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Computations versus bijections for tiling enumeration

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A B S T R A C T

The number of domino tilings of an Aztec rectangle is known to be $2^{\binom{n+1}{2}} \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j-i}$, while the number of lozenge tilings of a trapezoid is known to be $\prod_{1 \le i < j \le n} \frac{k_j - k_i}{j-i}$, where $k_1 < k_2 < \ldots < k_n$ prescribes the positions of certain defects along one side of the rectangle or trapezoid, respectively. It is shown that these objects can naturally be extended to all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ in such a way that the signed enumeration of the extended objects is given by the very same formula as the (restricted) straight enumeration. The main purpose of this article is to provide first combinatorial proofs of these facts. These proofs are derived from "computational" proofs, but we seek to compare them to known combinatorial constructions whenever possible. This reveals among other things that we have constructed an extension of urban renewal. This extension also played (in disguised form) a fundamental role in the recent first bijective proof of the alternating sign matrix theorem of Konvalinka and the author, and one important motivation for the results presented in this paper is to work towards a significant simplification of this proof to the effect that it has a more combinatorial and less computational flavor.

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1. Introduction

The main objective of this paper is a study of bijective proofs that were derived from non-bijective proofs, usually by translating algebraic manipulations. The latter type of proofs is sometimes referred to as "analytic proofs", sometimes as "algebraic proofs" — we will use the term "computational proofs". Our considerations will be based on two intimately related examples, namely *lozenge tilings* of trapezoids and *domino tilings* of rectangles, in both cases with defects on one side. Their enumerations are certainly known, what is less known is that these enumerations can naturally be extended to certain signed objects in a way such that their signed enumerations are given by the same formula as the original "unsigned" enumerations. In the case of domino tilings this is new, for lozenge tilings it has appeared before in disguise. In this paper, we will present first combinatorial proofs of these facts, and show how these proofs can be derived from computational proofs. The translation of certain computations into bijections appeared in the work of Garsia and Milne [9,10], where the technique (the so-called *involution principle*) was used to provide first bijective proofs of the Rogers-Ramanujan identities. Another example, where this principle was applied, is the bijective proof of Stanley's hook-content formula by Remmel and Whitney [16] (this example has relations to the current paper).

The study in this paper is also related to the recent first bijective proof of the alternating sign matrix theorem [8], and one other motivation for the work presented here is to eventually simplify and "combinatorialize" this proof (but we will not complete this task in the present paper). As a matter of fact, in this bijective proof, the framework of translating computations into bijections needed to be extended and also formalized, which is partly due to the complexity one has to face there. In the preparatory paper [6], we have introduced the concept of *sijections* as the natural generalization of bijections to signed sets (signed sets $\underline{S}$ are defined to be pairs (S^+, S^-) of disjoint sets, where S^+ is the positive part and S^- is the negative part). A sijection between the two signed sets $\underline{S} = (S^+, S^-)$, $\underline{T} = (T^+, T^-)$ is a manifestation of the fact that the *sizes* of two signed sets are the same (the size of $\underline{S}$ is defined to be $|S^+| - |S^-|$), and a sijection is simply a bijection between the disjoint unions $S^+ \sqcup T^-$ and $T^+ \sqcup S^-$. In the work of Garsia and Milne, the involution principle came into the play to be able to deal with signs when translating a computation into a bijection. Combinatorialists have their reservations about bijective proofs that are based on the involution principle because a priori there is no bound on the number of steps required to determine the image of a particular element under the bijection that does not involve the cardinality of the negative part of the underlying signed set. On the other hand, in [6], the involution principle has been identified as a special case of the composition of sijections (and such compositions are an important tool in the constructions in [6,8]), and the "naturalness" of the composition might let us argue that the involution principle is not as bad as its reputation. The special case of the composition that corresponds to the involution principle concerns the situation when only the "intermediate" signed set in the composition (that is the signed set $\underline{T}$ when we compose a sijection from $\underline{S}$ to $\underline{T}$ with a sijection from $\underline{T}$ to $\underline{U}$) has a non-empty negative part. In this sense, the involution principle seems to be unavoidable when working with signed sets, and, especially when an enumeration can naturally be extended to signed sets as will be the case in this paper, signed sets, sijections and the extension of the involution principle seem to be natural.

We give a brief overview of the paper.

- In *Section 2*, we start by recalling the enumeration formulas for domino tilings of Aztec rectangles (Theorem 1) and of lozenge tilings of trapezoids (Theorem 2) with defects on one side. We also demonstrate that the formula for lozenge tilings can be seen as a special case of a weighted count of domino tilings, see Theorem 3.
- In *Section 3*, we present natural generalizations of the theorems stated in Section 2. More specifically, we give combinatorial interpretations of the formulas in Theorems 1, 2 and 3 to arbitrary integer parameters in terms of signed enumerations. In this section, we also introduce some basic facts on signed sets. The rest of the paper is then devoted to proving the extended formulas in a combinatorial manner.
- In *Section 4*, we introduce an important combinatorial construction that is then used to reduce the (extended) Aztec diamond count to the (extended) lozenge tiling count. This construction is referred to as the *fundamental construction* throughout the paper. We also show that the fundamental construction is nothing else but a signed extension of a well-known combinatorial construction due to Kuperberg and Propp which is known as urban renewal [15]. It is also speculated that the recent complicated combinatorial proof of the alternating sign matrix theorem in [6,8,7] can be simplified considerably to one that is mainly based on this fundamental construction. This is done by discussing the case $n = 3$.
- *Section 5* is very short and uses the fundamental construction to reduce Theorem 5 to the special case $(u, v) = (1, 0)$ in a combinatorial manner. In particular, this reduces Theorem 3 on Aztec diamonds to Theorem 2 on lozenge tilings.
- In *Section 6*, we provide a combinatorial proof of the special case $(u, v) = (1, 0)$ of Theorem 5. This is done by translating a computational proof of that special case into a combinatorial proof. The constructions are admittedly involved, but the exposition is accompanied by extensive python code.

2. The Weyl dimension formula in tiling counting

The purpose of this section is to review known formulas for the number of Aztec diamonds and of lozenge tilings with defects on one side as well as a common generalization.

2.1. Domino tilings of Aztec rectangles

We denote the $n \times m$ *Aztec rectangle* by $\mathrm{AR}_{m,n}$ as indicated in Fig. 1 (left) for the case $m = 8, n = 5$. We are interested in partial *domino tilings* of $\mathrm{AR}_{m,n}$ in the sense that

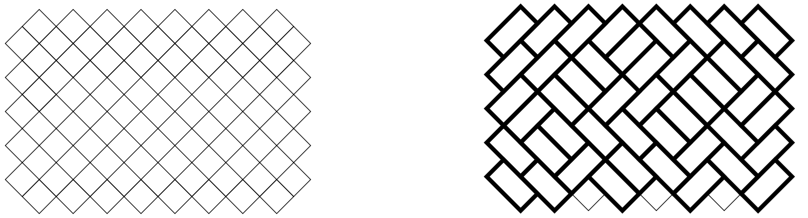


Fig. 1. $\mathrm{AR}_{8,5}$ (left) and a domino tiling with $k_1 = 1, k_2 = 2, k_3 = 4, k_4 = 6, k_5 = 8$ (right).

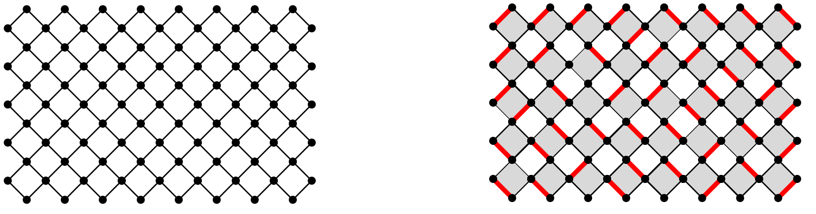


Fig. 2. "Dual" graph of $\mathrm{AR}_{8,5}$ and the matching corresponding to the domino tiling in Fig. 1.

not all unit squares in the bottom row need to be covered. An example is provided in Fig. 1 (right). There is the following well-known formula for the number of such domino tilings when prescribing the unit squares that are *not* removed.

Theorem 1. *[2] The number of domino tilings of* $\mathrm{AR}_{m,n}$*, where all unit squares in the bottom row are removed except for those in positions* $k_1, \ldots, k_n$ *is*

$$2^{\binom{n+1}{2}} \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}.$$

Note that the factor $\prod_{1 \le i < j \le n} \frac{k_j - k_i}{j-i}$ is a simple deformation of Weyl's dimension formula for representations of the general linear group. A discussion of this relation is provided in [3, Sec. 5].

Domino tilings of $\mathrm{AR}_{m,n}$ correspond to matchings of the "dual graph" (the outer face is omitted when taking the dual here), see Fig. 2. The reason for the checkerboad shading in the right figure will become clear next. The shaded squares are referred to as cells. We proceed by recalling a certain encoding of the matchings [3,1]: for each shaded square, we determine the number of matching edges on the boundary and subtract one. For our example in Fig. 2, we obtain the following $\{0, \pm 1\}$-matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In general, we obtain the following type of "partial" $n \times m$ *alternating sign matrices* (ASMs):

- The non-zero entries alternate in each row and column.
- All row sums are 1.
- The topmost non-zero entry of each column is 1 (if such an entry exists at all).
- The columns sum to 1 precisely for columns in positions $k_1, \ldots, k_n$.

This is not a bijection because there are $2^{\# \text{ of 1's}}$ matchings that correspond to a fixed partial ASM: in the cells that have two matching edges on their boundary (they correspond to the 1's in the matrix), we can "rotate" these edges along the boundary and still obtain the same matrix. It is also useful to consider the corresponding *monotone triangle*, which is obtained from the partial ASM by adding to each entry the entries in the same column above (this results in a $\{0,1\}$-matrix with i 1's in row i) and then recording row by row the positions of the 1's. In our example, we have

$$
\begin{array}{ccccccccc}
 & & & & 4 & & & & \\
 & & & 3 & & 7 & & & \\
 & & 1 & & 6 & & 8 & & \\
 & 1 & & 2 & & 6 & & 8 & \\
1 & & 2 & & 4 & & 6 & & 8
\end{array}
\quad .
$$

Monotone triangles have been introduced by Mills, Robbins and Rumsey [14] and are characterized by their weak increase along $\nearrow$-diagonals and $\searrow$-diagonals and strict increase along rows; in our case their bottom row is just $k_1, k_2, \ldots, k_n$. In order to obtain the number of Aztec diamonds, each entry that is different from its $\nwarrow$-neighbor and from its $\nearrow$-neighbor has to be weighted by 2 because such entries correspond to 1's in the partial alternating sign matrix (neighbors that actually do not exist are set to 0 for this purpose). This weighted enumeration is obviously equivalent to the straight enumeration of the objects defined next (which seem not to have appeared before in the literature).

Definition 2.1. A 2*-arrowed monotone triangle* of order n is a monotone triangle where each entry is assigned an element of $\{\nwarrow, \nearrow\}$ such that

- if an entry is assigned a $\nwarrow$, then it may not be equal to its $\nwarrow$-neighbor, and
- if an entry is assigned a $\nearrow$, then it may not be equal to its $\nearrow$-neighbor.

To see that the straight enumeration of 2-arrowed monotone triangles is equal to the 2-enumeration of partial ASMs, note that for each entry of the 2-arrowed monotone triangle that is equal to either its $\nwarrow$-neighbor or its $\nearrow$-neighbor, there is only one possible arrow assignment, while for those different from both of these neighbors, there are two possible arrow assignments.

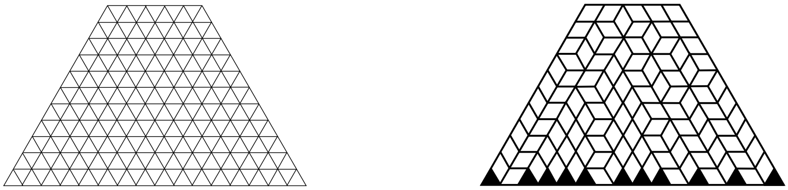


Fig. 3. $\mathrm{TT}_{16,11}$ (left) and a lozenge tiling of $\mathrm{TT}_{16,11}$ with $k_1 = 1, k_2 = 3, k_3 = 4, k_4 = 5, k_5 = 6, k_6 = 8, k_7 = 9, k_8 = 10, k_9 = 12, k_{10} = 14, k_{11} = 16$.

Note that, in the definition of 2-arrowed monotone triangles, the strict increase along rows different from the bottom row is also forced by the assignment of arrows, which implies that we could define 2-arrowed monotone triangles also simply as Gelfand-Tsetlin patterns with the arrow assignment and prescribe the bottom row (recall that Gelfand-Tsetlin patterns are defined as monotone triangles with the exception that rows need not be strictly increasing).

An example is given next, where the assigned arrows are placed above the entry. The example corresponds to the matching in Fig. 2 (right). Note that, for each entry, the orientation of the assigned arrow corresponds to the "orientation" of the matching edge among the two bottom edges of the cell that corresponds to the entry.

$$
\begin{array}{ccccccccc}
 & & & & \overset{\nearrow}{4} & & & & \\
 & & & \overset{\nwarrow}{3} & & \overset{\nwarrow}{7} & & & \\
 & & \overset{\nearrow}{1} & & \overset{\nwarrow}{6} & & \overset{\nearrow}{8} & & \\
 & \overset{\nwarrow}{1} & & \overset{\nwarrow}{2} & & \overset{\nearrow}{6} & & \overset{\nearrow}{8} & \\
\overset{\nwarrow}{1} & & \overset{\nwarrow}{2} & & \overset{\nwarrow}{4} & & \overset{\nearrow}{6} & & \overset{\nearrow}{8}
\end{array}.
$$

2.2. Lozenge tilings of trapezoids

For positive integers n, l, we refer to the isosceles trapezoid on the triangular lattice whose longer base is of length l and whose legs are of length n as an (l, n)-trapezoid and denote it by $\mathrm{TT}_{l,n}$. The $(16, 11)$-trapezoid is displayed in Fig. 3 (left). We are interested in partial lozenge tilings of $\mathrm{TT}_{l,n}$ in the sense that not all boundary unit triangles at the longer base need to be covered. An example is provided in Fig. 3 (right).

There is also a well-known formula for the number of such lozenge tilings when prescribing the unit triangles that are removed from the bottom. The appearance of the Weyl dimension formula is even more transparent here since these lozenge tilings correspond to Gelfand-Tsetlin patterns, as explained below.

Theorem 2. *The number of lozenge tilings of $\mathrm{TT}_{l,n}$ with unit triangles in positions $k_1, \ldots, k_n$ removed is*

$$\prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}.$$

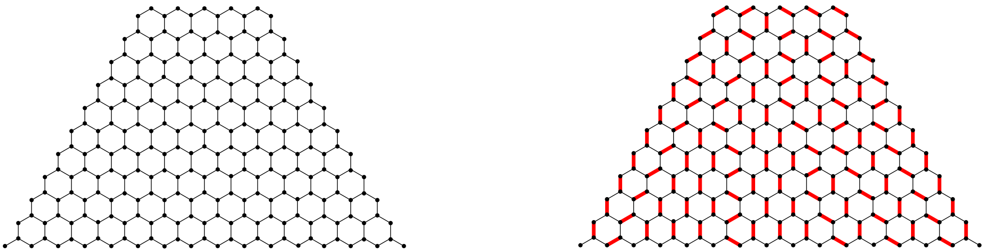


Fig. 4. The “dual” graph of $\mathrm{TT}_{16,11}$ (left) and the matching corresponding to the lozenge tiling in Fig. 3 (right).

Also in this case lozenge tilings correspond to matchings of the dual graph. This is illustrated in Fig. 4. It is also possible to encode the lozenge tilings as triangular arrays with n rows by simply recording the positions of vertical lozenges in each row. For our example we obtain

$$\begin{array}{ccccccccccccccccccc}
&&&&&&&&&3&&&&&&&&& \\
&&&&&&&&2&&5&&&&&&&& \\
&&&&&&&2&&4&&7&&&&&&& \\
&&&&&&1&&4&&6&&8&&&&&& \\
&&&&&1&&4&&5&&7&&10&&&&& \\
&&&&1&&3&&5&&6&&9&&11&&&& \\
&&&1&&3&&4&&6&&7&&10&&12&&& \\
&&1&&3&&4&&6&&7&&9&&11&&13&& \\
&1&&3&&4&&5&&7&&8&&10&&12&&14& \\
1&&3&&4&&5&&7&&8&&9&&11&&13&&15
\end{array} . \tag{2.1}$$

In general, we obtain triangular array with weak increase in $\nearrow$-direction and strict increase in $\searrow$-direction. Clearly, these patterns correspond to 2-arrowed monotone triangles where each entry is assigned $\nwarrow$. Gelfand-Tsetlin patterns are obtained by subtracting i from the i-th $\nearrow$-diagonal.

2.3. A common generalization

We introduce weights on the dual of $\mathrm{AR}_{m,n}$. For this purpose, we need the checkerboard shading indicated in Fig. 2 (right). Recall that the shaded squares are referred to as cells. For the boundary edges of cells, we introduce the following weights:

- The two top edges have weights 1.
- The bottom left edge has weight u.
- The bottom right edge has weight v.

Fig. 5. The case $v = 0$ and forced matching edges. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

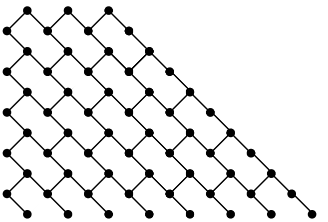


Fig. 6. Forced matching edges and adjacent edges deleted.

We denote this weighted graph by $\overline{\mathrm{AR}}_{m,n}(u,v)$. Clearly, $\overline{\mathrm{AR}}_{m,n}(1,1)$ is just the dual of $\mathrm{AR}_{m,n}$ with all edges weighted 1. On the other hand, the case $\overline{\mathrm{AR}}_{m,n}(1,0)$ corresponds to deleting the bottom right edge of each cell, see Fig. 5 (left). The deletion of these edge forces some edges to be contained in any matching, see Fig. 5 (right) where the forced edges are indicated in red (note that the bottom vertices of degree 1 do not have to be part of the matching). In Fig. 6 we have deleted all forced edges as well as all edges that are adjacent to such edges. Note that this is the dual graph of $\mathrm{TT}_{5,8}$ with one extra edge attached to all the vertices at the bottom. The latter takes into account the fact that, for the Aztec rectangle, $k_1, \ldots, k_n$ indicates which vertices are not deleted, while, for lozenge tilings, it indicates which vertices are deleted.

