}') + \tilde{\Phi}_{\geq 3}(z', \bar{z}', u'), \tag{2.8}$$

with $w' := u' + iv' \in \mathbb{C}^d$, $u' = \operatorname{Re} w \in \mathbb{R}^d$, $v' = \operatorname{Im} w' \in \mathbb{R}^d$ and $z' \in \mathbb{C}^n$. Where, as defined before, $Q(z', \bar{z}')$ is a map of quadric polynomial type which takes values in $\mathbb{R}^d$ and $\tilde{\Phi}_{\geq 3}(z', \bar{z}', u')$, the germ of an analytic map at the origin.
The variables z' and $\bar{z}'$ will be assigned the weights $p_1 = p_2 = 1$ and the variables w' as well as u and v will receive weight $p_3 = 2$.
Thus $\operatorname{Im} w = Q(z, \bar{z})$, the defining equation of a *modelquadric* is quasi-homogeneous of quasi-degree 2. The term of the higher order deformation $\tilde{\Phi}_{\geq 3}(z, \bar{z}, u)$ then has a quasi-order which is ≥ 3 and can be written as

$$\tilde{\Phi}_{\geq 3}(z', \bar{z}', u') = \sum_{p \geq 3} \tilde{\Phi}_p(z', \bar{z}', u')$$

where $\tilde{\Phi}_p(z', \bar{z}', u')$ stands for all polynomials which are quasi-homogeneous of degree p. Therefore $\tilde{M}$, the germ of the real analytic manifold, can be interpreted as a higher order perturbation of the quadric defined by a homogeneous equation of the type $v' = Q(z', \bar{z}')$. From this point on further restrictions on $Q(z', \bar{z}')$ can be made, such as being a Hermitian form and Levi nondegeneracy. These concepts were introduced in section 1.3.

Take the following formal holomorphic change of coordinates of the form

$$z' = Cz + f_{\geq 2}(z, w) =: f(z, w), \quad w' = sw + g_{\geq 3}(z, w) =: g(z, w)$$

where C is an invertible $n \times n$ matrix and s is an invertible real $d \times d$ matrix which satisfy

$$Q(Cz, \bar{C}\bar{z}) = sQ(z, \bar{z}).$$

Using this coordinate change, our given manifold 2.8 can be described by

$$v = Q(z, \bar{z}) + \Phi_{\geq 3}(z, \bar{z}, u).$$

Here the equation for the manifold M depends on the coordinates z and w. The goal is to find an expression for the terms $\Phi_{\geq 3}$. The conjugacy equation

$$\begin{aligned} sv + \mathrm{Im}(g_{\geq 3}(z,w)) &= Q(Cz + f_{\geq 2}(z,w), \bar{C}\bar{z} + \bar{f}_{\geq 2}(\bar{z}, \bar{w})) \\ &\quad + \tilde{\Phi}_{\geq 3}(Cz + f_{\geq 2(z,w)}, \bar{C}\bar{z} + \bar{f}_{\geq 2}(\bar{z}, \bar{w}), su + \mathrm{Re}(g_{\geq 3}(z,w))) \end{aligned}$$

Following the notation of *Lamel* and *Stolovitch*, we will set

$$f := f(z, u + iv) \text{ and } \bar{f} := \bar{f}(\bar{z}, u - iv),$$

knowing that $v := Q(z, \bar{z}) + \Phi_{\geq 3}(z, \bar{z}, u)$. In the following Q will be used to represent $Q(z, \bar{z})$. The conjugacy equation then takes the form

$$\frac{1}{2i}(g - \bar{g}) = Q(f, \bar{f}) + \tilde{\Phi}_{\geq 3}\left(f, \bar{f}, \frac{g + \bar{g}}{2}\right). \tag{2.9}$$

Analogously to the above notation we will set $f_{\geq 2} := f_{\geq 2}(z, u+iv)$ and $\bar{f}_{\geq 2} := \bar{f}_{\geq 2}(\bar{z}, u-iv)$. Furthermore we have that

$$\begin{aligned} \frac{1}{2i}(s(u+iv) - s(u-iv)) &= sQ(z,\bar{z}) + s\Phi_{\geq 3}(z, \bar{z}, v), \\ Q(f, \bar{f}) &= Q(Cz + f_{\geq 2}, \bar{C}\bar{z} + \bar{f}_{\geq 2} \\ &= Q(Cz, \bar{f}_{\geq 2}) + Q(f_{\geq 2}, \bar{C}\bar{z}) + Q(Cz, \bar{C}\bar{z}) + Q(f_{\geq 2}, \bar{f}_{\geq 2}), \\ \tilde{\Phi}_{\geq 3}\left(f, \bar{f}, \frac{1}{2}[g + \bar{g}]\right) &= \tilde{\Phi}_{\geq 3}(Cz, \bar{C}\bar{z}, su) \\ &\quad + \left(\tilde{\Phi}_{\geq 3}\left(f, \bar{f}, \frac{1}{2}[g + \bar{g}]\right) - \tilde{\Phi}_{\geq 3}(Cz, \bar{C}\bar{z}, su)\right). \end{aligned}$$

Using these properties the conjugacy equation 2.9 can be written as

$$\begin{aligned} &\frac{1}{2i}[g_{\geq 3}(z, u + iQ) - \bar{g}_{\geq 3}(\bar{z}, u - iQ)] - (Q(Cz, \bar{f}_{\geq 2}(\bar{z}, u - iQ)) + Q(f_{\geq 2}(z, u + iQ), \bar{C}\bar{z})) \\ &\quad = Q(f_{\geq 2}, \bar{f}_{\geq 2}) + \tilde{\Phi}_{\geq 3}(Cz, \bar{C}\bar{z}, su) - s\Phi_{\geq 3}(z, \bar{z}, u) \\ &\quad + \left(\tilde{\Phi}_{\geq 3}\left(f, \bar{f}, \frac{1}{2}(g + \bar{g})\right) - \tilde{\Phi}_{\geq 3}(Cz, \bar{C}\bar{z}, su)\right) \\ &\quad + \frac{1}{2i}(g_{\geq 3}(z, u + iQ) - g_{\geq 3}) - \frac{1}{2i}(\bar{g}_{\geq 3}(\bar{z}, u - iQ) - \bar{g}_{\geq 3}) \\ &\quad + (Q(Cz, \bar{f}_{\geq 2}) - Q(Cz, \bar{f}_{\geq 2}(z, u - iQ))) \\ &\quad + (Q(f_{\geq 2}, \bar{C}\bar{z}) - Q(f_{\geq 2}(z, u + iQ), \bar{C}\bar{z})). \end{aligned} \tag{2.10}$$

Setting the matrices to $C = id$ and $s = 1$, this equation can be written in a concise way to become

$$\mathcal{L}(f_{\geq 2}, g_{\geq 3}) = \mathcal{T}(z, \bar{z}, u; f_{\geq 2}, g_{\geq 3}, \Phi) - \Phi.$$

Here $\mathcal{L}(f_{\geq 2}, g_{\geq 3})$ stands for the left-hand side of the equation 2.10 and is a $\mathcal{L}$ linear operator. This operator $\mathcal{L}$ acts on the set of homogeneous holomorphic vector fields which have quasi-degree $k-2 \geq 1$, denoted by QH_{k-2}. Elements of this set are expressions of the form

$$f_{k-1}(z,w)\frac{\partial}{\partial z} + g_k(z,w)\frac{\partial}{\partial w} = f_{k-1}(z,w) \cdot \begin{pmatrix} \frac{\partial}{\partial z_1} \\ \vdots \\ \frac{\partial}{\partial z_n} \end{pmatrix} + g_k(z,w) \cdot \begin{pmatrix} \frac{\partial}{\partial w_1} \\ \vdots \\ \frac{\partial}{\partial w_d} \end{pmatrix},$$

with f_{k-1} being a quasi-homogeneous polynomial which takes values in the spaces $\mathbb{C}^n$ and g_k being of the same type but taking values in $\mathbb{C}^d$ and both map to the set of quasi-homogeneous polynomials of degree $k \geq 3$ taking values in $\mathbb{C}^d$. The operator $\mathcal{L}$ restricted to the space QH_{k-2} will henceforth be denoted by $\mathcal{L}_k$.
$\mathcal{T}(z, \bar{z}, u; f_{\geq 2}, g_{\geq 3}, \Phi) - \Phi$ represents the right-hand side 2.10 and is $\mathcal{T}$ of nonlinear type. Equation 2.10 can be extended into the homogeneous components and takes on the form

$$\begin{aligned} \mathcal{L}(f_{\geq 2}, g_{\geq 3}) &= \{\mathcal{T}(z, \bar{z}, u; f_{\geq 2}, g_{\geq 3}, \Phi)\}_k - \Phi_k \\ &= \{\mathcal{T}(z, \bar{z}, u; f_{\geq 2}^{<k-1}, g_{\geq 3}^{<k}, \Phi_{<k})\}_k - \Phi_k. \end{aligned} \tag{2.11}$$

The terms of $\{\mathcal{T}(z, \bar{z}, u; f_{\geq 2}^{<k-1}, g_{\geq 3}^{<k}, \Phi_{<k})\}_k$ stand for the quasi-homogeneous terms of degree $< k$ for $g_{\geq 3}$ and $< k-1$ for $f_{\geq 2}$ of the Taylor expansion of $\mathcal{T}(z, \bar{z}, u; f_{\geq 2}, g_{\geq 3}, \Phi)$ at the point 0.
The assumed conditions on the form Q, namely linear independence and nondegeneracy, have a grave impact on the linear operator $\mathcal{L}$. These assumptions have the consequence that $\mathcal{L}$, which acts on the space of formal holomorphic vector fields, has a finite-dimensional kernel. This kernel corresponds to the space of infinitesimal CR automorphisms of the standard quadric $\operatorname{Im} w = Q(z, \bar{z})$ where the origin is stabilized.
Therefore, a *formal normal form* can be generated for any $k \geq 3$ by examining the complimentary subspace $\mathcal{N}_k$ to the image of the operator $\mathcal{L}_k$.
Applying basic induction on k, to this principle, one can prove that a (f_{k-1}, g_k) as well as a $\Phi \in \mathcal{N}_k$ can be found, such that equation 2.11 is fulfilled. This means that a unique formal holomorphic change of coordinates can be found such that the rewritten defining function lies inside the space of normal forms $\mathcal{N} := \bigoplus_{k \geq 3} \mathcal{N}_k$. This successful change of coordinates works up to elements in the space of infinitesimal automorphisms of the model quadric.
The choice and construction of $\mathcal{N}$ is therefore very important. Moreover it would be beneficial if $\mathcal{N}$ would fulfill further properties, namely if the defining functions are analytic, the change of coordinates should also be analytic. *B.Lamel* and *L.Stolovitch*, choose an approach which rewrites the components of the operator $\mathcal{L}$ as a series of partial

differential operators.

Following notation, the terms in Taylor series expansion will be denoted by subscripts p and q, which represent exactly those terms which are of degree p in z and degree q in $\bar{z}$. For maps depending on on further variables, these terms will be analytic in the other variables in a fixed domain and remain independent of p and q. Furthermore the homogeneous polynomial of degree k in the Taylor expansion of a function f shall be denoted by $f_k(z,u)$.

A new set of conditions can be gained by taking a closer look at the expression $D_u^k g(z,u)(Q+\Phi)^k$. Under the assumption that $Q+\Phi$ is a scalar, which does not affect our expression, since the goal is to find a lower bound for the order at which a fix set of monomials in the variables z and $\bar{z}$ vanishes. Taking the opportunity to treat $D_u^k g$ as a symmetric multilinear form where the arguments are monomials of z and $\bar{z}$, one obtains equalities replacing the equations. These are

$$g_{\geq 3}(z,u+iQ) - g_{\geq 3}(z,u+iQ+i\Phi) = \sum_{k\geq 1} \frac{i^k}{k!} D_u^k g_{\geq 3}(z,u)(Q^k - (Q+\Phi)^k) \tag{2.12}$$

and

$$Q(f_{\geq 2} - f_{\geq 2}(z,u+iQ), \bar{C}\bar{z}) = Q\left(\sum_{k\geq 1} \frac{i^k}{k!} D_u^k f_{\geq 2}(z,u)((Q+\Phi)^k - Q^k), \bar{C}\bar{z}\right).$$

Thus, taking the Taylor coefficient yields

$$\{D_u^k g(z,u)(Q^k - (Q+\Phi)^k)\}_{p,q} = \sum_{l=0}^{p} D_u^k g_l(z,u)\{Q^k - (Q+\Phi)^k\}_{p-l,q}$$

and

$$\begin{aligned} &\{Q(f_{\geq 2} - f_{\geq 2}(z,u+iQ), \bar{C}\bar{z})\}_{p,q} \\ &= Q(\{f_{\geq 2} - f_{\geq 2}(z,u+iQ)\}_{p,q-1}, \bar{C}\bar{z}) \\ &= \sum_{l=0}^{p}\sum_{k\geq 1} \frac{i^k}{k!} Q(D_u^k f_l(z,u)\{(Q+\Phi)^k - Q^k\}_{p-l,q-1}, \bar{C}\bar{z}). \end{aligned} \tag{2.13}$$

Since the first condition on the $\Phi_{j,k}$ is that $\Phi_{p,0} = \Phi_{0,q} = 0$, $(Q+\Phi)^l$ does not have any $(p,q)-$terms where $p < l$ or $q < l$. Therefore

$$(2.12)_{p,0} = 0, \tag{2.14}$$

$$(2.12)_{p,1} = i\sum_{j<p} D_u g_{p-j}\Phi_{j,1} + iD_u g_{p-1}(u)\Phi_{1,1}, \tag{2.15}$$

$$
\begin{aligned}
(2.12)_{2,2} &= iD_u g_0(u)\Phi_{2,2} + iD_u g_1(u)\Phi_{1,2} \\
&+ \frac{1}{2}D_u^2 g_0(u)(2\Phi_{1,1}Q + \Phi_{1,1}^2),
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
(2.12)_{3,3} &= iD_u g_0(u)\Phi_{3,3} + iD_u g_1(u)\Phi_{2,3} + iD_u g_2(u)\Phi_{1,3} \\
&+ \frac{1}{2}D_u^2 g_0(u)(2\Phi_{2,2}Q + \{\Phi^2\}_{3,3} \\
&+ \frac{1}{2}D_u^2 g_1(u)(2\Phi_{1,2}Q + \{\Phi^2\}_{2,3} \\
&- \frac{i}{6}D_u^3 g_0(u)(3\Phi_{1,1}^2 Q + \Phi_{1,1}^3 + 3\Phi_{1,1}Q^2),
\end{aligned} \tag{2.17}
$$

$$
\begin{aligned}
(2.12)_{3,2} &= iD_u g_0(u)\Phi_{3,2} + iD_u g_1(u)\Phi_{2,2} + iD_u g_2(u)\Phi_{1,2} \\
&+ \frac{1}{2}D_u^2 g_0(u)(2\Phi_{2,1}Q + \{\Phi^2\}_{3,2}) \\
&+ \frac{1}{2}D_u^2 g_1(u)(2\Phi_{1,1}Q + \{\Phi^2\}_{2,2}),
\end{aligned} \tag{2.18}
$$

$$
(2.12)_{3,1} = iD_u g_0(u)\Phi_{3,1} + iD_u g_1(u)\Phi_{2,1} + iD_u g_2(u)\Phi_{1,1} \tag{2.19}
$$

Using these results and replacing the values g_k and $\bar{g_k}$ by i and $-i$, the terms of the form $\bar{g}_{\geq 3}(z, u - iQ) - \bar{g}_{\geq}(z, u - iQ - i\Phi)$ can be acquired.

$$
(2.13)_{p,1} = (2.13)_{p,0} = 0, \tag{2.20}
$$

$$
(2.13)_{2,2} = Q(iD_u f_0(u)\Phi_{2,1} + iD_u f_1(u)\Phi_{1,1}, \bar{C}\bar{z}), \tag{2.21}
$$

$$
\begin{aligned}
(2.13)_{3,3} &= Q(iD_u f_0(u)\Phi_{3,2} + iD_u f_1(u)\Phi_{2,2} + iD_u f_2(u)\Phi_{1,2}, \bar{C}\bar{z}) \\
&+ \frac{1}{2}Q(D_u^2 f_0(u)(2\Phi_{2,1}Q + \{\Phi^2\}_{3,2}) \\
&+ \frac{1}{2}D_u^2 f_1(u)(2\Phi_{1,1}Q + \{\Phi^2\}_{2,2}), \bar{C}\bar{z}),
\end{aligned} \tag{2.22}
$$

$$
(2.13)_{3,2} = Q(iD_u f_0(u)\Phi_{3,1} + iD_u f_1(u)\Phi_{2,1} + iD_u f_2(u)\Phi_{1,1}, \bar{C}\bar{z}). \tag{2.23}
$$

It holds that

$$
Q(f_{\geq 2}, \bar{f}_{\geq 2}) = \sum_{k,l\geq 0} \frac{i^{k+l}(-1)^l}{k!\,l!} Q(D_u^k f_{\geq 2}(z,u)(Q+\Phi)^k, D_u^l \bar{f}_{\geq 2}(\bar{z},u)(Q+\Phi)^l).
$$

