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**Visibility of Marginally Outer Trapped Surfaces**
A Study in Asymptotically De Sitter Spacetime

verfasst von / submitted by

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# Contents

# Preface

This thesis treats marginally outer trapped surfaces (or MOTS, for short) and their visibility in asymptotically de Sitter spacetimes. It is divided into five chapters: After a short introduction the second Chapter gives an overview of some of the geometry needed to understand this thesis. In the third chapter the main object of this thesis, the notion of marginally outer trapped surfaces, is introduced and several of its characteristics are studied. The fourth chapter presents visibility theorems for MOTS in asymptotically de Sitter spacetimes including an exhaustive proof of the underlying maximum principle for smooth null hypersurfaces. The fifth and final chapter is a fully detailed exposition of MOTS and their visibility concretely in de Sitter spacetime.

The prerequisites for understanding this thesis are a solid grasp on Lorentzian geometry and basic knowledge of General Relativity. Although the second Chapter introduces most of the geometry needed, it serves more as a short review of the needed concepts, rather than a detailed presentation of the subject. Two references that more than cover the required knowledge in both the mathematical and physical aspects would be [18] and [23]. While the former focuses more on the mathematical side, the latter is a standard physics textbook for General Relativity.

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## Notation and Conventions

In this section we fix the notation that will be used in this thesis. The notation will mostly be in line with [23], particularly abstract index notation will be used extensively. In a basis a tensor of type $(k, l)$ is characterized by its component functions $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l}$. The problem with using this notation throughout is the inherent choice of a basis, even if we want to state basis independent equation between tensors. That is why we introduce abstract index notation, which has all the advantages of the component notation but does not assume a basis. A tensor will be written as $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l}$ where the indices only signal the type and lets us use the summation conventions for true tensor equations. One key difference in notation between this thesis and [23] is the use of the distinction between greek and latin indices. In [23] greek indices signal a choice of coordinates, while latin indices are used for abstract index notation. In this thesis, unless coordinates are specified both indicate abstract index notation. We will often consider $n-2$ dimensional surfaces $S$ in a hypersurface $\mathscr{N}$ of the $n$ dimensional spacetime $\mathscr{M}$. The differentiation between greek, latin and capital latin indices is reserved for the distinction between these three layers in the following manner. Geometric objects associated with

- $\mathscr{M}$ use greek indices $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l}$
- $\mathscr{N}$ use latin indices $T^{i_1 \ldots i_k}{}_{j_1 \cdots j_l}$
- $S$ use capital latin indices $T^{A_1 \ldots A_k}{}_{B_1 \ldots B_l}$

On some occasions index free notation like in [18] will be used, especially in chapter 2, where index notation becomes tedious rather than a simplification, and obscures the geometric intuition.

We mostly consider Lorentzian manifolds $(\mathscr{M}, \bar{g})$ of signature $(-,+,+,+)$, as in most relativity texts. To make the notation even clearer beyond the use of different indices a bar as in the metric $\bar{g}$ signals objects associated with the full manifold $\mathscr{M}$, while $g$ denotes the metric of the hypersurface $\mathscr{N}$. The set of all vectorfields on $(\mathscr{M}, \bar{g})$ will be denoted $\mathfrak{X}(\mathscr{M})$ and the set of all one-forms $\Omega^1(\mathscr{M})$.

The Levi-Civita connection of $\bar{g}$ will be denoted $\bar{\nabla}$ and below is a list of the associated curvatures:

- Riemann tensor $\bar{R}^{\gamma}{}_{\mu\nu\sigma}\omega_\gamma = [\bar{\nabla}_\mu, \bar{\nabla}_\nu]\omega_\sigma$, for a one-form $\omega_\sigma$
- Ricci tensor $\bar{R}_{\mu\nu} = \bar{R}^{\sigma}{}_{\mu\sigma\nu}$
- Ricci scalar $\bar{R} = \bar{R}^{\sigma}{}_{\sigma}$

- Einstein tensor $\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}$

Note that the Einstein summation convention implies summation over equal indices. When index-free notation is used, $\mathbf{Ric}(X, Y)$ denotes the Ricci tensor for vectorfield $X, Y \in \mathfrak{X}(\mathscr{M})$ and $\mathbf{Ric}$ the scalar curvature.

# Chapter 1

# Introduction

In General Relativity spacetime is modelled by a Lorentzian manifold $(\mathscr{M}, g)$, where the metric $g$ satisfies the Einstein equations $\mathbf{G} + \Lambda\mathbf{g} = k\mathbf{T}$. A central contribution to our understanding of such key features of General Relativity as black holes and the history of our Universe are the singularity theorems by Roger Penrose and Stephen Hawking. The Penrose theorem (see [19]) predicts a singularity if several conditions are met, namely the existence of a trapped surface, a non-compact Cauchy surface and when the null energy condition ($\mathbf{Ric}(X, X) > 0 \quad \forall X \in T\mathscr{M}$ null) holds. Loosely speaking trapped surfaces are 2-dimensional closed surfaces within the spacetime where the future directed null geodesic congruences (which model light rays) converge locally. A measure for this convergence is given by the null expansion scalars $\theta^{\pm}$, where $\theta^{+}$ refers to the outgoing null direction and $\theta^{-}$ to the ingoing null direction. A positive sign of the null expansion scalar implies divergence and a negative sign convergence of future directed light rays. Trapped surfaces are characterized by $\theta^{\pm} < 0$, i.e. both null expansion scalars need to be negative. In this master thesis the case where $\theta^{+} = 0$ is studied extensively. These surfaces are called marginally outer trapped surfaces or MOTS, for short. In a sense MOTS are a generalisation of trapped surfaces, since only one direction (outgoing) is required to be zero. In the first part of this thesis the theory of MOTS will be compiled and comprehensively presented.

Next we focus on visibility theorems regarding MOTS and the related weakly trapped surfaces ($\theta^{\pm} \leq 0$). For the study of visibility one embeds the spacetime $(\mathscr{M}, g)$ into its conformal completion $(\widetilde{\mathscr{M}}, \tilde{g})$. More precisely $(\widetilde{\mathscr{M}}, \tilde{g})$ is called a conformal completion of $(\mathscr{M}, g)$ if $\mathscr{M}$ is the interior of $\widetilde{\mathscr{M}}$ and there exists a positive function $\Omega$ on $\widetilde{\mathscr{M}}$ that vanishes on the boundary, has non-vanishing differential and $\tilde{g} := \Omega^2 g$ extends smoothly to the boundary of $\widetilde{\mathscr{M}}$. This lets us define parts of the boundary as future null infinity $\mathscr{S}^{+}$, which has several crucial uses for the task at hand. In particular, the causal nature of $\mathscr{S}^{+}$ can be used to define the asymptotic behaviour of the spacetime. In this thesis the focus lies on manifolds with a spacelike $\mathscr{S}^+$, which corresponds to an asymptotically de Sitter behaviour. Visibility of a set $A \subset \mathscr{M}$ is then defined as whether or not $A$ lies within the timelike past of $\mathscr{S}^+$. Recent results on the visibility of MOTS will be presented (see [9]). The final part of this thesis contains a fully detailed exposition on MOTS and their visibility specifically in de Sitter spacetime.

# Chapter 2

# Elements of Lorentzian Geometry

In this chapter we will briefly review some notions from Lorentzian geometry that will be used extensively in this thesis. Hereby we follow mostly [18] and [7]. For brevity most proofs will be omitted but they can be found in the aforementioned texts. In the first section we consider semi-Riemannian manifolds $(\mathscr{M}, \bar{g})$ and submanifolds $\mathscr{N}$ of $\mathscr{M}$. After that we study the specific case of $(n-1)$ dimensional submanifolds called hypersurfaces, before moving on to a brief review of causality theory.

## 2.1 Semi-Riemannian submanifolds

In this section we consider submanifolds $(\mathscr{N}, g)$, where $g = j^*\bar{g}$ is the pull-back of $\bar{g}$ under the inclusion map $j : \mathscr{N} \hookrightarrow \mathscr{M}$. If $g$ is non-degenerate $(\mathscr{N}, g)$ is a *semi-Riemannian submanifold*. The dimension of $\mathscr{N}$ will be denoted by $n$. Our first goal is to study the tangent and normal geometry of $\mathscr{N}$.

For a submanifold $(\mathscr{N}, g)$ of $(\mathscr{M}, \bar{g})$ a vectorfield $X$ along the inclusion map $j$ is called an $\mathscr{N}$ vectorfield on $\mathscr{M}$. The set of these vectorfields is denoted by $\bar{\mathfrak{X}}(\mathscr{N})$. Since each tangent space $T_p\mathscr{N}$ is a non-degenerate subspace of $T_p\mathscr{M}$, the following decomposition holds.

$$T_p(\mathscr{M}) = T_p(\mathscr{N}) \oplus T_p(\mathscr{N})^\perp \tag{2.1}$$

where $T_p(\mathscr{N})^\perp$ denotes the also non-degenerate normal space of $T_p(\mathscr{N})$ of dimension $k$, called *codimension*. Similarly the index of $\bar{g}$ restricted to $T_p(\mathscr{N})^\perp$ is called *co-index* of $\mathscr{N}$ in $\mathscr{M}$. Every vector $X \in T_p(\mathscr{M})$ for $p \in \mathscr{N}$ then has a unique decomposition $X = \tan(X) + \mathrm{nor}(X)$ where $\tan(X) \in T\mathscr{N}$ denotes the tangential part and $\mathrm{nor}(X) \in T\mathscr{N}^\perp$ the normal part of $X$.

Our next goal is to find a map that lets us differentiate vectorfields $X \in \bar{\mathfrak{X}}(\mathscr{N})$ in direction of a vectorfield $V \in \mathfrak{X}(\mathscr{N})$. For $V \in \mathfrak{X}(\mathscr{N})$ and $X \in \bar{\mathfrak{X}}(\mathscr{N})$ the naive approach $\bar{\nabla}_V X$, where $\bar{\nabla}$ denotes the Levi-Civita connection associated with $\bar{g}$, does not work directly since neither vectorfield is an element in $\mathfrak{X}(\mathscr{M})$. However, let $V_{\text{ext}}$ and $X_{\text{ext}}$ be smooth local extensions of $V$ and $X$ to a neighbourhood $U$ in $\mathscr{M}$, then we can define the *induced connection* as $\bar{\nabla}_V X := \bar{\nabla}_{V_{\text{ext}}} X_{\text{ext}}$ restricted to $U \cap \mathscr{N}$. This map is well defined and fulfills the properties of the Levi-Civita connection, hence the use of the same symbol.

Note that the induced connection maps to $\bar{\mathfrak{X}}(\mathscr{N})$ and can therefore be decomposed into its tangential and normal parts. It turns out that the tangential projection is the Levi-Civita connection of $\mathscr{N}$ associated with the induced metric $g$, while the normal part is a new object which gives insight into how $\mathscr{N}$ lies within $\mathscr{M}$:

$$\bar{\nabla}_V X = \tan\big(\bar{\nabla}_V X\big) + \underbrace{\operatorname{nor}(\bar{\nabla}_V X)}_{=:K(V,X)} = \nabla_V X + K(V,X) \tag{2.2}$$

where $\nabla$ is the Levi-Civita connection of the induced metric $g$ and we introduced the (0,2) tensorfield $K$ called *second fundamental form.* Since the second fundamental form is a (0,2) tensor field on $\mathscr{N}$ with values in $T\mathscr{N}^{\perp}$, we can turn it into a vectorfield $H$ on $\mathscr{N}$ with values in $T\mathscr{N}^{\perp}$ by

$$H|_p = \sum_{i=1}^{n} \varepsilon_i K(e_i, e_j), \tag{2.3}$$

where $\{e_1, \ldots, e_n\}$ is a frame at $p \in \mathscr{N}$ and $\varepsilon_i = g(e_i, e_i)$.

The above decomposition in tangential and normal geometry of $\mathscr{N}$ leads us to the following fundamental result.

**Theorem 2.1.1.** (Gauß-Codazzi equation)
Let $V, W, X, Y$ be vectorfields tangent to $\mathscr{N}$, then

$$\begin{aligned}\langle \bar{\mathbf{R}}(V,W)X, Y\rangle = \langle \mathbf{R}(V,W)X, Y\rangle &+ \langle K(V,X), K(W,Y)\rangle \\ &- \langle K(V,Y), K(W,X)\rangle,\end{aligned} \tag{2.4}$$

where $\bar{\mathbf{R}}$ and $\mathbf{R}$ denote the Riemann tensor of $\mathscr{M}$ and $\mathscr{N}$ respectively.

## 2.2 Geometry of Hypersurfaces

In this section we turn our attention to semi-Riemannian submanifolds $\mathscr{N}$ of codimension 1, which will be called *semi-Riemannian hypersurfaces.* Depending on the signature of the induced metric we call a hypersurface

- spacelike, if the induced metric is Riemannian
- timelike, if the induced metric is Lorentzian

Note that for spacelike hypersurfaces the co-index is 1 for all points $p \in \mathscr{N}$ while it is 0 for timelike hypersurfaces. In other words the unit normal to $\mathscr{N}$ is timelike for Riemannian hypersurfaces and spacelike for Lorentzian hypersurfaces.

**Remark 2.2.1.** (Null hypersurfaces)
If the pullback of the metric on $\mathscr{N}$ is degenerate, it is not a semi-Riemannian hypersurface by the above definition, but by a slight abuse of language we still call $\mathscr{N}$ a *null hypersurface.*

For hypersurfaces the Gauß-Codazzi equation simplifies significantly, since the co-dimension is only 1.

**Theorem 2.2.2.** (Gauß-Codazzi equation for hypersurfaces)
For a hypersurface $\mathscr{N}$ with unit normal $m$ equation (2.4) simplifies to

$$-2\overline{\mathbf{G}}(m,m) = \mathbf{Ric} + K_{ij}K^{ij} - p^2, \tag{2.5}$$

where $\mathbf{G}$ is the Einstein-tensor, $p$ denotes the first (and only) components of the mean curvature vector and $K_{ij}K^{ij} := \sum_{j,i=1}^{n} \tilde{K}(E_j,E_i)\tilde{K}(E_i,E_j)$ for a frame field $\{E_i\}$ and $\tilde{K}$ is the only component function of the second fundamental form such that $K(X,Y) = \tilde{K}(X,Y)m$.

*Proof.* Let $\{E_i\}$ be a local orthonormal frame field. Contracting the $X$ and $W$ components of equation (2.4) yields

$$\begin{aligned}\overline{\mathbf{Ric}}(V,Y) = \mathbf{Ric}(V,Y) &+ \langle \overline{\mathbf{R}}(V,m)m, Y\rangle \\ &+ \sum_{i=1}^{n} \tilde{K}(V,E_i)\tilde{K}(E_i,Y) - p\tilde{K}(V,Y).\end{aligned} \tag{2.6}$$

By contracting the $V$ and $Y$ components we get

$$\overline{\mathbf{Ric}} = \mathbf{Ric} + 2\overline{\mathbf{Ric}}(m,m) + \sum_{j,i=1}^{n} \tilde{K}(E_j,E_i)\tilde{K}(E_i,E_j) + p^2. \tag{2.7}$$

And so

$$-2\overline{\mathbf{G}}(m,m) = \mathbf{Ric} + \tilde{K}_{ij}\tilde{K}^{ij} - p^2. \tag{2.8}$$

□

## 2.3 Causality

In this section, which is based on [7], we investigate the causal structure of a Lorentzian manifold $(\mathscr{M}, g)$. First, we need to equip the manifold with a smooth choice of future for every point, made precise by the following definition

**Definition 2.3.1.** (Time orientation)
A *time-orientation* of a Lorentzian manifold $(\mathscr{M}, g)$ is a map

$$\mathscr{T} : \mathscr{M} \mapsto \mathcal{P}(T\mathscr{M})$$

where $\mathcal{P}(T\mathscr{M})$ is the powerset of the tangentbundle, such that, for all $p \in \mathscr{M}$,

- $\mathscr{T}(p)$ belongs to one of the connected components of the set of timelike vectors in $T_p\mathscr{M}$
- there is a chart $(U, x)$ around $p$ such that $\frac{\partial}{\partial x^0}|_q \in \mathscr{T}(q)$ for all $q \in U$.

$(\mathscr{M}, g)$ is called *time-orientable* if it admits a time-orientation, and $(\mathscr{M}, g, \mathscr{T})$ is called a *time-oriented Lorentzian manifold.*

A connected, time-oriented Lorentzian manifold is called *spacetime*, and from now on every Lorentzian manifold will be a spacetime unless specified otherwise.

A causal curve $\gamma$ (i.e. $\dot{\gamma}(t)$ is causal for all $t$) is called future directed, if $\dot{\gamma}(t) \in \overline{\mathscr{T}(\gamma(t))}$ and past directed if $\dot{\gamma}(t) \in -\overline{\mathscr{T}(\gamma(t))}$. Next we define the essential relations which indicate the causal relation between two points.

**Definition 2.3.2.** (Causality relations)
For $p, q \in \mathscr{M}$, we define the following relations:

- $p << q \quad :\Longleftrightarrow \quad \exists$ future directed timelike curve from $p$ to $q$
- $p < q \quad :\Longleftrightarrow \quad \exists$ future directed causal curve from $p$ to $q$
- $p \leq q \quad :\Longleftrightarrow \quad p < q$ or $p = q$

and for $A \subset \mathscr{M}$, we define

- $I_+ := \{q \in \mathscr{M} | \exists p \in A : p << q\}$, the chronological future of A
- $J_+ := \{q \in \mathscr{M} | \exists p \in A : p \leq q\}$, the causal future of A.

The chronological and causal past are defined analogously.

**Definition 2.3.3.** (Achronal set)
Let $(\mathscr{M}, g)$ be a spacetime. A subset $A \subset \mathscr{M}$ is called *achronal* if there are no points $p, q \in A$ such that $p << q$.

Now we introduce a hypersurface called *Cauchy surface* that can be interpreted as an instance of time and serves as initial surface when formulating the Einstein equations as an evolutionary system. Furthermore Cauchy surfaces are of crucial importance for the singularity theorems discussed later.

**Definition 2.3.4.** (Cauchy surface)
A hypersurface $\mathcal{C}$ of $\mathscr{M}$ is called *Cauchy surface* if every inextendible causal curve hits $\mathcal{C}$ exactly once.

