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„Fitting martingales to given marginals and application in financial mathematics“

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Ema Muchová, Bc.

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Abstract

The problem of fitting martingales to given marginal distributions plays an important role in financial mathematics. In particular, in pricing exotic options. We therefore take a look at the work of Lowther [16], in which the existence and uniqueness of the desired martingale is proved. We mostly focus on the existence of the martingale. To this end we also point out the essential notion of convex ordering, which is together with the assumption of constant mean, required in order to fit martingales. By adding the condition of weak continuity of the marginals and restricting ourselves to processes that are strong Markov, we obtain uniqueness. Lastly, we clarify how fitting martingales to given marginals is related to option pricing. We explain why the Black-Scholes model is not optimal and furthermore we discuss stochastic volatility models and the local volatility model, as more realistic extensions of the Black-Scholes model.

Zusammenfassung

Die Zuordnung von Martingalen zu gegebenen Wahrscheinlichkeitsmaßen spielt eine wichtige Rolle in der Finannzmathematik, insbesonders bei der Bewertung von exotischen Optionen. Deswegen werfen wir einen Blick auf die Arbeit von Lowther [16], in welcher die Existenz und die Eindeutigkeit des entsprechenden Martingal bewiesen ist. Weiters betrachten wir den essentiellen Begriff der konvexen Ordnung, welche mit der Annahme von einem konstanten Mittelwert benötigt wird um Martingale mit vorgegebenen Marginalen finden zu können. Durch das Hinzufügen von schwacher Stetigkeit der Marginalen (als Bedingung) und der Einschränkung auf Prozessen, welche die starke Markow-Eigenschaft auffüllen, erlangen wir Eindeutigkeit. Zuletzt erklären wir, wie die Zuordnung von Martingalen zu gegebenen Marginalen mit Optionsbewertung zusammen hängt. Wir erklären weshalb das Black-

Scholes Modell nicht geeignet ist und weiters diskutieren stochastische Volatilitätsmodelle und das lokale Volatilitätsmodell, als realistischere Erweiterungen des Black-Scholes Modell.

Contents

1 Introduction

The paper of Lowther [16] deals with the problem of finding a martingale fitting given marginal distributions. This problem was already studied by many authors. Starting with Strassen [22] who showed that for a sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$, which have constant mean and are increasing in convex order, there is a martingale $(X_n)_{n\in\mathbb{N}}$ that fits these given marginal distributions. Kellerer [14] extended this result to the case where the marginal distributions μ_t and the martingale (X_t) are indexed by $t \in \mathbb{R}_+$.

Lowther's further contribution to these results is that he showed that these marginals can be fitted in a unique way by a martingale which lies in a particular class of strong Markov processes. Showing this however requires an extra condition on the marginal distributions. This condition is that they need to be weakly continuous i.e. if $t_n \to t$ then $\mu_{t_n}(f) \to \mu_t(f)$ for every continuous and bounded function f. Another consequence that comes with this is that the resulting map from the sets of marginals to the set of martingales is continuous. This means that a small change to marginal distributions results in only a small change to the corresponding martingale measure.

1.1 Structure of the thesis

The main focus of this thesis is the paper of Lowther [16]. It provides a detailed description of some of the results there and also explains the significance of these results in the context of financial mathematics.

In the second chapter we introduce the concept of convex ordering, which is an essential concept for this topic. Further, we introduce the call transform to represent the marginal distributions and briefly mention how the conditions on the call transform lead to different results regarding the existence and uniqueness of a martingale that fits the given marginal distributions.

The following third chapter gives a detailed description of the results of [16] concerning the existence of the martingale fitting given marginal distributions. Before diving into the proofs, we state all the essential definitions and describe the setup for solving our problem. At the end, we state and show in detail that the map that assigns the martingale measure to given marginal distributions is continuous under certain conditions.

In the fourth chapter we again follow the results of [16] and establish in detail why the class of almost-continuous diffusions is in fact not arbitrary.

The last chapter of the thesis connects the previously mentioned results with the field of financial mathematics. In particular, these results are related to the problem of pricing exotic options. At first we take a look at the Black-Scholes model and provide reasons why the assumptions of this model do not reflect behavior on the real market properly. Next, the definitions of stochastic volatility and local volatility models are given. These are more realistic extensions of the Black-Scholes model. The local volatility model is in fact a special case of the stochastic volatility model [19]. The chapter is closed by explaining Lowther's contribution to proving existence and uniqueness of the stock price process obtained from the local volatility model.

2 Convex order

2.1 Stochastic ordering

To be able to compare or order two random variables one can use different types of measures. One example are *location measures* e.g. mean or median. Second example are *dispersion measures* e.g. variance or standard deviation. In this case the result however depends on the choice of the measure and two different measures can lead to contradicting results. Therefore, it is convenient to consider the concept of stochastic ordering and order random variables after considering a whole class of measures producing the same result. Most of this section relies on [21] and [1]. The latter also provides an example of the use of convex ordering in finance.

First, we can consider stochastic orders that compare the location of random variables. One of the most common orders in this sense is the usual *stochastic order*. We say that the random variable X is stochastically larger than the random variable Y and write $X \leq_{st} Y$ if and only if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all increasing functions f. Intuitively, this means that Y is more likely to take on larger values than X [21].

An example of a stochastic order that compares the dispersion of random variables is the *convex order*. We say that the random variable X is smaller than Y in the convex order and write $X \leq_{cx} Y$ if and only if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all convex functions f. We can interpret this as Y being more likely to take on extreme values [21].

One can also notice that $X \leq_{cx} Y$ implies that $\mathbb{E}[X] = \mathbb{E}[Y]$. This can be shown by taking $f_1(x) = x$ and $f_2(x) = -x$ which are both convex functions and so by assumption both $\mathbb{E}[X] \leq \mathbb{E}[Y]$ and $\mathbb{E}[Y] \leq \mathbb{E}[X]$ must hold and therefore it follows that $\mathbb{E}[X] = \mathbb{E}[Y]$ [21].

Another consequence of $X \leq_{cx} Y$ is that $Var[X] \leq Var[Y]$ which we can obtain by applying the function $f(x) = x^2$. This shows that convex ordering strengthens ordering variables based on variance [1].

Convex order plays a key role in the problem of fitting martingales to given marginals. We know that a sequence of marginal distributions $(\mu_n)_{n\in\mathbb{N}}$ is increasing in the convex order if and only if there is a martingale $(X_n)_{n\in\mathbb{N}}$ that fits the given marginal distributions. The sufficiency follows from Strassen [22] and the necessity of this condition comes from Jensen's inequality by the following

$$\mathbb{E}[f(X_n)] = \mathbb{E}[f(\mathbb{E}[X_{n+1}|X_n])] \leq \mathbb{E}[\mathbb{E}[f(X_{n+1})|X_n]] = \mathbb{E}[f(X_{n+1})].$$

In particular, we can say that for two random variables X and Y, $X \leq_{cx} Y$ holds if and only if they admit a coupling (X', Y') such that it is a martingale i.e. $\mathbb{E}[Y'|X'] = X'$ a.s. [21].

2.2 Convex order in terms of call functions

The convex order can be characterized for example in terms of *call functions* or *potential functions*. First, we take a look at the characterization through call functions. In particular, the distribution μ can be represented by the function

$$C_\mu(x) = \int (y - x)_+ \, d\mu(y).$$

Furthermore, we can represent a family $(\mu_t)_{t\in\mathbb{R}_+}$ of marginal distributions by

$$C(t, x) = \int (y - x)_+ \, d\mu_t(y).$$

In financial terms, the function $C(t, x)$ describes prices of call options with strike price x and maturity t. This explains the term call function. The next Lemma is based on results in [21].

Lemma 2.1. *Given two random variables X and Y such that $\mathbb{E}[X] = \mathbb{E}[Y]$, $X \leq_{cx} Y$ holds if and only if*

$$\mathbb{E}(X - c)_+ \leq \mathbb{E}(Y - c)_+ \text{ for all } c \in \mathbb{R}.$$

This condition can be also written as

$$\int_c^\infty 1 - F(t)\ dt \leq \int_c^\infty 1 - G(t)\ dt \text{ for all } c \in \mathbb{R},$$

where F and G are the respective distribution functions.

Proof. $\Rightarrow$ This direction simply follows from the fact that $f(x) = (x-c)_+$ is a convex function.

$\Leftarrow$ We want to prove that $\mathbb{E}(X-c)_+ \leq \mathbb{E}(Y-c)_+$ for all $c \in \mathbb{R}$ implies $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for every convex function f. Assume $\mathbb{E}[f(X)],\ \mathbb{E}[f(Y)] < \infty$ and take wlog $f \geq 0$. Fix $\epsilon > 0$. Now, we can choose an interval $[-k,k]$ such that

$$\int_{[-k,k]^c} f(x)\ d\mu_X(x) < \epsilon,$$
$$\int_{[-k,k]^c} f(y)\ d\mu_Y(y) < \epsilon.$$

We approximate the function f on $[-k,k]$ similarly as in [15]. The function f is uniformly continuous inside the interval $[-k,k]$ and therefore $\forall\ \epsilon > 0\ \exists\ \delta > 0\ \forall x, y \in [-k,k] : |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

Now we can find $N \in \mathbb{N}$ such that the partition P of the interval $[-k,k]$ consists of points $\{i + m/2^N : i \in \{-k, ..., k-1\},\ m \in \mathbb{N} \cup \{0\}$ and $0 \leq m \leq 2^N\}$ and it holds that $\|P\| < \delta$.

Next, we can find a piece-wise linear convex function g with breakpoints in the points of the partition P, which equals f at these breakpoints. This function can be written as a linear combination of functions $a + bx, (x - a_1)_+, ..., (x-a_n)_+$ because these functions form a basis for linear splines defined on $[-k,k]$ with the given breakpoints [9].

For any $x \in [-k,k]$ there exists some i such that $x \in [x_i, x_{i+1}]$, where x_i, x_{i+1} are points of the partition P. Since f and g are equal at the breakpoints, we have $|g(x) - f(x)| = |g(x) - g(x_i) + f(x_i) - f(x))| \leq |g(x) - g(x_i)| + |f(x) - f(x_i)| < 2\epsilon$. This means that there is a sequence of piece-wise linear functions of the form $f_n(x) = \sum_{i=0}^{l} b_{n_i}(x - a_{n_i})_+ + ax + b$ that converges point-wise to the function f on $[-k,k]$.

By assuming $\mathbb{E}(X-c)_+ \leq \mathbb{E}(Y-c)_+$ for all c we obtain

$$\mathbb{E}[f_n(X)] \leq \mathbb{E}[f_n(Y)].$$

Applying dominated convergence theorem to the above inequality leads to $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for every f convex on $[-k, k]$. Finally, we get the alternative inequality by

$$\begin{aligned}
\mathbb{E}[(X-c)_+] &= \int_0^\infty \mathbb{P}((X-c)_+ \geq t)\ dt \\
&= \int_0^\infty \mathbb{P}(X-c \geq t)\ dt \\
&= \int_0^\infty \mathbb{P}(X \geq t+c)\ dt \\
&= \int_c^\infty \mathbb{P}(X \geq t)\ dt \\
&= \int_c^\infty 1 - F(t)\ dt.
\end{aligned}$$

□

Lemma 2.2. *If we represent a family of marginal distributions $(\mu_t)_{t\in\mathbb{R}_+}$ that are increasing in convex order and are weakly continuous in t by the function $C(t,x) = \int (y-x)_+\ d\mu_t(y)$, it satisfies the following properties:*
1. $C(t,x)$ is convex in x.
2. $C(t,x)$ is continuous and increasing in t.
3. $C(t,x) \to 0$ as $x \to \infty$, for every $t \in \mathbb{R}_+$.
4. There exists $a \in \mathbb{R}$ such that $C(t,x) + x \to a$ as $x \to -\infty$ for every $t \in \mathbb{R}_+$. The real number a is in fact the mean of the distribution μ_t.

