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Perspectives on Structuralism: Metaontology, Epistemology, and Dependence

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Perspectives on Structuralism:

Metaontology, Epistemology, and Dependence

Contents

Introduction

Structuralism in the philosophy of mathematics is a family of views tied together by the conviction that mathematics is the study of structure. For example, the natural numbers constitute the natural number structure, ℕ. The immediate promise of structuralism is that it ignores non-relevant properties, and captures the relevant ones. A relevant property of the natural number 2 is that it is the second natural number, so that 1 precedes it, and 3 follows. This means that positions of a structure have no 'inner nature', and that they are determined by their relationships to the structure to which they belong.

Structuralist ideas have notably been defended by Paul Benacerraf (1965), Geoffrey Hellman (1989, 2001), Stewart Shapiro (1997, 2000, 2011), Michael Resnik (1982, 1997) and Charles Parsons (1990, 2008). Structuralism comes in various guises with diametrically different views on ontology, spanning from fervent anti-realism to platonist realism (see Reck & Price 2000). While Hellman argues for a version of structuralism called *eliminative*, Parsons, Resnik, and Shapiro defend *non-eliminative* structuralism. Eliminative structuralism rejects the objective existence of abstract structures and their objects. Non-eliminative structuralism is, on the other hand, a realist position. The abstract structures are believed to exist objectively and independently of humans, and they also exist before any realisation of the structure by a particular system. For the non-eliminative structuralist, the ontological commitments to mathematical structures are thus not far removed from that of platonism. My research is placed firmly within non-eliminative structuralism, and the questions of ontology and epistemology that are raised in this thesis are decidedly spurred on by its realist ontology.[1]

This thesis approaches fundamental questions for structuralism from different angles. I clarify particular questions and difficulties and defend structuralism against a number of criticisms, thereby contributing to develop the position further. By doing a cumulative dissertation, I target specific issues that are more narrowly delineated. As my overall aim is to show that structuralism is a viable position, I can address particular hinders that question that viability. First of all, I can cast my net wider and concentrate on issues that are not in immediate disciplinary proximity, e.g., I can switch from ontological to epistemological hinders. Second, by having a more narrowly delineated scope for each individual article, I do not lose specificity in my work. My project of defending the viability of structuralism thus remains general in character, while each paper tackles particular issues in need of address.

[1] From now on, unless explicitly stated, all reference to 'structuralism' will be to non-eliminative structuralism.

An overarching theme of the dissertation is that each article concerns the legitimacy and sustainability of structuralism as a philosophical position. Whether structuralism is *overall defensible* turns on whether the justification offered up for it is all-purpose, i.e., it methodically accounts for the broad view, and whether it can also alleviate particular objections, i.e., serving as a buffer against specific attacks. For instance, if we are concerned with explaining our epistemic access to structures, but the arguments for their existence are severely lacking, the broad defence for structuralism suffers. I therefore argue that whatever *field* in which a problem occurs, be it epistemology, metaontology, or ontology, the proposed solution – and the justification for it – should not only address the specific problem, but also serve as raising the general likelihood of the position as a whole. In order to see how each individual paper adds to the overall defence for structuralism, I need to broach the contents and conclusions of each paper.

The dissertation consists of four chapters. Each chapter can be read as an autonomous text, as they will be free-standing articles intended for submission to relevant philosophical journals. Still, they are cumulative, so that each article builds upon the previous ones. They are all systematic in character, addressing issues relevant for the contemporary structuralist discussion. Articles 1, 3, and 4 have elements that draw on history of philosophy, as I believe they offer valuable insight for the contemporary discussion. While Article 1 looks to the historical background of metaontology in order to differentiate between two modern approaches, Article 3 uses an older account (Gödel 1944, 1964) as an example. Article 4 looks to a Husserlian relation of ontological dependence, but applies it to our present case in an original way.

Articles 1 and 2 are sister papers, where the second clearly builds upon the first. Article 1 examines the relationship between metaontology and ontology in the philosophy of mathematics. It clarifies what role metaontology can serve in formulating mathematical ontology, and how we can view it as rectifying a position's theoretical insufficiency. The first article is not specifically on structuralism, but argues that a mathematical realist should adopt a metaontology that puts mathematical ontology first. Arguments for mathematical realism have often been entangled with arguments for empirical science (see Putnam 1979). I argue that an appropriate metaontology disentangles itself from empirical science, as the relationship between philosophy, empirical science, and mathematics needs clarification, so that there is a 'division of labour' (see Husserl 2001a).

Article 2 takes as its starting point the conclusion that mathematical realism in ontology generally benefits from implementing an appropriate metaontology. In the context of structuralism, coherentist ideas have been defended (see Shapiro 1997). The second article develops metaontological coherentism, and investigates its relation to – and fit with – structuralist ontology. To this end, I look to coherence theories in analytic epistemology, as the notion of 'coherence' is unclear. The upshot is that coherentist structuralism better accounts for when we allow structures to exist, as it offers a framework in which existence claims can be expressed in terms of coherence.

Article 3 answers the question of when an epistemological account is deemed adequate with regard to the so-called access problem. I argue that we should differentiate between two approaches. Each approach has implicit metaepistemological leanings, which accounts for miscommunication in the epistemological debate generally, and within structuralism specifically. One approach takes an extra-mathematical stance on justification, while the other allows for the use of mathematical knowledge, the access to which we are out to explain. I argue for the latter, rejecting the prudence of pursuing a foundationalist account from an extra-mathematical starting point. We should rather explain our access from *within*, i.e., not denying our mathematical knowledge while trying to explain our access to it.

Article 4 takes on ontological dependence relations for structuralism. I argue we should look to a Husserlian relation of dependence – *foundation* – as it allows for infinite chains of dependence that cycle. Such a non-linear account of dependence fits with how mathematical objects are thought to be incomplete and dependent on the structure to which they belong. As they are also thought to depend on the other objects belonging to the same structure, a cyclical picture of how dependence relations hold among them, captures these structuralist intuitions. Article 4 thus advocates a non-linear account of dependence.

From these brief descriptions of each individual article's line of argument, we see a recurring theme. The topics of coherence and non-linearity run through Articles 2, 3, and 4. Coherence describes a quality of interconnectedness, consistency, and systematic unity (see Bender 1989). A more thorough discussion of the notion is to be found in Article 2. As for non-linearity, it can be applied to various disciplinary areas. The quality of being non-linear, in the case of epistemology, is strongly connected to the rejection of epistemological foundationalism (Article 3) (see Williams 2001). In the case of general metaphysics, it is connected to the rejection of metaphysical foundationalism (Article 4) (see Thompson 2018). As for metaontology of mathematics, it can be put in contrast to a neo-Fregean metaontology (consider Linnebo's asymmetrical abstractionism (2018)) (Article 2). Articles 2, 3, and 4 are thus thematically linked, not by their disciplinary proximity, as they range from metaontology to epistemology to dependence, but rather by what they invoke in order to make their case. Non-linearity and coherence are properties that describe a certain way of looking at a delimited field. That these qualities play an integral role in Articles 2, 3, and 4 – both in content and methodology – is apparent from the discrete conclusions of each article.

Moreover, there is also a thematic link between Articles 1 and 3, as they both argue for the need to implement *meta-perspectives*; metaontology for mathematical realism (Article 1), and the fruitfulness of classifying two approaches exhibiting metaepistemological leanings (Article 3). Article 1 clearly lays the ground for Article 2, which contains more direct contributions to the development of structuralism. Article 3, though it is not followed up in the same way (i.e., by a direct contribution to structuralist epistemology), adds to the overall framework, thus also providing some ground for the other articles. This is perhaps most clear when it comes to Articles 1 and 3's mutual use of the Carnapian distinction between internal and external questions (see Carnap 1950).

The chosen perspectives in this thesis are spurred on by identifying certain gaps in the structuralist literature. Metaontology for mathematics, still being a recent field, is developed so that a proper metaontology for structuralism, i.e., coherentism, is implemented (Articles 1-2). To overcome miscommunication in the literature on structuralist epistemology, a classification of two approaches clarifies the issue (Article 3). The final perspective argues for a historically inspired dependence relation, but, importantly, a relation that we also find to be present in structuralist epistemology, and, moreover, that fits with a coherentist picture of the metaphysical structure of reality (Article 4). These perspectives add to the overall justification for structuralism – as they converge thematically and methodologically – thus constituting a coherent and systematic defence, progressing the viability of structuralism.

List of manuscripts and their status

All articles are single-authored, in the English language, and will be submitted for publishing in international peer-reviewed journals.

- Article 1: "Metaontology for Mathematical Realism"
    - Not yet submitted.
- Article 2: "Coherentist Structuralism: Structures as Thin Objects"
    - Not yet submitted.

◊ Article 3: "Two Approaches to the Access Problem"
  - Not yet submitted.

◊ Article 4: "Ontological Dependence in Mathematical Structuralism"
  - Submitted for publishing – *European Journal of Philosophy of Science*, (28.02.2022).

Abstracts of papers

Article 1: "Metaontology for Mathematical Realism"

Questions of existence and ontological commitment are central to philosophy of mathematics. No matter which philosophical theory we ascribe to, there are mathematical objects we are ontologically committed to and not. While the Quinean question of *what there is* tends to be directly addressed within a philosophical view, there are also considerations as to *how* to go about such ontological questions. *Metaontology*, as introduced by Peter Van Inwagen (1998), targets exactly these considerations.

This first paper investigates the relationship between ontology and metaontology, and argues that metaontology can be construed as a *methodology* for – and *qualification* of – ontology. By construing metaontology as methodology, we can look at how we come to be ontologically committed to certain entities. It becomes a way of addressing theoretical underpinnings and philosophical assumptions that are otherwise only implicit in an ontology, but still shape our ontological views. In a way, we provide a vetting process, i.e., a framework in which we can trace how we ended up being committed to the existence of some entities and not to others. By construing metaontology as qualification, we consider metaontology as serving an ordering function. This would arguably result in our ontological views becoming more *uniform*. That is, by adopting a metaontology, we get some criteria for what entities to accept into our ontology. By adhering to these, we effectively put certain limiting conditions in place for what our theory should look like, thereby providing our ontology with a certain lawfulness.

While metaontology as a discipline was rather recently coined, its roots continue to be of interest to the contemporary discussion. The historical background can be traced to the Quine-Carnap debate, and their introduction of key notions such as 'ontological commitment' and 'linguistic frameworks'. The Quine-Carnap debate centres on the legitimacy of doing ontology, and whether it yields genuine knowledge. Moreover, it questions what it means to be ontologically committed to abstract entities, e.g., mathematical objects, and how we should go about justifying such commitment. We generally want to come up with a philosophical narrative in which mathematics has a subject-matter that is objective, and where we can use mathematics in our best empirical sciences without making mathematical knowledge subservient to them.

From the Quine-Carnap debate, we can draw a distinction between an *inflationary* and a *deflationary* attitude towards metaontology. This has to do with the relationship between ontology and science, where an inflationary attitude sees metaphysics as continuous with science, thus modelling ontology on science. A deflationary attitude puts mathematics first, and rather emphasises the limitations of ontological results. I argue that existence questions regarding mathematical ontology should be disentangled from their prospective applicability in empirical science, and that we should take a more modest approach towards our potential ontological insights. Rather, we should enlist metaontology as a way to provide a philosophical theory with the tools to qualify and specify the objects to which it is ontologically committed. As the burden of proof is on the mathematical realist rather than the nominalist, she should adopt such a qualifying tool, allowing her to streamline her ontological views.

Article 2: "Coherentist Structuralism: Structures as Thin Objects"

There is a metaontological view that has gotten recent traction within a realist setting, namely that of metaontological *minimalism*, defended by Øystein Linnebo (2018). It is not so that metaontological minimalism supports a minimal ontology, on the contrary, generous ontological views are very much compatible with this metaontological stance. It is rather the criteria for one's ontological commitments that are minimal. This allows for what Linnebo calls *thin objects*, where the idea is that an object is considered *thin* if it does not make substantial demands on the world. While a pure mathematical object is thin in an *absolute* sense (viz., sets, numbers, etc.), there are objects that are thin only in a *relative* sense, as well. An example is the set of two trees, where the set does not make any *further* substantial demands on the world, other than that of the spatio-temporal make-up of the trees in question.

While Linnebo pursues a Fregean abstractionist approach to metaontological minimalism and thin objects, there is also that of the *coherentist*. Coherentism is the view that given the *coherence* of a mathematical theory, the existence of the objects described by the theory in question is ensured. Coherentist approaches have roots in Hilbert's views, but have more recently been defended by Shapiro (1997). Coherentism is *minimalist* insofar that what is needed for an object to exist is very little. The existence of the intended object depends on the coherence of the theory that describes it, i.e., if the theory is coherent, all the objects described by it exist. As such, coherentism constitutes another approach to *thin* objects. The existence of objects resulting from this coherence is thin because, as Linnebo phrases it, their existence does not put any *further* metaphysical demands on the world, other than that of the theory providing their description (and thereby their existence).

In this second article I pursue a position I will call *coherentist structuralism*. It is a combination view that unites: i) a position's ontological commitments, and ii) metaontological considerations. The metaontological component consists of *coherentist minimalism*, while the ontological commitments are those of *non-eliminative structuralism*. The aim is to show that these are compatible with each other, and will reciprocally inform and clarify the other position. I argue that the structuralist benefits from a combination view with metaontological ambitions. The central claim of structuralism – that abstract structures exist – is given added justification, as the metaontological framework qualifies the ontological commitments made.

Crucial to Shapiro's structuralism, is the existence axiom for structures and their positions, viz., the *Coherence Principle*: "**Coherence**: If $\mathbf{\Phi}$ is a coherent formula in a second-order language, then there is a structure that satisfies $\mathbf{\Phi}$" (1997:95). Shapiro uses the coherence of a formula to assert that there exists a structure that satisfies the formula in question. This is clearly a position that commits itself to the existence of abstract structures *by way of coherence*. And while Shapiro uses the Coherence Principle as an existence criterion, he stops short of developing a larger metaontological framework. Moreover, what the notion 'coherence' really means, remains unclear. As Shapiro concedes: "The problem, of course, is that it is far from clear what 'coherent' comes to here", and also: "Coherence is not a rigorously defined mathematical notion, and there is no noncircular way to characterize it" (Shapiro 1997:95, 13).

The notion of coherence thus remains woolly. This opens up the possibility of looking somewhere else than to philosophy of mathematics. Coherence finds its most developed form in theories of justification. Coherence theories of justification in analytic epistemology provides a distinction between *systemic* and *relational* coherence, in order to characterise how the system's internal arrangement, and how the system as a whole, provides justification in different ways (see Bender (1989)). This distinction finds a correspondence in mathematical structuralism, where there is an ontological dependence relationship between a structure and the objects making up that structure. Epistemological theories of coherence are worth looking into, as the notion of coherence for metaontological coherentism needs clarification. If the distinction between systemic and relational coherence can inform our metaontological coherentist framework, developing coherentist structuralism holds some promise, as we might advance our understanding of how we think of structures and their objects, and what it takes for them to exist.

Article 3: "Two Approaches to the Access Problem"

Benacerraf's paper "Mathematical Truth" (1973) formulated the so-called *access problem*, an epistemological challenge faced by positions in philosophy of mathematics that endorse a realist ontology of mathematical objects. Given that we have mathematical knowledge, it becomes a problem to explain how mathematical propositions are reliably justified and knowable to us. Many attempts have been made to overcome this apparently fatal objection, but it remains one of the more dire obstacles for the contemporary mathematical realist.

To sort the different attempts made, I employ Audrey Yap's (2009) distinction between *external* and *internal* answers to the access problem. An *external* answer to the access problem is characterised by accepting the challenge as posed. This means that the gulf between physical and mathematical reality must be bridged, and that some means – capable of both breaching the causal limits of physical reality and probing into mathematical reality – must be endorsed. An *internal* answer does not need the gap between physical and mathematical reality to be filled in a metaphysically loaded way. Rather, we can explain our epistemic access using mathematical knowledge, thus providing an answer "in terms of *mathematical* adequacy" (Yap 2009:169). While the internal/external distinction describes different answers to the access problem, it does not sufficiently capture what is going on. For that, the distinction is too narrow, as it only indicates parts of a broader methodology in accounting for our epistemic access.

In this third paper I argue that the two kinds of answer lead to two distinct approaches, which I want to call the *Head On Approach* and the *Tweaking the Question Approach*. While the kind of answer we favour is reflected in the approach we pursue, each approach has other characteristics that determine what an appropriate epistemological story for mathematics should look like. As the two approaches diverge on the issue of *adequacy* in the face of the access problem, this frames the eligible positions sorted to each approach, and leads not only to different, but incompatible stories of how we acquire mathematical knowledge. This reveals that the two approaches have different *metaepistemological* tenets (i.e., fundamental aspects of epistemic theorising regarding implicit aims and standards), and that their respective accepted methodologies diverge in important ways. A consequence of this is miscommunication between the two camps, as the epistemological story of each approach violates implicit demands for what counts as adequate in the other.

To this end, I bring forth two example positions that have dealt with the access problem in paradigmatic ways. By looking at specific accounts, it becomes easier to draw out the methodological considerations of each approach. The Head On Approach involves the postulation of a special faculty that has a transcending quality. The postulation of this special faculty has often been made in the guise of mathematical intuition; a faculty described by analogy to sense perception. My example account of the Head On Approach is one of the more famous (and divisive) accounts of mathematical intuition, that of Kurt Gödel (1947, 1964). The second example account is Stewart Shapiro's stratified epistemology (1997, 2011), which does not involve a special faculty, and allows for mathematical means in pursuing an epistemological story of how we come to have mathematical knowledge. These two accounts serve to instantiate the characteristics prescribed by each approach.

This paper thus operates with different theoretic *levels*: we have the internal/external distinction, the two representative positions, and, most importantly, the two approaches manifesting different metaepistemological stances when confronted with the access problem. My aim in this paper is twofold: 1) to show how the kinds of answer that Yap presents – leading to the two corresponding approaches – deal with the epistemological challenge, and 2) to show that the Tweaking the Question Approach is superior to the Head On Approach and therefore should be pursued.

The classification of two approaches with different standards for adequacy allows us to understand the incommensurability tendencies present in realist responses to the access problem. The discussion of the two example positions makes this tendency clear; as the accounts are criticised on different *grounds*. While Shapiro is criticised for his *way of answering* the access problem, i.e., his chosen *approach*, Gödel is criticised for his proposed solution. We see this tendency also within the epistemological debate on structuralism. The objections raised by Fraser MacBride (2008) against Shapiro (1997) centres on how Shapiro fails to properly address the access problem. And in Shapiro's reply to MacBride (2011), it becomes evident that Shapiro considers MacBride to misconstrue his intentions, and setting the bar too high for when an epistemological account is deemed acceptable.

Article 4: "Ontological Dependence in Mathematical Structuralism"

Ontological dependence relations determine how objects and domains of objects depend on each other. While such relations are of continued importance to contemporary metaphysics, it is also an issue across local fields of philosophy, such as philosophy of mathematics. Mathematical objects can ontologically depend on other mathematical objects or domains of such. Ontological dependence relations are especially salient to the debate on mathematical structuralism, which is the view that mathematical objects have no 'inner nature', and that a mathematical object is what it is due to its mathematical context, i.e., the mathematical *structure* in which it appears. This means that the natural number 2 simply is the second place in the natural number structure, and so depends on the elements belonging to the same structure (the other natural numbers) and on the structure as a whole (the natural number structure, ℕ). This is why objects are deemed to be *incomplete* on a structuralist account (see Parsons 1980:149-150, 1990:334-5; Linnebo 2008:62-66). Linnebo calls this the *Incompleteness Claim* (2008:62-63). Linnebo further formulates the twofold *Dependence Claim*, consisting of two tenets, for some domain *D* of some mathematical structure:

> ODO. Each object in *D* depends on every other object in *D*.
> ODS. Each mathematical object depends on the structure to which it belongs. (2008:67-8)

The idea of the Dependence Claim is to cash out one of the characteristic features of structuralism; that mathematical objects are defined by their relationships to other mathematical objects belonging to the same structure. Mathematical objects fully *depend* on their context, i.e., their places within a certain structure. By being determined by its relational standing to other objects, a mathematical object is what it is at the mercy of those relationships. This is how the incompleteness of mathematical objects is thoroughly interlinked with their dependence.

While there is general consensus that there are ontological dependence relations within structuralism, there have been surprisingly few attempts to characterise the relation itself. Notable exceptions are Linnebo (2008) and Wigglesworth (2018). To further the work on the nature of ontological dependence in structuralism, other options should also be considered. One of the early discussions of ontological dependence is Edmund Husserl's discussion of *parts* and *wholes* and the relation of *foundation* that holds between them. This discussion makes up the third *Logical Investigation* and tries to render (more) clear how a part and whole have varying degrees of dependence and independence. Interestingly, Husserl shows in his philosophy of mathematics clear structuralist tendencies, especially in the *Logical Investigations* and *Formal and Transcendental Logic* where he discusses formal manifolds (see Centrone 2010; Hartimo 2021). According to Hartimo, Husserl fulfils both the Incompleteness Claim and the Dependence Claim (2021:162). This makes for the interesting case of considering the ontological dependence relation of foundation as a possible candidate for structuralism.

This fourth article argues that the relation of foundation has some natural affinity with the dependence relations relevant for non-eliminative structuralism. The immediate promise of the relation is that it allows for a more fine-grained analysis, due to its unifying character and its reference to essence: "One sees at once how such differences determine *essential divisions of the whole*" and how "*every content* [in the range of the whole] *is foundationally connected, whether directly or indirectly, with every content*" (2001b:34). The third *Investigation* is difficult to pin down, and an accurate analysis of foundation is equally challenging to come by. The relation does make certain distinctions, e.g., between *mediate* and *immediate* foundation (to capture a difference in transitivity) and, to follow Kit Fine's terminology, between *generic* and *objectual foundation* (Fine 1995b:465). While generic foundation is a relation between species *A* and *B*, objectual foundation is between individual objects either of the same species or of two different species. Sometimes, Husserl seemingly transitions from a generic relation to an objectual one, without making it clear, which complicates an accurate analysis (Fine 1995b:465). However, this distinction makes the relation of foundation even more relevant to our structuralist case, as there is a strong parallel between ODO and objectual foundation. ODS would then concern a relation of foundation between a species *A* and the object *a* of species *A*. Moreover, we also have the distinction between reciprocal and one-sided foundation, thus allowing for symmetrical relations of dependence. As the relation permits transitive and symmetrical relations of foundation, in the context of mathematical structuralism and ODO, we end up with chains of infinite founding relations. Foundation thus permits for *cyclical* relations of dependence. As we have chains of dependence relations that cycle, a linear structure of dependence is abandoned. The upshot for non-eliminative structuralism is twofold. First, it would account for the constitutive nature of a structure and its elements. Second, it would clarify the reciprocal dependence relation between the elements belonging to the same structure, and between a structure as a whole and the elements belonging to it. This suggests that the property of non-linearity generally meshes well with structuralism. Structuralism has clear anti-foundationalist (i.e., *linear*) tendencies both in epistemology and metaontology,[2] preferring a holistic approach that emphasises coherence. That the choice of an ontological dependence relation should fit with this general metaphysical picture, gives added incentive to pursue this non-linear account of ontological dependence.

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[2] See the second and third papers of this dissertation.

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Metaontology for Mathematical Realism*

Abstract:
This paper is on metaontology for mathematics. This means that the questions of ontology are those of numbers, sets, lines, and groups. The aim of the paper is twofold: 1) to show that doing metaontology and ontology of mathematics is generally worthwhile, and can provide genuine philosophical insight, and 2) that implementing an appropriate metaontology facilitates justifying mathematical realism as a viable position. If these claims are successfully defended, metaontological perspectives are shown to be welcome and fruitful additions to the endeavours of nuancing and furthering the debate on mathematical ontology.

1. Introduction

Existence questions take many forms. First, there are the questions of *which* entities exist and *what* their natures are. Second, one can question what the notion 'existence' means, whether there is such a thing as an existence predicate that all existing entities share, or discuss whether existence and reality are layered, so that an entity could have a degree of existence considered to be more *real* than another. While the former existence questions – the *which*'s and the *what*'s – are covered by ontology, i.e., the study of being and existence, the latter questions are rather part of *metaontology*. Metaontology concerns the nature and methodology of ontology, and it has seen increasing interest after the publication of Peter Van Inwagen's paper "Meta-Ontology" (1998), thus introducing a new field to the analytic philosophical stage. Since Van Inwagen (1998), metaontology has continued to raise scientific interest, with contemporary debates mainly focusing on two sets of questions: 1) the legitimacy of doing ontology – especially ontological realism – and 2) the philosophical depth and promise of ontology as such (Eklund 2013:229). In addition to these sets of questions, metaontological endeavours and positions – however diverging – have in common that they share an interest in the historical background of their field. While metaontological questions have implicitly formed part of ontology long before it became its own field of study, the historical background that draws the most attention is centred on the debate between W. V. O. Quine and Rudolf Carnap, each representing competing metaontological stands.[1] By now, metaontology has been a hot topic

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[1] For some recent discussions, see Alspector-Kelly (2001), Putnam (2004), Eklund (2006, 2009, 2013), Chalmers (2009), Soames (2009), Thomasson (2009, 2014), Blatti & Lapointe (2016), Turner (2016).

for a couple of decades, and it has branched into several narrower subcategories. The present paper is on metaontology for mathematics, so that the *which*'s and the *what*'s at stake – i.e., the questions of ontology – are those of numbers, sets, lines, and groups. The overarching aim of the paper is twofold, reflecting the two sets of questions mentioned above: 1) to show that doing metaontology and ontology of mathematics is generally worthwhile, and can provide genuine philosophical insight, and 2) that implementing an appropriate metaontology facilitates justifying mathematical realism as a viable position. If these claims are successfully defended, metaontological perspectives are shown to be welcome and fruitful additions to the endeavours of nuancing and furthering the debate on mathematical ontology.

In the next section, I will make some observations with regard to the general backdrop of the ontology for mathematics, and give a few reasons as to why ontological realism is attractive in the first place. In section 3, I go into the historical background for metaontology with the Quine-Carnap debate. In section 4, I look at the general relationship between ontology and metaontology, and how we can construe it as *methodology* (in section 4.1) and as *qualification* (in section 4.2), before looking at the genealogical relationship of ontology and metaontology (in section 4.3). Section 5 is on metaontology for mathematics in particular, where I apply the insights drawn from the previous general discussion. Two metaontological positions are presented, one inflationary and the other deflationary, both of which can be traced back to the Quine-Carnap debate. As such they can (to some extent) be categorised as either broadly neo-Quinean (inflationary) or neo-Carnapian (deflationary). Lastly, I argue that if one is a mathematical realist, one should adopt the deflationary metaontological attitude.

2. Mathematical ontology

One of the key questions in philosophy of mathematics is what mathematical objects *are*. Are they independent *abstracta*, fictional perhaps, or maybe they are referents of singular terms? The answers are diverse. Attendant to – and inseparable from – these issues, is the question of whether mathematical objects exist in the first place. A rough divide presents itself: realism or anti-realism in ontology. Realism in ontology comprise positions where mathematical objects exist independently of the human mind, while anti-realist positions deny such existence.[2] There

[2] There is also the difference between realism in truth-value and realism in ontology, where one can supposedly be a realist in truth-value and not in ontology (see Shapiro 2000). If one takes mathematical propositions at 'face value', the terms appearing in mathematical propositions are taken to refer, without defending a realm of mathematical objects as such. In a way, this allows for an ontologically more lightweight (albeit also more ambiguous) position without minimising how we take the content of mathematical propositions to hold real sway and purport to refer to objects belonging to some 'real' domain. For now, I operate with the bigger (and

is, of course, some element of *degree* in the answers to this question. One can, for example, think that the existence of mathematical objects depends on other facts, or try to circumvent the question by denouncing purported reference to, or quantification over a domain of, mathematical objects as a mere *façon à parler*.[3] However, some basic divide stands: If a position neither hails to a full-fledged recognition nor a repudiation of the existence of mathematical objects, but rather to a position placed among the degrees in between, some initial or naïve divide still remains. And this is our point of departure.

If one falls down on the realist side, the concept 'existence' poses trouble in itself. Or rather, we seem to falter as soon as existence concerns non-ordinary objects, by which I mean non-spatio-temporal objects. When we speak of non-ordinary objects, like mathematical ones, it suddenly becomes more difficult to defend a realist position. Suppose mathematical objects exist. Do they exist on a par with my favourite mug? Or with the earth? The difficulty these questions bring forth is linked to how we think of truth and facts. Their existence, in turn, forms an important part of our justification for the truth of those states of affairs.[4] From the fact that I am currently sitting here at my desk, we conclude that I, as a particular person, exist. Moreover, we implicitly accept that the words used, such as 'person', 'chair' or 'sitting', are meaningful and intersubjectively understood. Generally, we are happy to accept such inferences. However, we do so with a strong restriction in mind; namely, that these acceptances of inference apply only to 'ordinary objects'. When we do not talk of ordinary objects, but rather of mathematical objects, the previous inference becomes problematic.

It is tempting to extend the same generosity towards mathematical states of affairs as to spatio-temporal states of affairs, so that similar inferences can be made. From some mathematical proposition accepted as true, we would – in similar fashion – conclude that the words used i) are meaningful, and ii) refer to objects, relations, or properties that exist. And while the former is not particularly controversial, the latter certainly is.[5] If we do postulate that

metaphysically more heavyweight) chasm; realism and anti-realism in ontology, where the sole criterion is whether mathematical objects – whatever that may mean – exist or not.

[3] The first is an example of modal structuralism, where the existence of abstract structures depends on their realisation by a particular system. Geoffrey Hellman (1989) is a proponent of this view. It is important to note that Hellman wants to defend *nominalism* in his proposed structuralist account, and means to eliminate structures altogether, i.e., he is firmly placed within the anti-realist camp.

[4] There are, of course, genuine problems related to these sorts of inferences and judgments about the tangible world as well. There are always questions concerning our 'conceptual carving of reality' and whether the concepts used to designate certain objects, events, or phenomena capture, so to speak, the domain over which they theorise. These domains could be empirical sciences or fields of philosophy like philosophy of science, philosophy of art, or metaethics.

[5] That is not to say that mathematicians and philosophers of mathematics consistently have had the same view on what these meanings are – or whether they have meaning or content at all – on the contrary, this is one of the core tenets of philosophy of mathematics. But, I do make the much milder claim that there is general consensus that mathematical language and linguistic practices in mathematics are meaningful, and not nonsense. Interestingly,

the mathematical objects we purportedly quantify over exist, and that the relations holding between them describe certain states of affairs, we are suddenly faced with the issue of having posited the existence of a vast, infinite domain of objects.

What lies at the core of the queasiness felt when confronted by such a picture? In a way, the answer is simple: Our lack of hard-hitting evidence, that is, empirical evidence. The endless possibilities we have of producing true mathematical propositions conveying some state of affairs comprise such vastness, that our lack of any "*real*" evidence for their existence is troubling. However, such real evidence, if charged by the accusation that we lack it, is not possible to come by in the 'ordinary' sense. We will not be confronted by the spatio-temporal presence of a mathematical object, nor will we absent-mindedly walk into a mathematical proposition and thus be forced to acknowledge its existence (consider a lamp post for comparison).

