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Einstein-Theorien XIII.

Energiesatz und Divergenzidentitäten der Final Affine Fieldtheory.

Die Feldgleichungen entspringen aus dem Prinzip,

bei fester Grenze

$$I = \int \mathfrak{L}\, d\tau \;; \quad \delta I = \delta \int \mathfrak{L}\, d\tau = 0 \quad \text{mit} \quad \frac{\partial \mathfrak{L}}{\partial R_{ik}} = \mathfrak{g}^{ik} \quad \mathfrak{L} = \frac{2}{\lambda}\sqrt{-\mathrm{Det}\, R_{ik}}$$

$$\delta I = \int \mathfrak{g}^{ik}\, \delta R_{ik}\, d\tau = \int \left(\mathfrak{G}^{kl}{}_{\alpha} - \delta^{l}_{\alpha}\, \mathfrak{G}^{k\beta}{}_{\beta} \right) \delta \Gamma^{\alpha}_{kl}\, d\tau$$

Γ frei zu variieren.

$$\mathfrak{G}^{kl}{}_{\alpha} = \mathfrak{g}^{kl}{}_{,\alpha} + \mathfrak{g}^{\sigma l}\, {}^{*}\Gamma^{k}_{\sigma\alpha} + \mathfrak{g}^{k\sigma}\, {}^{*}\Gamma^{l}_{\alpha\sigma} - \frac{1}{2}\, \mathfrak{g}^{kl} \left({}^{*}\Gamma^{\sigma}_{\alpha\sigma} + {}^{*}\Gamma^{\sigma}_{\sigma\alpha} \right)$$

$${}^{*}\Gamma^{i}_{kl} = \Gamma^{i}_{kl} + \frac{2}{3}\, \delta^{i}_{k}\, \Gamma_{l} \qquad \Gamma_{l} = \frac{1}{2} \left(\Gamma^{\sigma}_{l\sigma} - \Gamma^{\sigma}_{\sigma l} \right).$$

$$R_{kl} = {}^{*}R_{kl} + F_{kl} \qquad F_{kl} = \frac{2}{3} \left(\frac{\partial \Gamma_{l}}{\partial x_{k}} - \frac{\partial \Gamma_{k}}{\partial x_{l}} \right).$$

$$\mathfrak{L} = \frac{1}{2}\, \mathfrak{g}^{ik} R_{ik}$$

Man betrachtet nun das Integral

$I^{*} = \int \mathfrak{g}^{ik}\, {}^{*}R_{ik}\, d\tau$, in welchem unter ${}^{*}R_{ik}$ der Riccitensor der Lösungen ${}^{*}\Gamma$ von

$$\mathfrak{g}^{kl}{}_{,\alpha} + \mathfrak{g}^{\sigma l}\, {}^{*}\Gamma^{k}_{\sigma\alpha} + \mathfrak{g}^{k\sigma}\, {}^{*}\Gamma^{l}_{\alpha\sigma} - \frac{1}{2}\, \mathfrak{g}^{kl} \left({}^{*}\Gamma^{\sigma}_{\alpha\sigma} + {}^{*}\Gamma^{\sigma}_{\sigma\alpha} \right) = 0$$

oder, was dasselbe ist von

$$g_{kl,\alpha} - g_{\sigma l}\, {}^{*}\Gamma^{\sigma}_{k\alpha} - g_{k\sigma}\, {}^{*}\Gamma^{\sigma}_{l\alpha} = 0$$

verstanden werden. Man berechnet die Variation von I^{*} für beliebige, <u>an keine Nebenbedingung gebundene</u> Variation der 16 $\mathfrak{g}^{ik}$ (oder g^{ik} oder g_{ik}):

$$\delta I^{*} = \int {}^{*}R_{ik}\, \delta \mathfrak{g}^{ik}\, d\tau + \int \mathfrak{g}^{ik}\, \delta\, {}^{*}R_{ik}\, d\tau. \qquad (1)$$

Setzt man jetzt voraus, dass die $\delta \mathfrak{g}^{ik}$ am Rande verschwinden und dass die ursprünglichen, unvariierten $\mathfrak{g}^{ik}$ der Bedingung genügen

$$\mathfrak{g}^{ik}{}_{,k} = 0 \qquad (2)$$

(was die $\delta \mathfrak{g}^{ik}$ bezügl. in gar keiner Weise beschränkt), dann wird

$$\delta I^{*} = \int {}^{*}R_{ik}\, \delta \mathfrak{g}^{ik}\, d\tau = \int \sqrt{-g} \left({}^{*}R_{ik} - \frac{1}{2}\, g_{ik}\, {}^{*}R \right) \delta g^{ik}\, d\tau \qquad (3)$$

wobei ${}^{*}R = g^{\mu\nu}\, {}^{*}R_{\mu\nu}$.

Jetzt formen wir das Integral I^* in bekannter Weise durch partielle Integration um:

$$I^* = \int g^{ik}\left(-\frac{\partial {}^*\Gamma^\alpha_{ik}}{\partial x_\alpha} + \frac{\partial {}^*\Gamma^\alpha_{i\alpha}}{\partial x_k} + \Lambda_{ik}\right) d\tau \quad , \text{ wobei}$$

(4) $$\Lambda_{ik} = {}^*\Gamma^\alpha_{i\beta}\,{}^*\Gamma^\beta_{\alpha k} - {}^*\Gamma^\alpha_{\beta\alpha}\,{}^*\Gamma^\beta_{ik} \quad \text{und später} \quad \Lambda = g^{ik}\Lambda_{ik}.$$

Durch partielle Integration:

(5) $$I^* = \int\Lambda\, d\tau + \int\left(g^{ik}{}_{,\alpha}\,{}^*\Gamma^\alpha_{ik} - g^{ik}{}_{,k}\,{}^*\Gamma^\alpha_{i\alpha}\right) d\tau + \int\left(-\frac{\partial(g^{ik}\,{}^*\Gamma^\alpha_{ik})}{\partial x_\alpha} + \frac{\partial g^{ik}\,{}^*\Gamma^\alpha_{i\alpha}}{\partial x_k}\right)d\tau$$

Der Integrand des ~~Das~~ zweiten Integrals rechter Hand wird nun, da daß

$$g^{ik}{}_{,\alpha} = -g^{\sigma k}\,{}^*\Gamma^i_{\sigma\alpha} - g^{i\sigma}\,{}^*\Gamma^k_{\alpha\sigma} + \tfrac12 g^{ik}\left({}^*\Gamma^\sigma_{\alpha\sigma} + {}^*\Gamma^\sigma_{\sigma\alpha}\right)$$

(6) $$g^{ik}{}_{,k} = -g^{\sigma k}\,{}^*\Gamma^i_{\sigma k} - g^{i\sigma}\,{}^*\Gamma^k_{k\sigma} + \tfrac12 g^{ik}\left({}^*\Gamma^\sigma_{k\sigma} + {}^*\Gamma^\sigma_{\sigma k}\right)$$

$$= -g^{\sigma k}\,{}^*\Gamma^i_{\sigma k} + \tfrac12 g^{ik}\left({}^*\Gamma^\sigma_{k\sigma} - {}^*\Gamma^\sigma_{\sigma k}\right)$$

folgendes:

$$-g^{\sigma k}\,{}^*\Gamma^i_{\sigma\alpha}\,{}^*\Gamma^\alpha_{ik} - g^{i\sigma}\,{}^*\Gamma^k_{\alpha\sigma}\,{}^*\Gamma^\alpha_{ik} + \tfrac12 \underline{g^{ik}\left({}^*\Gamma^\sigma_{\alpha\sigma} + {}^*\Gamma^\sigma_{\sigma\alpha}\right)}\,{}^*\Gamma^\alpha_{ik} +$$

(7) $$+ \underline{g^{\sigma k}\,{}^*\Gamma^i_{\sigma k}\,{}^*\Gamma^\alpha_{i\alpha}} - \tfrac12 g^{ik}\left({}^*\Gamma^\sigma_{k\sigma} - {}^*\Gamma^\sigma_{\sigma k}\right){}^*\Gamma^\alpha_{i\alpha} =$$

$$= -2\Lambda - \tfrac12 g^{ik}\left({}^*\Gamma^\sigma_{\alpha\sigma} - {}^*\Gamma^\sigma_{\sigma\alpha}\right){}^*\Gamma^\alpha_{ik} - \tfrac12 g^{ik}\left({}^*\Gamma^\sigma_{k\sigma} - {}^*\Gamma^\sigma_{\sigma k}\right){}^*\Gamma^\alpha_{i\alpha}$$

Wir schließen hieran sogleich eine für später benötigte analoge Umformung, nämlich diese:

$$g^{ik}{}_{,\alpha}\, d\,{}^*\Gamma^\alpha_{ik} - g^{ik}{}_{,k}\, d\,{}^*\Gamma^\alpha_{i\alpha} =$$

(8) $$= -g^{\sigma k}\,{}^*\Gamma^i_{\sigma\alpha}\, d\,{}^*\Gamma^\alpha_{ik} - g^{i\sigma}\,{}^*\Gamma^k_{\alpha\sigma}\, d\,{}^*\Gamma^\alpha_{ik} + \tfrac12 \underline{g^{ik}\left({}^*\Gamma^\sigma_{\alpha\sigma} + {}^*\Gamma^\sigma_{\sigma\alpha}\right)}\, d\,{}^*\Gamma^\alpha_{ik} +$$

$$+ \underline{g^{\sigma k}\,{}^*\Gamma^i_{\sigma k}\, d\,{}^*\Gamma^\alpha_{i\alpha}} - \tfrac12 g^{ik}\left({}^*\Gamma^\sigma_{k\sigma} - {}^*\Gamma^\sigma_{\sigma k}\right) d\,{}^*\Gamma^\alpha_{i\alpha} =$$

$$= -g^{ik}\, d\Lambda_{ik} - \tfrac12 g^{ik}\left({}^*\Gamma^\sigma_{\alpha\sigma} - {}^*\Gamma^\sigma_{\sigma\alpha}\right) d\,{}^*\Gamma^\alpha_{ik} - \tfrac12 g^{ik}\left({}^*\Gamma^\sigma_{k\sigma} - {}^*\Gamma^\sigma_{\sigma k}\right) d\,{}^*\Gamma^\alpha_{i\alpha}$$

Wir setzen zur Abkürzung $\tfrac12\left({}^*\Gamma^\sigma_{\alpha\sigma} - {}^*\Gamma^\sigma_{\sigma\alpha}\right) = {}^*\Gamma_\alpha$. Also dann

$$I^* = -\int\Lambda\, d\tau - \tfrac12\int\left(g^{ik}\,{}^*\Gamma_\alpha\,{}^*\Gamma^\alpha_{ik} + g^{ik}\,{}^*\Gamma_k\,{}^*\Gamma^\alpha_{i\alpha}\right)d\tau + \int\left[-\frac{\partial(g^{ik}\,{}^*\Gamma^\alpha_{ik})}{\partial x_\sigma} + \frac{\partial g^{ik}\,{}^*\Gamma^\alpha_{i\alpha}}{\partial x_k}\right]d\tau$$

