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„Foliations and Bypasses in Contact Topology‘

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Abstract

In this paper we present a brief introduction into both the theory of 2-dimensional foliations and 3-dimensional contact topology. Afterwards we develop the most important tools to discuss both Giroux's theory of convex surfaces [9] and Honda's method of bypass attachments [14]. The goal of this paper is to prove a conjecture by Honda [13, Theorem 11.1], Etnyre [6, Theorem 7.2.] and others that one can build up contact structures on 3-manifolds of the form $\Sigma \times [-1, 1]$ up to isotopy relative to the boundary entirely by bypass attachments.

In diesem Papier präsentieren wir eine kurze Einführung sowohl in die Theorie der Foliierungen auf 2-Mannigfaltigkeiten als auch in die 3-dimensionale Kontakttopologie. Anschließend entwickeln wir die wichtigsten Werkzeuge um Giroux's Theorie konvexer Flächen [9] und Honda's Technik der Beipässe [14] zu diskutieren. Ziel der Arbeit ist es eine Vermutung von Honda [13, Theorem 11.1], Etnyre [6, Theorem 7.2.] und anderen zu beweisen, dass Kontaktstrukturen auf 3-Mannigfaltigkeiten der Form $\Sigma \times [-1, 1]$ bis auf Isotopie, die den Rand fixiert, mit Hilfe von Beipässen beschrieben werden können.

Aknowledgements

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I would also like to thank my family and friends for supporting me throughout my studies and the weird few years we all had to go through. Surely I would not have progressed this quickly if I was not able to rely on them for their support and care.

Thanks to Patrick Massot [17] who allowed me to use some of his illustrations (Figure 3.2 and Figure 3.1). The beautiful diagrams of flows of vectorfields were created with the help of Mathematica while the remaining drawings were made using the tools of GIMP.

Contents

1 Introduction

3-dimensional contact topology is one of the major fields of study in modern low-dimensional topology. Roughly one considers 3-manifolds endowed with an extra structure, a 2-plane field fulfilling a non-degeneracy condition (see Definition 3.1). Meaning that at each point of the 3-dimensional manifold one chooses a 2-dimensional subspace. This could be done more generally for n-manifolds and k-plane fields. However, 3-dimensional contact structures are special in this area of mathematics, in so far as they do not have geometric properties but they indeed possess no local properties at all. For example, Riemannian metrics do possess a local invariant: its curvature. That a contact structure possesses no local structure is often called Darboux's Theorem (see Corollary 3.9).

Even though contact structures do not possess local structure, it is a fundamental result due to Bennequin [1, Théorème 1] that contact structures are topologically interesting. Roughly speaking they decompose into two big classes: tight- and overtwisted contact structures (see Definition 4.16). The classification of contact structures is especially interesting on closed and oriented 3-manifolds. Here there is a big difference between the tight and overtwisted world: Any 2-plane field is homotopic to an overtwisted contact structure and the isomorphism class of an overtwisted contact structure is equivalent to its isotopy class. This is a theorem by Eliashberg [3, Theorem 1.6.1.].

On the other hand, tight contact structures are rare in a certain sense: S^3 possesses a unique tight contact structure up to isotopy (this is a theorem due to Eliashberg [4, Theorem 2.1.1.]). In fact, some closed and orientable 3-manifolds admit no tight contact structure. So the existence and classification of tight contact structures is a very subtle and important topic.

One major approach is to decompose the 3-manifold into smaller pieces where contact structures are easier to understand. These pieces can, for instance be standard neighborhoods of points (see Corollary 3.9) or standard neighborhoods of certain embeddings of S^1 (see Theorem 3.11). However, more often one considers pieces of the form $\Sigma \times [-1, 1]$ where Σ is a closed and oriented surface. Both Giroux and Honda developed approaches to study tight contact structures on manifolds of this form up to isotopy relative to the boundary.

Given a surface Σ embedded in a contact manifold one obtains a structure called a characteristic foliation induced by restricting the plane field to the surface (see Definition 3.20). Using this information Giroux used deep results from the theory of foliations on closed surfaces to obtain a foliation-theoretic classification of contact structures on $\Sigma \times [-1, 1]$, see Theorem 4.19.

On the other hand, Honda developed a new technique called bypass attachment to classify those contact structures. The main objects are convex surfaces first considered by Giroux [9]. Roughly speaking these are surfaces which admit an I-invariant contact structure (see Definition 4.1). Isotopies of convex surfaces preserve an invariant which is given by a homotopy class of embedded curves on Σ. Honda's approach requires that there is a special convex annulus Σ' called a bypass which intersects Σ in a sufficiently nice way. Then one can show that one may attach Σ' to Σ to obtain a new convex surface Σ'' which is isotopic to Σ. The homotopy class of embedded curves is then obtained from the original one on Σ by a specific change.

The interesting thing to note is that the discrete changes of foliations described by Giroux's normal form theorem gives rise to a change of isotopy class of convex surfaces as if there was a bypass attached. The goal of this thesis is to prove in full detail an equivalent formulation of Giroux's normal form theorem (Theorem 4.19) in terms of bypasses, see Theorem 1.1.

To start, we will introduce one of our main tools; the theory of foliations on surfaces in Section 2. These are equivalence classes of vectorfields on compact and oriented surfaces Σ, see Definition 2.6. In Subsection 2.1, we introduce the most regular kind of foliations, the Morse-Smale foliations and discuss Peixoto's density theorem 2.18/ [19, Theorem 2] which says that Morse-Smale foliations are generic.

Then we proceed to study two classes of foliations, non-isochore foliations (see Definition 2.19) and divided foliations(see Defintion 2.21). Those foliation which appear as characteristic foliations of surfaces embedded in contact manifolds, respectively the characteristic foliations of convex surfaces embedded in contact manifolds. In particular, we provide a full proof of Theorem 2.22. This theorem has been claimed by Giroux [10, Proposition 2.5.] which claims that if X is non-isochore and fulfills a certain regularity condition then there are exactly two obstructions for the foliation to be divided.

To finish Section 2, we discuss Sotomayor's density Theorem 2.34 [21, Theorem II.2.]. In his work, he describes that given any family of vectorfields $(X_r)_{r\in[-1,1]}$, one finds a C^∞-close family of vectorfields $(X'_r)_{r\in[-1,1]}$ which are Morse-Smale on an open dense set $J \subset [-1,1]$ and otherwise X'_r contains a degeneration which can only be one of 5 different types. Sotomayor's density theorem is one of the key theorems needed for Giroux's normal form theorem.

In Section 3, we will give an introduction into the world of 3-dimensional contact structures. In Subsection 3.1 we will discuss the other important tool which we will use extensively: Moser's stability trick. If M is assumed to be closed (compact with empty boundary) then one can prove that a homotopy of contact structures ξ_t can be induced by an isotopy ϕ_t of M and this isotopy preserves the contact structure, see Theorem 3.7 In general, we will not work on closed manifolds M. However, if one is careful Moser's stability trick can still be used.

Afterwards, in Subsection 3.2 we proceed to consider surfaces Σ embedded in a contact manifold (M,ξ). There is a special kind of foliation which the contact structure induces on Σ. As it turns out, this foliation is non-isochore and encodes all local information of the contact structure close to Σ, see the Reconstruction Lemma 3.24. Even more, any non-isochore foliation can be induced by some contact structure on $\Sigma \times \mathbb{R}$, compare Theorem 3.23.

At last, in Section 4 we will come to the main theory: We first discuss the most important properties of convex surfaces. They are C^∞-generic by a Theorem of Giroux (see Lemma 3.25) and a surface is convex if and only if the induced foliation is divided, see Definition 4.1. We will observe that the aforementioned homotopy class of embedded arcs, also called the dividing curve Γ decides the convex isotopy class. In Subsection 4.1 we reprove some of the most important results which say that in the convex isotopy class of a convex surface one has great freedom to choose characteristic foliations. The main result to do so is Giroux's flexibility lemma 4.6.

We will briefly touch on one of the fundamental objects of contact topology in general, the so-called overtwisted disks 4.2. Omitting many details, we will only state Giroux's Criterion 4.15 which allows us to characterise the existence of overtwisted disks close to surfaces using only the dividing curve Γ. In addition, it tells us that certain phenomena such as contractible components of Γ cannot occur (if $\Sigma \neq S^2$) if there are no overtwisted disks arbitrarily close to Σ.

Having dealt with the preliminaries, we will proceed to discuss Giroux's normal form theorem 4.19/ [11, Lemme 15]. This theorem is based upon Sotomayors density theorem 2.34 and allows us to characterise all contact structures on $\Sigma \times [-1,1]$ where Σ_{-1}, Σ_1 are convex up to isotopy (relative to the boundary) by a single degeneration, a so-called retrograde connection. This is a connection between singularities of a foliation X such that it flows from a singularity with negative divergence to a singularity with positive divergence. By Theorem 2.22 these foliations cannot be divided. If the dividing curves of Σ_{-1} and Σ_1 are not homotopic then these levels have to occur. So Giroux's normal form theorem provides a foliation-theoretic description of contact structures on $\Sigma \times [-1,1]$.

The other picture which we will introduce is the theory of bypasses due to Honda [14]. Roughly speaking a bypass D^2 is a half-disk with a certain foliation transversal to Σ which can be attached to Σ which results in a convex surface $\Sigma + D^2$. The convex surfaces Σ and $\Sigma + D^2$ have dividing curves which differ by a certain kind of change. This change of the dividing curve looks the same as the one found by Giroux when a retrograde connection occurs. So it has been long believed that these approaches are equivalent. Thus, we will prove the following version of Giroux's normal form theorem:

Theorem 1.1. *Let $(\Sigma \times [-1,1], \xi)$ be a contact manifold such that Σ_{-1} and Σ_1 are convex. Then there is a contact structure ξ^0 on $\Sigma \times [-1,1]$ isotopic to ξ relative to the boundary such that:*

(i) Except for finitely many values $r_1, \ldots, r_n$ Σ_r is convex.

(ii) For each r_i there is a sufficiently small ϵ such that $\Sigma_{t_i-\epsilon}$ and $\Sigma_{t_i+\epsilon}$ are related by a bypass attachment.

2 Foliations on surfaces

First, we recount some of the fundamental results about 2-dimensional compact oriented manifolds, also called surfaces:

Theorem 2.1. *(Classification of closed surfaces) A surface Σ with empty boundary, is isomorphic to a sphere or a genus g surface. They are uniquely determined by their Euler characteristic $\chi(\Sigma) = 2 - 2g$.*

A proof of this can be found in the Differential Topology textbook by Hirsch [12, Theorem 9.3.11.].

Surfaces with empty boundary are also called **closed**. However, we will need to consider surfaces with non-empty and sometimes even non-smooth boundary:

Remark 2.2. We say that Σ is a **surface with polygonal boundary**, if Σ is orientable and compact and each point $p \in \Sigma$ has a neighborhood U which is diffeomorphic to an open subset $V \subset \mathbb{R}^2_{++}$ where $\mathbb{R}^2_{++}$ is the first quadrant $\{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$. We say that p is an interior point if V can be chosen to be disjoint from the x- and y- axis. p is a boundary point if V can be chosen disjoint from the origin. Otherwise p is called a corner of Σ. [1]

There is a special class of surfaces:

Definition 2.3. *A surface Σ is called **planar**, if there is an embedding $\Sigma \to \mathbb{R}^2$.*

Essentially all of them are spheres with polygons/disks removed:

Lemma 2.4. *Assume that Σ is a surface with corners embedded in $\mathbb{R}^2$ and $\partial\Sigma \neq \emptyset$ then there is an embedding of Σ into S^2.*

Proof. One merely has to note that $\mathbb{R}^2 \cup \{\infty\}$ can be endowed with a smooth structure making it isomorphic to a sphere using the stereographic projection. So Σ is naturally a subset of S^2. QED

In a certain sense all surfaces with polygonal boundary can be seen as subsets of closed surfaces:

Lemma 2.5. *Let Σ be a surface with polygonal boundary then there is an embedding of Σ into a genus g surface, where g is given by $g = \frac{-1}{2}(\chi(\Sigma) + b - 2)$ where b denotes the amount of polygonal- and smooth boundary component. In addition, there is no embedding of Σ into a surface with lower genus.*

[1] Note that some details have been omitted. To state this precisely, one needs to work with smooth atlases. But we will keep this section brief.

2.1 Morse-Smale foliations

To start, we will consider oriented foliations on surfaces, possibly with polygonal boundary. Foliations can either be described by their flow-lines or as equivalence classes of vectorfields, respectively 1-forms. For our purposes we will mainly stick to the later picture:

Definition 2.6. *Let Σ be a surface. An* ***(oriented) foliation*** *$\mathcal{F}$ on Σ is an equivalence class of vectorfields X under multiplication by functions $f : \Sigma \to \mathbb{R}_{>0}$. We say that $X \in \mathcal{F}$ orients the foliation.*

Dually if ω is a positive volume form on Σ, then $\beta = \iota_X \omega$ induces an equivalence class of 1-forms closed under multiplication of functions $f : \Sigma \to \mathbb{R}_{>0}$. We will refer to both equivalence classes as foliations, whichever is convenient. Before starting with theory, we will discuss some foliations in more detail:

Example 2.7. Let S^2 be the unit sphere in $\mathbb{R}^3$, and consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ the projection on the third coordinate. Then we can consider the restriction of f to S^2 and calculate its gradient foliation where the gradient is taken with respect to the restriction of the euclidean metric $<,>$.

The gradient of a function $g : S^2 \to \mathbb{R}$ is defined as the unique vectorfield $grad(g)$ of S^2 such that $< grad(g), Z >= dg(Z)$ for all Z tangent to S^2. Similarly $grad(f)_{\mathbb{R}^3} = \frac{d}{dz}$ fulfills the second property for all Z tangent to $\mathbb{R}^3$, so in particular for all Z tangent to (the embedding of) S^2. However $grad(f)_{\mathbb{R}^3}$ is not tangent to S^2.

If we restrict ourselves to S^2: $df(Z) =< grad(f), Z >$ for Z tangent to S^2 then we can change $grad(f)_{\mathbb{R}^3}$ to $grad(f)_{S^2}$: Consider $\mathfrak{n} := x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}$ the unit normal to the unit sphere, then we can observe that for any function $a : S^2 \to \mathbb{R}$: $< grad(f)_{\mathbb{R}^3} - a\mathfrak{n}, Z >=< grad(f)_{\mathbb{R}^3}, Z >$ for all tangent vectors of S^2. So we subtract $a\mathfrak{n} =< grad(f)^3_{\mathbb{R}}, \mathfrak{n} > \mathfrak{n}$ and obtain:

$$grad(f)_{S^2} = -xz\frac{d}{dx} - yz\frac{d}{dy} + (1 - z^2)\frac{d}{dz}$$

We observe that this vectorfield never vanishes, except if $z = \pm 1$ and thus $x = y = 0$. Otherwise the flow always points "upward", since the coefficient of $\frac{d}{dz}$ is always positive. In addition, we wish to consider the determinant and trace of the vectorfield at the points where it vanishes, the so-called singularities: We oberserve that

$$\begin{aligned} D(grad(f)) &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ div(grad(f)) = tr(grad(f)) &= -2 \\ det(D(grad(f)) &= 1 \end{aligned}$$

Similarly one calculates that the divergence and determinant of the south pole are positive. Singularities with positive determinant and non-vanishing trace are called **nodes**, for an illustration see Figure 2.1. A node has an associated sign given by the sign of its

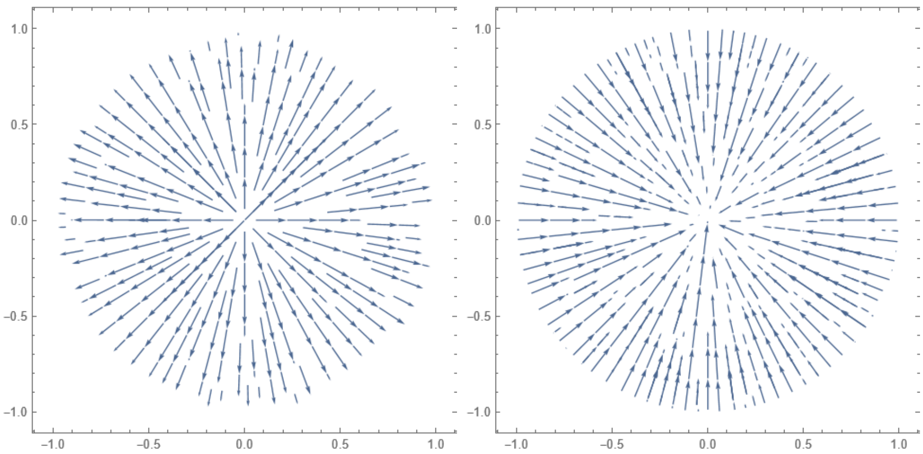

Figure 2.1: A positive and a negative node. The foliation is oriented away from positive node and oriented towards the negative node.

trace which happens to be + if all leaves point away from the node and − if all leaves point towards the sink.

A similar trick can be done to calculate such a flow for the torus:

Example 2.8. Consider the torus $T^2 = (S^1)^2$ as the image of $F : S^1 \times S^1 \to \mathbb{R}^3$:

$$\begin{aligned} F(\phi, \theta) &= ((R + rcos(\phi))cos(\theta), (R + rcos(\phi))sin(\theta), rsin(\phi)) \\ r &< R \end{aligned}$$

this map is 2π-periodic in both coordinates and this translates to an embedding of the torus after the quotient is taken. Now, we can consider the function f which is the projection to the y-coordinate and calculate the gradient vectorfield. As before one can calculate the outer normal, by first calculating DF, normalising the vectors and completing to a normal basis. Then one obtains the following:

$$\begin{aligned} DF(\phi, \theta) &= \begin{pmatrix} -rsin(\phi)cos(\theta) & -(R + rcos(\phi))sin(\theta) \\ -rsin(\phi)sin(\theta) & (R + rcos(\phi))cos(\theta) \\ rcos(\phi) & 0 \end{pmatrix} \\ \mathfrak{n}(\phi, \theta) &= \begin{pmatrix} -cos(\phi)cos(\theta) \\ cos(\phi)sin(\theta) \\ sin(\phi) \end{pmatrix} \end{aligned}$$

So one may calculate $grad(f)_{T^2}$:

$$\begin{aligned} grad(f)_{T^2} &= -cos(\phi)^2 cos(\theta)sin(\theta)\frac{d}{dx} + (1 - cos(\phi)^2 sin(\theta)^2)\frac{d}{dy} - sin(\phi)cos(\phi)sin(\theta)\frac{d}{dz} \\ &= -\frac{sin(\phi)sin(\theta)}{r}\frac{d}{d\phi} + \frac{cos(\theta)}{(R + rcos(\phi))}\frac{d}{d\theta} \end{aligned}$$

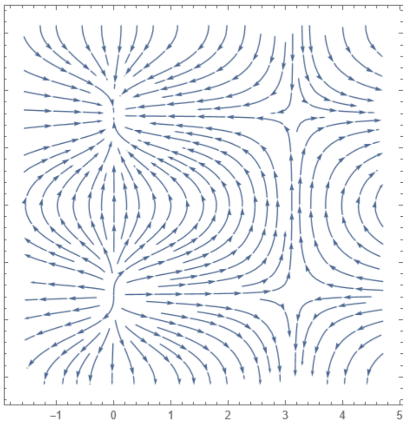

Figure 2.2: A representation of the gradient foliation of the torus.

In this case the dynamics are a lot more interesting, for an illustration see Figure 2.2. The singularities can only be at the values: $\phi \in \{0, \pi\}, \theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. So as above, we obtain:

$$Dgrad(f)(0, \frac{\pi}{2}) = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{1}{R+r} \end{pmatrix} \qquad det = \frac{1}{r(R+r)} \qquad div = \frac{-R-2r}{r(R+r)}$$

$$Dgrad(f)(\pi, \frac{\pi}{2}) = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & -\frac{1}{R-r} \end{pmatrix} \qquad det = \frac{-1}{r(R-r)} \qquad div = \frac{R-2r}{r(R-r)}$$

$$Dgrad(f)(0, \frac{3\pi}{2}) = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{R+r} \end{pmatrix} \qquad det = \frac{1}{r(R+r)} \qquad div = \frac{R+2r}{r(R+r)}$$

$$Dgrad(f)(\pi, \frac{3\pi}{2}) = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & \frac{1}{R-r} \end{pmatrix} \qquad det = \frac{-1}{r(R-r)} \qquad div = \frac{2r-R}{r(R-r)}$$

We have a positive node at $(0, \frac{3\pi}{2})$ and a negative node at $(0, \frac{\pi}{2})$. However the other two singularities have negative determinants and thus are not nodes. Such points are called **saddles**.

For the saddle at $(\pi, \frac{\pi}{2})$ we observe the following: The gradient along the line $\{\phi = \pi\}$ has vanishing $\frac{d}{d\phi}$ component, thus the direction of the gradient vectorfield is completely determined by the sign of $cos(\theta)$. Thus it is also always tangent to the line $\{\phi = \pi\}$. In fact, in this case they point away from the node at $(0, \frac{\pi}{2})$ and towards the saddle. So in a certain sense the node and the saddle are connected.

We denote by Fl_X the flow of X. This map is defined on an open subset of $M \times \mathbb{R}$ and the image of (p, t) is defined as the unique point $Fl_X(p, t)$, where $Fl_X(p, \cdot)$ is the solution to the differential equation $\dot{c}(t) = X(c(t))$ with initial value $c(0) = p$. See for instance the lecture notes on Analysis of manifolds by Cap [2].

For a fixed p, we call the image of $Fl_X(p, t)$ the **orbit** or **leaf** of p. Should the flow be defined for all positive, respectively all negative times, we can ask ourselves to what does this orbit limit? We define the set $\{p' \in M : \exists (t_n) : lim(t_n) = \infty, lim(Fl_X(p, t_n)) = p'\}$ to be the **stable limit set**. Replacing ∞ with $-\infty$ we obtain the **unstable limit set**, which can be interpreted as the future, respectively past limit of the orbit. It is important to note, that while the flow changes if one changes X to fX (as before $f : \Sigma \to \mathbb{R}_{>0}$) the orbits of a given point remain the same.

In the last example one can thus say that the saddle at $(\pi, \frac{\pi}{2})$ and the node at $(0, \frac{\pi}{2})$ are connected, via orbits whose stable limit set consists of the node and unstable limit set is the saddle. In fact, any saddle is the the unstable limit set of 2 orbits and the stable limit set of 2 orbits (not necessarily distinct, as two of these orbits may form a loop). These orbits are called called **separatrices**. Those orbits which limit into the saddles are called stable separatrices and those limiting away are called unstable separatrices. [2]

The above calculations for the type of singularities of a foliation are only partially true up to now since we did not show yet, that the sign of the determinant or the divergence are independent of the choice of X, the choice of volume element ω or the coordinate representation of X:

Lemma 2.9. *Let ω be a volume form on Σ, X a vectorfield and $f : \Sigma \to \mathbb{R}_{>0}$ a smooth function, then the following formulae hold:*

$$div_\omega(fX)\omega = f div_\omega(X)\omega + df \wedge \iota_X \omega \tag{2.1.1}$$
$$f div_{f\omega}(X) = div_\omega(fX) \tag{2.1.2}$$
$$div_\omega(fX) = f div_\omega(X) - df(X) \tag{2.1.3}$$

here $div_\omega X$ is defined by $d(\iota_X \omega) = div_\omega X \omega$.

Proof. The proof is just a straightforward computation:

$$div_\omega(fX)\omega = d(\iota_{fX}\omega) = d(f\iota_X\omega) = df \wedge \iota_X\omega + fd(\iota_X\omega)$$

The second follows from the same computation except that f has to be accounted for:

$$div_{f\omega}(X)f\omega = d(\iota_X f\omega) = df \wedge \iota_X\omega + f div_(\omega)(X)\omega$$

To proof the last equality, one may find a vectorfield X' such that $X' = 0$ if and only if $X = 0$. This is done using a metric $<, >$ then one obtains a vectorfield X' which fulfills that $< X', \cdot >= \omega(X, \cdot)$. In particular, if $X \neq 0$ then they form a basis of the tangent space if and only if $X \neq 0$ since $< X', X >= \omega(X, X) = 0$. If $X = 0$ then the equation is obvious, since both $\iota_X\omega = 0$ and $df(X) = 0$. So assume $X \neq 0$, then one obtains:

$$div_\omega(fX)\omega(X, X') = f div_\omega(X)\omega(X, X') + df(X)\omega(X, X') - df(X')\omega(X, X)$$

where the last term vanishes since $\omega(X, X) = 0$. Since ω_p is an antisymmetric bilinear in each point, this equation fully determines ω on all of Σ. QED

[2] In other conventions the stable separatrix of a saddle is the union of both of these orbits, we will stick to this naming convention as it is done by Sotomayor [21].

Theorem 2.10. *Let X be a vectorfield and p a singularity and $x = (x^1, \ldots, x^n)$, $y = (y^1, \ldots, y^n)$ coordinate charts around p. Then the matrix $D_x X = (\frac{d}{dx^i} X^j(p))$, where $\sum_{j=1}^n X^j e_j$ is the coordinate expansion with respect to x, is related to the corresponding matrix for y by conjugation by the coordinate change matrix $(\frac{dy^i}{dx^j})$*

Proof. This is again just a straightforward verification:

$$\frac{d}{dx^i} X_x^j(p) = \frac{d}{dx^i}(\sum_{l=1}^n \frac{dx^j}{dy^l} X_y^l)(p) = \sum_{l=1}^n \frac{d}{dx^i}(\frac{dx^j}{dy^l}) X_y^l(p) + \frac{dx^j}{dy^l}\frac{d}{dx^i} X_y^l(p) =$$

$$\sum_{l=1}^n \frac{d}{dx^i}(\frac{dx^j}{dy^l}) X_y^l(p) + \sum_{k=1}^n \frac{dx^j}{dy^l}\frac{dy^k}{dx^l}\frac{d}{dy^k} X_y^l(p)$$

Since $X(p) = 0$ the first sum vanishes and the last sum can be rewritten as:

$$D_x X = (\frac{dx^i}{dy_j}) D_y X (\frac{dy^i}{dx_j})$$

So $D_x X$ and $D_y X$ are related by conjugation. Thus the determinant and the trace are well-defined for fixed X. QED

The well-definedness of the sign of the determinant then follows from the chain rule: $D_x X$ and $D_x(fX)$ are related by $f D_x X = D_x(fX)$ at singular points, since f is strictly positive the sign remains unchanged by this transformation.