Our evidence for the truth of mathematical propositions and for the coherence of this field of knowledge is of a different kind, namely largely conceptual. This is why we cannot fully ease the nagging uncertainty that is seemingly always present: Why are we justified in positing the existence of any abstract mathematical objects, when we do not have empirical evidence for such existence?[6] How can we justifiably posit the existence of something we have never been in any *substantial* contact with, but only have vague intuitions of? These considerations serve as the backdrop of this paper, against which our theorising will be put in sharper relief.

2.1 Ontological realism in mathematics

Now that the problems pertaining to the existence of abstract mathematical objects have become somewhat clearer, this section offers some incentive for why one should still choose to embrace ontological realism. Various arguments have been given for accepting mathematical realism, I will only mention three: (i) Conforming with general usage and naïve understanding, (ii) the Indispensability of mathematics to the empirical sciences, and (iii) the Principle of epistemological parity.

Edmund Husserl treats exactly the topic of 'Unsinn' (nonsense) in relation to formal logic and the necessity for linguistic rules in order to prevent occurrences of nonsense (see Husserl (1969), (2001a:§15)).

[6] This is the point of departure in James Schwartz' (2015) paper on Stewart Shapiro's *ante rem* structuralism. There, Schwartz not only denounces Shapiro's evidence for not being independently compelling, but argues that the evidence offered by Shapiro actually works more in favour of Hellman's nominalist account.

(i) Conforming with general usage and naïve understanding

First of all, ontological realism conforms with some naïve conception of what mathematics is *about.* If one steps into some pre-philosophical way of thinking, one would arguably come to think that mathematics is about *something*, i.e., it is not devoid of content or meaning. Penelope Maddy (1990) argues exactly this, but also goes further: She says it conforms with how mathematicians speak and act when doing mathematics. Let us say that a mathematician has found a new proof for some theorem, perhaps simplifying it in some elegant and time-saving way. The point has already been somewhat indicated: One says that she has *found* a new proof for the theorem. Implicit in this linguistic practice is a general feeling of *discovery*, and in 'discovery' it is embedded that the thing discovered already had some existence before it was discovered. Coming-into-existence does not, usually, coincide with the discovery, given that we are still operating with a naïve and pre-philosophical understanding of these matters.[7]

(ii) Indispensability of mathematics to the empirical sciences

It is clear that mathematics is necessary for the practice and development of empirical science. From this uncontroversial fact springs an argument for ontological realism, commonly known as the Quine-Putnam indispensability argument. For our best empirical theories, reference to (or quantification over) mathematical entities is indispensable, and so we ought to accept the existence of mathematical entities. The argument surfaces in many of Quine's writings and is discussed at length in Hilary Putnam's (1979), thus becoming known as the Quine-Putnam argument.

> [Q]uantification over mathematical entities is indispensable for science, both formal and physical; therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities in question. This type of argument stems, of course, from Quine, who has for years stressed both the indispensability of quantification over mathematical entities and the intellectual dishonesty of denying the existence of what one daily presupposes. (Putnam 1979:347)

The applicability of mathematics to the empirical sciences is thus linked to our acceptance of the existence of the mathematical entities. The argument allows us to appeal to the success of our best scientific theories, and perhaps more importantly, to our reliance on them. As we do not want to own up to any "intellectual dishonesty", we draw the metaphysical conclusion necessary to keep our empirical sciences afloat. Ontological commitment to abstract

[7] If one steps away from this pre-philosophical perspective and allows for an example mathematician to have deep philosophical leanings, say, towards constructivism, and also be heavily occupied with the foundational debate, then, of course, it would be a different matter.

mathematical entities is not the worst of fates. Perhaps then, from a strictly empirical point of view, we are justified in relaxing the standards for existence, and not let the abstractness of them gnaw at us too much. In the end, it is a question of weighing the advantages and disadvantages. On the one hand, we are left with well-functioning empirical sciences at the price of ontological commitment to abstract mathematical entities, and on the other, we have the empirical sciences but without full access to the entirety of the mathematical toolbox, which must mean that the sciences eventually will suffer. The Quine-Putnam argument is very successful at inducing concessions and allowing for something to be the case in spite of previously held beliefs or scruples. And it does make sense; one feels the threat of being labelled "intellectually dishonest" as rather swaying, which is why it is one of the stronger arguments for accepting ontological realism and has been defended by many.[8]

(iii) Principle of epistemological parity (van Atten & Kennedy 2003)

There is another argument for ontological realism that also takes indispensability and success into the equation. But instead of appealing to the success of the empirical sciences, this argument appeals to the success of mathematics itself. It approaches the topic from epistemology and the success of our mathematical *knowledge*. Roughly, it goes as follows: On the basis of what we know, i.e., our mathematical knowledge, there is no reason to believe that mathematical objects do not exist. Hence, mathematical entities exist. To bring this out more clearly, let us consider a passage from Kurt Gödel's "Russell's Mathematical Logic".

> It seems to me that *the assumption of such objects* [classes and concepts] *is quite as legitimate as the assumption of physical bodies* and there is quite as much reason to believe in their existence. *They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions* and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the "data", i.e., in the latter case the actually occurring sense perceptions. (Gödel 1944:128, my emphasis)

Gödel argues by analogy: If we believe in the existence of the entities referred to in our empirical theories, so should we believe in the existence of the entities referred to in our mathematical theories. He also draws on indispensability; the existence of mathematical entities is deemed as necessary for our mathematical theories as physical bodies are for our physical theories. By putting the blocks of knowledge on equal footing, we should accept the existence of mathematical entities due to our acceptance of physical bodies. Mark van Atten and Juliette Kennedy call this line of argument the *principle of epistemological parity*, and argue that it was a

[8] See for example Maddy (1990, 1992), Resnik (1995, 1997), Colyvan (2001).

regulative principle in Gödel's thought (van Atten & Kennedy 2003:434). Kennedy argues that if we think of "physical objects on the one hand and abstract or mathematical objects on the other, from the point of view of what we know about them, there is no reason to be more (or less) committed to the existence of one than the other" (Kennedy 2014:6). The principle of epistemological parity argues for the existence of mathematical entities by drawing on the security and success of *mathematical* knowledge as a whole, and not in terms of mathematics' applicability to the empirical sciences.

The principle of epistemological parity shares another characteristic with the indispensability argument. Whereas the Quine-Putnam argument plays on our fear of losing full access to our best empirical sciences, the principle of epistemological parity plays on our fear of scepticism and the general erosion of the security of knowledge. This has to do with the status and place of mathematical knowledge as compared to other fields. If compared to other sciences, mathematical facts or truths are often deemed extra secure instances of knowledge. A true mathematical proposition is less exposed to revision than, for instance, facts based on fallible empirical observation. There are more ways in which an experiment, measurement, or observation can be tainted, imprecise, or misinterpreted, than for a proof or a theorem to be thusly challenged. If a theorem turns out to be false, it is not due to imprecision when measuring or failure of technical equipment. This is not to say that mathematical propositions – that were previously accepted as true – cannot be false; I am merely stating that there is added room and opportunity to make mistakes in the empirical sciences.

Now, if you compare mathematical knowledge to moral knowledge – which there is, in fact, quite a tradition of doing – ethicists sometimes appeal to the security of mathematical knowledge when defending the existence of moral facts.[9] If mathematical knowledge can enjoy the security of its knowledge – while being about abstract entities – perhaps moral realism can argue by analogy? The defences and rejections of this line of argument are many, but my point here remains valid: Mathematical knowledge is privileged as fields of knowledge go. The two arguments both play on something we would prefer not to entertain. With the Quine-Putnam indispensability argument, we contemplate the possibility of banning the use of mathematical knowledge in our best empirical sciences. With the principle of epistemological parity, we question mathematical knowledge by somehow considering it *lesser than* empirical knowledge, when we generally do not believe this to be the case. These are both argumentative moves that appeal to our fears of the prospect of diminishing our general bulk of knowledge.

[9] See Putnam (2004), Clarke-Doane (2014).

3. Historical background: Quine-Carnap debate

As mentioned above, the central historical background for the contemporary metaontological discussion is due to a debate between Quine and Carnap, particularly Quine's "On What There Is" (1948) and "Two Dogmas of Empiricism" (1951), and Carnap's "Empiricism, Semantics, and Ontology" (1950). The received view is that Quine emerged victorious, putting the analytic/synthetic distinction to bed, and reclaiming ontology as respectable and deserving of philosophical study. Hilary Putnam (2004) and David Chalmers (2009) both trace the development and rise of ontology as 'respectable' back to Quine's (1948) paper "On What There Is". Matti Eklund follows suit, declaring that the standard story of how ontological pursuits, and especially, ontological realism, again became respectable, is due to Quine (Eklund 2013:230). Also, Marc Alspector-Kelly sees it as the Quinean success of making ontological realism not fall prey to the so-called "disdain for the metaphysical", and to ensure that metaphysics "has a legitimate place within a generally naturalistic framework" (2001:93). It seems, then, to be some general consensus that the Quine-Carnap debate ended with the redemption of ontology, and that in the decades since – within the analytic tradition – ontology has reaped the benefits. Before we go into the consequences of this declared Quinean victory, let us focus on the central elements of the debate that will be relevant for our purposes.

First of all, following Jason Turner (2016), we can observe several points of agreement. They both believe being to be univocal; that is, there is no conceptual difference between *existing* and *being*, which means that if there *is* an entity, it equally *exists*. Moreover, there are no different *ways* of being or existing, and there is no use in contemplating the *different* ways something can exist. Pertaining to these points is viewing ontology as being "flat" (Schaffer 2009:354). This is a somewhat reductionist or simplifying view: Ontology is about figuring out which entities exist, and "once the list is done, ontological inquiry is done" (2016:6). Beyond their agreement on a "flat" interpretation of existing/being, there is, according to Scott Soames (2009), a further likening between the two. Quine and Carnap share the overarching goal of reforming metaphysics, considering it to be subsumed or in service of the investigation of science. They both, according to Soames, agree on "eschew[ing] metaphysics of the traditional apriori sort", but they disagree on the best way to do so (2009:424).

This is thus where their diverging metaphysical views begin: They disagree on how to go about doing ontology – and what ontology to adopt – so that it is conducive to the overarching goal of making philosophy of science the centre of philosophy. In what follows we will first discuss the Quinean idea of ontological commitment to abstract entities and then go on to the Carnapian use of linguistic frameworks.

3.1 Ontological commitment to abstract entities

To be able to come up with a list of which entities exist, the notion of 'ontological commitment' is crucial. In his (1948), Quine writes on ontological commitment: "A theory is committed to those and only those entities to which the bound variables of the theory must be capable of referring in order that the affirmations made in the theory be true" (Quine 1948:33). To be, then, is to be referred to by a bound variable of a theory, so that whatever the theory quantifies over is seen as existing. This makes existence *theory-dependent*: Depending on which theory we ascribe to, there are objects and entities whose existence we admit, and some whose existence we omit. Accepting a philosophical theory thus entails accepting the existence of the entities posited by the theory. That existence beliefs are formed by their entailment from a theory, makes it clear that Quine's concept of 'ontological commitment' already bears on metaontology rather than ontology (Van Inwagen 1998:233). This is because the threshold for existence is explicated in the metalanguage rather than the object-language; if our theory quantifies over or refers to certain variables, we are – by our acceptance of the theory – obliged to admit their existence. This is an integral part of the Quine-Putnam argument described above, where the existence of mathematical entities is accepted on the basis that they are indispensable to our best empirical sciences. As discussed in section 2.1, having full access to our best empirical sciences – with the necessary tools for furthering these sciences – seemingly outweighs the scruples we might have when it comes to accepting the existence of abstract entities. If the theory demands it, they must be admitted. It is thus a matter of considering whether ontological commitment to abstract entities is worth it, and for Quine and Putnam, the benefits of allowing empirical sciences to have their toolboxes fully equipped clearly was.

This brings the idea of calculating ontological cost to the fore. If the existence of an entity is accepted, it is by the authority of the theory's ontological commitments. By this same authority, we can now measure how demanding a theory's ontological commitments are, and thereby, we can compare theories with each other. The more entities allowed among a theory's ontological commitments, the higher the ontological *cost* (Quine 1960:270). The ontological cost of a theory can thus be considered as a basis for what theory to choose, as the ontological cost can simply be deemed too high. The theory-dependence present in ontological commitment is thus twofold. Not only are there direct consequences for what entities we admit into the domain of existing things, but we also judge which theories we should adhere to. The concept of ontological commitment not only tells us which entities exist (by the authority of the theory), but it also becomes a measure by which theories are deemed better or worse due to their cost. In this way, the concept of ontological commitment provides something more than a list of 'what there is' in the flat sense described above. By implementing the concept we add theoretic virtues into the mix.

3.2 Linguistic frameworks: Internal and external existence questions

Carnap (1950) introduces the concept of *linguistic frameworks* as a way of being able to safely use language referring to abstract entities and notions, e.g., numbers, properties, classes, and propositions. Empiricists, Carnap writes, are often sceptical toward such entities and try to avoid reference to them (1950:241). However, for mathematics, that strategy often results in characterising it as a formal system for which no interpretation is available (Carnap 1950:241). Carnap believes it is possible to keep reference to such entities, without being led to embracing some platonic realm, i.e., circumventing the 'pitfall' of platonism by acknowledging different linguistic frameworks.

> And now we must distinguish two kinds of questions of existence: first, questions of the existence of certain entities of the new kind *within the framework;* we call them *internal questions;* and second, questions concerning the existence or reality *of the system of entities as a whole,* called *external questions*. (Carnap 1950:242, emphasis original)

The difference between theory-internal and theory-external questions comes down to whether one operates from within or from the outside of a particular linguistic framework. For mathematics, the question of the reality or existence of numbers, for example, can be posed as an *internal* or *external* one. Let us consider the internal question first. To the question 'Are there numbers?', if posed from within the linguistic framework *the system of numbers*, Carnap answers in the affirmative:

> 'Is there a prime number greater than a hundred?' Here, however, the answers are found, not by empirical investigation based on observations, but by logical analysis based on the rules for the new expressions. Therefore the answers are here analytic, i.e., logically true. (Carnap 1950:244)

As the system of numbers relates to conceptual rather than empirical concerns, the methods by which we answer are also conceptual. Carnap does not use the word 'conceptual' but sticks to describing the answers as analytically or logically true, as his answer incorporates a defence of the analytic/synthetic distinction.[10] This is not necessary though. It is enough to note that the

[10] This is also one of the drawbacks of Carnap (1950), as Quine rather successfully attacked the analytic/synthetic distinction in his (1951).

methods vary across different fields, so that an internal question asked within an empirical science (e.g., chemistry) is answered by empirical investigation, while disciplines considered *logical* or *conceptual* (e.g., mathematics) answer questions by conceptual investigation.[11] As we see, internal questions are to be answered pretty straightforwardly – we appeal to the framework in question and see how to go about answering them by the nature of the framework (viz., conceptually or empirically).

What then about the external questions? Whereas we answer an internal question from within the framework – and by making use of the framework itself – this is not the case for theory-external questions. Carnap writes that from the internal question we must distinguish "the external question of the reality of the thing world [or system of numbers] itself", as this "is raised neither by the man in the street nor by scientists, but only by philosophers" (Carnap 1950:243). Philosophers occupied with the ontological question of numbers, i.e., the external question that is "*prior* to the acceptance of the [number] framework" are, according to Carnap, asking a pseudo-question devoid of any cognitive content (1950:245).[12]

How does linguistic frameworks fare when it comes to the question of whether numbers are part of *reality*? The philosophical question thus posed does not impress Carnap much. He considers it 'non-theoretical' and is very clear that while we still *speak* of accepting abstract entities, it remains just that, a way of speaking.

> We must still speak (and have done so) of 'the acceptance of the new entities' since this form of speech is customary; but one must keep in mind that this phrase does not mean for us anything more than acceptance of the new framework, i.e., of the new linguistic forms. Above all, it must not be interpreted as referring to as assumption, belief, or assertion of 'the reality of the entities'…. Thus it is clear that the acceptance of a linguistic framework must not be regarded as implying a metaphysical doctrine concerning the reality of the entities in question. (Carnap 1950:250)

While the distinction between internal and external questions of existence allows for the acceptance of abstract entities, there is no metaphysical claim about their existence involved. By implementing linguistic frameworks and the internal/external distinction,[13] we accept the entities that belong to a framework, and we allow for the methods that the frameworks necessitate (e.g., reference to abstract entities for mathematics and conceptual analysis of them), in order to further the different sciences (here: mathematics).

[11] See Frank Jackson (1998) for a discussion of conceptual analysis and its role for doing 'serious' metaphysics (Jackson 1998:41).
[12] See Flocke (2020) on Carnap's noncognitivism.
[13] See Audrey Yap's (2009) use of the distinction of internal and external questions regarding epistemic access to mathematical objects. Stewart Shapiro (2011) argues along similar lines, but without using the internal/external distinction explicitly.

4. Ontology and metaontology

Ontology and metaontology are undeniably interlinked, as the latter takes the former as its subject of study. David Chalmers (2009) points out that this is a characteristic that metaontology shares with another philosophical discipline, that of metaethics. By drawing this comparison, Chalmers views the relationship between ontology and metaontology as paralleling the relationship between ethics and metaethics. While the basic question of ethics is 'what is right?', the basic question of ontology is 'what exists?'. In abstracting from the basic questions of ethics and ontology, taking the disciplines as the starting point for new philosophical fields, we reach a higher level with two novel sets of basic questions. For metaethics, the question becomes whether there are "objective answers to the basic question of ethics". Similarly for metaontology, where we investigate whether there are "objective answers to the basic question of ontology". This basic question, of course, is 'what exists?' (Chalmers 2009:77). Metaontology thus operates on a higher level than ontology. The initial basic question of ontology is bracketed and abstracted on. The basic question – and thereby the discipline of ontology – has become the object of a new field of study, where the ambitions go beyond those of ontology.

The subject-matter of ontology can be considered evasive. This is in one way true, and in another not at all. The question of reality and what is in it, is complicated by the fact that our starting point is deeply entrenched in it. Whatever our position is on what there is, we can only contemplate reality *through* the ambiguous categories and concepts like 'exist', 'being', and 'object'. It seems that the only way to progress, is to continuously apply these concepts to the questions at hand, and see whether our understanding of the answers help us when we reapply them to the basic question 'What exists?'.

However, for many of the variants of the basic questions that are asked – and the answers we receive in return – the subject-matter is more straightforward. As we know, there are many answers to 'What exists?'. Quite a few of them are, supposedly, not in need of too much metaphysical heavy lifting. Questions that are common-sensical in nature, e.g., 'Does the floor on which I stand exist?', seem to be easily answered in the affirmative (given that one accepts some rudimentary realism about the external world). But alas, it is not that easy. While ontology is occupied with what exists, it is also an exercise in 'carving reality' in the appropriate and correct manner. And this is where the difficulties begin. We can deny the existence of the floor by arguing that 'floor' is an artificial concept and does not appropriately pick out any existing entity.[14] It turns out that there is no unison solution to this problem, since it is possible to deny the existence of entities that – from a common-sensical perspective – obviously do exist.

This tells us something about what we consider to be the task – and the risk – of formulating ontological questions; that sometimes, we end up defending the existence of objects whose non-existence appears absurd. But does this tell us something about the relationship between ontology and metaontology? I would argue it does. The above makes it salient that ontological disputes can easily end up as an exercise in philosophical scepticism – albeit a scepticism that challenges our conception of reality as consisting of separate entities that we can successfully refer to and that are conceptually delineated. The position is thus not an epistemological scepticism about reality and whether we can know anything about it. Rather, it constitutes a position on the far side of the anti-realist *metaontological* spectrum, where we question whether we can talk of the existence of entities at all, as our "conceptual carving" inevitably cannot capture reality. In this case, there are no objective answers to the basic question. Moreover, we cannot meaningfully say that basic things or objects exist (e.g., 'This table exists'), without automatically questioning that such statements can be easily accepted as true. But how could we ever confidently say that anything exists? We end up not seeing the wood for the trees, as our cautious behaviour towards existence questions prevent us from claiming that anything exists, as the risk of carving reality wrongly looms over us. For the purposes in this paper, I will not seriously consider such a scepticism. I will, that is, accept that there is an external world, that there are other people in it apart from myself, and that I can say with confidence that 'This table exists'.

In the remainder of this section, I will characterise metaontology as (i) *methodology*, and (ii) as *qualification*, before I look at the genealogical relationship between ontology and metaontology.

4.1 Metaontology as methodology

Let us first characterise metaontology as *methodology of ontology*, so that a metaontological line of inquiry provides an ontological theory with tools to study that to which it is ontologically committed. On this picture, the addition of a metaontological framework is, presumably, a

[14] This picks up on what I called 'ordinary objects' in the above introduction. The existence of an 'ordinary object' is not at all as easy as I indicated there. In *Ordinary Objects* (2007) Amie Thomasson provides a defence of common-sense realism when it comes to ordinary objects and combines that with a deflationary metaontology. Thomasson's defence of so-called 'ordinary objects' seeks to systematically disprove reductionist or simplificationist tendencies in ontology, on which common-sensical objects such as 'table or 'floor' are denied existence, because they are seen as unnecessary by-products of the ontological recarving of reality.

welcome addition to any position with ontological commitments. But are there any reasons to believe that a metaontology functions as an added toolbox that can improve an ontological view? Linnebo's suggestion – that metaontology investigates the key concepts of ontology – could be an example of this. Investigating 'objecthood' and 'existence' means clarifying two concepts that appear every time we want to assert that 'there *is* a mathematical *object* X'. Such a *process of sharpening* concepts can be seen as following a methodology, in order to figure out what mathematical entities exist on a given theory. And to commit to such a sharpening process (if the metaontology is good) could provide ontology with methodological rigour akin to the standards of a scientific process. Something similar comes up in Edmund Husserl's *Prolegomena* (2001a) when he describes the incompleteness of the sciences.

> Even the mathematician, the physicist and the astronomer need not understand the ultimate grounds of their activities in order to carry through even the most important scientific performances. Although their results have a power of rational persuasion for themselves and others, yet they cannot claim to have demonstrated all the last premisses in their syllogisms, nor to have explored the principles on which the success of their methods reposes. *The incomplete state of all sciences depends on this fact.* … Even mathematics, the most advanced of all sciences, can in this respect claim no special position. … Though the sciences have grown great despite these defects, and have helped us to a formerly undreamt of mastery over nature, they cannot satisfy us theoretically. *They are, as theories, not crystal-clear: the function of all their concepts and propositions is not fully intelligible, not all of their presuppositions have been exactly analysed, they are not in their entirety raised above all theoretical doubt.* (Husserl 2001a:15-16, my emphasis)

Husserl thus compares the empirical sciences to mathematics, and he sees the same sort of incompleteness in them, as relating to a proper explication of the concepts, propositions, and their underlying presuppositions. Husserl urges us to rectify this theoretical insufficiency, and if we are "[t]o reach this theoretical goal we first need, as is fairly generally admitted, a type of investigation which belongs to the metaphysical realm" (Husserl 2001a:16). Though he does not include such a metaphysical foundation for mathematics, for our case, we can extend this need so that a metaphysical investigation – i.e., a *metaontological* investigation – can be sought out for mathematical ontology as well.[15]

To pursue this point further, let us compare realist ontology for mathematics to an empirical science such as chemistry. Typical questions of ontology face us: What entities are there? What are atoms and their properties? Questions like these make up the starting point for any domain of knowledge. They are fundamental questions which one expects chemistry to

[15] He writes: "Such a metaphysical foundation is not, however, sufficient to provide the desired theoretical completion of the separate sciences. It concerns, moreover, only such sciences as have to do with actual reality, … certainly not the purely mathematical sciences whose objects are numbers, manifolds etc., things thought of as mere bearers of ideal properties independent of real being or non-being" (Husserl 2001a:16).

hold the answers to, as the answers describe the basic nature of the field. One of the decisive points in the development of chemistry was the implementation of scientific method, which effectively separated it from alchemy, the unscientific precursor to chemistry. The implementation of proper scientific method can certainly be thought of as a partial rectifying of theoretical insufficiency. There is no doubt that clarity and accuracy in methodology play a decisive role for the potential results of science. This is no less clear for chemistry. It seems obvious that if there occurs ever so slight a variance of conditions under which experiments are conducted, the results of these are contaminated. If this is discovered, one expects, due to good scientific practice, if not to nullify the results completely, at least to rerun and retest a number of times, in order to eliminate any disturbance that might alter the outcome. This, at any rate, seems to constitute a bare minimum.

The thought is that no one doubts the prudence of such measures for an empirical science like chemistry. And if metaontology is the methodology of ontology, we ought, by analogy, to rerun and retest our metaontological framework, and see how this potentially alters our ontological results. For example, if there is an ever so slight variance within the use of the concept of 'objecthood' in mathematical ontology, different entities would fall under the concept. It is clear how this would have massive implications for an ontological view: Entities previously considered objects are not, and entities previously not considered objects suddenly are! An example comes from mathematical structuralism, where you can either see the places of a structure as 'offices' to be filled by objects (say, the position "2" in the natural number structure is 'filled' by, say, the von Neumann ordinal "$\{\emptyset, \{\emptyset\}\}$"). Alternatively, we can regard the places as being objects in themselves, so that the natural number 2 simply is the second place in the natural number structure (Shapiro 1997:10-11). Another example comes from the conceptual realism of Gödel: "It seems to me that the assumption of such objects [classes and concepts] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence" (1944:137). Here, we see that Gödel actually identifies both classes *and concepts* as objects. Due to Gödel's very robust realism, his inclusion of mathematical concepts as worthy of objecthood and independent existence is not all that surprising. But it does radically change the ontological view in question by expanding the ontology and adding to the robustness of the realism. To have some sort of 'safety net' designed to catch the possible slipups of what is admitted into the ontology seems prudent, and if we need to formulate a metaontology to do this, that seems like the natural next step to take.

One might argue that there is no reason to accept an 'ought' from another discipline, another domain of knowledge. Why would prudent methodological measures for chemistry be applicable to the relation between ontology and metaontology for mathematics? However, arguing by analogy is not new when it comes to philosophy of mathematics. The comparison between the empirical sciences and mathematical platonism has been drawn to argue for the veracity and independence of mathematical knowledge, and, more importantly, for the existence of mathematical objects. But no matter how successful these lines of arguments were, a comparison between the *disciplines* is not the point here. My point is justificatory and relates to methodology. We saw that that which separates alchemy from chemistry is the implementation of scientific method. I argue that metaontology can serve the same principal methodological role in mathematical ontology. While the questions asked in chemistry and mathematical ontology are radically different, the process of sceptically inquiring into the most basic questions is the same. It has to do with eliminating all possible gaps of reasoning, and with making sure – even only for an extra check-up – that the process by which we ended up with the ontological view we endorse has been vetted and scrutinised.

4.2 Metaontology as qualification

So much for metaontology as methodology. To further explicate the relation between ontology and metaontology, let us look at another way of construing it, namely, to view metaontology as *qualification*. The main thought is that a metaontology is in some respect *restrictive*, so that the existence claims put forth by a theory meet certain limiting conditions. From what a theory claims to exist, there are criteria that the existence claims must share and fulfil. Unlike the rigour provided by methodology when we 'sharpen concepts', adhering to a schema of criteria for what entities we take to exist (in our theory) can provide rigour in a *unifying* sense. When a theory exhibits 'unifying rigour' it means that the theory in question (be it empirical, economic, linguistic, mathematical, or philosophical) adheres to certain commitments to ensure the coherence and unity of the theory. Let us consider a passage from Thomas Kuhn (2012), where he discusses what he calls a scientific paradigm.

> After about 1630, … and particularly after the appearance of Descartes's immensely influential scientific writings, most physical scientists assumed that the universe was composed of microscopic corpuscles and that all natural phenomena could be explained in terms of corpuscular shape, size, motion, and interaction. *That nest of commitments proved to be both metaphysical and methodological. As metaphysical, it told scientists what sorts of entities the universe did and did not contain: there was only shaped matter in motion. As methodological, it told them what ultimate laws and fundamental explanations must be like: laws must specify corpuscular motion and interaction, and explanation must reduce any given natural phenomenon to corpuscular action under these laws*. (Kuhn 2012:41, my emphasis)

We see that there were strong restrictions in place for what entities were allowed to exist at the time; there was only to be shaped matter in motion. The resultant theories would be *uniform* with respect to their ontological commitments, and no entity or phenomenon that was not reducible to shaped matter in motion would have passed the threshold. However limiting this might be, one can at least not fault the resultant theories for lacking in theoretically unifying rigour.

While Kuhn was trying to explain how scientific paradigms shape and limit what kinds of theories could be pursued within a scientific community, the idea of shared metaphysical commitments is reminiscent of how we can view criteria for what entities we allow to exist in an ontology. Interestingly, what Kuhn writes on the methodological commitments reveals another side to the existence criteria for entities; namely, that there is a normative component to it.[16] Not only are the existing entities restricted to shaped matter in motion, but we are told 'what the ultimate laws and fundamental explanations must be like', and that any natural phenomenon must be reducible to 'corpuscular action under these laws' (Kuhn 2012:41).

Let us apply this to our case at hand, the relationship of ontology and metaontology and the possible construal of metaontology as qualification. The formulation of shared commitments can provide an ontological view with lawfulness. We adopt a schema of criteria for why we should be ontologically committed to some entities and not to others. At the same time, we delineate the entities in question by our own conceptual apparatus (consider section 4.1). The commitments determine what something should be like for us to accept them in our theory, that is, they are rules of governance. They recommend ideal scenarios for what entities should be accepted, which makes them normative.

Let us also consider the descriptive side. As science is an activity, every scientific community shares common commitments of various sorts, viz., conceptual, methodological, instrumental, theoretical, and metaphysical (Kuhn 2012:42). Consequently, scientific communities often have a clear understanding of what their goals are, which methodological routes they pursue to advance in their research, and, to some extent, what they expect to find in terms of scientific results. Let us look at another passage from Kuhn.

> The scientist must … be concerned to understand the world and to extend the precision and scope with which it has been ordered. That commitment must, in turn, lead him to scrutinize, either for himself or through colleagues, some aspect of nature in great empirical detail. And, *if that scrutiny displays pockets of apparent disorder, then these*

[16] A critique often directed at Kuhn was that he conflated descriptive and normative considerations in describing the activity of science. The point I argue – that metaontology can be conceived of as qualification – does not depend on whether Kuhn did indeed conflate the two. For instance, see Kuhn (1983) for his remarks on this topic.

