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"On the Index of Saturated Equilibria"

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Abstract

This Master's Thesis provides a (geometrical) formula to determine the boundary index of an isolated, saturated equilibrium of a vector field in the plane. It is assumed that the non-negative orthant (or other state spaces, e.g., the probability simplex) is forward invariant and the equilibrium is part of the boundary of $\mathbb{R}^2_+$.

In the beginning, Brouwer degree theory is used to introduce the general concept of the index of an equilibrium. Then, we focus on biological systems and the boundary index, in particular we give a proof of the boundary index theorem by Hofbauer on the simplex. The first chapter is completed by a study of plane vector fields and the concept of rotation, which happens to be equivalent to the degree.
In the second chapter we prove an analogue of Bendixsons's index formula for the boundary index of an isolated, saturated equilibrium in the plane. We continue with a superficial treatment of homogeneous vector fields and conclude with a boundary index formula for non-degenerate systems.

Zusammenfassung

Diese Masterarbeit liefert eine (geometrische) Formel um den Randindex eines isolierten, saturierten Gleichgewichts eines Vektorfeldes in der Ebene zu bestimmen. Es wird angenommen, dass der erste Quadrant (oder andere Zustandsräume, z.B., der Wahrscheinlichkeits-Simplex) forwärts invariant belassen wird und das Gleichgewicht Teil des Randes von $\mathbb{R}^2_+$ ist.

Zu Beginn wird die Theorie des Abbildungsgrades von Brouwer verwendet um das generelle Konzept des Index eines Gleichgewichtes einzuführen. Als nächstes legen wir den Fokus auf biologische Systeme und den Randindex, insbesondere werden wir das Randindex-Theorem von Hofbauer am Simplex beweisen. Das erste Kapitel wird vervollständigt mit einer Untersuchung von Vektorfeldern in der Ebene und dem Konzept der Rotation, die wie sich zeigt äquivalent zum Abbildungsgrad ist.
Im zweiten Kapitel beweisen wir ein Analogon zur Indexformel von Bendixson für den Randindex eines isolierten, saturierten Gleichgewichts in der Ebene. Wir setzen fort mit einer oberflächlichen Betrachtung homogener Vektorfelder in der Ebene und beschließen mit einer Randindex-Formel für nicht degenerierte Systeme.

Contents

1 Mapping Degree and Index

1.1 Brouwer Degree

Consider a continuous function $f : [a, b] \to \mathbb{R}$ defined on a closed interval $[a, b]$. If $f(a)f(b) \leq 0$, then there exists some $c \in [a, b]$ such that $f(c) = 0$. This is a special case of the intermediate value theorem. It is mainly used to prove the existence of solutions of equations. When it comes to differential equations this may be employed to prove the existence of equilibria. However, the statement is restricted to one dimension. In order to generalize it, we follow [16, Chapter 12] and start by introducing the mapping degree of Brouwer. First, we will focus on the generic situations. In a second step, the range of applications will be enlarged by approximation arguments. To define what we mean by generic, we introduce the following notion of regularity.

Definition 1.1 (Regular Point)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded, $f : \overline{U} \to \mathbb{R}^n$ a function continuous on $\overline{U}$ and continuously differentiable on U. If $x \in U$ and $\det(Df|_x) \neq 0$, where $Df := (\frac{\partial f_i}{\partial x_j})_{1 \leq i,j \leq n}$ is the Jacobian of f, we say that x is a regular point of f.*

With this definition we can talk about the regularity of points in the domain of f. In the following we will do the same for the image.

Definition 1.2 (Regular Value)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded, $f : \overline{U} \to \mathbb{R}^n$ a function continuous on $\overline{U}$ and continuously differentiable on U. A point $y \in \mathbb{R}^n$ is called a regular value of f, if $y \notin f(\partial U)$ and every point in the preimage of y, with respect to f, is regular.*

Now we can define a mapping degree which assigns an integer to a triple containing a function, a domain and a value. This definition is due to Nagumo, see [14].

Definition 1.3 (Brouwer Degree)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded, $f : \overline{U} \to \mathbb{R}^n$ a function continuous on $\overline{U}$ and continuously differentiable on U, y a regular value. Then the integer*

$$\mathrm{Deg}(f, U, y) := \sum_{x \in U : f(x) = y} \mathrm{sgn}(\det(Df|_x))$$

is called the Brouwer degree.

Remark 1.4. If the preimage of y is empty we get $\mathrm{Deg}(f, U, y) = 0$ by definition. To make sure that the degree is well-defined, we have to show that the cardinality of the set $\{x \in U : f(x) = y\}$ is finite, see [15, p. 146]. Assume the contrary. Then there exists a sequence $(x_n)_{n \geq 1}$ with $x_n \neq x_m$ for $n \neq m$ and $f(x_n) = y$.

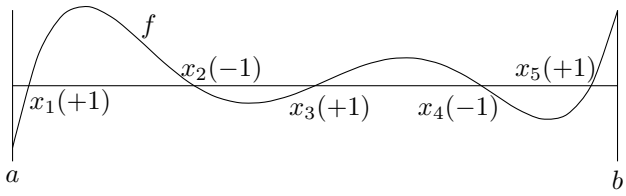


Figure 1: Intermediate value theorem via Brouwer degree

The set $\overline{U}$ is closed and bounded, therefore compact, hence there is a convergent subsequence $(x_{n_k})_{k\geq 1}$ with $\lim_{k\to+\infty} x_{n_k} = \tilde{x} \in \overline{U}$. From the continuity of f on $\overline{U}$ we can infer that $f(\tilde{x}) = y$. Since y is regular, it holds that $y \notin f(\partial U)$ and we can conclude that $\tilde{x} \in U$. Moreover, we deduce from the regularity of y that $\det(Df|_{\tilde{x}}) \neq 0$. The inverse function theorem informs us that there is a neighbourhood V of $\tilde{x}$ such that $f|_V$ is bijective. On the one hand we derived that $f(x_{n_k}) = y$ for all $k \geq 1$ and there exists $k_0 \in \mathbb{N}$ such that $x_{n_k} \in V$ for $k \geq k_0$. On the other hand f is one-to-one on V, so we arrived at a contradiction. Therefore the Brouwer degree is well-defined.

With Definition 1.3 we can also check that the Brouwer degree is normalized (D1), i.e., $\mathrm{Deg}(\mathrm{Id}_{\overline{U}}, U, y) = 1\ \forall y \in U$, where $\mathrm{Id}_{\overline{U}}$ is the identity map on $\overline{U}$. In addition, we see that it inherits a translation invariance property, i.e., $\mathrm{Deg}(f, U, y) = \mathrm{Deg}(f - y, U, 0)$. This we can verify by noticing that $\{x \in U : f(x) = y\} = \{x \in U : f(x) - y = 0\}$ and $Df = D(f(\cdot) - y)$. Moreover it is straight forward to check that if y is a regular value, then $\mathrm{Deg}(f, U, y) \neq 0$ implies the existence of a point $\tilde{x} \in U$ such that $f(\tilde{x}) = y$.
As an example we calculate $\mathrm{Deg}(f, (a, b), 0)$ in Figure 1. Inserting the definition gives $\mathrm{Deg}(f, (a, b), 0) = \sum_{i=1}^{5} \operatorname{sgn} \det Df|_{x_i} = 1 - 1 + 1 - 1 + 1 = +1 \neq 0$. Therefore we get the existence of at least one point $\tilde{x} \in (a, b)$ with $f(\tilde{x}) = 0$. We see that this is the same result we would get from the intermediate value theorem.
We want to drop the restriction that $y \notin f(\partial U)$ needs to be a regular value. A detailed treatment can be found in [15, p. 148]. Basically we use Sard's Lemma, telling us that the set of regular values lies dense in the image of f. So we can define $\mathrm{Deg}(f, U, y) := \lim_{n\to+\infty} \mathrm{Deg}(f, U, y_n)$ for any sequence of regular values y_n approaching y. It can be shown that this definition is independent of the choice of the sequence.
Now we can define the Brouwer degree uniquely by the following three properties. Note that, due to the definition above, we can also allow irregular values.

Proposition 1.5 (Defining properties of the Brouwer degree)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded, $f : \overline{U} \to \mathbb{R}^n$ continuous on $\overline{U}$ and continuously differentiable*

on U, $y \in \mathbb{R}^n \setminus f(\partial U)$. The following three properties define a unique function $\mathrm{Deg}(\cdot,\cdot,\cdot)$, *which assigns an integer* $\mathrm{Deg}(f,U,y)$ *to each triple* (f,U,y).

(D1) $\mathrm{Deg}(\mathrm{Id}_{\overline{U}},U,y) = 1$ *if* $y \in U$.

(D2) Let $U_1, U_2 \subset U$ *be open,* U_1 *and* U_2 *disjoint,* $y \notin f(\overline{U} \setminus (U_1 \cup U_2))$, *then*

$$\mathrm{Deg}(f,U,y) = \mathrm{Deg}(f,U_1,y) + \mathrm{Deg}(f,U_2,y).$$

(D3) Let $f, g : \overline{U} \to \mathbb{R}^n$ *be two functions continuously differentiable on* U *and continuous on* $\overline{U}$. *If there exists a homotopy* $H : [0,1] \times \overline{U}$ *which connects* f *and* g *such that* $y \notin H(\lambda, \partial U)$ *for all* $\lambda \in [0,1]$, *then for all* $\lambda \in [0,1]$

$$\mathrm{Deg}(H(\lambda,\cdot),U,y) = \mathrm{Deg}(f,U,y) = \mathrm{Deg}(g,U,y).$$

We use the following notion of homotopy.

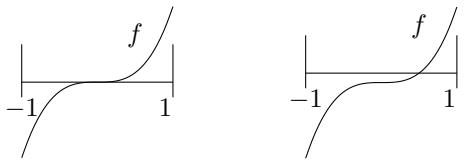
Definition 1.6 (Homotopy)**.** *Let* $f, g : X \to Y$ *be continuous maps between two topological spaces* X *and* Y. *We say that* f *and* g *are homotopic, if there exists a continuous map* $H : X \times [0,1] \to Y$ *with*

$$H(x,0) = f(x) \text{ and } H(x,1) = g(x) \ \forall x \in X.$$

The function H *is called the homotopy (connecting* f *and* g*).*

For a proof of Proposition 1.5 see [6, p. 5].
We have already named the first property in Proposition 1.5. The second one (D2) is called additivity and (D3) will be referred to as homotopy invariance. Property (D3) gives rise to a second approach to calculate the degree, if y is an irregular value. In fact, the homotopy invariance of the Brouwer degree allows us to perturb the function f in such a way that an irregular value becomes regular for the perturbation $\tilde{f}$. If we can connect these two functions via a homotopy H such that $H(0,x) = f(x)$, $H(1,x) = \tilde{f}(x)$ and $y \notin H(\lambda, \partial U)$ for all $\lambda \in [0,1]$, we can use (D3) to arrive at $\mathrm{Deg}(f,U,y) = \mathrm{Deg}(\tilde{f},U,y)$. The last expression is again straight forward to calculate. The following basic example may give some insight.

Example 1.7. Consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$ as in Figure 2a. We see that $y = 0$ is not a regular value, since $f'(0) = 0$. In order to determine $\mathrm{Deg}(f,(-1,1),0)$ we have to use either a perturbation or a sequence of regular values approaching 0. In the first case we can use $H_\epsilon(\lambda,x) = x^3 - \lambda\epsilon$, which is continuous and for $-1 < \epsilon < 1$ it holds that $0 \notin H_\epsilon(\lambda, \partial(-1,1))$ for all $\lambda \in [0,1]$. Choose $\epsilon = \frac{1}{8}$ as in Figure 2b and define $H(\lambda,x) := H_{\frac{1}{8}}(\lambda,x)$. From the homotopy



(a) $H(0,x) = f(x) = x^3$ (b) $H(1,x) = x^3 - \frac{1}{8}$

Figure 2: Degree of an irregular value

invariance we can deduce that $\mathrm{Deg}(f,(-1,1),0) = \mathrm{Deg}(H(0,\cdot),(-1,1),0) = \mathrm{Deg}(H(1,\cdot),(-1,1),0)$. Since the unique zero of $H(1,\cdot)$ is $x_1 = \frac{1}{2}$ and $DH = \frac{\partial H}{\partial x} = 3x^2$ we finally get that $\mathrm{Deg}(f,(-1,1),0) = \operatorname{sgn} DH|_{x_1} = +1$. In order to give an example for the second approach we need a sequence of regular values approaching 0, $(\frac{1}{n})_{n\geq 1}$ will do. From the definition above we see that

$$\begin{aligned}\mathrm{Deg}(f,(-1,1),0) &= \lim_{n\to+\infty} \mathrm{Deg}\left(f,(-1,1),\frac{1}{n}\right) = \lim_{n\to+\infty} \operatorname{sgn} f'\left(\frac{1}{\sqrt[3]{n}}\right)\\ &= \lim_{n\to+\infty} \operatorname{sgn} 3\left(\frac{1}{\sqrt[3]{n}}\right)^2 = +1.\end{aligned}$$

As mentioned above the definition is independent of the choice of the approaching sequence.

It remains to show that with the extension for irregular values, the statement on the existence of preimages remains unchanged. We may collect this result in the following proposition.

Proposition 1.8. *Let $U \subset \mathbb{R}^n$ be open and bounded, $f : \overline{U} \to \mathbb{R}^n$ a function continuous on $\overline{U}$ and continuously differentiable on U, $y \notin f(\partial U)$. Then, $\mathrm{Deg}(f,U,y) \neq 0$ implies the existence of a point $\tilde{x} \in U$ with $f(\tilde{x}) = y$.*

Proof. The case where y is a regular value is already done. Let y be irregular. From Sard's Lemma we get that there is a sequence $(y_m)_{m\geq 1}$ of regular values with $\lim_{m\to+\infty} y_m = y$. It follows from $0 \neq \mathrm{Deg}(f,U,y) = \lim_{m\to+\infty} \mathrm{Deg}(f,U,y_m)$ that there is $m_0 \in \mathbb{N}$ such that $\mathrm{Deg}(f,U,y_m) \neq 0$ for all $m > m_0$. Since $(y_m)_{m\geq m_0}$ is a sequence of regular values there is a corresponding sequence $(x_m)_{m\geq m_0} \in U$ of regular points such that $f(x_m) = y_m$. From the compactness of $\overline{U}$ and the continuity of f we can again, as in Remark 1.4, conclude that $\tilde{x} \in U$ and $f(\tilde{x}) = y$. □

We have now shown that the Brouwer degree enables us to generalize the statement of the intermediate value theorem to higher dimensions. Yet, we do not have a sufficiently easy way to calculate the degree. Recall that in Definition 1.3 it is even necessary to know the preimage of the value of interest in advance.

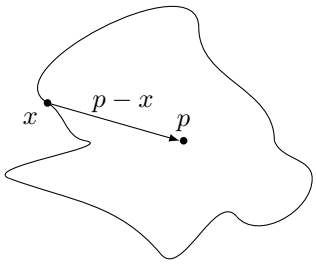


Figure 3: Degree on a star-shaped domain

Having this in mind, Proposition 1.8 is not revealing anything. The power of the intermediate value theorem, however, is the fact that it is sufficient to know the value of f on the two boundary points of the closed interval $[a, b]$. Therefore we are interested in statements, which draw conclusions towards the degree, without a priori knowledge of f in the interior of U.
To this end, we can use the homotopy invariance property. Given two continuously differentiable functions $f, g : U \to \mathbb{R}^n$, such that $f = g$ on ∂U and $f(\partial U) \neq y$, we get with $H(\lambda, x) = \lambda f(x) + (1 - \lambda)g(x)$ that $H(\lambda, \partial U) \neq y$ for all $\lambda \in [0, 1]$. As a consequence we have $\mathrm{Deg}(f, U, y) = \mathrm{Deg}(g, U, y)$, if f and g coincide on the boundary of U. Following this idea, we can now prove a first result on actually calculating the degree.

Lemma 1.9. *Let $U \subset \mathbb{R}^n$ be non-empty, open, bounded and star-shaped, $f : \overline{U} \to \mathbb{R}^n$ a function continuous on $\overline{U}$, continuously differentiable on U and pointing inwards on ∂U. Then $\mathrm{Deg}(f, U, 0) = (-1)^n$ holds.*

Proof. Since U is star-shaped and non-empty there is a point $p \in U$ such that this point can be connected to every point $x \in U$ via a straight line which is entirely in U, i.e., $\overline{px} \subset U$ for all $x \in U$. Since a star-shaped set remains star-shaped after closure, the same holds true for $\overline{U}$. We now define a homotopy $H(\lambda, x) = \lambda f(x) + (1 - \lambda)(p - x)$ for all $x \in \overline{U}$ and $\lambda \in [0, 1]$, which is clearly continuous.
In order to use the homotopy invariance property it remains to check that $0 \notin H(\lambda, \partial U)$ for all $\lambda \in [0, 1]$. As depicted in Figure 3 the vector $p - x$ connects every point x on the boundary with the point p in the interior. Since $f(x)$ is pointing inwards at these boundary points and $H(\lambda, x)$ is just the convex combination of these two vectors we have the desired result. Altogether we can now calculate the Brouwer degree of f, where we also use the translation

invariance.

$$\begin{aligned}\mathrm{Deg}(f,U,0) &= \mathrm{Deg}(p-\mathrm{Id}\,|_{\overline{U}},U,0) = \mathrm{Deg}(-Id|_{\overline{U}},U,-p)\\ &= \sum_{x\in U:-x=-p} \mathrm{sgn}\det D(-\mathrm{Id}\,|_{\overline{U}})|_x = \mathrm{sgn}\det diag(-1) = (-1)^n.\end{aligned}$$

□

Remark 1.10. An immediate consequence is that every domain U, together with a function f, meeting the assumptions of Lemma 1.9 contains a point $\tilde{x}$ such that $f(\tilde{x}) = 0$.

1.2 Index via Degree

In the qualitative analysis of vector fields, or equivalently differential equations, one is often challenged with the task of finding equilibria. Moreover, information on the behaviour of solutions near an equilibrium is needed. In the context of differential equations of the form $\dot{x} = f(x)$, the set of equilibria is given by $\{x \in U : f(x) = 0\}$. This suggests that $\mathrm{Deg}(f, U, 0)$ might contain some of the required information about equilibria of a differential equation. The following definition formalizes this idea.

Definition 1.11 (Isolated zero)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded, $f : U \to \mathbb{R}^n$ continuous on U. A point $\hat{x} \in U$ is an isolated zero of f (in the context of differential equations we speak of an isolated equilibrium), if $f(\hat{x}) = 0$ and there is an open neighbourhood V, called isolating neighbourhood, with $\hat{x} \in V$ such that for all $x \neq \hat{x} \in V$, $f(x) \neq 0$.*

Definition 1.12 (Index of an isolated zero)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded, $f : \overline{U} \to \mathbb{R}^n$ a function continuous on $\overline{U}$ and continuously differentiable on U. Let $\hat{x} \in U$ be an isolated zero of f with isolating neighbourhood V. Then we define the integer*

$$\mathrm{Ind}_f(\hat{x}) := \mathrm{Deg}(-f, V, 0)$$

as the index of $\hat{x}$ (with respect to f).

Remark 1.13. This naturally carries over the properties of the Brouwer degree in Proposition 1.5 to the index of an equilibrium. In particular, the additivity property of the degree implies that Definition 1.12 is independent of the isolating neighbourhood. In contrast to the classical definition, we added an extra minus sign in Definition 1.12. This results in assigning index $+1$ to asymptotically stable regular equilibria, independent of the dimension. A straightforward computation shows that in the case of a regular zero, we have $\mathrm{Ind}_f(\hat{x}) = (-1)^{\tilde{k}}$, where $\tilde{k}$ is defined as the number of positive eigenvalues of the Jacobian of f evaluated at

$\hat{x}$, i.e., $\tilde{k} := \#$ *positive eigenvalues of* $Df|_{\hat{x}}$. This implies for an asymptotically stable regular equilibrium $\mathrm{Ind}_f(\hat{x}) = +1$.

