Nonstandard Logicism

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This is an Accepted Manuscript of a book chapter published by Routledge/CRC Press in [*The Routledge Handbook of Logical Empiricism*] on [2021], available online: https://www.routledge.com/The-Routledge-Handbook-of-Logical-Empiricism/Uebel-Limbeck-Lilienau/p/book/9781138122000

Logicism was a dominant position in the foundations of mathematics of the late nineteenth and early twentieth century. Roughly put, it is the view that pure mathematics is reducible to higher-order logic. More specifically, the logicist thesis is usually taken to consist of two claims. First, all primitive terms of an axiomatized mathematical theory can be explicitly defined by using only logical vocabulary. Second, all axioms of the theory can be deduced from purely logical principles. It follows from these two claims that all theorems of a mathematical theory are also derivable from purely logical principles. Let us call this the classical or standard logicist thesis.

It is well known that the pioneering logicists Frege, Russell, Whitehead as well as subsequent philosophers such as Ramsey or Carnap defended variants of this view. However, the contributions of second generation logicists often differed from classical logicism in important respects, in particular concerning (i) the mathematical theories considered, (ii) the logical principles adopted, and (iii) the very concept of a logicist reduction. Thus, based on different accounts of what is meant by "logic," "mathematics," and "reducible," one can identify a number of nonstandard theories of logicism developed in the 1920s and later on.

Logicism should thus not be viewed as a monolithic research program, but rather as a family of different approaches on how the general project of reducing mathematics to logic can be made precise. The focus of this entry will be on different theories of logicism developed in the heyday of logical empiricism, that is, roughly between 1920 and 1940. The central aim here is to survey how

[1] Research on this project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 715222).

Frege's and Russell's logicist programs were modified in the period in question. The changes concern not only formal details of the underlying logic such as the adoption of a simple theory of logical types, but also the kind of mathematical theories considered for the reduction to logic. Whereas classical logicism focused mainly on arithmetic, logical empiricists such as Carnap and Hahn were interested in a generalized logicist thesis which is applicable to any axiomatic theory of pure mathematics.

Logicism and type theory

The origins of classical logicism can be traced back to foundational work in nineteenth century mathematics, in particular on the arithmetization and rigorization of analysis. Frege's own work on the logicist reduction of arithmetic is usually considered as a natural consequence of this line of foundational research. As is well known, Frege's logicism is developed most systematically in *Grundgesetze der Arithmetik* (1893). Part 1 of the book—titled "Exposition of the Begriffsschrift"—contains a description of the logical system used for the reduction of arithmetic. This is, roughly put, a higher-order logic complemented by a naive theory of sets or, in Frege's terminology, a theory of concept extensions. Frege's central objective in the book was to show that a version of the Dedekind-Peano axioms of arithmetic can be derived from his logical principles and that explicit logical definitions can be given for the primitive arithmetical terms. Unfortunately, this project was doomed to failure given the fact that his naive theory of classes turned out to be inconsistent. In his famous letter of 1902, Russell informed Frege that a class theoretic paradox can be derived from his logical system containing the infamous basic law V.

Subsequent research on logicism in the twentieth century was driven by the attempt to block Russel's paradox as well as related paradoxes based on a theory of logical types. Roughly put, this is a system of higher-order logic that describes the stratification of the logical universe into a hierarchy of typed objects. The inventor and main proponent of such a logic was, of course, Russell. A systematic development of his new system was given in the first edition of *Principia Mathematica* (Russell and Whitehead 1910-13). The ramified type theory presented there was modified substantially in work by a second generation of logicians, including Hilbert and Ackermann, Carnap, Gödel, Tarski, Ramsey, Chwistek, and Quine. Their work led to the *simplification* of type theory as well as to a purely *extensional* treatment of types. Another important modification of Russell and Whitehead's original framework concerns the distinction between the syntax and the semantics. The picture emerging in work by Carnap, Tarski, and others is that of type theory as a formal set theory, i.e. a theory describing a rich and stratified universe of objects.