~~Ferner~~ Ferner ist $g^{ik}\, d\Lambda_{ik} = d\Lambda - \Lambda_{ik}\, dg^{ik}$, also

nach (8)

$$-d\Lambda = -\Lambda_{ik}\, dg^{ik} + g^{ik}{}_{,\alpha}\, d\,{}^*\Gamma^\alpha_{ik} - g^{ik}{}_{,k}\, d\,{}^*\Gamma^\alpha_{i\alpha} + \cancel{\tfrac12}\, g^{ik}\,{}^*\Gamma_\alpha\, d\,{}^*\Gamma^\alpha_{ik} + \cancel{\tfrac12}\, g^{ik}\,{}^*\Gamma_k\, d\,{}^*\Gamma^\alpha_{i\alpha}$$

nach (7)

$$-2d\Lambda = {}^*\Gamma^\alpha_{ik}\, dg^{ik}{}_{,\alpha} - {}^*\Gamma^\alpha_{i\alpha}\, dg^{ik}{}_{,k} + \underline{g^{ik}{}_{,\alpha}\, d\,{}^*\Gamma^\alpha_{ik}} - \underline{g^{ik}{}_{,k}\, d\,{}^*\Gamma^\alpha_{i\alpha}} +$$

$$+ \underline{\cancel{\tfrac12}\, g^{ik}\,{}^*\Gamma_\alpha\, d\,{}^*\Gamma^\alpha_{ik} + \cancel{\tfrac12}\, g^{ik}\,{}^*\Gamma_k\, d\,{}^*\Gamma^\alpha_{i\alpha}} + \cancel{\tfrac12}\,{}^*\Gamma^\alpha_{ik}\, d(g^{ik}\,{}^*\Gamma_\alpha) + \cancel{\tfrac12}\,{}^*\Gamma^\alpha_{i\alpha}\, d(g^{ik}\,{}^*\Gamma_k)$$

Also: $$-d\Lambda = \Lambda_{ik}\, dg^{ik} + \cancel{\tfrac12}\,{}^*\Gamma^\alpha_{ik}\, d(g^{ik}\,{}^*\Gamma_\alpha) + \cancel{\tfrac12}\,{}^*\Gamma^\alpha_{i\alpha}\, d(g^{ik}\,{}^*\Gamma_k)$$

$$+ {}^*\Gamma^\alpha_{ik}\, d(g^{ik}{}_{,\alpha}) - {}^*\Gamma^\alpha_{i\alpha}\, d(g^{ik}{}_{,k})$$

Man muß nun wohl das ${}^*\Gamma_\alpha$ wieder hinausschaffen! (Ja, ja ?!). Da kommt g^{ik} herein. Nicht schön. — Wenn wirklich wäre:

$$\frac{\partial\Lambda}{\partial g^{ik}} = -\Lambda_{ik} \qquad \frac{\partial\Lambda}{\partial g^{ik}{}_{,\alpha}} = -{}^*\Gamma^\alpha_{ik} + \delta^\alpha_k\,{}^*\Gamma^\sigma_{i\sigma}$$

was wäre denn dann das:

$$\frac{\partial\Lambda}{\partial g^{ik}} - \frac{\partial}{\partial x_\alpha}\left(\frac{\partial\Lambda}{\partial g^{ik}{}_{,\alpha}}\right) = -{}^*\Gamma^\alpha_{i\beta}\,{}^*\Gamma^\beta_{\alpha k} + {}^*\Gamma^\alpha_{\beta\alpha}\,{}^*\Gamma^\beta_{ik} + \frac{\partial\,{}^*\Gamma^\alpha_{ik}}{\partial x_\alpha} - \frac{\partial\,{}^*\Gamma^\sigma_{i\sigma}}{\partial x_k}$$

ja das ist wirklich $= -{}^*R_{ik}$.

Die Sache ist so: es sind nicht die Derivierten. Es sind nicht einmal die Werte der Derivierten im Falle $g^{ik}{}_{,k} = 0$. Aber sie fungieren in diesem Falle als Derivierte. Daher genau nachsehen muß man, ob sich das Ganze wirklich in eine Ableitung verwandeln" läßt. Ja, Übergang zum „Nachbarpunkt, wo die Bedingung doch wieder erfüllt ist. —

Also wenn jetzt etwa

$$-{}^*\Gamma^\alpha_{ik} + \delta^\alpha_k\,{}^*\Gamma^\sigma_{i\sigma} = -\Lambda^\alpha_{ik} \quad \text{und dann}$$

gilt

$$d\Lambda = -\Lambda_{ik}\, dg^{ik} - \Lambda^\alpha_{ik}\, d(g^{ik}{}_{,\alpha})$$

sicher dann, wenn $g^{ik}{}_{,k} = 0$ erfüllt ist und auch bei der d-Veränderung erfüllt bleibt. Da nicht sicherlich die $d(g^{ik}{}_{,\alpha})$ nicht unabhängig sind, so kann man die Λ_{ik} und Λ^α_{ik} nicht als partielle Differentialquotienten ansprechen, aber das schadet nichts. Man hat daher:

$$\delta\int\Lambda\, d\tau = \int\delta\Lambda\, d\tau = -\int\left[\Lambda_{ik}\,\delta g^{ik} + \Lambda^\alpha_{ik}\,\delta(g^{ik}{}_{,\alpha})\right]d\tau =$$

$$= \int\left(\Lambda^\alpha_{ik,\alpha} - \Lambda_{ik}\right)\delta g^{ik}\, d\tau - \int\left(\Lambda^\alpha_{ik}\,\delta g^{ik}\right)_{,\alpha} d\tau$$

Durch Ausrechnen sieht man, daß $-{}^*R_{ik} = \Lambda^\alpha_{ik,\alpha} - \Lambda_{ik}$. Man kann daher auch schreiben

$$\delta\int\Lambda\, d\tau = \int\delta\Lambda\, d\tau = -\int\sqrt{-g}\left({}^*R_{ik} - \tfrac12 g_{ik}\,{}^*R\right)\delta g^{ik}\, d\tau - \int\left(\Lambda^\alpha_{ik}\,\delta g^{ik}\right)_{,\alpha} d\tau$$

$$= \int Z_{ik}\,\delta g^{ik}\, d\tau - \int\left(\Lambda^\alpha_{ik}\,\delta g^{ik}\right)_{,\alpha} d\tau \quad , \text{ wobei}$$

$$Z_{ik} = -\sqrt{-g}\left({}^*R_{ik} - \tfrac12 g_{ik}\,{}^*R\right).$$

Man fasst nun, ganz unabhängig davon, diejenige lokale Änderung der g^{ik} ins Auge, die wir $\delta^* g^{ik}$ nennen und die durch eine beliebige unendlich kleine Änderung des Bezugssystems zustande kommt:

$$x'_i = x_i + \xi^i$$

ξ^i sei klein, sonst beliebige Funktion der Koordinaten.

Man findet:

$$\delta^* g^{ik} = g^{rk}\frac{\partial \xi^i}{\partial x_r} + g^{ir}\frac{\partial \xi^k}{\partial x_r} - \xi^r \frac{\partial g^{ik}}{\partial x_r}$$

und ähnlich für g_{ik}, $\mathfrak{g}^{ik}$ u. dgl. Ferner betrachtet man das Integral mit beliebigem Integranden $\mathfrak{M}$, auf ein bestimmtes Gebiet G bezogen

$$\int_G \mathfrak{M}\, d\tau$$

und die Veränderung, die es bei dieser Änderung des Bezugssystems erleidet, die wir nennen

$$\delta^* \int_G \mathfrak{M}\, d\tau = \int_{G'} \mathfrak{M}'(x')\, d\tau' - \int_G \mathfrak{M}(x)\, d\tau =$$

$$= \int_G \mathfrak{M}'(x')\left|\frac{\partial x'}{\partial x}\right| d\tau - \int_G \mathfrak{M}(x)\, d\tau \quad ; \text{ da nun } \left|\frac{\partial x'}{\partial x}\right| = 1 + \frac{\partial \xi^i}{\partial x_i}$$

so

$$\delta^* \int_G \mathfrak{M}\, d\tau = \int_G \mathfrak{M}'(x')\, d\tau + \int_G \mathfrak{M}'(x')\frac{\partial \xi_i}{\partial x_i} d\tau - \int_G \mathfrak{M}(x)\, d\tau$$

$$= \int_G \mathfrak{M}'(x)\, d\tau + \int_G \frac{\partial \mathfrak{M}'}{\partial x_i}\xi_i\, d\tau + \int_G \mathfrak{M}'(x')\frac{\partial \xi_i}{\partial x_i} d\tau - \int_G \mathfrak{M}(x)\, d\tau$$

$$= \int_G \delta^* \mathfrak{M}\, d\tau + \int_G \frac{\partial \mathfrak{M}\xi_i}{\partial x_i} d\tau .$$

Dies wird immer dann verschwinden, wenn die Äußerlichkeit von solcher Art ist, dass sie das $\int_G \mathfrak{M}\, d\tau$ invariant lässt.

Für $\mathfrak{M} = \Lambda$ ist das erstens dann der Fall, wenn die Variation an den Grenzen verschwindet, weil in diesem Falle die Variation von $\delta\int \Lambda\, d\tau$ vom Vorzeichen abgesehen gleich der Variation des invarianten Integrals I^* ist (die $\mathfrak{g}^{ik}{}_{,k} = 0$ Bedingung bleibt ja sicherlich erhalten!). Zweitens aber ist es auch für jede lineare Substitution der Fall, weil eine solche gegenüber Λ selbst eine invariante Dichte ist. — Wir erhalten daher in diesen Fällen:

$$\delta^* \int \Lambda\, d\tau = \int \delta^* \Lambda\, d\tau + \int \frac{\partial \Lambda \xi^i}{\partial x_i} d\tau$$

$$= \int \mathfrak{T}_{ik}\, \delta^* \mathfrak{g}^{ik}\, d\tau + \int \left(\Lambda \xi^\alpha - \Lambda^\alpha_{ik}\, \delta^* \mathfrak{g}^{ik}\right)_{,\alpha} d\tau =$$

Nun ist

$$\int \mathfrak{T}_{ik}\, \delta^* \mathfrak{g}^{ik}\, d\tau = \int \mathfrak{T}_{ik}\left(g^{sk}\frac{\partial \xi_i}{\partial x_s} + g^{is}\frac{\partial \xi_k}{\partial x_s} - \xi_r \frac{\partial \mathfrak{g}^{ik}}{\partial x_r}\right) d\tau =$$

$$-\int \left(+\frac{\partial}{\partial x_s}\left(g^{sk}\mathfrak{T}_{rk} + g^{ks}\mathfrak{T}_{kr}\right) + \mathfrak{T}_{ik}\frac{\partial \mathfrak{g}^{ik}}{\partial x_r}\right)\xi_r\, d\tau + \int \frac{\partial}{\partial x_s}\left[\left(g^{sk}\mathfrak{T}_{rk} + g^{ks}\mathfrak{T}_{kr}\right)\xi_r\right] d\tau$$

Wir studieren den Klammerausdruck:

$$\mathfrak{T}_{ik}\frac{\partial \mathfrak{g}^{ik}}{\partial x_r} = \mathfrak{T}_{ik}\left(-g^{sk}\, {}^*\Gamma^i_{sr} - g^{is}\, {}^*\Gamma^k_{rs}\right) = -g^{sk}\mathfrak{T}_{ik}\, {}^*\Gamma^i_{sr} - g^{is}\mathfrak{T}_{ik}\, {}^*\Gamma^k_{rs}$$

Man kann leicht sehen, dass das invariant ist, aber wie bei, sondern einfache Angelegenheit ist es nicht. Sollte man die ξ am Rande verschwindend nehmen, so bleibt bloß dieses Integral stehen, und wegen der Willkürlichkeit der ξ_r muss die Klammer verschwinden. Wir lassen sie stehen weg und haben daher:

$$\delta^* \int \Lambda\, d\tau = \int \left[\Lambda \xi^\alpha - \Lambda^\alpha_{ik}\, \delta^* \mathfrak{g}^{ik} + \left(g^{\alpha k}\mathfrak{T}_{rk} + g^{k\alpha}\mathfrak{T}_{kr}\right)\xi^r\right]_{,\alpha} d\tau = 0 .$$

Wenn man jetzt die ξ^i als konstant voraussetzt, dann wird

$$\delta^* \mathfrak{g}^{ik} = -\xi^r \frac{\partial \mathfrak{g}^{ik}}{\partial x_r} \qquad \Lambda \xi^\alpha = \delta^\alpha_r \Lambda \xi^r .$$

Und da es beliebige Konstanten sein können und der Bereich beliebig klein, so folgt:

$$\left[\delta^\alpha_r \Lambda + \Lambda^\alpha_{ik}\frac{\partial \mathfrak{g}^{ik}}{\partial x_r} + g^{\alpha k}\mathfrak{T}_{rk} + g^{k\alpha}\mathfrak{T}_{kr}\right]_{,\alpha} = 0$$

$$\Lambda^\alpha_{ik} = {}^*\Gamma^\alpha_{ik} - \delta^\alpha_k\, {}^*\Gamma^\sigma_{i\sigma}$$

irgendwelche Ausdrücke. Diese Größen explizit zu berechnen, lohnt sich allgemein nicht.