In addition we can express $\mathrm{Deg}(f, U, 0)$ via the indices of the equilibria $\hat{x}_i \in U$. We collect this result in the following corollary.

Corollary 1.14. *Let $U \subset \mathbb{R}^n$ be open and bounded, $f : \overline{U} \to \mathbb{R}^n$ a function continuous on $\overline{U}$, continuously differentiable on U and non vanishing on ∂U. Assume in addition that all zeroes are isolated, i.e., all $\hat{x}_i \in U$, $i = 1, ..., s$, such that $f(\hat{x}_i) = 0$, have an isolating neighbourhood $V_i \subset U$ (this already implies that there are only finitely many zeroes). Then, by the additivity property of the degree and the fact that the V_i can be chosen pairwise disjoint*

$$\mathrm{Deg}(f, U, 0) = \sum_{i=1}^{s} \mathrm{Deg}(f, V_i, 0) = (-1)^n \sum_{i=1}^{s} \mathrm{Ind}_f(\hat{x}_i).$$

If, in addition, we also meet the assumptions of Lemma 1.9 we are left with $\sum_{i=1}^{s} \mathrm{Ind}_f(\hat{x}_i) = +1$. In the case of 0 being a regular value this implies that the number of zeroes of f is odd.

Definition 1.12 above also shows that, in the case of regular values, the index is bound to evaluate to ± 1. In the case of irregular values this is not necessarily true, as the following example shows.

Example 1.15. We may look at the complex function $f_n : \mathbb{C} \to \mathbb{C}$, $f_n(z) = z^n$, where $\mathbb{C}$ is interpreted as $\mathbb{R}^2$. Here the origin has index n. To see this we note that there is only one point $\hat{z} = 0 \in \mathbb{C}$ with $f_n(\hat{z}) = 0$, so we choose the unit disk D_1 as isolating neighbourhood. The case $n = 1$ is straight forward, since 0 is a regular value.
Let $n \geq 2$. Now 0 is not regular anymore, so we again define a homotopy $H(\lambda, z) = z^n - \lambda\epsilon$ with $0 \leq \lambda \leq 1$ such that $H(\lambda, \partial D_1) \neq 0$. To make things concrete, we fix $\epsilon = \frac{1}{4}$ and define $H(1, z) = z^n - \frac{1}{4} =: g_n(z)$. Let us rewrite the function and define $u_n : \mathbb{R}^2 \to \mathbb{R}$ and $v_n : \mathbb{R}^2 \to \mathbb{R}$ as the real- and imaginary part of g_n, respectively. The perturbed function $g_n = u_n + iv_n$ is holomorphic, therefore the Cauchy-Riemann equations hold and we are left with

$$\det\left(D\begin{pmatrix} u_n \\ v_n \end{pmatrix}\right) = \det\begin{pmatrix} \partial_x u_n & \partial_y u_n \\ -\partial_y u_n & \partial_x u_n \end{pmatrix} = (\partial_x u_n)^2 + (\partial_y u_n)^2 \geq 0.$$

The preimage of 0 (with respect to g_n), are the points $\hat{z}_k = \sqrt[n]{\frac{1}{4}} e^{2\pi i \frac{k}{n}}$, $k = 0, ..., n-1$. After differentiating the real part of g_n with respect to x and y, we see that $(\partial_x u_n)(\hat{z}_k) = n(\frac{1}{4})^{1-\frac{2}{n}} \mathrm{Re}(\hat{z}_k)$ and $(\partial_y u_n)(\hat{z}_k) = n(\frac{1}{4})^{1-\frac{2}{n}} \mathrm{Im}(\hat{z}_k)$. Since we have for all $k = 0, ..., n-1$ that $\hat{z}_k \neq 0$, i.e., real- and imaginary part do not

vanish simultaneously, the strict inequality

$$\det\left(D\begin{pmatrix}u_n\\v_n\end{pmatrix}\Big|_{\hat{z}_k}\right) > 0,$$

for all $k = 0, ..., n-1$, follows. Since all zeroes $\hat{z}_k$ are isolated, we can now use the additivity property of the degree, and therefore also for the index, to conclude with

$$\begin{aligned}\mathrm{Ind}_{f_n}(0) &= \mathrm{Deg}(-f_n, D_1, 0)\\ &= \mathrm{Deg}(-g_n, D_1, 0) = \sum_{k=0}^{n-1} \operatorname{sgn}\det\left(D\begin{pmatrix}u_n\\v_n\end{pmatrix}\Big|_{\hat{z}_k}\right) = n.\end{aligned}$$

Where we used in the third step that in the case of a matrix $M \in \mathbb{C}^{2\times 2}$ it holds that $\det(M) = \det(-M)$. Along the same lines one can show that for $f(z) = \bar{z}^n$ we have $\mathrm{Ind}_f(0) = -n$.

Example 1.15 shows that already in two dimensions the index can be any integer. In order to regain at least some control we are forced to add assumptions on the functions and vector fields under consideration. In the following we will examine semiflows generated by a differential equation $\dot{x} = f(x)$ with $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$. The following definition can be found in [8, p. 96].

Definition 1.16 (Dissipativity)**.** *A semi flow is called dissipative, if there exists a compact set $K \subset U$ such that it attracts all points of U. That is, for all $x \in U$ there is some time $t(x) > 0$ such that $x(t) \in K$ for all $t \geq t(x)$.*

With Definition 1.16 at hand we can now formulate our first result on the index sum of all equilibria arising from a differential equation defined on whole $\mathbb{R}^n$. The proof is taken from [9].

Lemma 1.17. *A differential equation $\dot{x} = f(x)$ on $U = \mathbb{R}^n$, generating a dissipative semiflow, has degree $(-1)^n$ with respect to any bounded open set $B \subset \mathbb{R}^n$ containing all its equilibria, i.e.,* $\mathrm{Deg}(f, B, 0) = (-1)^n$.
If, in addition, all equilibria are isolated we can conclude that the sum of their indices equals +1.

Proof. From the dissipativity it follows that there exists a compact set $K \subset \mathbb{R}^n$ with nonempty interior containing the ω-limits of all $x \in \mathbb{R}^n$. We define the function

$$\tau(x) = \inf\{t \geq 0 : x(t) \in \operatorname{int} K\} \tag{1.1}$$

for all $x \in \mathbb{R}^n$.
We show that, $x \mapsto \tau(x)$ is upper-semicontinuous and consequently it attains its maximum on the compact set K. Indeed, if $x_0 \in \operatorname{int} K$ then there is an open

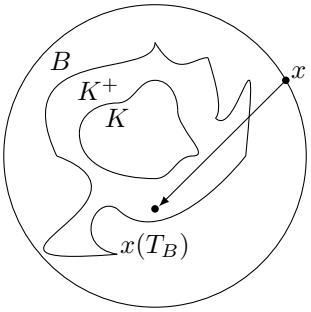


Figure 4: Degree of a dissipative semiflow

neighbourhood $N_{x_0} \subset \operatorname{int} K$ such that for all $x \in N_{x_0}$ $0 = \tau(x) < T$ for all $T > \tau(x_0) = 0$. Assume now that $x_0 \notin \operatorname{int} K$. From (1.1) and the dissipativity it follows that $x_0(\tau(x_0)) \in \partial K$ and for all $T > \tau(x_0)$ there exists $\tau(x_0) < \tilde{t} < T$: $x_0(\tilde{t}) \in \operatorname{int} K$. Hence there is a neighbourhood of $x_0(\tilde{t})$ contained in $\operatorname{int} K$, i.e., $B_\epsilon(x_0(\tilde{t})) \subset \operatorname{int} K$ for some $\epsilon > 0$. The continuous dependence on the initial condition of the flow map implies that there exists a $\delta > 0$ such that for all $x \in B_\delta(x_0) : x(\tilde{t}) \in B_\epsilon(x_0(\tilde{t})) \subset \operatorname{int} K$. Hence we have shown for all $x_0 \notin \operatorname{int} K$ and $T > \tau(x_0)$ that there exists an open neighbourhood $B_\delta(x_0)$ such that for all $x \in B_\delta(x_0)$ we have $\tau(x) < \tilde{t} < T$, which completes the proof of the upper semi-continuity of τ.
Therefore $T_K := \max\{\tau(x) : x \in K\} < \infty$, as the maximal return time of points in K, exists. As a result we have that

$$K^+ := \{x(t) : x(0) \in K \text{ and } 0 \leq t \leq T_K\}$$

is compact and forward invariant. Now consider an open ball B such that $K^+ \subset B$. Since $\overline{B}$ also contains the ω-limit set we have, with the same reasoning as before, that there exists $T_B := \max\{\tau(x) : x \in \overline{B}\} + 1 < \infty$. Again we will use the homotopy invariance of the degree and define

$$H(\lambda, x) := \begin{cases} f(x) & \lambda = 0 \\ \frac{x(T_B\lambda)-x}{T_B\lambda} & 0 < \lambda \leq 1, \end{cases}$$

for all $x \in B$. Clearly H is continuous and since $\omega(x) \subset K$ for all $x \in \mathbb{R}^n$, there are no equilibria or periodic orbits on ∂B resulting in $H(\lambda, \partial B) \neq 0$ for all $\lambda \in [0, 1]$.
Finally we point out that $H(1, x) = \frac{x(T_B)-x}{T_B}$, for all $x \in \partial B$ is just a vector pointing inwards, as shown in Figure 4, since K^+ is forward invariant. Together with B being convex and Lemma 1.9, we arrive at $\operatorname{Deg}(f, B, 0) = (-1)^n$. The additivity property of the degree allows us to replace the set B with any open

and bounded set containing all equilibria. If they are all isolated we can deduce from Corollary 1.14 that the sum of their indices equals +1. □

Lemma 1.17 also implies that every dissipative semiflow has at least one equilibrium. In some cases it is possible to use results like Lemma 1.17 to prove the existence of further equilibria. If, for example, we are in the situation that the sum of the indices of all known equilibria $\hat{x}_i$ of a system does not equal +1, i.e., $S := \sum_i \mathrm{Ind}_f(\hat{x}_i) \neq +1$, then this proves the existence of at least one more equilibrium. If $S < 1$ ($S > 1$) we even know that there exists at least one more equilibrium with positive (negative) index.

1.3 Index Theorem

So far we have considered functions f defined on $\mathbb{R}^n$. Anyway, many models in mathematical biology are defined only on the nonnegative orthant $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1, ..., n\}$ or other state spaces, e.g., the probability simplex $S_n = \{x \in \mathbb{R}^{n+1}_+ : \sum_{i=1}^{n+1} x_i = 1\}$. In the following we will focus on these domains and try to carry over some of the already established results. We will follow the ideas of [9].
Let

$$\dot{x} = f(x), \text{ where } f : \mathbb{R}^n_+ \to \mathbb{R}^n, \tag{1.2}$$

be an autonomous differential equation with continuously differentiable f. Throughout this section we will assume that $\mathbb{R}^n_+$ is forward invariant, i.e.,

$$\text{If } x = x(0) \in \mathbb{R}^n_+, \text{ then also } x(t) \in \mathbb{R}^n_+ \text{ for all } t \geq 0. \tag{1.3}$$

In mathematical biology, this restriction is fulfilled quite naturally. One often models systems, which describe the interaction of different species, keeps track of the densities of alleles or counts the number of infected individuals in a population. All these parameters make sense only if they remain non-negative and therefore a reasonable model in mathematical biology will often be forward invariant anyway. The second assumption we make is that f is dissipative, or equivalently, the semiflow generated by f is ultimately uniformly bounded, i.e., there exists a constant $k > 0$ such that for all $x \in \mathbb{R}^n_+$

$$\limsup_{t \to \infty} x_i(t) \leq k, \text{ for all } i = 1, ..., n. \tag{1.4}$$

In [1, p. 219] we can find the following equivalent, more applicable, formulation of assumption (1.3)

$$x_i = 0 \Rightarrow f_i(x) \geq 0 \text{ for all } i = 1, ..., n. \tag{1.5}$$

Intuitively this means that the non-negative orthant $\mathbb{R}^n_+$ is forward invariant if and only if the flow on the boundary does not point outwards. If we had asked for strict forward invariance of $\mathbb{R}^n_+$ ($x_i = 0 \Rightarrow f_i(x) > 0$ for all $i = 1, ..., n$), which implies that there are no equilibria on the boundary, then it would have followed from Lemma 1.17 that $\mathrm{Deg}(f, \operatorname{int} B, 0) = (-1)^n$. Here $B := \{x \in \mathbb{R}^n_+ : x_i \leq b\}$ is a box, similar to the ball in the proof of Lemma 1.17, that contains the $\omega-$limit set of $\mathbb{R}^n_+$ in its interior (by dissipativity) and the flow on ∂B points inwards. If all (interior) equilibria are isolated, then their index sum equals $+1$.
If we only assume forward invariance, however, some parts of the boundary of $\mathbb{R}^n_+$ may be invariant under the flow. There, the flow does not point inwards and may even contain equilibria. As it turns out, some of the boundary equilibria are more important to us than others. To tell these equilibria apart we investigate the Jacobian at these equilibria. We define the complement of the support of an equilibrium $\hat{x} \in \mathbb{R}^n_+$ as the set $I := \{i \in \{1, ..., n\} : \hat{x}_i = 0\}$. The support is given by the set $J := \{1, ..., n\} \setminus I$. We focus on two cases:

1) Let $i \in I$ and $j \in J$. To calculate $\frac{\partial f_i}{\partial x_j}(\hat{x})$, we first note that $\hat{x} + se_j = (\hat{x}_1, ..., \hat{x}_j + s, ..., \hat{x}_n)^T \in \mathbb{R}^n_+$, where e_j is the j-th unit vector, for all $s \in \mathbb{R}$ with $s \geq -\hat{x}_j$. Since $i \in I$, it follows from (1.5) that $f_i(\hat{x} + se_j) \geq 0$. Together with $f_i(\hat{x}) = 0$ this implies that $\frac{\partial f_i}{\partial x_j}(\hat{x}) = 0$.

2) Let $i, j \in I$ and $i \neq j$. In this case $\hat{x} + se_j = (\hat{x}_1, ..., \hat{x}_j + s, ..., \hat{x}_n)^T \in \mathbb{R}^n_+$, for all $s \geq 0$. Thus we can conclude that $\frac{\partial f_i}{\partial x_j}(\hat{x}) \geq 0$.

After rearranging the indices such that $I = \{1, ..., k\}$ for some $0 \leq k \leq n$, the Jacobian at an equilibrium has the form of a block matrix

$$Df(\hat{x}) = \left(\frac{\partial f_i}{\partial x_j}(\hat{x})\right)_{1 \leq i,j \leq n} = \left(\begin{array}{c|c} A & 0 \\ \hline B & C \end{array}\right), \tag{1.6}$$

where we will refer to A and C as the external- and internal part of the Jacobian, respectively. From property 2) above we infer that A is a quasipositive matrix, i.e., a matrix with only non-negative off-diagonal entries ($a_{ij} \geq 0$ if $i \neq j$). We are now able to define the identifying property of the somewhat more important equilibria.

Definition 1.18 (Saturated equilibrium)**.** *An equilibrium $\hat{x} \in \mathbb{R}^n_+$ is called (strictly) saturated, if all eigenvalues of the external part of the Jacobian at $\hat{x}$ have non-positive (negative) real part.*

We can now state our first theorem.

Theorem 1.19 (Index Theorem, Hofbauer 1990)**.** *Every semiflow on $\mathbb{R}^n_+$ generated by* (1.2)*, satisfying* (1.4) *and* (1.5)*, admits at least one saturated equilibrium. In the case that all saturated equilibria are regular, their index-sum equals +1.*

In the subsequent only the basic ideas of the proof are given, for details consider [9]. In order to sketch a proof of Theorem 1.19 we perturb the differential equation, by adding a suitably chosen inward flow $\rho(x)$ such that the flow on $\partial\mathbb{R}^n_+$ points inward, and we investigate the new system

$$\dot{x} = f_\epsilon = f(x) + \epsilon\rho(x),\ \epsilon > 0. \tag{1.7}$$

Then, by Lemma 1.17 we can conclude, for a large enough box $B = \{x \in \mathbb{R}^n_+ : x_i \leq b\}$, that $\mathrm{Deg}(f_\epsilon, \operatorname{int} B, 0) = (-1)^n$. This implies the existence of at least one equilibrium of (1.7) in $\operatorname{int} B \subset \mathbb{R}^n_+$. The following claim (a variant of it will be proven in the proof of Theorem 1.22) guarantees the existence of a saturated equilibrium $\hat{x} \in \mathbb{R}^n_+$ for (1.2).

A) A limit point $\hat{x}$ of equilibria of (1.7) is a saturated equilibrium of (1.2).

Finally, it remains to make sure that we do not lose boundary equilibria after the perturbation. This is important to justify the second part of Theorem 1.19. Together with the homotopy invariance of the degree (and therefore also of the index) the statement below ensures that in the case that all saturated equilibria are regular, the sum of their indices equals $+1$.

B) A regular, saturated equilibrium $\hat{x}$ of (1.2) gives rise to a unique C^1-family of equilibria of (1.7) $\hat{x}(\epsilon) \in \operatorname{int} \mathbb{R}^n_+$ for $\epsilon > 0$.

1.4 Boundary Index

In Theorem 1.19 it is assumed that all saturated equilibria of (1.2) are regular, which is necessary for the proof of assertion B) above. However, this regularity assumption restricts us to the case of saturated equilibria with index ± 1. In many cases, the transition of an equilibrium from saturated to non-saturated is of large interest. One may think of the situation, when an absent species can finally invade into a community being in equilibrium, as treated in [5]. This change takes place, when an external eigenvalue of (1.6) is zero and therefore the equilibrium is non-regular. Since we would like to understand these transitions we have to improve our result. First, we modify the definition of the index as suggested in [9].

Definition 1.20 (Boundary index)**.** *Let $\hat{x} \in \mathbb{R}^n_+$ be an isolated equilibrium of* (1.2)*. Let $V \subset \mathbb{R}^n$ be an isolating neighbourhood and $\epsilon : \mathbb{R}^n_+ \to \mathbb{R}^n$ a, in absolute value, sufficiently small, smooth perturbation. For all $x \in \partial\mathbb{R}^n_+$, ϵ must generate an inward flow, i.e., $x_i = 0 \Rightarrow \epsilon_i(x) > 0$. Define $V^+ := V \cap \mathbb{R}^n_+$. Then we say that*

$$\mathrm{Bd\_Ind}_f(\hat{x}) := \mathrm{Deg}(-(f+\epsilon), \operatorname{int} V^+, 0)$$

is the boundary index of $\hat{x}$.

The boundary index is independent of the perturbation, since they are all homotopic. Adding a small enough perturbation in Definition 1.20, ensures that there are no equilibria on $\partial\mathbb{R}^n_+$ and therefore also not on ∂V^+, i.e., $0 \notin (f+\epsilon)(\partial V^+)$. As a result, the degree of $-(f+\epsilon)$ with respect to $\operatorname{int} V^+$, and therefore the boundary index of $\hat{x}$, is well defined.
The concept of the boundary index is naturally very similar to the index. For a non-saturated equilibrium $\hat{x}$, we have that $\text{Bd\_Ind}_f(\hat{x}) = 0$. To justify this we use statement A) and see that for a small enough perturbation $f+\epsilon$ there is no equilibrium in V_+. From the definition of the degree it follows that $\text{Bd\_Ind}_f(\hat{x}) = 0$. In the case of a regular, saturated equilibrium it follows from statement B) and the homotopy invariance that $\text{Bd\_Ind}_f(\hat{x}) = \text{Ind}_f(\hat{x}) = \pm 1$. Since we introduced the boundary index to get rid of the regularity assumption in Theorem 1.19, we expect that the two concepts may differ for non-regular equilibria. The following example shows that this is indeed the case.