The example of the torus gives rise to some more important examples, first we will look at a case of the so-called saddle-node bifurcaton, which we will encounter later again (see Example 2.30):

Example 2.11. Let $X = -sin(\phi) sin(\theta)\frac{d}{d\phi} + cos(\theta)\frac{d}{d\theta}$ be a slightly changed version of the vectorfield from the above example. As we have seen above, there are 4 singularities: 2 nodes and 2 saddles. In general it is not possible to modify the vectorfield and remove singularities, however we may remove a node-saddle pair, if they are connected by a separatrix: Let $H(\theta)$ be a smooth cut-off function which is 1 around $\theta = \frac{\pi}{2}$ and 0 around $\theta = \frac{3\pi}{2}$. Then we may consider the s-dependent vectorfield X_s:

$$X_s = (H(\theta)s - sin(\phi)) sin(\theta)\frac{d}{d\phi} + cos(\theta)\frac{d}{d\theta}$$

We notice that the singularities on $\theta = \frac{\pi}{2}$ move towards $(\frac{\pi}{2}, \frac{\pi}{2})$ as s increases towards 1. At $s = 1$ there is exactly one singularity at $(\frac{\pi}{2}, \frac{\pi}{2})$ whose linearisation is a non-zero matrix with vanishing determinant. In addition, the second derivative of the vectorfield X in the direction of the eigenvector corresponding to 0 is non-vanishing. We call such points **saddle-nodes**. Similarly to Theorem 2.10, one may show that this definition is independent of the choice of X.

For $s > 1$ there are no more singularities, however the line $\theta = \frac{\pi}{2}$ is still an orbit of the foliation, called a **closed orbit**, see Figure 2.3. The orbit of p is a closed orbit, precisely if $Fl_X(p, t) = p$ for some t and p is not a singularity.

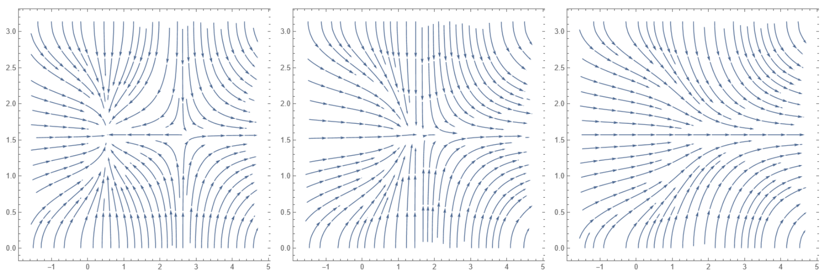

Figure 2.3: As s increases, the node and saddle start moving towards one another. Finally they merge into a saddle-node and then turn into a nonsingular attractive closed orbit.

As we have done for singularities there are also non-degeneracy conditions for closed orbits: Consider a closed orbit C of period 1 in the interior of Σ and a small (non-singular) neighborhood $S^1 \times [-1, 1]$ of $C = S^1 \times \{0\}$. Then by the continuity of the flow one can define the so-called **Poincaré-first return map**: Consider all points of the form $\{x\} \times [-1, 1]$, where $x \in S^1$ is arbitrary. Then one can try to define a map $\pi(s)$ which assigns to s the next intersection of $Fl_X((x, s), \cdot)$ with $\{x\} \times [-1, 1]$, where s is the coordinate on $[-1, 1]$, see Figure 2.4 This map is defined for $s = 0$, since it is a closed orbit. Furthermore, it will be defined for a small interval $(-\epsilon, \epsilon)$ around 0. An analogous construction can be done if C coincides with a boundary component of Σ, however one needs to restrict to one-sided neighborhoods. See for instance the introductory book by Teschl [22, Lemma 6.9.]. Essentially, this map measures the behaviour of orbits close-by, compare Figure. This leads us to the following definition:

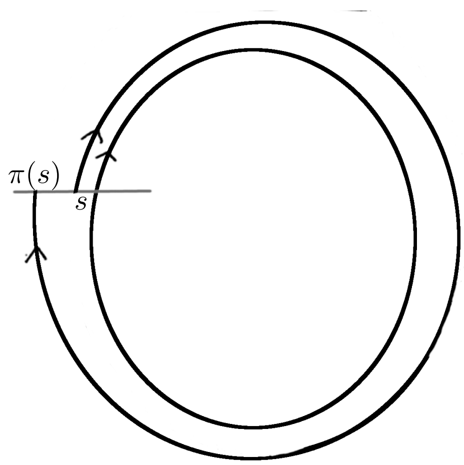

Figure 2.4: A repelling closed orbit and a nearby leaf. The grey line represents $\{x\} \times [-1, 1]$.

Definition 2.12. *Let Σ be a surface and $\mathcal{F}$ a foliation. Let C be a closed orbit, we say* *that C is an **attractive** closed orbit, if $\pi'(0) < 1$. It is **repelling**, if $\pi'(0) > 1$. We call it a **degenerate** closed orbit, otherwise.*

The upshot is that π' is independent of the choice of X and $x \in S^1$. However in practice, the first return map is unwieldy to use. However, there is a formula for its first derivative at 0, which is defined only in terms of the divergence:

Lemma 2.13. *([22, Lemma 12.6.]) Let Σ be a surface and $\mathcal{F}$ a foliation. Assume that C is a closed orbit and X a representative of $\mathcal{F}$. Denote by T the period of C with respect to X. Then the following relationship holds:*

$$\pi'(0) = e^{\int_0^T div(X(C(t)))dt}$$

QED

Using this definition, one can say that any closed orbit is attractive, repelling or degenerate, since it does not depend on the existence of orbits close-by. Consider for instance a closed orbit which intersects two different boundary components, in such a way that all orbits on either side leave the surface then the Poincaré-first return map does not exist, while the above integral can still be calculated.

As in Example 2.11 (replacing H with the identity) one can generate a fully non-singular foliation on the torus. In particular, a closed surface can only support a non-vanishing vectorfield if the Euler characteristic vanishes. Thus the torus is the only closed surface supporting such a foliation. These can however be very ill-behaved:

Example 2.14. Let r be a rational number. Then consider the foliation induced by:

$$X_r = \frac{d}{d\phi} + r\frac{d}{d\theta}$$

Changing from the torus to its universal cover $\mathbb{R}^2$, one easily sees that the orbits of X_r are precisely the lines $l_p = \{(x, p + xr) : x \in \mathbb{R}\}$. Since r is rational it can be represented as $r = \frac{n}{n'}$, where n and n' are coprime. Thus the line l_0 originating from $(0, 0)$ goes through (n', n), projecting it to the torus will thus create closed circles. Since all other orbits of X_r are obtained through translation, the torus is thus foliated by closed circles. If one calculates the divergence of this foliation, one obtains $div = 0$ so all closed orbits are degenerate.

However assume now that r is an irrational number. Then one can show that the orbits of X_r lie dense in the torus: The basic argument is to go to the universal cover $\mathbb{R}^2$ and show that there are whole numbers n, n' such that (n, nr) comes arbitrarily close to $(n, s + m)$ for each $s \in [0, 1]$. This observation is originally due to Kronecker [16, p. 50-51]. Thus in this case the whole torus is stable and unstable limit set of each orbit.

As can be seen this behaviour is not topologically stable, since the rational numbers lie dense in the reals and so in any small interval the change of topology of the foliation happens infinitely often. Additionally, foliations with dense leafs are very degenerate and so we wish to prevent such things from happening, this leads one to introduce the following regularity property:

Definition 2.15. *Let Σ be a surface and $\mathcal{F}$ a foliation on it. We say that $\mathcal{F}$ fulfills the* ***Poincaré-Bendixson property*** *if the stable, respectively unstable limit sets of a point are one of the following:*

- *A singular point;*
- *A closed orbit;*
- *A polygon of leafs, consisting of singularities and leafs in between them.*

Sadly, the Poincaré-Bendixson property is not stable under isotopies, since the rationally foliated torus fulfills the property, however the irrationally foliated torus does not. There is an important class of surfaces for which the Poincaré-Bendixson property is fulfilled in a special case:

Theorem 2.16. *(Poincaré-Bendixson, [22, Theorem 7.16.]) Let U be an open region of $\mathbb{R}^2$ with compact closure. Let the foliation $\mathcal{F}$ have only isolated singularities, then any leaf contained in U in positive(respectively negative) time has a stable (respectively unstable) limit set of either:*

- *A singular point;*
- *A closed orbit;*
- *A polygon of leafs, consisting of singularities and leafs in between them.*

QED

For an example of such a polygon, see Figure 2.5.

Naturally, should a foliation of a surface $\mathcal{F}$ decompose into planar regions and all its singularities have non-vanishing determinant(so they are isolated and stable under small isotopies) then the Poincaré-Bendixson property is stable. This leads us to introduce an additional property. To understand this definition, we need to discuss boundary conditions of foliations: The foliation $\mathcal{F}$ may be **parallel** to a boundary component e, so $X \in \mathcal{F}$ is tangent to the tangent space of e. Otherwise, it may be foliated **transverse** to the boundary. In this case, we differentiate between X **pointing outward** along e or X **pointing inward** depending on whether the flow of X would displace points out of Σ or into the interior of Σ.

Definition 2.17. *Let Σ be a surface endowed with a foliation $\mathcal{F}$ that is tangent to the boundary of Σ. We call $\mathcal{F}$* ***essentially planar****, if there is a decomposition of Σ into two subsurfaces Σ^+ and Σ^-, which are planar and $\mathcal{F}$ points out of Σ^+ and into Σ^- along the boundary which is in the interior of Σ.*

Our first two examples, where essentially planar. For the sphere it would have sufficed to take the equator and on the singular foliation of the torus from Example 2.8, set $\Sigma^- = \{(\phi, \theta) : \pi \geq \theta \geq 0\}$ and $\Sigma^+ = \{(\phi, \theta) : 2\pi \geq \theta \geq 0\}$: Recall the vectorfield:

$$X = -\frac{sin(\phi)sin(\theta)}{r}\frac{d}{d\phi} + \frac{cos(\theta)}{(R + rcos(\phi))}\frac{d}{d\theta}$$

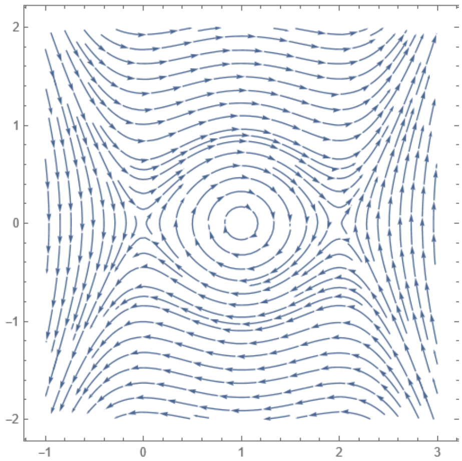

Figure 2.5: A triple of singularities with two saddles at $(0,0)$ and $(2,0)$ such that two separatrices connect. Note that each orbit in the interior is a degenerate closed orbit or the singularity at $(1,0)$. This may still happen, if the singularity was non-degenerate. Consider adding $X' = H((x-1)\frac{d}{dx} + y\frac{d}{dy})$ to the vectorfield generating this foliation, where H is a function with support inside the polygon. Then the singularity at $(2,0)$ becomes non-degenerate, however orbits sufficiently close to the orbit remain closed. So infinitely many closed orbits may appear even though there are only saddles and nodes present.

Then one notices that at $\theta = \pi$ the vectorfield is strictly negative in the θ-direction and for $\theta = 0, 2\pi$ the vectorfield is strictly positive in the θ-direction. Thus it points into Σ^- and out of Σ^+.

Recall, that the type of closed orbit is determined by the integral over the divergence. Using this, we may calculate the divergence of the non-parallely and non-singularly foliated torus from Example 2.11. We will do so for the closed orbit at $\theta = \frac{\pi}{2}$:

$$X_2 = \frac{d}{d\phi} + \frac{cos(\theta)}{(2 - sin(\phi))sin(\theta)}\frac{d}{d\theta}$$
$$div(X_2)(\cdot, \frac{\pi}{2}) = -\frac{1}{2 - sin(\phi)}$$

Thus the integral is negative, so the closed orbit at $\theta = \frac{\pi}{2}$ is attractive.

As it turns out one can remove all degenerate cases by an arbitrarily small change of the foliation and they exist in abundance:

Theorem 2.18. *(Peixoto's density theorem, [19, Theorem 2]) Any foliation $\mathcal{F}$ on a closed surface Σ, can be changed C^∞-small such that the resulting foliation $\mathcal{F}'$ fulfills the following:*

(i) All singularities are either saddles or nodes;

(ii) All closed orbits are non-degenerate;

(iii) All leafs limit to either a singularity or a closed orbit;

(iv) There is no leaf forming a saddle-saddle connection.

*Such foliations are called **Morse-Smale** foliations. Furthermore the set of Morse-Smale foliations form an open and dense subset of all foliations on a closed surface.*

QED

Note that the third condition implies the Poincaré-Bendixson property. Morse-Smale foliations will reappear again later though they are too restrictive for our purposes, so we will work with another property.

2.2 Non-isochore and divided foliations

In this section, we will discuss the relationship between two different foliations: Non-isochore foliations and divided foliations. The goal of this section is to provide a full proof of a claimed theorem by Giroux which seems to be missing from the literature up to now.

Definition 2.19. *Let Σ be a surface and $\mathcal{F}$ a foliation. Let p be a singularity then we say that p is a **positive/negative** singularity, if the divergence of $\mathcal{F}$ at p is positive/negative. If all singularities of $\mathcal{F}$ are either positive or negative, then we call $\mathcal{F}$ **non-isochore**.*

Recall that we have already encountered three types of singularities: nodes, whose determinant is non-zero and the real parts of the eigenvalues have the same sign. Saddles, whose determinant is non-zero and the eigenvalues have opposite sign. Saddle-nodes whose determinant vanishes, however there is one non-zero eigenvalue and the saddle-node is infinitesimally quadratic in the direction of the 0 eigenvalues.

So in fact, nodes and saddle-nodes always have a sign attached to them, while saddles might not. However, one can upgrade Peixoto's density theorem so that one has a non-isochore foliation: First take a Morse-Smale foliation and then around each saddle add a $H\epsilon x\frac{d}{dx}$ term, where H is some smooth cut-off function which is supported inside a small neighborhood of p, 1 inside a smaller neighborhood of p and $\frac{d}{dx}(p)$ is an eigenvector. Thus the eigenvalues are no longer negatives of one another and since the Morse-Smale condition is open, we can choose ϵ small enough so that it still fulfills the Morse-Smale conditions.

However, we will necessarily encounter some foliations which have more general non-isochore singularities. We will consider one typical example:

Example 2.20. Let $\mathcal{F}$ be a foliation on $T^2 = (S^1)^2$ given by $X = cos(\theta)\frac{d}{d\theta}$, where θ is the coordinate on the first copy of S^1. At $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{3}\}$ there is a line of singularities. Indeed their linearisation is given by:

$$DX = \begin{pmatrix} -sin(\theta) & 0 \\ 0 & 0 \end{pmatrix}$$

So, the determinant of the linearisation of these two lines of singularities vanishes while the traces do not so they come with an associated sign. Note that this foliation still fulfills the Poincaré-Bendixson property, since each orbit is a subset of the parallel line $\{(\theta, x)\}$ where x is an arbitrary value of S^1.

In general, the non-vanishing of the linearisation of a singularity implies that there is at least one non-zero eigenvalue. This in turn implies that for each singularity p of a foliation X there is a neighborhood U_p of p and a 1-dimensional submanifold l_p such that any other singularities in U_p most lie on l_p. Even more p divides l_p into two components, if there are orbits in U_p limiting to p in the "wrong" direction then they must be a subset of l_p:

Assume that $p = (0, 0) \in \mathbb{R}^2$ is not nodal. Then there is at least one non-zero real eigenvalue $\lambda \neq 0$, since the divergence of p does not vanish. After a possible chart-change, we may assume that the eigenvector corresponding to λ at p is $\frac{d}{dy}$. Possibly shrinking the area of definition of $X : U \to \mathbb{R}^2$, we may assume that the y-derivative of the second component of X does not vanish. Then denote by $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ the projection onto the y-coordinate, so the second derivative of $\pi_2 \circ X : U \to \mathbb{R}$ does not vanish. Thus 0 is a regular value of this map and so by the implicit function theorem $l := (\pi_2 \circ X)^{-1}(0)$ is a 1-dimensional submanifold of U. So singularities in a neighborhood of $p = (0, 0)$ may at most lie along l.

We also need to obtain control over the number of connections between singularities of the same sign: So assume that l' is a smooth arc such that X is parallel to l' and $p \in l'$. For easy of argument assume that p is a positive singularity and λ is the positive eigenvalue. If $\frac{d}{dy}(p) \in T_p(l')$ then denote by Z the vectorfield orienting l' such that $Z(p)$ is a positive multiple of $\frac{d}{dy}(p)$. Since X is paralell to l' it is a multiple of Z. Since $\frac{d}{dy}(p)(X) > 0$ one can observe that the coefficient function g such that $gZ = X$ has to have positive derivative at p. So X is pointing away from p. So l cannot contain any incoming orbits into p. So assume that $\frac{d}{dy}(p)$ is transversal to l' at p. Then on a small neighborhood of p in l' $\frac{d}{dy}$ is also transversal. So we can consider a neighborhood of $p \subset U$ with coordinates given by l' in the form (x', y) where x' is the coordinate on l' and y' is obtained by considering the flow of $Fl_{\frac{d}{dy}}(x')$. Now at $(x, 0)$ X is a multiple of $\frac{d}{dx}$ and the derivative of X in the direction of $\frac{d}{dy'}$ is a positive number at $p = (0, 0)$. Possibly shrinking the neighborhood we may assume that this derivative is positive on the whole neighborhood. Now we may observe that $X = g\frac{d}{dx'} + h\frac{d}{dy'}$ where $h > 0$ if $y' > 0$. So the flow originating at the point (a, b) with b positive has increasing second coordinate. So it may at most happen that flow has p as unstable limit set but not as positive one. Similarly for b negative and if p is a negative singularity.

If l' has p as a boundary component then one can repeat the same arguments as above on the half neighborhood U' which is obtained by considering splitting U into two components along $\gamma = \{(x, y) : x = 0\}$. If there are singularities on these possibly shrunken neighborhoods they have to lie on these arcs l' by the above arguments. So one observes that each singularity p has a neighborhood U where the only orbits which can limit to a singularity in the "wrong" direction lie along a curve l' which may be broken at $p \in l'$. In particular, we may choose U to be a neighborhood of p where the divergence already fulfills $sgn(div(X) = sgn(div(X(p)))$ and each such U has at most two flow lines which connect the outside of U with a singularity inside U.

We now come to one of the most important types of non-isochore foliations.

Definition 2.21. *Let Γ be a set of compact properly embedded arcs on a surface Σ. Furthermore let $\mathcal{F}$ be a non-isochore foliation. Then we say that Γ is a **dividing set** for $\mathcal{F}$ or Γ **divides** $\mathcal{F}$ if the following conditions are satisfied:*

(i) Γ is the regular preimage of $div_\omega(X)^{-1}(0)$ for some choice of orientation X of $\mathcal{F}$ and some volume form ω of Σ;

(ii) The foliation $\mathcal{F}$ points out of $R_+ = div_\omega(X)^{-1}([0, +\infty))$ and into $R_- = div_\omega(X)^{-1}((-\infty, 0]))$

So in particular, R^+ contains all positive singularities and R^- all negative singularities. In addition, since any orbit leaving R^+ never re-enters it, any closed orbit must either be completely contained in R_+ or R_-. Since the integral over the divergence on a degenerate closed orbit has to vanish a divided foliation cannot include any. Similarly, attractive closed orbits have to be in R_- and repelling closed orbits in R_+. Thus, we will also refer to attractive/repelling closed orbits as negative/positive.

The other important observation is that there are no orbits from the negative into the positive area. In particular, there are no **retrograde** connections which limit from a negative singularity to a positive singularity. A claimed theorem of Giroux [11, Proposition 2.5.], now reads as:

Theorem 2.22. *Let Σ be a surface endowed with a non-isochore foliation $\mathcal{F}$ which fulfills the Poincaré-Bendixson property and which is parallel to the boundary. Then the following are equivalent:*

(i) The foliation is divided.

(ii) The foliation does not contain any retrograde connections and no degenerate closed orbits.

One direction is clear by the above discussion. The other direction is a bit more subtle. This proof is based upon the one presented by Honda [14, Proposition 3.1.], there he proves the special case of non-isochore Morse-Smale foliations. A direct corollary of the Poincaré-Bendixson property is the following simplification: Any polygon which is a limit set consists of at most finitely many singularities. In addition since there can be at most 2 orbits which enter a singularity in the "wrong" direction the number of all orbits that these singularities have is bounded above. Moreover, since there are no retrograde connections the only possible polygons have either all positive or all negative singularities. Thus we will refer to these polygons as positive, respectively negative polygons. Even better positive polygons cannot be stable limit sets and negative polygons cannot be unstable limit sets, as is the case for positive/negative closed orbits.

The proof below is based upon an observation by Sotomayor [21, p. I.4.6.]. He provided the argument for polygons with one corner which is a saddle.

Lemma 2.23. *Let $\mathcal{F}$ be a foliation on a surface Σ. Then there are no points p whose stable limit set is a positive polygon. Similarly there are no points p whose unstable limit set is a negative polygon.*

Proof. Let l be a separatrix which is part of a positive polygon. Then one can construct an analogue of a Poincaré-Return map [21, p. I.4.6.]: There is an arc a which is transversal to the foliation and one endpoint is a point $p \in l$. Then one can define $\pi : a \backslash \{p\} \to a \backslash \{p\}$, by defining $\pi(x)$ to be the next intersection of the leaf originating from x with a.

The next observation is that given a volume form ω one can find a function $f : \Sigma \to \mathbb{R}_{>0}$ such that the divergence of $div_\omega(fX) > 0$ along the leafs forming the polygon. So let X be some vectorfield orienting $\mathcal{F}$ and let γ be a leaf which limits from one corner of the polygon p to another p'. For both of these we may find neighborhoods U_p and U'_p such that the divergence of some X is positive on both U_p and $U_{p'}$. Then we additionally observe that $\gamma : \mathbb{R} \to \Sigma$ fulfills that there is a sufficiently big R such that $\gamma([R, +\infty)) \subset U_{p'}$ and $\gamma(-\infty, -R]) \subset U_p$. Thus we have that $\int_\gamma div(X)\gamma = +\infty$. In particular (by possibly increasing R), we can assume $K := \int_{-R}^{R} div(X)(\gamma(s))ds > 0$. By Lemma 2.9 we need to solve $df(X) > -f div_\omega(X)$ to achieve that $div_\omega(fX)$ is strictly positive. ODE theory tells us a trivial solution to this problem by choosing:

$$f(s) = e^{\frac{s+R}{2R}K - \int_{-R}^{s} div(X)(\gamma(s))ds}$$

One observes that $f(-R) = f(R) = 1$ so, we may extend f to the rest of Σ by choosing a smooth cut-off function which smoothly normalises it to 1 outside a small neighborhood of the strip $[\gamma(-R), \gamma(R)]$ which is disjoint from the other edges of the polygon. We do this along all edges of the polygon.

Then consider the area Σ' formed by the positive polygon, the leaf γ' between x and $\pi(x)$ where $x \in a \setminus \{p\}$ and the piece a' of a between x and $\pi(x)$, compare Figure 2.6. This is a closed, planar area whose boundary is always part of the foliation, except along A'. The induced orientation of A' is in such a way that it forms a positive basis with X in the first slot if the flow lines leave Σ' along A', respectively with $-X$ if flow lines enter Σ' along A'. Thus, we observe by Stoke's theorem:

$$\int_{\Sigma'} div_\omega X = \int_{\partial \Sigma'} \iota_X \omega = \int_{A'} \iota_X \omega$$

The first integral is positive, and thus must be the third one. This means in particular, that there is at least one point along A' where the vectorfield X points out of A'. Since A was transversal to the foliation this has to be true for all of them. Thus $\pi(x)$ is farther away along A than x. This implies that all orbits close to γ must limit away from γ. Since the next intersection will either not hit A (which all close enough orbits on this side of the polygon do) or be farther away than the previous one.

QED

Since the point x has to move outward along a, we similarly observe that there are no closed orbits close to a positive/negative polygon:

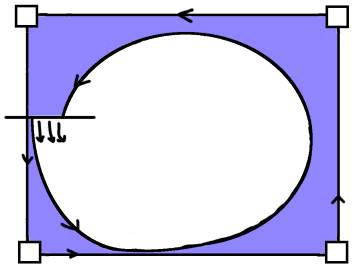

Figure 2.6: A polygon with 4 positive saddles as corners. The violet area is Σ'.

Corollary 2.24. *Let $\mathcal{F}$ be a non-isochore foliation which fulfills the Poincaré-Bendixson property. Assume that all polygons and closed orbits are either positive or negative. Then there are at most finitely many closed orbits.*

Proof. Since positive/negative orbits are isolated, closed orbits may not accumulate on each other. By the previous proof there are no closed orbits close to positive, resp. negative polygons. Since in this case $x = \pi(x)$ and so the integral would have to vanish. QED

Lemma 2.25. *Let $\mathcal{F}$ be a non-isochore foliation on a surface Σ which fulfills the Poincaré-Bendixson property. Furthermore let ω be a choice of volume form for Σ. Then there is an X orienting $\mathcal{F}$ such that the divergence of X is positive/negative on positive/negative singularities, repelling/attracting closed orbits, positive/negative polygons and orbits connecting such sets of the same sign.*

Proof. We will show the statement only for the positive case, the negative case follows analogously.

Let X be any vectorfield orienting $\mathcal{F}$. The case of positive singularities follows trivially. In addition, the set of singularities of X is a compact subset of Σ: Both the images of the vectorfield X and the 0-vectorfield are compact in $T\Sigma$, so their intersection is as well. Since the projection $T\Sigma \to \Sigma$ is continuous, the image of this compact intersection is also compact. Thus we may cover the set of singularities with finitely many neighborhoods U such that the divergence does not vanish on U and the foliation is nonsingular on U except perhaps along a line l_U, which follows from the discussion after Example 2.20.