Using a similar approach to the one above, the functions $D_u^k f_{j'}(z,u)(Q+\Phi)^k$ and $D_u^l \bar{f}_j(\bar{z},u)(Q+\Phi)^l$ only contain $(p,q)-$terms when $p \geq j' + k$ and $q \geq k$ or $p \geq l$

and $q \geq l + j$, respectively. These conditions can be merged and state that the bilinear form $Q(D_u^k f_{j'}(z,u)(Q+\Phi)^k, D_u^l \bar{f}_j(\bar{z},u)(Q+\Phi)^l$ includes $(p,q)-$ terms where $p \geq j'+k+l$ and $q \geq j + k + l$. This yields

$$Q(f_{\geq 2}, \bar{f}_{\geq 2})_{p,0} = Q(f_p, \bar{f}_0), \tag{2.24}$$

$$\begin{aligned} Q(f_{\geq 2}, \bar{f}_{\geq 2})_{p,1} = {} & Q(f_p, \bar{f}_1) + iQ(Df_{p-1}(Q + \Phi_{1,1}) + \sum D_u f_{p-j}\Phi_{j,1}, \bar{f}_0) \\ & - iQ(f_{p-1}, D_u\bar{f}_0(Q + \Phi_{1,1})), \end{aligned} \tag{2.25}$$

$$\begin{aligned} Q(f_{\geq 2}, \bar{f}_{\geq 2})_{2,2} = {} & Q(f_2, \bar{f}_2) + iQ(Df_1(Q + \Phi_{1,1}), \bar{f}_1) \\ & - iQ(f_1, D\bar{f}_1(Q + \Phi_{1,1})) - \frac{1}{2}(Q(f_0, D_u^2\bar{f}_0(Q + \Phi_{1,1})^2) \\ & + Q(D_u^2 f_0(Q + \Phi_{1,1})^2, \bar{f}_0)) \\ & - Q(D_u f_0(u)(Q + \Phi_{1,1}), D_u\bar{f}_0(u)(Q + \Phi_{1,1})), \end{aligned} \tag{2.26}$$

$$\begin{aligned} Q(f_{\geq 2}, \bar{f}_{\geq 2})_{3,3} = {} & Q(f_3, \bar{f}_3) + iQ(Df_0\Phi_{3,1} + Df_1\Phi_{2,1} + Df_2(Q + \Phi_{1,1}).\bar{f}_2) \\ & - iQ(f_2, D_u\bar{f}_0\Phi_{1,3} + D_u\bar{f}_1\Phi_{1,2} + D_u\bar{f}_2(Q + \Phi_{1,1})) \\ & + Q\bigg(i\bigg(D_u f_0\Phi_{3,2} + D_u f_1\Phi_{2,2} + D_u f_2(Q + \Phi_{1,1}) \\ & - \frac{1}{2}(D_u^2 f_0(Q + \Phi_{1,1})\Phi_{2,1} + D_u^2 f_1(Q + \Phi_{1,1})^2)\bigg), \bar{f}_1\bigg) \\ & + Q\bigg(f_1, -i\bigg(D_u\bar{f}_0\Phi_{2,3} + D_u\bar{f}_1\Phi_{2,2} + D_u\bar{f}_2(Q + \Phi_{1,1}) \\ & - \frac{1}{2}(D_u^2\bar{f}_0(Q + \Phi_{1,2} + D^2\bar{f}_1(Q + \Phi_{1,1}^2)\bigg)\bigg) \\ & + Q\bigg(\frac{-i}{3}D_u^3 f_0(Q + \Phi_{1,1})^3 + \frac{-1}{2}(D_u^2 f_0(Q + \Phi_{1,1})\Phi_{2,2} \\ & + D_u^2 f_1(Q, \Phi_{1,1})\Phi_{1,2}), \bar{f}_0\bigg) + Q\bigg(f_0, \frac{i}{3}D_u^3\bar{f}_0(Q + \Phi_{1,1})^3 \\ & + \frac{-1}{2}(D_u^2\bar{f}_0(Q + \Phi_{1,1})\Phi_{2,2} + D_u^2\bar{f}_1(Q, \Phi_{1,1})\Phi_{2,1})\bigg) \\ & + Q(-i(D_u f_0\Phi_{3,3} + D_u f_1\Phi_{2,3} + D_u f_2\Phi_{2,3}), \bar{f}_0) \\ & + Q(f_0, i(D_u\bar{f}_0\Phi_{3,3} + D_u\bar{f}_1\Phi_{3,2} + D_u\bar{f}_2\Phi_{3,2})) \\ & + \frac{-i}{2}Q(D_u f_0(u)Q + \Phi_{1,1}, D_u^2\bar{f}_0(u)(Q + \Phi_{1,1})^2) \\ & + \frac{i}{2}Q(D_u^2 f_0(u)(Q + \Phi_{1,1})^2, D_u\bar{f}_0(u)(Q + \Phi_{1,1})) \\ & + Q(iD_u f_1(z,u)(Q + \Phi_{1,1}), D_u\bar{f}_1(\bar{z},u)(Q + \Phi_{1,1})), \end{aligned} \tag{2.27}$$

$$\begin{aligned} Q(f_{\geq 2}, \bar{f}_{\geq 2})_{3,2} &= Q(f_3, \bar{f}_2) - iQ(f_2, D_u\bar{f}_1(Q + \Phi_{1,1}) + D_u\bar{f}_0\Phi_{1,2}) \\ &- iQ(f_1, D_u\bar{f}_0(Q + \Phi_{1,1})^2 + D_u\bar{f}_1\Phi_{2,1}) \\ &- iQ(f_0, D_u\bar{f}_1\Phi_{3,1} + D_u\bar{f}_0\Phi_{3,2}) \\ &+ Q(D_u f_1(z,u)(Q + \Phi_{1,1}), D_u\bar{f}_0(u)(Q + \Phi_{1,1})) \\ &- \frac{1}{2}Q(D_u^2 f_1(Q + \Phi_{1,1})^2, \bar{f}_0). \end{aligned} \tag{2.28}$$

And

$$\begin{aligned} &\tilde{\Phi}_{\geq 3}\left(f, \bar{f}, \frac{1}{2}(g + \bar{g})\right) - \tilde{\Phi}_{\geq 3}(Cz, \bar{C}\bar{z}, su) \\ &= \sum_{\substack{|\alpha|+|\beta|+|\gamma|=k \\ k\geq 1}} \frac{1}{\alpha!\,\beta!\,\gamma!} \frac{\partial^k \tilde{\Phi}_{\geq 3}}{\partial z^\alpha \bar{z}^\beta u^\gamma}(Cz, \bar{C}\bar{z}, su) f^\alpha_{\geq 2} \bar{f}^\beta_{\geq 2} \left(\frac{1}{2}(g_{\geq 3} + \bar{g}_{\geq})\right)^\gamma, \end{aligned} \tag{2.29}$$

where $\alpha, \beta \in \mathbb{N}^n$ and $\gamma \in \mathbb{N}^d$.

2.4 Computed $(p, q)-$terms of the conjugacy equation

In this section the results of the computed equations of the relevant $(p, q)-$terms of the conjugacy equation will be stated. A full computation can be found in *Convergence of the Chern−Moser−Beloshapka normal forms* by *Lamel* and *Stolovitch*, [8]. These equations will provide a basis for which the linear operator $\mathcal{L}$ can be split into a series of differential operators.

For p, q being non negative integers, let

$$T_{p,q} := \left\{\tilde{\Phi}_{\geq 3}\left(f, \bar{f}, \frac{1}{2}(g + \tilde{g})\right) - \tilde{\Phi}_{\geq 3}(Cz, \bar{C}\bar{z}, su)\right\}_{p,q},$$

where latter subscripts p and q, denote the terms in the Taylor series expansion of degree p in z and degree q in $\bar{z}$.
After computations using 2.14,2.20 and 2.24 the $(p, 0)-$ term of the conjugacy equation is

$$\begin{aligned} \frac{1}{2i}g_p &= Q(f_p, \bar{f}_0) + T_{p,0} + \tilde{\Phi}_{p,0}(Cz, \tilde{C}\tilde{z}, su) - s\Phi_{p,0}(z, \bar{z}, u) \\ &=: F_{p,0}, \end{aligned} \tag{2.30}$$

for $p \geq 2$. In the case where $p = 1$, an extra term $-Q(Cz, \bar{f}_0)$ is added to the previous equation by the operator $\mathcal{L}$. Thus the equation

$$\begin{aligned} \frac{1}{2i}g_1 - Q(Cz, \bar{f}_0) &= Q(f_1, \bar{f}_0) + T_{1,0} + \tilde{\Phi}_{1,0}(Cz, \tilde{C}\tilde{z}, su) - s\Phi_{1,0}(z, \bar{z}, u) \\ &=: F_{1,0} \end{aligned}$$

can be obtained. The final equation of $(p,0)-$type can be found in the case where $p=0$. Then the following equation holds

$$\begin{aligned}\operatorname{Im}(g_0) &= Q(f_0,\bar f_0) + T_{0,0} + \tilde\Phi_{0,0}(Cz,\tilde C\tilde z, su) - s\Phi_{0,0}(z,\bar z,u)\\ &=: F_{0,0}.\end{aligned}$$

For the $(p,1)-$ terms computations of the conjugacy equation using 2.15, 2.20 and 2.25 yield

$$\begin{aligned}\frac{1}{2}D_u g_{p-1}Q - Q(f_p,\bar C\bar z) = \operatorname{Im} i\sum_{j<p} D_u g_{p-j}\Phi_{j,1} + Q(f_p,\bar f_1)\\ + iQ(Df_{p-1}(Q+\Phi_{1,1}),\bar f_0) - iQ(f_{p-1},D_u\bar f_0(Q+\Phi_{1,1}))\\ + \tilde\Phi_{p,1}(Cz,su) - s\Phi_{p,1}(z,u) + T_{p,1} =: F_{p,1},\end{aligned} \tag{2.31}$$

for $p\geq 3$. If $p=2$ the equation is a slight variant of the previous equation in the sense that the right-hand side remains unchanged but the left-hand side gains a term of the type $iQ(Cz,D_u\bar f_0 Q)$ rendering

$$\frac{1}{2}D_u g_1 Q - Q(f_2,\bar C\bar z) + iQ(Cz,D_u\bar f_0 Q) = F_{2,1}. \tag{2.32}$$

For $p=1$ the following holds

$$D_u\operatorname{Re}(g_0(u))\cdot Q - Q(Cz,\bar f_1(\bar z,u)) - Q(f_1(z,u)\bar C\bar z) = F_{1,1}. \tag{2.33}$$

After computations the $(3,2)-$ terms turn into an equation of the type

$$\begin{aligned}&-\frac{1}{4i}D_u^2 g_1(z,u)Q^2 + \frac{1}{2}Q(Cz,D_u^2\bar f_0(u)Q^2) - iQ(D_u f_2(z,u)Q,\bar C\bar z)\\ &\quad= (2.28) + \frac{1}{2i}(2.18) + (2.23) + \tilde\Phi_{3,2}(Cz,\bar C\bar z,su) - s\Phi_{3,2}(z,\bar z,u)\\ &\quad- \frac{1}{2i}(2.\bar{1}8) + (2.\bar{2}3) + (2.29)_{3,2} := F_{3,2}\end{aligned} \tag{2.34}$$

where the reference $(2.29)_{3,2}$ denotes the $(3,2)-$ component of the equation (2.29), $(2.\bar{1}8)$ represents the $(3,2)-$ component of $(\bar g_{\geq 3}(\bar z,u-iQ)-\bar g_{\geq 3})$ and $(2.\bar{2}3)$ stands for the $(3,2)-$ component of $(Q(Cz,\bar f_{\geq 2}) - Q(Cz,\bar f_{\geq 2}(z,u-iQ)))$.
Applying a similar principle one yields the following for the $(2,2)-$ terms

$$\begin{aligned}&-\frac{1}{2}D_u^2\operatorname{Im}(g_0)\cdot Q^2 + iQ(Cz,D_u\bar f_1(\bar z,u)\cdot Q) - iQ(D_u f_1(z,u)\cdot Q,\bar C\bar z)\\ &\quad= (2.26) + \frac{1}{2i}(2.16) + (2.21) + \tilde\Phi_{2,2}(Cz,\bar C\bar z,su) - s\Phi_{2,2}(z,\bar z,u)\\ &\quad- \frac{1}{2i}(2.\bar{1}6) + (2.\bar{2}1) + (2.29)_{2,2} =: F_{2,2}.\end{aligned} \tag{2.35}$$

Lastly the $(3,3)-$ terms yield

$$
\begin{aligned}
-\frac{1}{6}D_u^3\operatorname{Re}(g_0)\cdot Q^3 + Q(Cz, D_u^2\bar{f}_1(\bar{z},u)\cdot Q^2) + Q(D_u^2 f_1(z,u)\cdot Q^2, \bar{C}\bar{z}) \\
= (2.27) + \frac{1}{2i}(2.17) + (2.22) + \tilde{\Phi}_{3,3}(Cz, \bar{C}\bar{z}, su) - s\Phi_{3,3}(z,\bar{z},u) \qquad (2.36) \\
- \frac{1}{2i}(\bar{2.17}) + (\bar{2.22}) + (2.29)_{3,3} =: F_{3,3}
\end{aligned}
$$

2.5 Construction of the operator $\mathcal{L}$

Using the results from the previous chapter the operator $\mathcal{L}$ will be defined using differential operators, following the construction of *Lamel* and *Stolovitch*. Starting with a basic transformation and applying the discovered equations will result in a proof of the first theorem, as seen in [8].

Take a transformation of the following type

$$z^* = z + \sum_{k\geq 0} f_k, \quad w^* = w + \sum_{k\geq 0} g_k$$

where both $f_k(z,w)$ and $g_k(z,w)$ are homogeneous of degree k in the variable z. A similar approach to the one outlined above can also be taken where f_k and g_k can be interpreted to be power series maps.
The equations yielded by the computations of the specific $(p,q)-$terms can be combined to result in a series of conditions. Using equations 2.30 and 2.31 of the $(p,0)-$ and the $(p,1)-$ terms one obtains

$$
\begin{aligned}
\operatorname{Im}(g_0) &= F_{0,0}, \\
\frac{1}{2i}g_1 - Q(Cz, \bar{f}_0) &= F_{1,0}, \\
\frac{1}{2i}g_p &= F_{p,0}, \quad p \geq 2, \\
\frac{1}{2}D_u g_p Q - Q(f_{p+1}, \bar{z}) &= F_{p+1,1}, \quad p \geq 2.
\end{aligned}
$$

Combining the equation for the $(p,1)$-term where $p = 2$ with the equation for the $(3,2)-$term, gives

$$
\begin{aligned}
\frac{1}{2}D_u g_1 Q - Q(f_2, \bar{z}) + iQ(z, D_u\bar{f}_0 Q) &= F_{2,1} \\
-\frac{1}{4i}D_u^2 g_1(z,u)Q^2 + \frac{1}{2}Q(z, D^2\bar{f}_0(u)Q^2) - iQ(D_u f_2(z,u)Q, \bar{z}) &= F_{3,2}.
\end{aligned}
$$

The final set of equations can be obtained by using the equations of the $(3,3)-$, the $(2,2)-$ and $(p,1)-$terms where $p = 1$. They are

$$\operatorname{Im}(g_0) = F_{0,0}$$

$$D_u \operatorname{Re}(g_0(u)) \cdot Q - Q(z, \bar{f}_1(\bar{z}, u)) - Q(f_1(z, u)Q, \bar{z}) = F_{1,1},$$
$$-\frac{1}{2} D_u^2 \operatorname{Im}(g_0) \cdot Q^2 + iQ(z, D_u \bar{f}_1(\bar{z}, u) \cdot Q) - iQ(D_u f_1(z, u) \cdot Q, \bar{z}) = F_{2,2}$$
$$-\frac{1}{6} D_u^3 \operatorname{Re}(g_0) \cdot Q^3 + Q(z, D_u^2 \bar{f}_1(\bar{z}, u) \cdot Q^2) + Q(D_u^2 f_1(z, u) \cdot Q^2, \bar{z}) = F_{3,3}$$

The operator which we are trying to construct acts on a space of maps and takes on values in the space of formal power series in $\mathbb{C}[\![z, \bar{z}, u]\!]^d$, with a corresponding Hermitian product. To achieve the operator necessary, the left-hand sides will be simplified, the linear occurrence of the terms $\Phi_{p,q}$ of the transformed manifold will be rewritten and the right-hand side will be adjusted along with it. This yields the following system of equations

$$\begin{aligned}
&\operatorname{Im} g_0 = \Phi_{0,0} + \tilde{F}_{0,0},\\
&\frac{1}{2i} g_p = \Phi_{p,0} + \tilde{F}_{p,0},\\
&- Q(f_{p+1}, \bar{z}) = \Phi_{p+1,1} + \tilde{F}_{p+1,1},\\
&- Q(f_2, \bar{z}) + iQ(z, D_u \bar{f}_0 Q) = \Phi_{2,1} + \tilde{F}_{2,1},\\
&\frac{1}{2} Q(z, D^2 \bar{f}_0(u) Q^2) - iQ(D_u f_2(z, u)Q, \bar{z}) = \Phi_{3,2} + \tilde{F}_{3,2},\\
&D_u \operatorname{Re}(g_0(u)) \cdot Q - Q(z, \bar{f}_1(\bar{z}, u)) - Q(f_1(z, u), \bar{z}) = \Phi_{1,1} + \tilde{F}_{1,1},\\
&iQ(z, D_u \bar{f}_1(\bar{z}, u) \cdot Q) - iQ(D_u f_1(z, u) \cdot Q, \bar{z}) = \Phi_{2,2} + \tilde{F}_{2,2},\\
&- \frac{1}{6} D_u^3 \operatorname{Re}(g_0) \cdot Q^3 + Q(z, D_u^2 \bar{f}_1(\bar{z}, u) \cdot Q^2) + Q(D_u^2 f_1(z, u) \cdot Q^2, \bar{z})\\
&= \Phi_{3,3} + \tilde{F}_{3,3},
\end{aligned} \tag{2.37}$$

with $p \geq 2$. To show existence of a normal form, it would be enough to look at the injectivity of the linear operator which appears on the left-hand side of the equation above 2.37. The equations stated here are the basis for the normalization conditions from section 2.2.

To get the conditions for the $\Phi_{p,0}$, where $p \geq 0$, the normalizing conditions 3.1 can be applied to the above equation 2.37, and using the results to substitute for $\operatorname{Im} g_0$ and g_p in the leftover equations. The normalizing conditions for the $\Phi_{p,1}$ terms, the operator $\mathcal{K}^*$ needs to be applied to the third and fourth line of 2.37. After applying the normalizing condition stated in 2.2, the system of equations becomes implicit in f_p for $p \geq 2$. The solution to this problem can be used to replace the f_p-terms in the other equations. This process leaves us with a new set of equations that take on the form

$$\begin{aligned}
-\frac{1}{2} \bar{\mathcal{K}} \Delta^2 f_0 &= \Phi_{3,2} - i\Delta\Phi_{2,1} + \hat{F}_{3,2},\\
\Delta \operatorname{Re}(g_0) - \bar{\mathcal{K}} \bar{f}_1 - \mathcal{K} f_1 &= \Phi_{1,1} + \hat{F}_{1,1},\\
i\bar{\mathcal{K}} \Delta f_1 - i \mathcal{K} \Delta f_1 &= \Phi_{2,2} + \hat{F}_{2,2},\\
-\frac{1}{6} \Delta^3 \operatorname{Re}(g_0) + \bar{\mathcal{K}} \Delta^2 \bar{f}_1 + \mathcal{K} \Delta^2 f_1 &= \Phi_{3,3} + \hat{F}_{3,3},
\end{aligned} \tag{2.38}$$

where the previously introduced operators are now used.

These equations now give us the operator we require and we can define the operator $\mathcal{L} : \mathbb{C}[\![u]\!]^n \times \mathbb{R}[\![u]\!]^d \times \mathbb{C}[\![u]\!]^{n^2} \to \mathscr{R}^d_{3,2} \bigoplus \mathscr{R}^d_{1,1} \bigoplus \mathscr{R}^d_{2,2} \bigoplus \mathscr{R}^d_{3,3}$ as

$$\mathcal{L}(f_0, \operatorname{Re} g_0, f_1) = \begin{pmatrix} -\frac{1}{2}\bar{\mathcal{K}}\Delta^2 f_0 \\ \Delta \operatorname{Re}(g_0) - \bar{\mathcal{K}}\bar{f}_1 - \mathcal{K} f_1 \\ i\bar{\mathcal{K}}\Delta f_1 - i\,\mathcal{K}\,\Delta f_1 \\ -\frac{1}{6}\Delta^3 \operatorname{Re}(g_0) + \bar{\mathcal{K}}\Delta^2 \bar{f}_1 + \mathcal{K}\,\Delta^2 f_1 \end{pmatrix}.$$

The space of normal forms is then the kernel of the adjoint operator, with respect to the Hermitian product on the individual spaces of $\mathcal{L}$. A formal solution to this problem exists and is in fact unique up to modulo $\ker \mathcal{L}$. Uniqueness occurs as soon as we require $(f_0, \operatorname{Re} g_0, f_1) \in \operatorname{Im} \mathcal{L}^*$. Constructing the normal form space is exactly what proves Theorem 1.