## 2.4 Decomposing Spacetime into a 3+1 Foliation

This section mainly follows chapter 3.4 in [12]. So far we have explained the geometry of hypersurfaces in a spacetime. The aim now is to decompose a 4-dimensional spacetime into non-intersecting hypersurfaces such that the hypersurfaces are level sets of a function. The complete set of the generating hypersurfaces is called a *spacetime foliation*, each individual hypersurface $\Sigma_t$ at time $t$ is called a *leaf*.
For example the Euclidean space $\mathbb{R}^3\backslash\{0\}$ can be foliated by 2-spheres of different radii centered on the same point. In this case each leaf $\Sigma_r$ corresponds to one 2-sphere with radius $r$. These hypersurfaces are non-intersecting and for $r \in \mathbb{R}^+$ generate the entire manifold $\mathbb{R}^3\backslash\{0\}$. The foliation is entirely determined by the function on the manifold since the leaves are its level sets, in this case the radius r.
In the context of General Relativity the foliation is given by $t$, which can often be interpreted as a global time function with spacelike level sets. In coordinates the foliation is given by $x^\mu = X^\mu(\zeta^\alpha, t)$. Given a hypersurface at $t = t_0$, a point $p \in \Sigma_{t_0}$ is fixed by its three coordinates $\zeta^\alpha$. This point $p$ corresponds to another point $p'$ in a different leaf $\Sigma_{t+dt}$, with the same intrinsic coordinates on the hypersurface. Of course the spacetime coordinates differ, since they depend on $t$. The vector connecting these two points is given by $\partial_t$ with the following components in the $\partial_\mu$ basis

$$\frac{\partial}{\partial t} = \frac{\partial X^\mu}{\partial t}\frac{\partial}{\partial x^\mu} = t^\mu \frac{\partial}{\partial x^\mu} \tag{2.9}$$

We can then decompose the vector $\partial_t$ with components $t^\mu$ into its normal and tangential part. Let $n$ a unit normal vector to $\Sigma_t$, $\{e_b\}$ a basis of $T\Sigma_t$ and let $g_{ab}$ denote the induced metric on $\Sigma_t$.

Then

- $\text{nor}(t^\mu) = -\underbrace{(t^\nu n_\nu)}_{:=N} n^\mu$
- $\tan(t^\mu) = g^{ab}(t_\nu e^\nu_b)X^\mu_a = g^{ab}\bar{g}_{\lambda\nu}X^\lambda_b t^\nu X^\mu_a = \underbrace{(X^a_\nu t^\nu)}_{:=N^a} X^\nu_a$

where we have introduced the *lapse function* $N$ and the *shift vector* $N^a$. The role of these objects in a foliation becomes quite clear when considering the example from before. The lapse function $N$ represents how far the next leaf, in this case a sphere, is separated. In general the lapse function does not have to be the same everywhere on $\Sigma_t$, although in this example it is. The shift vector $N^a$ corresponds to the deformation, or this case rotation of the leaf that is separated by the distance $N$ compared to $\Sigma_t$. In the example of $\mathbb{R}^3$ the north pole would only have the same coordinates $\zeta^a$ for future leaves if the shift vector were zero, otherwise the spheres would rotate.
In General Relativity where $t$ can be interpreted as a global time function and the leaves $\Sigma_t$ are spacelike timeslices $N$ measures the elapsed time between leaves while $N^a$ determines the deformation, this is demonstrated in Figure 2.1 below.

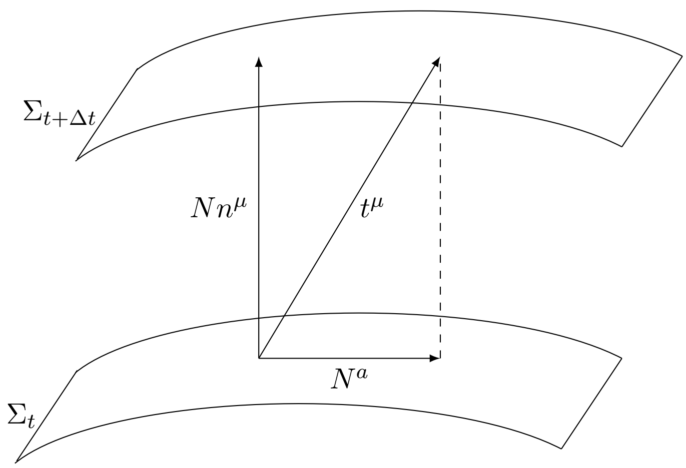


Figure 2.1: Lapse $N$ and shift vector $N^a$ in a spacetime foliation.

# Chapter 3

# Marginally Outer Trapped Surfaces

In this chapter we introduce the main object of study of this thesis, marginally outer trapped surfaces (MOTS). Although trapped surfaces in general are not a new concept, MOTS have recently become key in quasi-local descriptions of black holes. Since black holes are canonically defined via the visibility from future infinity, which requires knowledge of global properties of the spacetime and the existence of a conformal completion (see 4.1), it is of high interest to find local tools for the detection of black hole regions. Local properties in this context are properties which could in principle be measured by an observer with a finite life span[2]. The notions of trapped and marginally outer trapped surfaces let us achieve this goal. Trapped surfaces were first introduced by Roger Penrose in a 1965 paper for which he was awarded a Nobel Prize in 2020 [19]. For the interested reader a review paper concerning specifically this groundbreaking paper and the central idea of trapped surfaces can be found in [21]. A trapped surface is a closed (i.e. compact without boundary) spacelike 2-surface $S$ for which both congruences of future directed null geodesics emanating from $S$ orthogonally will converge locally in the future. Loosely speaking this convergence property can be given a numerical value by the null expansion scalars, which will be negative for converging geodesics and positive for expanding geodesics. If one of the null expansion scalars is zero, the surface $S$ is called marginally outer trapped surface (i.e. MOTS). In this chapter we study the theory of MOTS and introduce the above concepts in a precise manner.

## 3.1 The Null Expansion and Trapped Surfaces

This section mainly compiles information from [8] and [11]. As mentioned before, the null expansion will give us a measure of the convergence behaviour of congruences of null geodesics emanating orthogonally from a surface. To make this precise we start off with some definitions.

Let $(\mathscr{M}, \bar{g})$ be a spacetime of dimension $n \geq 3$ and $S$ an orientable, closed and spacelike submanifold of codimension two. Such submanifolds will be referred to as *surfaces*. In the case $n = 4$ the normal bundle $NS = (TS)^{\perp}$ of $S$ has two future directed orthogonal null directions and thus admits two unique (up to positive rescaling), smooth non-vanishing future directed orthogonal null vectorfields. These will be called $l^{\pm}$ respectively where $l^{+}$ denotes the outward and $l^{-}$ the inward null normal. It should be noted here that the choice of "outward" direction is often not obvious and arbitrary. Since we have the freedom of rescaling the null normals, the normalization is chosen such that $l_{+}^{\mu} l_{\mu}^{-} = -2$. In terms of the null normals the projector onto the surface $S$ is given by $P^{\mu\nu} = \bar{g}^{\mu\nu} + l_{+}^{(\mu} l_{-}^{\nu)}$, where $l_{+}^{(\mu} l_{-}^{\nu)}$ denotes the symmetrization. The induced metric on $S$ will be denoted by $h$. With this setup we can define the central object that lets us directly distinguish between different classes of trapped surfaces.

**Definition 3.1.1.** (Null expansion scalar)
Let $(\mathscr{M}, \bar{g})$ be a spacetime, $S$ a surface with Projectors $P^{\mu\nu}$ and future directed null normals $l^{\pm}$ as described above. We then call

$$\theta^{\pm} = P^{\mu\nu} \bar{\nabla}_{\mu} l_{\nu}^{\pm} \tag{3.1}$$

the *null expansion scalars.*

The magnitude of $\theta^{\pm}$ depends on the scaling of the null normals $l^{\pm}$, which was chosen arbitrarily. The sign however is independent of the scaling and has physical meaning, namely expansion (contraction) of $S$ for $\theta^{+} > (<)0$.

The null expansion scalars now lead to a natural definition of different types of trapped surfaces.

**Definition 3.1.2.** (Types of trapped surfaces)
Let $(\mathscr{M}, \bar{g})$ be a spacetime and let $S \subset \mathscr{M}$ be a surface with null expansions $\theta^{\pm}$, we then call $S$

- trapped, if $\theta^+ < 0$ and $\theta^- < 0$
- outer trapped, if $\theta^+ < 0$
- weakly trapped, if $\theta^+ \leq$ and $\theta^- \leq 0$
- weakly outer trapped, if $\theta^+ \leq 0$
- marginally outer trapped, if $\theta^+ = 0$

**Remark 3.1.3.** (Outward direction)
In the case where a marginally trapped surface exists for only one direction, we choose that direction as outwards.

Since index notation is often convenient for explicit calculations we define the second fundamental form previously introduced in 2.2 again.

**Definition 3.1.4.** (Second fundamental form vector)
For any vector $w$ normal to $S$ we call the $(1, 2)$ tensorfield $K^{\mu}{}_{\alpha\beta}$ defined by

$$K^{\mu}{}_{\alpha\beta} w_{\mu} = P_{\alpha}{}^{\sigma} P_{\beta}{}^{\tau} \bar{\nabla}_{\sigma} w_{\tau} \tag{3.2}$$

the *second fundamental form vector* of $S$.

It is not immediately obvious that the right hand side can be written as a tensor acting on $w$. To ensure that this is indeed possible we show by a short calculation that the right hand side is $C^{\infty}$-linear in $w$.

$$P_{\alpha}{}^{\sigma} P_{\beta}{}^{\tau} \bar{\nabla}_{\sigma}(f w_{\tau}) = P_{\alpha}{}^{\sigma} P_{\beta}{}^{\tau} \left( f \bar{\nabla}_{\sigma} w_{\tau} + w_{\tau} \bar{\nabla}_{\sigma} f \right) \tag{3.3}$$

$$= f P_{\alpha}{}^{\sigma} P_{\beta}{}^{\tau} \bar{\nabla}_{\sigma}(w_{\tau}). \tag{3.4}$$

The second term in (3.3) is zero since the projectors $P$ annihilate any vector normal to $S$, in particular $w$.

**Remark 3.1.5.** Note that $K^{\mu}{}_{\alpha\beta} w^{\alpha} n^{\beta} = 0$ for arbitrary vectors $w$,$n$ normal to $S$. Additionally the second fundamental form vector is only defined on $S$, thus it is natural to only consider vectors tangent to $S$ as possible input. This suggests the notational change $K^{\mu}{}_{AB}$ with $A, B \in \{2, 3\}$. This is made precise by requiring $K^{\mu}{}_{\alpha\beta} x^{\alpha} v^{\beta} = K^{\mu}{}_{AB} x^A v^B$, where $x^A = P^A{}_{\mu} x^{\mu} \in TS$ and similarly for $v$. For fixed normal direction $w$ the *second fundamental form* $K_{AB}$ is now a $(0, 2)$ tensorfield on $S$.

A completely analogous construction relates the projector $P_{\mu\nu}$ to the induced metric $h_{AB}$, by a slight abuse of notation they will be used interchangably from here on. Contracting the second fundamental form vector with the induced metric yields another essential curvature object.

**Definition 3.1.6.** (Mean curvature vector)
For a surface $S$ in a spacetime $(\mathscr{M}, \bar{g})$ with induced metric $h$ and second fundamental form vector $K^{\mu}{}_{AB}$ we call

$$H^{\mu} = h^{AB} K^{\mu}{}_{AB} \tag{3.5}$$

the *mean curvature vector.*

**Lemma 3.1.7.** (Mean curvature vector in terms of a null basis)
Let $S$ be a surface with null normals $l^{\pm}$ as before and corresponding null expansion scalars $\theta^{\pm}$. Then the mean curvature vector of $S$ can be written as

$$H^{\mu} = -\frac{1}{2}\left(\theta^{-} l_{+}^{\mu} + \theta^{+} l_{-}^{\mu}\right). \tag{3.6}$$

*Proof.* First we calculate $H^{\mu} l_{\mu}^{\pm}$,

$$H^{\mu} l_{\mu}^{\pm} = P^{\alpha\beta} P_{\alpha}{}^{\sigma} P_{\beta}{}^{\gamma} \bar{\nabla}_{\sigma} l_{\gamma}^{\pm} \tag{3.7}$$
$$= P^{\sigma\gamma} \bar{\nabla}_{\sigma} l_{\gamma}^{\pm} \tag{3.8}$$
$$= \theta^{\pm}. \tag{3.9}$$

Where the property $P^{\alpha}{}_{\beta} P_{\alpha}{}^{\mu} = P_{\beta}{}^{\mu}$ of projections was used. Recall the normalization of the null normal $l_{\mp}^{\mu} l_{\mu}^{\pm} = -2$. Expanding $H^{\mu}$ in terms of $\{l^{+}, l^{-}\}$ and correcting for the normalization yields

$$H^{\mu} = -\frac{1}{2}\left(\theta^{-} l_{+}^{\mu} + \theta^{+} l_{-}^{\mu}\right). \tag{3.10}$$

□

It is often convenient (for example in a leaf of a spacetime foliation, see 2.4) to consider trapped surfaces embedded in a spacelike hypersurface. Let $\mathscr{N}$ be a spacelike hypersurface in a spacetime $(\mathscr{M}, \bar{g})$ with unit normal $n$ and induced metric $g$. The second fundamental form of $\mathscr{N}$ is then defined analogously to 3.1.4 by replacing $w$ by $n$ and using the appropriate projectors $Q^{\mu\nu}$ from $\mathscr{M}$ to $\mathscr{N}$ as

$$K^{\mu}{}_{\alpha\beta} n_{\mu} = Q_{\alpha}{}^{\sigma} Q_{\beta}{}^{\tau} \bar{\nabla}_{\sigma} n_{\tau}, \tag{3.11}$$

where the identification of remark 3.1.5 is applied to yield the $(0,2)$-tensor $K_{ij}$ on $\mathscr{N}$.

**Remark 3.1.8.** Note that especially in equation (3.11) there is a slight abuse of notation since $K^{\mu}{}_{\alpha\beta}$ denotes both the second fundamental form vector of $\mathscr{N}$ and $S$. However after using the appropriate indices they are distinguished by $K_{ij}$ and $K_{AB}$ for $\mathscr{N}$ and $S$ respectively.

Let $S \subset \mathscr{N}$ be a two-sided submanifold of codimension one, which makes it a surface in a 4-dimensional spacetime. Because $S$ is two-sided there exist two unique globally defined unit vectorfields orthogonal to $S$ and tangent to $\mathscr{N}$. We arbitrarily choose one outwards pointing and call it $m$, its negative $-m$ is then the inwards pointing unit normal.

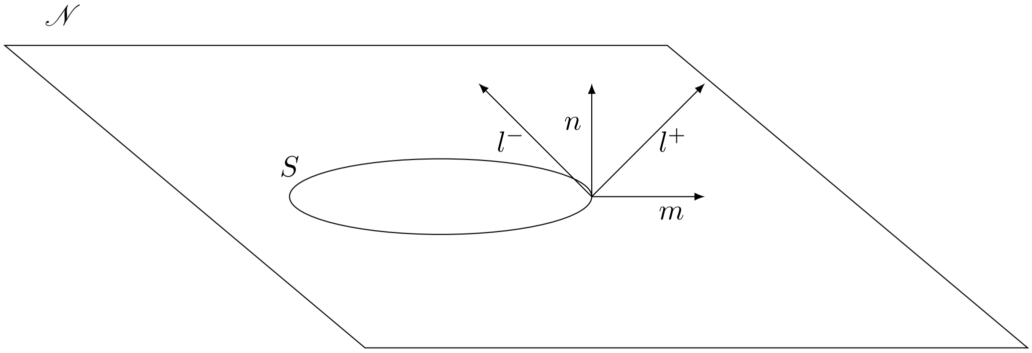


Figure 3.1: Normals to $S$ and $\mathscr{N}$ with one dimension suppressed.

**Definition 3.1.9.** (Mean curvature of $S$ in $\mathscr{N}$)
Let $S$ be a surface with mean curvature vector $H^\mu$ and unit outward normal $m_\mu$. Then

$$p = H^\mu m_\mu \tag{3.12}$$

is called the *mean curvature of $S$ in $\mathscr{N}$*.

In this setting the null expansion scalars can be expressed in terms of the mean curvature vector and the normal vectors, which is highly useful for practical calculations.

**Lemma 3.1.10.** (Curvature form of the null expansions)
Let $p = H^\mu m_\mu$ be the mean curvature of $S$ in $\mathscr{N}$ and $H := H^\mu n_\mu$. Then the null expansion scalars of $S$ can be written as

$$\theta^\pm = H \pm p. \tag{3.13}$$

*Proof.* First we calculate $H$,

$$H = H^\mu n_\nu = h^{AB} K^\mu{}_{AB} n_\mu \tag{3.14}$$

$$= P^{\alpha\beta} K^\mu{}_{\alpha\beta} n_\mu \tag{3.15}$$

$$= P^{\alpha\beta} P_\alpha{}^\mu P_\beta{}^\nu \bar{\nabla}_\mu n_\nu \tag{3.16}$$

$$= P^{\mu\nu} \bar{\nabla}_\mu n_\nu. \tag{3.17}$$

A similar calculation yields

$$p = P^{\mu\nu} \bar{\nabla}_\mu m_\nu \tag{3.18}$$

Next we decompose the null normals of $S$ as $l^{\pm}_{\mu} = n_\mu \pm m_\mu$. We first need to check that these vectors are indeed null and normal to $S$.

$$l^{\pm}_{\mu} l^{\mu}_{\pm} = (n_\mu \pm m_\mu)(n^\mu \pm m^\mu) = \underbrace{n_\mu n^\mu}_{-1} \pm \underbrace{2m_\mu n^\mu}_{0} + \underbrace{m_\mu m^\mu}_{1} = 0$$

where the first term has an additional negative sign because $\mathscr{N}$ is a space-like hypersurface with coindex 1. The second term is positive since $m$ is tangent to $\mathscr{N}$, which has a Riemannian induced metric. Orthogonality is also checked easily. Let $k$ be an arbitrary vector tangent to $S$ then

$$\bar{g}_{\mu\nu} l^{\mu}_{\pm} k^\nu = \bar{g}_{\mu\nu}(n^\mu \pm m^\mu)k^\nu = 0$$

since both $n^\mu$ and $m^\nu$ are normal to $S$ by definition. The statement then follows from linearity:

$$P^{\mu\nu}\bar{\nabla}_\mu(n_\nu \pm m_\nu) = P^{\mu\nu}\bar{\nabla}_\mu n_\nu \pm P^{\mu\nu}\bar{\nabla}_\mu m_\nu = H \pm p$$

□

The simplest case of a MOTS (i.e. $\theta^+ = H + p = 0$) is given if both $H$ and $p$ vanish identically. This leads us to consider the notion of minimal surfaces, since they are characterized by $p = 0$ as we will see in the next section.

## 3.2 Minimal Surfaces

In this section we study minimal surfaces as some of the MOTS we will encounter in future examples will be minimal surfaces. MOTS can be seen as spacetime analogues of minimal surfaces in Riemannian geometry, and as we will see, a lot of properties and equations are closely related. A minimal surface is a surface that locally extremizes area. Thus our first goal is to introduce a notion of varying a surface and its area to find extrema. To this end we consider a three dimensional Riemannian manifold $(\mathscr{N}, g)$. Let $S \subset \mathscr{N}$ be a closed two-sided 2-surface with unit outward normal $m$, projectors $P_{ij} = g_{ij} - m_i m_j$ and induced metric $h$. We proceed with defining a variation of $S$.