(NB: Gegen Pauli haben wir hier das Vorzeichen von $\mathfrak{T}_{ik}$ anders gewählt.) Identitäten sind es nicht, sie sind an $\mathfrak{g}^{ik}{}_{;k} = 0$ gebunden.

Jetzt setzt man die ξ^i als lineare Funktionen der Koordinaten voraus.

Wir schreiben jetzt zur Abkürzung

$$g^{\alpha k} T_{rk} + g^{k\alpha} T_{kr} = 2 T^{\alpha}_{r} \quad , \text{ also}$$

$$\delta^{*} \int \Lambda \, d\tau = \int \left[\Lambda \xi^{\alpha} - \Lambda^{\alpha}_{ik} \delta^{*} \mathfrak{g}^{ik} + 2 T^{\alpha}_{r} \xi^{r} \right]_{,\alpha} d\tau$$

$\mathfrak{g}$ sei wie üblich die Dichte. Die Funktionaldeterminante $\left| \frac{\partial x_i}{\partial x'_k} \right| = 1 - \frac{\partial \xi^{\nu}}{\partial x_{\nu}}$

Also ist das volle $\delta^{*} \mathfrak{g}^{ik}$

$$\delta^{*} \mathfrak{g}^{ik} = \mathfrak{g}^{rk} \frac{\partial \xi_i}{\partial x_r} + \mathfrak{g}^{ir} \frac{\partial \xi_k}{\partial x_r} - \underline{\xi^{r} \frac{\partial \mathfrak{g}^{ik}}{\partial x_r}} - \mathfrak{g}^{ik} \frac{\partial \xi^{r}}{\partial x_r} .$$

Die $\frac{\partial \xi_k}{\partial x_r}$ sind jetzt Konstante. ~~[illegible]~~ Die Glieder, die mit den früheren verschwinden, lassen wir fort. Es bleibt:

$$\delta^{*} \int \Lambda \, d\tau = \int \Big[\delta^{\alpha}_{r} \Lambda \frac{\partial \xi^{r}}{\partial x_{\alpha}} + \Lambda^{\alpha}_{ik} \frac{\partial \mathfrak{g}^{ik}}{\partial x_r} \frac{\partial \xi_r}{\partial x_{\alpha}} - \left(\Lambda^{\alpha}_{ik} \mathfrak{g}^{rk} \right)_{,\alpha} \frac{\partial \xi_i}{\partial x_r} - \left(\Lambda^{\alpha}_{ik} \mathfrak{g}^{ir} \right)_{,\alpha} \frac{\partial \xi_k}{\partial x_r} +$$

$$+ \left(\Lambda^{\alpha}_{ik} \mathfrak{g}^{ik} \right)_{,\alpha} \frac{\partial \xi^{r}}{\partial x_r} + 2 T^{\alpha}_{r} \frac{\partial \xi^{r}}{\partial x_{\alpha}} \Big] d\tau$$

$$= \int \Big\{ \Big(\delta^{\alpha}_{r} \Lambda + \Lambda^{\alpha}_{ik} \frac{\partial \mathfrak{g}^{ik}}{\partial x_r} + 2 T^{\alpha}_{r} \Big) \frac{\partial \xi_r}{\partial x_{\alpha}} - \Big(\left[\Lambda^{\sigma}_{rk} \mathfrak{g}^{\alpha k} \right]_{,\sigma} + \left[\Lambda^{\sigma}_{kr} \mathfrak{g}^{k\alpha} \right]_{,\sigma} -$$

$$- \delta^{\alpha}_{r} \left[\Lambda^{\sigma}_{ik} \mathfrak{g}^{ik} \right]_{,\sigma} \Big) \frac{\partial \xi_r}{\partial x_{\alpha}} \Big\} d\tau$$

Also wird

$$\tfrac{1}{2} \left(\delta^{\alpha}_{r} \Lambda + \Lambda^{\alpha}_{ik} \frac{\partial \mathfrak{g}^{ik}}{\partial x_r} \right) + T^{\alpha}_{r} = \tfrac{1}{2} \left(\Lambda^{\sigma}_{rk} \mathfrak{g}^{\alpha k} + \Lambda^{\sigma}_{kr} \mathfrak{g}^{k\alpha} - \delta^{\alpha}_{r} \Lambda^{\sigma}_{ik} \mathfrak{g}^{ik} \right)_{,\sigma}$$

$$\begin{array}{l|l}
\Lambda^{\sigma}_{ik} = {}^{\times}\Gamma^{\sigma}_{ik} - \delta^{\sigma}_{k} \, {}^{\times}\Gamma^{\beta}_{i\beta} & - \delta^{\alpha}_{r} \mathfrak{g}^{ik} \\
\Lambda^{\sigma}_{rk} = {}^{\times}\Gamma^{\sigma}_{rk} - \delta^{\sigma}_{k} \, {}^{\times}\Gamma^{\beta}_{r\beta} & \mathfrak{g}^{\alpha k} \\
\Lambda^{\sigma}_{kr} = {}^{\times}\Gamma^{\sigma}_{kr} - \delta^{\sigma}_{r} \, {}^{\times}\Gamma^{\beta}_{k\beta} & \mathfrak{g}^{k\alpha}
\end{array}$$

$${}^{\sigma\alpha}_{\ r} \quad \left.\begin{array}{l}
(\cdots) = - \delta^{\alpha}_{r} \, {}^{\times}\Gamma^{\sigma}_{ik} \mathfrak{g}^{ik} + \delta^{\alpha}_{r} \, {}^{\times}\Gamma^{\beta}_{i\beta} \mathfrak{g}^{i\sigma} \\
\quad + \underline{\mathfrak{g}^{\alpha k} \, {}^{\times}\Gamma^{\sigma}_{rk}} - {}^{\times}\Gamma^{\beta}_{r\beta} \mathfrak{g}^{\alpha\sigma} \\
\quad + \underline{\mathfrak{g}^{k\alpha} \, {}^{\times}\Gamma^{\sigma}_{kr}} - \delta^{\sigma}_{r} \, {}^{\times}\Gamma^{\beta}_{k\beta} \mathfrak{g}^{k\alpha}
\end{array}\right\}$$

~~Man setze: $\mathfrak{g}^{\alpha k} \, {}^{\times}\Gamma^{\sigma}_{rk} + \mathfrak{g}^{k\alpha} \, {}^{\times}\Gamma^{\sigma}_{kr} = 2 \Pi^{\alpha\sigma}_{r}$, Dann ist:~~

~~$$2 \Pi^{\alpha\sigma}_{\alpha} = \mathfrak{g}^{\alpha k} \, {}^{\times}\Gamma^{\sigma}_{\alpha k} + \mathfrak{g}^{k\alpha} \, {}^{\times}\Gamma^{\sigma}_{k\alpha} = 2 \mathfrak{g}^{ik} \, \Gamma^{\sigma}_{ik}$$~~

~~$$2 \Pi^{\alpha\sigma}_{\sigma} = \mathfrak{g}^{\alpha k} \, {}^{\times}\Gamma^{\sigma}_{\sigma k} + \mathfrak{g}^{k\alpha} \, {}^{\times}\Gamma^{\sigma}_{k\sigma}$$~~

Das gibt keine Vereinfachung. Jedoch ist

$$\left.\begin{array}{l}
\Lambda^{\sigma}_{i\sigma} = {}^{\times}\Gamma^{\sigma}_{i\sigma} - 4 \, {}^{\times}\Gamma^{\beta}_{i\beta} = - 3 \, {}^{\times}\Gamma^{\sigma}_{i\sigma} \\
\Lambda^{\sigma}_{\sigma k} = {}^{\times}\Gamma^{\sigma}_{\sigma k} - {}^{\times}\Gamma^{\beta}_{k\beta} = 0 .
\end{array}\right\} \text{ hat keine Anwendung. Aber, setzt man}$$

$$\mathfrak{g}^{\alpha k} \Lambda^{\sigma}_{rk} + \mathfrak{g}^{k\alpha} \Lambda^{\sigma}_{kr} = 2 M^{\alpha\sigma}_{r} \quad , \text{ Dann ist}$$

$$2 M^{\alpha\sigma}_{\alpha} = 2 \mathfrak{g}^{ik} \Lambda^{\sigma}_{ik}$$

Also kann man schreiben:

$$\tfrac{1}{2} \left(\delta^{\alpha}_{r} \Lambda + \Lambda^{\alpha}_{ik} \frac{\partial \mathfrak{g}^{ik}}{\partial x_r} \right) + T^{\alpha}_{r} = \left(M^{\alpha\sigma}_{r} - \tfrac{1}{2} \delta^{\alpha}_{r} M^{\beta\sigma}_{\beta} \right)_{,\sigma}$$

Man versucht noch zu berechnen:

$$\Lambda^{\alpha}_{ik} \frac{\partial \mathfrak{g}^{ik}}{\partial x_r} =$$

$$= \left({}^{\times}\Gamma^{\alpha}_{ik} - \delta^{\alpha}_{k} \, {}^{\times}\Gamma^{\beta}_{i\beta} \right) \left(- \mathfrak{g}^{\sigma k} \, {}^{\times}\Gamma^{i}_{\sigma r} - \mathfrak{g}^{i\sigma} \, {}^{\times}\Gamma^{k}_{r\sigma} + \mathfrak{g}^{ik} \, {}^{\times}\Gamma^{\sigma}_{r\sigma} \right) =$$

$$- \underline{\mathfrak{g}^{\sigma k} \, {}^{\times}\Gamma^{\alpha}_{ik} \, {}^{\times}\Gamma^{i}_{\sigma\alpha}} - \underline{\mathfrak{g}^{i\sigma} \, {}^{\times}\Gamma^{\alpha}_{ik} \, {}^{\times}\Gamma^{k}_{\alpha\sigma}} + \mathfrak{g}^{ik} \, \underline{{}^{\times}\Gamma^{\alpha}_{ik} \, {}^{\times}\Gamma^{\sigma}_{\alpha\sigma}}$$

$$+ \underline{\mathfrak{g}^{\sigma\alpha} \, {}^{\times}\Gamma^{i}_{\sigma\alpha} \, {}^{\times}\Gamma^{\beta}_{i\beta}} + \mathfrak{g}^{i\sigma} \, \cancel{{}^{\times}\Gamma^{\beta}_{i\beta} \, {}^{\times}\Gamma^{\alpha}_{\alpha\sigma}} - \mathfrak{g}^{i\alpha} \, \cancel{{}^{\times}\Gamma^{\beta}_{i\beta} \, {}^{\times}\Gamma^{\sigma}_{\alpha\sigma}} \Big) \qquad \binom{\alpha}{r}$$

Das scheint nicht einfacher zu sein.