2.6 General results

In this section the results of the paper *Convergence of the Chern−Moser−Beloshapka normal forms* by B. Lamel and L. Stolovitch will be outlined. The authors prove three important theorems related to the existence of a normal form and the convergence of the formal normal forms in this setting.

Take a Levi-nondegenerate hyperquadric $\operatorname{Im} w = Q(z, \bar{z})$. For perturbations, which take the form $\operatorname{Im} w = Q(z, \bar{z}) + \Phi_{\geq 3}(z, \bar{z}, \operatorname{Re} w)$, it is possible to find a *formal* normal form. This is what the first main result states in detail.

Theorem 1 *Let $Q(z, \bar{z})$ to be a non degenerate form of full rank on $\mathbb{C}^n$ with values in $\mathbb{C}^d$, i.e. Q takes the form $Q(z, \bar{z}) = (\bar{z}^t J_1 z, \ldots, \bar{z}^t J_d z)$, where the individual matrices J_k satisfy the nondegeneracy conditions 1.14. Then, a subspace $\hat{\mathcal{N}}_f \subset \mathbb{C}[\![z, \bar{z}, u]\!]$ defined in 2.6 exists such that the following can be stated. Let M be a manifold given near $0 \in \mathbb{C}^N$, defined by an equation of the following type*

$$\operatorname{Im} w' = Q(z', \bar{z}') + \tilde{\Phi}_{\geq 3}(z', \bar{z}', \operatorname{Re} w')$$

where $\tilde{\Phi} \in \mathbb{C}[\![z, \bar{z}, u]\!]$. Then a formal biholomorphic map, which is unique up to a finite-dimensional set of parameters, exists and takes the form $H(z, w) = (z + f_{\geq 2}, w + g_{\geq 3})$. In these new formal coordinates $(z, w) = H^{-1}(z', w')$ the given manifold M can be described by an equation of the type

$$\operatorname{Im} w = Q(z, \bar{z}) + \Phi_{\geq 3}(z, \bar{z}, \operatorname{Re} w)$$

where $\Phi_{\geq 3} \in \hat{\mathcal{N}}_f$.

This result can be interpreted as a specific description of the construction Beloshapka made for abstract normal forms.

The space $\hat{\mathcal{N}}_f$ defined in 2.6 is the set of all formal power series which fulfill all of the normalizing conditions from the previous section. The second defined space $\hat{\mathcal{N}}_f^w$ in 2.7 does not include the normalizing conditions of the $(2,3)-$ and $(3,2)-$ terms and thus only considers the transversal $d-$manifold $z = f_0(w)$ only as a parameter. This concept is what leads to the second theorem.

Theorem 2 *Take a nondegenerate form $Q(z,\bar{z})$ on $\mathbb{C}^n$ which has values in $\mathbb{C}^d$ and is of full rank, which means that Q takes the form $Q(z,\bar{z}) = (\bar{z}^t J_1 z, \ldots, \bar{z}^t J_d z)$, where the individual matrices J_k satisfy the nondegeneracy conditions 1.14. Using $\hat{\mathcal{N}}_f^w$ from 2.7, let $\mathcal{N}^w = \hat{\mathcal{N}}_f^w \cap \mathbb{C}\{z, \bar{z}, \operatorname{Re} w\}$. For the space $\mathcal{N}^w$ following holds. Take M to be a manifold defined near $0 \in \mathbb{C}^N$ by an equation of the following type*

$$\operatorname{Im} w' = Q(z,' \bar{z}') + \tilde{\Phi}_{\geq 3}(z, \bar{z}', \operatorname{Re} w).$$

Then for any $f_0 \in \mathbb{C}\{w\}^n$ which disappears at the origin subsequently, a biholomorphic map of the type

$$H(z,w) = (z + f_0 + f_{\geq 2}, w + g_{\geq 3})$$

can be found, which fulfills the condition that $f_{\geq 2}(0,w) = 0$ and is unique up to a finite-dimensional space of parameters. The manifold M can then be described in the new coordinates $(z,w) = H^{-1}(z',w')$ by an equation of the type

$$\operatorname{Im} w = Q(z,\bar{z}) + \Phi_{\geq 3}(z, \bar{z}, \operatorname{Re} w)$$

where $\Phi_{\geq 3} \in \mathcal{N}^w$.

The final result covers the conditions under which convergence for Theorem 1 holds. Convergence for every normal form cannot be guaranteed, so putting basic, algebraic conditions on the subset of formal normal forms, will force convergence of the transformation to the normal form under the condition that the data is convergent. This algebraic condition acts on the $(1,1)-$ and $(1,2)-$terms from the decomposition

$$\Phi(z,\bar{z},u) = \sum_{j,k} \Phi_{j,k}(z,\bar{z},u), \quad \Phi_{j,k}(tz, s\bar{z}, u) = t^j s^k \Phi_{j,k}(z,\bar{z},u).$$

Furthermore the notation $\Phi'_{j,k}$ will be used for

$$\Phi'_{j,k}(z,\bar{z},u) = \left(\frac{\partial \Phi_{j,k}}{\partial u_1}, \ldots, \frac{\partial \Phi'_{j,k}}{\partial u_d} \right),$$

where the left hand side is a $d \times d$ matrix with entries consisting of formal power series in u, which take values in the space of polynomials in z and $\bar{z}$.

Theorem 3 *Let $Q(z,\bar{z})$ to be a non degenerate form of full rank on $\mathbb{C}^n$ with values in $\mathbb{C}^d$, i.e. Q takes the form $Q(z,\bar{z}) = (\bar{z}^t J_1 z, \ldots, \bar{z}^t J_d z)$, where the individual matrices J_k satisfy the nondegeneracy conditions 1.14. Let M be a manifold given near $0 \in \mathbb{C}^N$, defined by an equation of the following type*

$$\operatorname{Im} w' = Q(z', \bar{z}') + \tilde{\Phi}_{\geq 3}(z', \bar{z}', \operatorname{Re} w')$$

where $\tilde{\Phi} \in \mathbb{C}\{z, \bar{z}, u\}$. Then the convergence of any formal biholomorphic from Theorem 1 is guaranteed of the following normal form

$$\operatorname{Im} w = Q(z, \bar{z}) + \Phi_{\geq 3}(z, \bar{z}, \operatorname{Re} w)$$

and fulfills

$$\Phi'_{1,1}\Phi_{1,2} + \Phi'_{1,2}(Q + \Phi_{1,1}) = 0 \tag{2.39}$$

The normalization in Theorem 3 differs from the approach Chern and Moser take.
However in the case of a hypersurface ($d = 1$) the normal form in Theorem 1 converges by default, without further restrictions. In this specific case equation 2.39 is always satisfied, since $\Phi_{1,1} = \Phi_{1,2} = 0$.
This different approach to the one Chern and Moser use is actually beneficial to dealing with higher codimensional manifolds and can be adapted to manifolds with lower codimensions.
The construction of the sets in these proofs and the full proofs are outlined in the later section of the paper *Convergence of the Chern − Moser − Beloshapka normal forms* by B. Lamel and L. Stolovitch.

3 Elliptic case

In this section the normalizing conditions of the elliptic case in $\mathbb{C}^{2\times 2}$, will be described. These have been solved using $Wolfram\ Mathematica$, in which a code as been written for the conditions 2.4 and 2.5 (see Bibliograpy [9]). The other two sets of normalizing conditions are of a simpler type and have been solved by hand.
In the case that $n = d = 2$, the matrices J_1 and J_2 can explicitly be written out for the elliptic, hyperbolic and parabolic cases listed above. Within the $Mathematica$ code, the matrix input can be adapted to yield the correct operators for the individual cases. This thesis outlines the elliptic case and in a similar way the other two cases can be retrieved by adapting the code.

We can now summarize the normalization conditions for an elliptic submanifold defined by

$$\operatorname{Im} w_1 = |z_1|^2 + |z_2|^2 + \Phi(z_1, z_2, \bar{z}_1, \bar{z}_2, \operatorname{Re} w_1, \operatorname{Re} w_2),$$
$$\operatorname{Im} w_2 = z_1\bar{z}_2 + z_2\bar{z}_1 + \Psi(z_1, z_2, \bar{z}_1, \bar{z}_2, \operatorname{Re} w_1, \operatorname{Re} w_2).$$

3.1 First set of normalizing conditions - (0,p)-terms

The first set of normalizing conditions 3.1 apply to power series of order $(0, p)-$ or $(p, 0)$ and state that

$$\Phi_{p,0} = \Phi_{0,p} = 0, \quad \text{for } p \geq 0. \tag{3.1}$$

We will start by decomposing the power series into

$$\Phi(z_1, z_2, \bar{z}_1, \bar{z}_2, \operatorname{Re} w_1, \operatorname{Re} w_2) = z_1\Phi_{1,0}(\bar{z}_1, \bar{z}_2, \operatorname{Re} w_1, \operatorname{Re} w_2) + z_2\Phi_{0,1}(\bar{z}_1, \bar{z}_2, \operatorname{Re} w_1, \operatorname{Re} w_2)$$

$$\Psi(z_1, z_2, \bar{z}_1, \bar{z}_2, \operatorname{Re} w_1, \operatorname{Re} w_2) = z_1\Psi_{1,0}(\bar{z}_1, \bar{z}_2, \operatorname{Re} w_1, \operatorname{Re} w_2) + z_2\Psi_{0,1}(\bar{z}_1, \bar{z}_2, \operatorname{Re} w_1, \operatorname{Re} w_2)$$

so that 3.1 becomes

$$\Psi_{0,1} = -\Phi_{1,0}, \quad \Psi_{1,0} = -\Phi_{0,1}.$$

3.2 Second normalizing conditions - (1,p)-terms

For the second set of normalizing conditions 2.2 we have that

$$\mathcal{K}^*\,\Phi_{p,1} = (\bar{\mathcal{K}})^*\Phi_{1,p} = 0, \quad \text{for } p \geq 0.$$

Which gives us the following two equations

$$\frac{\partial}{\partial \bar{z}_1}\Big|_0 b_1 + \frac{\partial}{\partial \bar{z}_2}\Big|_0 b_2 = 0 \tag{3.2}$$

$$\frac{\partial}{\partial \bar{z}_2}\Big|_0 b_1 + \frac{\partial}{\partial \bar{z}_1}\Big|_0 b_2 = 0 \tag{3.3}$$

We recall that $\Phi_{p,1} \in (\mathbb{C}[\![z, \bar{z}, u]\!]^2)/((\bar{z}^2))$, such that b_1 and b_2 are the first and second coordinates of this power series and thus take the form

$$b_1 = \sum_{\substack{|\alpha|=p \\ |\beta|=1}} \Phi^1_{\alpha,\beta}(u) z^\alpha \bar{z}^\beta \qquad\qquad b_2 = \sum_{\substack{|\alpha|=p \\ |\beta|=1}} \Phi^2_{\alpha,\beta}(u) z^\alpha \bar{z}^\beta$$

We can plug these into equations 3.2 and 3.3 to analyze what these normalizing conditions look like. For equation 3.2 we obtain the following.

$$\frac{\partial}{\partial \bar{z}_1}\Big|_0 \sum_{\substack{|\alpha|=p \\ |\beta|=1}} \Phi^1_{\alpha,\beta}(u) z^\alpha \bar{z}^\beta + \frac{\partial}{\partial \bar{z}_2}\Big|_0 \sum_{\substack{|\alpha|=p \\ |\beta|=1}} \Phi^2_{\alpha,\beta}(u) z^\alpha \bar{z}^\beta = 0$$

Splitting the sum into the cases e_1, where $\beta_1 = 1, \beta_2 = 0$ and e_2,where $\beta_1 = 0, \beta_2 = 1$ for β we obtain:

$$\sum_{|\alpha|=p} \Phi^1_{\alpha,e_1}(u) z^\alpha + \sum_{|\alpha|=p} \Phi^2_{\alpha,e_2}(u) z^\alpha = 0 \tag{3.4}$$

For equation 3.3 we get

$$\frac{\partial}{\partial \bar{z}_2}\Big|_0 \sum_{\substack{|\alpha|=p \\ |\beta|=1}} \Phi^1_{\alpha,\beta}(u) z^\alpha \bar{z}^\beta + \frac{\partial}{\partial \bar{z}_1}\Big|_0 \sum_{\substack{|\alpha|=p \\ |\beta|=1}} \Phi^2_{\alpha,\beta}(u) z^\alpha \bar{z}^\beta = 0$$

After splitting the sum into the different cases for β again, we obtain

$$\sum_{|\alpha|=p} \Phi^2_{\alpha,e_1}(u) z^\alpha + \sum_{|\alpha|=p} \Phi^1_{\alpha,e_2}(u) z^\alpha = 0. \tag{3.5}$$

3.3 Third set of normalizing conditions - diagonal terms

The third set of normalization conditions are expressed in terms of conditions on the

$$\Phi(z_1, z_2, \bar{z}_1, \bar{z}_2, \Re w_1, \Re w_2) = \sum_{\alpha,\beta,\gamma,\delta} \Phi_{\alpha,\beta,\gamma,\delta}(\Re w_1, \Re w_2) z_1^\alpha z_2^\beta \bar{z}_1^\gamma \bar{z}_2^\delta$$

where $\alpha + \beta + \gamma + \delta \leq 3$.

We start with solving the first of the two equations, 2.4, for the terms

$$\Phi_{0,1,0,1}(u_1, u_2), \quad \Phi_{0,1,1,0}(u_1, u_2), \quad \Psi_{1,0,0,1}(u_1, u_2), \quad \Psi_{1,0,1,0}(u_1, u_2),$$

where the dependencies on (u_1, u_2) will be omitted for sake of readability. This yields

$$\begin{aligned}
\Phi_{0,1,0,1} &= 36\Phi_{0,3,0,3}u_1^2 + 8\Phi_{1,2,1,2}u_1^2 + 4\Phi_{2,1,2,1}u_1^2 \\
&+ 12\Psi_{0,3,1,2}u_1^2 + 8\Psi_{1,2,2,1}u_1^2 + 12\Psi_{2,1,3,0}u_1^2 \\
&+ 4i\Phi_{0,2,0,2}u_1 + 24u_2\Phi_{0,3,1,2}u_1 + i\Phi_{1,1,1,1}u_1 \\
&+ 24u_2\Phi_{1,2,0,3}u_1 + 8u_2\Phi_{1,2,2,1}u_1 + 8u_2\Phi_{2,1,1,2}u_1 \\
&+ 2i\Psi_{0,2,1,1}u_1 + 24u_2\Psi_{0,3,2,1}u_1 + 2i\Psi_{1,1,2,0}u_1 \\
&+ 8u_2\Psi_{1,2,1,2}u_1 + 24u_2\Psi_{1,2,3,0}u_1 + 8u_2\Psi_{2,1,2,1}u_1 \\
&+ 2iu_2\Phi_{0,2,1,1} + 12u_2^2\Phi_{0,3,2,1} + 2iu_2\Phi_{1,1,0,2} \\
&+ 8u_2^2\Phi_{1,2,1,2} + 12u_2^2\Phi_{2,1,0,3} - \Psi_{0,1,1,0} \\
&+ 4iu_2\Psi_{0,2,2,0} + 36u_2^2\Psi_{0,3,3,0} + iu_2\Psi_{1,1,1,1} \\
&+ 8u_2^2\Psi_{1,2,2,1} + 4u_2^2\Psi_{2,1,1,2}
\end{aligned}$$

$$\begin{aligned}
\Phi_{0,1,1,0} &= 12\Phi_{0,3,1,2}u_1^2 + 8\Phi_{1,2,2,1}u_1^2 + 12\Phi_{2,1,3,0}u_1^2 \\
&+ 36\Psi_{0,3,0,3}u_1^2 + 8\Psi_{1,2,1,2}u_1^2 + 4\Psi_{2,1,2,1}u_1^2 \\
&+ 2i\Phi_{0,2,1,1}u_1 + 24u_2\Phi_{0,3,2,1}u_1 + 2i\Phi_{1,1,2,0}u_1 \\
&+ 8u_2\Phi_{1,2,1,2}u_1 + 24u_2\Phi_{1,2,3,0}u_1 + 8u_2\Phi_{2,1,2,1}u_1 \\
&+ 4i\Psi_{0,2,0,2}u_1 + 24u_2\Psi_{0,3,1,2}u_1 + i\Psi_{1,1,1,1}u_1 \\
&+ 24u_2\Psi_{1,2,0,3}u_1 + 8u_2\Psi_{1,2,2,1}u_1 + 8u_2\Psi_{2,1,1,2}u_1 \\
&+ 4iu_2\Phi_{0,2,2,0} + 36u_2^2\Phi_{0,3,3,0} + iu_2\Phi_{1,1,1,1} \\
&+ 8u_2^2\Phi_{1,2,2,1} + 4u_2^2\Phi_{2,1,1,2} - \Psi_{0,1,0,1} \\
&+ 2iu_2\Psi_{0,2,1,1} + 12u_2^2\Psi_{0,3,2,1} + 2iu_2\Psi_{1,1,0,2} \\
&+ 8u_2^2\Psi_{1,2,1,2} + 12u_2^2\Psi_{2,1,0,3}
\end{aligned}$$

$$\begin{aligned}
\Psi_{1,0,0,1} &= 4\Phi_{1,2,1,2}u_1^2 + 8\Phi_{2,1,2,1}u_1^2 + 36\Phi_{3,0,3,0}u_1^2 \\
&+ 12\Psi_{1,2,0,3}u_1^2 + 8\Psi_{2,1,1,2}u_1^2 + 12\Psi_{3,0,2,1}u_1^2 \\
&+ i\Phi_{1,1,1,1}u_1 + 8u_2\Phi_{1,2,2,1}u_1 + 4i\Phi_{2,0,2,0}u_1 \\
&+ 8u_2\Phi_{2,1,1,2}u_1 + 24u_2\Phi_{2,1,3,0}u_1 + 24u_2\Phi_{3,0,2,1}u_1 \\
&+ 2i\Psi_{1,1,0,2}u_1 + 8u_2\Psi_{1,2,1,2}u_1 + 2i\Psi_{2,0,1,1}u_1 \\
&+ 24u_2\Psi_{2,1,0,3}u_1 + 8u_2\Psi_{2,1,2,1}u_1 + 24u_2\Psi_{3,0,1,2}u_1 \\
&- \Phi_{1,0,1,0} + 2iu_2\Phi_{1,1,2,0} + 12u_2^2\Phi_{1,2,3,0} \\
&+ 2iu_2\Phi_{2,0,1,1} + 8u_2^2\Phi_{2,1,2,1} + 12u_2^2\Phi_{3,0,1,2} \\
&+ iu_2\Psi_{1,1,1,1} + 4u_2^2\Psi_{1,2,2,1} + 4iu_2\Psi_{2,0,0,2}
\end{aligned}$$

$$+ 8u_2^2\Psi_{2,1,1,2} + 36u_2^2\Psi_{3,0,0,3}$$

and

$$\begin{aligned}
\Psi_{1,0,1,0} &= 12\Phi_{1,2,0,3}u_1^2 + 8\Phi_{2,1,1,2}u_1^2 + 12\Phi_{3,0,2,1}u_1^2 \\
&+ 4\Psi_{1,2,1,2}u_1^2 + 8\Psi_{2,1,2,1}u_1^2 + 36\Psi_{3,0,3,0}u_1^2 \\
&+ 2i\Phi_{1,1,0,2}u_1 + 8u_2\Phi_{1,2,1,2}u_1 + 2i\Phi_{2,0,1,1}u_1 \\
&+ 24u_2\Phi_{2,1,0,3}u_1 + 8u_2\Phi_{2,1,2,1}u_1 + 24u_2\Phi_{3,0,1,2}u_1 \\
&+ i\Psi_{1,1,1,1}u_1 + 8u_2\Psi_{1,2,2,1}u_1 + 4i\Psi_{2,0,2,0}u_1 \\
&+ 8u_2\Psi_{2,1,1,2}u_1 + 24u_2\Psi_{2,1,3,0}u_1 + 24u_2\Psi_{3,0,2,1}u_1 \\
&- \Phi_{1,0,0,1} + iu_2\Phi_{1,1,1,1} + 4u_2^2\Phi_{1,2,2,1} \\
&+ 4iu_2\Phi_{2,0,0,2} + 8u_2^2\Phi_{2,1,1,2} + 36u_2^2\Phi_{3,0,0,3} \\
&+ 2iu_2\Psi_{1,1,2,0} + 12u_2^2\Psi_{1,2,3,0} + 2iu_2\Psi_{2,0,1,1} \\
&+ 8u_2^2\Psi_{2,1,2,1} + 12u_2^2\Psi_{3,0,1,2}.
\end{aligned}$$