Simple type theory came to serve as the standard logic in the 1920s and 1930s (see Ferreirós 1999 for detailed overview). Type theoretic systems usually discussed at the time usually contained two important higher-order axioms, namely an axiom scheme for comprehension and an axiom scheme of extensionality which states that properties are identical if they are co-extensional. In Russell's original presentation, three other axioms were taken to belong to the logical principles. The first one is the multiplicative axiom which is roughly equivalent to the set-theoretic axiom ofchoice. The second one is an axiom of infinity, which states that the there is a countable infinite number of objects in the individual domain of the language. A third axiom, relevant only for the original ramified treatment of types, is the axiom of reducibility.

With the adoption of type theory as a way to block the set-theoretic paradoxes deducible from Frege's logical system, it is easy to see how classical logicism can be reformulated in this new framework. A natural way to specify the logicist thesis can be given in terms of the notion of an *interpretation* of a theory in another one. Roughly put, given two axiomatic theories S and T (expressed in languages L_S and L_T respectively), an interpretation of T in S is given by a translation of the formulas in L_T into formulas of L_S that preserves (i) their logical structure of L_T sentences and (ii) the theorems of T. The second condition states that the translation of every theorem of the interpreted theory should also be*provable* from the axioms of the interpreting theory. Applied to the reduction of arithmetic to logic, one can show that the theory of Dedekind-Peano arithmetic (expressed in the second-order language) can be interpreted in type theory (expressed in the purely logical language). More specifically, there exists a translation of all arithmetical statements into purely logical statements based on Frege's logical definitions of the primitive arithmetical terms. This translation is theorem-preserving in the sense that for any arithmetical statement that is provable from the Dedekind-Peano axioms, its purely logical translation is derivable from the principles of type theory.

Logical truth in logical empiricism

Logical empiricism took shape as a philosophical movement in direct reaction to the foundational debates in mathematics at the turn of the last century. In particular, Frege's and Russell's thesis that mathematics is a branch of logic played a central role in the revival of empiricist philosophies in Vienna and elsewhere. While thinkers such as Hahn and Carnap took simple type theory to be the standard logical system for the logicist reduction of mathematics, their accounts differed from classical logicism in central respects. This is mainly due to the fact that, at their time, the conception of logic was subject to significant revision, largely in response to Wittgenstein's *Tractatus Logico-Philosophicus* (1922). Wittgenstein proposed a new analysis of logical truth in terms of the notion of a tautology. Such truths do not assert facts about the world, but concern only the logical form of statements. This new conception of logic as a systems of tautologies marks a sharp break with previous accounts, in particular the logical universalism shared by Frege and Russell.

As is well known, the members of the Vienna Circle wholeheartedly adopted the *Tractatus* conception of logic. How did the new understanding of logic as a set of tautologies transform the way in which logicism was understood by them? On first glance, both Carnap and Hahn seem to have embraced classical logicism in the sense outlined above. This is evident in a number of publications from the time, for instance, in several articles written by Carnap around 1930 that explicitly discuss Frege's and Russell's logicism (e.g., 1931). However, an important difference to their program becomes visible if one considers how the "fundamental logical sentences" are understood in these writings. Instead of characterizing logical laws as universal truths, they are conceived by Carnap as tautologies in Wittgenstein's sense. Consequently, assuming that mathematics is reducible to logic, it follows that all mathematical theorems are also purely tautological in character. This account of the nature of higher mathematics was widely shared among members of the Vienna Circle, as the following passage in the circle's official manifesto of 1929 indicates: "The conception of mathematics as tautological in character, which is based on the investigations of Russell and Wittgenstein, is also held by the Vienna Circle." (Verein Ernst Mach 1929/2012: 85)