Proof. For the first property we want to show that

$$C(t, \lambda x_1 + (1-\lambda)x_2) \leq \lambda\ C(t, x_1) + (1-\lambda)\ C(t, x_2)$$

for every $\lambda \in [0,1]$. This can be shown by using convexity of the function $f(x) = (x-c)_+$ as follows

$$\begin{aligned}
C(t, \lambda x_1 + (1-\lambda)x_2) &= \int (y - (\lambda x_1 + (1-\lambda)x_2))_+\ d\mu_t(y) \\
&\leq \lambda \int (y-x_1)_+\ d\mu_t(y) + (1-\lambda) \int (y-x_2)_+\ d\mu_t(y) \\
&= \lambda\ C(t,x_1) + (1-\lambda)\ C(t,x_2).
\end{aligned}$$

Continuity of $C(t,x)$ in t follows from weak continuity of the marginals. The fact that $C(t,x)$ is increasing follows from the sequence of distributions being increasing in convex order. The third property can be shown by the following

$$\lim_{x\to\infty}\int (y-x)_+ \; d\mu_t(y) = \lim_{x\to\infty}\int_x^\infty (y-x)\; d\mu_t(y) = 0.$$

Finally, we prove the last property and show that the real number a is the mean of the marginal distribution

$$\begin{aligned}\lim_{x\to-\infty}\int (y-x)_+ \; d\mu_t(y) + x &= \lim_{x\to-\infty}\int_x^\infty (y-x)\; d\mu_t(y) + x \\ &= \lim_{x\to-\infty}\left(\int_x^\infty y\; d\mu_t(y) - \int_x^\infty x\; d\mu_t(y)\right) + x \\ &= \lim_{x\to-\infty}\left(\int_x^\infty y\; d\mu_t(y) + \int_{-\infty}^x x\; d\mu_t(y)\right) \\ &= E[\mu_t] = a.\end{aligned}$$

□

2.3 Convex order in terms of potential functions

Here we consider the characterization through the potential function. More specifically, the distribution μ can be represented by the function

$$u_\mu(x) = \int |y-x|\; d\mu(y).$$

Furthermore, we can represent a family $(\mu_t)_{t\in\mathbb{R}_+}$ of marginal distributions by

$$u(t,x) = \int |y-x|\; d\mu_t(y).$$

Lemma 2.3. *Given $u_\mu(x) = \int |y-x|\; d\mu(y)$, we can obtain the following formula for the distribution function F_μ:*

$$F_\mu = \frac{u'_\mu + 1}{2}.$$

Proof.

$$\begin{aligned}
u_\mu(x) &= \mathbb{E}_\mu|y-x| \\
&= \int |y-x|\ d\mu(y) \\
&= \int_0^\infty \mathbb{P}(|y-x| \geq t)\ dt \\
&= \int_0^\infty \mathbb{P}(y-x > t)\ dt + \int_{-\infty}^0 \mathbb{P}(y-x \leq t)\ dt \\
&= \int_0^\infty \mathbb{P}(y > t+x)\ dt + \int_{-\infty}^0 \mathbb{P}(y \leq t+x)\ dt \\
&= \int_x^\infty \mathbb{P}(y > t)\ dt + \int_{-\infty}^x \mathbb{P}(y \leq t)\ dt \\
&= \int_x^\infty 1 - F(t)\ dt + \int_{-\infty}^x F(t)\ dt \\
&= \lim_{c\to\infty} \int_x^c 1 - F(t)\ dt + \lim_{c\to\infty} \int_{-c}^x F(t)\ dt
\end{aligned}$$

$$\begin{aligned}
u'_\mu(x) &= \lim_{c\to\infty} (1 - F(c) - (1 - F(x)) + F(x) - F(-c)) \\
&= 1 - 1 - (1 - F(x)) + F(x) - 0 \\
&= 2F(x) - 1
\end{aligned}$$

At last, we get the formula $F(x) = \frac{u'_\mu(x)+1}{2}$. □

Lemma 2.4. *(Theorem 3.A.2., [21]) Given two random variables X and Y such that $\mathbb{E}[X] = \mathbb{E}[Y]$, $X \leq_{cx} Y$ holds if and only if*

$$\mathbb{E}|X - c| \leq \mathbb{E}|Y - c| \text{ for all } c \in \mathbb{R}.$$

Proof. ⇒ If $X \leq_{cx} Y$, then $\mathbb{E}|X - c| \leq \mathbb{E}|Y - c|$ holds for all $c \in \mathbb{R}$ because $f(x) = |x - c|$ is a convex function.
⇐ We want to prove that $\mathbb{E}|X - c| \leq \mathbb{E}|Y - c|$ for all $c \in \mathbb{R}$ implies that $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for every convex function f. The function $u_\mu(x)$ can be

written in terms of the function $C_\mu(x)$ as follows

$$\begin{aligned} u_\mu(x) &= \int |y-x| \; d\mu(y) \\ &= \int 2 \left((y-x)_+ \; d\mu(y) - \frac{1}{2} \; (y-x) \right) \, d\mu(y) \\ &= 2 \int (y-x)_+ \; d\mu(y) - \int y \; d\mu(y) \; + \; x \\ &= 2 \; C_\mu(x) - \mathbb{E}[\mu] + x. \end{aligned} \tag{1}$$

From (1) we get the following

$$\begin{gathered} \mathbb{E}|X-c| \leq \mathbb{E}|Y-c| \\ 2\mathbb{E}[(X-c)_+] - \mathbb{E}[X] + c \leq 2\mathbb{E}[(Y-c)_+] - \mathbb{E}[Y] + c \\ \mathbb{E}[(X-c)_+] \leq \mathbb{E}[(Y-c)_+]. \end{gathered} \tag{2}$$

This means that $\mathbb{E}|X-c| \leq \mathbb{E}|Y-c|$ for all $c \in \mathbb{R}$ implies $\mathbb{E}[(X-c)_+] \leq \mathbb{E}[(Y-c)_+]$ for all $c \in \mathbb{R}$. Combining this with Lemma 2.1. gives the desired result. □

Similar properties as for the function $C(t,x)$ hold also for the function $u(t,x)$.

Lemma 2.5. *If we represent a family of marginal distributions $(\mu_t)_{t\in\mathbb{R}_+}$ that are increasing in convex order and are weakly continuous in t by the function $u(t,x) = \int |y-x| \; d\mu_t(y)$, it satisfies the following properties:*
1. $u(t,x)$ is convex in x.
2. $u(t,x)$ is continuous and increasing in t.
3. There exists a real number a such that $u(t,x) - x + a \to 0$ as $x \to \infty$ for every $t \in \mathbb{R}_+$.
4. There exists a real number a such that $u(t,x) + x - a \to 0$ as $x \to -\infty$ for every $t \in \mathbb{R}_+$.

The last two properties can be combined into a single property. There exists a real number a such that $u(t,x) - |x-a| \to 0$ as $x \to \pm\infty$, for every $t \in \mathbb{R}_+$.

Proof. The first two properties can be proved similarly as in Lemma 2.2. For the last property we do the following

$$\begin{aligned}\lim_{x\to-\infty}\int |y-x|-|x-a|\ d\mu_t(y) &= \lim_{x\to-\infty}\int_{-\infty}^{x}(x-y)\ d\mu_t(y)+\int_{x}^{\infty}(y-x)\ d\mu_t(y)\\ &\quad -|x-a|\\ &= \lim_{x\to-\infty}\int_{x}^{\infty} y-x\ d\mu_t(y)+x-a\\ &= \lim_{x\to-\infty}\int_{x}^{\infty} y\ d\mu_t(y)+\int_{-\infty}^{x} x\ d\mu_t(y)-a\\ &= \lim_{x\to-\infty}\int_{x}^{\infty} y\ d\mu_t(y)-a.\end{aligned}$$

This limit is 0 if $a = E[\mu_t]$. For $x \to \infty$ the computation follows similarly.

□

2.4 Call transform and martingale

Let $(\mu_t)_{t\in\mathbb{R}_+}$ be a sequence of marginal distributions with constant mean and let $C(t,x)$ be a function associated to the distribution μ_t via the call transform given by $C(t,x) = \int(y-x)_+\ d\mu_t(y)$.

As described in [2] and already hinted in the introduction, we get different results for the existence and uniqueness of the martingale measure when imposing different conditions on the function $C(t,x)$.

By assuming that $C(t,x)$ is increasing in t, we get that there exists a martingale fitting the given marginal distributions. This result was proved by Kellerer in [14]. This martingale is however not unique in general.

Adding the assumption that $C(t,x)$ is also continuous in t, we get that there exists a martingale fitting the given marginal distributions and furthermore it is also unique if we restrict ourselves to the class of martingales that are almost-continuous diffusions. This was proved by Lowther in [16] and it is the main interest of this thesis.

This topic is furthermore discussed in the paper [6] by Dupire. It is showed there that under more restrictive assumptions on $C(t,x)$ we obtain existence. These assumptions are $C(t,x)$ being increasing and also differentiable in t. It is mentioned there that under some technical assumptions, when restricting ourselves to martingales that are continuous diffusions, we obtain uniqueness. The prove was however provided by Lowther under weaker assumptions.

3 Existence of the martingale

This chapter is about the already mentioned result which says that given a sequence of marginal distributions which are weakly continuous and increasing in the convex order, there not only exists a martingale fitting these distributions, but it is also a strong Markov process. Before moving any further we define some essential notions.

3.1 Definitions

The below given definitions come from [4] or alternatively [16].

Definition 3.1. *(Definition Section 35, [4]) Let $(X_t)_{t\geq 0}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. The sequence $\{(X_t, \mathcal{F}_t) : t \geq 0\}$ is then a martingale if $\mathbb{E}[|X_t|] < \infty$ and if $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ for $0 \leq s \leq t$.*

Definition 3.2. *(Section 35., [4]) Let τ be a random variable on a probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$. Then τ is called a stopping time (with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$) if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.*

Definition 3.3. *(Chapter 9, [13]) Let X be a stochastic process indexed by $t \in [0, \infty)$. Given any time $t \in [0, \infty)$, X is said to be continuous in probability at time t if for all $\epsilon > 0$*

$$\lim_{s\to t} \mathbb{P}(\{\omega \in \Omega \mid |X_s(\omega) - X_t(\omega)| \geq \epsilon\}) = 0.$$

Definition 3.4. *(Definition 1.1., [16]) A real valued stochastic process X is strong Markov if for every bounded and measurable function $g : \mathbb{R} \to \mathbb{R}$ and every $t \in \mathbb{R}_+$ there exists a measurable function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that*

$$f(\tau, X_\tau) = E[g(X_{\tau+t})|\mathcal{F}_\tau]$$

for every finite stopping time τ.

Definition 3.5. *(Definition 1.1., [16]) A real valued stochastic process X is almost-continuous if it is càdlàg, continuous in probability and given any two independent càdlàg processes Y, Z each with the same distribution as X and for every $s < t \in \mathbb{R}_+$ we have*

$$\mathbb{P}(Y_s < Z_s,\ Y_t > Z_t \text{ and } Y_u \neq Z_u \text{ for every } u \in (s, t)) = 0.$$

This property says that $Y - Z$ cannot change sign without passing through zero [16]. Furthermore, it is clear that continuous processes satisfy this condition by the mean value theorem.

Definition 3.6. *(Definition 1.1., [16]) A real valued stochastic process X is an almost-continuous diffusion (ACD) if it is strong Markov and almost-continuous.*

Furthermore, to represent the marginal distributions we use the below defined space of functions.

Definition 3.7. *(Definition 1.2., [16]) Let CP be the set of functions $C : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that*
1. $C(t,x)$ is convex in x and continuous and increasing in t.
2. $C(t,x) \to 0$ as $x \to \infty$ for every $t \in \mathbb{R}_+$.
3. There exists $a \in \mathbb{R}$ such that $C(t,x) + x \to a$ as $x \to -\infty$ for every $t \in \mathbb{R}_+$.

The first property is the same as saying that the marginals are increasing in convex order and that they are weakly continuous. The last property is the same as saying that they have constant mean. If a process X has marginals consistent with some $C \in CP$, then

$$C(t,x) = \mathbb{E}[(X_t - x)_+]. \tag{3}$$

Lemma 3.1. *(Chapter 1, [16]) If X is a martingale which is continuous in probability then C given by $C(t,x) = \mathbb{E}[(X_t - x)_+]$ belongs to the space CP.*

Proof. Most of these properties were already proved in Chapter 2. Convexity of $C(t,x)$ follows from convexity of the function $f(x) = (y - x)_+$. The fact that C is increasing in t follows from X being a process that is increasing in the convex order, which follows from X being a martingale. This was also pointed out in Chapter 2.

The property that C is continuous in t requires more calculations. To prove this we need to show that if $t_n \to t$, it holds that $C(t_n, x) \to C(t,x)$ i.e. $\mathbb{E}[(X_{t_n} - x)_+] \to \mathbb{E}[(X_t - x)_+]$.