One observes that the amount of orbits between positive singularities which are not fully contained in one of these U is at most finite. If the leaf γ between two singularities p and p' is contained in such a U as above then the divergence is already strictly positive along it. So let γ limit between singularities p and p' in different U and U'. Using the construction in the proof of Lemma 2.23, we obtain a vectorfield X such that $div_\omega(X) > 0$ along γ. Choosing the cut-off function to be disjoint from the lines l_U and the finitely many other orbits between positive singularities, one can iteratively achieve that $div_\omega(X)$ is positive on all orbits between singularities.

Since positive closed orbits C fulfill that the integral of the divergence of X is positive, thus we may use a similar construction to obtain strictly positive divergence along C. Since there are at most finitely many orbits and finitely many U_p and finitely many orbits connecting singularities not entirely contained in U_p, we can choose the cut-off functions to be disjoint from these sets.

At last, we can repeat the same argument for orbits limiting from a positive polygon/ or positive closed orbit to a positive singularity. QED

So given a non-isochore foliation $\mathcal{F}$ without retrograde connections and degenerate closed orbits which fulfills the Poincaré-Bendixson property it has a representative X such that the divergence with respect to a fixed volume form ω is positive/negative on a neighborhood of the positive/negative singularities, positive/negative closed orbits, positive/negative polygons and orbits connecting them. So, we only have to extend this to the complement such that the divergence is 0 exactly along a 1-dimensional line. To do so, we observe that the already positive region retracts onto a neighborhood where the foliation is pointing outward, similarly for the negative region. So the complement of these retracts Σ' has a foliation which is nonsingular, contains no closed orbits, fulfills the Poincaré-Bendixson property and has boundary components which are either completely pointing outward (where it borders the negative region), completely pointing inward (where it borders the positive region) or transversal (where the boundaries of Σ and Σ' coincide). Thus, we must classify all those polygons which support such foliations:

Theorem 2.26. *Let Σ be a surface with polygonal boundary. Assume that there is a foliation $\mathcal{F}$ on Σ, which is nonsingular, contains no closed orbits, fulfills the Poincaré-Bendixson property and each edge, respectively smooth boundary component is either purely transversly or parallely foliated. Additionally no edges with in- and outgoing foliation are neighboring each other. Then Σ is either an annulus (possibly with polygonal boundary) where one boundary is strictly ingoing and one is strictly outgoing or a polygon with exactly two parallely foliated edges, compare Figure 2.7.*

Proof. The proof below is reconstructed from a discussion on math.stackexchange [20]. The discussion there was about surfaces with smooth boundaries, however if one is careful about removing vertices of polygonal boundaries the proof works in essentially the same way.

The main idea of the proof is to show that, we can turn a nonsingular foliation on Σ into a nonsingular foliation on a closed surface Σ''. Every step will either leave the euler characteristic unchanged or decrease it. Thus in the end, we can use the characterisation that every Σ is obtained from a closed genus g surface and the Euler characteristic of Σ calculates as $2 - 2g - (c + p)$, where $c + p$ is the sum over the circular and polygonal boundary components. Since the Euler characteristic has to vanish for a closed surface to support a nonsingular vectorfield, this leaves only the torus Σ''. Thus $2-2g-(c+p) \geq 0$ since we do not increase the Euler characteristic when building Σ''. The exact analysis is done at the end.

There are two main observations: First since Σ contains no closed orbits no smooth boundary component is parallely foliated, since this would be a closed orbit. Secondly,

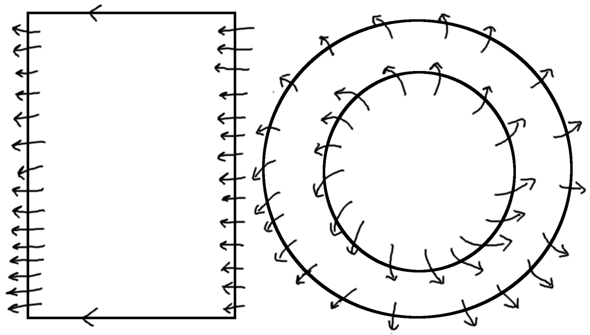

Figure 2.7: The only two surfaces (up to the addition of corners) which support a foliation which is nonsingular, circlefree and fulfills the Poincaré-Bendixson property.

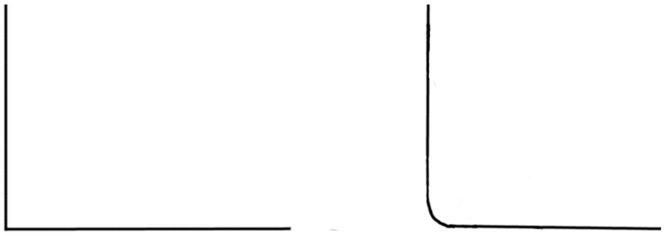

Figure 2.8: The procedure to smoothen out a corner. The black line represents the new boundary of Σ'.

a parallely foliated edge e of a polygonal boundary component bounds on one-side an ingoing boundary and on the other side an outgoing boundary: In coordinates let $(0,0)$ be a corner of the edge e (associated to $\{(x,y) : x \geq 0, y = 0\}$) then the foliation along $e_1 = \{(x,y) : y \geq 0, x = 0\}$ induces the sign of X along e. At $(0,0)$ $X = \phi \frac{d}{dx}$ and ϕ must be positive if e_1 is foliated inward. Since X is non-vanishing, the direction of X at the other corner p' of e is already defined. However in this case X has to point towards p', which means that the foliation has to be outward pointing on e_2 the edge neighboring p'.

The first step is then to remove non-essential corners, which are between edges that are both inward or outward pointing. Let p be such a corner where both edges meeting p are inward pointing. Again in coordinates let $p = (0,0)$ and the edges be the coordinate axis. Then we can consider the following alternative boundary curve:

$$\gamma(t) = \begin{pmatrix} H(t)(t+\epsilon) \\ H(-t)(-t-\epsilon) \end{pmatrix}$$

where H is some smooth cut-off function which is 1 if $t > \epsilon$ and 0 if $t < -\epsilon$, where $1 > \epsilon > 0$ is some number, compare Figure 2.8. One may calculate that γ has non-vanishing derivative and is strictly contained in the first quadrant. Thus we can consider Σ' which is the surface on the opposite side of γ. This surface has one corner less. However, we need to make sure that X is still pointing outward, for this we observe that originally at $p = (0,0)$ we had $X = \phi \frac{d}{dx} + \psi \frac{d}{dy}$, where both $\phi, \psi < 0$ at $p = (0,0)$ so we may have just assumed that this was true in the whole neighborhood. Thus one can try to solve the following equation:

$$\begin{pmatrix} \dot{\gamma}_x \\ \dot{\gamma}_y \end{pmatrix} = aX(\gamma)$$
$$a = \frac{\dot{\gamma}_x}{\phi(\gamma)}$$
$$a = \frac{\dot{\gamma}_y}{\psi(\gamma)}$$

where the first equation implies that a has the same sign as $\dot{\gamma}_x$ and the second that a has the same sign as $\dot{\gamma}_y$, which can only be satisfied by $a = 0$. However $a = 0$ implies that the derivative of γ needs to vanish, which does not happen. One can indeed show that if $t > -1 - \epsilon$ then the first derivative is non-zero. And for $t \leq -1 - \epsilon$: γ coincides with $(0, -t)$ for which $\dot{\gamma}_y$ is non-vanishing. Now, we know X' is either inward or outward pointing. For $t = -1$ one can now see that $\dot{\gamma}$ is $(0, -1)$ thus $X'(0, -1) = (2, 1)$ is pointing inward. Removing corners from outward pointing edges works analogously.

So we may assume that the only corners on Σ are between edges where the foliation is transversal and parallel. Now since transversal edges separate inward and outward pointing edges there must be an even number of them. So let $2l$ be the amount of transversal edges. Now around each outward pointing edge e there are two transversal edges e' and e''. We can glue these edges to one another. In such a way that the corner p' which separates e and e' and p'' which separates e and e'' are identified. In this way, we increase the amount of boundary components by 1, since e is now a closed smooth circle. On a neighborhood of e', respectively e'' we can normalise X (possibly changing it to agree with the vectorfield orienting e' on a neighborhood of e') and obtain that X glues to a smooth vectorfield which has no closed orbits nor singularities and along the smoothened edge e the vectorfield is outward pointing, while on the remaining polygonal boundary, where e, e', e'' originally belonged to there are now 2 transversal components less, compare Figure 2.9.

Iterating this process, yields a change in the euler characteristic $\chi(\Sigma') = \chi(\Sigma) - l$. However Σ' has only transversally foliated smooth boundary components now, of which there are exactly $c + p + l$-many where c is the original number of smooth boundaries and p was the amount of polygonal boundary components. Now change X close to the boundary components so that the derivative with respect to some neighborhood $C \times \mathbb{R}_+$ vanishes, i.e. it agrees with the $\frac{d}{dr}$, where r is the coordinate on $\mathbb{R}_+$. This can again be done in such a way that X remains nonsingular. Now glue a second copy of Σ' to Σ' (where the orientation on the second copy has to be reversed). This new surface Σ'' now has a nonsingular foliation induced by gluing X and $-X$ (on the second copy of Σ') together. This will be a smooth vectorfield since we smoothed X close to the boundary, by using the neighborhood $C \times \mathbb{R}_+$ to glue Σ' to itself we ensure that they fit together.

Now $\chi(\Sigma'') = 2\chi(\Sigma')$ and Σ'' is now a closed surface with a nonsingular foliation, thus

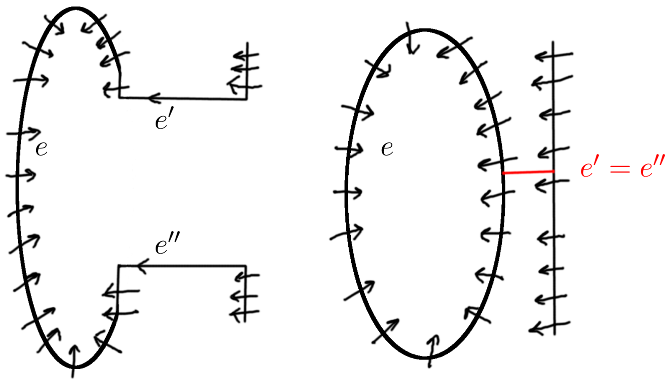

Figure 2.9: Gluing two edges together. Before the gluing e was an edge where the vectorfield was pointing outward. Afterwards e is a smooth boundary component such that the vectorfield is still pointing outward.

it must be the torus. So we have to find all solutions to the following equation:

$$\begin{aligned} 0 &= 2((2-2g)-(c+p+l)) \\ 0 &= 4-4g-2c-2p-2l \\ 4 &= 4g+2c+2p+2l \end{aligned}$$

If $g=1$, then $c=p=l=0$ thus Σ would have been a torus. The foliation on the torus was originally Poincaré-Bendixson, nonsingular and free of closed orbits, which cannot be. Since orbits need to either limit to a singularity, a closed orbit or a polygon.

So $g=0$, this leaves the options of $c=2$ and $p=l=0$, $c=1$ and $p=1, l=0$, $c=0$ and $p=1, l=1$. The first two options are annuli where possibly one boundary was polygonal, the second option is a polygon with exactly two transversal edges.

The last claimed statement is that there have to be both ingoing and outgoing edges, respectively circles. This follows again from the Poincaré-Bendixson property: If there where only ingoing boundaries then all orbits in forward time would be trapped inside surface and thus there must be some kind of limit set. However, if there are no singularities and no closed orbits then the limit sets cannot exist, which is a contradiction.

QED

Proof of Theorem 2.22. By Lemma 2.25, we may assume that $div(X) \neq 0$ on a neighborhood of all positive/negative singularities/polygons/closed orbits and orbits connecting these. So we only need to find an f such that $div(fX)^{-1}(0)$ divides the foliation on the complement Σ' which by Theorem 2.26 consists of annuli and strips. Using the methods used in the proof of Theorem 2.26, we may remove all edges and assume that the annuli are regular $[-1,1] \times S^1$ and the strips are $([-1,1])^2$. In addition, we choose coordinates such that the foliation is given by $\frac{d}{dx}$ where x is the first coordinate. Then, recall that $div(fX) = f div(X) + df(X)$ by Lemma 2.9 and so we must only choose f such that it fulfills the following ODE:

$$f' = (c + ax - div(X))f \tag{2.2.1}$$

$$c = \int_{-1}^{1} div(X)$$
$$|c| < a$$

which has a solution, which depends smoothly upon the second coordinate. After possibly smoothening f out close to the boundary. One easily verifies, that $f' + f div(X) = div(fx)$ has a regular 0-set. In addition, by the choice of f' the $\frac{d}{dx}$-vectorfield on each annulus and strip is transversal to this 0-set. Thus, we have verified all claimed statements. QED

Up to now, we started with a fixed volume form ω and found an X such that the divergence of X with respect to ω leads to a divided foliation. However, this property is independent of the choice of X. So let X be some vectorfield and ω some volume form. Then using the above procedure, we may find $f : \Sigma \to \mathbb{R}_{>0}$ such that $div_\omega fX$ is of the desired form. Using Lemma 2.9:

$$f div_{f\omega}(X) = div_\omega(fX)$$

we observe that X is adapted to $f\omega$ in the above way. This is not surprising since $\iota_{fX}\omega = \iota_X(f\omega)$ lead to the same 1-form. There is another description of a divided foliation which can be found in the convex surfaces lecture notes by Etnyre [6, Theorem 2.15.]. The proof presented below is based upon his arguments:

Lemma 2.27. *Let Σ be a surface endowed with a non-isochore foliation $\mathcal{F}$. The foliation $\mathcal{F}$ is divided if and only if for each β representing $\mathcal{F}$ there exists a function $u : \Sigma \to \mathbb{R}_{>0}$ such that $\beta \wedge du + ud\beta > 0$.*

Proof. First assume that $\mathcal{F}$ is divided then there is a volume form ω and a vectorfield X representing $\mathcal{F}$ such that 0 is a regular value of $div_(X)$. So let $\beta = \iota_X\omega$ and $u = \pm(div_\omega X)$ outside a sufficiently small neighborhood of $\Gamma = div_\omega(X)^{-1}(0)$ on a tubular neighborhood of the latter let u be a smooth transition function between ± 1 which decreases along leafs and is 0 exactly on $div_\omega(X)^{-1}(0)$. This is possible since the foliation is transversal to the 0-set. Then we observe the following:

$$\begin{aligned} ud\beta &= udiv_\omega X\omega \geq 0 \\ ud\beta &= 0 \Leftrightarrow u = d\beta = 0 \\ \beta \wedge du(X, Z) &= i_X\omega \wedge du(X, Z) = \omega(X, X)du(Z) - \omega(X, Z)du(X) \end{aligned}$$

The last inequality says precisely that if (X, Z) is a positive basis at some point p then $\omega(X, Z)du(X)$ has the sign of $-du(X)$. Our choice implies $du(X) \leq 0$, where $u(X) = 0$ outside a small neighborhood of the dividing curve. The above formula holds if $X \neq 0$

and if $X = 0$ then $\beta = \iota_X \omega = 0$. Thus both 2-forms are non-negative multiples of the volume form and for each point on Σ at least one of them is non-vanishing, which is exactly the statement we wanted.

On the other hand, assume that β represents $\mathcal{F}$ and there was a function $u : \Sigma \to \mathbb{R}_{>}0$ such that $\omega := ud\beta + \beta \wedge du > 0$ is a positive volume form. Then let X be dual to β ($\iota_X \omega = \beta$). Then one readily verifies:

$$div_\omega(\frac{1}{u}X) = \frac{1}{u^2}$$

So in particular, if $u \neq 0$ then $div_\omega(\frac{sign(u)}{u}X) = \frac{sign(u)}{u^2}$ which leads to an X' which almost has the correct divergence, however close to $u^{-1}(0)$ we need to do something different: Recall that if $u = 0$ then $\omega = \beta \wedge du > 0$, so in particular the vectorfield X orienting $\mathcal{F}$ is transversal to $u^{-1}(0)$, which in particular makes it into a submanifold. Now consider around each $u^{-1}(0)$ push-offs given by X which are so small that the neighborhood is nonsingular. Let $g = \frac{sign(u)}{u^2}$ outside these collections of annuli and strips around $u^{-1}(0)$ and on a neighborhood of their boundaries. Then consider an extension of g onto the interior as in Equation 2.2.1 where we choose the constants in such a way that they are exactly 0 if $u^{-1}(0)$. Thus, we have realised $u^{-1}(0)$ as the desired dividing curve. QED

2.3 Generic 1-parameter families of foliations

In this section, we will present the generalisation of Peixoto's density Theorem to families of foliations indexed by an interval. I.e. we consider $(X_r)_{r \in [-1,1]}$, where X_r is a vectorfield on a closed surface Σ. We recall Peixoto's density Theorem 2.18:

Let Σ be a closed surface. Then the set of Morse-Smale vectorfields X lies dense and open in the space of all vectorfields.

In other words, given a vectorfield X there is a C^∞-small vectorfield X' such that $X + X'$ is Morse-Smale. In addition, if X is Morse-Smale then $X + X'$ will be Morse-Smale for any sufficiently small vectorfield X'. Naturally, this cannot be true for families of vectorfields indexed by an interval. Consider for example some family where X_{-1} and X_1 are Morse-Smale and X_1 has two singularities more than X_{-1}. Then there is no C^∞ change of X_1 which removes these additional singularities, so for some t the singularities need to be brought into existence which necessarily needs to be a non-Morse Smale foliation. In this section, we will discuss Sotomayor's density Theorem, we will not go into any details and only discuss the so-called generic bifurcations. Most of this section is based upon Sotomayor's original work [21].

Definition 2.28. *Let X be a vectorfield on a closed surface Σ whose induced foliation $\mathcal{F}$ satisfies the following conditions:*

(i) All singularities of $\mathcal{F}$ are either saddles or nodes.

(ii) All closed orbits are non-degenerate.

(iii) There are no saddle-saddle connections.

*Then X is called a **Kupka-Smale** vectorfield.*

Note that the above notion is strictly weaker than Morse-Smale: A Kupka-Smale vectorfield which satisfies the Poincaré-Bendixson property is Morse-Smale. On the other hand the irrationally foliated torus from Example 2.14 is Kupka-Smale but not Morse-Smale.

Definition 2.29. *Let $(X_r)_{r\in[-1,1]}$ be a family of vectorfields on a closed surface Σ. We call $r \in [-1,1]$ an **ordinary value** of (X_r) if for all sufficiently close values r' to r there is a homeomorphism $\Sigma \to \Sigma$ carrying flow lines of X_r onto flow lines of $X_{r'}$.*

*If r is not ordinary, then r is called a **bifurcation value**.*

A trivial observation is that since Morse-Smale vectorfields are an open subset of all vectorfields that if a vectorfield X_r is Morse-Smale for some $r \in [-1,1]$ then r is an ordinary value. However, if X_r is only Kupka-Smale then it is not necessarily ordinary, as the example of the rationally/irrationally foliated torus from Example 2.14 shows. Considering a constant family of irrationally foliated tori leads to ordinary values. So one cannot trivially exclude them. We call this phenomenon the **Poincaré-Bendixson bifurcation**, for bifurcation values r when a vectorfield is Kupka-Smale, but not Morse-Smale.

Example 2.30. (saddle-node bifurcation) This special bifurcation is a phenomenon, we have already observed at the very beginning in Example 2.11: Let r be a value of $[-1,1]$ such that X_r is Morse-Smale, except for a single saddle-node p. Denote by U a small neighborhood such that p is the only singularity in U. Then we call r a **saddle-node bifurcation** if there is a neighborhood $(r-\epsilon, r+\epsilon)$ of r such that either U contains a saddle and a node for $r' < r$ or $r' > r$ and U contains no singularities for $r' > r$,respectively $r' < r$.

Example 2.31. (focus bifurcation) As it will turn out this bifurcation cannot happen in our setting. So we will not go into too many details:

Let r be a value of $[-1,1]$ such that X_r is Morse-Smale, except for a singularity p with two complex eigenvalues with vanishing real part, so in particular the trace of p vanishes. If p satisfies an additional non-degeneracy condition(see Sotomayor [21, Definition I.3.11.]) then we call r a **focus bifurcation**, if p has a neighborhood U such that p is the only singularity of U and there is a neighborhood $(r-\epsilon, r+\epsilon)$ of r such that U contains a single node with non-vanishing real part if $r' < r$ or $r' > r$ and U contains a non-degenerate closed orbit and a single node with non-vanishing real part if $r' > r$, respectively $r < r'$.

Note that the sign of the real parts has to flip during the focus bifurcation. Since, we do not provide all details, we give only an example below which looks like a focus bifurcation:

Consider the open unit disk $D^2 \subset R^2$. Then we define the following vectorfield X_r:

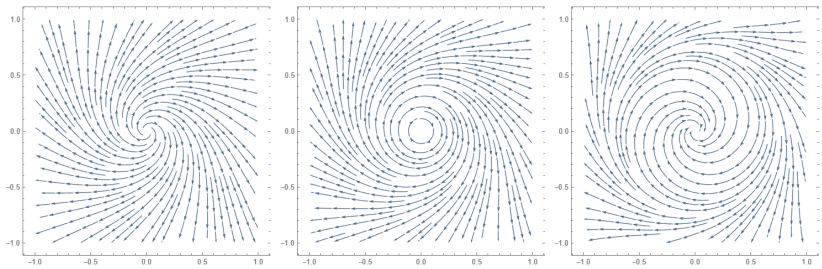

Figure 2.10: As r decreases the leafs close to the singularity start to rotate more and more. Until at $r = 0$ the singularity is no longer repelling. For $r < 0$ there is a repelling closed orbit and an attractive focus.

$$X_r = ((x^2 + y^2 + r)x + y)\frac{d}{dx} + ((x^2 + y^2 + r)y - x)\frac{d}{dy}$$

For $r \neq 0$, one observes that $p = (0, 0)$ has a Jacobi matrix with 0-s on the main diagonal and $-1, 1$ on the off-diagonal while for $r < 0$ the main diagonal is strictly positive and for $r > 0$ the main diagonal is strictly negative. For $r > 0$ there are no closed orbits, while for $r < 0$ there is a repelling orbit at $x^2 + y^2 = -r$, see also Figure 2.10.

Example 2.32. (birth-death bifurcation) Let X_r be a foliation which is Morse-Smale except for a single degenerate closed orbit C which is **quasi-generic** which means that the second derivative of the Poincaré-return map is non-vanishing. We then call r a **birth-death bifurcation**, if there is a non-singular neighborhood U of C and a neighborhood $(r - \epsilon, r + \epsilon)$ of r such that if $r < r'$(or $r > r'$) there are no closed orbits in U and if $r > r'$(or $r < r'$) there are two non-degenerate closed orbits, one attracting and one repelling, in U.

One such example is given on $\mathbb{R}^2$ by:

$$X_r = (((x^2 + y^2 - 2)^2 + r)x - y)\frac{d}{dx} + (((x^2 + y^2 - 2)^2 + r)y + x)\frac{d}{dy}$$

One observes that if $r > 0$ the vectorfield is always pointing outward, so there are no closed orbits. At $r = 0$ there appears a closed orbit at $x^2 + y^2 = 2$ which is necessarily degenerate. Then for $r < 0$ there are two closed orbits, one repelling at $x^2 + y^2 = 2 + \sqrt{-r}$ and one attracting at $x^2 + y^2 = 2 - \sqrt{-r}$, see also Figure 2.11.

Example 2.33. (saddle-saddle connection bifurcation) Let X_r be a foliation which is Morse-Smale except for a single saddle-saddle connection between p_1 and p_2 (possibly $p_1 = p_2$) where s_1 and s_2 denote the unstable separatrix of p_1 and the stable separatrix of s_2. We say that r is a **saddle-saddle connection bifurcation** if there is a neighborhood $(r - \epsilon, r + \epsilon)$ of r such that for $r' \neq r$ $(s_1)_r$ and $(s_2)_r$ do not coincide and there is an arc A transversal to s_1 such that the intersection point of A with $(s_1)_r$ (this exists by

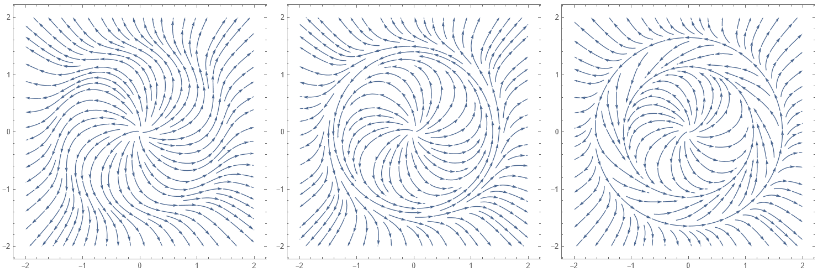

Figure 2.11: For $r > 0$ there are no closed orbits. As r decreases towards 0, the orbits close to $x^2+y^2 = 2$ start to spiral more and more, until at $r = 0$ they form a degenerate closed orbit. As r decreases further this degenerate closed orbit breaks into two non-degenerate closed orbits.

continuity) has non-vanishing derivative along A, compare Figure 2.12. An example of this is given by the following vectorfield:

$$X_r = (x^2 - 1)\frac{d}{dx} - (2xy + r(x^2 - 1))\frac{d}{d}dy$$

Note that for all r there are saddles at $(0, 0)$ and $(1, 0)$. For $r = 0$ the strip $\{(x, 0) : 0 < x < 1\}$ is part of the foliation. Thus the saddles are connected. For $r > 0$ the foliation always points downwards on this strip so the saddles cannot be connected. Similarly for $r > 0$.

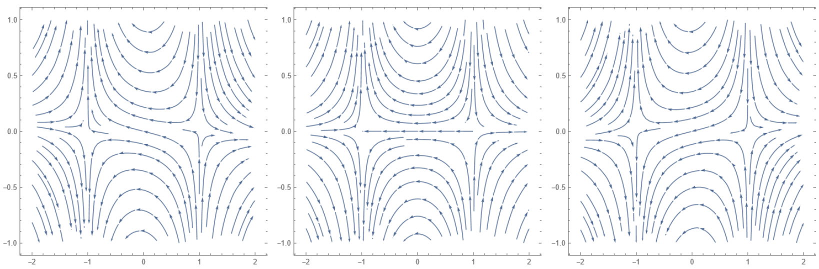

Figure 2.12: For negative r the separatrix of the saddle at $(1, 0)$ passes above the one at $(-1, 0)$. For $r = 0$ they conincide, until for $r < 0$ the separatrix of $(0, 1)$ passes below the one of $(-1, 0)$.