> *must challenge him to a new refinement of his observational techniques or to a further articulation of his theories*. (Kuhn 2012:42, my emphasis)

When the scientist is faced with apparent disorder in her findings, it spurs on further investigation. For our case, this would amount to accepting the existence of some entity that does not mesh well with our other ontological commitments, perhaps even contradicting them. If such an ontological dissonance is discovered, it is a given that 'a further articulation of [her] theories' is necessary. This further articulation will be tasked to make sense of the apparent disorder. As a general goal in formulating scientific theories, we wish to categorise, order, and discover relations previously unknown, in order to further our understanding of a field of study. In our efforts to advance theories of the world and thus our understanding of it, we have an uncanny knack for finding commonalities and patterns. And we prefer the subsequent theories to be exemplary in exhibiting characteristics of order. If a theory faces serious discrepancies, and our attempts at further articulation or accommodation are in vain, we tend to conclude that the theory under consideration is unlikely to be true, and we discard it. But the process of further articulating, and the directions given by certain criteria, are useful in weeding out instances of ontological dissonance. When we adopt a metaontology, we adopt a restrictive framework that limits our theoretical manoeuvres, and prescribes lawfulness for what is included in the ontology. This is one way of construing what metaontology is to ontology, and the task of metaontology as one of qualification. If this task is successful, the commitments are laid down, and pursuing a metaontology can thus be an act of ordering one's ontological views, and giving them unifying rigour.

4.3 Genesis of metaontology and ontology

Metaontological approaches are developed in union with ontological views and leanings. It is noteworthy that the metaontological approach one has adopted usually stems from already held views in ontology. More often than not, it is the need to qualify one's ontological view that provides the incentive to develop a metaontological framework. An example here would be Quine's concept of ontological commitment. By putting this metaontological constraint in place, he not only made belief in the existence of objects theory-dependent, but also formulated his justification as to why we ought to accept the existence of mathematical objects (see section 2.1 on Quine-Putnam Indispensability). Initially, then, there seems to be a methodological order to it. The ontological views come first, which then spurs on the development of an appropriate metaontology, in order to qualify and justify one's belief in the existence of objects.

However, one can also look at it in the reverse order. If we grant that the set of ontological beliefs and leanings held at the outset makes salient the need for a metaontological framework, who is to say that these beliefs and leanings were not the result of implicitly held metaontological beliefs? In this way, the development of a metaontological approach would simply be the exercise or process of making implicitly held beliefs – beliefs that have already shaped the ontological view in question – explicit. This would amount to a formulation of the background methodology of ontology, that is, the very activity of metaontology. Remember here, Husserl's point above, that the sciences are "not crystal-clear: the function of all their concepts and propositions is not fully intelligible, *not all of their presuppositions have been exactly analysed*, they are not in their entirety raised above all theoretical doubt" (Husserl 2001a:15-16, my emphasis). If metaontology can be construed as *both methodology* and *qualification*, we see how metaontology and ontology share a common goal – to give an account of what exists – but with more theoretical sufficiency. Then our beliefs are justified, and the underlying presuppositions examined and clarified.

Whatever the position one takes on the question of order – whether ontology or metaontology comes first, and which makes explicit what is implicit in the other – they do certainly influence each other. For example, it is rare to endorse robust realism in ontology (e.g., mathematical Platonism), only to argue that ontology does not provide genuine and 'deep' philosophical insight. It would be equally strange to defend ontology as capable of yielding substantial truths about reality, while eschewing the existence of all mind-independent entities (be they spatio-temporal or abstract). While a metaontology generated by certain ontological views can give input to the ontology in question, this is generally not in a radically *revisionary* sense. Likewise with an ontology generated by certain metaontological views, depending, of course, in both cases on the metaontological and ontological views in question. Ideally, they should both inform and improve each other, so that, no matter in which direction the views are generated (metaontology → ontology, or ontology → metaontology), the views reciprocally feed into each other and yield mutual improvement. If this is achieved, they can go hand in hand, and together, they can each add justificatory weight to the other.

5. Metaontology for mathematics

So much for a preliminary introduction to metaontology and mathematical realism, now we need to broach what metaontology for mathematics means. As has become clear, metaontology is on a level abstracted from ontology, and as such, metaontology for mathematics concerns the question of whether there are – to echo Chalmers (see section 4) – objective answers to the question 'What mathematical entities exist?'. A definitive reply to the question thus phrased is not, however, easily come by. In the case of mathematics, perhaps not surprisingly, it is mostly mathematical realists that have pursued metaontology. It is the task of the realist to provide evidence that mathematical objects exist; a task more precarious than that of the nominalist, whose task is to provide evidence that mathematical objects do *not* exist. This is simply because it is on the realist to provide an ontological account which makes it plausible that mathematical objects do indeed exist, i.e., the burden of proof is on the realist, not on the nominalist.[17] Most metaontologies for mathematics therefore have realist ambitions.[18] According to Øystein Linnebo (2018), metaontology for mathematics is concerned with the *key concepts* of ontology in mathematics, such as 'existence' or 'objecthood'. In order to list what objects there are, we need to determine what the concept of objecthood means. What are the criteria for a mathematical something to count as an object?[19]

The concept of existence poses different concerns. We briefly explored this in section 2, where the existence of a mathematical *abstractum* was compared to that of a medium-sized physical object, say, the chair I am sitting on. They obviously do not have the same properties or nature, the mathematical *abstractum* wholly lacking in causal powers and with no spatio-temporal make-up. But that is not the question. The question is rather whether the word 'exists' has the same meaning in both cases. That is, is existence univocal? We saw in section 3, that both Carnap and Quine considers this to be the case. Van Inwagen (following in Quine's footsteps, not Carnap's) also answers in the affirmative, citing the intimate connection between number and existence. It does not matter that the fourteen books on my desk are different objects than the fourteen favourite films you have on your watchlist; the *number* of these different things are the same, and the number of books on my desk simply *is* the same as the number of the films on your watchlist (Van Inwagen 1998:236). There is little sense to say that 'fourteen' changes its meaning dependent on what kind of objects it is ascribed to. According to Van Inwagen, the supposed existence of a mathematical *abstractum* thus follows the *normal* usage of the word 'exist', i.e., how we ascribe it to ordinary objects (1998:236). This thought

[17] For instance, consider Russell's teapot. Also, we do not expect the same kind of evidence for existence and non-existence. Whereas lack of evidence of something can be considered evidence for non-existence, this cannot be said for the case of existence, where a lack of evidence surely remains a deficit.

[18] See, for instance, MacBride (2003), Hawley (2007), Sider (2007), Hale & Wright (2009), and Eklund (2006, 2016) for discussion of neo-Fregean metaontology. Another metaontology from mathematical realism comes by way of Shapiro's (1997) *ante rem* structuralism, who pursues (though not explicitly) a metaontology based on coherence. Also, the second paper of this dissertation is on coherentist metaontology for non-eliminative structuralism.

[19] For Linnebo, objecthood is expressed by the Fregean triangle, where reference, identity criteria, and objecthood function together (2018:21-26).

is central to the neo-Fregeans' use of *Hume's Principle*, which says that the number of F's is identical to the number of G's if and only if there is a one-to-one correspondence holding between F and G, formalised as: $\#F\text{'s} = \#G\text{'s} \leftrightarrow F \approx G$. Here, it is the very fact that a number is ascribed to the objects that fall under the different concepts, F's and G's, and that these are identical if there is a one-to-one mapping from each F to each G. This means that in my example, the number of books on my desk and the number of films on your watchlist can be put in a one-to-one correspondence, and that it simply does not matter that the objects in question are different.

However, in this paper, I want to give a general defence of why the mathematical realist should adopt a metaontology. I will present two alternative metaontological approaches, which pick up on the historical roots of the contemporary metaontological discussion. First, we have the *inflationary attitude*, which follows a broadly neo-Quinean tradition. Second, we have the *deflationary attitude*, which follows a broadly neo-Carnapian tradition. I say 'broadly' here, as the two attitudes – while incorporating elements from each philosopher and the traditions they spurred on – cannot be said to faithfully represent the historical Quine and Carnap. I do believe though, that naming the metaontological attitudes thus is warranted by the general views professed in each.

5.1 Inflationary attitude: Neo-Quinean tradition

We saw in section 3.1 that adopting the Quinean concept of ontological commitment allows us to list what entities we take to exist on a given theory. Existence thus becomes theory-dependent, and, moreover, we are able to measure the ontological cost of a theory. This, in turn, can form a basis for theory-choice. If the costs of two theories are compared, that can give us the incentive to adopt the theory with the lower ontological cost. To value lower ontological costs is tantamount to evaluate theories on the basis of traditionally *scientific virtues* – like simplicity and explanatory power – or in this case, *ontological parsimony*. This is in line with the idea of regarding ontology as being continuous with science: "I see metaphysics, good and bad, as a continuation of science, good and bad, and grading off into meaninglessness" (Quine 1988:117). The view is, then, that ontology is considered good when it is modelled on science, so that the ontological view exhibits scientific virtues. We must also note Quine's end comment here – "…and grading off into meaninglessness" – as this makes it clear that sometimes, in Quine's view, metaphysics trails off and becomes something akin to the pseudo-questions described by Carnap. That is, Quine is not inclined to follow metaphysics' every whim, and there is some inherent distrust. This distrust, however, has to do with the overall task of philosophy, refashioned into philosophy of science. As Soames (2009) noted, Quine (and Carnap) both saw the main task of philosophy as one of servitude to the investigation of science, so that philosophy of science becomes the centre of philosophy. This is clearly in line with the continuity claim. Turner, in addressing the claim, characterises our attitudes toward science and philosophical ontology as *inflationary* thus:

> Our naïve attitude toward science is inflationary. Although some strands of scientific anti-realism demur, the naïve attitude is that long before anyone started theorizing, the world was as it was – either with electrons, or without. … An inflationary attitude toward more philosophical ontology is similar. Long before anyone started theorizing, reality was as it was – either with numbers, or without. … The inflationary attitude toward science is natural. Anyone who holds that attitude and is convinced by Quine that philosophical ontology is continuous with science should also have an inflationary attitude toward philosophical ontology. As a matter of fact, most contemporary self-styled Quineans in metaphysics *do* endorse inflationism, both about science and about ontology. Whether *Quine* held such a view, however, is less straightforward. (Turner 2016:5).[20]

Turner describes the inflationary attitude towards science as natural. An inflationary attitude has to do with the independent existence of i) the external world (for science), and ii) mathematics (for philosophical ontology, following the example above). The interesting step is taken when he writes that: If you are inflationary about science, and you believe in the Quinean thesis that philosophical ontology is continuous with science, then you *should* also have an inflationary view about ontology. That might very well be. It depends on whether you think philosophy should serve scientific investigation, or whether you think that philosophy of science is merely one part of a self-standing discipline. Let us, for now, further explore the idea that ontology is indeed continuous with science.

One way to consider this continuity claim is to look at it through the lens of the arguments of section 3.2 above, i.e., metaontology as qualification. Commitment to the existence of entities is integral to theoretical endeavours; this is common ground for science and ontology. If we take the continuity claim seriously – and the idea that philosophical ontology should be modelled on science – the resultant view can be seen as simply putting scientific values in action as means of *qualification*. That is, by entertaining ontological parsimony, we are actually implementing and sharpening a certain pattern of lawfulness. Traditionally scientific virtues thus draw up the limits for the operational and structural

[20] According to Turner, there is room to interpret Quine as being more deflationary toward science and ontology in general than the received neo-Quinean view of today. In that respect I am guilty of the same broad brushstrokes: I do not try to paint Quine as having held the inflationary view described in this section, but I do use certain broadly Quinean ideas as the figurehead for the inflationary approach (see also Price 2009). Soames (2009) also discusses the Quine-Carnap dispute as sometimes being misunderstood, in the way that views that today are called neo-Carnapian or neo-Quinean do not capture the views held by the historical Carnap or Quine.

framework. In order to safeguard from superfluous or pointless commitment, we are putting a principle of prudence in place. From a common-sensical point of view, it is better to commit to too little and remain agnostic about further commitment, than to embrace a too generous ontology, and thereby undervalue the commitments made. If one turns out to be wrong about some small part, it immediately diminishes the correctness of the rest, no matter how the justification for the rest stands. This is one way to look at a Quinean variant of metaontology as qualification; letting theoretical virtues constitute the lawfulness under which our ontological view is ordered.

5.2 Inflationary drawbacks

One of the drawbacks of the inflationary attitude is that it is formulated with the empirical sciences as a backdrop. The thought that metaontology for mathematics should conform to scientific virtues and standards means that what is inherently special to ontology of *mathematics* is put in second row. It becomes a view where metaphysics – as continuous and at the service of science – has priority over mathematics. This is clearly seen in the reliance on the Quine-Putnam indispensability argument. Maddy (2011), who previously embraced indispensability and shared a generally more Quinean outlook (see Maddy 1990), revokes her support of the indispensability argument.

> Though some take me to task for apostasy, I soon despaired of this position [naturalistic variant of Gödel's Robust Realism as from Maddy 1990], for three reasons: it relies on a Quine/Putnam indispensability argument that I couldn't continue to endorse; arguments for and against axiom candidates that seem compelling don't fit well with the metaphysics; and most importantly, just as a fundamentally naturalistic perspective counts against *criticizing* a bit of mathematics on the basis of extra-mathematical considerations, it counts just as heavily against *supporting* a bit of mathematics on the basis of extra-mathematical considerations. (Maddy 2011:ix, emphasis original)

Notice the first and third reasons. Let us look at the third first, which, according to Maddy, is the most important one. It says that one should not support *a bit of mathematics* on the basis of extra-mathematical consideration, just as one should not criticise a bit of mathematics on that basis. What does she mean here? It seems like while she always rejected the *criticism* of parts of mathematics on grounds not connected to mathematics, this is not the case for her support of parts of mathematics *on grounds not connected to mathematics*. To rephrase: we must not allow non-mathematical considerations to bias our view of parts of mathematics. Let us look at the first reason again; she can no longer endorse the Quine-Putnam indispensability argument. This argument can be seen as an instance of the third reason: We accept the existence of mathematical abstract entities on extra-mathematical grounds. In this case, these grounds are that referral to mathematical objects is considered necessary for our best empirical theories.

A consequence of considering the interests of the empirical sciences is that our justification for pure mathematics suffers. Considering to "pursue only the mathematics that's directly needed for natural science", is a "sentiment [that] is often based on a Quinean holism that sees mathematics in application as confirmed along with the rest of our overall web of belief, but *leaves the remaining pure mathematics without justification*" (Maddy 2011:87n45, my emphasis). By such a standard, we end up accepting mathematics not *en bloc*, but based on its applicability, and as we cannot find similar extra-mathematical functions for the pursuit of pure mathematics, it is left behind.[21]

> [T]he justification – or lack of justification – for mathematical methods is based on a metaphysical account of its subject matter. From the Second Philosopher's point of view, this gets things backwards: the order of justification goes the other way 'round, from the math to the metaphysics, not the metaphysics to the math. From her point of view, metaphysical considerations of this sort shouldn't be allowed to restrict the free pursuit of pure mathematics – and, in fact, they haven't. (Maddy 2011:87)

From a "mathematics first" perspective, the inflationary attitude to metaontology does indeed seem to get the order of things wrong.[22] Instead of putting mathematics first and providing justification for the whole of it, we end up i) modelling it on the empirical sciences (even though it is not itself one), ii) effectively devaluing parts of mathematics by its lack of applicability to empirical science, and iii) trying to figure out a metaontology for mathematical ontology from the point of view of general metaphysics and philosophy of science instead of letting our subject-matter – i.e., mathematics – take centre stage. There is one consolation to be had from Maddy's analysis however, that while the "metaphysics first, mathematics second" sort of view is popular, it has not succeeded at stifling the pursuit of pure mathematics, because, when it comes to it, most mathematicians do not let their research be restricted by the considerations of metaphysicians.

5.3 Deflationary attitude: The rise of neo-Carnapian alternatives

While the inflationary neo-Quinean attitude towards ontology has reigned supreme since ontology's redemption, deflationary approaches to metaontology have recently been on the rise (e.g., Hirsch 2002; Schiffer 2003; Rayo 2013; Thomasson 2014; Hofweber 2016; Linnebo

[21] See also Feferman (1993).
[22] According to Mirja Hartimo (2020) Edmund Husserl also endorsed a "mathematics first" sort of view.

2018) (Marschall & Schindler 2021:99). One of the motivations for Amie Thomasson is to offer an alternative to the "mainstream" neo-Quinean metaphysics.

> The neo-Quinean approach has become so dominant as to become almost invisible as a methodological choice. As Ted Sider puts it, "Recent work on ontology nearly always relies on the Quinean methodology" (2011, 169). The new metaphysics, dominated by ontology, is, from a methodological point of view, a neo-Quinean metaphysics—so much so that David Manley simply refers to this approach as "mainstream metaphysics" (2009). To those brought up on analytic philosophy over the past sixty years, the neo-Quinean conception of ontology has come to seem natural, even inevitable. (Thomasson 2014:3)

One of Thomasson's main critiques of the neo-Quinean view on metaphysics is that it is modelled on science, i.e., ontological theories are presented as adhering to a *scientific* standard, that metaphysicians are doing something *like* science, and therefore that they are capable of providing deep and general truths (Thomasson 2014:9). It boils down to a belief that, on the neo-Quinean 'mainstream metaphysics' attitude, the prospects and results of metaphysics are *overinflated*. The antidote to the inflationary metaontology is thus a deflationary one, historically traceable to Carnap and his distinction between internal and external questions. Thomasson wants to prescribe a more modest role for philosophy, where a Husserlian *division of labour* is again welcome.[23] Let us consider these two passages from Husserl's *Prolegomena*.

> It is not, fortunately, essential insight which makes science, in the common, practically most fruitful sense, possible, but scientific instinct and method. For this very reason the ingenious, methodical work of the special sciences, more concerned with practical results and mastery than with essential insight, is in need of a continuous 'epistemological' reflection which only the philosopher can provide …. Philosophical investigation has quite other ends [than the special sciences], and therefore presupposes quite other methods and capacities. It does not seek to meddle in the work of the specialist, but to achieve insight in regard to the sense and essence of his achievements as regards method and manner. (Husserl 2001a:159)
>
> Philosophical research so supplements the scientific achievements of the natural scientist and of the mathematician …. The *ars inventiva* of the special investigator and the philosopher's critique of knowledge, are mutually complementary scientific activities, through which complete theoretical insight, comprehending all relations of essence, first come into being. (Husserl 2001a:160)

The first passage makes a division of labour between the special sciences (empirical science and mathematics) and philosophy clear. Not only do they have different goals, but their methods are also different. While science is occupied with 'practical results and mastery',

[23] Also, consider earlier views on metaphysics, like the Humean perspective. Hume reflected on the limitations of metaphysical investigation, and would surely not present it as in the image of natural scientific investigation (Thomasson 2014:9-10).

philosophy takes on the role of giving epistemological reflection and insight. It is the task of the philosopher to investigate the scientific output and the 'method and manner', but not to 'meddle'. This limits the scope of the philosopher's task, and it does not attempt to paint a picture where the philosopher is doing something *like science*. The second passage highlights how philosophy is a supplement to scientific results – to make sense of them in a more fundamental 'theory of science' kind of way. The fact that philosophy under no circumstance is to have revisionary input on science, does not make philosophical investigation insignificant or render it incapable of providing insight. But it is important to acknowledge the differences between the special sciences and philosophy, thereby acknowledging their inherent limitations, both when it comes to method and resulting theories. By doing so, we do not overly inflate the role of philosophy or, in this case, ontology, but give it a more modest role.

To see ontology as something that interprets scientific achievements in order to gain insight of general theoretical underpinnings and knowledge as such, brings us to the next point: the order by which we consider metaphysics and mathematics. As opposed to the inflationary way of doing it – the "metaphysics first, mathematics second" sort of view, a deflationary attitude – which realises the limitations of metaphysical inquiry – has it the other way around. From the outset of the ontological investigation, the goal is a quest for insight "in regard to the sense and essence of [the special investigator's] achievements as regards method and manner" (Husserl 2001a:159). This means that we first consider the mathematics, and only then the metaphysics. By this order it is the task to analyse mathematics on the basis of mathematical methods and results, and not from a general metaphysics point of view that is modelled on natural science. An upshot of the deflationary attitude is that it does not put any undue influence on empirical science. This is reflected by the fact that it avoids relying on indispensability of mathematics for empirical science. Instead, I believe the principle of epistemological parity (see section 1.2) fits the deflationary attitude better. This argument is developed from an epistemological point of view (in line with Husserl's suggestion in the passage above), where we consider the success of mathematics and its achievements. If we think of mathematical knowledge, and compare it to the body of empirical knowledge, there is no more reason to be less committed to the entities described by mathematics than to those described by empirical science. In this case, there is no servitude of mathematics to empirical science. There is only the epistemological perspective of weighing mathematical knowledge to empirical knowledge, and finding them equal.

This is in line with Carnapian internal questions. If we consider mathematics from within the linguistic framework of mathematics, the question of whether we should take mathematical objects to exist can be answered straightforwardly by i) conceptual or ii) empirical means. Let us look at a passage from Soames (2009).

> Can there be any serious doubt that *there are prime numbers greater than a million, and hence that there are* numbers? Surely not. Perhaps, then, what needs to be abandoned is the idea that the existence of abstract objects is especially questionable, requiring an unusually demanding justification. This idea was, I would argue, one of Quine's central unexamined presuppositions. It is worth re-examining. Here, my sympathies are with Carnap. … [H]e was, I suspect, right in thinking that our ready appeal to them in mathematics and semantics is all the justification they need. (2009:442-43)

We can answer the internal questions of mathematical existence by conceptual means within the relevant framework. If this is posed as an external question, we cannot. Where there is room for 'empirical means' in this scenario, however, is our comparison between empirical and mathematical knowledge, and us finding them to be on a par. This we do by empirical considerations; we look at the fields' respective progress, the standing within the special investigators' community, and the overall success of the disciplines. But the question in mathematical ontology 'are there numbers?' is conceptual, as the threshold for mathematical existence is placed where it belongs, as a straightforwardly answerable question, if posed internally. The existence of mathematical objects is not in need of some *extra* justification, least of all an extra-mathematical justification.

6. Conclusion

In this paper we have seen how metaontology takes ontology as its subject-matter, and thus operates on a higher level. As such, it is for metaontology to put in place certain guidelines, within which we can actively reflect on the challenges mathematical ontology faces. To this end, we construed metaontology in two different ways, as methodology and qualification. First, by construing metaontology as methodology, we saw that it can serve as a process of sharpening. As a methodology, it formulates the need to ponder the framework in which we operate, and to consider how it potentially alters our ontological views. This is in some respect an exploratory process. By pursuing a metaontology for mathematics we are identifying an ontology's theoretical insufficiency, and by identifying implicit presuppositions, we complement the theoretical underpinnings that shape our ontological views. Second, by construing metaontology as qualification, we saw that metaontology takes on the task of prescribing lawfulness to ontological views. As we commit to the methodological aspect, we are also considering how commonalities and ontological dissonance appear in our theories. By putting limiting conditions in place, we effectively weed out the existence of entities which goes against our qualified ontological commitments. This contributes to a theory being more uniform, as we delineate what – and on which grounds – we are willing to accept into our ontology.

By tracing the historical debate of Quine and Carnap, we described two attitudes towards metaontology and ontology. We saw, by differentiating between an inflationary and a deflationary approach, that views on ontology and metaontology for mathematics is deeply entrenched in issues concerning empirical science, justificatory concerns when it comes to evidence for abstract entities, and in what way we should consider philosophy and ontology to be at the service of science. For a mathematical realist, who wants to argue for the objective existence of mathematical entities, the deflationary view on metaontology disentangles the status of mathematical objects' existence from other scientific concerns. This is arguably an advantage, as the upshot is that mathematical ontology is investigated on its own premises, and not investigated as analogous – neither in content nor results – to another discipline. Moreover, mathematical knowledge is considered privileged, and our study of mathematical ontology and metaontology should reflect that. As the burden of proof is borne by the realist to make their case rather than by the anti-realist, implementing a suitable metaontology – to serve as a methodological and qualifying framework – seemingly makes assumptions that were previously implicit in our theory explicit.

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Coherentist Structuralism: Structures as Thin Objects*

Abstract:
This paper explores the position of coherentist structuralism in philosophy of mathematics. The idea is that non-eliminative structuralism has a natural ally in coherentist minimalism, and that the positions have certain traits that suggest they should be combined. This paper aims at developing a combination view of the ontology of non-eliminative structuralism and the metaontology of coherentist minimalism. Moreover, I argue that the central claim of non-eliminative structuralism – that abstract structures exist – is given added justification by situating it within a broader philosophical framework; namely, by adopting metaontological coherentism.

1. Introduction

In philosophy of mathematics, the questions of what mathematical objects are and whether they exist continue to be of interest. These are topics pertaining to mathematical ontology. Non-eliminative structuralism is one position that provides answers to these issues. On the non-eliminative structuralist's view, mathematical ontology consists of abstract structures and their positions, which are thought to exist independently of humans, i.e., it is a variant of realism in ontology. Interestingly, non-eliminative structuralism has not been pursued from an explicitly metaontological perspective. Metaontology for mathematics has to do with methodology of ontology and the qualification of – or limiting restrictions on – one's ontological views. *Coherentist minimalism* is one such metaontological position, which holds that given a mathematical theory's *coherence*, the mathematical objects described by the theory exist. It is a metaontologically minimalist position because the criterion for mathematical existence – that a theory is coherent – is minimal, which thus allows for a generous ontology. The lower we set the bar for existence, the more entities exist. While coherentist ideas have been defended, it has not been developed as a metaontological background theory in which to frame one's ontological views.

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The notion of coherence remains woolly, which is why a connection to other coherence theories is useful. Coherence theories of justification in analytic epistemology provides a distinction between *systemic* and *relational* coherence, in order to characterise how the system's internal arrangement, and how the system as a whole, provides justification in different ways (see Bender (1989)). This distinction finds a correspondence in mathematical structuralism, where there is a dependence relationship between a structure and the elements making up that structure. This correspondence, I argue, is worth looking into.

This paper explores the position of *coherentist structuralism* in philosophy of mathematics. It is a combination view that unites the ontology of non-eliminative structuralism and the metaontology of coherentist minimalism. The idea is that non-eliminative structuralism has a natural ally in coherentist minimalism, and that the positions have certain traits that suggest they could benefit from being combined. Moreover, I argue that the central claim of non-eliminative structuralism – that abstract structures exist – is given added justification by situating it within a broader philosophical framework; namely, by adopting a suitable metaontology. Attendant to pursuing this combination view, is a general elucidation of coherentism and coherence, which I will strive to achieve by investigating possible areas of conceptual overlap with – or applicability from – coherence theories in analytic epistemology.

In section 2, I explain what is meant by a metaontology for mathematics, and sketch two approaches to it (inflationary and deflationary). Section 3 is on metaontological minimalism and *thin objects*, and how these ideas are instances of a deflationary approach to metaontology. In section 4, I outline what coherentist minimalism entails as a metaontological position, and I provide different explications of the notion '*coherence*'. In section 5, I move on to the combination view of coherentist structuralism and point to how the incompleteness and dependence of objects on a structuralist picture can be construed as *coherence claims*. On the combination view, structures can be conceived of as thin objects, thus making them metaphysically lightweight. This allows for a thinner realism, as opposed to a robust realism *à la* platonism in mathematics.

2. Metaontology for mathematics

Metaontology for mathematics has mathematical ontology as its subject-matter. When we ask 'what mathematical entities exist?', we are still within the field of ontology. When, on the other hand, we ask whether there are objective answers to the question 'what mathematical entities exist?' we are within the field of metaontology for mathematics (Chalmers 2009:77).[1] The questions of ontology are bracketed and abstracted on, and thereby made into a new field of study. When we pursue metaontology for mathematical ontology, we want to figure out what is required for mathematical entities to exist, and how we go about determining that. We are, then, in pursuit of a set of criteria that the entities to which we are ontologically committed must meet in order to exist. Metaontology is thus about the methodology of ontology, and can be characterised as an effort to investigate the legitimacy and standing of the field of ontology as such. According to Matti Eklund (2006) and Øystein Linnebo (2018), metaontology for mathematics is occupied with the basic concepts of ontology, like *existence* and *objecthood*. The answers to the question of when a mathematical 'something' counts as an object, and what it takes for a mathematical 'something' to exist, determine our mathematical ontology proper. Not only will our metaontological presuppositions have been clarified, but the resultant ontological view will be fixed. In this way, metaontology *is* concerned with the methodology of ontology, in the way the crucial concepts are subjected to a process of sharpening, leading to a better thought-out ontological account. Attendant to the idea that metaontology provides a methodology for mathematical ontology, is the idea that metaontology amounts to a sort of qualification of one's ontological views. Most metaontologies for mathematics are developed by ontologically realist positions, as the burden is on the realist to prove the existence of mathematical objects. It is the ontological realist that needs added justification for her views, and one way to go about that is to impose certain limiting requirements. If her ontological claims must submit to restrictive conditions, the claims will i) follow the same guiding principle, ii) become uniform, and iii) ensure that the ontology as a whole has a desired unity. Ontological claims thus receive qualification, making implicit theoretical underpinnings explicit, by articulating criteria for when we allow mathematical entities to exist.

While metaontology as a field was coined by Peter Van Inwagen as recently as (1998), there is some historical background that is useful to mention for the purposes of this paper. W. V. O. Quine (1948, 1951) and Rudolf Carnap (1950) engaged in a discussion about ontology, the analytic/synthetic distinction, and ontological commitment to abstract entities. For the case at hand, metaontology for mathematics, the issue of ontological commitment to abstract entities is especially relevant. In (1948) Quine describes what it takes to be committed to the existence of an entity: "A theory is committed to those and only those entities to which the bound variables of the theory must be capable of referring in order that the affirmations made in the theory

[1] For interesting discussions on metaontology, see, for instance, Eklund (2006) and the essays in *Metametaphysics* (2009), edited by David Chalmers, David Manley and Ryan Wasserman.

be true" (Quine 1948:33). Thus, we are ontologically committed to entities depending on which theories we accept, and what successful referring is needed for the theory to come out true. In the wake of the Quine-Carnap dispute,[2] two metaontological approaches were developed. There is the *inflationary* approach, which hails back to (broadly) Quinean ideas, and there is the *deflationary* approach, based on (broadly) Carnapian ideas.

Let us start off with the neo-Quinean inflationary approach. The received view is that Quine "won" the debate, and reinstated ontology and ontological realism, as legitimate enterprises with a natural place within the analytic tradition.[3] It is an inflationary attitude towards metaontology insofar as it models ontology on science, so that metaphysics is thought to be continuous with science (Quine 1988:117). As the existence of abstract entities is theory-dependent – due to Quine's criterion for ontological commitment – we have to look at the theories that include reference to abstract mathematical entities, specifically, the empirical sciences. Mathematics is indispensable for the empirical sciences, and thus reference to mathematical entities follows: We are ontologically committed to abstract mathematical entities. This brief statement is, in its spelled-out form, known as the *indispensability argument*.[4] We appeal to the applicability of mathematics to the empirical sciences and accept the existence of abstract entities, due to their indispensability for the sciences' full execution. The existence of mathematical entities thus relies on our desire to keep the toolbox for the empirical sciences fully equipped.

Let us move on to Carnap's contribution to the dispute and the deflationary neo-Carnapian approach to metaontology. Carnap (1950) introduces so-called *linguistic frameworks* to secure continued face-value talk of mathematical objects. He distinguishes between *internal* and *external* questions.