Das merkwürdigste ist die Art von Symmetrie, die T^{α}_{r} besitzt. Als gemischter Tensor kann er nicht symmetrisch sein. Auch ist T_{kr} nicht symmetrisch. Aber T^{α}_{r} bleibt unverändert, wenn man sowohl g^{ik} als auch Γ_{ik} durch ihr Spiegelbilder ersetzt.

NB.: wenn man den obigen Ausdruck bezüglich $\binom{\alpha}{r}$ verjüngt, dann kriegt man -2Λ, das ist ganz richtig. Die Klammern sind ganz als partielle Derivierte.

Bei

$$g^{\alpha k}\gamma_{\tau k} + g^{k\alpha}\gamma_{k\tau} = 2\gamma^\alpha_\tau$$

Lassen sich diese Gleichungen nach den $\gamma_{\tau k}$ auflösen?
Wir führen durch Rechnung. Rechnen wir mit $g^{\alpha k}$ mit der Form

$1 \; f^{12}$
$-f^{21} \; 1$
$1 \; f^{34}$
$-f^{34} \; 1$

$$2\gamma^1_1 = g^{1\cdot}\gamma_{1\cdot} + g^{\cdot 1}\gamma_{\cdot 1}$$
$$= \gamma_{11} + f^{12}\gamma_{12} - f^{12}\gamma_{21} + \gamma_{11}$$
$$\gamma^1_1 = \gamma_{11} + f^{12}\gamma_{\underset{\vee}{12}}$$
$$2\gamma^2_2 = g^{2\cdot}\gamma_{2\cdot} + g^{\cdot 2}\gamma_{\cdot 2}$$
$$= \gamma_{22} - f^{12}\gamma_{21} + \gamma_{22} + f^{12}\gamma_{12}$$
$$\gamma^2_2 = \gamma_{22} + f^{12}\gamma_{\underset{\vee}{12}}$$

Einfacher: allgemein ist

$$2\gamma^\alpha_\tau = g^{\alpha k}(\gamma_{\underline{\tau k}} + \gamma_{\underset{\vee}{\tau k}}) + g^{k\alpha}(\gamma_{\underline{\tau k}} - \gamma_{\underset{\vee}{\tau k}})$$
$$= g^{\alpha k} + g^{k\alpha}\gamma_{\underline{\tau k}} + (g^{\alpha k} - g^{k\alpha})\gamma_{\underset{\vee}{\tau k}}$$
$$\gamma^\alpha_\tau = g^{\underline{\alpha k}}\gamma_{\underline{\tau k}} + g^{\underset{\vee}{\alpha k}}\gamma_{\underset{\vee}{\tau k}}$$

Also in unserem Falle:

$$\gamma^\alpha_\tau = \gamma_{\underline{\tau\alpha}} - g^{\underset{\vee}{\alpha k}}\gamma_{\underset{\vee}{k\tau}} \qquad \gamma_{\underset{\vee}{k\tau}} = f_{k\tau}$$

$$\begin{pmatrix} 0 & f_{12} & 0 & 0 \\ -f_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & f^{34} \\ 0 & 0 & -f^{34} & 0 \end{pmatrix}\begin{pmatrix} 0 & f_{12} & -f_{13} & f_{14} \\ -f_{12} & 0 & f_{23} & f_{24} \\ +f_{13} & -f_{23} & 0 & f_{34} \\ -f_{14} & -f_{24} & -f_{34} & 0 \end{pmatrix} = \begin{pmatrix} -f_{12}f^{12}, & 0, & f^{12}f_{23}, & f^{12}f_{24} \\ 0, & -f_{12}f^{12}, & f^{12}f_{31}, & -f^{12}f_{14} \\ -f^{34}f_{14}, & f^{34}f_{24}, & -f^{34}f_{34} & 0 \\ -f^{34}f_{12}, & f^{34}f_{23}, & 0 & -f^{34}f_{34} \end{pmatrix}$$

$$\gamma^1_3 = \gamma_{\underline{13}} - f^{12}f_{23}$$
$$\gamma^3_1 = \gamma_{\underline{13}} + f^{34}f_{14} \qquad \gamma^2_4 = \gamma_{\underline{24}} + f^{12}f_{14}$$
$$\gamma^3_1 - \gamma^1_3 = f^{34}f_{14} + f^{12}f_{23} \qquad \gamma^4_2 = \gamma_{\underline{24}} - f^{34}f_{23}$$
$$\gamma^2_4 - \gamma^4_2 = \cancel{f^{12}f_{14} + f^{34}f_{23}} \;\; \cancel{f^{34}f_{23} + f^{12}} \text{ statt } \gamma^2_4 - \gamma^4_2 = f^{12}f_{14} + f^{34}f_{23}$$

Zufalls Grossmutter!

Rechnerische Verifizierung der Beziehung von $\delta\Lambda$ und $\delta\int\Lambda d\sigma$.

Man berechne direkt (die „Bedingung" angenommen!)

$$({}^\times\Gamma^\alpha_{ik} - \delta^\alpha_k\,{}^\times\Gamma^\sigma_{i\sigma})\,\delta(\mathfrak{g}^{ik}_{,\alpha}) =$$
$$({}^\times\Gamma^\alpha_{ik} - \delta^\alpha_k\,{}^\times\Gamma^\beta_{i\beta})\,\delta[-\mathfrak{g}^{\sigma k}\,{}^\times\Gamma^i_{\sigma\alpha} - \mathfrak{g}^{i\sigma}\,{}^\times\Gamma^k_{\alpha\sigma} + \mathfrak{g}^{ik}\,{}^\times\Gamma^\sigma_{\alpha\sigma}] =$$
$$-2({}^\times\Gamma^\alpha_{i\beta}\,{}^\times\Gamma^\beta_{\alpha k} - {}^\times\Gamma^\alpha_{\beta\alpha}\,{}^\times\Gamma^\beta_{ik})\,\delta\mathfrak{g}^{ik} - \mathfrak{g}^{ik}\,\delta({}^\times\Gamma^\alpha_{i\beta}\,{}^\times\Gamma^\beta_{\alpha k} - {}^\times\Gamma^\alpha_{\beta\alpha}\,{}^\times\Gamma^\beta_{ik})$$

Das erste ist sicher; das zweite:

$$\left.\begin{aligned} &-\mathfrak{g}^{\sigma k}\,{}^\times\Gamma^\alpha_{ik}\,\delta{}^\times\Gamma^i_{\sigma\alpha} + \mathfrak{g}^{\sigma\alpha}\,{}^\times\Gamma^\beta_{i\beta}\,\delta{}^\times\Gamma^i_{\sigma\alpha} \\ &-\mathfrak{g}^{i\sigma}\,{}^\times\Gamma^\alpha_{ik}\,\delta{}^\times\Gamma^k_{\alpha\sigma} + \cancel{\mathfrak{g}^{i\sigma}\,{}^\times\Gamma^\beta_{i\beta}\,\delta{}^\times\Gamma^\alpha_{\alpha\sigma}} \\ &+\mathfrak{g}^{ik}\,\Gamma^\alpha_{ik}\,\delta{}^\times\Gamma^\sigma_{\alpha\sigma} - \cancel{\mathfrak{g}^{i\alpha}\,{}^\times\Gamma^\beta_{i\beta}\,\delta{}^\times\Gamma^\sigma_{\alpha\sigma}} \end{aligned}\right\} = -\delta({}^\times\Gamma^\alpha_{i\beta}\,{}^\times\Gamma^\beta_{\alpha k} - {}^\times\Gamma^\alpha_{\beta\alpha}\,{}^\times\Gamma^\beta_{ik})\cdot\mathfrak{g}^{ik}$$

$\longleftrightarrow$

Eine Probe:

$$\delta^\times\mathfrak{g}^{ik} = \mathfrak{g}^{\tau k}\xi^i_{,\tau} + \mathfrak{g}^{i\tau}\xi^k_{,\tau} - \mathfrak{g}^{ik}\xi^\tau_{,\tau} - \xi^\tau\mathfrak{g}^{ik}_{,\tau}$$
$$\delta^\times\mathfrak{g}^{ki} = \mathfrak{g}^{\tau i}\xi^k_{,\tau} + \mathfrak{g}^{k\tau}\xi^i_{,\tau} - \mathfrak{g}^{ki}\xi^\tau_{,\tau} - \xi^\tau\mathfrak{g}^{ki}_{,\tau}$$
$$\delta^\times\mathfrak{g}^{\underset{\vee}{ik}} = \mathfrak{g}^{\underset{\vee}{i\tau}}\xi^k_{,\tau} + \mathfrak{g}^{\underset{\vee}{\tau k}}\xi^i_{,\tau} - \mathfrak{g}^{\underset{\vee}{ik}}\xi^\tau_{,\tau} - \xi^\tau\mathfrak{g}^{\underset{\vee}{ik}}_{,\tau}$$

Man wisse $\mathfrak{g}^{\underset{\vee}{ik}}_{,k} = 0$. Man kontrahiere nach k.

$$\delta^\times\mathfrak{g}^{\underset{\vee}{ik}}_{,k} = \cancel{\mathfrak{g}^{\underset{\vee}{i\tau}}\xi^k_{,\tau k}} + \cancel{\mathfrak{g}^{\underset{\vee}{i\tau}}_{,k}\xi^k_{,\tau}} + \cancel{\mathfrak{g}^{\underset{\vee}{\tau k}}\xi^i_{,\tau k}} - \mathfrak{g}^{\underset{\vee}{ik}}\xi^\tau_{,\tau k}$$
$$\cancel{-\xi^\tau_{,k}\mathfrak{g}^{\underset{\vee}{ik}}_{,\tau}}$$

stimmt!