Example 1.21. Consider the differential equation

$$\dot{x}_i = f_i(x) = -x_i^2, \;\; i = 1, ...n,$$

with the origin as unique and isolated equilibrium. Since the support of the origin is empty, the Jacobian, $Df(0) = 0^{n\times n}$, only consists of the external part. We conclude that the origin is a non-regular, saturated equilibrium. It remains to show that $\text{Ind}_f(0) = 0 \neq 1 = \text{Bd\_Ind}_f(0)$. We will restrict ourselves to the case $n = 2$. Let us define a homotopy componentwise as $H(\lambda, x) = (H_1, H_2)^T(\lambda, x)$, with $H_i(\lambda, x) = -x_i^2 + \lambda\epsilon_i^2$, for $i = 1, 2$. We choose the unit disk as isolating neighbourhood, $\epsilon_i > 0$ and $\epsilon_1^2 + \epsilon_2^2 < 1$, accordingly. This changes the equilibrium structure as shown in Figure 5. The four equilibria of $H(1, x)$ are $\hat{x}_1 = (\epsilon_1, \epsilon_2)^T$, $\hat{x}_2 = (-\epsilon_1, \epsilon_2)^T$, $\hat{x}_3 = (\epsilon_1, -\epsilon_2)^T$ and $\hat{x}_4 = (-\epsilon_1, -\epsilon_2)^T$. The Jacobian of $H(1, x)$ has the form

$$DH(1, x) = \begin{pmatrix} -2x_1 & 0 \\ 0 & -2x_2 \end{pmatrix}.$$

For that reason all equilibria $\hat{x}_i$, $i = 1, ..., 4$ are regular and we have

$$\text{Ind}_f(0) = \text{Deg}(-f, D_1, 0) = \sum_{i=1}^{4} \operatorname{sgn}\det DH(1, \hat{x}_i) = 1 - 1 - 1 + 1 = 0.$$

In contrast we only have one equilibrium, namely $\hat{x}_1$, in $D_1 \cap \mathbb{R}^2_+$. As a result

$$\text{Bd\_Ind}_f(0) = \text{Deg}(-(f+\epsilon^2), \operatorname{int}(D_1 \cap \mathbb{R}^2_+), 0) = \operatorname{sgn}\det DH(1, \hat{x}_1) = +1.$$

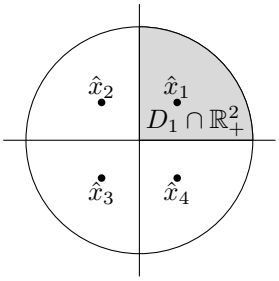


Figure 5: Equilibria of $\dot{x}_i = -x_i^2 + \epsilon_i^2$, with $\epsilon_i = \frac{1}{2}$.

If $n \neq 2$, the same ideas are used, but the bookkeeping and visualization becomes more tiring. After the perturbation there are 2^n regular equilibria. One half (2^{n-1}) consists of equilibria which contribute $+1$ to the index. Every remaining equilibrium (2^{n-1} are left) contributes -1 to the index. In total this implies that

$$\mathrm{Ind}_f(0) = 2^{n-1} - 2^{n-1} = 0.$$

For the boundary index we again have only one equilibrium in $B_1(0) \cap \mathbb{R}^n_+$. At this equilibrium, the determinant of the Jacobian of the perturbed field is positive, resulting in

$$\mathrm{Bd\_Ind}_f(0) = +1.$$

We can now improve the statement of Theorem 1.19. In the proof we used Lemma 1.17 to conclude that the sum of the indices of the saturated equilibria in the perturbed system is $+1$, or equivalently $\mathrm{Deg}(f_\epsilon, B, 0) = (-1)^n$. But if we assume that all saturated equilibria, $\hat{x}_i$, are isolated in the first place, we have per definition that

$$\sum_i \mathrm{Bd\_Ind}_f(\hat{x}_i) = \sum_i \mathrm{Deg}(-f_\epsilon, \mathrm{int}\, V_i^+, 0) = \mathrm{Deg}(-f_\epsilon, B, 0) = +1.$$

So we can drop the regularity assumption, since we no longer have to guarantee the uniqueness of the equilibria after perturbation, i.e, assertion B) above. The following result sums up the preceeding ideas.

Theorem 1.22 (Boundary Index Theorem, Hofbauer 1990)**.** *Every semiflow generated by* (1.2)*, satisfying* (1.4) *and* (1.5)*, admits at least one saturated equilibrium. In the case that all saturated equilibria are isolated, the sum of their boundary indices equals +1.*

Theorem 1.22 can also be extended to differentiable manifolds with corners, since they are locally diffeomorphic to $\mathbb{R}^{n-1}_+$. The probability simplex $S_{n-1} = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$ is an example. In the following we prove the boundary index

theorem on the simplex, where we use ideas from [9] and [10, p. 159].
Consider a semiflow $\dot{x} = f(x)$ on the simplex S_{n-1}, such that the two assumptions $\sum_{i=1}^{n} \dot{x}_i = 0$ and $x_i = 0 \Rightarrow f_i(x) \geq 0$ hold. Define

$$\dot{x}_i = g_i^{\epsilon}(x) := f_i(x) + \epsilon(1 - nx_i) \tag{1.8}$$

for all $x \in S_{n-1}$, $i = 1, ..., n$ and $\epsilon > 0$ small enough as perturbation. The orbits of the semiflow generated by the continuous perturbation (1.8) still remain on the simplex, i.e., $\sum_{i=1}^{n} \dot{x}_i = \sum_{i=1}^{n} g_i^{\epsilon}(x) = 0$. Moreover we see that $x_i = 0 \Rightarrow g_i^{\epsilon}(x) = f_i(x) + \epsilon > 0$ for all $\epsilon > 0$, and therefore the flow points inwards on ∂S_{n-1}. Since $S_{n-1} \subset \mathbb{R}^{n-1}$ is convex it follows at once from Lemma 1.9 that $\mathrm{Deg}(g^{\epsilon}, \mathrm{int}\, S_{n-1}, 0) = (-1)^{n-1}$.
It remains to show that a limit point $\hat{x} = \hat{x}(0)$ of equilibria $\hat{x}(\epsilon)$ for $\epsilon > 0$ of (1.8) (which exist since $\mathrm{Deg} \neq 0$ for all $\epsilon > 0$) is saturated. Let us reorder the indices as in (1.6), i.e., $I = \{i \in \{1, ..., n\} : \hat{x}_i = 0\} = \{1, ..., k\}$ for some $k \leq n$. For i and $j \in I$, we define

$$d_i(x_1, ..., x_n) = f_i(0, ..., 0, x_{k+1}, ..., x_n) \tag{1.9}$$

$$a_{ij}(x_1, ..., x_n) = \begin{cases} \frac{f_i(0,...,0,x_j,x_{j+1},...,x_n) - f_i(0,...,0,x_{j+1},...,x_n)}{x_j} & \text{if } x_j > 0 \\ \frac{\partial f_i}{\partial x_j} & \text{if } x_j = 0. \end{cases} \tag{1.10}$$

From the forward invariant assumption we infer that $d_i(x) \geq 0$ for all $x \in S_{n-1}$ and since $\hat{x}$ is an equilibrium $d_i(\hat{x}) = 0$. By assumption, f is continuously differentiable and therefore this also holds for a_{ij} in the interior of S_{n-1}. On the boundary of the simplex, the a_{ij}'s are still continuous. If we compare (1.10) with (1.6), we see that the matrix $A = A(\hat{x}) := a_{ij}(\hat{x})$ is simply the external part of the Jacobian of (1.8) for $\epsilon = 0$ and therefore has non-negative off-diagonal entries. For any point $x \in S_{n-1}$ and $i \in I$ it holds that

$$f_i(x) = \sum_{j=1}^{k} a_{ij}(x)x_j + d_i(x). \tag{1.11}$$

This follows from the definition of the a_{ij}'s in (1.10), which leaves us with a telescopic sum in (1.11). Assume now that $\hat{x} = \hat{x}(0)$ is not saturated, i.e., A has at least one eigenvalue with positive real part. We define the spectral bound of a real square matrix M as $s(M) := \max\{\mathrm{Re}(\mu) : \mu \text{ eigenvalue of M}\}$. Since A is quasi-positive, there exists $c > 0$ such that $A' = A + c\mathbb{I} \geq 0$ is a non-negative matrix and the Perron-Frobenius Theorem implies the existence of a non-negative eigenvalue $s(A') = \lambda \geq 0$ with corresponding non-negative left eigenvector $v \geq 0$ such that $vA' = \lambda v$, see [4, p. 26]. Inserting the definition of A' we see that v is

also a left eigenvector of A corresponding to the eigenvalue $\lambda - c$. Together with $\lambda = s(A') = s(A) + c$ this implies that $s(A) = \lambda - c$. We assume that $\hat{x}$ is not saturated and therefore $\lambda - c > 0$ must hold and consequently

$$\sum_{i,j=1}^{k} v_i a_{ij}(\hat{x}) = \sum_{j=1}^{k} (vA)_j = \sum_{j=1}^{k} (\lambda - c) v_j > 0 \tag{1.12}$$

since v is non-negative and not vanishing. In particular there is at least one $j \in \{1, ..., k\}$ such that $(vA)_j > 0$. Consider now an equilibrium $\hat{x}(\epsilon)$ of (1.8) with $\epsilon > 0$ small enough. Together with (1.11) we infer that

$$\sum_{j=1}^{k} a_{ij}(\hat{x}(\epsilon))\hat{x}_j(\epsilon) = -d_i(\hat{x}(\epsilon)) - \epsilon(1 - n\hat{x}_i(\epsilon)) < 0 \tag{1.13}$$

for all $\epsilon > 0$ small enough and $i = 1, ..., k$. But as shown above the a_{ij}'s are continuous on the simplex and therefore (1.12) also holds in a small neighbourhood of $\hat{x}$. For all $\epsilon > 0$ small enough, $\hat{x}(\epsilon)$ is in this neighbourhood. Since (1.8) has no equilibria on the boundary of S_{n-1}, $\hat{x}(\epsilon)$ has only positive entries for $\epsilon > 0$, which implies that $\sum_{i,j=1}^{k} v_i a_{ij}(\hat{x}(\epsilon))\hat{x}_j(\epsilon) > 0$ which contradicts (1.13). As a result, $\hat{x}$ is saturated. If the saturated equilibria $\hat{\mathbf{x}}_i$ are all isolated, with isolating neighbourhood $V_i^+ \subset S_{n-1}$, we obtain

$$\begin{aligned} \sum_i \mathrm{Bd\_Ind}_f(\hat{\mathbf{x}}_i) &= \sum_i \mathrm{Deg}(-g, \operatorname{int} V_i^+, 0) = \mathrm{Deg}(-g, \operatorname{int} S_{n-1}, 0) \\ &= (-1)^{n-1}(-1)^{n-1} = +1. \end{aligned}$$

1.5 Degree for Plane Vector Fields

1.5.1 Rotation of a Plane Vector Field

Let us now take a closer look at the degree and the index of vector fields in the plane, i.e., $\Phi : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, where U is bounded. We will establish another notion of the degree in two dimensions, called rotation of a vector field. Let us start with some preparatory steps, following [2, Chapter 5]. If not stated otherwise we assume that the vector field

$$\Phi(\mathbf{x}) = \begin{pmatrix} P(\mathbf{x}) \\ Q(\mathbf{x}) \end{pmatrix} \tag{1.14}$$

is continuous, i.e., $P : \mathbb{R}^2 \to \mathbb{R}$ and $Q : \mathbb{R}^2 \to \mathbb{R}$ are continuous, where $\mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2$. A vector field (1.14) is said to be non-vanishing on a set U, if $\Phi(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in U$.

We are interested in the angle ϕ between points of the vector field and the

positive x-axis, measured counterclockwise (in the positive direction). In the case of a non-vanishing vector field, this angle is defined for every point and given by

$$\cos\phi = \frac{P(\mathbf{x})}{\sqrt{P(\mathbf{x})^2 + Q(\mathbf{x})^2}}, \quad \sin\phi = \frac{Q(\mathbf{x})}{\sqrt{P(\mathbf{x})^2 + Q(\mathbf{x})^2}}. \tag{1.15}$$

From (1.15) we see that for every point $\mathbf{x} \in U$, the angle between the x-axis and the vector field there, is uniquely determined (up to multiples of 2π) and a continuous function of $\mathbf{x}$. In contrast we define the angle $\theta(\mathbf{v}_1, \mathbf{v}_2)$ between two vectors $\mathbf{v}_1, \mathbf{v}_2 \neq \mathbf{0} \in \mathbb{R}^2$ as the in absolute value smallest angle one has to rotate $\mathbf{v}_1$ to coincide in direction with $\mathbf{v}_2$. A positive (negative) angle corresponds to counterclockwise (clockwise) rotation of $\mathbf{v}_1$. If $\mathbf{v}_1$ and $\mathbf{v}_2$ have opposite directions, i.e., there exists $\lambda > 0$ such that $\mathbf{v}_1 + \lambda\mathbf{v}_2 = 0$, we define $\theta(\mathbf{v}_1, \mathbf{v}_2) = \pi$ and therefore we have

$$-\pi < \theta(\mathbf{v}_1, \mathbf{v}_2) \leq \pi, \text{ for all } \mathbf{v}_1, \mathbf{v}_2 \neq \mathbf{0} \in \mathbb{R}^2. \tag{1.16}$$

Definition 1.23 ((Parametrized) Simple Arc)**.** *Let $f_1, f_2 : [t_0, T] \to \mathbb{R}$ be continuous for all $t_0 \leq t \leq T$. Furthermore there must not be self-intersections, i.e., for all $t_1, t_2 \in [t_0, T]$ with $t_1 \neq t_2$ we demand $(f_1(t_1) - f_1(t_2))^2 + (f_2(t_1) - f_2(t_2))^2 \neq 0$. Then the set $l := \{\mathbf{x} \in \mathbb{R}^2 : x_1 = f_1(t), x_2 = f_2(t), t \in [t_0, T]\}$ is called a simple arc. If there are $t_1, t_2 \in [t_0, T]$ such that $\mathbf{x}_1 = \mathbf{x}(t_1)$ and $\mathbf{x}_2 = \mathbf{x}(t_2)$, we say that the set $\mathbf{x}_1\mathbf{x}_2 := \{\mathbf{x}(t) \in \mathbb{R}^2 : x_1 = f_1(t), x_2 = f_2(t), t \in [t_1, t_2]\} \subset l$ is a subarc of l.*
We speak of a parametrized simple arc, if we refer to a triple (f_1, f_2, l).

Definition 1.24 (Angle Function)**.** *Let Φ be a non-vanishing vector field on a simple arc l. A function $\alpha : l \to \mathbb{R}$ is called angle function (of the vector field Φ defined on the arc l), if α is continuous and for all $\mathbf{x} \in l$, $\alpha(\mathbf{x})$ is the angle between the positive x-axis and the vector $\Phi(\mathbf{x})$.*

In a first step we show that angle functions exist. Note that $|\theta(\mathbf{v}_1, \mathbf{v}_2)|$ is a metric for unit vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$. Since l is a compact set and Φ is non-vanishing, we can conclude that $\frac{\Phi}{|\Phi|}$ is uniformly continuous on l. As a consequence for all $\mathbf{x}, \mathbf{y} \in l$ there is a constant $\delta > 0$ such that from $\|\mathbf{x} - \mathbf{y}\| < 2\delta$ it follows that

$$\left|\theta\Big(\frac{\Phi(\mathbf{x})}{|\Phi(\mathbf{x})|}, \frac{\Phi(\mathbf{y})}{|\Phi(\mathbf{y})|}\Big)\right| < \pi.$$

We can now cover l with a finite number of open balls $B_\delta(\mathbf{x}_i)$ with radius δ and centers $\mathbf{x}_i$, $i = 0, ..., n$, where $\mathbf{x}_0$ and $\mathbf{x}_n$ are the start- and endpoints of l, respectively. Here we want to point out that we choose these points such that an

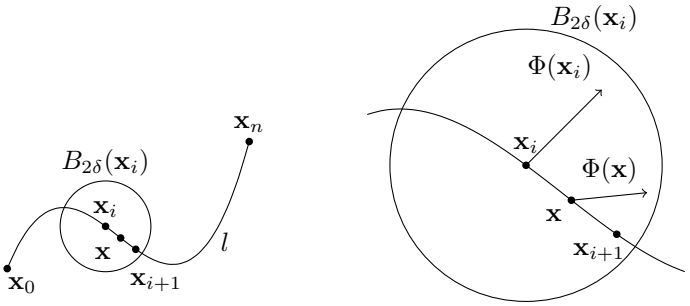


Figure 6: Construction of an angle function

increase in i corresponds to a definite direction on the arc, as shown in Figure 6. Since for all $\mathbf{x}$ on the subarc $\mathbf{x}_i\mathbf{x}_{i+1}$ we have $\|\mathbf{x}-\mathbf{x}_i\| < 2\delta$ we can now define an angular function inductively.

Let ϕ_0 be a fixed value of the angle between the positive x-axis and $\Phi(\mathbf{x}_0)$ and define $\alpha_0(\mathbf{x}) := \phi_0 + \theta(\Phi(\mathbf{x}_0), \Phi(\mathbf{x}))$ for all $\mathbf{x}$ in the subarc $\mathbf{x}_0\mathbf{x}_1$. Assume we have defined α_{i-1} for $i = 1, ..., n-1$, then $\alpha_i(\mathbf{x}) := \alpha_{i-1}(\mathbf{x}_i) + \theta(\Phi(\mathbf{x}_i), \Phi(\mathbf{x}))$ for all $\mathbf{x}$ in the subarc $\mathbf{x}_i\mathbf{x}_{i+1}$. Now $\alpha := \alpha_i$ on the subarc $\mathbf{x}_i\mathbf{x}_{i+1}$ for $i = 0, ..., n-1$ is an angle function. Indeed, since θ is continuous as long as $|\theta| < \pi$, it follows from the piecewise definiton that α is continuous. With the choice of α_0 it also holds that $\alpha(\mathbf{x})$ is the angle between the positive x-axis and $\Phi(\mathbf{x})$ for all $\mathbf{x} \in l$. Now we can define the rotation.