**Definition 3.2.1.** (Variation of a surface)
Let $S$ be a surface in the manifold $(\mathscr{N}, g)$ as above and $0 \in I \subset \mathbb{R}$ an open interval. We then define a variation of $S$ along an arbitrary (nowhere zero) vectorfield $v$ defined along $S$ as the map $\Phi : S \times I \to \mathscr{N}$ such that $\Phi(S, 0) = id$, for $t$ fixed $\Phi(\cdot, t)$ is an immersion and for $p \in S$ fixed $c_p(t) = \Phi(p, t)$ is a curve starting at $p \in \mathcal{S}$ with tangent vector $v(p)$.

Next we choose coordinates such that the line element near $S$ reads

$$ds^2 = \phi^2(r, x^i)dr^2 + g_{ij}(r, x^i)dx^i dx^j, \tag{3.19}$$

where $\phi$ is a function and $S$ is parametrized by $r = \text{const}$. The unit normal then has the form

$$m^i \frac{\partial}{\partial x^i} = \phi^{-1} \frac{\partial}{\partial r}. \tag{3.20}$$

Now we can introduce the notion of variation of a scalar object in the normal direction $\phi m = \frac{\partial}{\partial r}$.

**Definition 3.2.2.** (First variation of scalars in the normal direction)
Let $k$ be a scalar on $S$, $\phi : S \mapsto \mathbb{R}$ an arbitrary function and $\Phi$ a variation of $S$ along the normal vectorfield $m$. Then we define the $\phi$-variation of $k$ in direction $m$ as

$$\delta_{\phi m} = \phi m^i \nabla_i k. \tag{3.21}$$

The extrinsic curvature of $S$ is then given by $p_{ij} := P_i{}^k P_j{}^l \nabla_k m_l$ and the mean curvature by $p = g^{ij} p_{ij} = \nabla_i m^i$. Next we will calculate the first variation of the mean curvature.

**Theorem 3.2.3.** (First variation of the mean curvature)
Let $S$ be a surface as above in $\mathscr{N}$ with unit normal $m$. Furthermore let $p = g^{ij}p_{ij}$ be the mean curvature and $R$, $R_S$ the Ricci scalars of $\mathscr{N}$ and $S$, respectively. Then

$$\delta_{\phi m} p = -\Delta_S \phi - \phi \left( R - R_S + \frac{3}{2}p^2 + t_{ij}t^{ij} \right) := L_m(\phi), \tag{3.22}$$

where $t_{ij} = p_{ij} - \frac{p}{2}h_{ij}$ is the trace-free part of $p_{ij}$ and $\Delta_S$ denotes the Laplacian of $S$. Furthermore we introduced the self-adjoined operator $L_m$, which will play an important role in the stability of a minimal surface later on.

In order to prove this theorem we first need a technical lemma.

**Lemma 3.2.4.**

$$m^i \nabla_i m_k = -\phi^{-1} P_k{}^j \nabla_j \phi \tag{3.23}$$

*Proof.* First we have

$$\begin{aligned} P_{kj}\, m^i \nabla_i m^k &= P_{kj}\, m^i (\partial_i m^k + \Gamma^k{}_{il} m^l) && (3.24)\\ &= P_{kj}\, m^i \partial_i m^k + P_{kj}\, m^i \Gamma^k{}_{il} m^l. && (3.25) \end{aligned}$$

We now calculate each term separately and use that $m^1 = \phi^{-1}$ is the only non-zero component of $m^i$. Also note that $P_{11} = g_{11} - m_1 m_1 = \phi^2 - \phi^2 = 0$, and since $P_{ij}$ is diagonal $P_{1j} = 0$ for any $j$.

- First term

$$P_{kj}\, m^i \partial_i m^k = P_{1j}\, m^1 \partial_1 m^k = 0. \tag{3.26}$$

- Second term

$$\begin{aligned} P_{kj}\, m^i \Gamma^k{}_{il} m^l &= P_{kj}\, \phi^{-2} \Gamma^k{}_{11} && (3.27)\\ &= \phi^{-2} P_{kj}\, \frac{1}{2} g^{ks} (\partial_1 g_{1s} + \partial_1 g_{1s} - \partial_s g_{11}) && (3.28)\\ &= \phi^{-2} P_{kj}\, (g^{ks} \partial_1 g_{1s} - \frac{1}{2} g^{ks} \partial_s g_{11}) && (3.29)\\ &= \phi^{-2} P_{1j}\, g^{11} \partial_1 g_{11} - \frac{1}{2} \phi^{-2} P_{kj}\, \partial^k g_{11} && (3.30)\\ &= -\frac{1}{2} \phi^{-2} P_{kj}\, \partial^k g_{11}. && (3.31) \end{aligned}$$

Together this yields

$$P_{kj}\, m^i \nabla_i m^k = -\frac{1}{2} \phi^{-2} P_j{}^k \partial_k g_{11} = -\phi^{-1} P_j{}^k \partial_k \phi = -\phi^{-1} P_j{}^k \nabla_k \phi. \tag{3.32}$$

□

We proceed with the proof of theorem 3.2.3

*Proof.* We first define $a^i = m^k \nabla_k m^i - m^i \nabla_k m^k$ and calculate its divergence in two different ways.

- By using Lemma 3.2.4 we obtain

$$\nabla_i a^i = \nabla_i \left( -\phi^{-1} P^{ik} \nabla_k \phi \right) - \nabla_i (m^i \underbrace{\nabla_k m^k}_{p}) \tag{3.33}$$

$$= \nabla_i \left( -\phi^{-1} P^{ik} \nabla_k \phi \right) - m^i \nabla_i p - p^2 \tag{3.34}$$

$$= -\frac{\Delta_S \phi}{\phi} - m^i \nabla_i p - p^2. \tag{3.35}$$

- By the product rule we get

$$\nabla_i a^i = \underbrace{(\nabla_i m^k) \nabla_k m^i}_{p_{ij} p^{ij}} + m^k \nabla_i \nabla_k m^i - \underbrace{(\nabla_i m^i) \nabla_k m^k}_{p^2} - m^i \nabla_i \nabla_k m^k \tag{3.36}$$

$$= p_{ij} p^{ij} - p^2 + m^k \left( \nabla_k \nabla_i + [\nabla_i, \nabla_k] \right) m^i - m^i \nabla_i \nabla_k m^k \tag{3.37}$$

$$= p_{ij} p^{ij} - p^2 + m^i [\nabla_i, \nabla_k] m^k \tag{3.38}$$

$$= p_{ij} p^{ij} - p^2 + R_{ik} m^i m^k. \tag{3.39}$$

Comparing both expressions for $\nabla_i a^i$ yields

$$-\frac{\Delta_S \phi}{\phi} - m^i \nabla_i p - p^2 = p_{ij} p^{ij} - p^2 + R_{ij} m^i m^j, \tag{3.40}$$

and so

$$\phi m^i \nabla_i p = -\Delta_S \phi - \phi \left( p_{ij} p^{ij} + R_{ij} m^i m^j \right). \tag{3.41}$$

Now we use the Gauß-Codazzi equation for hypersurfaces (see 2.2.2)

$$R_{ij} m^i m^j = \frac{1}{2} \left( g_{ij} R m^i m^j + p^2 - R_S - p_{ij} p^{ij} \right) \tag{3.42}$$

$$= \frac{1}{2} \left( R - R_S + p^2 - p_{ij} p^{ij} \right) \tag{3.43}$$

and substitute back into equation (3.41) to obtain

$$\phi m^i \nabla_i p = -\Delta_S \phi - \phi \left[ p_{ij} p^{ij} + \frac{1}{2} \left( R - R_S + p^2 - p_{ij} p^{ij} \right) \right] \tag{3.44}$$

$$= -\Delta_S \phi - \frac{1}{2} \phi \left( R - R_S + p^2 + p_{ij} p^{ij} \right). \tag{3.45}$$

The final step is to express $p^2 + p_{ij}p^{ij}$ in terms of $t_{ij}$, which is easily done by an explicit calculation

$$
\begin{aligned}
t_{ij}t^{ij} &= \left(p_{ij} - \frac{p}{2}h_{ij}\right)\left(p^{ij} - \frac{p}{2}h^{ij}\right) && (3.46)\\
&= p_{ij}p^{ij} - pp_{ij}h^{ij} + \frac{p^2}{4}\underbrace{h_{ij}h^{ij}}_{=2} && (3.47)\\
&= p_{ij}p^{ij} - pp_{ij}\left(h^{ij} - m^i m^j\right) + \frac{1}{2}p^2 && (3.48)\\
&= p_{ij}p^{ij} - p^2 + \frac{1}{2}p^2 && (3.49)\\
&= p_{ij}p^{ij} - \frac{1}{2}p^2. && (3.50)
\end{aligned}
$$

And so

$$p^2 + p_{ij}p^{ij} = t_{ij}t^{ij} + \frac{3}{2}p^2. \tag{3.51}$$

Substituting this result in equation (3.45) yields equation (3.22) and thus concludes the proof. □

Next we study how the area changes along a variation of the surface. Let $S_t$ be a one parameter family of surfaces where $S_0 = S$ with variational vectorfield $v = \frac{\partial}{\partial r}$, which can be written as $v = \phi m$ by equation (3.20). Then the first variation of area, defined as $\delta_v \mathbf{vol}(S) := \frac{d}{dt}\mathbf{vol}(S_t)|_{t=0}$ is given by

$$\delta_v \mathbf{vol}(S) = \int_{S_0} \phi p \sqrt{\det g}. \tag{3.52}$$

A detailed derivation can be found in [16]. This leads to a natural definition of minimal surfaces, since we are looking for an extremal point in the first variation of area.

**Definition 3.2.5.** (Minimal surface)
A surface $S$ is called a minimal surface if its first variation of area vanishes for any $\phi$, i.e.

$$p \equiv 0$$

on $S$.

**Remark 3.2.6.** Note that the classical terminology of "minimal" is a bit misleading, since any extremum is (technically) included in the definition.

Next we turn to the second variation, which is vital for the stability of minimal surfaces. The second variation is defined as $\delta_v^2 \mathbf{vol}(S_t) = \frac{d^2}{dt^2}\mathbf{vol}(S)|_{t=0}$. Let again $S_t$ be a family of surfaces with variational vectorfield $v = \phi m$ and assume that $S_0$ is a minimal surface, i.e. $p = 0$, then

$$\delta_v^2 \mathbf{vol}(S) = \delta_v^2 \int_{S_0} \sqrt{\det g} \tag{3.53}$$

$$= \int_{S_0} \phi \delta_v p \sqrt{\det g} \tag{3.54}$$

$$= \int_{S_0} \phi L_m \phi \sqrt{\det g} \geq \lambda \int_{S_0} \phi^2 \sqrt{\det g} \tag{3.55}$$

where $\lambda$ denotes the lowest eigenvalue of $L_v$. Note that $\lambda$ exists since $L_v$ is a self-adjoined elliptic operator on a compact domain. In the second equality the first variation of the mean curvature 3.2.3 was used. Now the stability of minimal surfaces can be introduced.

**Definition 3.2.7.** (Stability of minimal surfaces)
Let $S$ be a minimal surface and $\Phi$ a variation of $S$ with variational vectorfield $v = \phi m$. Then $S$ is called stable if $\delta_v^2 \mathbf{vol}(S) \geq 0$ for all $\phi \geq 0$ with $\phi \not\equiv 0$.

**Lemma 3.2.8.** (Equivalent definitions of stability)
The following statements are equivalent

1. $\delta_v^2 \mathbf{vol}(S) \geq 0$
2. $\lambda \geq 0$
3. $\exists \phi \geq 0$ on $S$ such that $\delta_v p \geq 0$

*Proof.* The implications $(1) \Leftrightarrow (2)$ and $(1), (2) \Rightarrow (3)$ follow from equations (3.53) and (3.55). However how (1) and (2) follow from (3) is not obvious, for a proof see for example [3]. □

## 3.3 Stability and Marginally Outer Trapped Tubes

The following section is largely based on [3]. For a given MOTS $S$ in a leaf $\Sigma_0$ of a spacetime foliation $\{\Sigma_t\}$ of spacelike hypersurfaces it is natural to pose the question whether there exists a hypersurface $\mathcal{T}$ foliated by the time evolution of $S$ for every leaf $\Sigma_t$. One crucial criterion for the existence of such a smooth time evolution is the stability of the initial MOTS, which is the subject of this section. If it exists, the hypersurface $\mathcal{T}$ is then called marginally outer trapped tube (MOTT), a more precise definition can be found below.

### 3.3.1 First and Second Variation of the Null Expansion

The quantity used to classify the stability of MOTS is the first variation of the null expansion. First recall the definition of the variation of an arbitrary surface $S$ from 3.2.1. Although this definition was introduced in a Riemannian context, it works completely analogous for a spacetime. We denote the resulting family of surfaces under a variation $\Phi$ with respect to the vectorfield $q$ along $S$ as $S_t$. We can then (similarly to 3.2.2) define the first variation of the null expansion.

**Definition 3.3.1.** (First variation of the null expansion)
Let $S_t$ be a one parameter family of surfaces along the vectorfield $q$ as above, $l_t^+$ a differentiable (with respect to $t$) nowhere zero null vector in the normal bundle of $S_t$ and $\theta_t$ the null expansion scalar with respect to $l_t^+$ on each surface $S_t$. Then the first variation of the null expansion is defined as

$$\delta_q\theta = \partial_t\theta_t|_{t=0} \tag{3.56}$$

Note that $\delta_q\theta$ is linear, but not $C^\infty$linear in $q$. Our next aim is to introduce a linear elliptic operator that gives the first variation of the null expansion which is of great practical value. First we need the following technical lemma.

**Lemma 3.3.2.** (Explicit form of the first variation of the null expansion)
Let $S$ be a surface with second fundamental form vector $K^\mu_{AB}$ and induced covariant derivative $\nabla$. Let $l^+$ and $\theta$ be as before and decompose the variational vector $q$ in its tangential and normal parts $\tan(q)$ and $\mathrm{nor}(q)$, respectively. The normal component can also be expressed in terms of the null basis $\{l^+, l^-\}$ as $\mathrm{nor}(q) = bl^+ - \frac{u}{2}l^-$ for two functions $b$ and $u$ on $S$. Then the first variation of the second fundamental form is given by

$$\begin{aligned}\delta_q\theta =& a\theta + \tan(q)(\theta) - b\left(K^\mu_{AB}K^{\nu AB}l^+_\mu l^+_\nu + \bar{G}_{\mu\nu}l^\mu_+ l^\nu_+\right) - \Delta_S u + 2s^A\nabla_A u \\ &+ \frac{u}{2}\left(R - H^2 - \bar{G}_{\mu\nu}l^\mu_+ l^\nu_- - 2s^A s_A + 2\nabla_A s^A\right)\end{aligned} \tag{3.57}$$

where $a = -\frac{1}{2} l^-_\mu \partial_t l^\mu_{+t}|_{t=0}$ and $s_A = -\frac{1}{2} l^-_\mu \nabla_A l^\mu$.

**Remark 3.3.3.** This lemma is the analogon to 3.2.3. There we calculated the first variation of the mean curvature $\delta_v p$. Since the null expansion can be decomposed as $\theta^\pm = H \pm p$, some similarity is expected. The additional terms arise from the addition of $H$. The proof is rather technical and can be found in [3], appendix A.

### 3.3.2 The Stability Operator

Next we can define the stability operator. First we introduce a different decomposition of nor$(q)$ in terms of $l^+$ and an arbitrary vectorfield $v$ normal to $S$ with normalization $v^\mu l^+_\mu = 1$, which, as opposed to $l^-$, does not restrict the causal character of $v$ anywhere on $S$. The vector $v$ is then uniquely defined by a function $V$ according to $v^\mu = -\frac{1}{2} l^\mu_- + V l^\mu_+$. We use $\{v, l^+\}$ as a normal basis. Lastly we introduce a vector $u^\mu = \frac{1}{2} l^\mu_- + V l^\mu$, which is orthogonal to $v$ and satisfies $u^\mu u_\mu = -v^\mu v_\mu = -2V$.

**Definition 3.3.4.** (Stability operator)
For a function $\psi$ and any vector $v$ normal to $S$ satisfying $l^\mu_+ v_\mu = 1$ as above the stability operator is defined as

$$\begin{aligned} L_v \psi = & - \Delta_S \psi + 2 s^A \nabla_A \psi + \\ & \left( \frac{1}{2} R - V K^\mu_{AB} K^{\nu AB} l^+_\mu l^+_\nu - \bar{G}_{\mu\nu} l^\mu_+ u^\nu - s^A s_A + \nabla_A s^A \right) \psi \end{aligned} \tag{3.58}$$

**Remark 3.3.5.** Note that this is not the same operator as in equation (3.22), but the same name was chosen to highlight the analogous nature.

The following lemma will provide the connection between the stability operator and the first variation of the null expansion.

**Lemma 3.3.6.** (First variation of the null expansion in terms of stability operator)
Let $S$ be a MOTS, then the first variation of the null expansion in the direction of the null vector $\psi l^+$ and any normal vector $\psi v$ as above is then given by

- $\delta_{\psi l^+} \theta = -\psi \left( K^\mu_{AB} K^{\nu AB} l^+_\mu l^+_\nu + \bar{G}_{\mu\nu} l^\mu_+ l^\nu_+ \right)$
- $\delta_{\psi v} \theta = L_v \psi$

respectively.

*Proof.* Most of the work has already been done in Lemma 3.3.2. The result follows immediately from plugging the respective vectors into equation (3.57).

First recall the definitions of the various vectors used:

- $q = \tan(q) + bl^+ - \frac{u}{2}l^-$
- $v = Vl^+ - \frac{1}{2}l^-$
- $u = \frac{1}{2}l^- + Vl^+$

We proceed by checking the first equality of the lemma:

$$\begin{aligned}\delta_{\psi l^+}\theta =& a\theta + \tan\bigl(\psi l^+\bigr)(\theta) - \psi\left(K^{\mu}_{AB}K^{\nu AB}l^+_\mu l^+_\nu + \bar{G}_{\mu\nu}l^\mu_+ {l_+}^\nu\right) && (3.59)\\ &- \Delta_S u + 2s^A\nabla_A u + \frac{u}{2}\left(R - H^2 - \bar{G}_{\mu\nu}l^\mu_+ l^\nu_- - 2s^A s_A + 2\nabla_A s^A\right)\\ =& -\psi\left(K^{\mu}_{AB}K^{\nu AB}l^\nu_\mu l^\nu_+ + \bar{G}_{\mu\nu}l^\mu_+ l^\nu_+\right), && (3.60)\end{aligned}$$

where the first two terms vanish because $S$ is a MOTS and $l^+$ is normal to $S$. The last terms are zero since $u = 0$. Next we calculate the second identity, where the first two terms are omitted for the same reason

$$\begin{aligned}\delta_{\psi v}\theta =& -\psi V\left(K^{\mu}_{AB}K^{\nu AB}l^+_\mu l^+_\nu + \bar{G}_{\mu\nu}l^\mu_+ l^\nu_+\right) - \Delta_S\psi + 2s^A\nabla_A\psi && (3.61)\\ &+ \frac{\psi}{2}\left(R - H^2 - \bar{G}_{\mu\nu}l^\mu_+ l^\nu_- - 2s^A s_A + 2\nabla_A s^A\right)\\ =& -\Delta_S\psi + 2s^A\nabla_A\psi && (3.62)\\ &+ \left(\frac{1}{2}R - VK^{\mu}_{AB}K^{\nu AB}l^+_\mu l^+_\nu - \bar{G}_{\mu\nu}l^\mu_+ u^\nu_+ - s^A s_A + \nabla_A s^A\right)\psi\\ =& L_v\psi && (3.63)\end{aligned}$$

where $H^2$ vanishes after the second equality since the mean curvature can be written as $H^\mu = -\frac{1}{2}\left(\theta^- l^\mu_- + \theta^+ l^\mu_+\right)$ (see 3.1.7) thus $H^2 = 0$ because $S$ is a MOTS ($\theta^+ = 0$) and $\{l^+, l^-\}$ are null. □

### 3.3.3 Stability of MOTS

Similarly to the stability of minimal surfaces in 3.2.7 we can define stable MOTS in terms of the first variation of the null expansion and subsequently use the stability operator to find practical criteria for stability with the help of some PDE theory.