We differentiate between $p_1 = p_2$ and $p_1 \neq p_2$. The first kind is called a **loop** or sometimes a **homoclinic orbit**. If $p_1 \neq p_2$ then this connection is called a **heteroclinic orbit**.

A homoclinic orbit can be found using the equation $X = y\frac{d}{dx} + (x^2 - x)\frac{d}{dy}$. This is a classical example which is taken from the introductory book by Teschl [22, Problem 6.23.], for an illustration see Figure 2.13.

Now, we may state the main result of this section:

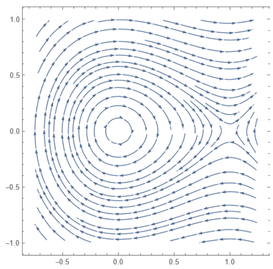

Figure 2.13: This is the standard example of a saddle where two separatrices of a saddle coincide, forming a homoclinic orbit.

Theorem 2.34. *(Sotomayor's density Theorem, [21, Theorem II.2.]) Consider the subset* $\mathfrak{X}_1$ *of* 1*-parameter vectorfields* $(X_r)_{r\in[-1,1]}$ *which fulfill the following properties:*

(i) The set of ordinary values J *of* (X_r) *is open and dense in* $[-1,1]$*. Furthermore* r *is an ordinary value if and only if* X_r *is Morse-Smale.*

(ii) Any bifurcation value r *is either a Poincaré-Bendixson bifurcation, a saddle-node bifurcation, a focus bifurcation, a birth-death bifurcation or a saddle-saddle connection bifurcation.*

Then $\mathfrak{X}_1$ *is dense in the space of all* 1*-parameter vectorfields.*

This result is probably the most important result in this section, as this is one of the key steps in showing the main Theorem of this thesis. Note that the set of bifurcation values J^c is the complement of an open dense set. So accumulation points of J^c must be bifurcation values. So it is of great importance to understand when and how they accumulate. Sotomayor has classified all accumulation conditions for each bifurcation except the Poincaré-Bendixson bifurcation:

Remark 2.35. (i) The set of saddle-node and focus bifurcations are isolated. ([21, Remark I.3.13.])

(ii) Let r be a birth-death bifurcation. Then there are 3 possibilities: ([21, Remarks I.2.8.b)-d)])

  a) r is isolated.

  b) r is an accumulation point of saddle-saddle connection bifurcations, if the quasi-generic closed orbit C is limit set of a stable and an unstable separatrix.

  c) r is an accumulation point of birth-death bifurcations, if the quasi-generic closed orbit C is both stable and unstable limit set of an orbit diffrent from C.

(iii) Let r be a saddle-saddle connection bifurcation. Then there are 2 possibilities([21, Remarks I.4.8.1.]):

a) r is isolated.

b) r is an accumulation point of saddle-saddle connection bifurcations, if the saddle-connection is contained in the limit set of some other saddle-separatrix.

3 3-dimensional contact topology

After the basic definition and some examples, we will introduce Moser's stablity trick in subsection 3.1. This is one of the most powerful tools in Contact Topology. Afterwards we will introduce the characteristic foliation (see Definition 3.20) of a surface embedded in a contact manifold and in subsection 3.2 discuss its connection to the surrounding contact structure.

Definition 3.1. *Let M be a smooth orientable 3-manifold, possibly with boundary. We call $\xi \subset TM$ a* ***contact structure*** *or a* ***contact distribution****, if there is a 1-form α such that for each $p \in M$ the planefield $\xi \cap T_x M =: \xi_p = \ker(\alpha_p)$ and $\alpha \wedge d\alpha$ is a positive volume form. A choice of α induces an* ***orientation*** *$d\alpha$ of ξ.*

Normally, such contact structures are called positively co-oriented contact structures and one drops the second condition (positivity) and only requires ξ to be the kernel of locally defined 1-forms(co-orientable). However, both conditions are usual assumptions for many applications. So we will not deal with these details.

One of the first observations, is that neither property depends on the exact α but are preserved under multiplication by non-vanishing scalar functions. However, if the scalar function is negative it reverses the induced orienation of ξ:

Lemma 3.2. *Let M be an orientable 3-manifold and ξ a contact structure on M. Furthermore let α be a 1-form positively co-orienting ξ and $f : M \to \mathbb{R} \setminus \{0\}$. Then $f\alpha$ is also a contact form with kernel ξ. If f is positive, then the orientation induced by $d\alpha$ agrees with the orientation induced by $d(f\alpha)$.*

Proof. Since f is non-vanishing, the kernels of α and $f\alpha$ coincide pointwise. Furthermore $f\alpha$ also gives rise to a positive volume form:

$$f\alpha \wedge d(f\alpha) = f\alpha \wedge (df \wedge \alpha + f d\alpha) = f\alpha \wedge (df \wedge \alpha) + f^2 \alpha \wedge d\alpha.$$

The first term vanishes, since $\wedge$ is skew-symmetric and thus if two entries agree it must be 0. The last term is then just a positive multiple of the positive volume form $\alpha \wedge d\alpha$ and thus also a positive volume form.

If f is positive, then $d(f\alpha) = df \wedge \alpha + f d\alpha$. If one inserts two vectors of ξ, the first term vanishes and thus the sign of $d\alpha$ and $d(f\alpha)$ agree. QED

Now we will discuss two classical examples on $\mathbb{R}^3$. First the standard contact structure and then the overtwisted contact structure. Afterwards, we will discuss an example of a contact structure arising from a foliation. Giroux has characterised when foliations give rise to contact structures, we will discuss this later.

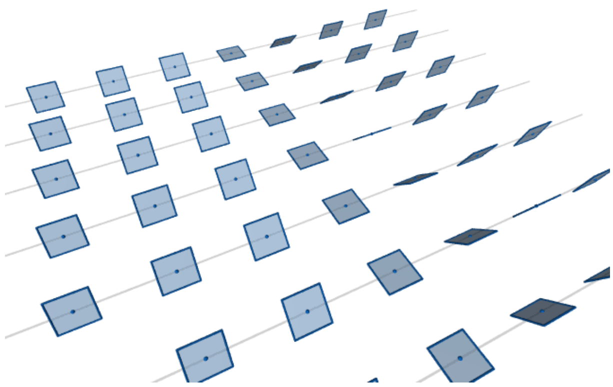

Figure 3.1: A visualisation of the standard contact structure. One notices that this structure is independent of shifts in the y and z directions. Along one ray $\{(x,y,z) : x \in \mathbb{R}^3\}$ the plane field does half a rotation in the limit. The image belongs to Patrick Massot and was published on his webpage [17].

Example 3.3. (Standard/tight $\mathbb{R}^3$) Consider $\mathbb{R}^3$ endowed with the contact structure ξ induced by $\alpha = dz + xdy$, where (x, y, z) are coordinates on $\mathbb{R}^3$. One easily observes that $d\alpha = dx \wedge dy$ and thus $\alpha \wedge d\alpha = dz \wedge dx \wedge dy = dx \wedge dy \wedge dz$. This is the so-called standard contact structure on $\mathbb{R}^3$, for a visualisation see Figure 3.1.

This contact structure is isotopic to the contact structure induced by $\alpha_1 = dz + xdy - ydx$, meaning that there is a diffeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\phi^*\alpha_1 = \alpha$. This diffeomorphism is given by:

$$\phi(x, y, z) = (\frac{x+y}{2}, \frac{x-y}{2}, z + \frac{xy}{2})$$

One readily verifies that this leads to the desired pullback.

Example 3.4. (Overtwisted $\mathbb{R}^3$) Normally the overtwisted $\mathbb{R}^3$ is given as the kernel of the 1-form $\alpha = cosrdz + rsinrd\phi$, where (r, ϕ, z) are cylindrical coordinates on $\mathbb{R}^3$. That this extends to a 1-form on all of $\mathbb{R}^3$ is shown in the Introductory book by Geiges [8, Example 2.1.6.]. In essence these are just some smoothness arguments at $r = 0$ and thus we will omit this. This contact structure is visualised in Figure 3.2

Example 3.5. (Overtwisted disk) Consider D^2 the unit disk in $\mathbb{R}^2$ endowed with the 1-form β given by:

$$\beta = (1 - x^2 - y^2)(-ydx + xdy)$$
$$X = (1 - x^2 - y^2)(x\frac{d}{dx} + y\frac{d}{dy})$$

where X is dual to β via $\omega = dx \wedge dy$. This induces a foliation with the following properties, see also Figure 3.3:

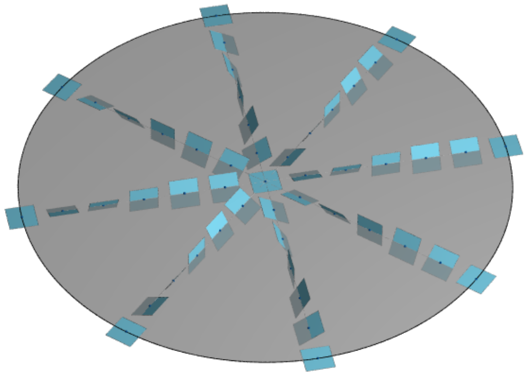

Figure 3.2: A visualisation of the overtwisted contact structure on the unit disk in the (x, y)-plane. In contrast to Figure 3.1 the contact structure does an infinite amount of full turns along each half-ray $\{(r, \theta, z) : r > 0\}$ for fixed θ and z. The image belongs to Patrick Massot and was published on his webpage [17].

(i) There is an isolated singularity at the origin $(x, y) = (0, 0)$.

(ii) There is a circle of singularities at $\{(x, y) : x^2 + y^2 = 1\}$.

(iii) The leafs of the foliation point radially outward from the origin towards the circle of singularities.

(iv) The limit set of each leaf is either the origin or a point of the circle of the circle $\{(x, y) : x^2 + y^2 = 1\}$.

(v) $d\beta$ is given by $(2 - 4x^2 - 4y^2)dx \wedge dy$

The first idea is now to find a function u such that $\alpha = \beta + udz$ is a contact form. This cannot be done for any β in general, as we shall see later. However, in this case it does work: Both at the origin and at the circle of singularities $d\beta$ does not vanish, in fact it only vanishes at the circle $x^2 + y^2 = \frac{1}{2}$. So if, we wish to find a real-valued function u such that $\alpha = \beta + udz$ satisfies the contact condition, this translates to the following:

$$\begin{aligned} \alpha \wedge d\alpha = (\beta + udz) \wedge (d\beta + du \wedge dz) = \\ \beta \wedge d\beta + \beta \wedge du \wedge dz + udz \wedge d\beta + udz \wedge du \wedge dz = \\ \beta \wedge du \wedge dz + ud\beta \wedge dz \end{aligned}$$

So we need $\beta \wedge du + ud\beta > 0$. By Lemma 2.27 this is equivalent to the foliation being divided. This foliation satisfies the conditions of Theorem 2.22 thus, with the methods presented in its proof one can construct such a u.

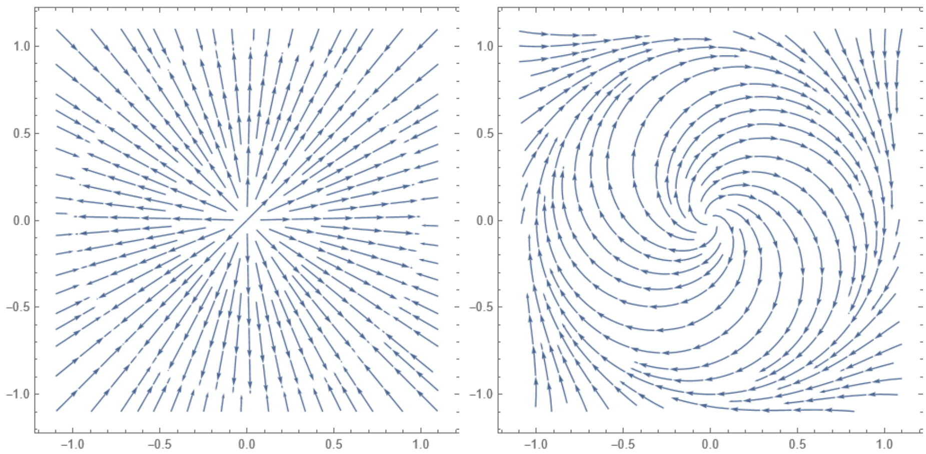

Figure 3.3: Two overtwisted disks. They have exactly one singularity in the interior and a limit of oppsite sign at the boundary.

A Morse-Smale version of this example can be constructed by slightly varying X:

$$X_\epsilon = (1 - x^2 - y^2)(x\frac{d}{dx} + y\frac{d}{dy}) + \epsilon(y\frac{d}{dx} - x\frac{d}{dy})$$

If $\epsilon \neq 0$ one obtains that the singularities at $x^2 + y^2 = 1$ disappear and instead give rise to an attracting closed orbit. The direction of rotation is given by the sign of ϵ. This foliation for $\epsilon > 0$ is depicted in Figure 3.3.

3.1 Moser's stability trick

Before starting to discuss Moser's stability trick there is one further ingredient which we will need:

Lemma 3.6. *Let (M, ξ) be a contact manifold and α a choice of co-orientation for ξ. Then there is a unique vectorfield R_α called the* ***Reeb vectorfield*** *which satisfies:*

$$\alpha(R_\alpha) = 1$$
$$d\alpha(R_\alpha, \cdot) = 0$$

Proof. This proof can be found in the introductory book by Geiges [8, Lemma/Definition 1.1.9.] though we provide many of the omitted details.

The first observation is that pointwise $d\alpha$ is an antisymmetric matrix and thus its kernel will be 1-dimensional. Since $\alpha \wedge d\alpha \neq 0$, it follows that there is a uniqe vector $R_\alpha(p)$ fulfilling the above equations. It only remains to be shown that R_α is smooth:

First choose a Riemannian metric $\langle \cdot, \cdot \rangle$ and denote by Y_1 the vectorfield such that $\langle Y_1, \cdot \rangle = \alpha$. Using a chart, we can complete $\{Y_1(p)\}$ to a basis $\{Y_1(p), Y_2(p), Y_3(p)\}$ around a the point p, where Y_2 and Y_3 are some of the coordinate vectorfields. Since linear independence is an open condition $\{Y_1, Y_2, Y_3\}$ will form a local frame of TM around p. Now, we may orthonormalise this frame and so (keeping notation) $\{Y_1, Y_2, Y_3\}$ is an orthonormal frame of TM around p. Observe that $\langle Y_1, \cdot \rangle = f\alpha$, up to a multiple of f. Thus Y_2 and Y_3 form a local frame of the kernel of α and thus of ξ. In particular, $d\alpha(Y_2, Y_3) \neq 0$, since $d\alpha$ defines a volume form on ξ. Now define:

$$R = Y_1 - \frac{d\alpha(Y_1, Y_3)}{d\alpha(Y_2, Y_3)} Y_2 - \frac{d\alpha(Y_1, Y_2)}{d\alpha(Y_3, Y_2)} Y_3$$

One now calculates that $d\alpha(R, \cdot) = 0$:

$$\begin{aligned} d\alpha(R, Y_1) &= -\frac{d\alpha(Y_1, Y_3) d\alpha(Y_1, Y_2)}{d\alpha(Y_2, Y_3)} - \frac{d\alpha(Y_1, Y_2) d\alpha(Y_1, Y_3)}{d\alpha(Y_3, Y_2)} \\ d\alpha(R, Y_2) &= d\alpha(Y_1, Y_2) - \frac{d\alpha(Y_1, Y_2)}{d\alpha(Y_3, Y_2)} d\alpha(Y_3, Y_2) \end{aligned}$$

The second line is 0 and the first line vanishes as well, after recognising that the denominators are related by -1. An analogous calculation shows that $d\alpha(R, Y_3)$ vanishes as well. Now R_α is obtained by rescaling R with $\frac{1}{\alpha(R)}$. Thus R_α is the desired vectorfield. QED

Now, we will present Moser's stability trick as it can be found in the introductory book by Geiges [8, Theorem 2.2.2.]:

Let ξ_t be a smooth family of contact distributions which are given by contact forms α_t. Then we can ask ourselves if there is an isotopy $\phi_t : M \to M$ such that $\phi_t^* \xi_t = \xi_0$. This can be rephrased as:

$$\phi_t^* \alpha_t = g_t \alpha_0$$

Differentiating this with respect to t, yields:

$$\frac{d}{dt}(\phi_t^* \alpha_t) = \dot{g}_t \alpha_0 \tag{3.1.1}$$

There is a formula expressing the left hand side in terms of Y_t, where Y_t is generating ϕ_t, in other words $Y_t = \frac{d}{dt}\phi_t$ where ϕ_t is also viewed as a function in t. See for instance the derivation of this formula by Geiges [8, Theorem 2.2.1.]:

$$\frac{d}{dt}(\phi_t^* \alpha_t) = \phi_t^* (\dot{\alpha}_t + \mathfrak{L}_{Y_t} \alpha_t)$$

Now substituting Cartan's magic formula into Equation 3.1.1 gives:

$$\phi_t^*(i_{Y_t} d\alpha_t) + \phi_t^* d(i_{Y_t} \alpha_t) + \phi_t^* \dot{\alpha}_t = \dot{g}_t \alpha_0$$

Now we pull this equation back along ϕ_t. The pullback of $\dot{g}_t$ will be labelled by h. Thus, we arrive at the following equation:

$$\iota_{Y_t} d\alpha_t + d(\iota_{Y_t}\alpha_t) + \dot{\alpha}_t = h\alpha_t \tag{3.1.2}$$

Since we have not made any choices, one could retrace the proof and see that any Y_t fulfilling the last equation gives rise to the desired isotopy. Thus, we will only need to care about finding Y_t and whether or not its flow is defined. The most powerful application is Gray's stability Theorem, the formulation below is taken from the introductory book by Geiges [8, Theorem 2.2.2.]

Theorem 3.7. *(Gray stability Theorem) Let M be a closed 3-manifold and α_t a smooth family of contact structures. Then there is an isotopy ϕ_t such that $\phi_t^*\alpha_t = g_t\alpha_0$.*

Proof. By the discussion of Moser's stability trick, we need only find a vectorfield Y_t satisfying Equation 3.1.2. We even have the freedom to assume that $Y_t \in \xi_t$. Then Equation 3.1.2 simplifies to:

$$\iota_{Y_t} d\alpha_t = \dot{\alpha}_t + h\alpha_t$$

Since $Y_t \in \xi_t$ and $d\alpha_t|_{\xi_t}$ is non-degenerate there is a unique solution to the above equation, if h is chosen correctly. To find the appropriate h, we insert the Reeb-vectorfield R_t of α_t:

$$0 = \dot{\alpha}_t(R_t) + h\alpha_t(R_t)$$
$$h = -\dot{\alpha}_t(R_t)$$

Now there is a unique solution to the above equation. Since M is closed Y_t has compact support and so we may integrate Y_t to the desired isotopy. QED

If M is not compact, we need to be a bit more careful with the existence of the flow of Y_t for time 1. One of the trivial ways to achieve this is shown below:

Theorem 3.8. *Let (M, ξ) be a contact manifold and let ξ_1 be another contact distribution defined on an open subset $U \subset M$. If there is a compact subset $A \subset U$ such that $\xi|_A = \xi_1|_A$ then there is an isotopy $\phi_t : M \to M$ such that $\phi_1^*\xi_1 = \xi$ on a neighborhood V of A and ϕ_t coincides with the identity outside of U.*

Proof. Let $H : [0,1] \to [0,1]$ be a smooth transition function that is constant on $[0, \epsilon]$ and $[1-\epsilon, 1]$. Then consider the 1-forms α and α_1 coorienting ξ and ξ_1 respectively. Since $\xi|_A = \xi_1|_A$, there is a non-vanishing function f such that $\alpha = f\alpha_1$. Extending f to an open neigborhood V' of A, we can assume that $\alpha = \alpha_1$ on A. For fixed β, the positive contact condition $\beta \wedge d\beta > 0$ is open and convex linear in $d\beta$. Thus $\alpha_t := H(t)\alpha_1 + (1-H(t))\alpha$ is contact on A and thus on the open neighborhood V'(after possibly shrinking V') of A. Possibly shrinking V' we may assume that it has compact closure.

We will use Moser's stability trick as in the proof of Theorem 3.7, so we observe that the following equation with the additional condition that $Y_t \in \xi_t$ has a unique solution:

$$\iota_{Y_t} d\alpha_t = -\dot{\alpha}_t + h\alpha_t$$

Since $\dot{\alpha}_t = 0$ on A, Y_t must vanish identically: Consider any $Y \in \xi_t$, inserting it into the above equation, will yield 0 on the right hand side and so Y_t can only be 0, since $d\alpha_t$ is non-degenerate on ξ_t. In addition, the flow of the 0 vectorfield exists for all times. So in particular, the flow of Y_t exists on A for time 1. Thus there exists a neighborhood V of A such that the flow ϕ exists for time $1-\epsilon$ and does not leave V'. Additionally, since $\dot{\alpha}_t = 0$ for $t > 1-\epsilon$, the flow exists for time 1 on V. Now let Y'_t coincide with Y_t on $Im(\phi|_{V\times[0,1]})$ and $Y'_t = 0$ outside of V'. Thus Y'_t is compactly supported and so we may integrate it to an isotopy ϕ_t of M. QED

A special case is Darboux's Theorem: All contact structures look locally the same, in the sense that for each point $x \in (M,\xi)$ one can find coordinates such that ξ is locally given by $\ker(dz - ydx)$ which is isomorphic to the contact structure given in Example 3.3. The idea is to choose coordinates in such a way that the contact structures agree at $p = (0,0)$ and then use the above Theorem 3.8.

Corollary 3.9. *(Darboux's Theorem) Let (M,ξ) be a contact manifold. Then for each point $p \in M$ there is a local chart (x,y,z) such that $\xi = \ker(dz - ydx)$.*

Proof. Recall the vectorfields R_α, Y_2, Y_3 which were constructed in the proof of Lemma 3.6: They formed a local frame around the point $p \in M$ and $\{Y_2, Y_3\}$ formed a frame of ξ around p. Now consider the following composition of flows:

$$Fl_{R_\alpha}(z, Fl_{Y_2}(y, Fl_{Y_1}(x,p)))$$

This map is defined on an open neighborhood of $(0,0,0) \in \mathbb{R}^3$ and fulfills that the pullback of α agrees with dz at $(0,0,0)$. Thus, we may apply Theorem 3.8 and this yields the isomorphism on the open neighborhood of $(0,0,0)$. After concatentation with the inverse of the composition of flows, we are done. QED

Before returning to the discussion of surfaces embedded in contact manifolds, we need to discuss some details of compact 1-submanifolds:

Definition 3.10. *Let C be a function from $[0,1]$ into (M,ξ). We say that C is a* ***Legendrian arc****, if C is an embedding and the image of T_xC is contained in ξ_x for all $x \in [0,1]$. If C descends to an embedding of a circle $S^1 = [0,1]/(0 \sim 1)$ then we call C a* ***Legendrian knot****.*

We have an important strengthening of Darboux's Theorem close to Legendrian knots, namely that all contact structures close to Legendrian knots are isomorphic. The proof is in essence just a normal form analysis of the contact structure, coupled with Theorem 3.8.

Theorem 3.11. *Let L be a Legendrian knot in a contact manifold (M,ξ) then there is a neighborhood of L isomorphic to $(\mathbb{R}^2 \times S^1, ker(cos(z)dx - sin(y)dy))$.*

Proof. In essence the proof follows the same line as Darboux's Theorem 3.9. The only difference is to find a complete frame of ξ along L. So let $p \in L$ be a point, then as in Lemma 3.6, we find an open neighborhood U of $L(p)$ such that $\{R_\alpha, Y_2, Y_3\}$ form a local frame for $TM|_U$ and $\{Y_2, Y_3\}$ form a frame for $\xi|_U$. Now the derivative of L, $\frac{d}{d\theta}$ also lies in ξ, so either Y_2 or Y_3 are not colinear to it at p. Assume then that $\{Y_2, \frac{d}{d\theta}\}$ form a local frame of $\xi|_{L\cap U}$. Now if $d\alpha(Y_2, \frac{d}{d\theta}) < 0$, we replace Y_2 by $-Y_2$. In this manner we obtain around each point $p \in L$ a vectorfield Y_2 which fulfills $d\alpha(Y_2, \frac{d}{d\theta}) > 0$, so any positive sum of such vectorfields also fulfills this condition. Since L is compact, we can cover L with finitely many such neighborhoods and use smooth partition functions to glue these vectorfields together to Y such that $Y \in \xi$ and $d\alpha(Y, \frac{d}{d\theta}) > 0$. Then we may consider the following composition of flows:

$$Fl_{R_\alpha}(y, Fl_Y(x, L(z)))$$

This flow is defined on an open neighborhood of $\{0,0\} \times S^1 \subset \mathbb{R}^2 \times S^1$, thus we may consider the pullback of α along it, which induces a contact structure on this neighborhood. Note that $\alpha = dy$ along $S^1 \times \{0,0\}$. Now repeat the same procedure along $\{0,0\} \times S^1 \subset (\mathbb{R}^2 \times S^1, ker(\cos(z)dx - sin(y)dy)$. Composing these two isomorphisms of contact structures, we obtain the desired isomorphism between the two structures. QED

Associated to a surface Σ bounding a Legendrian knot L there is an invariant called the Thurston-Bennequin invariant, which roughly measures the twisting of ξ along the boundary with respect to the tangent space of Σ:

Remark 3.12. Smooth maps $f : S^1 \to S^1$ can be assigned a number $deg(f)$ which counts the preimages of a regular value $y \in S^1$ with sign given by the sign of $d(f)(x)$, where $F(x) = y$. It is an essential observation from Differential Topology that this number is independent of the choice of $y \in S^1$ and invariant under smooth homotopies of f. See for example, the introductory notes on Differential Topology by Milnor [18, Chapter 4/5].