Definition 1.25 (Rotation of a subarc)**.** *Let l be a simple arc, Φ a non-vanishing vector field on l, $\mathbf{x}_1\mathbf{x}_2$ a subarc of l and α any angle function corresponding to the arc and the vector field. Then the rotation of the field Φ along the subarc $\mathbf{x}_1\mathbf{x}_2$ is defined as the real number*

$$\gamma(\Phi, \mathbf{x}_1\mathbf{x}_2) = \frac{1}{2\pi}[\alpha(\mathbf{x}_2) - \alpha(\mathbf{x}_1)].$$

Remark 1.26. The rotation in Definition 1.25 is independent of the choice of the angle function. Assume α and $\tilde{\alpha}$ are two angle functions. Then by Definition 1.24 we have that $\alpha(\mathbf{x}) - \tilde{\alpha}(\mathbf{x}) = 2\pi r(\mathbf{x})$ for all $\mathbf{x} \in l$, where $r(\mathbf{x})$ is an integer. From the continuity of both angle functions it follows that $r(\mathbf{x}) = const.$ for all $\mathbf{x} \in l$ and we are done. In addition we see that reversing the direction on the arc, changes the sign of the rotation, i.e.,

$$\gamma(\Phi, \mathbf{x}_2\mathbf{x}_1) = -\gamma(\Phi, \mathbf{x}_1\mathbf{x}_2). \tag{1.17}$$

Let C be a simple closed curve, i.e., a simple arc with the exception that there is a common start- and endpoint $\mathbf{x}(t_0) = \mathbf{x}(T)$. On a closed curve there are two possibilities to define a subarc $\mathbf{x}_1\mathbf{x}_2$. We therefore fix a direction on a curve such

that the interior points are to the left (anti-clockwise). We pick two different points $\mathbf{x}_1 \neq \mathbf{x}_2 \in C$ and define

$$\gamma(\Phi, C) := \gamma(\Phi, \mathbf{x}_1\mathbf{x}_2) + \gamma(\Phi, \mathbf{x}_2\mathbf{x}_1). \tag{1.18}$$

Note that since we fixed a direction on C, the subarc $\mathbf{x}_2\mathbf{x}_1$ of C is, in contrast to (1.17) not the subarc $\mathbf{x}_1\mathbf{x}_2$ traversed in the opposite direction.
Equation (1.18) is independent of the choice of the points, since $\gamma(\Phi, \mathbf{x}_1\mathbf{x}_3) = \gamma(\Phi, \mathbf{x}_1\mathbf{x}_2) + \gamma(\Phi, \mathbf{x}_2\mathbf{x}_3)$ for all $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ ordered in the direction of traversal on the arc $\mathbf{x}_1\mathbf{x}_3$. Since the start- and endpoints of a closed curve are identical, we can conclude that $\gamma(\Phi, C) \in \mathbb{Z}$. In [11, p. 6] one finds a simple geometric interpretation of the rotation $\gamma(\Phi, C)$ on a closed curve - it counts the number of full turns of $\Phi(\mathbf{x})$ as $\mathbf{x}$ runs through C.
If there is a parametrization of C, i.e., $C = \{\mathbf{x}(t) : t_0 \leq t \leq T, \mathbf{x}(t_1) \neq \mathbf{x}(t_2) \Leftrightarrow t_1 \neq t_0 \text{ or } t_2 \neq T\}$ where $\mathbf{x}(t) = (f_1(t), f_2(t))^T$ for fixed f_1, f_2, then there exists (like above but this time depending on the parametrization) an angle function $\alpha : [t_0, T] \to \mathbb{R}$ with $\alpha(t)$ being the angle between the x-axis and $\Phi(t) = \Phi(\mathbf{x}(t))$. It holds that $\gamma(\Phi, C) = \frac{1}{2\pi}[\alpha(T) - \alpha(t_0)]$. As above the rotation changes sign, when we change the orientation of the curve. We define

$$\gamma(\Phi, -C) := -\gamma(\Phi, C) = \frac{1}{2\pi}[\alpha(t_0) - \alpha(T)] \tag{1.19}$$

as the rotation of the set of points C, but traversed in reversed direction.

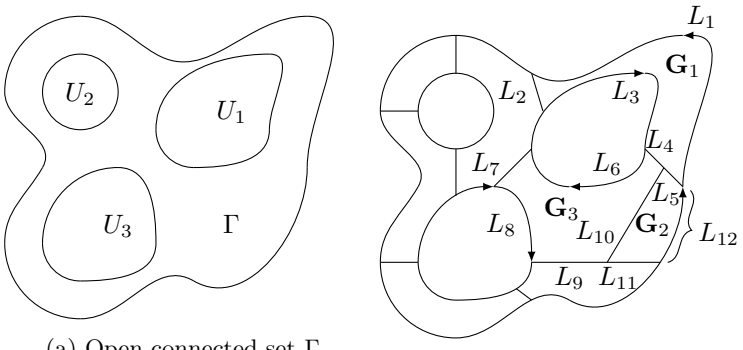
Example 1.27. Let us consider the unit circle S^1 with parametrization $\mathbf{x}(t) = (cos(2\pi t), sin(2\pi t))^T$ for $t \in [0, 1]$ and the vector field $\Phi(\mathbf{x}) = (x_1, x_2)^T$. An angle function is given by $\alpha(t) = 2\pi t$. We therefore get for the rotation $\gamma(\Phi, S^1) = \frac{1}{2\pi}[\alpha(1) - \alpha(0)] = +1$.

1.5.2 Rotation is Degree

Our goal now is to show that in the case of plane vector fields, the two concepts of rotation and degree (with respect to the value $\mathbf{0}$) coincide. Before doing this, we have to clarify what the rotation of an open set is. Until now, we have defined the rotation for simple closed curves only. Therefore we have to restrict to sets with nice enough boundary.

Definition 1.28. *Let $U \in \mathbb{R}^2$ be an open and bounded set with ∂U a simple closed curve. Let Φ be a vector field non-vanishing on ∂U. The rotation with respect to U is defined as $\gamma(\Phi, U) := \gamma(\Phi, \partial U)$.*

Definition 1.29 (Open-connected set)**.** *Let $C, C_1, ..., C_n$ be simple closed curves and denote $U, U_1, ..., U_n$ as the sets of points enclosed by $C, C_1, ..., C_n$ respectively.*



(a) Open-connected set Γ.

(b) Adding arcs $L_1, ..., L_k$.

Figure 7: Rotation of open-connected set without zero vectors

Assume that $\overline{U}_1, ..., \overline{U}_n \subset U$ *are pairwise disjoint, then we say that the set* $\Gamma := U \setminus \bigcup_{i=1}^{n} \overline{U_i}$, *is an open-connected set.*

An example of an open-connected set is shown in Figure 7a. The boundary of Γ is given by $\partial\Gamma = \bigcup_{i=1}^{n} C_i \cup C$. Given an open-connected set Γ and a vector field Φ non-vanishing on $\partial\Gamma$, we define the rotation as

$$\gamma(\Phi, \Gamma) := \gamma(\Phi, U) - \sum_{i=1}^{n} \gamma(\Phi, U_i). \tag{1.20}$$

The ideas in the proof of the following proposition are taken from [12, p. 21].

Proposition 1.30. *Let* $\Gamma \subset \mathbb{R}^2$ *be open-connected. Suppose* Φ *is non-vanishing on* $\overline{\Gamma}$, *then it holds that* $\gamma(\Phi, \Gamma) = 0$.

Proof. From the continuity of Φ it follows that Φ is uniformly continuous on the compact set $\overline{U}$. Therefore there exists $\delta > 0$ such that for all $\mathbf{x}_1, \mathbf{x}_2 \in U$ with $\|\mathbf{x}_1 - \mathbf{x}_2\| < \delta : |\theta(\Phi(\mathbf{x}_1), \Phi(\mathbf{x}_2))| < \frac{\pi}{2}$. We now decompose the bounded set Γ into subsets $\mathbf{G}_1, ..., \mathbf{G}_m$ such that for all $i = 1, ..., m$ we have that $\|\mathbf{x}_1 - \mathbf{x}_2\| < \delta$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{G}_i$. We denote the simple arcs, which form the boundaries of the sets $\mathbf{G}_1, ..., \mathbf{G}_m$, by $L_1, ..., L_k$. The direction on the arcs on the boundaries of the sets $\mathbf{G}_i$ shall be such that the interior of $\mathbf{G}_i$ is to the left, while traversing the boundary. As a result we have that ∂U is traversed anti-clockwise and ∂U_i clockwise for all $i = 1, ..., n$.

As shown in Figure 7b, we have two possible positions of an arc L_j. If such an arc is part of the boundary of Γ, it is part of the boundary of only one of the sets $\mathbf{G}_1, ..., \mathbf{G}_m$ and we are passing it once. If L_j is in the interior of Γ we are going through it twice, once in each direction. We may look at $\mathbf{G}_1$ and $\mathbf{G}_3$, for example. We are traversing the boundary of $\mathbf{G}_1$ via L_1, L_2, L_3, L_4 and L_5. If we follow $\partial\mathbf{G}_3$ we go through $L_6, L_7, L_8, L_9, L_{10}$ and L_4. We have passed the

interior arc L_4 in two different directions. For this reason, it follows from (1.17) and (1.19) that the rotation of the arcs in the interior of Γ in $\sum_{i=1}^{m} \gamma(\Phi, \mathbf{G}_i)$ cancel out. What remains is the boundary of Γ.
By construction it holds that for every $i = 1, ..., m$ $|\theta(\Phi(\mathbf{x}_1), \Phi(\mathbf{x}_2))| < \frac{\pi}{2}$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{G}_i$. This implies that

$$\begin{aligned} |\gamma(\Phi, \mathbf{G}_i)| &= |\gamma(\Phi, \mathbf{x}_1\mathbf{x}_2) + \gamma(\Phi, \mathbf{x}_2\mathbf{x}_1)| \leq |\gamma(\Phi, \mathbf{x}_1\mathbf{x}_2)| + |\gamma(\Phi, \mathbf{x}_2\mathbf{x}_1)| \\ &< \frac{1}{2\pi}(\pi + \pi) < 1. \end{aligned}$$

Since $\partial\mathbf{G}_i$ is a simple closed curve, its rotation must be an integer and therefore vanishes on each $\mathbf{G}_i$, which is leading us to

$$\begin{aligned} \gamma(\Phi, \Gamma) &= \gamma(\Phi, U) - \sum_{i=1}^{n} \gamma(\Phi, U_i) \\ &= \gamma(\Phi, U) + \sum_{i=1}^{n} \gamma(\Phi, -U_i) = \sum_{i=1}^{m} \gamma(\Phi, \mathbf{G}_i) = 0. \end{aligned}$$

□

A direct consequence of Proposition 1.30 is that the rotation fulfills the additivity property (D2) of Proposition 1.5 for open-connected sets Γ. Moreover, we notice that the vector field Φ, generated by the identity, has only one equilibrium - the origin. Given an open-connected set Γ such that Id is non-vanishing on its boundary, i.e., $\mathbf{0} \notin \partial\Gamma$, we have two possibilities. In the case of $\mathbf{0} \notin \Gamma$ it follows from Proposition 1.30 that $\gamma(\mathrm{Id}, \Gamma) = 0$. If $\mathbf{0} \in \Gamma$ it follows from the fact that the origin is the only zero of Id and (1.20) that $\gamma(\mathrm{Id}, \Gamma) = \gamma(\mathrm{Id}, U)$. But, if we traverse the boundary of a simple closed curve which has the origin in its interior, the vector field on that curve (with respect to the identity) makes one full turn in the anti-clockwise direction. As a consequence we have that $\gamma(\mathrm{Id}, \Gamma) = +1$ and property (D1) of Proposition 1.5 is fulfilled for open-connected sets Γ. It remains to show that the rotation is invariant under homotopies. We follow the ideas of [2, pp. 189–190].

Lemma 1.31. *Let Φ and Ψ be two non-vanishing vector fields on a simple closed curve C. Assume that the vectors $\Phi(\mathbf{x})$ and $\Psi(\mathbf{x})$ do not point in opposite directions for all $\mathbf{x} \in C$, then $\gamma(\Phi, C) = \gamma(\Psi, C)$.*

Proof. Let the curve C be given via a parametrization $\mathbf{x}(t) = (f_1(t), f_2(t))^T$ for $t \in [t_0, T]$. Let $\alpha = \alpha(t)$ be any angle function corresponding to Φ. By assumption it holds that $|\theta(t)| := |\theta(\Phi(\mathbf{x}(t)), \Psi(\mathbf{x}(t)))| < \pi$ for all $[t_0, T]$ and therefore $\theta(t)$ is continuous on that interval. As a consequence we can define $\alpha^*(t) := \alpha(t) + \theta(t)$ for all $t \in [t_0, T]$, which is continuous and by construction

an angle function of Ψ. Note that $\theta(t_0) = \theta(T)$, since $\mathbf{x}(t_0) = \mathbf{x}(T)$ and therefore

$$\begin{aligned}\gamma(\Psi, C) &= \frac{1}{2\pi}[\alpha^*(T) - \alpha^*(t_0)] = \frac{1}{2\pi}[\alpha(T) + \theta(T) - \alpha(t_0) - \theta(t_0)] \\ &= \frac{1}{2\pi}[\alpha(T) - \alpha(t_0)] = \gamma(\Phi, C).\end{aligned}$$

□

Lemma 1.32. *Let Φ and Ψ be two non-vanishing vector fields on a simple closed curve C. If there exists a homotopy $H : [0,1] \times C \to \mathbb{R}^2$ with $H(\lambda, \mathbf{x}) \neq 0$ for all $\lambda \in [0,1],\ \mathbf{x} \in C$ and $H(0,\cdot) = \Phi$ and $H(1,\cdot) = \Psi$, then $\gamma(\Phi, C) = \gamma(\Psi, C)$.*

Proof. From the continuity of H on the compact set $[0,1] \times C$, it follows again that H is uniformly continuous. Therefore there exists $\delta > 0$ such that $|\theta(H(\lambda^{'}, \mathbf{x}), H(\lambda^{''}, \mathbf{x}))| < \pi$ holds for all $\mathbf{x} \in C$ whenever $|\lambda^{'} - \lambda^{''}| < \delta$. On the interval $[0,1]$ we define a partition $0 = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 1$, such that $|\lambda_i - \lambda_{i+1}| < \delta$ for all $i = 0, ..., n-1$. From Lemma 1.31 we conclude that $\gamma(H(\lambda_i, \cdot), C) = \gamma(H(\lambda_{i+1}, \cdot), C)$ for all $i = 0, ..., n-1$. In particular we get that $\gamma(H, C)$ is constant for all $\lambda \in [\lambda_i, \lambda_{i+1}]$ and therefore $\gamma(H(0,\cdot), C) = \gamma(H(1,\cdot), C)$. □

So far we have shown the homotopy invariance only for a simple closed curve. Recall that in (1.20) the rotation of an open-connected set Γ is defined via the rotation of all its parts, i.e., a sum of rotations of simple closed curves. As a result, we can state an immediate corollary of Lemma 1.32.

Corollary 1.33. *Let Φ and Ψ be two vector fields, defined on an open-connected set Γ and non-vanishing on $\partial\Gamma$. If there exists a homotopy $H : [0,1] \times \partial\Gamma \to \mathbb{R}^2$ with $H(\lambda, \mathbf{x}) \neq 0$ for all $\lambda \in [0,1],\ \mathbf{x} \in \partial\Gamma$ and $H(0,\cdot) = \Phi$ and $H(1,\cdot) = \Psi$, then $\gamma(\Phi, \Gamma) = \gamma(\Psi, \Gamma)$.*

We have now shown, that the rotation has the properties (D1)-(D3) of Proposition 1.5. Since these three properties define the degree uniquely we can conclude that given an open-connected set $\Gamma \subset \mathbb{R}^2$ and a continuous vector field $\Phi : \overline{\Gamma} \to \mathbb{R}^2$, non-vanishing on $\partial\Gamma$, it holds that $\gamma(\Phi, \Gamma) = \mathrm{Deg}(\Phi, \Gamma, 0)$.

Remark 1.34. In Proposition 1.5 we have restricted the degree to continuously differentiable maps, which met the needs of our discussion there. However, at this point we want to mention that this definition can also be extended, via the Weierstrass approximation theorem, to continuous maps, see [15, p. 148].

Thus, all the results established for the degree, as well as the notion of the index of an isolated equilibrium, can be used for the rotation. Recall, from Definition 1.12, that for an equilibrium $\hat{\mathbf{x}}$ with isolating neighbourhood V we have defined $\mathrm{Ind}_\Phi(\hat{\mathbf{x}}) = \mathrm{Deg}(-\Phi, V, 0)$. Unlike the rotation, the degree in two dimensions

was defined for arbitrary open and bounded subsets of $\mathbb{R}^2$. However, given any isolating neighbourhood $V \subset \mathbb{R}^2$, we can inscribe a circle C with center $\hat{\mathbf{x}}$ with small enough radius such that $C \subset V$ and define $\mathrm{Ind}_\Phi(\hat{\mathbf{x}}) := \gamma(-\Phi, C) = \mathrm{Deg}(-\Phi, C, 0) = \mathrm{Deg}(-\Phi, V, 0)$. From Proposition 1.30 and Definition 1.12 we see that this definition is independent of the choice of C. A direct proof of the equivalence of the index defined via the rotation and the index defined via the degree is given in [6, p. 30]. It should be kept in mind that the rotation is an even function, i.e., $\gamma(-\Phi, C) = \gamma(\Phi, C)$, and therefore $\mathrm{Ind}_\Phi(\hat{\mathbf{x}}) = \gamma(\Phi, C)$ holds.

2 Index Formulas

In the following we will use some geometrical arguments in order to derive results on the rotation of a plane vector field. It will often be convenient to make a change to polar coordinates. We will study a differential equation of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \dot{\mathbf{x}} = \Phi(\mathbf{x}) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}, \tag{2.1}$$

with P and Q continuously differentiable. By substituting $r = \sqrt{x^2+y^2}$, $\tan\varphi = \frac{y}{x}$ and applying the chain rule we arrive at

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} = xP + yQ, \\ \dot{\varphi} &= \frac{1}{1+(\frac{y}{x})^2}\frac{\dot{y}x - \dot{x}y}{x^2} = \frac{Qx - Py}{r^2}. \end{aligned} \tag{2.2}$$

2.1 Phase Portrait near an Equilibrium

The material covered in this chapter can be found in [7, pp. 17–18].
Consider an isolated equilibrium $\hat{\mathbf{x}}$ of (2.1). Then there exists an isolating neighbourhood W such that ∂W is a parametrized simple closed curve. A characteristic orbit (of an equilibrium $\hat{\mathbf{x}}$) is an orbit $\mathbf{x}(t)$ of some point $\mathbf{x} \neq \hat{\mathbf{x}} \in W$ tending to $\hat{\mathbf{x}}$ in a fixed direction, i.e., $\lim_{t\to\infty} \|\mathbf{x}(t) - \hat{\mathbf{x}}\| = 0$ (respectively $\lim_{t\to-\infty} \|\mathbf{x}(t) - \hat{\mathbf{x}}\| = 0$) and the limit

$$\lim_{t\to\infty} \frac{\mathbf{x}(t) - \hat{\mathbf{x}}}{\|\mathbf{x}(t) - \hat{\mathbf{x}}\|} \qquad \left(\text{respectively } \lim_{t\to-\infty} \frac{\mathbf{x}(t) - \hat{\mathbf{x}}}{\|\mathbf{x}(t) - \hat{\mathbf{x}}\|}\right)$$

exists, and the orbit transversely intersects ∂W at a unique point $\mathbf{p} \in \partial W$. We say that $\hat{\mathbf{x}}$ has the finite sectorial decomposition property if there exists an isolating neighbourhood V of $\hat{\mathbf{x}}$ such that $C := \partial V$ is a parametrized simple closed curve and one of the following conditions is satisfied:

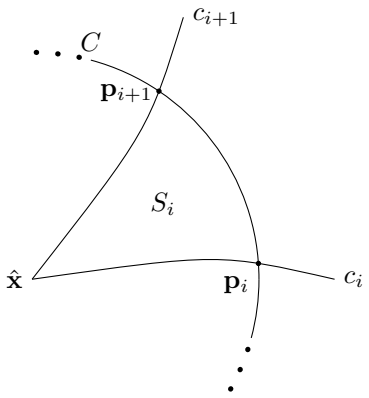


Figure 8: Sector S_i.

(i) If C is a periodic orbit, i.e., there exists $T \in \mathbb{R}$ such that $\mathbf{x}(T) = \mathbf{x} \in C$, and all orbits in V are periodic, we say that $\hat{\mathbf{x}}$ is a center.

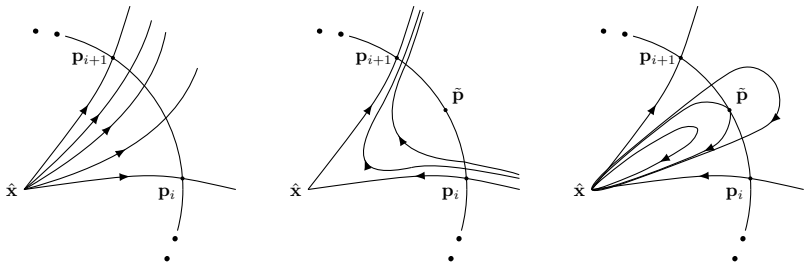
(ii) If at all points of C the vector field points inward, the ω-limit of all $\mathbf{x} \neq \hat{\mathbf{x}} \in V$ only consists of $\hat{\mathbf{x}}$, i.e., $\omega(\mathbf{x}) = \{\hat{\mathbf{x}}\}$, and there exists $t < 0$ such that $\mathbf{x}(t) \cap C \neq \emptyset$. Then we say that $\hat{\mathbf{x}}$ is an attracting spiral/node.