For the 2-arrowed monotone triangles, this corresponds to assigning to $\nwarrow$ the weight u, while we assign to $\nearrow$ the weight v, so that the total weight of an 2-arrowed monotone triangle is simply

$$u^{\#\text{ of }\nwarrow} v^{\#\text{ of }\nearrow}.$$

This simply follows from the above observation that, in the 2-arrowed monotone triangle, the orientation of an arrow corresponds to the "orientation" of the matching edge among the two bottom edges of the cell that corresponds to the entry that carries the arrow. Equivalently, this gives the following weight on the corresponding monotone triangle.

$$(u+v)^{\#\text{ entries different from }\nwarrow\text{- and }\nearrow\text{-neighbor}} \times u^{\#\text{ entries equal to }\nearrow\text{-neighbor}} v^{\#\text{ entries equal to }\nwarrow\text{-neighbor}}$$

A known common generalization of Theorems 1 and 2 is the following.

Theorem 3. *The generating function of the matchings of* $\overline{\mathrm{AR}}_{m,n}(u,v)$ *with all vertices in the bottom row removed except for those in positions* $k_1, \ldots, k_n$ *is*

$$(u+v)^{\binom{n+1}{2}} \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}.$$

The theorem can be reduced to the case $(u,v) = (1,0)$. There is a purely combinatorially proof of this reduction that is based on *urban renewal* [15], which is due to Kuperberg and Propp. In the following, we will present a generalization of Theorem 3 to arbitrary integers sequences $k_1, k_2, \ldots, k_n$ (i.e., not necessarily increasing) and present also here a combinatorial proof of the reduction to the case $(u,v) = (1,0)$. This combinatorial proof stems from a computational proof, however, we will argue that in case $k_1, \ldots, k_n$ is strictly increasing, this proof is actually equivalent to the one using urban renewal. In fact, this extension of urban renewal is one of the key ingredients in the bijective proof of the alternating sign matrix theorem presented in [8] (it appears in somewhat disguised form in Problem 18 in that paper).

3. An extension of Theorem 3

We are now working towards an alternative definition of 2-arrowed monotone triangles that will allow us to extend easily to bottom rows $\mathbf{k} = (k_1, \ldots, k_n)$ that are not necessarily increasing.

We define an *arrow row* of order n to be an element of $\{\nwarrow, \nearrow\}^n$. Such an arrow row $\mathbf{a} = (a_1, \ldots, a_n)$ deforms the Cartesian product

$$[k_1, k_2] \times [k_2, k_3] \times \ldots \times [k_{n-1}, k_n] \tag{3.1}$$

as follows:

- if $a_i = \nwarrow$, then k_i in $[k_{i-1}, k_i]$ is decreased by 1, and,
- if $a_i = \nearrow$, then k_i in $[k_i, k_{i+1}]$ is increased by 1.

This can be remembered easily as explained next: write $(a_1, \ldots, a_n)$ below the Cartesian product as follows

$$\begin{array}{cccccccccc} & [k_1,k_2] & \times & [k_2,k_3] & \times & \ldots & \times & [k_{n-1},k_n] & \\ a_1 & & a_2 & & a_3 & \ldots & a_{n-1} & & a_n \end{array}$$

and, if there is an arrow pointing to the right boundary of an interval, then the right boundary is decreased by 1, and, if there is an arrow pointing to the left boundary of an interval, then the left boundary is increased by 1. To understand the motivation for this definition, a_i should be seen as the arrow assigned to the i-th entry k_i in the bottom row of a 2-arrowed monotone triangle, because then the Cartesian product obtained by deforming (3.1) according to $\mathbf{a}$ is just the range of the penultimate row. We denote by $\mathbf{a}(\mathbf{k})$ this deformation of the Cartesian product.

Now 2-arrowed monotone triangles can also be defined by induction with respect to n: Let $\mathrm{Aztec}(k_1) = \{ \overset{\nwarrow}{k_1}, \overset{\nearrow}{k_1} \}$, and, for $n > 1$ and $\mathbf{k} = (k_1, \ldots, k_n)$,

$$\mathrm{Aztec}(\mathbf{k}) = \bigsqcup_{\mathbf{a} \in \{\nwarrow, \nearrow\}^n} \bigsqcup_{\mathbf{l} \in \mathbf{a}(\mathbf{k})} \mathrm{Aztec}(\mathbf{l}), \tag{3.2}$$

where $\bigsqcup$ denotes the disjoint union.

Similarly, for lozenge tilings, let $\mathrm{Lozenge}(k_1)$ be a one-element-set and, for $n > 1$ and $\mathbf{k} = (k_1, \ldots, k_n)$,

$$\mathrm{Lozenge}(\mathbf{k}) = \bigsqcup_{\mathbf{l} \in [k_1, k_2 - 1] \times [k_2, k_3 - 1] \times \ldots \times [k_{n-1}, k_n - 1]} \mathrm{Lozenge}(\mathbf{l}).$$

Our goal is to construct a bijection

$$\mathrm{Aztec}(\mathbf{k}) \Longrightarrow 2^{\left[\binom{n+1}{2}\right]} \times \mathrm{Lozenge}(\mathbf{k}), \tag{3.3}$$

where for a set M we denote by 2^M the power set of M and let $[N] = \{1, 2, \ldots, N\}$.

More generally, we introduce weights on $\mathrm{Aztec}(\mathbf{k})$ as follows: each element of $\mathrm{Aztec}(\mathbf{k})$ is accompanied by a sequence of arrow rows $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, where $\mathbf{a}_i$ is of length i and n is the length of $\mathbf{k}$. The weight of the element is

$$u^{\#\text{ of } \nwarrow} v^{\#\text{ of } \nearrow}, \tag{3.4}$$

where we count the total number of arrows in $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. We want our bijection to be weight-preserving, where the weights for elements in $\mathrm{Lozenge}(\mathbf{k})$ are all 1 and for sets S in $2^{\left[\binom{n+1}{2}\right]}$ is $u^{\binom{n+1}{2} - |S|} v^{|S|}$.

3.1. Signed sets and sijections

For the extension, we need the notion of signed sets (in particular signed intervals) and several constructions for signed sets such as the Cartesian product and the disjoint union. We adopt the definitions from [6] and summarize them in the following. We refer also to that paper for more details.

A *signed set* is a pair of disjoint finite sets: $\underline{S} = (S^+, S^-)$ with $S^+ \cap S^- = \emptyset$. An ordinary set S induces a signed set $\underline{S} = (S^+, S^-)$ through $S^+ = S$ and $S^- = \emptyset$ in which case we identify S and $\underline{S}$. Throughout the paper, signed sets are underlined, except when they are induced by ordinary sets as just explained. We will write $i \in \underline{S}$ to mean $i \in S^+ \cup S^-$.

Some basic notions are as follows.

- The *size* of a signed set $\underline{S}$ is $|\underline{S}| = |S^+| - |S^-|$.
- The *opposite* signed set of $\underline{S}$ is $-\underline{S} = (S^-, S^+)$. We have $|-\underline{S}| = -|\underline{S}|$.
- The *Cartesian product* of signed sets $\underline{S}$ and $\underline{T}$ is

$$\underline{S} \times \underline{T} = (S^+ \times T^+ \cup S^- \times T^-, S^+ \times T^- \cup S^- \times T^+),$$

  and we have $|\underline{S} \times \underline{T}| = |\underline{S}| \cdot |\underline{T}|$.
- The *disjoint union* of signed sets $\underline{S}$ and $\underline{T}$ is the signed set

$$\underline{S} \sqcup \underline{T} = (\underline{S} \times (\{0\}, \emptyset)) \cup (\underline{T} \times (\{1\}, \emptyset))$$

  with elements $(s, 0)$ for $s \in \underline{S}$ and $(t, 1)$ for $t \in \underline{T}$, and we have $|S \sqcup T| = |S| + |T|$.
- More generally, we can define the disjoint union of a family of signed sets $\underline{S}_t$, where the family is indexed with a signed set $\underline{T}$:

$$\bigsqcup_{t \in \underline{T}} \underline{S}_t = \bigcup_{t \in \underline{T}} (\underline{S}_t \times \underline{\{t\}}).$$

One of the crucial signed sets is the *signed interval*

$$\underline{[a, b]} = \begin{cases} ([a, b], \emptyset) & \text{if } a \leq b \\ (\emptyset, [b+1, a-1]) & \text{if } a > b \end{cases}$$

for $a, b \in \mathbb{Z}$, where $[a, b]$ stands for an interval in $\mathbb{Z}$ in the usual sense, noting that $\underline{[a, a-1]} = \emptyset$. We have $\underline{[b+1, a-1]} = -\underline{[a, b]}$ and $|\underline{[a, b]}| = b - a + 1$.

The usual properties such as associativity $(\underline{S} \sqcup \underline{T}) \sqcup \underline{U} = \underline{S} \sqcup (\underline{T} \sqcup \underline{U})$ and distributivity $(\underline{S} \sqcup \underline{T}) \times \underline{U} = \underline{S} \times \underline{U} \sqcup \underline{T} \times \underline{U}$ also hold.

The role of bijections for signed sets is played by "signed bijections", which we call *sijections*. A sijection φ from $\underline{S}$ to $\underline{T}$,

$$\varphi \colon \underline{S} \Longrightarrow \underline{T},$$

is essentially a bijection from $S^+ \sqcup T^-$ to $S^- \sqcup T^+$, but for us it will be convenient to view this bijection as an involution on the set $(S^+ \cup S^-) \sqcup (T^+ \cup T^-)$ with the property $\varphi(S^+ \sqcup T^-) = S^- \sqcup T^+$.

We can think of a sijection as a collection of a sign-reversing involution on a subset of $\underline{S}$, a sign-reversing involution on a subset of $\underline{T}$, and a sign-preserving matching between the remaining elements of $\underline{S}$ with the remaining elements of $\underline{T}$. When $S^- = T^- = \emptyset$, a sijection is simply a bijection. A sijection is clearly a manifestation of the fact that two signed sets have the same size. Indeed, if there exists a sijection $\varphi \colon \underline{S} \Longrightarrow \underline{T}$, we have $|\underline{S}| = |S^+| - |S^-| = |T^+| - |T^-| = |\underline{T}|$.

A simple sijection that is usually the building block of our more complicated sijections is the following.

Proposition 4. *Given $a, b, c \in \mathbb{Z}$, there is an explicit sijection*

$$\alpha = \alpha_{a,b,c} \colon \underline{[a,c]} \Longrightarrow \underline{[a,b]} \sqcup \underline{[b+1,c]}.$$

Construction. There are basically two cases: in the first case, either the negative parts of $\underline{[a,b]}$ and $\underline{[b+1,c]}$ are both empty (when $a \leq b < c$) or the positive parts of $\underline{[a,b]}$ and $\underline{[b+1,c]}$ are both empty (when $c \leq b < a$). In this case, the two intervals are disjoint, which implies the assertion. Otherwise, the negative part of one interval is empty, while the positive part of the other is empty. In this case, one is interval is contained the other, and the smaller interval "cancels" its copy in the bigger elements, due to the different sign. □

There are natural notions of *composition*, *Cartesian product* and *disjoint union* of sijections, see Proposition 2 in [6]. The composition is a generalization of the involution principle as indicated in the introduction.

3.2. *Extending* Aztec($\mathbf{k}$) *and* Lozenge($\mathbf{k}$)

We extend Aztec($\mathbf{k}$) to sequences $\mathbf{k} = (k_1, \ldots, k_n)$ that are not necessarily increasing as follows. For arrow rows $\mathbf{a} \in \{\nwarrow, \nearrow\}$, we define $\mathbf{a}(\mathbf{k})$ in a similar way, only (3.1) needs to be replaced by

$$\underline{[k_1, k_2]} \times \underline{[k_2, k_3]} \times \ldots \times \underline{[k_{n-1}, k_n]}.$$

The extension is denoted by $\underline{\text{Aztec}}(\mathbf{k})$. We set $\underline{\text{Aztec}}(k_1) = \text{Aztec}(k_1)$ and define inductively

$$\underline{\text{Aztec}}(\mathbf{k}) = \bigsqcup_{\mathbf{a} \in \{\nwarrow, \nearrow\}^n} \bigsqcup_{\mathbf{l} \in \mathbf{a}(\mathbf{k})} \underline{\text{Aztec}}(\mathbf{l}).$$

This is of course in perfect analogy to (3.2), except that we now work with signed intervals and Cartesian products of signed intervals as well as with disjoint unions of signed sets over index sets that are also signed sets. The weight is defined in exactly the same way as in (3.4), that is the exponents of u and v are the number of the two types of arrows in the n arrow rows.

For a signed set $\underline{S}$ and a weight function ω on $S^+ \cup S^-$, the generating function of $\underline{S}$ with respect to ω is defined as

$$\sum_{s \in S^+} \omega(s) - \sum_{s \in S^-} \omega(s).$$

We are now able to formulate the extension of Theorem 3.

Theorem 5. *For an integer sequence* $\mathbf{k} = (k_1, \ldots, k_n)$*, the generating function of* $\underline{\mathrm{Aztec}}(\mathbf{k})$ *is*

$$(u+v)^{\binom{n+1}{2}} \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j-i}.$$

This theorem is proved in Sections 5 and 6 roughly as follows. In Section 5 we use the construction from Section 4 to reduce Theorem 5 to the special case $u = 1$ and $v = 0$. In Section 6 we then deal with this special case.

Although the signed extension of the lozenge tiling count (denoted by $\underline{\mathrm{Lozenge}}(\mathbf{k})$) is in a sense the special case $u = 1$ and $v = 0$ of $\underline{\mathrm{Aztec}}(\mathbf{k})$, we still define it explicitly as follows for convenience. We let $\underline{\mathrm{Lozenge}}(k_1) = \mathrm{Lozenge}(k_1)$ and, for $n > 1$ and $\mathbf{k} = (k_1, \ldots, k_n)$,

$$\underline{\mathrm{Lozenge}}(\mathbf{k}) = \bigsqcup_{\mathbf{l} \in \underline{[k_1, k_2 - 1]} \times \underline{[k_2, k_3 - 1]} \times \ldots \times \underline{[k_{n-1}, k_n - 1]}} \underline{\mathrm{Lozenge}}(\mathbf{l}).$$

We have

$$\left|\underline{\mathrm{Lozenge}}(\mathbf{k})\right| = \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j-i}. \tag{3.5}$$

Note that Equation (3.5) was essentially proven in [4] in a non-combinatorial way.

4. A fundamental construction

Let $e_p(X_1, \ldots, X_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_p \le n} X_{i_1} X_{i_2} \cdots X_{i_p}$ denote the p-th elementary symmetric function and E_x the shift operator, i.e., $E_x p(x) = p(x+1)$. Our fundamental construction provides a sijective proof of

$$e_p(\mathrm{E}_{k_1}, \ldots, \mathrm{E}_{k_n}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j-i} = \binom{n}{p} \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j-i}, \tag{4.1}$$

where $\prod_{1 \le i < j \le n} \frac{k_j - k_i}{j-i}$ is interpreted as the size of $\underline{\mathrm{Lozenge}}(\mathbf{k})$. In [8, Problem 18], such a proof was given for

$$e_p(\Delta_{k_1}, \ldots, \Delta_{k_n}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j-i} = 0,$$

where $\Delta_x = \mathrm{E}_x - \mathrm{Id}$ is the difference operator. This identity can be seen to be equivalent to the one that is proved here. The advantage of the version that involves shift operators rather than difference operators is that it is a bijection if $k_1, \ldots, k_n$ is strictly increasing, i.e., the negative parts of the signed sets involved are empty.

In order to set up for a sijective proof, we define $\mathbf{e}(\mathbf{k}) \in \mathbb{Z}^n$ to be the following deformation of $\mathbf{k} = (k_1, \ldots, k_n)$ for $\mathbf{e} \subseteq [n]$:

$$\mathbf{e}(\mathbf{k})_i = \begin{cases} k_i + 1 & i \in \mathbf{e} \\ k_i & i \notin \mathbf{e} \end{cases}$$

Moreover, we let

$$\begin{aligned} &\mathbf{e}\left(\underline{[k_1, k_2 - 1]} \times \underline{[k_2, k_3 - 1]} \times \ldots \times \underline{[k_{n-1}, k_n - 1]}\right) \\ &\quad = \underline{[\mathbf{e}(\mathbf{k})_1, \mathbf{e}(\mathbf{k})_2 - 1]} \times \underline{[\mathbf{e}(\mathbf{k})_2, \mathbf{e}(\mathbf{k})_3 - 1]} \times \ldots \times \underline{[\mathbf{e}(\mathbf{k})_{n-1}, \mathbf{e}(\mathbf{k})_n - 1]}. \end{aligned} \tag{4.2}$$

It is crucial to note the following relation to $\mathbf{a}(\mathbf{k})$: Suppose $\mathbf{a} = \{\nwarrow, \nearrow\}^n$, then let $\mathbf{e} \subseteq [n]$ be such that $i \in \mathbf{e}$ if and only if $a_i = \nearrow$.

We present a sijection

$$\bigsqcup_{\mathbf{e} \in \binom{[n]}{p}} \underline{\text{Lozenge}}(\mathbf{e}(\mathbf{k})) \Longrightarrow \binom{[n]}{p} \times \underline{\text{Lozenge}}(\mathbf{k}),$$

and this establishes a sijective proof of (4.1). Here, $\binom{M}{p}$ is the set of all subsets of M of cardinality p. This sijection is referred to as the fundamental construction in the following. We present two descriptions of this sijection in Problems 6 and 8; more precisely, the two sijections actually differ in a minor detail.

The construction is inductively, with respect to n. By definition,

$$\begin{aligned} \bigsqcup_{\mathbf{e} \in \binom{[n]}{p}} \underline{\text{Lozenge}}(\mathbf{e}(\mathbf{k})) &= \bigsqcup_{\mathbf{e} \in \binom{[n]}{p}} \bigsqcup_{\mathbf{l} \in \mathbf{e}\left(\underline{[k_1,k_2-1]} \times \underline{[k_2,k_3-1]} \times \ldots \times \underline{[k_{n-1},k_n-1]}\right)} \underline{\text{Lozenge}}(\mathbf{l}) \\ &= \bigsqcup_{\mathbf{l} \in \bigsqcup_{\mathbf{e} \in \binom{[n]}{p}} \mathbf{e}\left(\underline{[k_1,k_2-1]} \times \underline{[k_2,k_3-1]} \times \ldots \times \underline{[k_{n-1},k_n]}\right)} \underline{\text{Lozenge}}(\mathbf{l}) \end{aligned} \tag{4.3}$$

We modify the index set of the disjoint union on the right-hand side, i.e.,

$$\bigsqcup_{\mathbf{e} \in \binom{[n]}{p}} \mathbf{e}\left(\underline{[k_1, k_2 - 1]} \times \underline{[k_2, k_3 - 1]} \times \ldots \times \underline{[k_{n-1}, k_n - 1]}\right).$$

In order to describe this modification, we need additional notation: for $\mathbf{e} \subseteq [n]$, we set

$$\widehat{\mathbf{e}}\left(\underline{[k_1, k_2 - 1]} \times \ldots \times \underline{[k_{n-1}, k_n - 1]}\right) = \prod_{i=1}^{n-1} \begin{cases} \underline{[k_i, k_{i+1} - 1]} & i \notin \mathbf{e} \\ \underline{[k_i + 1, k_{i+1}]} & i \in \mathbf{e} \end{cases}. \tag{4.4}$$

So here the effect of $\mathbf{e}$ is that it shifts the i-th interval in the Cartesian product by 1 if and only if $i \in \mathbf{e}$ rather than it shifts k_i. Since there are only $n-1$ intervals, the action does not depend on whether or not $n \in \mathbf{e}$.

