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Gaps in partially ordered sets

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Abstract

The topic of this thesis are gaps in partially ordered sets, where we in particular concentrate on the sets ${}^{\omega}\omega$ and $\mathcal{P}(\omega)$. We start with gaps in ${}^{\omega}\omega$, introduce important types of these gaps and state their basic properties. Then we investigate the behaviour of gaps under forcing and show that it is possible to both introduce gaps via forcing and destroy certain gaps using forcing. In Chapter 4 we switch our focus to gaps in $\mathcal{P}(\omega)$ and show that there are Special Gaps which are not Hausdorff Gaps. Then the influence of additional axioms, in particular versions of MA and PFA, is dealt with in chapter 5.

Abriss

Das Thema dieser Arbeit sind Lücken in partiell geordneten Mengen, wobei wir uns insbesondere auf die Mengen ${}^{\omega}\omega$ und $\mathcal{P}(\omega)$ konzentrieren. Wir beginnen mit Lücken in ${}^{\omega}\omega$, führen wichtige Typen dieser Lücken ein und geben ihre grundlegenden Eigenschaften an. Dann untersuchen wir das Verhalten von Lücken unter Forcing und zeigen, dass es möglich ist, sowohl Lücken durch Forcing einzuführen als auch einige Arten von Lücken durch Forcing zu zerstören. In Kapitel 4 wechseln wir unseren Fokus auf Lücken in $\mathcal{P}(\omega)$ und zeigen, dass es Spezielle Lücken gibt, die keine Hausdorff-Lücken sind. Dann wird der Einfluss zusätzlicher Axiome, insbesondere Versionen von MA und PFA, in Kapitel 5 behandelt.

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Chapter 1

Introduction

For any partially ordered set $(X, <_x)$ it is possible to define the notion of a gap in X as, roughly speaking, two sequences of elements of X that are somehow "asymptotically close". More specifically, a gap consists of two sequences $(x_i)_{i\in I}$ and $(x_j)_{j\in J}$, indexed by totally ordered index-sets I, J, such that $(x_i)_{i\in I}$ is increasing and $(x_j)_{j\in J}$ is decreasing with respect to the order inherited from I and J, respectively, such that $(x_i)_{i\in I}$ is pointwise below $(x_j)_{j\in J}$. The gap-property than states that there exists no element $y \in X$ which is "between" $(x_i)_{i\in I}$ and $(x_j)_{j\in J}$, i.e. $x_i <_x y <_x x_j$ for all $i \in I, j \in J$.

We will investigate mostly the case of $X = {}^{\omega}\omega$ with the order $\prec$, defined by letting $f \prec g$ if and only if the difference of g and f tends to infinity. Closely related to this is the space $\mathcal{P}(\omega)$, partially ordered by almost inclusion $\subset^*$. As index-sets we will almost always consider ordinals. For the scope of this work, interesting questions are in particular results on the existence of certain types of gaps (possibly under additional axioms), on the relation between different types of gaps and on the influence of forcing on gaps.

Pioneer work on these topics has been done by Hausdorff, for example in [1] or [2]. Hausdorff showed (in ZFC) that there exist gaps of a certain type, which we will name after him (see Theorem 12). An important role in the development of the theory of gaps also played Rothberger (see for example [3] or [4]), who will be the name-giver of the very important so-called Rothberger-Gaps. Later on it was among others Kunen who investigated in particular the influence of forcing on gaps.

This work is roughly structured as follows:

In Chapter 2 we will consider basic properties of gaps. We introduce certain types of gaps such as Hausdorff Gaps, Special Gaps and Rothberger Gaps (see Sections 2.3 and 2.4), for which we state and prove some of the most important properties. All results can be derived from ZFC without using any more delicate techniques such as forcing.

In Chapter 3 we bring forcing into play and investigate questions such as the possibilities of introducing and destroying gaps (see Sections 3.1 and 3.2, respectively), as well as circumstances under which gaps cannot be destroyed using forcing (Section 3.3).

Chapter 4 deals with gaps in $\mathcal{P}(\omega)$, where we consider similarities and differences to gaps in ${}^{\omega}\omega$. In Section 4.2 we introduce towers and consider their connections with gaps in $\mathcal{P}(\omega)$. This will lead to the result that the space of Hausdorff Gaps and Special Gaps do not coincide.

The last Chapter is dedicated to the investigation of the influence of additional axioms on the gaps. We will focus especially on Martin's Axiom in different versions (see Section 5.2) and the Open Coloring Axiom, where as a corollary we obtain results for the Proper Forcing Axiom (see Section 5.3).

Chapter 2

Gaps in ${}^{\omega}\omega$

In this chapter we introduce and discuss basic properties of gaps in $({}^{\omega}\omega, \prec)$, where ${}^{\omega}\omega$ is the set of functions $f\colon \omega \to \omega$, which we will call *reals*. By $\prec$ we denote a partial order on ${}^{\omega}\omega$, defined by

$$f \prec g \text{ if and only if } \lim_{n\to\infty} g(n) - f(n) = \infty.$$

Given some relation $R(n, f_0, ..., f_m)$, depending on $n \in \omega$ and reals $f_0, ..., f_m$, we say that R holds *eventually* if there exists $k \in \omega$ such that $R(n, f_0, ..., f_m)$ holds for all $n > k$.

Most of the time we will not explicitly say that we consider $({}^{\omega}\omega, \prec)$, however, all considerations and results in this chapter are with respect to $({}^{\omega}\omega, \prec)$, if not otherwise stated. This chapter mostly follows [5], however, we sometimes provide more detailed or slightly different proofs.

2.1 Definitions and First Interpolation Theorem

We start with defining the notion of a pregap and a gap, give some basic properties of them and prove the so called First Interpolation Theorem. The following definition is a more general version of the definition given in [5].

Definition 1 (Pregap)**.** Given two totally ordered sets $(I, <_I), (J, <_J)$ with minimal element and an ordered pair of sequences of reals $(\{f_i\}_{i\in I}, \{g_j\}_{j\in J})$, we say that $(\{f_i\}_{i\in I}, \{g_j\}_{j\in J})$ is an (I, J)-pregap if

$$f_{i_1} \prec f_{i_2} \prec g_{j_2} \prec g_{j_1}$$

for all $i_1 <_I i_2$ and $j_1 <_J j_2$, where $i_1, i_2 \in I$ and $j_1, j_2 \in J$.

This immediately leads us to the main thing of interest, the notion of a gap:

Definition 2 (Gap)**.** A pregap $(\{f_i\}_{i\in I}, \{g_j\}_{j\in J})$ is an (I,J)-gap if there is no real $h \in {}^\omega\omega$ such that

$$f_i \prec h \prec g_j$$

for all $i \in I$ and all $j \in J$. If there is such an h we say that h interpolates the pregap $(\{f_i\}_{i\in I}, \{g_j\}_{j\in J})$.

Remark. For the majority of cases we will use as index sets I, J ordinal numbers α, β with the usual ordering. Then we write $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ for an (α,β)-pregap. However, the definition above allows us to make some more general statements.

Intuitively, it is not surprising that we have some kind of symmetry:

Proposition 1. *Let α and β be ordinals. If there is an (α,β)-gap in $({}^\omega\omega, \prec)$, then there is an (β,α)-gap in $({}^\omega\omega, \prec)$.*

Proof. Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α,β)-gap in $({}^\omega\omega, \prec)$. Define $g'_\gamma = \max\{g_0 - f_\gamma, 0\}$ and $f'_\delta = \max\{g_0 - g_\delta, 0\}$ for each $\gamma < \alpha$ and each $\delta < \beta$. Then $(\{f'_\delta\}_{\delta<\beta}, \{g'_\gamma\}_{\gamma<\alpha})$ is a (β,α)-gap:

Since $f_\gamma \prec g_0$, eventually $g'_\gamma = g_0 - f_\gamma$ for all $\gamma < \alpha$. Also $g_\delta \prec g_0$, thus eventually $f'_\delta = g_0 - g_\delta$ for all $\delta < \beta$. So for $\gamma_1, \gamma_2 \in \alpha$ and $\delta_1, \delta_2 \in \beta$ with $\gamma_1 < \gamma_2$ and $\delta_1 < \delta_2$, eventually $g'_{\gamma_2} \prec g'_{\gamma_1}$ and $f'_{\delta_1} \prec f'_{\delta_2}$, because $f_{\gamma_1} \prec f_{\gamma_2}$ and $g_{\delta_2} \prec g_{\delta_1}$, respectively. Since $f_\gamma \prec g_\delta$, also $f'_\delta \prec g'_\gamma$ for all $\gamma \in \alpha$ and $\delta \in \beta$. Thus $(\{f'_\delta\}_{\delta<\beta}, \{g'_\gamma\}_{\gamma<\alpha})$ is a (β,α)-pregap.

If it was not a gap, let h interpolate it. But then $h' = \max\{g_0 - h, 0\}$ would interpolate $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$, since $f'_\delta \prec h \prec g'_\gamma$ implies $f_\gamma \prec h' \prec g_\delta$ for all $\delta \in \beta, \gamma \in \alpha$. This is a contradiction to the assumption that $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is a gap. □

Another intuitive and useful result is the following proposition, stating that it suffices to consider cofinal subsets to prove the gap-property for (α,β)-pregaps.

Proposition 2. *Let α and β be ordinals and $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α,β)-pregap. For $A \subseteq \alpha$ and $B \subseteq \beta$ both cofinal, the following are equivalent:*

1. $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ *is a gap,*

2. $(\{f_a\}_{a\in A}, \{g_b\}_{b\in B})$ *is a gap.*

Proof. (1) $\implies$ (2): The only thing to prove is that there is no $h \in {}^\omega\omega$ interpolating $(\{f_a\}_{a\in A}, \{g_b\}_{b\in B})$. But if there would be such an h, fix $\gamma < \alpha$ and $\delta < \beta$. Then there is an $a \in A$ and an $b \in B$ such that $\gamma < a$ and $\delta < b$, thus $f_\gamma \prec f_a$ and $g_b \prec g_\delta$. Since h interpolates $(\{f_a\}_{a\in A}, \{g_b\}_{b\in B})$, this implies $f_\gamma \prec f_a \prec h \prec g_b \prec g_\delta$. Because γ and δ were arbitrary, this holds for all $\gamma \in \alpha$ and all $\delta \in \beta$, which is a contradiction to 1.

(2) $\implies$ (1): Suppose h interpolates $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$. But then h interpolates $(\{f_a\}_{a\in A}, \{g_b\}_{b\in B})$, since $A \subseteq \alpha$ and $B \subseteq \beta$. This is a contradiction. □

Remark. If we consider an (α, β)-pregap for ordinals α, β, Proposition 2 ensures that we can assume α and β to be regular cardinals, whenever we want to prove that the pregap is a gap.

We now state and prove our first result, which is originally due to Hadamard [6]:

Theorem 3 (First Interpolation Theorem)**.** *There are no (α, β)-gaps in $({}^\omega\omega, \prec)$, if α and β are countable ordinals.*

Proof. Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α, β)-pregap and α, β be countable ordinals. By Proposition 2, we can assume that $\alpha, \beta \leq \omega$.

Suppose $\alpha = \beta = \omega$. Since $f_m \prec f_n \prec g_n \prec g_m$ for all $m < n \in \omega$, we can find a natural number $k \in \omega$ such that for all $j > k$ we have $f_m(j) \leq f_n(j) < g_n(j) \leq g_m(j)$. We can further choose this k such that $f_n(j) + 2 \cdot n < g_n(j)$ for all $j > k$. In particular, for each $n \neq 0$ we find k_n such that $f_{n-1}(j) \leq f_n(j) < g_n(j) \leq g_{n-1}(j)$ and $f_n(j) + 2 \cdot n < g_n(j)$ for all $j > k_n$. Inductively, we can build an increasing sequence $\{k_n\}_{n\geq 1}$ such that

$$f_0(j) \leq f_1(j) \leq \ldots \leq f_n(j) < g_n(j) \leq g_{n-1}(j) \leq \ldots \leq g_0(j)$$

and $f_n(j) + 2 \cdot n < g_n(j)$ for all $j > k_n$. Define $h \in {}^\omega\omega$ by $h(j) = f_n(j) + n$ whenever $j \in [k_n, k_{n+1})$ and $h(j) = 0$ for $j < k_1$. Then h interpolates $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$, so this pregap is not a gap.