Andere Form der invarianten Divergenzgleichung:
($\frac{1}{2}$ bleibt fort):

$$g^{sk} T_{ik,s} + g^{ks} T_{ki,s} + T_{sk}\, g^{sk}{}_{,i} + T_{ki}\, g^{ks}{}_{,i} + T_{ik}\, g^{ik}{}_{,i} = 0$$

Nun.

$$\sqrt{-g} = \frac{1}{\lambda^2}\sqrt{-\mathrm{Det}\, R_{ik}}$$

$$\lambda\sqrt{-g} = \frac{1}{2}\mathfrak{L} = \frac{1}{4}\left({}^*R_{ik}\,\mathfrak{g}^{ik} + F_{ik}\,\mathfrak{f}^{ik}\right)$$

Also:

$$T^i_k = \mathfrak{f}^{li} F_{lk} - \frac{1}{4}\delta^i_k\, \mathfrak{f}^{\mu\nu} F_{\mu\nu} + \frac{1}{2}\mathfrak{g}^{ik}\, {}^*R_{ik}$$

$$\delta\mathfrak{g}^{ik} = \delta\sqrt{-g}\, g^{ik} = \sqrt{-g}\,\delta g^{ik} + \frac{1}{2} g^{ik} (-)\frac{1}{\sqrt{-g}}\, g\, g^{\mu\nu}\delta g_{\mu\nu}$$

$$- \frac{1}{2}\sqrt{-g}\, g^{ik}\, \underbrace{g_{\mu\nu}\,\delta g^{\mu\nu}}$$

$$R_{ik}\,\delta\mathfrak{g}^{ik} = \sqrt{-g}\left(\delta g^{ik} - \frac{1}{2} g^{ik} g_{\mu\nu}\,\delta g^{\mu\nu}\right) R_{ik}$$

$$= \sqrt{-g}\left(R_{ik} - \frac{1}{2} g_{ik}\, g^{\mu\nu} R_{\mu\nu}\right)\delta g^{ik}$$

Rein Affine Betrachtungsweise.

$$I = \int \mathfrak{L}\, d\tau \qquad \delta I = \int \mathfrak{g}^{ik}\,\delta R_{ik}\, d\tau$$

$$\delta I = \int\left(\mathfrak{g}^{kl}{}_{\alpha} - \delta^l_\alpha\, \mathfrak{g}^{k\beta}{}_{\beta}\right)\delta\Gamma^\alpha_{kl}\, d\tau - \int\left[\left(\mathfrak{g}^{kl}\,\delta\Gamma^\alpha_{kl}\right)_{,\alpha} - \left(\mathfrak{g}^{kl}\,\delta\Gamma^\sigma_{k\sigma}\right)_{,l}\right] d\tau$$

$$'' \qquad - \int\left[\mathfrak{g}^{kl}\left(\delta\Gamma^\alpha_{kl} - \delta^\alpha_l\,\delta\Gamma^\sigma_{k\sigma}\right)\right]_{,\alpha} d\tau$$

$$\mathfrak{g}^{kl}{}_{\alpha} = \mathfrak{g}^{kl}{}_{,\alpha} + \mathfrak{g}^{\sigma l}\,{}^*\Gamma^k_{\sigma\alpha} + \mathfrak{g}^{k\sigma}\,{}^*\Gamma^l_{\alpha\sigma} - \frac{1}{2}\mathfrak{g}^{kl}\left({}^*\Gamma^\sigma_{\alpha\sigma} + {}^*\Gamma^\sigma_{\sigma\alpha}\right)$$

etz. ${}^*\Gamma = \dots$

$$\Gamma'^i_{kl} = \frac{\partial x'_i}{\partial x_r}\frac{\partial x_s}{\partial x'_k}\frac{\partial x_t}{\partial x'_l}\Gamma^r_{st} + \frac{\partial x'_i}{\partial x_m}\frac{\partial^2 x_m}{\partial x'_k\,\partial x'_l} \qquad x'_k = x_k + \xi_k$$

$$\delta^*\Gamma^i_{kl} = \Gamma^r_{kl}\frac{\partial\xi_i}{\partial x_r} - \Gamma^i_{rl}\frac{\partial\xi_r}{\partial x_k} - \Gamma^i_{kr}\frac{\partial\xi_r}{\partial x_l} - \frac{\partial^2\xi_i}{\partial x_k\,\partial x_l} - \frac{\partial\Gamma^i_{kl}}{\partial x_r}\xi_r$$

$$\delta^*\int\mathfrak{L}\, d\tau = \int\delta^*\mathfrak{L}\, d\tau + \int(\xi_\alpha\mathfrak{L})_{,\alpha}\, d\tau$$

Bei $\mathfrak{L}^{kl}_\alpha = \mathfrak{g}^{kl}{}_{\alpha} - \delta^l_\alpha\,\mathfrak{g}^{k\beta}{}_{\beta}$ Divr. $\quad \Gamma^\alpha_{kl} - \delta^\alpha_l\,\Gamma^\sigma_{k\sigma} = M^\alpha_{kl}$

$$\delta^*\int\mathfrak{L}\, d\tau = \int\mathfrak{L}^{kl}_\alpha\,\delta^*\Gamma^\alpha_{kl}\, d\tau - 2\int\left(\mathfrak{g}^{kl}\,\delta^* M^\alpha_{kl}\right)_{,\alpha} d\tau + \int(\xi_\alpha\mathfrak{L})_{,\alpha}\, d\tau =$$

$$= \int\left[-\left(\mathfrak{L}^{kl}_\alpha\Gamma^r_{kl}\right)_{,r}\xi_\alpha + \left(\mathfrak{L}^{kl}_\alpha\Gamma^\alpha_{rl}\right)_{,k}\xi_r + \left(\mathfrak{L}^{kl}_\alpha\Gamma^\alpha_{kr}\right)_{,l}\xi_r - \mathfrak{L}^{kl}_{\alpha,k,l}\,\xi_\alpha - \frac{\partial\Gamma^\alpha_{kl}}{\partial x_r}\xi_r\right] d\tau$$

$$+ \int\left[\left(\mathfrak{L}^{kl}_\alpha\Gamma^r_{kl}\xi_\alpha\right)_{,r} - \left(\mathfrak{L}^{kl}_\alpha\Gamma^\alpha_{rl}\xi_r\right)_{,k} - \left(\mathfrak{L}^{kl}_\alpha\Gamma^\alpha_{kr}\xi_r\right)_{,l} - \left(\mathfrak{L}^{kl}_\alpha\frac{\partial\xi_\alpha}{\partial x_l}\right)_{,k} +\right.$$

$$\left. + \left(\mathfrak{L}^{kl}_{\alpha,k}\,\xi_\alpha\right)_{,l} + 2\left(\mathfrak{g}^{kl}\,\delta^* M^\alpha_{kl}\right)_{,\alpha} + (\mathfrak{L}\xi_\alpha)_{,\alpha}\right] d\tau = \mathfrak{L}^{kl}_\alpha$$

$$= \int\left[-\left(\mathfrak{L}^{kl}_r\Gamma^\alpha_{kl}\right)_{,\alpha} + \left(\mathfrak{L}^{\alpha l}_k\Gamma^k_{rl}\right)_{,\alpha} + \left(\mathfrak{L}^{k\alpha}_l\Gamma^l_{kr}\right)_{,\alpha} - \mathfrak{L}^{k\alpha}_{r,k,\alpha} - \Gamma^\alpha_{kl,r}\right]\xi_r\, d\tau +$$

$$+ \int\left[\left(\mathfrak{L}^{kl}_r\Gamma^\alpha_{kl}\xi_r\right)_{,\alpha} - \left(\mathfrak{L}^{\alpha l}_k\Gamma^k_{rl}\xi_r\right)_{,\alpha} - \left(\mathfrak{L}^{k\alpha}_l\Gamma^l_{kr}\xi_r\right)_{,\alpha} - \left(\mathfrak{L}^{\alpha l}_r\frac{\partial\xi_r}{\partial x_l}\right)_{,\alpha} +\right.$$

$$\left. + \left(\mathfrak{L}^{k\alpha}_{r,k}\,\xi_r\right)_{,\alpha} + 2\left(\mathfrak{g}^{kl}\,\delta^* M^\alpha_{kl}\right)_{,\alpha} + (\mathfrak{L}\xi_\alpha)_{,\alpha}\right] d\tau,$$

Was ist:

$$\mathfrak{L}^{k\alpha}_{r;m} = \mathfrak{L}^{k\alpha}_{r,m} + \mathfrak{L}^{\sigma\alpha}_r\Gamma^k_{\sigma m} + \mathfrak{L}^{k\sigma}_r\Gamma^\alpha_{\sigma m} - \mathfrak{L}^{k\alpha}_\sigma\Gamma^\sigma_{rm} - \mathfrak{L}^{k\alpha}_r\Gamma^\sigma_{\sigma m}$$

$$\mathfrak{L}^{k\alpha}_{r;k} = \mathfrak{L}^{k\alpha}_{r,k} + \mathfrak{L}^{\sigma\alpha}_r\Gamma^k_{\sigma k} + \mathfrak{L}^{k\sigma}_r\Gamma^\alpha_{\sigma k} - \mathfrak{L}^{k\alpha}_\sigma\Gamma^\sigma_{rk} - \mathfrak{L}^{k\alpha}_r\Gamma^\sigma_{\sigma k}$$

Mittel!

$$\chi = -\mathfrak{L}^{k\alpha}_{r,k} + \mathfrak{L}^{\alpha\sigma}_k\Gamma^k_{r\sigma} - \mathfrak{L}^{k\sigma}_r\Gamma^\alpha_{k\sigma} + \mathfrak{L}^{k\alpha}_\sigma\Gamma^\sigma_{kr}$$

$$\mathfrak{L}^{\alpha k}_{r;k} = \mathfrak{L}^{\alpha k}_{r,k} + \mathfrak{L}^{\alpha\sigma}_r\Gamma^k_{\sigma k} + \mathfrak{L}^{\sigma k}_r\Gamma^\alpha_{\sigma k} - \mathfrak{L}^{\alpha k}_\sigma\Gamma^\sigma_{rk} - \mathfrak{L}^{\alpha k}_r\Gamma^\sigma_{\sigma k}$$

→ Man denke zuerst die ξ_k als Konstante. Dann:

$$\delta^*\int\mathfrak{L}\, d\tau = \int\left[\mathfrak{L}^{kl}_\alpha\frac{\partial\Gamma^\alpha_{kl}}{\partial x_r} + 2\left(\mathfrak{g}^{kl}\frac{\partial\Gamma^\alpha_{kl}}{\partial x_r} - \mathfrak{g}^{kl}\delta^\alpha_l\frac{\partial\Gamma^\sigma_{k\sigma}}{\partial x_r}\right)_{,\alpha} + \mathfrak{L}_{,r}\right]\xi_r\, d\tau$$

Ist das nicht trivial?

$$\mathfrak{L}_{,r} = \left(\mathfrak{L}^{kl}_\alpha - 2\mathfrak{g}^{kl}{}_{,\alpha} + 2\mathfrak{g}^{k\sigma}{}_{,\sigma}\,\delta^l_\alpha\right)\frac{\partial\Gamma^\alpha_{kl}}{\partial x_r}$$

$$-2\left(\delta^\alpha_\beta\,\mathfrak{g}^{kl} - \delta^l_\beta\,\mathfrak{g}^{k\alpha}\right)\frac{\partial}{\partial x_\alpha}\frac{\partial\Gamma^\beta_{kl}}{\partial x_r}$$

Das sind nun offenbar die partiellen Derivierten von $\mathfrak{L}$.
Soll man von Haus aus ${}^*\Gamma$ einführen?

Was sind die partiellen Derivierten von $\mathfrak{L}$?

$$d\mathfrak{L} = \mathfrak{g}^{ik}\, dR_{ik} = -(d\Gamma^{\alpha}_{kl})_{;\alpha} + (d\Gamma^{\alpha}_{k\alpha})_{;l} + 2\Gamma^{\alpha}_{\rho l}\, d\Gamma^{\rho}_{k\alpha}$$

Ist offenbar so. Ist genau dieselbe Umformung.

