



# A dual pair for the contact group

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## Abstract

Generalizing the canonical symplectization of contact manifolds, we construct an infinite dimensional non-linear Stiefel manifold of weighted embeddings into a contact manifold. This space carries a symplectic structure such that the contact group and the group of reparametrizations act in a Hamiltonian fashion with equivariant moment maps, respectively, giving rise to a dual pair, called the EPContact dual pair. Via symplectic reduction, this dual pair provides a conceptual identification of non-linear Grassmannians of weighted submanifolds with certain coadjoint orbits of the contact group. Moreover, the EPContact dual pair gives rise to singular solutions for the geodesic equation on the group of contact diffeomorphisms. For the projectivized cotangent bundle, the EPContact dual pair is closely related to the EPDiff dual pair due to Holm and Marsden.

**Keywords** Contact manifold · Contact diffeomorphism group · Coadjoint orbit · Dual pair · Homogeneous space · Symplectic manifold · Symplectization · Manifold of mappings · Infinite dimensional manifold · Non-linear Grassmannian · Non-linear Stiefel manifold

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## 1 Introduction

Every contact manifold gives rise to a symplectic manifold in a canonical way. If the contact structure is described by a 1-form  $\alpha$  on  $P$ , then this symplectic manifold can be described as  $P \times (\mathbb{R} \setminus 0)$  with the symplectic form  $d(t\alpha)$ , where  $t$  denotes the projection onto the second factor. Regarding the contact structure as a subbundle of hyperplanes,  $\xi \subseteq TP$ , and denoting the corresponding line bundle over  $P$  by  $L := TP/\xi$ , this symplectization can be described more naturally as  $M = L^* \setminus P$ , with the symplectic form induced from the canonical symplectic form on  $T^*P$  via the natural vector bundle inclusion  $L^* \subseteq T^*P$ .

The group of contact diffeomorphisms,  $\text{Diff}(P, \xi)$ , acts on  $M$  in a natural way, preserving the symplectic structure. This action is in fact Hamiltonian and admits an equivariant moment

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map. This moment map identifies (unions of) connected components of the symplectization  $M$  with certain coadjoint orbits of the contact group.

### 1.1 The EPContact dual pair

In this paper we will introduce a natural infinite dimensional generalization  $\mathcal{M}$  of the symplectization  $M = L^*\backslash P$  with similar features. To this end we fix a closed manifold  $S$ , we denote by  $|\Lambda|_S$  its line bundle of densities, and we consider the space  $\mathcal{M}$  of line bundle homomorphisms from  $|\Lambda|_S^* \rightarrow S$  to  $L^* \rightarrow P$  which restrict to a linear isomorphism on each fiber. Every volume density on  $S$  provides an identification  $\mathcal{M} \cong C^\infty(S, M)$  and permits to regard elements  $\Phi \in \mathcal{M}$  as pairs consisting of a map  $\varphi: S \rightarrow P$  together with a contact form for  $\xi$  along this map. This space  $\mathcal{M}$  can be equipped with the structure of a Fréchet manifold in a natural way, and admits a canonical (weakly non-degenerate) symplectic form. The symplectization  $M$  can be recovered by choosing  $S$  to be a single point.

The contact group acts on  $\mathcal{M}$  in a natural way, preserving the symplectic structure. This action is Hamiltonian and admits an equivariant moment map, see Proposition 2.4. Furthermore, the group of reparametrizations,  $\text{Diff}(S)$ , acts on  $\mathcal{M}$  in a Hamiltonian fashion, also admitting an equivariant moment map. On the non-linear Stiefel manifold of weighted embeddings,  $\mathcal{E} \subseteq \mathcal{M}$ , the latter action is free. We show that the restrictions of these moment maps to  $\mathcal{E}$ ,

$$\mathfrak{X}(P, \xi)^* \xleftarrow{J_L^\mathcal{E}} \mathcal{E} \xrightarrow{J_R^\mathcal{E}} \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^*, \tag{1}$$

constitute a symplectic dual pair in the sense of Weinstein [34], see Theorem 2.6. Here  $\mathfrak{X}(P, \xi)$  denotes the Lie algebra of contact vector fields on  $P$ ,  $\mathfrak{X}(S)$  denotes the Lie algebra of all vector fields on  $S$ , and  $\Omega^1(S, |\Lambda|_S)$  denotes the space of smooth 1-form densities on  $S$ . The moment maps are given by  $\langle J_L^\mathcal{E}(\Phi), X \rangle = \int_S \Phi(X \circ \varphi)$  for all  $X \in \mathfrak{X}(P, \xi)$ , and  $\langle J_R^\mathcal{E}(\Phi), Z \rangle = \int_S \Phi(T\varphi \circ Z)$  for all  $Z \in \mathfrak{X}(S)$ .

Actually, we will show a stronger statement: The group  $\text{Diff}(S)$  acts freely and transitively on the fibers of  $J_L^\mathcal{E}$ , and the group  $\text{Diff}(P, \xi)$  acts locally transitive on the level sets of  $J_R^\mathcal{E}$ , see Proposition 4.2 and Theorem 3.5. Moreover, we will see that the level sets of both moment maps are smooth submanifolds of  $\mathcal{E}$ . The dual pair in (1) will be referred to as the *EPContact dual pair*, because the left leg provides singular solutions of the EPContact equation, i.e., the Euler–Poincaré equation associated with the group of contact diffeomorphisms.

Recall that the projectivized cotangent bundle of a manifold  $Q$  admits a canonical contact structure. The EPContact dual pair corresponding to the projectivized cotangent bundle of  $Q$  is closely related to the EPDiff dual pair, due to Holm–Marsden [18], associated to the action of  $\text{Diff}(Q)$  and  $\text{Diff}(S)$  on  $T^*\text{Emb}(S, Q)$ , the cotangent bundle of embeddings of  $S$  into  $Q$ , see Sect. 5.1.

### 1.2 Coadjoint orbits of the contact group

The EPContact dual pair will be used to identify coadjoint orbits of the contact group via symplectic reduction for the reparametrization action, following the general principle: Symplectic reduction on one leg of a dual pair of moment maps leads to coadjoint orbits of the other group. The same principle was used in [12], where symplectic reduction on the right leg of the ideal fluid dual pair due to Marsden and Weinstein [26] led to coadjoint orbits of the

Hamiltonian group consisting of symplectic submanifolds [16], resp. of weighted isotropic submanifolds of the symplectic manifold [22, 35].

To make this more precise, consider the non-linear Grassmannian of weighted submanifolds,  $\mathcal{G} = \mathcal{E}/\text{Diff}(S)$ , consisting of pairs  $(N, \gamma)$  where  $N$  is a submanifold of type  $S$  in  $P$  and  $\gamma: |\Lambda|_N^* \rightarrow L|_N^*$  is an isomorphism of line bundles which may be regarded as being akin to a trivialization of the contact structure along  $N$ . This space  $\mathcal{G}$  is a Fréchet manifold in a natural way and the projection  $\mathcal{E} \rightarrow \mathcal{G}$  is a smooth principal bundle with structure group  $\text{Diff}(S)$ . The moment map  $J_L^\mathcal{E}$  descends to a  $\text{Diff}(P, \xi)$ -equivariant injective immersion  $\mathcal{G} \rightarrow \mathfrak{X}(P, \xi)^*$ , which permits to identify orbits of the contact group in  $\mathcal{G}$  with coadjoint orbits. Each 1-form density  $\rho \in \Omega^1(S, |\Lambda|_S)$  gives rise to a reduced space  $\mathcal{G}^\rho \subseteq \mathcal{G}$  given by

$$\mathcal{G}^\rho = (J_R^\mathcal{E})^{-1}(\mathcal{O}_\rho)/\text{Diff}(S) = (J_R^\mathcal{E})^{-1}(\rho)/\text{Diff}(S, \rho),$$

where  $\mathcal{O}_\rho$  denotes the  $\text{Diff}(S)$ -orbit through  $\rho$ , and  $\text{Diff}(S, \rho)$  is the isotropy group of  $\rho$ .

Reduction works best for the zero level. The corresponding reduced space  $\mathcal{G}^0$  coincides with the subset of weighted isotropic submanifolds,  $\mathcal{G}^{\text{iso}} \subseteq \mathcal{G}$ . We will see that  $\mathcal{G}^{\text{iso}}$  is a smooth submanifold of  $\mathcal{G}$  and that the action of the contact group on  $\mathcal{G}^{\text{iso}}$  admits local smooth sections. In particular, this action is locally transitive. Hence, the restriction of the moment map,  $\mathcal{G}^{\text{iso}} \rightarrow \mathfrak{X}(P, \xi)^*$ , identifies (unions of) connected components of  $\mathcal{G}^{\text{iso}}$  with coadjoint orbits of the contact group. Moreover, this identification intertwines the Kostant–Kirillov–Souriau symplectic form with the reduced symplectic form on  $\mathcal{G}^{\text{iso}}$ . These facts are summarized in Theorem 4.10.

The situation is more delicate with regard to reduction at more general levels. In this case the reduced spaces are more singular subsets of  $\mathcal{G}$  and it is unclear if the contact group acts locally transitive on them. If  $\rho$  is a contact 1-form density on  $S$ , i.e., if  $\ker \rho$  is a contact structure on  $S$ , then the reduced space  $\mathcal{G}^\rho$  consists of certain weighted contact submanifolds of  $P$  which are of type  $(S, \ker \rho)$ . This is an open condition on the submanifold in view of Gray’s stability theorem. The condition on the weight, however, is rather singular: The space of all admissible (for  $\mathcal{G}^\rho$ ) weights on a fixed contact submanifold may be identified with the  $\text{Diff}(S, \ker \rho)$ -orbit of  $\rho$ . The situation is tamer if we specialize to 1-dimensional  $S$ , see Example 4.16. In particular, (unions of) connected components in the spaces of weighted transverse knots of fixed length in a contact 3-manifold, may be identified with coadjoint orbits of the contact group.

### 1.3 Singular solutions of the Euler–Poincaré equation

Another motivation for studying the EPContact dual pair is the construction of singular solutions of the geodesic equation on the group of contact diffeomorphisms equipped with a right invariant Riemannian metric. This works analogous to the EPDiff equation, where the EPDiff dual pair has been used by Holm and Marsden [18] to construct singular solutions for the geodesic equation on the full diffeomorphism group. Similarly, point vortices in two dimensional ideal fluid flow, a geodesic equation on the group of volume preserving diffeomorphisms, have been described using a dual pair by Marsden–Weinstein [26]. The same kind of argument has been applied for the Vlasov equation in kinetic theory by Holm–Tronci [19] using the ideal fluid dual pair, and for the Euler–Poincaré equations on the group of automorphisms of a principal bundle in [13] using the EPAut dual pair [10].

In all these cases the singular solutions of the system are obtained, via a moment map, from a collective Hamiltonian dynamics on a symplectic manifold, referred to as Clebsch variables. This moment map turns out to be the left leg of a dual pair associated to commuting actions

on the manifold of embeddings, while the right leg moment map gives conserved quantities by Noether’s theorem. We show that for the group of contact diffeomorphisms the situation is similar.

To describe this in more detail, let us start by briefly reviewing the geodesic equation on a Lie group with respect to a right invariant Riemannian metric. We write the inner product on the Lie algebra  $\mathfrak{g}$  in the form  $(u, v) = \langle Qu, v \rangle$ , where the inertia operator  $Q: \mathfrak{g} \rightarrow \mathfrak{g}^*$  is symmetric and strictly positive. Formally, the right trivialized geodesic equation on the Lie algebra  $\mathfrak{g}$  is the Euler–Arnold equation,

$$\frac{d}{dt}u = -\text{ad}_u^\top u, \tag{2}$$

where the adjoint of the adjoint action can be characterized by  $(\text{ad}_u^\top v, w) := (v, \text{ad}_u w)$  for all  $u, v, w \in \mathfrak{g}$ . In other words,  $\text{ad}_u^\top = Q^{-1} \text{ad}_u^* Q$ , where  $\text{ad}_u^*: \mathfrak{g}^* \rightarrow \mathfrak{g}^*$  denotes the coadjoint action characterized by  $\langle \text{ad}_u^* m, v \rangle = \langle m, \text{ad}_u v \rangle$  for  $u, v \in \mathfrak{g}$  and  $m \in \mathfrak{g}^*$ .

Via Legendre transformation, using the momentum  $m := Qu$ , the Euler–Arnold equation (2) becomes the Lie–Poisson equation,

$$\frac{d}{dt}m = -\text{ad}_u^* m, \tag{3}$$

which is the Hamilton equation on the Poisson manifold  $\mathfrak{g}^*$  for the Hamiltonian

$$h: \mathfrak{g}^* \rightarrow \mathbb{R}, \quad h(m) := \frac{1}{2} \langle m, Q^{-1}m \rangle.$$

Its solutions are confined to coadjoint orbits, the symplectic leaves of  $\mathfrak{g}^*$ .

Let us now turn to the group of contact diffeomorphisms on a contact manifold  $(P, \xi)$ . Recall that its Lie algebra can be canonically identified with the space of contact vector fields,  $\mathfrak{X}(P, \xi) = \Gamma^\infty(L)$ , where  $L = TP/\xi$ . For simplicity, we will assume  $P$  to be closed. We consider  $\mathfrak{X}(P, \xi)^* = \Gamma^{-\infty}(L^* \otimes |\Lambda|_P)$ , the space of distributional sections of  $L^* \otimes |\Lambda|_P$ , where  $|\Lambda|_P$  denotes the bundle of densities on  $P$ . We assume that the inertia operator,  $Q: \Gamma^\infty(L) \rightarrow \Gamma^\infty(L^* \otimes |\Lambda|_P)$ , is a pseudo-differential operator of real order  $s$  which is symmetric, strictly positive, invertible, and its inverse,  $Q^{-1}: \Gamma^\infty(L^* \otimes |\Lambda|_P) \rightarrow \Gamma^\infty(L)$ , is a pseudo-differential operator of order  $-s$ . Hence, the corresponding inner product,  $(u, v) = \langle Qu, v \rangle$ , generates the Sobolev  $H^{s/2}$  topology on  $\Gamma(L)$ . Using elliptic theory, such inertia operators can be easily constructed. For instance, we may use  $Q = \phi(1 + \Delta)^{s/2}$ , where  $\Delta$  is a Laplacian acting on  $\Gamma(L)$  which is non-negative and formally self-adjoint with respect to a volume density on  $P$  and a fiberwise Euclidean metric on  $L$ , and  $\phi: L \rightarrow L^* \otimes |\Lambda|_P$  denotes the isomorphism of line bundles provided by these geometric choices.

The Hamiltonian function  $h(m) = \frac{1}{2} \langle m, Q^{-1}m \rangle$  is well defined on  $\Gamma^{-s/2}(L^* \otimes |\Lambda|_P)$ , the space of sections which are of Sobolev class  $-s/2$ . Note that the Sobolev space  $\Gamma^{-s/2}(L^* \otimes |\Lambda|_P)$  is invariant under the coadjoint action of  $\text{Diff}(P, \xi)$ . If  $k \in \Gamma^{-\infty}(L \boxtimes L)$  denotes the Schwartz kernel of  $Q^{-1}$ , then

$$h(m) = \frac{1}{2} \langle k, m \boxtimes m \rangle = \frac{1}{2} \int_{(x,y) \in P \times P} m(x)k(x, y)m(y)$$

extends continuously (regularization) to  $m \in \Gamma^{-s/2}(L^* \otimes |\Lambda|_P)$ . Assuming

$$s > \dim P - \dim S, \tag{4}$$

the moment map  $J_L^\mathcal{E}: \mathcal{E} \rightarrow \mathfrak{X}(P, \xi)^*$  takes values in  $\Gamma^{-s/2}(L^* \otimes |\Lambda|_P) = \Gamma^{s/2}(L)^*$ . Indeed, for  $\Phi \in \mathcal{E}$  the distribution  $J_L^\mathcal{E}(\Phi)$  is the push forward of a smooth section on  $S$  along a smooth embedding  $S \rightarrow P$ , cf. Remark 2.9. According to a standard property of the trace map on Sobolev spaces, see for instance [31, Proposition 1.6 in Chapter 4], it thus provides

a continuous functional on  $\Gamma^{s/2}(L)$ . The map  $J_L^\mathcal{E}$  is actually smooth into  $\Gamma^{-s/2}(L^* \otimes |\Lambda|_P)$ . Hence, the pull back of the Hamiltonian  $h$  to  $\mathcal{E}$ ,

$$H: \mathcal{E} \rightarrow \mathbb{R}, \quad H := h \circ J_L^\mathcal{E},$$

is smooth. Although the symplectic form on  $\mathcal{E}$  is only weakly non-degenerate, the function  $H$  gives rise to a Hamiltonian vector field  $X_H$  on (and tangential to)  $\mathcal{E}$ , cf. the discussion in [5, Section 4.2.2]. Indeed, since  $J_L^\mathcal{E}$  is a moment map, we formally have  $X_H(\Phi) = \zeta_{Q^{-1}J_L^\mathcal{E}(\Phi)}^\mathcal{E}(\Phi)$  and thus

$$X_H(\Phi) = \zeta_{Q^{-1}J_L^\mathcal{E}(\Phi)}^{L^*} \circ \Phi, \tag{5}$$

where  $\zeta^\mathcal{E}$  and  $\zeta^{L^*}$  denote the infinitesimal  $\text{Diff}(P, \xi)$ -actions on  $\mathcal{E}$  and  $L^*$ , respectively, cf. (25) and (20) below. By microlocal regularity,  $Q^{-1}J_L^\mathcal{E}(\Phi)$  is smooth along the submanifold  $N$  in  $P$  determined by  $\Phi$ , see for instance [32, Corollary 9.4 in Chapter 7] or [15, Proposition 3.11 in Chapter IV§3]. Furthermore, since  $\zeta^{L^*}: \Gamma^\infty(L) \rightarrow \Gamma^\infty(TL^*)$  is essentially given by a first order differential operator, it extends to distributional sections, and  $\zeta_{Q^{-1}J_L^\mathcal{E}(\Phi)}^{L^*}$  is smooth along  $L^*|_N$ . In particular, the latter is smooth along  $\Phi$  and thus  $X_H(\Phi)$  is a tangent vector to  $\mathcal{E}$  at  $\Phi$ , cf. (5).

Every solution  $\Phi_t \in \mathcal{E}$  of the Hamilton equation

$$\frac{d}{dt}\Phi_t = X_H(\Phi_t) \tag{6}$$

provides a singular solution (peakons, filaments, sheets)  $u_t := Q^{-1}J_L^\mathcal{E}(\Phi_t) \in \Gamma^{s/2}(L)$  of the Euler–Arnold equation (2) with momentum  $m_t := J_L^\mathcal{E}(\Phi_t) \in \Gamma^{-s/2}(L^* \otimes |\Lambda|_P)$ . The support of the distributional momentum  $m_t$  coincides with the smooth submanifold determined by  $\Phi_t$ , and this also coincides with the singular support of  $u_t$ . Due to the dual pair property, each solution  $\Phi_t$  of (6) remains in a level of the other moment map,  $J_R^\mathcal{E}: \mathcal{E} \rightarrow \mathcal{X}(S)^*$ , and is thus confined to a  $\text{Diff}(P, \xi)$  orbit in  $\mathcal{E}$ . Hence, its momentum  $m_t = J_L^\mathcal{E}(\Phi_t)$  is constrained to a coadjoint orbit.

If  $S$  is a single point, then the assumption in (4) implies that the distributional kernel  $k$  of  $Q^{-1}$  is continuous. In this case we have  $\mathcal{E} = L^* \setminus P$  and  $H$  is given by the (smooth) restriction of  $k$  to the diagonal.

The initial value problem for the EPContact equation has been studied by Ebin and Preston in [5]. They consider inertia operators of the form  $Q = 1 + \Delta$ , where the Laplacian is with respect to a Riemannian metric which is adapted to the contact structure.

It appears to be interesting [4] to replace the class of inertia operators considered above with operators in the Heisenberg calculus [3, 28, 30], a calculus of pseudo-differential operators which is closely linked to the contact geometry on  $P$ . Using the Rockland theorem, one can construct pseudo-differential operators  $Q: \Gamma^\infty(L) \rightarrow \Gamma^\infty(L^* \otimes |\Lambda|_P)$  of Heisenberg order  $s$  which are symmetric, strictly positive, invertible, and such that the inverse,  $Q^{-1}: \Gamma^\infty(L^* \otimes |\Lambda|_P) \rightarrow \Gamma^\infty(L)$ , is of Heisenberg order  $-s$ . For instance, we may use  $Q = \phi(1 + \Delta)^{s/2}$ , where  $\Delta$  is a subLaplacian. Everything mentioned above remains valid, provided the Sobolev spaces are being replaced with the corresponding spaces in the Heisenberg Sobolev scale and the assumption (4) is replaced by the stronger condition  $s/2 > \dim P - \dim S$ .

### 1.4 Structure of the paper

The remaining part of the paper is organized as follows. In Sect. 2 we construct the EPContact dual pair. In Sect. 3 we show that the level sets of the right moment map are submanifolds on

which the contact group acts locally transitive. In Sect. 4 we study the reduced spaces obtained by factoring out the group of reparametrizations. In Sect. 5.1 we compare the EPContact dual pair for the projectivized cotangent bundle with the EPDiff dual pair of Holm and Marsden. In Sect. 5.2 we provide a comparison with a dual pair due to Marsden and Weinstein for the Euler equation of an ideal fluid.

## 2 Weighted non-linear Stiefel manifolds

The aim of this section is to construct the EPContact dual pair, see Theorem 2.6.

### 2.1 Canonical symplectization of contact manifolds

In this section we set up our notation and recall some well known facts about the symplectization of contact manifolds. We emphasize the structure that will be generalized in the subsequent sections. For more details we refer to [1, Appendix 4.E] and [25, Section 12.3].