For this we use the property that X is continuous in probability, which means that for $t_n \to t$

$$\lim_{n\to\infty} \mathbb{P}(\{\omega \in \Omega \mid |X_{t_n}(\omega) - X_t(\omega)| \geq \epsilon\}) = 0.$$

First we show that the variable $(X-x)_+$ is continuous in probability i.e. that the following holds

$$\lim_{n\to\infty} \mathbb{P}(\{\omega \in \Omega \mid |(X_{t_n}(\omega)-x)_+ - (X_t(\omega)-x)_+| \geq \epsilon\}) = 0.$$

This can be shown by proving that for every $n \in \mathbb{N}$ we get

$$\mathbb{P}(\{\omega \in \Omega \mid |(X_{t_n}(\omega)-x)_+ - (X_t(\omega)-x)_+| \geq \epsilon\}) \leq \mathbb{P}(\{\omega \in \Omega \mid |X_{t_n}(\omega)-X_t(\omega)| \geq \epsilon\}).$$

The above is equivalent to showing

$$\mathbb{P}(\{\omega \in \Omega \mid |X_{t_n}(\omega)-X_t(\omega)| < \epsilon\}) \leq \mathbb{P}(\{\omega \in \Omega \mid |(X_{t_n}(\omega)-x)_+ - (X_t(\omega)-x)_+| < \epsilon\}).$$

We need to show that if ω is such that $|X_{t_n}(\omega) - X_t(\omega)| < \epsilon$ than also $|(X_{t_n}(\omega)-x)_+ - (X_t(\omega)-x)_+| < \epsilon$. There are two options, in the first case ω is such that $X_{t_n}(\omega), X_t(\omega) \geq x$ and so we get

$$\begin{aligned} |(X_{t_n}(\omega)-x)_+ - (X_t(\omega)-x)_+| &= |X_{t_n}(\omega) - x - X_t(\omega) + x| \\ &= |X_{t_n}(\omega) - X_t(\omega)| \\ &< \epsilon. \end{aligned}$$

The second case is when $X_{t_n}(\omega) \leq x$ and $X_t(\omega) \geq x$ or vice versa. Wlog assume the first case. We get $|(X_{t_n}(\omega)-x)_+ - (X_t(\omega)-x)_+| = |-X_t(\omega)+x| < \epsilon$ because x lies between X_{t_n} and X_t. The last case when $X_{t_n}(\omega), X_t(\omega) < x$ is clear. Because this holds for every n, we can conclude that the process $(X-x)_+$ is continuous in probability.

Furthermore, we assume that this variable is uniformly integrable and therefore we can conclude that $\mathbb{E}[(X_{t_n}-x)_+] \to \mathbb{E}[(X_t-x)_+]$ [3]. We proved that $C(t,x)$ is continuous in t. The last two properties are again proved in Chapter 2. □

Lemma 3.2. *The distribution function can be recovered from the function C by the following*

$$F_{\mu_t}(x) = \mu_t((-\infty, x]) = 1 + \frac{\partial C(t,x)}{\partial x_+}.$$

Proof. We obtain this result by the following calculation

$$\begin{aligned}
\frac{\partial C(t,x)}{\partial x_+} &= \lim_{h\to 0^+} \frac{C(t,x+h)-C(t,x)}{h} \\
&= \lim_{h\to 0^+} \frac{\lim_{k\to\infty}\int_{-k}^{k}(y-x-h)_+ - (y-x)_+}{h}\ d\mu_t(y) \\
&= \lim_{k\to\infty}\int_{-k}^{k} \lim_{h\to 0^+} \frac{(y-x-h)_+ - (y-x)_+}{h}\ d\mu_t(y) \\
&= \lim_{k\to\infty}\int_{x}^{k} -1\ d\mu_t(y) \\
&= -\mu_t((x,\infty)) \\
1+\frac{\partial C(t,x)}{\partial x_+} &= 1-\mu_t((x,\infty)) \\
&= \mu_t((-\infty,x]).
\end{aligned}$$

□

The main result of Lowther's paper is the following theorem. In this section we however focus mostly on the existence part.

Theorem 3.3. *(Theorem 1.3., [16]) For any $C \in CP$ there exists a unique measure $\mathbb{P}$ on $(D,\mathcal{F})$ under which X is an ACD martingale and $C(t,x) = E[(X_t - x)_+]$.*

The existence is going to be showed by taking limits of processes that are ACD martingales and are matching the marginals at finite sets of times and furthermore weak compactness will be used to ensure the existence of the limit [16]. The uniqueness part of this result will not be proved here. The proof can be however found in Chapter 4 of [16].

To begin with proving this result we are going to need some new definitions. We are going to prove the existence of the martingale measure and for this we introduce the measurable space of càdlàg real valued processes $(D,\mathcal{F})$ and the coordinate process X and X^S as described in [16]. Let

$$\begin{aligned}
&D = \{\text{càdlàg functions } \omega : \mathbb{R}_+ \to \mathbb{R}\}, \\
&X : \mathbb{R}_+ \times D \to \mathbb{R},\ (t,\omega) \mapsto X_t(\omega) \equiv \omega(t), \\
&\mathcal{F} = \sigma(X_t : t \in \mathbb{R}_+), \\
&\mathcal{F}_t = \sigma(X_s : s \in [0,t]).
\end{aligned}$$

X is a càdlàg process adapted to $(\mathcal{F}_t)_{t\geq 0}$. For a subset $S \subseteq \mathbb{R}_+$, we take $\mathbb{R}^S$ to be the set of real valued functions defined on S i.e. $\mathbb{R}^S = \{f : S \to \mathbb{R}\}$. We consider it as a topological space under the topology of pointwise convergence and we denote its Borel σ-algebra by $\mathcal{F}^S$. The coordinate process on $\mathbb{R}^S$ is denoted by X_t^S. Moreover,

$$X^S : S \times \mathbb{R}^S \to \mathbb{R}, \ (t, \omega) \mapsto X_t^S(\omega) \equiv \omega(t).$$

This process is adapted to the natural filtration $(\mathcal{F}_t^S)_{t\in\mathbb{R}_+}$, which is given by

$$\mathcal{F}_t^S = \sigma(X_s^S : s \in S, s \leq t).$$

$\mathbb{P}^S$ is used to denote the measure on $(\mathbb{R}^S, \mathcal{F}^S)$ obtained from the marginal law of X_t under $\mathbb{P}$ with $t \in S$. In the following we understand $\mathcal{P}(X)$ to be the collection of all probability measures on X.

Definition 3.8. *(Chapter 1, [3]) Suppose that $\mathbb{P}_n, \mathbb{P} \in \mathcal{P}(X)$ where X is a Polish space, i.e. a separable complete metric space. If*

$$\lim_{n\to\infty} \mathbb{P}_n(f) = \mathbb{P}(f)$$

for every continuous and bounded function f, then we say that $(\mathbb{P}_n)$ converges weakly to $\mathbb{P}$ and write $\mathbb{P}_n \xrightarrow{w} \mathbb{P}$.

The condition ensuring weak convergence can also be written as $\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}_n}[f] = \mathbb{E}[f]$ for every continuous and bounded function f. The topology of weak convergence on the probability measures on $(\mathbb{R}^S, \mathcal{F}^S)$ is the topology generated by the maps $\mathbb{P} \mapsto \mathbb{E}_{\mathbb{P}}[f]$ for all real valued continuous and bounded functions f in $\mathbb{R}^S$ [16].

The last thing to define before we move to the next result is convergence of measures in the sense of finite-dimensional distributions. Suppose $\mathbb{P}_n$ is a sequence of probability measures on $(D, \mathcal{F})$. We say that this sequence converges in the sense of finite dimensional distributions to the probability measure $\mathbb{P}$ if and only if $\mathbb{P}_n^S \to \mathbb{P}^S$ weakly for every finite subset S of $\mathbb{R}_+$ [16].

If S is countable then $\mathbb{R}^S$ is a Polish space i.e. a separable complete metric space as it has a countable dense subset consisting of ω with $\omega(t)$ rational for $t \in S$ and zero for all but finitely many t [16]. The topology is given by the following complete metric

$$d(\omega, \omega') = \sum_n 2^{-n} \min(|\omega(s_n) - \omega(s_n')|, 1),$$

where $S = \{s_1, s_2, ...\}$ [16].

Theorem 3.4. *(Theorem 15.39., [10]) Let $P \subset \mathcal{P}(X)$ and let X be a Polish space. Then under the topology of weak convergence, P is relatively compact if and only if P is tight.*

Proof. This result is due to Prokhorov. The proof can be found e.g. in [4]. □

A set of probability measures P on a Polish space is said to be tight if for every $\epsilon > 0$ there exists a compact set C with $\mathbb{P}(C) > 1 - \epsilon$ for all $\mathbb{P} \in P$ [10]. By the previous theorem, P is then weakly compact (compact under the topology of weak convergence) which means that for any sequence of probability measures $(\mathbb{P}_n)_{n\in\mathbb{N}}$ which is tight, there is a probability measure $\mathbb{P}$ and a subsequence $\mathbb{P}_{n_k}$ converging weakly to $\mathbb{P}$. This result is used to find martingale measures with specified marginals as limits of sequences [16]. Now that we have the essential definitions we can move on to the next result which is the first step in proving the main theorem of this section.

3.2 Existence of martingale measure

Lemma 3.5. *(Lemma 3.1., [16]) Let $C \in CP$ and $(\mathbb{P}_n)_{n\in\mathbb{N}}$ be a sequence of martingale measures on $(D, \mathcal{F})$ such that $\mathbb{E}_{\mathbb{P}_n}[(X_t - x)_+] \to C(t, x)$.*
Then there exists a subsequence $\mathbb{P}_{n_k}$ and a martingale measure $\mathbb{P}$ on $(D, \mathcal{F})$ such that $\mathbb{P}_{n_k} \to \mathbb{P}$ in the sense of finite-dimensional distributions. Furthermore, X is a martingale under $\mathbb{P}$, continuous in probability and satisfies $\mathbb{E}_{\mathbb{P}}[(X_t - x)_+] = C(t, x)$.

Proof. This proof provides a more detailed description of the proof in [16]. Choose $t \in \mathbb{R}_+$ and $\epsilon > 0$. We get that for every $K > 0$,

$$\begin{aligned}\mathbb{P}_n(|X_t| > K) &= \mathbb{E}_{\mathbb{P}_n}[\mathbb{1}_{\{|X_t|>K\}}] \\ &\le \mathbb{E}_{\mathbb{P}_n}[(X_t - K + 1)_+ + (X_t + K - 1)_+ - (X_t + K)_+ + 1] \\ &\to C(t, K - 1) + C(t, 1 - K) - C(t, -K) + 1.\end{aligned}$$

The inequality is depicted in Figure 1. This can be made arbitrarily small by making K large, hence we get that for every $\epsilon > 0$ there exists a $K > 0$ such that $\mathbb{P}_n(|X_t| > K) < \epsilon$ for every n.

Let $S = \{s_1, s_2, ...\}$ to be a countable dense subset of $\mathbb{R}_+$ and let $\epsilon > 0$. We get that there exists a sequence $K_n > 0$ such that

$$\mathbb{P}_n(|X_{s_n}| > K_n) < 2^{-n}\epsilon.$$

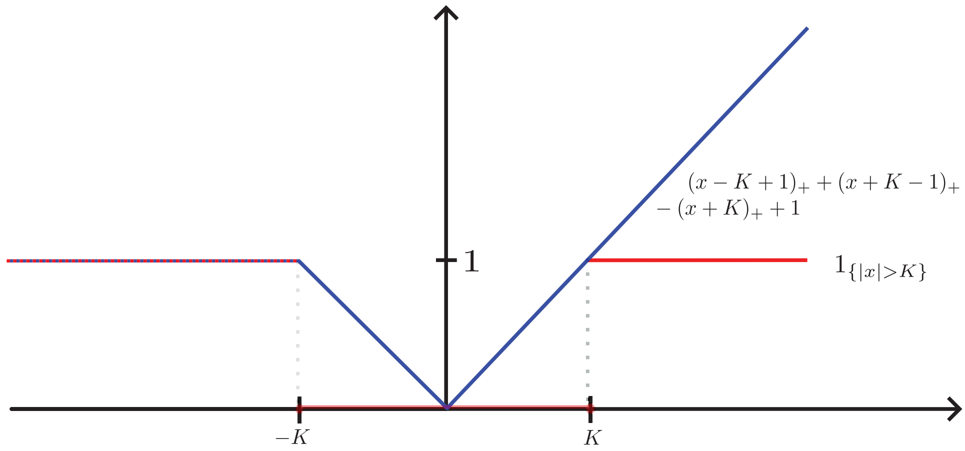

Figure 1: above inequality

Let A be a compact set consisting of all $\omega \in \mathbb{R}^S$ satisfying $|\omega(s_n)| \leq K_n$,

$$\mathbb{P}_n^S(\mathbb{R}^S \setminus A) \leq \sum_{n=0}^{\infty} \mathbb{P}_n(|X_{s_n}| > K_n) < \sum_{n=0}^{\infty} 2^{-n}\epsilon = \epsilon.$$

This implies that

$$\mathbb{P}_n^S(A) > 1 - \epsilon \ \forall \ n.$$

The sequence $\mathbb{P}_n^S$ is therefore tight and so there exists a probability measure $\mathbb{Q}$ on $(\mathbb{R}^S, \mathcal{F}^S)$ and subsequence $\mathbb{P}_{n_k}^S$ that converges weakly to $\mathbb{Q}$. From weak convergence we get that $\forall \ t \in S$ and $x, y \in \mathbb{R}$

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[(X_t^S - x)_+ - (X_t^S - y)_+] &= \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}_n}[(X_t - x)_+ - (X_t - y)_+] \\ &= C(t, x) - C(t, y). \end{aligned}$$

By letting y go to infinity and by using dominated convergence theorem we get the following

$$\mathbb{E}_{\mathbb{Q}}[(X_t^S - x)_+] = C(t, x),$$

where the right-hand side follows from the properties of elements in the set CP.