If α is finite, extend $\{f_\gamma\}_{\gamma<\alpha}$ by $f_{\gamma'} = f_\gamma$ for $\gamma' \geq \alpha$. Then apply the above argument to construct h, which will interpolate the pregap. The case that β is finite is similar. □

2.2 Second Interpolation Theorem

In this section we prove another interpolation theorem, which will be useful to prove the existence of a Hausdorff Gap. The notions introduced in this section as well as the first proof of the main results were given in [2], although we will follow [5] with the notation. We will focus on $(1, \alpha)$-pregaps for an ordinal α.

One of the key properties of a $(1, \alpha)$-pregap is being near to some subset of α:

Definition 3 (Nearness)**.** Let $(\{f\}, \{g_\gamma\}_{\gamma<\alpha})$ be a $(1, \alpha)$-pregap for an ordinal α. For $n \in \omega$, define $N_n^f = \{\gamma \in \alpha \mid \forall k > n \colon f(k) < g_\gamma(k)\}$. Then for $A \subseteq \alpha$, we say that f is near A if $A \cap N_n^f$ is finite for all $n \in \omega$.

Remark. The notation N_n^f is a bit misleading, since it suggests that N_n^f only depends on f and $n \in \omega$. But in fact, N_n^f depends on n and the $(1, \alpha)$-pregap $(\{f\}, \{g_\gamma\}_{\gamma<\alpha})$. Since it will always be clear from context witch $(1, \alpha)$-pregap needs to be considered, we avoid to use a more clear (but lengthy) notation.

We state basic properties of the sets N_n^f:

Proposition 4. *Let α be an ordinal and $(\{f\}, \{g_\gamma\}_{\gamma<\alpha})$ be a $(1, \alpha)$-pregap.*

1. *If $m < n \in \omega$, then $N_m^f \subseteq N_n^f$.*

2. *For $h \in {}^\omega\omega$, if there is a $k \in \omega$ such that $f(n) \leq h(n)$ for all $n > k$, then $N_n^h \subseteq N_n^f$ for all $n > k$.*

3. *If h is an interpolating real for $(\{f\}, \{g_\gamma\}_{\gamma<\alpha})$, then $N_n^h \subseteq N_n^f$ for all but finitely many $n \in \omega$.*

Proof. 1.: If $\gamma \in N_m^f$, then $f(k) < g_\gamma(k)$ for all $k > m$. Since $n > m$, this implies $f(j) < g_\gamma(j)$ for all $j > n$, so $\gamma \in N_n^f$.

2.: Let $h \in {}^\omega\omega$ and k be as in 2. Fix $n > k$ and suppose $\gamma \in N_n^h$. Then $h(j) < g_\gamma(j)$ for all $j > n$. Fix $j > n$. Since $j > n > k$, we have

$f(j) \leq h(j) < g_\gamma(j)$. Because $j > n$ was arbitrary, this holds for all $j > n$, so $\gamma \in N_n^f$. Since also $n > k$ was arbitrary, this implies 2.

3.: Let h be as in 3. Then $f \prec h$, so we find $k \in \omega$ such that $f(n) \leq h(n)$ for all $n > k$. Thus 2. implies 3. □

Proposition 5. *Let α be an ordinal and $(\{f\}, \{g_\gamma\}_{\gamma<\alpha})$ be a $(1, \alpha)$-pregap.*

1. *If α is finite, then f is near A for all $A \subseteq \alpha$.*
2. *If f is near $A, B \subseteq \alpha$, then f is near $A \cup B$.*
3. *If f is near $A \subseteq \alpha$ and $B \subseteq \alpha$ is such that $B \setminus A$ is finite, then f is near B.*

Proof. 1.: If α is finite, then trivially $\alpha \cap N_n^f$ is finite for all $n \in \omega$.

2.: If f is near A, B, then $A \cap N_n^f$ and $B \cap N_n^f$ are finite for all $n \in \omega$. Thus $(A \cup B) \cap N_n^f = (A \cap N_n^f) \cup (B \cap N_n^f)$ is finite for all $n \in \omega$.

3.: Let f be near A and $B \setminus A$ be finite. Note that $B = (B \setminus A) \cup A)$. Thus $B \cap N_n^f = ((B \setminus A) \cap N_n^f) \cup (A \cap N_n^f)$, which is finite for each $n \in \omega$. □

Proposition 6. *Let α be a countable ordinal and $(\{f\}, \{g_\gamma\}_{\gamma<\alpha})$ be a $(1, \alpha)$-pregap. If N_n^f is infinite and f is near γ for each $\gamma < \alpha$, then N_n^f is unbounded in α and of order type ω.*

Proof. Suppose N_n^f is bounded in α. Then there is $\gamma < \alpha$ such that $N_n^f < \gamma$, i.e. $N_n^f \subseteq \gamma$. But f is near γ, since $\gamma < \alpha$, so $N_n^f \cap \gamma = N_n^f$ is finite, a contradiction to the assumption that N_n^f is infinite.

To see that N_n^f is of order type ω, assume the contrary. Then there is $\beta \in N_n^f$ which has infinitely many predecessors in N_n^f. But $\beta \in \alpha$, so $N_n^f \cap \beta$ is finite by assumption of the proposition, which is a contradiction. □

Proposition 7. *Let α be an ordinal and $(\{f\}, \{g_\gamma\}_{\gamma<\alpha})$ be a $(1, \alpha)$-pregap. If f is near $A \subseteq \alpha$ and h interpolates $(\{f\}, \{g_\gamma\}_{\gamma<\alpha})$, then h is near A.*

Proof. Fix $n \in \omega$ and suppose $B = N_n^h \cap A$ is infinite. So we can find infinitely many $\gamma \in B$ such that for all $k > n$ we have $h(k) < g_\gamma(k)$. But, since $f \prec h$, we find n^* such that for all $j > n^*$ the inequality $f(j) < h(j)$ holds. But then for all $k > \max\{n^*, n\}$ we obtain $f(k) < h(k) < g_\gamma(k)$. Thus $N_{n^*}^f \cap B$ is infinite. Since $B \subseteq A$, this is a contradiction to f being near A. □

Proposition 8. *Let α be an ordinal, $(\{f\},\{g_\gamma\}_{\gamma<\alpha})$ be an $(1,\alpha)$-pregap and $\{f_n\}_{n\in\omega}\subseteq {}^\omega\omega$ be such that*

1. *$(\{f_n\}_{n\in\omega},\{g_\gamma\}_{\gamma<\alpha})$ is an (ω,α)-pregap and $f_0=f$,*

2. *f_{n+1} is near N_n^f for all $n\in\omega$.*

If $h\in{}^\omega\omega$ is such that

i $f(n)\leq h(n)$ for all $n\in\omega$,

ii h interpolates $(\{f_n\}_{n\in\omega},\{g_\gamma\}_{\gamma<\alpha})$,

then h is near α.

Proof. Let h be as in the statement. Note that $(\{f_{n+1}\},\{g_\gamma\}_{\gamma<\alpha})$ forms a $(1,\alpha)$-pregap for each $n\in\omega$, which is interpolated by h. By Proposition 7, we obtain that h is near N_n^f for each $n\in\omega$, i.e. $N_n^h\cap N_n^f$ is finite for each $n\in\omega$. By Proposition 4, we obtain that there is a $k\in\omega$ such that $N_n^h\subseteq N_n^f$ whenever $n>k$. But in this case $N_n^h\cap N_n^f=N_n^h$, implying that N_n^h is finite itself for all $n>k$.

If now $n\leq k$, N_n^h can not be infinite: Suppose N_n^h is infinite and fix such an $n\in\omega$. By Proposition 4, we know that $N_n^h\subseteq N_{k+1}^h$, where $k\in\omega$ is above. But N_{k+1}^h is finite by what we have just shown, so N_n^h can not be infinite. □

We can now prove the Second Interpolation Theorem, which is due to Hausdorff and will be very useful to construct a Hausdorff Gap (first proven by Hausdorff in [2], page 321):

Theorem 9 (Second Interpolation Theorem)**.** *Let α be a countable ordinal and let $(\{f\},\{g_\gamma\}_{\gamma<\alpha})$ be a $(1,\alpha)$-pregap.*

If f is near γ for all $\gamma<\alpha$, then there is an $h\in{}^\omega\omega$ that interpolates the pregap and is near α.

Remark. The important statement of Theorem 9 is that the interpolating real h is near α. The existence of such an element in ${}^\omega\omega$ is already clear by Theorem 3.

Proof. We use Proposition 8. So we construct a sequence of reals $\{f_n\}_{n\in\omega}$, such that

1. $(\{f_n\}_{n\in\delta}, \{g_\gamma\}_{\gamma<\alpha})$ is an (δ, α)-pregap with $f_0 = f$ for all $\delta \leq \omega$,

2. f_{n+1} is near N_n^f for all $n \in \omega$.

We construct $\{f_n\}_{n\in\omega}$ inductively:

Let $f = f_0$. Given we have already found $\{f_0, f_1, ...f_n\}$ we define f_{n+1} as follows:

If N_n^f is finite, we can use Theorem 3 and find an interpolating function f_{n+1}, which is near N_n^f by Proposition 5.

Now assume N_n^f is infinite. By Proposition 6, we obtain that N_n^f is of order type ω and cofinal in α. So we can enumerate N_n^f as $\{\gamma_i\}_{i\in\omega}$ so that $g_{\gamma_i} \prec g_{\gamma_j}$ whenever $j < i$. Then since $(\{f_l\}_{l\leq n}, \{g_\gamma\}_{\gamma<\alpha})$ is a pregap, we obtain $f_n \prec g_{\gamma_i} \prec g_{\gamma_{i-1}} \prec ... \prec g_{\gamma_0}$ for every $i \in \omega$. So for each $i \in \omega$ we can find $k_i \in \omega$ such that $f_n(j) < g_{\gamma_i}(j) < g_{\gamma_{i-1}}(j) < ... < g_{\gamma_0}(j)$ for all $j > k_i$. Inductively we obtain a sequence $\{k_i\}_{i\in\omega} \subseteq \omega$, which we can further ensure to be strictly increasing. Then we define f_{n+1} by

$$f_{n+1}(j) = \begin{cases} g_{\gamma_{i-1}}(j) & \text{if } j \in [k_i, k_{i+1}) \\ f_n(j) & \text{otherwise} \end{cases}$$

Then $f_n \prec f_{n+1} \prec g_{\gamma_i}$ for all $i \in \omega$:

By definition of k_i we obtain $f_n(j) \leq f_{n+1}(j)$ for all $j \in \omega$. For $j > k_i$ we have that $f_n(j) < g_{\gamma_i}(j) < f_{n+1}(j)$ and therefore $f_n \prec f_{n+1}$, because $f_n \prec g_{\gamma_i}$.

To see that $f_{n+1} \prec g_{\gamma_i}$ for all $i \in \omega$, fix $i \in \omega$ and recall that $\{k_i\}_{i\in\omega}$ is strictly increasing. For all $j \geq k_{i+2}$ we have that $f_{n+1}(j) = g_{\gamma_{l(j)}}(j)$, where $l(j) > i$. By definition of k_i, it follows that $f_{n+1}(j) = g_{\gamma_{l(j)}}(j) < g_{\gamma_i}(j)$ for $j \geq k_{i+2}$. Since $g_{\gamma_j} \prec g_{\gamma_i}$ for all $j \in \omega$ with $j > i$, we obtain $f_{n+1} \prec g_{\gamma_i}$.

Further f_{n+1} is near N_n^f:

Consider any $k \in \omega$. Suppose $N_k^{f_{n+1}} \cap N_n^f$ is infinite. Let $k \in [k_i, k_{i+1})$ and fix $j > i + 1$. Then for $l \in [k_j, k_{j+1})$ we obtain $f_{n+1}(l) = g_{\gamma_{j-1}}(l)$. By the choice of the k_i's we also have that $g_{\gamma_j}(l) < g_{\gamma_{j-1}}(l) = f_{n+1}(l)$, so $\gamma_j \notin N_k^{f_{n+1}} \cap N_n^f$. But since $j > i + 1$ was arbitrary, this is a contradiction to $N_k^{f_{n+1}} \cap N_n^f$ being infinite.