Consider a contact manifold  $(P, \xi)$  where  $\xi \subseteq TP$  denotes the contact subbundle. We write  $L := TP/\xi$  for the corresponding line bundle. The vector bundle projection of the dual line bundle will be denoted by  $\pi^{L^*} : L^* \rightarrow P$ . The canonical projection  $TP \rightarrow L$  permits to regard the dual bundle as a subbundle of the cotangent bundle,  $L^* \subseteq T^*P$ . We denote by  $\theta^{L^*} \in \Omega^1(L^*)$  the pull back of the canonical 1-form on  $T^*P$ .<sup>1</sup> Hence, the defining equation for  $\theta^{L^*}$  is

$$\theta^{L^*}_\beta(V) = \beta(T_\beta \pi^{L^*} \cdot V), \tag{7}$$

where  $\beta \in L^*_x, x \in P$ , and  $V \in T_\beta L^*$ . The pairing in (7) can be viewed either as a pairing between  $L^*_x$  and  $L_x$  by considering the class of  $T_\beta \pi^{L^*} \cdot V$  in  $L_x = T_x P/\xi_x$ , or as a pairing between  $T^*P$  and  $TP$  by considering  $\beta$  an element of  $L^*_x \subseteq T^*_x P$ . It is well known that the closed 2-form

$$\omega^{L^*} := d\theta^{L^*} \in \Omega^2(L^*)$$

restricts to a symplectic form on  $M := L^* \setminus P$ , which will be denoted by  $\omega^M = d\theta^M$ . The symplectic manifold  $(M, \omega^M)$  is called the symplectization of the contact manifold  $(P, \xi)$ . Note that both forms are homogeneous of degree one with respect to the fiberwise scalar multiplication  $\delta_t : L^* \rightarrow L^*$ , that is  $\delta_t^* \theta^{L^*} = t\theta^{L^*}$  and  $\delta_t^* \omega^{L^*} = t\omega^{L^*}$  for all  $t \in \mathbb{R}$ .

#### The action by the contact group

Let us write  $\text{Diff}(P, \xi)$  for the group of contact diffeomorphisms. Since contact diffeomorphisms preserve the contact subbundle  $\xi$ , the  $\text{Diff}(P, \xi)$ -action on  $P$  lifts to an action on the total space of  $L^*$ . For  $g \in \text{Diff}(P, \xi)$ , we let  $\Psi_g^{L^*} \in \text{Diff}(L^*)$  denote the corresponding (fiberwise linear) diffeomorphism on  $L^*$ . Clearly,  $\pi^{L^*} \circ \Psi_g^{L^*} = g \circ \pi^{L^*}, \delta_t \circ \Psi_g^{L^*} = \Psi_g^{L^*} \circ \delta_t$ , and  $\Psi_{g_2 g_1}^{L^*} = \Psi_{g_2}^{L^*} \Psi_{g_1}^{L^*}$  for all  $g, g_1, g_2 \in \text{Diff}(P, \xi)$  and  $t \in \mathbb{R}$ . Moreover, the contact group action preserves  $\theta^{L^*}$  and  $\omega^{L^*}$ , that is  $(\Psi_g^{L^*})^* \theta^{L^*} = \theta^{L^*}$  and  $(\Psi_g^{L^*})^* \omega^{L^*} = \omega^{L^*}$  for all  $g \in \text{Diff}(P, \xi)$ . Noticing that the symplectic piece  $M \subseteq L^*$  is invariant under the contact group action, we write  $\Psi_g^M$  for the restricted action.

<sup>1</sup> If  $\xi = \ker \alpha$  and  $L^* \cong P \times \mathbb{R}$  denotes the trivialization provided by  $\alpha$ , then  $\theta^{L^*} = t(\pi^{L^*})^* \alpha$ , where  $t$  denotes the projection onto the factor  $\mathbb{R}$ .

Let  $\mathfrak{X}(P, \xi)$  denote the Lie algebra of contact vector fields. Via the projection  $TP \rightarrow L$ , every (contact) vector field gives rise to a section of  $L$  which may in turn be regarded as a fiberwise linear function on the total space of  $L^*$ . This provides canonical identifications,

$$\mathfrak{X}(P, \xi) = \Gamma^\infty(L) = C^\infty_{\text{lin}}(L^*), \quad X \leftrightarrow X \bmod \xi \leftrightarrow h_X, \tag{8}$$

where  $h_X \in C^\infty_{\text{lin}}(L^*)$  is the fiberwise linear function given by  $h_X(\beta) = \beta(X_x)$  for  $\beta \in L^*_x$  and  $x \in P$ . Clearly, this identification is equivariant, i.e.,

$$(\Psi_g^{L^*})^* h_X = h_{g^*X} \tag{9}$$

for all  $g \in \text{Diff}(P, \xi)$  and  $X \in \mathfrak{X}(P, \xi)$ .

For  $X \in \mathfrak{X}(P, \xi)$ , we denote the corresponding fundamental vector field (infinitesimal action) on the total space of  $L^*$  by  $\zeta_X^{L^*} \in \mathfrak{X}(L^*)$ . Clearly,

$$T\pi^{L^*} \circ \zeta_X^{L^*} = X \circ \pi^{L^*}, \tag{10}$$

$T\delta_t \circ \zeta_X^{L^*} = \zeta_X^{L^*} \circ \delta_t$ ,  $(\Psi_g^{L^*})^* \zeta_X^{L^*} = \zeta_{g^*X}^{L^*}$ , and  $[\zeta_{X_1}^{L^*}, \zeta_{X_2}^{L^*}] = \zeta_{[X_1, X_2]}^{L^*}$  for all  $X, X_1, X_2 \in \mathfrak{X}(P, \xi)$ ,  $g \in \text{Diff}(P, \xi)$  and  $t \in \mathbb{R}$ . From the definition of  $\theta^{L^*}$  in (7) one immediately gets

$$i_{\zeta_X^{L^*}} \theta^{L^*} = h_X \tag{11}$$

for  $X \in \mathfrak{X}(P, \xi)$ . Invariance of  $\theta^{L^*}$  and  $\omega^{L^*}$  yields infinitesimal invariance  $L_{\zeta_X^{L^*}} \theta^{L^*} = 0$  and  $L_{\zeta_X^{L^*}} \omega^{L^*} = 0$ , respectively, for all  $X \in \mathfrak{X}(P, \xi)$ . Using Cartan’s formula and (11), we obtain

$$i_{\zeta_X^{L^*}} \omega^{L^*} = -dh_X \tag{12}$$

as well as the following formula for the bracket of contact vector fields,

$$h_{[X, Y]} = \zeta_X^{L^*} \cdot h_Y - \zeta_Y^{L^*} \cdot h_X = \omega^{L^*}(\zeta_X^{L^*}, \zeta_Y^{L^*}), \tag{13}$$

for all  $X, Y \in \mathfrak{X}(P, \xi)$ .

Over the symplectic piece  $M = L^* \setminus P$  the Hamiltonian vector field corresponding to  $h_X^M := h_X|_M$  coincides with  $\zeta_X^M := \zeta_X^{L^*}|_M$ , see (12). Moreover, (13) implies

$$h_{[X, Y]}^M = \{h_X^M, h_Y^M\}, \tag{14}$$

where the right hand side denotes the Poisson bracket on  $C^\infty(M)$ . The formulas (12) and (9) above imply that the action of  $\text{Diff}(P, \xi)$  on  $M$  is Hamiltonian with equivariant moment map

$$J^M : M \rightarrow \mathfrak{X}(P, \xi)^*, \quad \langle J^M(\beta), X \rangle := h_X^M(\beta) = \beta(X), \tag{15}$$

where  $\beta \in M$  and  $X \in \mathfrak{X}(P, \xi)$ .

**Remark 2.1** A slightly more explicit, yet less natural description is possible if the contact structure is described by a contact form  $\alpha \in \Omega^1(P)$ , that is, if  $\xi = \ker \alpha$ . Such a contact form provides a trivialization  $P \times \mathbb{R} \cong L^* \subseteq T^*P$ ,  $(x, t) \leftrightarrow t\alpha_x$ . Via this identification we have  $\theta^{L^*} = t(\pi^{L^*})^* \alpha$ , and the fiberwise linear function  $h_X$  from (8) becomes  $h_X(x, t) = t(i_X \alpha)(x)$  where  $x \in P$  and  $t \in \mathbb{R}$ . A diffeomorphism  $g$  of  $P$  is a contact diffeomorphism iff it preserves the contact form up to a conformal factor, i.e., iff there exists a (nowhere vanishing) function  $a_g$  on  $P$  such that  $g^* \alpha = a_g \alpha$ . Similarly, a vector field  $X$  on  $P$  is a contact vector field iff it satisfies  $L_X \alpha = \lambda_X \alpha$ , for a conformal factor  $\lambda_X \in C^\infty(P)$ . Both, the group action of  $\text{Diff}(P, \xi)$  and the Lie algebra action of  $\mathfrak{X}(P, \xi)$  on  $L^*$ , written

in the trivialization  $L^* \cong P \times \mathbb{R}$ , involve the conformal factors. More explicitly, we have  $\Psi_g^{L^*}(x, t) = (g(x), ta_g(x))$  and  $\zeta_X^{L^*}(x, t) = (X(x), t\lambda_X(x)\partial_t)$ .

**Coadjoint orbits**

It is well known that each connected component of a symplectic manifold is equivariantly symplectomorphic to a coadjoint orbit of its Hamiltonian group, see for instance [12]. We will now formulate a similar statement for the group  $\text{Diff}_c(P, \xi)$  of compactly supported contact diffeomorphisms which can be considered as a special case of Theorem 4.10 below.

For  $\beta \in M$ , the isotropy subgroup  $\text{Diff}_c(P, \xi; \beta)$  is a closed Lie subgroup of  $\text{Diff}_c(P, \xi)$ . Moreover, the map provided by the action,  $\text{Diff}_c(P, \xi) \rightarrow M, g \mapsto \Psi_g^M(\beta)$ , admits a local smooth right inverse defined in a neighborhood of  $\beta$ . In particular, the group  $\text{Diff}_c(P, \xi)$  acts locally and infinitesimally transitively on  $M$ , and the  $\text{Diff}_c(P, \xi)$ -orbit through  $\beta$  is open and closed in  $M$ . Denoting this orbit by  $M_\beta$ , the map  $\text{Diff}_c(P, \xi) \rightarrow M_\beta$  is a smooth principal bundle with structure group  $\text{Diff}_c(P, \xi; \beta)$ . Hence,

$$M_\beta = \text{Diff}_c(P, \xi) / \text{Diff}_c(P, \xi; \beta)$$

may be regarded as a homogeneous space. The moment map (15) induces an equivariant diffeomorphism between  $M_\beta$  and the coadjoint orbit of  $\text{Diff}_c(P, \xi)$  through  $J^M(\beta) \in \mathfrak{X}(P, \xi)^*$ . By infinitesimal equivariance of  $J^M$  and (13), this diffeomorphism intertwines the Kostant–Kirillov–Souriau symplectic form  $\omega^{\text{KKS}}$  with  $\omega^M$ . Indeed, for  $\beta \in M$  and  $X, Y \in \mathfrak{X}(P, \xi)$ , we get

$$\begin{aligned} & ((J^M)^* \omega^{\text{KKS}})(\zeta_X^M(\beta), \zeta_Y^M(\beta)) \\ &= \omega^{\text{KKS}} \left( \zeta_X^{\mathfrak{X}(P, \xi)^*}(J^M(\beta)), \zeta_Y^{\mathfrak{X}(P, \xi)^*}(J^M(\beta)) \right) \\ &= \langle J^M(\beta), [X, Y] \rangle \stackrel{(15)}{=} h_{[X, Y]}^M(\beta) \stackrel{(13)}{=} \omega^M(\zeta_X^M(\beta), \zeta_Y^M(\beta)), \end{aligned}$$

whence  $(J^M)^* \omega^{\text{KKS}} = \omega^M$ .

In particular, each connected component of  $M$  is equivariantly symplectomorphic to a coadjoint orbit of the identity component in  $\text{Diff}_c(P, \xi)$ . If  $P$  connected and the contact structure is not coorientable, then  $M$  is connected, hence a coadjoint orbit of  $\text{Diff}_c(P, \xi)$ .

**2.2 Moment maps on a manifold of weighted maps**

In this section we introduce an infinite dimensional generalization  $\mathcal{L}$  of  $L^*$  that also carries a canonical 1-form  $\theta^\mathcal{L}$  which is invariant under a natural  $\text{Diff}(P, \xi)$ -action.

To this end, we fix a closed manifold  $S$ . We let  $|\Lambda|_S$  denote the line bundle of densities [21, Chapter 16] on  $S$ , and we write  $\pi^{|\Lambda|_S} : |\Lambda|_S \rightarrow S$  for the corresponding vector bundle projection. Recall that sections of  $|\Lambda|_S$  can be integrated over  $S$  in a natural way. Every orientation of  $S$  provides an isomorphism of line bundles  $|\Lambda|_S \cong \Lambda^{\dim(S)} T^*S$ . A nowhere vanishing density, i.e., a section in  $\Gamma^\infty(|\Lambda|_S \setminus S)$ , will be referred to as a *volume density*.

We denote the space of line bundle homomorphisms from  $|\Lambda|_S^* \rightarrow S$  to  $L^* \rightarrow P$  by

$$\mathcal{L} := C_{\text{lin}}^\infty(|\Lambda|_S^*, L^*) := \left\{ \Phi \in C^\infty(|\Lambda|_S^*, L^*) \mid \forall t \in \mathbb{R} : \Phi \circ \delta_t^{|\Lambda|_S^*} = \delta_t^{L^*} \circ \Phi \right\}.$$

There is a canonical map  $\pi^\mathcal{L} : \mathcal{L} \rightarrow C^\infty(S, P)$ , characterized by

$$\pi^{L^*} \circ \Phi = \pi^\mathcal{L}(\Phi) \circ \pi^{|\Lambda|_S^*}, \tag{16}$$

for all  $\Phi \in \mathcal{L}$ . For the fiber over  $\varphi \in C^\infty(S, P)$  we have a canonical identification,

$$\mathcal{L}_\varphi := (\pi^\mathcal{L})^{-1}(\varphi) = \Gamma^\infty(|\Lambda|_S \otimes \varphi^* L^*). \tag{17}$$

The contact group  $\text{Diff}(P, \xi)$  acts from the left on  $\mathcal{L}$ , and the reparametrization group  $\text{Diff}(S)$  acts on  $\mathcal{L}$  from the right in an obvious way. More explicitly, these actions are given by

$$\Psi_g^\mathcal{L}(\Phi) := \Psi_g^{L^*} \circ \Phi \quad \text{and} \quad \psi_f^\mathcal{L}(\Phi) := \Phi \circ \psi_f^{|\Lambda|_S^*}, \tag{18}$$

where  $\Phi \in \mathcal{L}$ ,  $g \in \text{Diff}(P, \xi)$ ,  $f \in \text{Diff}(S)$ , and  $\psi_f^{|\Lambda|_S^*} \in \text{Diff}(|\Lambda|_S^*)$  denotes the induced (fiberwise linear) action of  $\text{Diff}(S)$  on the total space of  $|\Lambda|_S^*$ . The two actions on  $\mathcal{L}$  commute, and the map  $\pi^\mathcal{L}$  intertwines them with the corresponding actions on  $C^\infty(S, P)$  given by

$$\Psi_g^{C^\infty(S,P)}(\varphi) = g \circ \varphi \quad \text{and} \quad \psi_f^{C^\infty(S,P)}(\varphi) = \varphi \circ f,$$

where  $g \in \text{Diff}(P, \xi)$ ,  $f \in \text{Diff}(S)$ , and  $\varphi \in C^\infty(S, P)$ . More explicitly, we have  $\Psi_{g_2 g_1}^\mathcal{L} = \Psi_{g_2}^\mathcal{L} \circ \Psi_{g_1}^\mathcal{L}$ ,  $\pi^\mathcal{L} \circ \Psi_g^\mathcal{L} = \Psi_g^{C^\infty(S,P)} \circ \pi^\mathcal{L}$ ,  $\psi_{f_1 f_2}^\mathcal{L} = \psi_{f_2}^\mathcal{L} \circ \psi_{f_1}^\mathcal{L}$ ,  $\pi^\mathcal{L} \circ \psi_f^\mathcal{L} = \psi_f^{C^\infty(S,P)} \circ \pi^\mathcal{L}$ , and  $\Psi_g^\mathcal{L} \circ \psi_f^\mathcal{L} = \psi_f^\mathcal{L} \circ \Psi_g^\mathcal{L}$ , for  $g, g_1, g_2 \in \text{Diff}(P, \xi)$  and  $f, f_1, f_2 \in \text{Diff}(S)$ .

**Remark 2.2** Let  $\mu \in \Gamma^\infty(|\Lambda|_S \setminus S)$  be a volume density on  $S$ , i.e., a nowhere vanishing smooth section of  $|\Lambda|_S$ . Such a volume density provides an identification

$$\mathcal{L} \cong C^\infty(S, L^*), \quad \Phi \leftrightarrow \phi = \Phi \circ \hat{\mu},$$

where  $\hat{\mu} \in \Gamma^\infty(|\Lambda|_S^*)$  denotes the section dual to  $\mu$ , that is  $\hat{\mu}(\mu) = 1$ . In this picture the actions on  $\mathcal{L}$  take the form

$$\Psi_g^\mathcal{L}(\phi) = \Psi_g^{L^*} \circ \phi \quad \text{and} \quad \psi_f^\mathcal{L}(\phi) = \frac{f^* \mu}{\mu} \cdot (\phi \circ f),$$

where  $\phi \in C^\infty(S, L^*)$ ,  $g \in \text{Diff}(P, \xi)$  and  $f \in \text{Diff}(S)$ .

The space  $\mathcal{L}$  can be equipped with the structure of a smooth Fréchet manifold such that the identification  $\mathcal{L} \cong C^\infty(S, L^*)$  in Remark 2.2 becomes a diffeomorphism, for each choice of volume density  $\mu$ . The map  $\pi^\mathcal{L} : \mathcal{L} \rightarrow C^\infty(S, P)$  is a smooth vector bundle. The tangent space at  $\Phi \in \mathcal{L}$  can be canonically identified as

$$T_\Phi \mathcal{L} = \left\{ \eta \in C^\infty(|\Lambda|_S^*, TL^*) \left| \begin{array}{l} \pi^{TL^*} \circ \eta = \Phi \quad \text{and} \\ \forall t \in \mathbb{R} : \eta \circ \delta_t^{|\Lambda|_S^*} = T\delta_t^{L^*} \circ \eta \end{array} \right. \right\}. \tag{19}$$

The actions of  $\text{Diff}(P, \xi)$  and  $\text{Diff}(S)$  on  $\mathcal{L}$  are smooth. For  $X \in \mathfrak{X}(P, \xi)$  and  $Z \in \mathfrak{X}(S)$ , the corresponding fundamental vector fields are

$$\zeta_X^\mathcal{L}(\Phi) = \zeta_X^{L^*} \circ \Phi \quad \text{and} \quad \zeta_Z^\mathcal{L}(\Phi) = T\Phi \circ \zeta_Z^{|\Lambda|_S^*} \tag{20}$$

where  $\Phi \in \mathcal{L}$  and  $\zeta_Z^{|\Lambda|_S^*} \in \mathfrak{X}(|\Lambda|_S^*)$  denotes the fundamental vector field of the  $\text{Diff}(S)$ -action on the total space of  $|\Lambda|_S^*$ . Note that

$$T\pi^{|\Lambda|_S^*} \circ \zeta_Z^{|\Lambda|_S^*} = Z \circ \pi^{|\Lambda|_S^*} \quad \text{and} \quad (\delta_t^{|\Lambda|_S^*})^* \zeta_Z^{|\Lambda|_S^*} = \zeta_Z^{|\Lambda|_S^*}. \tag{21}$$

Clearly,  $(\Psi_g^\mathcal{L})^* \zeta_X^\mathcal{L} = \zeta_{g^* X}^{L^*}$ ,  $\zeta_{[X_1, X_2]}^\mathcal{L} = [\zeta_{X_1}^\mathcal{L}, \zeta_{X_2}^\mathcal{L}]$ ,  $T\pi^\mathcal{L} \circ \zeta_X^\mathcal{L} = \zeta_X^{C^\infty(S,P)} \circ \pi^\mathcal{L}$ ,  $(\psi_f^\mathcal{L})^* \zeta_Z^\mathcal{L} = \zeta_{f^* Z}^{|\Lambda|_S^*}$ ,  $\zeta_{[Z_1, Z_2]}^\mathcal{L} = -[\zeta_{Z_1}^\mathcal{L}, \zeta_{Z_2}^\mathcal{L}]$ ,  $T\pi^\mathcal{L} \circ \zeta_Z^\mathcal{L} = \zeta_Z^{C^\infty(S,P)} \circ \pi^\mathcal{L}$ , and  $[\zeta_X^\mathcal{L}, \zeta_Z^\mathcal{L}] = 0$ , where  $g \in \text{Diff}(P, \xi)$ ,  $X, X_1, X_2 \in \mathfrak{X}(P, \xi)$ ,  $f \in \text{Diff}(S)$ ,  $Z, Z_1, Z_2 \in \mathfrak{X}(S)$ .

### The canonical 1-form

Consider the 1-form  $\theta^{\mathcal{L}}$  on  $\mathcal{L}$  defined by

$$\theta^{\mathcal{L}}(\eta) := \int_S \theta^{L^*}(\eta), \tag{22}$$

where  $\eta \in T_{\Phi}\mathcal{L}$  and  $\Phi \in \mathcal{L}$ . Note here that, because of (19), inserting  $\eta$  into  $\theta^{L^*}$  leads to a fiberwise linear map  $\theta^{L^*}(\eta): |\Lambda|_S^* \rightarrow \mathbb{R}$  which, when regarded as a section of  $|\Lambda|_S$ , may be integrated over  $S$ . By invariance of  $\theta^{L^*}$ , the 1-form  $\theta^{\mathcal{L}}$  is invariant under both actions, i.e., we have

$$(\Psi_g^{\mathcal{L}})^*\theta^{\mathcal{L}} = \theta^{\mathcal{L}} \quad \text{and} \quad (\psi_f^{\mathcal{L}})^*\theta^{\mathcal{L}} = \theta^{\mathcal{L}}$$

for all  $g \in \text{Diff}(P, \xi)$  and  $f \in \text{Diff}(S)$ . The corresponding infinitesimal invariance reads

$$L_{\zeta_X^{\mathcal{L}}}\theta^{\mathcal{L}} = 0 \quad \text{and} \quad L_{\zeta_Z^{\mathcal{L}}}\theta^{\mathcal{L}} = 0,$$

where  $X \in \mathfrak{X}(P, \xi)$  and  $Z \in \mathfrak{X}(S)$ .