**Definition 3.3.7.** (Stability for MOTS)
Let $S$ be a MOTS and $v$ a vectorfield normal to $S$ satisfying $v_\mu l^\mu_+ = 1$. If there exists a function $\psi \geq 0$, $\psi \not\equiv 0$ on $S$, such that

- $\delta_{\psi v}\theta \geq 0$, $S$ is called stable
- $\delta_{\psi v}\theta \equiv 0$, $S$ is called marginally stable
- $\delta_{\psi v}\theta \geq 0$ and $\delta_{\psi v}\theta \not\equiv 0$, $S$ is called strictly stable

with respect to the direction $v$.

**Remark 3.3.8.** First note that the stability of the MOTS depends on the direction $v$, however when it is clear from the context the phrase "with respect to the direction $v$" is often omitted.
Secondly, in the literature the same definition can often be found as "strictly stably outermost MOTS". It is not immediately obvious why a stable mots needs to be outermost, but this follows from the so-called barrier property. Since the barrier property is not treated in this thesis the label "outermost" is omitted. For further reading on barrier properties of MOTS see for example [3] section 7.

For operators of the form of the stability operator the principal eigenvalue $\lambda$, which is defined as the eigenvalue with the smallest real part, exists. Additionally the eigenfunction $\phi$ corresponding to $\lambda$, called the principal eigenfunction, is positive everywhere. This result can be found in [13] in section 6.5.2. We can now, similarly to the above section, draw the connection between stability in terms of the first variation and the stability operator.

**Lemma 3.3.9.** (Stability of MOTS in terms of the stability operator)
Let $S \subset \Sigma_0$ be a MOTS and $\lambda$ the principal eigenvalue of the stability operator $L_v$. Then $S$ is stable iff $\lambda \geq 0$ and strictly stable iff $\lambda > 0$.

*Proof.* Suppose the principal eigenvalue satisfies $\lambda \geq 0$ for stability and $\lambda > 0$ for strict stability. Consider the variation $\delta_{\phi v}$ where $\phi$ is the principal eigenfunction. Then $\delta_{\phi v}\theta = L_v\phi = \lambda\phi$ by Lemma 3.3.6, which corresponds to being stable or strictly stable depending on whether $\lambda \geq 0$ or $\lambda > 0$.

For the converse direction note that the adjoint operator $L^\dagger$ with respect to the standard $L^2$ inner product $\langle \cdot, \cdot \rangle$ has the same principal eigenvalue $\lambda = \lambda^\dagger$ with principal eigenfunction $\phi^\dagger$. We can now calculate for (strictly) stable $S$:

$$\lambda \underbrace{\langle \phi^\dagger, \psi \rangle}_{>0} = \langle L_v^\dagger \phi^\dagger, \psi \rangle = \langle \phi^\dagger, L_v \psi \rangle = \langle \phi^\dagger, \delta_{\psi v}\theta \rangle \geq 0 (> 0), \tag{3.64}$$

which concludes the proof. □

### 3.3.4 Marginally Outer Trapped Tubes

As mentioned in the introduction of this section we want to define MOTTs and link their existence to the stability of a MOTS in the initial leaf $\Sigma_0$. This section will be kept short since the technicalities exceed the scope of this thesis, for a more detailed treatment see for example [3] or [2].

**Definition 3.3.10.** (Marginally Outer Trapped Tube)
Let $\left(\mathscr{M}, \bar{g}_{\mu\nu}\right)$ be a spactime foliated by hypersurfaces $\{\Sigma_t\}$. A co-dimension one submanifold $\mathcal{T}$ is called marginally outer trapped tube (MOTT) if $\mathcal{T} \cap \Sigma_t = S_t$, where each $S_t$ is a MOTS.

The following theorem demonstrates the importance of the stability of a given MOTS for the existence of a MOTT. The proof is omitted here, but can be found in [3] as Theorem 9.2.

**Theorem 3.3.11.** (Existence of MOTTS)
Let $\left(\mathscr{M}, \bar{g}_{\mu\nu}\right)$ be a spacetime foliated by smooth hypersurfaces $\Sigma_t$ for $t \in [0, T]$ and assume the leaf $\Sigma_0$ contains a smooth and strictly stable MOTS $S_0$. Then for some $\tau \in (0, T]$ there exists a smooth MOTT adapted to the foliation

$$\mathcal{T}_{[0,T)} = \Phi\left(S_0 \times [0, \tau)\right), \tag{3.65}$$

where $\Phi$ denotes the variation of $S_0$ as in Definition 3.2.1. Additionally every $S_t$ for $t \in [0, \tau)$, $S_t = \Phi\left(S_0 \times [0, t)\right)$ is a smooth and strictly stable MOTS.

# Chapter 4

# Visibility

## 4.1 Conformal Completions

For the remainder of this thesis conformal completions are of vital importance, since they are a main tool for the study of global properties of spacetimes. Introduced first by Roger Penrose, a conformal completion of a given spacetime is a manifold with boundary where its interior is the original spacetime. This lets us characterize infinities and later on visibility (or invisibility) of sets from these infinities. The following definitions and examples are based on section 3.1 in [8]. We start of with the central definition of this section.

**Definition 4.1.1.** (Conformal Completion)
A pair $(\widetilde{\mathscr{M}}, \tilde{g})$ is called a conformal completion at infinity of $(\mathscr{M}, g)$ if $\widetilde{\mathscr{M}}$ is a manifold with boundary and the following conditions hold:

- $\mathscr{M}$ is the interior of $\widetilde{\mathscr{M}}$.
- On $\widetilde{\mathscr{M}}$ there exists a function $\Omega$ such that the metric $\tilde{g}$, defined as $\Omega^2 g$ on $\mathscr{M}$, extends smoothly to the boundary of $\widetilde{\mathscr{M}}$, with the extended metric maintaining its signature on the boundary.
- $\Omega$ is positive on $\mathscr{M}$, smooth on $\widetilde{\mathscr{M}}$ and vanishes on

  $$\mathscr{S} := \partial\widetilde{\mathscr{M}},$$

  with $d\Omega$ vanishing nowhere on $\mathscr{S}$.

We set $\mathscr{S}^+ := \mathscr{S} \cap J^+(\mathscr{M})$ and $\mathscr{S}^- := \mathscr{S} \cap J^-(\mathscr{M})$. Conformal completions, specifically $\mathscr{S}^+$, let us introduce a notion of the asymptotic behaviour of spacetimes.

**Definition 4.1.2.** (Asymptotic behaviour of spacetimes)
Let $(\mathscr{M}, g)$ be a spacetime with conformal completion $(\widetilde{\mathscr{M}}, \tilde{g})$. Then $(\mathscr{M}, g)$ is called

- asymptotically flat, if $\mathscr{S}^+$ is null
- asymptotically de Sitter, if $\mathscr{S}^+$ is spacelike
- asymptotically anti de Sitter, if $\mathscr{S}^+$ is timelike

### 4.1.1 Examples of Conformal Completions

#### Minkowski spacetime

One of the simplest examples of a conformal completion is the one of $n$-dimensional Minkowski spacetime. In the following $n \geq 2$ is assumed. First introduce coordinates $u = t-r$, $x = 1/r$ and $x^A$ on $\mathbb{R}^{n+1}\backslash\{r = 0\}$, with $(r, x^A)$ being the usual spherical coordinates on $\mathbb{R}^n$. In these coordinates the metric takes the form

$$\eta = x^{-2}(x^2du^2 + 2dudx + d\omega^2), \tag{4.1}$$

where $d\omega^2$ is the standard round metric on the sphere $S^{n-1}$. Then $\Omega := x$ can be used as the conformal factor to rescale the metric, yielding

$$\tilde{\eta} := \Omega^2\eta = x^2du^2 + 2dudx + d\omega^2. \tag{4.2}$$

Clearly this metric extends smoothly to the boundary $\mathscr{S}^+ = \{x = 0\}$. Note that $\mathscr{S}^+ \approx \mathbb{R} \times S^{n-1}$. This construction is referred to as the *standard conformal completion of Minkowski spacetime at future null infinity.* We denote $\mathbb{R}^4$ with the hypersurface $\{x = 0\}$ attached $\widetilde{\mathscr{M}}^+$. An analogous construction of $\widetilde{\mathscr{M}}^-$ (by replacing $t$ with $-t$ in all formulae above) called *standard conformal completion of Minkowski spacetime at past null infinity* yields $\mathscr{S}^- = \{x = 0\}$ in the extended spacetime. The whole completion is then given by adding $\mathscr{S}^+$ and $\mathscr{S}^-$ simultaneously. It turns out a few additional boundary points can be added to the conformal completion, namely $i^0$, called *spatial infinity*, the *future timelike infinity* $i^+$ and the *past timelike infinity* $i^-$, for details see [8]. A sketch of the construction can be seen in figure 4.1.

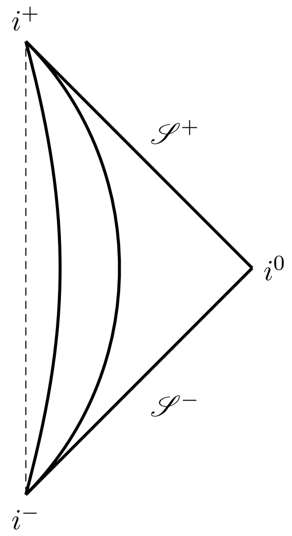


Figure 4.1: Conformal completion of Minkowski spacetime. The dashed line represents $\{r = 0\}$. Every point in the diagram represents a $(n-2)$-dimensional sphere, except for $r = 0$, $i^\pm$ and $i^0$, which are points.

Special care must be taken in adding the aforementioned points $\{i^0, i^+, i^-\}$, since the resulting topological space is no longer a manifold unless some identifications are made. We now proceed to construct these points and make the necessary identifications.

Let $x^\alpha$ be the standard coordinates in Minkowski spacetime for which the metric is diagonal with constant entries. Then we introduce new coordinates in the region where $\eta_{\mu\nu}x^\mu x^\nu < 0$ and $x^0 < 0$ by

$$y^\alpha = \frac{x^\alpha}{\eta_{\mu\nu}x^\mu x^\nu}. \tag{4.3}$$

Since $\eta_{\mu\nu}y^\mu y^\nu = \frac{1}{\eta_{\mu\nu}x^\mu x^\nu}$ and $y^0 > 0$ the map $x \mapsto y$ is a differomorphism from $\{\eta_{\mu\nu}x^\mu x^\nu < 0, x^0 < 0\}$ to $\{\eta_{\mu\nu}y^\mu y^\nu < 0, y^0 > 0\}$.

The past light cone of the origin of the $x$-coordinates is the future light cone of the origin of the $y$-coordinates. The point $i^-$ is then defined as the origin of the $y$-coordinates, and the point $i^+$ is obtained similarly by reversing the time orientation in the construction above.

The spatial infinity $i^0$ is again defined as the origin of new coordinates, which are obtained by using the transformation (4.3) on the set $\{\eta_{\mu\nu}x^\mu x^\nu > 0\}$. This set is then diffeomorphically mapped to $\{\eta_{\mu\nu}y^\mu y^\nu > 0\}$. Figure 4.2 below portrays the complete conformal completion of Minkowski spacetime.

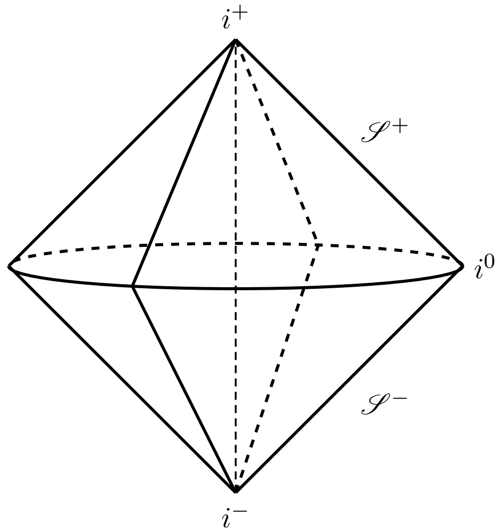


Figure 4.2: Entire conformal completion obtained by rotating figure 4.1 around the $\{r = 0\}$-axis. Note that $i^0$ is again only a point, even though it is drawn as a circle in the diagram.

**De Sitter spacetime**

This example is of special importance for this thesis since we consider applications of theorems in the next chapter specifically for de Sitter and asymptotically de Sitter spacetimes. Consider de Sitter spacetime

$$\mathscr{M} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times S^{n-1}, \quad g = \cos^{-2}(t)(-dt^2 + dS^2_{n-1}). \tag{4.4}$$

With the conformal factor $\Omega := \cos^{-1}(t)$ and the round metric $dS^2_{n-1}$ on the $(n-1)$-sphere $S^2_{n-1}$. De Sitter spacetime conformally embeds into the *Einstein static universe*

$$\mathscr{M}' = \mathbb{R} \times S^{n-1}, \quad g' = -dt^2 + dh^2. \tag{4.5}$$

The conformal completion is then given by $\widetilde{\mathscr{M}} = [-\pi/2, \pi/2] \times S^{n-1}$ with $\mathscr{I}^\pm = \{\pm\pi/2\} \times S^{n-1}$ and $\tilde{g} = -dt^2 + dS^2_{n-1}$.

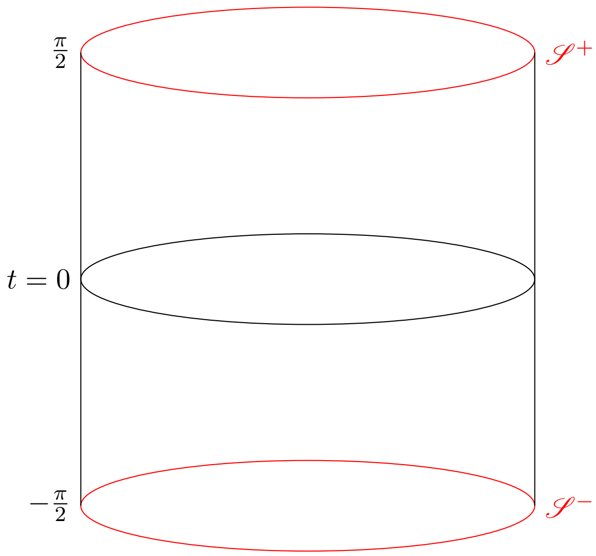


Figure 4.3: Conformal completion of de Sitter spacetime where every circle $\{t = \text{const.}\}$ represents a sphere $S^{n-1}$, including $\mathscr{I}^{\pm}$ as the top and bottom of the cylinder, respectively.

## 4.2 Visibility in Asymptotically De Sitter Spacetimes

In this section we study the visibility (or rather invisibility) of weakly trapped surfaces and marginally outer trapped regions, which will be defined later, in an asymptotically de Sitter setting. The theorems presented and their proofs can be found in [9]. Here some calculations are explained in some more detail.
In an asymptotically flat setting it has been shown that trapped surfaces are externally invisible (see Theorem 6.1 in [10]). That means given a future conformal completion and certain energy and causality conditions there are no trapped surfaces in $I^-(\mathscr{I}^+, \widetilde{\mathscr{M}})$. This result can be extended to weakly trapped surfaces in the asymptotically de Sitter case in the following manner.

**Theorem 4.2.1.** (Visibility of weakly trapped surfaces for $\mathscr{I}^+$ spacelike) Suppose $(\mathscr{M}, g)$ satisfies the null energy condition and admits a conformal completion $(\widetilde{\mathscr{M}}, \tilde{g})$ which is future causally simple with respect to $\mathscr{M}$ (i.e. $J^+(K, \widetilde{\mathscr{M}})$ is closed for all compact $K$). Suppose $\mathscr{I}^+$ is spacelike. Let $A \subset \mathscr{M}$ be a set such that $J^+(A, \widetilde{\mathscr{M}})$ does not contain all of $\mathscr{I}^+$. Then there are no weakly trapped surfaces in $J^+(A, \widetilde{\mathscr{M}}) \cap I^-(\mathscr{I}^+, \widetilde{\mathscr{M}})$.

*Proof.* We aim to construct two different null hypersurfaces that intersect on one null geodesic which connects a supposed weakly trapped surface to

$\mathscr{S}^+$ and then apply the maximum principle for smooth null hypersurfaces to reach a contradiction.
Suppose there exists a weakly trapped surface $S \subset J^+(A, \widetilde{\mathscr{M}}) \cap I^-(\mathscr{S}^+, \widetilde{\mathscr{M}})$ for some set $A \in \mathscr{M}$ such that $J^+(A, \widetilde{\mathscr{M}})$ does not contain all of $\mathscr{S}^+$. Since $J^+(S, \widetilde{\mathscr{M}})$ also does not contain all of $\mathscr{S}^+$ there exists a point

$$q_0 \in \partial \left( J^+(S, \widetilde{\mathscr{M}}) \cap \mathscr{S}^+ \right) = \partial J^+(S, \widetilde{\mathscr{M}}) \cap \mathscr{S}^+. \tag{4.6}$$

Our next goal is to define a geodesic sphere that meets $\partial J^+(S, \widetilde{\mathscr{M}})$ in exactly one point. To this end we introduce a Riemannian metric $h$ on $\mathscr{S}^+$, and let $U$ be a normal convex neighbourhood around $q_0$. Let $q_1$ be a point in $U \backslash J^+(S, \widetilde{\mathscr{M}})$, chosen such that some points of $J^+(S, \widetilde{\mathscr{M}})$ are within the injectivity radius of $q_1$. Let $q$ be the point on $\partial J^+(S, \widetilde{\mathscr{M}}) \cap U$ that minimizes the $h$-distance from $q_1$ to $\partial J^+(S, \widetilde{\mathscr{M}})$ in $\bar{U}$. Let $r > 0$ be this distance and $S^+$ the geodesic sphere centered around $q_1$. Note that $S^+$ is a smooth hypersurface in $\mathscr{S}^+$ that includes $q$ but does not meet $I^+(S, \widetilde{\mathscr{M}})$ since $\partial J^+(S, \widetilde{\mathscr{M}}) = J^+(S, \widetilde{\mathscr{M}}) \backslash I^+(S, \widetilde{\mathscr{M}})$. The last equation holds because $J^+(S, \widetilde{\mathscr{M}})$ is closed since we assumed causal simplicity and $I^+(S, \widetilde{\mathscr{M}})$ is the interior of $J^+(S, \widetilde{\mathscr{M}})$.
Since $q \in J^+(S, \widetilde{\mathscr{M}})$ there exists a null geodesic $\gamma : [a, b] \mapsto \widetilde{\mathscr{M}}$ satisfying $\gamma(a) \in S$ and $\gamma(b) = q$ emanating orthogonally from $S$. Since $\mathscr{S}^+$ is spacelike and $\gamma$ is a null-geodesic, $\gamma$ must intersect $\mathscr{S}^+$ transversally, and hence $\gamma([a, b)) \subset \mathscr{M}$. Additionally, since $\gamma$ does not enter the timelike future of $S$ there are no null focal points, and hence the null exponential map has full rank at all points along $\gamma$. This allows one to generate a smooth null hypersurface $N_1 \subset J^+(S, \widetilde{\mathscr{M}})$ containing the segment $\gamma|_{[a,b-\varepsilon]}$ for some $\varepsilon > 0$. Furthermore let $N_2$ be a smooth null hypersurface which is a subset of $J^-(S, \widetilde{\mathscr{M}})$ and contains $\gamma(a, b]$.
Below is a sketch of the basic idea of the construction of the null hypersurfaces $N_1$ and $N_2$, since this is a proof via contradiction the picture does not represent a possible scenario and should only help with visualisation.