So we have found our f_{n+1} as desired.

Continuing this construction, inductively we obtain a sequence $\{f_n\}_{n\in\omega}$ satisfying 1. and 2.

By Theorem 3, we find $h \in {}^{\omega}\omega$ that interpolates $(\{f_n\}_{n\in\omega}, \{g_\gamma\}_{\gamma<\alpha})$. Without loss of generality, we can also assume that $f(n) = f_0(n) \leq h(n)$ for all $n \in \omega$. Since $\{f_n\}_{n\in\omega}$ satisfies 1. and 2., we can apply Proposition 8 and obtain that h is near α. □

2.3 (ω_1, ω_1)-Gaps in $({}^{\omega}\omega, \prec)$

We consider the special case of (ω_1, ω_1)-gaps in this section. We present two different notions of such gaps, Hausdorff Gaps and Special Gaps, which will turn out to be closely related.

2.3.1 Hausdorff Gaps

We start with Hausdorff Gaps, which will be our first example of a gap. We will mainly follow [5] in this section, which is a reformulation of results originally proven in [2].

Definition 4 (Hausdorff Gap)**.** Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be a (ω_1, ω_1)-pregap. Then we say this pregap is a Hausdorff Gap, if $f_\gamma \leq g_\gamma$ pointwise and f_γ is near γ for all $\gamma \in \omega_1$.

The notion of a Hausdorff Gap seems a bit misleading at this point, since we a priori only know that a Hausdorff Gap is a pregap with some special properties. However, we can show that the name of this notion is indeed justified:

Proposition 10. *A Hausdorff Gap is a gap.*

Proof. Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be a Hausdorff Gap. Suppose it is not a gap. Then we find $h \in {}^{\omega}\omega$ interpolating $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$.

Fix $\gamma \in \omega_1$. We know $f_\gamma \prec h \prec g_\gamma$, so there is an $k_\gamma \in \omega$ such that $f_\gamma(n) < h(n) < g_\gamma(n)$ for all $n > k_\gamma$. Consider $\{k_\gamma\}_{\gamma\in\omega_1}$ and note that this is a subset of ω, so we find an uncountable $A \subseteq \omega_1$ such that $k_a = k$ for all $a \in A$, for some $k \in \omega$.

Because A is uncountable, we can find $a^* \in A$ such that there are infinitely many $b \in A$ with $b < a^*$. Fix such a b. Then $h(n) < g_b(n)$ for $n > k(= k_b)$. Since $k = k_b = k_{a^*}$, also $f_{a^*}(n) < h(n)$ for all $n > k$, so $f_{a^*}(n) < h(n) < g_b(n)$ for all $n > k$. Thus $b \in N_k^{f_{a^*}}$ and because there are infinitely many such $b < a^*$, we obtain that f_{a^*} is not near a^*. This is a contradiction to $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ being a Hausdorff Gap. $\square$

Remark. In the proof of Proposition 10 we did not use the property of Hausdorff Gaps that $f_\gamma \leq g_\gamma$ pointwise. Further we did not use that we have an (ω_1, ω_1)-pregap for ω_1, but just that we have an (α, α)-pregap for α uncountable. So this results generalizes as follows:

Proposition 11. *Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\alpha})$ be an (α, α)-pregap and let α be uncountable. If f_γ is near γ for all $\gamma \in \alpha$, then $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\alpha})$ is an (α, α)-gap.*

Although we already obtained some results about the notions of pregaps and gaps, we do not know that there exist gaps in $({}^\omega\omega, \prec)$. The following theorem by Hausdorff [2] ensures their existence, even for Hausdorff Gaps:

Theorem 12. *In $({}^\omega\omega, \prec)$ there exists a Hausdorff Gap.*

Proof. We prove the theorem by constructing a Hausdorff Gap using induction. For all $\alpha \in \omega_1$ we construct a pregap $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\alpha})$ such that

1. $f_\gamma \leq g_\gamma$ pointwise for all $\gamma < \alpha$,

2. f_γ is near γ for all $\gamma < \alpha$.

We start with f_0 and g_0 being arbitrary elements of ${}^\omega\omega$ such that $f_0 \prec g_0$.

Now suppose we have already constructed an (α, α)-pregap satisfying 1. and 2.

If α is a successor ordinal, then there is β such that $\alpha = \beta + 1$. Choose f_α and g_α in ${}^\omega\omega$ such that $f_\beta \prec f_\alpha \prec g_\alpha \prec g_\beta$, what we can do by Theorem 3. Without loss of generality, in fact by modifying at most countably many initial values of f_α and g_α, we can assume that $f_\beta \leq f_\alpha \leq g_\alpha \leq g_\beta$ pointwise. Since f_β is near β, by Proposition 7, we obtain that f_α is near β. Also, $\alpha \setminus \beta = \{\beta\}$ is finite, so Proposition 5 yields that f_α is near α.

If α is a limit ordinal, we can use Theorem 3 to obtain an h that interpolates $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\alpha})$. Note that $(\{f_\gamma\}, \{g_\delta\}_{\delta<\alpha})$ is a $(1, \alpha)$-pregap that is interpolated by h for each $\gamma \in \alpha$. Thus, by Proposition 7, we get that h is near γ for each $\gamma < \alpha$. Applying Theorem 9 to the $(1, \alpha)$-pregap $(\{h\}, \{g_\gamma\}_{\gamma<\alpha})$ gives us an interpolating $h' \in {}^\omega\omega$ that is near α. Then put

$f_\alpha = h'$. By Theorem 3, we also get an h'' interpolating $(\{f_\gamma\}_{\gamma\leq\alpha}, \{g_\gamma\}_{\gamma<\alpha})$. By modifying at most finitely many values of h'', we obtain $g_\alpha \in {}^\omega\omega$ such that $(\{f_\gamma\}_{\gamma\leq\alpha}, \{g_\gamma\}_{\gamma\leq\alpha})$ satisfies 1. and 2.

Thus we can construct an (ω_1, ω_1)-pregap satisfying 1. and 2. which is in consequence a Hausdorff Gap. □

2.3.2 Special Gaps

Another important class of (ω_1, ω_1)-gaps in $({}^\omega\omega, \prec)$ are Special Gaps, which were introduced by Kunen [7].

Definition 5 (Special Gap)**.** Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be an (ω_1, ω_1)-pregap. We say that $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ is a Special Gap if we find an $n \in \omega$ such that $f_\gamma(k) \leq g_\gamma(k)$ for all $k > n$, $\gamma < \omega_1$; and for all $\gamma < \delta < \omega_1$ we find an $l > n$ with $f_\gamma(l) > g_\delta(l)$ or $f_\delta(l) > g_\gamma(l)$.

As for Hausdorff Gaps, we show that the name Special Gap is indeed justified:

Proposition 13. *Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be a Special Gap. Then it is a gap.*

Proof. Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be a Special Gap. Suppose that we find an $h \in {}^\omega\omega$ that interpolates the pregap. For each $\gamma < \omega_1$ we have $f_\gamma \prec h \prec g_\gamma$, thus we can find a $n_\gamma \in \omega$ such that $f_\gamma(k) < h(k) < g_\gamma(k)$ for all $k > n_\gamma$. But then there exists an uncountable set $X \subseteq \omega_1$ such that for $\gamma, \delta \in X$ we have $n_\gamma = n_\delta =: n$.

Consider $\gamma \neq \delta \in X$. Let $i \in \{0, 1, ..., n-1\}$ and observe that for each such i there are only countably many possible values for $f_\gamma(i)$ and $f_\delta(i)$. So there must be an uncountable set $Y \subseteq X$ such that for all $\gamma, \delta \in Y$ and all $i \in \{0, 1, ..., n-1\}$ we have that $f_\gamma(i) = f_\delta(i)$. Using a similar argument for g_γ and g_δ, we can further find $Z \subseteq Y$ uncountable such that for all $\gamma, \delta \in Z$ we obtain $g_\gamma(i) = g_\delta(i)$ for $i \in \{0, 1, ..., n-1\}$. But then for any $\gamma, \delta \in Z$ with $\gamma < \delta$ we obtain that $f_\gamma(k) \leq g_\delta(k)$ and $f_\delta(k) \leq g_\gamma(k)$ for all $k \in \omega$. This contradicts the assumption that $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ is a Special Gap. □

We will show that Special Gaps are equivalent to Hausdorff Gaps. Therefore, we need to define what we mean when we call two gaps equivalent:

Definition 6 (Equivalence of Pregaps)**.** Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ and $(\{f'_\gamma\}_{\gamma<\alpha}, \{g'_\delta\}_{\delta<\beta})$ be (α, β)-pregaps for ordinals α, β. We say that the pregaps are equivalent if

1. for all $\gamma < \alpha$ there exists $\gamma' < \alpha$ such that $f_\gamma \prec f'_{\gamma'}$ and $f'_\gamma \prec f_{\gamma'}$,
2. for all $\delta < \beta$ there exists $\delta' < \beta$ such that $g_\delta \succ g'_{\delta'}$ and $g'_\delta \succ g_{\delta'}$.

Now the point is the following:

Proposition 14. *Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ and $(\{f'_\gamma\}_{\gamma<\alpha}, \{g'_\delta\}_{\delta<\beta})$ be equivalent (α, β)-pregaps. Then $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is an (α, β)-gap if and only if $(\{f'_\gamma\}_{\gamma<\alpha}, \{g'_\delta\}_{\delta<\beta})$ is an (α, β)-gap.*

Proof. Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be a gap and assume that $(\{f'_\gamma\}_{\gamma<\alpha}, \{g'_\delta\}_{\delta<\beta})$ is not a gap. Let $h \in {}^\omega\omega$ interpolate $(\{f'_\gamma\}_{\gamma<\alpha}, \{g'_\delta\}_{\delta<\beta})$.

For $\gamma < \alpha$ we obtain $\gamma' < \alpha$ such that $f_\gamma \prec f'_{\gamma'}$, thus $f_\gamma \prec h$. Now let $\delta < \beta$. We find δ' with $g_\delta \succ g'_{\delta'}$. Thus also $h \prec g_\delta$, which implies that h interpolates $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$, a contradiction.

The proof of the other direction is the same, switching the roles of $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ and $(\{f'_\gamma\}_{\gamma<\alpha}, \{g'_\delta\}_{\delta<\beta})$. □

We can show the first very important result about equivalence of gaps, which is taken from [5]:

Theorem 15. *Every Hausdorff Gap is equivalent to a Special Gap.*

For the proof we will need the following proposition, which can be found as Lemma 19.1 in [8]:

Proposition 16. *Let κ be an infinite regular cardinal, $\lambda < \kappa$ and $f : \kappa \to \mathcal{P}(\kappa)$ be such that $x \notin f(x)$ and $|f(x)| < \lambda$ for all $x \in \kappa$. Then there exists $X \subseteq \kappa$ such that $|X| = \kappa$ and for distinct $x, y \in X$ it holds that $x \notin f(y)$.*

Proof. We use transfinite recursion to construct a sequence of disjoint subsets of $\mathcal{P}(\kappa)$, $\{(X_\alpha)\}_{\alpha<\lambda}$, such that for all $\alpha < \lambda$ and $x, y \in X_\alpha$ we have $x \notin f(y)$:

Note that such X_α exist, since the singleton set $\{x\}$ satisfies $x \notin f(x)$ for all $x \in \kappa$ by assumption. If we have already constructed $\{(X_\beta)\}_{\beta<\alpha}$, let X_α be a maximal subset of $\kappa \setminus \bigcup\{X_\beta \mid \beta < \alpha\}$ such that for any $x, y \in X_\alpha$, $x \notin f(y)$. The existence of such an X_α is ensured by Zorn's lemma.