If we want to find explicit expressions for the normalization conditions 2.3 we substitute the solutions above into it and decompose each of the series into power series depending on the dependencies and if it is one of the terms to be solved for. For these rules the use of Φ will be used to denote both Φ and Ψ. We have following decompositions.

$$\begin{aligned}
\Phi_{\alpha,\beta,\gamma,\delta}(u_1,u_2) = \Phi_{\alpha,\beta,\gamma,\delta}(0,0) + u_1\Phi_{1,0,\alpha,\beta,\gamma,\delta}(u_1) + u_2\Phi_{0,1,\alpha,\beta,\gamma,\delta}(u_2) \\
+ u_1u_2\Phi_{1,1,\alpha,\beta,\gamma,\delta}(u_1,u_2)
\end{aligned}$$

for terms that will be solved for, as well as

$$\Phi_{1,0,\alpha,\beta,\gamma,\delta}(u_1) = \Phi_{1,0,\alpha,\beta,\gamma,\delta}(0) + \Phi_{2,0,\alpha,\beta,\gamma,\delta}(u_1)$$

for terms where $\alpha+\beta+\gamma+\delta = 2$. Furthermore we will decompose the term with u_2 in a similar fashion

$$\Phi_{0,1,\alpha,\beta,\gamma,\delta}(u_1) = \Phi_{0,1,\alpha,\beta,\gamma,\delta}(0) + \Phi_{0,2,\alpha,\beta,\gamma,\delta}(u_1)$$

where the same condition on $\alpha,\beta,\gamma,\delta$ hold as above and we have that

$$\Phi_{\alpha,\beta,\gamma,\delta}(0,0) = 0$$

when $\alpha+\beta+\gamma+\delta = 2$, where $\Phi_{\alpha,\beta,\gamma,\delta,\nu,\mu} \in \mathbb{C}$. After applying these expansions to the equation, where the first conditions have been substituted, we can start to solve for the individual terms. To compute the constant terms of the u_2^j, u_1 will be set to zero and the equation can be simplified. After setting u_2 to zero the constant terms are left and can be solved for

$$\Phi_{0,2,2,0}(0,0) \text{ and } \Psi_{2,0,0,2}(0,0).$$

These terms can then be expressed in the following way

$$
\begin{aligned}
\Phi_{0,2,2,0}(0,0) &= \frac{1}{4}i\Phi_{0,1,1,0,0,1}(0) - \frac{1}{4}\Phi_{1,1,1,1}(0,0) - \frac{1}{4}i\Psi_{0,1,0,1,0,1}(0) \\
&\quad - \frac{1}{2}\Psi_{0,2,1,1}(0,0) - \frac{1}{2}\Psi_{1,1,0,2}(0,0) \\
\Psi_{2,0,0,2}(0,0) &= -\frac{1}{4}i\Phi_{0,1,1,0,1,0}(0) - \frac{1}{2}\Phi_{1,1,2,0}(0,0) - \frac{1}{2}\Phi_{2,0,1,1}(0,0) \\
&\quad + \frac{1}{4}i\Psi_{0,1,0,1,1,0}(0) - \frac{1}{4}\Psi_{1,1,1,1}(0,0)
\end{aligned}
\tag{3.6}
$$

Working backwards, these equations can be applied to the equation used to solve for these, where u_2 has not been set to zero. Now this equation can be solved for

$$\Phi_{0,1,0,2,2,0}(u_2) \text{ and } \Psi_{0,1,2,0,0,2}\,(u_2)$$

and gives us

$$
\begin{aligned}
\Phi_{0,1,0,2,2,0}(u_2) &= \frac{15}{2}iu_2\Phi_{0,1,0,3,3,0}\,(u_2) - \frac{1}{4}\Phi_{0,1,1,1,1,1}\,(u_2) + \frac{3}{2}iu_2\Phi_{0,1,1,2,2,1}\,(u_2) \\
&\quad + \frac{1}{2}iu_2\Phi_{0,1,2,1,1,2}\,(u_2) - \frac{3}{2}iu_2\Phi_{0,1,3,0,0,3}\,(u_2) + \frac{1}{4}i\Phi_{0,2,1,0,0,1}\,(u_2) \\
&\quad - \frac{1}{2}\Psi_{0,1,0,2,1,1}\,(u_2) + 3iu_2\Psi_{0,1,0,3,2,1}\,(u_2) - \frac{1}{2}\Psi_{0,1,1,1,0,2}\,(u_2) \\
&\quad + 2iu_2\Psi_{0,1,1,2,1,2}\,(u_2) + 3iu_2\Psi_{0,1,2,1,0,3}\,(u_2) - \frac{1}{4}i\Psi_{0,2,0,1,0,1}\,(u_2) \\
&\quad + \frac{15}{2}i\Phi_{0,3,3,0}(0,0) + \frac{3}{2}i\Phi_{1,2,2,1}(0,0) + \frac{1}{2}i\Phi_{2,1,1,2}(0,0) \\
&\quad - \frac{3}{2}i\Phi_{3,0,0,3}(0,0) + 3i\Psi_{0,3,2,1}(0,0) + 2i\Psi_{1,2,1,2}(0,0) \\
&\quad + 3i\Psi_{2,1,0,3}(0,0)
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_{0,1,2,0,0,2}\,(u_2) &= -\frac{1}{2}\Phi_{0,1,1,1,2,0}\,(u_2) + 3iu_2\Phi_{0,1,1,2,3,0}\,(u_2) - \frac{1}{2}\Phi_{0,1,2,0,1,1}\,(u_2) \\
&\quad + 2iu_2\Phi_{0,1,2,1,2,1}\,(u_2) + 3iu_2\Phi_{0,1,3,0,1,2}\,(u_2) - \frac{1}{4}i\Phi_{0,2,1,0,1,0}\,(u_2) \\
&\quad - \frac{3}{2}iu_2\Psi_{0,1,0,3,3,0}\,(u_2) - \frac{1}{4}\Psi_{0,1,1,1,1,1}\,(u_2) + \frac{1}{2}iu_2\Psi_{0,1,1,2,2,1}\,(u_2) \\
&\quad + \frac{3}{2}iu_2\Psi_{0,1,2,1,1,2}\,(u_2) + \frac{15}{2}iu_2\Psi_{0,1,3,0,0,3}\,(u_2) + \frac{1}{4}i\Psi_{0,2,0,1,1,0}\,(u_2) \\
&\quad + 3i\Phi_{1,2,3,0}(0,0) + 2i\Phi_{2,1,2,1}(0,0) + 3i\Phi_{3,0,1,2}(0,0) \\
&\quad - \frac{3}{2}i\Psi_{0,3,3,0}(0,0) + \frac{1}{2}i\Psi_{1,2,2,1}(0,0) + \frac{3}{2}i\Psi_{2,1,1,2}(0,0) \\
&\quad + \frac{15}{2}i\Psi_{3,0,0,3}(0,0).
\end{aligned}
$$

To obtain the constant terms for u_1^j, this same principle can be applied to the expanded equation, where the constant terms have been substituted in. Therefore we can solve for the two terms

$$\Phi_{0,2,0,2}(0,0) \text{ and } \Psi_{2,0,2,0}(0,0)$$

and this gives us the following equations

$$\begin{aligned}
\Phi_{0,2,0,2}(0,0) &= \frac{1}{4} i\Phi_{1,0,1,0,1,0}(0) - \frac{1}{4}\Phi_{1,1,1,1}(0,0) - \frac{1}{4} i\Psi_{1,0,0,1,1,0}(0) \\
&\quad - \frac{1}{2}\Psi_{0,2,1,1}(0,0) - \frac{1}{2}\Psi_{1,1,2,0}(0,0) \\
\Psi_{2,0,2,0}(0,0) &= -\frac{1}{4} i\Phi_{1,0,1,0,0,1}(0) - \frac{1}{2}\Phi_{1,1,0,2}(0,0) - \frac{1}{2}\Phi_{2,0,1,1}(0,0) \\
&\quad + \frac{1}{4} i\Psi_{1,0,0,1,0,1}(0) - \frac{1}{4}\Psi_{1,1,1,1}(0,0)
\end{aligned} \tag{3.7}$$

for the constant terms. Continuing the system, we can find explicit solutions for the terms

$$\Phi_{1,0,0,2,0,2}\left(u_1\right) \text{ and } \Psi_{1,0,2,0,2,0}\left(u_1\right)$$

after substituting the constant solutions back in. They can be written as

$$\begin{aligned}
\Phi_{1,0,0,2,0,2}\left(u_1\right) &= \frac{15}{2} iu_1\Phi_{1,0,0,3,0,3}\left(u_1\right) - \frac{1}{4}\Phi_{1,0,1,1,1,1}\left(u_1\right) + \frac{3}{2} iu_1\Phi_{1,0,1,2,1,2}\left(u_1\right) \\
&\quad + \frac{1}{2} iu_1\Phi_{1,0,2,1,2,1}\left(u_1\right) - \frac{3}{2} iu_1\Phi_{1,0,3,0,3,0}\left(u_1\right) + \frac{1}{4} i\Phi_{2,0,1,0,1,0}\left(u_1\right) \\
&\quad - \frac{1}{2}\Psi_{1,0,0,2,1,1}\left(u_1\right) + 3iu_1\Psi_{1,0,0,3,1,2}\left(u_1\right) - \frac{1}{2}\Psi_{1,0,1,1,2,0}\left(u_1\right) \\
&\quad + 2iu_1\Psi_{1,0,1,2,2,1}\left(u_1\right) + 3iu_1\Psi_{1,0,2,1,3,0}\left(u_1\right) - \frac{1}{4} i\Psi_{2,0,0,1,1,0}\left(u_1\right) \\
&\quad + \frac{15}{2} i\Phi_{0,3,0,3}(0,0) + \frac{3}{2} i\Phi_{1,2,1,2}(0,0) + \frac{1}{2} i\Phi_{2,1,2,1}(0,0) \\
&\quad - \frac{3}{2} i\Phi_{3,0,3,0}(0,0) + 3i\Psi_{0,3,1,2}(0,0) + 2i\Psi_{1,2,2,1}(0,0) + 3i\Psi_{2,1,3,0}(0,0)
\end{aligned}$$

and

$$\begin{aligned}
\Psi_{1,0,2,0,2,0}\left(u_1\right) &= -\frac{1}{2}\Phi_{1,0,1,1,0,2}\left(u_1\right) + 3iu_1\Phi_{1,0,1,2,0,3}\left(u_1\right) - \frac{1}{2}\Phi_{1,0,2,0,1,1}\left(u_1\right) \\
&\quad + 2iu_1\Phi_{1,0,2,1,1,2}\left(u_1\right) + 3iu_1\Phi_{1,0,3,0,2,1}\left(u_1\right) - \frac{1}{4} i\Phi_{2,0,1,0,0,1}\left(u_1\right) \\
&\quad - \frac{3}{2} iu_1\Psi_{1,0,0,3,0,3}\left(u_1\right) - \frac{1}{4}\Psi_{1,0,1,1,1,1}\left(u_1\right) + \frac{1}{2} iu_1\Psi_{1,0,1,2,1,2}\left(u_1\right) \\
&\quad + \frac{3}{2} iu_1\Psi_{1,0,2,1,2,1}\left(u_1\right) + \frac{15}{2} iu_1\Psi_{1,0,3,0,3,0}\left(u_1\right) + \frac{1}{4} i\Psi_{2,0,0,1,0,1}\left(u_1\right)
\end{aligned}$$

$$+ 3i\Phi_{1,2,0,3}(0,0) + 2i\Phi_{2,1,1,2}(0,0) + 3i\Phi_{3,0,2,1}(0,0)$$
$$- \frac{3}{2}i\Psi_{0,3,0,3}(0,0) + \frac{1}{2}i\Psi_{1,2,1,2}(0,0) + \frac{3}{2}i\Psi_{2,1,2,1}(0,0)$$
$$+ \frac{15}{2}i\Psi_{3,0,3,0}(0,0).$$

The normalizing condition 2.3, after substituting the solutions for 2.4 and applying the four sets of equations listed above, gives us the complete solution for the rest of the power series for the terms

$$\Psi_{0,2,0,2}\left(u_1,u_2\right) \text{ and } \Phi_{2,0,2,0}\left(u_1,u_2\right)$$