Moreover, for Cartesian products of intervals $\underline{[a_1, b_1]} \times \ldots \underline{[a_n, b_n]}$ (signed boxes), overlining such as

$$\overline{\underline{[a_1, b_1]} \times \cdots \times \underline{[a_n, b_n]}}$$

indicates that we exclude elements $(l_1, \ldots, l_n) \in \underline{[a_1, b_1]} \times \cdots \times \underline{[a_n, b_n]}$ with $l_i = l_{i+1}$ for an i. Also note that for signed boxes either the positive part or the negative part is empty, which makes it possible to assign a sign to signed boxes; it will be convenient for us to assign both signs to empty signed boxes.

Problem 6. *Let $0 \leq p \leq n$ and $k_i \neq k_{i+1}$ for all $i \in \{1, 2, \ldots, n-1\}$. We construct an explicit sijection between*

$$\bigsqcup_{\mathbf{e} \in \binom{[n]}{p}} \overline{\mathbf{e}\left(\underline{[k_1, k_2 - 1]} \times \underline{[k_2, k_3 - 1]} \times \ldots \times \underline{[k_{n-1}, k_n - 1]}\right)}$$

and

$$\bigsqcup_{\mathbf{e} \in \binom{[n]}{p}} \overline{\widehat{\mathbf{e}}\left(\underline{[k_1, k_2 - 1]} \times \underline{[k_2, k_3 - 1]} \times \ldots \times \underline{[k_{n-1}, k_n - 1]}\right)}.$$

The sijection is the identity on the overlined boxes (but not on the indexing sets $\mathbf{e}$) in the following sense: each element on the left hand side is a pair $(\mathbf{l}, \mathbf{e})$ with $\mathbf{e} \in \binom{[n]}{p}$ with $\mathbf{l} \in \overline{\mathbf{e}\left(\underline{[k_1, k_2 - 1]} \times \ldots \times \underline{[k_{n-1}, k_n - 1]}\right)}$, and we will construct another set $\mathbf{f} \in \binom{[n]}{p}$ such that

$$\mathbf{l} \in \overline{\widehat{\mathbf{f}}\left(\underline{[k_1, k_2 - 1]} \times \ldots \times \underline{[k_{n-1}, k_n - 1]}\right)}.$$

This assignment is sign-preserving in the sense that the signed boxes $\mathbf{e}\left(\underline{[k_1, k_2 - 1]} \times \ldots \times \underline{[k_{n-1}, k_n - 1]}\right)$ and $\widehat{\mathbf{f}}\left(\underline{[k_1, k_2 - 1]} \times \ldots \times \underline{[k_{n-1}, k_n - 1]}\right)$ have the same sign.

Construction. Fix an element $(\mathbf{l}, \mathbf{e})$ from the left-hand side. Initially, we set $\mathbf{f} = \mathbf{e}$ and modify $\mathbf{f}$ while working from $i = 1$ to $i = n - 1$ through the intervals $\underline{[\mathbf{e}(\mathbf{k})_i, \mathbf{e}(\mathbf{k})_{i+1} - 1]}$ in the Cartesian product $\mathbf{e}\left(\underline{[k_1, k_2 - 1]} \times \underline{[k_2, k_3 - 1]} \times \ldots \times \underline{[k_{n-1}, k_n - 1]}\right)$.

Suppose we have reached $i \in \{1, \ldots, n-1\}$. We consider the deformation induced by the set $\mathbf{f}$ of the Cartesian product $\underline{[k_1, k_2 - 1]} \times \underline{[k_2, k_3 - 1]} \times \ldots \times \underline{[k_{n-1}, k_n - 1]}$ where we apply to $\underline{[k_j, k_{j+1} - 1]}$ the deformation given in (4.4) for $j < i$ and the deformation given in (4.2) for $j \geq i$; this set is denoted by $\mathcal{B}(\mathbf{f})$. By $\mathcal{B}'(\mathbf{f})$, we denote the deformation obtained when applying (4.4) for $j \leq i$ and (4.4) for $j > i$.

Now the crucial point is to observe that $\mathbf{l} \in \mathcal{B}(\mathbf{f})$ implies $\mathbf{l} \in \mathcal{B}'(\mathbf{f})$, except when $|\{i, i+1\} \cap \mathbf{f}| = 1$ and $l_i = k_{i+1}$ as explained next:

- If $|\{i, i+1\} \cap \mathbf{f}| = 0, 2$, then $\mathcal{B}(\mathbf{f}) = \mathcal{B}'(\mathbf{f})$.

- If $i \notin \mathbf{f}$ and $i+1 \in \mathbf{f}$, then the i-th interval in the Cartesian product $\mathcal{B}(\mathbf{f})$ is $\underline{[k_i, k_{i+1}]}$, while in $\mathcal{B}'(\mathbf{f})$ it is $\underline{[k_i, k_{i+1}-1]}$. If $k_{i+1} < k_i$, then the first interval is contained in the second. If $k_i \le k_{i+1}$, then the only element in the first interval that is not also in the second is k_{i+1}.
- If $i \in \mathbf{f}$ and $i+1 \notin \mathbf{f}$, then the i-th interval in the Cartesian product $\mathcal{B}(\mathbf{f})$ is $\underline{[k_i+1, k_{i+1}-1]}$, while in $\mathcal{B}'(\mathbf{f})$ it is $\underline{[k_i+1, k_{i+1}]}$. If $k_i < k_{i+1}$, then the first interval is contained in the second. If $k_{i+1} \le k_i$, then the only element in first interval that is not also in the second is k_{i+1}.

In the exceptional case (i.e., when $|\{i, i+1\} \cap \mathbf{f}| = 1$ and $l_i = k_{i+1}$), we replace $\mathbf{f}$ by the symmetric difference of $\mathbf{f}$ and $\{i, i+1\}$; otherwise we do nothing. This implies now that l_i is in the i-th interval of the modified Cartesian product $\mathcal{B}'(\mathbf{f})$, using the assumption $k_i \neq k_{i+1}$. The only other interval that was possibly affected by the change of $\mathbf{f}$ is the $(i+1)$-st, however, the fact that $k_{i+1} = l_i \neq l_{i+1}$ implies that l_{i+1} is in this interval also after the modification of $\mathbf{f}$.

It is not difficult to construct the inverse mapping. $\square$

Example 7. *We consider the example* $k_1 = 1, k_2 = 3, k_3 = 4, k_4 = 7, k_5 = 8$ *and* $\mathbf{e} = \{1, 3, 5\}$. *Then*

$$\mathbf{e}\left(\underline{[k_1, k_2-1]} \times \underline{[k_2, k_3-1]} \times \underline{[k_3, k_4-1]} \times \underline{[k_4, k_5-1]}\right) = \underline{[2,2]} \times \underline{[3,4]} \times \underline{[5,6]} \times \underline{[7,8]}.$$

We choose $\mathbf{l} = (2, 4, 5, 8)$ *as element of the Cartesian product. Applying the construction from Problem 6, we obtain* $\mathbf{f} = \{1, 2, 4\}$. *Observe that*

$$\widehat{\mathbf{f}}\left(\underline{[k_1, k_2-1]} \times \underline{[k_2, k_3-1]} \times \underline{[k_3, k_4-1]} \times \underline{[k_4, k_5-1]}\right) = \underline{[2,3]} \times \underline{[4,4]} \times \underline{[4,6]} \times \underline{[8,8]},$$

and that $\mathbf{l}$ *is also an element of this Cartesian product.*

We present a second version of Problem 6. It avoids the use of overlined boxes at the cost of involving non-empty negative parts also if $k_1, k_2, \ldots, k_n$ is strictly increasing. More importantly, the construction more obviously has a "computational flavor" although the sijection is essentially (up to overlining) the same as in Construction 6.

Problem 8. *Let* $n \ge 2, \mathbf{e} \subseteq [n]$ *and* $k_1, \ldots, k_n$ *be integers. We construct an explicit sijection*

$$\bigsqcup_{\mathbf{e} \in \binom{[n]}{p}} \mathbf{e}\left(\underline{[k_1, k_2-1]} \times \ldots \times \underline{[k_{n-1}, k_n-1]}\right)$$

$$\Longrightarrow \bigsqcup_{\substack{I\subseteq[2,n-1]\\ I\cap(I-1)=\emptyset}} \bigsqcup_{\mathbf{e}\in\binom{[n]\setminus I\cup(I-1)}{p-|I|}} (-1)^{|I|} \prod_{i=1}^{n-1} \begin{cases} \{k_{i+1}\} & i+1\in I \\ \{k_i\} & i\in I \\ \underline{[k_i,k_{i+1}-1]} & i,i+1\notin I, i\notin\mathbf{e} \\ \underline{[k_i+1,k_{i+1}]} & i,i+1\notin I, i\in\mathbf{e} \end{cases}.$$

Construction. We proceed by induction with respect to n. For $n=2$ the result is trivial if $p=0,2$. For $p=1$, the left hand side is

$$\underline{[k_1+1,k_2-1]} \sqcup \underline{[k_1,k_2]} \overset{\alpha}{\Longrightarrow} \underline{[k_1+1,k_2-1]} \sqcup \underline{[k_1,k_2-1]} \sqcup \underline{[k_2,k_2]}$$
$$\overset{\alpha}{\Longrightarrow} \underline{[k_1+1,k_2]} \sqcup \underline{[k_1,k_2-1]},$$

and we arrive at a right hand side after two applications of α, where here and throughout the paper α is the sijection from Proposition 4.

Now we distinguish between $1\in\mathbf{e}$ and $1\notin\mathbf{e}$. The left hand side is

$$\bigsqcup_{\mathbf{e}\in\binom{[2,n]}{p}} \underline{[k_1,\mathbf{e}(\mathbf{k})_2-1]} \times \left(\prod_{i=2}^{n-1} \underline{[\mathbf{e}(\mathbf{k})_i,\mathbf{e}(\mathbf{k})_{i+1}-1]}\right)$$
$$\sqcup \bigsqcup_{\mathbf{e}\in\binom{[2,n]}{p-1}} \underline{[k_1+1,\mathbf{e}(\mathbf{k})_2-1]} \times \left(\prod_{i=2}^{n-1} \underline{[\mathbf{e}(\mathbf{k})_i,\mathbf{e}(\mathbf{k})_{i+1}-1]}\right).$$

Applying α twice as well as $\underline{[a,b]} = -\underline{[b+1,a-1]}$, we obtain

$$\bigsqcup_{\mathbf{e}\in\binom{[2,n]}{p}} \underline{[k_1,k_2-1]} \times \left(\prod_{i=2}^{n-1} \underline{[\mathbf{e}(\mathbf{k})_i,\mathbf{e}(\mathbf{k})_{i+1}-1]}\right)$$
$$\sqcup \bigsqcup_{\mathbf{e}\in\binom{[2,n]}{p}} -\underline{[\mathbf{e}(\mathbf{k})_2,k_2-1]} \times \left(\prod_{i=2}^{n-1} \underline{[\mathbf{e}(\mathbf{k})_i,\mathbf{e}(\mathbf{k})_{i+1}-1]}\right)$$
$$\sqcup \bigsqcup_{\mathbf{e}\in\binom{[2,n]}{p-1}} \underline{[k_1+1,k_2]} \times \left(\prod_{i=2}^{n-1} \underline{[\mathbf{e}(\mathbf{k})_i,\mathbf{e}(\mathbf{k})_{i+1}-1]}\right)$$
$$\sqcup \bigsqcup_{\mathbf{e}\in\binom{[2,n]}{p-1}} -\underline{[\mathbf{e}(\mathbf{k})_2,k_2]} \times \left(\prod_{i=2}^{n-1} \underline{[\mathbf{e}(\mathbf{k})_i,\mathbf{e}(\mathbf{k})_{i+1}-1]}\right),$$

(Here we also use the construction for the Cartesian product of sijections as given in Proposition 4, applied to a simple case.) By induction, the first and third term can be combined to obtain the terms in the codomain with $2\notin I$.

The second term is empty unless $2\in\mathbf{e}$, while the fourth term is empty unless $2\notin\mathbf{e}$. We obtain

$$\bigsqcup_{\mathbf{e} \in \binom{[3,n]}{p-1}} \underline{[k_2,k_2]} \times \underline{[k_2+1, \mathbf{e}(\mathbf{k})_3 - 1]} \times \left(\prod_{i=3}^{n-1} \underline{[\mathbf{e}(\mathbf{k})_i, \mathbf{e}(\mathbf{k})_{i+1} - 1]} \right)$$
$$\sqcup \bigsqcup_{\mathbf{e} \in \binom{[3,n]}{p-1}} -\underline{[k_2,k_2]} \times \underline{[k_2, \mathbf{e}(\mathbf{k})_3 - 1]} \times \left(\prod_{i=3}^{n-1} \underline{[\mathbf{e}(\mathbf{k})_i, \mathbf{e}(\mathbf{k})_{i+1} - 1]} \right)$$

and using α and induction, this can be combined to the term in the codomain with $2 \in I$. $\square$

Now we can replace the index set in the disjoint union in (4.3) by the signed set in the codomain of the sijection in Problem 8. As $\underline{\text{Lozenge}}(\mathbf{l}) = \emptyset$ if $\mathbf{l}$ is in this codomain unless $I = \emptyset$, we may discard the cases when $I \neq \emptyset$. Alternatively, we can also use Problem 6, noting that the overlines can be removed as $\underline{\text{Lozenge}}(\mathbf{l}) = \emptyset$ if $l_i = l_{i+1}$ for an i. With this, we have constructed a sijection from the right-hand side of (4.3) to

$$\bigsqcup_{\mathbf{l} \in \bigsqcup_{\mathbf{e} \in \binom{[n]}{p}} \hat{\mathbf{e}}\left(\underline{[k_1,k_2-1]} \times \underline{[k_2,k_3-1]} \times \ldots \times \underline{[k_{n-1},k_n-1]}\right)} \underline{\text{Lozenge}}(\mathbf{l})$$
$$= \bigsqcup_{\mathbf{l} \in \underline{[k_1,k_2-1]} \times \underline{[k_2,k_3-1]} \times \ldots \times \underline{[k_{n-1},k_n-1]}} \bigsqcup_{\mathbf{e} \in \binom{[n]}{p}} \underline{\text{Lozenge}}(\mathbf{e}(\mathbf{l})),$$

where $\mathbf{e}(\mathbf{l})$ is defined in the obvious way, i.e., as $\mathbf{e}'(\mathbf{l})$ with $\mathbf{e}' = \mathbf{e} \cap [n-1]$. (Here we use the construction of the disjoint union of sijections as given by Propostion 2 in [6], again applied to a simple case.) As there is no effect on $\mathbf{e}(\mathbf{l})$ whether or not n is in $\mathbf{e}$, we obtain a sijection to

$$\bigsqcup_{\mathbf{l} \in \underline{[k_1,k_2-1]} \times \underline{[k_2,k_3-1]} \times \ldots \times \underline{[k_{n-1},k_n-1]}} \left(\bigsqcup_{\mathbf{e} \in \binom{[n-1]}{p}} \underline{\text{Lozenge}}(\mathbf{e}(\mathbf{l})) \sqcup \bigsqcup_{\mathbf{e} \in \binom{[n-1]}{p-1}} \underline{\text{Lozenge}}(\mathbf{e}(\mathbf{l})) \right). \tag{4.5}$$

By the induction hypothesis, we have sijections

$$\bigsqcup_{\mathbf{e} \in \binom{[n-1]}{p}} \underline{\text{Lozenge}}(\mathbf{e}(\mathbf{l})) \to \binom{[n-1]}{p} \times \underline{\text{Lozenge}}(\mathbf{l})$$
$$\bigsqcup_{\mathbf{e} \in \binom{[n-1]}{p-1}} \underline{\text{Lozenge}}(\mathbf{e}(\mathbf{l})) \to \binom{[n-1]}{p-1} \times \underline{\text{Lozenge}}(\mathbf{l}).$$

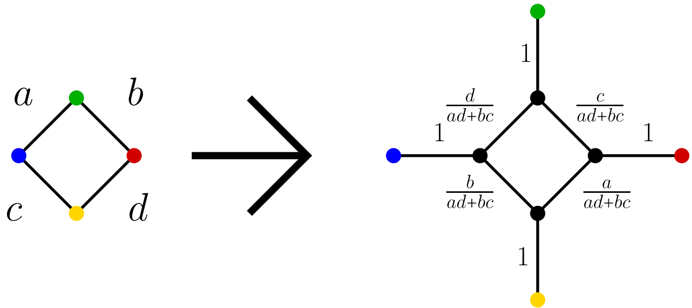
Using the distributivity and the obvious bijection

$$\binom{[n-1]}{p} \sqcup \binom{[n-1]}{p-1} \to \binom{[n]}{p},$$

the sijection is fully constructed.

Urban renewal

We indicate that in the special case when $k_1 < k_2 < \ldots < k_n$, this construction is obtained by applying urban renewal. This should also be compared with considerations of Lai, see for e.g. [13]. Urban renewal is a local replacement rule in a graph that preserves the matching generating function of that graph up to a multiplicative factor. Concretely, it is the following replacement rule. The quantities next to the edges indicate their weights.