Then there must be an α for which X_α has cardinality κ:
If not, we obtain $|X_\alpha| < \kappa$ for all $\alpha < \lambda$. For each $\alpha < \lambda$ consider $Y_\alpha = \bigcup\{f(x) \mid x \in X_\alpha\} \cup X_\alpha$ and let $Y = \bigcup\{Y_\alpha \mid \alpha < \lambda\}$. Then, since $|f(x)| < \lambda$ for all $x \in \kappa$ and $|X_\alpha| < \kappa$, we obtain $|Y_\alpha| < \kappa$ for each $\alpha < \lambda$ and consequently $|Y| < \kappa$, because κ is regular.

Now pick any $z \in \kappa \setminus Y$. Then $z \in \kappa \setminus \bigcup\{X_\beta \mid \beta < \alpha\}$ and $z \notin f(x)$ for any $x \in X_\alpha$, for all $\alpha < \lambda$. But $z \notin X_\alpha$, as $z \notin Y$, so there must be an $x \in X_\alpha$ with $x \in f(z)$, because otherwise we would have a contradiction to the maximality of X_α. This holds for all $\alpha < \lambda$. Thus $f(z) \cap X_\alpha \neq \emptyset$ and since $X_\alpha \cap X_\beta = \emptyset$ for $\alpha, \beta \in \lambda$, we obtain $|f(z)| \geq \lambda$. This is a contradiction. □

Remark. Mappings $f : X \to \mathcal{P}(X)$ such that $x \notin f(x)$ for all $x \in X$ are usually called *set mappings*.

Now we are ready to proof Theorem 15.

Proof of Theorem 15. Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be a Hausdorff Gap. Define a set mapping $h\colon \omega_1 \to [\omega_1]^{<\aleph_0}$ by

$$h(\gamma) = \{\delta < \gamma \mid f_\gamma(n) \leq g_\delta(n) \text{ for all } n \in \omega\}.$$

The fact that $h(\gamma)$ is finite for each $\gamma \in \omega_1$ holds because we consider a Hausdorff Gap, so f_γ is near γ.

Because h is a set mapping, we can apply Proposition 16 to h and obtain an uncountable $X \subset \omega_1$ such that for all $x, y \in X$ we have $x \notin h(y)$.

Consider the gap $(\{f_x\}_{x\in X}, \{g_x\}_{x\in X})$. Since $|X| = \omega_1$, this gap is equivalent to $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$. Further for $x, y \in X$ with $x < y$ we obtain $x \notin h(y)$, thus we can find an $n \in \omega$ for which $f_x(n) > g_y(n)$. Putting $n = 0$, where n is as in Definition 5, we obtain that $(\{f_x\}_{x\in X}, \{g_x\}_{x\in X})$ is a Special Gap. □

2.4 Rothberger Gaps

In this section we introduce Rothberger Gaps in $({}^\omega\omega \prec)$.

Definition 7. For a regular uncountable cardinal number κ, we say that every (κ, ω)-gap and every (ω, κ)-gap in $({}^\omega\omega, \prec)$ is a κ-Rothberger Gap.

Definition 8. For $f, g \in {}^\omega\omega$, we write $f =^* g$ for the case that f and g agree for all but finitely many values and say that f and g are almost equal.

Remark. The notion $=^*$ is an equivalence relation and respects $\prec$, that is, if $f_1 =^* f_2$ and $g_1 =^* g_2$, then $f_1 \prec g_1$ implies $f_2 \prec g_2$.

The next theorem, which is due to Hausdorff and Rothberger (see [2] and [4], modern formulation in [5]), shows that the existence of a κ-Rothberger Gap is equivalent to the existence of $(\kappa, 0)$-gap and a $(\kappa, 1)$-gap:

Theorem 17. *Let κ be a regular uncountable cardinal number. Then the following are equivalent:*

1. *There exists a κ-Rothberger Gap in $({}^\omega\omega, \prec)$.*
2. *There exists a $(\kappa, 0)$-gap in $({}^\omega\omega, \prec)$.*
3. *There exists a $(\kappa, 1)$-gap in $({}^\omega\omega, \prec)$.*

Remark. By Proposition 2, statement 1. in the theorem can be made more general to cover all (α, β)-gaps and (β, α)-gaps in $({}^\omega\omega, \prec)$ for an uncountable α and a countable ordinal β. Proposition 1 ensures that $(\kappa, 0)$ and $(\kappa, 1)$ could be replaced by $(0, \kappa)$ and $(1, \kappa)$, respectively.

Proof. 1. $\implies$ 2.:
Without loss of generality, assume that we have a (κ, ω)-Rothberger Gap $(\{f_\gamma\}_{\gamma<\kappa}, \{g_n\}_{n\in\omega})$. We can further assume that $\{g_n\}_{n\in\omega}$ is decreasing pointwise, that means $g_n(i) \geq g_m(i)$ for $n < m$ and all $i \in \omega$.

We want to construct a sequence of length κ in $({}^\omega\omega, \prec)$, which is unbounded and increasing with respect to $\prec$. Then this sequence is a $(\kappa, 0)$-gap. We first construct an $\prec$-unbounded sequence in ${}^\omega\omega$:

For all $\gamma < \kappa$, define the real h_γ as follows:

$$h_\gamma(n) = \begin{cases} \max\{k \in \omega \mid g_n(k) \leq f_\gamma(k)\} & \text{if defined} \\ 1 & \text{otherwise.} \end{cases}$$

Because $f_\gamma \prec g_n$ for each $\gamma < \kappa$ and each $n \in \omega$, h_γ is well-defined for each $\gamma < \kappa$.

We observe that h_γ is increasing, i.e. that $h_\gamma(n) \leq h_\gamma(m)$ for $n < m$, because the fact $\{g_n\}_{n\in\omega}$ is decreasing pointwise implies that $\max\{k \mid g_n(k) \leq f_\gamma(k)\} \leq \max\{k \mid g_m(k) \leq f_\gamma(k)\}$ if $n < m$.

Claim 1. *The set* $\{h_\gamma\}_{\gamma<\kappa}$ *is unbounded in* $({}^\omega\omega, \prec)$.

Proof. Suppose that $\{h_\gamma\}_{\gamma<\kappa}$ is bounded and let h be a witness. Without loss of generality, we can assume that h is strictly increasing.

Define a real f by

$$f(i) = \begin{cases} g_n(i) & \text{if } i \in [h(n), h(n+1)) \\ 1 & \text{otherwise.} \end{cases}$$

Then for a fixed $\gamma < \kappa$, we observe that there is an $i_\gamma \in \omega$ such that $h_\gamma(i) < h(i)$ for all $i > i_\gamma$, because $h_\gamma \prec h$. But by the definition of h_γ, this implies that $f_\gamma(j) < g_i(j)$ for all $i > i_\gamma$ and all $j \geq h(i)$. Let $j > h(i_\gamma + 1)$ and $j \in [h(l), h(l+1))$ for some $l > i_\gamma$. Then since $j \geq h(l)$, this implies $f_\gamma(j) < g_l(j) = f(j)$.

Then even $f_\gamma \prec f$ for all $\gamma < \kappa$: If not, let γ^* be a counterexample, i.e. $f_{\gamma^*} \not\prec f$. Let δ be such that $\gamma^* < \delta < \kappa$. Then applying the argument we have just given to δ, we obtain that there exists some $k \in \omega$ such that $f_\delta(j) < f(j)$ for all $j > k$. But since $f_{\gamma^*} \prec f_\delta$, this is a contradiction to $f_{\gamma^*} \not\prec f$. So $f_\gamma \prec f$ for all $\gamma < \kappa$.

But on the other hand also $f \prec g_n$ for all $n \in \omega$. To see this, let $n \in \omega$ be given. Then for $j > h(n+1)$ we obtain that $f(j) < g_n(j)$, because $\{g_n\}_{n\in\omega}$ is pointwise decreasing. Since this holds for every natural n, we must have $f \prec g_n$ for every $n \in \omega$.

Finally, we obtain that f interpolates $(\{f_\gamma\}_{\gamma<\kappa}, \{g_n\}_{n\in\omega})$, which is a contradiction. □

Now we show that we can thin out $\{h_\gamma\}_{\gamma<\kappa}$ so that we obtain a strictly increasing unbounded sequence.

Claim 2. *There exists a* $\beta < \kappa$ *such that* h_γ *is unbounded for all* γ *with* $\beta < \gamma < \kappa$.

Proof. Suppose the claim is false. Then we find a cofinal set $C \subseteq \kappa$ such that for each $\gamma \in C$, h_γ is bounded. Because h_γ is increasing, these h_γ's are eventually constant.

Fix $\gamma \in C$. Then there is a $n_\gamma \in \omega$ such that $h_\gamma(n) = h_\gamma(n_\gamma)$ whenever $n > n_\gamma$. By definition of h_γ, this implies that there is a k_γ such that

$f_\gamma(k) < g_n(k)$ for all $k > k_\gamma$ and all $n > n_\gamma$. Because $\{g_n\}_{n\in\omega}$ is decreasing pointwise, this implies $f_\gamma(k) < g_n(k)$ for all $k > k_\gamma$ and all $n \in \omega$.

We can find such a $k_\gamma \in \omega$ for all $\gamma \in C$. Since κ is regular and C is cofinal in κ, $|C| = \kappa$. So there is $D \subseteq C$ with $|D| = \kappa$ and $k_{\delta_1} = k_{\delta_2} := k_D$ for all $\delta_1, \delta_2 \in D$. We obtain that $f_\delta(k) < g_n(k)$ for all $\delta \in D$, all $k > k_D$ and all $n \in \omega$.

This allows us to define

$$f(i) = \begin{cases} \max\{f_\delta(i) \mid \delta \in D\} & \text{if } i > k_D \\ 1 & \text{otherwise.} \end{cases}$$

But then f interpolates $(\{f_\delta\}_{\delta\in D}, \{g_n\}_{n\in\omega})$ and thus, by Proposition 2, it interpolates $(\{f_\gamma\}_{\gamma<\kappa}, \{g_n\}_{n\in\omega})$, since D is cofinal in κ. This is a contradiction. □

Claim 2 allows to assume that h_γ is unbounded for all $\gamma < \kappa$, since otherwise we can remove all corresponding f_γ from $\{f_\gamma\}_{\gamma\in\kappa}$ and still have a (κ, ω)-Rothberger Gap.

Claim 3. *For all $\gamma < \delta < \kappa$ we find an $n \in \omega$ such that $h_\gamma(m) < h_\delta(m)$ for all $m > n$.*

Proof. Let γ, δ be as in the statement of the claim. Then we find $l \in \omega$ such that $f_\gamma(j) < f_\delta(j)$ for all $j > l$, since $f_\gamma \prec f_\delta$. We assumed that h_γ is unbounded, so we further find $n \in \omega$ such that $h_\gamma(n) > l$.

Let $m > n$. By definition, $h_\gamma(m) = \max\{k \in \omega \mid g_m(k) \le f_\gamma(k)\} = k^*$. But since $h_\gamma(m) > l$, we obtain $f_\gamma(k^*) < f_\delta(k^*)$ and in particular, $g_m(k^*) \le f_\gamma(k^*) < f_\delta(k^*)$, thus $h_\delta(m) = \max\{k \in \omega \mid g_m(k) \le f_\delta(k)\} \ge k^* = h_\gamma(m)$. □

We are now ready to prove the important claim:

Claim 4. *There exists $S \subseteq \kappa$ such that $|S| = \kappa$ and $\{h_s\}_{s\in S}$ satisfies that $h_{s_1} \neq^* h_{s_2}$ for any $s_1 \neq s_2 \in S$.*

Proof. Assume the claim is false, then we find a $\gamma < \kappa$ such that for all δ, ϵ with $\gamma \le \delta < \epsilon < \kappa$ we have that $h_\delta =^* h_\epsilon$. Otherwise we would have a cofinal subset of S with the property of the claim and this set would have cardinality κ, since κ is regular.

But then any function that bounds $\{h_\beta\}_{\beta\leq\gamma}$ also bounds $\{h_\beta\}_{\beta<\kappa}$. By Claim 3, it suffices to find g that bounds h_γ to obtain an upper bound (with respect to $\prec$) of $\{h_\beta\}_{\beta<\kappa}$. By Theorem 3, we can always find such a g, since this is equivalent to find an interpolating function for the $(1,0)$-pregap $(\{h_\gamma\},\{\emptyset\})$.