Moreover, we introduce the 2-form  $\omega^{\mathcal{L}} := d\theta^{\mathcal{L}}$  on  $\mathcal{L}$ . By invariance of  $\theta^{\mathcal{L}}$ , this 2-form is invariant under both actions too. More explicitly, we have  $(\Psi_g^{\mathcal{L}})^*\omega^{\mathcal{L}} = \omega^{\mathcal{L}}$  and  $(\psi_f^{\mathcal{L}})^*\omega^{\mathcal{L}} = \omega^{\mathcal{L}}$  for  $g \in \text{Diff}(P, \xi)$  and  $f \in \text{Diff}(S)$ , as well as infinitesimal invariance  $L_{\zeta_X^{\mathcal{L}}}\omega^{\mathcal{L}} = 0$  and  $L_{\zeta_Z^{\mathcal{L}}}\omega^{\mathcal{L}} = 0$  for  $X \in \mathfrak{X}(P, \xi)$  and  $Z \in \mathfrak{X}(S)$ . Clearly, see [9, 33],

$$\omega^{\mathcal{L}}(\eta_1, \eta_2) = \int_S \omega^{L^*}(\eta_1, \eta_2) \tag{23}$$

where  $\eta_1, \eta_2 \in T_{\Phi}\mathcal{L}$  and  $\Phi \in \mathcal{L}$ . As before, the fiberwise linear function  $\omega^{L^*}(\eta_1, \eta_2)$  on  $|\Lambda|_S^*$  may be regarded as a section of  $|\Lambda|_S$  which can be integrated over  $S$ .

The exact 2-form  $\omega^{\mathcal{L}} = d\theta^{\mathcal{L}}$  is not (weakly) non-degenerate, because  $\omega^{L^*}$  is not symplectic on all of  $L^*$ . In the subsequent section, we will restrict to an invariant open subset of  $\mathcal{L}$  on which  $\omega^{\mathcal{L}}$  is (weakly) symplectic. On this symplectic part, both actions are Hamiltonian with equivariant moment map. This is a well known formal consequence of the fact that these actions preserve the 1-form  $\theta^{\mathcal{L}}$ , see for instance [25, Section 12.3]. The corresponding Hamiltonian functions and moment maps are given by contraction of the fundamental vector fields with the canonical 1-form. However, these geometric objects make sense on all of  $\mathcal{L}$ . Hence, we will now formulate their fundamental relations on  $\mathcal{L}$ .

### The left moment map

For  $X \in \mathfrak{X}(P, \xi)$ , consider the function  $h_X^{\mathcal{L}}: \mathcal{L} \rightarrow \mathbb{R}$  defined by

$$i_{\zeta_X^{\mathcal{L}}}\theta^{\mathcal{L}} =: h_X^{\mathcal{L}}. \tag{24}$$

Using the infinitesimal invariance,  $L_{\zeta_X^{\mathcal{L}}}\theta^{\mathcal{L}} = 0$ , we obtain

$$i_{\zeta_X^{\mathcal{L}}}\omega^{\mathcal{L}} = -dh_X^{\mathcal{L}} \tag{25}$$

analogous to (12), as well as

$$h_{[X, Y]}^{\mathcal{L}} = \zeta_X^{\mathcal{L}} \cdot h_Y^{\mathcal{L}} = -\zeta_Y^{\mathcal{L}} \cdot h_X^{\mathcal{L}} = \omega^{\mathcal{L}}(\zeta_X^{\mathcal{L}}, \zeta_Y^{\mathcal{L}}), \tag{26}$$

for all  $X, Y \in \mathfrak{X}(P, \xi)$ , cf. (13). From the invariance of  $\theta^{\mathcal{L}}$  we obtain, cf. (9)

$$(\Psi_g^{\mathcal{L}})^* h_X^{\mathcal{L}} = h_{g^*X}^{\mathcal{L}} \quad \text{and} \quad (\psi_f^{\mathcal{L}})^* h_X^{\mathcal{L}} = h_X^{\mathcal{L}} \tag{27}$$

for all  $f \in \text{Diff}(S)$ ,  $g \in \text{Diff}(P, \xi)$ , and  $X \in \mathfrak{X}(P, \xi)$ . We introduce a smooth map

$$J_L^{\mathcal{L}}: \mathcal{L} \rightarrow \mathfrak{X}(P, \xi)^* \tag{28}$$

by putting  $\langle J_L^{\mathcal{L}}, X \rangle := h_X^{\mathcal{L}}$ , that is,

$$\langle J_L^{\mathcal{L}}(\Phi), X \rangle := h_X^{\mathcal{L}}(\Phi) = \theta^{\mathcal{L}}(\zeta_X^{\mathcal{L}}(\Phi)), \tag{29}$$

where  $\Phi \in \mathcal{L}$  and  $X \in \mathfrak{X}(P, \xi)$ . The equations in (27) may be written in the form

$$\langle J_L^{\mathcal{L}} \circ \Psi_g^{\mathcal{L}}, X \rangle = \langle \Psi_g^{\mathcal{L}}, g^*X \rangle \quad \text{and} \quad J_L^{\mathcal{L}} \circ \psi_f^{\mathcal{L}} = J_L^{\mathcal{L}}, \tag{30}$$

where  $g \in \text{Diff}(P, \xi)$ ,  $X \in \mathfrak{X}(P, \xi)$ , and  $f \in \text{Diff}(S)$ . Combining (24), (20), (22), (7), (10), and (16), we obtain

$$h_X^{\mathcal{L}}(\Phi) = \int_S \Phi(X \circ \varphi), \tag{31}$$

where  $\varphi \in C^\infty(S, P)$ ,  $\Phi \in \mathcal{L}_\varphi = \Gamma^\infty(|\Lambda|_S \otimes \varphi^*L^*)$ , and  $X \in \mathfrak{X}(P, \xi) = \Gamma^\infty(L)$ , cf. (17) and (8). Here we use the canonical contraction between  $L^* \subseteq T^*P$  and  $TP$  to obtain the density  $\Phi(X \circ \varphi) \in \Gamma^\infty(|\Lambda|_S)$ . More explicitly, the verification of (31) reads:

$$\begin{aligned} h_X^{\mathcal{L}}(\Phi) &\stackrel{(24)}{=} \theta^{\mathcal{L}}(\zeta_X^{\mathcal{L}}(\Phi)) \stackrel{(20)}{=} \theta^{\mathcal{L}}(\zeta_X^{L^*} \circ \Phi) \stackrel{(22)}{=} \int_S \theta^{L^*}(\zeta_X^{L^*} \circ \Phi) \\ &\stackrel{(7)}{=} \int_S \Phi(T\pi^{L^*} \circ \zeta_X^{L^*} \circ \Phi) \stackrel{(10)}{=} \int_S \Phi(X \circ \pi^{L^*} \circ \Phi) \\ &\stackrel{(16)}{=} \int_S \Phi(X \circ \varphi \circ \pi^{|\Lambda|_S^*}) = \int_S \Phi(X \circ \varphi). \end{aligned}$$

### The right moment map

For  $Z \in \mathfrak{X}(S)$ , consider the function  $k_Z^{\mathcal{L}}: \mathcal{L} \rightarrow \mathbb{R}$  defined by

$$i_{\zeta_Z^{\mathcal{L}}} \theta^{\mathcal{L}} =: k_Z^{\mathcal{L}}. \tag{32}$$

Using the infinitesimal invariance,  $L_{\zeta_Z^{\mathcal{L}}} \theta^{\mathcal{L}} = 0$ , we obtain

$$i_{\zeta_Z^{\mathcal{L}}} \omega^{\mathcal{L}} = -dk_Z^{\mathcal{L}} \tag{33}$$

as well as

$$k_{[Z_1, Z_2]}^{\mathcal{L}} = \zeta_{Z_1}^{\mathcal{L}} \cdot k_{Z_2}^{\mathcal{L}} - \zeta_{Z_2}^{\mathcal{L}} \cdot k_{Z_1}^{\mathcal{L}} = \omega^{\mathcal{L}}(\zeta_{Z_1}^{\mathcal{L}}, \zeta_{Z_2}^{\mathcal{L}}), \tag{34}$$

for all  $Z, Z_1, Z_2 \in \mathfrak{X}(S)$ . From the invariance of  $\theta^{\mathcal{L}}$  we obtain

$$(\Psi_g^{\mathcal{L}})^* k_Z^{\mathcal{L}} = k_Z^{\mathcal{L}} \quad \text{and} \quad (\psi_f^{\mathcal{L}})^* k_Z^{\mathcal{L}} = k_{f^*Z}^{\mathcal{L}} \tag{35}$$

for all  $g \in \text{Diff}(P, \xi)$ ,  $f \in \text{Diff}(S)$ , and  $Z \in \mathfrak{X}(S)$ . We introduce a smooth map

$$J_R^{\mathcal{L}}: \mathcal{L} \rightarrow \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^* \tag{36}$$

by putting  $\langle J_R^{\mathcal{L}}, Z \rangle := k_Z^{\mathcal{L}}$ , that is,

$$\langle J_R^{\mathcal{L}}(\Phi), Z \rangle := k_Z^{\mathcal{L}}(\Phi) = \theta^{\mathcal{L}}(\zeta_Z^{\mathcal{L}}(\Phi)), \tag{37}$$

where  $\Phi \in \mathcal{L}$  and  $Z \in \mathfrak{X}(S)$ . The equations in (35) may be written in the form

$$\langle J_R^{\mathcal{L}} \circ \psi_f^{\mathcal{L}}, Z \rangle = \langle \psi_f^{\mathcal{L}}, f_* Z \rangle \quad \text{and} \quad J_R^{\mathcal{L}} \circ \Psi_g^{\mathcal{L}} = J_R^{\mathcal{L}}, \tag{38}$$

where  $f \in \text{Diff}(S)$ ,  $Z \in \mathfrak{X}(S)$ , and  $g \in \text{Diff}(P, \xi)$ . In view of (32), (20), (22), (7), (16), and (21), we have

$$k_Z^{\mathcal{L}}(\Phi) = \int_S \Phi(T\varphi \circ Z), \tag{39}$$

where  $\varphi \in C^\infty(S, P)$ ,  $\Phi \in \mathcal{L}_\varphi = \Gamma^\infty(|\Lambda|_S \otimes \varphi^* L^*)$ , and  $Z \in \mathfrak{X}(S)$ , cf. (17). As before, we use the canonical contraction between  $L^* \subseteq T^*P$  and  $TP$  to obtain a density  $\Phi(T\varphi \circ Z) \in \Gamma^\infty(|\Lambda|_S)$ . More explicitly, the verification of (39) reads:

$$\begin{aligned} k_Z^{\mathcal{L}}(\Phi) &\stackrel{(32)}{=} \theta^{\mathcal{L}}(\zeta_Z^{\mathcal{L}}(\Phi)) \stackrel{(20)}{=} \theta^{\mathcal{L}}(T\Phi \circ \zeta_Z^{|\Lambda|_S^*}) \stackrel{(22)}{=} \int_S \theta^{L^*}(T\Phi \circ \zeta_Z^{|\Lambda|_S^*}) \\ &\stackrel{(7)}{=} \int_S \Phi(T\pi^{L^*} \circ T\Phi \circ \zeta_Z^{|\Lambda|_S^*}) \stackrel{(16)}{=} \int_S \Phi(T\varphi \circ T\pi^{|\Lambda|_S^*} \circ \zeta_Z^{|\Lambda|_S^*}) \\ &\stackrel{(21)}{=} \int_S \Phi(T\varphi \circ Z \circ \pi^{|\Lambda|_S^*}) = \int_S \Phi(T\varphi \circ Z). \end{aligned}$$

It follows from (37) and (39) that  $J_R^{\mathcal{L}}(\Phi)$  is indeed a smooth 1-form density as indicated in (36), i.e.,  $J_R^{\mathcal{L}}(\Phi) \in \Omega^1(S, |\Lambda|_S)$ . More precisely, we have

$$\lambda(J_R^{\mathcal{L}}(\Phi)(Z)) = (\Phi \circ \lambda)(T\varphi \circ Z) \tag{40}$$

for  $Z \in \mathfrak{X}(S)$  and  $\lambda \in \Gamma^\infty(|\Lambda|_S^*)$ . Note that  $J_R^{\mathcal{L}}(\Phi)$  can also be characterized as the smooth 1-form density on  $S$  corresponding to the 1-homogeneous vertical 1-form  $\Phi^* \theta^{L^*}$  on the total space of  $|\Lambda|_S^*$ . More explicitly, we have

$$J_R^{\mathcal{L}}(\Phi) = \Phi^* \theta^{L^*} \tag{41}$$

via the canonical identification

$$\Omega^1(S, |\Lambda|_S) = \left\{ \beta \in \Omega^1(|\Lambda|_S^*) \mid \beta \text{ is vertical and } (\delta_t^{|\Lambda|_S^*})^* \beta = t\beta \text{ for all } t \in \mathbb{R} \right\}. \tag{42}$$

Here  $\rho \in \Omega^1(S, |\Lambda|_S)$  corresponds to  $\beta \in \Omega^1(|\Lambda|_S^*)$  given by  $\beta(W) = w(\rho(T_w \pi^{|\Lambda|_S^*} \cdot W))$  where  $w \in |\Lambda|_S^*$  and  $W \in T_w |\Lambda|_S^*$ .

**Remark 2.3** Using a volume density  $\mu$  on  $S$  to identify  $\mathcal{L} \cong C^\infty(S, L^*)$  as in Remark 2.2, the differential forms  $\theta^{\mathcal{L}}$  and  $\omega^{\mathcal{L}}$  become, see (22) and (23),

$$\theta^{\mathcal{L}}(\eta) = \int_S \theta^{L^*}(\eta) \mu \quad \text{and} \quad \omega^{\mathcal{L}}(\eta_1, \eta_2) = \int_S \omega^{L^*}(\eta_1, \eta_2) \mu, \tag{43}$$

where  $\phi \in C^\infty(S, L^*)$  and  $\eta, \eta_1, \eta_2 \in T_\phi C^\infty(S, L^*) = \{\eta \in C^\infty(S, TL^*) : \pi^{TL^*} \circ \eta = \phi\}$ . For  $X \in \mathfrak{X}(P, \xi)$  and  $Z \in \mathfrak{X}(S)$ , the fundamental vector fields  $\zeta_X^{\mathcal{L}}$  and  $\zeta_Z^{\mathcal{L}}$  identify to

$$\zeta_X^{\mathcal{L}}(\phi) = \zeta_X^{L^*} \circ \phi \quad \text{and} \quad \zeta_Z^{\mathcal{L}}(\phi) = T\phi \circ Z + \text{div}_\mu(Z) \cdot (R \circ \phi), \tag{44}$$

where  $\operatorname{div}_\mu(Z) := \frac{L_Z\mu}{\mu}$  denotes the  $\mu$ -divergence, and  $R := \frac{\partial}{\partial t}|_{t=1}\delta_t^{L^*} \in \mathfrak{X}(L^*)$  denotes the Euler vector field of  $L^*$ . The functions  $h_X^{\mathcal{L}}$  and  $k_Z^{\mathcal{L}}$  become, see (31) and (39),

$$h_X^{\mathcal{L}}(\phi) = \int_S (\phi^* h_X)\mu \quad \text{and} \quad k_Z^{\mathcal{L}}(\phi) = \int_S (\phi^* \theta^{L^*})(Z)\mu. \tag{45}$$

Hence, the maps  $J_L^{\mathcal{L}}$  and  $J_R^{\mathcal{L}}$  identify to

$$J_L^{\mathcal{L}}: C^\infty(S, L^*) \rightarrow C^\infty(L^*)^* \rightarrow \mathfrak{X}(P, \xi)^*, \quad J_L^{\mathcal{L}}(\phi) = \phi_*\mu \tag{46}$$

$$J_R^{\mathcal{L}}: C^\infty(S, L^*) \rightarrow \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^*, \quad J_R^{\mathcal{L}}(\phi) = \phi^*\theta^{L^*} \otimes \mu \tag{47}$$

where  $\phi \in C^\infty(S, L^*)$  and we use the inclusion  $\mathfrak{X}(P, \xi) = C_{\text{lin}}^\infty(L^*) \subseteq C^\infty(L^*)$ , see (8).

### 2.3 The symplectic part

Let  $\mathcal{M} \subseteq \mathcal{L} = C_{\text{lin}}^\infty(|\Lambda|_S^*, L^*)$  denote the open subset of line bundle homomorphisms  $|\Lambda|_S^* \rightarrow L^*$  which restrict to a linear isomorphism on each fiber,

$$\mathcal{M} := C_{\text{lin, inj}}^\infty(|\Lambda|_S^*, L^*). \tag{48}$$

<sup>2</sup> We will denote the restriction to  $\mathcal{M}$  of any action, function, form, or vector field on  $\mathcal{L}$  considered above, by replacing the superscript  $\mathcal{L}$  with  $\mathcal{M}$ . Because  $L^* \setminus P$  is symplectic, the 2-form  $\omega^{\mathcal{M}} = d\theta^{\mathcal{M}}$  is (weakly) non-degenerate, whence symplectic, cf. (23).

The map  $\pi^{\mathcal{M}}: \mathcal{M} \rightarrow C^\infty(S, P)$  is a principal fiber bundle with structure group  $C^\infty(S, \mathbb{R}^\times)$ , provided we restrict to the connected components of  $C^\infty(S, P)$  in the image of  $\pi^{\mathcal{M}}$ . If  $\varphi$  is in one of these components, then the fiber  $\mathcal{M}_\varphi := (\pi^{\mathcal{M}})^{-1}(\varphi)$  may be canonically identified with the space of nowhere vanishing sections of the line bundle  $|\Lambda|_S \otimes \varphi^*L^*$ , cf. (17). Thus, disregarding the density part,  $\mathcal{M}_\varphi$  may be considered as the space of contact forms for  $\xi$  along the map  $\varphi: S \rightarrow P$ .

Clearly,  $\mathcal{M}$  is invariant under the action of the groups  $\operatorname{Diff}(P, \xi)$  and  $\operatorname{Diff}(S)$ . Since both actions preserve the 1-form  $\theta^{\mathcal{M}}$ , they are Hamiltonian with equivariant moment maps obtained by contraction of the 1-form with the infinitesimal generators, see for instance [25, Section 12.3]. We summarize these facts in the following proposition.

**Proposition 2.4** (a) *The action of the group  $\operatorname{Diff}(P, \xi)$  on  $\mathcal{M}$  is Hamiltonian with an equivariant moment map  $J_L^{\mathcal{M}}: \mathcal{M} \rightarrow \mathfrak{X}(P, \xi)^*$ , given by*

$$\langle J_L^{\mathcal{M}}(\Phi), X \rangle = (i_{\zeta_X^{\mathcal{M}}}\theta^{\mathcal{M}})(\Phi) = h_X^{\mathcal{M}}(\Phi) = \int_S \Phi(X \circ \varphi), \tag{49}$$

where  $\Phi \in \mathcal{M}_\varphi$  and  $X \in \mathfrak{X}(P, \xi)$ . Moreover, the moment map  $J_L^{\mathcal{M}}$  is  $\operatorname{Diff}(S)$ -invariant. More explicitly, we have  $(\Psi_g^{\mathcal{M}})^*\omega^{\mathcal{M}} = \omega^{\mathcal{M}}$ ,  $i_{\zeta_X^{\mathcal{M}}}\omega^{\mathcal{M}} = -d\langle J_L^{\mathcal{M}}, X \rangle$ ,  $\langle J_L^{\mathcal{M}} \circ \Psi_g^{\mathcal{M}}, X \rangle = \langle \Psi_g^{\mathcal{M}}, g^*X \rangle$ , and  $J_L^{\mathcal{M}} \circ \psi_f^{\mathcal{M}} = J_L^{\mathcal{M}}$  where  $g \in \operatorname{Diff}(P, \xi)$ ,  $X \in \mathfrak{X}(P, \xi)$ , and  $f \in \operatorname{Diff}(S)$ .

(b) *The action of the group  $\operatorname{Diff}(S)$  on  $\mathcal{M}$  is Hamiltonian with an equivariant moment map  $J_R^{\mathcal{M}}: \mathcal{M} \rightarrow \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^*$ , given by*

$$\langle J_R^{\mathcal{M}}(\Phi), Z \rangle = (i_{\zeta_Z^{\mathcal{M}}}\theta^{\mathcal{M}})(\Phi) = k_Z^{\mathcal{M}}(\Phi) = \int_S \Phi(T\varphi \circ Z), \tag{50}$$

<sup>2</sup> Using a volume density on  $S$  to identify  $\mathcal{L} \cong C^\infty(S, L^*)$  as in Remark 2.2, the space  $\mathcal{M}$  corresponds to  $C^\infty(S, L^* \setminus P)$ . When  $\xi = \ker \alpha$  for a contact form  $\alpha$ , then the corresponding trivialization  $L^* \cong P \times \mathbb{R}$  yields a further identification  $\mathcal{M} \cong C^\infty(S, P) \times C^\infty(S, \mathbb{R}^\times)$ .

where  $\Phi \in \mathcal{M}_\varphi$  and  $Z \in \mathfrak{X}(S)$ . Moreover, the moment map  $J_R^{\mathcal{M}}$  is  $\text{Diff}(P, \xi)$ -invariant. More explicitly, we have  $(\psi_f^{\mathcal{M}})^* \omega^{\mathcal{M}} = \omega^{\mathcal{M}}$ ,  $i_{\zeta_Z^{\mathcal{M}}} \omega^{\mathcal{M}} = -d\langle J_R^{\mathcal{M}}, Z \rangle$ ,  $\langle J_R^{\mathcal{M}} \circ \psi_f^{\mathcal{M}}, Z \rangle = \langle \psi_f^{\mathcal{M}}, f_* Z \rangle$ , and  $J_R^{\mathcal{M}} \circ \Psi_g^{\mathcal{M}} = J_R^{\mathcal{M}}$ , where  $f \in \text{Diff}(S)$ ,  $Z \in \mathfrak{X}(S)$ , and  $g \in \text{Diff}(P, \xi)$ .

**Proof** The statements in (a) follow immediately from (25), (29), (30), and (31). The statements in (b) follow immediately from (33), (37), (38), and (39).  $\square$

**Remark 2.5** If  $S$  is a single point, then we recover the symplectization discussed in Sect. 2.1. More precisely, in this case the canonical volume density on  $S$  provides a canonical isomorphism between the line bundles  $\pi^{\mathcal{L}}: \mathcal{L} \rightarrow C^\infty(S, P)$  and  $\pi^{L^*}: L^* \rightarrow P$ . Up to this identification, we have  $\Psi_g^{\mathcal{L}} = \Psi_g^{L^*}$ , for all  $g \in \text{Diff}(P, \xi)$ ,  $\theta^{\mathcal{L}} = \theta^{L^*}$  and  $\omega^{\mathcal{L}} = \omega^{L^*}$ . Moreover,  $\mathcal{M} = M$  and  $J_L^{\mathcal{M}} = J^M$ . Clearly, the  $\text{Diff}(S)$ -action is trivial in this case and  $J_R^{\mathcal{M}} = 0$ .