Falls $\mathfrak{L}^{\alpha l}{}_{\alpha} = 0$ genommen wird, hat man nur mit einer eigentlich kovarianten Identität folgendes:

$$\left(\delta^{\alpha}_{n}\mathfrak{L} + 2\mathfrak{g}^{kl}\frac{\partial \Gamma^{\alpha}_{kl}}{\partial x_n} - 2\mathfrak{g}^{k\alpha}\frac{\partial \Gamma^{\sigma}_{k\sigma}}{\partial x_n}\right)_{,\alpha} = 0$$

$$\Gamma^{\alpha}_{kl} = {}^{*}\Gamma^{\alpha}_{kl} - \frac{2}{3}\delta^{\alpha}_{k}\Gamma_{l}$$

$$\Gamma^{\sigma}_{k\sigma} = {}^{*}\Gamma^{\sigma}_{k\sigma} - \frac{2}{3}\Gamma_{k}$$

$$\mathfrak{g}^{kl}\frac{\partial \Gamma^{\alpha}_{kl}}{\partial x_n} = \mathfrak{g}^{kl}\frac{\partial\, {}^{*}\Gamma^{\alpha}_{kl}}{\partial x_n} - \frac{2}{3}\mathfrak{g}^{\alpha l}\frac{\partial \Gamma_l}{\partial x_n}$$

$$\mathfrak{g}^{k\alpha}\frac{\partial \Gamma^{\sigma}_{k\sigma}}{\partial x_n} = \mathfrak{g}^{k\alpha}\frac{\partial\, {}^{*}\Gamma^{\sigma}_{k\sigma}}{\partial x_n} - \frac{2}{3}\mathfrak{g}^{k\alpha}\frac{\partial \Gamma_k}{\partial x_n}$$

$$\mathfrak{g}^{kl}\frac{\partial \Gamma^{\alpha}_{kl}}{\partial x_n} - \mathfrak{g}^{k\alpha}\frac{\partial \Gamma^{\sigma}_{k\sigma}}{\partial x_n} = \mathfrak{g}^{kl}\frac{\partial\, {}^{*}\Gamma^{\alpha}_{kl}}{\partial x_n} - \mathfrak{g}^{k\alpha}\frac{\partial\, {}^{*}\Gamma^{\sigma}_{k\sigma}}{\partial x_n} - \frac{4}{3}\mathfrak{g}^{\underset{\vee}{\alpha l}}\frac{\partial \Gamma_l}{\partial x_n}$$

Falls auch $\mathfrak{g}^{\underset{\vee}{\alpha l}}{}_{,\alpha} = 0$ genommen wird, kann man das letzte Glied umformen:

$$-\frac{4}{3}\left(\mathfrak{g}^{\underset{\vee}{\alpha l}}\frac{\partial \Gamma_l}{\partial x_n}\right)_{,\alpha} = -\frac{4}{3}\mathfrak{g}^{\underset{\vee}{\alpha l}}\frac{\partial^2 \Gamma_l}{\partial x_n \partial x_\alpha} = -\frac{2}{3}\mathfrak{g}^{\underset{\vee}{\alpha l}}\frac{\partial}{\partial x_n}\left(\frac{\partial \Gamma_l}{\partial x_\alpha} - \frac{\partial \Gamma_\alpha}{\partial x_l}\right) =$$

$$= -\,\mathfrak{g}^{\underset{\vee}{\alpha l}}\frac{\partial}{\partial x_n}F_{\alpha l} = -\frac{\partial}{\partial x_n}\left(\mathfrak{g}^{\underset{\vee}{kl}}F_{kl}\right) \leftarrow \boxed{\text{falsch}}$$ daher lautet es:

2 korrigiert

$$\left[\delta^{\alpha}_{n}\left(\mathfrak{L} - 2\mathfrak{g}^{\underset{\vee}{kl}}F_{kl}\right) + 2\mathfrak{g}^{kl}\frac{\partial\, {}^{*}\Gamma^{\alpha}_{kl}}{\partial x_n} - 2\mathfrak{g}^{k\alpha}\frac{\partial\, {}^{*}\Gamma^{\sigma}_{k\sigma}}{\partial x_n}\right]_{,\alpha} = 0$$

wo jetzt die ${}^{*}\Gamma$ die Einstein-kovarianten sind. Da

$\mathfrak{L} = \frac{1}{2}\mathfrak{g}^{kl}R_{kl} = \frac{1}{2}\mathfrak{g}^{kl}\,{}^{*}R_{kl} + \frac{1}{2}\mathfrak{g}^{\underset{\vee}{kl}}F_{kl}$ würde man einen Faktor $\frac{1}{2}$ in unserer Gl. erwarten, so dass F_{kl} ganz herausfiele. Statt dessen ergibt sich ein Faktor 2. Dies ist merkwürdig.

$$-\frac{4}{3}\left(\mathfrak{g}^{\underset{\vee}{\alpha l}}\frac{\partial \Gamma_l}{\partial x_n}\right)_{,\alpha} = -\,\mathfrak{g}^{\underset{\vee}{\alpha l}}F_{\alpha l,n} = +\left(\mathfrak{g}^{\underset{\vee}{\alpha l}}F_{ln,\alpha} + \mathfrak{g}^{\underset{\vee}{\alpha l}}F_{n\alpha,l}\right)$$

$$= \left(\mathfrak{g}^{\underset{\vee}{\alpha l}}F_{ln}\right)_{,\alpha} + \left(\mathfrak{g}^{\underset{\vee}{\alpha l}}F_{n\alpha}\right)_{,l} = \left(\mathfrak{g}^{\underset{\vee}{\alpha l}}F_{ln}\right)_{,\alpha} + \left(\mathfrak{g}^{\underset{\vee}{l\alpha}}F_{nl}\right)_{,\alpha} =$$

$$= -2\left(\mathfrak{g}^{\underset{\vee}{\alpha l}}F_{nl}\right)_{,\alpha}$$ Also:

$$\left[\delta^{\alpha}_{n}\mathfrak{L} - 4\mathfrak{g}^{\underset{\vee}{\alpha l}}F_{nl} + 2\mathfrak{g}^{kl}\frac{\partial\, {}^{*}\Gamma^{\alpha}_{kl}}{\partial x_n} - 2\mathfrak{g}^{k\alpha}\frac{\partial\, {}^{*}\Gamma^{\sigma}_{k\sigma}}{\partial x_n}\right]_{,\alpha} = 0$$

Überlegung der Dimensionen.

(Eine Invariante muss auch von den Einheiten unabhängig sein)

Nach der Bestimmungsgleichung:

$$g_{ik,l} - g_{\sigma k}\Gamma^{\sigma}_{il} - g_{i\sigma}\Gamma^{\sigma}_{lk} = 0$$

hat Γ die Dimension x^{-1}. R also x^{-2}. Daher $\sqrt{-\mathrm{Det}\, R_{ik}}$ hat x^{-4}. Daher der Integrand x^{0}. Das ist recht. Aber was hätte denn der Lorentzsche? Denn das ist doch bei Einstein genau so:

$$\sqrt{-g}\, g^{ik}R_{ik} \qquad g_{ik}\,dx_i\,dx_k = \text{Invariante} \qquad g_{ik} \sim x^{-2}$$

$$g^{ik} \sim x^{2} \qquad \Gamma \sim g^{ik}\left(\frac{\partial g}{\partial x}\cdots\right) \sim x^{-1} \qquad R_{ik} \sim x^{-2}$$

$$g^{ik}R_{ik} \sim x^{0} \qquad \sqrt{-g} \sim x^{-4} \qquad \sqrt{-g}\, g^{ik}R_{ik}\,dx_1\,dx_2\,dx_3\,dx_4 \sim x^{0}.$$

Was ist eigentlich mit der Identität, die aussagt

$$\delta^{i}_{k}\mathfrak{L} = \frac{\partial \mathfrak{L}}{\partial R_{\sigma i}}R_{\sigma k} + \frac{\partial \mathfrak{L}}{\partial R_{i\sigma}}R_{k\sigma}$$

$$= \mathfrak{g}^{\sigma i}R_{\sigma k} + \mathfrak{g}^{i\sigma}R_{k\sigma} \qquad R_{ik} = \lambda g_{ik}$$

$$= \lambda\mathfrak{g}^{\sigma i}g_{\sigma k} + \lambda\mathfrak{g}^{i\sigma}g_{k\sigma} = 2\lambda\,\delta^{i}_{k}\sqrt{-g}$$ trivial.

Es gilt sogar:

$$\delta^{i}_{k}\mathfrak{L} = \frac{1}{2}\mathfrak{g}^{\sigma i}R_{\sigma k} = \frac{1}{2}\mathfrak{g}^{i\sigma}R_{k\sigma}$$

$$g^{\sigma i}g_{k\sigma} = g^{\sigma i}\left(g_{\sigma k} + 2g_{\underset{\vee}{k\sigma}}\right) = \delta^{i}_{k} + 2g^{\sigma i}g_{\underset{\vee}{k\sigma}}$$

$$g^{i\sigma}g_{\sigma k} = g^{i\sigma}\left(g_{k\sigma} - 2g_{\underset{\vee}{k\sigma}}\right) = \delta^{i}_{k} - 2g^{i\sigma}g_{\underset{\vee}{k\sigma}}$$

$$g^{\sigma i}g_{k\sigma} - g^{i\sigma}g_{\sigma k} = 2\left(g^{\sigma i} - g^{i\sigma}\right)g_{\underset{\vee}{k\sigma}} = -4g^{\underset{\vee}{i\sigma}}g_{\underset{\vee}{k\sigma}}.$$

Wie definiert man allgemein Kugelsymmetrie?

Ein Tensorfeld besitzt Kugelsymmetrie, wenn es bei beliebiger Koordinatenwahl eine mit der Drehgruppe isomorphe Transformationsgruppe gibt, welche die Tensoren, Komponenten als Funktionen der Koordinaten invariant läßt.

(Für eine Erwiderung an Weitzenböck):

$$g^{ik} \,\big|\, g_{ik,l} - g_{\sigma k}\,{}^\times\Gamma^\sigma_{il} - g_{i\sigma}\,{}^\times\Gamma^\sigma_{lk} = 0 \,\big|\, g^{in} g^{mk} \qquad (1)$$

$$\frac{1}{g} g_{,l} - {}^\times\Gamma^i_{il} - {}^\times\Gamma^\sigma_{l\sigma} = 0 .$$

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x_l} - \frac12\left({}^\times\Gamma^\sigma_{\sigma l} + {}^\times\Gamma^\sigma_{l\sigma}\right) = 0 \,\big|\, \sqrt{-g}\, g^{mn} \qquad (2)$$

$$-g^{in} g_{ik} g^{mk}{}_{,l} - g^{in}\,{}^\times\Gamma^m_{il} - g^{mk}\,{}^\times\Gamma^n_{lk} = 0$$

$$g^{mn}{}_{,l} + g^{in}\,{}^\times\Gamma^m_{il} + g^{mk}\,{}^\times\Gamma^n_{lk} = 0 \,\big|\, \sqrt{-g} \qquad (3)$$

W. beschränkt sich auf den Umgekehrten. Widerlegung vorläufig:

Man schreibe (1):

$$A_{ikl} = 0 \qquad (1)$$

Dann lautet (2)

$$\frac12 g^{ik} A_{ikl} = 0 \qquad (2)$$

und (3)

$$-g^{in} g^{mk} A_{ikl} = 0 \qquad (3)$$

Die Gleichung, auf die man eben hinaus wollte (ohne sie zuletzt wirklich aufzuschreiben) lautet

$$-\sqrt{-g}\, g^{in} g^{mk} A_{ikl} + \frac12 \sqrt{-g}\, g^{mn} g^{ik} A_{ikl} = 0 \qquad (4)$$

(nämlich: $\sqrt{-g}\cdot(3) + \sqrt{-g}\, g^{mn}\cdot(2)$).