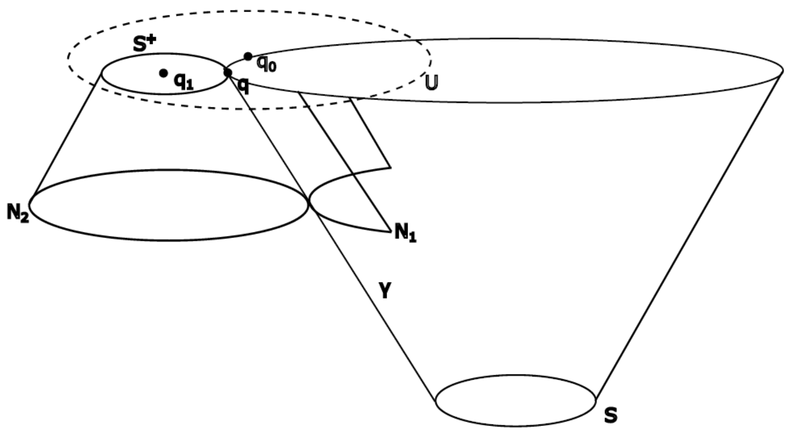


Figure 4.4: Construction of the null hypersurfaces $N_1$ and $N_2$

Since $S$ is future weakly trapped by assumption, both $\theta^{\pm}$ are non-positive on $S$. Let $\theta_1(s)$ be the null mean curvature of $N_1$ along $\gamma$, then $\theta_1(a) \leq 0$. The vorticity for orthogonally emanating congruences of null geodesics from a surface vanishes, hence the Raychaudhuri equation simplifies to

$$\theta_1' = -\mathbf{Ric}(K, K) - \mathrm{Tr}(\sigma^2) - \frac{\theta_1^2}{n-2}, \tag{4.7}$$

where $K$ is a null vectorfield proportional to $\dot{\gamma}$ and $\sigma$ denotes the shear tensor (see [17] equation 2.10). Applying the null-energy condition yields $\theta_1' \leq 0$, since both other terms are also obviously non-positive. Therefore $\theta_1(s) \leq 0$ for all $s \in [a, b - \varepsilon]$.
Our goal now is to find an expression and a bound for the null mean curvature on $N_2$ to subsequently apply a maximum principle to reach a contradiction. To this end we define a null vector $\tilde{K}$ at $q$ that is future outward pointing and orthogonal to $S^+$. Since $\tilde{K}$ lies in $\mathscr{S}^+$, it is null with regard to the unphysical metric $\tilde{g} = \Omega^2 g$. Since $\tilde{K}$ is proportional to $\dot{\gamma}$ and there exists a unique (up to positive rescaling) smooth future directed null vectorfield on $N_2$, we can extend $\tilde{K}$ to $N_2$. Because this extension is unique only up to a positive rescaling we can choose the normalization

$$\tilde{K}(\Omega) = \tilde{g}(\tilde{K}, \tilde{\nabla}\Omega) = -1. \tag{4.8}$$

along $\gamma$ near $q$. This normalization is possible because $\Omega \to 0$ as one approaches $\mathscr{S}^+$ and $\tilde{K}$ is null. Since $\gamma$ is a maximizing geodesic from $S$ to $q$, it has to start orthogonally to $S$. The same argument can be applied backwards to conclude that $\gamma$ also has to meet $S^+$ orthogonally.
Next we want to relate the null expansion scalar $\tilde{\theta}_2^+$ in the unphysical metric to $\theta_2^+$ in the physical metric. We now choose a spacelike 2-surface $S \subset N_2$

with projector $h_{\mu\nu} = g_{\mu\nu} + l_{(\mu}K_{\nu)}$, where $l$ denotes the second null direction with normalization $l_\mu K^\mu = -1$. Note that under the conformal transformation $g \to \tilde{g} = \Omega^2 g$ the vectors $l$ and $K$ turn into $\tilde{l} = \Omega^{-1}l$, $\tilde{K} = \Omega^{-1}K$ to satisfy $\tilde{g}_{\mu\nu}\tilde{l^\mu}\tilde{K^\nu} = -1$. Using equations (3.1.7) and (D.3) in [23] for the covariant derivative under conformal transformations we get

$$\tilde{\theta}_2^+ = \tilde{h}^{\mu\nu}\tilde{\nabla}_\mu \tilde{K}_\nu \tag{4.9}$$

$$= \left(\nabla_\mu \tilde{K}_\nu - 2\tilde{K}_\sigma \delta^\sigma_{(\mu}\nabla_{\nu)} \log\Omega - g_{\mu\nu}g^{\sigma\lambda}\tilde{K}_\sigma \nabla_\lambda \log\Omega\right)\tilde{h}^{\mu\nu} \tag{4.10}$$

$$= \left(\nabla_\mu(\Omega K_\nu) - 2\Omega K_\sigma \delta^\sigma_{(\mu}\nabla_{\nu)} \log\Omega - g_{\mu\nu}g^{\sigma\lambda}\Omega K_\sigma \nabla_\lambda \log\Omega\right)\tilde{h}^{\mu\nu} \tag{4.11}$$

$$= \frac{1}{\Omega}\nabla_\mu K_\nu h^{\mu\nu} + \Big(K_\nu \nabla_\mu \Omega - 2(\delta^\sigma{}_{(\mu}\nabla_{\nu)} \log\Omega)\Omega K_\sigma + \tag{4.12}$$

$$+ g^{\sigma\lambda}(\nabla_\lambda \log\Omega)\Omega K_\sigma\Big)\frac{1}{\Omega^2}h^{\mu\nu}. \tag{4.13}$$

Using $h^{\mu\nu}K_\nu = 0$ and $g_{\mu\nu}h^{\mu\nu} = (n-2)$ yields

$$\tilde{\theta}_2^+ = \frac{1}{\Omega}\theta_2^+ + g_{\mu\nu}g^{\sigma\lambda}(\nabla_\lambda \log\Omega)K_\sigma \frac{1}{\Omega}h^{\mu\nu} \tag{4.14}$$

$$= \frac{1}{\Omega}\theta_2^+ + (n-2)g^{\sigma\lambda}(\nabla_\lambda \log\Omega)K_\sigma \frac{1}{\Omega} \tag{4.15}$$

$$= \frac{1}{\Omega}\theta_2^+ + (n-2)\frac{1}{\Omega^2}g^{\sigma\lambda}K_\sigma \tag{4.16}$$

$$= \frac{1}{\Omega}\theta_2^+ + (n-2)\frac{1}{\Omega}\tilde{K}(\Omega). \tag{4.17}$$

Finally we can rearrange the last equation to obtain

$$\theta_2^+ = -(n-2)\tilde{K}(\Omega) + \Omega\tilde{\theta}_2^+. \tag{4.18}$$

On $N_2$ the null expansion scalar $\tilde{\theta}_2^+$ is bounded, therefore close to $q$ we have $\tilde{\theta}_2^+ > 0$ since the first term in equation (4.18) is positive and the second is arbitrarily small because $\Omega \to 0$ as one approaches $q$. On $\gamma$ near $q$ we now have the following relation between the null expansion scalars for both null hypersurfaces,

$$\theta_1^+ \leq 0 < \theta_2^+. \tag{4.19}$$

At this stage we introduce the following terminology, we say $N_1$ lies to the future of $N_2$ near $q$, if for a neighbourhood $U \in \widetilde{\mathscr{M}}$ around $q$, $N_1 \cap U \in J^+(N_1 \cap U, U)$. Since $\mathscr{S}^+$ does not meet $I^+(S, \widetilde{\mathscr{M}})$, this is clearly true. This and equation (4.19) lets us apply the maximum principle for smooth null hypersurfaces, which states that $N_1$ and $N_2$ coincide near $q$. However, this is not the case by construction, thus we have reached a contradiction and concluded the proof. □

As is apparent from the last line of the proof, understanding the maximum principle for smooth null hypersurfaces is of vital importance. The next section is dedicated to this theorem (see 4.3.1), but first we conclude our study of visibility.

Weakly trapped surfaces as in the previous theorem make assumptions on both null expansion scalars $\theta^{\pm}$. We can find an analogous statement for weakly outer trapped regions, which only require the null expansion scalar to be non-positive on the boundary for the outward direction.

**Definition 4.2.2.** (Weakly outer trapped region)
Let $(\mathscr{M}, g)$ be a spacetime and let $T$ be a compact connected spacelike hypersurface with $\partial T =: S \in \mathscr{M}$. If $S$ is a weakly outer trapped surface (i.e. $\theta^{+} \leq 0$) with respect to the outward direction of the region, we call $T$ a *weakly outer trapped region.*

With this definition at hand, we conclude this section with a theorem regarding the visibility of such weakly outer trapped regions.

**Theorem 4.2.3.** (Visibility of future weakly outer trapped regions for $\mathscr{S}^{+}$ spacelike)
Suppose $(\mathscr{M}, g)$ admits a conformal completion $(\widetilde{\mathscr{M}}, \tilde{g})$ that is asymptotically de Sitter, is causally simple w.r.t. $\widetilde{\mathscr{M}}$ and satisfies the null energy condition. Let $A \in \mathscr{M}$ be a set such that $J^{+}(A, \widetilde{\mathscr{M}})$ does not contain all of $\mathscr{S}^{+}$. Then there are no weakly outer trapped regions in $J^{+}(A, \widetilde{\mathscr{M}}) \cap I^{-}(\mathscr{S}^{+}, \widetilde{\mathscr{M}})$.

*Proof.* The proof is very similar to the one of Theorem 4.2.1. Suppose there exists a weakly outer trapped region $T$, with boundary $S$, contained in $J^{+}(A, \widetilde{\mathscr{M}}) \cap I^{-}(\mathscr{S}^{+}, \widetilde{\mathscr{M}})$ for an arbitrary set $A \in \mathscr{M}$. Then $S$ is a weakly outer trapped surface by Definition 4.2.2. We again construct a null geodesic $\gamma : [a, b] \mapsto \widetilde{\mathscr{M}}$ with $\gamma(a) \in S$ and $\gamma(b) \in \partial J^{+}(T, \widetilde{\mathscr{M}}) \cap \mathscr{S}^{+}$. Since $\gamma$ does not enter the timelike future of $T$, we know that $\dot{\gamma}(a)$ points in the outward direction. Since $S$ is weakly outer trapped and $(\mathscr{M}, g)$ satisfies the null energy condition, similarly to the proof in Theorem 4.2.1, $\theta^{+}(s) \leq 0$ along $\gamma(s)$ for all $s$. The rest of the proof is exactly the same, and we find a contradiction via the maximum principle for smooth null hypersurfaces. □

## 4.3 The Maximum Principle for Smooth Null Hypersurfaces

In this section we present the maximum principle for smooth null hypersurfaces, which can also be found in [14]. First we state the theorem, the proof of which will be the main objective of this section.

**Theorem 4.3.1.** (Maximum principle for smooth null hypersurfaces)
Let $N_1$ and $N_2$ be smooth null hypersurfaces in a spacetime $(\mathscr{M}, \bar{g})$. If

- $N_1$ and $N_2$ meet at $p \in \mathscr{M}$ and $N_2$ lies to the future of $N_1$ near $p$
- the null expansion scalars $\theta_1$ of $N_1$, and $\theta_2$ of $N_2$, satisfy $\theta_2 \leq 0 \leq \theta_1$

then $N_1$ and $N_2$ coincide near $p$ and this common null hypersurface satisfies $\theta = 0$.

The main idea of the proof will be to intersect both null hypersurfaces with a spacelike hypersurface through the point $p \in \mathscr{M}$, and then show that the spacelike intersections agree. In order to do that, we have to express the spacelike intersections as a graph over some base surface and reformulate the null expansion scalar as a quasi-linear elliptic operator acting on the graph function. Then we can use a maximum principle for quasi-linear elliptic operators to show that the spacelike intersections have to agree. But before we can proceed with the proof, some prerequisites are required to accomplish the above.

**Remark 4.3.2.** (Guide to literature)
The maximum principle for null hypersurfaces has been treated extensively in several papers, for example in [14] and [4]. In [14] the hypersurfaces $N_1$ and $N_2$ are intersected with a timelike hypersurface in order to prove the equality of the resulting spacelike intersections represented by $C^2$-functions. While we here we only deal with the smooth case we feel that our proof is slightly simpler since we intersect the hypersurfaces $N_1$ and $N_2$ with a spacelike hypersurface in order to show the equality of the resulting spacelike intersections.
To show the equality of the spacelike intersections a maximum principle for quasi-linear elliptic operators is used. [14] references a maximum principle by A.D. Alexandrov, which was difficult to find in the literature. In this thesis it is substituted with a more readily available alternative.

### 4.3.1 Maximum Principle for Quasi-Linear Operators

First we need to introduce quasi-linear operators and the appropriate maximum principle for the task at hand. The following mostly relies on [13].

**Definition 4.3.3.** ($k$-th order partial differential equations)
Let $F$ be a map

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \mapsto \mathbb{R} \tag{4.20}$$

for an integer $k \geq 1$ and an open subset $\Omega \in \mathbb{R}^n$. Furthermore let

$$D^k u(x) = \{D^\alpha u(x) : |\alpha| = k\} \tag{4.21}$$

be the set of $k$-th order derivatives of $u(x)$. Then the expression

$$F(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0 \quad (x \in \Omega) \tag{4.22}$$

for $u : \Omega \mapsto \mathbb{R}$ is called a $k$-th order partial differential equation.

**Definition 4.3.4.** ($U$-admissibility)
Let $\Omega \subset \mathbb{R}^n$ be compact and $U \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. We say $u \in C^2(\Omega)$ is $U$-*admissible* if $(x, u(x), \partial u(x)) \in U$ for all $x \in \Omega$, where $\partial u = (\partial_1 u, \ldots, \partial_n u)$.

**Definition 4.3.5.** (Uniformly elliptic Quasi-linear operator)
Let $u \in C^2(\Omega)$ be $U$-admissible and $a^{ij}, b \in C^1(U)$ with $a^{ij}$ symmetric and $1 \leq i, j \leq n$. We then call a second order partial differential operator $Q$ *quasi-linear* if it can be written as

$$Q[u] = a^{ij}(x, u, \partial u)\partial_{ij} u + b(x, u, \partial u). \tag{4.23}$$

Furthermore $Q$ is *uniformly elliptic* if for each $(x, r, p) \in U$, and for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \xi^i \neq 0$

- there is a constant $C_E > 0$ such that

$$a^{ij}(x, r, p)\xi_i \xi_j \geq C_E \|\xi\|^2 \tag{4.24}$$

- and

$$\left|\frac{\partial a^{ij}}{\partial p^k}\right|, \left|\frac{\partial a^{ij}}{\partial r}\right|, \left|\frac{\partial b}{\partial p^k}\right|, \left|\frac{\partial b}{\partial r}\right| \leq C_E. \tag{4.25}$$

We can now proceed with the main result of this section, which is needed in the proof of Theorem 4.3.1. This corollary including its proof can be found in [1] as Theorem 2.4. There it is stated for lower regularity, but for our purposes a slightly weakened version suffices since the functions we apply it to will be smooth.

**Corollary 4.3.6.** (Maximum principle)
Let $\Omega \subset \mathbb{R}^n$ be compact, and $U \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. Furthermore let $u_0, u_1 : \Omega \to \mathbb{R}$ be $U$-admissible $C^2(\Omega)$ functions and $Q$ a uniformly elliptic quasi-linear operator satisfying the following.

1. There is a constant $C_M$ such that $Q[u_0] \leq C_M \leq Q[u_1]$
2. $u_1 \leq u_0$ in $\Omega$, and there is at least one point $x \in \Omega$ with $u_1(x) = u_0(x)$.

Then $u_0 \equiv u_1$ in $\Omega$.

### 4.3.2 Graph Representation of Spacelike 2-Surfaces

We now proceed with our objective of expressing a spacelike surface as a graph and subsequently the null expansion as a quasi-linear elliptic operator. First let $(\mathscr{N}, g)$ be a spacelike hypersurface in a spacetime $(\mathscr{M}, \bar{g})$. Then we choose a surface $S \in \mathscr{N}$ that we want to express as the graph of a function over some base surface $R \in \mathscr{N}$ that is close to $S$ but otherwise arbitrary. Next we choose Gaussian normal coordinates such that the line element in $\mathscr{N}$ close to $R$ is locally given by

$$ds^2 = dr^2 + h_{AB} dx^A dx^B \tag{4.26}$$

and $R$ is parametrized by $r = r_0$. Next we define a function $f(x^A)$ on $R$ such that $S$ is given by $r' = r - f(x^A) = r_0$. This construction expresses the surface $S$ as a graph over $R$. Let $m$ be the unit normal of $R$ in $\mathscr{N}$, the associated mean curvature is then given by $p = \nabla_i m^i$. Primed quantities will in general refer to $S$ while their un-primed counterparts lie in $R$. Thus the mean curvature of $S$ is given by $p' = \nabla_i m'^i$ and its metric by $h'_{AB}$. Next we introduce some useful notation for the following calculations:

- $f_A = \frac{\partial f}{\partial x^A}$
- $F^2 = h^{AB} f_A f_B$
- $\psi^2 = 1 + F^2$

In order to calculate the null expansion as an operator acting on $f(x^A)$, we need the following three technical results.