### 2.4 A dual pair on the non-linear Stiefel manifold of weighted embeddings

We will now restrict to an open subset of  $\mathcal{M}$  on which the  $\text{Diff}(S)$ -action is free. Let

$$\mathcal{E} := \text{Emb}_{\text{lin}}(|\Lambda|_S^*, L^*) \tag{51}$$

denote the open subset of all (fiberwise linear) embeddings in  $\mathcal{L} = C^\infty_{\text{lin}}(|\Lambda|_S^*, L^*)$ . Elements of  $\mathcal{E}$  are automatically isomorphisms on fibers, so  $\mathcal{E} \subseteq \mathcal{M}$ . We consider  $\mathcal{E}$  as a non-linear Stiefel manifold of weighted embeddings.<sup>3</sup>

We will denote the restriction to  $\mathcal{E}$  of any action, function, form, or vector field on  $\mathcal{L}$  considered above, by replacing the superscript  $\mathcal{L}$  with  $\mathcal{E}$ . The map  $\pi^{\mathcal{E}}: \mathcal{E} \rightarrow \text{Emb}(S, P)$  is a principal fiber bundle with structure group  $C^\infty(S, \mathbb{R}^\times)$ , provided we restrict to the connected components of  $\text{Emb}(S, P)$  in the image of  $\pi^{\mathcal{E}}$ . Since  $\mathcal{E}$  is open in  $\mathcal{M}$ , the symplectic form  $\omega^{\mathcal{M}}$  restricts to a symplectic form  $\omega^{\mathcal{E}}$  on  $\mathcal{E}$ . Hence,  $(\mathcal{E}, \omega^{\mathcal{E}})$  is a (weakly) symplectic Fréchet manifold.

Note that  $\mathcal{E}$  is invariant under the actions of  $\text{Diff}(P, \xi)$  and  $\text{Diff}(S)$ . In view of Proposition 2.4, the restrictions of  $J_L^{\mathcal{M}}$  and  $J_R^{\mathcal{M}}$  to  $\mathcal{E}$  provide equivariant moment maps

$$\mathfrak{X}(P, \xi)^* \xleftarrow{J_L^{\mathcal{E}}} \mathcal{E} \xrightarrow{J_R^{\mathcal{E}}} \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^* \tag{52}$$

for the actions of  $\text{Diff}(P, \xi)$  and  $\text{Diff}(S)$  on  $\mathcal{E}$ , respectively.

A pair of equivariant moment maps for commuting Hamiltonian actions of (infinite dimensional) Lie groups  $G$  and  $H$  on an (infinite dimensional) symplectic manifold  $Q$ ,

$$\mathfrak{g}^* \xleftarrow{J_L} Q \xrightarrow{J_R} \mathfrak{h}^*,$$

is called a symplectic dual pair [34] if the distributions  $\ker T J_L$  and  $\ker T J_R$  are symplectic orthogonal complements of one another:  $(\ker T J_L)^\perp = \ker T J_R$  and  $(\ker T J_R)^\perp = \ker T J_L$ . Both identities are needed here, due to the weakness of the symplectic form. Let  $\mathfrak{g}_Q(x) := \{\zeta_X^Q(x) | X \in \mathfrak{g}\}$  denote the tangent space to the  $G$ -orbit at  $x \in Q$ . When

$$\mathfrak{g}_Q = \mathfrak{h}_Q^\perp \quad \text{and} \quad \mathfrak{h}_Q = \mathfrak{g}_Q^\perp, \tag{53}$$

<sup>3</sup> Using a volume density  $\mu$  on  $S$  to identify  $\mathcal{L} \cong C^\infty(S, L^*)$  as in Remark 2.2, the subset  $\mathcal{E}$  corresponds to  $C^\infty(S, L^* \setminus P) \cap (\pi^{\mathcal{L}})^{-1}(\text{Emb}(S, P))$ . If, moreover,  $\xi = \ker \alpha$ , we get a further identification  $\mathcal{E} \cong \text{Emb}(S, P) \times C^\infty(S, \mathbb{R}^\times)$ .

i.e., if the  $G$ -orbits and  $H$ -orbits are symplectic orthogonal complements of one another, then the actions are said to be mutually completely orthogonal [23]. Since  $\ker TJ_R = \mathfrak{h}_Q^\perp$ , the first identity in (53) can be rephrased as the transitivity of the  $\mathfrak{g}$ -action on level sets of the moment map  $J_R$ , and similarly for the second identity.

Mutually completely orthogonality of the actions implies that  $J_L$  and  $J_R$  form a dual pair. The reverse implication is not always true, due to the weakness of the symplectic form [11].

**Theorem 2.6** *The moment mappings  $J_L^\mathcal{E}$  and  $J_R^\mathcal{E}$  in (52) form a symplectic dual pair, called the EPContact dual pair. Moreover, the commuting actions of  $\text{Diff}(P, \xi)$  and  $\text{Diff}(S)$  on  $\mathcal{E}$  are mutually completely orthogonal, i.e., for each  $\Phi \in \mathcal{E}$  we have*

$$\begin{aligned} \{\zeta_X^\mathcal{E}(\Phi) \mid X \in \mathfrak{X}(P, \xi)\} &= \{\zeta_Z^\mathcal{E}(\Phi) \mid Z \in \mathfrak{X}(S)\}^\perp \\ &:= \{A \in T_\Phi \mathcal{E} \mid \forall Z \in \mathfrak{X}(S) : \omega_\Phi^\mathcal{E}(A, \zeta_Z^\mathcal{E}(\Phi)) = 0\} \end{aligned} \tag{54}$$

as well as

$$\begin{aligned} \{\zeta_Z^\mathcal{E}(\Phi) \mid Z \in \mathfrak{X}(S)\} &= \{\zeta_X^\mathcal{E}(\Phi) \mid X \in \mathfrak{X}(P, \xi)\}^\perp \\ &:= \{B \in T_\Phi \mathcal{E} \mid \forall X \in \mathfrak{X}(P, \xi) : \omega_\Phi^\mathcal{E}(\zeta_X^\mathcal{E}(\Phi), B) = 0\}. \end{aligned} \tag{55}$$

**Proof** Suppose  $\Phi \in \mathcal{E}$ . The inclusion

$$\{\zeta_X^\mathcal{E}(\Phi) \mid X \in \mathfrak{X}(P, \xi)\} \subseteq \{\zeta_Z^\mathcal{E}(\Phi) \mid Z \in \mathfrak{X}(S)\}^\perp$$

follows immediately from (27) and (25). To show the converse inclusion, suppose  $A \in \{\zeta_Z^\mathcal{E}(\Phi) \mid Z \in \mathfrak{X}(S)\}^\perp$ . The 1-form  $\beta := \Phi^* i_A \omega^{L^*} \in \Omega^1(|\Lambda|_S^*)$ , given by  $\beta(V) = \omega_{\Phi(y)}^{L^*}(A(y), T_y \Phi(V))$  for all  $V \in T_y |\Lambda|_S^*$ , satisfies  $(\delta_t^{|\Lambda|_S^*})^* \beta = t\beta$ , by homogeneity of  $\Phi$ ,  $A$ , and  $\omega^{L^*}$ . Thus, for all  $Z \in \mathfrak{X}(S)$ ,

$$0 = \omega^\mathcal{E}(A, \zeta_Z^\mathcal{E}(\Phi)) \stackrel{(23)}{=} \int_S \omega^{L^*}(A, T\Phi \circ \zeta_Z^{|\Lambda|_S^*}) = \int_S \beta(\zeta_Z^{|\Lambda|_S^*}),$$

where the integrands are fiberwise linear functions on the total space of  $|\Lambda|_S^*$ , which may be regarded as sections of  $|\Lambda|_S$  and integrated over  $S$ . By Lemma 2.7 below, there exists a fiberwise linear function  $u \in C_{\text{lin}}^\infty(|\Lambda|_S^*)$  such that  $\beta = du$ .

Because  $\Phi$  is a fiberwise linear embedding, one can construct  $h \in C_{\text{lin}}^\infty(L^*)$ , i.e.  $h \circ \delta_t^{L^*} = th$  for all  $t \in \mathbb{R}$ , such that  $h \circ \Phi = u$  and  $dh \circ \Phi = i_A \omega^{L^*}$ . Indeed, let  $\tilde{u} \in C_{\text{lin}}^\infty(L^*)$  be any fiberwise linear function with  $\tilde{u} \circ \Phi = u$  and write  $h = \tilde{u} + h'$ . Hence, it suffices to construct  $h' \in C_{\text{lin}}^\infty(L^*)$  which vanishes along  $\Phi$  and has prescribed derivative  $dh' \circ \Phi = i_A \omega^{L^*} - (d\tilde{u}) \circ \Phi$  along  $\Phi$ . This is possible, since  $\Phi^*(i_A \omega^{L^*}) - \Phi^*(d\tilde{u}) = \beta - du = 0$ .

According to the identification (8), there exists a contact vector field  $X \in \mathfrak{X}(P, \xi)$  such that  $h = -h_X$ , hence

$$i_A \omega^{L^*} = -dh_X \circ \Phi \stackrel{(12)}{=} i_{\zeta_X^{L^*} \circ \Phi} \omega^{L^*}.$$

Since  $\omega^{L^*}$  is non-degenerate over  $L^* \setminus P$ , we conclude  $A = \zeta_X^{L^*} \circ \Phi$ , and using (20) we get  $A = \zeta_X^\mathcal{E}(\Phi)$ , whence (54).

It remains to check the other equality (55). The inclusion

$$\{\zeta_Z^\mathcal{E}(\Phi) \mid Z \in \mathfrak{X}(S)\} \subseteq \{\zeta_X^\mathcal{E}(\Phi) \mid X \in \mathfrak{X}(P, \xi)\}^\perp$$

follows immediately from (35) and (33), or (54). To show the converse inclusion, suppose that  $B \in \{\zeta_X^\mathcal{E}(\Phi)|X \in \mathfrak{X}(P, \xi)\}^\perp$ . Hence, for all  $X \in \mathfrak{X}(P, \xi)$ ,

$$0 = \omega^\mathcal{E}(\zeta_X^\mathcal{E}(\Phi), B) \stackrel{(20)}{=} \omega^\mathcal{E}(\zeta_X^{L^*} \circ \Phi, B) \stackrel{(23)}{=} \int_S \omega^{L^*}(\zeta_X^{L^*} \circ \Phi, B) \stackrel{(12)}{=} - \int_S (dh_X \circ \Phi)(B),$$

and thus  $\int_S (dh \circ \Phi)(B) = 0$ , for all  $h \in C_{\text{lin}}^\infty(L^*)$ , cf. (8). This implies that  $B$  is tangential to  $\tilde{N} := \Phi(|\Lambda|_S^*)$ . To see this, consider  $\gamma: |\Lambda|_S^* \rightarrow \text{ann}(T\tilde{N}) \subseteq T^*L^*$  satisfying  $\pi^{T^*L^*} \circ \gamma = \Phi$  and  $(T\delta_t^{L^*})^* \circ \gamma \circ \delta_t^{|\Lambda|_S^*} = \gamma$  for all  $t$ . Since  $\Phi$  is a fiberwise linear embedding, there exists  $h \in C_{\text{lin}}^\infty(L^*)$  with  $h \circ \Phi = 0$  and  $\gamma = dh \circ \Phi$ , hence  $\int_S \gamma(B) = 0$  for all such  $\gamma$ . We conclude that  $B$  is tangential to  $\tilde{N}$ . Consequently, there exists a vector field  $W$  on the total space of  $|\Lambda|_S^*$  such that  $B = T\Phi \circ W$ . Clearly,  $\delta_t^*W = W$ , for all  $t \in \mathbb{R}$ . Using Lemma 2.8 below, we conclude that there exists  $Z \in \mathfrak{X}(S)$  such that  $W = \zeta_Z^{|\Lambda|_S^*}$ . In view of (20), we obtain  $B = \zeta_Z^\mathcal{E}(\Phi)$ . This completes the proof of (55).  $\square$

**Lemma 2.7** *Suppose  $\beta \in \Omega^1(|\Lambda|_S^*)$  is a 1-form on the total space of  $|\Lambda|_S^*$ , such that  $\delta_t^*\beta = t\beta$  for all  $t \in \mathbb{R}$  and*

$$\int_S \beta \left( \zeta_Z^{|\Lambda|_S^*} \right) = 0 \tag{56}$$

for all  $Z \in \mathfrak{X}(S)$ .<sup>4</sup> Then  $\beta = di_R\beta$  where  $R = \frac{\partial}{\partial t}|_{t=1}\delta_t \in \mathfrak{X}(|\Lambda|_S^*)$  denotes the radial vector field, i.e., the fundamental vector field of the action  $\delta_t$ .

**Proof** We fix a volume density  $\mu$  on  $S$  and identify  $|\Lambda|_S^* \cong S \times \mathbb{R}$  correspondingly. The two canonical projections shall be denoted by  $p: S \times \mathbb{R} \rightarrow S$  and  $t: S \times \mathbb{R} \rightarrow \mathbb{R}$ , respectively. The radial vector field becomes  $R = t\partial_t \in \mathfrak{X}(S \times \mathbb{R})$ . By homogeneity,  $\beta \in \Omega^1(S \times \mathbb{R})$  can be written in the form  $\beta = tp^*B + (p^*b)dt$  where  $B \in \Omega^1(S)$  and  $b \in C^\infty(S, \mathbb{R})$ . Moreover, for  $Z \in \mathfrak{X}(S)$ , we have

$$\zeta_Z^{|\Lambda|_S^*} = p^*Z + (p^*\text{div}(Z))t\partial_t, \tag{57}$$

where  $L_Z\mu =: \text{div}(Z)\mu$  and  $p^*Z \in \mathfrak{X}(S \times \mathbb{R})$  denotes the vector field which projects to  $Z$  on  $S$  and 0 on  $\mathbb{R}$ . Consequently,

$$\beta \left( \zeta_Z^{|\Lambda|_S^*} \right) = tp^*(i_ZB + b \text{div}(Z)).$$

Using Stokes' theorem, we obtain

$$\int_S \beta \left( \zeta_Z^{|\Lambda|_S^*} \right) = \int_S (i_ZB + b \text{div}(Z))\mu = \int_S (B - db) \wedge i_Z\mu.$$

In view of the assumption (56), we conclude that  $B = db$ , whence  $di_R\beta = d(tp^*b) = tp^*db + (p^*b)dt = tp^*B + (p^*b)dt = \beta$ , the desired relation.  $\square$

**Lemma 2.8** *Suppose  $W$  is a vector field on the total space of  $|\Lambda|_S^*$ , such that  $\delta_t^*W = W$  for all  $t \in \mathbb{R}$  and such that*

$$\int_S dh(W) = 0, \tag{58}$$

<sup>4</sup> Note that the integrand is a fiberwise linear function on the total space of  $|\Lambda|_S^*$ , which may be regarded as a section of  $|\Lambda|_S$  and integrated over  $S$ .

for all smooth, fiberwise linear functions  $h$  on the total space of  $|\Lambda|_S^*$ .<sup>5</sup> Then  $W$  is a fundamental vector field of the natural  $\text{Diff}(S)$  action on  $|\Lambda|_S^*$ , i.e., there exists  $Z \in \mathfrak{X}(S)$  such that  $W = \zeta_Z^{|\Lambda|_S^*}$ .

**Proof** As in the proof of the preceding lemma we fix a volume density  $\mu$  on  $S$ , we identify  $|\Lambda|_S^* \cong S \times \mathbb{R}$  correspondingly, and we denote the two canonical projections by  $p: S \times \mathbb{R} \rightarrow S$  and  $t: S \times \mathbb{R} \rightarrow \mathbb{R}$ . Hence, the vector field  $W$  can be written in the form  $W = p^*Z + (p^*w)t\partial_t$ , where  $Z \in \mathfrak{X}(S)$  and  $w \in C^\infty(S)$ . Every function  $\bar{h} \in C^\infty(S)$  gives rise to a fiberwise linear function  $h := tp^*\bar{h}$  on the total space of  $|\Lambda|_S^*$ . Then

$$dh(W) = tp^*(\bar{h}w + d\bar{h}(Z))$$

and Stokes' theorem yields

$$\int_S dh(W) = \int_S (\bar{h}w + d\bar{h}(Z))\mu = \int_S \bar{h}(w - \text{div}(Z))\mu.$$

Using the assumption (58), we conclude that  $w = \text{div}(Z)$ . Consequently, see (57), we obtain  $W = p^*Z + (p^*w)t\partial_t = p^*Z + (p^*\text{div}(Z))t\partial_t = \zeta_Z^{|\Lambda|_S^*}$ . □

**Remark 2.9** Let us give a more explicit description of the EPContact dual pair if the contact structure is described by a contact form,  $\xi = \ker \alpha$ , and a volume density  $\mu$  on  $S$  has been fixed. We have already pointed out before, see footnote 3, that these choices provide an identification of the non-linear Stiefel manifold  $\mathcal{E}$  with  $\text{Emb}(S, P) \times C^\infty(S, \mathbb{R}^\times)$ . Via this identification, the actions of  $\text{Diff}(P, \xi)$  from the left and  $\text{Diff}(S)$  from the right are

$$\Psi_g^\mathcal{E}(\varphi, h) = \left( g \circ \varphi, \left( \frac{g^*\alpha}{\alpha} \circ \varphi \right) h \right) \quad \text{and} \quad \psi_f^\mathcal{E}(\varphi, h) = \left( \varphi \circ f, (h \circ f) \frac{f^*\mu}{\mu} \right),$$

where  $g \in \text{Diff}(P, \xi)$ ,  $f \in \text{Diff}(S)$ , and  $(\varphi, h) \in \text{Emb}(S, P) \times C^\infty(S, \mathbb{R}^\times)$ . Using the identification  $\mathfrak{X}(P, \xi) = C^\infty(P)$  provided by the contact form  $\alpha$ , the EPContact dual pair (52) becomes

$$C^\infty(P)^* \xleftarrow{J_L^\mathcal{E}} \text{Emb}(S, P) \times C^\infty(S, \mathbb{R}^\times) \xrightarrow{J_R^\mathcal{E}} \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^* \tag{59}$$

with moment maps

$$J_L^\mathcal{E}(\varphi, h) = \varphi_*(h\mu) \quad \text{and} \quad J_R^\mathcal{E}(\varphi, h) = \varphi^*\alpha \otimes h\mu. \tag{60}$$

This follows readily from the formulas provided in Remarks 2.2 and 2.3.

In view of Theorem 2.6 one might expect [2, 12] that the contact group acts locally transitive on the level sets of  $J_R^\mathcal{E}$ . This is indeed the case, see Theorem 3.5 in the subsequent section. Moreover, one might expect that a coadjoint orbit  $\mathcal{O} \subseteq \mathfrak{X}(S)^*$  gives rise to a reduced symplectic structure on the quotient  $(J_R^\mathcal{E})^{-1}(\mathcal{O})/\text{Diff}(S)$  which is equivariantly symplectomorphic to a coadjoint orbit of  $\text{Diff}_c(P, \xi)$  via the symplectomorphism induced by the moment map  $J_L^\mathcal{E}$ . Below we will see that this can be made rigorous for coadjoint orbits corresponding to isotropic embeddings, see Theorem 4.10.

<sup>5</sup> Note that the integrand is a fiberwise linear function on the total space of  $|\Lambda|_S^*$ , which can be regarded as a section of  $|\Lambda|_S$  and integrated over  $S$ .

### 3 Level sets of the right moment map

In this section we will show that each level set of the right moment map

$$J_R^\mathcal{E} : \mathcal{E} \rightarrow \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^*$$

is a smooth splitting Fréchet submanifold in  $\mathcal{E}$ . Furthermore, we will see that the contact group acts locally transitive on each level set. More precisely, we will show that this action admits local smooth sections and the isotropy groups are Lie subgroups. Hence, (unions of) connected components of these level sets may be regarded as homogeneous spaces of the contact group. These results are summarized in Theorem 3.5 below.

A similar transitivity statement has been established in [9, Proposition 5.5] using methods quite different from the approach presented here.

Let  $\pi^{J^1L} : J^1L \rightarrow P$  denote the 1-jet bundle of sections of  $L$ . Recall that each section  $h \in \Gamma^\infty(L)$  gives rise to a section  $j^1h \in \Gamma^\infty(J^1L)$ . We equip the total space of  $J^1L$  with the contact structure uniquely characterized by the following property: A section  $s \in \Gamma^\infty(J^1L)$  has isotropic image iff there exists  $h \in \Gamma^\infty(L)$  such that  $s = j^1h$ .<sup>6</sup> In this case  $h = \pi_L^{J^1L} \circ s$ , where  $\pi_L^{J^1L} : J^1L \rightarrow L$  denotes the natural projection.

Consider the line bundle  $p : \text{hom}(p_1^*L, p_2^*L) \rightarrow P \times P$  where  $p_1, p_2 : P \times P \rightarrow P$  denote the two canonical projections. We let  $\mathcal{P} := \text{isom}(p_1^*L, p_2^*L)$  denote the open subset of fiberwise invertible maps. We equip the total space of  $\mathcal{P}$  with the contact structure

$$\xi_a^\mathcal{P} := \{A \in T_a\mathcal{P} \mid a((T_a(p_1 \circ p)A) \bmod \xi_{(p_1 \circ p)(a)}) = (T_a(p_2 \circ p)A) \bmod \xi_{(p_2 \circ p)(a)}\} \quad (61)$$

where  $a \in \mathcal{P}$ .<sup>7</sup> Note that a diffeomorphism  $g \in \text{Diff}(P)$  is contact if and only if there exists a smooth map  $a : P \rightarrow \mathcal{P}$  with isotropic image satisfying  $p_1 \circ p \circ a = \text{id}$  and  $p_2 \circ p \circ a = g$ . Moreover, in this case  $\Psi_{g,x}^L = a(x)$  in  $\text{hom}(L_x, L_{g(x)})$ , for all  $x \in P$ . Here  $\Psi_{g,x}^L$  denotes the restriction of  $\Psi_g^L$  to the fiber  $L_x$ .