Now let $s < t$ be in S, let $Z : \mathbb{R}^S \to \mathbb{R}$ be $\mathcal{F}_s^S$-measurable, continuous and such that ZX_s^S is bounded, and $0 \le Z \le 1$ then,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[Z(X_s^S - x)_+] &= \lim_{n\to\infty} \mathbb{E}_{P_n^S}[Z(X_s^S - x)_+] \\
&\le \lim_{n\to\infty} \mathbb{E}_{P_n^S}[Z(X_t^S - x)_+] \\
&\le \lim_{n\to\infty} \mathbb{E}_{P_n^S}[Z(X_t^S - x)_+] - \mathbb{E}_{\mathbb{Q}}[Z(X_t^S - y)_+] + \mathbb{E}_{\mathbb{Q}}[(X_t^S - y)_+] \\
&= \lim_{n\to\infty} \mathbb{E}_{P_n^S}[Z((X_t^S - x)_+ - (X_t^S - y)_+)] + C(t, y) \\
&= \mathbb{E}_{\mathbb{Q}}[Z((X_t^S - x)_+ - (X_t^S - y)_+)] + C(t, y).
\end{aligned}$$

The first and last steps follow from weak convergence of the sequence of measures $\mathbb{P}_n^S$, the second step follows from the sequence of measures being increasing in the convex order and the third step follows from the previous calculation and the fact that $0 \le Z \le 1$. By repeating the same procedure of letting y go to infinity and using dominated convergence theorem we get

$$\mathbb{E}_{\mathbb{Q}}[Z(X_s^S - x)_+] \le \mathbb{E}_{\mathbb{Q}}[Z(X_t^S - x)_+],$$

which means that $(X^S - x)_+$ is a submartingale with respect to the measure $\mathbb{Q}$. Proceeding further with the calculations we get

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[ZX_s^S] &= \lim_{x\to-\infty} \mathbb{E}_{\mathbb{Q}}[Z(X_s^S - x)_+ + x] \\
&\le \lim_{x\to-\infty} \mathbb{E}_{\mathbb{Q}}[Z(X_t^S - x)_+ + x] \\
&= \mathbb{E}_{\mathbb{Q}}[ZX_t^S].
\end{aligned}$$

The first step and last step follow from the properties of functions in CP and the middle step follows from the property of convex order. This shows that X^S is a submartingale with respect to $\mathbb{Q}$.

Because $C \in CP$ we get that $\mathbb{E}_{\mathbb{Q}}[X_t^S]$ is constant across $t \in S$ and because X^S is a submartingale with respect to $\mathbb{Q}$ we get that $\mathbb{E}_{\mathbb{Q}}[X_t^S|\mathcal{F}_s^S] - X_s^S \ge 0$. From this we get

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[X_t^S|\mathcal{F}_s^S]] - \mathbb{E}_{\mathbb{Q}}[X_s^S] = \mathbb{E}_{\mathbb{Q}}[X_t^S] - \mathbb{E}_{\mathbb{Q}}[X_s^S] = 0.$$

Because expected value of a non-negative random variable is zero also the variable equals zero a.s. and so X^S is a martingale with respect to $\mathbb{Q}$.

Now we can extend X_t^S to all $t \in \mathbb{R}_+$ by taking $X_t^S = \mathbb{E}_{\mathbb{Q}}[X_u^S|\mathcal{F}_t]$ for all $u \ge t$ in S. In the next step we want to show that X^S is continuous in probability. From the fact that it is a martingale it follows that it has a.s. left and right limits $X^S_{t^-}, X^S_{t^+}$ for $t \in \mathbb{R}_+$. By assuming that $0 \in S$, taking the difference of left and right limits of $C(t,x) = \mathbb{E}_{\mathbb{Q}}[(X^S_t - x)_+]$ in t and using continuity of C we get:

$$
\begin{aligned}
0 &= \mathbb{E}_Q[(X^S_{t_+} - x)_+ - (X^S_{t_-} - x)_+] \\
&= \mathbb{E}_Q[\mathbb{1}_{\{X^S_{t_+},\ X^S_{t_-} \geq x\}}(X^S_{t_+} - X^S_{t_-}) + \mathbb{1}_{\{X^S_{t_+} \geq x,\ X^S_{t_-} \leq x\}}(X^S_{t_+} - x) \\
&\quad + \mathbb{1}_{\{X^S_{t_+} \leq x,\ X^S_{t_-} \geq x\}}(-X^S_{t_-} + x)] \\
&= \mathbb{E}_Q[\mathbb{1}_{\{X^S_{t_-} > x\}}(X^S_{t_+} - X^S_{t_-}) + \mathbb{1}_{\{X^S_{t_-} > x > X^S_{t_+} \text{ or } X^S_{t_+} > x \geq X^S_{t_-}\}}(|X^S_{t_+} - x|)] \\
&= \mathbb{E}_Q[\mathbb{1}_{\{X^S_{t_-} > x > X^S_{t_+} \text{ or } X^S_{t_+} > x \geq X^S_{t_-}\}}(|X^S_{t_+} - x|)].
\end{aligned}
$$

This implies that $\mathbb{Q}(X^S_{t_-} > x > X^S_{t_+}) = \mathbb{Q}(X^S_{t_+} > x \geq X^S_{t_-}) = 0$ for every x because $|X^S_{t_+} - x|$ is non-negative and so we get $X^S_{t_+} = X^S_{t_-}$. We already know that X^S is a martingale and now we also know that it is right-continuous in probability. This implies that there is a càdlàg version and therefore there is a measure $\mathbb{P}$ on $(D, \mathcal{F})$ satisfying $\mathbb{P}^S = \mathbb{Q}$. We also get that X is a martingale under $\mathbb{P}$. It is furthermore continuous in probability. Because S is dense in $\mathbb{R}_+$, by taking limits of $t \in S$ we get $\mathbb{E}_{\mathbb{P}}[(X_t - x)_+] = C(t,x)$.

The next step is to show that $\mathbb{P}_n \to \mathbb{P}$ in the sense of finite dimensional distributions. Suppose that this does not hold. This would mean that there exists a random variable $Z = f(X_{t_1}, ..., X_{t_m})$ for some finite subset $F = \{t_1, t_2, ..., t_m\}$ of $\mathbb{R}_+$ where $f : \mathbb{R}^m \to \mathbb{R}$ for which $\mathbb{E}_{\mathbb{P}_n}[Z] \not\to \mathbb{E}_{\mathbb{P}}[Z]$. We may assume that there is some $\epsilon > 0$ such that for every n it holds that

$$\mathbb{E}_{\mathbb{P}_n}[Z] \geq \mathbb{E}_{\mathbb{P}}[Z] + \epsilon.$$

We set $S' = S \cup F$. By the previous results, passing to a further subsequence we get that there is a measure $\mathbb{P}'$ on $(D, \mathcal{F})$ such that $\mathbb{P}_n^{S'} \to (\mathbb{P}')^{S'}$. In particular, by restricting to S, $(\mathbb{P}')^S = \lim_{n\to\infty} \mathbb{P}_n^S = \mathbb{P}^S$ and by right-continuity in t, it follows that $\mathbb{P} = \mathbb{P}'$ and $\mathbb{P}_n^{S'} \to \mathbb{P}^{S'}$ which is a contradiction. □

The following lemma will be used to construct ACD martingale measures with specified marginals by taking limits of measures matching the marginals at finitely many times [16].

Lemma 3.6. *(Lemma 3.2., [16]) Let* $(\mathbb{P}_n)_{n\in\mathbb{N}}$ *be a sequence of ACD martingale measures on* $(D,\mathcal{F})$ *and* $C \in CP$ *be such that* $\mathbb{E}_{\mathbb{P}_n}[(X_t - x)_+] \to C(t,x)$*. Then there exists a subsequence* $(\mathbb{P}_{n_k})_{k\in\mathbb{N}}$ *converging in the sense of finite-dimensional distributions to an ACD martingale measure* $\mathbb{P}$ *satisfying* $\mathbb{E}_{\mathbb{P}}[(X_t - x)_+] = C(t,x)$*.*

Before we prove this we first state a few results from [18] that will help prove the above lemma.

Lemma 3.7. *(Lemma 3.2., [18]) Let* $\mathbb{P}$ *be a probability measure on* $(D,\mathcal{F})$ *under which* X *is continuous in probability. Then each of the following statements implies the next.*
1. X *is an almost-continuous diffusion.*
2. For every $s < t \in \mathbb{R}_+$*, non-negative* $\mathcal{F}_s$*-measurable random variables* U, V*, and real numbers* a *and* $b < c \leq d < e$ *we have*

$$\mathbb{E}[U\mathbb{1}_{\{X_s<a,d<X_t<e\}}]\mathbb{E}[V\mathbb{1}_{\{X_s>a,b<X_t<c\}}] \leq \mathbb{E}[U\mathbb{1}_{X_s<a,b<X_t<c}]\mathbb{E}[V\mathbb{1}_{X_s>a,d<X_t<e}]. \tag{4}$$

3. X *is almost-continuous.*

Proof. The proof can be found in [18]. □

Corollary 3.8. *(Corollary 3.12., [18]) Let* $(\mathbb{P}_n)_{n\in\mathbb{N}}$ *be probability measures on* $(D,\mathcal{F})$ *which satisfy property 2 of Lemma 3.7. If* $\mathbb{P}_n \to \mathbb{P}$ *in the sense of finite-dimensional distributions on a dense subset of* $\mathbb{R}_+$*, then* $\mathbb{P}$ *also satisfies this property.*

Proof. The proof can be found in [18]. □

Definition 3.9. *(Definition 4.1., [18]) Let* X *be any real valued and adapted stochastic process. We shall say that it satisfies the Lipschitz property if for all* $s < t \in \mathbb{R}_+$ *and every bounded Lipschitz continuous* $g : \mathbb{R} \to \mathbb{R}$ *with* $|g'| < 1$*, there exists a Lipschitz continuous* $f : \mathbb{R} \to \mathbb{R}$ *with* $|f^{'}| \leq 1$ *and,*

$$f(X_s) = \mathbb{E}[g(X_t)|\mathcal{F}_s]. \tag{5}$$

Corollary 3.9. *(Corollary 4.6., [18]) Let* $(\mathbb{P}_n)_{n\in\mathbb{N}}$ *and* $\mathbb{P}$ *be probability measures on* $(D,\mathcal{F})$ *such that* $\mathbb{P}_n \to \mathbb{P}$ *in the sense of finite-dimensional distributions on a dense subset of* $\mathbb{R}_+$*. If the Lipschitz property for* X *is satisfied under each* $\mathbb{P}_n$*, then it is also satisfied under* $\mathbb{P}$*.*

Proof. The proof can be found in [18]. □

Lemma 3.10. *(Lemma 4.2., [18]) Let X be a càdlàg adapted real valued process that satisfies the Lipschitz property. Then it is strong Markov.*

Proof. The proof can be found in [18]. □

Corollary 3.11. *(Corollary 4.4., [18]) Let X be an almost-continuous diffusion that decomposes as*

$$X_t = M_t + \int_0^t b(s, X_s)ds,$$

where M is a local martingale, $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is locally integrable and such that there exists $K \in \mathbb{R}$ satisfying

$$b(t, y) - b(t, x) \leq K(y - x)$$

for every $t \in \mathbb{R}_+$ and $x < y \in \mathbb{R}$. Then $e^{-Kt}X_t$ satisfies the Lipschitz property.