Man multipliziere nun (4) mit $\frac{1}{\sqrt{-g}} g_{mn}$. Das gibt:

$$-g^{ik}A_{ikl} + 2 g^{ik} A_{ikl} = 0 \quad \text{also im Wesentlichen (2).}$$

$$g^{ik} A_{ikl} = 0$$

Schreibt man also das (4) so:

$$B^{mn}{}_l = 0 , \quad \text{so ist}$$

$$B^{mn}{}_l - \frac12 g_{\mu\nu} g^{mn} B^{\mu\nu}{}_l \equiv -\sqrt{-g}\, g^{in} g^{mk} A_{ikl} \,\Big|\, -\frac{1}{\sqrt{-g}} g_{rn} g_{ms}$$

$$-\frac{1}{\sqrt{-g}} g_{rn} g_{ms}\left(B^{mn}{}_l - \frac12 g^{mn} g_{\mu\nu} B^{\mu\nu}{}_l\right) \equiv A_{rsl} \qquad \text{Qu. e. d.}$$

beide Seiten anders geschrieben: $\qquad g_{rn} g^{mn} = \delta^m_r$

$$\frac{1}{\sqrt{-g}}\left(-g_{rn} g_{ms} B^{mn}{}_l + \frac12 g_{rs} g_{mn} B^{mn}{}_l\right) \equiv A_{rsl} .$$

Also man multipliziere die (jetzige) Ausgangsgleichung mit g_{mn}:

$$\mathfrak{g}^{mn}{}_{,l} + \mathfrak{g}^{\sigma n}\,{}^\times\Gamma^m_{\sigma l} + \mathfrak{g}^{m\sigma}\,{}^\times\Gamma^n_{l\sigma} - \frac12 \mathfrak{g}^{mn}\left({}^\times\Gamma^\sigma_{l\sigma} + {}^\times\Gamma^\sigma_{\sigma l}\right) = 0 \,\big|\, g_{mn}$$

$$g_{mn}\left(\sqrt{-g}\, g^{mn}{}_{,l} + g^{mn}\frac{\partial\sqrt{-g}}{\partial x_l}\right) + \sqrt{-g}\left({}^\times\Gamma^\sigma_{\sigma l} + {}^\times\Gamma^\sigma_{l\sigma} - 2\,{}^\times\Gamma^\sigma_{l\sigma} - 2\,{}^\times\Gamma^\sigma_{\sigma l}\right) = 0$$

Nun ist $g_{mn} g^{mn}{}_{,l} = -g^{mn} g_{mn,l} = -\frac{1}{g}\frac{\partial g}{\partial x_l} = -\frac{\partial \lg g}{\partial x_l} = -2\frac{\partial \lg\sqrt{-g}}{\partial x_l}$. Also

$$2\sqrt{-g}\,\frac{\partial \lg\sqrt{-g}}{\partial x_l} - \sqrt{-g}\left({}^\times\Gamma^\sigma_{\sigma l} + {}^\times\Gamma^\sigma_{l\sigma}\right) = 0 \qquad (A) .$$

Zweitens multipliziere man die Ausgangsgleichung mit $g_{rn} g_{ms}$:

$$g_{rn} g_{ms}\left(\sqrt{-g}\, g^{mn}{}_{,l} + g^{mn}\frac{\partial\sqrt{-g}}{\partial x_l}\right) + \sqrt{-g}\left[g_{ms}\,{}^\times\Gamma^m_{rl} + g_{rn}\,{}^\times\Gamma^n_{ls} - \frac12 g_{rs}\left({}^\times\Gamma^\sigma_{l\sigma} + {}^\times\Gamma^\sigma_{\sigma l}\right)\right] = 0$$

$$-\sqrt{-g}\, g_{rs,l} + \cancel{g_{rs}\frac{\partial\sqrt{-g}}{\partial x_l}} + \sqrt{-g}\left[g_{ms}\,{}^\times\Gamma^m_{rl} + g_{rn}\,{}^\times\Gamma^n_{ls} - \cancel{\frac12 g_{rs}\left({}^\times\Gamma^\sigma_{l\sigma} + {}^\times\Gamma^\sigma_{\sigma l}\right)}\right] = 0$$

Qu. e. d.

$g_{ms} g_{rn}$

A. Einstein, Rev. of Mod. Phys. 20, 35, 1948.

"A Generalized Theory of Gravitation".

From this paper it may be inferred that Einstein in the mean time has not got one step further. For if he had he'd say so. — What he has been doing is, to strain his brain to avoid the "side conditions", as I advised him to do.

For preferring the Hermitian to the Real representation, he gives the reason, that you can construct a Hermitian second rank tensor from one (complex) vector, but not a non-symmetric second rank tensor from one real vector.

He thus takes refuge into ~~the~~ ludicrous formality. For

since he keeps to real coordinates x_k and thus to real coordinate transformations, one complex vector obviously corresponds to two real ones.

It is rather trivial to state that you cannot achieve with one real vector what you can with two of them.

And, by the way: even from a complex vector A_k you cannot construct the most general tensor of 2^{nd} rank. Only a special one, viz. $A_k \bar{A}_i$

~~But a special hermitic 2^{nd} rank tensor can even be formed from one real vector viz. let~~

A_k ~~be real and define~~

$$A_k^* = A_k \sqrt{-1}$$?

~~Then form~~

$$A_i A_k^* + A_k A_i^* \sqrt{-1}$$

$${}^*R_{ik} = -\frac{\partial {}^*\Gamma^\alpha_{ik}}{\partial x_\alpha} + \underbrace{\frac{\partial {}^*\Gamma^\alpha_{i\alpha}}{\partial x_k} + {}^*\Gamma^\alpha_{i\beta}\,{}^*\Gamma^\beta_{k\alpha} - {}^*\Gamma^\alpha_{k\alpha}\,{}^*\Gamma^\beta_{ik}}_{\Lambda_{ik}}$$

$$I^* = \int \mathfrak{g}^{ik}\,{}^*R_{ik}\,d\tau$$

$$\delta I^* = \int {}^*R_{ik}\,\delta\mathfrak{g}^{ik}\,d\tau + \cancel{\int \mathfrak{g}^{ik}\,\delta\,{}^*R_{ik}\,d\tau}$$

if $\delta\mathfrak{g}^{ik} = 0$ at boundary.

$$\mathfrak{g}^{ik}{}_{,l} + \mathfrak{g}^{\sigma k}\,{}^*\Gamma^i_{\sigma l} + \mathfrak{g}^{i\sigma}\,{}^*\Gamma^k_{l\sigma} - \tfrac{1}{2}\mathfrak{g}^{ik}\left({}^*\Gamma^\sigma_{l\sigma} + {}^*\Gamma^\sigma_{\sigma l}\right) = 0$$

$$g_{ik,l} - g_{\sigma k}\,{}^*\Gamma^\sigma_{il} - g_{i\sigma}\,{}^*\Gamma^\sigma_{lk} = 0$$

$$\boxed{\mathfrak{g}^{i\underset{\vee}{k}}{}_{,k} = 0} \qquad \left({}^*\Gamma^\sigma_{k\sigma} = {}^*\Gamma^\sigma_{\sigma k}\right).$$

$$\delta I^* = -\int \mathfrak{T}_{ik}\,\delta g^{ik}\,d\tau$$

$$\mathfrak{T}_{ik} = -\sqrt{-g}\left({}^*R_{ik} - \tfrac{1}{2}g_{ik}\,{}^*R\right).$$

$\int \Lambda\,d\tau$; $\Lambda = \mathfrak{g}^{ik}\Lambda_{ik}$; $\delta\mathfrak{g}^{ik} \neq 0$ (boundary)

$$\underbrace{\left({}^*\Gamma^\alpha_{ik} - \delta^\alpha_k\,{}^*\Gamma^\sigma_{i\sigma}\right)}_{\Lambda^\alpha_{ik}}\delta(\mathfrak{g}^{ik}{}_{,\alpha}) = -2\Lambda_{ik}\,\delta\mathfrak{g}^{ik} - \mathfrak{g}^{ik}\,\delta\Lambda_{ik}$$

$$\delta\Lambda = -\Lambda_{ik}\,\delta\mathfrak{g}^{ik} - \Lambda^\alpha_{ik}\,\delta(\mathfrak{g}^{ik}{}_{,\alpha}) \quad (!)$$

$$\delta\int\Lambda\,d\tau = \int\delta\Lambda\,\delta\tau = \int\underbrace{\left(\Lambda^\alpha_{ik,\alpha} - \Lambda_{ik}\right)}_{-{}^*R_{ik}}\delta\mathfrak{g}^{ik}\,d\tau - \int\left(\Lambda^\alpha_{ik}\,\delta\mathfrak{g}^{ik}\right)_{,\alpha}d\tau$$

$$\int\delta\Lambda\,d\tau = \int\mathfrak{T}_{ik}\,\delta g^{ik}\,d\tau - \int\left(\Lambda^\alpha_{ik}\,\delta\mathfrak{g}^{ik}\right)_{,\alpha}d\tau$$

$\longleftrightarrow$

$$x'_k = x_k + \xi_k \qquad \delta^* g^{ik} = g^{rk}\frac{\partial\xi_i}{\partial x_r} + g^{ir}\frac{\partial\xi_k}{\partial x_r} - \frac{\partial g^{ik}}{\partial x_r}\xi_r .$$

$$\delta^*\int\Lambda\,d\tau = \int\delta^*\Lambda\,d\tau + \int(\Lambda\xi_\alpha)_{,\alpha}\,d\tau =$$

$$= -\int\left[\left(g^{sk}\mathfrak{T}_{rk} + g^{ks}\mathfrak{T}_{kr}\right)_{,s} + \mathfrak{T}_{ik}\,g^{ik}{}_{,r}\right]\xi_r\,d\tau +$$

$$+ \int\left[\Lambda\xi_\alpha - \Lambda^\alpha_{ik}\,\delta^*\mathfrak{g}^{ik} + \left(g^{\alpha k}\mathfrak{T}_{rk} + g^{k\alpha}\mathfrak{T}_{kr}\right)\xi_r\right]_{,\alpha}d\tau .$$

$$ {}^{\times}\Gamma^{\alpha}_{kl} = \Gamma^{\alpha}_{kl} + \frac{2}{3}\,\delta^{\alpha}_{k}\,\Gamma_{l} $$

$$ \Gamma^{\alpha}_{kl} = {}^{\times}\Gamma^{\alpha}_{kl} - \frac{2}{3}\,\delta^{\alpha}_{k}\,\Gamma_{l} $$

$$ \Gamma^{\sigma}_{k\sigma} = {}^{\times}\Gamma^{\sigma}_{k\sigma} - \frac{2}{3}\,\Gamma_{k} $$

$$ \Gamma^{\alpha}_{kl} - \delta^{\alpha}_{l}\,\Gamma^{\sigma}_{k\sigma} = {}^{\times}\Gamma^{\alpha}_{kl} - \frac{2}{3}\,\delta^{\alpha}_{k}\Gamma_{l} - \delta^{\alpha}_{l}\,{}^{\times}\Gamma^{\sigma}_{k\sigma} + \frac{2}{3}\,\delta^{\alpha}_{l}\Gamma_{k} $$

$$ = {}^{\times}\Gamma^{\alpha}_{kl} - \delta^{\alpha}_{l}\,{}^{\times}\Gamma^{\sigma}_{k\sigma} + \frac{2}{3}\left(\delta^{\alpha}_{l}\Gamma_{k} - \delta^{\alpha}_{k}\Gamma_{l}\right) $$

The fundamental identities.