It is well known [24, Corollary in Section 1] that there exists a contact diffeomorphism

$$J^1L \supseteq V \xrightarrow{\Xi} U \subseteq \mathcal{P} \quad (62)$$

from an open neighborhood  $V$  of the zero section  $P \subseteq J^1L$  onto an open neighborhood  $U$  of the diagonal  $P \subseteq \mathcal{P}$  intertwining the contact structure obtained by restriction from  $J^1L$  with the contact structure obtained by restriction from  $\mathcal{P}$ . Moreover, for all  $x \in P$ , we have

$$\Xi(0_x) = \text{id}_{L_x}. \quad (63)$$

It is also well known, see [20, Theorem 43.19] for the coorientable case, that the map

$$\Gamma_c^\infty(L) \supseteq \mathcal{W} \xrightarrow{F} \text{Diff}_c(P, \xi), \quad F(h) := p_2 \circ p \circ \Xi \circ j^1h \circ (p_1 \circ p \circ \Xi \circ j^1h)^{-1}, \quad (64)$$

provides a chart for the Lie group  $\text{Diff}_c(P, \xi)$  at the identity, which is known as Lychagin chart. Here  $\mathcal{W}$  is a  $C^\infty$ -open neighborhood of zero such that, for each  $h \in \mathcal{W}$ , the image of

<sup>6</sup> If  $L \cong P \times \mathbb{R}$  is a trivialization of  $L$ , then  $J^1L \cong T^*P \times \mathbb{R}$ , and the contact structure can be described by the contact form  $p^*\theta - dt$ , where  $\theta$  denotes the canonical 1-form on  $T^*P$ , while  $p : T^*P \times \mathbb{R} \rightarrow T^*P$  and  $t : T^*P \times \mathbb{R} \rightarrow \mathbb{R}$  denote the canonical projections.

<sup>7</sup> If  $\xi = \ker \alpha$ , and  $\mathcal{P} \cong P \times P \times (\mathbb{R}^\times)$  denotes the corresponding trivialization, then the contact structure can be described by the contact form  $tp_1^*\alpha - p_2^*\alpha$  on  $P \times P \times (\mathbb{R}^\times)$ .

$j^1h$  is contained in  $V$  and  $p_1 \circ p \circ \Xi \circ j^1h$  as well as  $p_2 \circ p \circ \Xi \circ j^1h$  are diffeomorphisms of  $P$ . Clearly,  $F(0) = \text{id}_P$ , see (63). Moreover, for  $h \in \mathcal{W}$  and  $x \in P$ , we have

$$\Psi_{F(h),x}^L = \left( \Xi \circ j^1h \circ (p_1 \circ p \circ \Xi \circ j^1h)^{-1} \right)(x) \tag{65}$$

in  $\text{hom}(L_x, L_{F(h)(x)})$ . In particular,

$$j_x^1h = 0 \iff F(h)(x) = x \text{ and } \Psi_{F(h),x}^L = \text{id}_{L_x}. \tag{66}$$

**Lemma 3.1** *For  $\Phi \in \mathcal{E}$ , the isotropy subgroup*

$$\text{Diff}_c(P, \xi; \Phi) = \{g \in \text{Diff}_c(P, \xi) : \Psi_g^\mathcal{E}(\Phi) = \Phi\}$$

*is a splitting Lie subgroup of  $\text{Diff}_c(P, \xi)$ .*

**Proof** Put  $\varphi = \pi^\mathcal{E}(\Phi) \in \text{Emb}(S, P)$  and  $N := \varphi(S)$ . For the chart  $F$  in (64) we obtain

$$F^{-1}(\text{Diff}_c(P, \xi; \Phi)) = \{h \in \Gamma_c^\infty(L) \mid \forall x \in N : j_x^1h = 0\} \cap \mathcal{W},$$

see (66) and (18). Since  $N$  is a closed submanifold in  $P$ , the linear space on the right hand side admits a linear complement in  $\Gamma_c^\infty(L)$ . To construct such a complement, let  $\pi^W : W \rightarrow N$  denote the normal bundle of  $N$ , where  $W = TP|_N/TN$ ; fix a tubular neighborhood  $W \subseteq P$  of  $N$  such that  $N$  corresponds to the zero section in  $W$ ; and choose an isomorphism of line bundles  $L|_W \cong (\pi^W)^*L|_N$ . This provides a linear map

$$\Gamma^\infty(L|_N) \oplus \Gamma^\infty(L|_N \otimes W^*) \rightarrow \Gamma^\infty(L|_W), \tag{67}$$

by regarding sections of  $L|_N$  as  $\pi^W$ -fiberwise constant sections of  $L|_W$ , and by regarding sections of  $L|_N \otimes W^*$  as  $\pi^W$ -fiberwise linear sections of  $L|_W$ . Let  $\chi \in C_c^\infty(W, \mathbb{R})$  be a compactly supported bump function such that  $\chi \equiv 1$  in a neighborhood of the zero section. Multiplication with  $\chi$  and extension by zero provides a linear map  $\Gamma^\infty(L|_W) \rightarrow \Gamma_c^\infty(L)$ . Composing this with (67), we obtain a linear map we will denote by

$$\chi : \Gamma^\infty(L|_N) \oplus \Gamma^\infty(L|_N \otimes W^*) \rightarrow \Gamma_c^\infty(L). \tag{68}$$

The image of  $\chi$  provides a linear complement of  $\{h \in \Gamma_c^\infty(L) \mid \forall x \in N : j_x^1h = 0\}$  in  $\Gamma_c^\infty(L)$ . Hence,  $\text{Diff}_c(P, \xi; \Phi)$  is a splitting Lie subgroup of  $\text{Diff}_c(P, \xi)$ .  $\square$

Suppose  $\Phi_1, \Phi_2 \in \mathcal{M}$ , and write  $\varphi_i = \pi^\mathcal{M}(\Phi_i) \in C^\infty(S, P)$ . For  $x \in S$  consider the restrictions to the fibers,  $\Phi_{1,x} : |\Lambda|_{S,x}^* \rightarrow L_{\varphi_1(x)}^*$  and  $\Phi_{2,x} : |\Lambda|_{S,x}^* \rightarrow L_{\varphi_2(x)}^*$ , and define a smooth map  $G(\Phi_1, \Phi_2) : S \rightarrow \mathcal{P}$  by

$$G(\Phi_1, \Phi_2)(x) = (\Phi_{1,x} \circ \Phi_{2,x}^{-1})^* \in \text{hom}(L_{\varphi_1(x)}, L_{\varphi_2(x)}) \tag{69}$$

for  $x \in S$ . Clearly,

$$p_1 \circ p \circ G(\Phi_1, \Phi_2) = \varphi_1 \quad \text{and} \quad p_2 \circ p \circ G(\Phi_1, \Phi_2) = \varphi_2. \tag{70}$$

**Lemma 3.2** *The map  $G(\Phi_1, \Phi_2) : S \rightarrow \mathcal{P}$  has isotropic image iff  $J_R^\mathcal{M}(\Phi_1) = J_R^\mathcal{M}(\Phi_2)$ .*

**Proof** Suppose  $x \in S$ ,  $Z_x \in T_x S$ ,  $0 \neq \lambda_x \in |\Lambda|_{S,x}^*$ , and write  $a := G(\Phi_1, \Phi_2)(x)$ . Then:

$$\begin{aligned}
 T_x G(\Phi_1, \Phi_2) \cdot Z_x &\in \xi_a^{\mathcal{P}} \\
 &\stackrel{(61)}{\Leftrightarrow} a(T_a(p_1 \circ p)T_x G(\Phi_1, \Phi_2) \cdot Z_x \bmod \xi_{(p_1 \circ p)(a)}) \\
 &\quad = T_a(p_2 \circ p)T_x G(\Phi_1, \Phi_2) \cdot Z_x \bmod \xi_{(p_2 \circ p)(a)} \\
 &\stackrel{(70)}{\Leftrightarrow} a(T_x \varphi_1 \cdot Z_x \bmod \xi_{(p_1 \circ p)(a)}) = T_x \varphi_2 \cdot Z_x \bmod \xi_{(p_2 \circ p)(a)} \\
 &\stackrel{(69)}{\Leftrightarrow} \Phi_{1,x}^*(T_x \varphi_1 \cdot Z_x \bmod \xi_{(p_1 \circ p)(a)}) = \Phi_{2,x}^*(T_x \varphi_2 \cdot Z_x \bmod \xi_{(p_2 \circ p)(a)}) \\
 &\Leftrightarrow \lambda_x(\Phi_{1,x}^*(T_x \varphi_1 \cdot Z_x \bmod \xi_{(p_1 \circ p)(a)})) = \lambda_x(\Phi_{2,x}^*(T_x \varphi_2 \cdot Z_x \bmod \xi_{(p_2 \circ p)(a)})) \\
 &\Leftrightarrow \Phi_{1,x}(\lambda_x)(T_x \varphi_1 \cdot Z_x) = \Phi_{2,x}(\lambda_x)(T_x \varphi_2 \cdot Z_x) \\
 &\stackrel{(40)}{\Leftrightarrow} \lambda_x(J_R^{\mathcal{M}}(\Phi_1)(Z_x)) = \lambda_x(J_R^{\mathcal{M}}(\Phi_2)(Z_x)) \\
 &\Leftrightarrow J_R^{\mathcal{M}}(\Phi_1)(Z_x) = J_R^{\mathcal{M}}(\Phi_2)(Z_x)
 \end{aligned}$$

The lemma follows at once. □

For  $\rho \in \Omega^1(S, |\Lambda|_S)$  we let

$$\mathcal{E}^\rho := (J_R^\mathcal{E})^{-1}(\rho) = \{\Phi \in \mathcal{E} : J_R^\mathcal{E}(\Phi) = \rho\}$$

denote the corresponding level set of the moment map  $J_R^\mathcal{E} : \mathcal{E} \rightarrow \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^*$ .

**Lemma 3.3** *The level set  $\mathcal{E}^\rho$  is a smooth splitting Fréchet submanifold in  $\mathcal{E}$ , for each  $\rho \in \Omega^1(S, |\Lambda|_S)$ .*

**Proof** Fix  $\Phi_1 \in \mathcal{E}^\rho$ , put  $\varphi_1 := \pi^\mathcal{E}(\Phi_1) \in \text{Emb}(S, P)$ , and consider the submanifold  $N := \varphi_1(S)$  of  $P$ . Let  $\pi^W : W \rightarrow N$  denote its normal bundle,  $W := TP|_N/TN$ . Choose a tubular neighborhood  $W \subseteq P$  of  $N$  in  $P$  such that the zero section in  $W$  corresponds to  $N$ . As in the proof of Lemma 3.1, we fix an isomorphism of line bundles,

$$L|_W \cong (\pi^W)^*L|_N \tag{71}$$

and a compactly supported bump function  $\chi \in C_c^\infty(W, \mathbb{R})$  such that  $\chi \equiv 1$  on an open neighborhood  $\mathcal{X}$  of the zero section in  $W$ . The corresponding map (68) extends naturally to a linear map  $\tilde{\chi}$  such that the following diagram commutes:

$$\begin{array}{ccc}
 \Gamma_c^\infty(L) & \xrightarrow{j^1} & \Gamma_c^\infty(J^1L) \\
 \chi \uparrow & & \uparrow \tilde{\chi} \\
 \Gamma^\infty(L|_N) \oplus \Gamma^\infty(L|_N \otimes W^*) & \xrightarrow{j^1 \oplus \text{id}} & \Gamma^\infty(J^1(L|_N)) \oplus \Gamma^\infty(L|_N \otimes W^*)
 \end{array} \tag{72}$$

Here the component  $\Gamma^\infty(J^1(L|_N)) \rightarrow \Gamma_c^\infty(J^1L)$  of  $\tilde{\chi}$  is the tensorial map induced from the map  $\Gamma^\infty(L|_N) \rightarrow \Gamma_c^\infty(L)$  given by  $\pi^W$ -fiberwise extension and multiplication with  $\chi$ . The line bundle isomorphism in (71) also provides an isomorphism

$$\Gamma^\infty((J^1L)|_N) \cong \Gamma^\infty(J^1(L|_N)) \oplus \Gamma^\infty(L|_N \otimes W^*). \tag{73}$$

Using this isomorphism to replace the lower right corner in the diagram (72), we obtain linear maps  $\gamma$  and  $\Gamma^\infty((J^1L)|_N) \rightarrow \Gamma_c^\infty(J^1L)$ ,  $s \mapsto \tilde{s}$ , such that the following diagram

commutes:

$$\begin{array}{ccc}
 \Gamma_c^\infty(L) & \xrightarrow{j^1} & \Gamma_c^\infty(J^1L) & \begin{array}{c} \tilde{s} \\ \uparrow \\ \hline s \end{array} \\
 \uparrow x & & \uparrow & \\
 \Gamma^\infty(L|_N) \oplus \Gamma^\infty(L|_N \otimes W^*) & \xrightarrow{\gamma} & \Gamma^\infty((J^1L)|_N) & 
 \end{array} \tag{74}$$

For every  $v \in \Gamma^\infty(W)$  with  $v(N) \subseteq \mathcal{X}$  we obtain a linear isomorphism

$$\tilde{v}: \Gamma^\infty((J^1L)|_N) \rightarrow \Gamma^\infty(v^*(J^1L)), \quad \tilde{v}(s) := \tilde{s} \circ v. \tag{75}$$

Moreover,  $\tilde{v}$  and its inverse  $\tilde{v}^{-1}$  are given by first order differential operators depending smoothly on  $v$ . Furthermore, if  $v(N) \subseteq \mathcal{X}$  and  $s \in \Gamma^\infty((J^1L)|_N)$ , then

$$\tilde{s} \circ v \text{ has isotropic image in } J^1L \iff s \in \text{img}(\gamma), \tag{76}$$

since  $\tilde{s}$  is holonomic when restricted to fibers of  $\pi^W$ . Also note that  $\text{img}(\gamma)$  admits a closed complementary subspace in  $\Gamma^\infty((J^1L)|_N)$ . Indeed, the space of smooth sections in the kernel of the canonical projection  $J^1(L|_N) \rightarrow L|_N$  provides a closed complement for the image of  $j^1: \Gamma^\infty(L|_N) \rightarrow \Gamma^\infty(J^1(L|_N))$ . Taking the sum with  $\Gamma^\infty(L|_N \otimes W^*)$  and using (73), we obtain a complementary subspace of  $\text{img}(\gamma)$  in  $\Gamma^\infty((J^1L)|_N)$ .

Let  $\mathcal{V}$  denote the  $C^\infty$ -open neighborhood of zero in  $\Gamma^\infty((J^1L)|_N)$  consisting of all  $s \in \Gamma^\infty((J^1L)|_N)$  with the following five properties:

- (a) the image of  $\tilde{s}$  is contained in  $V$ , cf. (62),
- (b)  $p_1 \circ p \circ \Xi \circ \tilde{s}: P \rightarrow P$  is a diffeomorphism,
- (c)  $p_2 \circ p \circ \Xi \circ \tilde{s}: P \rightarrow P$  is a diffeomorphism,
- (d) the image of  $(p_1 \circ p \circ \Xi \circ \tilde{s})^{-1} \circ \varphi_1: S \rightarrow P$  is contained in  $\mathcal{X} \subseteq W$ , and
- (e)  $\psi_s := \pi^W \circ (p_1 \circ p \circ \Xi \circ \tilde{s})^{-1} \circ \varphi_1: S \rightarrow N$  is a diffeomorphism.

For  $s \in \mathcal{V}$  we define  $v_s := (p_1 \circ p \circ \Xi \circ \tilde{s})^{-1} \circ \varphi_1 \circ \psi_s^{-1} \in \Gamma^\infty(W)$ . Hence,

$$v_s \circ \psi_s = (p_1 \circ p \circ \Xi \circ \tilde{s})^{-1} \circ \varphi_1. \tag{77}$$

We will next show that the following map is a diffeomorphism

$$\begin{aligned}
 \Gamma^\infty((J^1L)|_N) \supseteq \mathcal{V} \rightarrow \mathcal{U} \subseteq \{G \in C^\infty(S, \mathcal{P}) : p_1 \circ p \circ G = \varphi_1\} \\
 s \mapsto G_s := \Xi \circ \tilde{s} \circ (p_1 \circ p \circ \Xi \circ \tilde{s})^{-1} \circ \varphi_1
 \end{aligned} \tag{78}$$

from  $\mathcal{V}$  onto the  $C^\infty$ -open subset  $\mathcal{U}$  in  $\{G \in C^\infty(S, \mathcal{P}) : p_1 \circ p \circ G = \varphi_1\}$  consisting of all  $G \in C^\infty(S, \mathcal{P})$  with the following five properties:

- (a)  $p_1 \circ p \circ G = \varphi_1$ ,
- (b) the image of  $G$  is contained in  $U$ , cf. (62),
- (c) the image of  $\pi^{J^1L} \circ \Xi^{-1} \circ G: S \rightarrow P$  is contained in  $\mathcal{X} \subseteq W$ ,
- (d)  $\psi_G := \pi^W \circ \pi^{J^1L} \circ \Xi^{-1} \circ G: S \rightarrow N$  is a diffeomorphism, and
- (e)  $s_G := \tilde{v}_G^{-1}(\Xi^{-1} \circ G \circ \psi_G^{-1}) \in \mathcal{V}$ , where  $v_G := \pi^{J^1L} \circ \Xi^{-1} \circ G \circ \psi_G^{-1} \in \Gamma^\infty(W)$ .

To see that (78) is a diffeomorphism, let  $s \in \mathcal{V}$  and observe that (77) and (78) yield

$$G_s = \Xi \circ \tilde{s} \circ v_s \circ \psi_s \tag{79}$$

as well as  $\psi_{G_s} = \psi_s$  and  $v_{G_s} = v_s$ . Hence,  $\Xi^{-1} \circ G_s \circ \psi_{G_s}^{-1} = \tilde{s} \circ v_{G_s}$  and (75) gives

$$s = \tilde{v}_{G_s}^{-1}(\Xi^{-1} \circ G_s \circ \psi_{G_s}^{-1}). \tag{80}$$

We conclude that  $G_s \in \mathcal{U}$  and  $s_{G_s} = s$ , for all  $s \in \mathcal{V}$ , see (j). This shows that the map  $\mathcal{U} \rightarrow \mathcal{V}$ ,  $G \mapsto s_G$ , is left inverse to the map (78). To show that it is right inverse too, consider  $G \in \mathcal{U}$  and note that (75) and (j) yield  $\tilde{s}_G \circ \nu_G = \Xi^{-1} \circ G \circ \psi_G^{-1}$ . Hence,

$$\Xi \circ \tilde{s}_G \circ \nu_G \circ \psi_G = G.$$

Composing with  $p_1 \circ p$  and using (b), (f) we obtain,

$$\nu_G \circ \psi_G = (p_1 \circ p \circ \Xi \circ \tilde{s}_G)^{-1} \circ \varphi_1.$$

Combining the latter two equations, we get

$$\Xi \circ \tilde{s}_G \circ (p_1 \circ p \circ \Xi \circ \tilde{s}_G)^{-1} \circ \varphi_1 = G.$$

In other words,  $G_{s_G} = G$ , for all  $G \in \mathcal{U}$ , cf. (78). This shows that (78) is indeed a diffeomorphism. Using (76), (79), and the fact that  $\Xi$  is a contact diffeomorphism we find

$$G_s \text{ has isotropic image in } \mathcal{P} \iff s \in \text{img}(\gamma). \tag{81}$$

The construction in (69), cf. also (70), provides a diffeomorphism

$$\mathcal{M} \cong \{G \in C^\infty(S, \mathcal{P}) : p_1 \circ p \circ G = \varphi_1\}, \quad \Phi_2 \mapsto G(\Phi_1, \Phi_2).$$

Combining this with the diffeomorphism in (78), we see that the map

$$\Gamma^\infty((J^1L)|_N) \supseteq \mathcal{V} \rightarrow \mathcal{E}, \quad s \mapsto \Phi_s, \tag{82}$$

characterized by  $G(\Phi_1, \Phi_s) = G_s$ , is a diffeomorphism from  $\mathcal{V}$  onto a  $C^\infty$ -open neighborhood of  $\Phi_1$  in  $\mathcal{E}$ . Combining Lemma 3.2 with (81) and  $J_R^\mathcal{E}(\Phi_1) = \rho$ , we obtain

$$J_R^\mathcal{E}(\Phi_s) = \rho \iff s \in \text{img}(\gamma). \tag{83}$$

This shows that (82) is a submanifold chart for  $\mathcal{E}^\rho$  in  $\mathcal{E}$ , centered at  $\Phi_1$ . □

**Lemma 3.4** *The action of  $\text{Diff}_c(P, \xi)$  on the level set  $\mathcal{E}^\rho$  admits local smooth sections, for each  $\rho \in \Omega^1(S, |\Lambda|_S)$ .*

**Proof** We continue to use the notation set up in the proof of Lemma 3.3. Using the commutativity of the diagram (74) we obtain a linear map  $\text{img}(\gamma) \rightarrow \Gamma_c^\infty(L)$ ,  $s \mapsto h_s$ , such that  $j^1h_s = \tilde{s}$ , for all  $s \in \text{img}(\gamma)$ . Using (65), (a), (b), and (c), see also (64), we find  $h_s \in \mathcal{W}$  and

$$\Psi_{F(h_s), \varphi_1(x)}^L = \left( \Xi \circ \tilde{s} \circ (p_1 \circ p \circ \Xi \circ \tilde{s})^{-1} \right) (\varphi_1(x))$$

in  $\text{hom}(L_{\varphi_1(x)}, L_{F(h_s)(\varphi_1(x))})$ , for all  $x \in S$  and  $s \in \text{img}(\gamma) \cap \mathcal{V}$ . Hence, see (78) and (82),

$$\Psi_{F(h_s), \varphi_1(x)}^L = G(\Phi_1, \Phi_s)(x).$$

Using (69) we obtain

$$\Phi_{s,x}^* \circ \Psi_{F(h_s), \varphi_1(x)}^L = \Phi_{1,x}^*,$$

and dualizing yields

$$\Psi_{F(h_s), \varphi_1(x)}^{L*} \circ \Phi_{1,x} = \Phi_{s,x}$$

for all  $x \in S$  and  $s \in \text{img}(\gamma) \cap \mathcal{V}$ . Hence, in view of (18), we get

$$\Psi_{F(h_s)}^\mathcal{E}(\Phi_1) = \Phi_s,$$

for all  $s \in \text{img}(\gamma) \cap \mathcal{V}$ . As (82) restricts to a chart,  $\text{img}(\gamma) \cap \mathcal{V} \rightarrow \mathcal{E}^\rho$ , for the manifold  $\mathcal{E}^\rho$ , the lemma follows. □

Combining Lemmas 3.1, 3.3, and 3.4 we obtain the following result:

**Theorem 3.5** *Suppose  $\rho \in \Omega^1(S, |\Lambda|_S)$ . Then the level set  $\mathcal{E}^\rho$  is a smooth splitting Fréchet submanifold of  $\mathcal{E}$ . For  $\Phi \in \mathcal{E}^\rho$ , the isotropy subgroup  $\text{Diff}_c(P, \xi; \Phi)$  is a closed Lie subgroup of  $\text{Diff}_c(P, \xi)$ . Moreover, the map provided by the action,  $\text{Diff}_c(P, \xi) \rightarrow \mathcal{E}^\rho, g \mapsto \Psi_g^\xi(\Phi)$ , admits a local smooth right inverse defined in a neighborhood of  $\Phi$  in  $\mathcal{E}^\rho$ . In particular, the group  $\text{Diff}_c(P, \xi)$  acts locally and infinitesimally transitive on  $\mathcal{E}^\rho$ , and the  $\text{Diff}_c(P, \xi)$ -orbit of  $\Phi$  is open and closed in  $\mathcal{E}^\rho$ . Denoting this orbit by  $\mathcal{E}_\Phi^\rho$ , the map  $\text{Diff}_c(P, \xi) \rightarrow \mathcal{E}_\Phi^\rho$  is a smooth principal bundle with structure group  $\text{Diff}_c(P, \xi; \Phi)$ . Hence,*

$$\mathcal{E}_\Phi^\rho = \text{Diff}_c(P, \xi) / \text{Diff}_c(P, \xi; \Phi)$$

may be regarded as a homogeneous space.