(iii) If at all points of C the vector field points outward, the α-limit of all $\mathbf{x} \neq \hat{\mathbf{x}} \in V$ only consists of $\hat{\mathbf{x}}$, i.e., $\alpha(\mathbf{x}) = \{\hat{\mathbf{x}}\}$, and there exists $t > 0$ such that $\mathbf{x}(t) \cap C \neq \emptyset$. Then we say that $\hat{\mathbf{x}}$ is a repelling spiral/node.

(iv) We say that $\hat{\mathbf{x}}$ has a non-trivial finite sectorial decomposition if we are not in one of the cases (i), (ii) or (iii) above and if there is a finite number of characteristic orbits $c_0, ..., c_{n-1}$ each intersecting C at a unique point $\mathbf{p}_0, ..., \mathbf{p}_{n-1}$ (ordered in such a way that the order induced by the parametrization of C is respected) such that within one sector S_i (the compact region bounded by $\mathbf{p}_i$, c_i, $\hat{\mathbf{x}}$, c_{i+1}, $\mathbf{p}_{i+1}$ and the piece of C between $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$, where c_0=c_n, see Figure 8) there is one of the following situations:

   (a) If at all points of the subarc $\mathbf{p}_i\mathbf{p}_{i+1} \subset C$ the vector field points inward (outward) and for all $\mathbf{x} \neq \hat{\mathbf{x}} \in S_i$ $\omega(\mathbf{x}) = \{\hat{\mathbf{x}}\}$ ($\alpha(\mathbf{x}) = \{\hat{\mathbf{x}}\}$) and there exists $t < 0$ ($t > 0$) such that $\mathbf{x}(t) \cap C \neq \emptyset$, then we say that S_i is an attracting (a repelling) parabolic sector. The phase portrait in a repelling parabolic sector is shown in Figure 9a. In the attracting case one may think of the same picture with reversed flow direction.

   (b) If there is a point $\tilde{\mathbf{p}} \in \mathbf{p}_i\mathbf{p}_{i+1} \subset C$ such that at all $\mathbf{p} \neq \tilde{\mathbf{p}} \in \mathbf{p}_i\tilde{\mathbf{p}}$ the flow points inwards (outwards) while at all $\mathbf{p} \neq \tilde{\mathbf{p}} \in \tilde{\mathbf{p}}\mathbf{p}_{i+1}$ the flow points outwards (inwards), the flow at $\tilde{\mathbf{p}}$ is tangent to C and $\tilde{\mathbf{p}}(t) \notin S_i$



(a) Parabolic sector (b) Hyperbolic sector (c) Elliptic sector

Figure 9: Different flows in a sector.

for all $t \neq 0$, while for all $\mathbf{x} \in \operatorname{int} S_i$ there are $t_1 < 0 < t_2$ such that $\mathbf{x}(t_i) \cap C \neq \emptyset$ for $i = 1, 2$, then we say that S_i is a hyperbolic sector. An example is shown in Figure 9b.

(c) If there exists a point $\tilde{\mathbf{p}} \in \mathbf{p}_i\mathbf{p}_{i+1}$ such that $\alpha(\tilde{\mathbf{p}}) = \omega(\tilde{\mathbf{p}}) = \{\hat{\mathbf{x}}\}$ and $\tilde{\mathbf{p}}(t) \subset S_i$ for all $t \in \mathbb{R}$; At all points $\mathbf{p} \neq \tilde{\mathbf{p}} \in \mathbf{p}_i\tilde{\mathbf{p}}$ the flow points inwards, $\omega(\mathbf{p}) = \{\hat{\mathbf{x}}\}$ and $\mathbf{p}(t) \subset \operatorname{int} S_i$ for all $t > 0$ while at all $\mathbf{p} \neq \tilde{\mathbf{p}} \in \tilde{\mathbf{p}}\mathbf{p}_{i+1}$ the flow points outwards, $\alpha(\mathbf{p}) = \{\hat{\mathbf{x}}\}$ and $\mathbf{p}(t) \subset \operatorname{int} S_i$ for all $t < 0$; Define $F_1 := \bigcup_{\mathbf{p} \in \mathbf{p}_i\tilde{\mathbf{p}}} \{\mathbf{p}(t) : t \geq 0\}$ and $F_2 := \bigcup_{\mathbf{p} \in \tilde{\mathbf{p}}\mathbf{p}_{i+1}} \{\mathbf{p}(t) : t \leq 0\}$, then for all $\mathbf{x} \in S_i \setminus (F_1 \cup F_2)$ it holds that $\omega(\mathbf{x}) = \alpha(\mathbf{x}) = \hat{\mathbf{x}}$, then we say that S_i is an elliptic sector, see Figure 9c for an example. The same is also true for reversed flow directions.

In the cases (i)-(iii) we speak of a trivial sectorial decomposition, while case (iv) is referred to as non-trivial sectorial decomposition. The definition of sectors used here is not the most general, for an extensive treatment one may see [2, Chapters 7-8].

2.2 Bendixson's Index Formula

Assume now that $\hat{\mathbf{x}}$ has a non-trivial finite sectorial decomposition property and that the characteristic orbits $c_0, ..., c_{n-1}$ are given by straight lines and divide C, which is assumed to be the boundary of a circular isolating neighbourhood, into $n \geq 1$ equally large parts.

Let $\mathbf{p}_i\mathbf{p}_{i+1}$ be a subarc of C corresponding to a repelling parabolic sector. Since the characteristic orbits are straight lines and the subarcs between two orbits on C are of the same length, it holds that $\theta(\mathbf{p}_i, \mathbf{p}_{i+1}) = \frac{2\pi}{n}$. A possible angle function on this arc is given by $\alpha(\mathbf{p}) = \theta(\mathbf{e}_1, \mathbf{p}_i) + \theta(\mathbf{p}_i, \Phi(\mathbf{p}))$, where $\mathbf{e}_1$ is the first unit vector. Since we have chosen $c_0, ..., c_{n-1}$ as straight lines we have for

all $k = 0, ..., n-1$ $\Phi(\mathbf{p}_k) = c(\mathbf{p}_k)\mathbf{p}_k$ for some $c(\mathbf{p}_k) > 0$ and therefore

$$\gamma(\Phi, \mathbf{p}_i\mathbf{p}_{i+1}) = \frac{1}{2\pi}[\alpha(\mathbf{p}_{i+1}) - \alpha(\mathbf{p}_i)] = \frac{1}{2\pi}\theta(\mathbf{p}_i, c(\mathbf{p}_{i+1})\mathbf{p}_{i+1}) = \frac{1}{n}. \tag{2.3}$$

If we reverse the flow direction, i.e., we look at the vector field $-\Phi$ instead of Φ, a repelling parabolic sector becomes attracting. The rotation is an even function and therefore we have for an attracting, as well as for a repelling, parabolic sector $\gamma(\Psi, \mathbf{p}_i\mathbf{p}_{i+1}) = \gamma(-\Psi, \mathbf{p}_i\mathbf{p}_{i+1}) = \frac{1}{n}$. In what follows, we therefore will not distinguish between these two cases.

Now consider a hyperbolic sector, as shown in Figure 9b. If the flow direction is reversed, we can again use the fact that the rotation is an even function. An angle function for $\mathbf{p} \in \mathbf{p}_i\mathbf{p}_{i+1}$ is given by $\alpha(\mathbf{p}) = \theta(\mathbf{e}_1, -\mathbf{p}_i) + \theta(-\mathbf{p}_i, \Phi(\mathbf{p}))$, which is leading us to

$$\gamma(\Phi, \mathbf{p}_i\mathbf{p}_{i+1}) = \frac{1}{2\pi}\theta(-\mathbf{p}_i, c(\mathbf{p}_{i+1})\mathbf{p}_{i+1}) = \frac{1}{2\pi}\Big[-\Big(\pi - \frac{2\pi}{n}\Big)\Big] = -\frac{1}{2} + \frac{1}{n}. \tag{2.4}$$

Given an elliptic sector, we can use the same angle function as in the hyperbolic case to arrive at

$$\gamma(\Phi, \mathbf{p}_i\mathbf{p}_{i+1}) = \frac{1}{2\pi}\theta(-\mathbf{p}_i, c(\mathbf{p}_{i+1})\mathbf{p}_{i+1}) = \frac{1}{2\pi}\Big[\Big(\pi + \frac{2\pi}{n}\Big)\Big] = \frac{1}{2} + \frac{1}{n}. \tag{2.5}$$

Define E as the number of elliptic sectors, H as the number of hyperbolic sectors and P as the number of parabolic sectors. Under the above assumptions we have for the vector field Φ that $n = P + H + E$. Combining now the results of (2.3), (2.4), (2.5) and using the additive definition of the rotation in (1.18) we arrive at

$$\begin{aligned}\gamma(\Phi, C) &= \sum_{i=0}^{n-1}\gamma(\Phi, \mathbf{p}_i\mathbf{p}_{i+1}) = \sum_{j=1}^{P}\frac{1}{n} + \sum_{k=1}^{H}\Big(-\frac{1}{2} + \frac{1}{n}\Big) + \sum_{l=1}^{E}\Big(\frac{1}{2} + \frac{1}{n}\Big) \\ &= \frac{P + H + E}{n} + \frac{E - H}{2} = 1 + \frac{E - H}{2}.\end{aligned} \tag{2.6}$$

We can also include the cases (i)-(iii) of the finite sectorial decomposition property. If $\hat{\mathbf{x}}$ is a node (a small neighbourhood of the equilibrium does not contain spiraling solutions), then it holds that $E = H = 0$ and we define $P = 1$. It can be shown that the index of a node is $+1$, which also fits (2.6). If there are spiraling solutions (center or spiral) then we say by definition that $E = H = P = 0$. Since the index of a center or a spiral is also $+1$, we are done.

This is a special case of an old result of Bendixson. The ideas in the derivation are taken from [7, Chapter 6.7]. We have made some simplifications to avoid technical difficulties, but the formula holds in a more general way, as shown in [3].

Theorem 2.1 (Bendixson's Index Formula)**.** *Let* $\hat{\mathbf{x}} \in \mathbb{R}^2$ *be an isolated equilibrium of the vector field* Φ *having the finite sectorial decomposition property. Let E, H and P denote the number of elliptic, hyperbolic and parabolic sectors respectively. Then* $\mathrm{Ind}_\Phi(\hat{\mathbf{x}}) = 1 + \frac{E-H}{2}$.

2.3 Geometric Index Formula for Biological Systems

Let us consider a differential equation $\dot{\mathbf{x}} = \Phi(\mathbf{x})$ on $\mathbb{R}^2_+$. Assume that $\mathbb{R}^2_+$ is forward invariant, i.e., (1.5) holds, and that $\hat{\mathbf{x}} \in \partial\mathbb{R}^2_+$ is an isolated, saturated equilibrium which has the finite sectorial decomposition property. Similar to Theorem 2.1, we want to draw conclusions on $\mathrm{Bd\_Ind}_\Phi(\hat{\mathbf{x}})$, by examining the phase portrait of the differential equation near $\hat{x}$. Since we are interested in the boundary index, we restrict our attention to the sectorial decomposition within the non-negative orthant $\mathbb{R}^2_+$. The main tools we use are Theorem 1.22 and homotopic transformations. In the current chapter the homotopic transformations (which are non-trivial) are not stated explicitly, rather we will content ourselves with a geometric discussion.

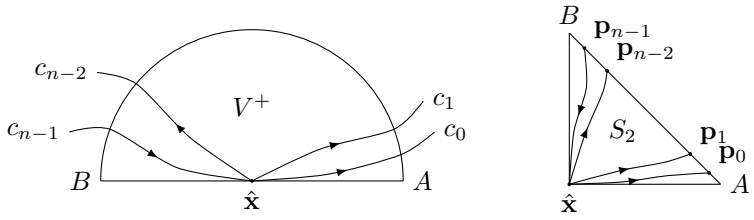
2.3.1 Preparatory steps

Since $\hat{\mathbf{x}}$ has the finite sectorial decomposition property, there exists an isolating neighbourhood V, such that ∂V is a simple closed curve. Define $V^+ := V \cap \mathbb{R}^2_+$. The forward invariance assumption excludes centers and spirals. Assume now that there is at least one characteristic orbit in V^+.

When it comes to the boundary index, we are only interested in the local behaviour of the vector field. We therefore compactify V^+, and the flow there, with a suitable homotopy to the simplex S_2. The characteristic orbits, $c_0, ..., c_{n-1} \subset \mathbb{R}^2_+$ with $n \geq 1$, will then lead to, or come from, isolated equilibria $\mathbf{p}_0, ..., \mathbf{p}_{n-1}$ respectively, on the boundary of S_2, see Figure 10. Moreover, we can choose the homotopy in such a way that the external eigenvalue at each equilibrium $\mathbf{p}_i$ is nonzero and therefore $\mathrm{Bd\_Ind}_\Phi(\mathbf{p}_i) \in \{-1, 0, +1\}$. In addition, the edge AB, which contains the equilibria $\mathbf{p}_i$, is invariant under the flow.

Now the special case of Theorem 1.22 on the simplex is applicable. The flow direction of the characteristic orbit c_i determines if the equilibrium $\mathbf{p}_i$ is saturated or not. This allows us to decide, whether we have to include $\mathbf{p}_i$ in the index sum. Since the index is invariant under homotopies, the transformations did not change it and all the results we obtain for $\hat{\mathbf{x}}$ and the compactified flow will also hold for the equilibrium in the original system.

Next we will show, that we can w.l.o.g. assume that the whole boundary of the simplex is invariant, i.e., forward and backward in time. Assume we have at least two characteristic orbits, i.e., $c_0, ..., c_{n-1}$ with $n \geq 2$. If c_0 and c_{n-1} coincide



(a) Flow in the isolating neighbourhood.

(b) Flow after compactification.

Figure 10: Homotopic compactification onto the simplex.

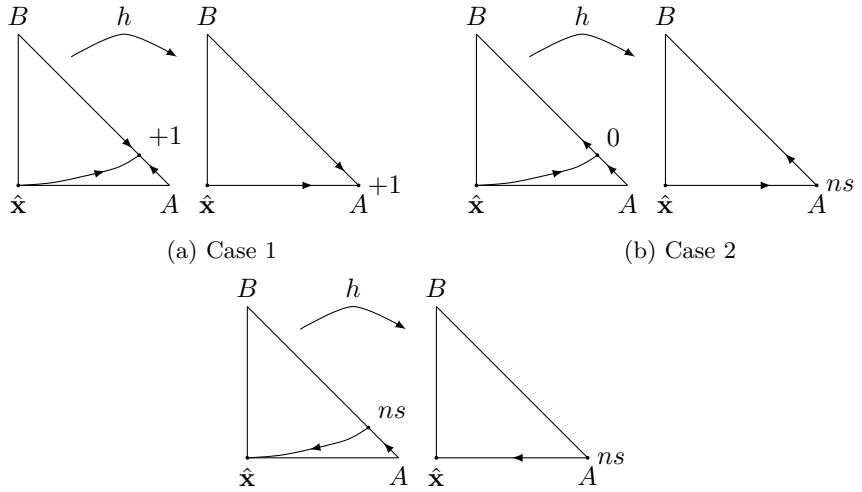
with the boundaries $\hat{\mathbf{x}}A$ and $\hat{\mathbf{x}}B$, we are done. Assume at least one of them does not. We deform the orbits c_0 and c_{n-1} via a homotopy, such that they coincide with the boundary $\hat{\mathbf{x}}A$ and $\hat{\mathbf{x}}B$, respectively.

Since we want to apply Theorem 1.22, it is necessary to determine the boundary index of the equilibria $\mathbf{p}_0, ..., \mathbf{p}_{n-1}$. Together with the result on the boundary index sum, it is then possible to find out the boundary index of $\hat{\mathbf{x}}$. To this end, we have to make sure that the index sum of all the equilibria, is invariant under the homotopic transformations.

If c_0 and c_{n-1} lead to (or come from) A and B, we may deform these orbits such that they overlap with $\hat{\mathbf{x}}A$ and $\hat{\mathbf{x}}B$, respectively. This has no influence on the equilibrium structure and therefore does not alter the index sum. Now assume that at least one of the two orbits does not lead to (or come from) one of the corners A or B, w.l.o.g. consider A and c_0. Then the corner A is no equilibrium and the flow direction on AB is from A to $\mathbf{p}_0$, by forward invariance. After applying an appropriate homotopy, $\mathbf{p}_0$ and $\mathbf{p}_{n-1}$ coincide with A and B, which are now equilibria. Moreover the orbits c_0 and c_{n-1} coincide with the edges $\hat{\mathbf{x}}A$ and $\hat{\mathbf{x}}B$, respectively.

These corner equilibria play an important role in the following analysis, since they have two external eigenvalues. This implies that they are not saturated (ns) as soon as one of these eigenvalues is positive. The deformation did not change the behaviour of the vector field near all the other equilibria. So we only have to check that the boundary index sum of $\mathbf{p}_0$ and $\mathbf{p}_{n-1}$ is invariant under the applied homotopies. In Figure 11 we show this for $\mathbf{p}_0$, the second case is completely analogous.

Let us now consider the case $n = 1$, i.e., only one characteristic orbit c_0. We split this orbit into two, such that a parabolic sector P appears. We will treat this sort of transition in more detail, but in reversed order, in the proof of Theorem 2.4 and Figure 14. As before we move the two emerged orbits to the boundary of the simplex and observe in Figure 12, that this does not change the index sum.

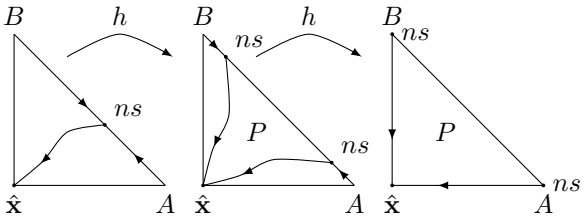


(a) Case 1

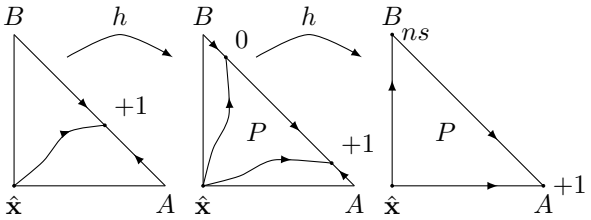
(b) Case 2

(c) Case 3

Figure 11: In Case 1, the equilibrium $\mathbf{p}_0$ is saturated before and after applying the homotopy h. The boundary index remains constant and equals $+1$ since the equilibrium is locally asymptotically stable.
Case 2 shows that before the deformation by h, $\mathbf{p}_0$ is saturated, but $\mathrm{Bd\_Ind}(\mathbf{p}_0) = 0$ since a small perturbation lets this equilibrium disappear. Afterwards it is not saturated (ns), since the flow on $\mathbf{p}_0 B$ points away from $\mathbf{p}_0$ and therefore $\mathbf{p}_0$ does not contribute to the index sum.
In Case 3, the equilibrium is not saturated in all possible configurations, since the flow on c_0 points from $\mathbf{p}_0$ to $\hat{\mathbf{x}}$. We omitted the flow direction on $\mathbf{p}_0 B$, because this has no effect on the justification above.



(a) Case 4



(b) Case 5

Figure 12: In Case 4, we begin with a non-saturated equilibrium $\mathbf{p}_0$. After splitting the characteristic orbit c_0, which connects $\hat{\mathbf{x}}$ and $\mathbf{p}_0$, a parabolic sector P appears. We are left with two non-saturated equilibria at the corners, which also do not contribute to the boundary index sum.
Case 5 covers the same steps as Case 4, but this time $\mathbf{p}_0$ is saturated. After the split, there are two saturated equilibria $\mathbf{p}'$ and $\mathbf{p}''$ with $\text{Bd\_Ind}(\mathbf{p}') = +1$ and $\text{Bd\_Ind}(\mathbf{p}'') = 0$. When moved to the corners, $\mathbf{p}''$ is not saturated anymore and does not contribute to the index sum. The second equilibrium $\mathbf{p}'$ remains saturated with index +1 and therefore the overall index sum is again constant. The same can, of course, be done with reversed flow direction on the arc AB.