They are

$$\begin{aligned}
\Psi_{0,2,0,2}\left(u_1,u_2\right) = {} & \frac{15}{2}i\Phi_{1,1,0,3,0,3}\left(u_1,u_2\right)u_1^3 + \frac{3}{2}i\Phi_{1,1,1,2,1,2}\left(u_1,u_2\right)u_1^3 + \frac{1}{2}i\Phi_{1,1,2,1,2,1}\left(u_1,u_2\right)u_1^3 \\
& - \frac{3}{2}i\Phi_{1,1,3,0,3,0}\left(u_1,u_2\right)u_1^3 + 3i\Psi_{1,1,0,3,1,2}\left(u_1,u_2\right)u_1^3 + 2i\Psi_{1,1,1,2,2,1}\left(u_1,u_2\right)u_1^3 \\
& + 3i\Psi_{1,1,2,1,3,0}\left(u_1,u_2\right)u_1^3 + \frac{15}{2}i\Phi_{1,0,0,3,1,2}\left(u_1\right)u_1^2 + \frac{9}{2}i\Phi_{1,0,1,2,0,3}\left(u_1\right)u_1^2 \\
& + 3i\Phi_{1,0,1,2,2,1}\left(u_1\right)u_1^2 + i\Phi_{1,0,2,1,1,2}\left(u_1\right)u_1^2 + \frac{3}{2}i\Phi_{1,0,2,1,3,0}\left(u_1\right)u_1^2 \\
& - \frac{3}{2}i\Phi_{1,0,3,0,2,1}\left(u_1\right)u_1^2 + \frac{15}{2}i\Phi_{0,1,0,3,0,3}\left(u_2\right)u_1^2 + \frac{3}{2}i\Phi_{0,1,1,2,1,2}\left(u_2\right)u_1^2 \\
& + \frac{1}{2}i\Phi_{0,1,2,1,2,1}\left(u_2\right)u_1^2 - \frac{3}{2}i\Phi_{0,1,3,0,3,0}\left(u_2\right)u_1^2 - \Phi_{1,1,0,2,0,2}\left(u_1,u_2\right)u_1^2 \\
& + \frac{15}{2}iu_2\Phi_{1,1,0,3,1,2}\left(u_1,u_2\right)u_1^2 - \frac{1}{4}\Phi_{1,1,1,1,1,1}\left(u_1,u_2\right)u_1^2 + \frac{9}{2}iu_2\Phi_{1,1,1,2,0,3}\left(u_1,u_2\right)u_1^2 \\
& + 3iu_2\Phi_{1,1,1,2,2,1}\left(u_1,u_2\right)u_1^2 + iu_2\Phi_{1,1,2,1,1,2}\left(u_1,u_2\right)u_1^2 + \frac{3}{2}iu_2\Phi_{1,1,2,1,3,0}\left(u_1,u_2\right)u_1^2 \\
& - \frac{3}{2}iu_2\Phi_{1,1,3,0,2,1}\left(u_1,u_2\right)u_1^2 + 9i\Psi_{1,0,0,3,0,3}\left(u_1\right)u_1^2 + 6i\Psi_{1,0,0,3,2,1}\left(u_1\right)u_1^2 \\
& + 4i\Psi_{1,0,1,2,1,2}\left(u_1\right)u_1^2 + 6i\Psi_{1,0,1,2,3,0}\left(u_1\right)u_1^2 + 3i\Psi_{1,0,2,1,2,1}\left(u_1\right)u_1^2 \\
& + 3i\Psi_{0,1,0,3,1,2}\left(u_2\right)u_1^2 + 2i\Psi_{0,1,1,2,2,1}\left(u_2\right)u_1^2 + 3i\Psi_{0,1,2,1,3,0}\left(u_2\right)u_1^2 \\
& - \frac{1}{2}\Psi_{1,1,0,2,1,1}\left(u_1,u_2\right)u_1^2 + 9iu_2\Psi_{1,1,0,3,0,3}\left(u_1,u_2\right)u_1^2 + 6iu_2\Psi_{1,1,0,3,2,1}\left(u_1,u_2\right)u_1^2 \\
& - \frac{1}{2}\Psi_{1,1,1,1,2,0}\left(u_1,u_2\right)u_1^2 + 4iu_2\Psi_{1,1,1,2,1,2}\left(u_1,u_2\right)u_1^2 + 6iu_2\Psi_{1,1,1,2,3,0}\left(u_1,u_2\right)u_1^2 \\
& + 3iu_2\Psi_{1,1,2,1,2,1}\left(u_1,u_2\right)u_1^2 + \frac{15}{2}iu_2\Phi_{1,0,0,3,2,1}\left(u_1\right)u_1 - \frac{1}{2}\Phi_{1,0,1,1,0,2}\left(u_1\right)u_1 \\
& - \frac{1}{2}\Phi_{1,0,1,1,2,0}\left(u_1\right)u_1 + 3iu_2\Phi_{1,0,1,2,1,2}\left(u_1\right)u_1 + \frac{9}{2}iu_2\Phi_{1,0,1,2,3,0}\left(u_1\right)u_1 \\
& + \frac{3}{2}iu_2\Phi_{1,0,2,1,0,3}\left(u_1\right)u_1 + iu_2\Phi_{1,0,2,1,2,1}\left(u_1\right)u_1 - \frac{3}{2}iu_2\Phi_{1,0,3,0,1,2}\left(u_1\right)u_1 \\
& + \frac{1}{4}i\Phi_{2,0,1,0,0,1}\left(u_1\right)u_1 - \Phi_{0,1,0,2,0,2}\left(u_2\right)u_1 + \frac{15}{2}iu_2\Phi_{0,1,0,3,1,2}\left(u_2\right)u_1
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{4}\Phi_{0,1,1,1,1,1}\left(u_2\right)u_1 + \frac{9}{2}iu_2\Phi_{0,1,1,2,0,3}\left(u_2\right)u_1 + 3iu_2\Phi_{0,1,1,2,2,1}\left(u_2\right)u_1 \\
&+ iu_2\Phi_{0,1,2,1,1,2}\left(u_2\right)u_1 + \frac{3}{2}iu_2\Phi_{0,1,2,1,3,0}\left(u_2\right)u_1 - \frac{3}{2}iu_2\Phi_{0,1,3,0,2,1}\left(u_2\right)u_1 \\
&+ \frac{15}{2}i\Phi_{0,3,1,2}(0,0)u_1 + \frac{9}{2}i\Phi_{1,2,0,3}(0,0)u_1 + 3i\Phi_{1,2,2,1}(0,0)u_1 \\
&+ i\Phi_{2,1,1,2}(0,0)u_1 + \frac{3}{2}i\Phi_{2,1,3,0}(0,0)u_1 - \frac{3}{2}i\Phi_{3,0,2,1}(0,0)u_1 \\
&+ \frac{15}{2}iu_2^2\Phi_{1,1,0,3,2,1}\left(u_1,u_2\right)u_1 + \frac{1}{4}i\Phi_{1,1,1,0,1,0}\left(u_1,u_2\right)u_1 - \frac{1}{2}u_2\Phi_{1,1,1,1,0,2}\left(u_1,u_2\right)u_1 \\
&- \frac{1}{2}u_2\Phi_{1,1,1,1,2,0}\left(u_1,u_2\right)u_1 + 3iu_2^2\Phi_{1,1,1,2,1,2}\left(u_1,u_2\right)u_1 + \frac{9}{2}iu_2^2\Phi_{1,1,1,2,3,0}\left(u_1,u_2\right)u_1 \\
&+ \frac{3}{2}iu_2^2\Phi_{1,1,2,1,0,3}\left(u_1,u_2\right)u_1 + iu_2^2\Phi_{1,1,2,1,2,1}\left(u_1,u_2\right)u_1 - \frac{3}{2}iu_2^2\Phi_{1,1,3,0,1,2}\left(u_1,u_2\right)u_1 \\
&+ 6iu_2\Psi_{1,0,0,3,1,2}\left(u_1\right)u_1 + 9iu_2\Psi_{1,0,0,3,3,0}\left(u_1\right)u_1 - \frac{1}{2}\Psi_{1,0,1,1,1,1}\left(u_1\right)u_1 \\
&+ 6iu_2\Psi_{1,0,1,2,0,3}\left(u_1\right)u_1 + 4iu_2\Psi_{1,0,1,2,2,1}\left(u_1\right)u_1 + 3iu_2\Psi_{1,0,2,1,1,2}\left(u_1\right)u_1 \\
&- \frac{1}{4}i\Psi_{2,0,0,1,0,1}\left(u_1\right)u_1 - \frac{1}{2}\Psi_{0,1,0,2,1,1}\left(u_2\right)u_1 + 9iu_2\Psi_{0,1,0,3,0,3}\left(u_2\right)u_1 \\
&+ 6iu_2\Psi_{0,1,0,3,2,1}\left(u_2\right)u_1 - \frac{1}{2}\Psi_{0,1,1,1,2,0}\left(u_2\right)u_1 + 4iu_2\Psi_{0,1,1,2,1,2}\left(u_2\right)u_1 \\
&+ 6iu_2\Psi_{0,1,1,2,3,0}\left(u_2\right)u_1 + 3iu_2\Psi_{0,1,2,1,2,1}\left(u_2\right)u_1 + 9i\Psi_{0,3,0,3}(0,0)u_1 \\
&+ 6i\Psi_{0,3,2,1}(0,0)u_1 + 4i\Psi_{1,2,1,2}(0,0)u_1 + 6i\Psi_{1,2,3,0}(0,0)u_1 \\
&+ 3i\Psi_{2,1,2,1}(0,0)u_1 - \frac{1}{4}i\Psi_{1,1,0,1,1,0}\left(u_1,u_2\right)u_1 + 6iu_2^2\Psi_{1,1,0,3,1,2}\left(u_1,u_2\right)u_1 \\
&+ 9iu_2^2\Psi_{1,1,0,3,3,0}\left(u_1,u_2\right)u_1 - \frac{1}{2}u_2\Psi_{1,1,1,1,1,1}\left(u_1,u_2\right)u_1 + 6iu_2^2\Psi_{1,1,1,2,0,3}\left(u_1,u_2\right)u_1 \\
&+ 4iu_2^2\Psi_{1,1,1,2,2,1}\left(u_1,u_2\right)u_1 + 3iu_2^2\Psi_{1,1,2,1,1,2}\left(u_1,u_2\right)u_1 + \frac{1}{4}i\Phi_{0,1,1,0,1,0}(0) \\
&+ \frac{1}{4}i\Phi_{1,0,1,0,0,1}(0) - u_2\Phi_{1,0,0,2,2,0}\left(u_1\right) + \frac{15}{2}iu_2^2\Phi_{1,0,0,3,3,0}\left(u_1\right) \\
&- \frac{1}{4}u_2\Phi_{1,0,1,1,1,1}\left(u_1\right) + \frac{3}{2}iu_2^2\Phi_{1,0,1,2,2,1}\left(u_1\right) + \frac{1}{2}iu_2^2\Phi_{1,0,2,1,1,2}\left(u_1\right) \\
&- \frac{3}{2}iu_2^2\Phi_{1,0,3,0,0,3}\left(u_1\right) + \frac{15}{2}iu_2^2\Phi_{0,1,0,3,2,1}\left(u_2\right) - \frac{1}{2}u_2\Phi_{0,1,1,1,0,2}\left(u_2\right) \\
&- \frac{1}{2}u_2\Phi_{0,1,1,1,2,0}\left(u_2\right) + 3iu_2^2\Phi_{0,1,1,2,1,2}\left(u_2\right) + \frac{9}{2}iu_2^2\Phi_{0,1,1,2,3,0}\left(u_2\right) \\
&+ \frac{3}{2}iu_2^2\Phi_{0,1,2,1,0,3}\left(u_2\right) + iu_2^2\Phi_{0,1,2,1,2,1}\left(u_2\right) - \frac{3}{2}iu_2^2\Phi_{0,1,3,0,1,2}\left(u_2\right) \\
&+ \frac{1}{4}iu_2\Phi_{0,2,1,0,1,0}\left(u_2\right) + \frac{15}{2}iu_2\Phi_{0,3,2,1}(0,0) - \frac{1}{2}\Phi_{1,1,0,2}(0,0) \\
&- \frac{1}{2}\Phi_{1,1,2,0}(0,0) + 3iu_2\Phi_{1,2,1,2}(0,0) + \frac{9}{2}iu_2\Phi_{1,2,3,0}(0,0) \\
&+ \frac{3}{2}iu_2\Phi_{2,1,0,3}(0,0) + iu_2\Phi_{2,1,2,1}(0,0) - \frac{3}{2}iu_2\Phi_{3,0,1,2}(0,0)
\end{aligned}
$$

$$
\begin{aligned}
&- \Phi_{0,2,1,1}(u_1,u_2) - u_2^2\Phi_{1,1,0,2,2,0}(u_1,u_2) + \frac{15}{2} i u_2^3 \Phi_{1,1,0,3,3,0}(u_1,u_2) \\
&+ \frac{1}{4} i u_2 \Phi_{1,1,1,0,0,1}(u_1,u_2) - \frac{1}{4} u_2^2 \Phi_{1,1,1,1,1,1}(u_1,u_2) + \frac{3}{2} i u_2^3 \Phi_{1,1,1,2,2,1}(u_1,u_2) \\
&+ \frac{1}{2} i u_2^3 \Phi_{1,1,2,1,1,2}(u_1,u_2) - \frac{3}{2} i u_2^3 \Phi_{1,1,3,0,0,3}(u_1,u_2) - \frac{1}{4} i \Psi_{0,1,0,1,1,0}(0) \\
&- \frac{1}{4} i \Psi_{1,0,0,1,0,1}(0) - \frac{1}{2} u_2 \Psi_{1,0,0,2,1,1}(u_1) + 3 i u_2^2 \Psi_{1,0,0,3,2,1}(u_1) \\
&- \frac{1}{2} u_2 \Psi_{1,0,1,1,0,2}(u_1) + 2 i u_2^2 \Psi_{1,0,1,2,1,2}(u_1) + 3 i u_2^2 \Psi_{1,0,2,1,0,3}(u_1) \\
&+ 6 i u_2^2 \Psi_{0,1,0,3,1,2}(u_2) + 9 i u_2^2 \Psi_{0,1,0,3,3,0}(u_2) - \frac{1}{2} u_2 \Psi_{0,1,1,1,1,1}(u_2) \\
&+ 6 i u_2^2 \Psi_{0,1,1,2,0,3}(u_2) + 4 i u_2^2 \Psi_{0,1,1,2,2,1}(u_2) + 3 i u_2^2 \Psi_{0,1,2,1,1,2}(u_2) \\
&- \frac{1}{4} i u_2 \Psi_{0,2,0,1,1,0}(u_2) + 6 i u_2 \Psi_{0,3,1,2}(0,0) + 9 i u_2 \Psi_{0,3,3,0}(0,0) \\
&- \frac{1}{2} \Psi_{1,1,1,1}(0,0) + 6 i u_2 \Psi_{1,2,0,3}(0,0) + 4 i u_2 \Psi_{1,2,2,1}(0,0) \\
&+ 3 i u_2 \Psi_{2,1,1,2}(0,0) - \Psi_{0,2,2,0}(u_1,u_2) - \frac{1}{4} i u_2 \Psi_{1,1,0,1,0,1}(u_1,u_2) - \frac{1}{2} u_2^2 \Psi_{1,1,0,2,1,1}(u_1,u_2) \\
&+ 3 i u_2^3 \Psi_{1,1,0,3,2,1}(u_1,u_2) - \frac{1}{2} u_2^2 \Psi_{1,1,1,1,0,2}(u_1,u_2) + 2 i u_2^3 \Psi_{1,1,1,2,1,2}(u_1,u_2) \\
&+ 3 i u_2^3 \Psi_{1,1,2,1,0,3}(u_1,u_2)
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_{2,0,2,0}(u_1,u_2) = {} & 3 i \Phi_{1,1,1,2,0,3}(u_1,u_2)\, u_1^3 + 2 i \Phi_{1,1,2,1,1,2}(u_1,u_2)\, u_1^3 + 3 i \Phi_{1,1,3,0,2,1}(u_1,u_2)\, u_1^3 \\
&- \frac{3}{2} i \Psi_{1,1,0,3,0,3}(u_1,u_2)\, u_1^3 + \frac{1}{2} i \Psi_{1,1,1,2,1,2}(u_1,u_2)\, u_1^3 + \frac{3}{2} i \Psi_{1,1,2,1,2,1}(u_1,u_2)\, u_1^3 \\
&+ \frac{15}{2} i \Psi_{1,1,3,0,3,0}(u_1,u_2)\, u_1^3 + 3 i \Phi_{1,0,1,2,1,2}(u_1)\, u_1^2 + 6 i \Phi_{1,0,2,1,0,3}(u_1)\, u_1^2 \\
&+ 4 i \Phi_{1,0,2,1,2,1}(u_1)\, u_1^2 + 6 i \Phi_{1,0,3,0,1,2}(u_1)\, u_1^2 + 9 i \Phi_{1,0,3,0,3,0}(u_1)\, u_1^2 \\
&+ 3 i \Phi_{0,1,1,2,0,3}(u_2)\, u_1^2 + 2 i \Phi_{0,1,2,1,1,2}(u_2)\, u_1^2 + 3 i \Phi_{0,1,3,0,2,1}(u_2)\, u_1^2 \\
&- \frac{1}{2} \Phi_{1,1,1,1,0,2}(u_1,u_2)\, u_1^2 + 3 i u_2 \Phi_{1,1,1,2,1,2}(u_1,u_2)\, u_1^2 - \frac{1}{2} \Phi_{1,1,2,0,1,1}(u_1,u_2)\, u_1^2 \\
&+ 6 i u_2 \Phi_{1,1,2,1,0,3}(u_1,u_2)\, u_1^2 + 4 i u_2 \Phi_{1,1,2,1,2,1}(u_1,u_2)\, u_1^2 + 6 i u_2 \Phi_{1,1,3,0,1,2}(u_1,u_2)\, u_1^2 \\
&+ 9 i u_2 \Phi_{1,1,3,0,3,0}(u_1,u_2)\, u_1^2 - \frac{3}{2} i \Psi_{1,0,0,3,1,2}(u_1)\, u_1^2 + \frac{3}{2} i \Psi_{1,0,1,2,0,3}(u_1)\, u_1^2 \\
&+ i \Psi_{1,0,1,2,2,1}(u_1)\, u_1^2 + 3 i \Psi_{1,0,2,1,1,2}(u_1)\, u_1^2 + \frac{9}{2} i \Psi_{1,0,2,1,3,0}(u_1)\, u_1^2 \\
&+ \frac{15}{2} i \Psi_{1,0,3,0,2,1}(u_1)\, u_1^2 - \frac{3}{2} i \Psi_{0,1,0,3,0,3}(u_2)\, u_1^2 + \frac{1}{2} i \Psi_{0,1,1,2,1,2}(u_2)\, u_1^2 \\
&+ \frac{3}{2} i \Psi_{0,1,2,1,2,1}(u_2)\, u_1^2 + \frac{15}{2} i \Psi_{0,1,3,0,3,0}(u_2)\, u_1^2 - \frac{3}{2} i u_2 \Psi_{1,1,0,3,1,2}(u_1,u_2)\, u_1^2 \\
&- \frac{1}{4} \Psi_{1,1,1,1,1,1}(u_1,u_2)\, u_1^2 + \frac{3}{2} i u_2 \Psi_{1,1,1,2,0,3}(u_1,u_2)\, u_1^2 + i u_2 \Psi_{1,1,1,2,2,1}(u_1,u_2)\, u_1^2
\end{aligned}
$$