> And now we must distinguish two kinds of questions of existence: first, questions of the existence of certain entities of the new kind *within the framework;* we call them *internal questions;* and second, questions concerning the existence or reality *of the system of entities as a whole, called* external questions. … In contrast to the former questions, this question is raised neither by the man in the street nor by scientists, but only by philosophers. (Carnap 1950:243)

Internal questions are answered within a particular linguistic framework about the entities belonging to that framework, e.g., mathematics. These questions can be answered straight-

[2] See Putnam (2004), Chalmers (2009), Soames (2009), Eklund (2013), for discussion on the Quine-Carnap dispute.
[3] I go more into the Quine-Carnap dispute and the inflationary and deflationary approach to metaontology in the first article of this dissertation.
[4] It is known as the Quine-Putnam indispensability argument, see Putnam (1979) for a formulation of it, for discussion, see for instance Maddy (1990, 1992), Resnik (1995, 1997), and Colyvan (2001).

forwardly, either by conceptual or empirical means. So, if I were to ask whether there are numbers, our knowledge of mathematics would provide the answer that there are numbers, for example the natural number 2. In this way, we appeal to mathematics itself and the inner workings of it. Now, an external question is something else entirely. The external question is *about* the framework, or thing-world, and the entities described as such. If I were to ask whether there are numbers, I could not answer as easily. The external question is metaphysically deeper. It is about whether the whole of mathematics captures some true aspect of reality or whether it is an artificial field, made up by mathematicians. The external question is only ever asked by philosophers, and has as such no bearing on the day-to-day practice of mathematics, nor on the layperson's everyday usage of elementary mathematics.

How does Carnap's idea of linguistic frameworks lead to a deflationary approach to metaontology? We could say that it relativises existence questions. It means that we can ask questions within different linguistic frameworks, and that within each and every framework the existence questions do not need external justification. To answer a question about mathematical existence, we simply answer from within the mathematical framework. And likewise for other frameworks. If we ask an existence question within linguistics, e.g., whether genitives exist, we can answer in the affirmative, only relying on the framework we operate within. If we were to ask whether genitives exist from an external perspective, we would engage in a futile debate, since questions concerning whether natural language concepts pick out existing qualities or categories are meaningless for Carnap. The neo-Carnapian deflationary approach to metaontology thus drastically lowers the bar for when we can adequately claim that certain entities exist; every syntactically well-defined framework comes with a domain of objects that exist relative to the framework.

Deflationary approaches to metaontology are on the rise (Marschall & Schindler 2021). Amie Thomasson (2014) argues for a deflationary approach, claiming that some existence questions should be easy. We should not demand extra special justification for claiming that certain entities exist, from the fear that they do not accurately 'carve reality at its joints'. If we successfully operate with certain concepts, and they pick out entities we have no problem referring to, then we should also be allowed to make easy existence claims about them. The Carnapian idea of linguistic frameworks is an important forerunner to this line of argument.

Where, then, does this leave us for our case of metaontology for mathematics? Let us consider Scott Soames' (2009) take on the Quine-Carnap dispute and the ontological commitment to abstract mathematical entities.

> Can there be any serious doubt that *there are prime numbers greater than a million, and hence that there are* numbers? Surely not. Perhaps, then, what needs to be abandoned is the idea that the existence of abstract objects is especially questionable, requiring an unusually demanding justification. This idea was, I would argue, one of Quine's central unexamined presuppositions. It is worth re-examining. Here, my sympathies are with Carnap. … [H]e was, I suspect, right in thinking that our ready appeal to them in mathematics and semantics is all the justification they need. (Soames 2009:442-43)

Soames charges Quine with holding that mathematical existence claims require "unusually demanding justification", and that this is one of Quine's "central unexamined presuppositions". Interestingly, such unexamined presuppositions are exactly what is addressed by metaontological efforts. This is at the centre of the Quine-Carnap dispute, and much of the contemporary metaontological debate departs from this point. Are we allowed to accept the existence of abstract mathematical objects without such extra demanding justification? And what would such extra demanding justification even look like? Due to the scope of this paper, I will not provide answers to these questions. For now, though, we have seen what characterises different metaontological approaches to mathematics, and become clearer on some of its historical roots.

3. Metaontological minimalism

A metaontological view that has gotten recent traction within an ontologically realist setting is that of metaontological *minimalism*, defended by Linnebo (2012, 2018). Linnebo characterises metaontology as the "study of the key concepts of ontology, such as existence and objecthood", and if these concepts have a minimal character, we have a form of metaontological minimalism on our hands (Linnebo 2018:4). This means that the bar for when mathematical entities exist is rather low, i.e., the requirements for mathematical existence are minimal. Metaontological minimalism does not support a minimal ontology, on the contrary, generous ontological views are very much compatible with this metaontological stance.

> Metaontological minimalism has consequences concerning ontology proper. *The thinner the concept of an object, the more objects there tend to be.* Metaontological minimalism thus tends to support a generous ontology. By contrast, a generous ontology does not by itself support a metaontological minimalism. The universe might just happen to contain an abundance of objects whose existence makes substantial demands on the world. (Linnebo 2018:5)

If the requirement for a mathematical 'something' to be an object is minimal, it leads to more objects. The more objects there are, the more generous an ontology do we have. It is important to note, as Linnebo does, that a generous ontology does not automatically support metaontological minimalism. But minimalism lends support the other way around. The neo-Fregeans Bob Hale and Crispin Wright are charged with being *maximalists*, as they support a generous (or "promiscuous" as Eklund puts it) ontology (Eklund 2016:79). However, they would rather characterise their position as minimalist (Hale & Wright 2009). We see here, then, that while it might be appropriate to call their ontology maximalist, minimalism is appropriate for their metaontology. This distinction is important.

Let us consider a neo-Fregean example of how existence and objecthood can have minimal character. It is based on Hume's principle,

$$\#F = \#G \leftrightarrow F \approx G$$

which says that the number of Fs is the same as the number of Gs if and only if there is a one-to-one correspondence between the Fs and Gs. If the equinumerosity claim is true, the number identity is also true, and the number ascribed to the Fs and Gs exists. For example, if you have set the table, and the number of forks match the number of knives, the set of forks is equinumerous to the set of knives. From this, we get that the number identity of forks and knives holds, and that the number of forks and knives exists. In this example, we have a view where we can get existence and objecthood from a metaphysical recarving of reality from already accepted objects. The truth of the right-hand side suffices for the truth of the left-hand side, thus ensuring the existence of the numbers figuring there. We see here that what is required for the numbers to exist is not that much. We have not said anything about the numbers before we claim that they exist. What we do say is that the truth of the one side is sufficient for the truth of the other, by using a mathematical principle. Much of the neo-Fregean program relies on this principle, and other so-called abstraction principles. The thought is to "get at" new or other objects by showing that what we have done so far is legitimate. This requires metaphysical recarving, and one has to get on board with the idea that reality can be carved at its joints in different – equally good – ways.

While this is one way to let existence and objecthood have minimal character, how can we think about the resultant objects? This is where the idea of *thin objects* come into play, which is central for Linnebo's metaontological minimalism. The thought is that an object is considered *thin* if it does not make substantial demands on the world (Linnebo 2018:xi). This is, as Linnebo acknowledges, still very vague. What does it mean for an object to not substantially alter the world? One way to look at it is illustrated by considering an object that does alter the world, e.g., the chair I am sitting on right now. The chair ensures my continued sitting, thereby substantially altering the world. But, if I were to consider the set of the chair I am sitting on, it does not ensure anything of the sort. The set of the chair would be considered *thin*, while the chair itself would be considered *thick*. The draw of thin objects is to find a way to have minimal requirements for existence. If this is achieved, we can make easy or non-demanding existence claims about mathematical entities.

According to Linnebo, metaontological positions are minimalist if they *allow* for thin objects (Linnebo 2018:5).

> Minimalists need not hold that *all* objects are thin. Their claim is that our concept of an object *permits* thin objects. Additional "thickness" can of course derive from the kind of object in question. Elementary particles, for example, are thick in the sense that their existence makes a substantial demand on the world. But their thickness derives from what it is to be an elementary particle, not from what it is to be an object. (Linnebo 2018:5)

The concept of an object needs extending if we are to permit thin objects. To avoid any confusion here; minimalism is not a claim about all objects in general or how we should conceive of spatio-temporal reality. We do not have to accept all objects as thin, or that we choose which objects that substantially alter the world. Metaontological minimalism is a specific thesis regarding the key concepts of mathematical ontology. If we widen the concepts of existence and objecthood – that is, make them minimal in character – they will include more 'somethings', and we end up with a more generous ontology. By doing that, we allow for the existence of more objects, some of which are thin. But whether thin or thick, objects do not get their thin- or thickness from what it means to be an object, but from being the *particular* objects they are, be it a set or a chair.

This brings us to levels of thinness. An object can be thin in an *absolute* sense, so that it does not make a substantial demand on the world, e.g., pure abstracta, sets, numbers. But objects can also be thin in a *relative* sense. An object is thin in a relative sense, if given the existence of some objects *X*, the existence of an additional object *Y* makes no *further* substantial demand on the world. An example is the set of two trees, where the set does not make any *further* substantial demands on the world, other than that of the spatio-temporal make-up of the trees in question. This distinction leads to a metaphysical gradation of objects. First, we have the metaphysically demanding thick objects. Spatio-temporal objects are always considered to be thick (e.g., the chair I'm sitting on). Second, we have the metaphysically least demanding objects, the absolutely thin objects. These are non-demanding and pure, insofar as they have no representational imprint or concrete quality to them. Examples are pure mathematical objects such as sets and numbers. Third, we have the relatively thin objects. This is a middle category with one foot in each camp. On the one hand, they make barely any demand on the world. On the other hand, they can have elements that are clearly thick, like in the example of the set of the two trees. A type might also qualify as a relatively thin object, whose tokens are concrete instantiations. Related to the idea of relatively thin objects are dependence relationships, which we will come back to in section 5.2.

For now, let us consider how metaontological minimalism and the idea of thin objects fit with the two metaontological approaches above (section 2). Metaontological minimalism is an effort to give the key concepts of ontology – existence and objecthood – a minimal character. This is pursued so that we can have a metaphysically lightweight ontology. A metaphysically lightweight ontology is one where there are thin objects, objects that do not substantially alter the world. The inclusion of thin objects is well-suited for mathematical ontology. It is easier to justify existence claims in mathematical ontology, if the realism is less "robust", i.e., the thinner the realism, the better. As the bar for mathematical existence is lowered, there is less need for extra special justification. Rather, the justification for mathematical existence claims can be found within the linguistic framework of mathematics, as straightforwardly answerable internal questions by conceptual means. From these considerations, it seems that metaontological minimalism and the idea of thin objects bear all the hallmarks of a deflationary metaontological account.

Before we move on to coherentism, let me first give a brief description of Linnebo's own approach to metaontological minimalism, namely, *abstractionism*. This is a neo-Fregean approach to thin objects, where the ingenuity of Hume's principle is transmitted to other abstraction principles, i.e., principles of the form: $\S\alpha = \S\beta \leftrightarrow \alpha \sim \beta$. Here, '§' is an operator, while '~' is an equivalence relation. An example of such a principle is that for the direction of lines: $l_1 \,||\, l_2 \Rightarrow d(l_1) = d(l_2)$. This means that if line 1 and line 2 are parallel, the direction of line 1 is the same as the direction of line 2. But, beware: This constitutes another metaphysical picture than that of the neo-Fregeans Hale & Wright. While Frege and the neo-Fregeans have a biconditional between the left-hand side and the right-hand side, Linnebo prefers the sufficiency operator '⇒' (Linnebo 2018:18-19). This means that Linnebo cannot go from the right to the left, but only from the left to the right. i.e., his abstraction principles are one-directional. This makes for an asymmetric picture of abstraction and metaphysical reality. From the left-hand side, that line 1 and line 2 are parallel, we can get to the right-hand side and thus realise that their directions are the same. We are thereby given genuinely *new* objects, i.e., the concept of direction. In favouring this asymmetric sufficiency operator, Linnebo dismisses the Fregean idea of recarving the same metaphysical content. The idea is that we can always get genuinely new objects from old and known ones.

4. Coherentist minimalism

Now, another approach to thin objects is that of *coherentist minimalism*.[5] While Fregean abstractionism has a one-directional or linear methodology to 'get at' the ontology we want, coherentist minimalism does not. Coherentism as a clear-cut *metaontological* approach has not been pursued,[6] though coherentist ideas have already been defended in non-eliminative structuralist accounts, e.g., Charles Parsons (1990), Michael Resnik (1997) and Stewart Shapiro (1997). Coherentism as an explicitly metaontological prism, through which we characterise mathematical ontology, has not – as of yet – claimed its proper standing as a potential background framework. The goal, surely, shoud be that the ontological, epistemological, and metaontological stories all fit together, so as to provide a general account of mathematical entities and knowledge, where the reasons for which we allow mathematical existence are explicated and made integral to the philosophical whole.

What are the specific traits of metaontological coherentism? And what makes it minimalist? As a metaontology, it aims at articulating criteria for when mathematical entities exist. Linnebo describes coherentism thus.

> The coherence of a mathematical theory suffices for the existence of the objects that the theory purports to describe. … All that the existence of these new mathematical objects involves, according to the view in question, is the coherence of the theories that describe the relevant structures. (Linnebo 2018:5)

Coherentism thus provides mathematical ontology with boundaries, and serves as a qualification of views. The threshold for the existence of mathematical entities is made explicit: If the theory is coherent, the entities described by the theory exist. This leads to ontological uniformity, for every ontological claim within a coherentist framework, we can be sure that the entity in question fulfils this criterion, it belongs to – or is described in – a coherent theory. Coherentism is thus minimalist, insofar as what is needed for an entity to exist is very little. Coherence suffices for mathematical existence. This makes metaontological coherentism a rather straightforward position. However, it is not that easy. The remaining difficulty lies ahead; to provide an adequate notion of coherence. This is what the next section will be occupied with.

[5] In (2018) Linnebo suggests coherentism as another (yet unexplored) road to thin objects. His abstractionism is not against coherentism, they can rather be seen as two different metaphysical pictures coming from different perspectives.

[6] This is probably because metaontology as a discipline was not yet named as such. Remember, 'metaontology' was first introduced be Peter Van Inwagen with the paper bearing that name (1998).

4.1 Coherence

While neo-Fregean abstractionism hails back to Gottlob Frege's logicist program, coherentist minimalism has its roots in David Hilbert's views.[7] The so-called Frege-Hilbert controversy was an exchange of letters between Frege and Hilbert at the turn of the century. In a response to Frege, Hilbert writes on the criterion for existence.

> As long as I have been thinking, writing and lecturing on these things, I have been the exact reverse: if the arbitrarily given axioms do not contradict each other with all their consequences, then they are true and the things defined by them exist. This is for me the criterion of truth and existence. (Hilbert in letter to Frege, Dec 29 1899, in (Frege 1980))

For Hilbert, consistency is enough. As long as we cannot, from a given set of axioms, deduce both a sentence Φ and its negation $\neg\Phi$, the theory defined by the axiom system is consistent.[8] According to Hilbert, consistency is the criterion for truth and existence, and the mathematical entities described by the theory exist. However, this seems too weak a criterion. We can have a set of propositions that are perfectly consistent without what is described by those propositions existing. For example, the set of propositions describing a pink elephant in various detailed ways might be consistent, without such an elephant existing. One could argue, continuing the idea of thin objects, that consistency is not a sufficient criterion only for *thick* objects, as the existence of the pink elephant would substantially alter the world. But perhaps consistency would be enough for thin objects. According to Stewart Shapiro, however, consistency does not work for the entities of mathematical ontology either (1997:13). On Shapiro's view, where the envisioned background metatheory is second-order, coherence cannot be defined as deductive consistency: "When it comes to structures, consistency does not imply existence, contra Hilbert. Some consistent second-order theories have no models…. Surely, such theories are not coherent" (Shapiro 1997:13). So, equating coherence with logical consistency does not seem to cut it as the relevant existence criterion.

Let us see how Shapiro considers coherence and its role in his version of non-eliminative structuralism, called *ante rem* structuralism. In (1997) he proposes a criterion for the existence of mathematical structures, which he calls the Coherence principle.

[7] For discussion on the Frege-Hilbert controversy, which can be seen as some historical origin to the contemporary metaontological debate, see Resnik (1974), Blanchette (1996, 2018), Doherty (2019), and Dean (2020).

[8] Hilbert's notion of consistency is not the same as the modern-day notion, as he was not explicit about this. For our purposes, Hilbert's views on consistency as *criterion for existence* is nonetheless relevant as a precursor to coherence as existence criterion in metaontological coherentism. For a historical and accurate discussion of Hilbert's notion of consistency, see Dean (2020).

> The main principle behind structuralism is that any coherent theory characterizes a structure, or a class of structures. For what it is worth, I state this much: **Coherence**: If **Φ** is a coherent formula in a second-order language, then there is a structure that satisfies **Φ**. The problem, of course, is that it is far from clear what "coherent" comes to here. (Shapiro 1997:95)

This is clearly a position that commits itself to the existence of abstract structures by way of coherence. Shapiro uses the coherence of a formula in a second-order language to assert that there is a structure that satisfies the formula in question. For Shapiro (1997) the Coherence principle thus plays an important role; it is the principle by which abstract structures exist. The main problem, as Shapiro acknowledges, is that it remains uncertain *what* coherence really amounts to. Shapiro suggests that coherence is an informal analogy to mathematical *satisfiability*.

> The relevant formal rendering of "coherence," then, is not "deductive consistency." A better analogue for coherence is something like "satisfiability." It will not do, of course, to *define* coherence as satisfiability. Normally, to say that a sentence Φ is satisfiable is to say that *there exists* a model of Φ. The locution "exists" here is understood as "is a member of the set-theoretic hierarchy," which is just another structure. What makes us think that set theory itself is coherent/satisfiable? (Shapiro 1997:135)

The main justification for and understanding of what coherence is rest on a similarity with satisfiability in mathematics, as set theory is taken to be the "ultimate court of appeal for existence questions" and doubts whether a particular mathematical entity exists are "resolved by showing that objects of this type can be found or modelled in the set-theoretic hierarchy" (Shapiro 1997:136). James Schwartz (2015) writes that it is not clear how heavily Shapiro's justification rests on this similarity. One thing is to point out this similarity and say that it makes for a nice analogy. The problem arises because their similarity is meant to support the Coherence principle, though they are explicitly not the same, as Shapiro refuses to define coherence in terms of satisfiability. So how helpful is the analogy really? And what is the analogy meant to point to? It seems muddled to on the one hand claim that their similarity offers support for the Coherence principle, but on the other hand shy away from an explicit explication of the relationship between the two. Shapiro should not play up the importance of their similarity as justification for the principle, while explaining what the coherence means by comparing it to satisfiability. Shapiro admits that his notion of coherence is circular, but argues that it is not viciously so: "Coherence is not a rigorously defined mathematical notion, and there is no noncircular way to characterize it" (Shapiro 1997:13). While reaffirming its familiarity with mathematical satisfiability, he chooses to consider it as an intuitive primitive that cannot be reduced to anything more fundamental or to something formal (Shapiro 1997:135). The circularity charge is also brought forth by Schwartz (2015), who argues that not only is Shapiro's reliance on the Coherence principle circular, it is viciously so.

> It may be true that the *coherence* of set theory is presupposed in mathematics, but that does not mean that mathematics also presupposes the applicability of the *coherence principle* to set theory, or that it presupposes the existence of a set-theoretic *ante rem* structure. And Shapiro nowhere claims that either of these things are presuppositions of *mathematics*.
>
> What follows is that Shapiro must look beyond mathematics in order to justify the coherence principle. Shapiro recognizes this himself, of course. For instance, he is perfectly aware that the satisfiability principle, which he claims 'underlies mathematical practice' [1997, p. 136], does not directly imply the existence of *ante rem* structures. Rather, what one produces when using the satisfiability principle are *systems*. (Schwartz 2015:372-73, emphasis original)

While Schwartz gives some plausibility to the initial thought that coherence is presupposed in mathematics, and that it is a quality that mathematics exhibits, that does not mean that the *Coherence principle* is found in mathematics. If we have to search beyond mathematics to justify the principle, the reliance on the similarity to satisfiability seems even less fruitful. While it may provide the necessary springboard to consider the principle seriously, once the informal analogy has reached its potential, we are in need of philosophical justification for relying on coherence for the existence of structures.[9] While consistency and satisfiability are properties best defined within mathematics, coherence is a property to be attributed on philosophical merit. The informal analogy Shapiro draws between coherence and mathematical satisfiability is, presumably, meant only as guidance as to the philosophical content of coherence. While Shapiro recognises that the Coherence principle requires philosophical justification, he stops short of providing its philosophical content.

4.2 Another concept of coherence

Now that we have seen some of the muddle that arises when the justification for the Coherence principle and the explication of coherence are taken together, let us consider another route. The goal must be to get a clearer understanding of what coherence amounts to. As a criterion for existence, we have seen that consistency is not enough, it can only be a minimum ingredient. We are thus in need of something *more*, which is why we should cast our net a bit wider and

[9] As Schwartz notes, Shapiro is well aware of this: "[Shapiro] is furthermore explicit about mathematical practice not providing the final say *à propos* of the ontology of mathematics, writing that 'at no time did the mathematical community don philosophical hats and decide that mathematical objects – numbers, for example – really do exist' [1997, p. 25], and that the 'true ontology' cannot be 'read off' of mathematical practice [1997, p. 34]" (Schwartz 2015:373).

see what comes up. In addition to being a view in metaontology, coherentism is also a view in another philosophical field; namely, epistemology.[10] Coherentism in epistemology is often considered as the refusal of one of the main theses of *foundationalism*,[11] viz.,

> … [the view] that justification is transmitted from one belief (or small set of beliefs) to another belief in a strongly directional or linear manner, and that all such lines of transmission find their source in one or another form of basic belief, a belief characterized by a kind of epistemic priority, in that its own justification is not, in turn, transmitted to it from other beliefs. (Bender 1989:1)

Foundationalism is linear, one-directional and reductionist in character, where some beliefs have epistemic priority over others. This is not the case for coherentism. In coherentism beliefs lend mutual support and justification to each other, as long as they all belong to the same system of beliefs.[12] John Bender characterises epistemological coherentism as the view where "no empirical beliefs enjoy epistemic priority, and all rely for their justification on their connection to, or membership in, the body of other things believed or accepted" (Bender 1989:1).

The notion of coherence also poses some difficulty in epistemological coherentism. There is a "tendency for coherence theorists to think of 'coherence' in two ways", and as these have been considered interchangeable, it has led to the notion being "obscured" (Williams 2001:117). The 'two ways' refer to a distinction between *relational* and *systemic* coherence, a distinction that was first drawn attention to by Bender (1989) (see Hansson 2006:95). First, let us have a look at systemic coherence.

> A large part of the initial, intuitive appeal of the coherence theory is traceable to the metaphor involved in the idea that a body of beliefs should "hang together" or "mesh", the threads of the fabric of knowledge being, as they are, interwoven. Thus envisioned, coherence is a *holistic* (and not obviously relational) property, ascribable, presumably in varying degrees, to the system as a whole, much as the property of strength applies to fabric. (Bender 1989:2)

Systemic coherence is thus a property of a whole, and, as suggested by Bender, it can be attributed in varying degrees. While coherence has also been treated as a categorical either-or property, in which a system is either altogether coherent or not at all, I will, following Elke Brendel (1999), assume a gradational approach to coherence here; it is not a property the system either fully exhibits or not, but a system can have more and less of it (see

[10] For some classic literature on epistemological coherentism, see Bonjour (1985), Davidson (1986), Bender (1989), Lehrer (1990), and Olsson (1999).
[11] Foundationalism has a long history of widespread accept (e.g., consider Descartes' first principles). For interesting contributions to resuscitate old-school foundationalism, see DePaul (2000).
[12] Consider Quine and Ullian's 'web of belief' metaphor and book (1970). Quine (1961) is also considered to have clear coherentist elements (Williams 2001:126n1).

Hansson:2006:94).[13] In describing Laurence BonJour's (1985) notion of systemic coherence, Bender writes:

> Overall coherence of the system is (at least in part) determined by its logical consistency, degree of probabilistic consistency, the number and strength of inferential interconnections among its members, its unity or lack of isolated subsystems, and its freedom from unexplained anomalies. (Bender 1989:2)

There are thus certain qualities and markers that are subsumed by the coherence notion, and that must – to a varying degree – be in place for the system to be systemically coherent. In contrast to systemic coherence, relational coherence is not a property ascribable to a whole, but rather a relation where a belief is *cohering with* a suitable system.

> Coherence here is being conceived in way subtly different from the "strength of the fabric" metaphor above. It has now become a *relation* between a given belief and a system of other beliefs, rather than a property of the whole system. Perhaps this distinction is not viewed as important, since the coherence of the system may simply be a function of the strength with which each member coheres with the rest, but I think there are … reasons for keeping the "relational" and the "systemic" notions of coherence clear. (Bender 1989:2)

The main difference between systemic and relational coherence is that while systemic coherence is ascribed due to a *totality's* internal arrangement, relational coherence describes the internal arranging itself. As such, we can conceive of systemic and relational coherence as two sides of the same coin, as one can be explained via the other. Sven Ove Hansson (2006) notes that whereas BonJour and Quine favour systemic coherence, Keith Lehrer favours relational coherence (2006:95). This raises the question of interdefinability; are they truly two equal sides of the same coin, or does one take precedence? There are different answers to this. Erik Olsson argues, against Lehrer (1997), that relational coherence presupposes systemic coherence, "as coherence in the systematic[14] sense is the holistic property in virtue of which cohering things fit together" and a "theory of relational coherence that does not employ systematic coherence cannot be a theory of 'fitting with in virtue of coherence'" (Olsson 1999:287). Bender (1989) and Hansson (2006) ask whether the converse holds; whether systemic coherence can only be had if there is relational coherence among the system's members. Whether they are mutually interdefinable is a question I will not attempt to settle in any finality here. The line of inquiry, however, does point to something worthy of note; that even in the coherentist debate, there are issues of fundamentality.

[13] This is not to deny, of course, that a system can also be completely devoid of any coherence whatsoever.
[14] Olsson prefers 'systematic' to 'systemic' coherence, however, I choose to follow Bender's coinage.

Hansson further remarks that while relational and systemic coherence both have a tradition in epistemology, "the distinction [between relational and systemic coherence] can easily be transferred to other subject-matter than knowledge, and it can therefore be included in a general theory of coherence" (2006:95). This is where we make our entrance: What is the insight begot from this epistemological notion of coherence? And how can it be relevant for our metaontological case of coherentism? First of all, let us keep the noted distinction. Whereas systemic coherence is attributed to a whole, relational coherence describes particular instances of cohering within the larger whole. This means that there is a clear part-whole difference to be preserved, and that the notion allows for nuance. We can individuate the particulars of a system that relationally cohere with each other, and, by the token of systemic coherence, the system as such is covered by one harmonious property as well. The distinction between systemic and relational coherence is thus relevant for recognising the parts of a whole, and explain how both the parts and the whole can exhibit coherence, as relation and property, respectively.

The spelling out of the conditions to be met in systemic coherence is enlightening. While Shapiro's notion relied on an informal analogue without explaining the notion itself, the qualities belonging to systemic coherence highlights the threads that must be in place to ensure the "strength of the fabric". Logical consistency, Hilbert's alternative, is included as an ingredient. It is not, however, sufficient for coherence, but merely one thread out of many. In addition, there must be unity and lack of isolated subsystems. This also seems a reasonable condition for coherence in mathematics. Unity is provided by implementing metaontology and setting a common threshold for when we take mathematical entities to exist, as it sets restrictive conditions for our ontological views. A related point goes for the lack of isolated subsystems. As we have construed metaontology as methodology of ontology, it would be worrisome if there was an abundance of isolated subsystems. A methodology that results in isolated subsystems cannot claim to yield to a regulative principle. We would not ascribe the property of systemic coherence to the metaontology if the ontological views were cleft.

For metaontological coherentism, the demand that we have freedom from unexplained anomalies, e.g., paradoxes, is also reasonable. While it is not unrelated to the ingredient of logical consistency, it suggests something different in this context. An anomaly does not have to pick out an instance of inconsistency, but can also characterise how something is unusual or not agreeing with the rest in a looser or informal sense.[15] Moreover, that the degree of coherence would depend on how many and how tightknit the inferential interconnections among the totality's members are, is again a call for unity and general regularity, not to say that it also

[15] Consider Thomas Kuhn's notion of anomaly in normal science (2012).

provides a constitutive quality, so that the overall system, i.e., mathematical theory, forms a whole out of its members. The spelling out of what systemic coherence means thus gives rise to the general property of a system being overall intelligible, where each part is contributing to form a united – and coherent – whole, whilst keeping the parts distinct enough to be individuated.

Coherentism might not be as straightforward a position as initially hoped (beginning of section 4). This is due to the difficulties attendant to get a clear understanding of the central notion of coherence. While the systemic/relational distinction borrowed from analytic epistemology contributes to our conceptual apparatus, thus making it easier to express nuances of coherence, it is still meant as the bar for which mathematical objects exist. That is, it is the epitome of our minimalist metaontology. While coherence has gotten more philosophical content, it is still problematic as an existence criterion. Let us reconsider the deflationary component of metaontological coherentism. The neo-Carnapian deflationary metaontology proclaims that we do not need extra special justification that certain entities exist, and that existence questions that can be answered by straightforward empirical and conceptual means should be considered easy.[16] A potential hurdle for our account of coherence is that we described it as a property which can be attributed in degrees, i.e., a theory can be more or less coherent (see Bender (1989), Brendel (1999)). While such a gradational account of coherence is useful when explaining degrees of justification, it is less ideal as an existence criterion in metaontology. We can still not insert our developed notion of coherence as the articulated threshold for mathematical existence, so that we get the straightforward answers we are seeking on our deflationary account. For while the philosophical content of coherence has been advanced, it stops short of being a criterion by which our ontological views are regulated.

Another upshot of borrowing from epistemological coherence theory is that it picks up on the historical roots of the two metaontologically minimalist views; neo-Fregean abstractionism and coherentism. We saw in section 4.1 that each hails back to the Frege-Hilbert controversy; the neo-Fregean abstractionism from Frege, and coherentism is indebted to Hilbert, due to his existence criterion of consistency, a minimum ingredient of the coherence notion. The neo-Fregean abstractionist project has a one-directional approach, in which an asymmetric sufficiency operator gives us new objects from old ones (e.g., equinumerosity claim). This approach picks up on the linear aspect of epistemological foundationalism. To make this clearer, let us consider a passage from Michael Williams (2001) on the difference between epistemological foundationalism and coherentism.

[16] See Thomasson (2014).