2.3.2 Boundary Index Formula

We have now shown that, after applying suitable homotopies, we can work in the framework of a simplex with invariant boundary. Since the deformations did not change the index of $\hat{\mathbf{x}}$, all the results we obtain for the system on the simplex remain valid in the initial case.

Definition 2.2 (Inward extremal characteristic orbits)**.** *Let $\dot{\mathbf{x}} = \Phi(\mathbf{x})$ be an autonomous differential equation on $\mathbb{R}^2_+$, such that $\mathbb{R}^2_+$ is forward invariant. Moreover, let $\hat{\mathbf{x}} \in \partial\mathbb{R}^2_+$ be an isolated equilibrium with isolating neighbourhood V and characteristic orbits $c_0, ..., c_{n-1} \subset V^+$. Then, the number of inward extremal characteristic orbits, $\eta \in \{0, 1, 2\}$, is the number of flows on the characteristic orbits c_0 and c_{n-1} pointing towards $\hat{\mathbf{x}}$.*

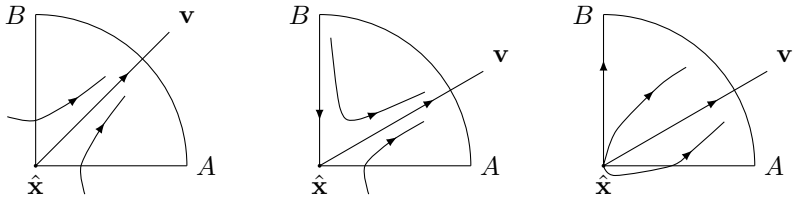
If $n = 1$, we define $\eta := \begin{cases} 2 & \text{if } c_0 \text{ points inwards.} \\ 0 & \text{if } c_0 \text{ points outwards.} \end{cases}$

Remark 2.3. In Figure 10a we see that $\eta = 1$. An Example for $\eta = 2$ and $\eta = 0$ is Figure 12a and Figure 12b, respectively.

Theorem 2.4 (Boundary Index Formula)**.** *Let $\dot{\mathbf{x}} = \Phi(\mathbf{x})$ be an autonomous differential equation on $\mathbb{R}^2_+$. Assume that Φ is continuously differentiable and leaves $\mathbb{R}^2_+$ forward invariant. Consider an isolated, saturated equilibrium $\hat{\mathbf{x}} \in \partial\mathbb{R}^2_+$ having the finite sectorial decomposition property and with isolating neighbourhood V. Assume that there exists at least one characteristic orbit $c_0 \subset V^+$. Let E, H and P be the number of elliptic, hyperbolic and parabolic sectors contained in V^+, respectively. Then* $\mathrm{Bd\_Ind}_\Phi(\hat{\mathbf{x}}) = \frac{\eta}{2} + \frac{E-H}{2}$ *holds.*

Remark 2.5. One may drop the assumption that the equilibrium is saturated. We include it in the boundary index formula, since we heavily exploit Theorem 1.22 in the proof. However, if we consider a non-saturated equilibrium it holds that there is at least one positive eigenvalue of A (the external part of the Jacobian). In particular it follows from the Perron-Frobenius Theorem that the spectral bound $s(A) > 0$ is the leading eigenvalue with a corresponding non-negative eigenvector $\mathbf{v} \in \mathbb{R}^2_+$. So we either get one characteristic orbit (a parabolic sector) or two characteristic orbits (spanning either a parabolic or a hyperbolic sector) as shown in Figure 13a, Figure 13b and Figure 13c respectively. All these cases fit the boundary index formula, since the boundary index of a non-saturated equilibrium is 0.

The assumption on the existence of at least one characteristic orbit in V^+ is not really restricting, since we can consider the remaining case separately. Assume that there is no characteristic orbit in V^+. The forward invariance of $\mathbb{R}^2_+$ ensures that V^+ is neither part of the interior of a hyperbolic sector nor of the interior of an elliptic sector. So it is part of a parabolic sector. It can be shown that if



(a) (b) (c)

Figure 13: Behaviour near a non-saturated equilibrium.

it is attracting, the boundary index of $\hat{\mathbf{x}}$ is $+1$. If it is repelling, the boundary index of $\hat{\mathbf{x}}$ equals 0.

Remark 2.6. Since we assume that $\mathbb{R}^2_+$ is forward invariant, $\hat{\mathbf{x}}$ is neither a center nor a spiral. It can be shown that the boundary index of an attracting (repelling) node is $+1$ (0), it holds that $E = H = 0$ and we define $\eta = 2$ ($\eta = 0$), which fits the boundary index formula. It remains to prove the boundary index formula in the case of a non-trivial sectorial decomposition.
The main steps in the proof will be the following: As the formula suggests, we show that we can neglect parabolic sectors via homotopic deformations. Next, we perform an induction step, which merges an elliptic and a hyperbolic sector into one characteristic orbit. Finally, the statement is verified by checking the induction hypothesis for each value of η separately. The key observation in all these steps is that this can be done without changing the index sum of the saturated equilibria. Therefore we are able to apply Theorem 1.22. All the characteristic orbits c_i in the figures of the proof are simplistically represented by straight lines. Since we are only interested in the types of sectors, this does not affect the argument.

Proof. Consider a non-trivial sectorial decomposition. We have shown above that we can w.l.o.g. assume that the vector field is given on the simplex S_2 with invariant boundary. A parabolic sector occurs if and only if two adjacent characteristic orbits c_i and c_{i+1} have the same flow direction. In this case we can apply a homotopy h, to merge this parabolic sector to one single characteristic orbit $\tilde{c}$. Therefore also the equilibria $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ on the edge AB will be replaced by a single equilibrium $\tilde{\mathbf{p}}$.
In Figure 14, all possible flow configurations are shown. We choose the homotopy h in such a way that the equilibria in all the remaining sectors are not affected. All possible flows have in common that the index sum is constant when applying h, i.e., $\mathrm{Bd\_Ind}_\Phi(\mathbf{p}_i) + \mathrm{Bd\_Ind}_\Phi(\mathbf{p}_{i+1}) = \mathrm{Bd\_Ind}_\Phi(\tilde{\mathbf{p}})$. If $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ are not saturated, then neither is $\tilde{\mathbf{p}}$ and consequently does not contribute to the index

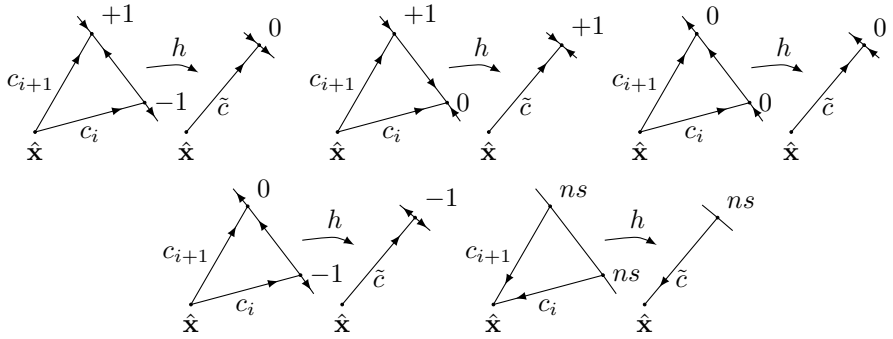


Figure 14: Merging a parabolic sector.

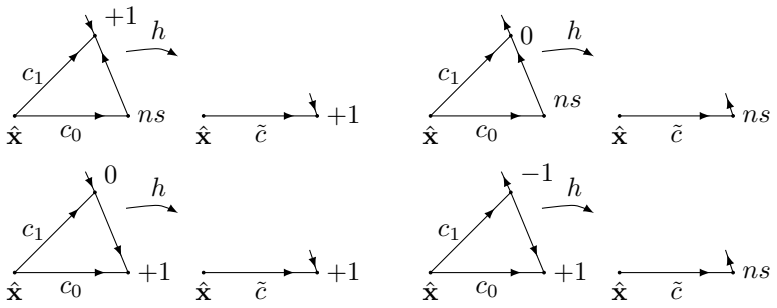


Figure 15: Merging a parabolic sector on the boundary.

sum. As mentioned above, the boundary equilibria have to be considered separately since they have two external eigenvalues.

In Figure 15, the situation on the boundary is shown. Again we observe that $\mathrm{Bd\_Ind}_\Phi(\mathbf{p}_0) + \mathrm{Bd\_Ind}_\Phi(\mathbf{p}_1) = \mathrm{Bd\_Ind}_\Phi(\tilde{\mathbf{p}})$, (here non-saturated equilibria are assigned with $\mathrm{Bd\_Ind} = 0$). If the flow on c_0 and c_1 is reversed, i.e., $\mathbf{p}_0$ and $\mathbf{p}_1$ are not saturated, then $\tilde{\mathbf{p}}$ is also not saturated and the index sum is constant under h. Therefore this procedure has no influence on the index of $\hat{\mathbf{x}}$, and we can get rid of all parabolic sectors, until we are left with a single characteristic orbit or a phase portrait with only hyperbolic and elliptic sectors.

We will perform an induction, therefore define E_{old} and H_{old} as the number of elliptic and hyperbolic sectors before the induction step respectively. After this step we will denote these numbers by E_{new} and H_{new}. Let now E_{old}, $H_{old} \geq 1$, meaning there are $n \geq 3$ characteristic orbits. Since we have already excluded parabolic sectors (by homotopic deformations), there exists a neighbouring pair consisting of an elliptic and a hyperbolic sector, as shown in Figure 16.

In Figure 16a, both sectors are bounded by the characteristic orbits c_E, c_H (leading to the saturated equilibria of the elliptic and hyperbolic sector, respectively) and the line $\mathbf{p}_E\mathbf{p}_H$. Since $\mathbf{p}_E$ is saturated and belongs to the elliptic sector,

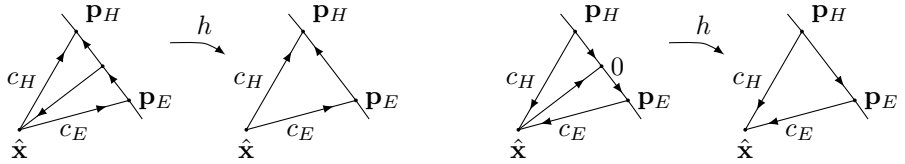


Figure 16: Merging hyperbolic and elliptic sectors.

the flow on $\mathbf{p}_E\mathbf{p}_H$ has to point away from it. The saturated equilibrium $\mathbf{p}_H$ of the hyperbolic sector must be attracting on this line. As a consequence, the non-saturated equilibrium between $\mathbf{p}_E$ and $\mathbf{p}_H$ has internal eigenvalue 0 and can be removed via a small perturbation h. The qualitative behaviour around the other two equilibria $\mathbf{p}_E$ and $\mathbf{p}_H$ does not change. Therefore also their index remains constant and we have created a parabolic sector. However, as we have shown above, we can merge this parabolic sector into a single orbit, without changing the index sum.

Consider now a flow as indicated in Figure 16b. Here, the equilibrium between $\mathbf{p}_E$ and $\mathbf{p}_H$ is saturated. Anyway, it has boundary index zero, since it can again be removed via a small perturbation. A parabolic sector remains, but as before we can merge it into a single characteristic orbit without changing the index sum.

Since the deformations did not change the index sum it holds by Theorem 1.22, that the index of $\hat{\mathbf{x}}$ is the same before and after the homotopic deformations, i.e., $\mathrm{Bd\_Ind}_\Phi(\hat{\mathbf{x}}) = \mathrm{Bd\_Ind}_{h(\Phi)}(\hat{\mathbf{x}})$. We assumed that we had $n \geq 3$ characteristic orbits and we have merged three of them into one. Consequently, there are $n-2$ orbits left and $E_{new} = E_{old} - 1$ and $H_{new} = H_{old} - 1$ holds. We can use the induction hypothesis on the transformed system to get

$$\begin{aligned}\mathrm{Bd\_Ind}_\Phi(\hat{\mathbf{x}}) &= \mathrm{Bd\_Ind}_{h(\Phi)}(\hat{\mathbf{x}}) \\ &= \frac{\eta}{2} + \frac{E_{new} - H_{new}}{2} = \frac{\eta}{2} + \frac{E_{old} - H_{old}}{2}.\end{aligned}$$

Until now we have shown that, given any flow in a non-negative isolated neighbourhood of the saturated boundary equilibrium $\hat{\mathbf{x}}$ with $n \geq 3$ characteristic orbits, we can first merge all parabolic sectors together. If, after that, we have H, $E \geq 1$, we must have a neighbouring pair of sectors, one of hyperbolic type and one of elliptic type. We can also merge these two sectors. If we continue in this manner, we will end up with $n = 1$ or $n = 2$ characteristic orbits, only hyperbolic or only elliptic sectors. It remains to show the statement for these four basic scenarios. We have to consider three different cases, depending on the

number of inward extremal orbits η.
Let us start with the case $\eta = 2$. In the scenarios $n = 1$ and $n = 2$, there is one single characteristic orbit or one parabolic sector, and therefore $E = H = 0$. As shown in Figure 12, $n = 1$ can be transformed into the case $n = 2$. Assume now we have already transformed the system, such that the boundary of S_2 is invariant. Since $\eta = 2$, the two corner equilibria $\mathbf{p}_0$ and $\mathbf{p}_1$ are both not saturated and $\hat{\mathbf{x}}$ is the only saturated equilibrium on the simplex. According to Theorem 1.22, it must hold that the index sum of all the saturated equilibria equals $+1$ and therefore $\text{Bd\_Ind}_\Phi(\hat{\mathbf{x}}) = +1 = \frac{\eta}{2}$. We have now shown the cases $n = 1$ and $n = 2$.
Assume that there are only elliptic sectors, i.e., $H = P = 0$. An elliptic sector can occur if and only if two adjacent characteristic orbits, c_i and c_{i+1}, have different flow direction and the saturated equilibrium on AB, belonging to this sector, is repelling on AB (restricted to this sector). Since we assume that $\eta = 2$, c_0 and c_{n-1} have the same flow direction (inward) and $n = 2k + 1$ for some $k \geq 1 \in \mathbb{N}$ must hold. Consequently there are $E = n - 1$ elliptic sectors. Since c_0 and c_{n-1} both lead to non-saturated equilibria, we have $\frac{n-1}{2} = k$ saturated equilibria $\mathbf{p}_i$ $i = 1, 3, ..., 2k - 1$ with $\text{Bd\_Ind}_\Phi(\mathbf{p}_i) = -1$ (repelling on AB). An application of Theorem 1.22 yields

$$\text{Bd\_Ind}_\Phi(\hat{\mathbf{x}}) = 1 - \sum_{i=1}^{k} \text{Bd\_Ind}_\Phi(\mathbf{p}_{2i-1}) = 1 + \frac{n-1}{2} = \frac{2}{2} + \frac{E}{2} = \frac{\eta}{2} + \frac{E}{2}.$$

Assume now that we only have hyperbolic sectors, i.e., $E = P = 0$. Such a sector can occur if and only if two adjacent orbits have different flow direction and the saturated equilibrium on AB, belonging to this sector, is attracting on AB (restricted to this sector). Again we have $k = \frac{n-1}{2}$ saturated equilibria $\mathbf{p}_i$. This time there are $H = n - 1$ hyperbolic sectors and $\text{Bd\_Ind}_\Phi(\mathbf{p}_i) = +1$ (attracting on AB), and therefore by Theorem 1.22 we conclude that

$$\text{Bd\_Ind}_\Phi(\hat{\mathbf{x}}) = 1 - \sum_{i=1}^{k} \text{Bd\_Ind}_\Phi(\mathbf{p}_{2i-1}) = 1 - \frac{n-1}{2} = \frac{2}{2} - \frac{H}{2} = \frac{\eta}{2} - \frac{H}{2}.$$

Let $\eta = 0$. As in the case $\eta = 2$, we can transform the case $n = 1$ and assume w.l.o.g that we have $n = 2$ characteristic orbits on the boundary of the simplex. Again these orbits bound a parabolic sector, i.e., $E = H = 0$. Since $\eta = 0$, depending on the flow on $\mathbf{p}_0\mathbf{p}_1$, one of the corner equilibria $\mathbf{p}_0$ and $\mathbf{p}_1$ is saturated with $\text{Bd\_Ind} = +1$ and the other one is not saturated. Assume w.l.o.g. that $\mathbf{p}_0$ is saturated. From Theorem 1.22 it follows that $\text{Bd\_Ind}_\Phi(\hat{\mathbf{x}}) = 1 - \text{Bd\_Ind}_\Phi(\mathbf{p}_0) = 0 = \frac{\eta}{2}$.
From $\eta = 0$, it follows that c_0 and c_{n-1} have the same flow direction (outward).

As above, there must be $n = 2k + 1$ characteristic orbits to have only elliptic, or only hyperbolic sectors. Assume we have only elliptic sectors, i.e., $E = n - 1$ and $P = H = 0$. Therefore, the flow on AB has to be repelling for all the equilibria with inward flow from their characteristic orbit. This implies that the two corner equilibria $\mathbf{p}_0$ and $\mathbf{p}_{n-1}$ are not saturated (even if the flow on c_0 and c_{n-1} points in their direction). Consequently there are $\frac{n-1}{2} - 1 = k - 1$ saturated equilibria $\mathbf{p}_i$, $i = 2, 4, ..., n - 3$ with $\text{Bd\_Ind}_\Phi(\mathbf{p}_i) = -1$. Theorem 1.22 implies

$$\text{Bd\_Ind}_\Phi(\hat{\mathbf{x}}) = 1 - \sum_{i=1}^{k-1} \text{Bd\_Ind}_\Phi(\mathbf{p}_{2i}) = 1 + \frac{2k-2}{2} = \frac{E}{2} = \frac{\eta}{2} + \frac{E}{2}.$$

If we have only hyperbolic orbits, i.e., $H = n - 1$ and $P = E = 0$, then the flow on AB has to be attracting for all the equilibria with inward flow on their characteristic orbit. Therefore the two corner equilibria are saturated with boundary index $+1$ in this case. The same is true for all the other saturated equilibria and we have $\frac{n-1}{2} + 1 = k + 1$ saturated equilibria $\mathbf{p}_i$, $i = 0, 2, ..., n - 1$ with $\text{Bd\_Ind}_\Phi(\mathbf{p}_i) = +1$ and therefore

$$\text{Bd\_Ind}_\Phi(\hat{\mathbf{x}}) = 1 - \sum_{i=0}^{k} \text{Bd\_Ind}_\Phi(\mathbf{p}_{2i}) = 1 - \frac{2k+2}{2} = \frac{H}{2} = \frac{\eta}{2} + \frac{H}{2}.$$

We conclude with the case $\eta = 1$. It follows from Definition 2.2, that one characteristic orbit, i.e., $n = 1$, is not possible. Consider the case $n = 2$, where we have either $H = 1$ or $E = 1$. Assume that $H = 1$ and w.l.o.g. that the corner equilibrium $\mathbf{p}_0$ is saturated, i.e., $\text{Bd\_Ind}_\Phi(\mathbf{p}_0) = +1$. Then $\text{Bd\_Ind}_\Phi(\hat{\mathbf{x}}) = 1 - \text{Bd\_Ind}_\Phi(\mathbf{p}_0) = 0 = \frac{\eta}{2} - \frac{H}{2}$. If $E = 1$, then both corner equilibria are not saturated and it must hold that $\text{Bd\_Ind}_\Phi(\hat{\mathbf{x}}) = 1 = \frac{\eta}{2} + \frac{E}{2}$. Consider the case of elliptic sectors only, i.e., $P = H = 0$. Since we have different flow directions on c_0 and c_{n-1} there must be an even number of characteristic orbits $n = 2k$ for some $k \geq 1 \in \mathbb{N}$. This leads to $E = n - 1$ elliptic sectors and again the flows of two adjacent orbits have to point in opposing directions. Assume w.l.o.g. that the flow on c_0 points away from $\hat{\mathbf{x}}$. However, since there are only elliptic sectors, $\mathbf{p}_0$ is not saturated and there are $\frac{n}{2} - 1$ saturated equilibria $\mathbf{p}_i$, $i = 2, 4, ..., n - 2$, with boundary index -1. Again we use Theorem 1.22 to arrive at

$$\text{Bd\_Ind}_\Phi(\hat{\mathbf{x}}) = 1 - \sum_{i=1}^{k-1} \text{Bd\_Ind}_\Phi(\mathbf{p}_{2i}) = 1 + \frac{n-2}{2} = \frac{1}{2} + \frac{n-1}{2} = \frac{\eta}{2} + \frac{E}{2}.$$

It remains to show the case of hyperbolic sectors only, i.e., $P = E = 0$ and all saturated equilibria have boundary index $+1$. Again it must hold that $n = 2k$

and $H = n-1$. Assume w.l.o.g. that $\mathbf{p}_0$ is saturated, then there are $\frac{n}{2}$ saturated equilibria $\mathbf{p}_i$, $i = 0, 2, ..., n-2$, leading us to

$$\text{Bd\_Ind}_\Phi(\hat{\mathbf{x}}) = 1 - \sum_{i=0}^{k-1} \text{Bd\_Ind}_\Phi(\mathbf{p}_{2i}) = 1 - \frac{n}{2} = \frac{1}{2} - \frac{n-1}{2} = \frac{\eta}{2} - \frac{H}{2}.$$

□

Remark 2.7. Of course one can use the same ideas to give an alternative proof of Theorem 2.1. However, there are minor simplifications that can be made in the arguments. We do not have to distinguish between different values of η, since we are then looking at a complete open neighbourhood of an equilibrium. Therefore we cannot compactify to the simplex, but have to compactify the flow to the unit disk with invariant boundary. With combinatorial considerations on the number of characteristic orbits, we can then verify the cases of only hyperbolic and elliptic sectors. The arguments in the induction step in the case of $E \geq 1$ and $H \geq 1$ remain unchanged, as well as the merging of a parabolic sector. Finally, we only have to consider the basic cases of one and two characteristic orbits separately.

