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„Foundations for a forward stability analysis of wave maps on the future light cone via hyperboloidal coordinates adapted to self-similarity“

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Andras Bonk

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Abstract

This master thesis lays the groundwork for a stability analysis of wave maps on the forward light cone which arise as solutions of a particular geometric wave equation. The main functional analytical tool to tackle this problem will be semigroup theory. Since the thesis is designed to be able to be read by a graduate student, after a short introduction to the problem we are concerned with, the first chapter is an introduction to the most important notions of elementary semigroup theory. The scope of the semigroup theory presented extends to the often called *Lumer-Phillips Theorem*, inter alia giving rise to solutions of abstract Cauchy problems. This will serve as the foundation for the second chapter. The second chapter will be original work. The stability analysis will be approached by the introduction of novel coordinates which will be called "forward self-similarity coordinates". Through these coordinates, energy bounds for solutions of the free wave equation in every odd dimension will be obtained, presented in semigroup language. This embodies the main result in this chapter, also being the main result of this thesis overall. The third and final chapter is a short discussion on how to place the achieved results in the overall analysis of the non-linear analysis and how one would proceed.

Zusammenfassung

Diese Masterarbeit legt das Fundament zur Stabilitätsanalyse von Wellenfunktionen auf dem Vorwärtslichtkegel welche als Lösungen einer bestimmten geometrischen Wellengleichung auftauchen. Das Werkzeug aus der Funktionalanalysis das vorwiegend benutzt wird ist die Halbgruppentheorie. Da die Arbeit für einen Masterstudenten verständlich sein sollte, werden nach einer kurzen Einführung in das zu behandelnde Problem, im ersten Kapitel die wichtigsten Begriffe der elementaren Halbgruppentheorie präsentiert. Der Umfang der präsentierten Halbgruppentheorie reicht bis zu dem *Lumer-Phillips Theorem*, mit welchem unter anderem Lösungen zu abstrakten Cauchy Problemen erzeugt werden können. Dieses Ergebnis legt den Grundstein für das zweite Kapitel. Das zweite Kapitel ist Originalwerk. Die Stabilitätsanalyse wird durch die Einführung von sogenannten "forwärts selbstähnlichen Koordinaten" angegangen. Durch diese Koordinaten werden obere Abschätzungen der Energie von Lösungen der freien Wellengleichung in jeder ungeraden Dimension herausgearbeitet, die in Halbgruppensprache verfasst werden. Dies verkörpert das Hauptresultat dieses Kapitels, welches zugleich das Hauptresult der gesammten Arbeit darstellt. Das dritte Kapitel ist eine kurze Erläuterung wie die erreichten Ergebnisse in der gesamten Analyse des nicht-linearen Problems einzuordnen sind und wie man fortfahren würde.

Contents

0.1. Notation

We fix some notation. In the first chapter some notions of functional analysis are necessary. Suppose E, F are Banach spaces, i.e. complete, normed vector spaces. A linear function $L : \mathcal{D}(L) \subset E \to F$, where $\mathcal{D}(L)$ is the *domain* of L, is called (linear) operator and $\mathrm{rg}(L) := L(\mathcal{D}(L))$ denotes the *range* of L. The space of linear operators from E to F is denoted by $\mathcal{L}(E, F)$, where we abbreviate by $\mathcal{L}(E)$ whenever $E = F$. One can equip this space with several notions of convergence. Most notable are

- convergence with respect to the uniform operator norm $\|\cdot\|_{\mathcal{O}(E,F)}$ given by $T_n \overset{n}{\to} T$ uniformly $\iff \sup_{\|x\|_F=1} \|T_n x - Tx\|_E \overset{n}{\to} 0.$
- strong (pointwise) convergence given by $S_n \overset{n}{\to} S$ strongly $\iff \|S_n x - Sx\|_F \overset{n}{\to} 0$ for every $x \in E$.

For $T \in \mathcal{L}(E, F)$ continuity with respect to the operator norm is easily seen to be equivalent to requiring that it is *bounded*, i.e. there exists $M < \infty$ such that $\|Tx\|_F \leq M \|x\|_E$ for all $x \in E$. Expressions like "the in general unbounded operator" are to be understood as not yet being able to say whether the operator in question is bounded, and possibly could be not bounded. A bounded, bijective operator whose inverse is also bounded is called a *homeomorphism*. We often use the composition $T \circ S$ of two operators T, S. Since we never multiply any operators we reserve the notation TS for this composition. For the most important fundamental results in functional analysis used in this thesis we invite the reader to check Appendix A, where also some further notions (closed operators, exponential operators, resolvents etc.) can be found.
In the second chapter we work on a subset of $\mathbb{R} \times \mathbb{R}^n$ where the first coordinate is the time variable and the second coordinate is the spatial variable. The underlying field is $\mathbb{R}$, e.g. the function space $C^\infty(\Omega)$ denotes the space of smooth functions mapping from Ω to $\mathbb{R}$. Slot derivatives ∂_i range from $0, \ldots, n$ where ∂_0 concerns the time variable and the other concern the spatial variable. Function tuples are denoted by bold letters, e.g. $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. A partial derivative acting on a function tuple is to be understood as component-wise application of the derivative, e.g. $\partial_j \mathbf{f} = \begin{pmatrix} \partial_j f_1 \\ \partial_j f_2 \end{pmatrix}$. Often times we will first introduce operators *formally*, that is describing an equation this operator should satisfy, without specifying the spaces on which said operator acts upon. This will be made evident by the use of Fraktur font letters in the formal definition. When turning this definition into a rigorous one, we will use the same letter in normal font, e.g. $\mathfrak{L}$ will turn into $\mathbf{L}$. We shorten our notation by introducing on $\mathbb{R}_+$ the relation $a \lesssim b$ iff there exists a constant C such that $a \leq Cb$. Accordingly we define $a \gtrsim b$ and the equivalence relation $\simeq$. The d-dimensional *open* ball centered at x of radius R is denoted by $\mathbb{B}^d_R(x)$. In the case of $x = 0$ we abbreviate by $\mathbb{B}^d_R$.

1. Semigroup Theory

1.1. Motivation

In this section the semigroup theory required for the later sections will be developed. It is mainly a revision of the first two chapters of [8], which is the standard literature for semigroup theory nowadays. The results and techniques are very much inspired by [8] and its origins in [9] and the author does not claim intellectual property of any of the ideas or techniques involved. Most results are simply adjusted to better fit our setting and most proofs are adjusted in a way the author found most intuitive.
To motivate the theory and establish the connections to our work, consider the following problem. We want to work on the wave equation

$$\begin{aligned} \Box u(t,x) := (-\partial_t^2 + \Delta_x)u(t,x) = 0, \\ u(0,x) = u_0(x), \\ \partial_0 u(0,x) = u_1(x), \end{aligned}$$

for some functions u_0, u_1. Although explicit solutions to this problem are well known, we tackle it in a different way to motivate our later approach.
We rewrite the problem into one where the emphasis lies on the spatial derivatives. This is achieved by first rewriting the equation such that on one side there are only time and on the other side only spatial derivatives, i.e.

$$\partial_t^2 u(t,x) = \Delta_x u(t,x),$$

and then introducing the variable $\partial_t u$ to rewrite this into the two dimensional problem

$$\partial_t \begin{pmatrix} u(t,x) \\ \partial_t u(t,x) \end{pmatrix} = \begin{pmatrix} \partial_t u(t,x) \\ \Delta_x u(t,x) \end{pmatrix}.$$

Seeing the right hand side as a spatial operator $\mathbf{L}$ defined by $\mathbf{L}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_2 \\ \Delta f_1 \end{pmatrix}$ acting on the tuple $\mathbf{u}(t,\cdot) := \begin{pmatrix} u(t,\cdot) \\ \partial_t u(t,\cdot) \end{pmatrix}$, we have $\partial_t \mathbf{u}(t,\cdot) = \mathbf{L}\mathbf{u}(t,\cdot)$, where we slightly abuse the notation by writing $\mathbf{L}\mathbf{u}(t,x) := \mathbf{L}(\mathbf{u}(t,\cdot))(x)$. The initial condition translates to $\mathbf{L}\mathbf{u}(0,x) = \mathbf{u}(0,x)$. This is the standard starting point for semigroup theory. That is, looking for solutions of the ordinary differential equation

$$\begin{aligned} \partial_t g(t) &= A g(t), \\ g(0) &= I. \end{aligned} \tag{1.1}$$

As we will see in the following, the specifications made on which objects A and g are, detrimentally vary the complexity of the problem. Let us take a step back and consider problem (1.1) in its most elementary form. That is, suppose we are given $A \in \mathbb{C}$ and g is a one dimensional function. Then (1.1) is solved by $g(t) = e^{At}$. We note that $\dot{g}(0) := \partial_t g(t) \mid_{t=0} = A$. Looking at this the other way around, the solution g emerges from the mapping $A \mapsto e^{At}$. This is exactly the approach we will successfully generalize for a much wider class of problems, where A and g will be operators, A possibly even being unbounded, as is the case for the differential operator $\mathbf{L}$ defined above. We will see that even in these settings A *generates* g in a sense specified later. Let us try to solve the problem in the case where the desired objects are operators. Suppose A and $T(t)$ are operators for every t and consider the operator problem

$$\begin{aligned} \partial_t T(t) &= AT(t), \\ T(0) &= I. \end{aligned} \tag{1.2}$$

Inspired by the most simple case (1.1), we predict that $T(t) := e^{At}$ is a solution to this problem. However, there is one major complication in even writing down this desired result. In contrast to bounded operators, in the case of unbounded operators it is not clear at all, what the exponential function of an operator even means. There is no general formula for the exponential function. We will establish assumptions on A for which we can define e^{At} in a proper way, to then show that the problem (1.2) is indeed solved by this exponential operator. This will be the content of the main result of this chapter, known as the *Lumer-Phillips Theorem.* We start with solving the problem in the case of bounded operators. We will see that in this case, all results resemble their counterparts in the most elementary scalar case (1.1).

1.2. Uniformly continuous semigroups

Definition 1.2.1. A family $(T(t))_{t\geq 0}$ of bounded, linear operators on a Banach space X is called a semigroup, if it satisfies the functional equation,

$$(FE) \qquad \begin{cases} T(t+s) = T(t)T(s) \text{ for all } t, s \geq 0, \\ T(0) = I, \end{cases}$$

which we call *semigroup law.*
Define $\xi : \mathbb{R}_+ \to \mathcal{L}(X), t \mapsto T(t)$. $(T(t))_{t\geq 0}$ is said to be

- uniformly continuous, if ξ is continuous when $\mathcal{L}(X)$ is endowed with the uniform operator topology, i.e. the topology on $\mathcal{L}(X)$ given by $S_n \overset{n}{\to} S \iff \|S_n - S\|_{\mathcal{O}(X)}$ $= \sup_{\|x\|_X = 1} \|S_n x - Sx\|_X \overset{n}{\to} 0$.
- strongly continuous, if ξ is continuous when $\mathcal{L}(X)$ is endowed with the strong operator topology, i.e. the topology on $\mathcal{L}(X)$ given by $S_n \overset{n}{\to} S \iff \|S_n x - Sx\|_X \overset{n}{\to} 0$, for every $x \in X$.

Remark 1.2.2.
- *Note that ξ describes $(T(t))_{t\geq 0}$ as a function. While mapping to bounded linear operators, ξ is not linear itself. Therefore we take caution in using the words linear and bounded when abbreviating with T for $(T(t))_{t\geq 0}$.*
- *It is evident that strong continuity is equivalently satisfied by requiring that the orbit maps at every $x \in X$, $\xi_x : t \mapsto T(t)x$ are continuous. We will often use this equivalent description.*

The following lemma convinces us that, when encountering problem (1.2), semigroups are the proper objects to deal with. It is a first result, which shows that uniform continuity added to satisfying the semigroup law already implies differentiability. We will later generalize this in a weaker sense to strong continuity. The following lemma contains differentiation and integration of Banach- valued functions. Since this topic may be lesser known to some readers we refer the interested reader to a short discussion in Appendix A, where the most important notions are introduced and sources for literature are mentioned.

Lemma 1.2.3. *Suppose that $(T(t))_{t\geq 0}$ is a uniformly continuous semigroup. Then it solves the differential equation*

$$\begin{cases} \partial_t T(t) = AT(t) \text{ for all } t \geq 0, \\ T(0) = I, \end{cases} \tag{DE}$$

for some bounded $A \in \mathcal{L}(X)$, which satisfies $AT(t) = T(t)A$ for all $t \geq 0$.

Proof. Define

$$V(t) := \int_0^t T(s)\ \mathrm{d}s,\ t \geq 0.$$

Since T is uniformly bounded on every interval $[0, t]$, this is well-defined and by A.3 continuously differentiable with $\dot{V}(t) = T(t)$. Since $\lim_{t\downarrow 0} \frac{V(t)}{t} = \dot{V}(0) = T(0) = I$, we infer that there is $t_0 > 0$ such that $\left\| \frac{V(t_0)}{t_0} - I \right\|_{\mathcal{O}(X)} < 1$, hence the operator $\frac{V(t_0)}{t_0}$ and therefore $V(t_0)$ are invertible with continuously differentiable inverse. With this we calculate

$$T(t) = V(t_0)^{-1}V(t_0)T(t) = V(t_0)^{-1}\int_0^{t_0} T(s+t)\mathrm{d}s = V(t_0)^{-1}(V(t+t_0) - V(t)).$$

Since the right hand is continuously differentiable we have proved that so is T and we compute

$$\dot{T}(t) = \lim_{h\downarrow 0} \frac{T(t+h) - T(t)}{h} = T(t) \lim_{h\downarrow 0} \frac{T(h) - T(0)}{h} = T(t)A,$$

where we have set $A := \lim_{h\downarrow 0} \frac{T(h)-T(0)}{h} = \dot{T}(0)$. Since T was shown to be continuously differentiable (as a map $\mathbb{R} \to (\mathcal{L}(X), \|\cdot\|_{\mathcal{O}(X)})$), A must be bounded. We also note that since $T(t+h) = T(h+t)$ in the above calculation we have that $T(t)A = AT(t)$. This proves the claim. □

The following two lemmata give a precise description of uniformly continuous semigroups. We will first show uniqueness and then give an explicit formula for the semigroup.

Lemma 1.2.4. *For given $A \in \mathcal{L}(X)$, two uniformly continuous semigroups $(T(t))_{t\geq 0}$, $(S(s))_{s\geq 0}$ satisfying (DE) must be the same.*

Proof. Suppose $(T(t))_{t\geq 0}, (S(s))_{s\geq 0}$ both solve (DE). Then consider

$$Q : t \mapsto T(t)S(s-t),\ 0 \leq t \leq s.$$

Q is differentiable with

$$\dot{Q}(t) = AT(t)S(s-t) - T(t)AS(s-t) = 0, \tag{1.3}$$

since A commutes with every $T(t)$ and $S(s)$. Hence Q is constant and

$$S(s) = Q(0) = Q(s) = T(s).$$

Thus $T = S$, as claimed. □

We are trying to find an - and by the previous result *the* - explicit solution to (DE). Inspired by the beginning of the chapter we suspect that the exponential operator plays an important role. It is introduced in A.4 together with some very useful properties. The importance of the exponential operator is that they are precisely the operators which are uniformly continuous semigroups. We will show this by proving that for a bounded operator A the family $(e^{tA})_{t\geq 0}$ is indeed a uniformly continuous semigroup, which by Lemma 1.2.3 means that it solves (DE). By uniqueness, the desired result immediately follows. The fact that the exponential operator is a semigroup is an immediate consequence of the property in Lemma A.5 and the continuity of the exponential function in the scalar case. For completeness we include it here.

Lemma 1.2.5. *For a bounded operator $A \in \mathcal{L}(X)$ the family $(e^{tA})_{t\geq 0}$ is a uniformly continuous semigroup.*

Proof. By A.5 we have

$$e^{0A} = I, e^{(t+s)A} = e^{tA}e^{sA}.$$

This means that $(e^{tA})_{t\geq 0}$ satisfies (FE). Hence it is a semigroup and to check uniform continuity, by

$$\left\| e^{(t+h)A} - e^{tA} \right\|_{\mathcal{O}(X)} \leq \left\| e^{tA} \right\|_{\mathcal{O}(X)} \left\| e^{hA} - I \right\|_{\mathcal{O}(X)},$$

it suffices to check that $\lim_{h\to 0} e^{hA} = I$. We compute

$$\left\| e^{hA} - I \right\|_{\mathcal{O}(X)} = \left\| \sum_{k=1}^{\infty} \frac{h^k}{k!} A^k \right\|_{\mathcal{O}(X)} \leq \sum_{k=1}^{\infty} \frac{|h|^k}{k!} \|A\|_{\mathcal{O}(X)}^k = e^{|h| \|A\|_{\mathcal{O}(X)}} - 1 \xrightarrow{h\to 0} 0.$$

Thus $(e^{tA})_{t\geq 0}$ is uniformly continuous and we are done. □

By uniqueness, see Lemma 1.2.4, we are now ready to summarize the precise form of uniformly continuous semigroups. This is the content of the following theorem, for which all the work has been done already.

Theorem 1.2.6. *Every uniformly continuous semigroup* $(T(t))_{t\geq 0}$ *is of the form*

$$T(t) = e^{tA},$$

where A *is precisely the bounded operator* $A = \dot{T}(0)$. *Conversely, for every bounded operator* $B \in \mathcal{L}(X)$ *on a Banach space* X, $(e^{tB})_{t\geq 0}$ *is a uniformly continuous semigroup.*

This result demonstrates the importance of the operator $\dot{T}(0)$. Since the operator *generates* its respective semigroup, in the sense clarified in the previous lemma, it is worth taking a closer look at the operator $\dot{T}(0)$ to understand the semigroup. Towards this end and to compare it with a similar construction in the strongly continuous case, we give it a fitting name.

Definition 1.2.7. Let $(T(t))_{t\geq 0}$ be a uniformly continuous semigroup. The bounded operator

$$\mathcal{L}(X) \ni \dot{T}(0) = \lim_{h\downarrow 0} \frac{1}{h}(T(h) - I)$$

is called the generator of $(T(t))_{t\geq 0}$.

Having obtained a very satisfactory result for solving (DE) in the case of bounded generators A, we immediately ask ourselves of its usefulness. Since we want to apply the theory to differential operators - being unbounded in general - in place of the generator we note that we have to generalize our theory.

1.3. Strongly continuous semigroups

The main result of the previous section was the inseparable relation between *bounded* generators and *uniformly continuous* semigroups. If we want to develop our theory for *unbounded* operators, we will therefore need another class of objects in place of uniformly continuous semigroups. In this section it will be shown that *strongly continuous* semigroups are the objects desired. For the definition of strong continuity, we refer the reader to Definition 1.2.1. Since uniform convergence is stronger than strong convergence, it is clear that every uniformly continuous semigroup is also a strongly continuous one. The converse is not true, as will be shown by an example further down in this section. It is evident that strong continuity can also be expressed by saying that for every $x \in X$ the orbit map $\xi_x : \mathbb{R}_{\geq 0} \to X, t \mapsto T(t)x$, is continuous. This should be compared with uniform continuity where it is required that the map $\xi : \mathbb{R}_{\geq 0} \to (\mathcal{L}(X), \|\cdot\|_{\mathcal{O}(X)}), t \mapsto T(t)$, is continuous. In this sense strong convergence is often referred to as pointwise convergence. In Lemma 1.2.3 it was established that uniform continuity together with (FE) implies differentiability of ξ. Hence we hope that strong continuity together with (FE) still implied differentiability but now in the pointwise sense, i.e. differentiability of ξ_x instead of ξ. To get used to strong continuity and to make our later life easier we first gather some useful facts about strongly continuous semigroups. Since we are dealing with pointwise convergence it is natural that some form of the uniform boundedness principle will be useful. It is presented in general form in the appendix (Lemma A.6) and the next lemmata are immediate consequences tailored to strongly continuous semigroups. First let us start with an auxiliary lemma which is also very useful in itself as we will see later. It states that semigroups which are strongly continuous in zero are uniformly bounded on every compact interval.

Lemma 1.3.1. *For a semigroup $(T(t))_{t\geq 0}$ on a Banach space X, suppose that $\lim_{t\downarrow 0} T(t)x = x$ for every $x \in X$. Then for every $t_0 \geq 0$ there exists M_{t_0} such that*

$$\|T(t)\|_{\mathcal{O}(X)} \leq M_{t_0}$$

for every $t \in [0, t_0]$. In particular, every strongly continuous semigroup is uniformly bounded on every interval $[0, t_0]$.

Proof. We first show that there exists $\delta > 0$ such that $\|T(\cdot)\|_{\mathcal{O}(X)}$ is uniformly bounded on $[0, \delta]$. Assume towards contradiction that there was no such δ. Then there is a sequence $(\delta_n) \downarrow 0$ such that $\|T(\delta_n)\|_{\mathcal{O}(X)} \xrightarrow{n\to\infty} \infty$. But by the uniform boundedness principle (Lemma A.6) there would then exist $x \in X$ such that $\|T(\delta_n)x\|_X \xrightarrow{n\to\infty} \infty$, which contradicts right continuity at zero for this particular x. Hence there exists $M_\delta > 1$ such that $\|T(t)\|_{\mathcal{O}(X)} \leq M_\delta$ for $t \in [0, \delta]$. The last step is to use the semigroup law (FE) to translate this boundedness to larger intervals. Indeed, for every $t_0 > 0$ there exists $N \in \mathbb{N}$ and $0 \leq t_1 < \delta$ such that $t_0 = \delta N + t_1$. Hence

$$\|T(t_0)\|_{\mathcal{O}(X)} \leq \|T(\delta)\|_{\mathcal{O}(X)}^N \|T(t_1)\|_{\mathcal{O}(X)} \leq M_\delta^{N+1}.$$

Since M_δ was chosen to be greater than 1 this estimate also holds for every $0 \leq t \leq t_0$. This is what we wanted to show. □

The next lemma is an immediate consequence. It gives us a simple way to check strong continuity. The lemma states that strong continuity for semigroups is already implied by strong right continuity at zero, which in hindsight shows that the assumption in the previous lemma is precisely that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup.

Lemma 1.3.2. *Let $(T(t))_{t\geq 0}$ be a semigroup on a Banach space X. The following are equivalent:*

(i) $(T(t))_{t\geq 0}$ *is strongly continuous.*

(ii) $\lim_{t\downarrow 0} T(t)x = x$.

Proof. It is clear that (i) implies (ii). For (ii) ⇒ (i) we note that right continuity is an immediate consequence of right continuity at 0 together with the semigroup property (FE). Indeed,

$$\lim_{h\downarrow 0} \|T(t+h)x - T(t)x\|_X \leq \|T(t)\|_{\mathcal{O}(X)} \lim_{h\downarrow 0} \|T(h)x - x\|_X = 0,$$

where we used the operator norm property $\|Sx\|_{\mathcal{O}(X)} \leq \|S\|_{\mathcal{O}(X)} \|x\|$. For left continuity, which we only have to show for $t > 0$, we observe for $-t < h < 0$ that

$$\lim_{h\uparrow 0} \|T(t+h)x - T(t)x\|_X \leq \lim_{h\uparrow 0} \|T(t+h)\|_{\mathcal{O}(X)} \|x - T(-h)x\|_X .$$

Since we have shown in the previous lemma that $\|T(\cdot)\|_{\mathcal{O}(X)}$ is bounded on $[0,t]$, the expression on the right hand side is zero as a consequence of right continuity at zero. In summary, $t \mapsto T(t)x$ is continuous at every $t \geq 0$, which is what we wanted to show. □

Lemma 1.3.1 also yields exponential boundedness on all of $\mathbb{R}_+$. This is concretized in the following lemma.

Lemma 1.3.3. *For every strongly continuous semigroup $(T(t))_{t\geq 0}$ there is $M > 0$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\|_{\mathcal{O}(X)} \leq Me^{\omega t}.$$

Proof. By Lemma 1.3.1 there is $M > 1$ such that $\|T(t)\|_{\mathcal{O}(X)} \leq M$ for all $t \in [0,1]$. Hence for $t' = N + \delta$ with $N \in \mathbb{N}, 0 \leq \delta < 1$ we have

$$\|T(t')\|_{\mathcal{O}(X)} \leq \|T(1)\|^N_{\mathcal{O}(X)} \|T(\delta)\|_{\mathcal{O}(X)} \leq M^{N+1} = Me^{log(M)N} \leq Me^{\omega t'},$$

where $\omega := log(M) > 0$ and the last step follows from $N \leq t'$ by construction. This proves the claim. □

In the proof of Lemma 1.3.1 we have already used the uniform boundedness principle. For later uses, we will now present a version for semigroups. It shows that, like in the case for scalar functions, pointwise convergence implies *locally* uniform convergence.

Lemma 1.3.4. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. Then the map*

$$\begin{aligned} K \times C &\to X, \\ (t,x) &\mapsto T(t)x \end{aligned}$$

is uniformly continuous on every pair of compact subsets $K \subset \mathbb{R}_+, C \subset X$.