This implies that $\{h_\gamma\}_{\gamma<\kappa}$ is bounded, a contradiction to Claim 1. $\square$

Using $S\subseteq\kappa$ as in Claim 4, we can define a sequence $\{d_s\}_{s\in S}$ by putting

$$d_s(n)=\sum_{i=0}^{n}h_s(i).$$

Then $(\{d_s\}_{s\in S},\emptyset)$ is a $(\kappa,0)$-gap: We can order $S\subseteq\kappa$ in the ordering it inherits from κ. Then for $s_1,s_2\in S$ with $s_1<s_2$ we obtain $d_{s_1}\prec d_{s_2}$ by Claim 4 and Claim 3. Also $\{d_s\}_{s\in S}$ is unbounded in $({}^\omega\omega,\prec)$, since $\{h_s\}_{s\in S}$ is. This proves 1. $\implies$ 2.

2. $\implies$ 3.:
Assume we have a $(\kappa,0)$-gap $(\{f_\gamma\}_{\gamma<\kappa},\emptyset)$. Without loss of generality, we can assume that each f_γ is strictly increasing.

We will make use of the following:

Claim 5. *There exists a $\beta<\kappa$ such that for γ,δ with $\beta<\gamma<\delta<\kappa$ it holds that*

$$\lim_{n\to\infty}|\min\{i\in\omega\mid f_\delta(i)\geq n\}-\min\{i\in\omega\mid f_\gamma(i)\geq n\}|=\infty.$$

Proof. Suppose the claim is false. Then for any $\beta<\kappa$ we find γ,δ with $\beta<\gamma<\delta<\kappa$ and $\limsup_{n\to\infty}|\min\{i\in\omega\mid f_\delta(i)\geq n\}-\min\{i\in\omega\mid f_\gamma(i)\geq n\}|\leq k_{\gamma,\delta}$ for $k_{\gamma,\delta}\in\omega$. Let (γ_0,δ_0) be any such pair γ,δ. For each $\beta<\kappa$, let $(\gamma_\beta,\delta_\beta)$ be the least pair in $\kappa\times\kappa$ with this property such that $(\gamma_\beta,\delta_\beta)\geq^\times\sup_{\alpha<\beta}\{(\gamma_\alpha,\delta_\alpha)\}$, where $\geq^\times$ is the pairwise order on $\kappa\times\kappa$, i.e. $(a,b)\geq^\times(c,d)$ if and only $a\geq b$ and $c\geq d$. Inductively, we get a sequence $((\gamma_\beta,\delta_\beta))_{\beta<\kappa}$ such that $\gamma_\beta<\delta_\beta$, $\gamma_\beta\geq\sup_{\alpha<\beta}\{\gamma_\alpha\}$ and $\delta_\beta\geq\sup_{\alpha<\beta}\{\delta_\alpha\}$.

Then for all $\beta<\kappa$ we observe that $f_{\gamma_\beta}(k_{\gamma_\beta,\delta_\beta}+i)\geq f_{\delta_\beta}(i)$ for all $i\in\omega$. Thus we find a set $C\subseteq\kappa$ of cardinality κ such that for all $\beta_1,\beta_2\in C$, $k_{\gamma_{\beta_1},\delta_{\beta_1}}=k_{\gamma_{\beta_2},\delta_{\beta_2}}=:k$. Since for all such β we have that $f_{\gamma_\beta}\prec f_{\delta_\beta}$, we find $l_{\gamma_\beta,\delta_\beta}\in\omega$ such that $f_{\gamma_\beta}(i)<f_{\delta_\beta}(i)$ for all $i>l_{\gamma_\beta,\delta_\beta}$. Thus we find $C'\subseteq C$

with $|C'| = \kappa$ so that $l_{\gamma_{c_1},\delta_{c_1}} = l_{\gamma_{c_2},\delta_{c_2}} =: l$ for any $c_1, c_2 \in C'$. Similarly, we find $C'' \subseteq C'$ with $|C''| = \kappa$ such that $f_{\gamma_{c_1}}(i) < f_{\gamma_{c_2}}(i)$ for $c_1 < c_2 \in C''$ and $i > l$. We enumerate $C'' = \{c_\beta\}_{\beta<\kappa}$ and obtain:

$$f_{\delta_{c_0}}(i) \leq f_{\gamma_{c_0}}(k+i) < f_{\gamma_{c_1}}(k+i) < f_{\delta_{c_1}}(k+i) \leq f_{\gamma_{c_1}}(2\cdot k+i) < ...$$

$$< f_{\gamma_{c_\beta}}((\beta+1)\cdot k+i) \leq f_{\delta_{c_\beta}}((\beta+1)\cdot k+i) < ...$$

for all $i > l$. This yields a strictly increasing sequence of natural numbers of length κ, a contradiction. □

Now define a sequence of strictly increasing functions $\{g_n\}_{n\in\omega}$ as follows: Let $g_0(k) = k$ for all $k \in \omega$ and let $g_n(k) = \max\{0, k-n\}$ for all $k, n \in \omega$. Then we obtain that $|g_0(i) - g_n(i)| = n < \infty$ for all $n \in \omega$ and all $i > n$.

Define

$$h_\gamma(n) = \begin{cases} g_i(n) & \text{if } n \in [f_\gamma(i), f_\gamma(i+1)) \text{ for some } i \in \omega \\ g_0(n) & \text{otherwise} \end{cases}$$

Then we claim that $(\{h_\gamma\}_{\gamma<\kappa}, \{g_0\})$ is our desired $(\kappa, 1)$-gap:

Claim 6. *$(\{h_\gamma\}_{\gamma<\kappa}, \{g_0\})$ is a $(\kappa, 1)$-gap.*

Proof. Fix any $\gamma < \kappa$ and $k \in \omega$. Note that $g_0(n) > g_1(n) > ... > g_{k+1}(n)$ for all natural $n > k$, thus $|g_0(n) - g_{k+1}(n)| \geq k$. Now we can find $m \in \omega$ such that $m > k$ and $m > f_\gamma(k+1)$. For $j > m$, $h_\gamma(j) = g_l(j)$ for some $l > k$. Thus in this case $h_\gamma(j) \leq g_{k+1}(j)$. Therefore $|g_0(j) - h_\gamma(j)| \geq k$ for all $j > m$. Since $h_\gamma \leq g_0$ pointwise and $k \in \omega$ was arbitrary, we have shown that $\lim_{n\to\infty}(g_0(n) - h_\gamma(n)) = \infty$, i.e. $h_\gamma \prec g_0$.

Now consider $\gamma < \delta < \kappa$. Fix $k \in \omega$. By Claim 5, we find $m \in \omega$ such that $|\min\{i \mid f_\delta(i) \geq j\} - \min\{i \mid f_\gamma(i) \geq j\}| \geq k$ for all $j > m$. But then for all $j > m$ it holds $h_\delta(j) - h_\gamma(j) \geq k$, since $h_\delta(j) = g_{i(j,\delta)}(j)$ and $h_\gamma(j) = g_{i(j,\gamma)}(j)$, where $i(j,\delta) = \min\{i \mid f_\delta(i) \leq j\}$ and $i(j,\gamma) = \min\{i \mid f_\gamma(i) \leq j\}$. As before, since $k \in \omega$ was arbitrary, we have shown that $h_\gamma \prec h_\delta$.

It is left to prove that $(\{h_\gamma\}_{\gamma<\kappa}, \{g_0\})$ can not be interpolated by any $g \in {}^\omega\omega$:
Assume this is false and let g be an interpolating function. Define

$$f(n) = \max\{i \mid g(i) \geq g_n(i)\}.$$

Then f is well-defined, since $g \prec g_0$ and $|g_0(i) - g_n(i)| = n < \infty$ for all $n \in \omega$ and all $i > n$.

Now we claim that f is an upper bound in $({}^\omega\omega, \prec)$ for any f_γ, which is a contradiction:

Let $\gamma < \kappa$. Fix an $m \in \omega$ such that $h_\gamma(j) < g(j)$ for all $j > m$. Then we find $n \in \omega$ such that $m \in [f_\gamma(n), f_\gamma(n+1))$.

Consider $k > n$. Then $f_\gamma(k) > m$, thus $h_\gamma(j) < g(j)$ for all $j > f_\gamma(k)$. Fix such a $j \in \omega$. We can find $l \in \omega$ with $l \geq k$ such that $h_\gamma(j) = g_l(j)$, i.e. $j \in [f_\gamma(l), f_\gamma(l+1))$. But now $g(j) > g_l(j) = h_\gamma(j)$. This holds for all such $j \in [f_\gamma(l), f_\gamma(l+1))$, which implies that $f(l+1) \geq f_\gamma(l+1)$. This argument shows that for all $l > k$ we have $f(l) \geq f_\gamma(l)$.

If f would not be an upper bound of f_γ for all $\gamma < \kappa$, we could find a $\gamma^* < \kappa$ such that $f_{\gamma^*} \not\prec f$. But for $\delta > \gamma^*$ we have $f_{\gamma^*} \prec f_\delta$. This implies $f(j) < f_\delta(j)$ for infinitely many $j \in \omega$, a contradiction to what we have just shown. So $f_\gamma \prec f$ for all $\gamma < \kappa$, a contradiction to $(\{f_\gamma\}_{\gamma<\kappa}, \emptyset)$ being a $(\kappa, 0)$-gap. □

3. $\implies$ 1.: Suppose we have a $(\kappa, 1)$-gap $(\{f_\gamma\}_{\gamma<\kappa}, \{g\})$. We define a (κ, ω)-pregap $(\{f'_\gamma\}_{\gamma<\kappa}, \{g_n\}_{n\in\omega})$ by

$$f'_\gamma(i) = i \cdot f_\gamma(i),$$

$$g_0 = g,$$

and for $n > 0$:

$$g_n(i) = \begin{cases} i \cdot g(i) - i \cdot 2^n & \text{if } g(i) > 2^n \\ 1 & \text{otherwise.} \end{cases}$$

Then the following holds:

Claim 7. *$(\{f'_\gamma\}_{\gamma<\kappa}, \{g_n\}_{n\in\omega})$ is a (κ, ω)-gap.*

Proof. It is clear that $f'_\gamma \prec f'_\delta$ for $\gamma < \delta < \kappa$ and $g_n \prec g_m$ whenever $m < n$.

Fix $\gamma < \kappa$ and $n \in \omega$. Since $f_\gamma \prec g$, we can find $m \in \omega$ such that $g(j) - f_\gamma(j) > 2^n$ for all $j > m$. But then for such a j, $g_n(j) - f'_\gamma(j) \geq j \cdot 2^n$, which tends towards infinity as $j \to \infty$. Thus $f'_\gamma \prec g_n$, so we obtain that $(\{f'_\gamma\}_{\gamma<\kappa}, \{g_n\}_{n\in\omega})$ is a pregap.

Now suppose there exists an $h \in {}^{\omega}\omega$ that interpolates this pregap. Then we can define h' by

$$h'(n) = \left\lceil \frac{h(n)}{n} \right\rceil.$$

Let $\gamma < \kappa$ and $k \in \omega$. Since $f'_\gamma \prec h$, we find $m_\gamma \in \omega$ such that $h(j) > f'_\gamma(j) + k$ for all $j > m_\gamma$. But then for all $j > m$ similarly $h'(j) > f_\gamma(j) + \frac{k}{j} \geq f_\gamma(j)$. Thus $f_\gamma \prec h'$. To see this, fix a $\delta > \gamma$ and note that we can find $m_\delta \in \omega$ such that $h'(j) \geq f_\delta(j)$ for $j > m_\delta$. But then, eventually we have the relation $f_\gamma \prec f_\delta \leq h'$, what implies $f_\delta \prec h'$.