Remark 3.13. (Normal bundle) Denote by $\langle , \rangle$ some choice of Riemannian inner product on M. Then one can define the normal bundle NL of $L \subset M^3$ as $NL := \{Z \in TM|_L : Z \perp TL\}$. This comes with a projection map $pr : TM|_L \to NL$ with $pr(Z) = Z - < \frac{d}{dr}, Z > \frac{d}{dr}$ where $\frac{d}{dr}$ is the coordinate of some orientation preserving embedding of $S^1 \hookrightarrow M$ with image L such that $||\frac{d}{dr}|| = 1$. The kernel of pr is exactly TL, so $pr : (TM|_L \setminus TL) \to (NL \setminus \{0_x\}_{x \in L})$.

Now a surface Σ provides us with a section σ of $T\Sigma|_{TL}$ the outward pointing normal, so $\{\sigma(x), \frac{d}{dr}(x)\}$ is a positive basis for each $x \in L$. So $pr(\sigma)$ is a non-vanishing section of NL thus $\frac{pr(\sigma)}{||pr(\sigma)||}$ provides us with a section of $SL := \{Z \in TM_L : Z \perp TL, ||Z|| = 1\}$. In particular SL is an S^1 bundle over L, so a non-vanishing section provides us with a trivialisation $SL \cong L \times S^1$, where $\frac{pr(\sigma)}{||pr(\sigma)||}(x) = (x, 1)$ for $1 \in S^1$.

Definition 3.14. *Let L be a Legendrian knot which is part of the boundary of a surface Σ embedded in a contact manifold (M, ξ). Define the* ***Thurston-Bennequin invariant*** *of L with respect to Σ as $tb(L, \Sigma) = deg(f)$ where $f : S^1 \to S^1$ is the function obtained* *from the section $Y|_L$ from Theorem 3.11 projected to the trivialisation of $SL = L \times S^1$ from Remark 3.13.*

This invariant is of fundamental importance to the classification of contact manifolds, for more details see Chapter 4.2. However before proceeding, let us calculate the Thurston-Bennequin number of the overtwisted disk from Example 3.5:

Example 3.15. Recall that the boundary was a circle of singularities, so the map f only hits one point of the trivialisation. Since all points which are not in the image of f are regular, it follows that $deg(f) = 0$ and thus the Thurston-Bennequin invariant of the overtwisted disk vanishes.

Another important tool coming from contact structures are special classes of vectorfields:

Definition 3.16. *Let (M, ξ) be a contact manifold with empty boundary. We call Y a* ***contact vectorfield*** *if the isotopy generated by Y preserves ξ under pushforward. Such an isotopy is called a* ***contact isotopy****.*

We already encountered one such vectorfield R_α. In fact, the Reeb vectorfield is the "standard" contact vectorfield using the following correspondence induced by a choice of α:

Lemma 3.17. *Let (M, ξ) be a contact manifold with empty boundary. Then the set of all contact vectorfields is in $1:1$-correspondence with the set of compactly supported functions $g : M \to \mathbb{R}$. In particular, any contact vectorfield defined on a compact subset may be extended to a global one.*

Proof. Starting from a contact vectorfield Y, we obtain a compactly supported function g via:

$$\alpha(Y) =: g$$

Y must be supported compactly since it integrates to an isotopy. On the other hand if, we have a compactly supported function g, we may try to find a vectorfield Z tangent to ξ such that $Y = gR_\alpha + Z$ is a contact vectorfield where R_α is the Reeb vectorfield of some co-orientation α of ξ. Recall Equation 3.1.2:

$$i_Y d\alpha + d(i_Y \alpha) = -\dot{\alpha} + h\alpha$$

Y is then a contact vectorfield if and only if $\dot{\alpha} = 0$. So by inserting $Y = gR_\alpha + Z$ the above equation simplifies to:

$$i_Z d\alpha = h\alpha - dg$$

This equation has to be true if we insert R_α. In which case we obtain $h = dg$, on the other hand this equation defines a uniquely defined vectorfield Z, if we enforce $Z \in \xi$. QED

The above characterisation leads us to a stronger notion of a contact vectorfield which also allows us to define it if the boundary is non-empty:

Definition 3.18. *Let (M, ξ) be a contact manifold. We call Y a* ***contact vectorfield*** *if $\alpha(Y) = -g$ is a comptactly supported function and $\iota_Y(d\alpha) = dg - dg(R_\alpha)\alpha$, where α is some choice of co-orientation for ξ and R_α is the Reeb vectorfield.*

The above notion is stronger in the sense that if the boundary is non-empty we must not necessarily obtain an isotopy when integrating Y. If Y is inward pointing along a boundary component then the isotopy generated by Y will not have the original boundary in its image.

3.2 Surfaces in contact manifolds

We can now return to the discussion of surfaces in contact manifolds.

Example 3.19. Recall the foliation from our discussion of the rational/irrational torus from Example 2.14:

$$X_r = \frac{d}{d\phi} + r\frac{d}{d\theta}$$

In fact, if we look at $T^2 \times \mathbb{R}$ with the 1-form $\alpha = -rd\phi + d\theta$, we see that $ker(\alpha)$ gives rise to a contact structure. Since $d\alpha = -dr \wedge d\phi = d\phi \wedge dr$. Then if we choose the orientation of $T^2 \times \mathbb{R}$ to be given by $d\theta \wedge d\phi \wedge dr$, we see that this gives rise to a contact structure.

Taking the pullback of α via the embedding $i_r : T^2 \to T^2 \times \mathbb{R}$, we recover the initial foliation. In fact, this can be done for any embedding of a compact surface:

Definition 3.20. *Let (M, ξ) be a contact 3-manifold, α a choice of co-orientation for ξ, Σ a surface in M and $i : \Sigma \hookrightarrow M$ an embedding. Then we call the foliation $i^*\alpha \in \mathcal{F}_\Sigma$ induced by α via the embedding i the* ***characteristic foliation*** *of the embedding i into (M, ξ).*

Since the co-orientation induced by α is consistent with multiplication by positive functions $f : M \to \mathbb{R}_{>0}$ this indeed gives rise to a well-defined oriented foliation.

As we will see a bit later, a foliation is the characteristic foliation of an embedding if and only if the foliation is non-isochore. So one might as well take this as a definition. However, before proceeding, we will discuss two crucial examples:

Example 3.21. (The tight sphere) Consider $(\mathbb{R}^3, \ker(dz + xdy - ydx))$ the tight $\mathbb{R}^3$ we encountered in Example 3.3. Furthermore let us look at the characteristic foliation induced from this structure on the unit sphere S^2. For $x, y \neq 0$ it is more convenient to consider the contact form in cylindrical coordinates $\alpha = dz + r^2 d\phi$. At the point $(0, 0, \pm 1)$ the contact structure is tangent to S^2, thus $\beta = 0$, away from it, we can parametrize the sphere by $F(\phi, z) = (\sqrt{1 - z^2}, \phi, z)$. So the pullback of α along F looks like $dz + (1 - z^2)d\phi$. Thus there are no further singular points of $\beta = i^*\alpha$ except at the north, resp. south pole. Additionally the characteristic foliation points from the north to the south pole. In fact, the foliation resembles that of Example 2.7 except that the leafs twist as they move downward.

Example 3.22. (Overtwisted disks) Recall the contact structure of the overtwisted $\mathbb{R}^3$ from Example 3.4 which was given by $\alpha = cos(r)dz + rsin(r)d\phi$. If we consider the disk $D^2 := \{(r, \phi, 0) : |r| \leq \pi\}$, then we observe that by continuity $\alpha(0) = dz$, so there is a positive singularity at D^2 and if $r = \pi$ then there is a circle of singularities, since $\alpha = -dz$. However, since if $0 < r < \pi$ we observe that the vectorfield $X = \frac{1}{rsin(r)} \frac{d}{dr}$ is dual to $\beta = \alpha|_{TD^2}$ and thus each ray $\{(r, \phi, 0) : 0 < r < \pi, \phi = const.\}$ is a leaf of the foliation. Thus this foliation coincides with the foliation obtained in Example 3.4.

In fact, as we will discuss later the existence of disks with such foliations is of fundamental importance to the study of contact structures.

Next, we consider an embedding i of $\Sigma \times \mathbb{R}$ into a contact manifold (M, ξ). This yields both a characteristic foliation β_r for each $\Sigma_r := \Sigma \times \{r\}$ and a function $u_r := \alpha(\frac{d}{dr})$, where r is the coordinate on $\mathbb{R}$. Thus, we may rewrite α as $\beta_r + u_r dr$ and the contact condition can be written as:

$$\begin{aligned} 0 < \alpha \wedge d\alpha = (\beta_r + u_r dr) \wedge (d\beta_r - \dot{\beta}_r \wedge dr + du_r \wedge dr) = \\ (\beta_r \wedge (du_r - \dot{\beta}_r) + u_r d\beta_r) \wedge dr \end{aligned}$$

which translates into:

$$0 < \beta_r \wedge (du_r - \dot{\beta}_r) + u_r d\beta_r \tag{3.2.1}$$

An immediate consequence of this is, that if $\beta_r = 0$ then $d\beta_r$ may not vanish. If we take the dual vectorfield X_r via a volume form ω this means, that at a singularity the divergence of X_r cannot vanish. This exactly means that the foliation is non-isochore. As it turns out non-isochore and characteristic foliations are essentially equivalent. However for the following statement to make sense, we need that each embedded surface Σ admits a neighborhood of the form $\Sigma \times (-\epsilon, \epsilon)$ (or at least one-sided $\Sigma \times [0, \epsilon)$) where $\epsilon \in (0, \infty]$. In the cases which we will consider this can be constrcuted from the following construction:

Theorem 3.23. *(Giroux, [9, Proposition II.1.2.]) Let Σ be a surface and $\mathcal{F}$ be a foliation on Σ. Then $\mathcal{F}$ is the characteristic foliation of an embedding of Σ into a contact manifold if and only if the divergence of any $X \in \mathcal{F}$ does not vanish at singularities.*

Proof. The proof presented here is in essence the same as the original proof by Giroux [9, Proposition II.1.2.], however some details have been appended.

By definition the characteristic foliation of $i : \Sigma \to M$ is given by $i^*\alpha = \beta$. Furthermore if $\beta(i(x)) = 0$ then $T\Sigma|_{i(x)} = \xi_{i(x))}$. Now by the contact condition $\alpha \wedge d\alpha > 0$ we have that $d\alpha|_\xi \neq 0$ and so $d\beta = d(i^*\alpha) = i^*d\alpha \neq 0$ if $\beta(x) = 0$.

To prove the converse, let X orient $\mathcal{F}$ then we wish to find appropriate functions u_r and β_r such that they fulfill Inequality (3.2.1) together with $\beta_0 = \iota_X\omega$, where ω is a fixed positive area form for Σ. First we choose u_r: Since $d\beta_0 = (div_\omega X)\omega$ does not vanish when $\beta_0 = 0$, we can choose $u_r = div_\omega X$ thus if $r = 0$ $u_r d\beta_r = (div_\omega X)^2\omega > 0$. So we are left to find an appropriate extension β_r of β_0: To do so we take any choice of metric $\langle\cdot,\cdot\rangle$ for Σ and consider the unique vectorfield Z such that $\beta_0 = \langle Z, \cdot\rangle$. Then $\iota_Z\omega =: \lambda$ provides us with another 1-form. Thus we have the following 3 relations.

$$\begin{aligned}\beta(\cdot) &= \langle Z, \cdot\rangle\\ \beta(\cdot) &= \omega(X, \cdot)\\ \lambda(\cdot) &= \omega(Z, \cdot)\end{aligned}$$

For $X, Z \neq 0$ we have:

$$\begin{aligned}\omega(X, Z) = -\lambda(X) = \beta_0(Z) &= \langle Z, Z\rangle > 0\\ (\beta_0 \wedge -\lambda)(X, Z) &= -\beta_0(X)\lambda(Z) + \beta_0(Z)\lambda(X)\\ &= -\omega(X, X)\omega(Z, Z) + \omega(X, Z)^2 > 0\end{aligned}$$

So $\beta_0 \wedge (-\lambda) > 0$ if $\beta_0 \neq 0$. Now $\beta_r := \beta_0 + r(du + \lambda)$ provides the necessary extension. So the contact condition is fulfilled at $r = 0$, we will verify this by inserting it into Equation 3.2.1;

$$\begin{aligned}0 < \beta_0 &\wedge (du - du + \lambda) + ud\beta_0\\ \beta_0 &\wedge \lambda + (div_\omega X)^2\omega\end{aligned}$$

where the first term vanishes only if $\beta_0 = 0$ and the second term is strictly positive at singularities.

Since the contact condition is open there is an open neighborhood U of $\Sigma \times \{0\}$ such that $(U, \ker(\beta_r + udr))$ is a contact manifold. QED

Let us move on to the next important question: Given a characteristic foliation $\mathcal{F}$ of a surface Σ embedded in (M, ξ), how much local information about the contact structure ξ is encoded in $\mathcal{F}$? The answer in the closed case has first been observed by Giroux [9, Proposition II.1.2.].

From now on, we will only consider a sufficiently nice subclass of surfaces: We presume $i(\Sigma)$ is an extensible surface in M, i.e. there exists a sligthly larger surface $\Sigma' \subset M$ such that $i(\Sigma)$ is contained in the interior of Σ'. Note that if Σ is a closed surface this condition is trivially true. In addition, each boundary component of Σ shall be either a Legendrian arc or a Legendrian knot. Furthermore, we will require that there is a vectorfield Y transverse to $i(\Sigma)$ whose flow exists at least for some time around 0.

Lemma 3.24. *(Reconstruction Lemma, [9, Proposition II.1.2.]) Let (M, ξ^0) be a contact manifold, $i : \Sigma \hookrightarrow M$ an embedding of a surface. Assume that ξ^1 is another contact structure defined on an open neighborhood of $i(\Sigma)$. If the characteristic foliations induced on $i(\Sigma)$ by ξ^0 and ξ^1 agree, then there is an isotopy $\phi_t : M \to M$ and two neighborhoods $V \subset U \subset M$ of $i(\Sigma)$ such that:*

(i) $\phi_0 = id_M$

(ii) $\phi_t(i(\Sigma)) = i(\Sigma)$

(iii) $(\phi_1)_*\xi^0 = \xi^1$ *on* V

(iv) $\phi_t|_{M\setminus U} = id$

Proof. Consider a slightly larger surface Σ' that contains $i(\Sigma)$ and a tubular neighborhood $\Sigma' \times \mathbb{R}$ where both $\xi^0 = ker(\alpha^0)$ and $\xi^1 = ker(\alpha^1)$ are defined. On this set, we can consider the following representations:

$$\alpha^0 = \beta^0_r + u^0_r dr$$
$$\alpha^1 = \beta^1_r + u^1_r dr$$

where $\beta^i_r = i^*_r \alpha^i$ is the pullback of α^i along the inclusions $i_r : \Sigma' \to M$ which are obtained by $Fl_{\frac{d}{dr}}(\Sigma')$ where r is the variable on $\mathbb{R}$. Recall Equation 3.2.1 then the main observation is that, if you fix β_r and $d\beta_r$, then this equation is convex in u_r and $\dot{\beta}_r$. Since the characteristic foliations of Σ agrees w.r.t. ξ_0 and ξ_1, we can rescale α_0 and α_1 such that $i^*\alpha^0$ agrees with $i^*\alpha^1$ on $i(\Sigma) \subset \Sigma' \times \{0\}$.

Thus, we can consider the family of contact structures $\alpha^t = t\alpha^1 + (1-t)\alpha^0$ (To be completely precise, we need the same trick with the cut-off function as in the proof of Theorem 3.8, however for brevity we will omit this). By the above argument, this 1-form induces a contact structure at least along $i(\Sigma)$. Since the contact condition is open, this is true for an open neighborhood of $i(\Sigma)$, so we may assume that it is true on $\Sigma' \times (-\epsilon, \epsilon)$ after possibly shrinking Σ'.

The main subtlety is to show that the flow of the vectorfield generated by Moser's stability trick exists on Σ. Recall the condition from Equation 3.1.2:

$$\iota_{Y^t}(d\alpha^t) + d(\iota_{Y^t}(\alpha^t)) = \dot{\alpha}^t + h\alpha^t$$

Where Y^t is the vectorfield that gives rise to the desired isotopy ϕ_t. First we will prove that we can choose Y^t such that (ii) is fulfilled: As before, we will set $Y^t \in \xi^t$ as an additional constraint.

Consider the restriction of the equation to $T\Sigma$: Since β^t_0 is constant, it follows that $\dot{\alpha}^t|_{T\Sigma}$ vanishes. So let X orient the characteristic foliation of Σ_r considered as the push-off of the image of i, then $\alpha^t(X)$ vanishes. This implies however that $d\alpha^t(Y^t, X) = 0$. Since $d\alpha$ is non-degenerate on $i(\Sigma)$, it follows that Y^t is a multiple of X, away from singularities. On singularities of the characteristic foliation α^t vanishes on all tangent vectors, thus $d\alpha^t(Y^t, \cdot)$ vanishes for all tangent vectors. Since at singularities the tangent space of Σ and ξ agree, it follows that Y^t must vanish.

This implies however that the embedding is isotoped along flow lines of the characteristic foliation. So the image remains fixed and its characteristic foliation is preserved. This also implies that the flow of Y_t exists on $i(\Sigma)$. Using openness of the existence of flows and the compactness of the image of the embedding, we know that there are numbers r

and R such that the flow also exists for all times for points in $\Sigma'' \times [-r, r]$ and they do not leave $\Sigma''' \times [-R, R]$, where $i(\Sigma) \subset \Sigma'' \subset \Sigma''' \subset \Sigma'$. So take a smooth cut-off function so that the vectorfield vanishes outside a compact neighborhood U of $\Sigma''' \times [-R, R]$.

Integrating the vectorfield Y^t now yields that ϕ_0 is the identity so (i) is fulfilled. Y^t fulfills the above equation for the flow originating from $\Sigma'' \times [-r, r]$ so (iii) is also true. Since Y^t vanishes outside of U (iv) follows. QED

Another important observation by Giroux allows us to extend Peixoto's density Theorem to characteristic foliations:

Lemma 3.25. *(Giroux-Peixoto Lemma, [9, Lemme II.1.3.]) Let $i : \Sigma \to M$ be an embedding of a closed surface into a contact manifold (M, ξ) then there is a C^∞-small isotopy of the embedding i to an embedding i' such that the induced characteristic foliation is Morse-Smale.*

Proof. Recall that one can rewrite α in a neighborhood $i(\Sigma) \times [-1, 1]$ as $\alpha = \beta_r + u_r dr$ and then the contact condition is equivalent to: (Equation 3.2.1):

$$0 < \beta_r \wedge (du_r - \dot{\beta}_r) + u_r d\beta_r$$

Where $u_r : \Sigma \to \mathbb{R}$ and β_0 is a representative of the characteristic foliation of Σ. According to Peixoto's density Theorem 2.18 we may choose a C^∞-small γ in such a way that $\gamma + \beta_0$ is Morse-Smale. In order to obtain the isotopy, we will deform α close to $i(\Sigma)$ using Moser's stability trick and then compose i with the obtained isotopy.

Since we need to smoothly deform α, let $H : [-1, 1] \to [0, 1]$ be a smooth cut off-function which is supported away from $\{-1, 1\}$ and is 1 at 0. Thus we can choose γ such that it fulfills the contact condition of $\alpha^1 = \beta_r + H(r)\gamma + u_r dr$, i.e. it fulfills the following inequality:

$$0 < \beta_r \wedge (du_r - \dot{\beta}_r - \dot{H}(r)\gamma) + u_r(d\beta_r + H(r)d\gamma) + H(r)\gamma \wedge (du_r - \dot{\beta}_r)$$

In addition, $\beta_0 + \gamma$ shall be Morse-Smale. Since the prior condition is open, we can find a C^∞-small γ. Thus, we can extend $H(r)\gamma$ by 0 to the rest of M and thus obtain a family $\alpha^t = \alpha + tH(r)\gamma$ which is constant outside a compact neighborhood. Thus we obtain an isotopy $\psi_t : M \to M$ and $\Sigma' := \psi_1^{-1}(i(\Sigma))$ is a C^∞ isotopy of $i(\Sigma)$ which has Morse-Smale characteristic foliation. Now let $i' := \psi_1 \circ i$ and $i'(\Sigma) = \Sigma'$ has the desired Morse-Smale characteristic foliation. QED

Recall that Peixoto's Theorem does not provide us automatically with a foliation whose singularities have non-vanishing derivative. Though the contact condition of $\beta_0 + \gamma_0 + u_0 dr$ automatically enforces it.

4 Convex surfaces

The following chapter serves as an introduction to the study of convex surfaces originally due to Giroux [9] for closed surfaces. The generalisation to surfaces with Legendrian boundary was first observed by Kanda [15]. We will roughly follow the lecture notes by Honda [13] and Etnyre [6].

In the picture of Section 2.2 we have up to now only talked about non-isochore foliations. However divided foliations appear naturally in our setting as well: Recall the equivalent formulation of divided foliations introduced in Lemma 2.27: A non-isochore foliation $\mathcal{F}$ of a surface possesses a dividing curve if and only if there is a β representing $\mathcal{F}$ and a function $u : \Sigma \to \mathbb{R}$ such that the following equation holds:

$$0 < ud\beta + du \wedge \beta \tag{4.0.1}$$

Compare this to equation :

$$0 < \beta_r \wedge (du_r - \dot{\beta}_r) + u_r d\beta_r$$

The essential difference is that in the first inequality both β and u are independent of the factor r. This precisely means that if $\mathcal{F}$ is induced by an embedding of Σ into a contact manifold, there is a contact vectorfield Y transversal to Σ which gives rise to the above representation.

Definition 4.1. *Let (M, ξ) be a contact manifold and $i : \Sigma \hookrightarrow M$ an embedding. We call $i(\Sigma)$ a **convex surface** if there is a contact vectorfield Y which is transversal to $i(\Sigma)$.*

Equivalently $i(\Sigma)$ is a convex surface if and only if the characteristic foliation induced by i is a divided foliation.

As it is useful, we will sometimes also refer to Σ as a convex surface itself. This is a shorthand for the property of the embedding i.

Theorem 4.2. *Let Σ be a convex surface. Assume that Γ_Σ and Γ'_Σ are two dividing curves for Σ. Then these curves are homotopic through dividing curves.*

Proof. Let $\alpha = \beta + udr = \beta' + u'dr'$ two different representations of the contact structure of $\Sigma \times [-1, 1]$. Since the characteristic foliation coincides, we can set $\beta = \beta'$. Thus we get by convex linearity of the defining conditions:

$$0 < \beta \wedge du + ud\beta$$

That $\alpha_t := \beta + (1 - t)udr + tu'dr$ is a homotopy of contact structures. By the arguments as in Theorem 3.24, this leads to an isotopy defined around $\Sigma \times \{0\}$. Now each $u_t = (1 - t)u + tu'$ induces a homotopy of dividing curves upon Σ. QED

The most important observation is that the dividing curve $\Gamma_i = \{u = 0\}$ captures all of the topological information of a convex surface. This was first observed by Giroux [10, Lemme 2.4.]:

Definition 4.3. *Let Σ be a surface and i_0, i_1 two embeddings into a contact manifold such that $i_0(\Sigma)$ and $i_1(\Sigma)$ are convex. We call $i_0(\Sigma)$ and $i_1(\Sigma)$ convexly isotopic if there is an isotopy $i_t : \Sigma \to M$ such that for all t $\psi_t(\Sigma)$ is an embedded convex surface.*

Theorem 4.4. *Let $i : \Sigma \hookrightarrow M$ and $i' : \Sigma \hookrightarrow M$ be embedded convex surfaces, which are convexly isotopic. Then their dividing curves are homotopic as properly embedded curves.*

Proof. Let $i : \Sigma \times [0,1] \to (M, \xi)$ be the convex isotopy. Then for each t we obtain a u_t such that $\alpha = \beta_t + u_t dr$. Since the contact condition is open, we may vary β_t slightly, so that $\alpha = \beta_{t'} + u_t dr$ is still contact for t' sufficiently close to t. Since the unit intervall is compact, we obtain a finite covering U_i of $[0,1]$. Now, we may iteratively change the dividing curve of $\Sigma \times \{0\}$ to the dividing curve of $\Sigma \times \{1\}$ since on the overlap of two U_i we have two different divding curves induced by u_i which are homotopic as divinding curves and thus as properly embedded arcs by the previous Theorem 4.2. QED

Lastly, if a surface Σ decomposes into two convex subsurfaces Σ_1, Σ_2 joined along a Legendrian boundary, we may extend a contact vectorfield Y transverse to Σ_1 to a contact vectorfield Y' transverse to Σ. This follows from the methods used in Subsection 2.2. To find a function $f : \Sigma \to \mathbb{R}_{>0}$ such that $div_\omega(fX)^{-1}(0)$ is a regular submanifold where X orients $\mathcal{F}$. If X already fulfills this on Σ_1 then the procedure used in the proof of Theorem 2.22 allows us to modify X only on Σ_2 supported away from the common boundary of Σ_1 and Σ_2. This leads to a representation of the contact structure as $\alpha = \beta + udr$, where $Y = \frac{d}{dr}|_{\Sigma_1}$. Thus the Reconstruction Lemma 3.24 provides us with an extension of Y to a contact vectorfield Y' which coincides with Y on $i(\Sigma)$. Thus we have proven the following lemma:

Lemma 4.5. *(Contact vectorfield extension) Assume $i : \Sigma \to M$ is a convex surface whose characteristic foliation fulfills the Poincaré-Bendixson property. Furthermore, let $\Sigma = \Sigma_1 \cup \Sigma_2$ where Σ_1 and Σ_2 meet along a Legendrian boundary. Then a contact vectorfield Y which is transverse to Σ_1 can be extended to a contact vectorfield Y' which is transverse to Σ and Y' agrees with Y on an open neighborhood U of Σ_1.*

In particular, this lemma makes sense of the statement that one can glue the dividing curves of Σ_1 and Σ_2 together to obtain a dividing curve of Σ. Since restricting u to either Σ_1 and Σ_2 provides dividing curves for either of them and the dividing curve is well-defined up to homotopy of dividing curves.

Before moving, on we will need to introduce a compact way to describe non-isochore foliations: Writing down explicit models of foliations containing several different structures at once is cumbersome, so we will use a pictorial way of describing them: Orbits will be drawn as straight lines with an arrow indicating their orientation. Nodes with two distinct eigenvalues will be drawn as circles while saddles are drawn as boxes. They are either coloured in if they are negative or their interior is white if they have positive sign.