Proof. Let $\epsilon > 0$. Since K is compact, there is t_0 such that $K \subset [0,t_0]$. Hence, by Lemma 1.3.1 there exists $M < \infty$ such that $\|T(t)\|_{\mathcal{O}(X)} \leq M$ for all $t \in K$. Since C is compact there exist n many $x_i \in C$ such that $C \subset \bigcup_{i=1}^n \mathbb{B}_{\epsilon/M}(x_i)$. Let $\delta' > 0$ such that $\|T(t)x_i - T(s)x_i\|_X < \epsilon$ for all $i = 1,\ldots,n$ whenever $|t-s| < \delta$, which is possible to find since $(T(t))_{t\geq 0}$ is strongly continuous and there are only finitely many x_i. For general $x \in C$ we find x_i such that $x \in \mathbb{B}_{\epsilon/M}(x_i)$ and compute

$$\begin{aligned} \|T(t)x - T(s)x\|_X &\leq \|T(t)x - T(t)x_i\|_X + \|T(t)x_i - T(s)x_i\|_X + \|T(s)x_i - T(s)x\|_X \\ &\leq \|T(t)\|_{\mathcal{O}(X)} \|x - x_i\|_X + \|T(t)x_i - T(s)x_i\|_X + \|T(s)\|_{\mathcal{O}(X)} \|x_i - x\|_X \\ &\leq 3\epsilon. \end{aligned}$$

For every $x, y \in C$ such that $\|x-y\|_X \leq \epsilon/M$ this yields

$$\|T(t)x - T(s)y\|_X \leq \|T(t)x - T(s)x\|_X + \|T(s)x - T(s)y\|_X \leq 4\epsilon.$$

Set $\delta := \min\{\delta', \epsilon/M\}$. The fact that $\mathbb{B}_\delta(t,x) \subset \mathbb{B}_\delta(t) \times \mathbb{B}_\delta(x)$ implies that for all $(t,x) \in K \times C$ and for all $(s,y) \in \mathbb{B}_\delta(t,x)$ we have $\|T(t)x - T(s)y\|_X \leq 4\epsilon$. This shows that the map $(t,x) \mapsto T(t)x$ is uniformly continuous on $K \times C$, which concludes the proof. □

We will soon see, that it is sometimes much easier to prove semigroup properties not on all elements of its domain, but on a dense subset. The next lemma should be seen as a preparatory tool towards this analysis.

Lemma 1.3.5. *Let $(T(t))_{t\geq 0}$ be a semigroup on a Banach space X. The following are equivalent:*

(i) $(T(t))_{t\geq 0}$ is strongly continuous.

(ii) There exist $\delta > 0, M > 1$ and a dense subset $D \subset X$ such that
- *(a) $\|T(t)\|_{\mathcal{O}(X)} \leq M$ for all $t \in [0,\delta]$,*
- *(b) $\lim_{t\downarrow 0} T(t)x = x$ for all $x \in D$.*

Proof. With Lemma 1.3.1 the implication $(i) \Rightarrow (ii)$ is clear. To prove the other direction, we note that by Lemma 1.3.2 we only have to prove that $\lim_{t\downarrow 0} T(t)x = x$ for all $x \in X$. Since D is assumed to be dense, we find a sequence $(x_n)_{n\geq 0} \subset D$ that converges to x. By triangle inequality we compute for every $n \geq 0$

$$\|T(t)x - x\|_X \leq \|T(t)\|_{\mathcal{O}(X)} \|x - x_n\|_X + \|T(t)x_n - x_n\|_X + \|x_n - x\|_X .$$

Using that the desired property is satisfied for all $x_n \in D$ and that $\|T(\cdot)\|_{\mathcal{O}(X)}$ is bounded on a right neighbourhood of 0, applying the limit $t \downarrow 0$ to the above inequality yields

$$\lim_{t\downarrow 0} \|T(t)x - x\|_X \leq (M+1) \|x - x_n\|_X .$$

By letting $n \to \infty$ on the right hand side, we obtain the desired result. □

Remember that our goal is to generalize the problem (DE) to unbounded operators A. Since (DE) is presented with derivatives we still need to include differentiability into our generalized theory. As commented on in the prelude to this chapter, we can only hope for differentiability of the orbit map ξ_x instead of ξ. First, we again use the (FE) property to make it easier to check for differentiability. This is the content of the next lemma, whose claim and proof are very similar to its counterpart in the continuous case (cf. Lemma 1.3.2).

Lemma 1.3.6. *Let $(T(t))_{t>0}$ be a strongly continuous semigroup on a Banach space X. Consider for every $x \in X$ the orbit map $\xi_x : t \mapsto T(t)x$. The following are equivalent.*

(i) ξ_x is differentiable on $\mathbb{R}_+$ with derivative $\dot{\xi}_x(t) = T(t)\dot{\xi}_x(0) = \dot{\xi}_x(0)T(t)$.

(ii) ξ_x *is right differentiable at* 0.

Proof. We only have to prove (ii) $\Rightarrow$ (i). Let $x \in X$. For right differentiability we use (FE) to obtain for $h > 0$,

$$\left\| \frac{1}{h}(\xi_x(t+h) - \xi_x(t)) \right\|_X \leq \|T(t)\|_{\mathcal{O}(X)} \left\| \frac{1}{h}(T(h)x - x) \right\|_X .$$

Letting $h \downarrow 0$ and using right differentiability at 0 we conclude that ξ_x is right differentiable at every $t \geq 0$. For left differentiability we compute for $-t < h < 0$

$$\left\| \frac{1}{h}(\xi_x(t+h) - \xi_x(t)) \right\|_X \leq \|T(t+h)\|_{\mathcal{O}(X)} \left\| \frac{1}{h}(x - T(-h)x) \right\|_X .$$

Since $\|T(\cdot)\|_{\mathcal{O}(X)}$ is bounded on $[0,t]$ letting $h \uparrow 0$ gives us left differentiability. In summary, ξ_x is differentiable on $\mathbb{R}_+$ for every $x \in X$ and

$$\dot{\xi}_x(t) = \lim_{h\downarrow 0} \frac{1}{h}(T(t+h)x - T(t)x) = T(t)\dot{\xi}_x(0) = \dot{\xi}_x(0)T(t).$$

□

Now we know how to check for differentiability. In the case of uniformly continuous semigroups, we showed in Lemma 1.2.3 that a uniformly continuous semigroup is actually already differentiable. As was said earlier, we now hope to have a similar pointwise result. It may be surprising that this will not be the case in general, at least not for every point. To give some motivation why this should not be as surprising, we will show in the following lemmata that if this was the case, then every strongly continuous semigroup was already uniformly continuous. This, in hindsight, justifies the care we take in the next definition. We now define the strongly continuous counterpart to the generator in the uniformly continuous case (cf. Definition 1.2.7). But we will do so only for a subset of our given Banach space, that is for the points where it exists.

Definition 1.3.7. Let $(T(t))_{t\geq 0}$ be strongly continuous semigroup on a Banach space X. The, in general unbounded, linear operator $A : \mathcal{D}(A) \subset X \to X$ defined pointwise by

$$Ax := \dot{\xi}_x(0) = \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x)$$

is called the generator of $(T(t))_{t\geq 0}$. Its domain $\mathcal{D}(A) := \{x \in X \mid \xi_x \text{ is right differentiable in } 0\}$ is the subspace of X where it is ensured that this pointwise definition makes sense.

The small claim in the definition that A is linear, follows from linearity of each $T(h)$ and is left to check to the thorough reader. It is time for an example where we calculate the generator of a semigroup. It also serves as an example to prove the long overdue statement that not every strongly continuous semigroup is uniformly continuous.

Example 1.3.8. *Consider the translation semigroup* $(T(t))_{t\geq 0}$ *acting on the Banach space* $(C(\mathbb{R}), \|\cdot\|_\infty)$ *by* $T(t)f = f(t+\cdot)$. *It is indeed a semigroup, since* $T(t+s)f = f(t+s+\cdot) = T(t)T(s)f$. *To check strong continuity we see that*

$$T(h)f = f(h+\cdot),$$

converges to f *for* $h \downarrow 0$ *since* f *was assumed to be bounded and hence uniformly continuous. Let us calculate the generator* A. *The limit*

$$\lim_{h\downarrow 0} \frac{T(h)f - f}{h} = \lim_{h\downarrow 0} \frac{f(h+\cdot) - f}{h}$$

exists exactly for those f *which are differentiable and for which* f' *converges uniformly. This shows that* A *is the differentiation operator* $Af = f'$ *with* $\mathcal{D}(A) = \{f \in (C(\mathbb{R}), \|\cdot\|_\infty) : f' \in (C(\mathbb{R}), \|\cdot\|_\infty)\}$. *One way to conclude that* $(T(t))_{t\geq 0}$ *cannot be uniformly continuous is to observe the obvious fact that* $\mathcal{D}(A)$ *is a proper subset of* $(C(\mathbb{R}), \|\cdot\|_\infty)$. *Another way, in view of Lemma 1.2.6, is to show directly that* A *is unbounded on* $(C(\mathbb{R}), \|\cdot\|_\infty)$, *which can be seen by using the sequence* $f_n := \sin(\cdot n)$, *for which we have that* $\|f_n\|_\infty = 1$ *for all* $n \in \mathbb{N}$ *but* $\|Af_n\|_\infty = \|n\cos(\cdot n)\|_\infty = n \to \infty$.

One question immediately coming to mind is how large the domain of A is. As commented on before, it is a proper subset of X, differing from the case of uniformly continuous semigroups. However if it was very small, or even empty, then we would have to make further assumptions on the semigroups we are interested in to make this theory usable, which was not desirable. The following lemmata reassure us that this cannot be the case. The first lemma gives us an idea on which elements are the right candidates to show that the domain is almost X, namely that $\mathcal{D}(A)$ is dense in X, which is the content of the second lemma. Other helpful properties are also collected.

Lemma 1.3.9. *Let* $(T(t))_{t\geq 0}$ *be a strongly continuous semigroup on a Banach space* X. *Its generator satisfies the following properties:*

(i) $T(t)$ *maps* $\mathcal{D}(A)$ *to itself for every* $t \geq 0$. *Indeed for* $x \in \mathcal{D}(A)$ *one has*

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t)x = AT(t)x = T(t)Ax.$$

(ii) For every $t \geq 0$, *we have*

$$\begin{aligned} T(t)x - x &= A\int_0^t T(s)x \,\mathrm{d}s \qquad && \text{if } x \in X, \\ &= \int_0^t T(s)Ax \,\mathrm{d}s \qquad && \text{if } x \in \mathcal{D}(A). \end{aligned}$$

In particular, for every $x \in X, t \geq 0$ *we have that* $\int_0^t T(s)x \,\mathrm{d}s \in \mathcal{D}(A)$.

Proof. (*i*) is just a reformulation of Lemma 1.3.6(i) whenever Ax exists. To show (*ii*), let $x \in X, 0 < h < t$. We compute

$$\begin{aligned}\frac{1}{h}\left(T(h)\int_0^t T(s)x \, \mathrm{d}s - \int_0^t T(s)x \, \mathrm{d}s\right) &= \frac{1}{h}\left(\int_h^{t+h} T(s)x \, \mathrm{d}s - \int_0^t T(s)x \, \mathrm{d}s\right) \\ &= \frac{1}{h}\left(\int_t^{t+h} T(s)x \, \mathrm{d}s - \int_0^h T(s)x \, \mathrm{d}s\right).\end{aligned}$$

The left hand side converges to $A\int_0^t T(s)x \, \mathrm{d}s$ while the right hand side converges to $T(t)x - x$ when $h \downarrow 0$ by A.3. Lastly let us prove the additional statement for $x \in \mathcal{D}(A)$. For all $s \in [0,t]$ we estimate

$$\begin{aligned}\left\| T(s)\frac{T(h)x - x}{h} - T(s)Ax \right\|_X &\leq \|T(s)\|_{\mathcal{O}(X)} \left\| \frac{T(h)x - x}{h} - Ax \right\|_X \\ &\leq \sup_{s\in[0,t]} \|T(s)\|_{\mathcal{O}(X)} \left\| \frac{T(h)x - x}{h} - Ax \right\|_X.\end{aligned}$$

Since $\|T(\cdot)\|_{\mathcal{O}(X)}$ is bounded on $[0,t]$ and $\lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x) = Ax$ we have on $[0,t]$ uniform convergence of $\left(T(\cdot)\frac{T(h)x-x}{h}\right)_{\frac{1}{h}\in\mathbb{N}}$ to the integrable $T(\cdot)Ax$. Hence we can exchange limit and integral in the last step in the following equation. We compute

$$\begin{aligned}A\int_0^t T(s)x \, \mathrm{d}s &= \lim_{h\downarrow 0} \frac{1}{h}\left(T(h)\int_0^t T(s)x \, \mathrm{d}s - \int_0^t T(s)x \, \mathrm{d}s\right) \\ &= \lim_{h\downarrow 0} \int_0^t T(s)\frac{1}{h}(T(h)x - x) \, \mathrm{d}s = \int_0^t T(s)Ax \, \mathrm{d}s,\end{aligned}$$

which is what we wanted to prove. □

This is now used to show that the generator $(A, \mathcal{D}(A))$ admits nice properties.

Lemma 1.3.10. *The generator $(A, \mathcal{D}(A))$ of a strongly continuous semigroup $(T(t))_{t\geq 0}$ is a densely defined, closed operator. It determines the semigroup uniquely.*

Proof. We start by showing that A is closed. Let $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ be such that $x_n \xrightarrow{n\to\infty} x$ and $Ax_n \xrightarrow{n\to\infty} y$, where $x, y \in X$. We need to show that $x \in \mathcal{D}(A)$ with $Ax = y$. By Lemma 1.3.9(ii) we have that for every $n \in \mathbb{N}$

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n \, \mathrm{d}s.$$

Since $\|T(\cdot)\|_{\mathcal{O}(X)}$ is bounded on $[0,t]$, very similar to the previous proof we have by uniform convergence of the integrand that by letting $n \to \infty$ we get

$$T(t)x - x = \int_0^t T(s)y \, \mathrm{d}s.$$

Multiplying by $1/t$ and then letting $t \downarrow 0$ yields $x \in \mathcal{D}(A)$ with $Ax = y$. Hence A is closed. Let us show that $\mathcal{D}(A)$ is dense in X. We have for every $t > 0$ and every $x \in X$ that $\int_0^t T(s)x \,\mathrm{d}s \in \mathcal{D}(A)$ as a consequence of Lemma 1.3.9(ii). Hence also $1/t \int_0^t T(s)x \,\mathrm{d}s \in \mathcal{D}(A)$. Letting $t \downarrow 0$ we have by A.3 that for any $x \in X$

$$\mathcal{D}(A) \ni \frac{1}{t}\int_0^t T(s)x \,\mathrm{d}s \xrightarrow{t\downarrow 0} x.$$

This shows that $\mathcal{D}(A) \subset X$ is dense. For uniqueness suppose A generated both $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$. Then consider $Q_x(s) := T(t-s)S(s)x$ for $0 \leq t \leq s$. For $x \in \mathcal{D}(A)$ we note that the set
$\left\{\frac{S(s+h)x - S(s)x}{h} : h \in (0,1]\right\} \cup \{AS(s)x\}$ is compact and hence by Lemma 1.3.4 we have that

$$\begin{aligned}\frac{1}{h}(Q_x(s+h) - Q_x(s)) &= \frac{1}{h}(T(t-s-h)S(s+h)x - T(t-s)S(s)x)\\ = \frac{1}{h}T(t-s-h)(S(s+h)x - S(s)x) &+ \frac{1}{h}(T(t-s-h) - T(t-s))S(s)x\end{aligned}$$

converge for $h \downarrow 0$ to

$$\dot{Q}_x(s) = T(t-s)AS(s)x - AT(t-s)S(s)x.$$

This is zero by the commutative property of the generator shown in Lemma 1.3.9(i). Hence we have that

$$T(t)x = Q_x(0) = Q_x(t) = S(t)x$$

for all $x \in \mathcal{D}(A)$. Since for every $t \geq 0$, $T(t), S(t)$ are continuous and $\mathcal{D}(A)$ is dense in X we conclude that $T(t) = S(t)$ for every $t \in \mathbb{R}_+$, which finishes the proof. □

The closed graph theorem Lemma A.10 now yields the aforementioned fact, that if a semigroup is strongly continuous and not uniformly continuous then the domain of the generator cannot be the full domain of the semigroup.

Lemma 1.3.11. *For a strongly continuous semigroup $(T(t))_{t\geq 0}$ and its generator $(A, \mathcal{D}(A))$ the following are equivalent:*

(i) The domain $\mathcal{D}(A)$ is all of X.

(ii) $(T(t))_{t\geq 0}$ is uniformly continuous.

Proof. $(ii) \Rightarrow (i)$ is clear since uniformly continuous operators are even differentiable in the uniform operator topology as shown in Lemma 1.2.3. Hence this is in particular true pointwise, which is the statement (i). For the other direction we have just shown that A is a closed, densely defined operator. Thus the closed operator theorem yields that A is bounded if it is defined everywhere. Then Theorem 1.2.6 together with the uniqueness shown in the previous lemma yield that $(T(t))_{t\geq 0}$ must be uniformly continuous. □

Another implication of Lemma 1.3.10 is that we can now develop a spectral theory for A, which is most times only meaningful when the operator of interest is closed. To this end let us introduce some important notions.

Definition 1.3.12. Let $A : \mathcal{D}(A) \subset E \to F$ be a linear operator between Banach spaces. $\lambda \in \mathbb{C}$ is said to be in the resolvent set $\rho(A)$ iff the operator $\lambda\mathbb{I} - A$, also defined on $\mathcal{D}(A)$, is bijective onto F and its inverse is bounded. In this case we call $(\lambda - A)^{-1}$ the resolvent of A at λ, and denote it by $R(\lambda, A)$. The complement of the resolvent set is the spectrum $\sigma(A)$.

Remark 1.3.13. *In A.13 and its beforehand discussion we show that it follows from the closed graph theorem that if A is closed, then existence of $(\lambda - A)^{-1}$ already implies its boundedness. Hence for our theory we do not need to check whether this inverse is bounded when considering the generator A, for we have shown that generators are closed.*

The resolvent will be the so far missing tool to connect generators to semigroups in the strongly continuous case. Towards being able to work with the resolvent we first mention two nice and helpful properties of semigroups which are used in many circumstances to shorten and ease a proof and also to generate new semigroups. The first one is generating new semigroups from given ones by conjugation.

Definition 1.3.14. Suppose $(T(t))_{t\geq 0}, (S(t))_{t\geq 0}$ are strongly continuous semigroups on Banach spaces X, Y respectively. They are called similar if there exists a linear homeomorphism $V : Y \to X$ such that

$$V^{-1}T(t)V = S(t),$$

for all $t \geq 0$.

The following corollary shows that conjugation is a tool to construct new semigroups.

Corollary 1.3.15. *Let $V : X \to Y$ be a homeomorphism between Banach spaces X, Y. Suppose $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on Y. Then $S(t) := V^{-1}T(t)V$ defines a strongly continuous semigroup on X.*

Proof. Since V and V^{-1} are assumed to be continuous, conjugation with V preserves strong continuity. For the semigroup law, we note that by the semigroup law of T we have

$$S(t+s) = V^{-1}T(t+s)V = V^{-1}T(t)VV^{-1}T(s)V = S(t)S(s),$$

for all $t, s \geq 0$. □

The following lemma shows that similarity preserves the one-to-one correspondence between semigroups and generators in Lemma 1.3.10 in the most natural way.

Lemma 1.3.16. *Suppose we are given a homeomorphism $V : Y \to X$ and strongly continuous semigroups $(T(t))_{t\geq 0}, (S(t))_{t\geq 0}$ on Banach spaces X, Y respectively with generators A_T, A_S. Then $V^{-1}T(t)V = S(t)$ for all $t \geq 0$ if and only if $V^{-1}A_TV|_{\{y\in Y : Vy\in\mathcal{D}(A_T)\}} = A_S$.*

Proof. By uniqueness (Lemma 1.3.10) we only have to show that $V^{-1}A_T V|_{V^{-1}\mathcal{D}(A_T)}$ generates $V^{-1}T(t)V$. Hence let us compute

$$\lim_{h\downarrow 0}\frac{V^{-1}T(h)Vy-y}{h}=V^{-1}\lim_{h\downarrow 0}\frac{T(h)Vy-Vy}{h}.$$

By definition of a generator, this limit exists exactly for the set $\{y\in Y : Vy\in\mathcal{D}(A_T)\}$ and we can write $V^{-1}A_T Vy$ for such points. This finishes the proof. □

The second property is the rescaling property.

Lemma 1.3.17. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A,\mathcal{D}(A))$. Then any $\lambda\in\mathbb{C}$ yields another strongly continuous semigroup $(S(t))_{t\geq 0}$ by setting $S(t):=e^{-\lambda t}T(t)$. The generator $(B,\mathcal{D}(B))$ of S is given by $B=A-\lambda$ and $\mathcal{D}(B)=\mathcal{D}(A)$.*

Proof. Since the exponential function in the scalar case is continuous and satisfies $e^{a+b}=e^a e^b$ we see that S is a semigroup which is strongly continuous. It is clear that $\lim_{h\downarrow 0}\frac{T(h)x-x}{h}-\lambda x$ exists iff $\lim_{h\downarrow 0}\frac{T(h)x-x}{h}$ does. Together with the computation

$$\lim_{h\downarrow 0}\frac{S(h)x-x}{h}=\lim_{h\downarrow 0}\frac{e^{-\lambda h}T(h)x-T(h)x}{h}+\lim_{h\downarrow 0}\frac{T(h)x-x}{h}=-\lambda x+Ax,$$

we conclude $(B,\mathcal{D}(B))=(A-\lambda,\mathcal{D}(A))$. □

Let us remark on some of the uses of rescaling. We remember from Lemma 1.3.3 that every strongly continuous semigroup T exhibits exponential boundedness, say $\|T(t)\|_{\mathcal{O}(X)}\leq Me^{\omega t}$, for some $M>0,\omega\in\mathbb{R}$. By the previous lemma multiplying $T(t)$ with $e^{-\omega t}$ yields a semigroup S which now is *bounded*, in the sense that, $\|S(t)\|_{\mathcal{O}(X)}\leq M$ for all t. In the special case where M can be chosen to be less than or equal to 1, S is called *contractive*. A semigroup that is contractive is also called *contraction semigroup*. It will not surprise the reader that many results regarding semigroups are much easier to prove if one assumes bounded- or even contractiveness. The pleasant surprise is that one often is able to generalize the desired result to arbitrary semigroups from contractive ones. Another use of Lemma 1.3.17 is that when proving a result which shall hold for any semigroup, one can assume a special form. This technique is made evident in the following lemma. The lemma gives a useful description of the resolvent.

Lemma 1.3.18. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. Then the resolvent of the generator $(A,\mathcal{D}(A))$ admits the following form. Let $\lambda\in\mathbb{C}$. If $R_{int}(\lambda)x:=\int_0^\infty e^{-\lambda t}T(t)x\ \mathrm{d}t$ exists for all $x\in X$, then $\lambda\in\rho(A)$ and $R(\lambda,A)=R_{int}(\lambda)$.*

Remark 1.3.19. *To avoid confusion, it should be noted that the implicit limit in the definition of $R_{int}(\lambda)$ is meant as a pointwise limit. That is, $R_{int}(\lambda)=\lim_{s\to\infty}\int_0^s e^{-\lambda t}T(t)\ \mathrm{d}t$ in the strong operator topology, i.e. $\left\|R_{int}(\lambda)x-\int_0^s e^{-\lambda t}T(t)x\ \mathrm{d}t\right\|_X\xrightarrow{s\to\infty}0$, for every $x\in X$. When writing $\int_0^\infty e^{-\lambda t}T(t)\ \mathrm{d}t$ we will always mean the operator defined pointwise*

as the pointwise limit
$\int_0^\infty e^{-\lambda t} T(t)\mathrm{d}t\ (x) := \lim_{s\to\infty} \int_0^s e^{-\lambda t} T(t)x\ \mathrm{d}t$. *This is not to be confused with the operator* C *for which* $\left\|C - \int_0^s e^{-\lambda t} T(t)\ \mathrm{d}t\right\|_{\mathcal{O}(X)} \xrightarrow{s\to\infty} 0$, *whose existence relies on the convergence of* $\int_0^s e^{-\lambda t} T(t)\ \mathrm{d}t$ *in the uniform operator topology, on which we make no assumptions. However, in cases where* C *exists,* C *and* R_{int} *coincide, since convergence with respect to the uniform operator norm is stronger than strong convergence.*

Proof. Without loss of generality, we can assume $\lambda = 0$, since $\lambda \in \rho(A)$ is equivalent to $0 \in \rho(A - \lambda)$, $A - \lambda$ being the generator of the rescaled semigroup defined by $e^{-\lambda t} T(t)$, as we saw in the previous lemma. Hence we want to show that $R_{int}(0) = (-A)^{-1}$, i.e. $AR_{int}(0)x = -x$ for $x \in X$ and $R_{int}(0)Ax = -x$ for $x \in \mathcal{D}(A)$. To this end, let $x \in X$ and compute for $h > 0$

$$\frac{T(h) - I}{h} R_{int}(0)x = \frac{1}{h}\left(\int_h^\infty T(t)x\ \mathrm{d}t - \int_0^\infty T(t)x\ \mathrm{d}t\right) = -\frac{1}{h}\int_0^h T(t)x\ \mathrm{d}t.$$

Taking the limit $h \downarrow 0$ yields $\operatorname{rg} R_{int}(0) \subset \mathcal{D}(A)$ with $AR_{int}(0)x = -x$. To show the other direction, by Lemma 1.3.9(ii), we have

$$\lim_{s\to\infty} A \int_0^s T(t)x\ \mathrm{d}t = \lim_{s\to\infty} \int_0^s T(t)Ax\ \mathrm{d}t = R_{int}(0)Ax$$

for $x \in \mathcal{D}(A)$. Since $\lim_{s\to\infty} \int_0^s T(t)x\ \mathrm{d}t = R_{int}(0)x$ and A is closed we have that $R_{int}(0)x \in \mathcal{D}(A)$ with $R_{int}(0)Ax = AR_{int}(0)x = -x$ for all $x \in \mathcal{D}(A)$. We conclude $R_{int}(0) = (-A)^{-1}$. □

The following lemma is an easy consequence of the previous lemma. Its significance arises from the fact that semigroups are exponentially bounded (cf. Lemma 1.3.3). It is a very useful result, stating that the resolvent set of a generator always includes a right halfplane. It also yields a scaling norm bound of the resolvent, which will be detrimental in the following pages. In contrast to the remark to the previous lemma, here we do show existence of the integral formula for the resolvent by proving its existence as an element of the uniform operator topology. This then obviously implies its pointwise existence.