Now fix $k \in \omega$. We know that $h \prec g_k$, so we can find $m \in \omega$ such that $h(j) < g_k(j)$ for all $j > m$. This is equivalent to $g_k(j) - h(j) = j \cdot g(j) - j \cdot 2^k - h(j) \geq 1$ for all $j > m$. This implies that $g(j) - h'(j) \geq 2^k$ for all $j > m$. Since $k \in \omega$ war arbitrary, this implies $h' \prec g$. But then h' interpolates $(\{f_\gamma\}_{\gamma<\kappa}, \{g\})$, a contradiction. □

Because every (κ, ω)-gap is a κ-Rothberger Gap by definition, Claim 7 proves 3. $\implies$ 1. □

Chapter 3

Forcing and gaps in $({}^{\omega}\omega, \prec)$

Up to this point only classical techniques such as induction were used to obtain the results we have presented in the previous chapters. From a set theoretical point of view, it is a natural question to ask about the behaviour of gaps in $({}^{\omega}\omega, \prec)$ under forcing. We will introduce a forcing notion that will produce a gap in $({}^{\omega}\omega, \prec)$ and present some results regarding the question in which cases gaps can be destroyed by forcing and in which cases not. We will mostly follow [5] and will often use a similar notation, whenever appropriate.

The reader is assumed to have basic knowledge about forcing. All needed preliminaries can for example be found in [9], however, this is certainly not the only source for this material.

There are two results we use quite often in this chapter, therefore we stress them. We start with a combinatorial fact, the well-known Δ-System Lemma:

Proposition 18 (Δ-System Lemma)**.** *Let X be a set and $\{A_\gamma \mid \gamma < \kappa\}$ be a collection of finite subsets of X, where κ is an uncountable regular cardinal. Then there is a κ-sized subset $I \subseteq \kappa$ and a finite R such that $A_i \cap A_j = R$ for distinct $i, j \in I$.*

In the situation of the lemma, we call $\{A_\gamma \mid \gamma \in I\}$ a Δ-system and the finite set R the *root* of $\{A_\gamma \mid \gamma \in I\}$.

Proof. Let $\{A_\gamma \mid \gamma < \kappa\}$ be as in the statement of the lemma. Each A_γ is finite, so we can find a natural number n such that the set $I = \{\gamma < \kappa \mid$

$|A_\gamma| = n\}$ has cardinality κ and let $A = \{A_\gamma \mid \gamma \in I\}$. Now we use induction on n:

For $n = 1$, we obtain that $A_i \cap A_j = \emptyset$ for all distinct $i, j \in I$. So for $R = \emptyset$, A is a Δ-system of size κ with root R and the lemma holds.

For the induction step, let $n > 1$ and assume the lemma holds for all $m < n$.

We consider two cases: First assume that we find an $x \in X$ such that $B = \{A_\gamma \in A \mid x \in A_\gamma\}$ has cardinality κ. Then take $B' = \{A_\gamma \setminus \{x\} \mid A_\gamma \in B\}$, which is a κ-sized collection of finite subsets of X each of which having cardinality $n-1$, thus by induction hypotheses B' has a subset which is a Δ-system of size κ.

Now assume we can not find an $x \in X$ which is in κ-many A_γ's where $\gamma \in I$. Then we can construct a Δ-system of size κ with root $R = \emptyset$ by induction: For the start choose any A_δ from A. Given $\{A_\delta\}_{\delta \in J_\alpha} \subseteq A$ for some $\alpha < \kappa$ and $J_\alpha \subset I$ with $|J_\alpha| < \kappa$ such that $A_\delta \cap A_{\delta'} = \emptyset$ for all $\delta, \delta' \in J_\alpha$, we observe that by assumption each element of $\bigcup_{\delta \in J_\alpha} A_\delta$ is in less than κ-many A_γ's, for $\gamma \in I$. Because $|\bigcup_{\delta \in J_\alpha} A_\delta| < \kappa$ and $|\bigcup_{\gamma \in I} A_\gamma| = \kappa$, we can find $\delta^* \in I \setminus J_\alpha$ such that $A_{\delta^*} \cap A_\delta = \emptyset$ for each $\delta \in J_\alpha$. So we can extend J_α by δ^* and obtain a bigger Δ-system. Thus, inductively we can construct a collection $\{A_\delta\}_{\delta \in J} \subseteq A$ which forms a Δ-system and $|J| = \kappa$. □

Another fact we will frequently use, most of the time even without explicitly stating it, is the following Proposition, stating that whenever there exists an element in a generic extension satisfying some property, we find a name for it. There are multiple proofs, the one given here is similar as in [9].

Proposition 19 (Maximal Principle)**.** *Suppose that $\mathbb{P}$ is a forcing notion. Let $p \in \mathbb{P}$ and $\tau_1, ..., \tau_n$ be $\mathbb{P}$-names such that*

$$p \Vdash \exists x \colon \phi(x, \tau_1, ..., \tau_n).$$

Then there is a $\mathbb{P}$-name τ such that

$$p \Vdash \phi(\tau, \tau_1, ..., \tau_n).$$

Proof. For the proof, we write $\overline{\tau}$ for $\tau_1, ..., \tau_n$ and $M^{\mathbb{P}}$ for the collection of $\mathbb{P}$-names. Let p be as in the statement. Consider the set $X = \{q \leq p \mid$

$\exists\sigma \in M^{\mathbb{P}}\colon q \Vdash \phi(\sigma, \overline{\tau})\}$. For each $q \in X$ we can pick $\sigma_q \in M^{\mathbb{P}}$ such that $q \Vdash \phi(\sigma_q, \overline{\tau})$. Using Zorn's Lemma, we can find $A \subseteq X$ such that A is a maximal antichain below p in $\mathbb{P}$.

Now let

$$\tau = \bigcup_{q \in A} \{(\nu, s) \mid \nu \in \mathrm{dom}\{\sigma_q\} \text{ and } s \leq q \text{ with } s \Vdash \nu \in \sigma_q\}.$$

We claim that τ is as desired:

Let G be generic and $p \in G$. Then $G \cap A \neq \emptyset$, since A is a maximal antichain, thus we find $t \in A \cap G$. This t is unique, because A is an antichain and G is a filter. Therefore, the valuation of τ in the generic extension, τ^G, consists of ν such that there is $s \leq t$ and $s \Vdash \nu \in \sigma_t$. Thus $\tau^G \subseteq \sigma_t^G$. On the other hand, any $\nu \in \sigma_t^G$ is such that $\nu \in \mathrm{dom}\{\sigma_t\}$ and we can find an $r \in G$ such that $r \Vdash \nu \in \sigma_t$. But since $r, t \in G$ we find $s \in G$ such that $s \leq r, t$, thus $s \Vdash \nu \in \sigma_t$. But then $\nu \in \tau^G$, what implies $\sigma_t^G \subseteq \tau^G$. We obtain that $t \Vdash \phi(\tau, \overline{\tau})$, which completes the proof, since $t \in G$.

□

3.1 Forcing gaps

We start by presenting a forcing notion that will introduce a gap in a generic extension. We follow [5] here.

For ordinals α and β let $\phi_{\alpha,\beta}$ be the set $\alpha \times \{0\} \cup \beta \times \{1\}$. We define a linear order $<$ on $\phi_{\alpha,\beta}$ by letting $(\gamma, i) < (\delta, j)$ if one of the following conditions hold:

- $i < j$ (i.e. $i = 0$ and $j = 1$),
- $i = j = 0$ and $\gamma < \delta$,
- $i = j = 1$ and $\gamma > \delta$.

Note that given a pregap $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$, the ordering of $\phi_{\alpha,\beta}$ reflects the ordering of the pregap with respect to $\prec$ by identifying f_γ with $(\gamma, 0)$ and g_δ with $(\delta, 1)$.

Now consider the collection F of finite partial functions $p\colon [\phi_{\alpha,\beta}]^{<\aleph_0} \times \omega \to \omega$, where as usual $[X]^{<\aleph_0}$ represents the collection of finite subsets of a given set X. Then we define the forcing notion $\mathbf{F}_{\alpha,\beta}$ to be $(F, \ll)$, where for $p, q \in F$ we let $p \ll q$ if

1. $q \subset p$,

   and, for $\text{dom}\{p\} = D_p \times n_p$, $\text{dom}\{q\} = D_q \times n_q$, where $D_p, D_q \in [\phi_{\alpha,\beta}]^{<\aleph_0}$ and $n_p, n_q < \omega$, we have

2. $p(d_1, i) < p(d_2, i)$ for $d_1, d_2 \in D_q$ such that $d_1 < d_2$ and $i \in [n_q, n_p)$.

The maximal element of $\mathbf{F}_{\alpha,\beta}$ is $\emptyset$, denoted by $\mathbf{1}_{\mathbf{F}_{\alpha,\beta}}$. To get some insight in the structure of $\mathbf{F}_{\alpha,\beta}$, we give a condition for compatibility of two elements $p, q \in \mathbf{F}_{\alpha,\beta}$:

Proposition 20. *Let α, β be ordinals and $p, q \in \boldsymbol{F}_{\alpha,\beta}$. Suppose dom$\{p\} = D_p \times n_p$ and dom$\{q\} = D_q \times n_q$, where $n_q \leq n_p$. Let $D_p = \{d_1^p, d_2^p, ..., d_{m_p}^p\}$, $D_q = \{d_1^q, d_2^q, ..., d_{m_q}^q\}$ and $D_p \cap D_q = \{d_1, ..., d_n\}$, all enumerated in order. Then p and q are compatible if and only if*

1. *$q(d, i) = p(d, i)$ for all $d \in D_p \cap D_q$ and all $i < n_q$,*

   and

2. *$p(d_1, i) \geq |\{d' \in D_q \setminus D_p \mid d' < d_1\}|$ and $p(d_{k+1}, i) - p(d_k, i) \geq |\{d' \in D_q \setminus D_p \mid d_k < d' < d_{k+1}\}|$ for all $i \in [n_q, n_p)$ and all $k \in \{1, ..., n-1\}$.*

Proof. First assume that $n_q = n_p$. Then, since $[n_q, n_p) = \emptyset$, condition 2. of the proposition is always true.

$\Longrightarrow$: Suppose p and q are compatible, then there is $r \ll p, q$, so r extends both p and q. In particular, p and q must agree on their common domain; this is 1.

$\Longleftarrow$: If now 1. holds, any r that extends p and q satisfies $r \ll p, q$, since condition 2. in the definition of $\ll$ always holds if $n_p = n_q$.

Now assume $n_q < n_p$.

$\Longrightarrow$: If p and q are compatible, there is r with $r \ll p, q$. Then r extends both p and q, so 1. holds. Let $\text{dom}\{r\} = D_r \times n_r$. Now since $r \ll q$ and $r \ll p$, by condition 2. in the definition of $\ll$, for any $i \in [n_q, n_r)$ it must hold that $r(d, i) < r(d', i)$ for all $d < d' \in D_q$. Now let $\{d' \in D_q \setminus D_p \mid d' < d_1\} = \{e_1, ..., e_l\}$, enumerated in the order of $\phi_{\alpha,\beta}$. Then for $i \in [n_q, n_r)$

$$r(e_1, i) < ... < r(e_l, i) < r(d_1, i) = p(d_1, i),$$

what implies that $p(d_1, i) \geq l$, as desired.

If we enumerate $\{d' \in D_q \setminus D_p \mid d_k < d' < d_{k+1}\}$ as $\{e'_1, ..., e'_{l'}\}$, from a similar argument as above we obtain that

$$p(d_k, i) = r(d_k, i) < r(e'_1, i) < ... < r(e'_{l'}, i) < r(d_{k+1}, i) = p(d_{k+1}, i),$$

and this implies $p(d^p_{k+1}, i) - p(d^p_k, i) \geq l'$:

$\Longleftarrow$: We define an $r \in \mathbf{F}_{\alpha,\beta}$ that witnesses the compatibility of p and q. Let $\operatorname{dom}\{r\} = (D_p \cup D_q) \times n_p$.

First, let $r(x) = p(x)$ for $x \in \operatorname{dom}\{p\}$ and $r(x) = q(x)$ for $x \in \operatorname{dom}\{q\}$. We can do this by condition 1. All left to do is to define r on $(D_q \setminus D_p) \times [n_q, n_p)$. But then condition 2. guarantees us that there is enough space to do this. Namely, we can for example define r as follows:

Let $r(d, i) = p(d_k, i) + j$ if d is the j-th element in the set $\{d' \in D_q \setminus D_p \mid d_k < d' < d_{k+1}\}$. If $d < d_1$, let $r(d, i) = j$ if d is the j-th element in $\{d' \in D_q \setminus D_p \mid d' < d_1\}$. Finally, if $d > d_n$, let $r(d, i) = p(d, i) + j$ where d is the j-th element in $\{d' \in D_q \setminus D_p \mid d' > d_n\}$.