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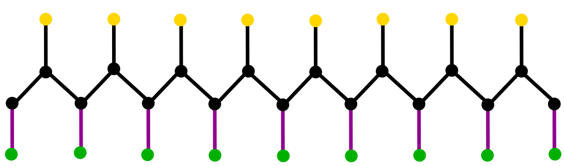
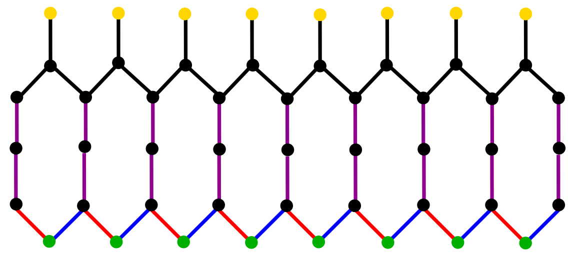
The blue, green, red and yellow vertices are the points of connections to the rest of the graph. The matching generating function of the original graph can be obtained from the matching generating function of the modified graph by multiplication with $ad + bc$. The proof is simply by considering all (not necessarily perfect) matchings of the two subgraphs, where in the right graph all black vertices need to be covered, and observing that the matching generating function coincides up to the factor when the corresponding blue, green, red and yellow vertices are covered. We will apply this to the case $a = b = 1$.

We consider the trapezoidal honeycomb graph, i.e., the dual of $\mathrm{TT}_{l,n}$. For $l = 9$, the bottom layer of this graph is



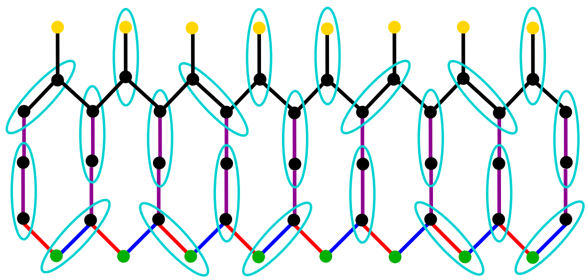
where we have added vertical edges (purple) at the bottom and the yellow vertices connect the layer to the remaining graph. Not all green vertices will be matching covered, more precisely those in positions $k_1, k_2, \ldots, k_n$ are the ones that are covered, which clearly corresponds to removing the bottom vertices in these positions in the original graph (see Theorem 2). Now we attach another edge to each bottom edge (purple) and then a zig-zag of red and blue edges at the bottom. For $l = 9$, this gives the following extended bottom layer.



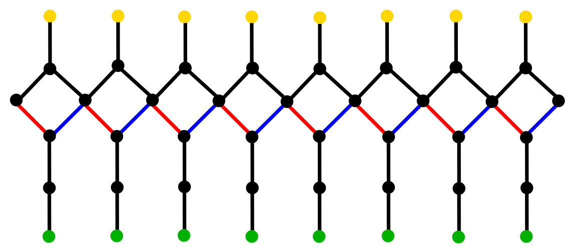
We assign to all red edges the weight u and to all blue edges the weight v. Then it is not hard to see that the coefficient of $u^{n-p}v^p$ of the matching generating function of this extended graph (including the part above the layer) is

$$e_p(\mathrm{E}_{k_1}, \ldots, \mathrm{E}_{k_n}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i},$$

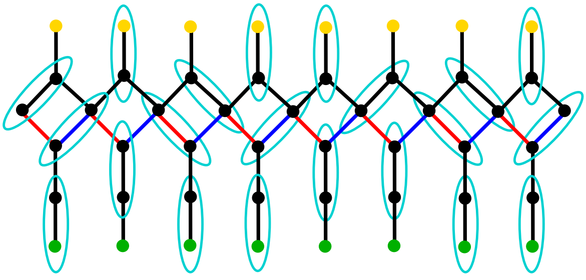
where $k_1, k_2, \ldots, k_n$ are the positions (counted from the left) of the green vertices that are covered with matching edges. Example 7 corresponds to the following: the green vertices that are matching covered are precisely those in positions $(1, 3, 4, 7, 8) = (k_1, k_2, k_3, k_4, k_5)$, among them are those in positions $1, 3, 5$ (here we are talking about the relative positions in $\mathbf{e}$) are covered by an /-edge, and the yellow matching covered vertices are those in positions $(2, 4, 5, 8) = (l_1, l_2, l_3, l_4)$.



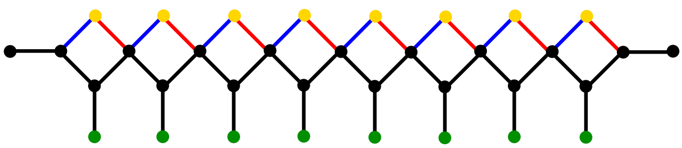
The matching generating function surely does not change when contracting all the purple edges and adding paths of length two at the bottom. This results in the following layer.



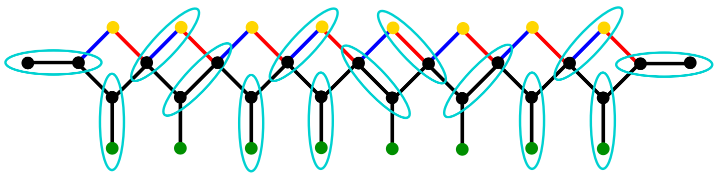
In our example, this leads to the following matching.



Next we apply urban renewal to the $l-1$ squares. We gain a factor of $(u+v)^{l-1}$ and note that the red edges carry the weight $\frac{u}{u+v}$, while the blue edge carry the weight $\frac{v}{u+v}$; the black edges in the squares carry the weight $\frac{1}{u+v}$. We contract the paths of length 2 between the squares, towards the top and towards the bottom.

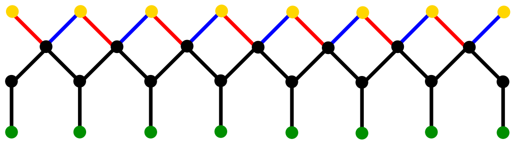


There is now a certain choice to determine the matching after the transformation, which we adjust according to Problem 6. For our example, this will give the following.

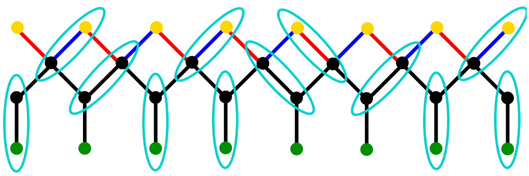


The underlying choice is the following. In principle, there are three types of squares according to the number of matching edges they contain: $0, 1$ or 2. Urban renewal transforms a square with i edges into a square with $2-i$ edges. For $i=1$, an /-edge is transformed into the other /-edge, while an \-edge is transformed into the other \-edge; for $i=2$ the transformation is clear. To determine what we do if $i=0$, note first that there is in principal one more square with 2 matching edges than with 0 matching edges before the application of urban renewal. In the matching obtained after the application of urban renewal, we choose for the squares that had previously $i=0$ matching edges the same configuration of the two edges (there are two possible configurations) then in the closest square to the left that had $i=2$ matching edges before urban renewal. The rightmost square with 2 matching edges will determine which we pick from u and v for the extra factor.

The two horizontal edges on the left and right must be contained in the matching. Since their weight is 1 we can simplify delete them as well as all adjacent edges. From the factor $(u+v)^{l-1}$ we "take" $(u+v)^{l-2}$ to multiply the weights of edges incident with the $l - 2$ vertices of degree 4 in the subgraph by $u + v$. After this modification, red edges again have weight u, while blue edges have weight v. We obtain the following reduced graph.



The original generating function can be obtained from this by multiplying with $u + v$: As indicated, the factor $u + v$ encodes the two possible configurations for the rightmost square with two matching edges in the original graph. In our example, we obtain the following.



The matching covered yellow vertices are of course still those in positions $(2, 4, 5, 8) = (l_1, l_2, l_3, l_4)$, but among them those covered by a /-edge are in position $1, 2, 4$, which corresponds to the set $\mathbf{f}$ determined in Example 7.

This is the relation between Problem 6 and urban renewal that holds true also beyond our particular example. Note that there is another natural choice for choosing the matching by letting the $i = 0$ squares copying the configuration of the closest square with $i = 2$ to the *right*, and this would correspond to working from right to left instead of working from left to right in Problem 6. We have reached a "state" that corresponds to (4.5), and we can apply induction now.

4.1. Outlook: towards a simplified bijective proof of the alternating sign matrix theorem

A crucial tool in [8] was the operator formula for the number of monotone triangles with prescribed bottom row [5]. The formula states that the number of monotone triangles with bottom row $k_1, \ldots, k_n$ is

$$\prod_{1 \le p < q \le n} (\mathrm{E}_{k_p} + \mathrm{E}_{k_q}^{-1} - \mathrm{E}_{k_p} \mathrm{E}_{k_q}^{-1}) \prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}.$$

In a sense, the operator formula follows from the recursion underlying monotone triangles. More concretely, if $\alpha(n; k_1, \ldots, k_n)$ is the number of monotone triangles with bottom row $(k_1, \ldots, k_n)$, then

$$\alpha(n;k_1,\ldots,k_n) = \sum_{\substack{1 \le l_1 \le k_1 \le l_2 \le k_2 \le \ldots \le l_{n-1} \le k_n \\ l_1 < l_2 < \ldots < l_{n-1}}} \alpha(n-1;l_1,\ldots,l_{n-1}).$$

It helps to write the summation "operator" $\sum_{\substack{1 \le l_1 \le k_1 \le l_2 \le k_2 \le \ldots \le l_{n-1} \le k_n \\ l_1 < l_2 < \ldots < l_{n-1}}}$ in terms of sums over intervals $\sum_{i=a}^{b} f(i)$. One such possibility that has been used so far is

$$\begin{aligned} &\alpha(n;k_1,\ldots,k_n) \\ &= \left[\prod_{i=1}^{n} (\mathrm{E}_{k_i'} + \mathrm{E}_{k_i}^{-1} - \mathrm{E}_{k_i'} \mathrm{E}_{k_i}^{-1}) \sum_{l_1=k_1'}^{k_2} \sum_{l_2=k_2'}^{k_3} \ldots \sum_{l_{n-1}=k_{n-1}'}^{k_n} \alpha(n-1;l_1,\ldots,l_{n-1}) \right]_{k_i=k_i'} . \end{aligned}$$

Another possibility to write this is given next: For each $i \in \{2,3,\ldots,n-1\}$, we define two operators L_i and R_i that are applied to the summation operator $\sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \ldots \sum_{l_{n-1}=k_{n-1}}^{k_n}$ as follows:

$$\begin{aligned} &L_i \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \ldots \sum_{l_{n-1}=k_{n-1}}^{k_n} s(l_1,\ldots,l_{n-1}) \\ &\quad := \mathrm{E}_{k_1} \mathrm{E}_{k_2} \cdots \mathrm{E}_{k_{i-1}} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \ldots \sum_{l_{n-1}=k_{n-1}}^{k_n} \mathrm{E}_{l_1}^{-1} \mathrm{E}_{l_2}^{-1} \cdots \mathrm{E}_{l_{i-1}}^{-1} s(l_1,\ldots,l_{n-1}), \\ &R_i \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \ldots \sum_{l_{n-1}=k_{n-1}}^{k_n} s(l_1,\ldots,l_{n-1}) \\ &\quad := \mathrm{E}_{k_{i+1}}^{-1} \mathrm{E}_{k_{i+2}}^{-1} \cdots \mathrm{E}_{k_n}^{-1} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \ldots \sum_{l_{n-1}=k_{n-1}}^{k_n} \mathrm{E}_{l_i} \mathrm{E}_{l_{i+1}} \ldots \mathrm{E}_{l_{n-1}} s(l_1,\ldots,l_{n-1}). \end{aligned}$$

Now the summation operator underlying the recursion (omitting the summand) can also be written as follows.

$$\prod_{i=2}^{n-1} (L_i + R_i - L_i R_i) \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \ldots \sum_{l_{n-1}=k_{n-1}}^{k_n}$$

This is well-defined because the relevant operators commute.

We exemplify on the case $n=3$ why our fundamental construction may lead to a combinatorial proof of the operator formula. Indeed, what we need to prove is that

$$\begin{aligned} &\left[\mathrm{E}_{k_1} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \mathrm{E}_{l_1}^{-1} + \mathrm{E}_{k_3}^{-1} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \mathrm{E}_{l_2} - \mathrm{E}_{k_1} \mathrm{E}_{k_3}^{-1} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \mathrm{E}_{l_1}^{-1} \mathrm{E}_{l_2} \right] \\ &\quad \times (\mathrm{E}_{l_1} + \mathrm{E}_{l_2}^{-1} - \mathrm{E}_{l_1} \mathrm{E}_{l_2}^{-1})(l_2 - l_1 + 1) \end{aligned} \tag{4.6}$$

is equal to

$$(\mathrm{E}_{k_1}+\mathrm{E}_{k_2}^{-1}-\mathrm{E}_{k_1}\mathrm{E}_{k_2}^{-1})(\mathrm{E}_{k_2}+\mathrm{E}_{k_3}^{-1}-\mathrm{E}_{k_2}\mathrm{E}_{k_3}^{-1})(\mathrm{E}_{k_1}+\mathrm{E}_{k_3}^{-1}-\mathrm{E}_{k_1}\mathrm{E}_{k_3}^{-1})\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3}(l_2-l_1+1), \tag{4.7}$$

where we also use

$$\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3}(l_2-l_1+1)=\frac{1}{2}(k_2-k_1+1)(k_3-k_2+1)(k_3-k_1+1).$$

We rewrite

$$\left[\mathrm{E}_{k_1}\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3}\mathrm{E}_{l_1}^{-1}+\mathrm{E}_{k_3}^{-1}\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3}\mathrm{E}_{l_2}-\mathrm{E}_{k_1}\mathrm{E}_{k_3}^{-1}\sum_{l_1=k_1}^{k_2}\sum_{l_2=k_2}^{k_3}\mathrm{E}_{l_1}^{-1}\mathrm{E}_{l_2}\right] \\ \times(\mathrm{E}_{l_1}+\mathrm{E}_{l_2}^{-1}-\mathrm{E}_{l_1}\mathrm{E}_{l_2}^{-1})$$

from (4.6): We may omit sums by using the following.

$$\sum_{l_i=k_i}^{k_{i+1}}a(l_i)=\left[\sum_{x=k_i}^{k_{i+1}}\mathrm{E}_{l_i}^{x}\,a(l_i)\right]\Bigg|_{l_i=0}=\left[\frac{\mathrm{E}_{l_i}^{k_{i+1}+1}-\mathrm{E}_{l_i}^{k_i}}{\mathrm{E}_{l_i}-\mathrm{Id}}a(l_i)\right]\Bigg|_{l_i=0}$$

Here, $\frac{1}{\mathrm{E}_{l_i}-\mathrm{Id}}a(l_i)$ stands for the following formal sum.

$$\frac{1}{\mathrm{E}_{l_i}-\mathrm{Id}}a(l_i)=-\sum_{j=0}^{\infty}\mathrm{E}_{l_i}^{j}\,a(l_i)=-a(l_i)-a(l_i+1)-a(l_i+2)-\dots$$

Letting

$$P(X_1,X_2)=\frac{1-X_1+X_1X_2}{(X_1-1)(X_2-1)X_2}(X_1^{k_1}X_2^{k_2}-X_1^{k_2}X_2^{k_2}+X_1^{k_2}X_2^{k_2+1}-X_1^{k_2+1}X_2^{k_2+1} \\ -X_1^{k_1}X_2^{k_3+1}+X_1^{k_2+1}X_2^{k_3+1}),$$

we see that (4.6) is equal to

$$[\,P(\mathrm{E}_{l_1},\mathrm{E}_{l_2})(l_2-l_1+1)]|_{l_i=0}$$

We can get rid of the second, third and fourth monomial in the long factor of $P(X_1,X_2)$. Indeed, this part can be written as

$$\frac{1-X_1+X_1X_2}{(X_1-1)(X_2-1)X_2}(-X_1^{k_2}X_2^{k_2}+X_1^{k_2}X_2^{k_2+1}-X_1^{k_2+1}X_2^{k_2+1})$$
$$=-\frac{(1-X_1+X_1X_2)(1-X_2+X_1X_2)}{(X_1-1)(X_2-1)X_2}X_1^{k_2}X_2^{k_2}=:P_1(X_1,X_2).$$

As $P_1(X_1,X_2)X_2$ is symmetric in X_1,X_2, the fundamental construction can be used to show bijectively that

$$P_1(\mathrm{E}_{l_1},\mathrm{E}_{l_2})\,\mathrm{E}_{l_2}(l_2-l_1+1)=c(l_2-l_1+1)$$

for some constant c, and, therefore,

$$P_1(\mathrm{E}_{l_1},\mathrm{E}_{l_2})(l_2-l_1+1)|_{l_i=0}=0.$$

Thus, it suffices to consider

$$P(X_1,X_2)-P_1(X_1,X_2)=\frac{1-X_1+X_1X_2}{(X_1-1)(X_2-1)X_2}(X_1^{k_1}X_2^{k_2}-X_1^{k_1}X_2^{k_3+1}+X_1^{k_2+1}X_2^{k_3+1})$$
$$=P_2(X_1,X_2).$$

As for (4.7), we can use the analogous approach to obtain

$$Q(X_1,X_2)=\frac{1}{(X_1-1)(X_2-1)X_2}\Big(X_1^{k_1}X_2^{k_2}-X_1^{k_1+1}X_2^{k_2}+X_1^{k_1+1}X_2^{k_2+1}$$
$$-X_1^{k_2+1}X_2^{k_2+1}-X_1^{k_1}X_2^{k_3+1}+X_1^{k_1+1}X_2^{k_3+1}+X_1^{k_2+1}X_2^{k_3+1}$$
$$-X_1^{k_2+2}X_2^{k_3+1}-X_1^{k_1+1}X_2^{k_3+2}+X_1^{k_2+2}X_2^{k_3+2}\Big).$$

Here, the monomial $X_1^{k_2+1}X_2^{k_2+1}$ in the long factor can be deleted for similar reasons. What remains then is just $P(X_1,X_2)-P_1(X_1,X_2)$.

The simplified combinatorial proof of the alternating sign matrix theorem we have in mind should essentially only be based on the fundamental construction, which is quite appealing because this construction is just a signed extension of urban renewal. As such also the description (and coding) of the combinatorial proof will most likely be considerably simpler than the one given in [6,8].