Proof. The proof can be found in [18]. □

Corollary 3.12. *(Corollary 1.3., [18]) Let $(\mathbb{P}_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $(D, \mathcal{F})$ under which X is an ACD martingale. If $\mathbb{P}_n \to \mathbb{P}$ in the sense of finite dimensional distributions on a dense subset of $\mathbb{R}_+$ and X is continuous in probability under $\mathbb{P}$, then it is an almost-continuous diffusion under $\mathbb{P}$.*

Proof. Sketch of proof as found in [18]. In the first part, we prove that under the above conditions X is almost-continuous under $\mathbb{P}$. X is an almost-continuous diffusion under the measures $\mathbb{P}_n$. It then follows by Lemma 3.7. that the second condition of the lemma is also satisfied. Corollary 3.8. then implies that the measure $\mathbb{P}$ also satisfies this property. Therefore, by using Lemma 3.7. again, we get that X is almost-continuous under $\mathbb{P}$. By Corollary 3.11., $e^{-Kt}X_t$ satisfies the Lipschitz property under $\mathbb{P}_n$ and therefore by corollary 3.9. it satisfies the Lipschitz property also under $\mathbb{P}$. Lemma 3.10. then indicates that $e^{-Kt}X_t$ is strong Markov under $\mathbb{P}$ and so it follows that X is also strong Markov under $\mathbb{P}$. Hence, we proved that X is an almost-continuous diffusion under $\mathbb{P}$. □

Proof. (Lemma 3.6.) This proof is based on the proof found in [16]. First, we get from Lemma 3.5. that there exists a subsequence $\mathbb{P}_{n_k}$ converging in the sense of finite-dimensional distributions to a martingale measure $\mathbb{P}$. Furthermore, we get that X is continuous in probability and it satisfies $\mathbb{E}[(X_t - x)_+] = C(t, x)$. From Corollary 3.12. it follows that X is an almost-continuous diffusion. □

In the last step to prove the existence of the ACD martingale measure we need to show that it is possible to fit the marginals arbitrarily closely [16]. One way to match the marginals at any finite set of times is to use a Skorokhod embedding to time-change a Brownian motion [16]. This procedure is described below and relies on [11]. The Skorokhod embedding problem was first described and solved by Skorokhod, but since then it was studied and extended by many authors. An overview of the history of the solutions to different extensions of the problems can be found in [20].

The Skorokhod embedding theorem concerns the embedding of a given law in Brownian motion by construction of a suitably minimal stopping time [11]. It is showed there that for a martingale $(X_t)_{0\leq t\leq 1}$ with $X_0 \sim \mu_0$ and $X_1 \sim \mu_1$, there is an upper bound with respect to stochastic ordering on the law of $S = \sup_{0\leq t\leq 1} X_t$. This upper bound denoted by $\mu^*_{0,1} \in \mathcal{P}(\mu_0, \mu_1) = \{\nu | \nu \text{ is law of S and } X \in \mathcal{M}(\mu_0, \mu_1)\}$ is attained and furthermore, the martingale the maximum of which attains the upper bound is a (time-change of) Brownian motion.

The following procedure of defining a suitable stopping time is based on [16]. First, we fix $C \in \text{CP}$ and take t_0, t_1 such that $t_0 < t_1$. We consider μ_t to be the measure satisfying $\int (y - x)_+ \, d\mu_t = C(t, x)$ for $t = t_0, t_1$. Next, we define the distribution function $F_1(x) = C_{,2}(t_1, x+) + 1 = \mu_{t_1}((-\infty, x])$. For $u \in (0, 1)$ set $\beta(u) = \inf\{x \in \mathbb{R} : F_1(x) \geq u\}$, and let $g_u : [\beta(u), \infty) \to \mathbb{R}$ be

$$g_u(x) = C(t_1, \beta(u)) + (x - \beta(u))(u - 1).$$

Next, $\alpha(u) \geq \beta(u)$ is chosen to be such that $g_u(\alpha(u)) = C(t_0, \alpha(u))$. $\alpha(u)$ is then uniquely defined in case when $C(t_1, \beta(u)) > C(t_0, \beta(u))$, otherwise set $\alpha(u) = \beta(u)$. Next, we get another distribution function $F^*_{0,1}(x) = \inf\{u \in (0, 1) : \alpha(u) > x\}$. It can be shown that it is right-continuous and increasing from 0 to 1 so it is indeed another distribution function.

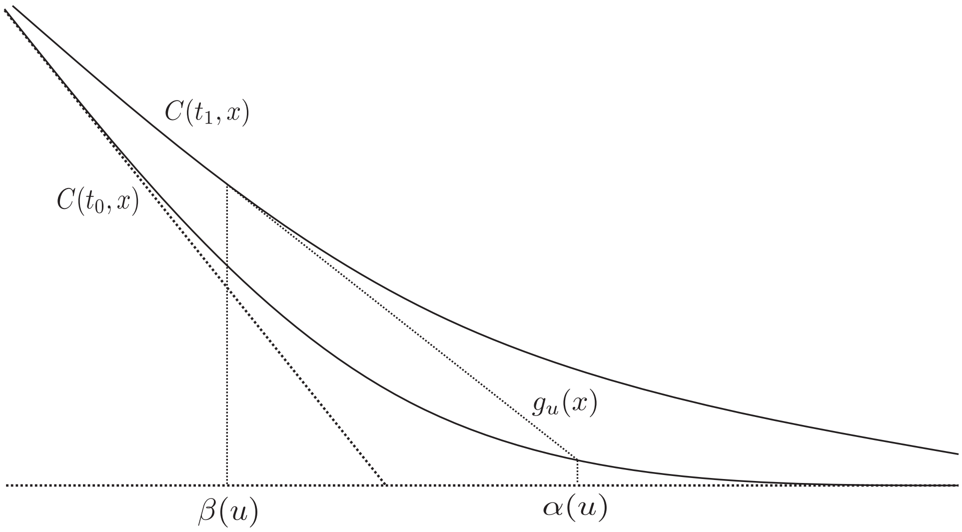

Figure 2: Fitting marginals to C(t,x)

To define a stopping time τ we first consider a Brownian motion B with initial distribution μ_{t_0} and $S_t = \sup_{s \le t} B_s$ is its maximum process. The stopping time τ can then be defined by

$$\tau = \inf\{t \in \mathbb{R}_+ : F_1(B_t) \le F^*_{0,1}(S_t)\}. \tag{6}$$

This means that B^τ is a uniformly integrable martingale and B_τ has the law μ_{t_1}. This result is proved in [11] in Proposition 2.2 and Corollary 2.1.

Lemma 3.13. *(Lemma 3.3., [16]) Let $C \in CP$ and $t_0 < t_1 \in \mathbb{R}_+$. Then, there exists an ACD martingale X such that $\mathbb{E}[(X_t - x)_+] = C(t, x)$ for $t = t_0, t_1$.*

Proof. This proof is again based on the proof in [16]. Assume B as a Brownian motion with initial measure μ_{t_0} and let τ be a stopping time as in (6). We consider a function $\theta : (t_0, t_1) \to \mathbb{R}$ which can be any continuous function increasing from $-\infty$ to ∞. Specifically, we can consider

$$\theta(t) = \begin{cases} -\infty, & t \le t_0 \\ (t_1 - t)^{-1} - (t - t_0)^{-1}, & (t_0, t_1) \\ \infty, & t \ge t_1. \end{cases}$$

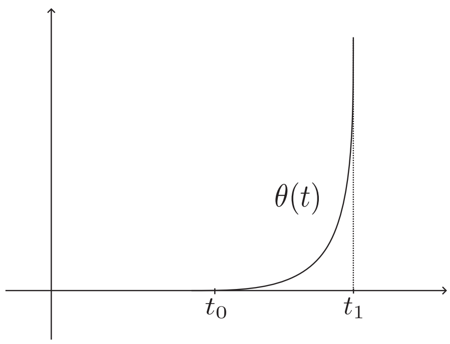

Figure 3: $\theta(t)$ on (t_0, t_1)

Next, we define stopping times

$$\tau_t = \inf\{s \in \mathbb{R}_+ : B_s \geq \theta(t)\}.$$

With this we can construct the ACD martingale by taking $X_t = B_{\tau_t \wedge \tau}$. From Proposition 2.2 in [11] it follows that X_{t_0} has the distribution μ_{t_0} and X_{t_1} has the distribution μ_{t_1}. Since B^τ is uniformly integrable, X will be a martingale. If we consider T to be the first time for which $\tau_T \geq \tau$, then

$$X_t = \begin{cases} \max(B_0, \theta(t)), & t < T \\ B_\tau, & t \geq T. \end{cases}$$

X can only have a single jump at time T, where $X_{T-} \geq X_T$. The last thing that needs to be shown is that X is an almost-continuous diffusion. First we check continuity in probability. It holds that $X_{t-} \geq X_t$ and because X is a martingale we also have $\mathbb{E}[X_{t-}] = \mathbb{E}[X_t]$. This implies that $X_{t-} = X_t$ a.s.

The stopping times $\tau_t \wedge \tau$ are hitting times of the strong Markov process (B_t, S_t), where $S_t = \sup_{s \leq t} B_s$, so the time changed process $(X_t, S_{\tau_t \wedge \tau})$ will be also strong Markov. X must be strong Markov because $S_{\tau_t \wedge \tau} = X_t$ if $X_t \geq \theta(t)$ and X is constant as soon as $X_t < \theta(t)$.

In the last step we show that X is almost-continuous. To this end we choose a càdlàg process Y independent and identically distributed as X. If we assume that $Y_s > X_s$ and $Y_t < X_t$ for $s < t$, then we can set T as the first time at which $Y_T < X_T$. It must hold that $Y_{T-} = X_{T-} = \theta(T)$. In fact, T must be the first time at which $Y_{T-} = X_{T-}$, and therefore it is previsible. From X being a martingale we obtain $\mathbb{E}[X_{T-}] = \mathbb{E}[X_T]$ and $\mathbb{E}[Y_{T-}] = \mathbb{E}[Y_T]$ and, as $X_{T-} \geq X_T$, $Y_{T-} \geq Y_T$, we have $Y_T = X_T$. □

We can extend the above result to match marginals at a finite set of times.

Corollary 3.14. *(Corollary 3.4., [16]) Let $C \in CP$ and $A \subset \mathbb{R}_+$ be finite. Then, there exists an ACD martingale measure $\mathbb{P}$ on $(D, \mathcal{F})$ such that $\mathbb{E}[(X_t - x)_+] = C(t, x)$ for all $t \in A$.*

Proof. This proof is based on the proof in [16]. First we consider a set containing single a time $A = \{t\}$. We can take $\mathbb{P}$ to be the measure under which X_t is independent of t with the required distribution. For a larger set of times $A = \{t_0 < t_1 < ... < t_n\}$ we use induction on n. With this we assume that there is an ACD martingale measure $\mathbb{P}_1$ matching the required marginals at times $t_0, t_1, ..., t_{n-1}$. By Lemma 3.13. there is an ACD martingale measure $\mathbb{P}_2$ matching the marginals at times t_{n-1}, t_n. $X_{t_{n-1}}$ must have the same distribution under both $\mathbb{P}_1$ and $\mathbb{P}_2$. We can therefore join these two measures together at time t_{n-1} to get $\mathbb{P}$, which is the unique measure on $(D, \mathcal{F})$ such that

$$\mathbb{E}_{\mathbb{P}}[AB] = \mathbb{E}_{\mathbb{P}_1}[A\mathbb{E}_{\mathbb{P}_2}[B|X_{t_{n-1}}]] = \mathbb{E}_{\mathbb{P}_2}[\mathbb{E}_{\mathbb{P}_1}[A|X_{t_{n-1}}]B]$$

for all bounded random variables A, B where A is $\mathcal{F}_{t_{n-1}}$-measurable and B is $\sigma(X_t : t \geq t_{n-1})$-measurable. X is an ACD martingale under $\mathbb{P}$ on both intervals, $[0, t_{n-1}]$ and $[t_{n-1}, \infty)$, and so it follows that it is an ACD martingale under $\mathbb{P}$ on $[0, \infty)$. □

Finally, we apply Corollary 3.14. to construct the required ACD martingale measure. With the next result we conclude the proof of the existence part of Theorem 3.3.

Lemma 3.15. *For every $C \in CP$ there is an ACD martingale measure $\mathbb{P}$ on $(D, \mathcal{F})$ satisfying $\mathbb{E}_{\mathbb{P}}[(X_t - x)_+] = C(t, x)$.*

Proof. This proof is based on the proof in [16]. It follows by Corollary 3.14., that for every n there is an ACD martingale measure $\mathbb{P}_n$ satisfying $\mathbb{E}_{\mathbb{P}_n}[(X_{k/n} - x)_+] = C(k/n, x)$ for $k = 0, 1, ..., n$. This implies that $\mathbb{E}_{\mathbb{P}_n}[(X_t - x)_+] \to C(t, x)$, so the existence of the ACD martingale measure $\mathbb{P}$ follows from Lemma 3.6. □

3.3 Continuity of the map to martingale measures

Using the fact that the ACD martingale measure is unique we prove the following theorem.