Now, both Hahn and Carnap were aware that this revised logicist thesis built on the notion of logical tautology is problematic. It is difficult to see how logical type theory (let alone theories of higher mathematics) can be tautological in character, that is, without any ontological commitments. How did the logical empiricists aim to vindicate the logicist thesis in light of this fact? Two lines of reasoning can be mentioned here. The first concerns different attempts to *generalize* the notion of a tautology in order to make it applicable to type-theoretic logic and a fortiori also to mathematics. Hahn fully embraced the *Tractatus*-style conception of logic in his philosophical writings from the 1920s and early 1930s (e.g. 1929). In particular, he defended the view that logical laws concern only the logical form of statements and have no representational function. However, in contrast to Wittgenstein's "thin" account of tautologies, Hahn was interested in formulating a "wider" conception of tautologies (1933). This is based on the fact that he, in contrast to Wittgenstein, adopted an early conventionalism about the choice of logical principles (see Uebel 1995). Thus, Hahn argued that one can freely adopt our logical system for the study of inferences in our language. This includes the possibility to adopt set-theoretic principles such as the axioms of infinity or choice. Consequently, what counts as a valid tautological transformation is specified relative to the particular choice of a logical framework.

A second strategy to vindicate logicism is also based on the reassessment of the logical status of certain axioms of type theory. As is well known, Russell's and Whitehead's axioms of choice, infinity, and reducibility were viewed critically by many proponents of logicism, including also philosophers of logical empiricism. Consider, for instance, Carnap's *Abriss der Logistik*, where the status of these axioms is discussed in detail. As Carnap points out, neither the axioms of choice nor infinity "should not be included among the basic principles of logic, since its admissibility has been problematic" (1929, §24b). The decidedly existential and thus non-logical status of these axioms thus presented a central problem to type-theoretic logicism. It was clear to Carnap and others that these axioms are independent of the other logical principles of type theory but at the same time indispensable for the logicist reduction of mathematics. A possible solution to this problem was to view these axioms not as proper logical principles but rather as hypothetical assumptions in logical reasoning. More specifically, that central idea here was to reformulate those mathematical statements whose proof depends on these axioms in terms of conditional statements where the axioms in question occur in the antecedent.

This "conditional" logicism presents a weakened form of classical logicism that can be found in several works of the time (see Musgrave 1977 and Coffa 1981). The first systematic formulation of this approach was given in Russell's *Introduction to Mathematical Philosophy* (1919). This approach was also adopted by several members of logical empiricism. Carnap, in particular, formulated variants of the strategy to conditionalize mathematical statements in his writings from the period in question. Thus he wrote: "[Russell] . . . transformed a mathematical sentence, say *S*, the proof of which required the axiom of infinity, *I*, or the axiom of choice, *C*, into a conditional sentence; hence *S* is taken to assert not *S*, but $I \supset S$ or $C \supset S$, respectively. This conditional sentence is then derivable from the axioms of logic." (1931/1964: …)

The central motivation for this logical reconstruction was to reduce mathematics to logic without having to assert the logical truth of existential axioms such as choice and infinity. Applied to the program of reducing arithmetic to logic, this method yields a nonstandard form of logicism: arithmetical statements are also translated into purely logical statements here, but not based on explicit "logicist" definitions of the primitive terms. Instead, they are translated into conditional statements in the language of type theory.

If-thenism and general axiomatics

Logical empiricists including Hahn and Carnap formulated variants of a "conditional logicism" based on the critique of the non-tautological nature of axioms such as infinity and choice. Interestingly, the if-thenist reconstruction of mathematical statements is also adopted to generalize the logicist thesis in a different way, namely to make it applicable to non-arithmetical theories in mathematics. This approach is again rooted in Russell's foundational work.

In his *Principles of Mathematics* (1903), the method of conditionalization was originally introduced in the discussion of non-Euclidian geometries (see Gandon 2009). Russell argued that the axioms of mutually inconsistent geometrical theories should not be viewed as assertative statements but rather as hypothetical claims about possible structures of space. Geometrical theorems, in turn, are to be expressed as quantified conditional statements that contain the *ramsified* axioms as the antecedent. This logical reconstruction has an important consequence: a geometry, conceived now as class of conditional statements, is expressible in a pure logical language and thus does not express any factual content about the world. Russell was well aware of this fact and suggested if-thenism as a natural approach to describe theories of pure mathematics: "Pure mathematics is the class of all propositions of the form '*p* implies *q*', where *p* and *q* are propositions containing one or more variables, the same in two propositions, and neither *p* nor *q* contains any constants except logical constants." (1903: 3) This logical reconstruction exercised an important influence on the philosophers of logical empiricism. In particular, independent of the classical logicist project to reduce arithmetic to a firm logical basis, Russell's account was viewed by Carnap and Hahn, among others, as a way to capture modern axiomatic reasoning in non-arithmetical branches of mathematics.