5. Sijective proof relating Theorem 5 to (3.5)

We apply the construction of Problem 8 to construct a weight-preserving sijection from $\underline{\mathrm{Aztec}}(\mathbf{k})$ to $2^{\left[\binom{n}{2}\right]}\times\underline{\mathrm{Lozenge}}(\mathbf{k})$. The construction is by induction with respect to n. The case $n=1$ is clear. By the definition of $\underline{\mathrm{Aztec}}(\mathbf{k})$ and of $\mathbf{a}(\mathbf{k})$,

$$\underline{\mathrm{Aztec}}(k_1,\dots,k_n)=\prod_{i=1}^{n}(\mathrm{Id}+\mathrm{E}_{k_i})\bigsqcup_{\mathbf{l}\in\underline{[k_1,k_2-1]}\times\cdots\times\underline{[k_{n-1},k_n-1]}}\underline{\mathrm{Aztec}}(\mathbf{l}).$$

By the induction hypothesis, there is a sijection from $\underline{\mathrm{Aztec}}(\mathbf{l})$ to $2^{[\binom{n}{2}]}\times\underline{\mathrm{Lozenge}}(\mathbf{l})$, and so we can perform this replacement and reach

$$2^{\left[\binom{n}{2}\right]}\times\prod_{i=1}^{n}(\mathrm{Id}+\mathrm{E}_{k_i})\bigsqcup_{\mathbf{l}\in\underline{[k_1,k_2-1]}\times\cdots\times\underline{[k_{n-1},k_n-1]}}\underline{\mathrm{Lozenge}}(\mathbf{l})$$

by a sijection. This can also be written as

$$2^{\left[\binom{n}{2}\right]}\times\bigsqcup_{p\in[n]}\bigsqcup_{\mathbf{l}\in\bigsqcup_{\mathbf{e}\in\binom{[n]}{p}}\mathbf{e}(\underline{[k_1,k_2-1]}\times\cdots\times\underline{[k_{n-1},k_n-1]})}\underline{\mathrm{Lozenge}}(\mathbf{l}).$$

Now, as $\mathrm{Lozenge}(\mathbf{l})=\emptyset$ if $l_i=l_{i+1}$ for an i, we can overline $\mathbf{e}(\underline{[k_1,k_2-1]}\times\cdots\times\underline{[k_{n-1},k_n-1]})$.

$$2^{\left[\binom{n}{2}\right]}\times\bigsqcup_{p\in[n]}\bigsqcup_{\mathbf{l}\in\bigsqcup_{\mathbf{e}\in\binom{[n]}{p}}\overline{\mathbf{e}(\underline{[k_1,k_2-1]}\times\cdots\times\underline{[k_{n-1},k_n-1]})}}\underline{\mathrm{Lozenge}}(\mathbf{l})$$

Now we use the construction of Problem 8 to obtain a sijection to

$$2^{\left[\binom{n}{2}\right]}\times\bigsqcup_{p\in[n]}\bigsqcup_{\mathbf{l}\in\bigsqcup_{\mathbf{e}\in\binom{[n]}{p}}\overline{\widehat{\mathbf{e}}(\underline{[k_1,k_2-1]}\times\cdots\times\underline{[k_{n-1},k_n-1]})}}\underline{\mathrm{Lozenge}}(\mathbf{l}),$$

where we have actually used the construction of the disjoint union of sijections. Here is also the crucial point for the fact that the sijection preserves the weights: the sijection fixes p. We can rewrite this as

$$2^{\left[\binom{n}{2}+1\right]}\times\bigsqcup_{\mathbf{l}\in\underline{[k_1,k_2-1]}\times\cdots\times\underline{[k_{n-1},k_n-1]}}\prod_{i=1}^{n-1}(\mathrm{Id}+\mathrm{E}_{l_i})\underline{\mathrm{Lozenge}}(\mathbf{l}).$$

Now we can apply the construction of Problem 6 to $\prod_{i=1}^{n-1}(\mathrm{Id}+\mathrm{E}_{l_i})\underline{\mathrm{Lozenge}}(\mathbf{l})$, and after $n-1$ applications, one obtains a sijection to

$$2^{[n-1]}\times\underline{\mathrm{Lozenge}}(\mathbf{l}).$$

In total, we obtain

$$2^{\left[\binom{n+1}{2}\right]} \times \bigsqcup_{\mathbf{l} \in \underline{[k_1,k_2-1]} \times \cdots \times \underline{[k_{n-1},k_n-1]}} \underline{\text{Lozenge}}(\mathbf{l}) = 2^{\left[\binom{n+1}{2}\right]} \times \underline{\text{Lozenge}}(\mathbf{k})$$

as desired.

6. Sijective proof of $\left|\underline{\mathbf{Lozenge}}(\mathbf{k})\right| = \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j-i}$

For strictly increasing sequences of positive integers $k_1, k_2, \ldots, k_n$, the set $\underline{\text{Lozenge}}(k_1, \ldots, k_n)$ is in easy bijective correspondence with semistandard tableaux of shape $(k_n - n, k_{n-1} - n + 1, \ldots, k_1 - 1)$ with entries in $\{1, 2, \ldots, n\}$: This follows from considering the triangular array associated with the lozenge tiling and subtracting i from the i-th $\nearrow$-diagonal. Reversing each row of the triangular array gives clearly a partition and the skew shape with row i being the outer shape and row $i-1$ being the inner shape comprises the cells containing i in the corresponding semistandard tableau. To give an example, observe that the triangular array in (2.1) is transformed into

$$\begin{array}{ccccccccccccccccccc}
 & & & & & & & & & 2 & & & & & & & & & \\
 & & & & & & & & 1 & & 3 & & & & & & & & \\
 & & & & & & & 1 & & 2 & & 4 & & & & & & & \\
 & & & & & & 0 & & 2 & & 3 & & 4 & & & & & & \\
 & & & & & 0 & & 2 & & 2 & & 3 & & 5 & & & & & \\
 & & & & 0 & & 1 & & 2 & & 2 & & 4 & & 5 & & & & \\
 & & & 0 & & 1 & & 1 & & 2 & & 2 & & 4 & & 5 & & & \\
 & & 0 & & 1 & & 1 & & 2 & & 2 & & 3 & & 4 & & 5 & & \\
 & 0 & & 1 & & 1 & & 1 & & 2 & & 2 & & 3 & & 4 & & 5 & \\
0 & & 1 & & 1 & & 1 & & 2 & & 2 & & 2 & & 3 & & 4 & & 5
\end{array}$$

when subtracting i from the i-th $\nearrow$-diagonal, and then we obtain the following tableau.

1	1	2	3	5
2	3	4	6	
3	4	8		
5	5			
6	8			
7	10			
8				
9				
10				

The formula $\prod_{1 \le i < j \le n} \frac{k_j - k_i}{j-i}$ is a rewriting of Stanley's hook-content formula [18] for which bijective proofs have been given in [17,11,12]. In this sense, it can be argued that there is a bijective proof for the case of strictly increasing positive integers $k_1, k_2, \ldots, k_n$, although the "rewriting part" may not be satisfying, because it introduces many additional factors in the numerator and denominator. More importantly, it is not at all clear how to extend these proofs to all integer sequences $k_1, k_2, \ldots, k_n$.

It should be noted that there is a sijective option that reduces to the case $0 < k_1 < k_2 < \ldots < k_n$: first, by reducing to strictly increasing sequences of integers using the sijection that shows

$$\underline{\text{Lozenge}}(k_1, \ldots, k_n) = -\underline{\text{Lozenge}}(k_1, \ldots, k_{i-1}, k_{i+1}, k_i, k_{i+2}, \ldots, k_n).$$

This sijection has been constructed in [6, Problem 5]. Second, we need to reduce to positive integers by using the trivial sijection underlying

$$\underline{\text{Lozenge}}(k_1 + c, \ldots, k_n + c) = \underline{\text{Lozenge}}(k_1, \ldots, k_n)$$

for any integer c. However, a more uniform approach would be desirable, which also avoids the manipulations that are necessary when rewriting the formula. This is accomplished in this section at the cost of involving signed sets with non-empty negative part also in the case when $k_1, k_2, \ldots, k_n$ is strictly increasing. We will first discuss the case $n = 3$, where this can actually still be avoided.

6.1. The case $n = 3$

For a positive integer m and a signed set $\underline{S}$, we define

$$m\,\underline{S} = \underbrace{\underline{S} \sqcup \ldots \sqcup \underline{S}}_{m \text{ times}}.$$

We construct a sijection

$$2\,\underline{\text{Lozenge}}(k_1, k_2, k_2) \Longrightarrow \underline{[k_1, k_2 - 1]} \times \underline{[k_2, k_3 - 1]} \times \underline{[k_1, k_3 - 1]}.$$

We use the following construction, which corresponds to reversing the summation when summing over an interval. Suppose $\underline{S}(l)$ is a family of signed sets indexed by an integer l. Then there is a sijection

$$\bigsqcup_{l \in \underline{[a, b-1]}} \underline{S}(l) \Longrightarrow \bigsqcup_{l \in \underline{[a, b-1]}} \underline{S}(a + b - 1 - l).$$

This is shown by considering the two cases $a < b$ and $a \geq b$. We will also use the even more trivial sijection $\underline{[a, b]} \Longrightarrow \underline{[a + c, b + c]}$. We apply this as follows.

$$\begin{aligned}&\bigsqcup_{(l_1,l_2)\in\underline{[k_1,k_2-1]}\times\underline{[k_2,k_3-1]}} \underline{[l_1,l_2-1]}\\ &\qquad \Longrightarrow \bigsqcup_{(l_1,l_2)\in\underline{[k_1,k_2-1]}\times\underline{[k_2,k_3-1]}} \underline{[k_1+k_2-1-l_1,k_2+k_3-2-l_2]}\\ &\qquad\qquad \Longrightarrow \bigsqcup_{(l_1,l_2)\in\underline{[k_1,k_2-1]}\times\underline{[k_2,k_3-1]}} \underline{[l_2,k_3-k_1-1+l_1]}.\end{aligned}$$

Now

$$\begin{aligned}&2\bigsqcup_{(l_1,l_2)\in\underline{[k_1,k_2-1]}\times\underline{[k_2,k_3-1]}} \underline{[l_1,l_2-1]}\\ &\qquad \Longrightarrow \bigsqcup_{(l_1,l_2)\in\underline{[k_1,k_2-1]}\times\underline{[k_2,k_3-1]}} \underline{[l_1,l_2-1]}\sqcup\underline{[l_2,k_3-k_1-1+l_1]}\\ &\qquad\quad \Longrightarrow \bigsqcup_{(l_1,l_2)\in\underline{[k_1,k_2-1]}\times\underline{[k_2,k_3-1]}} \underline{[l_1,k_3-k_1-1+l_1]}\\ &\qquad\qquad \Longrightarrow \bigsqcup_{(l_1,l_2)\in\underline{[k_1,k_2-1]}\times\underline{[k_2,k_3-1]}} \underline{[k_1,k_3-1]}\\ &\qquad\qquad\qquad \Longrightarrow \underline{[k_1,k_2-1]}\times\underline{[k_2,k_3-1]}\times\underline{[k_1,k_3-1]}.\end{aligned}$$

Note that for $k_1 < k_2 < k_3$, this construction does not involve signed sets with non-empty negative parts.

6.2. An auxilliary sijection: combinatorics versus computation

We define the following natural "incarnation" of *falling factorials* as signed sets:

$$\underline{[k]}_i = \underline{[1,k]}\times\underline{[1,k-1]}\times\cdots\times\underline{[1,k-i+1]} \qquad \text{with} \quad \underline{[k]}_0 = (\{1\},\{\}) \tag{6.1}$$

From now on most of the procedures are implemented in `python`. They can be downloaded from

http://www.mat.univie.ac.at/~ifischer/dominolozenge.py

and some of them use

https://www.fmf.uni-lj.si/~konvalinka/ASMcodePartI.py.

For instance, the falling factorials are implemented through `ssfallingfactorial(k,i)`, so for instance, calling `print(ssfallingfactorial(-1,3))` gives the following output:

$$([],[(0,-1,-2),(0,-1,-1),(0,-1,0),(0,0,-2),(0,0,-1),(0,0,0)])$$

We have $|\underline{[k]}_i| = k\cdot(k-1)\cdots(k-i+1)$. A sijection that will be applied repeatedly in our construction is between that following signed sets, with $i \geq 0$.

$$\underline{[1, i+1]} \times \bigsqcup_{l \in \underline{[0,k-1]}} \underline{[l]}_i \Longrightarrow \underline{[k]}_{i+1} \tag{6.2}$$

Next we sketch an obvious sijection. We need to distinguish whether $k \geq 0$ or not, and work from right to left in (6.2).

Case $k \geq 0$: Take an $(i+1)$-tuple $(c_1, \ldots, c_{i+1})$ from the right hand side. The first coordinate on the left hand side is the minimal index j such that $c_j + j - 1$ is maximal. This maximum can be at most k and we subtract 1 to obtain l. The element on the left hand side from $\underline{[l]}_i$ is simply $(c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_{i+1})$.

Case $k < 0$: For an element $(c_1, \ldots, c_{i+1})$ from the right hand side, determine the maximal index j such that $c_j + j - 1$ is minimal to obtain the first coordinate on the left hand side. The minimum is at least $k+1$, and to obtain l, we subtract 1 from the minimum. To obtain the element from $\underline{[l]}_i$, delete c_j from the tuple.

Out of curiosity, we aim at constructing another sijection for (6.2), this time by translating computations into combinatorics. We will then compare to the above "natural" construction. Telescoping is the principle that is underlying the following sijection.

Problem 9. *Suppose $\underline{S}(l)$ is a signed set indexed by an integer $l \in \underline{[a, b-1]}$. Then there is a sijection*

$$\bigsqcup_{l \in \underline{[a,b-1]}} (\underline{S}(l+1) \sqcup -\underline{S}(l)) \Longrightarrow \underline{S}(b) \sqcup -\underline{S}(a).$$

Construction. We can assume $a \leq b$, since there are obvious sijections

$$\bigsqcup_{l \in \underline{[a,b-1]}} (\underline{S}(l+1) \sqcup -\underline{S}(l)) \Longrightarrow - \bigsqcup_{l \in \underline{[b,a-1]}} (\underline{S}(l+1) \sqcup -\underline{S}(l))$$

and

$$\underline{S}(b) \sqcup -\underline{S}(a) \Longrightarrow - \left(\underline{S}(a) \sqcup -\underline{S}(b)\right).$$

For $a \leq b$, we define a sign reversing involution for elements on the left-hand side of the form $((s_{l+1}, 0), l)$, $s_{l+1} \in \underline{S}(l+1), l \in \underline{[a, b-1]}$ with $l < b-1$ and $((s_l, 1), l), s_l \in -\underline{S}(l), l \in \underline{[a, b-1]}$ with $l > a$ as follows

$$((s_{l+1}, 0), l)) \longleftrightarrow ((s_{l+1}, 1), l+1).$$

Elements of the form $((s_b, 0), b-1)$, $s_b \in \underline{S}(b)$ are mapped to $(s_b, 0)$ on the right hand side, while elements of the form $((s_a, 1), a), s_a \in -\underline{S}(a)$ are mapped to $(s_a, 1)$ on the right hand side. $\square$

Problem 9 is implemented in `telescoping(a,b)`. Calling

```
print(telescoping(3,8)((((103,0),4),0)))
```

gives

$$(((\texttt{103},\texttt{1}),\texttt{5}),\texttt{0}).$$

Note that the input is a pair. By our definition sijections, the first element is in the domain if the second coordinate of the pair is 0 and it is in the codomain if the second coordinate is 1.

The following is an application of telescoping.

Problem 10. *Let $a, b, i \in \mathbb{Z}$ and $i \geq 0$. Then there is an explicit sijection between the following signed sets.*

$$\underline{[1, i+1]} \times \bigsqcup_{l \in \underline{[a,b-1]}} \underline{[l]}_i \Longrightarrow \underline{[b]}_{i+1} \sqcup (-\underline{[a]}_{i+1})$$

Construction. We have the following chain of sijections. The description is extremely detailed to be able to compare with the code easily. We use the sijection α from Proposition 4.

$$\begin{aligned}
\underline{[1, i+1]} \times \underline{[l]}_i \Rightarrow \underline{[l-i+1, l+1]} \times \underline{[l]}_i \Rightarrow -\underline{[l+2, l-i]} \times \underline{[l]}_i \\
\overset{\alpha_{l+2,0,l-i}}{\Rightarrow} -(\underline{[l+2, 0]} \sqcup \underline{[1, l-i]}) \times \underline{[l]}_i \\
\Rightarrow (-\underline{[l+2, 0]} \sqcup -\underline{[1, l-i]}) \times \underline{[l]}_i \Rightarrow (\underline{[1, l+1]} \sqcup -\underline{[1, l-i]}) \times \underline{[l]}_i \\
\overset{\text{distr.}}{\Rightarrow} \underline{[1, l+1]} \times \underline{[l]}_i \sqcup -\underline{[1, l-i]} \times \underline{[l]}_i \\
\Rightarrow \underline{[l+1]}_i \sqcup -\underline{[l]}_i \times \underline{[1, l-i]} \Rightarrow \underline{[l+1]}_{i+1} \sqcup -\underline{[l]}_{i+1}
\end{aligned} \tag{6.3}$$

Now we apply Lemma 9 to $\underline{S}(l) = \underline{[l]}_i$. □

Problem 10 is implemented in `ffsum(i,a,b)`. For

$$\texttt{print(ffsum(3,0,10)(((2,((7,5,4),8)),0)))},$$

we obtain $(((\texttt{4},\texttt{7},\texttt{7},\texttt{5}),\texttt{0}),\texttt{1})$.

Setting $a = 0$ and $b = k$, we obtain a sijection for (6.2) which is different from the one described first. The sijection can be described as follows.

Case $k \geq 0$: Given an element in $\mathbf{c} \in \underline{[k]}_{i+1}$, we consider the first coordinate c_1. This coordinate is supposed to be in $\underline{[1, k]}$, however, if $c_1 \in \underline{[k-i, k]}$, then we let the element in $\underline{[1, i+1]}$ be $c_1 - k + i + 1$, $l = k - 1$ and the element in $\underline{[l]}_i$ be $\mathbf{c}$ with the first coordinate deleted. Otherwise we move the first coordinate (which is in $\underline{[1, k-i-1]}$) to the end and obtain an element of $\underline{[k-1]}_{i+1}$. We consider the new first coordinate and if it is in the top $i+1$ segment of $\underline{[1, k-1]}$ then we shift to obtain an element of $\underline{[1, i+1]}$, set $l = k - 2$ and delete the first coordinate; otherwise we move the first coordinate to the end and repeat. It is not hard to see that this procedure will terminate at some point. Again it is not hard to see that this procedure terminates.

Case $k < 0$: Given an element

$$\mathbf{c} \in (-1)^{i+1}\underline{[k+1,0]} \times \underline{[k,0]} \times \cdots \times \underline{[k-i+1,0]},$$

we consider the last coordinate c_{i+1}. This coordinate is supposed to be in $\underline{[k-i+1,0]}$, however, if $c_{i+1} \in \underline{[k-i+1,k+1]}$, then we let the element in $\underline{[1,i+1]}$ be $c_{i+1}-k+i$, $l=k$ and the element in $\underline{[k]}_i$ be $\mathbf{c}$ with the last coordinate deleted. Otherwise we move the last coordinate (which is in $\underline{[k+2,0]}$) to the beginning and obtain an element of $\underline{[k+1]}_{i+1}$. We consider the new last coordinate and if it is in the bottom $i+1$ segment of its interval then we shift to obtain an element of $\underline{[1,i+1]}$, otherwise we move the last coordinate to the beginning and repeat.