Theorem 3.16. *(Theorem 1.4., [16]) For every $C \in$CP denote the unique ACD martingale measure given by Theorem 3.3. by $\mathbb{P}_C$. Then the function*

$$CP \to \mathcal{M}(D),\ C \mapsto \mathbb{P}_C$$

is continuous, under pointwise convergence on CP and convergence in the sense of finite-dimensional distributions on $\mathcal{M}(D)$.

This theorem says that given $C_n \to C$ pointwise, $\mathbb{E}_{\mathbb{P}_{C_n}}[Z] \to \mathbb{E}_{\mathbb{P}_{\mathbb{C}}}[Z]$ for every $Z = f(X_{t_1}, ..., X_{t_m})$ with $t_1, ..., t_m \in \mathbb{R}_+$ and $f : \mathbb{R}^m \to \mathbb{R}$ continuous, bounded.

Proof. This proof is based on the proof in [16]. To prove continuity we show that for $C_n, C \in$ CP, if $C_n \to C$, then $\mathbb{P}_{C_n} \to \mathbb{P}_C$. This will be proved by contradiction. Therefore assume that $\mathbb{P}_{C_n} \nrightarrow \mathbb{P}_C$. This means that there exists $\epsilon > 0$ and a variable Z of the form $Z = f(X_{t_1}, ..., X_{t_m})$ for $t_1, ..., t_m \in \mathbb{R}_+$ and $f : \mathbb{R}^m \to \mathbb{R}$ continuous and bounded, such that $\mathbb{E}_{\mathbb{P}_{C_n}}[Z] \geq \mathbb{E}_{\mathbb{P}_C}[Z] + \epsilon$ infinitely often. We may assume that this holds for every n by passing to a further subsequence if necessary. We already showed in Lemma 3.6 that by passing to a further subsequence there exists an ACD martingale measure $\mathbb{P}$ such that $\mathbb{P}_{C_n} \to \mathbb{P}$ and such that $\mathbb{E}_{\mathbb{P}}[(X_t - x)_+] = C(t, x)$. By uniqueness we get that $\mathbb{P} = \mathbb{P}_C$ and that is a contradiction. □

4 The class of almost-continuous diffusions is not arbitrary

In the previous chapter we showed that there exists an ACD martingale measure fitting the marginal distributions. It can be furthermore shown that the measure fitting the marginal distributions is in fact unique. This is proved in Chapter 4 of [16]. In this section we are going to prove the following result.

Theorem 4.1. *(Theorem 1.5., [16]) Suppose that we have a continuous map from a dense subset S of CP to the martingale measures*

$$S \to \mathcal{M}(D),\ C \mapsto \mathbb{Q}_C$$

such that for every $C \in S$ the equality $\mathbb{E}_{\mathbb{Q}_C}[(X_t - x)_+] = C(t, x)$ is satisfied. Then $\mathbb{Q}_C = \mathbb{P}_C$.

4.1 Extremal marginals

This result suggests that the initial choice of the class of almost-continuous diffusions was not random. The proof uses the fact that there are certain marginal distributions for which there is only one possible martingale measure [16]. These correspond to extremal elements of CP and they form a dense subset of CP.

First, by checking that every convex combination of elements of CP still belongs to CP, we note that the space CP is a convex subset of the space of all real-valued functions on $\mathbb{R}_+ \times \mathbb{R}$ [16]. For a convex subset of a vector space we can define extremal points as points which cannot be expressed as a convex combination of other elements in the set.

Definition 4.1. *(Chapter 5, [16]) For a convex subset S of a vector space V, an element $x \in S$ is said to be extremal if given any $y, z \in S$ and any $\lambda \in (0,1)$ such that $x = \lambda y + (1-\lambda)z$, then $y = z = x$.*

In our setup, we call an element of CP extremal if it is extremal among the convex set of elements of CP with the same initial value i.e. value at $t = 0$.

Definition 4.2. *(Definition 5.1., [16]) An element $C \in CP$ is extremal if given any $C_1, C_2 \in CP$ and $\lambda \in (0,1)$ such that $C_1(0,x) = C_2(0,x) = C(0,x)$ $\forall x \in \mathbb{R}$ and $C = \lambda C_1 + (1-\lambda)C_2$, then $C_1 = C_2 = C$.*

In the next step we are going to show that for this specific type of elements in CP, there exists a unique martingale measure $\mathbb{P}$.

Lemma 4.2. *(Lemma 5.2., [16]) Let $C \in CP$ be extremal. Then, there exists a unique martingale measure $\mathbb{P}$ on $(D, \mathcal{F})$ satisfying $\mathbb{E}_{\mathbb{P}}[(X_t - x)_+] = C(t,x)$.*

Proof. This proof is provides a more detailed explanation of the proof in [16]. In the previous chapter we already proved the existence part of this measure in Theorem 3.3. Now we only need to prove the uniqueness part. First, we are going to show that any measure $\mathbb{P}$ satisfying the conditions of the lemma is Markov. In particular, we show that $\mathbb{E}_{\mathbb{P}}[(X_t - x)_+ | \mathcal{F}_T] = \mathbb{E}_{\mathbb{P}}[(X_t - x)_+ | X_T]$ for any $T \geq 0$.

Taking $T \geq 0$ and $\mathcal{F}_T$-measurable random variable Z with $0 \leq Z \leq 1$, we define C_1, C_2 to be such that $C_1(t,x) = C_2(t,x) = C(t,x)$ for $t < T$ and

$$C_1(t,x) = \mathbb{E}_{\mathbb{P}}[Z\mathbb{E}_{\mathbb{P}}[(X_t - x)_+ | X_T]] + \mathbb{E}_{\mathbb{P}}[(1-Z)(X_t - x)_+],$$
$$C_2(t,x) = \mathbb{E}_{\mathbb{P}}[(1-Z)\mathbb{E}_{\mathbb{P}}[(X_t - x)_+ | X_T]] + \mathbb{E}_{\mathbb{P}}[Z(X_t - x)_+]$$

for $t \geq T$. It can be checked that $C_1, C_2 \in \mathrm{CP}$, $C_1(0,x) = C_2(0,x) = C(0,x)$ and $(C_1 + C_2)/2 = C$. Because C is extremal we get that $C_1 = C = C_2$. For $t \geq T$, substituting $C_1(t,x) = \mathbb{E}_{\mathbb{P}}[(X_t - x)_+]$ into the above equation for C_1 we get

$$\mathbb{E}_{\mathbb{P}}[(X_t - x)_+] = \mathbb{E}_{\mathbb{P}}[Z\mathbb{E}_{\mathbb{P}}[(X_t - x)_+|X_T]] + \mathbb{E}_{\mathbb{P}}[(1-Z)(X_t - x)_+].$$

This can be simplified to

$$\mathbb{E}_{\mathbb{P}}[Z(X_t - x)_+] = \mathbb{E}_{\mathbb{P}}[Z\mathbb{E}_{\mathbb{P}}[(X_t - x)_+|X_T]].$$

From this we get that $\mathbb{E}_{\mathbb{P}}[(X_t - x)_+|\mathcal{F}_T] = \mathbb{E}_{\mathbb{P}}[(X_t - x)_+|X_T]$ and so it follows that X is Markov.

In the next step we are going to show there is only one such measure. Assume $\mathbb{P}, \mathbb{Q}$ are two martingale measures satisfying the conditions of the lemma. Again choose $T \geq 0$ and X_T-measurable random variable Z with $0 \leq Z \leq 1$, define C_1, C_2 by $C_1(t,x) = C_2(t,x) = C(t,x)$ for $t < T$ and

$$\begin{aligned} C_1(t,x) &= \mathbb{E}_{\mathbb{P}}[Z(X_t - x)_+] + \mathbb{E}_{\mathbb{Q}}[(1-Z)(X_t - x)_+], \\ C_2(t,x) &= \mathbb{E}_{\mathbb{Q}}[Z(X_t - x)_+] + \mathbb{E}_{\mathbb{P}}[(1-Z)(X_t - x)_+] \end{aligned}$$

for $t \geq T$. It can be again checked that $C_1, C_2 \in \mathrm{CP}$, $C_1(0,x) = C_2(0,x) = C(0,x)$ and $(C_1 + C_2)/2 = C$. And because C is extremal we get that $C_1 = C = C_2$. For $t \geq T$, substituting $C_1(t,x) = \mathbb{E}_{\mathbb{Q}}[(X_t - x)_+]$ into the above equation for C_1 we get

$$\mathbb{E}_{\mathbb{Q}}[(X_t - x)_+] = \mathbb{E}_{\mathbb{P}}[Z(X_t - x)_+] + \mathbb{E}_{\mathbb{Q}}[(1-Z)(X_t - x)_+].$$

This can be simplified to

$$\mathbb{E}_{\mathbb{P}}[Z(X_t - x)_+] = \mathbb{E}_{\mathbb{Q}}[Z(X_t - x)_+].$$

for all $t \geq T$. By the above we get that $\mathbb{P}$ and $\mathbb{Q}$ are both Markov measures for X with the same pairwise and initial distributions and this implies $\mathbb{P} = \mathbb{Q}$. □

Because there exists a unique martingale measure for the extremal points, we know this measure is in fact an ACD martingale measure. The next step is to show that the extremal elements are dense in CP. To show this, we will construct for a $C \in \mathrm{CP}$ an extremal element that matches C at any increasing sequence of times [16]. First, we show that there is an extremal element matching C at two times $t = t_0, t_1$. The construction method used will be the same as in Chapter 3 in Lemma 3.12. This was based on the Skorokhod embedding described in [11]. The following procedure can be found in [16].

For $C \in$ CP let μ_t be the corresponding marginal distributions with $\int (y-x)_+ \, d\mu_t = C(t,x)$. Fixing $t_0 < t_1$, define the distribution function $F_1(x) = C_{,2}(t_1, x+) + 1 = \mu_{t_1}((-\infty, x])$ and for $u \in (0,1)$ set
$\beta(u) = \inf\{x \in \mathbb{R} : F_1(x) \geq u\}$. For $x \geq \beta(u)$ set

$$g_u(x) = C(t_1, \beta(u)) + (x - \beta(u))(u-1)$$

and define $\alpha(u) \geq \beta(u)$ by $g_u(\alpha(u)) = C(t_0, \alpha(u))$. $\alpha(u)$ is uniquely defined whenever $C(t_1, \beta(u)) > C(t_0, \beta(u))$, otherwise set $\beta(u) = \alpha(u)$.

We define $\tilde{C} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by setting $\tilde{C}(t,x)$ equal to $C(t_0, x)$ for $t \leq t_0$, $C(t_1, x)$ for $t \geq t_1$ and

$$\tilde{C}(t,x) = \begin{cases} C(t_1, x), & x \leq \beta(u) \\ C(t_0, x), & x \geq \alpha(u) \\ g_u(x), & \beta(u) < x < \alpha(u) \end{cases} \tag{7}$$

for $t_0 < t < t_1$, where $u(t) \equiv (t - t_0)/(t_1 - t_0)$. Next we show that $\tilde{C}$ defines an extremal element of CP matching C at t_0 and t_1.

Lemma 4.3. *(Lemma 5.3., [16]) Suppose that $C \in CP$ and $t_0 < t_1 \in \mathbb{R}_+$. Then $\tilde{C}$ defined by (7) is an extremal element of CP such that $\tilde{C}(t,x)$ equals $C(t_0, x)$ for $t \leq t_0$ and $C(t_1, x)$ for $t \geq t_1$.*

Proof. This proof follows the steps of this proof in [16]. It is not hard to see that for $t_0 < t < t_1$, $\tilde{C}(t,x)$ is a function convex in x between $C(t_0, x)$ and $C(t_1, x)$. To show that $\tilde{C} \in$ CP it is enough to show that it is continuous and increasing in t. We will focus on the interval (t_0, t_1), as this is where it gets interesting.

First, we show that $\tilde{C}$ is increasing in t. For this, choose any $s < t \in (t_0, t_1)$. Then, $u(s) \leq u(t)$ and so $\beta(u(s)) \leq \beta(u(t))$ follows from its definition. For the case when $x \leq \beta(u(t))$ we get by combining the results of the case $x \leq \beta(u(s))$ and $\beta(u(s)) \leq x \leq \beta(u(t))$ the following

$$\tilde{C}(t,x) = C(t_1, x) \geq \tilde{C}(s,x).$$

For the case where $x > \beta(u(t))$, we have

$$\tilde{C}(t,x) = \max(g_{u(t)}(x), C(t_0,x)) \geq \max(g_{u(s)}(x), C(t_0,x)) = \tilde{C}(s,x).$$

By combining all the results we see that $\tilde{C}(t,x)$ is is increasing in t.