Then r is such that $r \ll p, q$. □

We state an important property of $\mathbf{F}_{\alpha,\beta}$:

Proposition 21. *Let α, β be ordinal numbers. Then $\boldsymbol{F}_{\alpha,\beta}$ is ccc.*

Proof. Suppose we can find an uncountable antichain A in $\mathbf{F}_{\alpha,\beta}$. Enumerate A as $\{a_i\}_{i<\omega_1}$, so that each a_i is a finite partial function $[\phi_{\alpha,\beta}]^{<\aleph_0} \times \omega \to \omega$. We denote the domain of a_i with $D_i \times n_i$, where $D_i \in [\phi_{\alpha,\beta}]^{<\aleph_0}$ and $n_i \in \omega$.

By the uncountability of A we find an uncountable set $A' \subseteq A$ for which $n_i = n_{i'} := n$ for $i, i' \in A'$. By the Δ-system lemma we can assume that $\{D_i\}_{i \in A'}$ forms a Δ-system with root D. Note that the restriction $a_i \restriction D \times n$ is finite for each a_i, thus there are at most countably many different such restrictions. Therefore, we can find $A'' \subseteq A'$ for which the restrictions agree, i.e. $a_i \restriction D \times n = a_{i'} \restriction D \times n$ for $i, i' \in A''$. But then any two elements $a_i, a_{i'}$ for $i, i' \in A''$ are compatible by Proposition 20, which is a contradiction. □

The following technical proposition will be helpful when proving the main result of this sector later on:

Proposition 22. *Let α, β be regular uncountable cardinals and $\gamma \leq \alpha$ and $\delta \leq \beta$ be limit ordinals such that $\gamma < \alpha$ or $\delta < \beta$. Consider $p \in \boldsymbol{F}_{\alpha,\beta}$ and let*

$p' \in \boldsymbol{F}_{\gamma,\delta}$ be the restriction of p to $[\phi_{\gamma,\delta}]^{<\aleph_0} \times \omega$.
Then there is a $q \in \boldsymbol{F}_{\gamma,\delta}$ that satisfies

1. *$q \ll p'$,*

2. *p is compatible with any r such that $r \ll q$ and $r \in \boldsymbol{F}_{\gamma,\delta}$.*

Proof. For the proof, we suppose that $\gamma < \alpha$ and $\delta = \beta$. Further, we may assume that $p \neq p'$, since otherwise the statement is trivially true.

For $\text{dom}\{p\} = D_p \times n_p$ and $\text{dom}\{p'\} = D_{p'} \times n_{p'}$, first note that $n_p = n_{p'}$, since we only restrict D_p when moving from p to p'.

Enumerate $D_p \cap D_{p'}$ by $\{d_1, d_2, ..., d_n\}$ in the order of $\phi_{\gamma,\delta}$. We observe that D_p is of the form $\{d_1, ..., d_i, e_1, ..., e_m, d_{i+1}, ..., d_n\}$, listed in the order of $\phi_{\alpha,\beta}$, where $D_p \setminus D_{p'} = \{e_1, ..., e_m\}$ for some $m \in \omega$. Note that all e_i's are of the form $(\nu, 0)$ for $\gamma < \nu < \alpha$, all d_j's are of the form $(\mu, 0)$ for $\mu < \gamma$, $j \leq i$ and all d_j's are of the form $(\rho, 1)$ for $\rho < \beta$ if $j > i$.

Now we use the assumption that γ is a limit ordinal and choose m-many ν_k such that $\gamma < \nu_1 < ... < \nu_m < \alpha$. Then we define the desired element $q \in \mathbf{F}_{\gamma,\delta}$ by letting $\text{dom}\{q\} = (D_{p'} \cup \{(\nu_1, 0), ..., (\nu_m, 0)\}) \times n_p$ and let $q(x) = p'(x)$ for $x \in \text{dom}\{p'\}$ and $q(y)$ be some natural number elsewhere.

Then $q \ll p'$, since the only thing we must consider is that $p' \subset q$. This is because condition 2. in the definition of $\ll$ is trivially true, because $\text{dom}\{p'\} = D_{p'} \times n_p$ and $\text{dom}\{q\}$ is of the form $D_q \times n_p$.

Now consider any r such that $r \in \mathbf{F}_{\gamma,\beta}$ and $r \ll q$. Let the domain $\text{dom}\{r\}$ of r be $D_r \times n_r$. Then $r(x) = p(x)$ for any $x \in D_p \cap D_r$, because $r(x) = p'(x)$ for any such x and p' is the restriction of p to $[\phi_{\gamma,\delta}]^{<\aleph_0} \times \omega$. Thus, by Proposition 20, if $n_r = n_p$, we are done.

So assume $n_r \neq n_p$, i.e. $n_r > n_p$. Let $d' \in D_p \setminus D_r$ and note that $D_r \cap D_p \supseteq D_{p'} \cap D_p = \{d_1, d_2, ..., d_n\}$. Thus the sets $\{d' \in D_p \setminus D_r \mid d_j < d' < d_{j+1}\}$ are empty for all $j \neq i$, since this holds for the sets $\{d' \in D_p \setminus D_{p'} \mid d_j < d' < d_{j+1}\}$. Now consider $X = \{d' \in D_p \setminus D_r \mid d_i < d' < d_{i+1}\}$, which is of cardinality m, by the definition of r. In fact, $X = \{(\nu_1, 0), ..., (\nu_m, 0)\}$. But then for $i \in [n_p, n_r)$

$$r(d_i, i) = p(d_i, i) < r((\nu_1, 0), i) < ... < r((\nu_m, 0), i) < r(d_{i+1}, i) = p(d_{i+1}, i),$$

thus we can apply Proposition 20 to obtain that r and p are compatible. □

Now we define $\mathbf{F}_{\alpha,\beta}$-names that will form a gap in the generic extension:

For $\gamma < \alpha$ and $\delta < \beta$ define the $\mathbf{F}_{\alpha,\beta}$-names

$$\dot{f}_\gamma = \{((m,\check{}\,n), p) \mid m, n \in \omega, p \in \mathbf{F}_{\alpha,\beta}, ((\gamma,0),m) \in \text{dom}\{p\} \subset \{(\gamma,0)\} \times \omega \text{ and } p((\gamma,0),m) = n\}$$

and

$$\dot{g}_\delta = \{((m,\check{}\,n), p) \mid m, n \in \omega, p \in \mathbf{F}_{\alpha,\beta}, ((\delta,1),m) \in \text{dom}\{p\} \subset \{(\delta,1)\} \times \omega \text{ and } p((\delta,1),m) = n\}.$$

Observe that for a $\mathbf{F}_{\alpha,\beta}$-generic filter G, the evaluation $\dot{f}_\gamma^G$ is a subset of $\omega \times \omega$. Further, if $(m,n) \in \dot{f}_\gamma^G$ then there is $p \in G$ such that $p((\gamma,0),m) = n$. We will later see that $\dot{f}_\gamma^G$ is actually a function $\omega \to \omega$. Similarly, this observations also hold for $\dot{g}_\delta$.

Now the main theorem, similar as most of the section form [5], is

Theorem 23 (Generic Gap Theorem)**.** *If α and β are regular uncountable cardinal numbers, then*

$$\mathbf{1}_{\mathbf{F}_{\alpha,\beta}} \Vdash \text{"}(\{\dot{f}_\gamma\}_{\gamma<\alpha}, \{\dot{g}_\delta\}_{\delta<\beta}) \text{ is an } (\alpha,\beta)\text{-gap."}$$

Proof. Let α and β be as in the theorem. We prove the Theorem in a couple of steps and start with the following useful observation:

Claim 8. *For $\gamma < \alpha$, $\delta < \beta$ and $k \in \omega$, the following sets are dense open:*

- $D_{\gamma,k} = \{p \in \boldsymbol{F}_{\alpha,\beta} \mid dom\{p\} = D \times n \text{ such that } n > k \text{ and } (\gamma,0) \in D\}$
- $D'_{\delta,k} = \{p \in \boldsymbol{F}_{\alpha,\beta} \mid dom\{p\} = D \times n \text{ such that } n > k \text{ and } (\delta,1) \in D\}$

Proof. We prove for $D_{\gamma,k}$ for some $\gamma < \alpha$ and $k \in \omega$, since the proof for $D'_{\delta,k}$ is almost the same. Consider any $q \in \mathbf{F}_{\alpha,\beta}$. Suppose that the domain of q is $D_q \times n_q$ and pick any n such that $n > k$ and $n > n_q$. Let $D = D_q \cup \{(\gamma,0)\}$ and let $\{d_1, ..., d_m\}$ be the ordered enumeration of D. Then we define $p \in \mathbf{F}_{\alpha,\beta}$ by setting $p(x) = q(x)$ for $x \in \text{dom}\{q\}$ and $p(d_i,j) = i$ for $(d_i,j) \notin \text{dom}\{q\}$. Then $p \ll q$ and $p \in \mathbf{F}_{\alpha,\beta}$.

The fact that both sets are open, follows since if p is in $D_{\gamma,k}$ and $q \ll p$, then $p \subset q$, so in particular $q \in D_{\gamma,k}$. □

Claim 9. *Let $\gamma < \alpha$ and $\delta < \beta$ be given. Then the following hold:*

1. $\mathbf{1}_{\mathbf{F}_{\alpha,\beta}} \Vdash$ "$\dot{f}_\gamma$ *is a function* $\check{\omega} \to \check{\omega}$"

2. $\mathbf{1}_{\mathbf{F}_{\alpha,\beta}} \Vdash$ "$\dot{g}_\delta$ *is a function* $\check{\omega} \to \check{\omega}$"

Proof of Claim 9. We only prove 1., as 2. is almost the same.

Fix $\gamma < \alpha$, let G be a $\mathbf{F}_{\alpha,\beta}$-generic filter and note that $\dot{f}_\gamma^G$ is a subset of $\omega \times \omega$. If $\dot{f}_\gamma^G$ would not be a function, we could find $p, q \in G$ such that there are $m \in \omega$ and $n_1 \neq n_2 \in \omega$ for which $p((\gamma, 0), m) = n_1$ and $q((\gamma, 0), m) = n_2$. But since $p, q \in G$, they are compatible, which means there exists an element that extends both p and q, thus such m, n_1 and n_2 can not exist. Now since G is a filter, $G \cap D_{\gamma,k} \neq \emptyset$ for all $k \in \omega$. But then $k \in \mathrm{dom}\{\dot{f}_\gamma^G\}$ for all $k \in \omega$, so that $\dot{f}_\gamma^G$ is a function $\omega \to \omega$. Since G was arbitrary, this gives 1. of the claim. □

Claim 10. *Let* $\gamma_1 < \gamma_2 < \alpha$ *and* $\delta_1 < \delta_2 < \beta$ *be given. Then the following hold:*

1. $\mathbf{1}_{\mathbf{F}_{\alpha,\beta}} \Vdash$ "$\dot{f}_{\gamma_1} \prec \dot{f}_{\gamma_2}$"

2. $\mathbf{1}_{\mathbf{F}_{\alpha,\beta}} \Vdash$ "$\dot{g}_{\delta_2} \prec \dot{g}_{\delta_1}$"

3. $\mathbf{1}_{\mathbf{F}_{\alpha,\beta}} \Vdash$ "$\dot{f}_{\gamma_2} \prec \dot{g}_{\delta_2}$"

In other words,

$$\mathbf{1}_{\mathbf{F}_{\alpha,\beta}} \Vdash \text{"}(\{\dot{f}_\gamma\}_{\gamma<\alpha}, \{\dot{g}_\delta\}_{\delta<\beta}) \textit{ is a pregap.}\text{"}$$