Lemma 1.3.20. *Let* $(T(t))_{t\geq 0}$ *be a strongly continuous semigroup on a Banach space* X *with generator* $(A, \mathcal{D}(A))$*, satisfying the growth bound*

$$\|T(t)\|_{\mathcal{O}(X)} \leq Me^{\omega t} \tag{1.4}$$

for some constants $M > 0, \omega \in \mathbb{R}$. *If* $\operatorname{Re}\mu > \omega$, *then*

(i) $\mu \in \rho(A)$ *and* $R_{int}(\mu) = R(\mu, A)$ *and*

(ii) $\|R(\mu, A)\|_{\mathcal{O}(X)} \leq \frac{M}{\operatorname{Re}\mu - \omega}$.

Proof. Since we have

$$\left\| \int_0^s e^{-\mu t} T(t) \; \mathrm{d}t \right\|_{\mathcal{O}(X)} \leq M \int_0^s e^{(-\operatorname{Re}\mu + \omega)t} \; \mathrm{d}t \xrightarrow{s \to \infty} \frac{M}{\operatorname{Re}\mu - \omega}, \tag{1.5}$$

we even get convergence of $\int_0^s e^{-\mu t} T(t) \; \mathrm{d}t$ in the uniform operator topology. We invoke Lemma 1.3.18 and its attached Remark 1.3.19 to conclude (i). The resolvent $R(\mu, A)$ equaling $R_{int}(\mu)$ together with (1.5) then prove (ii). □

Remark 1.3.21. *By Lemma 1.3.3 there always exists a pair M, ω such that T satisfies the above growth bound. In the context of spectral theory the previous lemma encourages one to find the sharpest choice for these constants.*

An important consequence is the following convergence property of the resolvent, which suggests which elements to consider to define an exponential function for unbounded operators.

Lemma 1.3.22. *Let $(A, \mathcal{D}(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space X. Suppose T is bounded, i.e. there exists $M > 0$ such that for all $t > 0$ it holds that $\|T(t)\|_{\mathcal{O}(X)} \leq M$. Then the resolvent admits the following convergence properties.*

(i) $\lim_{\lambda \to \infty} \|\lambda R(\lambda, A)x - x\|_X = 0$ *for all* $x \in X$.

(ii) $\lim_{\lambda \to \infty} \|\lambda A R(\lambda, A)x - Ax\|_X = 0$ *for all* $x \in \mathcal{D}(A)$.

Proof. The boundedness assumption on T is equivalent to saying that ω can be chosen to be 0 in Lemma 1.3.20. The identity

$$\lambda R(\lambda, A)x - x = R(\lambda, A)(\lambda + A - \lambda)x = R(\lambda, A)Ax$$

for $x \in \mathcal{D}(A)$ together with the norm estimate for the resolvent in Lemma 1.3.20(ii) then yield

$$\|\lambda R(\lambda, A)x - x\|_X = \|R(\lambda, A)Ax\|_X \leq \|R(\lambda, A)\|_{\mathcal{O}(X)} \|Ax\|_X \leq \frac{M}{\lambda} \|Ax\|_X \,,$$

which converges to 0 for $\lambda \to \infty$. This proves (i) for $x \in \mathcal{D}(A)$. Since the set $\{\lambda R(\lambda, A) : \lambda > 0\}$ is uniformly bounded by M we may use the well-known fact, that for a bounded sequence of operators on a Banach space, strong continuity on the whole space coincides with strong continuity on a dense subset (see A.7), to conclude (i) for all $x \in X$. To prove (ii) we use the commutative property of the resolvent on $\mathcal{D}(A)$

$$AR(\lambda, A)x = R(\lambda, A)Ax,$$

which is proved in A.14. With this for every $x \in \mathcal{D}(A)$ we have

$$\|\lambda A R(\lambda, A)x - Ax\|_X = \|\lambda R(\lambda, A)Ax - Ax\|_X \,,$$

which converges to 0 for $\lambda \to \infty$ by (i). This completes the proof. □

Remember that our goal is to define the exponential operator e^{tA} for the unbounded generator A of a semigroup. Item (ii) in the previous lemma yields a sequence of -as we will soon see- bounded operators that can be defined on all of X, which on $\mathcal{D}(A)$ approach A. Since we already know how to define the exponential function for bounded operators, in turn yielding *uniformly* continuous semigroups (cf. Theorem 1.2.6), we hope that defining the exponential of an unbounded operator as the limit of exponentials of bounded ones will serve our purpose. The following theorem shows that this is indeed true, at least when imposing some restrictions on A or equivalently on its resolvent $R(\lambda, A)$. The theorem together with a following consequence in the special case of Hilbert spaces, are the main results of this chapter. There are different ways to restrict A to get different satisfactory results. For our purpose, the following assumptions will serve best. They are rather strict, consequently even yielding *contraction* semigroups, but will be satisfied by our differential operator in the next chapter.

Theorem 1.3.23. *Let $(A, \mathcal{D}(A))$ be a linear operator on a Banach space X. The following are equivalent:*

(i) $(A, \mathcal{D}(A))$ *generates a strongly continuous contraction semigroup.*

(ii) $(A, \mathcal{D}(A))$ *is closed, densely defined and for every $\lambda > 0$ one has $\lambda \in \rho(A)$ and $\|\lambda R(\lambda, A)\|_{\mathcal{O}(X)} \leq 1$.*

(iii) $(A, \mathcal{D}(A))$ *is closed, densely defined and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ one has $\lambda \in \rho(A)$ and $\|R(\lambda, A)\|_{\mathcal{O}(X)} \leq \frac{1}{\operatorname{Re}\lambda}$.*

Proof. Since (iii) trivially implies (ii), while (i) implies (iii) by Lemma 1.3.10 and Lemma 1.3.20 (remember that contractive means that (M, ω) can be chosen to be $(1, 0)$ in (1.4)), we only have to show that (ii) implies (i). To this end we define on X the aforementioned operators

$$A_n := nAR(n, A)$$

for every $n \in \mathbb{N}$. We compute

$$A_n = n(n + A - n)(n - A)^{-1} = n^2 R(n, A) - nI$$

and

$$A_n A_m = nmAAR(n, A)R(m, A) = mAR(m, A)nAR(n, A) = A_m A_n, \qquad (1.6)$$

where we used the commutativity of resolvents with one another on all of X and of the resolvent with A on $\mathcal{D}(A)$. This shows that the A_n are bounded operators for each $n \in \mathbb{N}$ which commute with one another. In 1.3.22(ii) we have seen that $(A_n)_n$ converges strongly to A. As mentioned before we are now interested in the existence of the strong limit of the uniformly continuous semigroups

$$T_n(t) := e^{tA_n},$$

for then we hope that the family $(T(t))_{t\geq 0}$ defined on X by $T(t)x := \lim_{n\to\infty} T_n(t)x$ is a strongly continuous semigroup with generator A. We mention that by A.5 $T_m(s)$ and $T_n(t)$ commute for any $n, m \in \mathbb{N}, s, t \geq 0$. Let us first prove that for every $x \in X$, $T_n(t)x$ converges for $n \to \infty$. By noticing that

$$\|T_n(t)\|_{\mathcal{O}(X)} \leq e^{-nt} e^{n^2 t \|R(n,A)\|_{\mathcal{O}(X)}} \leq 1, \tag{1.7}$$

we again have a uniformly bounded sequence of operators, hence strong convergence on all of X coincides with strong convergence on $\mathcal{D}(A)$ (A.7). So we only need to prove convergence of $T_n(t)x$ for $x \in \mathcal{D}(A)$. To this end for every $x \in \mathcal{D}(A), t \geq 0$ consider

$$Q_x(s) := T_m(t-s)T_n(s)x, \;\; 0 \leq s \leq t. \tag{1.8}$$

An application of the fundamental theorem of calculus for functions on Banach spaces yields

$$\begin{aligned} T_n(t)x - T_m(t)x = Q_x(t) - Q_x(0) &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} Q_x(s) \;\mathrm{d}s \\ &= \int_0^t T_m(t-s)T_n(s)A_n x - T_n(s)T_m(t-s)A_m x \;\mathrm{d}s \\ &= \int_0^t T_m(t-s)T_n(s)(A_n x - A_m x) \;\mathrm{d}s, \end{aligned}$$

where the last two equalities hold by the commutativity of the family $(T_n(t))_{n\in\mathbb{N}, t\geq 0}$. Taking norms gives

$$\|T_n(t)x - T_m(t)x\|_X \leq t \,\|A_n x - A_m x\|_X \,. \tag{1.9}$$

Since by item *(ii)* in Lemma 1.3.22, $(A_n x)_n$ converges to Ax, in particular it is a Cauchy sequence. By the fact that X is complete together with (1.9) we have that for every $x \in \mathcal{D}(A)$, $(T_n(\cdot)x)_n$ converges uniformly on every interval $[0, t_0]$. In particular we have shown that for any $t \geq 0, x \in \mathcal{D}(A)$, $\lim_{n\to\infty} T_n(t)x$ exists, which is as we mentioned before enough to conclude that it already exists for all $x \in X$. By (1.7) we see that $\|T(t)\|_{\mathcal{O}(X)} \leq 1$ for every $t \geq 0$. In particular, $T(t)$ is a bounded operator for every $t \geq 0$. As the pointwise limit of semigroups, $(T(t))_{t\geq 0}$ satisfies (FE), hence is a semigroup itself. For strong continuity we note that by (1.9) on every interval $[0, t_0]$ the orbit maps at every $x \in \mathcal{D}(A)$

$$\xi_x : t \mapsto T(t)x \tag{1.10}$$

are uniform limits of continuous functions, hence continuous itself. By Lemma 1.3.5 this already shows that this is true for all $x \in X$. Altogether this means that $(T(t))_{t\geq 0}$ is a strongly continuous contractive semigroup. Lastly we want to show that A generates $(T(t))_{t\geq 0}$. First we highlight that the uniform convergence of $(T_n(\cdot)x)_n$ for $x \in \mathcal{D}(A)$ already implies that this is true for all $x \in X$, by Appendix A Lemma A.7. Let us denote the generator of $(T(t))_{t\geq 0}$ by $(B, \mathcal{D}(B))$. For $x \in \mathcal{D}(A)$ we have that on every interval $[0, t_0]$

$$\xi_{x,n} : t \mapsto T_n(t)x \tag{1.11}$$

converges uniformly to ξ_x. For the differentiated functions

$$\dot{\xi}_{x,n} : t \mapsto T_n(t)A_n x \tag{1.12}$$

we have that they converge uniformly on $[0, t_0]$ to

$$\eta_x : t \mapsto T(t)Ax. \tag{1.13}$$

This is seen by the computation

$$\begin{aligned} \sup_{t\in[0,t_0]} \|T(t)Ax - T_n(t)A_n x\|_X &\leq \sup_{t\in[0,t_0]} \|T(t)Ax - T_n(t)Ax\|_X \\ &\quad + \sup_{t\in[0,t_0]} \|T_n(t)Ax - T_n(t)A_n x\|_X \\ &\leq \sup_{t\in[0,t_0]} \|T(t)Ax - T_n(t)Ax\|_X + \|Ax - A_n x\|_X , \end{aligned} \tag{1.14}$$

where the right hand side converges to zero by the earlier remarks that $T_n(\cdot)$ converges uniformly at $Ax \in X$ and the strong convergence of $(A_n)_n$ to A on $\mathcal{D}(A)$. Together this implies that ξ_x is differentiable with $\dot{\xi}_x(0) = \eta_x(0)$. Hence $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $Ax = Bx$ for $x \in \mathcal{D}(A)$. Choose any $\lambda > 0$. By assumption $\lambda - A$ is a bijection from $\mathcal{D}(A)$ onto X. Since B is the generator of a strongly continuous contraction semigroup, Lemma 1.3.20 implies that $\lambda \in \rho(B)$, whence $\lambda - B$ is a bijection on $\mathcal{D}(B)$. Therefore and since $\lambda - A = \lambda - B$ on $\mathcal{D}(A)$, $\mathcal{D}(A)$ cannot be a proper subset of $\mathcal{D}(B)$ and we conclude $\mathcal{D}(A) = \mathcal{D}(B)$ and $A = B$. This means that $(A, \mathcal{D}(A))$ generates $(T(t))_{t\geq 0}$. This finishes the proof. □

This result will be sufficient for us. It should be mentioned that from here on out, one could generalize this result in several directions, as is done in [8].

1.3.1. The Lumer-Phillips Theorem

We take a different route. Since our differential operators in the next chapter will (almost) meet the assumptions, we are satisfied with the above form. However in the context of differential operators one does not always start with a closed operator defined on the domain given in the definition of our generator, but on some "nicer" spaces. The drawback is that one needs to include one step to be able to construct a semigroup, namely taking the closure of said operator. One type of operators where this works very nicely are the so called *dissipative* operators which will be introduced in the following. Since the overlying space in the next chapter will be a Hilbert space, the rest of this chapter is formulated in terms of Hilbert spaces.

Definition 1.3.24. Let $(A, \mathcal{D}(A))$ be a linear operator on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. A is called dissipative if

$$\operatorname{Re}\langle Ax, x\rangle \leq 0. \tag{1.15}$$

for all $x \in \mathcal{D}(A)$.

This is a special definition for operators on Hilbert spaces. Usually one defines dissipativity in a more general way, for operators on Banach spaces. The content of the next lemma includes the usual definition.

Lemma 1.3.25. *Let $A : \mathcal{D}(A) \subset H \to H$ be a linear operator on a Hilbert space H. A is dissipative if and only if for every $\lambda > 0$ and $x \in \mathcal{D}(A)$ the norm estimate*

$$\|(\lambda - A)x\|_H \geq \lambda \|x\|_H \tag{1.16}$$

holds.

Proof. Let $\lambda > 0, x \in \mathcal{D}(A)$. We calculate

$$\|(\lambda - A)x\|_H^2 = \lambda^2 \|x\|_H^2 - 2\lambda \operatorname{Re}\langle Ax, x\rangle + \|Ax\|_H^2 .$$

Since $\lambda \operatorname{Re}\langle Ax, x\rangle \leq 0$ by assumption, taking roots proves one direction. For the other direction, we first note that since the statements are trivially satisfied for $x = 0$, we can w.l.o.g. assume that $\|x\|_H = 1$ (otherwise consider $z = x/\|x\|_H$). For every $\lambda > 0$, define $x_\lambda := (\lambda - A)x/\|(\lambda - A)x\|_H$. Then we have by (1.16)

$$\lambda \leq \|(\lambda - A)x\|_H = \langle(\lambda - A)x, x_\lambda\rangle = \lambda\langle x, x_\lambda\rangle - \langle Ax, x_\lambda\rangle = \lambda \operatorname{Re}\langle x, x_\lambda\rangle - \operatorname{Re}\langle Ax, x_\lambda\rangle.$$

Using Cauchy-Schwarz once in the first and once in the second inner product yields the estimates

$$0 \geq \operatorname{Re}\langle Ax, x_\lambda\rangle, \tag{1.17}$$

$$1 - \frac{1}{\lambda} \|Ax\|_H \leq \operatorname{Re}\langle x, x_\lambda\rangle, \tag{1.18}$$

for all $\lambda > 0$. Suppose $x' \in H$ is a weak accumulation point of $(x_\lambda)_{\lambda\in\mathbb{N}}$, which exists since the unit ball is weakly compact in Hilbert spaces. We have

$$\|x'\|_H^2 = \langle x', x'\rangle \leq \limsup_{\lambda\to\infty} |\langle x_\lambda, x'\rangle| \leq \limsup_{\lambda\to\infty} \|x_\lambda\|_H \|x'\|_H = \|x'\|_H , \tag{1.19}$$

which implies $\|x'\|_H \leq 1$. (1.17) and (1.18) imply for $\lambda' \to \infty$, where $(x_{\lambda'})_{\lambda'}$ is the subsequence for which x' is the weak limit, that $0 \geq \operatorname{Re}\langle Ax, x'\rangle$ and $1 \leq \operatorname{Re}\langle x, x'\rangle$. With the second inequality and Cauchy-Schwarz, we compute

$$1 \leq \operatorname{Re}\langle x, x'\rangle \leq |\langle x, x'\rangle| \leq \|x'\|_H \leq 1, \tag{1.20}$$

which means that we can write equality signs instead, yielding

$$\langle x, x'\rangle = \|x\|_H^2 = \|x'\|_H^2 . \tag{1.21}$$

But this means that $\|x - x'\|_H^2 = \langle x - x', x - x'\rangle = 0$, hence $x' = x$ and by the above $\operatorname{Re}\langle Ax, x\rangle \leq 0$. This is what we wanted to show. ☐

We will mostly use this dissipativity property. To show that this is the right class of operators to consider we collect several very useful properties.

Lemma 1.3.26. *Let $A : \mathcal{D}(A) \subset H \to H$ be a dissipative operator on a Hilbert space H. It admits the following properties.*

(i) *For every $\lambda > 0$ the operator $(\lambda - A)$ is injective and*

$$\left\|(\lambda - A)^{-1} z\right\|_H \leq \frac{1}{\lambda} \|z\|_H$$

for every $z \in \operatorname{rg}(\lambda - A) = (\lambda - A)\mathcal{D}(A)$.

(ii) *$(\lambda - A)$ is surjective for some $\lambda > 0$ iff it is surjective for every $\lambda > 0$.*

(iii) *A is closed iff $\operatorname{rg}(\lambda - A)$ is closed for all $\lambda > 0$.*

(iv) *A is closable if $\operatorname{rg}(A) \subset \overline{\mathcal{D}(A)}$. The closure $\overline{A}$ is again dissipative and satisfies $\operatorname{rg}(\lambda - \overline{A}) = \overline{\operatorname{rg}(\lambda - A)}$.*

Proof. (i) Injectivity immediately follows from (1.16) in Lemma 1.3.25, since only 0 is mapped to 0. The estimate then is just a reformulation of (1.16) for $x = (\lambda - A)^{-1} z$.
(ii) Suppose $\lambda > 0$ is such that $(\lambda - A)$ is surjective. By (i) the inverse of $\lambda - A$ is bounded, hence $R(\lambda, A)$ exists. Since for $\mu \in (0, 2\lambda)$ we have that $|\mu - \lambda| < \lambda \leq 1/\|R(\lambda, A)\|_{\mathcal{O}(X)}$, the series representation for the resolvent (see Lemma A.15) yields that $\mu \in \rho(A)$. With (i) we have the estimate $\|R(\mu, A)\|_{\mathcal{O}(X)} \leq 1/\mu$. With the same arguments, one shows that $(0, 4\lambda) \subset \rho(A)$ (for $\mu \in [2\lambda, 4\lambda)$ consider $\mu/2+\epsilon$). Proceeding this way, we eventually get that $(0, \infty) \subset \rho(A)$, which is what we wanted to prove.
(iii) The operator A is closed if and only if $\lambda - A$ is closed for any $\lambda > 0$. Since $\lambda - A$ is invertible on $\operatorname{rg}(\lambda - A)$ by (i) and the graph of $\lambda - A$ is closed if and only if the graph of $(\lambda - A)^{-1}$ is closed, the previous is equivalent to requiring that $(\lambda - A)^{-1}$ is closed. By (i) this operator is bounded and hence by Lemma A.9 we conclude that this is equivalent to $\mathcal{D}((\lambda - A)^{-1}) = \operatorname{rg}(\lambda - A)$ being closed.
(iv) By Corollary A.11 and its beforehand discussion, to show closability of A it suffices to show that if $(x_n)_n \subset \mathcal{D}(A)$ is such that $x_n \to 0$ and Ax_n converges to some $y \in H$, it must be that $y = 0$. Let us be given such a sequence $(x_n)_n \subset \mathcal{D}(A)$. By (1.16) we have for $n \in \mathbb{N}, w \in \mathcal{D}(A)$ and $\lambda > 0$

$$\left\|\lambda^2 x_n - \lambda A x_n + (\lambda - A) w\right\|_H = \|(\lambda - A)(\lambda x_n + w)\|_H \geq \lambda \|\lambda x_n + w\|_H .$$

Letting $n \to \infty$ and dividing by λ yields

$$\left\|-y + w - \frac{1}{\lambda} A w\right\|_H \geq \|w\|_H .$$

By letting $\lambda \to \infty$ we obtain

$$\|-y + w\|_H \geq \|w\|_H .$$

Since we have $y \in \overline{\mathrm{rg}(A)} \subset \overline{\mathcal{D}(A)}$ by assumption, there exists a sequence $(w_m)_m \subset \mathcal{D}(A)$ converging to y. Hence the above inequality implies for $m \to \infty$

$$0 \geq \|y\|_H .$$

Hence we have shown that $y = 0$ and thus by Corollary A.11 closability. By construction (cf. A.11 and its beforehand remarks) we have that $\mathcal{D}(A)$ is not only dense in $\mathcal{D}(\overline{A})$ but for a given $x \in \mathcal{D}(\overline{A})$ there always exists a sequence $(x_n)_n \subset \mathcal{D}(A)$ converging to x for which $(Ax_n)_n$ converges. The equivalent notion of dissipativity (1.16) is stable when taking limits, which implies dissipativity for $\overline{A}$. Also by the graph property $G(\overline{\lambda - A}) = \overline{G(\lambda - A)}$ we have for every $\lambda > 0$ that $\mathrm{rg}(\lambda - A)$ is dense in $\mathrm{rg}(\lambda - \overline{A})$. By (iii) the latter is closed in H, hence $\overline{\mathrm{rg}(\lambda - A)} = \mathrm{rg}(\lambda - \overline{A})$, which was the only claim left to prove. □

With these properties we can formulate the generation theorem in the final, most handy form for us. Often times the following theorem is called the Lumer-Phillips-Theorem. Since we already have done all the work, it can be formulated as almost a corollary of the previous.

Theorem 1.3.27 (Lumer-Phillips)**.** *Let $A : \mathcal{D}(A) \subset H \to H$ be a densely defined, dissipative operator on a Hilbert space H. The following are equivalent.*

(i) The closure $\overline{A}$ of A generates a strongly continuous contraction semigroup.

(ii) $\mathrm{rg}(\lambda - A)$ *is dense in H for some (hence all)* $\lambda > 0$.

Proof. $(i) \Rightarrow (ii)$: By Theorem 1.3.23 we have that $(0, \infty) \subset \rho(\overline{A})$, which means that $(\lambda - \overline{A})$ is surjective for all $\lambda > 0$. The claim then follows by item (iv) in Lemma 1.3.26 which states that $\overline{\mathrm{rg}(\lambda - A)} = \mathrm{rg}(\lambda - \overline{A})$.
$(ii) \Rightarrow (i)$: Let $\lambda > 0$ be such that $\mathrm{rg}(\lambda - A)$ is dense. By 1.3.26 (iv), we have that A is closable and its dissipative closure $\overline{A}$ satisfies $\mathrm{rg}(\lambda - \overline{A}) = H$. Hence by item (ii) in the same lemma, this property translates to all $\lambda > 0$, which means that $(0, \infty) \subset \rho(\overline{A})$. $\overline{A}$ being closed, densely defined and dissipative together with Lemma 1.3.26(i) imply that $\|\lambda R(\lambda, \overline{A})\|_{\mathcal{O}(X)} \leq 1$. Hence we invoke Theorem 1.3.23 to conclude (i). □

Note that the assumptions in the theorem do not explicitly call for properties of the resolvent, in contrast to Theorem 1.3.23. The needed information is covered by requiring that A shall be dissipative.
In the next chapter we will start with a differential operator defined on some tangible space. By introducing an inner product on this space we will produce the overlying Hilbert space as the completion of the original space with respect to the inner product. Thus by construction the differential operator will be densely defined. We will have to prove that it is dissipative and $(\lambda - A)$ has dense range for some λ. By the Lumer-Phillips theorem we are then provided with a strongly continuous semigroup generated by the closure of said differential operator.