$$
\begin{aligned}
&- \Psi_{1,1,2,0,2,0}\left(u_1,u_2\right)u_1^2 + 3iu_2\Psi_{1,1,2,1,1,2}\left(u_1,u_2\right)u_1^2 + \frac{9}{2}iu_2\Psi_{1,1,2,1,3,0}\left(u_1,u_2\right)u_1^2 \\
&+ \frac{15}{2}iu_2\Psi_{1,1,3,0,2,1}\left(u_1,u_2\right)u_1^2 - \frac{1}{2}\Phi_{1,0,1,1,1,1}\left(u_1\right)u_1 + 3iu_2\Phi_{1,0,1,2,2,1}\left(u_1\right)u_1 \\
&+ 4iu_2\Phi_{1,0,2,1,1,2}\left(u_1\right)u_1 + 6iu_2\Phi_{1,0,2,1,3,0}\left(u_1\right)u_1 + 9iu_2\Phi_{1,0,3,0,0,3}\left(u_1\right)u_1 \\
&+ 6iu_2\Phi_{1,0,3,0,2,1}\left(u_1\right)u_1 - \frac{1}{4}i\Phi_{2,0,1,0,1,0}\left(u_1\right)u_1 - \frac{1}{2}\Phi_{0,1,1,1,0,2}\left(u_2\right)u_1 \\
&+ 3iu_2\Phi_{0,1,1,2,1,2}\left(u_2\right)u_1 - \frac{1}{2}\Phi_{0,1,2,0,1,1}\left(u_2\right)u_1 + 6iu_2\Phi_{0,1,2,1,0,3}\left(u_2\right)u_1 \\
&+ 4iu_2\Phi_{0,1,2,1,2,1}\left(u_2\right)u_1 + 6iu_2\Phi_{0,1,3,0,1,2}\left(u_2\right)u_1 + 9iu_2\Phi_{0,1,3,0,3,0}\left(u_2\right)u_1 \\
&+ 3i\Phi_{1,2,1,2}(0,0)u_1 + 6i\Phi_{2,1,0,3}(0,0)u_1 + 4i\Phi_{2,1,2,1}(0,0)u_1 \\
&+ 6i\Phi_{3,0,1,2}(0,0)u_1 + 9i\Phi_{3,0,3,0}(0,0)u_1 - \frac{1}{4}i\Phi_{1,1,1,0,0,1}\left(u_1,u_2\right)u_1 \\
&- \frac{1}{2}u_2\Phi_{1,1,1,1,1,1}\left(u_1,u_2\right)u_1 + 3iu_2^2\Phi_{1,1,1,2,2,1}\left(u_1,u_2\right)u_1 + 4iu_2^2\Phi_{1,1,2,1,1,2}\left(u_1,u_2\right)u_1 \\
&+ 6iu_2^2\Phi_{1,1,2,1,3,0}\left(u_1,u_2\right)u_1 + 9iu_2^2\Phi_{1,1,3,0,0,3}\left(u_1,u_2\right)u_1 + 6iu_2^2\Phi_{1,1,3,0,2,1}\left(u_1,u_2\right)u_1 \\
&- \frac{3}{2}iu_2\Psi_{1,0,0,3,2,1}\left(u_1\right)u_1 - \frac{1}{2}\Psi_{1,0,1,1,0,2}\left(u_1\right)u_1 - \frac{1}{2}\Psi_{1,0,1,1,2,0}\left(u_1\right)u_1 \\
&+ iu_2\Psi_{1,0,1,2,1,2}\left(u_1\right)u_1 + \frac{3}{2}iu_2\Psi_{1,0,1,2,3,0}\left(u_1\right)u_1 + \frac{9}{2}iu_2\Psi_{1,0,2,1,0,3}\left(u_1\right)u_1 \\
&+ 3iu_2\Psi_{1,0,2,1,2,1}\left(u_1\right)u_1 + \frac{15}{2}iu_2\Psi_{1,0,3,0,1,2}\left(u_1\right)u_1 + \frac{1}{4}i\Psi_{2,0,0,1,1,0}\left(u_1\right)u_1 \\
&- \frac{3}{2}iu_2\Psi_{0,1,0,3,1,2}\left(u_2\right)u_1 - \frac{1}{4}\Psi_{0,1,1,1,1,1}\left(u_2\right)u_1 + \frac{3}{2}iu_2\Psi_{0,1,1,2,0,3}\left(u_2\right)u_1 \\
&+ iu_2\Psi_{0,1,1,2,2,1}\left(u_2\right)u_1 - \Psi_{0,1,2,0,2,0}\left(u_2\right)u_1 + 3iu_2\Psi_{0,1,2,1,1,2}\left(u_2\right)u_1 \\
&+ \frac{9}{2}iu_2\Psi_{0,1,2,1,3,0}\left(u_2\right)u_1 + \frac{15}{2}iu_2\Psi_{0,1,3,0,2,1}\left(u_2\right)u_1 - \frac{3}{2}i\Psi_{0,3,1,2}(0,0)u_1 \\
&+ \frac{3}{2}i\Psi_{1,2,0,3}(0,0)u_1 + i\Psi_{1,2,2,1}(0,0)u_1 + 3i\Psi_{2,1,1,2}(0,0)u_1 \\
&+ \frac{9}{2}i\Psi_{2,1,3,0}(0,0)u_1 + \frac{15}{2}i\Psi_{3,0,2,1}(0,0)u_1 + \frac{1}{4}i\Psi_{1,1,0,1,0,1}\left(u_1,u_2\right)u_1 \\
&- \frac{3}{2}iu_2^2\Psi_{1,1,0,3,2,1}\left(u_1,u_2\right)u_1 - \frac{1}{2}u_2\Psi_{1,1,1,1,0,2}\left(u_1,u_2\right)u_1 - \frac{1}{2}u_2\Psi_{1,1,1,1,2,0}\left(u_1,u_2\right)u_1 \\
&+ iu_2^2\Psi_{1,1,1,2,1,2}\left(u_1,u_2\right)u_1 + \frac{3}{2}iu_2^2\Psi_{1,1,1,2,3,0}\left(u_1,u_2\right)u_1 + \frac{9}{2}iu_2^2\Psi_{1,1,2,1,0,3}\left(u_1,u_2\right)u_1 \\
&+ 3iu_2^2\Psi_{1,1,2,1,2,1}\left(u_1,u_2\right)u_1 + \frac{15}{2}iu_2^2\Psi_{1,1,3,0,1,2}\left(u_1,u_2\right)u_1 - \frac{1}{4}i\Phi_{0,1,1,0,0,1}(0) \\
&- \frac{1}{4}i\Phi_{1,0,1,0,1,0}(0) - \frac{1}{2}u_2\Phi_{1,0,1,1,2,0}\left(u_1\right) + 3iu_2^2\Phi_{1,0,1,2,3,0}\left(u_1\right) \\
&- \frac{1}{2}u_2\Phi_{1,0,2,0,1,1}\left(u_1\right) + 2iu_2^2\Phi_{1,0,2,1,2,1}\left(u_1\right) + 3iu_2^2\Phi_{1,0,3,0,1,2}\left(u_1\right) \\
&- \frac{1}{2}u_2\Phi_{0,1,1,1,1,1}\left(u_2\right) + 3iu_2^2\Phi_{0,1,1,2,2,1}\left(u_2\right) + 4iu_2^2\Phi_{0,1,2,1,1,2}\left(u_2\right) \\
&+ 6iu_2^2\Phi_{0,1,2,1,3,0}\left(u_2\right) + 9iu_2^2\Phi_{0,1,3,0,0,3}\left(u_2\right) + 6iu_2^2\Phi_{0,1,3,0,2,1}\left(u_2\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{4}iu_2\Phi_{0,2,1,0,0,1}\left(u_2\right)-\frac{1}{2}\Phi_{1,1,1,1}(0,0)+3iu_2\Phi_{1,2,2,1}(0,0)\\
&+4iu_2\Phi_{2,1,1,2}(0,0)+6iu_2\Phi_{2,1,3,0}(0,0)+9iu_2\Phi_{3,0,0,3}(0,0)\\
&+6iu_2\Phi_{3,0,2,1}(0,0)-\Phi_{2,0,0,2}\left(u_1,u_2\right)-\frac{1}{4}iu_2\Phi_{1,1,1,0,1,0}\left(u_1,u_2\right)\\
&-\frac{1}{2}u_2^2\Phi_{1,1,1,1,2,0}\left(u_1,u_2\right)+3iu_2^3\Phi_{1,1,1,2,3,0}\left(u_1,u_2\right)-\frac{1}{2}u_2^2\Phi_{1,1,2,0,1,1}\left(u_1,u_2\right)\\
&+2iu_2^3\Phi_{1,1,2,1,2,1}\left(u_1,u_2\right)+3iu_2^3\Phi_{1,1,3,0,1,2}\left(u_1,u_2\right)+\frac{1}{4}i\Psi_{0,1,0,1,0,1}(0)\\
&+\frac{1}{4}i\Psi_{1,0,0,1,1,0}(0)-\frac{3}{2}iu_2^2\Psi_{1,0,0,3,3,0}\left(u_1\right)-\frac{1}{4}u_2\Psi_{1,0,1,1,1,1}\left(u_1\right)\\
&+\frac{1}{2}iu_2^2\Psi_{1,0,1,2,2,1}\left(u_1\right)-u_2\Psi_{1,0,2,0,0,2}\left(u_1\right)+\frac{3}{2}iu_2^2\Psi_{1,0,2,1,1,2}\left(u_1\right)\\
&+\frac{15}{2}iu_2^2\Psi_{1,0,3,0,0,3}\left(u_1\right)-\frac{3}{2}iu_2^2\Psi_{0,1,0,3,2,1}\left(u_2\right)-\frac{1}{2}u_2\Psi_{0,1,1,1,0,2}\left(u_2\right)\\
&-\frac{1}{2}u_2\Psi_{0,1,1,1,2,0}\left(u_2\right)+iu_2^2\Psi_{0,1,1,2,1,2}\left(u_2\right)+\frac{3}{2}iu_2^2\Psi_{0,1,1,2,3,0}\left(u_2\right)\\
&+\frac{9}{2}iu_2^2\Psi_{0,1,2,1,0,3}\left(u_2\right)+3iu_2^2\Psi_{0,1,2,1,2,1}\left(u_2\right)+\frac{15}{2}iu_2^2\Psi_{0,1,3,0,1,2}\left(u_2\right)\\
&+\frac{1}{4}iu_2\Psi_{0,2,0,1,0,1}\left(u_2\right)-\frac{3}{2}iu_2\Psi_{0,3,2,1}(0,0)-\frac{1}{2}\Psi_{1,1,0,2}(0,0)\\
&-\frac{1}{2}\Psi_{1,1,2,0}(0,0)+iu_2\Psi_{1,2,1,2}(0,0)+\frac{3}{2}iu_2\Psi_{1,2,3,0}(0,0)\\
&+\frac{9}{2}iu_2\Psi_{2,1,0,3}(0,0)+3iu_2\Psi_{2,1,2,1}(0,0)+\frac{15}{2}iu_2\Psi_{3,0,1,2}(0,0)\\
&-\Psi_{2,0,1,1}\left(u_1,u_2\right)+\frac{1}{4}iu_2\Psi_{1,1,0,1,1,0}\left(u_1,u_2\right)-\frac{3}{2}iu_2^3\Psi_{1,1,0,3,3,0}\left(u_1,u_2\right)\\
&-\frac{1}{4}u_2^2\Psi_{1,1,1,1,1,1}\left(u_1,u_2\right)+\frac{1}{2}iu_2^3\Psi_{1,1,1,2,2,1}\left(u_1,u_2\right)-u_2^2\Psi_{1,1,2,0,0,2}\left(u_1,u_2\right)\\
&+\frac{3}{2}iu_2^3\Psi_{1,1,2,1,1,2}\left(u_1,u_2\right)+\frac{15}{2}iu_2^3\Psi_{1,1,3,0,0,3}\left(u_1,u_2\right)
\end{aligned}
$$

3.4 Fourth set of normalizing conditions - mixed terms

The fourth set of normalizing conditions 2.5, apply to the mixed terms $\Phi_{2,3}$, $\Phi_{1,2}$, $\Phi_{3,2}$,$\Phi_{2,1}$. To solve equation 2.5 the terms will again be split into their power series with the following rules, where Φ stands for both Φ and Ψ.

$$
\begin{aligned}
\Phi_{\alpha,\beta,\gamma,\delta}(u_1,u_2) &= \Phi_{\alpha,\beta,\gamma,\delta}(0,0)+u_1\Phi_{1,0,\alpha,\beta,\gamma,\delta}(u_1)+u_2\Phi_{0,1,\alpha,\beta,\gamma,\delta}(u_2)\\
&+u_1u_2\Phi_{1,1,\alpha,\beta,\gamma,\delta}(u_1,u_2)
\end{aligned}
$$

applied first to terms that will be solved for and then to the other terms appearing in equation 2.5.
Furthermore the terms consisting of derivatives with respect to u_1 and u_2, denoted by

the superscripts $(1,0)$ and $(0,1)$, respectively, will be expanded in the following way

$$\begin{aligned}\Phi^{(1,0)}_{\alpha,\beta,\gamma,\delta}(u_1,u_2) &= u_1u_2\Phi^{(1,0)}_{1,1,\alpha,\beta,\gamma,\delta}(u_1,u_2) + u_1\Phi'_{1,0,\alpha,\beta,\gamma,\delta}(u_1)\\ &+ \Phi_{1,0,\alpha,\beta,\gamma,\delta}(u_1) + u_2\Phi_{1,1,\alpha,\beta,\gamma,\delta}(u_1,u_2)\end{aligned}$$

and

$$\begin{aligned}\Phi^{(0,1)}_{\alpha,\beta,\gamma,\delta}(u_1,u_2) &= u_1u_2\Phi^{(0,1)}_{1,1,\alpha,\beta,\gamma,\delta}(u_1,u_2) + u_2\Phi'_{0,1,\alpha,\beta,\gamma,\delta}(u_2)\\ &+ \Phi_{0,1,\alpha,\beta,\gamma,\delta}(u_2) + u_1\Phi_{1,1,\alpha,\beta,\gamma,\delta}(u_1,u_2).\end{aligned}$$

Using the same structure as in section 3.3, we will first solve the constant terms

$$\Phi(0,0)(0,2,3,0) \text{ and } \Psi(0,0)(0,2,3,0) \tag{3.8}$$

and then the terms

$$\Phi(u_2)(0,1,0,2,3,0) \text{ and } \Psi(u_2)(0,1,0,2,3,0). \tag{3.9}$$

To obtain results for the constant terms, in the expanded equation 2.5 u_1 will be set to zero and the equation will be simplified. Then it is possible to solve for the terms in 3.8. This yields

$$\begin{aligned}\Phi_{0,2,3,0}(0,0) = &-\frac{4}{3}i\Phi^{(0,1)}_{0,1,2,0}(0,0) - \frac{2}{3}i\Phi^{(0,1)}_{1,0,1,1}(0,0) - \frac{1}{3}i\Phi^{(1,0)}_{0,1,1,1}(0,0)\\ &- \frac{1}{3}i\Phi^{(1,0)}_{1,0,0,2}(0,0) - \frac{1}{3}i\Phi^{(1,0)}_{1,0,2,0}(0,0) - \frac{1}{3}\Phi_{1,1,2,1}(0,0)\\ &- \frac{1}{3}\Phi_{2,0,1,2}(0,0) - \frac{2}{3}i\Psi^{(0,1)}_{0,1,1,1}(0,0) - \frac{4}{3}i\Psi^{(0,1)}_{1,0,0,2}(0,0)\\ &- \frac{1}{3}i\Psi^{(1,0)}_{0,1,0,2}(0,0) - \frac{1}{3}i\Psi^{(1,0)}_{0,1,2,0}(0,0) - \frac{1}{3}i\Psi^{(1,0)}_{1,0,1,1}(0,0)\\ &- \frac{1}{3}\Psi_{0,2,2,1}(0,0) - \frac{1}{3}\Psi_{1,1,1,2}(0,0) - \Psi_{2,0,0,3}(0,0)\end{aligned}$$

and

$$\begin{aligned}\Psi_{0,2,3,0}(0,0) = &-\frac{2}{3}i\Phi^{(0,1)}_{0,1,1,1}(0,0) - \frac{4}{3}i\Phi^{(0,1)}_{1,0,0,2}(0,0) - \frac{1}{3}i\Phi^{(1,0)}_{0,1,0,2}(0,0)\\ &- \frac{1}{3}i\Phi^{(1,0)}_{0,1,2,0}(0,0) - \frac{1}{3}i\Phi^{(1,0)}_{1,0,1,1}(0,0) - \frac{1}{3}\Phi_{0,2,2,1}(0,0)\\ &- \frac{1}{3}\Phi_{1,1,1,2}(0,0) - \Phi_{2,0,0,3}(0,0) - \frac{4}{3}i\Psi^{(0,1)}_{0,1,2,0}(0,0)\\ &- \frac{2}{3}i\Psi^{(0,1)}_{1,0,1,1}(0,0) - \frac{1}{3}i\Psi^{(1,0)}_{0,1,1,1}(0,0) - \frac{1}{3}i\Psi^{(1,0)}_{1,0,0,2}(0,0)\\ &- \frac{1}{3}i\Psi^{(1,0)}_{1,0,2,0}(0,0) - \frac{1}{3}\Psi_{1,1,2,1}(0,0) - \frac{1}{3}\Psi_{2,0,1,2}(0,0)\end{aligned}$$

By substituting these solutions into the expanded equation one can solve for the second pair of terms 3.9 and obtain

$$\begin{aligned}\Phi_{0,1,0,2,3,0}(u_2) = &-\frac{4}{3}i\Phi^{(0,1)}_{0,1,0,1,2,0}(u_2) - \frac{2}{3}i\Phi^{(0,1)}_{0,1,1,0,1,1}(u_2) - \frac{1}{3}i\Phi^{(1,0)}_{0,1,0,1,1,1}(u_2)\\ &- \frac{1}{3}i\Phi^{(1,0)}_{0,1,1,0,0,2}(u_2) - \frac{1}{3}i\Phi^{(1,0)}_{0,1,1,0,2,0}(u_2) - \frac{1}{3}\Phi_{0,1,1,1,2,1}(u_2)\\ &- \frac{1}{3}\Phi_{0,1,2,0,1,2}(u_2) - \frac{2}{3}i\Psi^{(0,1)}_{0,1,0,1,1,1}(u_2) - \frac{4}{3}i\Psi^{(0,1)}_{0,1,1,0,0,2}(u_2)\\ &- \frac{1}{3}i\Psi^{(1,0)}_{0,1,0,1,0,2}(u_2) - \frac{1}{3}i\Psi^{(1,0)}_{0,1,0,1,2,0}(u_2) - \frac{1}{3}i\Psi^{(1,0)}_{0,1,1,0,1,1}(u_2)\\ &- \frac{1}{3}\Psi_{0,1,0,2,2,1}(u_2) - \frac{1}{3}\Psi_{0,1,1,1,1,2}(u_2) - \Psi_{0,1,2,0,0,3}(u_2)\end{aligned}$$

and

$$\begin{aligned}\Psi_{0,1,0,2,3,0}(u_2) = &-\frac{2}{3}i\Phi^{(0,1)}_{0,1,0,1,1,1}(u_2) - \frac{4}{3}i\Phi^{(0,1)}_{0,1,1,0,0,2}(u_2) - \frac{1}{3}i\Phi^{(1,0)}_{0,1,0,1,0,2}(u_2)\\ &- \frac{1}{3}i\Phi^{(1,0)}_{0,1,0,1,2,0}(u_2) - \frac{1}{3}i\Phi^{(1,0)}_{0,1,1,0,1,1}(u_2) - \frac{1}{3}\Phi_{0,1,0,2,2,1}(u_2)\\ &- \frac{1}{3}\Phi_{0,1,1,1,1,2}(u_2) - \Phi_{0,1,2,0,0,3}(u_2) - \frac{4}{3}i\Psi^{(0,1)}_{0,1,0,1,2,0}(u_2)\\ &- \frac{2}{3}i\Psi^{(0,1)}_{0,1,1,0,1,1}(u_2) - \frac{1}{3}i\Psi^{(1,0)}_{0,1,0,1,1,1}(u_2) - \frac{1}{3}i\Psi^{(1,0)}_{0,1,1,0,0,2}(u_2)\\ &- \frac{1}{3}i\Psi^{(1,0)}_{0,1,1,0,2,0}(u_2) - \frac{1}{3}\Psi_{0,1,1,1,2,1}(u_2) - \frac{1}{3}\Psi_{0,1,2,0,1,2}(u_2),\end{aligned}$$

This same system can be applied where we solve for the constant terms

$$\Phi_{0,2,0,3}(0,0) \text{ and } \Psi_{0,2,0,3}(0,0) \tag{3.10}$$

in the expanded equation, where u_2 is set to zero. We get the following results

$$\begin{aligned}\Phi_{0,2,0,3}(0,0) = &-\frac{1}{3}i\Phi^{(0,1)}_{0,1,1,1}(0,0) - \frac{1}{3}i\Phi^{(0,1)}_{1,0,0,2}(0,0) - \frac{1}{3}i\Phi^{(0,1)}_{1,0,2,0}(0,0)\\ &- \frac{4}{3}i\Phi^{(1,0)}_{0,1,0,2}(0,0) - \frac{2}{3}i\Phi^{(1,0)}_{1,0,1,1}(0,0) - \frac{1}{3}\Phi_{1,1,1,2}(0,0)\\ &- \frac{1}{3}\Phi_{2,0,2,1}(0,0) - \frac{1}{3}i\Psi^{(0,1)}_{0,1,0,2}(0,0) - \frac{1}{3}i\Psi^{(0,1)}_{0,1,2,0}(0,0)\\ &- \frac{1}{3}i\Psi^{(0,1)}_{1,0,1,1}(0,0) - \frac{2}{3}i\Psi^{(1,0)}_{0,1,1,1}(0,0) - \frac{4}{3}i\Psi^{(1,0)}_{1,0,2,0}(0,0)\\ &- \frac{1}{3}\Psi_{0,2,1,2}(0,0) - \frac{1}{3}\Psi_{1,1,2,1}(0,0) - \Psi_{2,0,3,0}(0,0)\end{aligned}$$

and

$$\begin{aligned}\Psi_{0,2,0,3}(0,0) = &-\frac{1}{3}i\Phi^{(0,1)}_{0,1,0,2}(0,0) - \frac{1}{3}i\Phi^{(0,1)}_{0,1,2,0}(0,0) - \frac{1}{3}i\Phi^{(0,1)}_{1,0,1,1}(0,0)\\ &- \frac{2}{3}i\Phi^{(1,0)}_{0,1,1,1}(0,0) - \frac{4}{3}i\Phi^{(1,0)}_{1,0,2,0}(0,0) - \frac{1}{3}\Phi_{0,2,1,2}(0,0)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{3}\Phi_{1,1,2,1}(0,0)-\Phi_{2,0,3,0}(0,0)-\frac{1}{3}i\Psi^{(0,1)}_{0,1,1,1}(0,0)\\
&-\frac{1}{3}i\Psi^{(0,1)}_{1,0,0,2}(0,0)-\frac{1}{3}i\Psi^{(0,1)}_{1,0,2,0}(0,0)-\frac{4}{3}i\Psi^{(1,0)}_{0,1,0,2}(0,0)\\
&-\frac{2}{3}i\Psi^{(1,0)}_{1,0,1,1}(0,0)-\frac{1}{3}\Psi_{1,1,1,2}(0,0)-\frac{1}{3}\Psi_{2,0,2,1}(0,0)
\end{aligned}
$$

Continuing the process as above, these solutions can be substituted into the expanded equation and we can then solve for the terms

$$\Phi_{1,0,0,2,0,3}\left(u_1\right) \text{ and } \Psi_{1,0,0,2,0,3}\left(u_1\right).$$