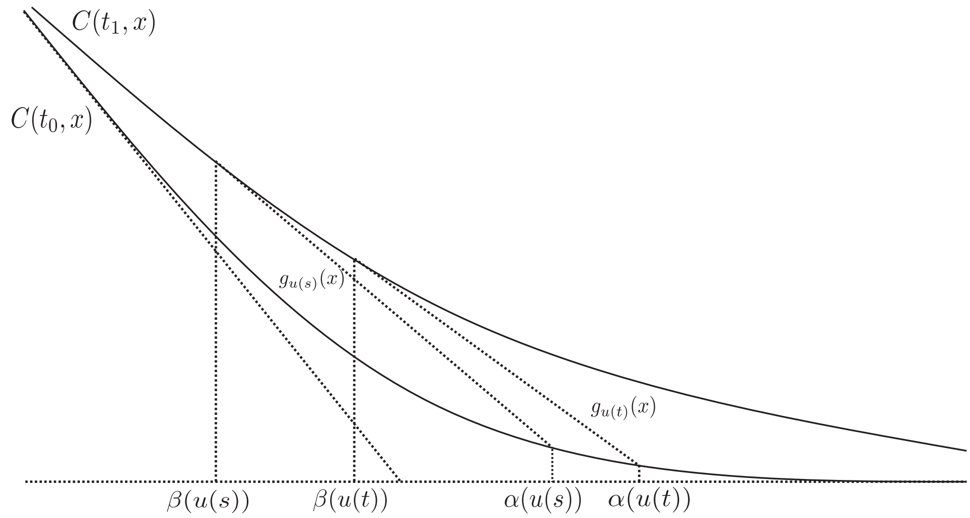

Figure 4: $\tilde{C}(s,x)$ and $\tilde{C}(t,x)$

Next, we show that $\tilde{C}$ is continuous in t by showing that $t \mapsto \tilde{C}(t,x)$ maps the interval $[t_0,t_1]$ onto $[C(t_0,x), C(t_1,x)]$ and therefore must be continuous. This is done by constructing $\tilde{C}$ such that $\tilde{C}(t,x) = y$ for given $x \in \mathbb{R}$ and y such that $C(t_0,x) < y < C(t_1,x)$.

Choose any $x, y \in \mathbb{R}$ with $C(t_0,x) < y < C(t_1,x)$, and choose $b < x$ to minimize $u = (C(t_1,b) - y)/(x-b)$. To show that this exists we note that choosing b small enough so that $C(t_1,b) - C(t_0,b) < y - C(t_0,x)$ gives

$$(C(t_1,b) - y)/(x-b) < (C(t_0,b) - C(t_0,x))/(x-b) \leq 1$$

where the first step follows from the inequality $C(t_1,b) - C(t_0,b) < y - C(t_0,x)$. The limit as $b \to -\infty$ is 1 and so it must have a minimum by continuity.

By choosing t for which $u(t) = 1-(C(t_1,x)-y)/(b-x)$ gives $C_{,2}(t_1,b+) \geq u-1 \geq C_{,2}(t_1,b-)$. By this it follows that $\tilde{C}(t,x) = C(t_1,b) + (x -$

$b)(1-u) = y$. Hence, the map $t \mapsto \tilde{C}(t,x)$ maps the interval $[t_0, t_1]$ onto $[C(t_0,x), C(t_1,x)]$ and it must be continuous.

Next, we need to show that $\tilde{C}$ is extremal. Suppose that $\tilde{C} = \lambda C_1 + (1-\lambda)C_2$ for $C_1, C_2 \in \mathrm{CP}$, $\lambda \in (0,1)$, and

$$C_1(0,x) = C_2(0,x) = \tilde{C}(0,x) = C(t_0,x).$$

To show that $C_1 = C_2 = \tilde{C}$, we use the fact that if a non-trivial convex combination of two increasing functions is constant, then those functions must also be constant. In particular, $\tilde{C}(t,x) = \tilde{C}(t_1,x) = C(t_1,x)$ for all $t \geq t_1$ so we must also have $C_i(t,x) = C_i(t_1,x)$ for $i = 1,2$ and $t \geq t_1$ and $\tilde{C}(t,x) = \tilde{C}(t_0,x) = C(t_0,x)$ for $t \leq t_0$ so $C_i(t,x) = C_i(0,x) = C(t_0,x)$ for $i = 1,2$ and $t \leq t_0$.

It remains to check the interval (t_0, t_1). We choose $t \in (t_0, t_1)$, set $\alpha = \alpha(u(t))$ and $\beta = \beta(u(t))$ and look separately at cases $x \leq \beta$, $x \geq \alpha$ and $x \in (\beta, \alpha)$.

If $x \leq \beta$, $\tilde{C}(t,x) = \tilde{C}(t_1,x)$ by definition of $\tilde{C}$ and so $C_i(t,x) = C_i(t_1,x)$ for $i = 1,2$. Alternatively, if $x \geq \alpha$ then $\tilde{C}(t,x) = \tilde{C}(t_0,x)$. Therefore, $C_i(t,x) = C_i(t_0,x) = C(t_0,x)$.

For $x \in (\beta, \alpha)$

$$\lambda (C_1)_{,2}(t, x+) + (1-\lambda)(C_2)_{,2}(t, x+) = \tilde{C}_{,2}(t, x+) = u(t) - 1.$$

Because the functions $(C_i)_{,2}(t, x+)$ are increasing functions of x, this means that they are constant for $x \in (\beta, \alpha)$. With this we get that $C_i(t,x)$ are linear functions of x over (β, α). So far, we have shown that

$$C_i(t,x) = \begin{cases} C_i(t_1,x), & x \leq \beta \\ C(t_0,x), & x \geq \alpha \\ ((\alpha - x)C_i(t_1,\beta) + (x-\beta)C(t_0,\alpha))/(\alpha-\beta), & \beta < x < \alpha. \end{cases}$$

Suppose that there exists an $x \in \mathbb{R}$ for which $C_1(t_1,x) < C(t_1,x)$. Choose t such that $u(t) = C_{,2}(t_1, x+) + 1$ and set $\beta = \beta(u(t)),\ \alpha = \alpha(u(t))$. From this it follows that $\beta \leq x < \alpha$, so by using the fact that $C_1(t,x)$ and $C(t,x)$ are convex in x and increasing in t we get

$$\begin{aligned} (C_1)_{,2}(t_1, x+) &\geq (C(t_0,\alpha) - C_1(t_1,\beta))/(\alpha-\beta) \\ &\geq (C(t_0,\alpha) - C(t_1,\beta))/(\alpha-\beta) \\ &\geq C_{,2}(t_1, x-). \end{aligned}$$

Taking the right hand limits in x we obtain $(C_1)_{,2}(t_1, x+) \geq C_{,2}(t_1, x+)$.

For $C_1(t_1, x) > C(t_1, x)$, the above argument and $C_2(t_1, x) < C(t_1, x)$ give $(C_2)_{,2}(t_1, x+) \geq C_{,2}(t_1, x+)$ and this implies $(C_1)_{,2}(t_1, x+) \leq C_{,2}(t_1, x+)$. The function $g(x) = (C(t_1, x) - C_1(t_1, x))^2$ has a non-positive derivative everywhere and must be therefore decreasing. The limit $g(x) \to 0$ as $x \to \pm\infty$ implies that g is identically 0. So, $C_1(t_1, x) = C(t_1, x)$. It follows that $C_1 = C$. □

By the previous lemma, we may construct an extremal element of CP matching C at an increasing sequence of times [16].

Corollary 4.4. *(Corollary 5.4., [16]) Suppose that $C \in CP$ and $t_0 < t_1 < ... \uparrow \infty$ are in $\mathbb{R}_+$. There is an extremal $\tilde{C} \in CP$ such that $\tilde{C}(t_k, x) = C(t_k, x)$ for each k.*

Proof. This proof is based on the proof in [16]. First, let us assume that wlog $t_0 = 0$. By the previous lemma we know that there exists an extremal $\tilde{C}_k \in$ CP such that $\tilde{C}_k(t, x)$ equals $C(t_{k-1}, x)$ for $t \leq t_{k-1}$ and equals $C(t_k, x)$ for $t \geq t_k$ (k = 1, 2, ...).

Now, define $\tilde{C}$ by $\tilde{C}(t, x) = \tilde{C}_k(t, x)$ for $t_{k-1} \leq t < t_k$. Clearly, $\tilde{C} \in$ CP. It only needs to be shown that it is extremal. To this end, suppose that

$$\tilde{C} = \lambda C_1 + (1 - \lambda)C_2 \tag{8}$$

for $\lambda \in (0, 1)$, $C_1, C_2 \in$ CP, and $C_1(0, x) = C_2(0, x) = C(0, x)$.

Using induction on k to show that $C_1(t, x) = C_2(t, x) = C(t, x)$ for $t \leq t_k$. For $k = 0$ it is a requirement. Assume $k \geq 1$ and that this holds for $t \leq t_{k-1}$. If we take the function $\theta(t) \equiv (t \wedge t_k) \vee t_{k-1}$, which takes values in $[t_{k-1}, t_k]$, then $\tilde{C}(\theta(t), x) = \tilde{C}_k(t, x)$ is extremal and by the equality in (8) we get that $\tilde{C}_k(t, x) = C_1(t, x) = C_2(t, x)$ for $t_{k-1} \leq t \leq t_k$. It follows by induction that $\tilde{C} = C_1 = C_2$. □

Finally, we complete the proof of Theorem 4.1. We show that $C \mapsto \mathbb{P}_C$ given by Theorem 4.1 is the unique continuous map to the martingale measures matching all possible sets of marginals [16].

4.2 Concluding the results

Proof. (Theorem 4.1.) This proof is based on the proof in [16]. Choose any $C \in S$ and let Z be a random variable of the form

$$Z = f(X_{t_1}, ..., X_{t_m})$$

for $t_1, ...t_m \in \mathbb{R}_+$ and $f : \mathbb{R}^m \to \mathbb{R}$ be continuous and bounded. We need to show that $\mathbb{E}_{\mathbb{Q}_C}[Z] = \mathbb{E}_{\mathbb{P}_C}[Z]$.

From Corollary 4.4. we obtain a sequence $(C_n)_{n\in\mathbb{N}}$ of extremal elements of CP such that $C_n(k/n, x) = C(k/n, x)$ for all $k \in \mathbb{N}$ such that $k \leq n$. This gives $C_n \to C$.

Because S is dense in CP, there is a sequence $(C_{n,m})_{m\in\mathbb{N}}$ such that $d(C_{n,m}, C_n) \to 0$ as $m \to \infty$, where d is the metric on CP given by

$$d(C_1, C_2) = \sup\{|C_1(t, x) - C_2(t, x)| \wedge 2^{-|x|-t} : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}.$$

Now, fix $n \in \mathbb{N}$. By $\mathbb{E}_{\mathbb{Q}_{C_{n,m}}}[(X_t - x)_+] \to C_n(t, x)$, Lemma 3.5. shows that by potentially passing to a subsequence, there exists a martingale measure $\mathbb{P}$ satisfying the equation

$$C_n(t, x) = \mathbb{E}_{\mathbb{P}}[(X_t - x)_+],$$

and such that $\mathbb{Q}_{C_{n,m}} \to \mathbb{P}$ in the sense of finite-dimensional distributions as $m \to \infty$. However, C_n is extremal and therefore by Lemma 4.2., $\mathbb{P} = \mathbb{P}_{C_n}$. For every n we can choose an $m_n \in \mathbb{N}$ such that

$$d(C_{n,m_n}, C_n) < 2^{-n}, \ \left|\mathbb{E}_{\mathbb{Q}_{C_{n,m_n}}}[Z] - \mathbb{E}_{\mathbb{P}_{C_n}}[Z]\right| < 2^{-n}.$$

In particular, $d(C, C_{n,m_n}) \leq d(C_n, C_{n,m_n}) + d(C, C_n) \leq d(C, C_n) + 2^{-n}$ so the continuity of $C \mapsto \mathbb{Q}_C$ gives that $\mathbb{Q}_{C_{n,m_n}}$ tends to $\mathbb{Q}_C$ in the sense of finite-dimensional distributions as $n \to \infty$. Similarly, $\mathbb{P}_{C_n}$ tends to $\mathbb{P}_{\mathbb{C}}$. This gives

$$|\mathbb{E}_{\mathbb{P}_C}[Z] - \mathbb{E}_{\mathbb{Q}_C}[Z]| = \lim_{n\to\infty} \left|\mathbb{E}_{\mathbb{P}_{C_n}}[Z] - \mathbb{E}_{\mathbb{Q}_{C_{n,m_n}}}[Z]\right| \leq \lim_{n\to\infty} 2^{-n} = 0.$$

□

5 Application in financial mathematics

The already mentioned result of Lowther (Theorem 3.3.) states that given a sequence of marginal distributions satisfying certain assumptions, there exists a a unique ACD martingale which is consistent with the marginals. Furthermore, in [18] it is stated under which assumption the martingale is continuous. The result is presented in the lemma below. The proof can be found in [18] at the end of Chapter 3.