However, it can be checked that the first mentioned, "natural" sijection can be obtained when replacing (6.3) by

$$\begin{aligned}
\underline{[1,i+1]} \times \underline{[l]}_i \overset{\text{shift}}{\Longrightarrow} \underline{[0,i]} \times \underline{[l]}_i &\overset{\alpha_{0,0,i},\text{distr.}}{\Longrightarrow} \underline{[0,0]} \times \underline{[l]}_i \sqcup \underline{[1,i]} \times \underline{[l]}_i \\
&\overset{\text{shift,assoc.,com.}}{\Longrightarrow} \underline{[l+1,l+1]} \times \underline{[l]}_i \sqcup \underline{[1,l]} \times \underline{[1,i]} \times \underline{[l-1]}_{i-1} \\
&\overset{\text{induction}}{\Longrightarrow} \underline{[l+1,l+1]} \times \underline{[l]}_i \sqcup \underline{[1,l]} \times \left(\underline{[l]}_i \sqcup -\underline{[l-1]}_i\right) \\
&\overset{\text{distr.}}{\Longrightarrow} \underline{[l+1,l+1]} \times \underline{[l]}_i \sqcup \underline{[1,l]} \times \underline{[l]}_i \sqcup -\underline{[l]}_{i+1} \overset{\text{distr.}}{\Longrightarrow} \underline{[l+1]}_{i+1} \sqcup -\underline{[l]}_{i+1}.
\end{aligned}$$

Note that we use induction with respect to i and for $i=0$ we have

$$\underline{[1,1]} \times \underline{[l]}_0 \Longrightarrow \underline{[l+1]}_1 \sqcup -\underline{[l]}_1.$$

This is implemented in `ffsum1(i,a,b)`. For

$$\texttt{print(ffsum1(3,0,10)(((2,((7,5,4),8)),0)))},$$

we obtain $\texttt{(((7,8,5,4),0),1)}$.

6.3. Determinants of matrices of signed sets

The following definition has already appeared in [8]. It involves the signed set of permutations of n elements, which is denoted by $\underline{\mathfrak{S}}_n$ and where a permutation is in the positive part if and only if it has an even number of inversions.

Following conventions from `python`, we start indexing at 0. Also the symmetric group $\mathfrak{S}_n$ is understood as the set of all bijections $\{0,1,\ldots,n-1\} \to \{0,1,\ldots,n-1\}$.

Definition 6.1. Let $\underline{\mathcal{S}} = (\underline{S}_{i,j})_{0 \le i,j \le n-1}$ be an $n \times n$ matrix of signed sets, i.e., $\underline{S}_{i,j}$ are signed sets for all i, j. The determinant of $\underline{\mathcal{S}}$ is the following signed set.

$$\det(\underline{\mathcal{S}}) = \bigsqcup_{\sigma \in \underline{\mathfrak{S}}_n} \underline{S}_{0,\sigma(0)} \times \underline{S}_{1,\sigma(1)} \times \cdots \times \underline{S}_{n-1,\sigma(n-1)}$$

A typically element of an order n determinant could be as follows:

$$((8, 2, -3, 5, 10, -17), (5, 3, 0, 2, 4, 1)), \tag{6.4}$$

assuming that $8 \in \underline{S}_{0,5}, 2 \in \underline{S}_{1,3}, -3 \in \underline{S}_{2,0}, 5 \in \underline{S}_{3,2}, 10 \in \underline{S}_{4,4}, -17 \in \underline{S}_{5,1}$.

Many of the usual properties of determinants also hold for this analogue and the proofs can usually be copied verbatim (some of them are given in [8]). In particular, we will need the following.

(1) *Transposing:* There is a sijection

$$\det_{0 \le i,j \le n-1} \left(\underline{S}_{i,j}\right) \Longrightarrow \det_{0 \le i,j \le n-1} \left(\underline{S}_{j,i}\right).$$

The map is in fact a sign preserving bijection, which maps a permutation to its inverse and reorders the entries accordingly. The procedure in our code is called `transpose`. If we apply it to (6.4), we obtain

$$((-3, -17, 5, 2, 10, 8), (2, 5, 3, 1, 4, 0)),$$

independent of which of the two elements are thought of being in the domain and the codomain.

(2) *Linearity in each row and column:* Suppose $\underline{\mathcal{R}}_0, \ldots, \underline{\mathcal{R}}_{n-1}$ are the rows of an $n \times n$ square matrix, then let $\det(\underline{\mathcal{R}}_0, \ldots, \underline{\mathcal{R}}_{n-1})$ denote its determinant, abusing the notation. Suppose $\underline{\mathcal{R}}_i, \underline{\mathcal{R}}'_i$ are two "vectors" of length n of signed sets, $\underline{S}$ is a signed set and

$$\underline{\mathcal{R}}_i \sqcup \underline{\mathcal{R}}'_i \qquad \text{and} \qquad \underline{S} \times \underline{\mathcal{R}}_i$$

are understood componentwise. Using distributivity and commutativity, we have the following sijections.

$$\begin{aligned}
&\det(\underline{\mathcal{R}}_0, \ldots, \underline{\mathcal{R}}_i \sqcup \underline{\mathcal{R}}'_i, \ldots \underline{\mathcal{R}}_{n-1}) \\
&\quad \Longrightarrow \det(\underline{\mathcal{R}}_0, \ldots, \underline{\mathcal{R}}_i, \ldots \underline{\mathcal{R}}_{n-1}) \sqcup \det(\underline{\mathcal{R}}_0, \ldots, \underline{\mathcal{R}}'_i, \ldots \underline{\mathcal{R}}_{n-1}) \\
&\det(\underline{\mathcal{R}}_0, \ldots, \underline{S} \times \underline{\mathcal{R}}_i, \ldots \underline{\mathcal{R}}_{n-1}) \Longrightarrow \underline{S} \times \det(\underline{\mathcal{R}}_0, \ldots, \underline{\mathcal{R}}_i, \ldots \underline{\mathcal{R}}_{n-1})
\end{aligned}$$

The first rule is implemented as `decomposedisjointunionrow(i)`. Applying it to the following element from the domain

$$((8,2,-3,(5,1),10,-17),(5,3,0,2,4,1)) \tag{6.5}$$

with $i=3$, we obtain

$$(((8,2,-3,5,10,-17),(5,3,0,2,4,1)),1) \tag{6.6}$$

from the codomain. The second rule is implemented as `pulloutfrontfactorrow(i)`. Applying it to the following element from the domain

$$((-3,-17,5,2,(41,10),8),(2,5,3,1,0,4)) \tag{6.7}$$

with $i=4$, we obtain

$$(41,((-3,-17,5,2,10,8),(2,5,3,1,0,4))) \tag{6.8}$$

from the codomain. For columns, the analogue statement follows from transposing and from the statement for rows. The rules are implemented as

`decomposedisjointunioncolumn(i)` and `pulloutfrontfactorcolumn(i)`.

Applying the first column rule to (6.5) for $i=2$ gives also (6.6), while applying the second column rule to (6.7) for $i=0$ gives (6.8).

(3) Suppose rows i and j are identical for a signed matrix. Then there is a sijection from its determinant to the empty set. The sign reversing involution on the determinant is induced by the application of the transposition (i,j) on the permutation. The respective procedure in our `python` code is `equalrow(i,j)`. Applying `equalrow(3,5)` to

$$(((-3,-17,5,2,10,8),(2,5,3,1,0,4)),0),$$

we obtain

$$(((-3,-17,5,8,10,2),(2,5,3,4,0,1)),0).$$

(The input of a sijection is always a pair, where the second element is either 0 or 1 and indicates whether we have an element from the domain (0) or from the codomain (1).) Note we are "thrown back" to the domain. The analogue statement is true for columns by transposing. The procedure in `python` is `equalcolumn(i,j)`. When applying `equalcolumn(3,5)` to the same element, then only the permutation is changed, namely to $(2,3,5,1,0,4)$, while $(-3,-17,5,2,10,8)$ is not changed.

(4) *Expansion along rows and columns:* Again let $\underline{\mathcal{S}}=(\underline{S}_{i,j})_{0\le i,j\le n-1}$ be an $n\times n$ square matrix of signed sets and denote by $\underline{\mathcal{S}}^{i,j}$ the $(n-1)\times(n-1)$ square matrix obtained by deleting the i-th row and j-th column. Then, for fixed i, there is a sijection

$$\det\left(\underline{\mathcal{S}}\right) \Longrightarrow \bigsqcup_{j=0}^{n-1} (-1)^{i+j} \underline{\mathcal{S}}_{i,j} \times \det\left(\underline{\mathcal{S}}^{i,j}\right).$$

This is again actually a sign-preserving bijection: Suppose $\sigma \in \underline{\mathfrak{S}}_n$, then we consider all possible images $\sigma(i) = j$ and transform σ into a permutation in $\underline{\mathfrak{S}}_{n-1}$ by restricting to $\{0, 1, 2, \ldots, n-1\} \setminus \{i\}$ in an obvious way. For columns, the sijection is again constructed by transposing. The `python` procedures are `expanddetrow(i)` and `expanddetcolumn(i)`, respectively. Applying `expanddetrow(2)` to

$$(((-3, -17, 5, 2, 10, 8), (2, 5, 3, 1, 0, 4)), 0)$$

gives

$$(((5, ((-3, -17, 2, 10, 8), (2, 4, 1, 0, 3))), 3), 1),$$

while applying `expanddetcolumn(2)` to the same element gives

$$(((-3, ((-17, 5, 2, 10, 8), (4, 2, 1, 0, 3))), 0), 1).$$

Note that the combinations of (2) and (3) give the invariance under row and column operations. The corresponding procedures are `addrowfrontfactor(i,j)` and `addcolumnfrontfactor(i,j)`. So, applying

`addrowfrontfactor(2,4)` or `addcolumnfrontfactor(3,0)`

to

$$(((-3, -17, 5, 2, 10, 8), (2, 5, 3, 1, 0, 4)), 0)$$

gives

$$(((-3, -17, (5, 0), 2, 10, 8), (2, 5, 3, 1, 0, 4)), 1).$$

On the other hand, applying

`addrowfrontfactor(2,4)` or `addcolumnfrontfactor(3,0)`

to

$$(((-3, -17, ((11, 5), 1), 2, 10, 8), (2, 5, 3, 1, 0, 4)), 1)$$

gives

$$(((-3, -17, ((11, 10), 1), 2, 5, 8), (2, 5, 0, 1, 3, 4)), 1)$$

and

$$(((-3,-17,5,2,((11,10),1),8),(2,5,0,1,3,4)),1),$$

respectively.

6.4. Sijection $\prod_{i=1}^{n-1} \underline{[i]}_i \times \underline{\mathrm{Lozenge}}(k_1,\ldots,k_n) \Longrightarrow \prod_{i=1}^{n} \prod_{j=i+1}^{n} \underline{[k_i,k_j-1]}$

Now we have all ingredients to construct the sijective proof of $|\underline{\mathrm{Lozenge}}(k_1,\ldots,k_n)| = \prod_{1\le i<j\le n} \frac{k_j-k_i}{j-i}$. More precisely, we will construct a sijection

$$\prod_{i=1}^{n-1} \underline{[i]}_i \times \underline{\mathrm{Lozenge}}(k_1,\ldots,k_n) \Longrightarrow \prod_{i=1}^{n} \prod_{j=i+1}^{n} \underline{[k_i,k_j-1]}.$$

In the classical case $k_1 < k_2 < \ldots < k_n$, the sijection is clearly a bijection as domain and codomain have empty negative parts. In this case, we will also describe it using the classical involution principle. The underlying auxiliary signed set is $\det_{0\le i,j\le n-1}\left(\underline{[k_{i+1}]}_j\right)$ and there will be

(1) a sign reversing involution on a subset of $\det_{0\le i,j\le n-1}\left(\underline{[k_{i+1}]}_j\right)$ together with a sign preserving bijection from the complement of this set to $\prod_{i=1}^{n} \prod_{j=i+1}^{n} \underline{[k_i,k_j-1]}$ (see Section 6.4.1), and
(2) another sign reversing on another subset of $\det_{0\le i,j\le n-1}\left(\underline{[k_{i+1}]}_j\right)$ together with another sign preserving bijection from the complement of this other set to $\prod_{i=1}^{n-1} \underline{[i]}_i \times \underline{\mathrm{Lozenge}}(k_1,\ldots,k_n)$ (see Section 6.4.2).

According to the involution principle, these maps are sufficient to construct a bijection. Note that the maps in (1) constitute a sijection from $\det_{0\le i,j\le n-1}\left(\underline{[k_{i+1}]}_j\right)$ to $\prod_{i=1}^{n} \prod_{j=i+1}^{n} \underline{[k_i,k_j-1]}$, and the maps in (2) constitute a sijection from $\det_{0\le i,j\le n-1}\left(\underline{[k_{i+1}]}_j\right)$ to $\prod_{i=1}^{n-1} \underline{[i]}_i \times \underline{\mathrm{Lozenge}}(k_1,\ldots,k_n)$. These sijections are special in the sense that their codomains have empty negative parts (assuming $k_1 < k_2 < \ldots < k_n$). They are special cases of the sijections we obtain for arbitrary $(k_1,\ldots,k_n) \in \mathbb{Z}^n$ (also described in Sections 6.4.1 and 6.4.2), and the general sijection from $\prod_{i=1}^{n-1} \underline{[i]}_i \times \underline{\mathrm{Lozenge}}(k_1,\ldots,k_n)$ to $\prod_{i=1}^{n} \prod_{j=i+1}^{n} \underline{[k_i,k_j-1]}$ is constructed by composing the two general sijections. Note that also in the general setting, either the negative or the positive part of $\prod_{i=1}^{n} \prod_{j=i+1}^{n} \underline{[k_i,k_j-1]}$ is empty.

Before we come to the actual construction of the two sijections, let us briefly give an outlook on a detail of our bijection in the special case $k_1 < k_2 < \ldots < k_n$. Suppose we have two sets $S \subseteq T$, and we know that

$$|S| = \frac{|T|}{d} \tag{6.9}$$

for some positive integer d. In our case $S = \text{Lozenge}(\mathbf{k})$, $T = \prod_{1 \le i < j \le n} \underline{[k_i, k_j - 1]}$ and $d = 1! \cdot 2! \cdots (n-1)!$, noting that we can indeed interpret $S \subseteq T$: For a particular element of Lozenge($\mathbf{k}$), the entry in the i-th $\nearrow$-diagonal and the j-th $\searrow$-diagonal (both counted from the left) is in $\underline{[k_i, k_j - 1]}$. In fact, the element is actually in $\underline{[k_i, k_j - j + i]}$ by the strictness along $\searrow$-diagonals.

To show (6.9), it is natural to ask for a 1-to-d map from T to S. Since $S \subseteq T$, then one could find it natural that the following two conditions are satisfied for this map.

- The map is the identity on S.
- The map is "continuous" in the sense that little changes of elements of T cause little changes of their images.

For our bijection, the following variation of the first condition will be satisfied: Take an element of Lozenge($\mathbf{k}$) and add $n-1-l$ to all elements of row l, $l = 1, 2, \ldots, n-1$. The so-obtained triangular array can also be interpreted as an element of $\prod_{1 \le i < j \le n} \underline{[k_i, k_j - 1]}$. The corresponding element of $\underline{[n-1]}_{n-1} \times \underline{[n-2]}_{n-2} \times \ldots \times \underline{[0]}_0$ is the one with all 1's. The second property is by no means defined precisely, but, in a sense, whether or not it is satisfied depends on how often the two sijections need to be applied (the two sijections themselves can probably be considered as continuous), which is a priori not clear when the involution principle is underlying.

6.4.1. *Sijection* $\det_{0 \le i,j \le n-1}\left(\underline{[k_{i+1}]}_j\right) \Longrightarrow \prod_{i=1}^{n} \prod_{j=i+1}^{n} \underline{[k_i, k_j - 1]}$

Problem 11. *There is an explicit sijection*

$$\det_{0 \le i,j \le n-1}\left(\underline{[k_{i+1}]}_j\right) \Longrightarrow \prod_{i=1}^{n} \prod_{j=i+1}^{n} \underline{[k_i, k_j - 1]}.$$

Concerning the order of factors and "parenthesizing", the right hand side will be interpreted as follows:

$$\begin{aligned}(\underline{[k_1, k_2 - 1]} \times \underline{[k_1, k_3 - 1]} \times \cdots \times \underline{[k_1, k_n - 1]}) \times (\underline{[k_2, k_3 - 1]} \times \ldots \\ \times \underline{[k_2, k_n - 1]}) \times \cdots \times (\underline{[k_{n-1}, k_n - 1]})\end{aligned}$$

Before we give the general construction that is derived from a computation, we give a description of the case $0 < k_1 < k_2 < \ldots < k_n$, which does not unravel its connection to a computation. An element of $\det_{0 \le i,j \le n-1}\left(\underline{[k_{i+1}]}_j\right)$ is a pair $((D_0, D_1, \ldots, D_{n-1}), (p_0, p_1, \ldots, p_{n-1}))$ such that

- $p_0, p_1, \ldots, p_{n-1}$ is a permutation of $\{0, 1, \ldots, n-1\}$, and
- D_i is a p_i tuple with $(D_i)_j \in \underline{[1, k_{i+1} + 1 - j]}$, $j = 1, 2, \ldots, p_i$.

Our running example will be $\mathbf{k} = (1, 6, 7, 9, 14, 17, 21, 25, 28)$ and

$$D_0 = (),\ D_1 = (3,1,4),\ D_2 = (4,2,2,3,1),\ D_3 = (9,1,3,2),$$
$$D_4 = (8,10),\ D_5 = (15,12,11,6,13,10),$$
$$D_6 = (5,),\ D_7 = (17,20,3,15,21,2,18,1),\ D_8 = (26,21,25,2,13,17,2),$$

and, therefore, $(p_0, p_1, \ldots, p_{n-1}) = (0, 3, 5, 4, 2, 6, 1, 8, 7)$.