Carnap, in particular, did not view logicism and formal axiomatics as opposing programs in the foundations of mathematics. One can view his early work on the philosophy of mathematics as an attempt to reconcile modern axiomatic mathematics with a generalized version of the Fregean or Russellian logicist thesis (compare Awodey and Carus 2001 and Reck, 2004). This is most explicit in his work on "general axiomatics" from the late 1920s. In particular, in Part II of Carnap's *Abriss*— titled "Applied Logistic"—Carnap suggests the following type-theoretic formalization of axiomatic theories: the primitive terms of a theory are expressed as free variables (each of a given arity and type). Axioms and theorems are expressed as sentential functions, that is, as open formulas in modern sense. Carnap argues that an axiomatic theory does not only give an *implicit definition* of the primitive terms occurring in the axioms in the sense specified by Hilbert, but also an explicit definition of a higher-order concept, the so-called *Explizitbegriff* of an axiom system. More specifically, he holds that:

> For instance, if $x, y, \ldots \alpha, \beta, \ldots P, Q, \ldots$ are the primitive variables of the AS and if we name the conjunction of axioms (that is a propositional function) AS($x, y, \ldots \alpha, \beta, \ldots P, Q, \ldots$), then the definition of the explicit concept of this AS is:
>
> $$\hat{x}, \hat{y}, \ldots \; \alpha, \beta, \ldots \hat{P}, \hat{Q}, \ldots \{AS(x, y, \ldots \alpha, \beta, \ldots P, Q, \ldots)\}$$
>
> (1929: 72)

How is this approach of formalizing axiomatic theories related to the if-thenism described in Russell's *Principles of Mathematics*? Interestingly, Carnap's understanding of mathematical statements is highly similar to Russell's in this respect. While the if-thenist reconstruction is not mentioned in *Abriss*, Carnap explicitly discusses it in a related paper titled "Proper and Improper Concepts" (1927). He argues there that the mathematical content of a theorem is best expressed by a closed formula, namely a quantified conditional statement that contains the "logical product" of the axioms of a given theory in the antecedent. The theorems of a given theory are thus to be translated into purely logical statements of the form:

$$\forall x, y, \ldots \alpha, \beta, \ldots P, Q, \ldots [AS(x, y, \ldots \alpha, \beta, \ldots P, Q, \ldots) \rightarrow \varphi(x, y, \ldots \alpha, \beta, \ldots P, Q, \ldots)]$$

where variables $x, y, \ldots \alpha, \beta, \ldots P, Q, \ldots$ present the primitive vocabulary of the theory, *AS* presents the axioms of a theory, and φ the ramsified theorem in question. (This if-thenist construction can also be found in the writings of other logical empiricists, e.g. Hempel 1945.)

A central philosophical motivation underlying Carnap's adoption of this Russelian if- thenism was to defend some form of non-classical logicism. This generalized version of a logicist reduction is usually characterized in the modern literature in terms of two conditions (Musgrave 1977: 117-8):

(1) All mathematical statements have the logical form of conditional statements with the logical product of the axioms in the antecedent and a ramsified theorem in the consequent.

(2) All true mathematical statements are derivable from logical axioms.

The first condition states that all mathematical statements can be reformulated in purely logical terms. This language logicism corresponds to the weak logicist thesis discussed in Carnap's *Abriss*. Given that the explicit concept of an axiom system can be expressed in purely logical terms, it follows that any mathematical theory (including non-arithmetical ones such as geometry or topology) "can be represented as a branch of logistic itself" (1929: …). Moreover, any theorem of a given theory can be translated into a purely logical sentence based on the if-thenist reconstruction.