### 4 Weighted non-linear Grassmannians

We continue to consider a manifold  $P$  endowed with a contact structure  $\xi$ , and a closed manifold  $S$ . Recall that the  $\text{Diff}(S)$  action is free on the non-linear Stiefel manifold  $\mathcal{E}$  of weighted embeddings. We will now factor out this action and consider the corresponding space  $\mathcal{G} = \mathcal{E} / \text{Diff}(S)$  of unparametrized weighted submanifolds of  $P$ .

#### 4.1 Principal bundles over non-linear Grassmannians

Let  $\text{Gr}_S(P)$  denote the non-linear Grassmannian of all smooth submanifolds of  $P$  which are diffeomorphic to  $S$ . It is well known that  $\text{Gr}_S(P)$  can be equipped with the structure of a Fréchet manifold such that the canonical map  $\text{Emb}(S, P) \rightarrow \text{Gr}_S(P)$  becomes a principal bundle with structure group  $\text{Diff}(S)$ .

Consider the space of weighted submanifolds

$$\mathcal{G} := \left\{ (N, \gamma) \mid \begin{array}{l} N \in \text{Gr}_S(P) \text{ and} \\ \gamma \in \Gamma^\infty(|\Lambda|_N \otimes L|_N^*) \text{ a nowhere vanishing section} \end{array} \right\}. \tag{84}$$

The  $\text{Diff}(P, \xi)$ -actions on  $P$  and on  $L^*$  induce a left action on  $\mathcal{G}$ . For  $g \in \text{Diff}(P, \xi)$  we let  $\Psi_g^\mathcal{G}$  denote the corresponding action on  $\mathcal{G}$ , that is,  $\Psi_g^\mathcal{G}(N, \gamma) = (g(N), g_*\gamma)$ .

**Remark 4.1** If  $\xi = \ker \alpha$ , then the contact form  $\alpha$  provides a trivialization  $L^* \cong P \times \mathbb{R}$  which permits to identify  $\mathcal{G}$  with a weighted non-linear Grassmannian,

$$\mathcal{G} \cong \text{Gr}_S^{\text{wt}}(P) := \{ (N, \nu) \mid N \in \text{Gr}_S(P) \text{ and } \nu \in \Gamma^\infty(|\Lambda|_N \setminus N) \}, \tag{85}$$

by identifying  $(N, \nu)$  with  $(N, \nu \otimes \alpha|_N) \in \mathcal{G}$ . The weighted Grassmannian can be equipped with a smooth structure such that the canonical forgetful map  $\text{Gr}_S^{\text{wt}}(P) \rightarrow \text{Gr}_S(P)$  is a smooth fiber bundle. Indeed, it can be canonically identified with the bundle associated to the principal fiber bundle  $\text{Emb}(S, P) \rightarrow \text{Gr}_S(P)$  via the  $\text{Diff}(S)$ -action on the space  $\Gamma^\infty(|\Lambda|_S \setminus S)$  of volume densities on  $S$ . Note that the induced smooth structure on  $\mathcal{G}$  does not depend on the contact form  $\alpha$  for  $\xi$ . Via the identification (85), the  $\text{Diff}(P, \xi)$ -action becomes

$$\Psi_g^\mathcal{G}(N, \nu) = \left( g(N), \frac{g_*\alpha}{\alpha}|_{g(N)} g_*\nu \right), \tag{86}$$

where  $g \in \text{Diff}(P, \xi)$  and  $(N, \nu) \in \text{Gr}_S^{\text{wt}}(P)$ . Indeed,  $g_*(\nu \otimes \alpha|_N) = \frac{g_*\alpha}{\alpha}|_{g(N)} g_*\nu \otimes \alpha|_{g(N)}$ .

The space  $\mathcal{G}$  in (84) can be equipped with the structure of a smooth manifold such that the canonical forgetful map

$$\pi^{\mathcal{G}} : \mathcal{G} \rightarrow \text{Gr}_S(P)$$

becomes a smooth fiber bundle with typical fiber  $\Gamma^\infty(|\Lambda|_S \setminus S)$ . Indeed, if  $(N, \gamma) \in \mathcal{G}$ , then locally around  $N$ , the contact structure on  $P$  is coorientable and can be described by a contact form. We can therefore use Remark 4.1 to equip  $\mathcal{G}$  with a smooth structure. In view of (86) the  $\text{Diff}_c(P, \xi)$ -action on  $\mathcal{G}$  is smooth.

To an element  $\Phi \in \mathcal{E} = \text{Emb}_{\text{lin}}(|\Lambda|_S^*, L^*)$  over the embedding  $\varphi = \pi^{\mathcal{E}}(\Phi) \in \text{Emb}(S, P)$  we associate a pair  $(N, \gamma) \in \mathcal{G}$  in the following way:  $N = \varphi(S)$  and  $\gamma$  is the composition of  $\Phi$  (corestricted to  $L^*|_N$ ) with the isomorphism  $|\Lambda|_N^* : |\Lambda|_N^* \rightarrow |\Lambda|_S^*$  induced by the diffeomorphism  $\varphi : S \rightarrow N$ . It is easy to see that the map  $q : \mathcal{E} \rightarrow \mathcal{G}$ , given by  $q(\Phi) = (N, \gamma)$ , is a smooth principal bundle with structure group  $\text{Diff}(S)$ . We summarize this in the following  $\text{Diff}(P, \xi)$ -equivariant commutative diagram:

$$\begin{CD} \mathcal{E} @>\pi^{\mathcal{E}}>> \text{Emb}(S, P) \\ @VqVV @VVV \\ \mathcal{G} @>\pi^{\mathcal{G}}>> \text{Gr}_S(P) \end{CD} \tag{87}$$

By  $\text{Diff}(S)$  invariance, see Proposition 2.4(a), the moment map  $J_L^{\mathcal{E}}$  descends to a smooth map

$$J_L^{\mathcal{G}} : \mathcal{G} \rightarrow \mathfrak{X}(P, \xi)^*, \quad J_L^{\mathcal{G}} \circ q = J_L^{\mathcal{E}}. \tag{88}$$

In view of (49) we have the explicit formula

$$\langle J_L^{\mathcal{G}}(N, \gamma), X \rangle = \int_N \gamma(X|_N), \tag{89}$$

where  $(N, \gamma) \in \mathcal{G}$  and  $X \in \mathfrak{X}(P, \xi)$ . On the right hand side  $X$  is regarded as a section of  $L$ , see (8), restricted to  $N$  and contracted with  $\gamma$  to produce a density on  $N$  which can be integrated.<sup>8</sup>

**Proposition 4.2** *The following assertions hold true:*

- (a) *The map  $J_L^{\mathcal{G}} : \mathcal{G} \rightarrow \mathfrak{X}(P, \xi)^*$  is a  $\text{Diff}(P, \xi)$ -equivariant injective immersion.*
- (b) *We have  $\text{Diff}(P, \xi; (N, \gamma)) = \text{Diff}(P, \xi; J_L^{\mathcal{G}}(N, \gamma))$ , where the left hand side denotes the isotropy group of  $(N, \gamma) \in \mathcal{G}$  and the right hand side denotes the isotropy group of  $J_L^{\mathcal{G}}(N, \gamma) \in \mathfrak{X}(P, \xi)^*$  for the coadjoint action.*
- (c) *The group  $\text{Diff}(S)$  acts freely and transitively on level sets of  $J_L^{\mathcal{E}} : \mathcal{E} \rightarrow \mathfrak{X}(P, \xi)^*$ .*

**Proof** In view of Proposition 2.4(a), the smooth map  $J_L^{\mathcal{G}}$  is  $\text{Diff}(P, \xi)$ -equivariant. It follows from the dual pair symplectic orthogonality condition (55) that  $J_L^{\mathcal{G}}$  is immersive. To check injectivity, suppose  $(N_1, \gamma_1)$  and  $(N_2, \gamma_2)$  are two elements in  $\mathcal{G}$  such that  $J_L^{\mathcal{G}}(N_1, \gamma_1) = J_L^{\mathcal{G}}(N_2, \gamma_2)$ . Since  $\gamma_i$  is nowhere vanishing, we have  $\text{supp}(J_L^{\mathcal{G}}(N_i, \gamma_i)) = N_i$ ,

<sup>8</sup> Using a contact form  $\alpha$  to identify  $\mathcal{G} \cong \text{Gr}_S^{\text{wt}}(P)$  as in Remark 4.1, the map (89) is simply

$$\langle J_L^{\mathcal{G}}(N, \nu), X \rangle = \int_N \alpha(X)|_N \nu.$$

see (89), whence  $N_1 = N_2$ . Assume, for the sake of contradiction,  $\gamma_1 \neq \gamma_2$ . Then there exists  $\bar{X} \in \Gamma^\infty(L|_N)$  such that  $\langle \gamma_1, \bar{X} \rangle \neq \langle \gamma_2, \bar{X} \rangle$  with respect to the canonical pairing between  $\Gamma^\infty(|\Lambda|_N \otimes L|_N^*)$  and  $\Gamma^\infty(L|_N)$ . Extending  $\bar{X}$  to a global section  $X \in \Gamma^\infty(L)$ , we obtain  $\langle J_L^{\mathcal{G}}(N_1, \gamma_1), X \rangle \neq \langle J_L^{\mathcal{G}}(N_2, \gamma_2), X \rangle$  using (89). Since this contradicts our assumption  $J_L^{\mathcal{G}}(N_1, \gamma_1) = J_L^{\mathcal{G}}(N_2, \gamma_2)$ , we must have  $\gamma_1 = \gamma_2$ . This shows that  $J_L^{\mathcal{G}}$  is injective.

The assertion about the isotropy groups in (b) follows readily from the injectivity and equivariance of  $J_L^{\mathcal{G}}$ . The assertion in (c) also follows from the injectivity statement in (a), since the  $\text{Diff}(S)$ -action on the fibers of  $q: \mathcal{E} \rightarrow \mathcal{G}$  is free and transitive.  $\square$

### 4.2 Right leg symplectic reduction

In this section we study the spaces obtained by symplectic reduction for the right moment map  $J_R^{\mathcal{E}}: \mathcal{E} \rightarrow \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^*$ . For a 1-form density  $\rho \in \Omega^1(S, |\Lambda|_S)$  we put

$$\mathcal{G}^\rho := q(\mathcal{E}^\rho),$$

where  $\mathcal{E}^\rho = (J_R^{\mathcal{E}})^{-1}(\rho)$ . By  $\text{Diff}(S)$ -equivariance of  $J_R^{\mathcal{E}}$ , and since  $\text{Diff}(S)$  acts transitively on the fibers of  $q: \mathcal{E} \rightarrow \mathcal{G}$ , the definition of  $\mathcal{G}^\rho$  may be rephrased equivalently as

$$q^{-1}(\mathcal{G}^\rho) = \mathcal{E}^\rho \cdot \text{Diff}(S) = (J_R^{\mathcal{E}})^{-1}(\rho \cdot \text{Diff}(S)). \tag{90}$$

Here  $\rho \cdot \text{Diff}(S) \subseteq \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^*$  denotes the coadjoint orbit through  $\rho$ . Note that  $q$  induces a bijection

$$\mathcal{G}^\rho = (J_R^{\mathcal{E}})^{-1}(\rho \cdot \text{Diff}(S)) / \text{Diff}(S) = \mathcal{E}^\rho / \text{Diff}(S, \rho), \tag{91}$$

where  $\text{Diff}(S, \rho) = \{f \in \text{Diff}(S) : f^*\rho = \rho\}$  denotes the isotropy group of  $\rho$ . Thus,  $\mathcal{G}^\rho$  is the underlying set of the symplectically reduced space at  $\rho$ .

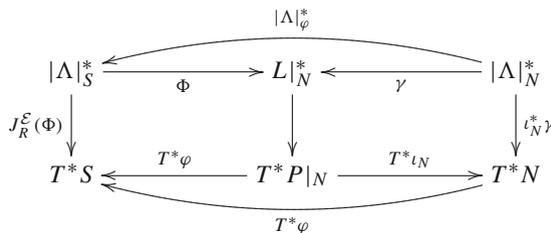
We have the following more explicit description of  $\mathcal{G}^\rho$ :

**Lemma 4.3** *For each  $\rho \in \Omega^1(S, |\Lambda|_S)$  we have*

$$\mathcal{G}^\rho = \{(N, \gamma) \in \mathcal{G} \mid (N, \iota_N^* \gamma) \cong (S, \rho)\}.$$

Here  $\iota_N: N \rightarrow P$  denotes the inclusion and the pull back  $\iota_N^* \gamma \in \Omega^1(N, |\Lambda|_N) = \Gamma^\infty(|\Lambda|_N \otimes T^*N)$  is defined as the composition  $|\Lambda|_N^* \xrightarrow{\gamma} L|_N^* \subseteq T^*P|_N \xrightarrow{T^*\iota_N} T^*N$ .<sup>9</sup>

**Proof** Consider  $\Phi \in \mathcal{E}$  over  $\varphi := \pi^{\mathcal{E}}(\Phi) \in \text{Emb}(S, P)$  and put  $(N, \gamma) := q(\Phi)$ . By definition of  $q$ , we have  $\varphi(S) = N$  and the ‘‘triangle’’ on the top of the following diagram commutes:



The left rectangle in this diagram commutes in view of the formula for  $J_R^{\mathcal{E}}$  in (50); the right rectangle commutes in view of the definition of  $\iota_N^* \gamma$ ; and the ‘‘triangle’’ at the bottom

<sup>9</sup> Because  $\gamma$  is nowhere vanishing, the kernel of  $\iota_N^* \gamma: TN \rightarrow |\Lambda|_N$  coincides with  $\xi|_N \cap TN$ .

commutes trivially. We conclude that  $(N, \iota_N^* \gamma) \cong (S, J_R^\mathcal{E}(\Phi))$  via  $\varphi$ . Hence,  $(N, \iota_N^* \gamma) \cong (S, \rho)$  iff  $(S, J_R^\mathcal{E}(\Phi)) \cong (S, \rho)$ . The latter, in turn, holds iff there exists  $f \in \text{Diff}(S)$  with  $J_R^\mathcal{E}(\Phi) = f^* \rho$ , i.e., iff  $\Phi \in (J_R^\mathcal{E})^{-1}(\rho \cdot \text{Diff}(S))$ . Using the description (90) of  $\mathcal{G}^\rho$  we obtain the lemma.  $\square$

**Remark 4.4** We have seen in Remark 4.1 that the choice of a contact form  $\alpha$  on  $P$  permits to identify  $\mathcal{G}$  with a weighted Grassmannian. Under this identification, the reduced space becomes

$$\mathcal{G}^\rho \cong \{(N, \nu) \in \text{Gr}_S^{\text{wt}}(P) : (N, \iota_N^* \alpha \otimes \nu) \cong (S, \rho)\}. \tag{92}$$

**Remark 4.5** A general fiber of the forgetful map  $\pi^\mathcal{G} : \mathcal{G} \rightarrow \text{Gr}_S(P)$  will intersect several of the spaces  $\mathcal{G}^\rho$ , for many different  $\rho$ . A notable exception are fibers over isotropic submanifolds, cf. (95) in the subsequent section.

**Remark 4.6** We do not expect  $\mathcal{G}^\rho$  to be a (smooth) submanifold in  $\mathcal{G}$  for general  $\rho$ .

### 4.3 Weighted isotropic non-linear Grassmannians

We will now specialize to the isotropic case,  $\rho = 0$ . Let us introduce the notation

$$\mathcal{E}^{\text{iso}} := (\pi^\mathcal{E})^{-1}(\text{Emb}^{\text{iso}}(S, P)) = (J_R^\mathcal{E})^{-1}(0) = \mathcal{E}^0, \tag{93}$$

where  $\text{Emb}^{\text{iso}}(S, P)$  denotes the space of isotropic embeddings, cf. (41), (47), or (60). This can equivalently be characterized as the elements in  $\mathcal{E} = \text{Emb}_{\text{lin}}(|\Lambda|_S^*, L^*)$  which restrict to isotropic embeddings  $|\Lambda|_S^* \rightarrow L^* \setminus P = M$ .<sup>10</sup>

Let  $\text{Gr}_S^{\text{iso}}(P)$  denote the space of isotropic submanifolds of type  $S$  and consider the space of all weighted isotropic submanifolds of type  $S$ ,

$$\begin{aligned} \mathcal{G}^{\text{iso}} &:= (\pi^\mathcal{G})^{-1}(\text{Gr}_S^{\text{iso}}(P)) \\ &= \{(N, \gamma) | N \in \text{Gr}_S^{\text{iso}}(P), \gamma \in \Gamma^\infty(|\Lambda|_N \otimes L^*|_N) \text{ nowhere vanishing}\}. \end{aligned} \tag{94}$$

In view of (87) and (93) we have  $q^{-1}(\mathcal{G}^{\text{iso}}) = \mathcal{E}^{\text{iso}} = (J_R^\mathcal{E})^{-1}(0)$ . Hence,  $\mathcal{G}^{\text{iso}}$  coincides with the reduced space  $\mathcal{G}^\rho$  for  $\rho = 0$ , i.e.,

$$\mathcal{G}^0 = (J_R^\mathcal{E})^{-1}(0) / \text{Diff}(S) = \mathcal{G}^{\text{iso}} = (\pi^\mathcal{G})^{-1}(\text{Gr}_S^{\text{iso}}(P)). \tag{95}$$

**Remark 4.7** If  $\alpha$  is a contact form for  $\xi$ , then isotropic submanifolds  $N$  are characterized by  $\iota_N^* \alpha = 0$  and the identification in Remark 4.4 becomes

$$\mathcal{G}^0 = \mathcal{G}^{\text{iso}} \cong \{(N, \nu) : N \in \text{Gr}_S^{\text{iso}}(P) \text{ and } \nu \in \Gamma^\infty(|\Lambda|_N \setminus N)\}. \tag{96}$$

**Lemma 4.8** *The subset  $\text{Gr}_S^{\text{iso}}(P)$  is a smooth splitting submanifold of  $\text{Gr}_S(P)$ .*

**Proof** This follows from the tubular neighborhood theorem for contact structures near isotropic submanifolds, see [14, Theorem 2.5.8] or [24, Theorem 1]. Since we were not able to locate this statement in the literature, we will sketch a proof below.