However, we will need singularities of a less regular kind: Lines of singularities which terminate in either a nodal or a saddle-like end. This means that close-by orbits behave as if the singularities where isolated. So a negative nodal end has all orbits close-by entering it and a negative saddle-like end has one orbit leaving it while others are deflected. The full collection of these singularities and their diagrammatical representation is shown in Figure 4.1.

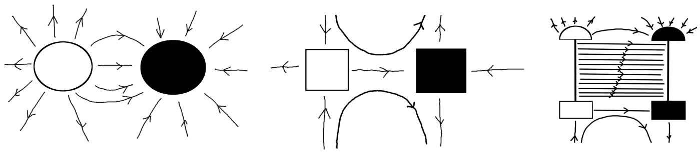

Figure 4.1: From left to right: A positive node, a negative node, a positive saddle, a negative saddle, a line of positive singularities with a nodal and a saddle-like end and a line of negative singularities with a nodal and a saddle-like end.

4.1 Flexibility of convex surfaces

The most important tool in the study of flexibiltiy of convex surfaces is Giroux's flexibility Theorem. The Theorem essentially says that any two characteristic foliations which are divided by a dividing curve Γ can be transformed into one another, remaining divided by Γ:

Theorem 4.6. *(Giroux's flexibility Theorem, [9, Proposition 3.6.])*
Let $i : \Sigma \to M$ be a convex surface and let Γ_i be a dividing curve corresponding to the contact vectorfield Y. Assume that $\mathcal{F}_1$ is another foliation divided by Γ_i and that $\mathcal{F}_\xi$ on the boundary and on a tubular neighborhood of the dividing curve. Then for each ϵ there is a convexly isotopic convex surface $i' : \Sigma \to M$ contained in $\Sigma \times (-\epsilon, \epsilon)$ such that i' induces the characteristic foliation $\mathcal{F}_1$. In addition, the convex isotopy $i_t(\Sigma)$ is transverse to Y for all t.

Proof. The proof we present here is the one presented by Etnyre in his introductory lecture notes [6, Theorem 2.26.].

We will first deal with the case $\epsilon = +\infty$. The main observation is that given an r-invariant representation of $\alpha = \beta + udr$, we may choose u in such a way that du has support inside a neighborhood of the dividing curve: This is simply done by choosing a tubular neighborhood of the dividing curve Γ and then define a function which is defined by $f := \frac{1}{u}$ outside the tubular neighborhood of the dividing curve and then normalises to $f = 1$ on a smaller neighborhood of the dividing curve. Then $f\alpha = f\beta + (fu)dr$ is as desired.

Then let β_0 and β_1 co-orient the foliation $\mathcal{F}_\xi = \mathcal{F}_0$, respectively the foliation of $\mathcal{F}_1$ together with functions u_0, u_1 as above. Now consider a tubular neighborhood of the dividing curve, where β_0 is a multiple of β_1 and du_i are supported. There we may consider the function $\frac{u_1}{u_0}$, which can be extended to Γ_i by L'Hôpitals law since the derivatives of u_1 and u_0 do not vanish at the dividing curve. So multiplying β_1(still denoted by β_1) by this function, we obtain that $\alpha_i = \beta_i + udr$ is a contact form, where $u = u_0$. Now let $\omega = \beta_0 \wedge du + ud\beta_0$ and let X_i be the dual to β_i. Then we may consider the following family of 1-forms:

$$\alpha_t = \omega(X_t, \cdot) + udr$$

where

$$X_t = tX_1 + (1-t)X_0$$

One observes that this is contact for all t, since Inequality 4.0.1 is convex linear in β. Applying Moser's stabily trick, we obtain the desired isotopy i_t. It remains to observe that $\frac{d}{dr} = Y$ is still a contact vectorfield transversal to i_t. The below proof technique is taken from the Contact topology lecture notes of Honda [13]. Recall Equation 3.1.2 of Moser's stability trick.

$$\iota_{Z_t} d\alpha_t + d(\iota_{Z_t}\alpha_t) + \dot{\alpha}_t = h\alpha_t$$

By construction if $u \neq \pm 1$ then $\dot{\alpha}_t = 0$, so $Z_t = 0$ if $u \neq \pm 1$. On the other hand if $\dot{\alpha}_t \neq 0$ then $du = 0$. Assume $u = 1$, then we may specialise to the following equation for $\alpha_t = \beta_t + udr$ and split $Z_t = g_t \frac{d}{dr} + Z_t'$ where Z_t' is tangent to Σ.

$$d\beta_t(Z_t', \cdot) + d(\beta_t(Z_t') + g_t) + \beta_0 - \beta_1 = h\beta_t + hdr$$

This has a solution given by:

$$d\beta_t(Z_t', \cdot) = -\beta_0 + \beta_1 \tag{4.1.1}$$
$$g_t = -\beta_t(Z_t') \tag{4.1.2}$$

The first line has a unique solution since the divergence of X_t was non-vanishing if u is positive. So we also obtain a unique g_t. One calculates that for this choice $h = 0$. The key observation is that the above choice of Z_t is independent of r, so the isotopy is transverse to $Y = \frac{d}{dr}$ as desired.

Now, we only have that $i'(\Sigma) \subset \Sigma \times \mathbb{R}$ and we observe that we may at most observe that $i'(\Sigma) \subset \Sigma \times [-K, K]$ where $K = max|g_t| + 1$. To obtain the convex isotopy that leads to $i'(\Sigma) \subset \Sigma \times (-\epsilon, \epsilon)$ with the desired characteristic foliation we need to do a preliminary isotopy: Let $H : (-\epsilon, \epsilon) \to \mathbb{R}_{>0}$ be a smooth transition function which is $\frac{3K}{\epsilon}$ if $|r| < \frac{\epsilon}{3}$ and 1 if $|r| > \frac{2\epsilon}{3}$. Then we consider the intermediary contact structure $\alpha' = \beta_0 + Hdr$. As shown in the proof of the Reconstruction Lemma 3.24 this isotopy preserves the level sets of $\Sigma \times (-\epsilon, \epsilon)$ and even more is r-independent for $|r| < \frac{\epsilon}{3}$. So the image of $\frac{d}{dr}$ under this isotopy is a multiple of $\frac{d}{dr'}$ for r in this range. Now we redo the steps above and obtain that Equation 4.1.1 leads to:

$$d\beta_t(Z'_t, \cdot) = -\beta_0 + \beta_1 \tag{4.1.3}$$

$$g'_t = -\frac{\beta_t(Z'_z)}{H} \tag{4.1.4}$$

So $||g'_t|| = \frac{||g_t||}{H}$ and thus the flow is contained in $\Sigma \times (-\frac{\epsilon}{3}, \frac{\epsilon}{3})$. QED

One of the main questions we need to answer is that: Given an embedded surface $\Sigma \subset M$, how much of the surface needs to be isotoped to obtain an embedded surface $\Sigma' \subset M$ with the desired foliation:

Lemma 4.7. *Let $i : \Sigma \hookrightarrow M$ be a convex surface. Let $\mathcal{F}_1$ be another foliation of Σ which satisfies the conditions of Theorem 4.6. In addition, let both $\mathcal{F}$ and $\mathcal{F}_1$ satisfy the Poincaré-Bendixson property. Denote by A the set where the foliations do not agree: Then let B be the set which contains A and for each $p \in A$ the orbit of p with respect to X. If the orbit of p intersects the dividing curve then B only has to contain the orbit of p up to its intersection with the dividing curve.*

Then $i(B^C) \subset i(\Sigma) \cap i'(\Sigma)$, so the isotopy of the embedded surfaces is supported inside B.

Proof. Let $\alpha = \beta + udr$ and X be the vectorfield orienting $\mathcal{F}$ which is dual to β via the volume form $\omega = du \wedge \beta + ud\beta$ and such that du is contained in a sufficiently small tubular neighborhood of the dividing curve disjoint from A.

Now, we wish to construct a vectorfield X' orienting $\mathcal{F}_1$ such that $X|_{B^C} = X'|_{B^C}$ and $\iota_{X'}\omega + udr$ is contact. To begin let $X'|_{A^C} = X|_{A^C}$ and X' orients $\mathcal{F}_1$. Let l be a leaf of the foliation which intersects A and denote by p any point of $l \cap B$. Say that the whole orbit of p is contained in the positive region of Σ. Then we need to ensure that $div(fX') > 0$ for some positive function $f : \Sigma \to \mathbb{R}_{>0}$. By the methods introduced in the proof of Theorem 2.22 we find such an f. If the orbit of p limits in either positive or negative time to a limit set of X where $X = X'$ then $div(X') > 0$ so we may choose $f = 1$ on a neighborhood of this limit set, so $X = X'$ close to this limit set.

If l intersects the dividing curve, then there is a point q along l such that $q = l \cap \Gamma_i$. Even more there is a point p' such that $X = X'$ along l for all points after p' such that $Fl_{X'}(p, t) = p'$ for $t < T$. Using the technique from the proof of Theorem 2.22 we obtain that $f(p')$ might have to be some large number C to achieve $div(fX') > 0$ along the flow of X' contained between p and p'. However along l between p' and q, we have that $X' = X$, so in particular we may calculate that: $div(\frac{1}{u}X) = \frac{1}{u^2}$ (an observation by Etnyre [6]) and so in particular $div(\frac{f}{u}X) = \frac{f}{u^2} + \frac{1}{u}df(X)$ by Lemma 2.9. So we only need a solution for the inequality $df(X) > -\frac{f}{u}$, however since u goes to 0 as the flow approaches q, we can achieve that $f = 1$ after some point p'' which lies in between p' and q. We may extend $f = 1$ along the rest of the leaf l. Since all of these considerations depend smoothly upon the point, we can use this to construct a smooth function f which is 1 except inside B. Thus $X = X'$ outside of B and in particular the family of 1-forms $\alpha^t = \omega(X_t, \cdot) + udr$ is t-invariant for points of Σ not in $B \times \mathbb{R}$.

The above set $i(B)$ is left invariant by the preliminary isotopy such that $i'(\Sigma) \subset \Sigma \times (-\epsilon, \epsilon)$ as in the proof of Giroux's flexiblity Theorem 4.1. Note that the isotopy $\alpha = \beta + udr$ to $\alpha = \beta + Kudr$ fulfills that $\dot{\alpha} = 0$ if $u = 0$. So it is sufficient that B contains the orbit of a point $p \in A$ up to its intersection with the dividing curve since this isotopy fixes the intersection point and the isotopy is chosen to be disjoint from the dividing curve. QED

Giroux's flexibility Theorem 4.1 now allows us to prove two very important techniques to modify the characteristic foliation of a convex surface:

Lemma 4.8. *(Node Flexibility) Let Σ be a convex surface whose characteristic foliation fulfills the Poincaré-Bendixson property and n a node with in the interior of Σ. Furthermore let l and l' be two leafs of $\mathcal{F}_i$ which limit to n. After an isotopy of Σ contained in a neighborhood of the node one can assume that on the characteristic foliation l and l' join smoothly. In particular the isotopy preserves convexity, the Poincaré-Bendixson property and remains transversal to a pre-assigned contact vectorfield Y for all times t.*

Proof. This is a simple corollary of the previous Lemma 4.7: Assume n is a positive node then there is a neighborhood U of n such that all leafs are pointing outward along the boundary of U and n is the only singularity in U. Then one can find an explicit local model such that l and l' are joined smoothly, this is done for example in the dissertation of Etnyre [7, Lemma 2.25.]. Since this is a change close to an isolated node, the Poincaré-Bendixson property is fulfilled on U and since the rest of Σ is unperturbed the stable limit sets of all orbits leaving U are unchanged it also remains true on $\Sigma \setminus U$. QED

There are several versions of the so-called Elimination Lemma, see Giroux [9, Lemme II.2.3.] or Eliashberg-Fraser [5, Lemma 2.1.]. The Elimination Lemma states that given a node and saddle of the same sign connected by a separatrix in the characteristic foliation of $\mathcal{F}_i$ there is a C^0-small isotopy of i with support close to the separatrix such that the resulting surface has no singularities in this area. The Creation Lemma [5, Lemma 2.3.] states the reverse, given any section of a leaf and a nonsingular neighborhood there is a C^0-small isotopy of the surface close to the leaf such that the neighborhood contains a node-saddle singularity pair of the same sign. We only need this Creation Lemma in a more specific situation, so we will only state it for the precise case we need.

Lemma 4.9. *(Creation Lemma) Let Σ be a convex surface with a characteristic foliation $\mathcal{F}$ which fulfills the Poincaré-Bendixson property and let Y be a contact vectorfield. Given a leaf l which limits between two limit sets of opposite sign then one can convexly isotop i to an embedding i' such that i and i' coincide except on a neighborhood U of a strip of l where U is nonsingular with respect to $\mathcal{F}$ and the characteristic foliation of i' has exactly two singularities one node and one saddle of the same sign where the sign can be arbitrarily chosen. In addition, one may choose the isotopy in such a way that the Poincaré-Bendixson property is fulfilled for all times. Compare Figure 4.2.*

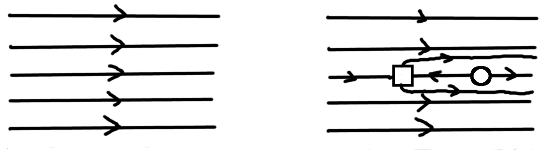

Figure 4.2: Adding a positive saddle-node pair along a chosen leaf l.

Proof. We will consider the case where we wish to add a positive pair. The negative pair follows analogously.

Consider a small rectangle $[-1,1]\times[-1,2]$ where $[-1,1]\times\{2\}$ is a piece of the dividing curve and $l\cap([-1,1]\times[-1,2]) = \{0\}\times[-1,2]$ and where the foliation $X = \frac{d}{dy}$. Then let $X' = X$ except inside the rectangle $[-1,1]^2$, where X' is given by $X_\epsilon = x\frac{d}{dx} + (y^2-\epsilon^2)\frac{d}{dy}$ (using a smooth cut-off function so that X' is C^∞). For $\epsilon = 0$ there is a positive saddle-node at $(0,0)$ thus the divergence of X_0 is positive around $(0,0)$ thus there is a sufficiently small ϵ such that $div(X_\epsilon|_{\{0\}\times[-\epsilon,\epsilon]}) > 0$. Choosing this ϵ, we may consider the condition of Lemma 4.7 and observe that the rectangle $[-1,1]\times[-1,2]$ fulfills the criteria for B. So we obtain that the isotopy which gives us i' is supported inside this rectangle.

Regarding the Poincaré-Bendixson property: We choose ϵ so small that X_ϵ has positive divergence between the singularities at $\{0\}\times\{\epsilon\}$ and $\{0\}\times\{-\epsilon\}$ so the function f which we used to achieve that $div(fX') > 0$ can be chosen constantly 1 along the leaf connecting the new singularities. In fact the change from X to X' is either nonsingular on U or there is exactly 1 saddle-node for for $t = \sqrt{1-\epsilon^2}$ since the transition vectorfields X_t are given by:

$$(1-t) + t(y^2-\epsilon^2)\frac{d}{dy} + (1-t)x\frac{d}{dx}$$

close to the arc $\{0\}\times[-\epsilon,\epsilon]$. And for times $t > \frac{1}{1-\epsilon^2}$ there are two singularities. One node at $\{0\}\times\{\epsilon\}$ and one saddle at $\{0\}\times\{-\epsilon\}$. Thus this rectangle fulfills the conditions of the Poincaré-Bendixson Theorem. Since all orbits which leave this rectangle do so through the dividing curve they do not re-enter the rectangle. Thus their limit set is unaffected by this change. The same is true for orbits entering the rectangle. So the characteristic foliation of i_t fulfills the Poincaré-Bendixson Theorem for all times t. QED

Example 4.10. If the Theorem is used properly one can even generate non-convex foliations: Consider the unit sphere S^2 in the standard contact structure $(\mathbb{R}^3, \ker(dz + r^2 d\theta))$. Recall that this has a characteristic foliation with exactly two singularities: A

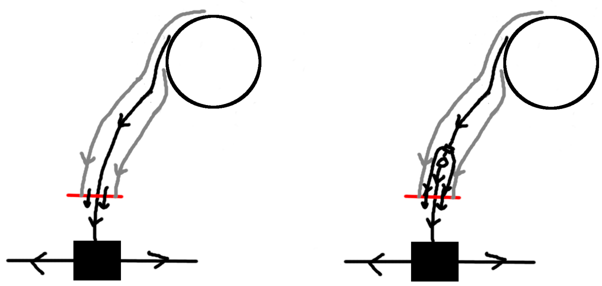

Figure 4.3: A positive node-saddle pair added along the stable separatrix of a negative saddle which connects to a repelling cloed orbit. The red line indicated the dividing curve, while the grey orbits indicate those orbits which bounded the rectangle used in the Creation Lemma 4.9

positive node at the north pole and a negative node at the south pole. Now consider a leaf l which winds from the north to the south pole and two different choices of dividing curve given by transversal contact vectorfields Y and Y' respectively such that on one the dividing curve is close to the south pole and on the other it is close to the north pole. Now using Y, we can generate a positive node-saddle pair close to the south pole where one of the stable separatrices of the saddle coincides with a strip of l. By the Theorem above the convex isotopy can be chosen such that only a neighborhood of the south pole is unchanged.

The same can be done for a negative node-saddle pair close to the north pole with respect to Y', with an isotopy supported away from a neighborhood of the north pole. These isotopies can be composed and thus one can create a retrograde saddle-saddle connection on S^2.

Remark 4.11. Using the Creation Lemma 4.9 one can convexly deform $i(\Sigma)$ to a convex surface Σ' such that $i(\Sigma)$ and Σ' coincide outside an open triangle Δ where one side is the dividing curve and the other two sides are leafs close to l. In Figure 4.3 the case of adding a saddle-node pair along a stable separatrix of a negative hyperbolic point which limits from a repelling closed orbit is illustrated. Lemma 4.7 shows that $i(\Sigma)$ and Σ' coincide outside Δ. In addition, if $l' \neq l$ is any leaf leaving Δ through the side which is part of the dividing curve, one may use the node flexiblity lemma 4.8 to ensure that l and l' form a smooth curve going through the new node n_+.

The following Theorem was first observed by Honda [14, Theorem 3.7.]. Essentially, one can realise any family of properly embedded curves as leafs of a characteristic foliation. The only restriction to this is that one must be able to extend this characteristic foliation to the complement, this for example prohibits that a curve cuts-off a region which has trivial intersection with the dividing curve. Since then the region would have transverse boundary and could only contain positive/negative nodes but not both, so the leafs do not have a valid limit set. Additionally, we need that the characteristic foliation close to the dividing curve and the boundary is respected:

Theorem 4.12. *(Realisation Lemma, [14, Theorem 3.7.]) Let Σ be a convex surface and let Γ_Σ be a dividing curve with a contact vectorfield X. Furthermore, let $C \subset \Sigma$ be a collection of properly embedded curves such that $C \pitchfork \Gamma_\Sigma$ and each component of $\Sigma \setminus C$ intersects Γ_Σ. Then there is a convexly isotopic surface $i_1(\Sigma)$, such that $i_1(C')$ is part of the characteristic foliation, where C' is a family of properly embedded curves having the same amount of intersection points with Γ_Σ as C and C' is homotopic to C.*

Proof. We follow the proof as it is outlined in the introductory lecture notes by Etnyre [6, Theorem 2.28].

Homotop C in such a way that C' coincides with existing leafs close to its intersections with the dividing curve and the boundary. Then we wish to construct a foliation which fits together with C' and respects the previous foliation close to the boundary and close to the dividing curve. So we consider the boundaries of the polygons formed by $\partial\Sigma\ C'$ and Γ_Σ. Now each of these polygons has a boundary component which is part of the dividing curve, since C' is non-isolating. Assume we are considering a planar polygon in the positive region of Σ, then along each boundary component which comes from C' either put a saddle whose unstable separatrices limit to its intersection with Γ_Σ and if there are none put a repelling closed orbit there. If there is an node at the boundary of Σ (or a graph with nodal end) or a closed orbit of C' inside the polygon, then we are fine. If there are none add an node in the interior. Connect all stable separatrices to that repelling orbit, respectively node. Now the last step is to separate boundary polygons from one another, since they are both outgoing there needs to be a separatrix to separate the zones of influence of these boundary polygons. So add positive saddles such that the unstable separatrices divide the polygon into regions, where there is exactly one polygonal boundary component.

Iterate this construction for each component and this yields a foliation which is divided by the dividing curve. The isotopy is guaranteed by Giroux's flexibility Theorem 4.6.

Should a component be non-planar then one needs a different method to generate a foliation for such a component. In the notes by Etnyre [6, Theorem 2.28] he does so using the theory of Morse functions, but we will omit this approach. QED

The Theorem below, will come in handy in a special situation, which we will need later: Recall the classification of closed orientable surfaces. These say that any closed orientable surface is uniquely (up to diffeomorphism) classified by its Euler characteristic. Now consider the unit disk Σ with n interior disks removed, then we can glue a second copy of Σ to itself and this will double the characteristic foliation $\chi(\Sigma') = 2\chi(\Sigma) = 2(1-n)$ and produces a closed surface Σ'. In this manner we can produce any closed orientable surface Σ', together with $\partial\Sigma \subset \Sigma'$ which decompose Σ' into planar regions. Using this, we can deform any convex surface to admit an essentially planar characteristic foliation(see Definition 2.17. This was first observed by Giroux [10, Lemme 2.9.]:

Lemma 4.13. *(Essential planarity realisation) Let Σ be a closed convex surface embedded in a contact manifold (M, ξ). Then there is a convex isotopy of Σ such that the characteristic foliation is essentially planar.*

Proof. The idea is to take the set of embedded curves C above, which decomposes Σ into two regions(if it is connected, otherwise iterate this argument) and use the Realisation Lemma. However, this does not yield the desired decomposition, we need to make a slight modification.

First deform C so that it is non-separating, by adding at least one intersection with the dividing curve on each C. This homotopy of C does not change the condition that C bisects Σ into planar regions, if we ensure that C does not add intersection with itself. So it is just a deformation of the disk with n-holes. Then instead of adding positive saddles at the appropriate points of C, we add a node, which is shielded from the interior of each polygonal region by a saddle whose unstable separatrices leave the polygon parallel to the arcs of C.

In this way, we realise C in such a way that it is foliated by positive nodes, negative saddles and their stable separatrices. Thus there is a small tubular neighborhood A of C such that all orbits leave transversally to its boundary. Now A and its complement are both planar where A is a collection of annuli while the complement is a deformation retract of $\Sigma \setminus X$. In addition the characteristic foliation points out of A and into the complement, thus fulfilling the conditions of an essentially planar foliation. QED

4.2 Overtwisted disks

We will only briefly cover this topic and omit most details. A good ressource for this discussion is the introductory book by Geiges [8, Chapter 4.5.].

Consider a convex surface Σ with a homotopically trivial component C of the dividing curve. If Γ consists of more components than C then one can Legendrian realize a slightly larger curve C' using the Realisation lemma 4.12. This will lead to a disk D^2 with boundary C' which does not intersect the dividing curve, such a disk or the non-existence of such disks is of fundamental importance:

Definition 4.14. *Let D^2 be a convex disk embedded into (M, ξ). If $tb(D^2, L) = 0$ where L is the boundary of D^2. Then we call D^2 an **overtwisted disk**.*

We already encountered such disks in Example 3.5. The above method is one of the major ways to find such disks. However reversely, if in each neighborhood of a surface Σ there is an overtwisted disk and Σ has at least smooth boundary. Then one can prove that Σ has a homotopically trivial component. Only the so-called tight sphere (which we encountered in Example 3.21) has a dividing curve with a homotopically trivial component. If it has exactly 1 component then the sphere is not overtwisted.

Lemma 4.15. *(Giroux Criterion) Let $\Sigma \neq S^2$ be an embedded convex surface. If Σ has a homotopically trivial component of the dividing curve, then in every neighborhood of Σ*

there is an overtwisted disk. In other words, if there are no overtwisted disks in a neighborhood of Σ then the dividing curve of Σ contains no homotopically trivial components.

Proof. We will only prove a partial result, assuming that the dividing curve of Σ has at least two connected components. For the general case, see the lecture notes by Etnyre [6, Theorem 3.1.] where the following proof is taken from.

Let $\Sigma \times \mathbb{R}$ be a $\mathbb{R}$-invariant neighborhood of Σ. Then let γ be a curve parallel to the homotopically trivial component. Now one component of $\Sigma \setminus \gamma$ which contains the homotopically trivial component contains a component of the dividing curve and the other piece of $\Sigma \setminus \gamma$ also contains a component of the dividing curve, so we may realise γ as a piece of the characteristic foliation using the Realisation lemma 4.12. This γ then intersects no component of the dividing curve. Now γ bounds a disk and $\alpha = \beta \pm dt$ close to γ. Thus the Thurston-Bennequin map of γ does not map to either positive, respectively negative preimages of the map defined in Definition 3.14. Thus the disk bound by γ is an overtwisted disk.

Using Giroux's flexibility theorem 4.1 we can realise an overtwisted disk in any neighborhood $\Sigma \times (-\epsilon, \epsilon)$ of Σ. QED

As noted in the Introduction 1 the existence or lack of overtwisted disks is of fundamental importance.

Definition 4.16. *Let U be an open neighborhood of a contact manifold (M, ξ), we call U a* ***tight*** *neighborhood, if there are no overtwisted disks in U. We call (M, ξ)* ***tight*** *if $U = M$ is a tight neighborhood.*

If M is not tight, it is called ***overtwisted****.*

Remark 4.17. The dichotomy of tight and overtwisted contact structures(those which contain overtwisted disks) is of fundamental importance: Overtwisted contact structures exist abundandly on every closed manifold and are essentially classified by homotopy (a result due to Eliashberg [3, Theorem 1.6.1.]). On the other hand tight contact structures need not exist on all manifolds. So there are several interesting classification results, the most fundamental one is due to Bennequin [1, Théorème 1]: The standard contact structure on $\mathbb{R}^3$ is tight.

One of the most important results is the following classification results due to Eliashberg:

Theorem 4.18. *([4, Theorem 2.1.3.]) Two tight contact structures ξ^0 and ξ^1 on B^3, which agree on a neighborhood of ∂B^3 are isotopic relative to ∂B^3.*

4.3 Giroux's normal form

Now, we come to one of the main theorems of this thesis: Giroux's normal form.