These then take the form

$$
\begin{aligned}
\Phi_{1,0,0,2,0,3}\left(u_1\right)=&-\frac{1}{3}i\Phi^{(0,1)}_{1,0,0,1,1,1}\left(u_1\right)-\frac{1}{3}i\Phi^{(0,1)}_{1,0,1,0,0,2}\left(u_1\right)-\frac{1}{3}i\Phi^{(0,1)}_{1,0,1,0,2,0}\left(u_1\right)\\
&-\frac{4}{3}i\Phi^{(1,0)}_{1,0,0,1,0,2}\left(u_1\right)-\frac{2}{3}i\Phi^{(1,0)}_{1,0,1,0,1,1}\left(u_1\right)-\frac{1}{3}\Phi_{1,0,1,1,1,2}\left(u_1\right)\\
&-\frac{1}{3}\Phi_{1,0,2,0,2,1}\left(u_1\right)-\frac{1}{3}i\Psi^{(0,1)}_{1,0,0,1,0,2}\left(u_1\right)-\frac{1}{3}i\Psi^{(0,1)}_{1,0,0,1,2,0}\left(u_1\right)-\frac{1}{3}i\Psi^{(0,1)}_{1,0,1,0,1,1}\left(u_1\right)\\
&-\frac{2}{3}i\Psi^{(1,0)}_{1,0,0,1,1,1}\left(u_1\right)-\frac{4}{3}i\Psi^{(1,0)}_{1,0,1,0,2,0}\left(u_1\right)-\frac{1}{3}\Psi_{1,0,0,2,1,2}\left(u_1\right)\\
&-\frac{1}{3}\Psi_{1,0,1,1,2,1}\left(u_1\right)-\Psi_{1,0,2,0,3,0}\left(u_1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_{1,0,0,2,0,3}\left(u_1\right)=&-\frac{1}{3}i\Phi^{(0,1)}_{1,0,0,1,0,2}\left(u_1\right)-\frac{1}{3}i\Phi^{(0,1)}_{1,0,0,1,2,0}\left(u_1\right)-\frac{1}{3}i\Phi^{(0,1)}_{1,0,1,0,1,1}\left(u_1\right)\\
&-\frac{2}{3}i\Phi^{(1,0)}_{1,0,0,1,1,1}\left(u_1\right)-\frac{4}{3}i\Phi^{(1,0)}_{1,0,1,0,2,0}\left(u_1\right)-\frac{1}{3}\Phi_{1,0,0,2,1,2}\left(u_1\right)\\
&-\frac{1}{3}\Phi_{1,0,1,1,2,1}\left(u_1\right)-\Phi_{1,0,2,0,3,0}\left(u_1\right)-\frac{1}{3}i\Psi^{(0,1)}_{1,0,0,1,1,1}\left(u_1\right)\\
&-\frac{1}{3}i\Psi^{(0,1)}_{1,0,1,0,0,2}\left(u_1\right)-\frac{1}{3}i\Psi^{(0,1)}_{1,0,1,0,2,0}\left(u_1\right)-\frac{4}{3}i\Psi^{(1,0)}_{1,0,0,1,0,2}\left(u_1\right)\\
&-\frac{2}{3}i\Psi^{(1,0)}_{1,0,1,0,1,1}\left(u_1\right)-\frac{1}{3}\Psi_{1,0,1,1,1,2}\left(u_1\right)-\frac{1}{3}\Psi_{1,0,2,0,2,1}\left(u_1\right).
\end{aligned}
$$

Using these found equations and substituting them into the expanded equation, we can solve for the rest of the terms of the power series. We will be solving for

$$\Phi_{1,1,0,3}\left(u_1,u_2\right) \text{ and } \Psi_{1,1,0,3}\left(u_1,u_2\right)$$

and obtain the following two solutions

$$
\begin{aligned}
\Phi_{1,1,0,3}\left(u_1,u_2\right)=&-\Phi_{1,1,0,2,0,3}\left(u_1,u_2\right)u_1^2-\frac{1}{3}\Phi_{1,1,1,1,1,2}\left(u_1,u_2\right)u_1^2-\frac{1}{3}\Phi_{1,1,2,0,2,1}\left(u_1,u_2\right)u_1^2\\
&-\frac{1}{3}\Psi_{1,1,0,2,1,2}\left(u_1,u_2\right)u_1^2-\frac{1}{3}\Psi_{1,1,1,1,2,1}\left(u_1,u_2\right)u_1^2-\Psi_{1,1,2,0,3,0}\left(u_1,u_2\right)u_1^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{3}i\Phi^{(0,1)}_{1,1,0,1,1,1}(u_1,u_2)\,u_1^2-\frac{1}{3}i\Phi^{(0,1)}_{1,1,1,0,0,2}(u_1,u_2)\,u_1^2-\frac{1}{3}i\Phi^{(0,1)}_{1,1,1,0,2,0}(u_1,u_2)\,u_1^2\\
&-\frac{1}{3}i\Psi^{(0,1)}_{1,1,0,1,0,2}(u_1,u_2)\,u_1^2-\frac{1}{3}i\Psi^{(0,1)}_{1,1,0,1,2,0}(u_1,u_2)\,u_1^2-\frac{1}{3}i\Psi^{(0,1)}_{1,1,1,0,1,1}(u_1,u_2)\,u_1^2\\
&-\frac{4}{3}i\Phi^{(1,0)}_{1,1,0,1,0,2}(u_1,u_2)\,u_1^2-\frac{2}{3}i\Phi^{(1,0)}_{1,1,1,0,1,1}(u_1,u_2)\,u_1^2-\frac{2}{3}i\Psi^{(1,0)}_{1,1,0,1,1,1}(u_1,u_2)\,u_1^2\\
&-\frac{4}{3}i\Psi^{(1,0)}_{1,1,1,0,2,0}(u_1,u_2)\,u_1^2-\Phi_{0,1,0,2,0,3}(u_2)\,u_1-\frac{1}{3}\Phi_{0,1,1,1,1,2}(u_2)\,u_1\\
&-\frac{1}{3}\Phi_{0,1,2,0,2,1}(u_2)\,u_1-\frac{2}{3}\Phi_{1,0,0,2,1,2}(u_1)\,u_1-\frac{1}{3}\Phi_{1,0,1,1,2,1}(u_1)\,u_1\\
&-\frac{2}{3}\Phi_{1,0,2,0,1,2}(u_1)\,u_1-\frac{2}{3}u_2\Phi_{1,1,0,2,1,2}(u_1,u_2)\,u_1-\frac{1}{3}u_2\Phi_{1,1,1,1,2,1}(u_1,u_2)\,u_1\\
&-\frac{2}{3}u_2\Phi_{1,1,2,0,1,2}(u_1,u_2)\,u_1-\frac{1}{3}\Psi_{0,1,0,2,1,2}(u_2)\,u_1-\frac{1}{3}\Psi_{0,1,1,1,2,1}(u_2)\,u_1\\
&-\Psi_{0,1,2,0,3,0}(u_2)\,u_1-\frac{2}{3}\Psi_{1,0,0,2,2,1}(u_1)\,u_1-\frac{1}{3}\Psi_{1,0,1,1,1,2}(u_1)\,u_1\\
&-\frac{2}{3}\Psi_{1,0,2,0,2,1}(u_1)\,u_1-\frac{2}{3}u_2\Psi_{1,1,0,2,2,1}(u_1,u_2)\,u_1-\frac{1}{3}u_2\Psi_{1,1,1,1,1,2}(u_1,u_2)\,u_1\\
&-\frac{2}{3}u_2\Psi_{1,1,2,0,2,1}(u_1,u_2)\,u_1-\frac{1}{3}i\Phi^{(0,1)}_{0,1,0,1,1,1}(u_2)\,u_1-\frac{1}{3}i\Phi^{(0,1)}_{0,1,1,0,0,2}(u_2)\,u_1\\
&-\frac{1}{3}i\Phi^{(0,1)}_{0,1,1,0,2,0}(u_2)\,u_1-\frac{5}{3}i\Phi^{(0,1)}_{1,0,0,1,0,2}(u_1)\,u_1-\frac{1}{3}i\Phi^{(0,1)}_{1,0,0,1,2,0}(u_1)\,u_1\\
&-i\Phi^{(0,1)}_{1,0,1,0,1,1}(u_1)\,u_1-\frac{5}{3}iu_2\Phi^{(0,1)}_{1,1,0,1,0,2}(u_1,u_2)\,u_1-\frac{1}{3}iu_2\Phi^{(0,1)}_{1,1,0,1,2,0}(u_1,u_2)\,u_1\\
&-iu_2\Phi^{(0,1)}_{1,1,1,0,1,1}(u_1,u_2)\,u_1-\frac{1}{3}i\Psi^{(0,1)}_{0,1,0,1,0,2}(u_2)\,u_1-\frac{1}{3}i\Psi^{(0,1)}_{0,1,0,1,2,0}(u_2)\,u_1\\
&-\frac{1}{3}i\Psi^{(0,1)}_{0,1,1,0,1,1}(u_2)\,u_1-i\Psi^{(0,1)}_{1,0,0,1,1,1}(u_1)\,u_1-\frac{1}{3}i\Psi^{(0,1)}_{1,0,1,0,0,2}(u_1)\,u_1\\
&-\frac{5}{3}i\Psi^{(0,1)}_{1,0,1,0,2,0}(u_1)\,u_1-iu_2\Psi^{(0,1)}_{1,1,0,1,1,1}(u_1,u_2)\,u_1-\frac{1}{3}iu_2\Psi^{(0,1)}_{1,1,1,0,0,2}(u_1,u_2)\,u_1\\
&-\frac{5}{3}iu_2\Psi^{(0,1)}_{1,1,1,0,2,0}(u_1,u_2)\,u_1-\frac{4}{3}i\Phi^{(1,0)}_{0,1,0,1,0,2}(u_2)\,u_1-\frac{2}{3}i\Phi^{(1,0)}_{0,1,1,0,1,1}(u_2)\,u_1\\
&-i\Phi^{(1,0)}_{1,0,0,1,1,1}(u_1)\,u_1-\frac{5}{3}i\Phi^{(1,0)}_{1,0,1,0,0,2}(u_1)\,u_1-\frac{1}{3}i\Phi^{(1,0)}_{1,0,1,0,2,0}(u_1)\,u_1\\
&-iu_2\Phi^{(1,0)}_{1,1,0,1,1,1}(u_1,u_2)\,u_1-\frac{5}{3}iu_2\Phi^{(1,0)}_{1,1,1,0,0,2}(u_1,u_2)\,u_1-\frac{1}{3}iu_2\Phi^{(1,0)}_{1,1,1,0,2,0}(u_1,u_2)\,u_1\\
&-\frac{2}{3}i\Psi^{(1,0)}_{0,1,0,1,1,1}(u_2)\,u_1-\frac{4}{3}i\Psi^{(1,0)}_{0,1,1,0,2,0}(u_2)\,u_1-\frac{1}{3}i\Psi^{(1,0)}_{1,0,0,1,0,2}(u_1)\,u_1\\
&-\frac{5}{3}i\Psi^{(1,0)}_{1,0,0,1,2,0}(u_1)\,u_1-i\Psi^{(1,0)}_{1,0,1,0,1,1}(u_1)\,u_1-\frac{1}{3}iu_2\Psi^{(1,0)}_{1,1,0,1,0,2}(u_1,u_2)\,u_1\\
&-\frac{5}{3}iu_2\Psi^{(1,0)}_{1,1,0,1,2,0}(u_1,u_2)\,u_1-iu_2\Psi^{(1,0)}_{1,1,1,0,1,1}(u_1,u_2)\,u_1-\frac{2}{3}\Phi_{0,2,1,2}(0,0)\\
&-\frac{1}{3}\Phi_{1,1,2,1}(0,0)-\frac{2}{3}\Phi_{2,0,1,2}(0,0)-\frac{2}{3}\Psi_{0,2,2,1}(0,0)\\
&-\frac{1}{3}\Psi_{1,1,1,2}(0,0)-\Psi_{1,1,3,0}(u_1,u_2)-\frac{2}{3}\Psi_{2,0,2,1}(0,0)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{5}{3}i\Phi^{(0,1)}_{0,1,0,2}(0,0)-\frac{1}{3}i\Phi^{(0,1)}_{0,1,2,0}(0,0)-i\Phi^{(0,1)}_{1,0,1,1}(0,0)\\
&-i\Psi^{(0,1)}_{0,1,1,1}(0,0)-\frac{1}{3}i\Psi^{(0,1)}_{1,0,0,2}(0,0)-\frac{5}{3}i\Psi^{(0,1)}_{1,0,2,0}(0,0)\\
&-i\Phi^{(1,0)}_{0,1,1,1}(0,0)-\frac{5}{3}i\Phi^{(1,0)}_{1,0,0,2}(0,0)-\frac{1}{3}i\Phi^{(1,0)}_{1,0,2,0}(0,0)\\
&-\frac{1}{3}i\Psi^{(1,0)}_{0,1,0,2}(0,0)-\frac{5}{3}i\Psi^{(1,0)}_{0,1,2,0}(0,0)-i\Psi^{(1,0)}_{1,0,1,1}(0,0)\\
&-\frac{2}{3}u_2\Phi_{0,1,0,2,1,2}(u_2)-\frac{1}{3}u_2\Phi_{0,1,1,1,2,1}(u_2)-\frac{2}{3}u_2\Phi_{0,1,2,0,1,2}(u_2)\\
&-\frac{1}{3}u_2\Phi_{1,0,0,2,2,1}(u_1)-\frac{1}{3}u_2\Phi_{1,0,1,1,1,2}(u_1)-u_2\Phi_{1,0,2,0,0,3}(u_1)\\
&-\frac{1}{3}u_2^2\Phi_{1,1,0,2,2,1}(u_1,u_2)-\frac{1}{3}u_2^2\Phi_{1,1,1,1,1,2}(u_1,u_2)-u_2^2\Phi_{1,1,2,0,0,3}(u_1,u_2)\\
&-\frac{2}{3}u_2\Psi_{0,1,0,2,2,1}(u_2)-\frac{1}{3}u_2\Psi_{0,1,1,1,1,2}(u_2)-\frac{2}{3}u_2\Psi_{0,1,2,0,2,1}(u_2)\\
&-u_2\Psi_{1,0,0,2,3,0}(u_1)-\frac{1}{3}u_2\Psi_{1,0,1,1,2,1}(u_1)-\frac{1}{3}u_2\Psi_{1,0,2,0,1,2}(u_1)\\
&-u_2^2\Psi_{1,1,0,2,3,0}(u_1,u_2)-\frac{1}{3}u_2^2\Psi_{1,1,1,1,2,1}(u_1,u_2)-\frac{1}{3}u_2^2\Psi_{1,1,2,0,1,2}(u_1,u_2)\\
&-\frac{5}{3}iu_2\Phi^{(0,1)}_{0,1,0,1,0,2}(u_2)-\frac{1}{3}iu_2\Phi^{(0,1)}_{0,1,0,1,2,0}(u_2)-iu_2\Phi^{(0,1)}_{0,1,1,0,1,1}(u_2)\\
&-\frac{2}{3}iu_2\Phi^{(0,1)}_{1,0,0,1,1,1}(u_1)-\frac{4}{3}iu_2\Phi^{(0,1)}_{1,0,1,0,0,2}(u_1)-\frac{2}{3}iu_2^2\Phi^{(0,1)}_{1,1,0,1,1,1}(u_1,u_2)\\
&-\frac{4}{3}iu_2^2\Phi^{(0,1)}_{1,1,1,0,0,2}(u_1,u_2)-iu_2\Psi^{(0,1)}_{0,1,0,1,1,1}(u_2)-\frac{1}{3}iu_2\Psi^{(0,1)}_{0,1,1,0,0,2}(u_2)\\
&-\frac{5}{3}iu_2\Psi^{(0,1)}_{0,1,1,0,2,0}(u_2)-\frac{4}{3}iu_2\Psi^{(0,1)}_{1,0,0,1,2,0}(u_1)-\frac{2}{3}iu_2\Psi^{(0,1)}_{1,0,1,0,1,1}(u_1)\\
&-\frac{4}{3}iu_2^2\Psi^{(0,1)}_{1,1,0,1,2,0}(u_1,u_2)-\frac{2}{3}iu_2^2\Psi^{(0,1)}_{1,1,1,0,1,1}(u_1,u_2)-iu_2\Phi^{(1,0)}_{0,1,0,1,1,1}(u_2)\\
&-\frac{5}{3}iu_2\Phi^{(1,0)}_{0,1,1,0,0,2}(u_2)-\frac{1}{3}iu_2\Phi^{(1,0)}_{0,1,1,0,2,0}(u_2)-\frac{1}{3}iu_2\Phi^{(1,0)}_{1,0,0,1,0,2}(u_1)\\
&-\frac{1}{3}iu_2\Phi^{(1,0)}_{1,0,0,1,2,0}(u_1)-\frac{1}{3}iu_2\Phi^{(1,0)}_{1,0,1,0,1,1}(u_1)-\frac{1}{3}iu_2^2\Phi^{(1,0)}_{1,1,0,1,0,2}(u_1,u_2)\\
&-\frac{1}{3}iu_2^2\Phi^{(1,0)}_{1,1,0,1,2,0}(u_1,u_2)-\frac{1}{3}iu_2^2\Phi^{(1,0)}_{1,1,1,0,1,1}(u_1,u_2)-\frac{1}{3}iu_2\Psi^{(1,0)}_{0,1,0,1,0,2}(u_2)\\
&-\frac{5}{3}iu_2\Psi^{(1,0)}_{0,1,0,1,2,0}(u_2)-iu_2\Psi^{(1,0)}_{0,1,1,0,1,1}(u_2)-\frac{1}{3}iu_2\Psi^{(1,0)}_{1,0,0,1,1,1}(u_1)\\
&-\frac{1}{3}iu_2\Psi^{(1,0)}_{1,0,1,0,0,2}(u_1)-\frac{1}{3}iu_2\Psi^{(1,0)}_{1,0,1,0,2,0}(u_1)-\frac{1}{3}iu_2^2\Psi^{(1,0)}_{1,1,0,1,1,1}(u_1,u_2)\\
&-\frac{1}{3}iu_2^2\Psi^{(1,0)}_{1,1,1,0,0,2}(u_1,u_2)-\frac{1}{3}iu_2^2\Psi^{(1,0)}_{1,1,1,0,2,0}(u_1,u_2)
\end{aligned}
$$

and

$$
\Psi_{1,1,0,3}(u_1,u_2)=-\frac{1}{3}\Phi_{1,1,0,2,1,2}(u_1,u_2)\,u_1^2-\frac{1}{3}\Phi_{1,1,1,1,2,1}(u_1,u_2)\,u_1^2-\Phi_{1,1,2,0,3,0}(u_1,u_2)\,u_1^2
$$