> Whereas foundationalist theories are *atomistic*, coherence theories are *holistic*. For the coherence theorist, there is not a question of a belief's being justified all by itself, as the foundationalist's basic beliefs are supposed to be. To be justified, a belief must fit into a justified system; and the system is more or less justified depending on how well it 'hangs together' *considered as a whole*. This reference to whole systems is crucial. … The difference between foundationalism and the coherence theory is sometimes explained in terms of two competing models of justifying inference. The foundationalist conceives justifying inference on a *linear* model, in which justification proceeds from given 'premises' to 'conclusions' by justification-transmitting rules. The coherence theorist's holistic model of justification is decidedly *non-linear*. While the beliefs that comprise a given system will be logically interconnected in various ways, these connections are not in themselves relations of justification. … The coherence theory's distinctiveness arises from tracing the epistemic status of the background system to the way the entire system fits together. This *systematic*[17] coherence implies radical holism. (Williams 2001:117, emphasis original)

On the one side, there is some structural overlap between abstractionism and epistemological foundationalism, and on the other, between metaontological coherentism and epistemological coherentism. The latter is not directly shocking. Coherence theories, whether in metaontology or epistemology, have the rejection of linearity in common, whether it concerns the transmission of justification between beliefs, or how we get at mathematical entities and how their existence is given to us. Shapiro, in describing how we should i) accept the circular aspect of coherence, and ii) treat coherence as an intuitive primitive, writes:

> There is no getting around this situation. We cannot ground mathematics in any domain or theory that is more secure than mathematics itself. All attempts to do so have failed, and once again, foundationalism is dead …. The circle that we are stuck with … is not vicious and we can live with it. (Shapiro 1997:135)

The circularity of coherence and the non-linearity of coherentist theories have therefore a likely explanation. Coherence theories are motivated by giving a non-linear account of how things are and can be explained, that is, coherentism provides a competing picture, choosing to highlight the many interconnections and how those add to the whole system, be it a system of beliefs or of mathematical entities.

5. Coherentist structuralism

As staked out in the introduction, this paper aims at developing a combination view of metaontological coherentism and non-eliminative structuralism. Now that we have covered

[17] Williams also uses 'systematic' instead of 'systemic', but he refers to the same distinction described by Bender.

metaontological coherentism and gotten a better understanding of what coherence is, it is time to apply our metaontological insights and see where it gets us. We have characterised coherentism as a deflationary approach to metaontology, due to the reduced requirements for mathematical existence. The goal is to make the existence of mathematical entities more palatable, by conceiving structures as *thin*. This section sets out to show some of the natural connections between coherentism and structuralism, picking up on the gradation of thinness of objects (as described in section 3) and relating the distinction between systemic and relational coherence (section 4.2) to one of the main tenets of structuralism; mathematical objects are incomplete, which leads to dependence relations between a structure and its elements.

5.1 Non-eliminative structuralism

Structuralism is the philosophical position that mathematics is the study of abstract structures, for example, that the natural numbers constitute the natural number structure. Structuralist ideas have notably been defended by Paul Benacerraf (1965), Geoffrey Hellman (1989, 2001), Shapiro (1997, 2011), Resnik (1982, 1997) and Parsons (1980, 1990, 2008).[18] An official starting point for mathematical structuralism came with Benacerraf 's "What Numbers Could Not Be" (1965), in which he concluded that numbers are not objects.

> Therefore, numbers are not objects at all, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an *abstract structure* – and the distinction lies in the fact that the 'elements' of the structure have no properties other than those relating them to other 'elements' of the same structure. (Benacerraf 1965:291)

This means that particular mathematical objects have no more inner nature than what can be captured by their structural properties, so that the natural number 2, for instance, has the properties it does have due to its *place* in the natural number structure. Structuralism comes in various guises with diametrically different views on ontology. While Benacerraf and Hellman argue for an *eliminative* version of structuralism, Parsons, Resnik and Shapiro defend *non-eliminative* structuralism. Eliminative structuralism rejects the objective existence of abstract structures. Non-eliminative structuralism is, on the other hand, a realist position. The abstract structures are believed to exist objectively and independently of humans, and they also exist before any realisation of the structure by a particular mathematical system. For the non-eliminative structuralist, the ontological commitment to mathematical structures and their elements is thus not far removed from that of platonism. Compared to platonism however,

[18] See Reck & Price (2000).

realist structuralism holds the promise of ignoring properties such as '7 is Inger's favourite number', and rather focus on the properties that are mathematically relevant.

When it comes to mathematical objecthood, I hold the view that positions in a structure are mathematical objects in their own right. That is, I will not regard the positions in a structure as offices that must be filled or instantiated by objects belonging to a particular system.[19] Here I follow Shapiro (1997), who writes that "mathematical objects – places in a structure – are abstract and causally inert" (112). When it comes to structures, the question is whether these also should be considered as mathematical objects, or whether they are *sui generis* entities. Here I do not, again following Shapiro, characterise structures as mathematical objects, but rather as a "one-over-many entity", and as the pattern or form of a system, which, "in turn, is a collection of related objects" (1997:84).

On the structuralist picture, then, what is the nature of the natural number 2? Given that structuralism considers mathematical objects as positions in a structure, let us see a couple of characterisations of objects by non-eliminative structuralists. Resnik describes these positions as follows: "A position is like a geometrical point. It has no distinguishing features other than those it has in virtue of being the particular position it is in the pattern to which it belongs" (1997:203). Parsons characterises structuralism in terms of an object's lack of 'nature': "The idea behind the structuralist view of mathematical objects is that such objects have no more 'nature' than is given by the basic relations of a structure to which they belong" (2004:57). And finally, Shapiro points to the dependency mathematical objects exhibit: "The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other" (2000:258). These quotes all point to an important property; that mathematical objects *qua* positions are *incomplete*.[20] They are at the mercy of their relations, so that their nature is exhausted by their mathematical context. Talk of mathematical objects must refer to their structural backdrop. But if a mathematical object is inconceivable alone, what does this incompleteness lead to? Enter: ontological dependence. Ontological dependence is a relation that tracks metaphysical fundamentality and priority. An object can ontologically depend on something for its existence, its identity, or its essence. It can also be conceived of as a metaphysical primitive; in which case it is

[19] This is the view of *in re* structuralism, where the abstract structure does not exist unless there is a system that exemplifies it. On this view structure is not ontologically prior to a system, rather, there would be no natural number structure if all systems exemplifying the natural numbers were destroyed. This makes systems ontologically prior to structure (Folina 2020:276-77).

[20] The incompleteness of mathematical objects on a structuralist picture is not to be confused with mathematical incompleteness, as in Kurt Gödel's two Incompleteness Theorems (see Gödel 1931).

not possible to reduce the relation to anything more fundamental or give it a further definition. I will not venture into determining the nature of the ontological dependence relation fitting for structuralism here,[21] and will settle for treating it as a metaphysical primitive. Linnebo (2008) formulates what he calls the Dependence Claim for structuralism thus:

> ODO. Each object in *D* depends on every other object in *D*.
>
> ODS. Each mathematical object depends on the structure to which it belongs. (2008:67-8)

A mathematical object stands in a twofold dependence relation: 1) it depends on the other objects belonging to the same structure, and 2) it depends on the structure as such. As we saw from the citations above, the idea of incompleteness and dependence of objects is defended by structuralists (Parsons 1990, 2004; Resnik 1997; Shapiro 1997). In addition to ODO and ODS, there is also the issue of whether a *structure* depends on its positions or is only depended on. According to Janet Folina, *ante rem* structuralism "asserts the ontological *priority* and *independence* of structures from objects and systems" (2020:278). And, consider Shapiro:

> Each mathematical object is a place in a particular structure. There is thus a certain priority in the status of mathematical objects. The structure is prior to the mathematical objects it contains, just as any organization is prior to the offices that constitute it. The natural-number structure is prior to 2, just as "baseball defense" is prior to "shortstop" and "U.S. Government" is prior to "vice president." (Shapiro 1997:78).

If we compare this to set theory, the dependence idea is that a set depends upon its members, e.g., how the singleton of Socrates depends on Socrates. Here it seems like the larger whole depends upon its parts. Should we not say that a structure, as the larger whole, also depends upon its parts, that is, the positions belonging to it? We could, if we hold the view that positions *constitute* the structure. But it seems to go the other way around. Consider the incompleteness of mathematical objects on the structuralist view: the number 2 is incomplete in terms of being what it is due to its relational backdrop. And the relational backdrop is the structure to which it belongs and the other objects belonging to it. Moreover, a *position* is, by its nature, *situated* – the property a position has is exactly that of standing in structural relationship with other positions. Without the number 2, there would be no position 6, as they are irrevocably interdependent. Thus, the structure is ontologically prior to its positions. When it comes to the set

[21] The fourth paper of this dissertation concerns the ontological dependence relation present in structuralism. Two accounts have been given as to what the ontological dependence relation of structuralism should be, Linnebo (2008) describes it as a relation of individuation, and John Wigglesworth (2018) as a grounding relation. I sketch another alternative, and consider the Husserlian relation of *foundation*.

depending on its members, we are here talking of how a specific mathematical object, a set, depends upon its members, so the relevant structure – the set-theoretical hierarchy – keeps its role as ontologically prior to the particular set. The idea is that a structure need not be instantiated by any objects, it exists independently.

So far, we have seen that on the structuralist picture, mathematical objects stand in a twofold ontological dependence relation; they depend on the other objects belonging to the same structure (ODO), and they depend on the structure as a whole (ODS). But how do these dependencies relate to metaontological coherence? As seen in section 4, given coherentist minimalism, *coherence* is the criterion for existence. If a mathematical theory is coherent, the objects described by the theory exist. We borrowed from epistemological coherentism, and saw how there are two notions of coherence, where one is a relation, while the other is a property. While relational coherence is "how a belief is justified if it 'fits in' or coheres with a suitable background system", systemic coherence "is how the epistemic status of the background system is traced to the way the entire system fits together or is internally arranged" (Williams 2001:117). We also saw that while our analysis of coherence has made its philosophical content clearer, it has yet to function as a criterion. To fully work as an existence criterion in metaontology – so that if a theory is coherent, the entities described by the theory exist – a gradational account is not ideal. How would we determine whether a theory is coherent *enough*, so that the described entities is granted existence? However, when it comes to analysing coherence, there are reasons why we consider systems to be more or less coherent. In our application of epistemological coherence to metaontological coherentism, we saw how the distinction between systemic and relational coherence allows for a more accurate analysis. It is clear that if we ascribe systemic coherence to a whole, without its parts being arranged in a relationally coherent way, the stamp of systemic coherence holds less sway, and the whole is *less* coherent than it otherwise would have been. This is one reason to accept the gradational approach to coherence.

A related point, also from section 4.2, is how the interdefinability of relational and systemic coherence brings out issues of fundamentality. For instance, characterising the particulars of a system as relationally coherent adds to the systemic coherence of the system. Olsson (1999) argues that relational coherence presupposes systemic coherence, as systemic coherence is "the holistic property in virtue of which cohering things fit together" (1999:287). Relational instances of coherence can also be seen as recognition of the particulars of a systemically coherent system, so that a twofold coherence is expressed. Interestingly, there is a clear parallel between how we have analysed coherence, and how, on the structuralist's view, mathematical objects stand in a twofold dependence relation. If we compare relational coherence to ODO (how objects depend on objects), there is some correspondence. As the object ontologically depends on the other objects belonging to the same structure, we have an instance of relational coherence if a belief 'fits in' or coheres with the other beliefs belonging to the same system of beliefs.

As we want a coherentist metaontology, and there are already ontological dependence relations present in structuralism, we can combine the two, and formulate the ontological dependence relation in terms of coherence, thereby formulating coherence as our existential criterion.

> ODO ⇢ Relational coherence:
> If a mathematical object coheres with the other objects belonging to the same structure, the object exists.

And if we consider ODS and systemic coherence, we see something similar.

> ODS ⇢ Systemic coherence:
> If the internal arrangement of objects in a structure is relationally coherent, so that the objects form a united whole, the structure is systemically coherent.
> If the structure is systemically coherent, the structure exists.

We have now reformulated the dependence relations of mathematical objects as coherence claims. The parallel between relational coherence and ODO is formally straightforward: they both characterise how a part stands in a specific relation to other parts, and where all the parts are of the same whole. How ODS can be reformulated in terms of systemic coherence, is slightly more complicated, as systemic coherence is a property ascribable only to wholes. A system exhibits systemic coherence when the internal arranging of the system's objects is relationally coherent. This means that each and every object must cohere with the others.[22] Remembering the conditions for systemic coherence in section 4.2, a system is systemically coherent if it exhibits unity and logical consistency, there is freedom from unexplained anomalies and a lack of separated subsystems, and the inferential interconnections are strong and many. In combining ODS and systemic coherence, we can reformulate ODS as a systemic coherence claim about mathematical structure, in which these conditions are all met. As coherentist structuralism is a combination view of metaontological coherentism and ontological structuralism, we have now explicated the threshold for the existence of structures and their objects by

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[22] We are, then, to some extent following Bender (1989) and Hansson (2006) when they ask whether a system can be systemically coherent, only insofar as the objects belonging to the structure are relationally coherent. Olsson (1999) holds the other view, where objects relationally cohere in virtue of being instances of the systemic coherent system to which they belong. I believe that a reciprocal view of the interdefinability of systemic and relational coherence is the better option, and that they are equally enlightening sides of the same coin.

merging a main thesis of structuralism – that mathematical objects are incomplete and stand in ontological dependence relationships – with the main thesis of coherentism – that if a theory is coherent, the entities described by that theory exist.

It is important to remember that there are different levels of coherence. Coherence can be a property of a whole and a relation between entities. To quote Bender: "Coherence cycles but is not circular", and while the "threat of circularity certainly exists at this juncture", there is a "more promising avenue that opens up if we begin to think of the coherence theory *dynamically* rather than purely structurally" (1989:8). While Bender is talking of belief acceptance and justification from prior acceptances, he has a point that is also relevant for us: That depending on the perspective, an entity can exhibit both relational and systemic coherence. This means that we, supposedly, can have a structure that is systemically coherent, but which can also be relationally coherent with other structures within the larger frame of a mathematical theory. And, as we know, the systemic coherence of the structure depends on the relational coherence of the internal arrangement of interconnections. We should, then, be allowed to talk of coherence at different levels, as both property and relation.

5.2 Structures as thin objects

In the above section we saw that one of the main tenets of structuralism – the incompleteness of mathematical objects and their resultant ontological dependency relationships – can be reformulated as coherence claims. This shows that the ontological dependence from our ontology, and the two-level notion of coherence from our metaontology have some natural affinity. In this section I want to draw attention to a facet of Parsons' (1990) structuralism; the distinctions he draws between different levels of abstractness of mathematical objects. The idea is to make a plausible connection to the gradation thinness described in section 4, so that the levels correspond.

How does the idea of thin objects translate to a structuralist setting? First of all, as structures are not considered 'mathematical objects', I will refrain from identifying a structure with a thin *object*, but merely call a structure *thin*, by which I mean its existence is thin. To the structuralist, a coherentist approach means that the structures and the mathematical objects they consist of are regarded as thin. In order to see what this might mean, let us briefly look at Parsons' account of mathematical intuition with regard to mathematical objects. Parsons develops an account of mathematical intuition where we contemplate objects that are said to be *quasi-concrete* (as opposed to pure mathematical objects). This is a variety of Hilbertian intuition, where you take certain *strings of strokes* to inhabit the properties of the natural numbers (see Hilbert 1926). A string of strokes, e.g. "|||||... " and "|, ||, |||, ||||", can then realise the properties of the natural number structure. From our experience with these strings of strokes, we realise that the tokens are of the same type, and that the string can always be extended by an additional stroke (Parsons 2008:159-162). By arranging a pattern consisting of strings of strokes where each string is extended by one stroke, the properties of the natural number structure are realised. In this case, then, the strings of strokes exemplifying the pattern can be viewed as *thick*, relative to the natural number structure that is considered *thin.* Let us see what these quasi-concrete objects are meant to be.

> Pure mathematical objects are to be contrasted not only with concrete objects, but also with certain abstract objects, that I call quasi-concrete, because they are directly 'represented' or 'instantiated' in the concrete. Examples might be geometric figures (as traditionally conceived), symbols whose tokens are physical utterances or inscriptions, and perhaps sets or sequences of concrete objects. (Parsons 1990:304)
>
> The most important examples of quasi-concrete objects are those in which the concrete representations are the sort of objects that can be perceived. … Although sets are in general not quasi-concrete, it does seem that sets of concrete objects should count as such; here the relation of representation would be just membership. As I am understanding the notion, sets of physical objects that are inaccessible to observation would also count as quasi-concrete. (Parsons 2008:35)

We see here that Parsons contrasts pure mathematical objects with concrete objects (i.e., objects that have concrete representation, e.g., spatio-temporal objects). Additionally, there is another type of abstract objects, the *quasi-concrete* objects, which have instantiations in the concrete. An example here is a set whose members are physical objects. There are, then, three levels described: 1) pure mathematical objects, 2) quasi-concrete objects, and 3) concrete objects. If we remember how thin objects also allow for a gradation in thinness, a natural correspondence between the layers presents itself. Thin objects can be thin in an absolute sense, i.e., pure mathematical objects, but they can also be thin in a relative sense, e.g., the set of five books. And lastly, we have thick objects, which are objects that make a substantial demand on the world, e.g., spatio-temporal objects. The pure mathematical objects of Parsons naturally relate to objects that are thin in an absolute sense. Concrete objects correspond to thick objects. And lastly, the quasi-concrete objects correspond to objects that are thin in a relative sense. Parsons (1990) structuralist account thus finds a natural ally in the idea of thin objects, as it provides a conceptual framework to express metaphysically graded objects. For instance, the existence of a structure is thin relative to the existence of any systems that might exemplify the structure. But, as a structure does not need to be instantiated by any particular system, a structure is actually thin in an absolute sense. By allowing for relatively thin objects, or quasi-concrete, we can accommodate and explain how particular systems exemplify abstract structures. This feature is connected to ontological dependency relationships. We saw above that ontological dependence is a relation to analyse ontological priority and fundamentality. And, the quasi-concrete objects, while a type of abstract objects, do depend on their representations. What makes the structuralism we are pursuing here decidedly realist when it comes to ontology, is that the structures *need* not be instantiated by particular systems. It is still a nice feature of our framework that it is able to express this distinction. This potential for conceptual explanation is another point on which the ontology of structuralism and the metaontology of coherentism find common ground, and strengthens their complementary companionship.

Coherentism thus constitutes an alternative approach to thin objects. The existence of objects resulting from the coherence of a mathematical theory is thin because, as Linnebo phrases it, their existence does not put any *further* metaphysical demands on the world, other than that of the theory providing their description. As a metaontological position, the coherentist version of minimalism is attractive exactly because of this *descriptive* character. The coherentist brings to the fore a mathematical system's capacity for describing objects in a coherent manner, and thus ties an object's existence to its description. Moreover, it provides a metaphysical picture of the existence of mathematical objects that is not linear and foundationalist. This is an upshot, since it is difficult to see what the 'basic belief' or first principles should be for a structuralist view of mathematics. Structuralism, along with coherentism, is committed to a view where the whole is considered a crucial notion, and has priority to particular mathematical objects. A formulation of a metaontological criterion for mathematical existence should follow this non-linearity. But aside from these nice features, we might still want to ask whether the coherentist criterion is in fact true. Do all coherent structures really exist? Here we ought to remember though, that our inquiry is metaontological. Coherentist structuralism is a metaontologically minimalist account of mathematics, and as such, the entities it describes are not only an altogether different case from that of our pink elephant, but our metaontological inquiry is also decidedly distinct from the ontological inquiry. For our present case, it is not the coherentist structuralist's task to prove from indubitable beliefs that mathematical structures and objects exist. What is at stake, and what I have tried to show, is that a coherentist approach to thin objects is compatible with a structuralist account of mathematics.

6. Conclusion

In this paper we have seen how metaontological coherentism fits with non-eliminative structuralist ontology. Non-eliminative structuralism is a position that is committed to the existence of abstract mathematical structures; hence, it is a variant of realism in mathematical ontology. The question is whether mathematical objects can be construed so as to not be existentially too demanding, so that we can more readily accept them as existing entities, without supporting a robust platonic heaven. Coherentism advocates a lightweight metaphysics, by providing an approach to thin objects. This makes it a deflationary meta-ontology, as it lowers the bar for existence and does not demand extra special justification. Rather, mathematical existence questions are answerable as internal questions in the metaontological framework provided. The criterion for existence – a mathematical theory's *coherence* – constitutes a minimal requirement. We saw from coherence theories in epistemology, that a distinction between systemic and relational coherence allows us to differentiate levels or cycles of coherence. Moreover, in applying this notion of coherence to our structuralist ontology, we could reformulate one of its main tenets: The twofold ontological Dependence Claim is reflected in – or is possible to spell out as – relational and systemic coherence claims. Also, the relative and absolute thinness of abstract objects have a corresponding distinction in Parsons (1990, 2008), between pure and quasi-concrete abstract objects. There are thus important theoretical areas in which structuralism and coherentism overlap and allow for reformulation and explanation. This lends support to our thesis that a structuralist with ontologically realist ambitions can also be a metaontological coherentist, as it provides her with complementary theoretical underpinnings. The investigation into coherence and coherentism pursued a philosophical elucidation of the criterion, but one in which emphasis was put on its opposition to linear accounts of justification and metaontology. While our coherentist inquiry of structuralism has shown that there is a natural affinity, there are open questions worth pursuing; for instance, whether a coherentist structuralist really is better off than her purely structuralist counterpart in justifying the existence of abstract structures. And, how we spell out the conditions of systemic coherence so that we can determine whether a mathematical theory is adequately coherent, so that the thin existence of structures and their entities is ensured. These will have to be addressed elsewhere. For now, I have tried to develop another notion of coherence, one that draws on distantly related coherence theories, so that structuralist claims of dependence can be construed as metaontological coherence claims, and we can formulate a cyclical notion of systemic coherence as a suitable existence criterion for structures and their objects.

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Two Approaches to the Access Problem*

Abstract:
In this paper I argue that there are two approaches to the access problem – *the Head On Approach* and *the Tweaking the Question Approach*. These are reflective of two kinds of answer to the problem, viz., *internal* and *external* answers, a Carnapian distinction made salient for epistemology by Audrey Yap (2009). The kind of answer one favours, determines the approach one pursues in order to account for mathematical knowledge. The two approaches are characterised by different views on what counts as an adequate epistemological account. A consequence of this is miscommunication between the two camps, as the epistemological story of each approach violates implicit demands and premises for what counts as adequate for the other. This paper demonstrates how the two kinds of answer that Yap presents deal with the epistemological challenge, and shows that the Tweaking the Question Approach is superior to the Head On Approach.

1. Introduction

In the paper "Mathematical Truth" (1973) Paul Benacerraf formulated the so-called *access problem*. This is an epistemological challenge faced by positions in philosophy of mathematics that pursue a platonist ontology. If mathematical objects exist outside of space and time, it is not clear how we can, from our spatio-temporal starting point, epistemically access and come to have knowledge of them. While Benacerraf's paper was not intended to refute mathematical platonism specifically, it is often taken to present a fatal objection to realist positions in mathematical ontology. There have been several efforts to overcome this challenge. To sort the different attempts made, I employ Audrey Yap's (2009) distinction between *external* and *internal* answers to the access problem.[1] While an external answer means that we must ensure epistemic access to mathematical objects by *bridging* the apparent gulf between them and us, an internal answer argues that there is no need for extra-mathematical justification to explain this access. In this paper I argue that the two kinds of answers lead to two distinct approaches, which I want to call *the Head On Approach* and *the Tweaking the Question Approach*. While the kind of answer we favour is reflected in the approach we pursue, each approach has other

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[1] This distinction hails from Carnap (1950), and is used to differentiate between existence questions. I will come back to this origin in section 4.

characteristics that determine what an appropriate epistemological story for mathematics should look like. As the two approaches diverge on the issue of *adequacy* in the face of the access problem, this frames the eligible positions sorted to each approach, and leads to not only different, but incompatible stories of how we acquire mathematical knowledge. This reveals that the two approaches have different *metaepistemological* tenets (i.e., fundamental aspects of epistemic theorising regarding implicit aims and standards), and that their respective accepted methodologies diverge in important ways. A consequence of this is miscommunication between the two camps, as the epistemological story of each approach violates implicit demands for what counts as adequate for the other. To this end, I bring forth two specific accounts that have dealt with the access problem in paradigmatic ways, in order to exemplify and make the methodological considerations of each approach explicit. This paper thus operates with different theoretic *levels*: we have the internal/external distinction, the two representative positions, and, most importantly, the two approaches manifesting different metaepistemological stances when confronted with the access problem. My aim in this paper is thus twofold: 1) to show how the kinds of answer that Yap presents – leading to the two corresponding approaches – deal with the epistemological challenge, and 2) to show that the Tweaking the Question Approach is superior to the Head On Approach and therefore should be pursued.

In section 2 I introduce the access problem and explain why it poses a problem for positions with a realist ontology of mathematical objects. In section 3 I outline two example positions that attempt to overcome the access problem in different ways, Kurt Gödel's account of mathematical intuition and Stewart Shapiro's stratified epistemology. In section 4 I look at the kinds of answer outlined by Yap (2009) and develop the two corresponding approaches, before I show how these approaches are exemplified by the two positions described in section 3. In section 5 I look at how the classification of the two approaches reveals different standards for when an account is deemed epistemologically adequate, which accounts for the incommensurability tendencies in the epistemological debate on structuralism (e.g., between MacBride (2008) and Shapiro (1997, 2011)). I end this section by arguing that the Head On Approach should be discarded in favour of the Tweaking the Question Approach, due to justificatory concerns.

2. The access problem

Generally, platonist positions in the philosophy of mathematics share three commitments: i) mathematical objects *exist*, ii) they exist *independently* of us and our language, thought, action, etc., and iii) they are *abstract*. The abstractness component means that they are in some realm causally closed off from our own. And, as mathematical objects exist, the axioms, theorems, and propositions referring to them are true or false independently of human language, thought, or actions. For example, the natural number 2 is an abstract mathematical object that exists, and the statement '2 + 2 = 4' is true even if no rational agent ever had knowledge of its truth. All objects that mathematical theories deal with, such as natural numbers, real numbers, sets, etc., have this same sort of independent existence, which is why we can discover their properties and relations, and also why we can express our knowledge of them in our language. A true mathematical proposition successfully refers to some mathematical objects and accurately describes their properties and the relations that hold between them. This also means that mathematics can only be discovered, as opposed to being constructed or extended by our minds. This is not to say that our mathematical *knowledge* cannot be furthered or extended (as it most certainly can), but rather that the domain of true mathematical propositions is, and always has been, exhaustive.

In "Mathematical Truth" (1973) Benacerraf writes that accounts of mathematical truth have been motivated by two different concerns, a semantical and an epistemological concern. The semantical concern is described as "the concern for having a homogenous semantical theory in which semantics for the propositions of mathematics parallel the semantics for the rest of the language" while the epistemological concern is that "the account of mathematical truth [must] mesh with a reasonable epistemology" (1973:661).[2] For the case at hand – i.e., the possibility of epistemic access to abstract, mind-independent entities – it is the latter that occupies us. The epistemological concern is reflected in the formulation of a condition that must be met, to secure a "coherent over-all philosophic account of truth and knowledge" (Benacerraf 1973:666).

> [A] satisfactory account of mathematical truth must be consistent with the possibility that some such [mathematical] truths be knowable. To put it more strongly, the concept of mathematical truth … must fit into an over-all account of knowledge in a way that makes it intelligible how we have the mathematical knowledge that we have. (1973:667)

Later in the paper, Benacerraf states how it is exactly lack of attention to the above condition that leads the realist into trouble.

[2] The semantic concern is spelled out as a semantic condition, where "any theory of mathematical truth [should] be in conformity with a general theory of truth – a theory of truth theories, if you like – which certifies that the property of sentences that the account calls "truth" is indeed truth. This, it seems to me, can be done only on the basis of some general theory for at least the language as a whole" (Benacerraf 1973:666).

> [O]n a realist (i.e., standard) account of mathematical truth our explanation of how we know the basic postulates must be suitably connected with how we interpret the referential apparatus of the theory. … For what is missing is *precisely* what my second principle demands: an account of the link between our cognitive faculties and the objects known. … We accept as knowledge only those beliefs which we can appropriately relate to our cognitive faculties. (Benacerraf 1973:674, emphasis original)

Justin Clarke-Doane notes that while Benacerraf's "Mathematical Truth" has been "deeply influential", this is "more for its theme than for its detail" (2016:18).[3] This is because Benacerraf demands that there must be *causal* connection between the objects of knowledge and the knowers, as he favours a causal theory of knowledge (1973:671). Since a causal theory of knowledge is no longer much supported, one could argue, following Øystein Linnebo, that Benacerraf's considerations are biased against mathematics at the outset (Linnebo 2006:546). By demanding a causal connection, we are treating "platonistic mathematics much like physics and the other garden-variety empirical sciences", but, as Linnebo notes, mathematics is different, and therefore "philosophers have no right to subject it to epistemological standards that have their home in contingent empirical knowledge" (Linnebo 2006:546).[4]

There is an improvement of the problem that is due to Hartry Field (1989), which avoids invoking *any* specific theory of knowledge (Field 1989:232-33). As is generally noted, most contemporary discussions of the problem take Field's improved formulation (1989) as their point of departure.[5]

> Perhaps the most widely discussed challenge to the platonist position is epistemological. … Benacerraf's formulation of the challenge relied on a causal theory of knowledge which almost no one believes anymore; but I think that he was on to a much deeper difficulty with platonism. … Benacerraf's challenge—or at least, the challenge which his paper suggests to me—is to provide an account of the mechanisms that explain how our beliefs about those remote entities can so well reflect the facts about them. The idea is that *if it appears in principle impossible to explain this*, then that tends to *undermine* the belief in mathematical entities, *despite* whatever reason we might have for believing in them. (Field 1989:25-26, emphasis original)

Field avoids causality, but still makes the challenge for the platonist clear: If we cannot justify or explain the reliability of our mathematical beliefs – even if we grant that mathematical objects exist and that we have knowledge of them – that is bad news. As Field sagely argues, we should be able to give an account of mathematical knowledge, where the correctness of our

[3] See Pataut (2016) for essays on Benacerraf's philosophy.
[4] This is dubbed the "natural response" by Linnebo (2006:546).
[5] For instance, Clarke-Doane (2016). Clarke-Doane further argues that there is no satisfying characterisation of what Benacerraf's Problem *really* amounts to, and moreover, that problems in Beneacerraf's vein appear in many other areas than philosophy of mathematics (Clarke-Doane 2016:17-18). For a response to Clarke-Doane, see Liggins (2018), who argues that the problem remains a serious objection to mathematical platonism and cannot be dismissed.

beliefs about mathematical objects is not made into a "huge coincidence" (Field 2005:77). If we cannot provide an account which avoids this coincidence component, we have failed to explain our epistemic access to mathematical objects. As David Liggins points out: "Field is seeking to cast doubt on platonist theories by calling attention to a phenomenon which they are committed to recognizing but seem unable to explain" (Liggins 2010:74). That is, the platonist theory that claims that mathematical objects exist is undermined, as it seems we have no way of explaining how our beliefs about these entities are *reliable*. Thus, we end up with the access problem, and while many attempts have been made to overcome it, it remains a dire obstacle for the contemporary mathematical realist.[6]

3. Two exemplifying positions

Let us now look into some attempts that have been made to explain our access to mathematical entities. I will introduce two positions with a platonist ontology, which have dealt with the problem in different ways; Kurt Gödel's mathematical intuition and Stewart Shapiro's *ante rem* structuralism.[7] There are a few reasons for this choice. First of all, Hale & Wright (2002), in describing platonist responses to the access problem as *conservative*, bring forth Gödel's and Shapiro's accounts as two such instances (2002:104).[8] There is thus some precedent for choosing the two as example positions. Second, Gödel's account of mathematical intuition is in some sense paradigmatic, modelled as it is on sense perception, and driven by an analogy between mathematics and empirical science. Moreover, Benacerraf (1973) himself uses Gödel as the representative for what he calls the "standard view" of mathematical truth.[9] Third, Shapiro's structuralism makes for an interesting contemporary case; it is unrepentingly realist in its ontology, but nevertheless tries to render the epistemological story *naturalistic*. Fourth and finally, Shapiro likens his own methodology to that of Gödel: "The methodology is consonant…with that sketched by Kurt Gödel and a host of others, most of whom see a clean

[6] See Liggins (2006, 2010), Linnebo (2006) for discussion.