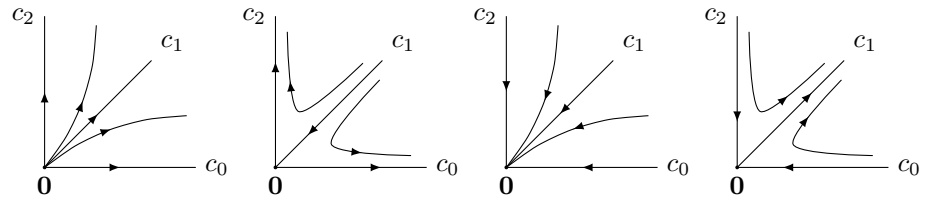
Example 2.8. Consider the degenerate Lotka-Volterra system

$$\begin{aligned} \dot{x} &= x(ax + by) \\ \dot{y} &= y(cx + dy) \end{aligned} \tag{2.7}$$

where $a, b, c, d \in \mathbb{R}$. The Jacobian of (2.7) at the origin is given by $J = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and therefore has a double zero eigenvalue. As a consequence the origin is a saturated equilibrium. Assume that $a \neq 0 \neq d$, otherwise the origin is not isolated. An equilibrium of (2.7), not located on one of the axes, must satisfy $ax + by = 0 = cx + dy$, which is equivalent to $ad = bc$. In this case, there is a line of equilibria through the origin with slope $k = -\frac{a}{b}$. So let $ad \neq bc$, such that Theorem 2.4 is applicable.

The phase portrait of the system depends on the choice of the parameters, we show here only some cases. Let $a, d > 0$ and $b, c < 0$, then the phase portraits are shown in Figure 17a and 17b for the cases $ad > bc$ and $ad < bc$, respectively. In Figure 17a we see two parabolic sectors and in Figure 17b there are two hyperbolic sectors. At the same time the flow on c_0 and c_2 points away from the origin and therefore $\eta = 0$. According to Theorem 2.4 the boundary indices of the origin in Figure 17a and 17b are 0 and -1, respectively.

Consider now the case $a, d < 0$ and $b, c > 0$. If in addition $ad > bc$ holds, then we observe from Figure 17c that there are two parabolic sectors and $\eta = 2$. We infer from the boundary index formula that $\text{Bd\_Ind}(\mathbf{0}) = +1$. In the subcase of



(a) $ad > bc$ (b) $ad < bc$ (c) $ad > bc$ (d) $ad < bc$

Figure 17: Phase portraits of (2.7). In (a) and (b) we have $a, d > 0$ and $b, c < 0$. In (c) and (d) we have $a, d < 0$ and $b, c > 0$.

$ad < bc$, shown in Figure 17d, we have two hyperbolic sectors with $\eta = 2$ and therefore boundary index 0.

2.4 An analytical Approach

As shown in Example 2.8, we can easily determine the boundary index of an isolated and saturated equilibrium with the help of Theorem 2.4. However, it is necessary to know the phase portrait in a neighbourhood of the equilibrium. This is sometimes a quite complicated task and we are therefore interested to develop results where this is not needed. In a first step we will look at a simple class of vector fields - homogeneous systems. In particular, these vector fields are holomorphic and in [13, Chapter 10] we see that this implies that homogeneous systems have the finite sectorial decomposition property.
After treating this special case, the results are generalized to non-homogeneous fields with additional regularity assumptions on the vector fields under consideration. Similar conclusions were already made in [12] for the rotation. We will employ their ideas to derive further results on the boundary index.

2.4.1 Homogeneous Systems

Definition 2.9 (Homogeneous polynomial vector field of degree m)**.** *Consider a vector field of the form* (2.1)*. We say that* Φ *is a homogeneous polynomial vector field of degree* m *if* P *and* Q *are homogeneous polynomials of degree* m*, i.e.,*

$$\begin{aligned} P(x,y) &= \sum_{i=0}^{m} a^i_{m-i} x^{m-i} y^i, \ \textit{with } a^i_{m-i} \in \mathbb{R}, \\ Q(x,y) &= \sum_{i=0}^{m} b^i_{m-i} x^{m-i} y^i, \ \textit{with } b^i_{m-i} \in \mathbb{R}. \end{aligned} \tag{2.8}$$

With a change to polar coordinates and (2.2), we compute for a homogeneous

(a) (b)

Figure 18: Detecting hyperbolic and elliptic sectors

polynomial vector field

$$\begin{aligned}\dot{r} &= r^m \sum_{i=0}^{m} [a^i_{m-i}\cos(\varphi) + b^i_{m-i}\sin(\varphi)]\cos^{m-i}(\varphi)\sin^i(\varphi)\\ \dot{\varphi} &= r^{m-1}\sum_{i=0}^{m}[b^i_{m-i}\cos(\varphi) - a^i_{m-i}\sin(\varphi)]\cos^{m-i}(\varphi)\sin^i(\varphi).\end{aligned} \tag{2.9}$$

We are only interested in forward invariant systems. For homogeneous vector fields this is guaranteed if we assume that $a^m_0 \geq 0$ and $b^0_m \geq 0$. The y-axis is invariant if and only if $\dot{\varphi} = 0$ for all $(r, \frac{\pi}{2})$ with $r > 0$. From (2.9) it follows that this is equivalent to $a^m_0 = 0$. Let $\varphi \neq \frac{\pi}{2}$ then $\dot{\varphi} = 0 \iff \sum_{i=0}^{m}[b^i_{m-i} - a^i_{m-i}\tan(\varphi)]\tan^i(\varphi) = 0$, which is independent of the radius r. Therefore the characteristic orbits, which separate two adjacent sectors are given by straight lines. When it comes to the boundary index, we are only interested in the behaviour in the first quadrant. Therefore the characteristic orbits c_j are given by straight lines through the origin with angle $\varphi_j = \arctan(\tilde{x}_j)$, where $\tilde{x}_j$ are the non-negative solutions of

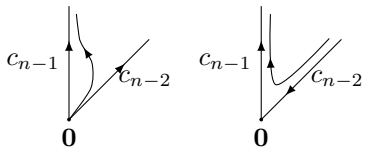
$$Z(x) = \sum_{i=0}^{m} b^i_{m-i}x^i - a^i_{m-i}x^{i+1} = 0. \tag{2.10}$$

The standing assumption is that the origin is isolated, consequently $\dot{r}(r, \varphi_j) \neq 0$ for all $r > 0$ small enough. Since $\cos(\varphi) > 0$ for all $0 \leq \varphi < \frac{\pi}{2}$, the flow direction on c_j is given by the sign of $Y(\tilde{x}_j)$, where

$$Y(x) = \sum_{i=0}^{m} a^i_{m-i}x^i + b^i_{m-i}x^{i+1}. \tag{2.11}$$

In the case of $a^m_0 = 0$, the flow direction on the invariant y-axis is determined by $\operatorname{sgn} b^m_0$.

In order to employ Theorem 2.4, it is necessary to detect hyperbolic and elliptic sectors. We do so by looking at the zeroes of $P(x, y)$ and the behaviour of $Q(x, y)$, from (2.9), at these points. An elliptic or a hyperbolic orbit, as shown in Figure 18, has one point $(\tilde{x}, \tilde{y})$ where $P(\tilde{x}, \tilde{y}) = 0$. Since P is homogeneous,



(a) (b)

Figure 19: Problematic Cases

there is in fact a line of zeroes through the origin in this sector with slope $\frac{\tilde{y}}{\tilde{x}}$. It holds that, $P(x,y) = x^m P(1, \frac{y}{x})$, and therefore we may as well look for zeroes of the polynomial in k, $P(1,k) = \sum_{i=0}^{m} a_{m-i}^i k^i$. However, we only want to detect zeroes in $\mathbb{R}_+^2$, so we restrict ourselves to non-negative solutions of $P(1,k) = 0$. The same is true for the homogeneous polynomial Q. Of course, we lose the possible solution $x = 0$, but this is not of interest for us. Indeed, if the y-axis is invariant ($a_0^m = 0$), then $x = 0$ is not part of a hyperbolic or elliptic orbit. If there is an inward flow on the y-axis ($a_0^m > 0$), then this sector is neglected since we are only interested in the behaviour of the flow in the first sector which is completely contained in $\operatorname{int} \mathbb{R}_+^2$.
Let $\tilde{k}$ be a zero of $P(1,k)$ in an elliptic sector as shown in Figure 18a, then either $P(1,k)$ was positive before and negative afterwards (left), i.e., $P(1, \tilde{k} - \epsilon) > 0$ and $P(1, \tilde{k} + \epsilon) < 0$, while $Q(1, \tilde{k}) > 0$. Or the other way round Figure 18a (right), i.e., $P(1, \tilde{k} - \epsilon) < 0$, $P(1, \tilde{k} + \epsilon) > 0$ and $Q(1, \tilde{k}) < 0$.
Let $P(1, \tilde{k}) = 0$ in a hyperbolic sector. In the first case, Figure 18b (left), we observe $P(1, \tilde{k} - \epsilon) > 0$ and $P(1, \tilde{k} + \epsilon) < 0$, while $Q(1, \tilde{k}) < 0$. In Figure 18b (right) we have $P(1, \tilde{k} - \epsilon) < 0$ and $P(1, \tilde{k} + \epsilon) > 0$, while $Q(1, \tilde{k}) > 0$. We want to count $+1$ and -1 for each elliptic and hyperbolic sector, respectively. Together with the observations above this is done by

$$\sum_{i=1}^{s} \frac{1}{2} \operatorname{sgn} Q(1, k_i)[\operatorname{sgn} P(1, k_i - \epsilon) - \operatorname{sgn} P(1, k_i + \epsilon)], \tag{2.12}$$

where we sum over all zeroes $k_1, ..., k_s$ of $P(1,k)$.
However, there are two cases which are not included in (2.12). A problem may arise in the sector which contains the positive y-axis, if the positive y-axis is invariant. There is also a zero $\tilde{k}$ of $P(1,k)$ in Figure 19a, with $P(1, \tilde{k} - \epsilon) > 0$, $P(1, \tilde{k} + \epsilon) < 0$ and $Q(1, \tilde{k}) > 0$ and therefore this parabolic sector would falsely be counted as an elliptic sector. The second problem could be a hyperbolic sector, next to the positive y-axis, as shown in Figure 19b. In this sector there is no zero of $P(1,k)$ and consequently we would overlook this hyperbolic sector. Anyway, if one of the two problematic cases above appears, (2.12) is almost correct, i.e., $E - H + 1 = \sum_i \frac{1}{2} \operatorname{sgn} Q(1, k_i)[\operatorname{sgn} P(1, k_i - \epsilon) - \operatorname{sgn} P(1, k_i + \epsilon)]$.

Definition 2.10 (Ecological differential equation)**.** *We say that a differential equation $\dot{\mathbf{x}} = \Phi(\mathbf{x})$ is ecological, if the non-negative x-axis and the non-negative y-axis are invariant.*

Consider the case of an ecological differential equation, which is, in the framework of homogeneous systems, equivalent to $a_0^m = 0 = b_m^0$. From $P(x,0) = a_m^0 x^m$ and $Q(0,y) = b_0^m y^m$ we see that a_m^0 and b_0^m determine the flow direction on the positive x- and y-axis, respectively. It follows, that $\eta = 1 - \frac{\operatorname{sgn} a_m^0 + \operatorname{sgn} b_0^m}{2}$, where we assume that $a_m^0 \neq 0 \neq b_0^m$ (otherwise the origin is not isolated). In view of Theorem 2.4, we define

$$\begin{aligned}\Omega_P(\mathbf{0}) := \frac{1}{2} - \frac{\operatorname{sgn} a_m^0 + \operatorname{sgn} b_0^m}{4} \\ + \sum_{i=1}^{s} \frac{1}{4} \operatorname{sgn} Q(1,k_i)[\operatorname{sgn} P(1,k_i - \epsilon) - \operatorname{sgn} P(1,k_i + \epsilon)],\end{aligned} \tag{2.13}$$

where $k_1, ..., k_s$ are all the positive zeroes of $P(1,k)$. With the two exceptional cases from above and the fact that the boundary index is an integer, we can now formulate a Corollary of Theorem 2.4.

Corollary 2.11. *Let $\dot{\mathbf{x}} = \Phi(\mathbf{x})$ be an ecological, homogeneous polynomial differential equation and Φ of the form* (2.9)*. Let the origin be an isolated, saturated equilibrium of Φ. Then*

$$\operatorname{Bd\_Ind}_\Phi(\mathbf{0}) = \begin{cases} \Omega_P(\mathbf{0}) & \text{if } \Omega_P(\mathbf{0}) \in \mathbb{Z} \\ \Omega_P(\mathbf{0}) - \frac{1}{2} & \text{else.} \end{cases}$$

Remark 2.12. The same can be done (with minor adjustments) if we consider the zeroes of $Q(1,k)$. In this case the problems from above may arise in the sector which contains the x-axis. Instead of formula (2.13), we then get

$$\begin{aligned}\Omega_Q(\mathbf{0}) = \frac{1}{2} - \frac{\operatorname{sgn} a_m^0 + \operatorname{sgn} b_0^m}{4} \\ + \sum_{i=1}^{r} \frac{1}{4} \operatorname{sgn} P(1,k_i)[\operatorname{sgn} Q(1,k_i + \epsilon) - \operatorname{sgn} Q(1,k_i - \epsilon)],\end{aligned} \tag{2.14}$$

where $k_1, ..., k_r$ are the positive zeroes of $Q(1,k)$. Corollary 2.11 may as well be formulated with $\Omega_Q(\mathbf{0})$.

Example 2.13. Consider again the degenerate Lotka-Volterra system (2.7) in Example 2.8, i.e.,

$$\begin{aligned} P(x,y) = x(ax+by) \quad ; \quad P(1,k) = a + bk \\ Q(x,y) = y(cx+dy) \quad ; \quad Q(1,k) = k(c + dk). \end{aligned} \tag{2.15}$$

Assume that $a \neq 0 \neq d$ and $ad \neq bc$. It is immediate that $Q(x,0) = 0$ and $P(0,y) = 0$ or equivalently $a_0^m = 0 = b_m^0$, hence (2.15) is an ecological homogeneous system. In Example 2.8 we have already checked that the origin is an isolated and saturated equilibrium, so we can employ Corollary 2.11. The unique zero of $P(1,k)$ is given by $k_1 = -\frac{a}{b}$, which is positive if and only if $\operatorname{sgn}(ab) = -1$, (the possible non-negative solution $a = 0$ cannot be included, since then there is a line of equilibria on the x-axis and the origin is no longer isolated). First, let $a, d > 0$ and $c, b < 0$. Observe that $a = a_2^0$, $b = a_1^1$, $c = b_1^1$, $d = b_0^2$ and $m = 2$ which gives

$$\begin{aligned}
\Omega_P(\mathbf{0}) &= \frac{1}{2} - \frac{\operatorname{sgn} a + \operatorname{sgn} d}{4} \\
&\quad + \sum_{i=1}^{1} \frac{1}{4} \operatorname{sgn} Q(1,k_i)[\operatorname{sgn} P(1,k_i - \epsilon) - \operatorname{sgn} P(1,k_i + \epsilon)] \\
&= 0 + \frac{1}{4} \operatorname{sgn}\Big(\frac{a}{b}\Big(-c + d\frac{a}{b}\Big)\Big)[1 - (-1)] \\
&= \frac{1}{2} \underbrace{\operatorname{sgn}\Big(\frac{a}{b}\Big)}_{-1} \underbrace{\operatorname{sgn}\Big(-c + \frac{da}{b}\Big)}_{\operatorname{sgn}(bc-ad)} = \frac{1}{2} \operatorname{sgn}(ad - bc).
\end{aligned}$$

Since $\Omega_P(\mathbf{0})$ is not an integer, we are in one of the problematic cases and we have to substract $\frac{1}{2}$ to arrive at the boundary index

$$\operatorname{Bd\_Ind}(\mathbf{0}) = \begin{cases} -1 & \text{if } ad < bc \\ 0 & \text{if } ad > bc. \end{cases}$$

If $a, d < 0$ and $b, c > 0$ we get

$$\begin{aligned}
\Omega_P(\mathbf{0}) &= \frac{1}{2} - \frac{\operatorname{sgn} a + \operatorname{sgn} d}{4} + \frac{1}{4} \underbrace{\operatorname{sgn}\Big(\frac{a}{b}\Big(-c + d\frac{a}{b}\Big)\Big)}_{\operatorname{sgn}(bc-ad)}[-1 - 1] \\
&= 1 - \frac{1}{2} \operatorname{sgn}(bc - ad)
\end{aligned}$$

and consequently

$$\operatorname{Bd\_Ind}(\mathbf{0}) = \begin{cases} +1 & \text{if } ad > bc \\ 0 & \text{if } ad < bc, \end{cases}$$

which of course coincides with the results in Example 2.8.