2. Wave equation in forward self-similarity coordinates

2.1. Introduction

2.1.1. Radiality

Let us start with a short discussion on radial functions. These are d-dimensional functions whose whole information can be equivalently described on a 1-dimensional domain. To be more precise, a d-dimensional function $u : \mathbb{R}^d \to \mathbb{R}$ is called radial, if there exists a 1-dimensional, even function $\widehat{u} : \mathbb{R} \to \mathbb{R}$ for which

$$\widehat{u}(|x|) = u(x) \tag{2.1}$$

for all $x \in \mathbb{R}^d$. As the name suggests, this means that u is only dependent on the radius. $\widehat{u}$ is called the *radial representative* of u and is uniquely determined by u. This is since on the positive reals it is given by (2.1) and on the negatives it is given by $\widehat{u}(-r) = \widehat{u}(r)$ since we assumed it to be even. It is an easy exercise to see that $\widehat{u}$ inhibits the regularity of u and vice-versa. That is, $u \in C^k(\mathbb{R}^d, \mathbb{R})$ is equivalent to $\widehat{u} \in C^k(\mathbb{R}, \mathbb{R})$. Most results in this chapter will first be presented with respect to the radial representative. Slightly abusing the language, we will still say that $\widehat{u}$ may be d-dimensional, emphasizing the fact that it was generated by or will generate a d-dimensional function, despite being one-dimensional itself. Operators will also be extended by radiality. That is, suppose $\widehat{T}$ acts on the one-dimensional *even* function $\widehat{u}$. Then we can define an operator T acting on the d-dimensional radial function u, whose radial representative is $\widehat{u}$ by

$$Tu = (\widehat{T}\widehat{u})(|\cdot|).$$

We will stay consistent in always using the "hat"-notation ($\widehat{\ }$) when radial representatives or equivalently spaces of even functions are involved.

2.1.2. Wave maps in (1+3)-dimensional Minkowski space

We are interested in the wave maps equation, a generalization of the wave equation where the unknown takes values in some Riemannian manifold. We are concerned with the case in which the manifold is the 3-dimensional sphere. In this special case, a map $U : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{S}^3 \subset \mathbb{R}^4$ is called wave map if it solves

$$\partial^\mu \partial_\mu U + (\partial^\mu U \cdot \partial_\mu U)U = 0, \tag{2.2}$$

where μ ranges from 0 to 3 with $\partial^j = \partial_j$ for $j = 1, 2, 3$ and $\partial^0 = -\partial_0$ and the Einstein summation convention is employed. With the corotational ansatz

$$U(t,x) = \begin{pmatrix} \sin(|x|\, u(t,x)) \frac{x}{|x|} \\ \cos(|x|\, u(t,x)) \end{pmatrix}$$

where $u(t,\cdot)$ is radial for every $t \geq 0$, the system of equations (2.2) reduces to a single semilinear wave equation. It is given by

$$(\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r)\widehat{u}(t,r) + \frac{\sin(2r\widehat{u}(t,r)) - 2r\widehat{u}(t,r)}{r^3} = 0, \tag{2.3}$$

where $\widehat{u}(t,\cdot)$ is the radial representative of $u(t,\cdot)$ at every $t \geq 0$. This interestingly is a 5-dimensional rather than a 3-dimensional radial wave equation, which is seen by noticing that the above can be written as

$$(\partial_t - \Delta_x)v(t,x) + \frac{\sin(|x|\, v(t,x)) - 2\,|x|\, v(t,x)}{|x|^3} = 0,$$

where $v : \mathbb{R} \times \mathbb{R}^5 \to \mathbb{R}$ has 5-dimensional spatial domain such that $v(t,x) = \widehat{u}(t,|x|)$. It is readily seen that the geometric radial wave operator in (2.3) preserves symmetry, hence the above equation also holds for $r < 0$, for we assumed that $\hat{u}(t,\cdot)$ is even at every $t \geq 0$. We also note by the series expansion of the sine that the singularity in the second summand is removable and thus the second summand can be seen as a perturbation of the cubic nonlinearity $N(\widehat{v}) = \widehat{v}^3$.
This paper is mainly concerned with laying the groundwork for a forward stability analysis of the geometric wave equation (2.3) in the future light cone. To this end we will introduce novel coordinates which will be called "forward similarity coordinates". In these coordinates we will develop energy bounds for solutions of the free radial wave equation in every odd spatial dimension, focusing on dimension 1,3 and 5. In more detail, we will first produce an energy bound in dimension 1 which will then be lifted in 2-dimensional steps by the *descent method* introduced for a related problem on the complement of the future light cone in [1] and structurally revised in [6].

2.2. Coordinates

There exists a self-similar function explicitly given by $\phi^*(t,r) = 2\arctan(r/t)$, such that $\psi^*(t,r) := \phi^*(t,r)/r$ is a solution to (2.3). ϕ^* is usually used to demonstrate finite time blow-up in the related problem where the ansatz to the wave maps equation (2.2) is given by $\tilde{U}(t,x) = \begin{pmatrix} \sin(u(t,x)) \frac{x}{|x|} \\ \cos(u(t,x)) \end{pmatrix}$. It has been shown that ϕ^* is stable as a blow-up solution in this case, first numerically in [2] and then rigorously as an accumulated result of [5],[7],[3]. A more elegant proof was given in [4].
One could also consider ψ^* for $t \to \infty$ and ask whether ψ^* is stable as a semi-global solution to (2.3). In related problems, it has proved advantageous to adapt the coordinate

system to ϕ^*. Our approach in this direction will be the introduction of novel coordinates, which we will call forward self-similarity coordinates.
Let us introduce them step by step. Inspired by the properties of ϕ^* we try to find a coordinate pair $\Psi(s,y) = (\Psi_1(s,y), \Psi_2(s,y))$, satisfying the following properties:

(1) (a) Since our analysis will depend on the use of differential operators, the coordinates should be smooth.

(b) To avoid having too complicated coordinates, we want the dependence of Ψ_1, Ψ_2 from time and space to be multiplicative separable. That is, there exist functions $\Psi_{1,s}, \Psi_{1,y}, \Psi_{2,s}, \Psi_{2,y}$ such that $\Psi_i(s', y') = \Psi_{i,s}(s')\Psi_{i,y}(y')$ for $i = 1, 2$.

(c) They should be adapted to self-similarity. That is, the quotient of the new variables should be independent of time, i.e. there exists a spatial function W such that $\Psi_1(s,y)/\Psi_2(s,y) = W(y)$.

(d) We are interested in forward stability. By the underlying wave equation structure, which means finite speed of propagation, it is sufficient to consider the problem in a forward light cone. The behaviour which we want to understand is when the coordinates approach infinity. There are 2 possibilities for this. One, when approaching timelike infinity, which would also be perfectly covered by the time variable in the usual cartesian coordinates. The other however is when we approach lightlike infinity, which in cartesian coordinates is approached when t and x grow proportionately. Since we are not interested at all in spacelike infinity the cartesian space variable is not well suited to our analysis, and can be seen as wasted. We would like to have access to lightlike infinity through the new spatial variable alone. That is why the spatial dependence functions $\Psi_{1,y}$ respectively $\Psi_{2,y}$ should behave like $|y|, y$.

(e) To translate our results to the future we would like that for $s > 0$ each level set $\{(\Psi_1(s,y), \Psi_2(s,y)) : y \in Y\}$ lives (asymptotically) in its *own* future light cone. Here Y denotes the domain of the spatial variable, which will be specified later.

(2) We prefer to do our computations on a compact space, hence we would like to compactify our spatial domain. This property is of rather technical nature and its use is mainly to have easy access to the domain's endpoints which then do not have to be described as "at $\pm\infty$".

Let us construct our coordinates in two steps. For step (1) Property (c) in view of (b) means that $\Psi_{1,s}/\Psi_{2,s} \equiv C$. We could choose $\Psi_{1,s}(s') = \Psi_{2,s}(s') = s'$. However, using exponential time instead better suits our purpose, which is just attributed to the fact that when using differential operators on the coordinates the exponential function's invariance under differentiation comes in handy. Hence we set $\Psi_{1,s}(s') = \Psi_{2,s}(s') = e^{s'}$. Property (d) in view of (a) naturally gives rise to the *Japanese bracket* defined by $\langle y \rangle := \sqrt{1 + |y|^2}$, which is a smoothened version of the absolute value. Hence we try

$\Psi_{1,y}(y') = \langle y' \rangle, \Psi_{2,y}(y') = y'$ and check whether the corresponding coordinates already satisfy property (e). It turns out that the level sets are hyperboloids which converge asymptotically to the boundary of a future light cone. However, they all converge to the same one, given by $\{(|y|, y) : y \in \mathbb{R}\}$. To include a margin between two given level sets, we manipulate $\Psi_{1,y}$ by adding 1, which would -by their multiplicative relation- be equivalent to adding $e^{s'}$ in $\Psi_{1,s}$. That is $\Psi_{1,y}(y') := 1 + \sqrt{1 + |y|^2}$. With this trick, one calculates that the margin between two level sets, say of s_1, s_2, converges to $|e^{s_1} - e^{s_2}|$. By this, the level set corresponding to time s_0 has its own light cone given by $\{(e^{s_0} + |y|, y) : y \in \mathbb{R}\}$, which shows that these coordinates satisfy property (e). We have created the coordinates $\Psi^{(1)}(s, y) := \left(e^s(1 + \sqrt{1 + |y|^2}), e^s y\right)$. The final step is to bring these coordinates into a form such that they satisfy (2). This is achieved by compactifying $\mathbb{R}^d$ into $\mathbb{B}^d_{\pi/2}$ by the introduction of the tangent. By writing $y = |y| \frac{y}{|y|}$ we find the conformal compactification $K(y) = \tan(|y|)\frac{y}{|y|}$ which just stretches $\mathbb{B}^d_{\pi/2}$ to $\mathbb{R}^d$. Together with the function $\Psi^{(1)}$ constructed in step (1) we get $\Psi := \Psi^{(1)} \circ (\mathrm{id} \times K)$. We have $\Psi(s, y) = (e^s(1 + \sqrt{1 + \tan(|y|)^2}), e^s \tan(|y|)\frac{y}{|y|})$ where $\sqrt{1 + \tan(|y|)^2}$ is easily seen to be equal to $\frac{1}{\cos(|y|)}$. To summarize, we make the following definition.

Definition 2.2.1. In each spatial dimension $d \geq 1$ the forward hyperboloidal self-similarity coordinates (FHSC) are given by the map

$$\begin{aligned}\Psi_d :&(0,\infty) \times \mathbb{B}^d_{\pi/2} \to \mathbb{R} \times \mathbb{R}^d, \\ &(s, y) \mapsto \left(e^s(1 + \sqrt{1 + \tan(|y|)^2}), e^s \tan(|y|)\frac{y}{|y|}\right) = \left(e^s + \frac{e^s}{\cos(|y|)}, e^s \tan(|y|)\frac{y}{|y|}\right).\end{aligned}$$

Remark 2.2.2. *Let us comment on some of the properties of these coordinates. We invite the reader to also consult figure 2.1.*

- Ψ_d *is a diffeomorphism onto the subset of the future lightcone* $\{(t, x) \in \mathbb{R} \times \mathbb{R}^d : t \geq 1 + |x|\}$ *explicitly given by* $\mathrm{im}(\Psi_d) = \left\{(t, x) \in \mathbb{R} \times \mathbb{R}^d : t \geq 1 + \sqrt{1 + |x|^2}\right\}$. *Its inverse is*

$$\Psi_d^{-1}(t, x) = \left(\log\left(\frac{t^2 - |x|^2}{2t}\right), \arctan\left(\frac{2t\,|x|}{t^2 - |x|^2}\right)\frac{x}{|x|}\right).$$

- *FHSC are well adapted to self-similarity. In particular,* ϕ^* *in new coordinates is of the form*

$$\begin{aligned}\phi^*(t, r) = 2\arctan\left(\frac{e^s \tan(y)}{e^s(1 + \frac{1}{\cos(y)})}\right) &= 2\arctan\left(\frac{\sin(y)}{\cos(y) + 1}\right) \\ &= 2\arctan\left(\tan\left(\frac{y}{2}\right)\right) = y.\end{aligned}$$

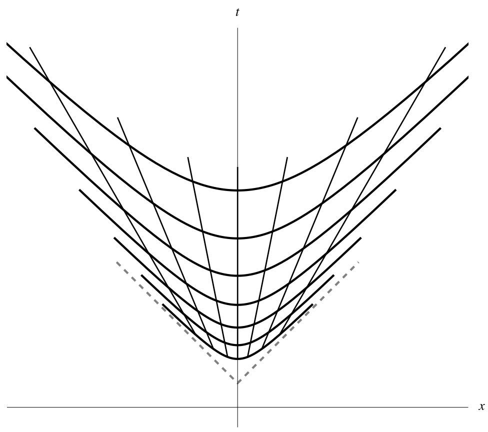

Fig. 2.1.: Forward self similarity coordinates. The hyperboloids are level sets of s, the straight lines emerging from the origin are level sets of y. The dashed line depicts the lower boundary of the future lightcone $\{(t,x) \in \mathbb{R}\times\mathbb{R}^d : t \geq 1+|x|\}$. The lowest hyperboloid depicts the lower boundary of the coordinates and we see that it asymptotically converges to the boundary of the lightcone.

- Ψ_d *is compatible with radiality, that is we have* $(\mathrm{id}\times|\cdot|)\circ\Psi_d = \Psi_1\circ(\mathrm{id}\times|\cdot|)$. *One helpful implication is that when working with radial functions, the change of coordinates does not change their radiality. Indeed, suppose we are given* u *in usual cartesian coordinates* (t,x) *with radial representatives* $\widehat{u}(t,\cdot)$ *for every* t*. Then for every* s *the function* $v(s,\cdot)$ *is radial, where* $v := u\circ\Psi_d^{-1}$*. This is seen by defining the radial representative of* $v(s,\cdot)$ *by*

$$\begin{aligned} v(s,y) &= u\left(e^s + \frac{e^s}{\cos(|y|)}, e^s\tan(|y|)\frac{y}{|y|}\right) \\ &= \widehat{u}\left(e^s + \frac{e^s}{\cos(|y|)}, e^s\tan(|y|)\right) =: \widehat{v}(s,|y|). \end{aligned}$$

The aforementioned is summarized in the following diagram:

$$\begin{array}{ccc} u(t,x) & \xrightarrow{\Psi_d^{-1}} & v(s,y) \\ {\scriptstyle \mathrm{id}\times|\cdot|}\downarrow & \circ & \downarrow{\scriptstyle \mathrm{id}\times|\cdot|} \\ \widehat{u}(t,|x|) & \xrightarrow[\Psi_1^{-1}]{} & \widehat{v}(s,|y|) \end{array} \tag{2.4}$$

In these coordinates (use Ψ_1) the 5-dim. radial wave equation in (2.3) becomes

$$\frac{1}{e^{2s}(1+\frac{1}{\cos(y)})^2}\tilde{\Box}_5^{rad}v(s,y) + \frac{\sin(2e^s\tan(y)v(s,y)) - 2e^s\tan(y)v(s,y)}{(e^s\tan(y))^3} = 0, \tag{2.5}$$

where $v = u \circ \Psi_1$ is the function u in the new coordinates and $\tilde{\Box}_d^{rad}$ is the free radial wave operator in forward similarity coordinates given explicitly by

$$\begin{aligned}\tilde{\Box}_d^{rad} v(s,y) =& \partial_s^2 v(s,y) - 2\cos(y)(\cos(y)+1)\partial_y^2 v(s,y) + 2\sin(y)\partial_{sy} v(s,y)\\ &+ \left(1 + (d-1)\frac{\cos(y)+1}{\cos(y)}\right)\partial_s v(s,y)\\ &+ \left(2\sin(y)(\cos(y)+1) - (d-1)\frac{(\cos(y)+1)^2}{\sin(y)}\right)\partial_y v(s,y).\end{aligned} \tag{2.6}$$

In the context of semigroup theory we rewrite the equation $\tilde{\Box}_d^{rad} v(s,y) = 0$ into a system of two equations by considering the tuple $\mathbf{v}(s,\cdot) := \begin{pmatrix} v(s,\cdot) \\ \partial_s v(s,\cdot) \end{pmatrix}$ for every $s \geq 0$. Then an equivalent description of v solving the free radial wave equation can be given as

$$\partial_s \mathbf{v}(s,\cdot) = \mathfrak{L}_d \mathbf{v}(s,\cdot)$$

where $\mathfrak{L}_d$ is with (2.6) formally defined via

$$\mathfrak{L}_d \mathbf{f} = \begin{pmatrix} [\mathfrak{L}_d \mathbf{f}]_1 \\ [\mathfrak{L}_d \mathbf{f}]_2 \end{pmatrix},$$

where

$$\begin{aligned} [\mathfrak{L}_d \mathbf{f}]_1 :=& f_2 \\ [\mathfrak{L}_d \mathbf{f}]_2 :=& 2(\cos(\cdot)^2 + \cos(\cdot)) f_1'' - (2\sin(\cdot)(\cos(\cdot)+1) - (d-1)\frac{(\cos(\cdot)+1)^2}{\sin(\cdot)}) f_1' \\ &- 2\sin(\cdot) f_2' + (1 + (d-1)\frac{\cos(\cdot)+1}{\cos(\cdot)}) f_2, \end{aligned} \tag{2.7}$$

for $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.

2.3. One dimensional free wave equation

We start in the simplest case. That is, we consider the one dimensional free wave equation

$$\begin{aligned} 0 = \tilde{\Box}_1^{rad} v(s,y) =& \partial_s^2 v(s,y) - 2\cos(y)(\cos(y)+1)\partial_y^2 v(s,y) + 2\sin(y)\partial_{sy} v(s,y) \\ &+ \partial_s v(s,y) + 2\sin(y)(\cos(y)+1)\partial_y v(s,y). \end{aligned} \tag{2.8}$$

Multiplying this equation by $\frac{1}{1+\cos(y)}$ and testing with $\partial_s v(s,y)$ we formally find the *energy identity*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(y)\partial_y v(s,y)^2 \,\mathrm{d}y + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2(1+\cos(y))}\partial_s v(s,y)^2 \,\mathrm{d}y\right] \\ = -\frac{1}{2}\partial_s v(s,-\frac{\pi}{2})^2 - \frac{1}{2}\partial_s v(s,\frac{\pi}{2})^2. \end{aligned}$$

The computation that leads to this identity is part of the next lemma and can be verified there.
This motivates the definition of the *energy*.

Definition 2.3.1. The *energy* $E : C^1[-\frac{\pi}{2}, \frac{\pi}{2}] \times C[-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ is given by

$$E(f_1, f_2) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(y) f_1'(y)^2 \, \mathrm{d}y + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2(1+\cos(y))} f_2(y)^2 \, \mathrm{d}y.$$

The previous energy identity formulated in terms of the energy means, that if v solves the 1-dimensional wave equation in new coordinates, then the map $s \mapsto E\left(v(s,\cdot), \partial_0 v(s,\cdot)\right)$ is positive and bounded by $E\left(v(0,\cdot), \partial_0 v(0,\cdot)\right)$. For later purposes, we want to reformulate this 1-dimensional bound into semigroup language. This will be done via an application of the Lumer-Phillips-Theorem. Towards this end we define appropriate functions spaces and a differential operator suitable to our problem for which we will then show dissipativity and density of its range.

Definition 2.3.2. The vector space

$$C^\infty_{odd}\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)^2 := \left\{\mathbf{f} = (f_1, f_2) \in C^\infty\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) \times C^\infty\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) : f_1, f_2 \text{ odd}\right\}$$

equipped with the inner product inspired by the energy

$$(\mathbf{f} \mid \mathbf{g})_{\mathcal{H}_1} := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(y) f_1'(y) g_1'(y) \, \mathrm{d}y + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2(1+\cos(y))} f_2(y) g_2(y) \, \mathrm{d}y \tag{2.9}$$

is a pre-Hilbert space. By $\mathcal{H}_1$ we denote its completion, which makes it a Hilbert space.
We remember the operator $\mathfrak{L}_1$ formally defined in (2.7)

$$\mathfrak{L}_1 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_2 \\ 2(\cos(\cdot)^2 + \cos(\cdot)) f_1'' - 2\sin(\cdot)(\cos(\cdot)+1) f_1' - 2\sin(\cdot) f_2' - f_2 \end{pmatrix}.$$

and define the operator $\mathbf{L}_1 : \mathcal{D}(\mathbf{L}_1) \subset \mathcal{H}_1 \to \mathcal{H}_1$ by $\mathcal{D}(\mathbf{L}_1) := C^\infty_{odd}\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)^2$ and $\mathbf{L}_1 \mathbf{f} := \mathfrak{L}_1 \mathbf{f}$. By construction $\mathbf{L}_1$ is densely defined.

Lemma 2.3.3. *The operator $\mathbf{L}_1$ is closable and its closure $\overline{\mathbf{L}}_1$ generates a strongly continuous contraction semigroup $(\mathbf{S}_1(s))_{s \geq 0}$ on $\mathcal{H}_1$. In particular, we have the estimate $\|\mathbf{S}_1(s)\mathbf{f}\|_{\mathcal{H}_1} \leq \|\mathbf{f}\|_{\mathcal{H}_1}$ for all $s \geq 0$, for all $\mathbf{f} \in \mathcal{H}_1$.*

Proof. Our strategy will be to show dissipativity of $\mathbf{L}_1$ and density of $\lambda - \mathbf{L}_1$ for some positive λ to then invoke the Lumer-Phillips Theorem to obtain the result. For dissipativity we have to show that $(\mathbf{L}_1 \mathbf{f} \mid \mathbf{f})_{\mathcal{H}_1} \leq 0$ for all $\mathbf{f} \in C^\infty_{odd}\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)^2$. We compute

$$\begin{aligned}(\mathbf{L}_1 \mathbf{f} \mid \mathbf{f})_{\mathcal{H}_1} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(y) f_1'(y) f_2'(y) \mathrm{d}y \\ &+ \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(2\cos(y) f_1''(y) - 2\sin(y) f_1'(y) - \frac{2\sin(y) f_2'(y) + f_2(y)}{1+\cos(y)}\right) f_2(y) \mathrm{d}y\end{aligned}$$

and by partially integrating the first term we note that it cancels with the second and third term, which leaves us with

$$(\mathbf{L}_1\mathbf{f} \mid \mathbf{f})_{\mathcal{H}_1} = -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f_2(y)}{1+\cos(y)}(2f_2'(y)\sin(y) + f_2(y))\mathrm{d}y.$$

By the fundamental theorem of calculus we have

$$\begin{aligned} f_2\left(-\frac{\pi}{2}\right)^2 + f_2\left(\frac{\pi}{2}\right)^2 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \partial_y\left(\frac{\sin(y)}{1+\cos(y)}f_2(y)^2\right)\mathrm{d}y \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f_2(y)}{1+\cos(y)}(2f_2'(y)\sin(y) + f_2(y))\mathrm{d}y, \end{aligned}$$

hence we conclude that $(\mathbf{L}_1\mathbf{f} \mid \mathbf{f})_{\mathcal{H}_1} = -\frac{1}{2}f_2\left(-\frac{\pi}{2}\right)^2 - \frac{1}{2}f_2\left(\frac{\pi}{2}\right)^2 \leq 0$. For density we show that there exists $\lambda > 0$ such that $\mathrm{rg}(\lambda - \mathbf{L}_1) = \mathcal{D}(\mathbf{L}_1) = C^\infty_{odd}\left(\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)^2$. Let $\lambda = 2$ and assume that $\mathbf{g}$ is such that $(2-\mathbf{L}_1)\mathbf{f} = \mathbf{g}$ for $\mathbf{f} \in C^\infty_{odd}\left(\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)^2$. That is

$$\begin{cases} 2f_1 - f_2 & = g_1, \\ -2(\cos(\cdot)^2 + \cos(\cdot))f_1'' + 2\sin(\cdot)(\cos(\cdot)+1)f_1' + 2\sin(\cdot)f_2' + 3f_2 & = g_2, \end{cases} \tag{2.10}$$

from which we see that $2-\mathbf{L}_1$ preserves smoothness and oddness. That is rg(2-$\mathbf{L}_1$) $\subset$ $\mathcal{D}(\mathbf{L}_1)$. To show that these sets in fact are equal, we consider the inverse problem and assume that $\mathbf{g} \in C^\infty_{odd}\left(\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)^2$. We try to find $\mathbf{f} \in C^\infty_{odd}\left(\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)^2$ such that $(2-\mathbf{L}_1)\mathbf{f} = \mathbf{g}$. Towards this end, we equivalently write the above system as

$$\begin{cases} f_2(y) & = 2f_1(y) - g_1(y), \\ f_1''(y) - f_1'(y)\frac{\sin(y)(3+\cos(y))}{\cos(y)(\cos(y)+1)} - f_1(y)\frac{3}{\cos(y)(\cos(y)+1)} & = -\frac{3g_1(y)+2\sin(y)g_1'(y)+g_2(y)}{2\cos(y)(\cos(y)+1)}. \end{cases}$$

Two fundamental solutions to the ODE in the second line are given by

$$\phi_1(y) = \frac{(1-\sin(y))(1+\cos(y))}{\cos(y)^2}, \qquad \phi_2(y) = \frac{\sin(y)(1+\cos(y))}{\cos(y)^2}.$$