[7] Gödel proposed his account of mathematical intuition before the publication of "Mathematical Truth" and the improved argument by Field (1989). However, it is still an account that tries to explain how we come to have knowledge of mathematical objects existing in an abstract realm.

[8] Hale & Wright call them *intuitional* and *intellectual* responses: Gödel's account involves a special faculty of *intuition*, as opposed to the *intellectual* tactic, where "access to the objects of pure mathematics is afforded by our general abilities of reason and understanding" (Hale & Wright 2002:104). Hale & Wright favour the intellectual response themselves.

[9] That is, platonistic accounts in which the sentence "there are at least three perfect numbers greater than 17" is analysed as being of the form "there are at least three *FG*'s that bear *R* to *a*" (Benacerraf 1973:663, 674). That is, the account "assimilates the logical form of mathematical propositions to that of apparently similar empirical ones: empirical and mathematical propositions alike contain predicates, singular terms, quantifiers, etc." (Benacerraf 1973:668). As such, the "standard view" complies with the semantical condition from footnote 2.

separation between mathematics and empirical science" (Shapiro 2011:140). As the two positions are introduced as *representative* positions, the choice seems justified, considering their paradigmatic role in previous discussions, their immediate differences (faculty of intuition vs. naturalistic story), and the fact that there seems to be areas of methodological overlap. In section 4, we will see how these two positions exemplify two basic approaches to the access problem.

3.1 Gödelian mathematical intuition

Kurt Gödel, who is perhaps most famous for the Completeness theorem and the two Incompleteness theorems, defended a robust realist position within philosophy of mathematics. Not only did he argue for the existence of mathematical objects such as sets and numbers, but also for the independent existence of mathematical *concepts*. By mathematical *concepts*, Gödel meant the relations and properties of set theory, such as "property of set", the primitive notion of membership (as denoted by "∈") and "the concept of set itself" (Parsons 1995:48). This means that he endorsed conceptual realism as well as objectual realism. This is most clearly expressed in the Gibbs Lecture of 1951:

> What is wrong, however, is that the meaning of the terms (that is, the concepts they denote) is asserted to be something man-made and consisting merely in semantical conventions. The truth, I believe, is that these concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe. (*1951:320)

It is the meaning of the primitive terms of set theory, then, that are deemed to exist as part of mathematical reality. The world of concepts (mathematical reality) exists separately from the world of things (physical reality). As a consequence, he also rejects Aristotelian realism, that is, the view that "concepts are parts or aspects of things" (Gödel *1951:321). Gödel thus proposes a mathematical realm in the vein of Plato's realm of Forms, so that mathematical objects and concepts need not be instantiated by physical objects, but where they exist in a realm wholly unrelated to physical reality.[10]

Gödelian concepts are also described as being *perceived* by reason: "For while with that latter [the senses] we perceive particular things, with reason we perceive concepts (above all primitive concepts) and their relations" (version IV of (*1953/9), quoted from Parsons

[10] There is a similar distinction in mathematical structuralism between *in re* and *ante rem* structuralism. *In re* structuralism requires that a structure be instantiated by a particular system in order to exist, while *ante rem* structuralism does not require such instantiation. *In re* structuralism is closer to the Aristotelian view, while *ante rem* structuralism is platonist. However, 'instantiation' is not understood in physical terms *à la* Aristotelian realism (though there can also be cases of physical instantiations of structures).

1995:63). Gödel, in his 1964 paper "What is Cantor's Continuum Problem?", calls the ability to perceive abstract concepts through reason *mathematical intuition*:[11]

> But, despite their remoteness to sense experience, we do have a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see why we should have less confidence in this kind of perception, i.e., mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. (1964:268)

Here we see that mathematical intuition is likened to sense perception.[12] Its role in building mathematical theories is compared to how the senses are necessary for sense perception, upon which physical theories are built. This perception of mathematical reality apparently must be the case because of how "the axioms force themselves upon us as being true".

This is an instance of what has become known as *intrinsic* evidence for axioms in set theory, as opposed to *extrinsic* evidence, a distinction introduced earlier in the 1964 paper.[13] Intrinsic evidence for axioms is characterised by some aspect of self-evidence, together with having a thorough understanding of the concepts underlying mathematics. It is instances of mathematical intuition that constitute intrinsic evidence. Extrinsic evidence for axioms are instances of abductive reasoning. The "success" of an axiom, in the form of "fruitfulness in consequences", such as the contraction of many proofs into one and making them simpler and easier to discover, is taken as evidence for that axiom being true (Gödel 1964:261). The axiom is then said to have "verifiable consequences". If there is an abundance of such consequences, and the axiom is "shedding so much light on a whole field, and yielding … powerful methods for solving problems", the axiom should be accepted, even if it is not supported by intrinsic evidence (Gödel 1964:261).

Mathematical intuition is meant to ensure epistemic access to mathematical reality. It is a faculty described as analogous to sense perception, in that they inhabit the same role when we build up our mathematical and physical theories of reality. Gödel writes that the success of our best mathematical theories is evidence in favour of such a faculty, leading him to propose mathematical intuition as a "psychological fact" (1964:268). Furthermore, mathematical

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[11] For an excellent discussion of Gödel's platonism and mathematical intuition, see Parsons (1995). Parsons makes the distinction between intuition *of* objects and intuition *that* a proposition is true. In light of Gödel's conceptual intuition, and how "the axioms force themselves upon us as being true", this is a useful distinction, as it brings out the immediacy component of *grasping* that is inherent in intuition.

[12] For contemporary accounts of mathematical intuition, see Maddy (1990), Parsons (1980, 2008), Tieszen (1989). See also Føllesdal (1992) and Tieszen (2005) for a likening of Gödel's mathematical intuition and Edmund Husserl's phenomenology. Gödel's admiration for Husserl and phenomenology is well known, notably expressed in (Gödel *1961/?).

[13] See Russell (1906), Maddy (1988).

intuition is not meant to yield immediate mathematical knowledge, but rather works as a source of it:

> It should be noted that mathematical intuition need not be conceived of as a faculty giving an *immediate* knowledge of the objects concerned. Rather, it seems that, as in the case of physical experience, we *form* our ideas also of those objects on the basis of something else which *is* immediately given. … That something besides the sensations actually is immediately given follows (independently from mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations, e.g., the idea of objects itself … Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas. … [T]he data of this second [mathematical] kind … may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality. (Gödel 1964:268)

This move ensures that the comparison between mathematical intuition and the senses is strengthened. In "Russell's Mathematical Logic" Gödel considers the analogy between physical and mathematical reality:

> It seems to me that the assumption of such objects [classes and concepts] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the "data", i.e., in the latter case the actually occurring sense perceptions. (1944:128)

We see here that sense perceptions are compared to mathematical data. We are capable of "having" these due to the objective reality of physical and mathematical objects. Sense perceptions constitute our evidence for the natural laws, on the one hand, and mathematical data constitute our evidence for the truth of the axioms, on the other. Moreover, our ability to have sense perceptions is due to our senses, on the one hand, and our ability to have mathematical data is due to our mathematical intuition, on the other. For while the senses give us sense perception caused by physical reality, mathematical intuition leads us to perceive "mathematical data" of mathematical reality (Gödel 1964:268).

Gödel's arguments for his realist views largely rest on two assumptions, i) that there is an analogy between physical and mathematical reality, and ii) the principle of epistemological parity. The analogy holds between physical and mathematical reality, between the senses and mathematical intuition and between sense perception and "mathematical data". Our best mathematical theories are about existing mathematical objects, just as our best physical theories are about existing physical objects. Mathematical theories do not purport to describe mathematical data, just as physical theories do not describe sense perception. This analogy is meant to make the existence of mathematical objects and our perceiving of them more palatable. According to Penelope Maddy, if our acquisition of mathematical knowledge mirrors our acquisition of knowledge of physical reality, some pre-philosophical attitude is captured (1990:ch. 1). This pre-philosophical attitude is the belief – held by both mathematicians and laypeople – that mathematical truths are eternal and unchanging, i.e., they are *discoverable* and not constructed by us (e.g., the statement '2 + 2 = 4' always has been and always will be true).

Mathematical objects are said to be just as necessary to build up our mathematical theories, as physical objects are necessary to build up our theories of physical reality (Gödel 1944:137). The success of our mathematical theories should verify both the existence of mathematical intuition and its capability to "have" mathematical data of mathematical objects. Mark van Atten and Juliette Kennedy (2003) call this appeal to mathematics itself the *principle of epistemological parity* (van Atten & Kennedy 2003:434). It brings the legitimacy of the claim that mathematical objects exist to the fore. If we take physical objects and mathematical objects, and consider our knowledge of them, there is "no reason to be more (or less) committed to the existence of one than the other" (van Atten & Kennedy 2003:434; Kennedy 2014:6). It can be seen as an appeal to scepticism. As we definitely do not want to doubt the existence of physical reality, we should also not doubt the existence of mathematical reality. Gödel induces us to make the connection that if we compare our knowledge, and if we find that mathematical knowledge is *as good as* our knowledge of physical reality, then we ought to accept that they both constitute knowledge of existing domains, one mathematical and one physical.

Gödel has faced harsh criticism for the postulation of mathematical intuition. One of the harsher judgements to be passed came from Charles Chihara, who described Gödel's appeal to mathematical perceptions to justify the existence of sets to be "strikingly similar to the appeal to mystical experiences that some philosophers have made to justify their belief in God" (1990:21).[14] Benacerraf also points to problems pertaining to the analogy:

> What troubles me is that without an account of how the axioms 'force themselves upon us as being true,' the analogy with sense perception and physical science is without much content. For what is missing is precisely what my second principle demands: an account of the link between our cognitive faculties and the objects known. (Benacerraf 1973:674)

While the faculty of mathematical intuition is described as giving us "mathematical data" akin to how we are given sense perceptions, what mathematical intuition *is* or how it *provides* this

[14] See also Chihara (1973:ch. 2) for Gödel's lack of justification for his mathematical intuition.

access to mathematical reality is left completely open. There are, then, serious deficits in Gödel's attempt at explaining our mathematical knowledge.

3.2 Shapiro's stratified epistemology

Let us now turn to a second realist position, *ante rem* structuralism. *Ante rem* structuralism is a version of mathematical structuralism, whose slogan is that *mathematics is the science of structures* (Shapiro 1997:5).[15] A structure is distinguished from systems, in that a particular mathematical system is the realisation of an abstract structure.

> I define a *system* to be a collection of objects with certain relations…. A *structure* is the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system. (Shapiro 1997:73-4)

While there are different versions of mathematical structuralism with different ontological views (spanning from anti-realism to realism), *ante rem* structuralism, as coined by Stewart Shapiro (1997), endorses a platonist ontology of mathematical reality. While Gödel believed that mathematical knowledge is knowledge of mathematical objects and concepts, Shapiro's position is structuralist, so that mathematical knowledge is knowledge of structures.

On the *ante rem* structuralist view, the structures in question exist independently of human thought and action, are abstract and exist independently of their realisation by a particular system. Shapiro also thinks that there are mathematical objects such as the number 2. These objects are construed as *places* of an *ante rem* structure, so that the number 2 is the second place in the natural-number structure, which is a particular infinite pattern (Shapiro 1997:77). The structures exist *before the thing*, and so an Aristotelian version, in which the structure exists as an aspect or part of a physical *system*, is rejected.[16] That *ante rem* structuralism[17] is subject to the access problem, is made clear by Shapiro and Michael Resnik.

> The realist… owes some account of how a physical being located in a physical universe can come to know about *abstracta* like mathematical objects…. The burden is on the realist to show how realism in ontology is compatible with naturalized epistemology. (Shapiro 1997:110)

[15] This goes for other variants of structuralism as well, even if they do not hold that structures have a platonist kind of existence. See for example Hellman (1989), Benacerraf (1965).
[16] See footnote 10.
[17] From now on I will use only 'structuralism' when talking of *ante rem* structuralism.

> If mathematics is about abstract entities, then how is mathematical knowledge possible? In particular, how can we have access to the subject matter of mathematics? (Resnik 1982:95)

Are there any advantages of endorsing a structuralist picture when dealing with the access problem? There might be, but for now that will not be my main objective. What is clear is that the challenge changes. It is no longer a question of how we can have knowledge of particular natural numbers such as the number 2, but rather how we can have knowledge of the natural-number structure. It is knowledge of this structure that will yield knowledge of the natural number 2. In the case of structuralism, then, the question becomes: How can we have knowledge of abstract structures, when the structures we describe and quantify over exist in a realm so wholly closed off from our own? Given that we do have mathematical knowledge, it becomes a problem for structuralists to explain how mathematical propositions are reliably justified and knowable to us.

As mathematical knowledge is about structures, it allows Shapiro to talk of patterns as a means to explain the genesis of mathematical knowledge. Resnik's version also focuses on the recognition of patterns, especially on experiencing something as *patterned* (Resnik 1982:97). This idea is also shared by Shapiro: "Structure is to structured as pattern is to patterned, as universal is to subsumed particular, as type is to token" (1997:84). Shapiro presents an account where our acquiring of mathematical knowledge is *stratified* (Shapiro 1997:12-13, 112-118, 129-132; 2011:138-40). Shapiro's stratified story of how we acquire mathematical knowledge is characterised by the following layers:

1. **Pattern recognition:** Our ability to recognise small, finite patterns. Recognition of the 2-pattern (all systems consisting of two things), the 3-pattern (all systems consisting of three things), and so on. Limited to the grasping of small finite structures.
2. **Projection:** The faculty of arranging the small, finite cardinality patterns, then projecting a larger pattern on them to create an overarching pattern. Realisation that each pattern within the overarching pattern is succeeded by a next-longest pattern and projecting that this property goes for all structures, so that one grasps the natural-number structure. From this structure, other infinite structures are also grasped.
3. **Characterisation:** By giving an implicit definition of a structure, the relations between the positions in the structure are described, and so the structure is made "available as an object of intellection" (Hale & Wright 2002:112).

According to Shapiro, we acquire mathematical knowledge by a step-by-step process. As soon as we have managed one level, we abstract and generalise so that our mathematical understanding is extended. The last stage, implicit definition, is where the *Coherence principle* comes in: "**Coherence:** If **Φ** is a coherent formula in a second-order language, then there is a structure that satisfies **Φ**" (Shapiro 1997:95). As the "main principle behind structuralism is that any coherent theory characterizes a structure, or a class of structures", the Coherence principle – and its elucidation – is crucial (Shapiro 1997:95). It is the central existence axiom in his axiomatic structure theory, and moreover, it also plays an integral role on the epistemological side of things.[18]

> One underlying theme of my book, repeated several times, is that the ability to discuss a given structure coherently is evidence that the structure exists. This plays a role at every level in the stratified epistemology, starting with pattern recognition, but it comes to the fore more centrally with the fourth[19] and most powerful epistemic strategy, implicit definition. The theme is codified in the coherence principle in the formal development. (Shapiro 2011:147)

It is in terms of us being able to coherently discuss a structure that a structure is said to exist. Shapiro appeals to mathematical knowledge to explain: "[I]t follows from the coherence of set theory that if we show that any proposed implicit definition *D* is satisfiable, then *D* is itself coherent, and thus *D* describes a structure" (2011:149). Shapiro thus appeals to the coherence of set theory. But set theory is not assumed to be a background ontology for structuralism, as the set-theoretic hierarchy is considered "just another structure" (Shapiro 1997:135). Shapiro uses a mathematical notion within that particular structure (viz., satisfiability within the set-theoretical hierarchy), as a model of structure theory, in order to argue that we have knowledge of structures.

In "Can 'Ante Rem' Structuralism Solve the Access Problem?" (2008), Fraser MacBride criticises Shapiro for not answering the access problem. MacBride brings forth Shapiro's admission that some epistemic access to the *abstracta* of mathematics must be secured. While MacBride does acknowledge we might be capable of the first step, abstraction and recognition of small finite patterns, he objects to Shapiro's second and third step, that of *projection* and *characterisation*. The first charge has to do with going from particular to general knowledge, i.e., how the projection of an overarching pattern onto finite ones can make us realise that every finite structure has its own distinct successor.

[18] See the second paper of this dissertation for discussion on the Coherence principle in Shapiro (1997, 2011) and on coherence in metaontology for mathematics.
[19] Shapiro says characterisation is the fourth strategy in his (2011), as he also includes holism as a general epistemic strategy. However, in (1997), characterisation is the third step.

> How the mathematical novice may legitimately pass from (1) *particular* knowledge that a given structure has a successor distinct from it, to (2) *general* knowledge that all structures … have their own distinct successors. One way… is to derive (2) from (1).… But however … this transition is effected, (2) cannot be got out of (1) without appeal being made to truths which are no less general in character than (2). (MacBride 2008:160)

As this move cannot be made without appeal to truths containing the generality it seeks to establish, the move is illegitimate, MacBride argues. More specifically, the idea that a law of generalisation from finite patterns holds for *general* finite patterns (and not only for the specific one of Shapiro's example), presupposes the exact amount of generality that the law itself seeks to explain. Shapiro, on his side, denies ever claiming that (2) is deducible from premises of pattern recognition. Rather he sees the mathematical novice as postulating (2) as a *hypothesis*: "A proposition can start life as a working hypothesis, or even a blind guess, and can later become an established belief or a known fact if it proves fruitful, serving a central essential role in a successful system" (Shapiro 2011:140). This is tantamount to how Gödel describes the role of extrinsic evidence for axioms and how consequences are verifiable in the sense that they come to serve "a central essential role" by "shedding so much light on a whole field" (Shapiro 2011:140; Gödel 1964:261). Shapiro also mentions how the hypothesis eventually can be adopted as an axiom, if it turns out to be fruitful enough. It seems that Shapiro – like Gödel – appeals to indispensability within mathematics itself, in his proposed solution for how we acquire mathematical knowledge.

In the same passage we can also find some trace of intrinsic evidence. "Perhaps *S* thinks (2) is true as a result of something innate, or by being somehow compelled to think it true, or perhaps just having a hunch" (Shapiro 2011:140). This innateness component is reminiscent of the appeal to the mathematical data given to us by mathematical intuition. For Gödel, it was exactly these instances of mathematical intuition that constituted intrinsic evidence. What exactly is going on here is difficult to pinpoint. But, Shapiro's account for why *S* has reasons to believe that (2), at least seems to suggest that there is some room for intrinsic evidence as well.

Let us consider the charge against the third step, characterisation. MacBride criticises the use of mathematics and lack of extra-mathematical security: "For even if coherent categorical descriptions are guaranteed to be non-empty, there still remains the epistemological issue about how it can be established that descriptions are coherent and categorical" (2008:162). As the notions of categoricity and coherence are to be explicated within mathematics, and thus require a lot of mathematical knowledge, MacBride argues that Shapiro's picture is viciously circular. Shapiro concedes that his account is circular, but not that it is viciously so. The reason is that Shapiro is not out to "justify mathematics from non-mathematical premises" (Shapiro 2011:149). The use of set theory in explaining coherence and categoricity is perfectly fine because, in trying to give an interpretation of mathematics as *ante rem* structures, Shapiro does not have to show that mathematics as a discipline is true and secure.

4. Two answers and two approaches

Now that the two positions have been introduced, we can turn to their paradigmatic role. The positions exhibit certain methodological traits, which makes it clear that they operate with different rules for deciding what makes for an acceptable epistemological story. To draw out this tension, I believe that we are best served by classifying them as belonging to two *approaches*. By belonging to distinct approaches, I mean that they manoeuvre within different epistemological frameworks, so that how they respond to the access problem is determined by structural requirements or conditions. By characterising the approaches to which the positions are instances, meta-epistemological leanings are brought to the fore, viz., epistemological aims and standards for epistemic theorising.[20] The advantage of characterising two approaches is that they are more general than any specific attempts at solving the access problem. Moreover, we can use the specific attempts (Gödel's and Shapiro's) to discuss how the approaches handle the problem, and thereby reveal how they determine what counts as an adequate epistemological story.

The two approaches I outline find their basis in the Carnapian internal/external distinction for existence questions (1950). Audrey Yap (2009) proposes that this distinction is applicable to epistemology, and specifically, to how we deal with the access problem (2009:168).[21] I follow Yap in this, and further argue that this distinction serves as a stepping stone for the Head On and the Tweaking the Question Approaches. Why, we may ask, go further than the two kinds of answer to the access problem? I believe there are reasons for this. By characterising an *approach*, we can draw out more general and implicit theoretical underpinnings than we could with a distinction between kinds of answer. Baked into our notion of approach, we have the idea that we, as epistemologists of mathematics, operate within a theoretical space that is delimited. This means that the accounts that are eligible on an approach are effectively

[20] Metaepistemology is a rising field, modelling itself on the success of metaethics. See Gerken (2018), Kyriacou, & McKenna (2018), McHugh, Way & Whiting (2018).

[21] From what I can gather, Yap is the only one to have applied the distinction to our case at hand.

restricted. We are, by describing the approaches, characterising general traits and issues for epistemological accounts of mathematical knowledge.

Before we broach our epistemological approaches, we look at how the distinction between internal and external questions appears in Rudolf Carnap's paper "Empiricism, Semantics, and Ontology" (1950). It does not, originally, pertain to epistemology, but is rather a way of distinguishing between different kinds of existence questions.[22]

> And now we must distinguish two kinds of questions of existence: first, questions of the existence of certain entities of the new kind *within the framework;* we call them *internal questions;* and second, questions concerning the existence or reality *of the system of entities as a whole, called* external questions. (Carnap 1950:242, emphasis original)

The notion of a *linguistic framework* is put forth to safeguard domains of knowledge whose language refers to abstract objects – e.g., mathematics – from "implying a metaphysical doctrine concerning the reality of the entities in question" (Carnap 1950:250). Accepting a linguistic framework "does not need any theoretical justification because it simply does not imply any assertion of reality" (Carnap 1950:250). To differentiate between questions that are internal or external *relative to a framework* is thus a way of weeding out unduly metaphysically laden questions, which we have no means of determining as they have not (yet) been formulated in "the common scientific language" (Carnap 1950:245).

Internal questions are meant to be answered straightforwardly, either by empirical or by conceptual means. This depends on the framework in question. In the framework *the world of things*, questions are to be answered by empirical investigation (e.g., "Did King Arthur actually live?") (Carnap 1950:242-43). Here, the existence or reality component is of an "empirical, scientific, non-metaphysical" character (Carnap 1950:243). In the framework *the system of numbers*, on the other hand, which is of a logical rather than factual nature, the methods are ruled by logical and conceptual analysis (Carnap 1950:244-45). If we ask, then, whether there are numbers from within the framework of mathematics, the answer is a resounding 'yes'. For by the acceptance of the framework, we already have that "Five is a number" is recognised as a true statement, and thus internally there is a number five.

If, on the other hand, we ask the external question (viz., 'does the system of numbers as such exist?'), the question is deemed a "pseudo-question", as "[philosophers] have not succeeded in giving to the external question and to the possible answers any cognitive content"

[22] Carnap (1950) along with Quine (1948, 1951) form the historical background for the contemporary field of metaontology. See the first paper of this dissertation for more on metaontology for mathematics.

(Carnap 1950:245).[23] The external question is only ever asked by philosophers, and is supposed to be "*prior* to the acceptance of the new framework" (Carnap 1950:245). This means that we are not given access to the linguistic tools of the framework – nor to our knowledge of the system of things described by the framework – when we attempt to answer it. The external question is the philosophical question of the ontological status of numbers, and is thus not straightforwardly answerable.

How does the distinction between internal and external existence questions fare when applied to epistemology? Carnap briefly mentions the task of a "pure, as distinguished from a psychological, epistemology" in relation to carrying out the empirical investigation in answering an internal question about *the world of things* (Carnap 1950:243). While the evaluation of the results of the empirical investigation follows certain rules of confirming or disconfirming, this is "usually carried out … as a matter of habit rather than as a deliberate, rational procedure" (Carnap 1950:243). One of the main tasks of epistemology, according to Carnap, is to lay down explicit rules for the evaluation in a rational reconstruction (Carnap 1950:243). This seems acceptable: When we formulate an epistemology, we lay down rules for the evaluation of evidence, thus restricting the format that acceptable evidence can take in order to be knowledge-conducive. But this seems manageable only if we are dealing with factual knowledge. For the case at hand – providing a story of how we come to have knowledge of mathematics – we have not gotten much further. To make matters worse, we are operating within an ontologically realist perspective, i.e., we are proclaiming the reality of mind-independent mathematical entities, or in Carnap's words, we are answering the external question in the affirmative.

Yap (2009) employs the Carnapian distinction in characterising Richard Dedekind as a logical structuralist.[24] According to her, Dedekind provides an answer to the access problem if we construe it as an internal question (2009:168).

> Benacerraf's problem is originally posed as a metaphysical problem. An account of mathematical objects ought to explain what they are, as well as how it is that we have epistemic access to them. Yet there are two distinct *kinds* of answers to the question, "what are natural numbers?"
>
> Benacerraf's own presentation of the dilemma invites an external answer, more metaphysically substantive than "the smallest system of objects containing 0 and closed under the successor function" (for instance). But this latter [internal] answer is essentially Dedekind's. The reason why he finds it acceptable, is that an answer in the same vein is given to the second question, "how can we have knowledge of them?" We can have knowledge of them because the axioms defining them are categorical, which implies that every proposition in the language of arithmetic, or its negation, is a semantic consequence of those axioms. This is the sense in which we can say that knowledge of the axioms

[23] See Flocke (2020) for a paper on Carnap's noncognitivism about ontology.

[24] According to Yap, Erich Reck (2003) provides an improved reconstruction of Dedekind's position, a reconstruction that is similar to Yap's own interpretation of Dedekind as a logical structuralist.

> suffices for knowledge of the objects' properties. The reason why this counts as an internal answer is because it is given in terms of the *mathematical* adequacy of the natural number definition. (Yap 2009:168-69)

Yap uses the internal/external distinction in a double manner; on the ontological question ("what natural numbers *are*") and on the epistemological question ("*how* we can have knowledge of them"). That the distinction is applicable to the epistemological question is taken as added support for the internal answer to the ontological question. That is, the ontological and epistemological story of the natural numbers both prefer the same kind of answer, which means that the overall account is rendered more cohesive. According to Yap, Dedekind prefers an internal answer, as it is less metaphysical and more mathematical in spirit. Yap argues that this fits with her construal of Dedekind as a logical structuralist, as one of the main motivations for structuralism originally was an "acknowledgment of the axiomatic character of modern mathematics, and an attempt to give a philosophical account accordingly" (Yap 2009:169). So far, we have some indication that there are different philosophical and epistemological aims attendant to the kind of answer one favours.

The subsequent sections stake out two different approaches to the access problem. By characterising these approaches, metatheoretical considerations will be made explicit, i.e., we can show how the approaches exhibit general – but diverging – metaepistemological outlooks as to *how* we should go about responding to the access problem. To this end, the earlier discussion of our two specific example positions will come in handy, as they instantiate some of the general traits of each approach. First, I outline what an external answer amounts to and how it leads to *the Head On Approach*. For this purpose, I will bring up Gödelian intuition. Favouring an external answer shapes what accounts are eligible, as the Head On Approach lays down certain conditions that restrict how we deal with the access problem. Second, I give an account of the internal type of answer and the corresponding approach of *Tweaking the Question*. Shapiro's stratified picture will be shown to be in the vein of the Tweaking the Question Approach. Interestingly, MacBride's critique of Shapiro and Shapiro's response to it bring out underlying metaepistemological leanings. The tension between the Head On Approach and that of Tweaking the Question is made explicit, as the aspects MacBride criticises go directly at Shapiro's methodology, i.e., his chosen approach.

4.1 External answer and the Head On Approach

An external answer to the access problem is characterised by accepting the challenge as posed. According to Yap, Benacerraf formulated it as an external question, demanding a metaphysical answer (2009:168). If we remember Carnap's characterisation of an external question and answer, it has to do with "*prior* acceptance" of the "reality of the thing world itself" (Carnap 1950:243, 245). If applied to epistemology and the access problem, it seems that we must find a way of explaining how we come to have mathematical knowledge without *using* mathematical knowledge. We must, then, find a wholly *philosophical* account for our access to mathematical entities. This means that the metaphysical gulf between physical and mathematical reality must be bridged by some means or another. One must, in Fraser MacBride's words, "do the impossible: [Mathematicians] must transcend their own concrete natures to pass over to the abstract domain" (2008:156). So, how are we capable of both breaching the causal limits of physical reality and probing into the abstract mathematical reality? We cannot fill the gap with something solely from the mathematicians' sphere (i.e., physical reality) or solely from the abstract domain (i.e., mathematical reality). Something *else* must be added, something capable of breaching the respective spheres and establishing an epistemic connection.

The approach favouring the external kind of answer I call the Head On Approach, as it rises to the challenge and charges at the access problem head on.[25] On this approach, the choice of possible solutions is limited, as certain rules are laid down. First of all, we are not allowed to use the tools of the domain of knowledge we have set out to explain, i.e., mathematical notions and facts. Second, as the gulf to be filled is between two layers of reality, whatever "filling" found will be metaphysical (as opposed to scientific). And third, the metaphysical connection sought after will have a transcending quality, as it must defy the limits of the two spheres it tries to connect. A consequence of these conditions is that a naturalised epistemological account is more or less ruled out.[26] That is, the Head On Approach and an external answer feature *a priori* considerations, as figured out from a "philosophy first" point of view.

We must, then, somehow "possess" an abstract component, so as to explain some affinity with the abstract domain. An immaterial and eternal soul would provide this, such as Plato's account of remembrance in *Meno*. Another alternative would be to say that mathematical objects have some imprint in the concrete, and that we are capable of detecting the abstract elements instantiated. An example could be something akin to the view put forth by Penelope Maddy (1990), in which she argues that we can perceive sets that contain spatio-temporal elements. If there are three eggs in the carton, we have perceived a set of three eggs (Maddy 1990:58).

[25] Another reason might be the image of banging one's head against the barrier of physical reality, hoping to reach into some abstract domain.

[26] By this, I take naturalised epistemology to mean the Quinean view that the methods by which we acquire knowledge should be explainable in terms of natural science, e.g., by analysing our faculties in line with cognitive science. See Maddy (1997).