We assume $\mathfrak{L} = \mathfrak{L}(\Gamma^\sigma_{\mu\nu,\alpha}, \Gamma^\sigma_{\mu\nu})$.

Then $\underline{\delta\mathfrak{L} = \frac{\partial}{\partial x_\alpha}\left(\frac{\partial\mathfrak{L}}{\partial\Gamma^\sigma_{\mu\nu,\alpha}}\delta\Gamma^\sigma_{\mu\nu}\right) - \phi^{\mu\nu}_\sigma \delta\Gamma^\sigma_{\mu\nu}}$ putting $\underline{\phi^{\mu\nu}_\sigma = \frac{\partial}{\partial x_\alpha}\left(\frac{\partial\mathfrak{L}}{\partial\Gamma^\sigma_{\mu\nu,\alpha}}\right) - \frac{\partial\mathfrak{L}}{\partial\Gamma^\sigma_{\mu\nu}}}$

If $\mathfrak{I} = \int\mathfrak{L}\,dx$, then $\delta\mathfrak{I} = \int\delta^*\mathfrak{L}\,dx + \int\frac{\partial}{\partial x_\alpha}(\varepsilon\mathfrak{L}\xi^\alpha)\,dx = 0.$ $(\bar{x}_i = x_i + \varepsilon\xi^i \text{ etc.})$

The transformation properties of $\Gamma^\sigma_{\mu\nu}$ give: $\underline{\delta^*\Gamma^\sigma_{\mu\nu} = \varepsilon\left(\frac{\partial\xi^\sigma}{\partial x_\beta}\Gamma^\beta_{\mu\nu} - \frac{\partial\xi^\beta}{\partial x_\mu}\Gamma^\sigma_{\beta\nu} - \frac{\partial\xi^\beta}{\partial x_\nu}\Gamma^\sigma_{\mu\beta} - \frac{\partial\Gamma^\sigma_{\mu\nu}}{\partial x_\beta}\xi^\beta - \frac{\partial^2\xi^\sigma}{\partial x_\mu\partial x_\nu}\right)}$

So: $\delta\mathfrak{I} = \int\frac{\partial}{\partial x_\alpha}\left(\frac{\partial\mathfrak{L}}{\partial\Gamma^\sigma_{\mu\nu,\alpha}}\delta^*\Gamma^\sigma_{\mu\nu} + \varepsilon\mathfrak{L}\xi^\alpha\right)dx - \int\phi^{\mu\nu}_\sigma\delta^*\Gamma^\sigma_{\mu\nu}\,dx = 0.$

The last term gives: $-\varepsilon\int\phi^{\mu\nu}_\sigma\left(\frac{\partial\xi^\sigma}{\partial x_\beta}\Gamma^\beta_{\mu\nu} - \frac{\partial\xi^\beta}{\partial x_\mu}\Gamma^\sigma_{\beta\nu} - \dots\right).dx$

$$= \varepsilon\int\left\{-\frac{\partial}{\partial x_\beta}\left(\phi^{\mu\nu}_\sigma\Gamma^\beta_{\mu\nu}\xi^\sigma\right) + \frac{\partial}{\partial x_\mu}\left(\phi^{\mu\nu}_\sigma\Gamma^\sigma_{\beta\nu}\xi^\beta\right) + \frac{\partial}{\partial x_\nu}\left(\phi^{\mu\nu}_\sigma\Gamma^\sigma_{\mu\beta}\xi^\beta\right) + \frac{\partial}{\partial x_\mu}\left(\phi^{\mu\nu}_\sigma\frac{\partial\xi^\sigma}{\partial x_\nu}\right) - \frac{\partial}{\partial x_\nu}\left[\frac{\partial}{\partial x_\mu}(\phi^{\mu\nu}_\sigma).\xi^\sigma\right]\right\}dx$$

$$+\varepsilon\int\left\{\frac{\partial}{\partial x_\beta}\left(\phi^{\mu\nu}_\sigma\Gamma^\beta_{\mu\nu}\right)\xi^\sigma - \frac{\partial}{\partial x_\mu}\left(\phi^{\mu\nu}_\sigma\Gamma^\sigma_{\beta\nu}\right)\xi^\beta - \frac{\partial}{\partial x_\nu}\left(\phi^{\mu\nu}_\sigma\Gamma^\sigma_{\mu\beta}\right)\xi^\beta + \phi^{\mu\nu}_\sigma\frac{\partial\Gamma^\sigma_{\mu\nu}}{\partial x_\beta}\xi^\beta - \frac{\partial}{\partial x_\mu}(\phi^{\mu\nu}_\sigma)\frac{\partial\xi^\sigma}{\partial x_\nu} + \frac{\partial^2}{\partial x_\mu\partial x_\nu}(\phi^{\mu\nu}_\sigma).\xi^\sigma\right\}dx$$

For an arbitrary but vanishing at the boundary variation ξ^i we get from the last row the first identity (the other terms of $\delta\mathfrak{I}$ have the form of a divergence and vanish):

$$\boxed{\frac{\partial}{\partial x_\alpha}\left[\phi^{\mu\nu}_\sigma\Gamma^\alpha_{\mu\nu} - \phi^{\alpha\nu}_\mu\Gamma^\mu_{\sigma\nu} - \phi^{\mu\alpha}_\nu\Gamma^\nu_{\mu\sigma} + \frac{\partial\phi^{\alpha\mu}_\sigma}{\partial x_\mu}\right] + \phi^{\mu\nu}_\alpha\frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x_\sigma} = 0.} \qquad (1)$$

$\delta\mathfrak{I}$ reduces then to
$$\delta\mathfrak{I} = \varepsilon\int\frac{\partial}{\partial x_\alpha}\left\{\frac{\partial\mathfrak{L}}{\partial\Gamma^\sigma_{\mu\nu,\alpha}}\left(\frac{\partial\xi^\sigma}{\partial x_\beta}\Gamma^\beta_{\mu\nu} - \frac{\partial\xi^\beta}{\partial x_\mu}\Gamma^\sigma_{\beta\nu} - \frac{\partial\xi^\beta}{\partial x_\nu}\Gamma^\sigma_{\mu\beta} - \frac{\partial\Gamma^\sigma_{\mu\nu}}{\partial x_\beta}\xi^\beta - \frac{\partial^2\xi^\sigma}{\partial x_\mu\partial x_\nu}\right) + \mathfrak{L}\xi^\alpha\right.$$
$$\left. - \phi^{\mu\nu}_\sigma\Gamma^\alpha_{\mu\nu}\xi^\sigma + \phi^{\alpha\nu}_\sigma\Gamma^\sigma_{\beta\nu}\xi^\beta + \phi^{\mu\alpha}_\sigma\Gamma^\sigma_{\mu\beta}\xi^\beta + \phi^{\alpha\nu}_\sigma\frac{\partial\xi^\sigma}{\partial x_\nu} - \frac{\partial\phi^{\mu\alpha}_\sigma}{\partial x_\mu}\xi^\sigma\right\}dx.$$

The variation $\xi^i = \text{const}$ gives now the second identity:

$$\boxed{\frac{\partial}{\partial x_\alpha}\left[\phi^{\mu\nu}_\beta\Gamma^\alpha_{\mu\nu} - \phi^{\alpha\nu}_\mu\Gamma^\mu_{\beta\nu} - \phi^{\mu\alpha}_\nu\Gamma^\nu_{\mu\beta} + \frac{\partial\phi^{\alpha\mu}_\beta}{\partial x_\mu} + \frac{\partial\mathfrak{L}}{\partial\Gamma^\sigma_{\mu\nu,\alpha}}\frac{\partial\Gamma^\sigma_{\mu\nu}}{\partial x_\beta} - \delta^\alpha_\beta\mathfrak{L}\right] = 0.} \qquad (2)$$

Then the variation $\frac{\partial\xi^i}{\partial x_k} = \text{const}$, if we write simply $[\dots]$ for the corresponding expression of (2):

$$\boxed{[\dots] = +\frac{\partial}{\partial x_\gamma}\left\{\frac{\partial\mathfrak{L}}{\partial\Gamma^\beta_{\mu\nu,\gamma}}\Gamma^\alpha_{\mu\nu} - \frac{\partial\mathfrak{L}}{\partial\Gamma^\sigma_{\alpha\nu,\gamma}}\Gamma^\sigma_{\beta\nu} - \frac{\partial\mathfrak{L}}{\partial\Gamma^\sigma_{\mu\alpha,\gamma}}\Gamma^\sigma_{\mu\beta} + \phi^{\gamma\alpha}_\beta\right\}} \qquad (3)$$

For $\frac{\partial^2\xi^i}{\partial x_k\partial x_l} = \text{const}$:

$$\boxed{\left\{\frac{\partial\mathfrak{L}}{\partial\Gamma^\beta_{\mu\nu,\gamma}}\Gamma^\alpha_{\mu\nu} - \frac{\partial\mathfrak{L}}{\partial\Gamma^\sigma_{\alpha\nu,\gamma}}\Gamma^\sigma_{\beta\nu} - \frac{\partial\mathfrak{L}}{\partial\Gamma^\sigma_{\mu\alpha,\gamma}}\Gamma^\sigma_{\mu\beta} + \phi^{\gamma\alpha}_\beta\right\} - \frac{\partial}{\partial x_\delta}\left(\frac{\partial\mathfrak{L}}{\partial\Gamma^\beta_{\alpha\gamma,\delta}}\right) = \text{skew in } \alpha,\gamma} \qquad (4)$$

And finally for an arbitrary transformation:

$$\boxed{\frac{\partial\mathfrak{L}}{\partial\Gamma^\beta_{\alpha\gamma,\delta}}\cdot\frac{\partial^3\xi^\beta}{\partial x_\alpha\partial x_\gamma\partial x_\delta} = 0.} \qquad (5)$$

(That is: the sum of the 6 ~~terms~~ terms $\alpha\gamma\delta + \delta\alpha\gamma + \gamma\delta\alpha + \gamma\alpha\delta + \delta\gamma\alpha + \alpha\delta\gamma = 0$)

In our case: $\delta\mathfrak{L} = \delta\Gamma^\sigma_{\mu\nu,\alpha}\underbrace{\left(-g^{\mu\nu}\delta^\alpha_\sigma + g^{\mu\alpha}\delta^\nu_\sigma\right)}_{\partial\mathfrak{L}/\partial\Gamma^\sigma_{\mu\nu,\alpha}} + \delta\Gamma^\sigma_{\mu\nu}\underbrace{\left(g^{\varrho\nu}\Gamma^\mu_{\varrho\sigma} + g^{\mu\varrho}\Gamma^\nu_{\sigma\varrho} - g^{\mu\nu}\Gamma^\varrho_{\sigma\varrho} - g^{\varrho\tau}\Gamma^\mu_{\varrho\tau}\delta^\nu_\sigma\right)}_{\partial\mathfrak{L}/\partial\Gamma^\sigma_{\mu\nu}}$

$$\phi^{\mu\nu}_\sigma = -g^{\mu\nu}{}_{,\sigma} + g^{\mu\alpha}{}_{,\alpha}\delta^\nu_\sigma - g^{\varrho\nu}\Gamma^\mu_{\varrho\sigma} - g^{\mu\varrho}\Gamma^\nu_{\sigma\varrho} + g^{\mu\nu}\Gamma^\varrho_{\sigma\varrho} + g^{\varrho\tau}\Gamma^\mu_{\varrho\tau}\delta^\nu_\sigma.$$

$\boxed{\text{Field equations: } \phi^{\mu\nu}_\sigma = 0}$