Suppose  $S \cong N \subseteq P$  is an isotropic submanifold, and let  $E := TN^\perp / TN$  denote its conformal symplectic normal bundle, see [14, Definition 2.5.3]. Using the relative Poincaré

<sup>10</sup> Using a volume density  $\mu$  on  $S$  to identify  $\mathcal{L} \cong C^\infty(S, L^*)$  as in Remark 2.2, the subset  $\mathcal{E}^{\text{iso}}$  corresponds to  $C^\infty(S, M) \cap (\pi^\mathcal{L})^{-1}(\text{Emb}^{\text{iso}}(S, P))$ . If moreover  $\xi = \ker \alpha$ , then the corresponding diffeomorphism  $\mathcal{L} \cong C^\infty(S, P) \times C^\infty(S)$  provides an identification  $\mathcal{E}^{\text{iso}} \cong \text{Emb}^{\text{iso}}(S, P) \times C^\infty(S, \mathbb{R}^\times)$ .

lemma, one easily constructs a 1-form  $\varepsilon$  on the total space of  $E$  such that (1)  $\varepsilon$  vanishes along the zero section; (2)  $i_X d\varepsilon = 0$  for every vector  $X$  tangent to the zero section; and (3) such that  $(d\varepsilon)|_N$  represents the conformal symplectic structure on each fiber of  $E$ , cf. the proof of [24, Proposition in Section 4]. Hence  $\alpha := p_1^*\varepsilon + p_2^*\theta + dt$  is a contact form in a neighborhood of the zero section of  $E \oplus T^*N \times \mathbb{R}$ , where  $p_1, p_2, t$  denote the canonical projections onto the three summands, and  $\theta$  denotes the canonical 1-form on  $T^*N$ . Assuming, for simplicity, that the contact structure on  $P$  is coorientable near  $N$ , the tubular neighborhood theorem for isotropic submanifolds asserts that there exists a contact diffeomorphism  $\psi$  between an open neighborhood of the zero section in  $E \oplus T^*N \times \mathbb{R}$  and an open neighborhood of  $N$  in  $P$  which restricts to the identity along  $N$ . Using this diffeomorphism, we obtain a manifold chart for  $\text{Gr}_S(P)$  centered at  $N$  by assigning to a smooth section  $\sigma$  of  $E \oplus T^*N \times \mathbb{R}$ , which is sufficiently  $C^1$ -close to the zero section, the submanifold  $\psi(\sigma(N))$  in  $P$ . As  $\psi$  is contact, the part of  $\text{Gr}_S^{\text{iso}}(P)$  covered by this chart corresponds to sections  $\sigma \in \Gamma^\infty(E \oplus T^*N \times \mathbb{R})$  such that  $\sigma^*\alpha = 0$ . Identifying  $\Gamma^\infty(E \oplus T^*N \times \mathbb{R}) = \Gamma^\infty(E) \times \Omega^1(N) \times C^\infty(N)$  and writing  $\sigma = (s, \beta, f)$  accordingly, the latter condition is equivalent to  $s^*\varepsilon + \beta + df = 0$ . Hence,  $\text{Gr}_S^{\text{iso}}(P)$  corresponds to the part of the chart domain contained in the splitting linear subspace

$$\Gamma^\infty(E) \times C^\infty(N) \subseteq \Gamma^\infty(E) \times \Omega^1(N) \times C^\infty(N) = \Gamma^\infty(E \oplus T^*N \times \mathbb{R}),$$

$$(s, f) \mapsto (s, -s^*\varepsilon - df, f).$$

This shows that  $\text{Gr}_S^{\text{iso}}(P)$  is a splitting smooth submanifold of  $\text{Gr}_S(P)$ . □

**Remark 4.9** Lemma 4.8 implies that  $\text{Emb}^{\text{iso}}(S, P)$  is a smooth splitting submanifold of  $\text{Emb}(S, P)$ , because the natural map  $\text{Emb}(S, P) \rightarrow \text{Gr}_S(P)$  is a (locally trivial) smooth principal bundle with typical fiber  $\text{Diff}(S)$ . Since  $\pi^\mathcal{E} : \mathcal{E} \rightarrow \text{Emb}(S, P)$  is a (locally trivial) smooth fiber bundle, this also implies that  $\mathcal{E}^{\text{iso}}$  is a smooth submanifold of  $\mathcal{E}$ , see (93). Using the isotropic isotopy extension theorem for contact manifolds, see [14, Theorem 2.6.2] for instance, one can show that the group  $\text{Diff}_c(P, \xi)$  acts locally and infinitesimally transitive on  $\mathcal{E}^{\text{iso}}$ . Hence, for  $\rho = 0$ , Theorem 3.5 is essentially known.

As mentioned before, one expects that connected components of  $\mathcal{G}^{\text{iso}}$ , endowed with a reduced symplectic form, are symplectomorphic to coadjoint orbits of  $\text{Diff}_c(P, \xi)$  via the restriction of  $J_L^\mathcal{G} : \mathcal{G} \rightarrow \mathfrak{X}(P, \xi)^*$ . The following theorem makes this precise.

**Theorem 4.10** (a) *The subset  $\mathcal{G}^{\text{iso}}$  is a smooth splitting submanifold of  $\mathcal{G}$ . Moreover, the map provided by the action,  $\text{Diff}_c(P, \xi) \rightarrow \mathcal{G}^{\text{iso}}, g \mapsto \Psi_g^\mathcal{G}(N, \gamma)$ , admits a local smooth right inverse defined in a neighborhood of  $(N, \gamma)$  in  $\mathcal{G}^{\text{iso}}$ . In particular, the group  $\text{Diff}_c(P, \xi)$  acts locally and infinitesimally transitive on  $\mathcal{G}^{\text{iso}}$ , and the  $\text{Diff}_c(P, \xi)$ -orbit of  $(N, \gamma)$  is open and closed in  $\mathcal{G}^{\text{iso}}$ . Denoting this orbit by  $\mathcal{G}_{(N, \gamma)}^{\text{iso}}$ , the smooth map  $\text{Diff}_c(P, \xi) \rightarrow \mathcal{G}_{(N, \gamma)}^{\text{iso}}$  is locally trivializable with structure group  $\text{Diff}_c(P, \xi; (N, \gamma))$  and induces a bijection*

$$\mathcal{G}_{(N, \gamma)}^{\text{iso}} = \text{Diff}_c(P, \xi) / \text{Diff}_c(P, \xi; (N, \gamma)).$$

(b) *The projection  $q$  restricts to a smooth principal bundle  $q^{\text{iso}} : \mathcal{E}^{\text{iso}} \rightarrow \mathcal{G}^{\text{iso}}$  with structure group  $\text{Diff}(S)$ . The restriction of the symplectic form  $\omega^\mathcal{E}$  to  $\mathcal{E}^{\text{iso}}$  descends to a (reduced) symplectic form  $\omega^{\mathcal{G}^{\text{iso}}}$  on  $\mathcal{G}^{\text{iso}}$ . The  $\text{Diff}(P, \xi)$ -equivariant injective immersion*

$$J_L^{\mathcal{G}^{\text{iso}}} : \mathcal{G}^{\text{iso}} \rightarrow \mathfrak{X}(P, \xi)^*, \quad \langle J_L^{\mathcal{G}^{\text{iso}}}(N, \gamma), X \rangle = \int_N \gamma(X|_N),$$

provided by restriction of  $J_L^{\mathcal{G}}$  from (89), identifies  $\mathcal{G}_{(N,\gamma)}^{\text{iso}}$  with the coadjoint orbit through  $J_L^{\mathcal{G}}(N, \gamma)$  of the contact group  $\text{Diff}_c(P, \xi)$ , such that

$$(J_L^{\mathcal{G}^{\text{iso}}})^* \omega^{\text{KKS}} = \omega^{\mathcal{G}^{\text{iso}}}, \tag{97}$$

where  $\omega^{\text{KKS}}$  denotes the Kostant–Kirillov–Souriau symplectic form on the coadjoint orbit through  $J_L^{\mathcal{G}}(N, \gamma)$ , cf. Remark 4.11 below.

**Remark 4.11** To avoid discussing differential forms on coadjoint orbits, we consider the Kostant–Kirillov–Souriau form on the coadjoint orbit through  $J_L^{\mathcal{G}}(N, \gamma)$  as a formal object only. We actually work with its pull back along  $J_L^{\mathcal{G}^{\text{iso}}}$ , that is, the well defined smooth 2-form on  $\mathcal{G}^{\text{iso}}$  characterized by

$$((J_L^{\mathcal{G}^{\text{iso}}})^* \omega^{\text{KKS}})(\zeta_X^{\mathcal{G}^{\text{iso}}}(N, \gamma), \zeta_Y^{\mathcal{G}^{\text{iso}}}(N, \gamma)) := \langle J_L^{\mathcal{G}^{\text{iso}}}(N, \gamma), [X, Y] \rangle, \tag{98}$$

where  $X, Y \in \mathfrak{X}(P, \xi)$  and  $(N, \gamma) \in \mathcal{G}^{\text{iso}}$ . To motivate this definition, recall that for a Lie algebra  $\mathfrak{g}$  the Kostant–Kirillov–Souriau symplectic form on the coadjoint orbit through  $\lambda \in \mathfrak{g}^*$  is (formally) given by

$$\omega^{\text{KKS}}(\zeta_X^{\mathfrak{g}^*}(\lambda), \zeta_Y^{\mathfrak{g}^*}(\lambda)) = \langle \lambda, [X, Y] \rangle,$$

where  $X, Y \in \mathfrak{g}$  and  $\zeta_X^{\mathfrak{g}^*}$  denotes the infinitesimal coadjoint action. Since  $J_L^{\mathcal{G}^{\text{iso}}}$  is equivariant, we are being lead to (98).

**Proof of Theorem 4.10** We have already observed that  $\text{Gr}_S^{\text{iso}}(P)$  is a smooth submanifold of  $\text{Gr}_S(P)$ , see Lemma 4.8. Since the forgetful map  $\pi^{\mathcal{G}}: \mathcal{G} \rightarrow \text{Gr}_S(P)$  is a smooth fiber bundle, we conclude that  $\mathcal{G}^{\text{iso}}$  is a smooth submanifold of  $\mathcal{G}$ , see (94). In particular, the map provided by the action  $p: \text{Diff}_c(P, \xi) \rightarrow \mathcal{G}^{\text{iso}}$ ,  $p(g) = \Psi_g^{\mathcal{G}}(N, \gamma)$ , is smooth. Using local sections of  $\mathcal{E}^{\text{iso}} \rightarrow \mathcal{G}^{\text{iso}}$  and the fact that the  $\text{Diff}_c(P, \xi)$ -action on  $\mathcal{E}^{\text{iso}}$  admits local smooth sections, see Theorem 3.5, we readily see that the  $\text{Diff}_c(P, \xi)$ -action on  $\mathcal{G}^{\text{iso}}$  admits local smooth sections. If  $U$  is an open subset in  $\mathcal{G}_{(N,\gamma)}^{\text{iso}}$  and if  $\sigma: U \rightarrow \text{Diff}_c(P, \xi)$  is such a local section, then

$$p^{-1}(U) \rightarrow U \times \text{Diff}_c(P, \xi; (N, \gamma)), \quad g \mapsto (p(g), \sigma(p(g))^{-1}g) \tag{99}$$

is a  $\text{Diff}_c(P, \xi; (N, \gamma))$ -equivariant local trivialization with inverse  $(x, h) \mapsto (\sigma(x)h)$ .<sup>11</sup>

In view of  $\mathcal{E}^{\text{iso}} = q^{-1}(\mathcal{G}^{\text{iso}})$ , the smooth principal bundle  $q: \mathcal{E} \rightarrow \mathcal{G}$  restricts to a smooth principal bundle  $q^{\text{iso}}: \mathcal{E}^{\text{iso}} \rightarrow \mathcal{G}^{\text{iso}}$  with structure group  $\text{Diff}(S)$ . By Proposition 4.2 the map  $J_L^{\mathcal{G}^{\text{iso}}}$  is a  $\text{Diff}(P, \xi)$ -equivariant injective immersion. In view of (the trivial inclusion in) Eq. (55), we have  $\omega^{\mathcal{E}}(\zeta_X^{\mathcal{E}}, \zeta_Z^{\mathcal{E}}) = 0$  for all  $X \in \mathfrak{X}(P, \xi)$  and  $Z \in \mathfrak{X}(S)$ . Since  $\text{Diff}_c(P, \xi)$  acts infinitesimally transitive on  $\mathcal{E}^{\text{iso}}$ , the 1-form  $\omega^{\mathcal{E}}(-, \zeta_Z^{\mathcal{E}})$ , thus, vanishes when pulled back to  $\mathcal{E}^{\text{iso}}$ . Hence, the restriction of  $\omega^{\mathcal{E}}$  to  $\mathcal{E}^{\text{iso}}$  is vertical. We conclude that there exists a unique 2-form  $\omega^{\mathcal{G}^{\text{iso}}}$  on  $\mathcal{G}^{\text{iso}}$  such that  $(q^{\text{iso}})^* \omega^{\mathcal{G}^{\text{iso}}}$  coincides with the pull back of  $\omega^{\mathcal{E}}$  to  $\mathcal{E}^{\text{iso}}$ . Clearly,  $\omega^{\mathcal{G}^{\text{iso}}}$  is closed. The 2-form  $\omega^{\mathcal{G}^{\text{iso}}}$  is (weakly) non-degenerate in view of (the non-trivial inclusion in) Equation (55). From (98), (29), (26) and the equivariance of  $q$  we immediately obtain  $(q^{\text{iso}})^*(J_L^{\mathcal{G}^{\text{iso}}})^* \omega^{\text{KKS}} = (q^{\text{iso}})^* \omega^{\mathcal{G}^{\text{iso}}}$ , whence (97). The remaining assertions are now obvious.  $\square$

**Remark 4.12** We expect that the isotropy group  $\text{Diff}_c(P, \xi; (N, \gamma))$  in Theorem 4.10(a) is a closed Lie subgroup in  $\text{Diff}_c(P, \xi)$ . If this is the case then the local trivializations in (99) are

<sup>11</sup> We do not claim that  $\text{Diff}_c(P, \xi; (N, \gamma))$  is a submanifold of  $\text{Diff}_c(P, \xi)$ .

diffeomorphisms, hence the map  $\text{Diff}_c(P, \xi) \rightarrow \mathcal{G}_{(N, \gamma)}^{\text{iso}}$  is a smooth principal bundle with structure group  $\text{Diff}_c(P, \xi; (N, \gamma))$  and  $\mathcal{G}_{(N, \gamma)}^{\text{iso}}$  may be regarded as a homogeneous space in the category of smooth manifolds.

In any case, one can equip  $\text{Diff}_c(P, \xi; (N, \gamma))$  with the smooth Frölicher structure, see [20, Section 23] and [6–8], induced from the ambient Lie group  $\text{Diff}_c(P, \xi)$ . Then the local trivializations in (99) are diffeomorphisms of Frölicher spaces and  $\mathcal{G}_{(N, \gamma)}^{\text{iso}}$  may be regarded as a homogeneous space in the sense of Frölicher spaces.

**Example 4.13** If  $S$  is the circle  $S^1$  and  $P$  is a 3-dimensional contact manifold, then the weighted non-linear Grassmannian  $\mathcal{G}$  becomes the manifold of weighted (unparametrized) knots in  $P$ , and  $\mathcal{G}^{\text{iso}}$  is the (symplectic) manifold of weighted Legendrian knots in  $P$ . By Theorem 4.10, its connected components can be identified with coadjoint orbits of the identity component of the contact group.

### 4.4 Weighted contact non-linear Grassmannians

Let us now consider a 1-form density  $\rho \in \Omega^1(S, |\Lambda|_S)$  of contact type, i.e.,  $\ker \rho \subseteq TS$  is assumed to be a contact hyperplane distribution. Then the reduced space  $\mathcal{G}^\rho$ , see (91), consists of weighted contact submanifolds. More precisely, according to Lemma 4.3 we have

$$\mathcal{G}^\rho \subseteq (\pi^{\mathcal{G}})^{-1}(\text{Gr}_{(S, \ker \rho)}^{\text{contact}}(P, \xi)), \tag{100}$$

where  $\text{Gr}_{(S, \ker \rho)}^{\text{contact}}(P, \xi) \subseteq \text{Gr}_S(P)$  denotes the subset of contact submanifolds which are of type  $(S, \ker \rho)$ . In contrast to the isotropic case, see (95), the inclusion (100) is strict.

The maps in (87) restrict to a  $\text{Diff}(P, \xi)$ -equivariant commutative diagram

$$\begin{CD} \mathcal{E}^\rho @>{\pi^{\mathcal{E}^\rho}}>> \text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi) \\ @V{q^\rho}VV @VVV \\ \mathcal{G}^\rho @>{\pi^{\mathcal{G}^\rho}}>> \text{Gr}_{(S, \ker \rho)}^{\text{contact}}(P, \xi) \end{CD} \tag{101}$$

where  $\text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi) \subseteq \text{Emb}(S, P)$  denotes the subset of contact embeddings inducing the contact structure  $\ker \rho$  on  $S$ .

**Lemma 4.14** *If  $\rho \in \Omega^1(S, |\Lambda|_S)$  is a contact 1-form density, then the following hold true:*

- (a)  $\text{Gr}_{(S, \ker \rho)}^{\text{contact}}(P, \xi)$  is an open subset of  $\text{Gr}_S(P)$ .
- (b)  $\text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi)$  is an initial Fréchet submanifold of  $\text{Emb}(S, P)$ .
- (c) The natural map

$$\text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi) \rightarrow \text{Gr}_{(S, \ker \rho)}^{\text{contact}}(P, \xi)$$

is a smooth principal bundle with structure group  $\text{Diff}(S, \ker \rho)$ .

- (d) The natural map

$$\mathcal{L}|_{\text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi)} \xrightarrow{(\pi^{\mathcal{L}}, J_R^{\mathcal{L}})} \text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi) \times \Gamma^\infty((TS/\ker \rho)^* \otimes |\Lambda|_S)$$

is a diffeomorphism of Fréchet manifolds, providing a  $\text{Diff}(S, \ker \rho)$ -equivariant trivialization of the bundle  $\pi^{\mathcal{L}}: \mathcal{L} \rightarrow C^\infty(S, P)$  over  $\text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi)$ .

(e) The map  $\pi^{\mathcal{E}} : \mathcal{E} \rightarrow \text{Emb}(S, P)$  restricts to a diffeomorphism of Fréchet manifolds,

$$\mathcal{E}^{\rho} \cong \text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi).$$

**Proof** (a) follows from the Gray stability theorem, see [14, Theorem 2.2.2]. Locally around points in  $\text{Gr}_{(S, \rho)}^{\text{contact}}(P, \xi)$ , the Gray stability theorem permits to construct cross sections of the  $\text{Diff}(S)$ -bundle  $\text{Emb}(S, P) \rightarrow \text{Gr}_S(P)$  which take values in  $\text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi)$ . Such a local cross section, defined on an open subset  $\mathcal{U}$  in  $\text{Gr}_S(P)$ , provides a local trivialization of  $\text{Diff}(S)$ -bundles,  $\mathcal{U} \times \text{Diff}(S) \cong \text{Emb}(S, P)|_{\mathcal{U}}$ , which maps  $\mathcal{U} \times \text{Diff}(S, \ker \rho)$  onto  $\text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi)|_{\mathcal{U}}$ . Recall that  $\text{Diff}(S, \ker \rho)$  is a Fréchet Lie group, and the natural inclusion into  $\text{Diff}(S)$  is initial, see [20, Theorem 43.19]. Whence (b) and (c).

Since  $\rho$  is nowhere vanishing, the map in (d) is a bijection. This map is smooth because the inclusion  $\text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi) \subseteq \text{Emb}(S, P)$  is initial. To see that its inverse is smooth too, we fix a vector bundle homomorphism  $\sigma : TS/\ker \rho \rightarrow TS$  splitting the canonical projection  $TS \rightarrow TS/\ker \rho$ . Let  $\mathcal{W}$  denote the set of embeddings  $\varphi \in \text{Emb}(S, P)$  for which the composition

$$TS/\ker \rho \xrightarrow{\sigma} TS \xrightarrow{T\varphi} \varphi^*TP \rightarrow \varphi^*L$$

is an isomorphism of line bundles over  $S$ . Clearly,  $\mathcal{W}$  is an open neighborhood of  $\text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi)$  in  $\text{Emb}(S, P)$ . We obtain a smooth map

$$s : \mathcal{W} \times \Gamma^{\infty}((TS/\ker \rho)^* \otimes |\Lambda|_S) \rightarrow \mathcal{L},$$

characterized by  $\pi^{\mathcal{L}}(s(\varphi, \beta)) = \varphi$  and  $J_R^{\mathcal{L}}(s(\varphi, \beta)) \circ \sigma = \beta$ , for all  $\varphi \in \mathcal{W}$  and  $\beta \in \Gamma^{\infty}((TS/\ker \rho)^* \otimes |\Lambda|_S)$ . Its restriction provides the smooth inverse for the map in (d).

Restricting the diffeomorphism in (d) to the level set  $\mathcal{E}^{\rho}$ , we obtain a diffeomorphism  $\mathcal{E}^{\rho} \cong \text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi) \times \{\rho\}$ , whence (e). □

A preliminary extended version of this paper contains an example [17, Proposition 4.20] which shows that for general contact 1-form densities  $\rho \in \Omega^1(S, |\Lambda|_S)$  the continuous bijection

$$\mathcal{E}^{\rho} / \text{Diff}(S, \rho) \rightarrow (J_R^{\mathcal{E}})^{-1}(\rho \cdot \text{Diff}(S)) / \text{Diff}(S) = \mathcal{G}^{\rho} \tag{102}$$

induced by the natural inclusion is not a homeomorphism with respect to the quotient topologies. Note that since  $q : \mathcal{E} \rightarrow \mathcal{G}$  admits local smooth sections, the quotient topology (Frölicher structure) on the right hand side in (102) coincides with the one induced from  $\mathcal{G}$ . Hence, for  $(N, \gamma) \in \mathcal{G}^{\rho}$  the map provided by the action,  $\text{Diff}(P, \xi) \rightarrow \mathcal{G}^{\rho}$ ,  $g \mapsto \Psi_g^{\mathcal{G}}(N, \gamma)$ , does not admit a continuous local (with respect to the trace topology induced from  $\mathcal{G}$ ) right inverse defined in a neighborhood of  $(N, \gamma)$ .

The diffeomorphism in Lemma 4.14(e) induces a natural homeomorphism:

$$\mathcal{E}^{\rho} / \text{Diff}(S, \rho) \cong \text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi) \times_{\text{Diff}(S, \ker \rho)} \frac{\text{Diff}(S, \ker \rho)}{\text{Diff}(S, \rho)}.$$

Note that the isotropy group  $\text{Diff}(S, \rho)$  is akin to the group of strict contact diffeomorphisms. The diffeomorphism in Lemma 4.14(d) induces a diffeomorphism

$$\mathcal{G}|_{\text{Gr}_{(S, \ker \rho)}^{\text{contact}}(P, \xi)} \cong \text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi) \times_{\text{Diff}(S, \ker \rho)} \Gamma^{\infty}((TS/\ker \rho)^* \otimes |\Lambda|_S \setminus S)$$

which restricts to a natural homeomorphism,

$$\mathcal{G}^{\rho} \cong \text{Emb}_{(S, \ker \rho)}^{\text{contact}}(P, \xi) \times_{\text{Diff}(S, \ker \rho)} \mathcal{O}_{\rho}, \tag{103}$$

where  $\mathcal{G}^\rho$  is equipped with the topology induced from  $\mathcal{G}$  and  $\mathcal{O}_\rho$  denotes the  $\text{Diff}(S, \ker \rho)$ -orbit of  $\rho$  equipped with the topology induced from  $\Omega^1(S, |\Lambda|_S)$ . The fact that the map in (102) fails to be a homeomorphism is reflected by the fact that the canonical continuous bijection  $\frac{\text{Diff}(S, \ker \rho)}{\text{Diff}(S, \rho)} \rightarrow \mathcal{O}_\rho$  is not homeomorphic. The remarks in this paragraph remain true if the topologies are replaced with the corresponding Frölicher structures.