Gödelian mathematical intuition is another way of answering the access problem head on. We saw in section 3.1 that the belief in such a faculty is troublesome. Where does it belong? Is it part of reason? If so, why differentiate it from normal intellectual powers? Or does it have physical place in the brain?[27] Gödel took mathematical intuition to be a psychological fact, a statement he was confident to make because of our capability "to produce the axioms of set theory and an open series of extensions of them" (Gödel 1964:268). The success of mathematics is taken to justify the faculty by which we come to have mathematical knowledge. Gödel's arguments for the existence of mathematical intuition thus amounts to an inference to the best explanation. We can see that Gödelian mathematical intuition satisfies the condition of being a wholly philosophical account of how we epistemically access mathematical entities, as no mathematical knowledge is used to explain what the special faculty consists of. But this is hardly a benefit, seeing that it is also not given a philosophical elucidation as to how it *works*. It is only given an approximate characterisation by way of its analogy to sense perception. This makes for a meagre explanation of how we come to have mathematical knowledge. While the faculty adheres to the philosophical standard of being prior to the acceptance of the thing world, Gödel still relies on the success of mathematics (i.e., our mathematical knowledge), in order to argue that we simply must possess the faculty that gave us this knowledge.

4.2 Internal answer and the Tweaking the Question Approach

The internal type of answer does not require that the gap between physical and mathematical reality be bridged in a metaphysically loaded way. As Yap argues, an answer to the access problem can also be "given in terms of *mathematical* adequacy of the natural number definition" (2009:169). This amounts to answering the access problem in an *internal* fashion, which allows for the use of mathematical tools to explain our epistemic access to mathematical objects.

An internal answer leads to the *Tweaking the Question Approach*. It is important to note that an internal answer is not an outright rejection of the external answer. If we find that an external answer still gives us the best explanation for our acquisition of mathematical knowledge, that is perfectly fine. But Tweaking the Question *is* a rejection of the Head On Approach. The boxes that must be checked in order to deal with the access problem Head On, are considered not only unnecessary, but hindering. On a Tweaking the Question kind of view, there is no reason why the epistemology should not be in line with a naturalistic outlook.

[27] According to Hao Wang (1987), Gödel toyed with the idea of mathematical intuition being a "physical organ" (Folina 2014:55).

Consider Quine's use of Neurath's science-as-boat image, in which he also includes philosophy: "Neurath has likened science to a boat which, if we were to rebuild it, we must rebuild plank by plank while staying afloat in it. The philosopher and the scientist are in the same boat" (Quine 1960:3). This fits well with the Tweaking the Question Approach. We cannot demand that we explain our mathematical knowledge from a perspective that is *prior* to the knowledge that we have. This is opposed to the foundationalist tendencies of the Head On Approach, where we are tasked to explain epistemic access in a linear fashion, from indubitable *philosophical* principles, devoid of mathematical content.

Moreover, not only are we allowed to use mathematical notions in our epistemological account, the metaphysical requirement for "bridging of the gap" is relaxed. By the Carnapian meaning of an internal question for the system of numbers, we should answer the internal question "by logical analysis based on rules for the new expressions" (Carnap 1950:244). For our epistemological purposes, we should take this to mean that we are *already* operating within a domain of knowledge, and we should not disregard the fact that the linguistic expressions of that domain are coherent and meaningful to us. That is, we *should*, on the other hand, disregard the possibility that we do not have access to this domain. Tweaking the Question positions thus tend to downplay the severity of the metaphysical chasm. One such attempt at diminishing our metaphysical distance to the mathematical entities has been made by Øystein Linnebo (2018). He argues that we must lighten the burden placed on the epistemologist. That means to lower the standard for what counts as an adequate epistemological account and lessen our ontological commitment. Linnebo's project of reducing the ontological commitment is pursued with the notion of a *thin object*. The idea is that the abstract objects of mathematics are metaphysically undemanding. For instance, the existence of the set of two trees does not make a substantial *further* demand on the world than that of the existence of the trees themselves. This route will perhaps explain the problem of ontological abundance in the ontology of mathematics.

How is Shapiro's stratified picture an instance of the Tweaking the Question Approach? First of all, he allows for the use of mathematical notions in the explanation of mathematical knowledge, namely satisfiability in set theory as a model of the coherence of structure theory. Shapiro advocates a holistic approach to epistemology (and philosophy), where our epistemic access to mathematical objects is not explained from the ground and up. Rather, we must settle for a view where the coherence of mathematics itself is valued, and where our ability to correctly pick out mathematical objects because they are sufficiently determined by our theories is granted.

> *I present an account of the existence of structures, according to which an ability to coherently discuss a structure is evidence that the structure exists….* This account is perspicuous and accounts for much of the 'data' – mathematical practice and common intuitions about mathematical and ordinary objects. The argument for realism is an inference to the best explanation. (Shapiro 1997:118, my emphasis)

If successful referring is what is needed for securing our mathematical knowledge, and where mathematical *adequacy* is made the standard for an acceptable story, then that amounts to a definite Tweaking of Question.

The main focus should be on mathematics, and what is needed in order to make sense of and successfully reason about mathematics. The "philosophy first" sort of view – as the Head On Approach agrees with – is not supported here. The emphasis put on mathematical practice meshes well with Shapiro's penchant for naturalised epistemology.

> [I]n setting up the epistemological task I endorsed the naturalistic thesis that 'any faculty that the knower has and can invoke in pursuit of knowledge must involve only natural processes amenable to ordinary scientific scrutiny'. Even putting indispensability theses aside, it is clear that ordinary scientific scrutiny of just about anything is going to involve mathematics, and the philosopher can make use of mathematics just as anyone else can. (Shapiro 2011:133)

Shapiro puts an additional constraint on the realist: we must make sure that our access to the abstract structures is compatible with naturalised epistemology. Epistemology of mathematics should follow mathematics' *lead*, i.e., it is a "mathematics first" sort of view. For Shapiro, naturalised epistemology has to do with delimiting the role of faculties, certainly ruling out a "special" faculty *à la* Gödelian intuition, which is not 'amenable to ordinary scientific scrutiny'.

Moreover, he argues that mathematical tools and notions are available to us in our epistemological efforts, as mathematics is *already* part of 'ordinary scientific scrutiny'. From Shapiro's naturalism, we get a defence for operating from an internal perspective. This amounts to a rejection of foundationalism and a reinforcement of Shapiro's holistic approach: "There is no getting around this situation. We cannot ground mathematics in any domain or theory that is more secure than mathematics itself. All attempts to do so have failed, and once again, foundationalism is dead" (Shapiro 1997:135).[28] Interestingly, Shapiro brings up his anti-foundationalism in relation to his explication of the Coherence principle. We saw in section 3.2 that *coherence* is a central notion in Shapiro's third step, i.e., characterisation. MacBride's criticism of characterisation turns on the issue of how we come to know that a description is coherent

[28] See also Shapiro (1991).

and categorical, as these are explained by our knowledge of mathematics. That is, MacBride argues that Shapiro's account is viciously circular, as an account of "how finite creatures can reliably access truths about the abstract and infinite is just as wanting in the case of set theory as it is in any other branch of mathematics" (MacBride 2008:163). Let us consider Shapiro's reply.

> [*A*]*nte rem* structuralism is itself no more secure than is set theory…. So if I were looking to provide some sort of extra-mathematical justification or security for set theory, making use of set theory would be viciously circular, and structures would have dropped out of the picture…. But that is not my game.
>
> My game, again, is to provide a justification for a philosophical interpretation of mathematics…This… is not a deductive enterprise, where I would have to start with non-mathematical, self-evident premises. *This is a different game from showing a sceptic that mathematics itself is true and known.* (Shapiro 2011:149, my emphasis)

It seems that MacBride and Shapiro disagree on exactly this last point. Should we show that mathematics itself is true or not within the philosophical account of mathematics, and can we use mathematical tools in answering the epistemological challenge? MacBride demands that Shapiro should give an account that "provid[es] a source of extra-mathematical certainty to underpin mathematical practice" (MacBride 2008:164). This, MacBride argues, is necessary to answer the task set up by Shapiro, namely how a physical being can come to have knowledge of abstract mathematical objects.

MacBride's negative answer relies on the attitude that the access problem should be answered *externally*. That is, the only type of answer that would satisfy MacBride is one that provides a metaphysical bridging between mathematical and physical reality.

> [I]t does not follow from the death of foundationalism that we cannot intelligibly ask, given our understanding of the contrasting natures of mathematicians and the subject-matter of their discipline, how it is ever more than a coincidence that our beliefs about mathematical objects are true. Of course, if it is not just foundationalism but epistemology more generally whose death-knell has sounded, then such questions cannot be intelligibly raised. (MacBride 2008:164)

Shapiro's response is to clarify that he rejects the need to answer in this way, and that he accepts a different approach, namely that of Tweaking the Question. His goal is not to ensure that structuralism is more secure than set theory, but to provide an epistemological story that supports his position that mathematics is the study of *ante rem* structures, where importance is placed on coherence within mathematics.[29]

[29] Øystein Linnebo (2018) also efforts to give an epistemological story of how mathematical objects are made available to us that is within the Tweaking the Question Approach. It is not, as Shapiro's is, based on coherence,

5. Untangling and comparing

We have seen how the two kinds of answer spurs on two distinct approaches to the access problem. Gödelian mathematical intuition and Shapiro's stratified picture were shown as representative positions for each approach. While Yap's application of the internal/external distinction to epistemology suggests a difference in preferred solutions to the access problem (Dedekind preferring an internal answer), it does not capture the larger philosophical assumptions that come with it. The two approaches have different views as to what counts as an acceptable epistemological story of mathematical knowledge, which is reflected in the nature of the objections raised against the representative positions. Criticism therefore attacks different aspects; whereas some objections are directed at justificatory failings of the account in question, some are also directed at the structural set-up of the account, i.e., the *approach* or methodology the account instantiates. This is apparent in the two approaches and the two representative positions we have covered. The most common objections against Gödelian intuition have been that neither is the existence of such a faculty sufficiently backed up, nor is the nature of the faculty sufficiently explicated. The objections raised against Shapiro's stratified account rather focus on how he 'misses the mark' and does not properly address the access problem. That is, Gödel is criticised for his *proposed solution*, while Shapiro is criticised for *how he attempts to solve the problem*.

To get a better grasp of this difference of objections raised, let us turn to Gödel and Shapiro's accounts once more. Interestingly, now that we have covered some of the arguments for and objections against the accounts, as well as the respective approaches of Gödel and Shapiro, we see that there is some common ground. They both use extrinsic evidence in the form of the success of mathematics and the appeal to indispensability to argue for their respective positions. Gödel argues that mathematical objects are necessary for our best mathematical theories. Shapiro prompts us to accept "(2) *general* knowledge that all structures… have their own distinct successors" from "(1) *particular* knowledge that a given structure has a successor distinct from it", and so get general knowledge of a structure's distinct successor from particular knowledge of a particular structure's distinct successor (MacBride 2008:160). Both bring up some level of success of mathematics. Also, they both use the secure position of mathematical knowledge as a means to explain how we acquire mathematical knowledge.

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but is rather an abstraction-based method. The idea is that you have abstraction principles, where the abstraction is asymmetric, so that one is given "new" objects from "old" ones. As our access to the "old" is deemed uncontroversial, we are led to the "new" objects from uncontroversial moves. See also Linnebo & Pettigrew (2014).

To this end, Shapiro brings up Ernst Zermelo's views on why we should accept the Axiom of Choice. If it is necessary for science, it must be accepted, as "principles must be judged from the point of view of science, and not science from the point of view of principles fixed once and for all" (quoted from Shapiro 2011:141). Furthermore, in describing what Shapiro calls his 'holistic' view on epistemology, Shapiro likens his methodology to that of Gödel: "The methodology is consonant … with that sketched by Kurt Gödel and a host of others, most of whom see a clean separation between mathematics and empirical science" (Shapiro 2011:140). While Gödel tries to establish an analogy between mathematics and empirical science, it is not so that mathematics is to be infused with principles from empirical science, e.g., causal connections to the subject-matter, need for observation, etc. If anything, it is the other way around, where the idea of objecthood and formal ideas are projected onto our experience of physical reality. Gödel develops this point in an infamous passage from his 1964 paper:

> That something besides the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g. the idea of object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only reproduce and combine those that are given. Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas. (Gödel 1964:268)

This idea is also picked up by Shapiro, in explaining how the applicability of mathematics in empirical sciences can be seen as physical reality exemplifying the pure structures of mathematics: "There is no sharp distinction between the mathematical and the mundane. To speak of objects at all is to impose structure on the material world, and this is to broach the mathematical" (1997:17). There are, then, some overlap when it comes to conceiving physical reality as exhibiting structural or formal elements.[30]

How does abstraction and conceptualising of mathematical reality come up in the two accounts? Let us start off with structuralism. Resnik describes pattern recognition as going "through a series of stages during which we conceptualize our experience in successively more abstract terms. At the last stage we leave experience far enough behind that our theories are best construed as theories of abstract entities" (1982:99). Shapiro describes this conceptualisation as happening when our experience with tokens results in our understanding the type, and

[30] Carnap makes a related point about the difference between mathematical and factual knowledge: "Quine does not acknowledge the distinction [between accepting a linguistic framework and the reality of the thing-world] which I emphasize above, because according to his general conception there are no sharp boundary lines between logical and factual truth, between questions of meaning and questions of fact, between the acceptance of a language structure and the acceptance of an assertion formulated in the language." (1950:250n6).

when we grasp some structures through their systems (1997:11). But experiencing something *as patterned* does have some element of immediacy to it. When we experience something as patterned, it seems to already involve some understanding that there is a pattern to be experienced. And seeing something as a type suggests that the something has already been conceptualised in some way. Perhaps Gödel's intuition of concepts, e.g., the intuition of the concept of set itself, has more in common with pattern recognition after all. We could, at least, imagine that if we implement the abstraction process on sets, as described by Resnik, this is how we come to grasp the concept of set itself. And perhaps, if we continue the abstraction process on structures, we come to grasp the concept of structure itself.

This line of thought is supported by Charles Parsons, who describes a variant of mathematical intuition for structuralism, in which we can intuit the tokens instantiated in the concrete.

> One has to approach [the string of stroke-tokens] with the *concept* of a type, first of all to have the capacity to recognize other tokens as of the same type or not. Something more than the mere capacity is involved, which might be described as seeing something *as* the type. (Parsons 1980:154)

The pure mathematical objects (viz., sets, numbers, etc.) and structures cannot be intuited in this way; mathematical intuition can only yield knowledge of concrete tokens.[31] It is thus resemblant of how Shapiro describes our experience with tokens in order to grasp the type. Parsons links this kind of intuition, based on Hilbertian intuition of strings of strokes (viz., '| , || , |||'), and the Husserlian notion of *foundation*, a dependence relation introduced in the third *Logical Investigation*.[32] Parsons suggests that we can think of the intuition of a type as being *founded* on the perception of a token (Parsons 2008:161).

There is seemingly more in common between mathematical intuition and the experience with tokens, resulting in knowledge of a type. It is important to note, however, that the stages described by Shapiro are meant as a contribution to naturalised epistemology: "Any faculty that the knower has and can invoke in pursuit of knowledge must involve only natural processes amenable to ordinary scientific scrutiny" (1997:110). He thus underlines the difference between his *strata* and Gödelian mathematical intuition. When it comes to the two approaches, and how Gödelian mathematical intuition and Shapiro's picture fall under the Head On Approach and the Tweaking the Question Approach respectively, there is still more that

[31] Parsons call these tokens "quasi-concrete objects" (2008:151).

[32] Elijah Chudnoff (2013) also relates this kind of intuitive knowledge to the Husserlian relation of foundation (Chudnoff 2013:370). See also Tieszen (1989, 2005). See the fourth paper of this dissertation for more on the relation of foundation and ontological dependence in structuralism.

divides than unifies them. This is manifest in the conditions that must be fulfilled to be deemed epistemologically acceptable, which seems to constitute the roots of the debate between MacBride and Shapiro. Despite Shapiro's efforts to provide a philosophical interpretation of how we come to have mathematical knowledge, MacBride is, to borrow Shapiro's turn of phrase, 'playing a different game'. MacBride is playing the game of Head On, while Shapiro is engaged in Tweaking the Question. Their apparent miscommunication, or – depending on one's view – their lack of understanding for the other's perspective, stems from their different epistemological standards.

The classification of the two approaches accounts for these incommensurability tendencies, as they each demand that the other follow their approach in resolving the access problem. The Head On Approach demands a metaphysical answer, in order to fully bridge the gap of the realms. MacBride seems to agree, as "the access problem which Shapiro set out to solve cannot be dealt with by appeal to our grasp of set theory" (MacBride 2008:163). That is, MacBride rejects *how* Shapiro attempts to solve the problem. Shapiro's naturalistic account, responsive to our ordinary scientific means (i.e., including mathematics), rejects a foundationalist, metaphysically laden, account of epistemic access. To argue against foundationalism and championing naturalism is not the same as foregoing the epistemological enterprise. It does, however, wave off the need for an ultimate ground on which to build human knowledge, i.e., some indubitable Cartesian first principles. Moreover, it professes 'philosophical modesty',[33] as it urges us to take the limitations of human understanding and our abilities for knowledge seriously. This is not a failing of human reason, nor for the Tweaking the Question approach, but rather a strength. The project of providing epistemic access to mathematical entities from non-mathematical justification seems futile, and should therefore be adjusted.

One final point needs mentioning. As we have seen, both approaches lean heavily on abductive reasoning and indispensability arguments. With the Head On Approach there seems to be a gap between the types of arguments used and that which is argued for. On the Head On Approach, here exemplified by Gödel's account, there must be something capable of transcending the limits for our existence and reaching into the abstract domain. This is explained in terms of mathematical intuition. In this respect, the faculty becomes a sort of placeholder for the mystery component, that of transcendence. To postulate – as a psychological fact – that we have a faculty that happens to fit the bill perfectly, seems arbitrary. As the arguments for the faculty are not only intrinsic evidence within mathematics or indubitable evidence that we have instances of mathematical intuition, it seems prudent to forgo this special faculty. With the

[33] Maddy equates naturalism in mathematics with 'philosophical modesty' (1997:161).

types of arguments we have seen, ontological parsimony and less metaphysically speculative proposals seem in order.[34] That means to uphold Shapiro's naturalised epistemology and pursue an epistemological account where the processes involved are "amenable to ordinary scientific scrutiny". And this task cannot be achieved unless the Head On Approach is discarded, and concerns and concepts are allowed to be explicated from within mathematics itself.

6. Conclusion

We saw that the internal/external distinction serves as a stepping stone for two approaches to the access problem. The characterisation of The Head On and Tweaking the Question Approaches revealed different views as to what counts as an acceptable epistemological story of mathematical knowledge. The discussion of the two specific accounts – Gödelian intuition and Shapiro's stratified picture – highlights how the approaches operate with different sets of metaepistemological outlooks. From this general metatheoretical level, the representative positions serve a paradigmatic role by instantiating these outlooks. While the Head On Approach demands that we must propose something that is able to transcend physical reality and reach into mathematical reality, this is not easy to justify. At least, it is not easy to argue for such a special faculty without appealing to indispensability or the success of mathematics. The arguments for both accounts amount to an inference to the best explanation. Given the similarity and plausibility of the arguments made for each specific account, and, more importantly, the type of justification that is *available to each approach*, the approach which is less metaphysically demanding should be preferred. This is not to say that Shapiro's account is not open to objections, nor that it is the best possible example of a Tweaking the Question Approach. Rather, if we must choose a route to pursue, the one where the processes involved are amenable to ordinary scientific scrutiny is preferable to the one that demands a metaphysically transcending bridging faculty. The outlining of the approaches, and the use of two specific accounts as instantiating these, have shown that there are philosophical assumptions at play. These assumptions are best seen as metaepistemological guides; that is, by aligning ourselves with an approach, we are also taking on board general outlooks as to what is deemed epistemologically adequate. From the discussion between Shapiro and MacBride, this became apparent; they support different approaches, and therein lies the main reason for their disagreement. Moreover,

[34] One could, of course, object to how a theoretical virtue like ontological parsimony squares with realism in mathematical ontology in the first place. However, there is no reason to liken object realism in mathematics to general ontological promiscuity, nor to extend such generosity to include human faculties.

while Gödel's account primarily has been criticised for its lack of justification for and explanation of the faculty of mathematical intuition, Shapiro is criticised for the way he attempts to answer. The classification of the two approaches thus serves to explain some of what is going on when we are dealing with the access problem. Different attempts made, endorse different metatheoretical premises. The support of particular metaepistemological outlooks is reflected in the resulting accounts, as they must adhere to the epistemological standard prescribed.

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Ontological Dependence in Mathematical Structuralism*

Abstract:
Mathematical objects can ontologically depend on other mathematical objects or domains of such. Ontological dependence relations are therefore integral to the ongoing effort to characterise *non-eliminative structuralism*. While there is general consensus that there are such dependence relations, there have been surprisingly few attempts to characterise the relation itself. Interestingly, due to the recent construals of Husserl as a proponent of early mathematical structuralism, his *relation of foundation* presents itself as a candidate. This paper explains how the relation of foundation can be a suitable dependence relation for non-eliminative structuralism, as it can (i) account for the constitutive nature of a structure and its elements, and (ii) clarify the reciprocal dependence relation among the elements belonging to the same structure on the one hand, and between a structure as a whole and the elements belonging to it on the other.

1. Introduction

Ontological dependence relations determine how objects and domains of objects depend on each other. In identifying what objects are dependent and what they depend on, we describe the metaphysical structure of reality. For instance, the orthodox view of metaphysical *foundationalism* holds that reality is hierarchical, structured by chains of dependence relations ending in fundamental things that do not depend on anything further (Bliss & Priest 2018:2). Naomi Thompson's (2018) suggestion of metaphysical *interdependence* or *coherentism*, on the other hand, allows for symmetrical relations of dependence, and rejects the requirement that chains of dependence are well-founded. These constitute very different pictures of how reality is metaphysically structured.

While such relations are of continued importance to contemporary metaphysics, they are also an issue in philosophy of mathematics. Ontological dependence relations are especially salient to *non-eliminative structuralism*, a realist position in ontology, whose slogan is that 'mathematics is the science of structure'.[1] On this view, particular mathematical objects are

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[1] This is not only the slogan of non-eliminative structuralism, but of eliminative structuralism as well.

characterised by the web of relations in which they stand. The relations holding between a structure and its elements are considered instances of such ontological dependence, thus prompting an analysis of them.

There have been a couple of attempts at describing the nature of ontological dependence in the case of non-eliminative structuralism. Øystein Linnebo (2008) puts forth a Dependence Claim, where the relevant dependence relation is construed as the possible individuation of mathematical objects. A more recent account is given by John Wigglesworth (2018), who construes the dependence as differentiating between a *partial* and *full* grounding relation, where mathematical objects belonging to the same structure partially ground each other, while a structure fully grounds the mathematical objects it consists of. Due to recent interpretations of Edmund Husserl as a proponent of early mathematical structuralism, there is another option that should be considered. Husserl's discussion of parts and wholes in the third *Logical Investigation* (2001),[2] introduces the ontological dependence relation *foundation* ('Fundierung') that holds between them. This paper argues that the relation of foundation has some natural affinity with the dependence relations relevant for non-eliminative structuralism. The immediate promise of the relation is that it allows for a more fine-grained analysis, due to its unifying character, and, moreover, that it permits for *cyclical* relations of dependence, as the foundation relation can be symmetrical. This means that we can have chains of dependence relations that cycle, thereby eschewing a linear structure of dependence. The upshot for non-eliminative structuralism is twofold. First, it would account for the constitutive nature of a structure and its elements. Second, it would clarify the reciprocal dependence relation between the elements belonging to the same structure, and between a structure as a whole and the elements belonging to it. These benefits for structuralism mesh well with the general metaphysical picture of interdependence or coherentism. Non-eliminative structuralism has clear anti-foundationalist tendencies both in epistemology and metaontology,[3] preferring a holistic approach that emphasises coherence. That the choice of an ontological dependence relation should fit within this general metaphysical picture, gives added incentive to pursue this non-linear account of ontological dependence.

In the next section, I outline non-eliminative structuralism, and show why ontological dependence relations play an integral part, due to their view that mathematical objects are *incomplete*. In section 3, I move on to the Husserlian relation of foundation and argue that it can be a suitable dependence relation for non-eliminative structuralism. In section 4, I connect the

[2] Originally 1900/1901. All references to Husserl are to the English translation of 2001.
[3] See the second and third papers of this dissertation.

benefits for non-eliminative structuralism with a general coherentist view on the metaphysical structure of reality.

2. Structuralism and incomplete objects

Mathematical structuralism is the view that mathematical objects have no 'inner nature', and that a mathematical object is what it is due to its mathematical context, i.e., the mathematical structure in which it appears. Paul Benacerraf's paper "What Numbers Could Not Be" (1965) is considered an official starting point for mathematical structuralism, in which he concludes that numbers are not objects.

> Therefore, numbers are not objects at all, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an abstract structure – and the distinction lies in the fact that the 'elements' of the structure have no properties other than those relating them to other 'elements' of the same structure. (Benacerraf 1965:291)

Benacerraf's characterisation of how elements are made up within a larger relational structure shows how we are no longer concerned with the particular natures of each mathematical object, but rather abstract on their particularities, so that we consider the structure in which they appear to have priority. Whether one endorses realism in ontology or not (Benacerraf did not), the issue of referring to an object's structural backdrop remains of importance in structuralism, as most structuralists hold that there is some relation of dependence at play between the different places in a structure and/or between a place and the whole of the structure. However, the question of the nature of ontological dependence relations is more pressing for the non-eliminative structuralist, i.e., where the existence of the 'elements' in question is not eliminated and thus endorses realism in ontology (see Parsons 2008; Shapiro 1997). This paper concerns itself only with the non-eliminative realist perspective.[4]

The relations of ontological dependence advertised and the referring to mathematical objects' structural backdrop are related to how objects are deemed *incomplete* on a structuralist account.[5] Linnebo (2008) dubs this the Incompleteness Claim, the thesis that mathematical objects lack any intrinsic properties.[6] Mathematical objects are known by their context, i.e., by

[4] From now on, all mention of 'structuralism' will be referring to non-eliminative structuralism.
[5] See Parsons (1980, 1990), Resnik (1997), Shapiro (1997, 2000), Linnebo (2008), Hartimo (2019).
[6] Linnebo differentiates between two strains of the Incompleteness Claim. The first characterisation is that mathematical objects do not have any non-structural properties (i.e., NS-incompleteness), while the second is the one stated above (i.e., I-incompleteness). Linnebo settles for the latter, as there are several counter-examples to NS-incompleteness (e.g., having the property of being abstract), and is rejected by at least one prominent structuralist (Shapiro in his (2006)) (Linnebo 2008:64).

being positions within a certain structure. A mathematical object is defined by its relations to other objects, and so is what it is at the mercy of its relationships.

> A position is like a geometrical point. It has no distinguishing features other than those it has in virtue of being the particular position it is in the pattern to which it belongs. (Resnik 1997:203)

> The idea behind the structuralist view of mathematical objects is that such objects have no more 'nature' than is given by the basic relations of a structure to which they belong. (Parsons 2004:57)

> The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other. (Shapiro 2000:258)

On the structuralist picture, a mathematical object is, in a sense, inconceivable alone. What does this incompleteness lead to? The answer comes readily enough: We must consider them against their structural backdrop. The incompleteness of objects thus naturally leads to, and is thoroughly interlinked with, their standing within webs of relations of ontological dependence.

While there is general consensus that there are ontological dependence relations within mathematical structuralism, there have been surprisingly few attempts to characterise the relation itself. Notable exceptions are made by Linnebo (2008) and Wigglesworth (2018). Linnebo formulates a twofold Dependence Claim, and spells out its two tenets, for some domain *D* of some mathematical structure:

> ODO. Each object in *D* depends on every other object in *D*.
> ODS. Each mathematical object depends on the structure to which it belongs. (2008:67-8)

Structuralists like Michael Resnik (1997) and Stewart Shapiro (1997, 2000) are committed to these, while Charles Parsons (2008) argues that structuralism holds of pure mathematical objects, but not of objects that are said to be quasi-concrete, in that they are partly given by intuition by their representation in the concrete, e.g., geometrical figures and linguistic types (Linnebo 2008:72; Parsons 1990:337-8). The idea of the Dependence Claim is to cash out one of the characteristic features of structuralism, that mathematical objects are defined by their relationships to other mathematical objects belonging to the same structure, and that a mathematical object has no other properties than structural ones.

While Linnebo's Dependence Claim (ODO and ODS) spells out relations of dependence between different types of objects (object~object and object~structure), it does not sufficiently specify the relation itself. The notion Linnebo develops, relies on the possible *individuation* of objects, so that an object *a* depends on object *b* if the process of individuation is so that in order for object *a* to be individuated, object *b* must be individuated (Linnebo 2008:78). The relation thus seems to take on an epistemological quality, so that it has to do with the possibility of our individuating the different objects. This brings our reasoning about mathematical objects and domains of such more to the fore and seems to be an attempt to alleviate metaphysically (more) heavyweight notions of ontological dependence.

Wigglesworth's (2018) construal of the dependence relation as one of partial and full grounding accommodates the twofoldness of Linnebo's Dependence Claim. ODO becomes a claim about partial grounding: the mathematical objects belonging to the same structure partially ground each other. And ODS becomes a claim about full grounding: a structure fully grounds every mathematical object belonging to it. Wigglesworth views the relevant grounding relation as one of metaphysical explanation of the identity of an object or structure, and thus picks up on Linnebo's preferred dependence relation as one of individuation. According to Wigglesworth, the relation of ground is generally thought to "have certain structural properties: irreflexivity, asymmetry, transitivity, and well-foundedness" (2018:217). Interestingly, Wigglesworth argues that the case of ground for non-eliminative structuralism serves as a counter-example, as the partial grounding relation of mathematical objects is shown to be reflexive, symmetrical, and transitive. These features lead to chains of partial ground that progresses infinitely, which means that the relation, in the context of mathematical structuralism, is non-well-founded (Wigglesworth 2018:233). However, it is still "bounded from below", i.e., "there is some fact *F* (not necessarily in the chain) such that each fact in the chain is either partially grounded by *F* or identical to *F*" (Wigglesworth 2018:233). The mathematical objects of a structure that partially ground each other give rise to infinite chains of partial ground. However, the identity of the structure fully grounds every mathematical object in its domain, and so "serve[s] as a lower bound for each chain" (Wigglesworth 2018:234). As "each chain is bounded from below, each chain has a foundation" (Wigglesworth 2018:234). Wigglesworth thus allows for a combination of infinite chains of ground and the inclusion of something fundamental.

In the next section, I want to pursue a dependence relation taken from Husserl's third *Logical Investigation*. Like Wigglesworth's account of ground, it has the structural properties of symmetry and transitivity. Wigglesworth's relation of ground tracks a notion of metaphysical explanation that holds between different hierarchical levels of the world, so that lower levels ground higher levels (Wigglesworth 2018:217-18). The relation, thus described, has a rather narrow area of applicability; it is only concerned with domains that are metaphysically structured in a hierarchical way. In turning to a Husserlian relation of dependence, its range is extended, as it pervades other areas than the metaphysical structure of the world.[7] This is an advantage for a suitable dependence relation for mathematical structuralism. If the relation tracking dependence (between a structure and its objects and among the objects belonging to the same structure) can also be used to explain other aspects of structuralism, it would give the relation an edge. We would possibly observe regularities in the position, as, say, ontology would share structural properties with epistemology, thus yielding an overall more cohesive account.