Thus, by variation of constants we get that

$$f_1(y) = \phi_1(y)\int_0^y \frac{\phi_2(x)}{W(x)}G(x)\mathrm{d}x + \phi_2(y)\int_y^{\frac{\pi}{2}} \frac{\phi_1(x)}{W(x)}G(x)\mathrm{d}x,$$

where $W(x) := \phi_1'(x)\phi_2(x) - \phi_2'(x)\phi_1(x) = \frac{(1+\cos(x))^2}{\cos(x)^3}$ is the Wronskian of ϕ_1, ϕ_2 and $G(x) := -\frac{3g_1(x)+2\sin(x)g_1'(x)+g_2(x)}{2\cos(x)(\cos(x)+1)}$ is the nonlinear part. That is

$$\begin{aligned} f_1(y) = &\frac{(1-\sin(y))(1+\cos(y))}{\cos(y)^2}\int_0^y \frac{\sin(x)}{2(1+\cos(x))^2}h(x)\mathrm{d}x \\ &+ \frac{\sin(y)(1+\cos(y))}{\cos(y)^2}\int_y^{\frac{\pi}{2}} \frac{1-\sin(x)}{2(1+\cos(x))^2}h(x)\mathrm{d}x, \end{aligned} \tag{2.11}$$

where $h(x) = 3g_1(x) + 2\sin(x)g_1'(x) + g_2(x)$. Note that h is an odd function and smooth, by the assumptions on g_1, g_2. To see that f_1 is odd we first rewrite it in a form where this is more visible. Indeed, expanding the $1 - \sin(\cdot)$ terms yields,

$$\begin{aligned} f_1(y) =& \frac{1+\cos(y)}{\cos(y)^2} \int_0^y \frac{\sin(x)}{2(1+\cos(x))^2} h(x)\ \mathrm{d}x \\ &+ \frac{\sin(y)(1+\cos(y))}{\cos(y)^2} \int_y^{\frac{\pi}{2}} \frac{1}{2(1+\cos(x))^2} h(x)\mathrm{d}x \\ &- \frac{\sin(y)(1+\cos(y))}{\cos(y)^2} \int_0^{\frac{\pi}{2}} \frac{\sin(x)}{2(1+\cos(x))^2} h(x)\mathrm{d}x. \end{aligned}$$

Hence by symmetry properties of the trigonometric functions and h, we compute

$$\begin{aligned} f_1(-y) =& \frac{1+\cos(y)}{\cos(y)^2} \int_0^{-y} \frac{\sin(x)}{2(1+\cos(x))^2} h(x)\ \mathrm{d}x \\ &- \frac{\sin(y)(1+\cos(y))}{\cos(y)^2} \int_{-y}^{\frac{\pi}{2}} \frac{1}{2(1+\cos(x))^2} h(x)\mathrm{d}x \\ &+ \frac{\sin(y)(1+\cos(y))}{\cos(y)^2} \int_0^{\frac{\pi}{2}} \frac{\sin(x)}{2(1+\cos(x))^2} h(x)\mathrm{d}x \\ =& - \frac{1+\cos(y)}{\cos(y)^2} \int_0^{y} \frac{\sin(x)}{2(1+\cos(x))^2} h(x)\ \mathrm{d}x \\ &- \frac{\sin(y)(1+\cos(y))}{\cos(y)^2} \int_{y}^{\frac{\pi}{2}} \frac{1}{2(1+\cos(x))^2} h(x)\mathrm{d}x \\ &+ \frac{\sin(y)(1+\cos(y))}{\cos(y)^2} \int_0^{\frac{\pi}{2}} \frac{\sin(x)}{2(1+\cos(x))^2} h(x)\mathrm{d}x \\ =& - f_1(y), \end{aligned}$$

where the second step follows from $\int_{-y}^{y} \frac{1}{2(1+\cos(x))^2} h(x)\mathrm{d}x = 0$, since h is odd. Thus f_1 is odd. It is also smooth on $(-\frac{\pi}{2}, \frac{\pi}{2})$ as a composition of smooth functions. To see smoothness at the endpoints, we note that by oddness it suffices to consider one of them. Hence, let $y \uparrow \pi/2$ in (2.11). Since $\frac{1-\sin(y)}{\cos(y)^2} = 1 + \sin(y)$ we note that the first summand is smooth at $\pi/2$. For the second summand of f_1 in (2.11),

$$f_{12}(y) := \frac{\sin(y)(1+\cos(y))}{\cos(y)^2} \int_y^{\frac{\pi}{2}} \frac{1-\sin(x)}{2(1+\cos(x))^2} h(x)\mathrm{d}x$$

a heuristic counting of zeros against singularities yields that we have 2 singularities from $\cos(\cdot)^2$ against 3 zeros, 1 from the integral border and the other 2 from $1 - \sin(\cdot)$ in the integrand, by which we expect smoothness. To make this more rigorous, consider the function h defined by

$$h(y) := f_{12}\left(y + \frac{\pi}{2}\right) = \frac{\sin(y+\frac{\pi}{2})(1+\cos(y+\frac{\pi}{2}))}{\cos(y+\frac{\pi}{2})^2} \int_y^0 \frac{1-\sin(x+\frac{\pi}{2})}{2(1+\cos(x+\frac{\pi}{2}))^2} h(x+\frac{\pi}{2})\mathrm{d}x,$$

where the second equality follows from the change of variable $\Phi(x) = x + \frac{\pi}{2}$. Since $\cos(\cdot + \frac{\pi}{2})^2$ and $1 - \sin(\cdot + \frac{\pi}{2})$ both have a zero of order 2 at 0, Lemma B.3 shows that h is smooth at 0. Hence f_{12} and thus f_1 are smooth at $\frac{\pi}{2}$. Since $f_2 = 2f_1 - g_1$, oddness and smoothness for f_2 immediately follow. Hence we have shown density and an invocation of the Lumer-Phillips Theorem finishes the proof.

□

Lemma 2.3.3 gives an energy estimate for a solution of the 1-dimensional wave equation in FHSC. We want to lift this estimate first to 3 and then to 5 dimensions. We will do this step by step.

2.4. Connection to higher dimensions via stepwise descent

The following method relies on a particular property of solutions of the d-dimensional radial wave equation. We will show that every d-dimensional solution induces a 1-dimensional one. First we want to understand this procedure in usual cartesian coordinates.

2.4.1. Formal descent in cartesian/ spherical coordinates

To this end we rewrite the cartesian wave equation into radial coordinates, since these are evidently best suited to radial solutions. It should be mentioned that the expressions "radial coordinates" and "spherical coordinates" are used interchangeably, radial being more fitting in our setting, while spherical is the wider-used expression with respect to coordinates.
To shorten the notation, let us first introduce two important operators.

Definition 2.4.1. The radial d-dimensional Laplacian Δ_d^{rad} and wave operator $\Box_d^{rad}$ are formally defined by

$$\Delta_d^{rad} := \partial_r^2 + \frac{d-1}{r}\partial_r,$$
$$\Box_d^{rad} := -\partial_t^2 + \Delta_d^{rad}.$$

The following Lemma and Corollary validate the names of the introduced operators. Since for radial functions a change in angle does not change the value of the function, in the following we are not interested in the specific form of spherical coordinates other than its radius. We only need that such a spherical transformation exists, which is well-known.

Lemma 2.4.2. *For the radial coordinate transformation*

$$\Phi \begin{pmatrix} x_1 \\ \dots \\ x_d \end{pmatrix} = \begin{pmatrix} \sqrt{x_1^2 + \dots + x_d^2} \\ f_1(x_1, \dots, x_d) \\ \dots \\ f_{d-1}(x_1, \dots, x_d) \end{pmatrix} =: \begin{pmatrix} r \\ \phi_1 \\ \dots \\ \phi_{d-1} \end{pmatrix}$$

the Laplacian of a radial function $u : \mathbb{R}^d \to \mathbb{R}$, $\Delta_x u(x_1, \ldots, x_d) := \sum_{i=1}^d \partial_{x_i}^2 u(x_1, \ldots, x_d)$ *becomes in radial coordinates*

$$\Delta_d^{rad} v(r, \phi_1, \ldots, \phi_{d-1}),$$

where $v(\Phi(x_1, \ldots, x_d)) = u(x_1, \ldots, x_d)$. *In particular, since* u *is radial and therefore* v *is,* $\Delta_x u(x) = 0$ *is equivalent to* $\Delta_d^{rad} v(r, \phi_1, \ldots, \phi_{d-1}) = \Delta_d^{rad} v(r, 0, \ldots, 0) = 0$, *which means that the 1-dimensional radial representative* $\tilde{v}$ *of* v, *solves* $\Delta_d^{rad} \tilde{v}(r) = 0$.

Proof. Let $u : \mathbb{R}^d \to \mathbb{R}$ be radial and v such that $v \circ \Phi = u$. We are interested in $\Delta_x u(x) = \sum_{i=1}^d \partial_i^2 v(\Phi(x_1, \ldots, x_d))$. Hence we calculate

$$\partial_i^2 (v \circ \Phi) = \sum_{j=1}^d \sum_{k=1}^d \partial_k \partial_j v \circ \Phi \cdot \partial_i \Phi_k \cdot \partial_i \Phi_j + \partial_j v \circ \Phi \cdot \partial_i^2 \Phi_j,$$

where Φ_l is the l-th component of Φ. Now, since v is radial we have $\partial_l v = 0$ for all $l \neq 1$, which reduces the sum to $\partial_i^2 (v \circ \Phi) = \partial_1^2 v \circ \Phi \cdot (\partial_i \Phi_1)^2 + \partial_1 v \circ \Phi \cdot \partial_i^2 \Phi_1$. With

$$\begin{aligned} \partial_i \Phi_1(x_1, \ldots, x_d) &= \frac{x_i}{r}, \\ \partial_i^2 \Phi_1(x_1, \ldots, x_d) &= \frac{r - \frac{x_i^2}{r}}{r^2} = \frac{r^2 - x_i^2}{r^3}, \end{aligned}$$

we get

$$\begin{aligned} \Delta_x u(x) &= \sum_{i=1}^d \partial_i^2 v(\Phi(x_1, \ldots, x_d)) \\ &= \sum_{i=1}^d \partial_1^2 v(\Phi(x_1, \ldots, x_d)) \frac{x_i^2}{r^2} + \partial_1 v(\Phi(x_1, \ldots, x_d)) \frac{r^2 - x_i^2}{r^3} \\ &= \partial_1^2 v(r, \phi_1, \ldots, \phi_{d-1}) + \partial_1 v(r, \phi_1, \ldots, \phi_{d-1})) \frac{dr^2 - r^2}{r^3}, \end{aligned}$$

which proves the claim. □

The following corollary just extends the previous result for the Laplacian to the wave operator.

Corollary 2.4.3. *The d-dimensional wave operator* $\Box_d u(t, x) = (-\partial_t^2 + \Delta_x) u(t, x)$ *takes in radial coordinates the form* $\Box_d^{rad} v(t, r, \phi_1, \ldots, \phi_{d-1})$ *whenever for all* t *the function* $x \mapsto u(t, x)$ *is radial.*

We are still due an explanation of how solutions of different dimensions are related. One possible answer to this question is that one can construct 1-dimensional solutions from higher dimensional ones in one go, as made evident in the following lemma. We will call this operation the *full descent in radial coordinates.*

Lemma 2.4.4. *Let $u : \mathbb{R} \to \mathbb{R}$ be radial, smooth and a solution to the 3-dimensional problem, i.e. $\Delta_3^{rad} u(r) = 0$ and $v : \mathbb{R} \to \mathbb{R}$ be radial, smooth and a solution to the 5-dimensional problem, i.e. $\Delta_5^{rad} v(r) = 0$. Then solutions to the 1-dimensional problem are given by $r \mapsto ru(r)$ and $r \mapsto 3rv(r) + r^2\partial_r v(r)$, i.e. $\Delta_1^{rad}(ru(r)) = 0 = \Delta_1^{rad}(3rv(r) + r^2\partial_r v(r))$.*

Proof. In three dimensions we have

$$\partial_r^2(r\tilde{u}(r)) = r(\partial_r^2\tilde{u}(r) + \frac{2}{r}\partial_r\tilde{u}(r)) = 0,$$

by the assumption on u. In five dimensions, multiplying and differentiating the assumption on v leads to $(4\partial_r + 6r\partial_r^2 + r^2\partial_r^3)\tilde{v}(r) = 0$. With this, we calculate

$$\begin{aligned} &\partial_r^2(3r\tilde{v}(r) + r^2\partial_r\tilde{v}(r)) = 8\partial_r\tilde{v}(r) + 7r\partial_r^2\tilde{v}(r) + r^2\partial_r^3\tilde{v}(r) \\ =& \left[4\partial_r\tilde{v}(r) + 6r\partial_r^2\tilde{v}(r) + r^2\partial_r^3\tilde{v}(r)\right] + r\left[\frac{4}{r}\partial_r\tilde{v}(r) + \partial_r^2\tilde{v}(r)\right] \\ =&0, \end{aligned}$$

since the expressions in the square brackets vanish. □

Remark 2.4.5. *This lemma can be extended to any odd dimension. The full descent operator in arbitrary odd dimension d mapping a function f to the function $r \mapsto (r^{-(d-3)}\partial_r)^{(\frac{d-3}{2})}(r^{d-2}\tilde{f}(r))$ gives the desired result. Note that the functions in 3 and 5 dimensions from above coincide with this function, i.e. that this really is an extension of the lemma. However, since we only need the lemma in the three- and five- dimensional case, this will not be proved.*

Since the dimension of the radial wave operator is fully encoded in its spatial derivatives but not in the time derivatives and the full descent operator is independent of time, replacing the Laplacian with the wave operator in the previous result is still valid. We gather this observation in a corollary.

Corollary 2.4.6. *Suppose $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are such that $\Box_3^{rad} u = 0$ and $\Box_5^{rad} v = 0$. Then $g_1(t, r) := ru(t, r)$ and $g_2(t, r) := 3rv(t, r) + r^2\partial_r v(t, r)$ solve $\Box_1^{rad} g = 0$.*

The key insight which heavily simplifies the analysis of higher dimensional solutions is that the full descent in Lemma 2.4.4 can be broken down into step-wise descent. The dimension is then not reduced from d to 1 in one go, but from d to $d - 2$ in each step. The following definition exists towards this analysis.

Definition 2.4.7. For odd $d \geq 3$, the d-dimensional descent operator D_d^{rad} is defined via

$$D_3^{rad} f(r) = rf(r),$$

and

$$D_d^{rad} f(r) = (d-2)f(r) + r\partial_r f(r),$$

for $d \geq 5$, odd.

With this definition at hand, we can formulate the intertwining identity for radial coordinates, which validates the name of the descent operator by showing that it maps d-dimensional to (d-2)-dimensional solutions of the wave equation.

Lemma 2.4.8. *For $d \geq 3$, odd, the radial Laplacian commutes with the radial descent operator in that*

$$\Delta_{d-2}^{rad} D_d^{rad} = D_d^{rad} \Delta_d^{rad}.$$

Proof. We only proof the cases $d \in \{3, 5\}$. The proof for $d > 5$ is similar straightforward. Let f be a suitable function. For $d = 3$, we have

$$D_3^{rad} \Delta_3^{rad} f(r) = r\partial_r^2 f(r) + 2\partial_r f(r) = \Delta_1^{rad}[(\cdot)f(\cdot)](r) = \Delta_1^{rad} D_3^{rad} f(r).$$

For $d = 5$, we have on the one side

$$\begin{aligned}
\Delta_3^{rad} D_5^{rad} f(r) &= (\partial_r^2 + \frac{2}{r}\partial_r)(3f(r) + r\partial_r f(r)) \\
&= (3\partial_r^2 + r\partial_r^3 + 2\partial_r^r + \frac{6}{r}\partial_r + 2\partial_r^2 + \frac{2}{r}\partial_r)f(r) \\
&= (r\partial_r^3 + 7\partial_r^2 + \frac{8}{r}\partial_r)f(r),
\end{aligned}$$

and on the other side

$$\begin{aligned}
D_5^{rad} \Delta_5^{rad} f(r) &= S_5(\partial_r^2 + \frac{4}{r})f(r) \\
&= (3\partial_r^2 + \frac{12}{r}\partial_r + r\partial_r^3 + r(-\frac{4}{r^2})\partial_r + 4\partial_r^2)f(r) \\
&= (r\partial_r^3 + 7\partial_r^2 + \frac{8}{r}\partial_r)f(r).
\end{aligned}$$

□

Again we extend this result to the wave operator.

Corollary 2.4.9. *For $d \geq 3$, odd, the radial wave operator commutes with the radial descent operator in that*

$$\Box_{d-2}^{rad} D_d^{rad} = D_d^{rad} \Box_d^{rad}.$$

Remark 2.4.10. *Note that this is a pure operator identity and no assumptions (other than differentiability) have to be made on f. In particular f does not have to be a solution of any kind.*

2.4.2. Descent and ascent in FHSC

Now we want to have a similar intertwining identity in similarity coordinates. The equality which emerges by just rewriting the operators from the previous section into operators in FHSC does not hold true. However the descent operators will be key in our later estimates, hence we want an identity including them. Therefore we start by rewriting them into the self similar setting. As we did in the previous section we formally introduce the operators. But since this is the setting we are actually interested in, we also define them properly afterwards, meaning that we also introduce the spaces on which the operators act upon. This step was omitted in radial coordinates in the previous section, since the previous section should only motivate the approach in a well-known setting.
In new coordinates we have

$$D_3^{rad} g(t,\cdot)(r) = rg(t,r) = e^s \tan(y) u(s,y) =: u_3^{\downarrow}(s,y) \tag{2.12}$$

and for odd $d \geq 5$

$$\begin{aligned} D_d^{rad} g(t,\cdot)(r) &= (d-2)g(t,r) + r\partial_r g(t,r) \\ &= (d-2)u(s,y) + e^s \tan(y) \left(-\frac{e^{-s}\tan(y)}{1+\frac{1}{\cos(y)}}\partial_s + e^{-s}\cos(y)\partial_y \right) u(s,y) \\ &= (d-2)u(s,y) + \left(\frac{\cos(y)-1}{\cos(y)} \right) \partial_s u(s,y) + \sin(y)\partial_y u(s,y) =: u_d^{\downarrow}(s,y), \end{aligned} \tag{2.13}$$

where in both cases $u = g \circ \Psi_1$, with Ψ_1 being the transformation into similarity coordinates

$$(s,y) \mapsto \left(e^s(1+\sqrt{1+\tan(y)^2}), e^s \tan(y) \right)$$

from the start of the chapter. We also gather the facts that

$$\partial_s u_3^{\downarrow}(s,y) = e^s \tan(y)(u(s,y) + \partial_s u(s,y)) \tag{2.14}$$

and

$$\partial_s u_d^{\downarrow}(s,y) = (d-2)\partial_s u(s,y) + \left(\frac{\cos(y)-1}{\cos(y)} \right) \partial_s^2 u(s,y) + \sin(y)\partial_y \partial_s u(s,y). \tag{2.15}$$

As before it is advantageous to only work with spatial operators. This is done by reformulating the problem into a two dimensional operator problem, by restricting ourselves to solutions of the wave equation. We remember the start of this chapter (2.7) where we showed that u solves the d-dimensional wave equation in self similar coordinates if and only if

$$\partial_s \begin{pmatrix} u(s,y) \\ \partial_s u(s,y) \end{pmatrix} = \mathfrak{L}_d \begin{pmatrix} u(s,\cdot) \\ \partial_s u(s,\cdot) \end{pmatrix} (y),$$

where the operator $\mathfrak{L}_d$ was given by

$$\begin{aligned}\left[\mathfrak{L}_d \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right]_1 =& f_2, \\ \left[\mathfrak{L}_d \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right]_2 =& 2(\cos(\cdot)^2 + \cos(\cdot))f_1'' - \left(2\sin(\cdot)(\cos(\cdot)+1) - (d-1)\frac{(\cos(\cdot)+1)^2}{\sin(\cdot)}\right) f_1' \\ & - 2\sin(\cdot)f_2' - \left(1 + (d-1)\frac{\cos(\cdot)+1}{\cos(\cdot)}\right) f_2.\end{aligned}$$

Then we *formally* define the linear *descent operators* $\mathfrak{D}_3^{\downarrow}$, $\mathfrak{D}_d^{\downarrow}$ for odd $d \geq 5$, by

$$\begin{aligned}\mathfrak{D}_3^{\downarrow} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &:= \tan(\cdot) \begin{pmatrix} f_1 \\ f_1 + f_2, \end{pmatrix} \\ \mathfrak{D}_d^{\downarrow} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &:= (d-2) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \sin(\cdot) \begin{pmatrix} f_1' \\ f_2' \end{pmatrix} + \frac{\cos(\cdot)-1}{\cos(\cdot)} \mathfrak{L}_d \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.\end{aligned}$$

For the tuples $\mathbf{u}(s,y) := \begin{pmatrix} u(s,y) \\ \partial_s u(s,y) \end{pmatrix}$, $\mathbf{u}_3^{\downarrow}(s,y) := \begin{pmatrix} u_3^{\downarrow}(s,y) \\ \partial_s u_3^{\downarrow}(s,y) \end{pmatrix}$, $\mathbf{u}_d^{\downarrow}(s,y) := \begin{pmatrix} u_d^{\downarrow}(s,y) \\ \partial_s u_d^{\downarrow}(s,y) \end{pmatrix}$ equation (2.12) with (2.14) and (2.13) with (2.15) then read

$$\mathbf{u}_3^{\downarrow}(s,y) = e^s \mathfrak{D}_3^{\downarrow}\mathbf{u}(s,\cdot)(y), \qquad \mathbf{u}_d^{\downarrow}(s,y) = \mathfrak{D}_d^{\downarrow}\mathbf{u}(s,\cdot)(y). \tag{2.16}$$

With these definitions at hand, we are now prepared to formulate the intertwining identity in self similar coordinates which is the self similar counterpart to Lemma 2.4.8. It should be no surprise that the intertwining identity for the first step, i.e. from dimension 1 to 3 is different to all the others, since the descent operator is scaled with exponential time in this case, as you can see in (2.16).

Lemma 2.4.11. *The descent operators satisfy the intertwining identities*

$$\begin{aligned}\mathfrak{D}_3^{\downarrow}\mathfrak{L}_3 - \mathfrak{L}_1\mathfrak{D}_3^{\downarrow} &= -\mathfrak{D}_3^{\downarrow}, \\ \mathfrak{D}_d^{\downarrow}\mathfrak{L}_d &= \mathfrak{L}_{d-2}\mathfrak{D}_d^{\downarrow},\end{aligned}$$

for odd $d \geq 5$.

Proof. We will only sketch the proof, which is attributed to the fact that there are no ingenious steps involved but an admittedly tedious computation. In dimension 3 we have

$$\mathfrak{D}_3^{\downarrow}\mathfrak{L}_3\mathbf{f}(y) = \tan(y) \begin{pmatrix} [\mathfrak{L}_3\mathbf{f}]_1(y) \\ [\mathfrak{L}_3\mathbf{f}]_2(y) + [\mathfrak{L}_3\mathbf{f}]_1(y) \end{pmatrix} = \tan(y) \begin{pmatrix} f_2(y) \\ [\mathfrak{L}_3\mathbf{f}]_2(y) + f_2(y) \end{pmatrix},$$

and

$$\mathfrak{L}_1\mathfrak{D}_3^{\downarrow}\mathbf{f}(y) = \begin{pmatrix} \tan(y)(f_2(y) + f_1(y)) \\ [\mathfrak{L}_1(\tan(\cdot)\mathbf{f})]_2(y) + [\mathfrak{L}_1(\tan(\cdot) \begin{pmatrix} 0 \\ f_1 \end{pmatrix})]_2(y) \end{pmatrix},$$

which shows that the equality is satisfied in the first component. Hence let us concentrate on the second component. A straightforward computation yields the product rule

$$\begin{aligned}[\mathfrak{L}_1(\tan(\cdot)\mathbf{g})]_2(y) = \tan(y)[\mathfrak{L}_1\mathbf{g}]_2(y) + 2\cos(y)(\cos(y)+1)(\tan''(y)g_1(y) + 2\tan'(y)g_1'(y))\\ -2\sin(y)(\cos(y)+1)\tan'(y)g_1(y) - 2\sin(y)\tan'(y)g_2(y),\end{aligned}$$

which we use to find that

$$\begin{aligned}[\mathfrak{D}_3^{\downarrow}\mathfrak{L}_3\mathbf{f} - \mathfrak{L}_1\mathfrak{D}_3^{\downarrow}\mathbf{f}]_2(y) =& \tan(y)([(\mathfrak{L}_3 - \mathfrak{L}_1)\mathbf{f}]_2(y) + f_2(y)\\ &- 2\cos(y)(\cos(y)+1)(\tan''(y)f_1(y) + 2\tan'(y)f_1'(y))\\ &+ 2\sin(y)(\cos(y)+1)\tan'(y)f_1(y) + 2\sin(y)\tan'(y)f_2(y)\\ &+ 2\sin(y)\tan(y)f_1'(y) + (\tan(y) + 2\sin(y)\tan'(y))f_1(y),\end{aligned}$$

which after a careful computation finally leads to

$$[\mathfrak{D}_3^{\downarrow}\mathfrak{L}_3\mathbf{f} - \mathfrak{L}_1\mathfrak{D}_3^{\downarrow}\mathbf{f}]_2(y) = [-\mathfrak{D}_3^{\downarrow}\mathbf{f}]_2(y).$$

This is what was left to show in dimension 3. In odd dimension greater than or equal to 5 the proof is a similar straightforward, but even lengthier computation and will be omitted. □

This identity is an integral part of lifting our 1 dimensional estimates to 3 and 5 dimensional ones, as we will see in the following. First we need to specify on which spaces the descent operator acts. In context of the intertwining identity and to connect to our theory developed in the one dimensional case, we require the range of the descent operator to be the domain of the operator $\mathbf{L}_1$, i.e. smooth functions on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ which are odd. To find the appropriate domain for the descent operator, we first show that it has a formal inverse and choose the domain of the descent operator as the range of its formal inverse with domain $C^\infty_{odd}([-\frac{\pi}{2}, \frac{\pi}{2}])$. These remarks show that the inverse of the descent is at least as useful to our construction as the descent operator itself. Hence we give its inverse its own name and develop the theory from there.