**Lemma 4.3.7.** (Metric on $S$)
In the setting described above the metric on $S$ and its inverse are given by

- $h'_{AB} = h_{AB} + f_A f_B$
- $h'^{AB} = h^{AB} - \frac{f_A f_B}{\psi^2}$.

*Proof.* Using the definition of $r'$ in the metric yields

$$\begin{aligned} ds^2 =& dr^2 + h_{AB} dx^A dx^B = (dr' + f_A dx^A)^2 + h_{AB} dx^A dx^B && (4.27) \\ =& dr'^2 + 2 f_A dx^A dr' + f_A dx^A f_B dx^B + h_{AB} dx^A dx^B. && (4.28) \end{aligned}$$

And thus

$$h'_{AB} = h_{AB} + f_A f_B. \tag{4.29}$$

We can now easily see that $h'^{AB}$ has the above form by

$$h'_{AB}h'^{BC} =(h_{AB}+f_Af_B)\left(h^{BC}-\frac{f^Bf^C}{\psi^2}\right) \tag{4.30}$$

$$=\delta_A^C - h_{AB}\frac{f^Bf^C}{\psi^2}+h^{BC}f_Af_B-\frac{f_Af_Bf^Bf^C}{\psi^2} \tag{4.31}$$

$$=\delta_A^C-\frac{f_Af^C}{\psi^2}+f_Af^C-\frac{F^2f_Af^C}{\psi^2} \tag{4.32}$$

$$=\delta_A^C-\frac{f_Af^C}{\psi^2}+\frac{(1+F^2)f_Af^C}{\psi^2}-\frac{F^2f_Af^C}{\psi^2} \tag{4.33}$$

$$=\delta_A^C. \tag{4.34}$$

□

**Lemma 4.3.8.** (Relation of the metric determinants)
Let $|\cdot|$ denote the determinant of the respective metric, then for the setting above

$$|h'|=\psi^2|h|=\psi^2|g|. \tag{4.35}$$

*Proof.* This proof is also a straightforward computation using the two dimensional Levi-Civita symbol $\varepsilon^{AB}$.

$$|h'| =\varepsilon^{AC}\varepsilon^{BD}h'_{AB}h'_{CD} \tag{4.36}$$

$$=\varepsilon^{AC}\varepsilon^{BD}(h_{AB}+f_Af_B)(h_{CD}+f_Cf_D) \tag{4.37}$$

$$=|h|+2\varepsilon^{AC}\varepsilon^{BD}h_{AB}f_Cf_D, \tag{4.38}$$

where in the last equality $\varepsilon^{AC}\varepsilon^{BD}f_Af_Bf_Cf_D$ vanishes since $\varepsilon^{AC}$ is anti-symmetric and $f_Af_C$ is symmetric. So

$$|h'| =|h|+|h|\underbrace{h^{CD}f_Cf_D}_{F^2} \tag{4.39}$$

$$=\psi^2|h|. \tag{4.40}$$

□

The next goal is to express the null expansion scalar $\theta' = H' \pm p'$ in terms of geometrical objects on $R$ and the graph function $f$.

**Lemma 4.3.9.** ($p'$ and $H'$ in terms of $f$ and quantities on $R$)
The mean curvatures $p$ of $R$ and $p'$ of $S$ satisfy the following relation

$$p' = \frac{p}{\psi} - \psi \Delta_R f - h^{AB} f_B \partial_A \frac{1}{\psi} \tag{4.41}$$

where $\Delta_R$ denotes the Laplacian on the reference surface $R$. Furthermore $H'$ is given by

$$H' = K - \frac{1}{\psi^2} K_{ij} g^{ik} g^{jl} (\delta_{kr} - f_A \delta_{Ak})(\delta_{lr} - f_B \delta_{Bl}) \tag{4.42}$$

for $K = g^{ij} K_{ij}$.

*Proof.* Recall that the metric on $\mathscr{N}$ has the form (4.26) in these coordinates. Then from the condition $g^{ij} m_i m_j = 1$ it is clear that the unit normal on $R$ is given by $m_i = \delta_{ri}$ where $m_r$ denotes the component in the $r$-direction. The unit normal on $S$ is $m'_i = \frac{1}{\psi} \frac{\partial r'}{\partial x^i} = \frac{1}{\psi}(\delta_{ri} - f_A \delta_{iA})$, which is also easily checked by $g^{ij} m'_i m'_j = \frac{1}{\psi^2} \psi^2 = 1$. The mean curvature on $S$ then reads

$$p' = \nabla_i m'^i = \frac{1}{\sqrt{|g|}} \partial_i \left[ \sqrt{|g|} g^{ij} \frac{1}{\psi} (\delta_{rj} - f_A \delta_{Aj}) \right] \tag{4.43}$$

$$= \frac{1}{\sqrt{|h|}} \partial_r \frac{\sqrt{|h|}}{\psi} - \frac{1}{\sqrt{|h|}} \partial_A \left( \frac{\sqrt{|h|} h^{AB} f_B}{\psi} \right) \tag{4.44}$$

$$= \frac{p}{\psi} + \underbrace{\partial_r \frac{1}{\psi}}_{=0} - \frac{1}{\sqrt{|h|}} \partial_A \left( \frac{\sqrt{|h|} h^{AB} f_B}{\psi} \right) \tag{4.45}$$

$$= \frac{p}{\psi} - \frac{1}{\psi} \Delta_R f - h^{AB} f_B \partial_A \frac{1}{\psi}. \tag{4.46}$$

Next we calculate $H'$:

$$H' = (g^{ij} - m'^i m'^j) K_{ij} \tag{4.47}$$

$$= \underbrace{g^{ij} K_{ij}}_{=:K} - K_{ij} m'^i m'^j \tag{4.48}$$

$$= K - \frac{1}{\psi^2} K_{ij} g^{ik} g^{jl} (\delta_{kr} - f_A \delta_{Ak})(\delta_{lr} - f_B \delta_{Bl}) \tag{4.49}$$

□

With the previous results we can now write the null expansion as a quasi-linear elliptic operator $\theta$ acting on a function $f$.

**Definition 4.3.10.** (Null expansion operator)
For a surface $S$ represented by a graph function $f$ on a reference surface $R$ we define the *null expansion operator* by

$$\begin{aligned}\theta[f] =& \frac{p}{\psi} - \frac{1}{\psi}\Delta_R f - h^{AB} f_B \partial_A \frac{1}{\psi} + \\ &+ K - \frac{1}{\psi^2} K_{ij} g^{ik} g^{jl} (\delta_{kr} - f_A \delta_{Ak})(\delta_{lr} - f_B \delta_{Bl})\end{aligned} \tag{4.50}$$

where $K = g^{ij} K_{ij}$.

**Theorem 4.3.11.** (Null expansion as quasi-linear elliptic operator)
The null expansion operator is quasi-linear and elliptic. For a surface $S$ represented by a graph $f$ the following holds.

$$\theta(S) = \theta[f] \tag{4.51}$$

*Proof.* Comparing the definition of $\theta[f]$ with Lemma 4.3.9 immediately yields

$$\theta[f] = p' + H' \tag{4.52}$$

showing the equality $\theta(S) = \theta[f]$. It remains to prove that $\theta[f]$ is indeed a quasi-linear elliptic operator as in Definition 4.3.5. The only second order terms we have to check to ensure quasi-linearity are

- $\frac{1}{\psi^2}\Delta_R f = \frac{1}{\psi^2} h^{AB} f_{AB}$ and
- $h^{AB} f_B \partial_A \frac{1}{\psi}$.

Here $\psi$ is a function of first order derivatives of $f$ as can be seen from Definition 4.3.2. Thus the first term is clearly quasi-linear. For the second term a short calculation is required.

$$\begin{aligned}h^{AB} f_B \partial_A \frac{1}{\psi} =& -\psi^{-2} h^{AB} f_B \partial_A \psi &(4.53)\\ =& -\frac{1}{2}\psi^{-2} h^{AB} f_B \partial_A \sqrt{1 + f^C f_C} &(4.54)\\ =& -\frac{1}{2}\psi^{-3} h^{AB} f_B \partial_A \left(f^C f_C\right) &(4.55)\\ =& -\frac{1}{2}\psi^{-3} h^{AB} f_B \left({f^C}_A f_C + f^C f_{CA}\right), &(4.56)\end{aligned}$$

where $f_{CA} = \partial_C \partial_A f$. This is also clearly quasi-linear second order.
Concerning ellipticity we can view equation (4.50) as a Laplacian perturbed by terms containing $f$ and its derivatives up to second order. Choosing the reference surface $R$ close enough to $S$ ensures that $f, f_A$ and $f_{AB}$ are sufficiently small and such that $\theta[f]$ is elliptic. Generally speaking (4.50) is a "prescribed mean curvature equation", which are treated in detail in [15]. □

To conclude this section we finally prove Theorem 4.3.1.

*Proof.* Let $\mathscr{N}$ be a spacelike hypersurface passing through the point $p$ as described in the theorem. Choose a spacelike reference surface $R \in \mathscr{N}$ close to $p$ and represent the spacelike intersections $S_1 := N_1 \cap \mathscr{N}$ and $S_2 := N_2 \cap \mathscr{N}$ as graphs of the functions $f_1$ and $f_2$ on $R$, respectively. If necessary shrink $\mathscr{N}$ such that $S_i$ are close enough to $R$. From the assumptions in the theorem we know the following:

- $f_1 \leq f_2$ (because $N_1$ lies to the future of $N_2$)
- $f_1(p) = f_2(p)$
- $\theta[f_2] \leq \theta[f_1]$ by assumption and Theorem 4.3.11.

Thus we can apply Corollary 4.3.6 to find $f_1 = f_2$ and hence $S_1 = S_2$. Furthermore let $l_i^+$ be the outward pointing null normal of $S_i$, then the null hypersurfaces $N_i$ are locally obtained by the exponentional map acting on $l_i^+$. We can write the null normal as $l_i^+ = n + m_i$ where $n$ is the timelike unit normal to $\mathscr{N}$ and $m_i$ the unit normal to $S_i$ in $\mathscr{N}$. We see that since $f_1 = f_2$ and $S_1 = S_2$ also $m_1 = m_2$ and hence $l_1^+ = l_2^+$. It follows that $N_1$ and $N_2$ agree near $p$ and this common hypersurface has null mean curvature $\theta = 0$ since by assumption $\theta_2 \leq 0 \leq \theta_1$. □

# Chapter 5

# MOTS in De Sitter Spacetime

In this chapter we study MOTS specifically in 4-dimensional de Sitter spacetime. Consider the vacuum Einstein equations with positive cosmological constant $\Lambda$

$$\bar{G}_{\mu\nu} = -\Lambda \bar{g}_{\mu\nu}. \tag{5.1}$$

Then 4-dimensional de Sitter spacetime is defined as the solution to the above equation which in local coordinates can be written as

$$\mathscr{M} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times S^3, \quad \bar{g} = \cos^{-2}(t)(-dt^2 + dS_3^2) \tag{5.2}$$

where $dS_3^2$ denotes the round metric on the 3-sphere.
Before investigating MOTS we investigate why the singularity theorems fail for de Sitter spacetime. First we state both classical singularity theorems regarding timelike and null geodesic incompleteness. An overview of these theorems can be found in e.g. [22].

**Theorem 5.1.** (Penrose singularity theorem)
Let $(\mathscr{M}, \bar{g})$ be a spacetime such that

1. $\bar{R}_{\mu\nu}x^\mu x^\nu \geq 0$ for all null vectors $x$, i.e., the null energy condition holds.
2. there is a non-compact Cauchy surface $\mathcal{C}$, and
3. there is an achronal trapped surface $S$.

Then $\mathscr{M}$ is future null geodesically incomplete.

Condition 2. is not satisfied in de Sitter spacetime because every spacelike Cauchy surface is compact and timelike/null hypersurfaces contain causal curves (see Definition 2.3.4) .

**Theorem 5.2.** (Hawking singularity theorem)
Let $(\mathscr{M}, \bar{g})$ be a spacetime such that

1. $\bar{R}_{\mu\nu}x^\mu x^\nu \geq 0$ for all timelike vectors $x$, i.e., the strong energy condition holds
2. there is a compact spacelike Cauchy surface $\mathcal{C}$ in $\mathscr{M}$, with
3. its mean curvature vector $H^\mu$ everywhere past pointing timelike.

Then $\mathscr{M}$ is future timelike geodesically incomplete.

This theorem does not apply because the strong energy condition does not hold, which can be seen from the following calculation. First we take the trace of equation (5.1), yielding

$$\bar{g}^{\mu\nu}\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}^{\mu\nu}\bar{g}_{\mu\nu} = -\Lambda\bar{g}^{\mu\nu}\bar{g}_{\mu\nu}. \tag{5.3}$$

Using $\bar{g}^{\mu\nu}\bar{g}_{\mu\nu} = 4$ we get

$$\bar{R} - 2\bar{R} = -4\Lambda. \tag{5.4}$$

And so

$$\Lambda = \frac{1}{4}\bar{R}. \tag{5.5}$$

Now let $x \in T\mathscr{M}$ be a timelike vector and use the Einstein equations (5.1) to calculate

$$\bar{R}_{\mu\nu}x^\mu x^\nu = \left(\frac{1}{2}\bar{R}\bar{g}_{\mu\nu} - \Lambda\bar{g}_{\mu\nu}\right)x^\mu x^\nu \tag{5.6}$$

$$= \left(\frac{1}{2}\bar{R} - \Lambda\right)\bar{g}_{\mu\nu}x^\mu x^\nu = \frac{1}{4}\bar{R}\bar{g}_{\mu\nu}x^\mu x^\nu < 0. \tag{5.7}$$

The inequality holds because de Sitter spacetime has constant positive curvature ($\bar{R} > 0$) and $x$ is timelike by assumption.

Since the remainder of this chapter is dedicated to MOTS in de Sitter spacetime the question arises why there are trapped surfaces but no singularities in $\mathscr{M}$. Intuitively this can be explained by the positive cosmological constant which is responsible for an accelerated expansion of spacetime. This expansion counteracts the convergence of geodesics prohibiting the formation of a singularity.

We proceed with the study of MOTS in de Sitter spacetime. For the remainder of this chapter we fix $n = 4$ and $\Lambda = 3$. To find MOTS $S$ with induced metric $h_{AB}$ in spacelike slices $\mathscr{N} = \{t = \text{const.}\}$, we need to calculate the null expansion scalars

$$\theta^{\pm} = H \pm p = h^{AB} K_{AB} \pm p. \tag{5.8}$$

First we derive a useful alternative expression for the second fundamental form on $\mathscr{N}$.

**Lemma 5.3.** (Second fundamental form for foliation with vanishing shift) Let $\mathscr{N} = \{t = \text{const.}\}$ be a leaf in a spacetime foliation with induced metric $g$, normal vector $n$ and lapse $N$ as in section 2.4. Furthermore assume the foliation has vanishing shift vector $N^{\alpha} = 0$. Then the second fundamental form $K_{ij}$ on $\mathscr{N}$ can be written as

$$K_{ij} = \frac{1}{2N} \frac{d}{dt} g_{ij}. \tag{5.9}$$

*Proof.* First we calculate as in [5] equation (2.52)

$$\begin{aligned} K_{\mu\nu} &= h_{\mu}{}^{\alpha} h_{\nu}{}^{\beta} \nabla_{\alpha} n_{\beta} && (5.10)\\ &= (\delta_{\mu}{}^{\alpha} + n_{\mu} n^{\alpha})(\delta_{\nu}{}^{\beta} + n_{\nu} n^{\beta}) \nabla_{\alpha} n_{\beta} && (5.11)\\ &= (\delta_{\mu}{}^{\alpha} + n_{\mu} n^{\alpha}) \delta_{\nu}{}^{\beta} \nabla_{\alpha} n_{\beta} && (5.12)\\ &= \nabla_{\mu} n_{\nu} + n_{\mu} n^{\alpha} \nabla_{\alpha} n_{\nu} && (5.13) \end{aligned}$$

where we used the identity $n^{\mu} \nabla_{\nu} n_{\mu} = 0$. Recall the action of the Lie derivative in direction of a vectorfield $X$ on the metric $g$ and an arbitrary vectorfield $V$

- $\mathscr{L}_X g_{\mu\nu} = 2\nabla_{(\mu} X_{\nu)}$
- $\mathscr{L}_X V^{\mu} = X^{\sigma} \nabla_{\sigma} V^{\mu} + V^{\sigma} \nabla_{\sigma} X^{\mu}$.

Also note that the metric on $\mathscr{N}$ can be decomposed as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + n_{\mu} n_{\nu}. \tag{5.14}$$

With this in mind we can calculate the Lie derivative of $g$:

$$\begin{aligned} \mathscr{L}_n g_{\mu\nu} &= \mathscr{L}_n (\bar{g}_{\mu\nu} + n_{\mu} n_{\nu}) && (5.15)\\ &= 2\nabla_{(\mu} n_{\nu)} + n_{\mu} \mathscr{L}_n n_{\nu} + n_{\nu} \mathscr{L}_n n_{\mu} && (5.16)\\ &= 2(\nabla_{(\mu} n_{\nu)} + n_{(\mu} n^{\sigma} \nabla_{\sigma} n_{\nu)}) && (5.17)\\ &= 2K_{\mu\nu}, && (5.18) \end{aligned}$$

where the last equality holds because $K_{\mu\nu}$ is symmetric. The result now follows from a calculation that can be also be found in [23] as equation

E.2.30,

$$\begin{aligned}
K_{\mu\nu} &= \frac{1}{2}\mathscr{L}_n g_{\mu\nu} && (5.19)\\
&= \frac{1}{2}\left(n^\sigma \nabla_\sigma g_{\mu\nu} + g_{\mu\sigma}\nabla_\nu n^\sigma + g_{\sigma\nu}\nabla_\mu n^\sigma\right) && (5.20)\\
&= \frac{1}{2}N^{-1}\left[N n^\sigma \nabla_\sigma g_{\mu\nu} + g_{\mu\sigma}\nabla_\nu (N n^\sigma) + g_{\sigma\nu}\nabla_\mu (N n^\sigma)\right] && (5.21)\\
&= \frac{1}{2}N^{-1} g_\mu{}^\sigma g_\nu{}^\gamma \left(\mathscr{L}_t g_{\sigma\gamma} - \mathscr{L}_N g_{\sigma\gamma}\right) && (5.22)\\
&= \frac{1}{2}N^{-1}\left(\frac{d}{dt}g_{\mu\nu} - \nabla_\mu N_\nu - \nabla_\nu N_\mu\right). && (5.23)
\end{aligned}$$

The last two terms are zero since $N^\mu = 0$ by assumption, yielding

$$K_{\mu\nu} = \frac{1}{2N}\frac{d}{dt}g_{\mu\nu}. \tag{5.24}$$

By similar reasoning as in Remark 3.1.5 we get

$$K_{ij} = \frac{1}{2N}\frac{d}{dt}g_{ij}. \tag{5.25}$$

□

**Lemma 5.4.** (Second fundamental form on $\mathscr{N}$ for de Sitter spacetime)
In de Sitter spacetime the second fundamental form on spacelike slices $\mathscr{N} = \{t = \text{const.}\}$ reads

$$K_{ij} = \frac{\sin t}{\cos^2 t}\left(dS_3^2\right)_{ij} \tag{5.26}$$

*Proof.* Comparing the definition of the lapse in equation (2.4) with the metric (5.2) yields $N = \cos^{-1}(t)$. It is obvious that the foliation has vanishing shift and the result follows immediately from Lemma 5.3, i.e.,

$$\begin{aligned}
K_{ij} =& \frac{1}{2}\cos t \frac{d}{dt}g_{ij} = \frac{1}{2}\cos t \frac{2\sin t}{\cos^3 t}\left(dS_3^2\right)_{ij} && (5.27)\\
=& \frac{\sin t}{\cos^2 t}\left(dS_3^2\right)_{ij}. && (5.28)
\end{aligned}$$

□

This explicit form is key to the calculations which are the contents of the following sections.