Step 1. We start our algorithm by arranging the determinant as follows: We put $\mathbf{k}$ in the bottom row. Then tuple D_i will appear as $\nwarrow$-diagonal "above" k_{i+1}, where we arrange the elements from bottom to top. We then add $n-2-i$ to row i, $i = 1, 2, \ldots, n-1$, which gives

$$\begin{array}{cccccccccccccccccc}
 & & & & & & & 7 & & & & & & & & & & \\
 & & & & & & & & 23 & & 7 & & & & & & & \\
 & & & & & 14 & & & & 6 & & 21 & & & & & & \\
4 & & & & & & 16 & & & & 24 & & 16 & & & & & \\
 & 5 & & 4 & & & & 8 & & & & 17 & & 4 & & & & \\
5 & & 3 & & 4 & & & & 12 & & & & 4 & & 26 & & & \\
 & 1 & & 2 & & 1 & & 10 & & 12 & & & & 20 & & 21 & & \\
 & & 2 & & 3 & & 8 & & 7 & & 14 & & 4 & & 16 & & 25 & \\
 & \mathbf{1} & & \mathbf{6} & & \mathbf{7} & & \mathbf{9} & & \mathbf{14} & & \mathbf{17} & & \mathbf{21} & & \mathbf{25} & & \mathbf{28}
\end{array}$$

in our example. In case we have reached an element of $\prod_{i=1}^{n} \prod_{j=i+1}^{n} \underline{[k_i, k_j - 1]}$, that is $(p_0, \ldots, p_{n-1}) = (0, 1, \ldots, n-1)$ and the element in the i-th $\nearrow$-diagonal and the j-th $\searrow$-diagonal is in $\underline{[k_i, k_j - 1]}$ for all $1 \le i < j \le n$, we stop. Otherwise we construct a sign-reversing involution as described in the following steps.

Step 2. If the algorithm did not stop in Step 1, then one of the following is true (or both): There is an i such that the i-th $\searrow$-diagonal is not of length i (including the bottom row in the counting) or the i-th element from the top in a $\nearrow$-diagonal is not contained in $\underline{[k_i, k_j - 1]}$ where j is the position of $\nearrow$-diagonal, counted from the left. Note that in the latter case, the element is therefore less than k_i because it is by construction less than k_j. We choose i minimal with this property.

In our example, we have $i = 2$: On the one hand, the length of the second $\searrow$-diagonal is $4 \neq 2$, but the length of the first $\searrow$-diagonal is 1. On the other hand, the second entries of the $\searrow$-diagonals in positions 2, 3 and 4 are less then $k_2 = 6$ and all first elements are greater than or equal to $k_1 = 1$.

We then delete the top $i-1$ elements of all $\searrow$-diagonals. (This implies that we delete *all* entries in the first $i-1$ diagonals since, by the minimality of i, the lengths of these diagonals are $1, 2, \dots, i-1$.)

Step 3. *Later we will see that this step can be greatly simplified by modifying the computation where this comes from very slightly (see* **"Modified Step 3"** *below). However, in order to understand the relation between the computation and the algorithm, we choose to describe the complicated version first. The building blocks are in fact the same, but in the modified version several steps can be merged.* We start this step by interchanging the first diagonal with the empty diagonal (the interchanging does not include the bottom elements k_i), and mark the former empty diagonal as special by underlining the corresponding k_i:

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&23&&&&&&&&& \\
&&&&&&&&6&&21&&&&&& \\
&&&&&16&&&&24&&16&&&&& \\
5&&&&&&8&&&&17&&4&&&& \\
&3&&4&&&&12&&&&4&&26&&& \\
&&2&&1&&&&12&&1&&20&&21&& \\
&&&3&&8&&7&&14&&2&&16&&25& \\
&&\mathbf{6}&&\mathbf{7}&&\mathbf{9}&&\mathbf{14}&&\mathbf{17}&&\underline{\mathbf{21}}&&\mathbf{25}&&\mathbf{28}
\end{array}
$$

Next we have two alternating "phases", a loosening-or-fixing-phase (LF) and a moving-phase (M). We start with LF, and stop when we are supposed to do M next but there is nothing to move (because there is no "loose" element; such elements will be overlined) and at most one special diagonal. We describe the two phases.

LF: We choose the leftmost diagonal whose top element is less than k_i (note that k_i is now the first element in the bottom row). We overline these elements if it is not already overlined; otherwise we remove the overline. The only exception happens, when there are just two elements in this diagonal, including the bottom element k_j. In this case, we underline k_j if it was not already underlined; otherwise we remove the underline. In our example, we obtain the following.

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&23&&&&&&&&& \\
&&&&&&&&6&&21&&&&&& \\
&&&&&16&&&&24&&16&&&&& \\
\overline{5}&&&&&&8&&&&17&&4&&&& \\
&3&&4&&&&12&&&&4&&26&&& \\
&&2&&1&&&&12&&1&&20&&21&& \\
&&&3&&8&&7&&14&&2&&16&&25& \\
&&\mathbf{6}&&\mathbf{7}&&\mathbf{9}&&\mathbf{14}&&\mathbf{17}&&\underline{\mathbf{21}}&&\mathbf{25}&&\mathbf{28}
\end{array}
$$

M: If there is an overlined element, we choose the bottommost, i.e., the one that is the top element of the shortest diagonal with this property (only top elements of a diagonal can be overlined). We move it (with the overline) on top of the diagonal that has exactly one element less than the diagonal where it came from. In our example, we obtain the following.

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&23\\
&&&&&&&&6&&21\\
&&&&&16&&&&24&&16\\
&&\overline{5}&&&&8&&&&17&&4\\
&3&&4&&&&12&&&&4&&26\\
&&2&&1&&&&12&&1&&20&&21\\
&&&3&&8&&7&&14&&2&&16&&25\\
&&\mathbf{6}&&\mathbf{7}&&\mathbf{9}&&\mathbf{14}&&\mathbf{17}&&\underline{\mathbf{21}}&&\mathbf{25}&&\mathbf{28}
\end{array}
$$

On the other hand, if there are no overlined elements, but two underlined diagonals, we interchanged them (without interchanging the corresponding k_j). If also this is not possible we stop with this step.

We apply two more instances of M and LF until we reach the following.

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&23\\
&&&&&&&&6&&21\\
&&&&&16&&&&24&&16\\
&&\overline{5}&&&&8&&&&17&&4\\
&&&4&&&&12&&\overline{3}&&4&&26\\
&&&&1&&\overline{2}&&12&&1&&20&&21\\
&&&3&&8&&7&&14&&2&&16&&25\\
&&\mathbf{6}&&\mathbf{7}&&\mathbf{9}&&\mathbf{14}&&\mathbf{17}&&\underline{\mathbf{21}}&&\mathbf{25}&&\mathbf{28}
\end{array}
$$

Now we are in an instance of LF, where we would want to overline 3, but since we would not be able to move 3 to a diagonal of length one less (that is the empty diagonal — we want to keep the first diagonal empty), we underline the k_j of this diagonal.

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&23\\
&&&&&&&&6&&21\\
&&&&&16&&&&24&&16\\
&&\overline{5}&&&&8&&&&17&&4\\
&&&4&&&&12&&\overline{3}&&4&&26\\
&&&&1&&\overline{2}&&12&&1&&20&&21\\
&&&3&&8&&7&&14&&2&&16&&25\\
&&\mathbf{6}&&\underline{\mathbf{7}}&&\mathbf{9}&&\mathbf{14}&&\mathbf{17}&&\underline{\mathbf{21}}&&\mathbf{25}&&\mathbf{28}
\end{array}
$$

After three more applications of M and of LF, we obtain the following.

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&23\\
&&&&&&&&6&&21\\
&&&&&16&&&&24&&16\\
5&&&&&&8&&&&17&&4\\
&3&&4&&&&12&&&&4&&26\\
&&2&&1&&&&12&&1&&20&&21\\
&&&3&&8&&7&&14&&2&&16&&25\\
&&\mathbf{6}&&\underline{\mathbf{7}}&&\mathbf{9}&&\mathbf{14}&&\mathbf{17}&&\underline{\mathbf{21}}&&\mathbf{25}&&\mathbf{28}
\end{array}
$$

Observe that we have reached the object we have started with in Step 3, except that we have another diagonal underlined. Since we are now in the M-phase and there is no element overlined, we interchange the two underlined diagonals, except for the k_j's.

$$
\begin{array}{ccccccccccccccc}
& & & & & 23 \\
& & & & & & 6 & & 21 \\
& & & 16 & & & & 24 & & 16 \\
& & & & 8 & & 5 & & 17 & & 4 \\
& 4 & & & & 12 & & 3 & & 4 & & 26 \\
1 & & 1 & & & & 12 & & 2 & & 20 & & 21 \\
& 2 & & 8 & & 7 & & 14 & & 3 & & 16 & & 25 \\
\mathbf{6} & & \underline{\mathbf{7}} & & \mathbf{9} & & \mathbf{14} & & \mathbf{17} & & \underline{\mathbf{21}} & & \mathbf{25} & & \mathbf{28}
\end{array}
$$

After applying LF three more times and M two more times we obtain the following.

$$
\begin{array}{ccccccccccccccc}
& & & & & 23 \\
& & & & & & 6 & & 21 \\
& & & 16 & & & & 24 & & 16 \\
& & & & 8 & & 5 & & 17 & & 4 \\
& 4 & & & & 12 & & 3 & & 4 & & 26 \\
1 & & 1 & & & & 12 & & 2 & & 20 & & 21 \\
& 2 & & 8 & & 7 & & 14 & & 3 & & 16 & & 25 \\
\mathbf{6} & & \mathbf{7} & & \mathbf{9} & & \mathbf{14} & & \mathbf{17} & & \underline{\mathbf{21}} & & \mathbf{25} & & \mathbf{28}
\end{array}
$$

Next we would be in the M-phase, but there is no overlined element nor there is a more than one underlined diagonal and so we stop with this step.

Finally, if there is an underlined diagonal, we move the entries — except for the k_j — to the first diagonal, and remove the underline.

Step 4. We move back the entries we had deleted, on top the diagonals they had been deleted from (a diagonal is "labeled" by its k_j). Finally, we subtract $n-2-i$ from row i, $1, 2, \ldots, n-1$.

$$
\begin{array}{cccccccccccccccccccc}
& & & & & & & & & 1 \\
& & & & & & & & & & 18 & & 2 \\
& & & & & & & 10 & & & & 2 & & 17 \\
2 & & & & & & & & 13 & & & & 21 & & 13 \\
& 3 & & & & 2 & & & & 6 & & & & 15 & & 2 \\
& & 2 & & 3 & & 3 & & & & 11 & & & & 3 & & 25 \\
& & & 2 & & 1 & & 1 & & 10 & & 12 & & & & 20 & & 21 \\
& & & & 4 & & 3 & & 9 & & 8 & & 15 & & 5 & & 17 & & 26 \\
& & & \mathbf{1} & & \mathbf{6} & & \mathbf{7} & & \mathbf{9} & & \mathbf{14} & & \mathbf{17} & & \mathbf{21} & & \mathbf{25} & & \mathbf{28}
\end{array}
$$

Comparing to the object we have started with, we see that essentially the diagonal above 6 and the diagonal above 7 have interchanged their places, except for each of the top elements of these diagonals. They have been added 1 for every step down and subtracted 1 for every step up.

We leave it to the reader to check that this is indeed a sign reversing involution. $\square$

Next we will see that this sign-reversing involution can be replaced by another, much simpler sign-reversing involution. Clearly, any sign-reversing involution on the same subset of $\det_{0 \le i,j \le n-1}\left(\underline{[k_{i+1}]}_j\right)$ (i.e., the set of elements that are not mapped to $\prod_{i=1}^{n}\prod_{j=i+1}^{n}\underline{[k_i, k_j - 1]}$ in Step 1) will do the trick. It is interesting to see how this simplification comes from a very minor modification in the computation, although it has great impact on the algorithm.

The key observation is that the "prioritizing" when there is more than one choice in the LF-phase and in the M-phase is not the same: in the LF-phase, we choose the leftmost diagonal, in the M-phase we choose the shortest diagonal. This is actually not necessary, as we can prioritize in the LF-phase is also with respect to the length of the diagonals. With this modification, the algorithm in Step 3 (excluding the first and the last step) has a much easier description. There are essentially two cases:

(1) The shortest diagonal whose top element is less than k_i has at least one element that is greater than or equal to k_i: Suppose l is the number of elements in this block (without the k_j) and b is the number of top elements that are less than k_i. Move these elements horizontally to the diagonal of length $l - b$.
(2) The shortest diagonal whose top element is less than k_i has only elements that are less than k_i (except for the bottom element of course). Then there are three further cases.
   (a) There is another diagonal that is underlined: In this case, we interchange the two diagonals (without interchanging the k_j's). The underline moves with the diagonal.
   (b) The shortest "contradicting" diagonal is underlined: in this case, only the underline is removed.
   (c) There is no diagonal underlined: in this case, the shortest "contradicting" diagonal gets underlined.

Here is an example, where the original algorithm indeed leads to a different result than the modified algorithm.

$$\begin{array}{cccccccc}
1 & & & & & & & \\
 & 7 & & 1 & & & & \\
 & & 8 & & 6 & & 1 & \\
 & \mathbf{6} & & \mathbf{8} & & \mathbf{10} & & \mathbf{15}
\end{array}$$

The original algorithm gives

$$\begin{array}{ccccccc}
 & 1 & & & & & \\
7 & & 1 & & & & \\
 & 8 & & 6 & & 1 & \\
\mathbf{6} & & \mathbf{8} & & \mathbf{10} & & \mathbf{15}
\end{array},$$

while the modified algorithm results in

$$\begin{array}{cccccccc}
1 & & & & & & & \\
 & 7 & & 1 & & & & \\
1 & & 8 & & 6 & & & \\
 & \mathbf{6} & & \mathbf{8} & & \mathbf{10} & & \mathbf{15}
\end{array}.$$

Finally, it turns out that we can also include the first and the last step of Step 3 and things are really easy then in Step 3.

Modified Step 3: Find the shortest diagonal whose top element is less than k_i. Suppose l is the number of elements in this diagonal (without the k_j) and b is the (maximal) number of top elements that are less than k_i. Move these elements horizontally to the diagonal of length $l - b$.

It can be checked that the algorithm is the special case $k_1 < k_2 < \ldots < k_n$ of the following construction.

Construction for Problem 11. The construction is by induction with respect to n. The case $n = 0$ is trivial. We add the negative of the first row to all other rows, where adding means that we take the disjoint union.

$$\begin{vmatrix} \underline{[k_1]}_0 & \underline{[k_1]}_1 & \underline{[k_1]}_2 & \ldots & \underline{[k_1]}_{n-1} \\ \emptyset & \underline{[k_2]}_1 \sqcup -\underline{[k_1]}_1 & \underline{[k_2]}_2 \sqcup -\underline{[k_1]}_2 & \ldots & \underline{[k_2]}_{n-1} \sqcup -\underline{[k_1]}_{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \emptyset & \underline{[k_n]}_1 \sqcup -\underline{[k_1]}_1 & \underline{[k_n]}_2 \sqcup -\underline{[k_1]}_2 & \ldots & \underline{[k_n]}_{n-1} \sqcup -\underline{[k_1]}_{n-1} \end{vmatrix}.$$

Then we expand with respect to the first column.

$$\begin{vmatrix} \underline{[k_2]}_1 \sqcup -\underline{[k_1]}_1 & \underline{[k_2]}_2 \sqcup -\underline{[k_1]}_2 & \ldots & \underline{[k_2]}_{n-2} \sqcup -\underline{[k_1]}_{n-2} & \underline{[k_2]}_{n-1} \sqcup -\underline{[k_1]}_{n-1} \\ \underline{[k_3]}_1 \sqcup -\underline{[k_1]}_1 & \underline{[k_3]}_2 \sqcup -\underline{[k_1]}_2 & \ldots & \underline{[k_3]}_{n-2} \sqcup -\underline{[k_1]}_{n-2} & \underline{[k_3]}_{n-1} \sqcup -\underline{[k_1]}_{n-1} \\ \ldots & \ldots & \ldots & \ldots & \\ \underline{[k_n]}_1 \sqcup -\underline{[k_1]}_1 & \underline{[k_n]}_2 \sqcup -\underline{[k_1]}_2 & \ldots & \underline{[k_n]}_{n-2} \sqcup -\underline{[k_1]}_{n-2} & \underline{[k_n]}_{n-1} \sqcup -\underline{[k_1]}_{n-1} \end{vmatrix}.$$

We multiply the $(n-2)$-nd column by $-\underline{[1, k_1 - n + 2]}$ and add it to the $(n-1)$-st column, then we multiply the $(n-3)$-rd column by $-\underline{[1, k_1 - n + 3]}$ and add it to the $(n-2)$-nd column etc. For the entry in row i and column j this gives, by distributivity, associativity, commutativity, application of α (see Proposition 4) and shifting intervals,

$$\begin{aligned} &(\underline{[k_{i+1}]}_j \sqcup -\underline{[k_1]}_j) \sqcup (-\underline{[1, k_1 - j + 1]}) \times (\underline{[k_{i+1}]}_{j-1} \sqcup -\underline{[k_1]}_{j-1}) \\ &\qquad \Longrightarrow (\underline{[k_{i+1}]}_j \sqcup -\underline{[k_1]}_j) \sqcup (\underline{[k_1]}_j \sqcup -\underline{[k_{i+1}]}_{j-1} \times \underline{[1, k_1 - j + 1]}) \\ &\qquad \Longrightarrow \underline{[k_{i+1}]}_j \sqcup -\underline{[k_{i+1}]}_{j-1} \times \underline{[1, k_1 - j + 1]} \\ &\qquad \Longrightarrow \underline{[k_{i+1}]}_{j-1} \times \underline{[1, k_{i+1} - j + 1]} \sqcup -\underline{[k_{i+1}]}_{j-1} \times \underline{[1, k_1 - j + 1]} \\ &\qquad \Longrightarrow \underline{[k_1, k_{i+1} - 1]} \times \underline{[k_{i+1}]}_{j-1} \end{aligned} \tag{6.10}$$

Now we pull out $\underline{[k_1, k_{i+1} - 1]}$ from the i-th row, $1 \le i \le n-1$, and what remains is

$$\begin{vmatrix} \underline{[k_2]}_0 & \underline{[k_2]}_1 & \ldots & \underline{[k_2]}_{n-2} \\ \underline{[k_3]}_0 & \underline{[k_3]}_1 & \ldots & \underline{[k_3]}_{n-2} \\ \ldots & \ldots & \ldots & \ldots \\ \underline{[k_n]}_0 & \underline{[k_n]}_1 & \ldots & \underline{[k_n]}_{n-2} \end{vmatrix},$$

to which we apply the induction hypothesis. $\square$

The simplification coming from changing how to prioritize can be understood as follows. In the LF-phase, the prioritizing follows from how the Cartesian product of sijections is defined (see [6, Proposition 2 (2)]). However, one can simply transpose the matrix before applying the transformation from (6.10), and transpose again after this step, because then the prioritizing in the LF-phase is also with respect to the length of the diagonals.