Lemma 2.4.12. *There is a bijective, linear operator, called the 1-dimensional ascent operator,* $\mathbf{A}_1^{\uparrow} : C^\infty_{odd}([-\frac{\pi}{2}, \frac{\pi}{2}])^2 \to C^\infty_{even,0}([-\frac{\pi}{2}, \frac{\pi}{2}])^2 := \{\widehat{\mathbf{f}} \in C^\infty_{even}([-\frac{\pi}{2}, \frac{\pi}{2}])^2 : \widehat{f}_1(\frac{\pi}{2}) = 0 = \widehat{f}_2(\frac{\pi}{2})\}$ *such that* $\mathbf{A}_1^{\uparrow} = (\mathfrak{D}_3^{\downarrow})^{-1}$. *In particular, the intertwining identity in Lemma 2.4.11 manifests itself as*

$$\widehat{\mathbf{L}}_3\mathbf{A}_1^{\uparrow} - \mathbf{A}_1^{\uparrow}\mathbf{L}_1 = -\mathbf{A}_1^{\uparrow},$$

where $\widehat{\mathbf{L}}_3 : C^\infty_{even,0}([-\frac{\pi}{2}, \frac{\pi}{2}])^2 \to C^\infty_{even,0}([-\frac{\pi}{2}, \frac{\pi}{2}])^2$ *is given by* $\widehat{\mathbf{L}}_3\widehat{\mathbf{f}} := \mathfrak{L}_3\widehat{\mathbf{f}}$.

Proof. To construct $\mathbf{A}_1^{\uparrow}$ let $\widehat{\mathbf{f}}$ and $\mathbf{g}$ be such that $\mathfrak{D}_3^{\downarrow}\widehat{\mathbf{f}} = \mathbf{g}$. That is

$$\begin{cases}\tan(\cdot)\widehat{f}_1 = g_1,\\ \tan(\cdot)(\widehat{f}_1 + \widehat{f}_2) = g_2.\end{cases} \tag{2.17}$$

The inverse operation is then explicitly given by

$$\widehat{\mathbf{f}} = \cot(\cdot)\begin{pmatrix} g_1 \\ g_2 - g_1 \end{pmatrix} =: \mathbf{A}_1^{\uparrow}\mathbf{g}. \tag{2.18}$$

To show surjectivity of $\mathbf{A}_1^{\uparrow}$ assume that $\widehat{\mathbf{f}} \in C^\infty_{even,0}([-\frac{\pi}{2}, \frac{\pi}{2}])^2$. We need to find $\mathbf{g} \in C^\infty_{odd}([-\frac{\pi}{2}, \frac{\pi}{2}])^2$ such that $\mathbf{A}_1^{\uparrow}\mathbf{g} = \widehat{\mathbf{f}}$. We start by considering the first component

$$\frac{\sin(y)}{\cos(y)}\widehat{f}_1(y) = g_1(y). \tag{2.19}$$

From this we see that g_1 is odd and smooth on $(-\frac{\pi}{2}, \frac{\pi}{2})$ with $g_1(0) = 0$. Using the fundamental theorem of calculus with the fact that $\widehat{f}_1(\frac{\pi}{2}) = 0$ and the change of variables $\Phi(x) = (y - \frac{\pi}{2})(x + \frac{\pi}{2})$ we write the above as

$$g_1(y) = \frac{\sin(y)}{\cos(y)}\int_{\frac{\pi}{2}}^{y} \widehat{f}_1'(x)\,\mathrm{d}x = \frac{\sin(y)(y-\frac{\pi}{2})}{\cos(y)}\int_0^1 \widehat{f}_1'\left(\left(y - \frac{\pi}{2}\right)x + \frac{\pi}{2}\right)\,\mathrm{d}x. \tag{2.20}$$

We see that g_1 is also smooth in a left neighbourhood of $\frac{\pi}{2}$ since $\frac{y-\frac{\pi}{2}}{\cos(y)}$ describes a smooth function and the integral term is smooth which is seen by an application of Lemma B.1 to the function $g_1(\cdot + \frac{\pi}{2})$. By the same arguments we show the claims for g_2, which yields surjectivity. Injectivity of $\mathbf{A}_1^{\uparrow}$ follows by assuming that $\widehat{f}_1 \equiv 0 \equiv \widehat{f}_2$ in (2.17) since smoothness of g_1, g_2 then dictates that $g_1 \equiv 0 \equiv g_2$.
The intertwining identity follows from the intertwining identity for the descent operator in Lemma 2.4.11 by composition with $\mathbf{A}_1^{\uparrow}$ from the left and from the right. $\widehat{\mathbf{L}}_3$ maps to $C^\infty_{even,0}([-\frac{\pi}{2}, \frac{\pi}{2}])^2$ since by the intertwining identity we have that $\mathrm{rg}(\widehat{\mathbf{L}}_3) = \mathrm{rg}\left(\mathbf{A}_1^{\uparrow}(\mathbf{L}_1 - \mathbf{I})(\mathbf{A}_1^{\uparrow})^{-1}\right) \subset \mathrm{rg}(\mathbf{A}_1^{\uparrow}) = C^\infty_{even,0}([-\frac{\pi}{2}, \frac{\pi}{2}])^2$. This ends the proof. □

This result enables us to turn our formal definition of the 3-dimensional descent operator into a rigorous one.

Corollary 2.4.13. *The 3-dimensional descent operator* $\widehat{\mathbf{D}}_3^{\downarrow} : C^\infty_{even,0}([-\frac{\pi}{2}, \frac{\pi}{2}])^2 \to C^\infty_{odd}([-\frac{\pi}{2}, \frac{\pi}{2}])^2$ *given by* $\widehat{\mathbf{D}}_3^{\downarrow}\widehat{\mathbf{f}} := \mathfrak{D}_3^{\downarrow}\widehat{\mathbf{f}}$ *is bijective with inverse* $\mathbf{A}_1^{\uparrow}$.

By radiality we extend this result to radial functions whose domain is 3-dimensional.

Corollary 2.4.14. *The 3-dimensional descent operator for 3-dimensional radial functions* $\mathbf{D}_3^{\downarrow} : C^\infty_{rad}\left(\overline{\mathbb{B}^3_{\frac{\pi}{2}}}\right)^2 \to C^\infty_{odd}([-\frac{\pi}{2}, \frac{\pi}{2}])^2$ *given by* $\mathbf{D}_3^{\downarrow}\mathbf{f} := \widehat{\mathbf{D}}_3^{\downarrow}\widehat{\mathbf{f}}$ *is bijective with inverse* $\breve{\mathbf{A}}_1^{\uparrow}$, *where* $\breve{\mathbf{A}}_1^{\uparrow} : C^\infty_{odd}([-\frac{\pi}{2}, \frac{\pi}{2}])^2 \to C^\infty_{rad}\left(\overline{\mathbb{B}^3_{\frac{\pi}{2}}}\right)^2$ *is the operator mapping a function tuple* $\mathbf{f}$ *to the 3-dimensional radial function* $\breve{\mathbf{A}}_1^{\uparrow}\mathbf{f}$ *whose radial representative is* $\mathbf{A}_1^{\uparrow}\mathbf{f}$, *i.e.* $\breve{\mathbf{A}}_1^{\uparrow}\mathbf{f} := \mathbf{A}_1^{\uparrow}\mathbf{f}(|\cdot|)$. *It satisfies the intertwining identity*

$$\mathbf{D}_3^{\downarrow}\mathbf{L}_3 - \mathbf{L}_1\mathbf{D}_3^{\downarrow} = -\mathbf{D}_3^{\downarrow},$$

where $\mathbf{L}_3 : C^\infty_{rad}\left(\overline{\mathbb{B}^3_{\frac{\pi}{2}}}\right)^2 \to C^\infty_{rad}\left(\overline{\mathbb{B}^3_{\frac{\pi}{2}}}\right)^2$ *is given by* $\mathbf{L}_3\mathbf{f} := \widehat{\mathbf{L}}_3\widehat{\mathbf{f}}(|\cdot|)$.

We want to use a similar construction for the next step, i.e. transferring the 3-dimensional result we just obtained into 5 dimensions. In contrast to the step from 1 to 3 dimensions it is not evident what the range of the 3-dimensional ascent actually is, which is due to the fact that regularity at the endpoints will not be fully preserved. Therefore we will not further specify it other than providing an overlying target space where we ensure that the range can be fully embedded into.

Lemma 2.4.15. *There is a linear operator, called the 3-dimensional ascent operator,* $\widehat{\mathbf{A}}_3^{\uparrow} : C^\infty_{even,0}([-\frac{\pi}{2}, \frac{\pi}{2}])^2 \to C^1_{even,0}\left([-\frac{\pi}{2}, \frac{\pi}{2}]\right)^2 \cap C^\infty_{even}((-\frac{\pi}{2}, \frac{\pi}{2}))^2$ *which is bijective onto its image and such that* $\widehat{\mathbf{A}}_3^{\uparrow} = (\mathfrak{D}_5^{\downarrow})^{-1}$. *In particular, the intertwining identity in Lemma 2.4.11 manifests itself as*

$$\widehat{\mathbf{L}}_5 \widehat{\mathbf{A}}_3^{\uparrow} = \widehat{\mathbf{A}}_3^{\uparrow} \widehat{\mathbf{L}}_3,$$

where $\widehat{\mathbf{L}}_5 : \mathrm{rg}(\widehat{\mathbf{A}}_3^{\uparrow}) \to \mathrm{rg}(\widehat{\mathbf{A}}_3^{\uparrow})$ *is given by* $\widehat{\mathbf{L}}_5 \widehat{\mathbf{f}} := \mathfrak{L}_5 \widehat{\mathbf{f}}$.

Proof. Since this is a longer proof we subdivide it into several steps.

(i) Construction of $\widehat{\mathbf{A}}_3^{\uparrow}$:

To construct $\widehat{\mathbf{A}}_3^{\uparrow}$, let $\widehat{\mathbf{g}} \in C^\infty_{even,0}([-\frac{\pi}{2}, \frac{\pi}{2}])^2$. We want to find $\widehat{\mathbf{f}}$ such that $\mathfrak{D}_5^{\downarrow} \widehat{\mathbf{f}} = \widehat{\mathbf{g}}$. That is

$$\begin{cases} 3\widehat{f}_1 + \sin(\cdot)\widehat{f}_1' + \frac{\cos(\cdot)-1}{\cos(\cdot)}\widehat{f}_2 = \widehat{g}_1, \\ 3\widehat{f}_2 + \sin(\cdot)\widehat{f}_2' + \frac{\cos(\cdot)-1}{\cos(\cdot)}[\mathfrak{L}_5\widehat{\mathbf{f}}]_2 = \widehat{g}_2. \end{cases} \tag{2.21}$$

Inserting the definition of $\mathfrak{L}_5$ and putting the first into the second equation yields

$$\begin{cases} \widehat{f}_2 & = \frac{\cos(\cdot)}{\cos(\cdot)-1}(-3\widehat{f}_1 - \sin(\cdot)\widehat{f}_1' + \widehat{g}_1), \\ \widehat{f}_1'' + \frac{4+\cos(\cdot)^2}{\cos(\cdot)\sin(\cdot)}\widehat{f}_1' + \frac{3(2-\cos(\cdot)^2)}{\cos(\cdot)^2\sin(\cdot)^2}\widehat{f}_1 & = G, \end{cases} \tag{2.22}$$

where $G(y) := \frac{1}{\cos(y)\sin(y)}\left(\frac{2-\cos(y)^2}{\cos(y)\sin(y)}\widehat{g}_1(y) + (2-\cos(y))\widehat{g}_1'(y) + \frac{1-\cos(y)}{\sin(y)}\widehat{g}_2(y)\right)$ denotes the non-linear part in the second equation. When solving the second equation towards $\widehat{f}_1$, we find the two fundamental solutions

$$\widehat{\phi}_{11}(y) := \frac{\cos(y)^3}{\sin(y)^3}, \qquad \widehat{\phi}_{12}(y) := \frac{\cos(y)^2(\cos(y)-1)}{\sin(y)^3}.$$

The Wronskian of $\widehat{\phi}_{11}$ and $\widehat{\phi}_{12}$ is $W(y) := \widehat{\phi}_{11}'(y)\widehat{\phi}_{12}(y) - \widehat{\phi}_{11}(y)\widehat{\phi}_{12}'(y) = -\frac{\cos(y)^4}{\sin(y)^5}$. Then variation of parameters yields the solution to the non-linear ODE

$$\begin{aligned} \widehat{f}_1(y) =& a_{11}\widehat{\phi}_{11}(y) + a_{12}\widehat{\phi}_{12}(y) - \widehat{\phi}_{11}(y)\int_0^y \frac{\widehat{\phi}_{12}(x)}{W(x)}G(x)\,\mathrm{d}x - \widehat{\phi}_{12}(y)\int_y^0 \frac{\widehat{\phi}_{11}(x)}{W(x)}G(x)\,\mathrm{d}x \\ =& a_{11}\widehat{\phi}_{11}(y) + a_{12}\widehat{\phi}_{12}(y) + \frac{\cos(y)^3}{\sin(y)^3}\int_0^y \frac{\cos(x)-1}{\cos(x)^3}H(x)\,\mathrm{d}x \\ &+ \frac{\cos(y)^2(\cos(y)-1)}{\sin(y)^3}\int_y^0 \frac{1}{\cos(x)^2}H(x)\,\mathrm{d}x, \end{aligned} \tag{2.23}$$

where $H(x) := \cos(x)\sin(x)^2 G(x) = \frac{2-\cos(x)^2}{\cos(x)}\widehat{g}_1(x) + \sin(x)(2-\cos(x))\widehat{g}_1'(x) + (1-\cos(x))\widehat{g}_2(x)$ and $a_{11}, a_{12} \in \mathbb{R}$ are to be specified later. From (2.22) we see that the second components of the fundamental system are

$$\widehat{\phi}_{21}(y) := \frac{\cos(y)}{\cos(y)-1}(-3\widehat{\phi}_{11}(y) - \sin(y)\widehat{\phi}_{11}'(y)) = \frac{3\cos(y)^3}{\sin(y)^3},$$
$$\widehat{\phi}_{22}(y) := \frac{\cos(y)}{\cos(y)-1}(-3\widehat{\phi}_{12}(y) - \sin(y)\widehat{\phi}_{12}'(y)) = \frac{\cos(y)^2(2-4\cos(y))}{\sin(y)^3}.$$

This in turn means that $\widehat{f}_2$ is given by

$$\begin{aligned}\widehat{f}_2(y) =& a_{21}\widehat{\phi}_{21}(y) + a_{22}\widehat{\phi}_{22}(y) + \widehat{\phi}_{21}(y)\int_0^y \frac{\widehat{\phi}_{12}(x)}{W(x)}G(x)\,\mathrm{d}x + \widehat{\phi}_{22}(y)\int_y^0 \frac{\widehat{\phi}_{11}(x)}{W(x)}G(x)\,\mathrm{d}x \\ &+ \frac{\cos(y)}{\cos(y)-1}\widehat{g}_1(y) \\ =& a_{21}\widehat{\phi}_{21}(y) + a_{22}\widehat{\phi}_{22}(y) + \frac{3\cos(y)^3}{\sin(y)^3}\int_0^y \frac{\cos(x)-1}{\cos(x)^3}H(x)\,\mathrm{d}x \\ &+ \frac{\cos(y)^2(2-4\cos(y))}{\sin(y)^3}\int_y^0 \frac{1}{\cos(x)^3}H(x)\,\mathrm{d}x + \frac{\cos(y)}{\cos(y)-1}\widehat{g}_1(y),\end{aligned} \tag{2.24}$$

with $a_{21}, a_{22} \in \mathbb{R}$. Hence to find $\widehat{\mathbf{A}}_3^{\uparrow}$, we consider the family of linear operators $(\mathbf{A}_{a_{11},a_{12},a_{21},a_{22}})_{a_{ij}\in\mathbb{R}}$ defined by (2.23) and (2.24). We note that there is some choice to be made now. Indeed, choosing the precise form of $\widehat{\mathbf{A}}_3^{\uparrow}$ is heavily connected to restricting the space in which $\widehat{\mathbf{A}}_3^{\uparrow}$ maps into, i.e. requiring some properties of the target functions. We want the target space to be so restrictive that there is only one possible quadruple $(a_{11}, a_{12}, a_{21}, a_{22})$ for which $\mathbf{A}_{a_{11},a_{12},a_{21},a_{22}}$ satisfies all the assumptions. For our purpose we require that the functions in the range of $\widehat{\mathbf{A}}_3^{\uparrow}$ should be smooth at 0. This is a sensible requirement, for we do not expect irregular behaviour in the interior of $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and the functions in the domain of $\widehat{\mathbf{A}}_3^{\uparrow}$ are assumed to be smooth in particular in the interior of $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We will see that this choice already yields a unique choice for $(a_{11}, a_{12}, a_{21}, a_{22})$. This is an interesting fact, which should be kept in mind, since this means that choosing the behaviour at 0 already dictates the behaviour at the boundary. We claim that $\mathbf{A}_{0,0,0,0}$ is the operator we are searching for. If we can show that $\mathbf{A}_{0,0,0,0}$ meets all the requirements, this already shows that this is the *only* choice for $(a_{11}, a_{12}, a_{21}, a_{22})$, since every added non-trivial linear combination of $\widehat{\phi}_{11}$ and $\widehat{\phi}_{12}$ and of $\widehat{\phi}_{21}$ and $\widehat{\phi}_{22}$ is singular at 0. So let us henceforth consider $A_{0,0,0,0}$.

(ii) $\widehat{\mathbf{A}}_3^{\uparrow} = \mathbf{A}_{0,0,0,0}$:

We start with the first component

$$\widehat{f}_1(y) = \frac{\cos(y)^3}{\sin(y)^3}\int_0^y \frac{\cos(x)-1}{\cos(x)^3}H(x)\,\mathrm{d}x + \frac{\cos(y)^2(\cos(y)-1)}{\sin(y)^3}\int_y^0 \frac{1}{\cos(x)^2}H(x)\,\mathrm{d}x, \tag{2.25}$$

with $H(x) = \frac{2-\cos(x)^2}{\cos(x)}\widehat{g}_1(x) + \sin(x)(2-\cos(x))\widehat{g}_1'(x) + (1-\cos(x))\widehat{g}_2(x)$. Note that H is an even function, which means that the integrands are and finally $\widehat{f}_1$ is. Thus for the smoothness claim made in the statement of the lemma we only need to show that $\widehat{f}_1|_{(-\frac{\pi}{2},\frac{\pi}{2})} \in C^\infty((-\frac{\pi}{2},\frac{\pi}{2}))$ and that $\widehat{f}_1$ is continuously differentiable in a left neighbourhood of $\frac{\pi}{2}$ with $\widehat{f}_1(\frac{\pi}{2}) = 0$. In the interior of $[0,\frac{\pi}{2}]$, $\widehat{f}_1$ is smooth as a composition of smooth functions. The task is to show smoothness at 0 and left continuous differentiability at $\frac{\pi}{2}$ with $\widehat{f}_1(\frac{\pi}{2}) = 0$. Towards this end, let us consider the part of $\widehat{f}_1$ that depends on $\widehat{g}_1$ and the one that depends on $\widehat{g}_2$ separately. For the $\widehat{g}_1$-part that is

$$\begin{aligned}
\widehat{f}_{11}(y) :=& \frac{\cos(y)^3}{\sin(y)^3}\int_0^y \frac{(\cos(x)-1)(2-\cos(x)^2)}{\cos(x)^4}\widehat{g}_1(x)\ \mathrm{d}x \\
&+ \frac{\cos(y)^3}{\sin(y)^3}\int_0^y \frac{(\cos(x)-1)\sin(x)(2-\cos(x))}{\cos(x)^3}\widehat{g}_1'(x)\ \mathrm{d}x \\
&+ \frac{\cos(y)^2(\cos(y)-1)}{\sin(y)^3}\int_y^0 \frac{(2-\cos(x)^2)}{\cos(x)^3}\widehat{g}_1(x)\ \mathrm{d}x \\
&+ \frac{\cos(y)^2(\cos(y)-1)}{\sin(y)^3}\int_y^0 \frac{\sin(x)(2-\cos(x))}{\cos(x)^2}\widehat{g}_1'(x)\ \mathrm{d}x.
\end{aligned}$$

Integration by parts in the terms with the derivative and the fact that the occurring boundary terms cancel out each other yield

$$\begin{aligned}
\widehat{f}_{11}(y) =& \frac{\cos(y)^3}{\sin(y)^3}\int_0^y \frac{-2(\cos(x)-1)(1+\sin(x)^2)}{\cos(x)^4}\widehat{g}_1(x)\ \mathrm{d}x \\
&+ \frac{\cos(y)^2(\cos(y)-1)}{\sin(y)^3}\int_y^0 \frac{\cos(x)-1-\sin(x)^2}{\cos(x)^3}\widehat{g}_1(x)\ \mathrm{d}x.
\end{aligned}$$

Since $\cos(\cdot)-1$ and $\sin(\cdot)^2$ vanish quadratically at zero, the first item of Lemma B.3 enables us to conclude that $\widehat{f}_{11}$ is smooth at 0. By the boundary assumption on $\widehat{g}_1$ we can invoke the second item of Lemma B.3 for the function $q := \widehat{f}_{11}(\cdot - \frac{\pi}{2})$. It yields that $q \in C^1(-\frac{\pi}{2}, 0])$, with $q(0) = 0$, which means that $\widehat{f}_{11} \in C^1((0,\frac{\pi}{2}])$ with $\widehat{f}_{11}(\frac{\pi}{2}) = 0$. We emphasize the important fact, that Lemma B.3(ii) can also be applied in the case where $\widehat{g}_1$ is only continuously differentiable instead of smooth in a left neighbourhood of $\frac{\pi}{2}$. This is of importance when generalizing this proof into higher dimensions. For the part of $\widehat{f}_1$ that depends on $\widehat{g}_2$ we compute

$$\begin{aligned}
\widehat{f}_{12}(y) :=& \frac{\cos(y)^3}{\sin(y)^3}\int_0^y \frac{-(1-\cos(x))^2}{\cos(x)^3}\widehat{g}_2(x)\ \mathrm{d}x \\
&+ \frac{\cos(y)^2(\cos(y)-1)}{\sin(y)^3}\int_y^0 \frac{1-\cos(x)}{\cos(x)^2}\widehat{g}_2(x)\mathrm{d}x.
\end{aligned}$$

Lemma B.3 again yields that $\widehat{f}_{12}(y)$ is smooth at 0. At $\frac{\pi}{2}$ we even have two times continuous differentiability with $\widehat{f}_{12}(\frac{\pi}{2}) = 0$. We conclude that $\widehat{f}_1$ is smooth at 0 as

a sum of the smooth functions $\widehat{f}_{11}$ and $\widehat{f}_{12}$ and continuously differentiable at $\frac{\pi}{2}$ with $\widehat{f}_1(\frac{\pi}{2}) = 0$.
Let us turn to the second component

$$\begin{aligned}\widehat{f}_2(y) =&\widehat{\phi}_{21}(y) \int_0^y \frac{\widehat{\phi}_{12}(x)}{W(x)} G(x) \, \mathrm{d}x + \widehat{\phi}_{22}(y) \int_y^0 \frac{\widehat{\phi}_{11}(x)}{W(x)} G(x) \, \mathrm{d}x + \frac{\cos(y)}{\cos(y)-1} \widehat{g}_1(y) \\ =& \frac{3\cos(y)^3}{\sin(y)^3} \int_0^y \frac{\cos(x)-1}{\cos(x)^3} H(x) \, \mathrm{d}x + \frac{\cos(y)^2(2-4\cos(y))}{\sin(y)^3} \int_y^0 \frac{1}{\cos(x)^3} H(x) \, \mathrm{d}x \\ &+ \frac{\cos(y)}{\cos(y)-1} \widehat{g}_1(y).\end{aligned}$$