## 5.1 Spherical MOTS in de Sitter Spacetime

We choose spherical coordinates for $g$ on $\mathscr{N} = \{t = \text{const.}\}$:

$$g_{ij}dx^i dx^j = \cos^{-2} t \left( d\psi^2 + \sin^2 \psi (d\zeta^2 + \sin^2 \zeta d\phi^2) \right), \tag{5.29}$$

$$\psi \in [0, \pi], \quad \zeta \in [0, \pi], \quad \phi \in [0, 2\pi). \tag{5.30}$$

The volume element is then given by

$$\sqrt{g} := \sqrt{\det g_{ij}} = \frac{\sin^2 \psi \sin \zeta}{\cos^3 t}. \tag{5.31}$$

We can now investigate the null expansion scalars for 2-surfaces $S$ given by $\{\psi = \text{const.}\} \subset \mathscr{N}$ in order to find MOTS. On such surfaces the induced metric reads

$$h_{AB} = \cos^{-2} t \left( dS_2^2 \right)_{AB}, \tag{5.32}$$

where $dS_2^2$ denotes the standard metric on the 2-sphere. Recall the curvature form of the null expansions scalars (3.1.10)

$$\theta^{\pm} = h^{AB} K_{AB} \pm p. \tag{5.33}$$

We first calculate the mean curvature $p$. The unit outward normal of $S$ in $\mathscr{N}$ takes the form $m = (\cos t, 0, 0)$. Then

$$p = \nabla_i m^i = \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g}\, m^i \right) \tag{5.34}$$

$$= \frac{\cos^3 t}{\sin^2 \psi \sin \zeta} \partial_\psi \left( \frac{\sin^2 \psi \sin \zeta}{\cos^3 t} \cos t \right) \tag{5.35}$$

$$= \frac{\cos^3 t}{\sin^2 \psi \sin \zeta} \frac{2 \sin \psi \cos \psi \sin \zeta}{\cos^2 t} \tag{5.36}$$

$$= \frac{2 \cos t \cos \psi}{\sin \psi} \tag{5.37}$$

$$= 2 \cot \psi \cos t. \tag{5.38}$$

Now we calculate the first term in equation (5.33). First we project $K_{ij}$ onto the surface $S$ to get $K_{AB}$ in the following manner

$$h_a{}^i h_b{}^j K_{ij} = \frac{\sin t}{\cos^2 t} \left( dS_2^2 \right)_{ab}, \tag{5.39}$$

hence

$$K_{AB} = \frac{\sin t}{\cos^2 t} \left( dS_2^2 \right)_{AB}. \tag{5.40}$$

Where ${h_a}^i = h^{ij}h_{aj} = \left(dS_2^2\right)_a^{\ i}$. Next we contract $K_{AB}$ with the metric on $S$ to yield $H$

$$H = h^{AB}K_{AB} = \cos^2 t \frac{\sin t}{\cos^2 t} \left(dS_2^2\right)^{AB} \left(dS_2^2\right)_{AB} \tag{5.41}$$

$$= {\delta^A}_A \sin t \tag{5.42}$$

$$= 2 \sin t. \tag{5.43}$$

Combining the previous calculations yields the following equation for the null expansion scalars:

$$\theta^{\pm} = \pm 2 \cot \psi \cos t + 2 \sin t \tag{5.44}$$

For the surface to be marginally outer trapped the null expansion w.r.t. the outward direction has to be zero. In this case both $\theta^{\pm} = 0$ yield MOTS as per Remark 3.1.3. In the first case $\theta^{+} = 0$, we calculate

$$2 \cot \psi \cos t + 2 \sin t = 0 \tag{5.45}$$

$$- \cot \psi = \frac{\sin t}{\cos t} \tag{5.46}$$

$$\cot \psi = - \tan t \tag{5.47}$$

$$\psi = t + \frac{\pi}{2}. \tag{5.48}$$

For the other case ($\theta^{-} = 0$), similarly we get

$$\psi = -t + \frac{\pi}{2}. \tag{5.49}$$

We next picture the situation in the conformal completion of de Sitter (see 4.1.1). It is apparent that the MOTS travel "outward" in the conformal completion as $t$ increases. Recall that the time evolution of MOTS is called a marginally outer trapped tube (MOTT). They are depicted in the figure below.

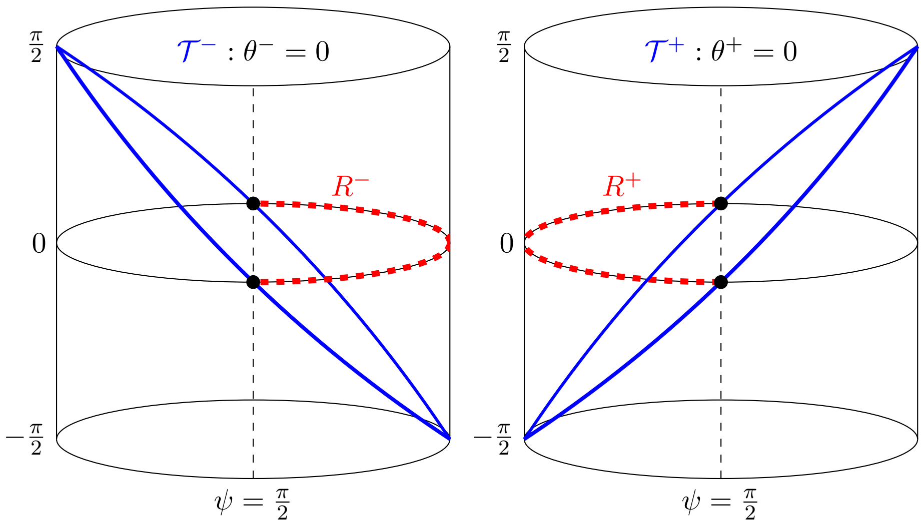


Figure 5.1: MOTTs $\mathcal{T}^{\pm}$ and marginally outer trapped regions $R^{\pm}$ in compactified de Sitter spacetime.

In this picture two dimensions are suppressed, a circle $\{t = \text{const.}\}$ represents a 3-sphere and two antipodal points a 2-sphere. The equator at $(t = 0, \psi = \pi/2)$ satisfies both (5.48) and (5.49) and is therefore a marginally trapped surface. Marginally outer trapped regions $R^{\pm}$ are defined as compact hypersurfaces $R^{\pm}$ with $\partial R^{\pm} = S^{\pm}$, where $S^{\pm}$ is a MOTS (analogously to Definition 4.2.2). The "outward" traveling of MOTS in the picture above intuitively looks like the surfaces are shrinking and converging to a single point on the north pole and south pole respectively at $t = \pi/2$. This is however an artifact of the conformal completion, how this looks in the physical metric is demonstrated in the figure below.

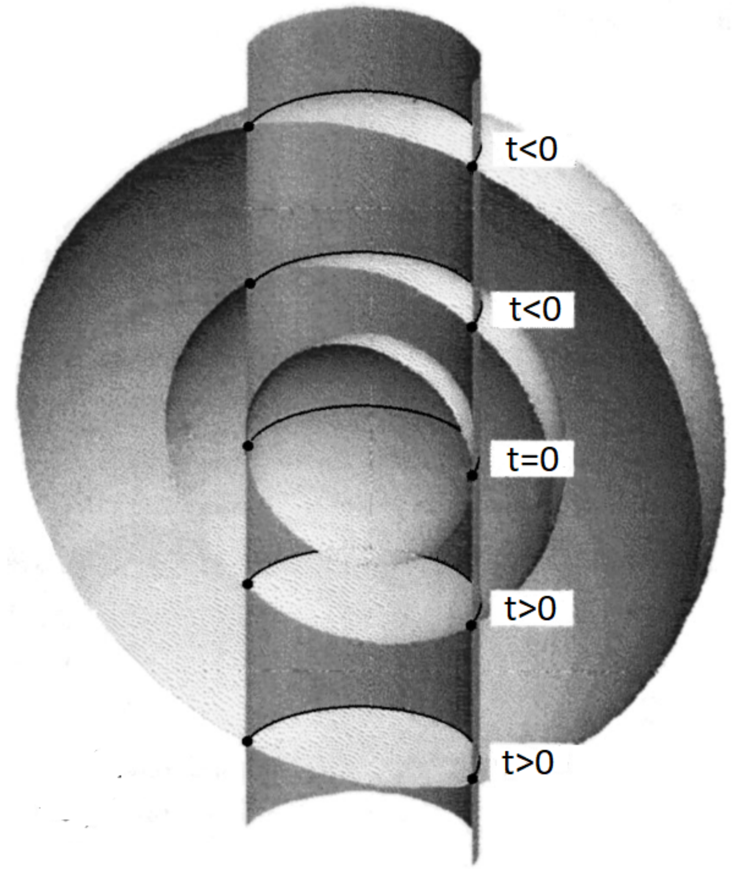


Figure 5.2: MOTS for different times[20]

The figure depicts cross-sections of 3-spheres growing in size with $t$ in de Sitter spacetime. At the neck ($\{t = 0\}$) the MOTS lies on the equator. The size of the MOTS stays constant and counteracts the growth of the 3-spheres by traveling toward the poles. This can be seen by explicitly calculating the area of MOTS $S$ for $\theta^+ = 0$ as

$$\mathbf{vol}(S) = \int_S \sqrt{h} = \int_0^\pi \int_0^{2\pi} \frac{\sin^2\psi \sin\zeta}{\cos^2 t} d\zeta d\phi \tag{5.50}$$

$$= 2\pi \frac{\sin^2\psi}{\cos^2 t} \int_0^\pi \sin\zeta d\zeta \tag{5.51}$$

$$= 4\pi \frac{\sin^2\psi}{\cos^2 t} \tag{5.52}$$

$$= 4\pi \frac{\sin^2(t + \frac{\pi}{2})}{\cos^2 t} \tag{5.53}$$

$$= 4\pi \frac{\cos^2 t}{\cos^2 t} \tag{5.54}$$

$$= 4\pi, \tag{5.55}$$

where we used equation (5.32) to calculate $\sqrt{h}$ and equation (5.48) to substitute for $\psi$ since $S$ is a MOTS. Clearly $\mathbf{vol}(S)$ does not depend on $t$ and equals the usual volume of 2-spheres. Note that the case $\theta^-$ works analogously since $\cos^2(-t) = \cos^2 t$.

Next we investigate the existence and visibility of weakly trapped surfaces in the spherical foliation in the context of Theorem 4.2.1. Note that the global conditions of the theorem are fulfilled for the conformal completion at hand. The crucial aspect is the choice of the set $A$, specifically when $J^+(A, \widetilde{\mathscr{M}})$ does not contain all of $\mathscr{S}^+$, the theorem asserts that there are no weakly trapped surfaces in $J^+(A, \widetilde{\mathscr{M}}) \cap I^-(\mathscr{S}^+, \widetilde{\mathscr{M}})$. Choosing $A$ as either MOTS $S_t^\pm := \{t, \psi = \pm t + \pi/2\}$ for any $t > 0$, the causal future of $A$ does not contain all of $\mathscr{S}^+$, except for $t = 0$. Thus the theorem can be applied for $S_t^\pm$ for any $t > 0$. Therefore there are no weakly trapped surfaces in $J^+(A, \widetilde{\mathscr{M}}) \cap I^-(\mathscr{S}^+, \widetilde{\mathscr{M}})$. This is in accordance with our findings since none of the MOTS are weakly trapped surfaces except at $t = 0$ (see figure 5.1). The theorem does not apply at $t = 0$, because $J^+(S_0^+, \widetilde{\mathscr{M}})$ contains all of $\mathscr{S}^+$ as mentioned before.

We can also study the visibility of marginally outer trapped regions in the context of Theorem 4.2.3. Similarly to the previous paragraph the global conditions are satisfied. The marginally outer trapped region $R_t^-$ at time $t$ corresponding to the MOTS $S_t^-$ is given by $\psi \in [0, \psi^-]$ where $\psi^- = t + \pi/2$, in accordance with equation (5.49). The marginally outer trapped region for the other direction $R_t^+$ works analogously and they clearly exist for all times $t$. However, this does not contradict the theorem since the causal future of every MOTS does not contain an entire marginally outer trapped region, as is clear from figure 5.1.

Furthermore, we can illustrate that the marginally outer trapped regions do not contain trapped surfaces with the following figure.

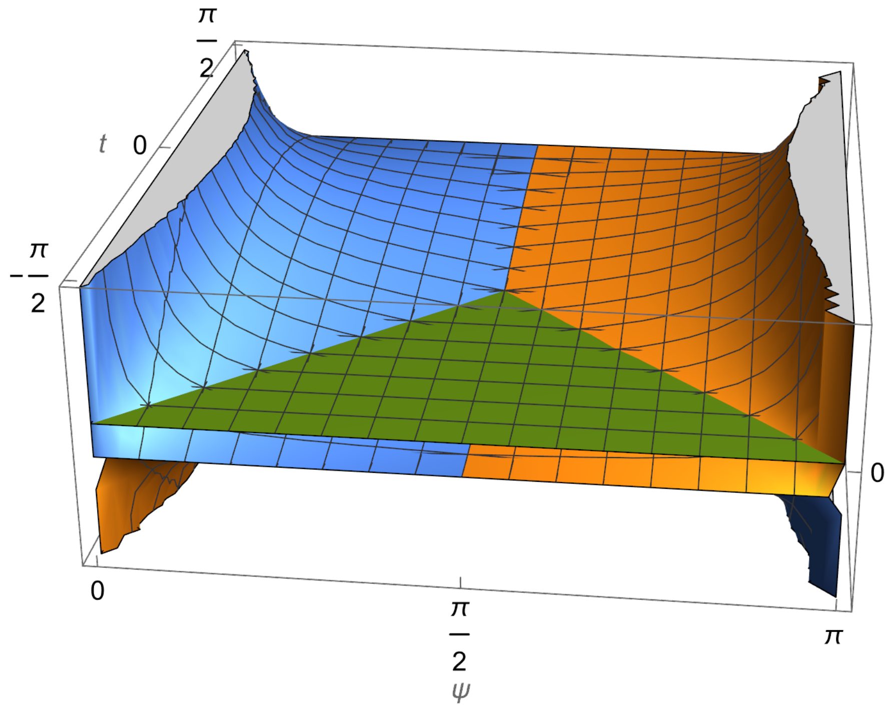


Figure 5.3: $\theta^{\pm}$ depending on $t \in (-\pi/2, \pi/2)$ and $\psi \in (0, \pi)$. The green surface depicts the 0-plane, the blue surface $\theta^+$ and the orange $\theta^-$.

MOTS are found at the intersection of either $\theta^+$ or $\theta^-$ with the green surface. Both $\theta^{\pm} = 0$ at $(t = 0, \psi = \pi/2)$, as in figure 5.1. Recall a surface is trapped if both $\theta^{\pm} < 0$, which is the case where the green surface is visible. This is the only zone not covered by marginally outer trapped regions. It is also apparent that the respective null expansions of the marginally outer trapped regions are positive inside the region and negative outside (this is also explicitly calculated in the proof of Theorem 5.1.1 below).

Lastly, we investigate the stability of the spherical MOTS. If they were stable the marginally outer trapped regions would contain trapped surfaces. Since there are no trapped surfaces within a marginally outer trapped region $R^{\pm}$ we expect all respective MOTS to be unstable, which holds true by the following theorem.

**Theorem 5.1.1.** (Stability of spherical MOTS)
MOTS of spherical topology in the foliation (5.29) of de Sitter spacetime are unstable.