As pointed out above, this condition is usually accompanied by a second thesis, namely that all true mathematical statements so construed become derivable from the logical (i.e. type theoretic) axioms in question. Expressed in modern logical terminology, (2) states that the if-thenist translation is also theorem-preserving, that is, it induces an *interpretation* of a mathematical theory in type theory in the sense specified earlier. Although this second thesis of conditional logicism is usually not described explicitly in published work from the 1920s and early 1930s, it is likely that this view shared by Carnap and his fellow logical empiricists. (A version of this thesis can be found in his (2000).) Obviously, this if-thenism presents a weaker form of logicism than Frege's and Russell's original programs. In particular, what is missing here are explicit logicist definitions of the primitive terms of a mathematical theory. Moreover, mathematical axioms are not supposed to be derived from purely logical principles in the present account. What is derived from the principles of type theory are the ramsified conditional statements described above. Thus, the conditional logicism effectively shows that all proofs of theorems can be formalized within a general type-theoretic system. This is given by the fact that for any axiomatic mathematical theory *A* and every statement φ in the language L_A, the following equivalence holds:

$$TT \cup \{A\} \vdash \varphi \leftrightarrow TT \vdash \forall \vec{X} (A(\vec{X}) \rightarrow \varphi(\vec{X}))$$

Thus, whenever a statement is derivable from theory *A*(plus the logical axioms of *TT*, then the universal ramsification of ($A \rightarrow \varphi$) is derivable from the logical axioms alone (compare again Musgrave 1977 and Coffa 1981).

Logical pluralism

Logicism lost much of its philosophical significance in the course of the 1930s, mainly as a result of Gödel's incompleteness results. Roughly put, Gödel's results show that arithmetical truth cannot be identified with logical provability. This was a serious blow for the traditional logicist thesis that arithmetic is reducible to higher-order logic. A second reason for the gradual demise of the Frege's and Russell's program was that the scope of logic changed significantly in the period in question. First-order logic was eventually established as the standard logical system and replaced the logical theory of types. Moreover, logic also underwent a metatheoretic turn in work by Tarski, Carnap, and Gödel (among others). The new metalogical approach and the clear syntax-semantics distinction implied by it was clearly incompatible with the logical universalism present in the work of Frege, Russell, but also in Wittgenstein's *Tractatus*.

While logicism was challenged by these developments, it would be wrong to conclude that it was given up at the time. In fact, it remained a central position in work by philosophers affiliated with logical empiricism well after the 1930s. This is true, in particular, of Carnap's work. His project on general axiomatics—originally devised as two volumes of *Untersuchungen zur* *allgemeinen Axiomatik*—was eventually abandoned in 1930, mainly in response to Tarski's metatheoretic definition of notions such as categoricity, truth, and logical consequence (see Awodey and Carus 2001). As is well known, Tarski emphasized the distinction between the formulation of axiomatic (or "deductive") theories in an object language and the specification of their metatheoretic properties in a separate and richer metalanguage. Given this background, Carnap eventually developed a similar approach that presented a sharp break with Wittgenstein's position. Moreover, influenced by a correspondence with Gödel in 1932, he also adopted a purely syntactic conception of logic and, more importantly, the idea of logical syntax as the study of metalinguistic properties of "logico-mathematical" languages. This new approach culminated in his *Logical Syntax of Language*, first published in 1934.

Carnap's work on logical syntax is marked by a number of important innovations. First, his account of logico-mathematical systems is decidedly metatheoretic: mathematical theories such as arithmetic are presented axiomatically in a fully specified object language. In the case of Peano arithmetic this is the higher-order language of simple types L_{II}. As in Tarski's work, the formation and transformation rules of this language are expressed in separate syntax language. Given this setup, a central syntactic concept introduced in *Logical Syntax* is the notion of analyticity which is explicitly introduced by Carnap as an explication of logical truth. Thus, Wittgenstein's notion of tautological truth is replaced here by a decidedly metatheoretical concept. Roughly put, analyticity for sentences in L_{II} is defined analogously to Tarski's treatment of formal truth, namely in terms of several recursive clauses for the valuation (or satisfaction) of open formulas.