Theorem 4.19. *(Giroux normal form, [11], Lemma 15) Let $(\Sigma \times [-1, 1], \xi)$ be a contact manifold, where Σ is a closed surface such that Σ_{-1} and Σ_1 are convex. Then there is an* *isotopy relative to the boundary such that Σ_r is convex for all t, except for finitely many r_i, where the characteristic foliation on Σ_{r_i} fulfill:*

(i) Each limit set is either a singularity or a closed orbit;

(ii) All singularities are either nodes or saddles;

(iii) All orbits are non-degenerate;

(iv) There is one and only one retrograde saddle-saddle connection.

In other words the characteristic foliation on Σ_{r_i} has a saddle-saddle connection bifurcation whose saddle connection is retrograde.

Where $\Sigma_{\pm 1}$ is convex in the sense that there is a transversal contact vectorfield. By possibly changing from Y to $-Y$ this contact vectorfield can be assumed to be inward pointing.

The most important steps of the proof are highlighted here, however we will not prove the theorem in full detail. For example, we will omit some analytical considerations which are due to Giroux [11].

A priori, we need the following lemma to obtain control over contact structures on $\Sigma \times [-1, 1]$ where each $\Sigma_r := \Sigma \times \{r\}$ is divided by the same dividing curve $\Gamma_r := \Gamma_r$. The following lemma is a combination of two results by Giroux [11, Lemme 2.4. and Lemme 2.7.]. The proof is a variant of the one found in his original work.

Lemma 4.20. *(Special case of the Realisation and Uniqueness Lemmas, [11, Lemme 2.4. and Lemme 2.7.]) Let $\xi^0 = ker(\alpha^0)$ be a contact structure on $\Sigma \times [-1, 1]$ such that for $\beta = \alpha^0|_{\Sigma^r}$ there is a function $v : \Sigma \to \mathbb{R}$ such that $\beta_r \wedge dv + vd\beta_r > 0$. Then ξ^0 is isotopic to a contact structure ξ^1 relative to the boundary (i.e. there is a neighborhood of Σ_{-1} and Σ_1 such that $\xi^0 = \xi^1$) such that Σ_r for $|r| < \epsilon$ has any foliation $\mathcal{F}$ with a representation β such that $\beta \wedge dv + vd\beta > 0$.*

Proof. Let ξ^0 be represented by $\alpha^0 = \beta_r + u_r^0 dr$. Then denote by β'_r the foliation which is given by: $\beta'_r = \beta_{H(r)}$ where H is a smooth transition function which agrees with r close to $r = -1$ and $r = 1$ and is constantly 0 on $r \in [-\frac{1}{2}, \frac{1}{2}]$. Denote by γ_r the family of 1-forms which is obtained by realising $\mathcal{F}$ using Giroux's flexibility theorem 4.6. This can be done in such a way that $\gamma_0 = \beta_0$ and $\gamma_0 + vdr$ is a contact form. Then set $\beta_r^1 = \gamma_{\tilde{H}(r)}$ where $\tilde{H} : [-\frac{1}{2}, \frac{1}{2}] \to [0, 1]$ is a smooth transition function which is constantly 0 on a neighborhood of $-\frac{1}{2}$ constantly 1 on a neighborhood of 0, constantly 1 on a neighborhood of $\frac{1}{2}$ and which coincides with β'_r outside of $[-\frac{1}{2}, \frac{1}{2}]$.

By construction both β_r^0 and β_r^1 fulfill that $\beta_r^i + vds$ is a contact form for r constant. Thus $\beta_r^0 + vdr$ and $\beta_r^1 + vdr$ are contact forms, if necessary by multiplying v by some large constant λ, compare Equation 3.2.1. In addition, $\beta_r^0 = \beta_r^1$ for r close to -1 and 1, so there are neighborhoods $[-1, -1 + \epsilon]$ and $[1 - \epsilon, 1]$ such that $\beta_r^1 + u_r^0 dr$ satisfies the contact condition as well. Now denote by u_r^1 the function which is given by u_r^0 close to -1 and 1 and by λv for all r such that $\beta_r^0 \neq \beta_r^1$.

Now $\alpha^0 = \beta_r^0 + u_r^0 dr$ is convexly isotopic to $\beta_r^1 + u_r^1 dr$, since the contact condition for such expansions (see Inequality 3.2.1) is convex linear in the u-component. Now set $\beta_r^t = (1-t)\beta_r^0 + t\beta_r^1$. This is constantly $\beta_r^0 = \beta_r^1$ if $u_r^1 \neq \lambda v$ and otherwise we observe the following condition:

$$0 < \beta_r^t \wedge (\lambda dv - \dot{\beta}_r^t) + \lambda v d\beta_r^t$$

which is fulfilled for all r and t if λ is sufficiently large. So $\beta_r^0 + u_r^1 dr$ is isotopic to $\alpha^1 = \beta_r^1 + u_r^1 dr$. Putting both isotopies together α^0 is convexly isotopic to α^1 and both isotopies where supported away from $r = -1, 1$. QED

The second main tool is Sotomayor's density Theorem which we restate here for convenience:

Theorem 4.21. *(Sotomayor's density Theorem, [21, Theorem II.2.]) Denote by $\mathfrak{X}^1$ the set of 1-parameter vectorfields $(X_r)_{r\in[-1,1]}$ such that:*

(i) There is an open dense set $J \subset [-1,1]$ such that Σ_r is Morse-Smale for $r \in \Lambda$.

(ii) J^C decomposes into three subsets $J_1 \cup J_2 \cup J_3 \cup J_4 \cup J_5$:

- *a) $r \in J_1$ is Kupka-Smale, i.e. all singularities and closed orbits are non-degenerate and there are no saddle connections;*
- *b) $r \in J_2$ is Morse-Smale except for exactly 1 saddle-node;*
- *c) $r \in J_3$ is Morse-Smale except for exactly 1 composed focus;*
- *d) $r \in A_4$ is Morse-Smale except for exactly 1 degenerate closed orbit whose second derivative is non-vanishing;*
- *e) $r \in A_5$ is Morse-Smale except for exactly 1 saddle-saddle connection. This connection is either homoclinic or heteroclinic.*

Then $\mathfrak{X}^1$ is C^∞-dense in the set of all 1-parameter vectorfields.

Now, we are ready to prove Giroux's Dynamic Banalisation Lemma:

Theorem 4.22. *(Giroux's Dynamic Banalisation Lemma, [11, Lemme 2.10] Assume that $\Sigma \times [-1,1]$ is endowed with a contact structure such that $\Sigma_{\pm 1}$ are convex. Then there is an isotopy relative to the boundary such that either Σ_r is convex or fulfills the Poincaré-Bendixson property.*

Proof. We outline here the original proof:

The main problem is that the Poincaré-Bendixson property is not an open condition. However, essential planarity and a foliation having only isolated singularities are both open conditions which are stronger than the Poincaré-Bendixson property so we wish to change ξ to a contact structure such that:

- Each Σ_r is essentially planar for $|r| \leq 1 - \epsilon$;

- The singularities of Σ_r are isolated for each $|r| \leq 1 - \epsilon$.

Since both Σ_{-1} and Σ_1 are convex, there is a function $v_{\pm 1} : \Sigma \to \mathbb{R}$ such that $\beta_{\pm 1} + v_{\pm 1} dr$ are contact. After a C^∞-small perturbation we may assume that $v_1^{-1}(0)$ is transversal to $v_{-1}^{-1}(0)$ and the contact condition together with $\beta_{\pm 1}$ is still fulfilled. Since the contact condition is open, we may assume without loss of generality that $\beta_r + v_1 dr$ is a contact form for $r \in [\frac{1}{8}, 1]$, similarly for v_{-1}. Now let C be a set of curves which decompose Σ into two planar regions, possibly isotoping C we may assume that each component of C intersects both $v_{\pm 1}^{-1}(0)$ and coincides with leaves of $\Sigma_{\pm \frac{1}{2}}$ close to $v_{\pm 1}^{-1}(0)$. Using the essentially planar realisation lemma 2.17, we may assume that there are foliations $\mathcal{F}_\pm$ which contain C as a collection of leafs and are divided by $v_{\pm 1}^{-1}(0)$. This can be achieved using Lemma 4.20.

Now using the following construction one can reparametrise $\Sigma \times [-1, 1]$ in such a way that Σ_r is convex or it is essentially planar: Denote by $\Pi^\pm$ a retract of $\Sigma^\pm$ such that the characteristic foliation of $\Sigma_{\pm \frac{1}{2}}$ are still transversal to $\partial \Pi^\pm$. This is an open condition on the characteristic foliations, thus we assume this is fulfilled on $\Sigma \times [-\frac{3}{4}, -\frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{4}]$. Then we choose a strictly increasing function $g : [-1, 1] \to [-1, 1]$ which coincides with the identity on $[-1, -\frac{3}{4}] \cup [\frac{3}{4}, 1]$ and maps $[0, 1]$ into $[\frac{1}{2}, 1]$. Denote by $h : \Sigma \times [-1, 1] \to [-1, 1]$ a smooth function which coincides with g on Π^+ and with $-g(-\cdot)$ on Π^- such that $h(x, \cdot)$ is strictly increasing and coincides with the identity on $[-1, -\frac{3}{4}] \cup [\frac{3}{4}, 1]$. Then we can define the following isotopy:

$$\phi_t(x, r) = (id, th(r, x) + (1 - t)r)$$

Now for $|r| \geq \frac{3}{4}$ Σ_r is convex, since ϕ_t coincides with the identity. On the other hand if $|r| \leq \frac{3}{4}$, say r is positive, then $\partial \Pi_r^+$ decomposes Σ_r into planar regions such that the characteristic foliation points out of Π_r^+.

To use the Poincaré-Bendixson Theorem 2.16 on $\Sigma_r^\pm$ for each $|r| \leq \frac{3}{4}$, we only need that the singularities are isolated: So we apply Sotomayor's density Theorem 4.21 on $\Sigma \times [-\frac{3}{4}, \frac{3}{4}]$ with the following C^∞-smallness conditions:

(i) $\alpha^1 = \beta_r + H(r)t\gamma + u_r dr$ is contact for each $t \in [0, 1]$ and r where $H(r)$ is some smooth cut off function which vanishes at $\frac{1}{2}$;

(ii) The foliation given by $\beta_r + H(r)t\gamma$ is divided for $|r| \leq \frac{1}{2}$ and $t \in [0, 1]$:

(iii) The foliation induced by $\beta_r + H(r)t\gamma$ is essentially planar for $|r| \leq \frac{1}{2}$.

Now, we may use Moser's stability trick to obtain a contact isotopy from α to α^1. Since the level sets Σ_r for which $|r| \geq \frac{1}{2}$ are convex and the level sets for which $|r| \leq \frac{1}{2}$ fulfill the Poincaré-Bendixson property. QED

From this proof, we actually obtain more information: Namely that we can decompose $[-\frac{1}{2}, \frac{1}{2}]$ into the regular values and the 4 different bifurcation sets J^2, J^3, J^4 and J^5,

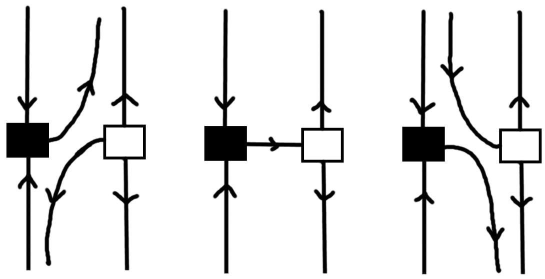

Figure 4.4: The evolution of a retrograde saddle-saddle connection.

where $J^1 = \emptyset$ since the foliations fulfill the Poincaré-Bendixson property. In addition $J^3 = \emptyset$ since a composed focus is an isochore singularity and the contact condition of α^1 prevents those from appearing.

Now Theorem 2.22, tells us that Σ_r is only non-convex if $r \in J^4 \cup J^5_{ret}$ where J^5_{ret} are those values of J^5 which possess a retrograde saddle-saddle connection. So, we need a technique to achieve that $J^4 = \emptyset$. This follows from the following isotopy theorem due to Giroux:

Lemma 4.23. *(Giroux [10, Lemme 15/16]) Let T be filled torus around a degenerate closed orbit $C_{r'}$ such that there are no other closed orbits and singularities inside for r close to r'. Then there is an isotopy of $\Sigma \times [-1, 1]$ supported inside T and for r close to r' such that there is a retrograde saddle-saddle connection at r' and no degenerate closed orbits for r close to r' on Σ_r. The move creates 4 singularities e^+,e^-, h^+ and h_- along $C_{r'}$. In addition, the retrograde saddle-saddle connection is the unique degeneration of $C_{r'}$.*

QED

In addition, a retrograde saddle-saddle connection is unstable in 1-parameter families of characteristic foliations:

Theorem 4.24. *(Giroux's Crossing Lemma, [10, Lemme 2.14]) Let $(\Sigma \times [-1, 1], \xi)$ be a contact manifold. Assume for $r = 0$ there is a retrograde saddle-saddle connection C_0 between h^- and h^+ in the characteristic foliation of Σ_0. Furthermore let A be a transverse arc to C_0 that is positively co-orienting the foliation.*

Denote by C^-_r and C^+_r the separatrices of h^- and h^+ respectively such that $C^-_0 = C^+_0$. Then the intersection point of A with $C^- + r$ moves in the direction of the orientation of A for small r, while the intersection point of A and C^-_r moves in the opposite direction, see Figure 4.4.

Before finishing the proof, we will restate Remark 2.35:

Remark 4.25. (i) J_2 is isolated [21, Proposition 3.5.].

(ii) $r \in J_4$ is either isolated or one of the following happens: [21, Remarks 2.8.b]

a) The degenerate closed orbit is stable and unstable limit set of saddle separatrices. Then saddle-separatrices accumulate;

b) The degenerate closed orbit is not a limit of saddle separatrices but stable and unstable limit set of another orbit. In this case degenerate closed orbits accumulate.

(iii) $r \in J_5$ the saddle separatrix is isolated or the following happens: [21, Remarks 4.8.1.]

a) The saddle connections has the same saddle as stable and unstable limit set. Additionally another separatrix has the saddle connection as its limit set. Then saddle connections accumulate.

Proof of Theorem 4.19. First we apply the Dynamics Banalisation Lemma 4.22 and obtain an isotopy of the contact structure such that ξ is either convex or fulfills the Poincaré-Bendixson Theorem. So Σ_r is convex if $r \notin J^4 \cup J^5_{ret}$. Now by (iii)(a) above J^5_{ret} is isolated and may only accumulate around J^4. In addition J^4 is finite since a closed orbit divides either Σ^+ or Σ^- into two different components where the foliation does not return to. So we may apply Lemma 4.23 to achieve a contact structure ξ^1 such that $J^4 = \emptyset$. In addition, since this isotopy replace all degenerate closed orbits with retrograde saddle connection we only need to care about J^5_{ret}. However, due to the Crossing Lemma 4.24 J^5_{ret} may only consist of points which may only be finitely many since by Remark 4.25 saddle connections may only accumulate around homoclinic orbits. Since homoclinic orbits are prograde connections the level set r' where one occurs has a neighborhood where all foliations are divided. Thus J^5_{ret} is isolated and thus finite.

QED

4.4 Bypasses

In this chapter, we will introduce the powerful technique of bypasses due to Honda [14]. Roughly speaking, a bypass on a convex surface is a transversal convex half-disk. These bypasses can be attached to the convex surface and quantize its change to the dividing curve. As we will prove at the end of this chapter, one can equivalently describe Giroux's normal form Theorem through bypass attachments (see Theorem 4.33). Though first, we will make precise what "attaching" a convex surface means. The procedure to attach bypasses presented here is due to Honda [14]. Our definition of what a bypass is, is different from the original definition, however Honda has proven that both definitions are actually equivalent. So, we will not deal with such details.

Consider two convex surfaces Σ and Σ' meeting transversaly along a smooth boundary component L. In general, we cannot just glue them, though if a neighborhood of L is sufficiently regular, then we can do so:

Definition 4.26. *Let Σ be an embedded surface with smooth boundary. We say that a boundary component L of Σ has a **standard neighborhood**, if the characteristic foliation* *on a neighborhood of L coincides with the foliation of $\{(x,y,z) : x = 0, y \geq 0\}$ in $(\mathbb{R}^2 \times S^1, \ker(cos(2\pi nz)dx + sin(2\pi nz)dy))$ for some n.*

*If L is a smooth Legendrian curve in the interior of Σ, then we say that L has a **standard neighborhood**, if the characteristic foliation coincides with the foliation of $\{(x,y,z) : x = 0\}$ in $(\mathbb{R}^2 \times S^1, \ker(cos(2\pi nz)dx + sin(2\pi nz)dy))$ for some n.*

Remark 4.27. One may calculate that the Thurston-Bennequin invariant of L with respect to a surface Σ which bounds L is $-n$ if L has a standard neighborhood in Σ. If Σ is convex then one may calculate the Thurston-Bennequin number of L as $-\frac{|\Gamma \cap L|}{2}$.

Now using this local model of $L \subset \Sigma$ we see that there are lines of positive singularities for $z = \frac{k}{n}$ and lines of negative singularties for $z = \frac{2k+1}{2n}$ on this neighborhood of L.

Now if both Σ and Σ' exhibit such neighborhoods of $L = \Sigma \cap \Sigma'$. Then L can be locally modelled by the following standard neighborhood $(\mathbb{R}^2 \times S^1, ker(cos(2\pi nz)dx + sin(2\pi nz)dy))$ where $L = \{(x,y,z) : x = y = 0\}$ and $\Sigma = \{(x,y,z) : x = 0, y \geq 0\}$ and $\Sigma' = \{(x,y,z) : x \geq 0, y = 0\}$ (see [6, Exercise 4.2.]). Note that in this local model both $\frac{d}{dx}$ and $\frac{d}{dy}$ are contact vectorfields which are transversal to Σ, respectively Σ' close to L. Now one observes that the dividing curve of Σ induced by $\frac{d}{dx}$ is along the lines $\{(x,y,z) : x = 0, y \geq 0, z = \frac{k}{2n}\}$ for $k = 1, \ldots, 2n$. Similarly for Σ' where the dividing curve induced by $\frac{d}{dy}$ is given through $\{(x,y,z) : x \geq 0, y = 0, z = \frac{k}{2n} - \frac{1}{4n}\}$ for $k = 1, \ldots, 2n$. Recall that the vectorfield extension lemma 4.5 justifies only looking at partially defined contact vectorfields given that the foliations fulfill the Poincaré-Bendixson property (which we always assume going forward).

Now one may connect Σ and Σ' by removing a δ-neighborhood of L and replacing it with $\{(x-\delta)^2 + (y-\delta)^2 = \delta^2\}$ which is convex as well (using the contact vectorfield $-\frac{d}{dr}$ which is induced by cylindrical coordinates (r, θ, z) instead of (x, y, z)). One may approximate the resulting convex C^1-surface by a C^∞-surface with the same characteristic foliation. The important information however, is how the dividing curves link up: In the above model Σ has dividing curves at $\{(x,y,z) : x = 0, y \geq 0, z = \frac{k}{2n}\}$ for $k = 1, \ldots, 2\pi$ and Σ' has dividing curves at $\{(x,y,z) : x \geq 0, y = 0, z = \frac{k}{2n} - \frac{1}{4n}\}$. Besides the area close to the original boundary, $\Sigma + \Sigma'$ will have a decomposition into positive and negative areas that respects the ones coming from Σ and Σ'. So for k even there is a positive area between $\frac{k}{2n}$ and $\frac{k+1}{2n}$ on Σ and a positive area on Σ' between $\frac{k}{2n} - \frac{1}{4n}$ and $\frac{k+1}{2n} - \frac{1}{4n}$. So the dividing curve on $\Sigma + \Sigma'$ must connect $\frac{k}{2n}$ to $\frac{k}{2n} - \frac{1}{4n}$. Since this neighborhood is bounded by Legendrian curves, one may use Lemma 4.5 to see that one may extend the description of the dividing curve on this piece of $\Sigma + \Sigma'$ to the rest of this convex surface.

Attaching an annulus is of special importance: Assume that Σ and $\Sigma' = S^1 \times [0,1]$ intersect in the interior of Σ along a Legendrian knot L. Furthermore assume that $S^1 \times [0,1]$ has a standard neighborhood in Σ' and $S^1 \times 0 = L$ has one as well in Σ, then one can do a series of edge-roundings to attach the annulus to Σ': The intersection of Σ and Σ' can be modelled in $(\mathbb{R}^2 \times S^1, ker(cos(2\pi nz)dx + sin(2\pi nz)dy))$ as the intersection of $\Sigma = \{(x,y,z) : x = 0\}$ and $\Sigma' = \{(x,y,z) : x \geq 0, y = 0\}$. The idea then is thicken Σ' to $\Sigma' \times [-\epsilon, \epsilon]$ via an extension of $\frac{d}{dy}$, where ϵ is chosen sufficiently small such that $\Sigma' \times \{-\epsilon, \epsilon\}$ still intersects Σ along a knot which has a standard neighborhood. This is possible since $L \subset \Sigma$ has a small neighborhood L where it is normal, so a push-off of L by $\frac{d}{dy}$ (which is tangent to Σ) for sufficiently small values will still have a normal neighborhood.

Then we can attach $\Sigma' \times \{-\epsilon, \epsilon\}$ to $\Sigma \setminus L \times (-\epsilon, \epsilon)$. This leaves the upper boundaries of $\Sigma' \times \{-\epsilon, \epsilon\}$ where one can attach $S^1 \times \{1\} \times [-\epsilon, \epsilon]$ which is also convex since it consists of parallely foliated Legendrian curves. Since this will be of importance, Figures 4.5 and 4.6 illustrate attaching an annulus with a negative boundary-parallel region:

The above example is already the main ingredient one needs to discuss bypasses:

Definition 4.28. *Let Σ be a convex surface and Σ' a convex annulus which intersects Σ transversally along a Legendrian knot L and Σ' is standardly foliated on a neighborhood of its boundary components. We say that Σ' is a **bypass** for Σ, if the dividing curve of Σ' has exactly one component whose both ends are on L while all other components connect different boundary components.*

By a result of Kanda it is sufficient that Σ' has standardly foliated neighborhoods of its boundaries:

Lemma 4.29. *(Kanda [15, Lemma 5.10.]) Let Σ be a convex surface and Σ' an annulus such that they intersect transversally along a Legendrian knot $L \subset \Sigma$. If Σ' is standard around L then there is a C^0-small convex isotopy of Σ which is contained in a neighborhood of $L \subset \Sigma$ such that L is standard in Σ, Σ and Σ' still intersect transversally along L.*

A bypass attachment is then the series of edge-roundings described above. In particular, from Figure 4.6 one can see the following theorem:

Theorem 4.30. *(Bypass attachment, [14, Lemma 3.12.]) Let Σ' be a bypass for a convex surface Σ, then there is a neighborhood $(\Sigma \times [0,1], \xi)$ of $\Sigma \cup \Sigma'$ such that ξ is r-invariant on $\Sigma \times [0, \epsilon]$, $\Sigma \times \{\epsilon\}$ is the original embedding and the dividing curve of $\Sigma \times \{1\}$ is related to the dividing curve of Σ by the bypass attachment deformation, see Figure 4.7*

For our purposes, we will need a definition of bypass attachment which is a bit more general:

Definition 4.31. *Let Σ and Σ' be convex surfaces in a contact structure. We say that Σ and Σ' are related by a bypass attachment, if there are convex isotopies taking Σ to Σ_1 and Σ' to Σ'_1 such that $\Sigma_1 + A$ is Σ'_1, where A is a bypass for Σ_1.*

4.5 Proof of Theorem 1.1

One of the first tools found by Honda is that an annulus with several components of the dividing curve which connect the same boundary to itself can be reduced to a bypass:

Theorem 4.32. *(Imbalance Principle, Honda [14, Proposition 3.17.]) Let $\Sigma' = S^1 \times [0,1]$ be a convex annulus inside a tight contact manifold with a boundary which has a standard neighborhood. If $tb(S^1 \times \{0\}) < tb(S^1 \times \{1\}) < 0$ then there is a bypass along $S^1 \times \{0\}$ and thus on any convex surface Σ which intersects Σ' transversally along $S^1 \times \{0\}$.*

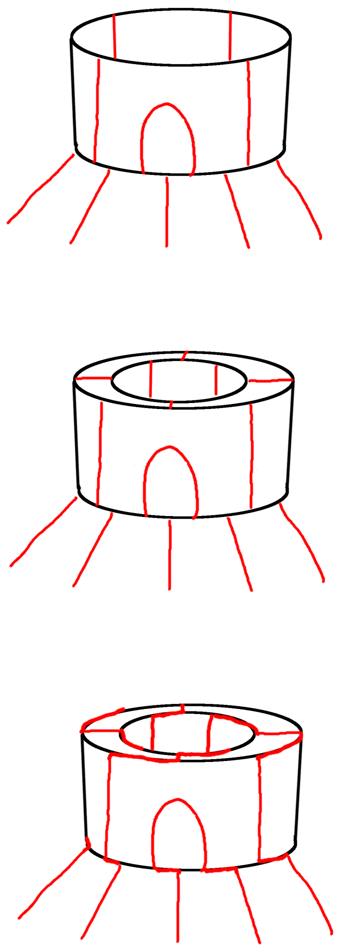

Figure 4.5: Attaching a bypass: First there is only Σ and a transversal annulus. Then we double the annulus and close of the top with a contact push-off of of the upper boundary of the annulus. Then we use Edge-Rounding and glue all components together.