Lemma 5.1. *(Lemma 1.4., [18]) Let X be an almost-continuous process. If the support of X_t is connected for every t in $\mathbb{R}_+$ outside of a countable set then X is continuous.*

This result of Lowther can be translated into finance by taking the marginal distributions to be the risk-neutral probability densities φ_t of the stock price S_t that are obtained from the prices of European call options through the equality $C(t,x) = \mathbb{E}_{\mu_t}[(S_t - x)_+]$. Finding the full risk-neutral diffusion process of the stock price then leads to being able to price American and exotic i.e. path-dependent options [6].

This chapter relies mostly on [12] (5.1), Gatheral's book [8] (5.2), Dupire's work in [6] (5.2, 5.3), [19] (5.3) (contains extensive descriptions of the models defined below) and [5] (5.4). Before going further, let us briefly define some necessary notions. The definitions below can be found in e.g. [7].

We understand a European contingent claim to be a non-negative random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$. It is an asset, the future payoff of which is contingent (dependent) on the price behaviour of the underlying securities, which is an uncertain event. Furthermore, we call it a derivative of the underlying asset X if it is measurable with respect to $\sigma(X) = \sigma(X_1, ..., X_T)$. European call options are of the form $C^{call} = (X_T - K)_+$ and European put options are of the form $C^{put} = (K - X_T)_+$. They are modelled easily as the right to exercise the option only comes at a fixed time of maturity T, as opposed to e.g. American options for which the holder has the right to exercise the option at any time before or at time of maturity.

5.1 Black-Scholes model

In the Black-Scholes model, the stock price is assumed to follow dynamics given by the stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t B_t,$$

where μ is a drift parameter, $\sigma > 0$ is a constant volatility parameter and B_t is the standard Brownian motion. It can be shown that under the risk-neutral measure, $\mu = r - q$, where r is the constant interest rate and q is the continuous dividends function [12]. The analytical solution to this SDE is given by

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma dB_t\right).$$

Assuming that the market is arbitrage free and complete i.e. there exists an equivalent martingale measure and every contingent claim is attainable (can be hedged with the traded asset and a risk-free asset), we can determine the price of a contingent claim as it's discounted expected values under the unique equivalent martingale measure [19]. In particular, the price of a call and put option is then given by

$$\begin{aligned} C(S, K, T-t) &= D(t, T)\mathbb{E}_{\mathbb{Q}}[(S_T - K)_+] \\ P(S, K, T-t) &= D(t, T)\mathbb{E}_{\mathbb{Q}}[(K - S_T)_+], \end{aligned}$$

where $\mathbb{Q}$ is the risk-neutral measure and

$$D(t, T) = \exp\left(-\int_t^T r(k)dk\right)$$

is the discount factor and $r = r(t)$ is a deterministic interest rate [12].

It is assumed that all trading strategies are self-financing and admissible i.e. the value of the replicating portfolio is bounded below by zero [19]. Here, the martingale representation theorem is used to construct the replicating strategy. Furthermore, the discounted value of a contingent claim is given by the initial cost of setting up the replicating strategy and the gains from trading [19].

Knowing S_T we obtained the option prices. Reversing the process and first observing the market price $p_{T,K}$ of a chosen option $C_{T,K}$ where T is the maturity and K is the strike price, we can determine the unique $\sigma = \sigma(T, K)$ such that the Black-Scholes formula reproduces the given market price $p_{T,K}$. The resulting σ is the so called *implied volatility* and the collection of all such implied volatilities is called *volatility surface* [12, 19]. Applying this process on real option prices we may notice that the volatility is increasing with respect to T and convex with respect to K which contradicts the assumption of the Black-Scholes model that the volatility is constant. As stated in [12], by plotting the implied volatility with respect to the varying parameter K we get a graph shape know as the *volatility smile* if it slopes upward on both ends or *volatility skew* if it slopes upward only on the left side. A volatility smile is more typical for stock options and volatility skew is more usual for index options [12].

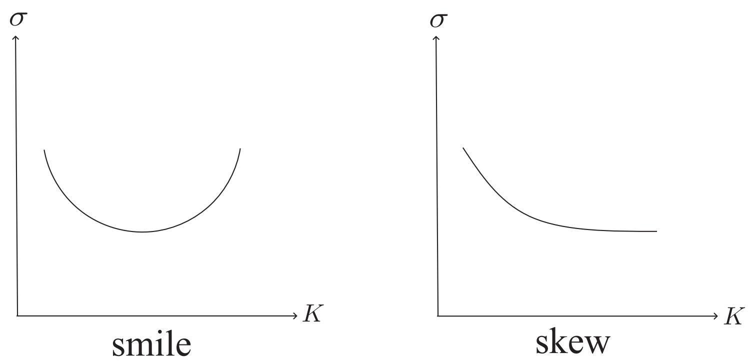

Figure 5: volatility smile and volatility skew

5.2 Stochastic volatility models

The above leads to the question if one can find a model that would overcome this gap between the assumptions of the Black-Scholes model and the real market behavior i.e. take into account the non-constant nature of volatility. From a practical point of view, the motivation for finding a better model is that using the Black-Scholes model requires continuously changing the volatility assumption in order to match the market prices [8]. The class of *stochastic volatility models* deals with the issue of constant volatility in the Black-Scholes model [8].

Here it is assumed that not only the asset price is random but also the volatility. It furthermore assumes that the dynamics is separate for the stock price and the volatility and it is given by the SDEs

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sqrt{\sigma_t} S_t dB_t^S \\ d\sigma_t &= \alpha(t, \sigma_t, S_t)dt + \eta\beta(t, \sigma_t, S_t)\sqrt{\sigma_t} dB_t^\sigma \end{aligned}$$

with

$$\langle dB_t^S, dB_t^\sigma \rangle = \rho dt,$$

where μ_t is instantaneous drift of stock price returns, η is the volatility of volatility, ρ is the correlation between random stock price returns and changes in σ_t and B_t is again the standard Brownian motion [8].

5.3 Local volatility model

As stated in [6] adding a non-traded source of risk such as the already mentioned stochastic volatility or e.g. jumps or transaction costs leads to losing completeness of the market i.e. losing the ability to hedge options with the underlying asset of the model. In the stochastic volatility model we added an extra source of randomness to the volatility. Therefore, without volatility being a traded asset, the model becomes incomplete [19].

Because completeness is important for arbitrage pricing and hedging, Dupire shows how to build a model that maintains completeness and also is compatible with the observed smiles at all maturities. He shows that it is possible to find a risk-neutral process the dynamics of which follows the SDE

$$dS_t = \mu_t S_t dt + \sigma(S_t, t) S_t B_t. \tag{9}$$

Here, the volatility σ is a deterministic function of time t and the price of the underlying asset S_t. The model which he introduced is called the *local volatility model*. The results in [6] show how to hedge and price any American or path-dependent options by observing European option prices.

Knowing the prices of all path-dependent options is equivalent to knowing the full (risk-neutral) diffusion process of the stock price [6]. Knowing all European option prices is equivalent to knowing only the probability densities of the stock price at different times, conditional on its current value [6]. The full diffusion contains much more information than the conditional laws, as distinct diffusions may generate identical conditional laws [6]. Dupire however shows that restricting ourselves to risk-neutral diffusions, we can retrieve from the conditional laws the unique risk-neutral diffusion which generates the prices.

The process of obtaining exotic option prices from European option prices is described below and is based on [8]. In the following we assume the stock price diffuses with risk-neutral drift $\mu(t) = r(t) - D(t)$ and local volatility $\sigma(S_t, t)$ as in (9). Knowing the European option prices we can determine the risk-neutral probability density of the stock price being equal to K at time T from the equation

$$C(S_0, K, T) = \int_0^\infty (S_T - K)_+ \varphi_T(S_T, T; S_0) dS_T \tag{10}$$

where $\varphi_T(S_T, T; S_0)$ is the pseudo-probability density of the final stock price at time T, by differentiating twice with respect to K and thus obtaining

$$\varphi_T(K,T;S_0) = \frac{\partial^2 C}{\partial K^2}(K,T;S_0).$$

In this context, the function $C(t,x) = \int (y-x)_+ \, d\mu_t(y)$ represents the option prices for different strikes and maturities. The marginal distributions μ_t are the marginal probabilities of the underlying stock price under the risk-neutral measure.

Differentiating (10) with respect to T gives

$$\begin{aligned}
\frac{\partial C}{\partial T} &= \int_0^\infty dS_T \left\{ \frac{\partial}{\partial T}\varphi(S_T,T;S_0) \right\} (S_T - K)_+ \\
&= \int_0^\infty dS_T \left\{ \frac{1}{2}\frac{\partial^2}{\partial S_T^2}(\sigma^2 S_T^2 \varphi) - \frac{\partial}{\partial S_T}(\mu S_T \varphi) \right\} (S_T - K)_+,
\end{aligned}$$

as $\varphi(S_T,T;S_0)$ evolves according to the Fokker-Planck equation

$$\frac{1}{2}\frac{\partial^2}{\partial S_T^2}(\sigma^2 S_T^2 \varphi) - S\frac{\partial}{\partial S_T}(\mu S_T \varphi) = \frac{\partial \varphi}{\partial T}.$$

Integrating by parts twice gives

$$\begin{aligned}
\frac{\partial C}{\partial T} &= \frac{\sigma^2 K^2}{2}\varphi + \int_K^\infty dS_T \mu S_T \varphi \\
&= \frac{\sigma^2 K^2}{2}\frac{\partial^2 C}{\partial K^2} + \mu(T)\left(-K\frac{\partial C}{\partial K} \right).
\end{aligned}$$

The above is the Dupire equation in case the underlying stock has risk-neutral drift μ. The forward price of the stock at time T is given by

$$F_T = S_0 \exp\left\{ \int_0^T dt \mu_t \right\}.$$

Expressing the option price as a function of the forward price, we would get

$$\frac{\partial C}{\partial T} = \frac{\sigma^2 K^2}{2}\frac{\partial C^2}{\partial K^2},$$

where C now represents $C(F_T,K,T)$. This leads to

$$\sigma^2(K,T,S_0) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}.$$

This derivation is borrowed from [8]. Knowing the European option prices we can compute the right-hand side and so given all European option prices for all strikes and maturities we obtain unique local volatilities. Thus we found a formula for the diffusion parameter (local volatility) of the unique risk neutral diffusion process which generates these prices i.e. the diffusion parameter of (9).

Knowing all these marginal densities therefore leads to finding a unique diffusion if we restrict ourselves to risk-neutral diffusions. This determines the prices of all path-dependent options. Knowledge of the whole process allows for the pricing of path-dependent options by e.g. Monte-Carlo methods and American options by e.g. dynamic programming [6].

5.4 Lowther's contribution

As described in the previous subsections, knowing the option prices $C(t, x)$ for all maturities $0 \leq t \leq T$ and strikes x is equivalent to knowing the marginal distributions $(\mu_t)_{0 \leq t \leq T}$ for the stock price process. The goal is to find a martingale measure under which the stock price process fits the given marginal distributions. Furthermore, we also restrict ourselves to processes that are continuous and Markov and assume that $C(t, x)$ is increasing in t, convex in x and sufficiently smooth. We have seen in the previous subsection that Dupire introduced the diffusion

$$dS_t = \mu_t S_t dt + \sigma(S_t, t) S_t B_t$$

for $0 \leq t \leq T$, where

$$\sigma^2(K, T, S_0) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}.$$

This serves as means for showing existence of the desired process under the given conditions. In the following we are concerned about uniqueness of such a process. We expect the unique solution to be equal to Dupire's solution. We know that Dupire stated in [6] that uniqueness is obtained under certain technical assumptions, this was however not proved.

Dupire's process is a continuous martingale satisfying the Markov property. Lowther showed in [16] and [17] that Dupire's solution is unique within the class of processes that are strong Markov martingales and admit continuous paths. His result in [17] is the following.

Theorem 5.2. *(Theorem 1.2., [17]) Let X and Y be $\mathbb{R}$-valued continuous, strong Markov martingales. If for every $t \in \mathbb{R}_+$, X_t and Y_t have the same one-dimensional marginal distributions, they also have the same joint distribution.*

In [5] it is discussed if it is not sufficient to consider only Markov processes instead of strong Markov processes in Lowther's results. The mentioned question is if there exist two processes, one a continuous strong Markov martingale and second a continuous Markov martingale not satisfying the strong Markov property with the same absolutely continuous (or even more regular) marginals and the answer turns out to be yes. Furthermore Theorem 4.3 of [5] provides a sufficient set of regularity assumptions to reason Dupire's statement in [6] that under some regularity assumptions there is a unique diffusion process.

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