Consider now the case of non-ecological equations, i.e., a_0^m or b_m^0 does not vanish. Let $\mathbb{R}^2_+$ be forward invariant and assume that $a_0^m \neq 0$ and therefore positive (otherwise we have to make some obvious changes in the following arguments). From the forward invariance assumption, the intermediate value theorem and

$a_0^m > 0$, it follows that there exists a $0 \leq \tilde{k} < \infty$ such that $Z(\tilde{k}) = 0$, where Z is from (2.10). In other words, there is at least one characteristic orbit, which is by assumption not the positive y-axis. Since Z is a polynomial of order $m+1$ we have finitely many zeroes $0 \leq \tilde{k}_1 < ... < \tilde{k}_s < \infty$, written in ascending order and $s \leq m+1$. In the following we will construct a sequence of polynomials, which allows us to draw conclusions on $E - H$, without computing the phase portrait. Define $T_0(k) := P(1,k)$, $T_1(k) := Q(1,k)$ and apply the euclidean algorithm with negative remainders, i.e.,

$$\begin{aligned} T_0(k) &= q_1(k)T_1(k) - T_2(k) \\ &\vdots \\ T_{i-1}(k) &= q_i(k)T_i(k) - T_{i+1}(k) \\ &\vdots \\ T_{l-1}(k) &= q_l(k)T_l(k). \end{aligned} \tag{2.16}$$

Remark 2.14. Here, $T_l(k) = \gcd(T_0(k), T_1(k))$ and therefore has no non-negative real roots. Indeed, assume otherwise, then $P(1,k)$ and $Q(1,k)$ would have a common real root $c \geq 0$. Since $P(x,y)$ and $Q(x,y)$ are homogeneous, this implies that for all $(x,y)^T \in \mathbb{R}^2_+$ with $\frac{y}{x} = c$, $P(x,y) = Q(x,y) = 0$ and therefore we have a line of equilibria. This is a contradiction to the standing assumption that the origin is isolated.

From (2.16), we can extract a sequence of polynomials

$$T_0(k), T_1(k), ..., T_l(k). \tag{2.17}$$

Define a function $\sigma : [0, \infty) \longrightarrow \mathbb{N}_0$ such that $\sigma(k)$ is equal to the number of sign changes in (2.17), where we ignore possible zeroes. It is easy to see that σ changes its value only at a zero of one of the polynomials T_i in (2.17). Let $0 \leq k_0$ be a zero of one of the polynomials T_i, where $i = 1, ..., l-1$. Then it follows from (2.16), that $T_{i-1}(k_0) = -T_{i+1}(k_0) \neq 0$. Indeed, from (2.16) we get that if $T_{i+1}(k_0) = 0$, then also $T_j(k_0) = 0$ for all $j \leq i+1$. As in Remark 2.14 above, it would follow that the origin is not isolated and we arrive at a contradiction. Consequently we are in one of the following situations:

1) $\operatorname{sgn} T_{i-1}(k_0) = \operatorname{sgn} T_{i-1}(k_0 \pm \epsilon) < 0$ and $\operatorname{sgn} T_{i+1}(k_0) = \operatorname{sgn} T_{i+1}(k_0 \pm \epsilon) > 0$

2) $\operatorname{sgn} T_{i-1}(k_0) = \operatorname{sgn} T_{i-1}(k_0 \pm \epsilon) > 0$ and $\operatorname{sgn} T_{i+1}(k_0) = \operatorname{sgn} T_{i+1}(k_0 \pm \epsilon) < 0$

where $0 < \epsilon$ is sufficiently small.
This implies that σ does not change its value at the zeroes of $T_1, ..., T_{l-1}$. The possible effects on (2.17) are shown in Figure 20.

$$
\begin{aligned}
&\text{i)} \quad \ldots \pm - \mp \ldots \longrightarrow \ldots \pm 0 \mp \ldots \longrightarrow \ldots \pm + \mp \ldots \\
&\text{ii)} \quad \ldots \pm + \mp \ldots \longrightarrow \ldots \pm 0 \mp \ldots \longrightarrow \ldots \pm - \mp \ldots \\
&\text{iii)} \quad \ldots \pm - \mp \ldots \longrightarrow \ldots \pm 0 \mp \ldots \longrightarrow \ldots \pm - \mp \ldots \\
&\text{iv)} \quad \ldots \pm + \mp \ldots \longrightarrow \ldots \pm 0 \mp \ldots \longrightarrow \ldots \pm + \mp \ldots
\end{aligned}
$$

Figure 20: Possible sign changes in (2.17), when passing through a zero k_0 of T_i, $i = 1, ..., l-1$. An ordered triple of signs $\in \{0, +, -\}$ corresponds to $(\operatorname{sgn} T_{i-1}, \operatorname{sgn} T_i, \operatorname{sgn} T_{i+1})$. The first, second and third column evaluates the triples at $k_0 - \epsilon$, k_0 and $k_0 + \epsilon$ for ϵ small enough, respectively.

As mentioned above, T_l does not have any non-negative roots and therefore σ may only change its value at a zero k_0 of $T_0(k) = P(1, k)$. Define $\Delta\sigma(k_0)$ as the change of σ, when passing through k_0 from below, i.e., $\Delta\sigma(k_0) := \sigma(k_0 + \epsilon) - \sigma(k_0 - \epsilon)$, with $\epsilon > 0$. We have to consider three different cases with $P^- := P(1, k_0 - \epsilon)$ and $P^+ := P(1, k_0 + \epsilon)$.

$$
\Delta\sigma(k_0) = \begin{cases} +1 & \text{if } \operatorname{sgn} P^- \operatorname{sgn} P^+ < 0 \text{ and } \operatorname{sgn} P^+ \operatorname{sgn} Q(1, k_0) < 0 \\ -1 & \text{if } \operatorname{sgn} P^- \operatorname{sgn} P^+ < 0 \text{ and } \operatorname{sgn} P^+ \operatorname{sgn} Q(1, k_0) > 0 \\ 0 & \text{if } \operatorname{sgn} P^- \operatorname{sgn} P^+ > 0. \end{cases} \tag{2.18}
$$

It is easily seen, that we can write (2.18) simply as

$$
\Delta\sigma(k_0) = \frac{1}{2} \operatorname{sgn} Q(1, k_0)[\operatorname{sgn} P(1, k_0 - \epsilon) - \operatorname{sgn} P(1, k_0 + \epsilon)].
$$

Consider two values $0 \leq a < b < \infty$, such that $P(1, a)$ and $P(1, b)$ do not vanish. Between two consecutive zeroes k_i and k_{i+1} of $P(1, k)$ in (a, b), the value of σ is constant. It follows that $\sigma(k_i + \epsilon) = \sigma(k_{i+1} - \epsilon)$, and therefore

$$
\begin{aligned}
\sigma(b) - \sigma(a) &= \sum_{i=1}^{n} \sigma(k_i + \epsilon) - \sigma(k_i - \epsilon) = \sum_{i=1}^{n} \Delta\sigma(k_i) \\
&= \frac{1}{2} \sum_{i=1}^{n} \operatorname{sgn} Q(1, k_i)[\operatorname{sgn} P(1, k_i - \epsilon) - \operatorname{sgn} P(1, k_i + \epsilon)],
\end{aligned} \tag{2.19}
$$

holds where $k_1 < ... < k_n$ are all the zeroes of $P(1, k)$ in the open interval (a, b). Recall the derivation of (2.12) and the fact that the smallest and largest non-negative zeroes $0 \leq \tilde{k}_1 < \tilde{k}_s < \infty$ of (2.10) bound the region of interest for the boundary index. In fact, $\arctan(\tilde{k}_1)$ and $\arctan(\tilde{k}_s)$ are the angles of the first and last characteristic orbit with the x-axis in $\mathbb{R}^2_+$, respectively. We observe

that from (2.19) and the assumption $a_0^m \neq 0$, it follows that

$$E - H = \sigma(\tilde{k}_s) - \sigma(\tilde{k}_1). \tag{2.20}$$

It remains to determine the flow direction on these two characteristic orbits. Recall from (2.11), that this is done by $\operatorname{sgn} Y(\tilde{k}_1)$ and $\operatorname{sgn} Y(\tilde{k}_s)$. The results can be combined in the following Corollary of Theorem 2.4.

Corollary 2.15. *Let $\dot{\mathbf{x}} = \Phi(\mathbf{x})$ be a homogeneous polynomial differential equation and Φ of the form (2.9), with the origin an isolated, saturated equilibrium. Assume that $a_0^m \neq 0$ and $\mathbb{R}^2_+$ is forward invariant, then*

$$\text{Bd\_Ind}_\Phi(\mathbf{0}) = \frac{1}{2} - \frac{\operatorname{sgn} Y(\tilde{k}_1) + \operatorname{sgn} Y(\tilde{k}_s)}{4} + \frac{\sigma(\tilde{k}_s) - \sigma(\tilde{k}_1)}{2}.$$

Remark 2.16. To avoid the restriction $a_0^m \neq 0$ one can either use the case distinction made in Corollary 2.11, or work out the results for $b_m^0 \neq 0$. An analogous result on the index of the origin on $\mathbb{R}^2$ was established in [12, p. 86]. There the assumption is that the origin is isolated and that $a_0^m \neq 0$. Then the index with respect to the homogeneous polynomial vector field Φ is given by

$$\text{Ind}_\Phi(\mathbf{0}) = \sigma(\infty) - \sigma(-\infty),$$

where $\sigma(\pm\infty) := \lim_{k\to\infty} \sigma(\pm k)$.

2.4.2 Towards non-homogeneous Systems

So far, the class of vector fields on which we can apply Corollaries 2.11 and 2.15 is rather small. In the following, we enlarge the range of the results above. The standing assumption is that the vector field Φ has an isolated, saturated equilibrium at the origin, i.e., $\Phi(\mathbf{0}) = 0$. If the equilibrium under consideration is not the origin, we have to perform a change of coordinates in advance.

Definition 2.17 (Principal Part)**.** *Let Φ and Φ_0 be two vector fields defined on $\mathbb{R}^2_+$. We say that Φ_0 is a principal part of Φ, if there exists a $\delta > 0$ such that for all $(x, y)^T \neq \mathbf{0} \in \mathbb{R}^2_+$ with $\|(x, y)^T\| < \delta$*

$$\|\Phi(x, y) - \Phi_0(x, y)\| < \|\Phi_0(x, y)\|.$$

Proposition 2.18. *Let Φ be a continuous vector field, leaving $\mathbb{R}^2_+$ forward invariant, and Φ_0 its principal part. Assume that in the set $A_\delta := \{(x, y)^T \in \mathbb{R}^2_+ : x^2 + y^2 \leq \delta^2\}$ there is no equilibrium except the origin. Then it holds that* $\text{Bd\_Ind}_\Phi(\mathbf{0}) = \text{Bd\_Ind}_{\Phi_0}(\mathbf{0})$

Proof. Recall from Definition 1.20 that the boundary index is invariant under homotopic deformations. So it remains to show that every vector field is homotopic to its principal part. From Definition 2.17 and the reversed triangle inequality it follows, that $\|\Phi_0\| > \frac{\|\Phi\|}{2}$ and therefore A_δ does not contain an equilibrium different from the origin for Φ_0 as well. If Φ and Φ_0 do not point in opposite directions on A_δ, we can connect these two vector fields homotopically via $H(\mathbf{x}, \lambda) = \lambda\Phi(\mathbf{x}) + (1-\lambda)\Phi_0(\mathbf{x})$, on A_δ. Assume towards a contradiction that Φ and Φ_0 are opposing on A_δ, i.e., there exists a point $(\tilde{x}, \tilde{y}) \in A_\delta$ and $\alpha > 0 : \Phi(\tilde{x}, \tilde{y}) = -\alpha\Phi_0(\tilde{x}, \tilde{y})$. Since Φ_0 is a principal part of Φ it holds that

$$\|\Phi(\tilde{x}, \tilde{y}) - \Phi_0(\tilde{x}, \tilde{y})\| < \|\Phi_0(\tilde{x}, \tilde{y})\|$$

and it follows that

$$(1+\alpha)\|\Phi_0(\tilde{x}, \tilde{y})\| < \|\Phi_0(\tilde{x}, \tilde{y})\|,$$

and we arrived at the desired contradiction. □

With Proposition 2.18 at hand, we are now able to enlarge the class of vector fields where we can apply the developed theory. Assume that Φ is sufficiently smooth such that we can apply Taylor's Theorem, i.e.,

$$\begin{aligned} P(x,y) &= a_1(x,y) + ... + a_m(x,y) + \omega_1(x,y) \\ Q(x,y) &= b_1(x,y) + ... + b_m(x,y) + \omega_2(x,y), \end{aligned} \tag{2.21}$$

with a_i and b_i homogeneous polynomials of order i and $\omega_1, \omega_2 = o\big((x^2+y^2)^{\frac{m}{2}}\big)$, i.e.,

$$\lim_{x^2+y^2\to 0} \frac{\omega_1(x,y)}{(x^2+y^2)^{\frac{m}{2}}} = \lim_{x^2+y^2\to 0} \frac{\omega_2(x,y)}{(x^2+y^2)^{\frac{m}{2}}} = 0. \tag{2.22}$$

The strategy is to show that the vector field

$$\Phi_0^m(x,y) := \begin{pmatrix} a_1(x,y) + ... + a_m(x,y) \\ b_1(x,y) + ... + b_m(x,y) \end{pmatrix} \tag{2.23}$$

in (2.21) has the same boundary index as Φ. This will be done by checking that (2.23) is a principal part of Φ, if we add one more assumption.

Definition 2.19 (Non-degeneracy)**.** *A vector field* (2.23) *on* $\mathbb{R}^2_+$ *is non-degenerate if there exist* $\alpha > 0$ *and* $\delta > 0$ *such that*

$$\|\Phi_0^m(x,y)\| \geq \alpha(x^2+y^2)^{\frac{m}{2}} \tag{2.24}$$

for all $(x,y)^T \in \mathbb{R}^2_+$ *with* $x^2 + y^2 \leq \delta^2$.

Lemma 2.20. *Let Φ be a vector field of the form (2.21), leaving $\mathbb{R}^2_+$ forward invariant and assume that Φ_0^m is non-degenerate. Then it holds that* $\mathrm{Bd\_Ind}_\Phi(\mathbf{0}) = \mathrm{Bd\_Ind}_{\Phi_0^m}(\mathbf{0})$.

Proof. From (2.21) and (2.23) we infer that $\|\Phi - \Phi_0^m\| = \|(\omega_1, \omega_2)^T\|$. Since $\omega_i = o\big((x^2+y^2)^{\frac{m}{2}}\big)$ we can find for all $\epsilon > 0$ some $\varrho \leq \delta$ such that $|\omega_i(x,y)| < \epsilon(x^2+y^2)^{\frac{m}{2}}$ for all $x^2+y^2 \leq \varrho^2$ and $i = 1, 2$. Therefore

$$\|\Phi - \Phi_0^m\| = (\omega_1^2 + \omega_2^2)^{\frac{1}{2}} < \sqrt{2}\epsilon(x^2+y^2)^{\frac{m}{2}} \leq \|\Phi_0^m\|$$

holds for all $x^2 + y^2 \leq \varrho^2 \leq \delta^2$ if we choose $\epsilon = \frac{\alpha}{\sqrt{2}}$. Now we have shown that Φ_0^m is a principal part of Φ and it follows that the origin has an isolating neighbourhood. Indeed, for all $(x,y)^T \neq \mathbf{0} \in \mathbb{R}^2_+$ such that $x^2+y^2 \leq \varrho^2$, we see that the triangle inequality implies

$$\|\Phi_0^m\| \leq \|\Phi - \Phi_0^m\| + \|\Phi\| < \|\Phi_0^m\| + \|\Phi\|$$

and therefore $0 < \|\Phi\|$. Finally, the claim follows from Proposition 2.18. □

Now we can combine Lemma 2.20 with Corollary 2.15. Given a vector field Φ, with Taylor expansion (2.21) and a_m or b_m is the first non-vanishing term in the expansion, i.e.,

$$\begin{aligned} a_1 \equiv a_2 \equiv ... \equiv a_{m-1} \equiv 0 \\ b_1 \equiv b_2 \equiv ... \equiv b_{m-1} \equiv 0, \end{aligned} \tag{2.25}$$

and additionally Φ_0^m is non-degenerate, we can infer the boundary index of Φ by considering the homogeneous vector field Φ_0^m.

Example 2.21. We may demonstrate the above results for a vector field with $m = 2$. Consider again the degenerate Lotka-Volterra system, but with higher order terms, i.e.,

$$\Phi(x,y) = \Phi_0^2(x,y) + \omega(x,y) = \begin{pmatrix} x(ax+by) \\ y(cx+dy) \end{pmatrix} + \begin{pmatrix} \omega_1(x,y) \\ \omega_2(x,y) \end{pmatrix} \tag{2.26}$$

where $\omega_1(x,y), \omega_2(x,y) = o(x^2+y^2)$. In addition let $\omega_1(0,y) \geq 0$ and $\omega_2(x,0) \geq 0$ to retain the forward invariance. In Example 2.8 we have already shown that the origin is isolated for Φ_0^2 if and only if $a \neq 0 \neq d$ and $ad \neq bc$. Since Proposition 2.18 only requires an isolating neighbourhood in the non-negative orthant, we can be less restrictive. It is enough to assume $ad \neq bc$ or that the slope of the line of equilibria is negative, i.e., $k = -\frac{a}{b} < 0$. Let now $C := \{\mathbf{x} \in \mathbb{R}^2_+ : x^2+y^2 = 1\}$ be the positive unit circular arc and define $\alpha_0 := \min_{\mathbf{x} \in C} \|\Phi_0^2\|$. Since $\|\Phi_0^2\| > 0$ on

the compact set C by assumption and the vector field is continuous, its minimum is attained there. Consequently $\alpha_0 > 0$ holds. Via a change to polar coordinates we see that

$$\|\Phi_0^2(r,\varphi)\| = r^2 \left\| \begin{pmatrix} a\cos^2(\varphi) + b\cos(\varphi)\sin(\varphi) \\ c\cos(\varphi)\sin(\varphi) + d\sin^2(\varphi) \end{pmatrix} \right\| \geq r^2\alpha_0$$

for all (r,φ) with $\varphi \in [0, \frac{\pi}{2}]$. In other words, Φ_0^2 is non-degenerate. Now we can apply Lemma 2.20, which implies that $\mathrm{Bd\_Ind}_\Phi(\mathbf{0}) = \mathrm{Bd\_Ind}_{\Phi_0^2}(\mathbf{0})$ holds. Which means that we may use the results of Example 2.13 to draw conclusions on the boundary index of the perturbed system (2.26).

References

[1] H. Amann. *Ordinary Differential Equations. An Introduction to Nonlinear Analysis*. Walter de Gruyter, 1990.

[2] A. A. Andronov et al. *Qualitative Theory of Second-order Dynamic Systems*. Wiley, 1973.

[3] I. Bendixson. "Sur les courbes définies par des équations différentielles". In: *Acta Mathematica* 24 (1901), pp. 1–88.

[4] A. Berman and R. J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Academic Press, 1979.

[5] B. Boldin. "Introducing a population into a steady community: The critical case, the center manifold, and the direction of bifurcation". In: *SIAM Journal on Applied Mathematics* 66.4 (2006), pp. 1424–1453.

[6] K. Deimling. *Nonlinear Functional Analysis*. Springer, 1985.

[7] F. Dumortier, J. Llibre, and J. C. Artés. *Qualitative Theory of Planar Differential Equations*. Springer, 2006.

[8] J. K. Hale. *Theory of Functional Differential Equations*. Springer, 1977.

[9] J. Hofbauer. "An index theorem for dissipative semiflows". In: *Rocky Mountain Journal of Mathematics* 20.4 (1990), pp. 1017–1031.

[10] J. Hofbauer and K. Sigmund. *Evolutionary Games and Population Dynamics*. Cambridge University Press, 1998.

[11] M. A. Krasnosel'skiy and P. P. Zabreiko. *Geometrical Methods of Nonlinear Analysis*. Springer, 1984.

[12] M. A. Krasnosel'skiy et al. *Plane Vector Fields*. Academic Press, 1966.

[13] S. Lefschetz. *Differential Equations: Geometric Theory*. Second Edition. Pure and Applied Mathematics, Vol. VI. Wiley, 1963.

[14] M. Nagumo. "A theory of degree of mapping based on infinitesimal analysis". In: *American Journal of Mathematics* 73.3 (July 1951), pp. 485–496.

[15] M. Růžička. *Nichtlineare Funktionalanalysis*. Springer, 2020.

[16] E. Zeidler. *Nonlinear Functional Analysis and its Applications I. Fixed-Point Theorems*. Springer, 1986.