3. The Husserlian relation of foundation

In this section we look at a Husserlian relation of ontological dependence. Our interest in the relation is systematic, as we want to see how it can be a suitable dependence relation for structuralism. We should therefore observe some affinity with the position. For instance, it should capture some structuralist traits, like the Incompleteness of objects. Moreover, it should be capable of accommodating both ODO and ODS, as well as formulating relations of ontological priority. Let us begin by taking a brief look at Husserl, before we move on to the relation itself.

In recent years, Husserl's structuralist tendencies in his philosophy of mathematics have been highlighted.[8] Mirja Hartimo (2021) argues that Husserl's notion of *manifold* is to be identified with *structure* in the structuralist sense. In describing the objects of a manifold, Husserl writes:

> The objects remain quite indefinite as regards their matter, to indicate which the mathematician prefers to speak of them as 'thought-objects'. They are not determined directly as individual or specific singulars, nor indirectly by way of their material species or genera, but solely by the *form* of the connections attributed to them. (Husserl 2001a:156)

We see that Husserl's structuralism fulfils the Incompleteness Claim as put forth by Linnebo (Hartimo 2021:167). The particular natures of mathematical objects are indefinite, as they must be considered by their belonging to a certain manifold, or, translated to contemporary talk: they are incomplete, as they have "no more 'nature' than is given by the basic relations of a structure to which it belongs" (Parsons 2004:57). Husserl's position also satisfies the Dependence Claim,

[7] Barry Smith and David Woodruff Smith argue that Husserl's relation of dependence, i.e., *foundation*, holds in all aspects of his thinking, including ontology, epistemology, and phenomenology (1995:13-14).
[8] See Centrone (2010), Hartimo (2012, 2019, 2021).

as mathematical objects are solely determined by the form of the connections attributed to them (Hartimo 2021:162). As Husserl is a realist in ontology, this makes his structuralism of the non-eliminative variety (Hartimo 2021:162).[9] Husserl's structuralism can also be connected to his discussion of parts and wholes in the third *Logical Investigation*, where he differentiates between dependent and independent parts. Instead of connecting 'structure' with manifold, Simone Aurora identifies Husserl's notion of 'whole' with the structuralist notion of structure (2015:8).[10] Though the structuralism referred to in this quote is of the French structuralist movement, I believe the remarks are still relevant to our case, as structuralism encompasses a larger historical tendency than specific developments in philosophy of mathematics.[11] In interpreting Husserl as a non-eliminative structuralist, Hartimo pursues the mathematical notion of manifold, rather than the mereological notion.

What we are after in this paper, however, is a historically inspired ontological dependence relation that is suitable for contemporary mathematical structuralism. To this end, we have two reasons why we should look to Husserl's third *Logical Investigation*. First of all, as Peter Simons writes: "Although advertised as a theory of whole and part, Husserl's [third] investigation spends as much time on the concepts of dependence and independence, which, while they bear crucially on Husserl's particular brand of whole-part theory, cannot be counted as purely mereological notions" (Simons 1982:115). As the notions of dependence are not *purely* mereological, it suggests that they can be applicable in other fields than mereology as well, e.g., philosophy of mathematics, and, moreover, that it might be relevant for specific positions in which dependence relations matter, i.e., non-eliminative structuralism. Second, according to Fabrice Correia, the dependence relation introduced to account for the relations between parts and wholes is "actually the main source of the contemporary interest in ontological dependence" (2008:1020). That Husserl's discussion of manifold can be directly construed as an instance of non-eliminative structuralism, in addition to the notion of 'whole' being treated as 'structure' within a larger structuralist tradition, gives us sufficient motivation and historical

[9] Hartimo compares it to Parsons' non-eliminative structuralism from his (2008) (2021:162).

[10] In (2018) Aurora identifies the mathematical notion of 'manifold' with 'structure' as conceived of by Bourbaki and Jean Piaget.

[11] According to Richard Tieszen, "the [modern] understanding of the world in natural science … involves a shift to formal or structural features of experience in which we abstract from content or certain aspects of meaning. These shifts … are attended by a kind of idealization" (Tieszen 2013:111). Also, consider Ernst Cassirer: "Strcturalism is no isolated phenomenon; it is, rather the expression of a general tendency of thought that, in these last decades, has become more and more prominent in almost all fields of scientific research" (1945:120).

The French structuralist tradition was at its height when mathematical structuralism was given its official starting point (with Benacerraf (1965)). Connections and lines of influence between these two structuralisms is a field worthy of study in its own right, and the lack of research on the field is mainly due to the lack of constructive engagement between the so-called analytic and continental tradition (see Tieszen 2013).

incentive to pursue the ontological dependence relation described in the third *Logical Investigation* as a candidate for non-eliminative structuralism.

Thus advertised, we can take on the ontological dependence relation used to describe the relationship between wholes and parts, i.e., the relation of *foundation* ('Fundierung').[12] The interpretations of foundation are inconclusive, as some have considered it to be of the modal-existential variety (Simons 1982; Fine 1995a, 1995b), while others have emphasised its essentialist flair (Correia 2004, 2008).[13] According to Correia, the third *Investigation* is mainly concerned with the difference between "the *dependent parts* or 'moments' or 'particularized properties', like the redness of a visual datum …, and the *independent parts* or 'pieces', like the head of a horse or a brick in the wall" (2004:349). This distinction must be understood "in terms of the more fundamental notion of foundation, a form of ontological dependence" (Correia 2004:349). The main distinction takes a visual datum as an example, which recurs throughout Husserl's treatment of dependent and independent parts.

However, as Kit Fine notes: "Although Husserl begins his study with examples from the psychological sphere, he intends his conclusions to have universal validity and to be applicable to all objects whatever" (1995b:463). The universal applicability of the relation of foundation is achieved by "ris[ing], in the case of any type of whole, to its pure form, its categorial type, by abstracting from the specificity of the sorts of content in question" (Husserl 2001b:39). Structural or formal aspects are thus given emphasis. This is reminiscent of Shapiro's definition of structure, and how it is "the abstract form of a system … highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system" (Shapiro 1997:74). Structure is on an abstracted level, similarly to the pure form of whole, where "its categorial type" is abstracted from the specificities. The relation of foundation plays a unificatory role, and is thus constitutive and regulatory, as the parts form something more together, i.e., a whole.

There is a key passage from the third *Investigation* that allows us to highlight features of foundation relevant to our case of structuralism.

> A 'founded' content … depends on the specific 'nature' of its 'founding' contents: there is a pure law which renders the Genus of the 'founded' content dependent on the definitely indicated Genus of the 'founding' contents. A whole in the full and proper sense is, in general, a combination determined by the lowest Genera of its parts. A law corresponds to each material unity. There are different sorts of whole corresponding to these

[12] In the foreword to the second edition, Husserl wrote that he believed the third *Logical Investigation* to be "all too little read", and that "it is also an essential presupposition for the full understanding of the Investigations which follow" (Husserl 2001a:7).

[13] There have been several attempts at formalising the relations of foundation, see Simons (1982), Fine (1995b), Casari (2000), Correia (2004).

> different laws, or, otherwise put, to the different sorts of contents that are to serve as parts. We cannot at will make the same content at one time part of one sort of whole, at another time part of another sort. To be a part, and, more exactly, to be a part of some determinate sort (a metaphysical, physical or logical part or whatever) is rooted in the pure generic nature of the contents in question, and is governed by laws which in our sense are *a priori* laws or 'laws of essence'. (Husserl 2001b:38-39)

This passage contains three related, but distinct points. Let us untangle them in turn.

First, there is the distinction, following Fine's terminology, between *generic* and *objectual* foundation (Fine 1995b:465). While generic foundation is a relation between species A and B, objectual foundation is between individual objects either of the same species or of two different species. The distinction between generic and objectual foundation thus finds a parallel in the twofold Dependence Claim formulated by Linnebo. ODO (objects depending on objects) would correspond to objectual foundation, while ODS (objects depending on structure) would concern a relation of foundation between a species A and the objects a of species A. This is one way in which foundation accommodates the Dependence Claim. This passage clearly describes generic foundation (or a case of ODS): That a founded content "depends on the specific 'nature' of its 'founding' contents", and that "there is a pure law which renders the Genus of the 'founded' content dependent on the definitely indicated Genus of the 'founding' contents" means, in a structuralist setting, that the positions belonging to a structure are of the same type, e.g., the mathematical objects belonging to the natural number structure are all natural numbers (Husserl 2001b:38). However, sometimes Husserl seemingly transitions from a generic relation to an objectual one, without making the shift clear, which complicates an accurate analysis (Fine 1995b:465). This is a general setback of Husserl's explication of foundation, and the different formalisations of the relations thus differ in their analysis.[14]

Second, we have the Incompleteness of objects and its relation to their Dependence. Consider again how a "whole in the full and proper sense is … a combination determined by the lowest Genera of its parts" (Husserl 2001b:38). The Incompleteness of objects is baked into the definition of *part*: "To be a part, and, more exactly, to be a part of some determinate sort (a metaphysical, physical or logical part or whatever) is rooted in the pure generic nature of the contents in question" (Husserl 2001b:38). On the structuralist picture, this means that *what it means to be a mathematical object* is to depend on a structure for being determined. To further explain this point, let us consider Robert Sokolowski's take on 'moments' or dependent parts, again considering a visual datum. The dependent part is considered in relation to the datum in

[14] See footnote 13.

which it appears. This is akin to mathematical objects' Incompleteness, if considered in relation to the structure in which they belong.

> Moments are parts that permeate each other. They are inseparable from one another and from their wholes. … The necessity of blending these different parts is not due to any psychological disposition in me or in my culture, but is grounded in the sense of the parts (III #7). Each part, by virtue of what it is, contains within itself a *rule* dictating the necessary progression of supplements that it must possess, the necessary series of horizons within which it must rest: brightness entails color, color entails surface, surface entails extension. These are essentially dependent parts, moments of a whole (III #10). (Sokolowski 1977:96)[15]

The incompleteness of moments is made salient, as is their resultant dependence. This is clear from the fact that dependent parts permeate each other, and are inseparable from one another and from their wholes. Moreover, this is *due to the sense of the parts*, as they are *essentially dependent parts of a whole*.

Third, let us consider the reference to *essence*, specifically how it is that "to be a part of some determinate sort (a metaphysical, physical or logical part or whatever) is rooted in the pure generic nature of the contents in question, and is governed by laws which in our sense are *a priori* laws or 'laws of essence'" (Husserl 2001b:39). These 'laws of essence', or the *rule* contained in each part (Sokolowski 1977:96), are expanded on in a following passage.

> [T]he Idea of Unity or the Idea of a whole is based on the Idea of 'Founding', and the latter Idea upon the Idea of Pure Law; the Form of Law is further as such categorial … and that *to this extent* the notion of a Founded Whole is a categorial notion. But the *content* of the law governing each such whole is determined by the material specificity of the 'founding' contents and consequently of the 'founded' types of content, and it is this law, definite in its content, which gives the whole its unity. For this reason we rightly call each ideally possible specification of the Idea of such unity a material or also a real (*reale*) unity. (Husserl 2001b:39)

We see that the law governing a founded whole and it parts is connected to how the relation of foundation contributes to the whole's *unity*. This law is categorial, or essential, and to this extent, the founded whole is also categorial. It is, however, its definiteness, that gives the whole its unity. Translated to our contemporary structuralist talk, this suggests that the unity of a structure is provided by how the mathematical objects serve to constitute the structure, which in turn, constitutes the parts and provides them with their nature. The law guiding each structure is different, depending on the structure in question: "There are different sorts of whole corresponding to these different laws, or, otherwise put, to the different sorts of contents that are to

[15] 'III' refers to the third investigation, while '#7' and '#10' refer to chapter 7 and 10.

serve as parts. We cannot at will make the same content at one time part of one sort of whole, at another time part of another sort" (Husserl 2001b:39). This also seems to suggest an opposition to cross-structural identities, on which the real number 2 is identified with the natural number 2 (see Parsons 2008:§18). Moreover, we also capture the distinction between a system and a structure: "For this reason we rightly call each ideally possible specification of the Idea of such unity a material or also a real (*reale*) unity" (Husserl 2001b:39). The possible specifications would, on this interpretation, be the possible instantiations of a structure. And, possible instantiations of a structure are – in structuralist terms – the possible particular systems that realise a structure's properties.

We looked at a central passage on foundation, which led to certain features being highlighted: its accordance with the Incompleteness Claim and the twofold Dependence Claim, its essentialist features, and its affinity with the contemporary notion of 'structure' and possible accommodation of 'system'. There is another feature that deserves highlighting, namely, its *anti-reductionism* (Sokolowski 1997:xvi). By anti-reductionism, I mean the view that we should not, for reasons of ontological simplicity, overly reduce an entity to another by way of ontological dependence. For instance, we should not exclude artificial objects (e.g., 'chair') from our ontology. Even though their existence is wholly dependent on their molecular makeup, reducing it to a section of atoms, is tantamount to deny the object in question its importance or relevance. According to Gian-Carlo Rota (1989), this is one of the great advantages of the relation of foundation. Rota differentiates between the *facticity* and the *function* of a foundation relation, where function depends on facticity. In order to bring out the difference, let us consider Rota's example of reading.[16] Rota compares the actual *text* that we read to the text's *content*. Us learning the content of the text or reading it depends upon the text. This makes the text the facticity, i.e., the depended on, whereas the content of the text constitutes the function, i.e., the dependent.

> The text is the *facticity* that *lets* the content *function* as relevant. … Facticity is the essential support (it is *selbsständig*), but it is meant not to upstage the function it *founds*. … Function is relevant; facticity is not. Nonetheless, function lacks autonomous standing (it is *unselbsständig*): take away the facticity, and the function also disappears. This tenuous umbilical chord between relevant function and irrelevant facticity is a source of anxiety. It is awkward to admit that what matters most, namely functions, is *unselbsständig*; we might feel more comfortable if we succeeded in *reducing* functions to *selbsständig* facticities. (Rota 1989:73)

[16] Rota takes his example from Wittgenstein's *Philosophical Investigations* (1958:§§156-171).

To read some book, we need a copy of it. But we do not need a specific copy, we might as well read the same book, i.e., the same content, in a different copy. This is how function, i.e., the content, depends on facticity, i.e., the actual text, even though *what is relevant* is the content, not the specific copy of the text. This is where the issue of ontological anti-reductionism makes its entrance, as it "is awkward to admit that what matters most, namely functions, is *unselbsständig*", thus feeding the need for ontological reduction (Rota 1989:73). According to Rota, foundation is a primitive logical relation that cannot be reduced to anything simpler, and conflating function and facticity would be "an instance of *reduction*, an error in reasoning" (Rota 1989:73). How do we determine what is relevant and what is not, what is function and what is facticity in a foundation relation? The clue is to be found in *context-dependence*.

> The point is that there is no single "what" that we necessarily "see" while watching four people around a table—or while watching *anything*. All *what*'s whatsoever are functions in *Fundierung*-relations. All *what*'s "are" by the grace of some *Fundierung*-relation whose *context-dependence* cannot be shoved under a rug. The *context-dependence* of Fundierung is taken by the reductionist for "arbitrariness." But this is a mistake. The rules of bridge are *dependent* upon the *context* of the game of bridge, but they are in no way *arbitrary*. (Rota 1989:76)

According to Rota, what serves as facticity and function in a foundation relation depends on what we are looking for, and what our anticipations are.[17] If, say, we ask what makes the copy of the text an instance of some book, we can say that it is factically related to the writing, which is factically related to sheets of paper being bound, imprinted with ink, and being arranged in a certain way. What about the ink? It is factically related to the specific makeup of ingredients. But again, there are other "chain[s] of motivations" we could follow: We could, for instance, say that our reading the text is factically related to our being able to see, etc. (Rota 1989:76). The point is that foundation relations may be layered; so that what is the facticity in one foundation relation is the function of another (Rota 1989:76).

We might object that there is no reason this is applicable to mathematical structuralism. Rota's examples are (at least partly) taken from the spatio-temporal world, and do not seem to be directly concerned with ontological dependence relation for mathematics. However, let us again consider the twofold Dependence Claim, and a recent objection to ODO.

> This objection is based on the claim that there cannot be circular relations of dependence. The objector can attempt to support this claim by observing that when an object *b* depends on another object *a*, then *a* must be 'prior' to *b*. But then *a* cannot in turn depend on *b*, since two objects cannot be 'prior' to each other in one and the same sense. The

[17] The topic of anticipation and filling is integral to Husserl's view, see Føllesdal (1988).

objector thus arrives at the claim that there cannot be any cyclical relations of dependence. (Linnebo 2008:68-69)

As the objects belonging to the same structure all depend on each other, a symmetrical notion of dependence fails to indicate which objects are ontologically prior. We saw in section 2 that Wigglesworth (2018) allows for symmetrical relation of ground, which – together with reflexivity and transitivity – lead to infinite chains of dependence and non-well-foundedness. As the relation of foundation share these structural properties, we end up with the same infinite chains and non-well-foundedness (Husserl 2001b:27-28).[18] While the objection correctly points out that we cannot determine ontological priority among the objects belonging to the same structure, this is rather like directing attention to a trait of structuralism, than identifying a disadvantage.

There is also an objection of circularity raised against ODS, which takes as an assumption that relations of dependence must be non-circular, or even well-founded (Linnebo 2008:69).[19] It continues by defending a general metaphysical picture of downwards dependence where "any structure whatsoever depends on the objects that it involves" (Linnebo 2008:69-70). ODS is meant as a case of upwards dependence, as the objects belonging to a structure depend on that structure. Downwards dependence is given initial plausibility if we consider the intuitive idea that a structure without the objects that constitute it does not make much sense. What is it a structure of, if it does not highlight the interrelationships between its objects, i.e., its positions? According to Shapiro, this is not the case: "The structure is prior to the mathematical objects it contains, just as any organization is prior to the offices that constitute it. The natural-number structure is prior to 2, just as … 'U.S. Government' is prior to vice president'" (Shapiro 1997:78). However, a way to accommodate the objection, is to consider Rota's emphasis on foundation's anti-reductionism and the distinction between facticity and function. We saw that while the function is founded on the facticity, it is the function that is *relevant*. The anti-reductionism of foundation means that "[c]onflating function with facticity … is an instance of *reduction*, an error in reasoning" (Rota 1989:73). Moreover, the function and facticity in a foundation relation is context-dependent and layered, so that what is the facticity in one foundation can be the function in another (Rota 1989:76). This opens up for the

[18] Reflexivity, that something is founded on itself, is – to my knowledge – not addressed. However, as foundation is clearly symmetrical and transitive, we do end up with infinite chains, e.g., '1 ~ 2 ~ 3 ~ 1 ~ 3 ~ 2 ~…'. Husserl differentiates between *mediate* and *immediate* foundation, which supposedly captures a difference in transitivity (Husserl 2001b:28). We can think of mediate foundation as transitive, while immediate as not (this is suggested by Bliss & Priest (2018), though not in reference to Husserl). This allows for explaining containment of parts by other parts of a whole. See also Fine (1995a).

[19] This is due to Hellman (2001) and MacBride (2006).

view that while structure might be prior to its objects, i.e., ODS, we can also consider the structure to be founded on its objects, if we consider the structure to be the function, and the objects to constitute the facticity. The structure is still the relevant part of this foundation relation, which we could take to mean that it has priority over its objects, as reducing it to its factical objects would be an error, and go against the nature of foundation.

4. Metaphysical interdependence and links to coherentism

We saw that the objection against ODS rests on a general metaphysical picture of downwards dependence, according to which a structure depends on the objects it involves (Linnebo 2008:69-70). This general metaphysical picture is an instance of *metaphysical foundationalism*, the view that reality is hierarchical (i.e., the hierarchy thesis), and structured by chains of dependence relations ending in some fundamental thing(s) that do(es) not depend on anything further (i.e., the fundamentality thesis) (Bliss & Priest 2018:2). Metaphysical foundationalism has been the standard view among contemporary analytic metaphysicians. Concomitantly, the standard view on metaphysical dependence is that it is irreflexive, asymmetrical, and transitive. Ricki Bliss and Graham Priest argue that we ought to pay more attention to alternative accounts to both, as "[r]eality may well not have the metaphysical structure of a well-founded chain, but a much more complex and fascinating one" (2018:1, 31).[20]

One such alternative is explored by Thompson (2018), who develops the view of metaphysical *interdependence* or *coherentism*.[21] Thompson rejects two theses of metaphysical foundationalism: *well-foundedness* and *asymmetry*. While well-foundedness "guarantees that each fact is ultimately grounded in some foundational fact or facts" (i.e., the fundamentality thesis), asymmetry "guarantees that grounding hierarchies run only in one direction; from the more fundamental to the less fundamental" (i.e., the hierarchy thesis) (Thompson 2018:109). We thus end up with infinite chains of ground in *both directions*. Not only does her account allow for non-well-founded chains of ground, but reality also is not construed as hierarchically structured in more or less fundamental levels. As the structural property of transitivity is preserved, this leads to chains of ground that *cycle*, i.e., there are ontological loops of ground.

It has been observed before that metaphysical coherentism – though a *general* metaphysical picture of reality – is relevant for mathematical structuralism. As a mathematical

[20] See Bliss & Priest (2018) for a collection of papers that challenge aspects of metaphysical foundationalism.
[21] Thompson prefers 'interdependence' in her particular view, while Bliss & Priest call it 'coherentism' (2018:3, 14, 31). We will use both, where it is explicitly Thompson's formulation, 'interdependence' will be appropriate.

object stands within a web of ontological dependence relations, the web's interconnections should exhibit some overall coherence. This idea turns up in Shapiro (1997), for whom *coherence* is the existence criterion by which structures exist. As long as a theory is coherent, the structures described by it exist (Shapiro 1997:95). A coherentist metaphysical picture seems appropriate for structuralism, as a holistic approach is generally preferred.[22] Fittingly, Thompson uses mathematical structuralism as an example position where interdependence might be better suited than foundationalism (2018:118). The choice of an ontological dependence relation that favours cyclical chains of dependence, thus referring to an overall system of dependence, should fit within this general coherentist metaphysical picture.

Thompson's view of metaphysical coherentism is, like Wigglesworth's account of grounding and our account of foundation, non-well-founded and allows for symmetrical and transitive relations of dependence. But, she goes one step further than Wigglesworth. Wigglesworth holds that as a structure fully grounds its objects, it provides them with a foundation. He distinguishes between the non-well-foundedness of ODO and the well-foundedness of ODS, where, as "each chain in the structure is bounded from below, each chain has a foundation" (2018:234). In contrast to Thompson, Wigglesworth does not reject the fundamentality thesis, as the upward dependence of ODS *is* well-founded, and there is something fundamental that does not depend on anything further. Thompson calls this a *weak* form of interdependence, where all that is required is "that there be at least one counterexample to both well-foundedness and asymmetry", in which case, we might get "hybrids featuring chains that bottom out in foundational facts alongside small pockets of interdependence" (2018:110). This seems to fit Wigglesworth's bill: There is a counterexample in the non-well-foundedness and symmetry of partial ground for ODO, while the well-foundedness of ODS provides each infinite chain in the structure with a foundation.

Thompson discusses *integrated wholes* (as compared to *mere aggregates*), and considers the example of a circle. There are good reasons, she writes, to consider any divisions of the circle, i.e., parts, as grounded in the whole (Thompson 2018:111). This would constitute ODS, i.e., how the objects depend on the structure. However, she also argues that there are good reasons to believe that the whole, e.g., circle, is grounded in its parts, e.g., two semi-circles. This is the upshot of metaphysical interdependence: A completely cyclical picture of reality can "simultaneously account for both these seemingly competing intuitions" (Thompson 2018:111). When applied to our case, it would mean that the structure is founded on its objects. From our above discussion of function and facticity, we saw that a reduction of the function to

[22] See Shapiro (2011).

the facticity in a foundation relation would be an error of reasoning. The same applies here. There seems to be something wrong about saying that the circle is founded on two semi-circles being adjoined. The division into two semi-circles seems to happen after the construction of the circle, thus suggesting that the circle is indeed prior to its parts.[23] A structure is similarly prior to its objects. However, the intuition that two semi-circles do form a circle by a unificatory relation of foundation also has merit. A way, then, to account for both directions of dependence is to consider the context-dependence of foundation as it appears in Rota (1989). We may consider the structure as that which is constituted by objects, but, importantly, we cannot reduce it to its objects.

There are two further points that link Thompson's interdependence to our case at hand, both epistemological. First, Thompson takes epistemological coherentism as inspiration for her view (Thompson 2018:109-110). Just as there is epistemological foundationalism and coherentism, there is, on the metaphysical side, foundationalism and interdependence. The thought is that just as justification is not transferred linearly but rather cycles and must be considered in reference to the system of beliefs as a whole, so do dependence relations cycle. This is relevant, as the metaontological coherentism developed for structuralism above also takes epistemological coherentism as analogous to some extent. Whereas the twofold coherence is taken as an identity criterion in metaontology, the coherentism of interdependence is evident from the non-linearity of how dependence cycles.

Second, Husserlian foundation *is* explicitly mentioned as a dependence relation in structuralism, but in relation to epistemology and mathematical intuition. Charles Parsons writes that we "can indeed talk, after Husserl, of intuition of a type as *founded* on perception of a token" (2008:161).[24] This is reminiscent of Shapiro's stratified epistemology, a step of which describes how our experience with tokens lead to understanding of the type (Shapiro 1997:11). Moreover, Richard Tieszen writes that "[t]he general idea of founding and founded structures in mathematical cognition is a fundamental part of Husserl's view" and Husserl's foundation-relation is essential to understand "the independence and nonindependence of objects" in general (Tieszen 2005:259).

That foundation pops up in epistemology for structuralism, as well as being used as a dependence relation that differentiates between founding and founded structures, suggests that

[23] The case of the table depending on its atoms for its existence is more straightforward, as there is causality at play. Metaphysical dependence in the case of circles and mathematical structure and objects seems different. And still, due to our anti-reductionist dependence relation, we would not reduce the table to its atoms, as that would deny the table its relevance as an artefact.

[24] Though the discussion of Husserl is not extensive, Parsons proclaims that Husserl has "the most developed and in some ways the clearest philosophical statements [of a structuralist view] from before World War II" (Parsons 2008:41).

foundation encompasses more than mere ontological dependence. Not only can we implement it as the ontological dependence relation appropriate for the twofold Dependence Claim, but we can also implement it in our epistemological account for structuralism. This is surely a benefit, as it makes structuralist ontology and epistemology mesh well together, i.e., the different aspects of structuralism are given over-all coherence, thus rendering structuralism as a more unified position.

5. Conclusion

We have seen that dependence relations play an integral part in characterising mathematical structuralism. A suitable dependence relation must be able to explain both how mathematical objects depend on their structure (ODS), and how they depend on each other (ODO). To this end, we took on the Husserlian relation of foundation and saw how it accommodates both. Foundation allows for cyclical chains of dependence, as it is asymmetrical, transitive, and non-well-founded. As foundation is explicitly anti-reductionist and context-dependent, the distinction between function and facticity permits us to still make sense of the intuition that a whole is founded on its parts, while at the same time keeping the view that a structure is ontologically prior to its objects. Moreover, by looking at the general metaphysical picture of coherentism, we found that abandoning a linear structure of reality fits with the dependence relation appropriate for structuralism, and that, in this respect, it has commonalities with other non-linear aspects of structuralism. To construe the dependence relation in mathematical structuralism as foundation thus provides a more complete picture as to how a structure and its objects are irrevocably interlinked.

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Abstract

This thesis defends non-eliminative structuralism in philosophy of mathematics. Non-eliminative structuralism combines the view that mathematics is the study of abstract structures with a realist ontology. It is a cumulative thesis, and in four independent articles I investigate questions of metaontology, epistemology, and ontology. The particular perspective of each article adds to the overall justification for structuralism – as they converge thematically and methodologically – thus constituting a coherent and systematic defence, and progressing the viability of the position.

Articles 1 and 2 are sister papers, where the second clearly builds upon the first. The first article examines the relationship between metaontology and ontology in the philosophy of mathematics. I argue that metaontology can serve a useful role in formulating mathematical ontology, and that we can view it as rectifying a position's theoretical insufficiency. Article 2 takes as its starting point the conclusion that mathematical realism in ontology generally benefits from implementing an appropriate metaontology. It develops metaontological coherentism, and investigates its relation to – and fit with – structuralist ontology. Article 3 answers the question of when an epistemological account is deemed adequate with regard to the so-called access problem. I argue that two approaches can be differentiated. Each approach has implicit metaepistemological leanings, which accounts for miscommunication in the epistemological debate generally, and within structuralism specifically. Article 4 takes on ontological dependence relations for structuralism, between a structure and its objects and among the objects belonging to the same structure. I defend a Husserlian relation of dependence – *foundation* – as it allows for infinite chains of dependence that cycle. Such a non-linear account of dependence fits with how mathematical objects are thought to be incomplete and dependent on the structure to which they belong.

Zusammenfassung

Die vorliegende Dissertation verteidigt den sogenannten nicht-eliminativen Strukturalismus („non-eliminative structuralism“) als Position in der Mathematikphilosophie. Der nicht-eliminative Strukturalismus verbindet die Auffassung, dass Mathematik das Studium abstrakter Strukturen sei, mit einer realistischen Ontologie. In den vier unabhängigen Artikeln dieser kumulativen Arbeit untersuche ich Fragen der Metaontologie, Epistemologie und Ontologie. Die jeweiligen Perspektiven der einzelnen Artikel tragen zur allgemeinen Rechtfertigung des Strukturalismus als Ganzes bei – da sie thematisch und methodisch konvergieren – und bilden somit eine kohärente und systematische Verteidigung des Strukturalismus, welche die Tragfähigkeit dieser Position fördert.

Die Artikel 1 und 2 sind „Schwesterartikel“, wobei der zweite Artikel auf den ersten aufbaut. Der erste Artikel untersucht die Beziehung zwischen Metaontologie und Ontologie in der Philosophie der Mathematik. Ich argumentiere, dass die Metaontologie eine nützliche Rolle bei der Formulierung der mathematischen Ontologie spielen kann und dass wir sie zur Vervollständigung einer ontologischen Position betrachten können. Artikel 2 bezieht diese Schlussfolgerung auf den mathematischen Realismus, und argumentiert, dass diese ontologische Position im Allgemeinen von der Implementierung einer geeigneten Metaontologie profitiert. Der Aufsatz entwickelt einen metaontologischen Kohärentismus und untersucht dessen Beziehung zur strukturalistischen Ontologie – und dessen Kompabilität mit ihr. Artikel 3 beantwortet die Frage, wann ein erkenntnistheoretischer Ansatz im Hinblick auf das sogenannte „access problem“ als angemessen gilt. Ich argumentiere, dass zwei Ansätze mit verschiedenen impliziten metaepistemologischen Neigungen, welche zu Missverständnissen in der erkenntnistheoretischen Debatte im Allgemeinen und im Strukturalismus im Besonderen führen, unterschieden werden können. Artikel 4 befasst sich mit den ontologischen Abhängigkeitsverhältnissen im Strukturalismus. Ich verteidige ein Husserlsches Abhängigkeitsverhältnis – die *Fundierung* – da es unendliche, zyklische Abhängigkeitsketten zulässt. Eine solche nicht-lineare Darstellung der Abhängigkeit entspricht der Betrachtung von mathematischen Objekten als unvollständig und abhängig von der Struktur, zu der sie gehören.