The most significant innovation in Carnap's book is the fact that logical universalism of the traditional logicists is replaced by a form of logical pluralism. According to this view, there exist no unique or correct logic. Rather, according to Carnap, one can freely choose between different logical systems for the task of formalizing mathematics or the sciences. The different frameworks are equally valid or acceptable. Moreover, the choice between them should be based purely on pragmatic or instrumental considerations. Carnap's adoption of this logical pluralism is best expressed in his famous remark on the *principle of tolerance*: "In logic, there are no morals.

Everyone is at liberty to build up his own logic, i.e. his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments." (1934/1937: 52) The principle expressed here also presents a fundamental break with the *Tractatus*-style conception of logic. It is no longer the case that the nature and role of logic is ultimately grounded in metaphysical considerations concerning the relation between our language and the world. Instead, different logical systems can be adopted for different theoretical purposes and studied by metatheoretical means.

Given Carnap's new framework for the study of logic and mathematics, the question arises what residual role is assigned to logicism in *Logical Syntax* (see Friedman 1999 and the papers in Wagner 2009). Frege's and Russell's original program of reducing mathematics to higher-order logic is discussed in §84 of the book. However, Carnap clearly does not ascribe to it the importance he did in his pre-*Syntax* work. In fact, the study of the logical syntax of formal languages is viewed here as a way to reconcile logicism with other foundational views, in particular Hilbert's program. As a consequence, the particular understanding of the classical logicist project is significantly changed.

Notice first that Carnap is still working with a simplified version of Russell's logical theory of types in 1934. Type theory is expressed here in language L_{II} which, in contrast to his previous work on general axiomatics, is explicitly treated as an object language now. Surprisingly, L_{II} is no longer considered for the logical reduction of arithmetic. This is due to the fact that all arithmetical terms are already contained as primitive signs in the language. Moreover, the axioms of Peano arithmetic are not supposed to be deducible from the logical principles of type theory here, but they already belong to axiom base of the logical system (1934/1937: §30). Thus, Peano arithmetic is no longer interpreted in type theory, it is now taken to be a part of it. Given Carnap's account of "logico-mathematical systems," classical logicism obviously becomes irrelevant in this context. The logicist project of reducing mathematics to logic is replaced here by the more general project of showing that both logic and mathematics can be characterized as analytic in Carnap's new sense of the term.

A second issue that distinguishes Carnap's approach in *Logical Syntax* from classical logicism relates to his logical pluralism. We have seen that the definition of the notion of analyticity given in 1934 is a relative one: analytic truth, understood as an explication of logical or tautological truth, is specified relative to a particular language or logical system. Which mathematical statements count as analytic therefore depends on the prior choice of a logical system with its formation and transformation rules. For instance, statements of classical analysis will turn out as analytic relative to the type theoretic system L_{II}, but not analytic relative to the weaker language L_I of primitive recursive arithmetic also discussed in the book. Since one is free to choose between such logical frameworks, it follows that whether certain branches of mathematics count as logical also becomes a question of pragmatic choice. Thus, given Carnap's new principle of logical tolerance, the logicist reduction of mathematics to logic is no longer "a question of philosophical significance, but only one of technical expedience" (ibid.: §84). To show that the logicist thesis holds, it suffices to adopt a sufficiently strong background system that (i) either contains the mathematical axioms and primitive terms in question or (ii) that allows one to deduce these axioms in terms of sufficiently strong transformation rules.

References

Awodey, S. and Carus, A. (2001) "Carnap, completeness, and categoricity: the *Gabelbarkeitssatz* of 1928," *Erkenntnis* 54: 145–172.

Carnap, R. (1927) "Eigentliche und uneigentliche Begriffe," *Symposion* 1: 355– 374.

----- (1929) *Abriss der Logistik*, Vienna: Springer.