The more complicated version of this sijection is implemented as `dettoprod(k)`, while the simplified version is `dettoprod1(k)`. In order to recover our example, we need to set `k=[1,6,7,9,14,17,21,25,28]` and

$$
\begin{aligned}
\mathtt{x} = (((), (3,1,4), (4,2,2,3,1), (9,1,3,2), (8,10), (15,12,11,6,13,10), (5,), \\
(17,20,3,15,21,2,18,1), (26,21,25,2,13,17,2)), (0,3,5,4,2,6,1,8,7)),
\end{aligned}
$$

and call `dettoprod(k)((x,0))` or `dettoprod1(k)((x,0))`, respectively. (In this particular case, there is no difference in the result.)

6.4.2. Sijection $\prod_{i=1}^{n-1} \underline{[i]}_i \times \underline{\mathrm{Lozenge}}(k_1,\ldots,k_n) \Longrightarrow \det_{1\le i,j\le n}\left(\underline{[k_i]}_{j-1}\right)$

Problem 12. *There is an explicit sijection*

$$
\underline{[n-1]}_{n-1} \times \underline{[n-2]}_{n-2} \times \ldots \times \underline{[0]}_0 \times \mathrm{Lozenge}(k_1,\ldots,k_n) \Longrightarrow \det_{0\le i,j\le n-1}\left(\underline{[k_{i+1}]}_j\right).
$$

We again start with a description of the sijection for the special case $k_1 < k_2 < \ldots < k_n$. Recall that the negative part of $\underline{[n-1]}_{n-1} \times \underline{[n-2]}_{n-2} \times \ldots \times \underline{[0]}_0 \times \mathrm{Lozenge}(k_1,\ldots,k_n)$ is empty, and so the sijection consists of a sign-reversing involution on a subset of $\det_{0\le i,j\le n-1}\left(\underline{[k_{i+1}]}_j\right)$ and a sign-preserving bijection between the complement of the subset and $\underline{[n-1]}_{n-1} \times \underline{[n-2]}_{n-2} \times \ldots \times \underline{[0]}_0 \times \mathrm{Lozenge}(k_1,\ldots,k_n)$.

Roughly speaking, the procedure is as follows: We start with an element of $\det_{0\le i,j\le n-1}\left(\underline{[k_{i+1}]}_j\right)$ and try to transform it to an element of $\underline{[n-1]}_{n-1} \times \underline{[n-2]}_{n-2} \times \ldots \times \underline{[0]}_0 \times \mathrm{Lozenge}(k_1,\ldots,k_n)$. If this procedure fails at some point, we apply a certain involution and then perform the reverse steps of our transformation to reach a determinant again.

We will first consider the following example, where we will reach indeed an element of $\underline{[n-1]}_{n-1} \times \underline{[n-2]}_{n-2} \times \ldots \times \underline{[0]}_0 \times \mathrm{Lozenge}(k_1,\ldots,k_n)$ and need not apply the involution.

$$
\begin{array}{ccccccccccc}
 & & & & & 17 & & & & & \\
 & & & & 13 & & 20 & & & & \\
 & & & 12 & & 15 & & 16 & & & \\
 & & 10 & & 12 & & 16 & & 15 & & \\
 & 7 & & 7 & & 9 & & 14 & & 21 & \\
\mathbf{5} & & \mathbf{8} & & \mathbf{12} & & \mathbf{14} & & \mathbf{19} & & \mathbf{25}
\end{array}
$$

We add $n-2-i$ to row i for $i = 1,2,\ldots,n-1$.

$$\begin{array}{ccccccccccc}
 & & & & & 20 \\
 & & & & 15 & & 22 \\
 & & & 13 & & 16 & & 17 \\
 & & 10 & & 12 & & 16 & & 15 \\
 & 6 & & 6 & & 8 & & 13 & & 20 \\
\mathbf{5} & & \mathbf{8} & & \mathbf{12} & & \mathbf{14} & & \mathbf{19} & & \mathbf{25}
\end{array}$$

Now we work through the rows, bottom to top (not including the bottommost row). The procedure can only be applied if the array is already of triangular shape (i.e., the entries are accumulated at the end of their rows).

Suppose we have reached row i. We neglect all entries below that row and consider the $\searrow$-diagonals of the i entries in row i. In each diagonal, move the maximum down to row i, push the entries below the former position of the maximum by one unit up in $\nwarrow$-direction and subtract 1 from the elements above this position. We record the position where the maximum came from in a new triangular array (to determine the position, we count bottom to top). If there are more occurrences of a maximum, then we choose the bottommost occurrence. Applying this to row $n-1=5$ in our example, we obtain

$$\begin{array}{ccccccccccc}
 & & & & & 19 \\
 & & & & 14 & & 17 \\
 & & & 12 & & 15 & & 15 \\
 & & 6 & & 8 & & 13 & & 20 \\
 & 6 & & 10 & & 13 & & 16 & & 22 \\
\mathbf{5} & & \mathbf{8} & & \mathbf{12} & & \mathbf{14} & & \mathbf{19} & & \mathbf{25}
\end{array}
\quad
\begin{array}{ccccccccccc}
 & & & & & \bullet \\
 & & & & \bullet & & \bullet \\
 & & & \bullet & & \bullet & & \bullet \\
 & & \bullet & & \bullet & & \bullet & & \bullet \\
 & 1 & & 2 & & 3 & & 2 & & 4 \\
\mathbf{5} & & \mathbf{8} & & \mathbf{12} & & \mathbf{14} & & \mathbf{19} & & \mathbf{25}
\end{array}.$$

This process is clearly reversible. For row 4, this gives

$$\begin{array}{ccccccccccc}
 & & & & & 18 \\
 & & & & 13 & & 16 \\
 & & & 8 & & 13 & & 14 \\
 & & 6 & & 12 & & 15 & & 20 \\
 & 6 & & 10 & & 13 & & 16 & & 22 \\
\mathbf{5} & & \mathbf{8} & & \mathbf{12} & & \mathbf{14} & & \mathbf{19} & & \mathbf{25}
\end{array}
\quad
\begin{array}{ccccccccccc}
 & & & & & \bullet \\
 & & & & \bullet & & \bullet \\
 & & & \bullet & & \bullet & & \bullet \\
 & & 1 & & 2 & & 2 & & 1 \\
 & 1 & & 2 & & 3 & & 2 & & 4 \\
\mathbf{5} & & \mathbf{8} & & \mathbf{12} & & \mathbf{14} & & \mathbf{19} & & \mathbf{25}
\end{array}.$$

In the end, we obtain the following two triangular arrays.

$$\begin{array}{ccccccccccc}
 & & & & & 14 \\
 & & & & 12 & & 16 \\
 & & & 8 & & 13 & & 18 \\
 & & 6 & & 12 & & 15 & & 20 \\
 & 6 & & 10 & & 13 & & 16 & & 22 \\
\mathbf{5} & & \mathbf{8} & & \mathbf{12} & & \mathbf{14} & & \mathbf{19} & & \mathbf{25}
\end{array}
\quad
\begin{array}{ccccccccccc}
 & & & & & 1 \\
 & & & & 1 & & 2 \\
 & & & 1 & & 1 & & 3 \\
 & & 1 & & 2 & & 2 & & 1 \\
 & 1 & & 2 & & 3 & & 2 & & 4 \\
\mathbf{5} & & \mathbf{8} & & \mathbf{12} & & \mathbf{14} & & \mathbf{19} & & \mathbf{25}
\end{array}$$

The element from $\underline{[5]}_5 \times \underline{[4]}_4 \times \ldots \times \underline{[0]}_0$ is obviously encoded in the second triangular array. In the following, we will address this as the *normal* "det-to-loz" procedure. For general determinants, it may not be applicable for the following two reasons: (1) when we reach row i, the entries are not "accumulated" towards the end of row i, or (2) after applying the procedure to row i, an element of row i is strictly less than its $\swarrow$-neighbor (it must be less than its $\searrow$-neighbor by construction). We choose the minimal i with this property, and then apply the following to the array that consists of rows $1, 2, \ldots, i$. We denote by $l_1, \ldots, l_{i+1}$ the entries in row $i+1$. Again we will see later that everything can be simplified — in this case even without really changing anything, but just by analyzing the algorithm and accumulating certain steps. The inflated steps presented next correspond to the steps in the computation.

Step 1. We switch the empty diagonal with its left neighbor until the empty diagonal is above l_1. In principal, a non-empty diagonal can be special or non-special. Before Step 1, all diagonals are non-special, but now all diagonals that have been switched with the empty diagonal are special. This is indicated by underlining the bottom element.

Step 2. Here we have again two phases: assigning (A) and switching (S). The phases alternate, starting with A, and end when we have reached S at some point and there is nothing to switch (which happens precisely when all special diagonals are left of all non-special diagonals — not taking the empty diagonal into account).

All diagonals whose maximum element is strictly less than the $\swarrow$-neighbor of the bottom element of the diagonal are contradicting. Note that at this point all special diagonals are contradicting since they have been moved from left to right.

A: We search for the leftmost contradicting diagonal and change its status: if it is special we make it non-special and vice-versa.

S: We search for a pair of neighboring diagonals, where the left diagonal is non-special and the right diagonal is special and switch them. After this move, the left diagonal is assigned to be non-special and the right diagonal to be special. When we switch two diagonals, we need to switch the according diagonals in the recording array that remembered the positions when applying det-to-loz below row i.

Step 3. If there are special diagonals left, they must be accumulated at the left of the row. We switch the empty diagonal with all of them. In the end, all diagonals are non-special.

Step 4. Now we redo the det-to-loz procedure in rows $i+1, i+2, \ldots, n-1$, in this order. For this, we use the array that marked the positions of the maxima.

We consider the following example.

$$
\begin{array}{cccccccccccccccc}
 & & & & & & & & 11 & & & & & & & \\
 & & & 2 & & & & & & 13 & & & & & & \\
2 & & & & 1 & & & & & & 19 & & & & & \\
 & 1 & & & & 7 & & 9 & & & & 8 & & & & \\
2 & & 2 & & & & 8 & & 3 & & & & 15 & & & \\
 & 1 & & 4 & & 7 & & 9 & & 1 & & & & 17 & & \\
 & & 2 & & 2 & & 7 & & 9 & & 2 & & 17 & & 3 & \\
 & \mathbf{1} & & \mathbf{3} & & \mathbf{6} & & \mathbf{8} & & \mathbf{10} & & \mathbf{14} & & \mathbf{20} & & \mathbf{26}
\end{array}
$$

The diagonal above **1** is empty. We add $n-2-i = 6-i$ to row i, $i = 1, 2, \ldots, n-1$. Then, in each diagonal we move down the maxima, push up the elements below the previous position of the maximum and subtract 1 from the elements above. This gives

$$
\begin{array}{cccccccccccccccc}
 & & & & & & & & 15 & & & & & & & \\
 & & & 5 & & & & & & 16 & & & & & & \\
3 & & & & 3 & & & & & & 10 & & & & & \\
 & 3 & & & & 8 & & 4 & & & & 16 & & & & \\
1 & & 4 & & & & 8 & & 1 & & & & 17 & & & \\
 & 1 & & 1 & & 6 & & 8 & & 1 & & & & 2 & & \\
 & & 3 & & 5 & & 7 & & 9 & & 11 & & 16 & & 22 & \\
 & \mathbf{1} & & \mathbf{3} & & \mathbf{6} & & \mathbf{8} & & \mathbf{10} & & \mathbf{14} & & \mathbf{20} & & \mathbf{26}
\end{array}
$$

The positions of the maxima in the previous array as recorded in the following vector.

$$(3_3, 5_5, 2_2, 2_6, 4_4, 1_1, 5_7)$$

We move to the next row. The diagonal above 11 is the only contradicting diagonal. We move the empty diagonal to the first position and underline the bottom elements of the diagonals with which it has been switched. We are then in phase A. The leftmost contradicting diagonal is the one above 5, and we remove the underline.

$$
\begin{array}{cccccccccccccccc}
&&&&&&&&15&&&&&&& \\
&&&&&5&&&&16&&&&&& \\
&&3&&&&3&&&&10&&&&& \\
&&&3&&&&8&&4&&16&&&& \\
&&1&&4&&&&8&&1&&17&&& \\
&&&1&&1&&6&&8&&1&&2&& \\
&&3&&5&&\underline{7}&&\underline{9}&&\underline{11}&&\underline{16}&&22& \\
&\mathbf{1}&&\mathbf{3}&&\mathbf{6}&&\mathbf{8}&&\mathbf{10}&&\mathbf{14}&&\mathbf{20}&&\mathbf{26}
\end{array}
$$

We are now in phase S. We switch the diagonals above 5 and 7.

$$
\begin{array}{cccccccccccccccc}
&&&&&&&&15&&&&&&& \\
&&&&&5&&&&16&&&&&& \\
3&&&&&&3&&&&10&&&&& \\
&3&&&&&&8&&4&&16&&&& \\
&&4&&1&&&&8&&1&&17&&& \\
&&&1&&1&&6&&8&&1&&2&& \\
&&3&&5&&\underline{7}&&\underline{9}&&\underline{11}&&\underline{16}&&22& \\
&\mathbf{1}&&\mathbf{3}&&\mathbf{6}&&\mathbf{8}&&\mathbf{10}&&\mathbf{14}&&\mathbf{20}&&\mathbf{26}
\end{array}
$$

After 12 applications of A and 11 of applications of B, we finally obtain the following.

$$
\begin{array}{cccccccccccccccc}
&&&&&&&&15&&&&&&& \\
&&&&&5&&&&16&&&&&& \\
&&&&&&3&&3&&10&&&&& \\
&&&4&&&&8&&3&&16&&&& \\
&&1&&1&&&&8&&4&&17&&& \\
&&&1&&1&&6&&8&&1&&2&& \\
&&3&&\underline{5}&&\underline{7}&&\underline{9}&&\underline{11}&&\underline{16}&&22& \\
&\mathbf{1}&&\mathbf{3}&&\mathbf{6}&&\mathbf{8}&&\mathbf{10}&&\mathbf{14}&&\mathbf{20}&&\mathbf{26}
\end{array}
$$

We apply **Step 3** to obtain the following.

$$
\begin{array}{cccccccccccccccc}
&&&&&&&&15&&&&&&& \\
&&&5&&&&&&16&&&&&& \\
&&&&3&&3&&&&10&&&&& \\
&4&&&&8&&3&&&&16&&&& \\
1&&1&&&&8&&4&&&&17&&& \\
&1&&1&&6&&8&&1&&&&2&& \\
&&3&&5&&7&&9&&11&&16&&22& \\
&\mathbf{1}&&\mathbf{3}&&\mathbf{6}&&\mathbf{8}&&\mathbf{10}&&\mathbf{14}&&\mathbf{20}&&\mathbf{26}
\end{array}
$$

We have to perform the same switches to the position recording row.

$$(3_3, 4_4, 2_2, 2_6, 5_5, 1_1, 5_7)$$

We move back the maxima accordingly, and then subtract $n - 2 - i$ from row $i = 1, 2, \ldots, n - 1$.

$$\begin{array}{cccccccccccccccc}
&&&&&&&&11&&&&&&& \\
&&&2&&&&&&13&&&&&& \\
&&&&1&&8&&&&19&&&&& \\
&3&&&&7&&1&&&&8&&&& \\
2&&3&&&&8&&2&&&&15&&& \\
&1&&1&&7&&9&&4&&&&17&& \\
&&2&&2&&7&&9&&2&&17&&3& \\
&\mathbf{1}&&\mathbf{3}&&\mathbf{6}&&\mathbf{8}&&\mathbf{10}&&\mathbf{14}&&\mathbf{20}&&\mathbf{26}
\end{array}$$

Analyzing Steps 1-3, it turns out that overall we only do the following: *We search for the leftmost diagonal that can be moved left and exchange it with the leftmost diagonal above the i-th row to which position it can be moved.*

The construction in this special case stems from the following computation.

Construction for Problem 12. By definition,

$$\begin{aligned}
&\underline{[n-1]}_{n-1} \times \underline{[n-2]}_{n-2} \times \ldots \times \underline{[0]}_0 \times \operatorname{Lozenge}(k_1, \ldots, k_n) \\
&\qquad = \underline{[n-1]}_{n-1} \times \underline{[n-2]}_{n-2} \times \ldots \times \underline{[0]}_0 \\
&\qquad\quad \times \bigsqcup_{(l_1,\ldots,l_{n-1}) \in \underline{[k_1,k_2-1]} \times \ldots \times \underline{[k_{n-1},k_n-1]}} \operatorname{Lozenge}(l_1, \ldots, l_{n-1}).
\end{aligned}$$

By induction, there is a sijection

$$\underline{[n-2]}_{n-2} \times \ldots \times \underline{[0]}_0 \times \operatorname{Lozenge}(l_1, \ldots, l_{n-1}) \Longrightarrow \det_{0 \le i,j \le n-2} \left(\underline{[l_{i+1}]}_j \right),$$

and this induces a sijection from the left hand side in the statement to

$$\underline{[n-1]}_{n-1} \times \bigsqcup_{(l_1,\ldots,l_{n-1}) \in \underline{[k_1,k_2-1]} \times \ldots \times \underline{[k_{n-1},k_n-1]}} \det_{0 \le i,j \le n-2} \left(\underline{[l_{i+1}]}_j \right).$$

Using linearity in the rows, we obtain a sijection to

$$\underline{[n-1]}_{n-1} \times \det_{0 \le i,j \le n-2} \left(\bigsqcup_{l_i \in \underline{[k_{i+1},k_{i+2}-1]}} \underline{[l_i]}_j \right),$$

and using the linearity in the columns we arrive at

$$\det_{0 \le i,j \le n-2} \left(\underline{[1,j]} \times \bigsqcup_{l_i \in \underline{[k_{i+1},k_{i+2}-1]}} \underline{[l_i]}_{j-1} \right).$$

Now we apply Lemma 10 and obtain

$$\det_{0 \le i,j \le n-2} \left(\underline{[k_{i+2}]}_{j+1} \sqcup -\underline{[k_{i+1}]}_{j+1} \right).$$

We can construct a sijection from this by applying elementary row and column operations and expansion with respect to the first column to the right-hand side. $\square$

The procedure is implemented in `loztodet(k)`. For the "non-human" version of Problem 10, there is also `loztodet1(k)`. For recover our example, we need to set `k=[1,3,6,8,10,14,20,26]`,

$$\mathtt{x} = (((), (2,1,2), (2,4,2,1,2), (7,7), (9,9,8,7,1,2), (2,1,3,9), (17,), (3,17,15,8,19,13,11)), (0,3,5,2,6,4,1,7)),$$

and call `loztodet(k)((x,1))`.

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