$$
\begin{aligned}
&- \Psi_{1,1,0,2,0,3}\left(u_1,u_2\right)u_1^2 - \frac{1}{3}\Psi_{1,1,1,1,1,2}\left(u_1,u_2\right)u_1^2 - \frac{1}{3}\Psi_{1,1,2,0,2,1}\left(u_1,u_2\right)u_1^2 \\
&- \frac{1}{3}i\Phi^{(0,1)}_{1,1,0,1,0,2}\left(u_1,u_2\right)u_1^2 - \frac{1}{3}i\Phi^{(0,1)}_{1,1,0,1,2,0}\left(u_1,u_2\right)u_1^2 - \frac{1}{3}i\Phi^{(0,1)}_{1,1,1,0,1,1}\left(u_1,u_2\right)u_1^2 \\
&- \frac{1}{3}i\Psi^{(0,1)}_{1,1,0,1,1,1}\left(u_1,u_2\right)u_1^2 - \frac{1}{3}i\Psi^{(0,1)}_{1,1,1,0,0,2}\left(u_1,u_2\right)u_1^2 - \frac{1}{3}i\Psi^{(0,1)}_{1,1,1,0,2,0}\left(u_1,u_2\right)u_1^2 \\
&- \frac{2}{3}i\Phi^{(1,0)}_{1,1,0,1,1,1}\left(u_1,u_2\right)u_1^2 - \frac{4}{3}i\Phi^{(1,0)}_{1,1,1,0,2,0}\left(u_1,u_2\right)u_1^2 - \frac{4}{3}i\Psi^{(1,0)}_{1,1,0,1,0,2}\left(u_1,u_2\right)u_1^2 \\
&- \frac{2}{3}i\Psi^{(1,0)}_{1,1,1,0,1,1}\left(u_1,u_2\right)u_1^2 - \frac{1}{3}\Phi_{0,1,0,2,1,2}\left(u_2\right)u_1 - \frac{1}{3}\Phi_{0,1,1,1,2,1}\left(u_2\right)u_1 \\
&- \Phi_{0,1,2,0,3,0}\left(u_2\right)u_1 - \frac{2}{3}\Phi_{1,0,0,2,2,1}\left(u_1\right)u_1 - \frac{1}{3}\Phi_{1,0,1,1,1,2}\left(u_1\right)u_1 \\
&- \frac{2}{3}\Phi_{1,0,2,0,2,1}\left(u_1\right)u_1 - \frac{2}{3}u_2\Phi_{1,1,0,2,2,1}\left(u_1,u_2\right)u_1 - \frac{1}{3}u_2\Phi_{1,1,1,1,1,2}\left(u_1,u_2\right)u_1 \\
&- \frac{2}{3}u_2\Phi_{1,1,2,0,2,1}\left(u_1,u_2\right)u_1 - \Psi_{0,1,0,2,0,3}\left(u_2\right)u_1 - \frac{1}{3}\Psi_{0,1,1,1,1,2}\left(u_2\right)u_1 \\
&- \frac{1}{3}\Psi_{0,1,2,0,2,1}\left(u_2\right)u_1 - \frac{2}{3}\Psi_{1,0,0,2,1,2}\left(u_1\right)u_1 - \frac{1}{3}\Psi_{1,0,1,1,2,1}\left(u_1\right)u_1 \\
&- \frac{2}{3}\Psi_{1,0,2,0,1,2}\left(u_1\right)u_1 - \frac{2}{3}u_2\Psi_{1,1,0,2,1,2}\left(u_1,u_2\right)u_1 - \frac{1}{3}u_2\Psi_{1,1,1,1,2,1}\left(u_1,u_2\right)u_1 \\
&- \frac{2}{3}u_2\Psi_{1,1,2,0,1,2}\left(u_1,u_2\right)u_1 - \frac{1}{3}i\Phi^{(0,1)}_{0,1,0,1,0,2}\left(u_2\right)u_1 - \frac{1}{3}i\Phi^{(0,1)}_{0,1,0,1,2,0}\left(u_2\right)u_1 \\
&- \frac{1}{3}i\Phi^{(0,1)}_{0,1,1,0,1,1}\left(u_2\right)u_1 - i\Phi^{(0,1)}_{1,0,0,1,1,1}\left(u_1\right)u_1 - \frac{1}{3}i\Phi^{(0,1)}_{1,0,1,0,0,2}\left(u_1\right)u_1 \\
&- \frac{5}{3}i\Phi^{(0,1)}_{1,0,1,0,2,0}\left(u_1\right)u_1 - iu_2\Phi^{(0,1)}_{1,1,0,1,1,1}\left(u_1,u_2\right)u_1 - \frac{1}{3}iu_2\Phi^{(0,1)}_{1,1,1,0,0,2}\left(u_1,u_2\right)u_1 \\
&- \frac{5}{3}iu_2\Phi^{(0,1)}_{1,1,1,0,2,0}\left(u_1,u_2\right)u_1 - \frac{1}{3}i\Psi^{(0,1)}_{0,1,0,1,1,1}\left(u_2\right)u_1 - \frac{1}{3}i\Psi^{(0,1)}_{0,1,1,0,0,2}\left(u_2\right)u_1 \\
&- \frac{1}{3}i\Psi^{(0,1)}_{0,1,1,0,2,0}\left(u_2\right)u_1 - \frac{5}{3}i\Psi^{(0,1)}_{1,0,0,1,0,2}\left(u_1\right)u_1 - \frac{1}{3}i\Psi^{(0,1)}_{1,0,0,1,2,0}\left(u_1\right)u_1 \\
&- i\Psi^{(0,1)}_{1,0,1,0,1,1}\left(u_1\right)u_1 - \frac{5}{3}iu_2\Psi^{(0,1)}_{1,1,0,1,0,2}\left(u_1,u_2\right)u_1 - \frac{1}{3}iu_2\Psi^{(0,1)}_{1,1,0,1,2,0}\left(u_1,u_2\right)u_1 \\
&- iu_2\Psi^{(0,1)}_{1,1,1,0,1,1}\left(u_1,u_2\right)u_1 - \frac{2}{3}i\Phi^{(1,0)}_{0,1,0,1,1,1}\left(u_2\right)u_1 - \frac{4}{3}i\Phi^{(1,0)}_{0,1,1,0,2,0}\left(u_2\right)u_1 \\
&- \frac{1}{3}i\Phi^{(1,0)}_{1,0,0,1,0,2}\left(u_1\right)u_1 - \frac{5}{3}i\Phi^{(1,0)}_{1,0,0,1,2,0}\left(u_1\right)u_1 - i\Phi^{(1,0)}_{1,0,1,0,1,1}\left(u_1\right)u_1 \\
&- \frac{1}{3}iu_2\Phi^{(1,0)}_{1,1,0,1,0,2}\left(u_1,u_2\right)u_1 - \frac{5}{3}iu_2\Phi^{(1,0)}_{1,1,0,1,2,0}\left(u_1,u_2\right)u_1 - iu_2\Phi^{(1,0)}_{1,1,1,0,1,1}\left(u_1,u_2\right)u_1 \\
&- \frac{4}{3}i\Psi^{(1,0)}_{0,1,0,1,0,2}\left(u_2\right)u_1 - \frac{2}{3}i\Psi^{(1,0)}_{0,1,1,0,1,1}\left(u_2\right)u_1 - i\Psi^{(1,0)}_{1,0,0,1,1,1}\left(u_1\right)u_1 \\
&- \frac{5}{3}i\Psi^{(1,0)}_{1,0,1,0,0,2}\left(u_1\right)u_1 - \frac{1}{3}i\Psi^{(1,0)}_{1,0,1,0,2,0}\left(u_1\right)u_1 - iu_2\Psi^{(1,0)}_{1,1,0,1,1,1}\left(u_1,u_2\right)u_1 \\
&- \frac{5}{3}iu_2\Psi^{(1,0)}_{1,1,1,0,0,2}\left(u_1,u_2\right)u_1 - \frac{1}{3}iu_2\Psi^{(1,0)}_{1,1,1,0,2,0}\left(u_1,u_2\right)u_1 - \frac{2}{3}\Phi_{0,2,2,1}(0,0) \\
&- \frac{1}{3}\Phi_{1,1,1,2}(0,0) - \Phi_{1,1,3,0}\left(u_1,u_2\right) - \frac{2}{3}\Phi_{2,0,2,1}(0,0)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2}{3}\Psi_{0,2,1,2}(0,0) - \frac{1}{3}\Psi_{1,1,2,1}(0,0) - \frac{2}{3}\Psi_{2,0,1,2}(0,0) \\
&- i\Phi^{(0,1)}_{0,1,1,1}(0,0) - \frac{1}{3}i\Phi^{(0,1)}_{1,0,0,2}(0,0) - \frac{5}{3}i\Phi^{(0,1)}_{1,0,2,0}(0,0) \\
&- \frac{5}{3}i\Psi^{(0,1)}_{0,1,0,2}(0,0) - \frac{1}{3}i\Psi^{(0,1)}_{0,1,2,0}(0,0) - i\Psi^{(0,1)}_{1,0,1,1}(0,0) \\
&- \frac{1}{3}i\Phi^{(1,0)}_{0,1,0,2}(0,0) - \frac{5}{3}i\Phi^{(1,0)}_{0,1,2,0}(0,0) - i\Phi^{(1,0)}_{1,0,1,1}(0,0) \\
&- i\Psi^{(1,0)}_{0,1,1,1}(0,0) - \frac{5}{3}i\Psi^{(1,0)}_{1,0,0,2}(0,0) - \frac{1}{3}i\Psi^{(1,0)}_{1,0,2,0}(0,0) \\
&- \frac{2}{3}u_2\Phi_{0,1,0,2,2,1}\left(u_2\right) - \frac{1}{3}u_2\Phi_{0,1,1,1,1,2}\left(u_2\right) - \frac{2}{3}u_2\Phi_{0,1,2,0,2,1}\left(u_2\right) \\
&- u_2\Phi_{1,0,0,2,3,0}\left(u_1\right) - \frac{1}{3}u_2\Phi_{1,0,1,1,2,1}\left(u_1\right) - \frac{1}{3}u_2\Phi_{1,0,2,0,1,2}\left(u_1\right) \\
&- u_2^2\Phi_{1,1,0,2,3,0}\left(u_1,u_2\right) - \frac{1}{3}u_2^2\Phi_{1,1,1,1,2,1}\left(u_1,u_2\right) - \frac{1}{3}u_2^2\Phi_{1,1,2,0,1,2}\left(u_1,u_2\right) \\
&- \frac{2}{3}u_2\Psi_{0,1,0,2,1,2}\left(u_2\right) - \frac{1}{3}u_2\Psi_{0,1,1,1,2,1}\left(u_2\right) - \frac{2}{3}u_2\Psi_{0,1,2,0,1,2}\left(u_2\right) \\
&- \frac{1}{3}u_2\Psi_{1,0,0,2,2,1}\left(u_1\right) - \frac{1}{3}u_2\Psi_{1,0,1,1,1,2}\left(u_1\right) - u_2\Psi_{1,0,2,0,0,3}\left(u_1\right) \\
&- \frac{1}{3}u_2^2\Psi_{1,1,0,2,2,1}\left(u_1,u_2\right) - \frac{1}{3}u_2^2\Psi_{1,1,1,1,1,2}\left(u_1,u_2\right) - u_2^2\Psi_{1,1,2,0,0,3}\left(u_1,u_2\right) \\
&- iu_2\Phi^{(0,1)}_{0,1,0,1,1,1}\left(u_2\right) - \frac{1}{3}iu_2\Phi^{(0,1)}_{0,1,1,0,0,2}\left(u_2\right) - \frac{5}{3}iu_2\Phi^{(0,1)}_{0,1,1,0,2,0}\left(u_2\right) \\
&- \frac{4}{3}iu_2\Phi^{(0,1)}_{1,0,0,1,2,0}\left(u_1\right) - \frac{2}{3}iu_2\Phi^{(0,1)}_{1,0,1,0,1,1}\left(u_1\right) - \frac{4}{3}iu_2^2\Phi^{(0,1)}_{1,1,0,1,2,0}\left(u_1,u_2\right) \\
&- \frac{2}{3}iu_2^2\Phi^{(0,1)}_{1,1,1,0,1,1}\left(u_1,u_2\right) - \frac{5}{3}iu_2\Psi^{(0,1)}_{0,1,0,1,0,2}\left(u_2\right) - \frac{1}{3}iu_2\Psi^{(0,1)}_{0,1,0,1,2,0}\left(u_2\right) \\
&- iu_2\Psi^{(0,1)}_{0,1,1,0,1,1}\left(u_2\right) - \frac{2}{3}iu_2\Psi^{(0,1)}_{1,0,0,1,1,1}\left(u_1\right) - \frac{4}{3}iu_2\Psi^{(0,1)}_{1,0,1,0,0,2}\left(u_1\right) \\
&- \frac{2}{3}iu_2^2\Psi^{(0,1)}_{1,1,0,1,1,1}\left(u_1,u_2\right) - \frac{4}{3}iu_2^2\Psi^{(0,1)}_{1,1,1,0,0,2}\left(u_1,u_2\right) - \frac{1}{3}iu_2\Phi^{(1,0)}_{0,1,0,1,0,2}\left(u_2\right) \\
&- \frac{5}{3}iu_2\Phi^{(1,0)}_{0,1,0,1,2,0}\left(u_2\right) - iu_2\Phi^{(1,0)}_{0,1,1,0,1,1}\left(u_2\right) - \frac{1}{3}iu_2\Phi^{(1,0)}_{1,0,0,1,1,1}\left(u_1\right) \\
&- \frac{1}{3}iu_2\Phi^{(1,0)}_{1,0,1,0,0,2}\left(u_1\right) - \frac{1}{3}iu_2\Phi^{(1,0)}_{1,0,1,0,2,0}\left(u_1\right) - \frac{1}{3}iu_2^2\Phi^{(1,0)}_{1,1,0,1,1,1}\left(u_1,u_2\right) \\
&- \frac{1}{3}iu_2^2\Phi^{(1,0)}_{1,1,1,0,0,2}\left(u_1,u_2\right) - \frac{1}{3}iu_2^2\Phi^{(1,0)}_{1,1,1,0,2,0}\left(u_1,u_2\right) - iu_2\Psi^{(1,0)}_{0,1,0,1,1,1}\left(u_2\right) \\
&- \frac{5}{3}iu_2\Psi^{(1,0)}_{0,1,1,0,0,2}\left(u_2\right) - \frac{1}{3}iu_2\Psi^{(1,0)}_{0,1,1,0,2,0}\left(u_2\right) - \frac{1}{3}iu_2\Psi^{(1,0)}_{1,0,0,1,0,2}\left(u_1\right) \\
&- \frac{1}{3}iu_2\Psi^{(1,0)}_{1,0,0,1,2,0}\left(u_1\right) - \frac{1}{3}iu_2\Psi^{(1,0)}_{1,0,1,0,1,1}\left(u_1\right) - \frac{1}{3}iu_2^2\Psi^{(1,0)}_{1,1,0,1,0,2}\left(u_1,u_2\right) \\
&- \frac{1}{3}iu_2^2\Psi^{(1,0)}_{1,1,0,1,2,0}\left(u_1,u_2\right) - \frac{1}{3}iu_2^2\Psi^{(1,0)}_{1,1,1,0,1,1}\left(u_1,u_2\right)
\end{aligned}
$$

4 Outlook

Within this thesis the explicit solutions of the normalization conditions of the elliptic case in $\mathbb{C}^{2\times 2}$ have been described. These individual conditions are the foundation on which the Theorems of chapter 2.6 are based.
Furthermore, the code which has been created to solve the individual normalizing conditions has been made adaptable such that not only the elliptic case has been solved, but also the parabolic and hyperbolic cases.
To do this, the input matrices simply need to be changed to the matrices of the case one would like to study, based on the following information

For the hyperbolic case the equation

$$\begin{aligned} Q_{-1} : v_1 &= |z_1|^2 - |z_2|^2 \\ v_2 &= z_1\bar{z_2} + z_2\bar{z_1} \end{aligned} \tag{4.1}$$

can be rewritten as

$$\begin{aligned} Q_{-1} : v_1 &= \begin{pmatrix} \bar{z_1} & \bar{z_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ v_2 &= \begin{pmatrix} \bar{z_1} & \bar{z_2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{aligned} \tag{4.2}$$

where the input matrices are $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. As for the parabolic case the general equations take the form

$$\begin{aligned} Q_0 : v_1 &= |z_1|^2 \\ v_2 &= z_1\bar{z_2} + z_2\bar{z_1} \end{aligned} \tag{4.3}$$

where using matrix notation yields

$$\begin{aligned} Q_0 : v_1 &= \begin{pmatrix} \bar{z_1} & \bar{z_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ v_2 &= \begin{pmatrix} \bar{z_1} & \bar{z_2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \end{aligned} \tag{4.4}$$

Thus the matrices are $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

By computing also the outputs for the other two cases, the space of manifolds in $\mathbb{C}^{2\times 2}$ has been covered, since in $\mathbb{C}^{2\times 2}$ there always exists an isomorphism which maps to one of the cases above.

It is of interest to not only solve the normalizing conditions listed in this thesis, but also to look at the results of the $\mathcal{L}$ operator used to prove Theorem 1. To do this the coordinate transformation needs to be solved, which can be done based on the construction of the operator $\mathcal{L}$. A result for this would finish the cases in $\mathbb{C}^{2\times 2}$.
Another fascinating aspect is what happens in higher dimensions where $n \geq 3$ and/or $d \geq 3$. Particularly cases where $n \neq d$ would be of great interest to analyse, as they might show further aspects on this topic. This task poses of higher complication since the manifolds cannot be classified into cases as in the $n = 2 = d$ situation.

Bibliography

[1] M. Salah Baouendi, Peter Ebenfelt, and Linda Preiss Rothschild. *Real Submanifolds in Complex Space and Their Mappings (PMS-47)*. Princeton University Press, 2016.

[2] GR Belitskii. Invariant normal forms of formal series. *Functional Analysis and Its Applications*, 13(1):46–47, 1979.

[3] V. Beloshapka. Construction of the normal form of the equation of a surface of high codimension. *Mathematical notes of the Academy of Sciences of the USSR*, 48:721–725, 1990.

[4] S. S. Chern and J. K. Moser. Real hypersurfaces in complex manifolds. *Acta Mathematica*, 133(none):219 – 271, 1974.

[5] Vladimir V. Ežov and Gerd Schmalz. Holomorphic automorphisms of quadrics. *Math. Z.*, 216(3):453–470, 1994.

[6] E. Fischer. Über die Differentiationsprozesse der Algebra. *J. Reine Angew. Math.*, 148:1–78, 1918.

[7] Martin Kolar, Ilya Kossovskiy, and Dmitri Zaitsev. Normal forms in cauchy-riemann geometry. *Analysis and Geometry in Several Complex Variables*, 681:153, 2015.

[8] Bernhard Lamel and Laurent Stolovitch. Convergence of the Chern-Moser-Beloshapka normal forms. *J. Reine Angew. Math.*, 765:205–247, 2020.

[9] Bernhard Lamel and Frances Teichmann. Hyperlink to access the updated wolfram mathematica code: https://complex.univie.ac.at/research/student-research/normalform-conditions/.

Zusammenfassung

Diese Arbeit befasst sich mit Chern-Moser-Beloshapka-Normalformen im $\mathbb{C}^{2\times2}$. Am Anfang wird dieses Thema mit einer Beschreibung des Fischer Inneren Produkts, der Gruppe von Isomorphismen in $\mathbb{C}^{2\times2}$ und dem Gebiet der Normalformen eingeleitet. In den weiteren Kapiteln werden die Resultate von *Convergence of the Chern–Moser–Beloshapka normal forms*, von Bernhard Lamel and Laurent Stolovitch hervorgehoben, welche die Basis für die Forschung dieser Arbeit sind. Um zu den Ergebnissen im $\mathbb{C}^{2\times2}$ zu gelangen ist ein Wolfram Mathematica Programm geschrieben worden, mit dem die Normalisierungsbedingungen vom elliptischen Fall im $\mathbb{C}^{2\times2}$ gelöst worden sind. Durch eine Adaption des Programms ist es auch möglich die beiden anderen Fälle, nämlich hyperbolisch und parabolisch, zu analysieren.