From equation (2.22) we see that $\widehat{f}_1$ being even implies that $\widehat{f}_2$ is even, too. For the smoothness claimed, let us again consider the part that depends on $\widehat{g}_1$ first. That is

$$\begin{aligned}\widehat{f}_{21}(y) :=& \frac{3\cos(y)^3}{\sin(y)^3} \int_0^y \frac{(\cos(x)-1)(2-\cos(x)^2)}{\cos(x)^3} \widehat{g}_1(x) \, \mathrm{d}x \\ &+ \frac{3\cos(y)^3}{\sin(y)^3} \int_0^y \frac{\sin(x)(\cos(x)-1)(2-\cos(x))}{\cos(x)^3} \widehat{g}_1'(x) \, \mathrm{d}x \\ &+ \frac{\cos(y)^2(2-4\cos(y))}{\sin(y)^3} \int_y^0 \frac{2-\cos(x)^2}{\cos(x)^3} \widehat{g}_1(x) \, \mathrm{d}x \\ &+ \frac{\cos(y)^2(2-4\cos(y))}{\sin(y)^3} \int_y^0 \frac{\sin(x)(2-\cos(x))}{\cos(x)^2} \widehat{g}_1'(x) \, \mathrm{d}x \\ &+ \frac{\cos(y)}{\cos(y)-1} \widehat{g}_1(y).\end{aligned}$$

Since the integrands are the same in $\widehat{f}_{21}$ and $\widehat{f}_{11}$, partial integration yields the same integral term. The slight difference is that the boundary terms do not cancel out each other at y and we get

$$\begin{aligned}\widehat{f}_{21}(y) =& \frac{3\cos(y)^3}{\sin(y)^3} \int_0^y \frac{-2(\cos(x)-1)(1+\sin(x)^2)}{\cos(x)^4} \widehat{g}_1(x) \, \mathrm{d}x \\ &+ \frac{\cos(y)^2(2-4\cos(y))}{\sin(y)^3} \int_y^0 \frac{\cos(x)-1-\sin(x)^2}{\cos(x)^3} \widehat{g}_1(x) \, \mathrm{d}x \\ &+ \left(\frac{3(\cos(y)-1)(2-\cos(y))}{\sin(y)^2} - \frac{(2-4\cos(y))(2-\cos(y))}{\sin(y)^2} + \frac{\cos(y)}{\cos(y)-1} \right) \widehat{g}_1(y).\end{aligned}$$

A small computation shows that the coefficient of $\widehat{g}_1$ equals the non-singular $\frac{8\cos(y)-10}{\cos(y)+1}$. Thus we can conclude again by the first item in Lemma B.3 that $\widehat{f}_{21}$ is smooth at 0 and hence in all of $(-\frac{\pi}{2}, \frac{\pi}{2})$. The same arguments as for $\widehat{f}_{11}$ with the additional fact that $g_1(\frac{\pi}{2}) = 0$ show that $\widehat{f}_{21}$ is continuously differentiable at $\frac{\pi}{2}$ with $\widehat{f}_{21}(\frac{\pi}{2}) = 0$. For the part of $\widehat{f}_2$ that depends on $\widehat{g}_2$ we compute

$$\widehat{f}_{22}(y) := \frac{3\cos(y)^3}{\sin(y)^3}\int_0^y \frac{-(1-\cos(x))^2}{\cos(x)^3}\widehat{g}_2(x)\,\mathrm{d}x + \frac{\cos(y)^2(\cos(y)-1)}{\sin(y)^3}\int_y^0 \frac{1-\cos(x)}{\cos(x)^2}\widehat{g}_2(x)\,\mathrm{d}x,$$

and by the same arguments we conclude that $\widehat{f}_{22}$ is smooth at 0 and hence in $(-\frac{\pi}{2},\frac{\pi}{2})$, continuously differentiable at $\frac{\pi}{2}$ and $\widehat{f}_{22}(\frac{\pi}{2}) = 0$. Hence we can say the same for $\widehat{f}_2$ and this finally shows the desired result, namely that $\mathrm{rg}(\mathbf{A}_{0,0,0,0}) = C^\infty_{even}((-\frac{\pi}{2},\frac{\pi}{2}))^2 \cap C^1_{even,0}([-\frac{\pi}{2},\frac{\pi}{2}])^2$. Together with the short discussion before this step we thus found the desired operator and will henceforth write $\widehat{\mathbf{A}}_3^\uparrow$ for $\mathbf{A}_{0,0,0,0}$.

(iii) $\widehat{\mathbf{A}}_3^\uparrow$ is injective:

The last step is to show that $\widehat{\mathbf{A}}_3^\uparrow$ is injective. This is an easy consequence of the construction of $\widehat{\mathbf{A}}_3^\uparrow$ since we constructed it in a way such that it is a formal inverse of $\mathfrak{D}_5^\downarrow$ for functions defined in the interior of $[-\frac{\pi}{2}.\frac{\pi}{2}]$. Hence assuming that $\widehat{f}_1 \equiv 0 \equiv \widehat{f}_2$ in (2.21) together with the smoothness assumption on $\widehat{\mathbf{g}}$ is readily seen to imply that $\widehat{g}_1 \equiv 0 \equiv \widehat{g}_2$.

We conclude that $\widehat{\mathbf{A}}_3^\uparrow$ is bijective onto its range and we can define $\widehat{\mathbf{L}}_5$ the way we did, since like in the 3-dimensional case (cf. Lemma 2.4.12) its image is a subset of the range of the ascend operator as a consequence of the intertwining identity. This ends the proof. □

Again let us formulate this in terms of the descent operator.

Corollary 2.4.16. *The 5-dimensional descent operator* $\widehat{\mathbf{D}}_5^\downarrow : \mathrm{rg}(\widehat{\mathbf{A}}_3^\uparrow) \to C^\infty_{even,0}([-\frac{\pi}{2},\frac{\pi}{2}])$ *given by* $\widehat{\mathbf{D}}_5^\downarrow\widehat{\mathbf{f}} := \mathfrak{D}_5^\downarrow\widehat{\mathbf{f}}$ *is bijective with inverse* $\widehat{\mathbf{A}}_3^\uparrow$.

And also by radiality for 5-dimensional radial functions.

Corollary 2.4.17. *The 3-dimensional ascent operator for 3-dimensional radial functions* $\breve{\mathbf{A}}_3^\uparrow : C^\infty_{rad}\left(\overline{\mathbb{B}^3_{\frac{\pi}{2}}}\right)^2 \to C^\infty_{even}\left(\mathbb{B}^5_{\frac{\pi}{2}}\right)^2 \cap C^1_0\left(\overline{\mathbb{B}^5_{\frac{\pi}{2}}}\right)^2$ *mapping a function tuple* $\mathbf{f}$ *with radial representative* $\widehat{\mathbf{f}}$ *to the 5-dimensional radial function* $\breve{\mathbf{A}}_3^\uparrow\mathbf{f}$ *whose radial representative is* $\widehat{\mathbf{A}}_3^\uparrow\widehat{\mathbf{f}}$, *i.e.* $\breve{\mathbf{A}}_3^\uparrow\mathbf{f} := \widehat{\mathbf{A}}_3^\uparrow\widehat{\mathbf{f}}(|\cdot|)$, *is bijective onto its image. Its inverse* $\mathbf{D}_5^\downarrow : \mathrm{rg}(\breve{\mathbf{A}}_3^\uparrow) \to C^\infty_{rad}(\overline{\mathbb{B}^3_{\frac{\pi}{2}}})$ *given by* $\mathbf{D}_5^\downarrow\mathbf{g} := \widehat{\mathbf{D}}_5^\downarrow\widehat{\mathbf{g}}(|\cdot|)$ *satisfies the intertwining identity*

$$\mathbf{D}_5^\downarrow\mathbf{L}_5 = \mathbf{L}_3\mathbf{D}_5^\downarrow,$$

where $\mathbf{L}_5 : \mathrm{rg}(\breve{\mathbf{A}}_3^\uparrow) \to \mathrm{rg}(\breve{\mathbf{A}}_3^\uparrow)$ *is given by* $\mathbf{L}_5\mathbf{f} := \widehat{\mathbf{L}}_5\widehat{\mathbf{f}}(|\cdot|)$.

In general, for odd dimension $d \geq 7$ we can copy the proof of the 5-dimensional case, which is due to the fact that $\mathfrak{D}_d^\downarrow$ is almost the same as $\mathfrak{D}_5^\downarrow$ only differing in the first coefficient. The other difference is that we now already start with a function space which contains functions which do not have to be smooth but only continuously differentiable in $\frac{\pi}{2}$. In the proof of the 5-dimensional case we remarked that this does not change the result however. That is, the $(d-2)$-dimensional ascent preserves continuous differentiability at the endpoints. Let us present the result for general $d \geq 7$.

Lemma 2.4.18. *For odd $d \geq 5$ one can successively find a linear operator, called the d-dimensional ascent operator, $\widehat{\mathbf{A}}_d^{\uparrow} : \mathrm{rg}(\widehat{\mathbf{A}}_{d-2}^{\uparrow}) \subset C^1_{even,0}\left([-\frac{\pi}{2}, \frac{\pi}{2}]\right)^2 \cap C^{\infty}_{even}((-\frac{\pi}{2}, \frac{\pi}{2}))^2 \to C^1_{even,0}\left([-\frac{\pi}{2}, \frac{\pi}{2}]\right)^2 \cap C^{\infty}_{even}((-\frac{\pi}{2}, \frac{\pi}{2}))^2$, which is bijective onto its image such that $\widehat{\mathbf{A}}_d^{\uparrow} = (\mathfrak{D}_{d+2}^{\downarrow})^{-1}$. In particular, the intertwining identity in Lemma 2.4.11 manifests itself as*

$$\widehat{\mathbf{L}}_{d+2}\widehat{\mathbf{A}}_d^{\uparrow} = \widehat{\mathbf{A}}_d^{\uparrow}\widehat{\mathbf{L}}_d,$$

where $\widehat{\mathbf{L}}_{d+2} : \mathrm{rg}(\widehat{\mathbf{A}}_d^{\uparrow}) \to \mathrm{rg}(\widehat{\mathbf{A}}_d^{\uparrow})$ is given by $\widehat{\mathbf{L}}_{d+2}\widehat{\mathbf{f}} := \mathfrak{L}_{d+2}\widehat{\mathbf{f}}$.

Hence we can introduce the d-dimensional descent operator for $d \geq 7$.

Corollary 2.4.19. *For odd $d \geq 7$ the d-dimensional descent operator $\widehat{\mathbf{D}}_d^{\downarrow} : \mathrm{rg}(\widehat{\mathbf{A}}_{d-2}^{\uparrow}) \to \mathrm{rg}(\widehat{\mathbf{A}}_{d-4}^{\uparrow})$ given by $\widehat{\mathbf{D}}_d^{\downarrow}\mathbf{f} := \mathfrak{D}_d^{\downarrow}\mathbf{f}$ is bijective with inverse $\widehat{\mathbf{A}}_{d-2}^{\uparrow}$.*

And also by radiality for d-dimensional radial functions.

Corollary 2.4.20. *For odd $d \geq 5$, the d-dimensional ascent operator for d-dimensional radial functions successively defined by $\check{\mathbf{A}}_d^{\uparrow} : \mathrm{rg}(\check{\mathbf{A}}_{d-2}^{\uparrow}) \to C^{\infty}_{even}\left(\mathbb{B}^{d+2}_{\frac{\pi}{2}}\right)^2 \cap C^1_0\left(\overline{\mathbb{B}^{d+2}_{\frac{\pi}{2}}}\right)^2$ mapping a function tuple $\mathbf{f}$ with radial representative $\widehat{\mathbf{f}}$ to the (d+2)-dimensional radial function $\check{\mathbf{A}}_d^{\uparrow}\mathbf{f}$ whose radial representative is $\widehat{\mathbf{A}}_d^{\uparrow}\widehat{\mathbf{f}}$, i.e. $\check{\mathbf{A}}_d^{\uparrow}\mathbf{f} := \widehat{\mathbf{A}}_d^{\uparrow}\widehat{\mathbf{f}}(|\cdot|)$, is bijective onto its image. Its inverse $\mathbf{D}_{d+2}^{\downarrow} : \mathrm{rg}(\check{\mathbf{A}}_d^{\uparrow}) \to \mathrm{rg}(\check{\mathbf{A}}_{d-2}^{\uparrow})$ given by $\mathbf{D}_{d+2}^{\downarrow}\mathbf{g} := \widehat{\mathbf{D}}_{d+2}^{\downarrow}\widehat{\mathbf{g}}(|\cdot|)$ satisfies the intertwining identity*

$$\mathbf{D}_{d+2}^{\downarrow}\mathbf{L}_{d+2} = \mathbf{L}_d\mathbf{D}_{d+2}^{\downarrow},$$

where $\mathbf{L}_{d+2} : \mathrm{rg}(\check{\mathbf{A}}_d^{\uparrow}) \to \mathrm{rg}(\check{\mathbf{A}}_d^{\uparrow})$ is given by $\mathbf{L}_{d+2}\mathbf{f} := \widehat{\mathbf{L}}_{d+2}\widehat{\mathbf{f}}(|\cdot|)$.

2.4.3. Main result via full descent

It is finally time to lift the semigroups estimate to higher dimensions. To be able to formulate them, we need to introduce norms on the function spaces we found in the previous section. To this end, first let us summarize what we have shown so far by defining the *full descent operator*, which is just the successive application of each 2-step descent operator.

Lemma 2.4.21. *For odd $d \geq 3$, there is a bijective, linear operator $\mathbf{D}_d : \mathrm{rg}(\check{\mathbf{A}}_{d-2}^{\uparrow}) \to C^{\infty}_{odd}([-\frac{\pi}{2}, \frac{\pi}{2}])$ such that*

$$\mathbf{L}_d\mathbf{f} = \mathbf{D}_d^{-1}(\mathbf{L}_1 - \mathbf{I})\mathbf{D}_d\mathbf{f}. \tag{2.26}$$

Proof. Define $\mathbf{D}_d := \mathbf{D}_3^{\downarrow} \circ \mathbf{D}_5^{\downarrow} \circ \cdots \circ \mathbf{D}_d^{\downarrow}$. Successive application of the intertwining identities then show that

$$\mathbf{D}_d^{\downarrow} \mathbf{L}_d \mathbf{f} + \mathbf{D}_d^{\downarrow} \mathbf{f} = \mathbf{L}_3 \mathbf{D}_d^{\downarrow} \mathbf{f} + \mathbf{D}_d^{\downarrow} \mathbf{f} = \mathbf{L}_1 \mathbf{D}_d^{\downarrow} \mathbf{f}.$$

Subtracting $\mathbf{D}_d^{\downarrow} \mathbf{f}$ and applying the inverse of $\mathbf{D}_d^{\downarrow}$ then yields the claim. □

Note that $\mathbf{D}_d^{-1} = \breve{\mathbf{A}}_{d-2}^{\uparrow} \circ \breve{\mathbf{A}}_{d-4}^{\uparrow} \circ \cdots \circ \breve{\mathbf{A}}_1^{\uparrow}$. Now we lift the space $\mathcal{H}_1$ (cf. Def. 2.3.2) into higher dimensions using the above defined full descent operator. Since the full descent is linear and bijective the following definition indeed defines an inner product.

Definition 2.4.22. For odd $d \geq 3$ consider the inner product

$$(\mathbf{f} \mid \mathbf{g})_{\mathcal{H}_d} := (\mathbf{D}_d \mathbf{f} \mid \mathbf{D}_d \mathbf{g})_{\mathcal{H}_1}$$

on $\operatorname{rg}(\breve{\mathbf{A}}_{d-2}^{\uparrow})$ and denote by $\mathcal{H}_d$ the completion of the space, making it a Hilbert space. By $\|\cdot\|_{\mathcal{H}_d}$ we denote the norm induced by the inner product.

It is a straightforward functional analytical exercise to see that $\mathbf{D}_d$ extends to a homeomorphism from $\mathcal{H}_d$ to $\mathcal{H}_1$. For its construction, consider the equivalence class $\mathbf{f} \in \mathcal{H}_d$, i.e. there is a representative $(\mathbf{f}_k)_k \subset \operatorname{rg}(\breve{\mathbf{A}}_{d-2}^{\uparrow})$ of $\mathbf{f}$ such that $(\mathbf{f}_k)_k$ is Cauchy w.r.t. $\|\cdot\|_{\mathcal{H}_d}$. By definition of $\mathcal{H}_d$, $(\mathbf{D}_d \mathbf{f}_k)_k$ is a Cauchy sequence in $\mathcal{H}_1$, i.e. is a representative of some $\mathbf{g} \in \mathcal{H}_1$. One checks that setting $\mathbf{D}_d \mathbf{f} := \mathbf{g}$ then yields the desired well-defined, homeomorphic extension. Let us gather this fact in a corollary.

Corollary 2.4.23. *The full descent operator $\mathbf{D}_d$ extends to a homeomorphic operator in $(\mathcal{L}(\mathcal{H}_d, \mathcal{H}_1), \|\cdot\|_{\mathcal{O}(\mathcal{H}_d, \mathcal{H}_1)})$.*

The 1-dimensional semigroup estimate (cf. Lemma 2.3.3) then translates in the following way, producing the main result of this master thesis.

Theorem 2.4.24. *Let $d \geq 3$ be odd. The operator $\mathbf{L}_d : \operatorname{rg}(\breve{\mathbf{A}}_{d-2}^{\uparrow}) \subset \mathcal{H}_d \to \mathcal{H}_d$ is closable and its closure*

$$\overline{\mathbf{L}}_d = \mathbf{D}_d^{-1}(\overline{\mathbf{L}}_1 - \mathbf{I})\mathbf{D}_d, \qquad \mathcal{D}(\overline{\mathbf{L}}_d) = \mathbf{D}_d^{-1}\mathcal{D}(\overline{\mathbf{L}_1}) \tag{2.27}$$

generates the strongly continuous semigroup

$$\mathbf{S}_d(s) = e^{-s}\mathbf{D}_d^{-1}\mathbf{S}_1(s)\mathbf{D}_d \tag{2.28}$$

with growth bound

$$\|\mathbf{S}_d(s)\mathbf{f}\|_{\mathcal{H}_d} \leq e^{-s}\|\mathbf{f}\|_{\mathcal{H}_d} \tag{2.29}$$

for all $\mathbf{f} \in \mathcal{H}_d, s \geq 0$.

Proof. By Lemma 2.4.21 and Lemma A.12 we have that $\overline{\mathbf{L}}_d = \mathbf{D}_d^{-1}(\overline{\mathbf{L}}_1 - \mathbf{I})\mathbf{D}_d$ with $\mathcal{D}(\overline{\mathbf{L}}_d) = \{\mathbf{f} \in \mathcal{H}_d : \mathbf{D}_d\mathbf{f} \in \mathcal{D}(\overline{\mathbf{L}}_1)\}$. Hence $\overline{\mathbf{L}}_d$ generates the conjugated, rescaled semigroup $\mathbf{S}_d(s) = e^{-s}\mathbf{D}_d^{-1}\mathbf{S}_1(s)\mathbf{D}_d$ by Lemmata 1.3.16 and 1.3.17. The growth bound then follows from the growth bound of $\mathbf{S}_1$ in Lemma 2.3.3 by

$$\begin{aligned}\|\mathbf{S}_d(s)\mathbf{f}\|_{\mathcal{H}_d} = e^{-s}\left\|\mathbf{D}_d^{-1}\mathbf{S}_1(s)\mathbf{D}_d\mathbf{f}\right\|_{\mathcal{H}_d} &= e^{-s}\left\|\mathbf{S}_1(s)\mathbf{D}_d^{-1}\mathbf{f}\right\|_{\mathcal{H}_1} \\ &\leq e^{-s}\left\|\mathbf{D}_d^{-1}\mathbf{f}\right\|_{\mathcal{H}_1} = e^{-s}\|\mathbf{f}\|_{\mathcal{H}_d}\end{aligned}$$

for all $s \geq 0$. □

3. Outlook

As mentioned in the introduction, estimating the free wave evolution is but a starting point for the actual analysis of the underlying geometric wave equation. This chapter will serve as a future prospect of the next steps. Since this is still currently researched by the author, this chapter should rather be seen as a heuristic explanation of the main ideas.

3.1. Understanding the main result

While Theorem 2.4.24 gives a full description of the free wave evolution in every odd dimension, it needs some work to be unpacked into an explicit form. For one that is, giving an explicit description of the Hilbert spaces $\mathcal{H}_d$ involved in the estimates in Theorem 2.4.24. Remember that norms on these Hilbert spaces were introduced implicitly as the norm of the explicit Hilbert space $\mathcal{H}_1$ after application of the descent operator. Since $\mathcal{H}_1$ was introduced as a weighted L^2-based Hilbert space we expect that $\mathcal{H}_d$ is also some weighted L^2- based Hilbert space. However finding its explicit form proves more difficult than one may anticipate. The solution to this problem would enable one to really calculate with these norms. Connected to this may also be an explicit description of the range of the ascend operators, instead of only providing an overlying function space the way we did. Completing both of these steps would end the analysis on the free wave equation and one would be ready to introduce the geometric non-linearity into the analysis.

3.2. Introduction of the non-linearity

This will most likely be done by invoking Duhamel's formula to dissect the problem into the linear and non-linear part. The analysis on the linear part is then readily accessible by the results in this work while the analysis of the non-linear part generally is handled by a fix point argument. It is hoped and expected that these topics appear in future works by the author.

Appendix

A. Functional analytical background of semigroup theory

Banach space valued differentiation and integration

When working with semigroups, not only differentiation, which is easily understood as the limit of a difference quotient, but also integration can be very useful. There are several ways to extend scalar integration to its Banach-valued counterpart. When wanting to extend the Lebesgue integral the most widely used notion is the so called Bochner integral. However, for our purpose, we will be satisfied with an extension of the Riemann integral. Just like in the scalar case, in the Banach valued case we define the integral as the limit of Riemann sums. Since we are only working with continuous operators (either $T : \mathbb{R}_+ \to \mathcal{L}(X), t \mapsto T(t)$ or $T_x : \mathbb{R}_+ \to X, t \mapsto T(t)x$) and we are only interested in integrals on a compact interval, the same arguments for continuous functions in the scalar case, show that this limit exists. All properties of Riemann integrals used in this paper are proved exactly as in the scalar case, just by replacing the absolute values by the respective norms. We list them in the following. The interested reader is invited to delve into a more involved discussion of this topic in e.g. [[9], III.].

Definition A.1. Suppose E is a Banach space and consider an operator $S : \mathbb{R}_+ \to E$.

- We define formal differentiation by

$$\dot{S}(t) := \frac{\mathrm{d}}{\mathrm{d}t}S(t) := \lim_{h\to 0}\frac{1}{h}(S(t+h) - S(t)),$$

  where the limiting process takes place in the topology of E. If this limit exists, we call S differentiable at t. The usual terminology concerning differentiability follows. We will use the dot notation interchangeably with $\frac{\mathrm{d}}{\mathrm{d}t}$ and ∂_t. We also have the *product rule* $\partial_t(T(t)S(t)) = \dot{T}(t)S(t) + T(t)\dot{S}(t)$ where we remind the reader that contrary to the name of this rule, the operators are composed and not multiplied.

- Let $t_0 \geq 0$. A tagged partition P of $[0, t_0]$ is a tuple of finite sequences $(x_k)_{k=0}^n$ and $(s_k)_{s=0}^{n-1}$ such that $0 = x_0 < x_1 < \cdots < x_n = t_0$ and $s_k \in [x_k, x_{k+1}]$. The mesh of P, denoted by $|P|$, is the length of the largest interval, i.e. $\max_{1\leq i\leq n}(x_i - x_{i-1})$. A refinement of P is a tagged partition $Q = ((y_k)_{k=0}^m, (r_k)_{k=0}^{m-1})$ of $[0, t_0]$ such that $m > n$ and for every $0 \leq i \leq n$ we have $x_i \in (y_k)_{k=0}^m$ and $s_i \in (r_k)_{k=0}^{m-1}$. The Riemann sum of a Banach-valued function $S : [0, t_0] \to E$ with respect to P is defined as

$$\sum(S, P) := \sum_{k=0}^{n-1} S(s_k)(x_{k+1} - x_k).$$

The formal Riemann integral of S, denoted by $\int_0^{t_0} S(t)\ \mathrm{d}t$, is the limit of the Riemann sums in E, when the partitions get finer. That is

$$\int_0^{t_0} S(t)\ \mathrm{d}t := \lim_{n\to\infty} \sum(S, P_n)$$

where $(P_n)_n$ is a refining sequence, that is a sequence of tagged partitions such that each partition refines its predecessor and $|P_n| \xrightarrow{n} 0$. If this limit exists and is independent of the refining sequence used, we call S integrable on $[0, t_0]$.

Corollary A.2. *Suppose $S : \mathbb{R}_+ \to E$ is continuous. Consistent with the scalar case the integral admits its usual properties (linearity, triangle inequality, etc.) and in particular the following properties.*

- *S is integrable on $[0, t_0]$ for all $t_0 > 0$.*
- *The fundamental theorem of calculus.*
- *The dominated convergence theorem.*
- *Suppose $E = (\mathcal{L}(X), \|\cdot\|_{\mathcal{O}(X)})$ for a Banach space X. It follows that $S_x : \mathbb{R}_+ \to X, t \to S(t)x$ is continuous for every $x \in X$. We also have that $(\int_0^{t_0} S(t)\ \mathrm{d}t)x = \int_0^{t_0} S(t)x\ \mathrm{d}t$. Note that this is non-trivial fact, since on the one side the limiting process of the Riemann sums happens in $L(X)$ while on the other side it happens in X.*

We collect one important convergence property of the integral which we will use several times.