*Proof.* Recall Definition 3.3.7, saying that a MOTS $S$ is stable if there exists a function $\psi \geq 0$ with $\psi \not\equiv 0$ on $S$ such that

$$\delta_{\psi v}\theta^{\pm} \geq 0, \tag{5.56}$$

for a vector $v$ normal to $S$ and satisfying $v_\mu l^\mu_\pm = 1$. Since $\psi \geq 0$ and $v$ is directed outwards of the region $R^\pm$ (because of the condition $v_\mu l^\mu_\pm = 1$), $\psi v$ is also directed outwards. To satisfy equation (5.56) the null expansion has to be positive close to the MOTS $S^\pm$ but also outside of $R^\pm$. We can check the sign of $\theta^\pm$ close to $S^\pm$ by differentiating (5.44) with respect to $\psi$.

$$\frac{\partial\theta^+}{\partial\psi} = -\frac{2\cos t}{\sin^2\psi} < 0, \quad \text{since } \cos t > 0 \text{ for all } t \tag{5.57}$$

$$\frac{\partial\theta^-}{\partial\psi} = \frac{2\cos t}{\sin^2\psi} > 0 \tag{5.58}$$

Since $\theta^\pm = 0$ on $\partial R^\pm = S^\pm$, this implies the respective null expansion $\theta^\pm$ is positive inside the marginally outer trapped regions $R^\pm$ and negative outside. This is also illustrated by figure 5.3. However this contradicts equation (5.56) and thus concludes the proof. □

## 5.2 Toroidal MOTS in de Sitter Spacetime

In this section we extent our analysis to MOTS in de Sitter of toroidal topology. To this end we introduce the following coordinates on $\mathscr{N} = \{t = \text{const.}\}$

$$g_{ij}dx^i dx^j = \cos^{-2}(t)\left(d\tau^2 + \sin^2\tau d\psi^2 + \cos^2\tau d\phi^2\right), \tag{5.59}$$

$$\tau \in [0, \pi/2], \quad \psi, \phi \in [0, 2\pi). \tag{5.60}$$

The metric on the 2-Surfaces $S = \{\tau = \text{const.}\}$ reads

$$h_{AB} = \cos^{-2} t \left(dT_2^2\right)_{AB}, \tag{5.61}$$

where $\left(dT_2^2\right)$ denotes the metric on the 2-torus. We again need to calculate both terms of the null expansions scalars in equation (5.33). Starting with the mean curvature, the unit outward normal of $S$ in $\mathscr{N}$ takes the form

$m = (\cos t, 0, 0)$. Using the volume element $g = \frac{\sin^2 \tau \cos^2 \tau}{\cos^3 t}$ we can calculate

$$p = \nabla_i m^i = \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g} m^i \right) \tag{5.62}$$

$$= \frac{\cos^3 t}{\sin \tau \cos \tau} \partial_\tau \left( \frac{\sin \tau \cos \tau}{\cos^3 t} \cos t \right) \tag{5.63}$$

$$= \frac{\cos^3 t}{\frac{1}{2} \sin 2\tau} \partial_\tau \left( \frac{\frac{1}{2} \sin 2\tau}{\cos^3 t} \cos t \right) \tag{5.64}$$

$$= \frac{\cos^3 t}{\sin 2\tau} \cdot \frac{2 \cos 2\tau}{\cos^2 t} \tag{5.65}$$

$$= \frac{2 \cos t \cos 2\tau}{\sin 2\tau} \tag{5.66}$$

$$= 2 \cos t \cot 2\tau. \tag{5.67}$$

Next we calculate the second term in (5.33). The second fundamental form of $S$ reads $K_{AB} = \frac{\sin t}{\cos^2 t} \left( dT_2^2 \right)$, which is obtained by a similar calculation as in the spherical case in (5.39). We proceed with the calculation to obtain

$$h^{AB} K_{AB} = \cos^2 t \frac{\sin t}{\cos^2 t} \underbrace{\delta^A{}_A}_{2} = 2 \sin t. \tag{5.68}$$

Combining the previous results yields

$$\theta^\pm = H \pm p = 2(\pm \cos t \cot 2\tau + \sin t). \tag{5.69}$$

Finally the toroidal MOTS w.r.t. the $l^+$-direction (i.e. $\theta^+ = 0$) are then parametrized by

$$\cot 2\tau = -\tan t \tag{5.70}$$

$$\Rightarrow \quad t = 2\tau - \frac{\pi}{2}, \tag{5.71}$$

and w.r.t. the other direction ($\theta^- = 0$) by

$$t = 2\tau + \frac{\pi}{2}. \tag{5.72}$$

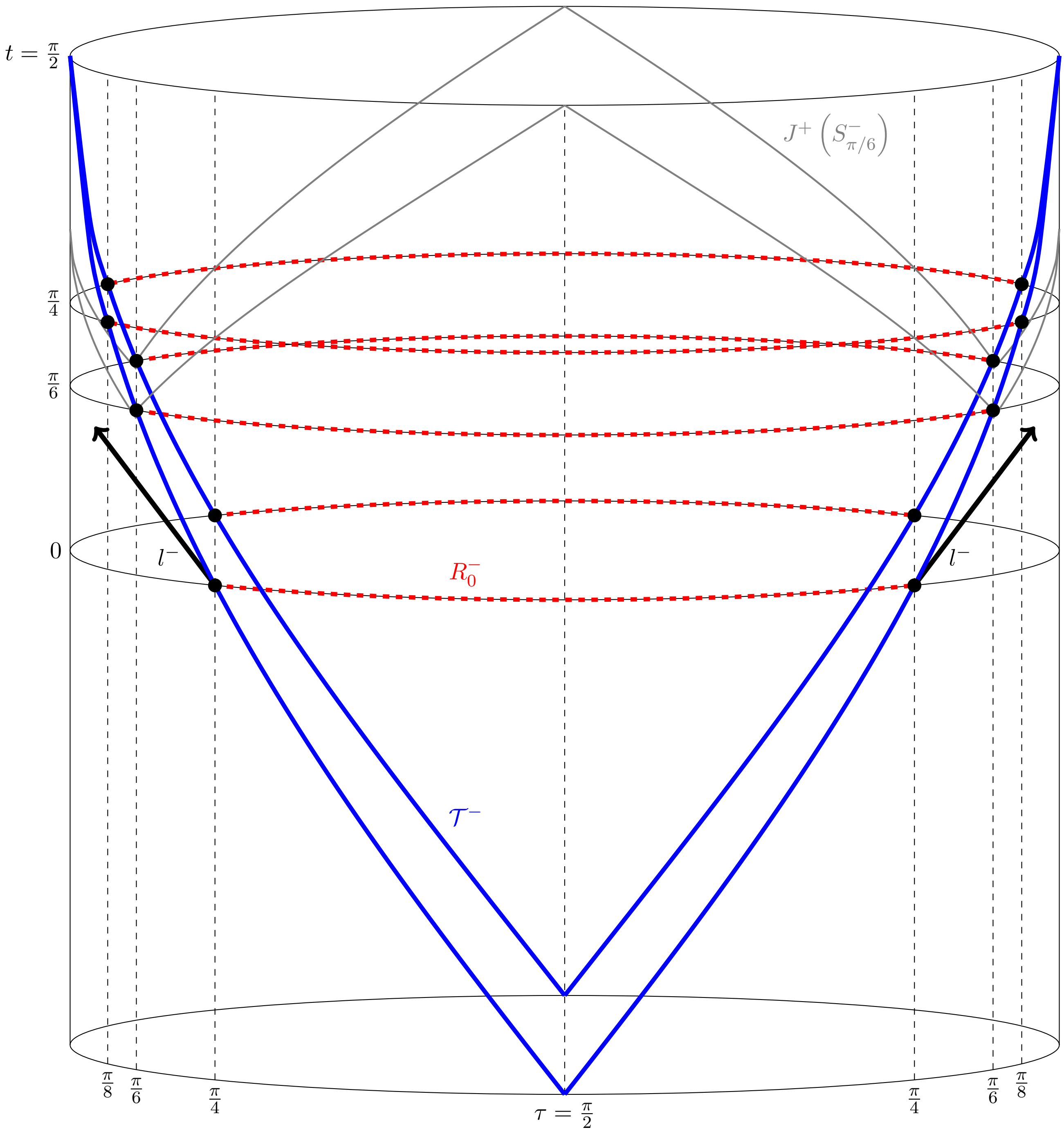


Figure 5.4: MOTT $\mathcal{T}^-$ : $\theta^- = 0$ with two dimensions suppressed. Four points at equal times along $\mathcal{T}^-$ represent a torus, the red dotted lines indicate marginally outer trapped regions $R^-$ and the future lightcone of the MOTS at $t = \pi/6$ is drawn in gray.

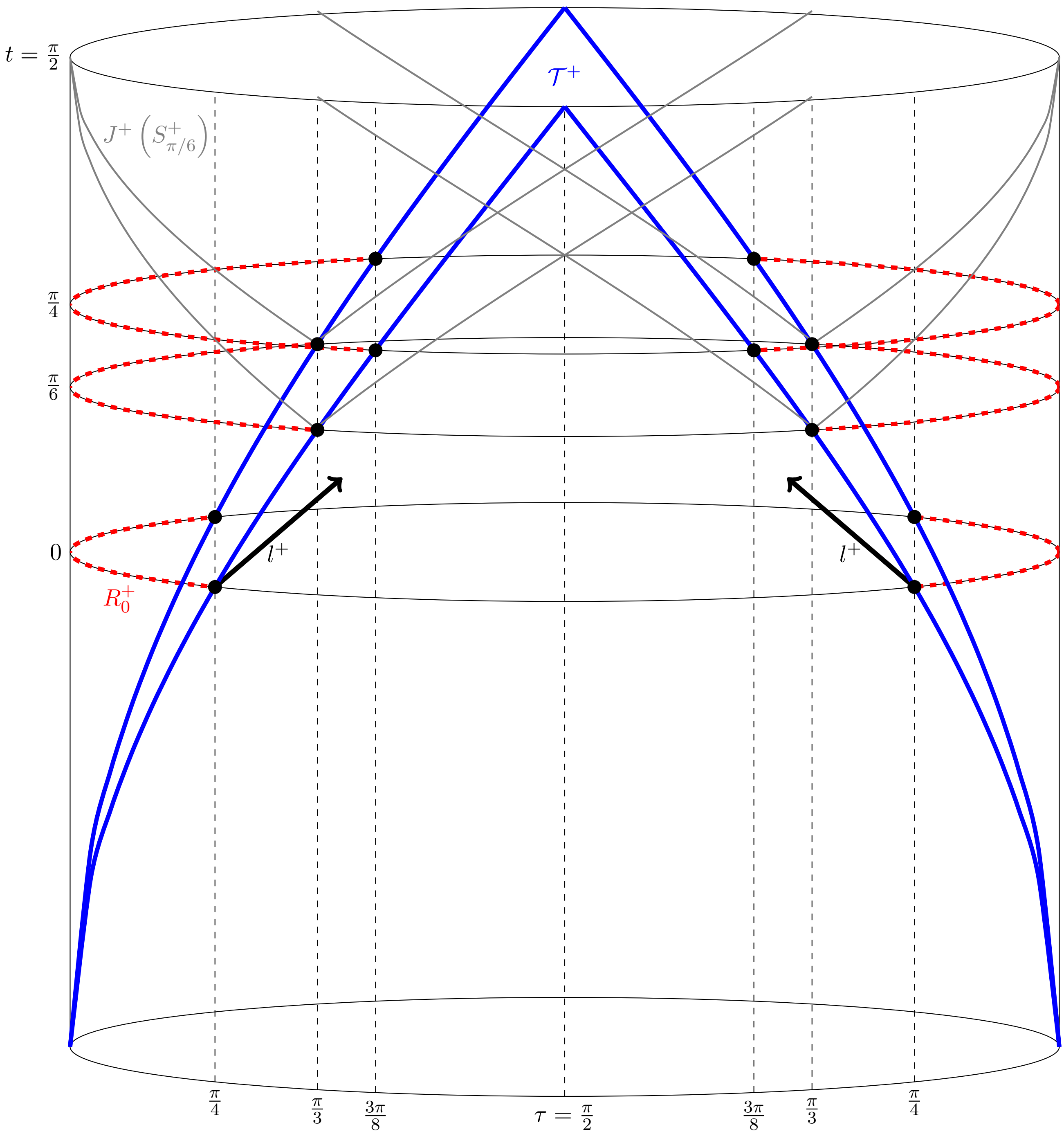


Figure 5.5: MOTT $\mathcal{T}^+ : \theta^+ = 0$ similarly to the previous figure. Dotted red lines indicate marginally outer trapped regions $R^+$ and the future lightcone of the MOTS at $t = \pi/6$ is drawn in gray.

For $\mathcal{T}^+$ the marginally outer trapped region $R_t^+$ corresponding to each MOTS at time $t$ is given by $\tau \in [0, \tau^+]$, where $\tau^+ = t/2 + \pi/4$ in accordance with equation (5.71). Similarly the marginally outer trapped regions $R_t^-$ for $\mathcal{T}^-$ are given by $\tau \in [\tau^-, \pi/2]$ for $\tau^- = t/2 - \pi/4$.

We can now, similarly to the previous section, apply Theorem 4.2.1 and Theorem 4.2.3. A similar analysiscan be found in [6]. The global conditions in the theorem are again fulfilled, the crucial issue is whether $J^+(A, \widetilde{\mathscr{M}})$ contains all of $\mathscr{S}^+$. However, $A$ can still be chosen freely, in particular as a MOTS. The first theorem, concerning the existence of weakly trapped surfaces, works similarly to the spherical case. The only weakly trapped surface exists at $t = 0$, however the causal future of the MOTS $S_0^\pm$ contains all of $\mathscr{S}^+$ and the theorem does not apply.

Regarding Theorem 4.2.3 we first consider marginally outer trapped regions $R_t^+$ corresponding to $\mathcal{T}^+$. The marginally outer trapped regions associated with $\mathcal{T}^-$ work analogously. Let $S_t^+$ be the MOTS in the timeslice $\{t = \text{const.}\}$. When the set $A$ is chosen to be the initial MOTS $S_0^+$, already every slice $t \geq \pi/4$, in particular $\mathscr{S}^+$ (i.e. $\{t = \pi/2\}$), is contained in $J^+(S_0^+, \widetilde{\mathscr{M}})$. Thus the theorem cannot be applied, which is consistent with our findings since marginally outer trapped regions exist for all times. The earliest MOTS where its causal future does not contain all of $\mathscr{S}^+$ is given on the timeslice $t = \pi/6$, as can be seen in Figure 5.5. Choosing this MOTS as the set A, at first glance seems to contradict the theorem, since it can be applied and states no weakly outer trapped regions are contained in $J^+(S^+_{\pi/6}, \widetilde{\mathscr{M}}) \cap I^-(\mathscr{S}^+, \widetilde{\mathscr{M}})$. But we know that there are marginally outer trapped regions for all times, in particular for timeslices after $t = \pi/6$. However no timeslice to the future of the MOTS at $t = \pi/6$ is completely contained in $J^+(S^+_{\pi/6}, \widetilde{\mathscr{M}})$, especially no marginally outer trapped region. This can easily be seen as each marginally outer trapped region contains the set $\tau = 0$ which is not included in the causal future of any MOTS $S_t^+$ for $t > \pi/6$, as is clear from figure 5.5. This is again consistent with Theorem 4.2.3.

An analogous statement to Theorem 5.1.1 on stability holds here.

**Theorem 5.2.1.** (Stability of toroidal MOTS)
MOTS of toroidal topology in the foliation (5.29) of de Sitter spacetime are unstable.

*Proof.* The proof works completely analogous to Theorem 5.1.1. The derivatives of $\theta^{\pm}$ with respect to $\tau$ read

$$\frac{\partial\theta^+}{\partial\tau} = -\frac{4\cos t}{\sin^2 2\tau} < 0 \tag{5.73}$$

$$\frac{\partial\theta^-}{\partial\tau} = \frac{4\cos t}{\sin^2 2\tau} > 0. \tag{5.74}$$

Which concludes the proof by the same reasoning as in the spherical case. □

The sign of $\theta^{\pm}$ inside and outside the marginally trapped outer regions is illustrated in the figure below.

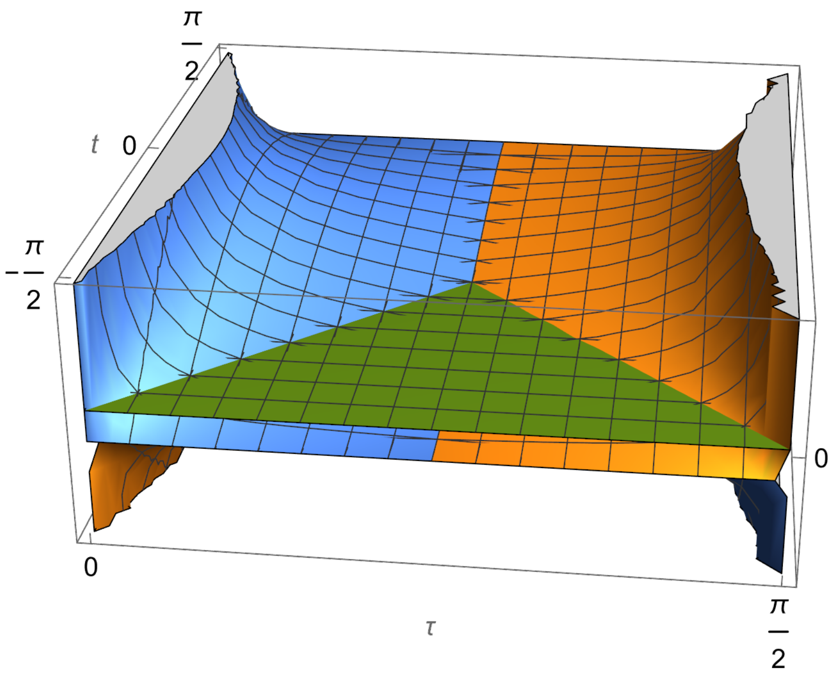


Figure 5.6: $\theta^{\pm}$ depending on $t \in (-\pi/2, \pi/2)$ and $\tau \in (0, \pi/2)$. The green surface depicts the 0-plane, the blue surface $\theta^+$ and the orange $\theta^-$.

This figure is very similar to Figure 5.3 and the same conclusions can be drawn with the only exception that both $\theta^{\pm} = 0$ now at $(t = 0, \tau = \pi/2)$.

## Abstract

In this thesis we study the theory of marginally outer trapped surfaces (or MOTS, for short) and their visibility. A closed spacelike surface in a spacetime $(\mathscr{M}, g)$ is called trapped if both congruences of normal (future directed) null geodesics are converging. If $\mathscr{M}$ contains such a trapped surface, satisfies the null energy condition and admits a non-compact Cauchy surface the spacetime is singular by Roger Penrose's classical singularity theorem[19]. Trapped surfaces mark the point of no return when a singularity forms as the result of a gravitational collapse. MOTS are a generalisation of trapped surfaces, in the sense that only one of the congruences has zero convergence. As such they are an integral part in the mathematical study of black holes.

In the first part of this thesis we review some of the Lorentzian geometry needed and then delve into the theory of marginally outer trapped surfaces. There, several important notions such as the stability of MOTS and the closely related minimal surfaces are explored. Afterwards we present visibility theorems regarding MOTS in asymptotically de Sitter spacetimes based on the recent work by Piotr T. Chruściel, Gregory J. Galloway and Eric Ling [9]. The final part gives a detailed exposition of MOTS and their visibility, specifically in de Sitter spacetime.

In summary, the present work serves as a basis for further investigations, for example, in the field of MOTTs (Marginally Outer Trapped Tubes).

## Zusammenfassung

In dieser Arbeit untersuchen wir die Theorie der marginal nach außen gefangenen Flächen (kurz MOTS) und deren Sichtbarkeit. Eine geschlossene raumartige Fläche in einer Raumzeit $(\mathscr{M}, g)$ wird als gefangen bezeichnet, wenn beide Scharen von normalen zukunftsgerichteten Nullgeodäten konvergieren. Wenn $\mathscr{M}$ eine solche gefangene Fläche enthält, die Null-Energie-Bedingung erfüllt und eine nicht-kompakte Cauchy-Fläche zulässt, ist die Raumzeit singulär - das ist die Aussage des klassischen Singularitätstheorems von Roger Penrose [19]. Bildet sich als Folge eines Gravitationskollapses eine Singularität, begrenzen gefangene Flächen den Punkt ohne Wiederkehr. MOTS dienen als Verallgemeinerung der gefangenen Flächen und sind somit ein integraler Bestandteil der mathematischen Theorie schwarzer Löcher.

Im ersten Teil dieser Arbeit geben wir einen Überblick über die benötigten Themen der Lorentzsche Geometrie und vertiefen uns dann in die Theorie der MOTS. Dort werden wichtige Begriffe wie die Stabilität von MOTS und die der eng verwandten Minimalflächen behandelt. Anschließend stellen wir Sichtbarkeitstheoreme bezüglich MOTS in asymptotischen de Sitter-Raumzeiten vor, die auf einer neuen Arbeit von Piotr T. Chruściel, Gregory J. Galloway und Eric Ling beruhen [9]. Der letzte Teil ist eine ausführliche Darstellung von MOTS und ihrer Sichtbarkeit speziell in der de Sitter Raumzeit.

Zusammenfassend dient die vorliegende Arbeit als Grundlage für weitere Untersuchungen, zum Beispiel auf dem Gebiet MOTTs (Marginally Outer Trapped Tubes).

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