**Remark 4.15** Let  $\rho \in \Omega^1(S, |\Lambda|_S)$  be a contact 1-form density. Since  $\mathcal{G}^\rho$  may not be a manifold, we refrain from considering the Kostant–Kirillov–Souriau form on  $\mathcal{G}^\rho$ . However, formally pulling back the Kostant–Kirillov–Souriau form along  $J_L^{\mathcal{E}^\rho} : \mathcal{E}^\rho \rightarrow \mathfrak{X}(P, \xi)^*$ , we obtain a well defined smooth 2-form  $(J_L^{\mathcal{E}^\rho})^* \omega^{\text{KKS}}$  on  $\mathcal{E}^\rho$ , characterized by

$$((J_L^{\mathcal{E}^\rho})^* \omega^{\text{KKS}})(\xi_X^{\mathcal{E}^\rho}(\Phi), \xi_Y^{\mathcal{E}^\rho}(\Phi)) := \langle J_L^{\mathcal{E}^\rho}(\Phi), [X, Y] \rangle,$$

where  $\Phi \in \mathcal{E}^\rho$  and  $X, Y \in \mathfrak{X}(P, \xi)$ , cf. Remark 4.11 and Theorem 3.5. Proceeding exactly as in the proof of Theorem 4.10, we see that this coincides with  $\omega^{\mathcal{E}^\rho}$ , the pull back of the symplectic form  $\omega^{\mathcal{E}}$  to  $\mathcal{E}^\rho$ , i.e.,

$$(J_L^{\mathcal{E}^\rho})^* \omega^{\text{KKS}} = \omega^{\mathcal{E}^\rho}.$$

For 1-dimensional  $S$  the situation is as nice as one could wish for:

**Example 4.16** Let us specialize to the circle,  $S = S^1$ . In this case, any contact 1-form density  $\rho \in \Omega^1(S, |\Lambda|_S)$  gives rise to an orientation and a Riemannian metric on  $S$ . We write  $\sqrt{|\rho|}$  for the induced volume density on  $S$ , and denote the total volume by  $\text{vol}(\rho) := \int_S \sqrt{|\rho|}$ . Using parametrization by arc length it is easy to see that two contact 1-form densities lie in the same  $\text{Diff}(S)$ -orbit iff they have the same total volume. In particular, these orbits are closed submanifolds in  $\Omega^1(S, |\Lambda|_S)$ . Moreover, parametrization by arc length provides local smooth sections for the  $\text{Diff}(S)$ -action on said orbits. Note that  $\text{Diff}(S, \ker \rho) = \text{Diff}(S)$  in this case.

Suppose  $(P, \xi)$  is a contact manifold and let  $\rho \in \Omega^1(S, |\Lambda|_S)$  be a contact 1-form density on  $S = S^1$ . Using (103) we conclude that  $\mathcal{G}^\rho$  is a closed submanifold of  $\mathcal{G}$ . Parametrization by arc length provides local smooth sections of  $\mathcal{E}^\rho \rightarrow \mathcal{G}^\rho$  and the latter is a locally trivial smooth principal bundle. Note that the structure group  $\text{Diff}(S, \rho) \cong \text{SO}(1)$  is a closed Lie subgroup of  $\text{Diff}(S)$ . Using Theorem 3.5, we conclude that the  $\text{Diff}_c(P, \xi)$ -action on  $\mathcal{G}^\rho$  admits local smooth sections. Hence, its orbits are open and closed subsets in  $\mathcal{G}^\rho$  which may be identified with coadjoint orbits of the contact group via the restriction of  $J_L^{\mathcal{G}}$ . The symplectic form on  $\mathcal{E}$  gives rise to a reduced symplectic form on  $\mathcal{G}^\rho$  which coincides with the pull back of the Kostant–Kirillov–Souriau symplectic form via  $J_L^{\mathcal{G}}$  as in Theorem 4.10(b). If  $P$  is 3-dimensional, then  $\mathcal{G}^\rho$  is a (symplectic) manifold of weighted transverse knots.

A slightly more explicit description can be given if the contact structure is admits a contact form,  $\xi = \ker \alpha$ . Then, via the identification in Remark 4.4, we have

$$\mathcal{G}^\rho \cong \{ (N, \nu) \in \text{Gr}_{S^1}^{\text{wt}}(P) \mid \iota_N^* \alpha \neq 0, \text{ vol}(\iota_N^* \alpha \otimes \nu) = \text{vol}(\rho) \},$$

for every contact 1-form density  $\rho$ .

## 5 Relation with other dual pairs

### 5.1 Comparison with the EPDiff dual pair

A pair of moment maps has been introduced by Holm and Marsden [18] in relation to the EPDiff equations, describing geodesics on the group of all diffeomorphisms. The left moment

map provides singular solutions of these equations, whereas the right moment map provides a constant of motion for the collective dynamics of these singular solutions. In this section we relate the EPDiff dual pair of a manifold with the EPContact dual pair of its projectivized cotangent bundle.

Recall that the projectivized cotangent bundle,

$$P := \mathbb{P}(T^*Q) = (T^*Q \setminus Q) / \mathbb{R}^\times \xrightarrow{p} Q,$$

admits a canonical contact structure [1, Appendix 4] given by

$$\xi_\ell = (T_\ell p)^{-1}(\ker \beta), \tag{104}$$

where  $\ell \in P$  and  $\beta \in T^*Q$  is any non-zero element of  $\ell$ . As the natural action of  $\text{Diff}(Q)$  on  $P$  preserves the contact structure  $\xi$ , we obtain an injective group homomorphism

$$\text{Diff}(Q) \rightarrow \text{Diff}(P, \xi).$$

The line bundle  $L^*$  associated with the projectivized cotangent bundle, see Sect. 2.1, is naturally isomorphic to the canonical line bundle over  $P$ :

$$\gamma = \{(\ell, \beta) | \ell \in P, \beta \in \ell\}.$$

Indeed, the vector bundle homomorphism  $\chi : \gamma \rightarrow T^*P$  over the identity on  $P$ , given by  $\chi(\ell, \beta) := \beta \circ T_\ell p$ , induces an isomorphism of line bundles,  $\chi : \gamma \rightarrow L^*$ . Furthermore,

$$\chi^* \theta^{L^*} = \text{pr}_2^* \theta^{T^*Q}, \tag{105}$$

where  $\text{pr}_2 : \gamma \rightarrow T^*Q$  denotes the canonical projection, i.e., the blow-up of the zero section in  $T^*Q$ , and  $\theta^{T^*Q}$  denotes the canonical 1-form on  $T^*Q$ . One readily checks:

**Lemma 5.1** *The map  $\kappa : L^* \rightarrow T^*Q$ ,  $\kappa := \text{pr}_2 \circ \chi^{-1}$  is a vector bundle homomorphism over the bundle projection  $p$ ,*

$$\begin{array}{ccc} L^* & \xrightarrow{\kappa} & T^*Q \\ \pi^{L^*} \downarrow & & \downarrow \pi^{T^*Q} \\ P & \xrightarrow{p} & Q \end{array}$$

with the following properties:

- (a)  $\kappa$  is equivariant over the homomorphism  $\text{Diff}(Q) \rightarrow \text{Diff}(P, \xi)$ .
- (b)  $\kappa$  restricts to a diffeomorphism from  $L^* \setminus P$  onto  $T^*Q \setminus Q$ .
- (c)  $\kappa^* \theta^{T^*Q} = \theta^{L^*}$ .

Composition with  $\kappa$  provides a map

$$\mathcal{L} = C_{\text{lin}}^\infty(|\Lambda|_S^*, L^*) \xrightarrow{\kappa^*} C_{\text{lin}}^\infty(|\Lambda|_S^*, T^*Q) = T^*C^\infty(S, Q)_{\text{reg}}$$

where  $T^*C^\infty(S, Q)_{\text{reg}}$  denotes the regular part of the cotangent bundle. The identification on the right hand side is provided by the canonical pairing between  $\Gamma^\infty(|\Lambda|_S \otimes \eta^* T^*Q)$  and the tangent space  $T_\eta C^\infty(S, Q) = \Gamma^\infty(\eta^* TQ)$  at  $\eta \in C^\infty(S, Q)$ . Via this identification, the canonical 1-form on  $T^*C^\infty(S, Q)_{\text{reg}}$  can be written in the form

$$\theta^{T^*C^\infty(S, Q)_{\text{reg}}}(A) = \int_S \theta^{T^*Q}(A), \tag{106}$$

where  $A$  is a tangent vector at  $\Phi \in T^*C^\infty(S, Q)_{\text{reg}}$ . The differential  $d\theta^{T^*C^\infty(S, Q)_{\text{reg}}}$  is the canonical (weakly non-degenerate) symplectic form on  $T^*C^\infty(S, Q)_{\text{reg}}$ .

The cotangent lifted actions of the groups  $\text{Diff}(Q)$  and  $\text{Diff}(S)$  on the manifold  $C^\infty(S, Q)$  preserve the canonical 1-form  $\theta^{T^*C^\infty(S, Q)_{\text{reg}}}$ . In particular, these actions are Hamiltonian with equivariant moment maps [18],  $J_{\text{Sing}} : T^*C^\infty(S, Q)_{\text{reg}} \rightarrow \mathfrak{X}(Q)^*$ ,

$$\langle J_{\text{Sing}}(\Phi), Y \rangle = \theta^{T^*C^\infty(S, Q)_{\text{reg}}} \left( \zeta_Y^{T^*C^\infty(S, Q)_{\text{reg}}}(\Phi) \right) = \int_S \Phi(Y \circ \eta), \tag{107}$$

and  $J_S : T^*C^\infty(S, Q)_{\text{reg}} \rightarrow \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^*$ ,

$$\langle J_S(\Phi), Z \rangle = \theta^{T^*C^\infty(S, Q)_{\text{reg}}} \left( \zeta_Z^{T^*C^\infty(S, Q)_{\text{reg}}}(\Phi) \right) = \int_S \Phi(T\eta \circ Z), \tag{108}$$

respectively, where  $\eta \in C^\infty(S, Q)$  and  $\Phi \in T_\eta^*C^\infty(S, Q)_{\text{reg}} = \Gamma^\infty(|\Lambda|_S \otimes \eta^*T^*Q)$ . Here  $\zeta_Y^{T^*C^\infty(S, Q)_{\text{reg}}}$  and  $\zeta_Z^{T^*C^\infty(S, Q)_{\text{reg}}}$  denote the fundamental vector fields on  $T^*C^\infty(S, Q)_{\text{reg}}$  corresponding to the (infinitesimal) action of  $Y \in \mathfrak{X}(Q)$  and  $Z \in \mathfrak{X}(S)$ , respectively.

These maps and the moments maps in (28) and (36) fit into the following diagram:

$$\begin{array}{ccccc} \mathfrak{X}(P, \xi)^* & \xleftarrow{J_L^\mathcal{L}} & \mathcal{L} & \xrightarrow{J_R^\mathcal{L}} & \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^* \\ \downarrow i^* & & \downarrow \kappa_* & & \parallel \\ \mathfrak{X}(Q)^* & \xleftarrow{J_{\text{Sing}}} & T^*C^\infty(S, Q)_{\text{reg}} & \xrightarrow{J_S} & \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^* \end{array} \tag{109}$$

Here  $i^*$  denotes the dual of the Lie algebra homomorphism  $i : \mathfrak{X}(Q) \rightarrow \mathfrak{X}(P, \xi)$  corresponding to the homomorphism of groups  $\text{Diff}(Q) \rightarrow \text{Diff}(P, \xi)$ . Clearly,  $i^*$  is equivariant over the homomorphism  $\text{Diff}(Q) \rightarrow \text{Diff}(P, \xi)$ . Note that via (8) and  $\kappa$ , the Lie algebra  $\mathfrak{X}(P, \xi) = C_{\text{lin}}^\infty(L^*)$  may be regarded as the space of homogeneous functions on  $T^*Q \setminus Q$ , while the image of  $i$  consists of those which extend to fiberwise linear functions on  $T^*Q$ .

Recall the open symplectic part  $\mathcal{M} = C_{\text{lin, inj}}^\infty(|\Lambda|_S^*, L^*)$  in  $\mathcal{L}$  and let  $T^*C^\infty(S, Q)_{\text{reg}}^\times$  denote the open subset of the regular cotangent bundle that corresponds to the space  $C_{\text{lin, inj}}^\infty(|\Lambda|_S^*, T^*Q)$  of smooth maps which are linear and injective on fibers.

**Proposition 5.2** *The diagram (109) commutes. The map  $\kappa_*$  is equivariant over the homomorphism  $\text{Diff}(Q) \rightarrow \text{Diff}(P, \xi)$  and also  $\text{Diff}(S)$ -equivariant. It restricts to a symplectic diffeomorphism from  $\mathcal{M}$  onto  $T^*C^\infty(S, Q)_{\text{reg}}^\times$ .*

**Proof** The map  $\kappa_*$  is equivariant over the homomorphism  $\text{Diff}(Q) \rightarrow \text{Diff}(P, \xi)$  since  $\kappa$  has the same property, see Lemma 5.1(a). Clearly,  $\kappa_*$  is  $\text{Diff}(S)$ -equivariant too. Hence, the fundamental vector fields are  $\kappa_*$ -related, that is,

$$T\kappa_* \circ \zeta_{i(Y)}^\mathcal{L} = \zeta_Y^{T^*C^\infty(S, Q)_{\text{reg}}} \circ \kappa_* \quad \text{and} \quad T\kappa_* \circ \zeta_Z^\mathcal{L} = \zeta_Z^{T^*C^\infty(S, Q)_{\text{reg}}} \circ \kappa_* \tag{110}$$

for  $Y \in \mathfrak{X}(Q)$  and  $Z \in \mathfrak{X}(S)$ . Using Lemma 5.1(c), (22), and (106), we obtain

$$(\kappa_*)^* \theta^{T^*C^\infty(S, Q)_{\text{reg}}} = \theta^\mathcal{L}. \tag{111}$$

Combining the latter with the first equation in (110), we see that the square on the left hand side in (109) commutes, cf. (107) and (29). Combining (111) with the second equation in (110), we see that the square on the right hand side in (109) commutes, cf. (108) and (37). Using Lemma 5.1(b) we see that  $\kappa_*$  restricts to a diffeomorphism from  $\mathcal{M}$  onto  $T^*C^\infty(S, Q)_{\text{reg}}^\times$  which is symplectic in view of (111).  $\square$

Restricting the actions and moment maps in the first row in the diagram (109) to the non-linear Stiefel manifold  $\mathcal{E} = \text{Emb}_{\text{lin}}(|\Lambda|_S^*, L^*)$  of weighted embeddings, we get the EPContact dual pair from Theorem 2.6. The pair of moment maps in the second row, when restricted to the open subset  $T^* \text{Emb}(S, \mathcal{Q})_{\text{reg}}^\times$ , form a symplectic dual pair [9]:

$$\mathfrak{X}(\mathcal{Q})^* \xleftarrow{J_{\text{Sing}}} T^* \text{Emb}(S, \mathcal{Q})_{\text{reg}}^\times \xrightarrow{J_S} \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^*, \tag{112}$$

namely the EPDiff dual pair of Holm and Marsden [18].

### 5.2 Comparison with the dual pair for the Euler equation

A dual pair of moment maps associated to the Euler equations of an ideal fluid has been described by Marsden and Weinstein [26]; it justifies the existence of Clebsch canonical variables for ideal fluid motion and also explains the Hamiltonian structure of point vortex solutions in a geometric way. In this section we relate the EPContact dual pair to the ideal fluid dual pair, via the symplectization of the contact manifold.

Recall the symplectic manifold  $\mathcal{M} = C_{\text{lin, inj}}^\infty(|\Lambda|_S^*, L^*)$  with Hamiltonian actions of the groups  $\text{Diff}(P, \xi)$  and  $\text{Diff}(S)$  and moment maps  $J_L^{\mathcal{M}}$  and  $J_R^{\mathcal{M}}$  from Sect. 2.3. We fix a volume density  $\mu$  on  $S$ . The latter provides an identification

$$\iota_\mu: \mathcal{M} \rightarrow C^\infty(S, M) \tag{113}$$

given by  $\iota_\mu(\Phi) := \Phi \circ \hat{\mu}$ , where  $\hat{\mu} \in \Gamma^\infty(|\Lambda|_S^*)$  denotes the section dual to  $\mu$ . Here  $M = L^* \setminus P \subseteq T^*P$  is the symplectization of the contact manifold  $(P, \xi)$ , equipped with the exact symplectic form  $\omega^M = d\theta^M$  obtained by restricting the cotangent bundle symplectic form, cf. Sect. 2.1. As the action of the contact group on  $M$  is symplectic, we have an (injective) group homomorphism  $\text{Diff}(P, \xi) \rightarrow \text{Diff}(M, \omega^M)$ .

There is a natural (exact) symplectic form on  $C^\infty(S, M)$ , that can be described by

$$\omega_\phi^{C^\infty(S, M)}(U, V) = \int_S \omega^M(U, V)\mu, \tag{114}$$

where  $U, V \in \Gamma^\infty(\phi^*TM) = T_\phi C^\infty(S, M)$  are vector fields along  $\phi \in C^\infty(S, M)$ . The right action of  $\text{Diff}(S, \mu)$  on  $C^\infty(S, M)$  is Hamiltonian, with equivariant moment map:

$$J_R^{C^\infty(S, M)}: C^\infty(S, M) \rightarrow \Omega^1(S) \stackrel{\mu}{=} \Omega^1(S, |\Lambda|_S) \subseteq \mathfrak{X}(S)^* \rightarrow \mathfrak{X}(S, \mu)^*.$$

Here the first arrow is given by pull back of  $\theta^M$ ; the second identification is via the volume density  $\mu$ ; the third is the inclusion of smooth sections into distributional sections of  $T^*S \otimes |\Lambda|_S$ ; and the fourth map is the dual of the canonical inclusion  $\mathfrak{X}(S, \mu) \subseteq \mathfrak{X}(S)$ . We write this as

$$J_R^{C^\infty(S, M)}(\phi) = \phi^* \theta^M \otimes \mu \quad \text{i.e.} \quad \langle J_R^{C^\infty(S, M)}(\phi), X \rangle = \int_S (\phi^* \theta^M)(X)\mu, \tag{115}$$

where  $\phi \in C^\infty(S, M)$  and  $X \in \mathfrak{X}(S, \mu)$ .

Via the Lie algebra homomorphism  $C^\infty(M) \rightarrow \mathfrak{X}_{\text{ham}}(M, \omega^M)$ , the Poisson algebra  $C^\infty(M)$  acts from the left on  $C^\infty(S, M)$  in a Hamiltonian fashion with equivariant moment map  $J_L^{C^\infty(S, M)}: C^\infty(S, M) \rightarrow C^\infty(M)^*$  given by

$$J_L^{C^\infty(S, M)}(\phi) := \phi_* \mu \quad \text{i.e.} \quad \langle J_L^{C^\infty(S, M)}(\phi), h \rangle = \int_S (\phi^* h)\mu, \tag{116}$$

where  $\phi \in C^\infty(S, M)$  and  $h \in C^\infty(M)$ .

The four moment maps mentioned in this section fit into the following diagram:

$$\begin{array}{ccccc}
 \mathfrak{X}(P, \xi)^* & \xleftarrow{J_L^{\mathcal{M}}} & \mathcal{M} & \xrightarrow{J_R^{\mathcal{M}}} & \mathfrak{X}(S)^* \\
 j^* \uparrow & & \cong \downarrow \iota_\mu & & \downarrow i^* \\
 C^\infty(M)^* & \xleftarrow{J_L^{C^\infty(S, M)}} & C^\infty(S, M) & \xrightarrow{J_R^{C^\infty(S, M)}} & \mathfrak{X}(S, \mu)^*
 \end{array} \tag{117}$$

Here  $j: \mathfrak{X}(P, \xi) \rightarrow C^\infty(M)$ ,  $j(X) := h_X^M$ , denotes the Lie algebra homomorphism provided by (8), see also (14). In view of (9),  $j$  is equivariant over the homomorphism  $\text{Diff}(P, \xi) \rightarrow \text{Diff}(M, \omega^M)$ . Note that the composition of  $j$  with the action  $C^\infty(M) \rightarrow \mathfrak{X}_{\text{ham}}(M, \omega^M)$  yields a Lie algebra homomorphism  $\mathfrak{X}(P, \xi) \rightarrow \mathfrak{X}_{\text{ham}}(M, \omega^M) \subseteq \mathfrak{X}(M, \omega^M)$  corresponding to the homomorphism of groups  $\text{Diff}(P, \xi) \rightarrow \text{Diff}(M, \omega^M)$ , see (12). Finally,  $i: \mathfrak{X}(S, \mu) \rightarrow \mathfrak{X}(S)$  denotes the natural inclusion, which is clearly equivariant over the inclusion  $\text{Diff}(S, \mu) \subseteq \text{Diff}(S)$ .

**Proposition 5.3** *The diagram (117) commutes. The map  $\iota_\mu$  in (113) is a symplectic diffeomorphism which is equivariant over the inclusion  $\text{Diff}(S, \mu) \subseteq \text{Diff}(S)$  and equivariant over the homomorphism  $\text{Diff}(P, \xi) \rightarrow \text{Diff}(M, \omega^M)$ .*

**Proof** Clearly,  $\iota_\mu$  is an equivariant diffeomorphism, see Remark 2.2. It is symplectic in view of (43) and (114). The right hand side of the diagram commutes in view of (47) and (115). The left hand side of the diagram commutes in view of (46) and (116). □

The first row in (117) becomes the EPContact dual pair from Theorem 2.6 when restricted to the non-linear Stiefel manifold  $\mathcal{E} = \text{Emb}_{\text{lin}}(|\Lambda|_S^*, L^*)$  of weighted embeddings. In the second row, by restricting the actions and moment maps to the open subset  $\text{Emb}(S, M) \subseteq C^\infty(S, M)$  of embeddings, we obtain a symplectic dual pair, see [9] and [11, Section 4.2]:

$$C^\infty(M)^* \xleftarrow{J_L^{\text{Emb}(S, M)}} \text{Emb}(S, M) \xrightarrow{J_R^{\text{Emb}(S, M)}} \mathfrak{X}(S, \mu)^*, \tag{118}$$

namely the ideal fluid dual pair of Marsden and Weinstein [26].

Note that the image  $\iota_\mu(\mathcal{E})$  is an open subset (strict, in general) of  $\text{Emb}(S, M)$ :

$$\iota_\mu(\mathcal{E}) = \{ \phi \in C^\infty(S, M) : \pi^M \circ \phi \in \text{Emb}(S, P) \},$$

where  $\pi^M: M \rightarrow P$  denotes the restriction of the canonical projection  $\pi^{L^*}: L^* \rightarrow P$ .

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