----- (1931) "Die logizistische Grundlegung der Mathematik," *Erkenntnis*, 2: 91–105. Trans. "The Logicist Foundations of Mathematics" in P. Bernacerraf and H. Putnam (eds.), *The Philosophy of Mathematics*, Englewood Cliffs: Prentice-Hall, 1964, pp. 31-41

----- (1934) *Logische Syntax der Sprache*, Vienna: Springer. Trans. *The Logical Syntax of Language*, London: Kegan, Paul, Trench Teubner & Cie, repr. Chicago: Open Court, 2002.

----- (2000) *Untersuchungen zur allgemeinen Axiomatik*, Darmstadt: Wissenschaftliche Buchgesellschaft.

Coffa, A. (1981) "Kant and Russell," *Synthese* 46: 247–263.

Ferreirós, J. (1999) *Labyrinth of Thought - a History of Set Theory and its Role in Modern Mathematics*, Basel: Birkhäuser.

Frege, G. (1893) *Grundgesetze der Arithmetik. I. Band*. Hildesheim: Georg Olms Verlagsbuchhandlung. Trans. *The Basic Laws of Arithmetic*, Oxford: Oxford University Press, 2013.

Friedman, M. (1999) "Tolerance and Analyticity in Carnap's Philosophy of Mathematics," in Friedman, *Reconsidering Logical Positivism*. Cambridge: Cambridge University Press, pp. 198-233.

Gandon, S. (2009). Toward a topic-specific logicism? Russell's theory of Geometry in the *Principles of Mathematics*," *Philosophia Mathematica*, 17: 35–72.

Hahn, H. (1929) "Empirizismus, Mathematik, Logik," *Forschungen und Fortschritte* 5. Trans. "Empiricism, Mathematics, Logic" in Hahn, *Empiricism, Logic and Mathematics* (ed. by B. McGuinness), Dordrecht: Reidel, 1980, pp. 20-30.

----- (1933) Logik, Mathematik, Naturerkennen, Vienna: Gerold. Trans. "Logic, Mathematics and Knowledge of Nature" in B. McGuinness (ed.) *Unified Science*, Dordrecht: Reidel, 1987, pp. 24-45

Hempel, G. (1945) "On the nature of mathematical truth," *The American Mathematical Monthly* 52: 543–56.

Musgrave, A. (1977) "Logicism revisited," *British Journal of Philosophy of Science*, 28:99–127.

Reck, E. (2004) "From Frege and Russell to Carnap: Logic and Logicism in the 1920s," In S. Awodey and C. Klein (eds.), *Carnap Brought Home: The View from Jena*, Chicago: Open Court, pp. 151-180.

Russell, B. (1903) *The Principles of Mathematics*, London: Routledge.

Russell, B. (1919) *Introduction to Mathematical Philosophy*. London: George Allen & Unwin.

Russell, B. and Whitehead, N. A. (1910-13) *Principia Mathematica, 3 vols.,* Cambridge: Cambridge University Press, 2nd ed. 1927.

Uebel, T. (2005) "Learning Logical Tolerance: Hans Hahn on the Foundations of Mathematics," *History and Philosophy of Logic* 26: 175–209.

Verein Ernst Mach (ed.) (1929) *Wissenschaftliche Weltauffassung. Der Wiener Kreis*, Vienna: Wolf. Repr. and trans. "The Scientific World-Conception. The Vienna Circle" in F. Stadler and T. Uebel (eds.), *Wissenschaftliche Weltauffassung. Der Wiener Kreis. Hrsg. vom Verein Ernst Mach (1929)*, Vienna: Springer, 2012, pp. 75-115. (Part trans. "The Scientific Conception of the World. The Vienna Circle" in O. Neurath, *Empicism and Sociology*, Dordrecht: Reidel, 1973, pp. 299-318.)

Wagner, P. (ed.) (2009) *Carnap's Logical Syntax of Language*, Basingstoke: Palgrave Macmillan.

Wittgenstein, L. (1922). *Tractatus Logico-Philosophicus*, London: Routledge and Kegan Paul.