Lemma A.3. *Suppose X is a Banach space. Let $(T(t))_{t\geq 0}$ be a uniformly continuous semigroup and $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on X. Then for every $t \geq 0$*

$$\lim_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} T(s)\ \mathrm{d}s = T(t)$$

and

$$\lim_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} S(s)x\ \mathrm{d}s = S(t)x$$

for every $x \in X$.

Proof. We only prove it in the case of uniform continuity, since the other case is a straightforward adjustment. For $h > 0$, we note that $\frac{1}{h}\int_t^{t+h} T(s)\ \mathrm{d}s = T(t)\frac{1}{h}\int_0^h T(s)\ \mathrm{d}s$ from which it suffices to prove $\lim_{h\downarrow 0}\frac{1}{h}\int_0^h T(s)\ \mathrm{d}s = I$ to show the claim. We compute

$$\left\|\frac{1}{h}\int_0^h T(s)\ \mathrm{d}s - I\right\|_{\mathcal{O}(X)} = \frac{1}{h}\left\|\int_0^h T(s)\ \mathrm{d}s - hI\right\|_{\mathcal{O}(X)} = \frac{1}{h}\left\|\int_0^h T(s) - I\ \mathrm{d}s\right\|_{\mathcal{O}(X)}$$
$$\leq \frac{1}{h}\int_0^h \sup_{s'\in[0,h]}\left\|T(s') - I\right\|_{\mathcal{O}(X)}\ \mathrm{d}s \leq \sup_{s'\in[0,h]}\left\|T(s') - I\right\|_{\mathcal{O}(X)},$$

which converges to 0 for $h \downarrow 0$, since $(T(t))_{t\geq 0}$ was assumed to be a uniformly continuous semigroup. In the case of strong continuity, one can do the same computation pointwise with the appropriate norm. □

Exponential operators

Lemma A.4. *For a bounded operator $A : X \to X$ on a Banach space X define for every $t \in \mathbb{R}$,*

$$e^{tA} := \sum_{k=0}^{\infty}\frac{t^k}{k!}A^k.$$

For every $t \in \mathbb{R}$, the series converges in $(\mathcal{L}(X), \|\cdot\|_{\mathcal{O}(X)})$ with the estimate

$$\left\|e^{tA}\right\|_{\mathcal{O}(X)} \leq e^{|t|\|A\|_{\mathcal{O}(X)}}.$$

Proof. Consider for $N \in \mathbb{N}$ the partial sum $S_N := \sum_{k=0}^N \frac{t^k}{k!}A^k$. For every $M < N \in \mathbb{N}$ we have by the submultiplicativity of the operator norm

$$\|S_N - S_M\|_{\mathcal{O}(X)} \leq \sum_{k=M+1}^{N}\frac{|t|^k}{k!}\|A\|_{\mathcal{O}(X)}^k.$$

The later converges to 0 for $M, N \to \infty$, since it can be estimated by the tail of $\sum_{k=0}^\infty \frac{|t|^k}{k!}\|A\|_{\mathcal{O}(X)}^k = e^{|t|\|A\|_{\mathcal{O}(X)}}$. Hence $(S_N)_N$ is Cauchy, thus converges, for $(\mathcal{L}(X), \|\cdot\|_{\mathcal{O}(X)})$ is complete. Its limit e^{tA} then satisfies $\left\|e^{tA}\right\|_{\mathcal{O}(X)} \leq e^{|t|\|A\|_{\mathcal{O}(X)}}$, which is seen by defining $S_{-1} := 0$ and taking $M = -1$ and $N \to \infty$ in the above inequality. □

Lemma A.5. *Let X be a Banach space. For bounded operators $B_1, B_2 \in \mathcal{L}(X)$ for which $B_1B_2 = B_2B_1$, it holds that*

$$e^{B_1}e^{B_2} = e^{B_1+B_2}$$

In particular for a bounded operator $A \in \mathcal{L}(X)$,

$$e^{(t+s)A} = e^{tA}e^{sA}.$$

Proof. With the same steps as in the scalar case, one shows that for bounded, linear operators C, D of the form

$$C = \sum_{k=0}^{\infty} C_k, D = \sum_{k=0}^{\infty} D_k,$$

such that at least one of the series absolutely converges, the formula for the Cauchy product

$$CD = \sum_{k=0}^{\infty} \sum_{i=0}^{k} C_{k-i} D_i$$

holds. With this we observe

$$e^{B_1} e^{B_2} = \sum_{k=0}^{\infty} \frac{1}{k!} B_1^k \sum_{k=0}^{\infty} \frac{1}{k!} B_2^k = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{1}{(k-i)!} B_1^{k-i} \frac{1}{i!} B_2^i = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} B_1^{k-i} B_2^i$$
$$= e^{B_1 + B_2},$$

where the last step follows from the binomial formula for operators, which holds when the operators in question commute. This ends the proof. □

Uniform boundedness

The following fundamental result in functional analysis establishes a connection between pointwise boundedness and uniform boundedness for linear operators, i.e. strong continuity to uniform continuity. It is often referred to as the uniform boundedness principle or Banach-Steinhaus Theorem after its original publishers. The following formulation is a special result where the operators act on Banach spaces, in which case the result can be formulated rather strongly. For the proof of this well-known result we refer to reader to any literature covering elementary functional analysis, for example ([10], p.45, Theorem 2.6).

Lemma A.6 (uniform boundedness principle)**.** *Suppose X, Y are Banach spaces and let $(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{O}(X)})$ be the space of all bounded, linear operators from X to Y. Suppose $F \subset \mathcal{L}(X,Y)$ is a collection of bounded, linear operators. Then if*

$$\sup_{T \in F} \|Tx\|_Y < \infty \text{ for all } x \in X,$$

already

$$\sup_{T \in F} \|T\|_{\mathcal{O}(X,Y)} < \infty.$$

The next lemma is of similar nature, in that it also establishes a stronger form of convergence by a seemingly weaker form. To be more precise, it shows that in the event of a uniform bound convergence in the uniform and also strong operator norm is already given by convergence on a dense subset.

Lemma A.7. *Let $t_0 > 0$ be such that $\{S_n(t)\}_{n\in\mathbb{N},t\in[0,t_0]}$ is a family of uniformly bounded operators on a Banach space X, i.e. there exists an $M > 0$ such that for all $t \in [0, t_0], n \in \mathbb{N}$*

$$\|S_n(t)\|_{\mathcal{O}(X)} \leq M. \tag{.1}$$

Let $D \subset X$ be a dense subspace. Then

(i) Let $t \in [0, t_0]$. For $n \to \infty$, $S_n(t)$ converges strongly in $\mathcal{L}(X, X)$ iff $S_n(t)$ converges strongly in $\mathcal{L}(D, X)$.

(ii) For all $x \in X$, for $n \to \infty$, $S_n(\cdot)x$ converges uniformly on $[0, t_0]$ iff it does so for all $x \in D$.

Proof. In both cases we only need to prove from right to left. For (i) let $t \in [0, t_0]$ and let us denote by $S(t)$ the element in $\mathcal{L}(D, X)$ to which $S_n(t)$ strongly converges. The operator norm of $S(t)$ (on D) is bounded by M since

$$\begin{aligned}\|S(t)\|_{\mathcal{O}(D,X)} = \sup_{\|x\|_X=1,x\in D} \|S(t)x\|_X \leq \sup_{\|x\|_X=1} \left\|\lim_{n\to\infty} S_n(t)x\right\|_X &\leq \lim_{n\to\infty} \sup_{\|x\|_X=1} \|S_n(t)x\|_X \\ &\leq M,\end{aligned}$$

as a consequence of (.1). We claim that $S(t)$ can be extended to all of X by $S(t)x := \lim_{m\to\infty} S(t)x_m$ for a sequence $(x_m)_m \subset D$ converging to $x \in X$. Existence of this limit follows from the completeness of X by noticing that $(S(t)x_m)_m$ is Cauchy, since $S(t)$ is bounded on D. $S(t)$ being bounded on D also implies that the limit is independent of the converging sequence used. Hence $S(t)$ is well-defined. One observes that by taking limits, the operator norm of $S(t)$ is bounded by M on X. We claim that the extended $S(t)$ is the strong limit of $S_n(t)$ in $\mathcal{L}(X, X)$. Indeed, for $D \supset (x_m)_m \to x \in X$, by triangle inequality we have

$$\begin{aligned}\|S(t)x - S_n(t)x\|_X \leq& \|S(t)x - S(t)x_m\|_X + \|S(t)x_m - S_n(t)x_m\|_X \\ &+ \|S_n(t)x_m - S_n(t)x\|_X \\ \leq& 2M \|x - x_m\|_X + \|S(t)x_m - S_n(t)x_m\|_X .\end{aligned} \tag{.2}$$

Since we assumed strong convergence on D the expression on the right hand side can be made arbitrary small by choosing m and n large enough. Hence we have proven that $S_n(t)$ attains a strong limit even in $\mathcal{L}(X, X)$, which is what is needed for (i). The proof for (i) is easily adjusted to prove (ii), by again using (.2) to translate convergence on all of X to convergence only on D. □

Closed operators

Definition A.8. Let $A : \mathcal{D}(A) \subset X \to Y$ be a linear operator between Banach spaces X and Y. Define the graph of A as the set

$$G(A) := \{(x, Ax) : x \in \mathcal{D}(A)\}.$$

A is called closed, if $G(A)$ is closed in $X \times Y$ with respect to the product topology.

For operators on Banach spaces, an equivalent description of closedness is given by sequential closedness. That is, A is closed if and only if for every sequence $(x_n)_n \subset \mathcal{D}(A)$ which converges to some $x \in X$ and for which $(Ax_n)_n$ converges to some $y \in Y$, it must be that $x \in \mathcal{D}(A)$ and $Ax = y$. In the case of operators which are bounded on their domain, it is equivalent to look for closedness of the domain.

Lemma A.9. *Let $A : \mathcal{D}(A) \subset X \to Y$ be a linear operator between Banach spaces X and Y. Suppose A is bounded, i.e. there exists $M > 0$ such that $\|Ax\|_Y \leq M \|x\|_X$ for all $x \in \mathcal{D}(A)$. The following are equivalent.*

(i) $\mathcal{D}(A)$ is closed.

(ii) A is closed.

Proof. $(i) \Rightarrow (ii)$: Suppose $\mathcal{D}(A) \ni x_n \xrightarrow{n\to\infty} x \in X$ and $Ax_n \xrightarrow{n\to\infty} y \in Y$. Since $\mathcal{D}(A)$ is closed we have $x \in \mathcal{D}(A)$ and $\|y - Ax\|_Y = \lim_{n\to\infty} \|A(x_n - x)\|_Y = 0$, by boundedness of A. Hence A is closed.
$(ii) \Rightarrow (i)$: Let $\mathcal{D}(A) \ni x_n \xrightarrow{n\to\infty} x \in X$. Since A is bounded we have that $(Ax_n)_n$ is a Cauchy sequence. It converges since Y is complete. Since A is closed, we conclude that $x \in \mathcal{D}(A)$. □

We include the following well-known result in functional analysis, called the closed graph theorem. It establishes a very important connection between closed operators and continuous operators. For the proof we refer the interested reader to ([10], pp. 50-51, Proposition 2.14, Theorem 2.15).

Lemma A.10 (closed graph theorem)**.** *Suppose $A : X \to Y$ is a linear operator between Banach spaces X and Y. The following are equivalent:*

(i) A is closed.

(ii) A is bounded.

An operator A is called closable if there exists a closed extension, i.e. a closed operator B with $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $B|_{\mathcal{D}(A)} = A$. One readily sees, by sequential closedness, that this is exactly the case whenever for two sequences in $\mathcal{D}(A)$, $(x_n^1)_n, (x_n^2)_n$ which converge to some $x \in X$, and for which the limits of $(Ax_n^1)_n, (Ax_n^2)_n$ exist, these limits must coincide. The minimal closed extension is called the closure of A, denoted by $\overline{A}$. The procedure to obtain this operator should be intuitive. Take the closure of $G(A)$ and define $\overline{A}$ via the added points. That is, suppose $\mathcal{D}(A) \ni x_n \to x$ such that $(Ax_n)_n$ converges. Then $x \in \mathcal{D}(\overline{A})$ with $\overline{A}x := \lim_{n\to\infty} Ax_n$, which is well-defined in the case of closability. Then $\overline{A}$ is necessarily a closed operator with $G(\overline{A}) = \overline{G(A)}$. By linearity, that is by considering the sequence $(x_n^1 - x_n^2)_n$, one can restrict the assumption above to null sequences. We collect this fact in the following corollary.

Corollary A.11. *Suppose $A : \mathcal{D}(A) \subset X \to Y$ is a linear operator between Banach spaces X and Y. A is closable iff for every sequence $(x_n)_n \subset \mathcal{D}(A)$ for which $x_n \to 0$ and existence of the limit $Ax_n \to y \in Y$ are satisfied, we have $y = 0$. If A is closable, its closure $\overline{A}$ satisfies $G(\overline{A}) = \overline{G(A)}$.*

The following Lemma states that closability is stable under similarity.

Lemma A.12. *Let $C : Y \to X$ be a linear homeomorphism between Banach spaces X, Y and suppose $B : \mathcal{D}(B) \subset X \to X$ is a closable linear operator. Then the operator A defined by*

$$A := C^{-1}BC : \mathcal{D}(A) \subset Y \to Y,$$

$\mathcal{D}(A) := \{y \in Y : Cy \in \mathcal{D}(B)\}$, is closable with

$$\overline{A} = C^{-1}\overline{B}C,$$

and $\mathcal{D}(\overline{A}) = \{y \in Y : Cy \in \mathcal{D}(\overline{B})\}$.

Proof. By the above corollary, for closability it suffices to consider null sequences. Therefore assume $\mathcal{D}(A) \ni y_n \to 0$ such that $\lim_{n\to\infty} Ay_n =: z$ exists. We need to show that $z = 0$. By definition of $\mathcal{D}(A)$ we have that $(Cy_n)_n \subset \mathcal{D}(B)$. Since C is assumed to be continuous and linear it follows that $Cy_n \xrightarrow{n\to\infty} 0$ and that $Cz = C\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} BCy_n$. In particular, $\lim_{n\to\infty} BCy_n$ exists. Hence $\lim_{n\to\infty} BCy_n = 0$, for B is assumed to be closable. By continuity and linearity of C^{-1} we conclude $z = 0$.
With very similar arguments one proves that conjugation with a linear homeomorphism preserves closedness. That means, $C^{-1}\overline{B}C|_{\{y\in Y:Cy\in\mathcal{D}(\overline{B})\}}$ is indeed closed, since $\overline{B}$ is closed. Hence what is left to prove to show that $\overline{A}$ is indeed of the given form, is that $C^{-1}\overline{B}C|_{\{y\in Y:Cy\in\mathcal{D}(\overline{B})\}}$ is the minimal closed extension of B. Assuming the contrary, conjugation with $C^{-1}|_{\{x\in X:x\in\mathcal{D}(\overline{B})\}}$ would generate a closed extension of B whose graph was a proper subset of the graph of $\overline{B}$, contradicting the minimality of $\overline{B}$. This finishes the proof. □

Closedness is an assurance that the operator in question is in several ways well-behaved. While closedness and continuity do not imply each other in general, the closed graph theorem (Lemma A.10) states that they do, if the operator is everywhere-defined. In the general case, operators satisfying one or the other still share nice properties. The inverse of a closed, injective operator defined on the operators range is necessarily closed again, which is a simple consequence of the fact that their graphs are flipped versions of each other. This gains its significance when trying to develop a meaningful spectral theory. Indeed, suppose A is a closed operator such that $\lambda - A$ is bijective for some $\lambda \in \mathbb{C}$. Since $\lambda - A$ is necessarily closed too, injectivity implies that its inverse is closed. Since $(\lambda - A)^{-1}$ is everywhere-defined, for $\lambda - A$ is surjective, the closed graph theorem states that it must be bounded. Conversely, suppose that $\lambda \in \mathbb{C}$ is in the resolvent set of A, i.e. $\lambda - A$ is bijective and its inverse is bounded. The other direction of the closed graph theorem then implies that $(\lambda - A)^{-1}$ must be closed. It in particular being injective, means that $\lambda - A$, hence also A, are closed. We summarize the above in the following corollary.

Corollary A.13. *Let $A : \mathcal{D}(A) \subset X \to X$ be a linear operator on a Banach space X. Suppose there exists $\lambda \in \mathbb{C}$ such that $\lambda - A$ is bijective. The following are equivalent.*

(i) A is closed.

(ii) $(\lambda - A)^{-1}$ is bounded.

Having developed the connection between closedness and resolvents, let us collect some facts about the latter.

Resolvents

Lemma A.14. *Let $(A, \mathcal{D}(A))$ be a linear operator on a Banach space X. For $\lambda \in \rho(A)$ there is on $\mathcal{D}(A)$ the commutative property*

$$AR(\lambda, A) = R(\lambda, A)A.$$

Proof. We have that $\mathcal{D}(A) = \mathcal{D}(\lambda - A)$ and $R(\lambda, A)$ maps $\mathcal{D}(\lambda - A)$ to itself, for it is the inverse of $\lambda - A$. Hence for $x \in \mathcal{D}(A)$ we compute

$$\begin{aligned}\lambda R(\lambda, A)x - R(\lambda, A)Ax = R(\lambda, A)(\lambda - A)x = x &= (\lambda - A)R(\lambda, A)x \\ &= \lambda R(\lambda, A)x - AR(\lambda, A)x.\end{aligned}$$

Subtraction of $\lambda R(\lambda, A)x$ yields the desired result. □

Lemma A.15. *Let $(A, \mathcal{D}(A))$ be a linear operator on a Banach space X. Suppose $\lambda \in \rho(A)$. Then for every $\mu \in \mathbb{C}$ such that $|\mu - \lambda| < 1/\left\|R(\lambda, A)\right\|_{Op}$ it follows that $\mu \in \rho(A)$ with*

$$R(\mu, A) = \sum_{k=0}^{\infty} (\mu - \lambda)^k R(\lambda, A)^{k+1}.$$

Proof. Since

$$\mu - A = \mu - \lambda + \lambda - A = ((\mu - \lambda)R(\lambda, A) - I)(\lambda - A)$$

is invertible for $\left\|(\mu - \lambda)R(\lambda, A)\right\|_{Op} < 1$, which meets the assumption, the inverse is given by

$$(\mu - A)^{-1} = \sum_{k=0}^{\infty} (\mu - \lambda)^k R(\lambda, A)^k R(\lambda, A).$$

We use Corollary A.13 first for λ from (ii) to (i) to conclude that A is closed and then for μ from (i) to (ii) to conclude that $(\mu - A)^{-1}$ is bounded. Hence $\mu \in \rho(A)$ and we are allowed to write $R(\mu, A)$ for the above sum, which finishes the proof. □

B. Smoothness and singularities

Lemma B.1. *Let $g \in C^{k+1}(\mathbb{R})$. Then the function f defined by*

$$f(y) := \int_0^1 g'(yx) \, \mathrm{d}x$$

is k- times continuously differentiable.

Proof. We first show that f is continuously differentiable. Towards this end, consider for $0 < h < 1$

$$\int_0^1 \frac{g'((y+h)x) - g'(yx)}{h} \mathrm{d}x.$$

Since for all $x \in [0,1], y \mapsto g'(yx)$ is continuously differentiable, the mean value theorem states that there is $\xi \in [y, (y+h)]$ such that $\frac{g'((y+h)x)-g'(yx)}{h} = g''(\xi x)x$. For all $h < 1$ this is uniformly bounded on $[0,1]$ by $C := \sup_{x \in [0,1]} \|g''\|_{L^\infty([yx,(y+1)x])}$. Since constant functions are integrable on $[0,1]$ the dominated convergence theorem tells us that we can exchange limit and integration and thus we get

$$f'(y) = \int_0^1 g''(yx)x \, \mathrm{d}x,$$

which describes a continuous function. For general $k \in \mathbb{N}$ we get by the same arguments that

$$f^{(k)}(y) = \int_0^1 g^{(k+1)}(yx)x^k \, \mathrm{d}x.$$

Hence f is as smooth as claimed. □

Lemma B.2. *Suppose $g \in C^\infty(\mathbb{R})$ has a zero of order $k \in \mathbb{N}$ at $a \in \mathbb{R}$. Then there exists $f \in C^\infty(\mathbb{R})$ such that $g(y) = (y-a)^k f(y)$ and $f(a) \neq 0$.*

Proof. Otherwise considering the function $g(\cdot - a)$ we can w.l.o.g. assume that $a = 0$. By the fundamental theorem of calculus and the change of variables $\Phi(x) = yx$, we write

$$g(y) = \int_0^y g'(x) \, \mathrm{d}x = y \int_0^1 g'(yx) \, \mathrm{d}x.$$

By Lemma B.1, we have that $f_1(y) := \int_0^1 g'(yx) \, \mathrm{d}x$ is smooth. We also note that $f_1(y) = g(y)/y$ has a zero of order $k-1$ at 0. Repeating this procedure k times yields a smooth f_k such that $g(y) = y^k f_k(y)$. It follows that $f_k(0) \neq 0$ and we are done. □

Lemma B.3. *Let $m, n \in \mathbb{N}$ such that $-m + n \geq -1$. Suppose $g, h \in C^\infty((-\frac{\pi}{2}, \frac{\pi}{2}))$ such that g, h have zeroes at 0 of order m, n respectively.*

(i) *Then* f *defined by*

$$f(y) := \frac{1}{g(y)} \int_0^y h(x) \, \mathrm{d}x,$$

is smooth at 0.

(ii) *Suppose additionally* $n \geq 1$, $a < 0$ *and let* $p \in C^1((-\frac{\pi}{2}, 0])$ *with* $p(0) = 0$. *Then the function* k *defined by*

$$k(y) := h(y) \int_a^y \frac{p(x)}{g(x)} \, \mathrm{d}x$$

is left continuously differentiable at 0 *with* $k(0) = 0$.

Proof. (i): By smoothness and Lemma B.2 there are smooth functions ϕ_g, ϕ_h which do not vanish in a neighbourhood U of 0 such that $g(y) = y^m \phi_g(y), h(y) = y^n \phi_k(y)$. Hence via the substitution $\Phi(x) = yx$ we get

$$f(y) = \frac{1}{y^m \phi_g(y)} \int_0^y x^n \phi_h(x) \, \mathrm{d}x = \frac{y^{n+1-m}}{\phi_g(y)} \int_0^1 x^n \phi_h(yx) \, \mathrm{d}x,$$

which is smooth on U, proving the first claim.
(ii) : Use the same notation as in part (i). Now we have

$$k(y) = y^n \phi_h(y) \int_a^y \frac{p(x)}{x^m \phi_g(x)} \, \mathrm{d}x.$$

To prove that k is as smooth as claimed it suffices to show the claim for the function $\tilde{k} := k/\phi_h$, by smoothness of ϕ_h. We first show that $\tilde{k}$ is left continuous at 0 with $\tilde{k}(0) = 0$. Towards this end, we write $\tilde{k} = \frac{\tilde{k}_n}{\tilde{k}_d}$ where the numerator and denominator of $\tilde{k}$ are given by $\tilde{k}_n(y) := \int_a^y \frac{p(x)}{x^m \phi_g(x)} \, \mathrm{d}x, \tilde{k}_d(y) := \frac{1}{y^n}$. We note that $\lim_{y \uparrow 0} \left| \tilde{k}_n(y) \right| \leq \infty = \lim_{y \uparrow 0} \left| \tilde{k}_d(y) \right|$. To apply L'Hôspital's rule, we compute

$$\lim_{y \uparrow 0} \frac{\tilde{k}_n'(y)}{\tilde{k}_d'(y)} = \lim_{y \uparrow 0} \frac{p(y) y^{-m} \phi_g(y)^{-1}}{\frac{-n}{y^{n+1}}} = \lim_{y \uparrow 0} - \frac{p(y) y^{n+1-m}}{n \phi_g(y)},$$

which is 0 for $n \geq m - 1$ since $p(0) = 0$. Hence $\tilde{k}$ is left continuous at 0 with $\tilde{k}(0) = 0$. To show the claimed differentiability, we compute

$$\tilde{k}'(y) = n y^{n-1} \int_a^y \frac{p(x)}{x^m \phi_g(x)} \mathrm{d}x + \frac{y^{n-m} p(y)}{\phi_g(y)}.$$

By again invoking L'Hôspital's rule and a computation similar to above, the continuity of the first summand is equivalent to the continuity of the function $y \mapsto p(y) y^{n-m}$ at 0.

For $n \geq m$ this is a consequence of the continuity of p. For $n = m - 1$ we have by the fundamental theorem of calculus that since $p(0) = 0$

$$\frac{p(y)}{y} = \frac{1}{y} \int_0^y p'(x) \, \mathrm{d}x = \int_0^1 p'(yx) \, \mathrm{d}x.$$

Hence $y \mapsto p(y)/y$ is left continuous at 0 by Lemma B.1. This also shows the claim for the second summand, which finishes the proof. □

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