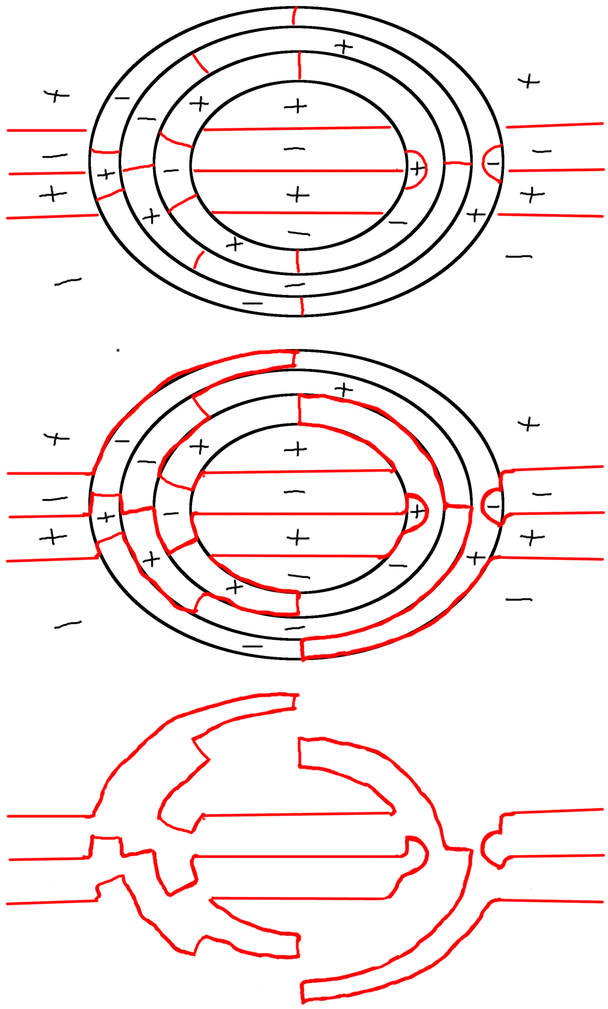

Figure 4.6: Attaching a bypass from bird's-eye view. The second and third picture both describe $\Sigma + \Sigma'$. The second picture has the old boundaries marked while they are removed in the third picture to highlight the new dividing curve. Straightening out this curve leads to the picture in Figure 4.7

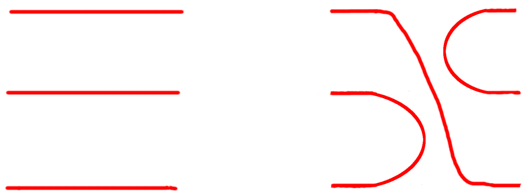

Figure 4.7: The evolution of a retrograde saddle-saddle connection.

Proof. Since the annulus in contained in a tight manifold it may not have any components of the dividing curve which bound a disk. Furthermore $S^1 \times \{0\}$ has $-2tb(S^1 \times \{0\})$ intersections with the dividing curve, while $S^1 \times \{1\}$ has $-2tb(S^1 \times \{1\})$ many. Thus one component of the dividing curve must start and end on $S^1 \times \{0\}$. In particular there must be a ∂-parallel one. Using the Legendrian realisation principle one can isotop Σ' such that a curve C parallel to $S^1 \times \{0\}$ is standardly foliated and Σ' restricted to the area between C and $S^1 \times \{0\}$ has exactly one component whose dividing curve intersects $S^1 \times \{0\}$ twice while all others connect C and $S^1 \times \{0\}$. QED

Now we will come to the original content of this thesis. Compare Theorem 4.19:

Theorem 4.33. *Let $(\Sigma \times [-1, 1], \xi)$ be a contact manifold such that Σ_{-1} and Σ_1 are convex. Then there is an isotopy relative to the boundary such that Σ_r is convex except for finitely many $r_1, \ldots, r_n$. Additionally $\Sigma_{r_i-\epsilon}$ and $\Sigma_{r_i+\epsilon}$ are related by a bypass attachment for sufficiently small $\epsilon > 0$.*

This is a direct corollary of the following lemma which we will prove in the rest of the paper:

Lemma 4.34. *Let $(\Sigma \times [-1, 1], \xi)$ be a contact manifold such that Σ_r for $r \neq 0$ is convex. Additionally, assume the characteristic foliation of Σ_0 fulfills:*

- *Each limit set is either a singularity or a closed orbit;*
- *All singularities are either nodes or saddles;*
- *All orbits are non-degenerate;*
- *There is exactly one saddle-saddle connection which is retrograde;*
- *The foliation is essentially planar.*

Then for ϵ sufficiently small there is a bypass relating $\Sigma_{-\epsilon}$ and Σ_ϵ.

The first step is to normalise a neighborhood of the retrograde saddle-connection and make sure that it changes the dividing curve as we would expect from a bypass:

Theorem 4.35. *Assume $(\Sigma \times [-1,1], \xi)$ is in Giroux normal form, with exactly one non-convex level set at 0. Then for a sufficiently small $\epsilon > 0$ there exists an isotopy of $\Sigma \times [-\epsilon, \epsilon]$ such that:*

(i) Σ_r for $r \neq 0$ is moved by a convex isotopy;

(ii) There is a disk $D \subset \Sigma$ with Legendrian boundary such that D_0 contains the retrograde saddle connection. Its foliation is depicted in Figure 4.8;

(iii) The contact structure is r-invariant on $(\Pi \times [\epsilon, \epsilon], \xi)$, where Π denotes the complement of D.

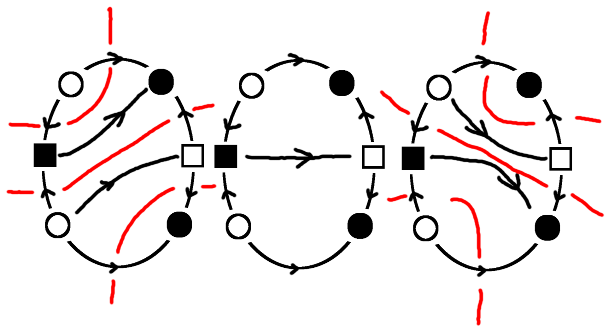

Figure 4.8: The foliation of D_-, D_0 and D_ϵ. Note that the change of the dividing curve coincides with the way we would expect it to change if it came from a bypass with a half-disk over D_{-1}.

Proof. Denote by h_+ and h_- the saddles connected by the retrograde saddle connection. Let p be a point on the retrograde saddle connection and denote by U a small neighborhood of p. Choose coordinates on U such that the characteristic foliation is given by $\beta = gdy$ where $Y > 0$ now isotop β_0 slightly contained in this neighborhood (using a smooth cut-off function) so that β_0 agrees with $\beta_0 = gdy + \epsilon dx$ in this neighborhood. For ϵ small enough the corresponding $\alpha_\epsilon = \beta_r + H\epsilon dx + u_r dr$ will still fulfill the contact condition for H an appropriate cut-off function. Denote by Σ' the surface obtained from Σ by the isotopy gained from Moser's stability trick as used in Lemma 3.25. Then Σ' will be a convex surface: Since there was only one heteroclinic saddle-saddle connection there is no accumulation of separatrices close to the retrograde saddle connection. So we forced the separatrices of h_- and h_+ to miss one another and did not introduce any new saddle-saddle connections for ϵ sufficiently small. In addition outside of $\bar{U}$, Σ and Σ' overlap. So we obtain a contact vectorfield Y transverse to Σ'. Now consider the vectorfield Y' which agrees with Y outside a neighborhood $V \subset \Sigma \times [-1,1]$ of h_+, h_- and the retrograde saddle connection and agrees with $\frac{d}{dr}$ on a smaller neighborhood of the retrograde saddle connection. Thus Y' is transverse to Σ. Now, consider the flow of Σ under Y' for small times. The pullback contact structure on $\Sigma \times (-\epsilon, \epsilon)$ will fix the characteristic foliation on $\Sigma \times \{0\}$. By the Reconstruction Lemma 3.24, the original contact structure on $\Sigma \times (-\epsilon, \epsilon)$ (viewed as a subset of $\Sigma \times [-1, 1]$) and the pullback contact structure given by the flow are related by a contact isotopy (after possibly shrinking ϵ).

Possibly rescaling Y' by a constant factor this gives a new trivialisation of $\Sigma \times [-1, 1]$ as a product (where r still denotes the coordinate on the second factor). We notice that this isotopy fixes the characteristic foliation of Σ_0, which is essentially planar and has only isolated singularities. Both are open conditions, thus this is true for the whole isotopy on $\Sigma \times (-\epsilon, \epsilon)$(possibly shrinking ϵ again). Now by the Poincaré-Bendixson Theorem 2.16 all these characteristic foliations fulfill the Poincaré-Bendixson property. Since two saddles not being connected by a saddle connection is an open condition, if there are no degenerate closed orbits or Legendrian polygons, for possibly smaller times no new saddle connections are introduced. Additionally by Giroux's Crossing Lemma 4.24, the only retrograde saddle connection on $\Sigma \times \{0\}$ is always unstable in contact structures. Thus for sufficiently small times Σ_r for $r \neq 0$ is moved by a convex isotopy.

Now we have that $\frac{d}{dr}$ is contact away from V which is a neighborhood of the retrograde saddle connection and its saddles on Σ_0. Now let us return to Σ': This is a convex surface with no saddle connections, thus the stable separatrices of h_- and the unstable separatrices of h_+ cross the dividing curve associated to Y. Possibly shrinking the chosen neighborhood V above, the dividing curve crosses these separatrices before it possibly enters the neighborhood V. Now we use the Creation Lemma 4.9 and add positive singularities along the stable separatrices of h_+ and negative singularities along the unstable separatrices of h_+. The neighborhoods chosen can be made disjoint from V and thus this isotopy of Σ' is also an isotopy of Σ, compare Figure 4.9. Using Remark 4.11, one can achieve that this isotopy is supported inside the area where Σ and Σ' overlap.

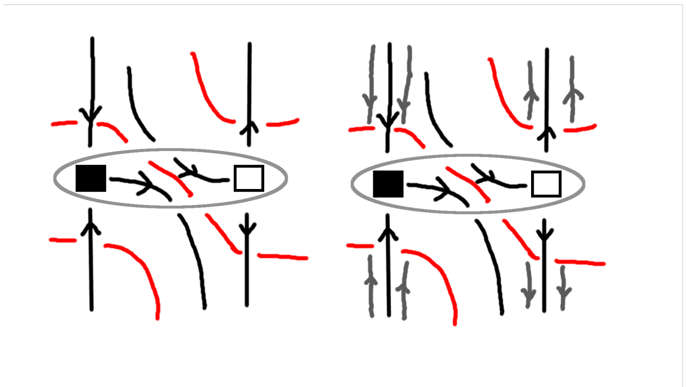

Figure 4.9: A pictorial representation of the neighborhood V in light grey) while the dark grey orbits indicate the rectangles used in the Creation Lemma 4.9 and contain the isotopy away from V.

Now consider this isotopy on surfaces close to Σ_0 in the contact structure created

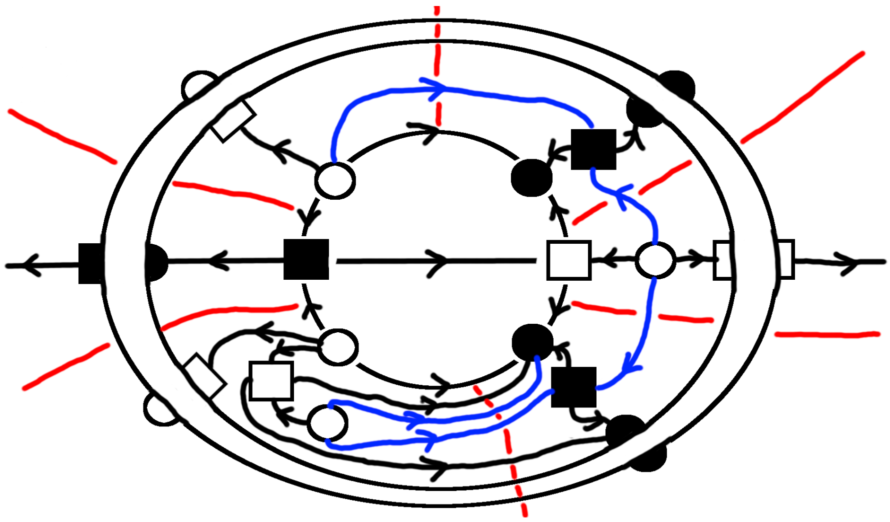

Figure 4.10: One possible characteristic foliation which contains D and can be realised on a tight sphere S^2. The blue arc denotes a possible choice of C.

above. This isotopy is transverse at all times to $\frac{d}{dr}$, so we may consider its push-off in this direction. Our version of the Creation Lemma 4.9 preserves the Poincaré-Bendixson character on Σ_0 for all times thus a push-off of this isotopy in the r-direction will also preserve the Poincaré-Bendixson property for nearby surfaces since the contact structure is r-invariant there. The same argument as above, yields that there is a small neighborhood of Σ_0 such that each level set $\Sigma \times [-\epsilon, \epsilon]$ is moved by a convex isotopy except for the 0-level.

Iterating this argument, we achieve the characteristic foliation depicted in Figure 4.8 on D_0. By Remark 4.11 we may choose the isotopies in such a way that this disk has a smooth boundary. QED

Using the contact vectorfield extension lemma 4.5, we can now see that the retrograde saddle connection gives rise to the same change of dividing curve as a bypass attachment. We wish to find a convex annulus which gives us the desired bypass. To do so, we will first need to find a tight neighborhood of D_0:

Lemma 4.36. *After an isotopy of $\Sigma \times [-1, 1]$, we may assume that Σ_0 has a characteristic foliation as is given by Figure 4.10 and Σ_r for $r \neq 0$ has the same characteristic foliation which only differs inside D_r from this foliation. We call this larger disk D'_r.*

Proof. This is a direct application of Giroux's flexibility Theorem. By Lemma 4.7 this isotopy can be chosen to be supported outside a neighborhood of D_0. Since $\Sigma \times [-1, 1]$ is $\mathbb{R}$-invariant on $\Pi \times [-1, 1]$ it does not matter whether we do the change on Σ_0 or Σ_r. Since D_r is convex and is not moved this isotopy induces a convex isotopy on Σ_r. QED

Lemma 4.37. *The disk D'_0 has a tight neighborhood.*

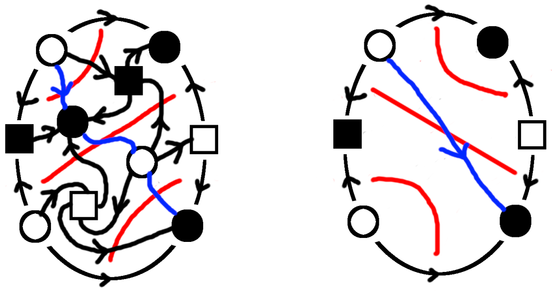

Figure 4.11: One possible characteristic foliation inside D_-, respectively D_+. The extension of C is drawn in blue.

Proof. As observed in Example 4.10, we have a retrograde saddle connection on an S^2 in the standard contact structure which is tight by Remark 4.17. Redoing all the previous steps on S^2 leads to an embedding of a sphere S^2 which contains a disk whose foliation agrees with the one on D'_0. Thus D'_0 has a tight neighborhood. QED

After possibly rescaling, we may assume that all $D' \times [-1, 1]$ have a tight neighborhood.

Lemma 4.38. *For $0 < \epsilon < 1$ there is a convex isotopy of D_ϵ and $D_{-\epsilon}$ which fixes a neighborhood of their boundary. Extending the isotopy to $\Sigma_{-\epsilon}$ and Σ_ϵ we may assume that there are Legendrian knots L_- and L_+ on $\Sigma_{-\epsilon}$ and Σ_ϵ such that $tb(L_-, \Sigma_{-\epsilon}) = -8$ and $tb(L_+, \Sigma_\epsilon) = -6$. Additionally L_- and L_+ are extensions of $C_{\pm\epsilon}$.*

Proof. This is achieved by using Giroux's flexibility Theorem, using the foliations depicted in Figure 4.11. QED

Lemma 4.39. *There is a convex annulus A whose boundaries are L_+ and L_-. This annulus has a ∂-parallel region over $D_{-\epsilon}$ which leads to a bypass inducing the desired change of the dividing curve.*

Proof. Now, we wish to consider an annulus A between L_+ and L_-: This annulus shall be generated by a vectorfield Z which agrees with $\frac{d}{dr}$ on a neighborhood of Π_r, with Y_- close to Σ_- and with Y_+ close to Σ_+, where Y_- and Y_+ are extensions of $\frac{d}{dr}|_{\Pi\times[-1,1]}$ given by the contact vectorfield extension lemma 4.5.

The characteristic foliation of the annulus A, where it is either generated by $\frac{d}{dr}$, Y_- or Y_+ is already determined: Let Σ' be parametrized by $S^1 \times [0, 3]$ where $S^1 \times [0, 1]$ is the push-off of L_- by Y_-, $S^1 \times [2, 3]$ is the push-off of L_+ by Y_+ and if $S^1 \cong [0, 8]/\sim$ then it is generated by $\frac{d}{dr}$ on an open neighborhood of $[0, 3] \times [0, 3]$, where $\{0\} \times \{0, 3\}$ is the strip above the negative node n_- of the boundary of D_- and $\{3\} \times \{0, 3\}$ is the strip above the positive node n_+ of the boundary of D_+.

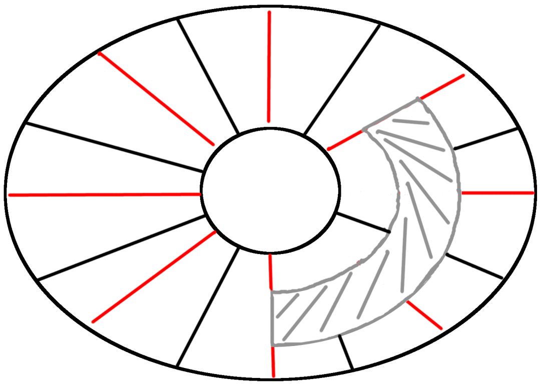

Figure 4.12: The characteristic foliation on the annulus. Outside of the grey area we know exactly how the characteristic foliation looks like and the red lines indicate the 0-set of u' while the black lines indicate lines of singularities.

Restricting A to the section $S^1 \times [0,1]$ where the annulus is a push-off of L_- under a convex vectorfield. So let $\alpha = \beta + udY_-$ be the convex representation on Σ_-. From this we see immediately that: $\alpha|_{S^1\times[0,1]} = udY_-$ so there are lines of singularities above $u = 0$ with signs given by the sign of du. Thus there are positive singularities on Σ' if L_+ crossed from a negative area of Σ into a positive one. Since there are 8 intersections with the dividing curve, we have 8 such lines of singularities. WLOG, these lines of singularities are at $\{k + \frac{1}{2}\}$ on $S^1 = [0,8]/\sim$. In fact, this is a standard neighborhood of the boundary L_- and thus there is a representation of α over $S^1 \times [0,1] \times \mathbb{R}$ as $\ker(cos(\pi z)dy - sin(\pi z)dx)$. This representation remains true on a neighborhood of $[3,8]/\sim \times[0,3]$, since $\frac{d}{dr}$ is a contact vectorfield. The only area that we do not have such a representation yet on is $(0,3) \times [2,3]$. There we have a single line of singularities, as there is only one intersection with the dividing curve of L_+ inside $D_{+\epsilon}$. Considering everything, we obtain a representation of $\alpha = \beta' + u'ds$, where $u' = 0$ if $x = 0,3,4,5,6,7$ or $x = 1,2$ and $y \in [0,1]$ and β' is a 1-form on Σ'' which induces the foliation depicted in Figure 4.12.

Now we will use a slight variation of C^∞-genericity of Morse-Smale vectorfields on $[0,3] \times [1,2]$ and change β' to β'' such that β'' agrees with β' outside a neighborhood of this rectangle. That this kind of change is possible was first observed by Honda [14, Proposition 3.1.]: First we need to put nodal ends on the lines of singularities at $x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ and $y = [0,1]$, respectively $x = \frac{3}{2}, y = [2,3]$. To do so, we add $\delta H(x,y)\frac{d}{dx}$ to X where H is supported inside a neighborhood of the lines $\{\frac{k}{2}\} \times [\frac{1}{2}, 1]$ in such a way that there is a half-nodal singularity at $\{\frac{k}{2}\} \times \{\frac{1}{2}\}$. Depending on the sign of the singularities one has to adapt the sign of H. In addition, one chooses $\delta > 0$ sufficiently small such that the change induced on β' still leads to a contact form. Now, using the Giroux-Peixoto Lemma 3.25 we obtain a C^∞-small γ such that $\beta + \tilde{H}\gamma$ is Morse-Smale on the interior of $[0,3] \times [1,2]$ and which does not create new singularities on a small annulus A' around $[0,3] \times [1,2]$. By construction there was such a non-singuar annulus already and being non-singular is an open condition. $\tilde{H}$ is then a smooth cut-off function supported outside of $A' \cup [0,3] \times [1,2]$ and thus $H\gamma$ is preserved on the complement of $A' \cup (0,3) \times (1,2)$.

Now observe that this new foliation on A is Poincaré-Bendixson. Indeed either any half-orbit remains in $A' \cup [0,3] \times [1,2]$ which is planar and has only finitely many singularities and thus is Poincaré-Bendixson. If the other half-orbit limits to a singularity on the complement, the limit is exactly 1 singularity by the previous characterisation. In addition, one observes that any closed orbit on Σ' must be contractible (the lines of singularities away from $A' \cup [0,3] \times [1,2]$ prevent any such orbits) and thus it would bound an overtwisted disk which by the tightness assumption is impossible. Finally should a retrograde saddle connection appear in $A' \cup [0,3] \times [1,2]$ one may choose a C^∞-small change to disconnect them, similarly to the one done in the beginning of the proof of Theorem 4.35. Saddle connections completely contained in $[0,3] \times [1,2]$ are prevented but they may appear in A'.

Now after this isotopy the annulus A is convex and fulfills the conditions of the Imbalance Principle so there must be at least one ∂-parallel component at the boundary $S^1 \times \{0\}$ by Theorem 4.32. We will show there must be one enclosing the Legendrian divide $\{1\frac{1}{2}\} \times [0, \frac{1}{2}]$. There are two main observations:

- There can be no ∂-parallel components of the dividing curve of A enclosing the lines of singularities at $\frac{1}{2}$ and $2\frac{1}{2}$.
- The lines of singularities originating at $\{k+\frac{1}{2}\} \times \{0,3\}$ are in the same component of the dividing curve and there are no others in the same component for $k = 3,4,5,6,7$.

The second point implies that for $k = 3,4,5,6,7$ there must be a component of (some choice of) the dividing curve originating in $(k - \frac{1}{2}, k + \frac{1}{2}) \times \{0\}$ and terminating in $(k - \frac{1}{2}, k + \frac{1}{2}) \times \{3\}$. The first point eliminates the other options. There are then two possible options left for the dividing curve. Both of which include a ∂-parallel component enclosing the line of singularities at $\{1 + \frac{1}{2}\} \times [0,1]$ which leads to the desired bypass attachment.

The first point is easy to see: If there was a ∂-parallel component above these lines of singularities this would result in bypass attachments. However if one where to attach those bypasses these would result in overtwisted disk components of the dividing curve of Σ in the tight neighborhood which is impossible, compare Figure 4.13. The second bullet point follows directly by the initial discussion of the foliation on this part of the annulus which remained unchanged by the isotopy. QED

After this very technical result, we are not yet done. All we have achieved up to now, is that we have two convex surfaces $\Sigma_- + A$ and Σ_+ which have the same dividing curve, however we still need to find a convex isotopy which transports one into the other. To do so, we will once again use the tightness of $D' \times [-1,1]$:

Lemma 4.40. *(Conclusion of Theorem 4.33)* $\Sigma_- + A$ *is convexly isotopic to* Σ_+.

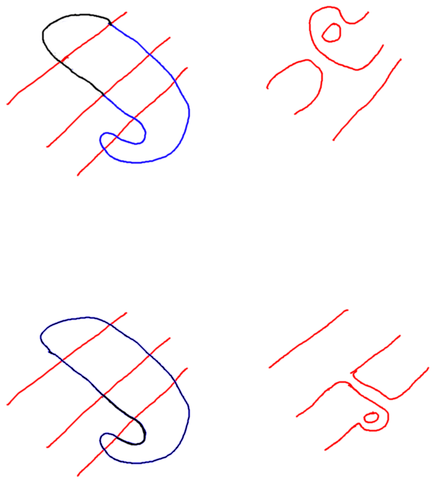

Figure 4.13: Attaching a bypass along the other components will result in homotopically trivial components. The attachment arcs are drawn in black, while the rest of L is drawn in blue.

Proof. We know $\Sigma_- + A =: \tilde{\Sigma}$ and Σ_+ have the same dividing curve and their characteristic foliations agree outside the disk D'. So we may isotop $\tilde{\Sigma}$ such that it has the same characteristic foliation as Σ_+ with an isotopy contained away from the boundary of $L := \partial D'$. Now there is a standardly foliated annulus $L' \times [-\frac{1}{2}, \frac{1}{2}]$ between Σ and Σ_+, for which the disks cut-off by L' agree. As before, we may use the edge rounding result and glue these disks to the annulus to obtain an S^2 which bounds a tight B^3. Analogously one may consider $D^2 \times [-1, 1]$ with the $\mathbb{R}$-invariant contact structure which induces the characteristic foliations of D'_+ on each level set. This also leads to a tight B^3.

Using Eliashbergs Theorem 4.18 we may isotop the original contact structure of B^3 to coincide with the contact structure which has the same foliation on each level set of D'. After this isotopy there is an extension Y of $\frac{d}{dr}|_{(\Sigma \setminus D')} \times [-\epsilon, \epsilon]$ which is a contact vectorfield and whose flow transports $\tilde{\Sigma}$ to Σ_+. QED

Proof of Theorem 4.33. First, we use Giroux's normal form Theorem 4.19 to isotop the contact structure relative to the boundary such that each level set of $\Sigma \times [-1, 1]$ is convex except for finitely many values $r_1, \ldots, r_n$ where the characteristic foliation of Σ_{r_i} fulfills the following:

(i) All singularities are either saddles or nodes;

(ii) All closed orbits are non-degenerate;

(iii) There is a single saddle-saddle connection which is convex;

(iv) The foliation is essentially planar.

So for an i, we must now find a bypass relating $\Sigma_{r_i-\epsilon_i}$ to $\Sigma_{r_i+\epsilon_i}$ for a sufficiently small number ϵ_i such that the only bifurcation value of the family of foliations in the range $[r_i - \epsilon_i, r_i + \epsilon_i]$ is at r_i.

One notices that each level-set Σ_r for $r \in [r_i - \epsilon_i, r_i)$ is convexly isotopic. The same is true for the positive half-interval $(r_i, r_i + \epsilon_i]$. Thus it does not matter for which exact value we find the bypass attachment. After we apply the sequence of convex isotopies to some Σ_- and Σ_+ given in Lemma 4.35, Lemma 4.36 and Lemma 4.38, we use Lemma 4.39 to find a bypass A which we may attach to Σ_-. Finally Lemma 4.40 tells us that $\Sigma_- + A$ is convexly isotopic to Σ_+.

QED

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