Proof of Claim 10. 1.: Let G be a $\mathbf{F}_{\alpha,\beta}$-generic filter. We know that the sets $D_{\gamma_1,k}$ and $D_{\gamma_2,k}$ are dense open for any $k \in \omega$, thus so is their intersection. This implies that there is some $p \in G \cap D_{\gamma_1,k} \cap D_{\gamma_2,k}$ and this means $((\gamma_1, 0), k), ((\gamma_2, 0), k) \in \mathrm{dom}\{p\}$. Suppose that the domain of p is given by $D_p \times n_p$ and consider any $m > n_p$. Now let $q \ll p$ be such that $\mathrm{dom}\{q\} = D_q \times n_q$ for $n_q > m > n_p$. Then, by condition 2. in the definition of $\ll$, it follows that $q((\gamma_1, 0), m) < q((\gamma_2, 0), m)$. This implies that

$$q \Vdash \text{"}\dot{f}_{\gamma_1}(m) < \dot{f}_{\gamma_2}(m)\text{"},$$

thus also

$$p \Vdash \text{"}\dot{f}_{\gamma_1}(m) < \dot{f}_{\gamma_2}(m)\text{"}.$$

Since $m > n_p$ was arbitrary, it follows that

$$p \Vdash "\dot{f}_{\gamma_1}(m) < \dot{f}_{\gamma_2}(m) \text{ for all m} > \check{n_p}".$$

The generic filter G was arbitrary, so we obtain that actually

$$\mathbf{1}_{\mathbf{F}_{\alpha,\beta}} \Vdash "\dot{f}_{\gamma_1} < \dot{f}_{\gamma_2} \text{ eventually pointwise}". \tag{3.1}$$

To obtain that even

$$\mathbf{1}_{\mathbf{F}_{\alpha,\beta}} \Vdash "\dot{f}_{\gamma_1} \prec \dot{f}_{\gamma_2}", \tag{3.2}$$

note that we can w.l.o.g. assume that there are infinitely many γ' such that $\gamma_1 < \gamma' < \gamma_2 < \alpha$, because α is a regular uncountable cardinal number and it suffices to prove the theorem (and thus the claim) for a cofinal subset of α. Then we can apply the argument that lead to 3.1 to n such γ''s, so that we obtain

$$\mathbf{1}_{\mathbf{F}_{\alpha,\beta}} \Vdash "\dot{f}_{\gamma_1} < \dot{f}_{\gamma'_1} < ... < \dot{f}_{\gamma'_n} < \dot{f}_{\gamma_2} \text{ eventually pointwise}",$$

for each $n \in \omega$. But this implies 3.2.

2. and 3.: We leave out a detailed argument for 2 and 3., since it is essentially the same argument, just using $D'_{\delta,k}$ in the one or the other place. □

Now suppose that Theorem 23 is false, then we can find a $p \in \mathbf{F}_{\alpha,\beta}$ and a $\mathbf{F}_{\alpha,\beta}$-name τ such that

$$p \Vdash "\tau \text{ interpolates } (\{\dot{f}_\gamma\}_{\gamma<\alpha}, \{\dot{g}_\delta\}_{\delta<\beta})".$$

Since p is a finite partial function $[\phi_{\alpha,\beta}]^{<\aleph_0} \times \omega \to \omega$, we find limit ordinals $\gamma \leq \alpha$ and $\delta \leq \beta$ such that at least one of the inequalities $\gamma < \alpha$ and $\delta < \beta$ holds and $p \in \mathbf{F}_{\gamma,\delta}$. For the proof, let us assume that $\delta < \beta$. Then we can also assume w.l.o.g. that $\tau \in \mathbf{F}_{\gamma,\delta}$.

Now fix $p' \in \mathbf{F}_{\alpha,\beta}$ such that $p' \ll p$ and $n \in \omega$ and

$$p' \Vdash "\tau(i) < \dot{g}_\delta(i) \text{ for all } i > \check{n}". \tag{3.3}$$

Let p'' be the restriction of p' to $\mathbf{F}_{\gamma,\delta}$. Since γ is limit and p' is a finite partial set, we can find $\epsilon < \gamma$ such that $p'' \in \mathbf{F}_{\epsilon,\delta}$. Then $p'' \ll p' \ll p$ and this implies that

$$p'' \Vdash "\tau \text{ interpolates } (\{\dot{f}_\gamma\}_{\gamma<\alpha}, \{\dot{g}_\delta\}_{\delta<\beta})"$$

and thus

$$p'' \Vdash \text{"}\dot{f}_\epsilon \prec \tau\text{"}.$$

We apply Proposition 22 and find $q \in \mathbf{F}_{\epsilon,\delta}$ such that $q \ll p''$ and p' is compatible with any $r \in \mathbf{F}_{\epsilon,\delta}$ if $r \ll q$.

Now choose $r \ll q$ for which there is $n' \in \omega$ such that

$$r \Vdash \text{"}\dot{f}_\epsilon(i) < \tau(i) \text{ for all } i > \check{n'}\text{"}. \tag{3.4}$$

For $\mathrm{dom}\{r\} = D_r \times n_r$ and $\mathrm{dom}\{p''\} = D_{p''} \times n_{p''}$, pick some arbitrary $n_s > 1 + \max\{n, n', n_r, n_{p''}\}$ and define $s \in \mathbf{F}_{\alpha,\beta}$ as follows:

- $\mathrm{dom}\{s\} = D_s \times n_s$,
- $s \ll p'$ and $s \ll r$,
- $s((\epsilon, 0), n_s - 1) > s((\delta, 1), n_s - 1)$.

This is possible, since r and p' are compatible and $(\epsilon, 0) \notin D_r$ and $(\delta, 1) \notin D_{p'}$. Note that

$$s \Vdash \text{"}\dot{f}_\epsilon(n_s - 1) > \dot{g}_\delta(n_s - 1)\text{"} \tag{3.5}$$

and, because $s \ll r, p'$ by 3.3 and 3.4 also

$$s \Vdash \text{"}\dot{f}_\epsilon(i) < \tau(i) < \dot{g}_\delta(i) \text{ for all } i > \max\{\check{n}, \check{n'}\}\text{"}.$$

But this is a contradiction to 3.5, what finishes the proof. □

3.2 Layer's interpolation order

In the last section we discussed a forcing notion that introduces a gap in a generic extension - now we consider the following question:

> *Given an (α, β)-gap, can we find a forcing notion that destroys it, i.e. that introduces an interpolating real in the generic extension?*

The answer to this question is "yes" and the forcing involved is the following, introduced by Laver in [10]:

Definition 9 (Layer's Interpolation Order.)**.** Suppose $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is an (α, β)-pregap in $({}^\omega\omega, \prec)$ for infinite ordinal numbers α and β.

Then the Layer's interpolation order $(\mathbf{L}_{\alpha,\beta}, \lhd)$ is the forcing notion defined as follows:

- Elements of $\mathbf{L}_{\alpha,\beta}$ are quadruples (X, Y, s, n) such that $X \in [\alpha]^{<\aleph_0}$, $Y \in [\beta]^{<\aleph_0}$, s is a finite sequence of natural numbers, $n \in \omega$ and for any $\gamma \in X, \delta \in Y$ and every $m > \mathrm{dom}\{s\}$ we have that $f_\gamma(m) + n < g_\delta(m) - n$. The maximal element $\mathbf{1}_{\mathbf{L}_{\alpha,\beta}}$ is $(\emptyset, \emptyset, \emptyset, \emptyset)$.
- For (X, Y, s, n) and (X', Y', s', n') in $\mathbf{L}_{\alpha,\beta}$, we say that $(X, Y, s, n) \lhd (X', Y', s', n')$ if
  1. $X' \subseteq X$, $Y' \subseteq Y$, $s' \subseteq s$ and $n' \leq n$,

     and
  2. for $\gamma \in X, \delta \in Y$ it holds that $f_\gamma(m) + n' \leq s(m) \leq g_\delta(m) - n'$ for all $m \in \mathrm{dom}\{s\} \setminus \mathrm{dom}\{s'\}$.

Remark. Note that Definition 9 is explicitly dependent from the underlying pregap. For different (α, β)-pregaps, there are different $\mathbf{L}_{\alpha,\beta}$'s.

Of course, the point is the following:

Theorem 24. *Let α, β be infinite ordinals and $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α, β)-pregap. Let $\mathbf{L}_{\alpha,\beta}$ be the corresponding Layer interpolation order. Then*

$$\mathbf{1}_{\mathbf{L}_{\alpha,\beta}} \Vdash \text{"}(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta}) \text{ is not a gap"}.$$

Proof. Let G be a $\mathbf{L}_{\alpha,\beta}$-generic filter. We claim that

$$s^* = \bigcup\{s \in {}^{<\omega}\omega \mid \exists X \in [\alpha]^{<\aleph_0} : \exists Y \in [\beta]^{<\aleph_0} : \exists n \in \omega : (X, Y, s, n) \in G\}$$

is an interpolation real for $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$, what proves the proposition.

Claim 11. *s^* is a real.*

Proof. Let $S = \{s \in {}^{<\omega}\omega \mid \exists X \in [\alpha]^{<\aleph_0} : \exists Y \in [\beta]^{<\aleph_0} : \exists n \in \omega : (X, Y, s, n) \in G\}$ and $s, s' \in S$. Then there are X, X', Y, Y', n, n' such that $(X, Y, s, n) \in G$ and $(X', Y', s', n') \in G$. Since G is a filer, this implies that they are compatible, thus there is r with $r \supseteq s, s'$. Therefore, s^* is a sequence of natural numbers. We are left with proving that $\mathrm{dom}\{s^*\} = \omega$, i.e. that s^* is of length ω.

Therefore, we show that $A_k = \{(X, Y, s, n) \in \mathbf{L}_{\alpha,\beta} \mid k \in \mathrm{dom}\{s\}\}$ is dense for each $k \in \omega$. Let $k \in \omega$ and $(X, Y, s, n) \in \mathbf{L}_{\alpha,\beta}$ such that, without loss of generality, $k \notin \mathrm{dom}\{s\}$.

Let $s' \in {}^{<\omega}\omega$ be such that $\mathrm{dom}\{s'\} = k+1$ and $s' \restriction \mathrm{dom}\{s\} = s$, where as usual $s' \restriction \mathrm{dom}\{s\}$ is the restriction of s' to $\mathrm{dom}\{s\}$. For $i \in \mathrm{dom}\{s'\} \setminus \mathrm{dom}\{s\}$ let $s'(i) = \frac{1}{2} \cdot \lfloor(\min\{g_\delta(i) \mid \delta \in Y\} - \max\{f_\gamma(i) \mid \gamma \in X\})\rfloor$. But then, since for any pair of $\gamma \in X, \delta \in Y$ we have that $f_\gamma(i) - n < g_\delta(i) + n$, we obtain that $(X, Y, s', n) \lhd (X, Y, s, n)$.

Thus A_k is dense for any $k \in \omega$, so $G \cap A_k \neq \emptyset$, thus s^* is indeed a real. $\square$

Claim 12. *$f_\gamma < s^* < g_\delta$ for all $\gamma < \alpha$ and all $\delta < \beta$.*

Proof. Let $\gamma < \alpha$ and $\delta < \beta$. Then we claim that

1. $B_\gamma = \{(X, Y, s, n) \in \mathbf{L}_{\alpha,\beta} \mid \gamma \in X\}$
2. $B_\delta = \{(X, Y, s, n) \in \mathbf{L}_{\alpha,\beta} \mid \delta \in Y\}$

are dense and open. Since the proofs for B_γ and B_δ are similar, we prove only for B_γ. Let $(X, Y, s, n) \in \mathbf{L}_{\alpha,\beta}$ and without loss of generality assume that $\gamma \notin X$. But then $(X \cup \{\gamma\}, Y, s, n) \lhd (X, Y, s, n)$, thus B_γ is dense. The fact that B_γ is open is clear by definition of $\lhd$.

Since B_γ, B_δ, A_k are dense open, where A_k is as in the proof of Claim 11, their intersection is dense open. This implies that, for S as in the proof of Claim 11, we find $s \in S$ such that there are X, Y, n for which $(X, Y, n, s) \in G \cap B_\gamma \cap B_\delta \cap A_k$. But then $f_\gamma(i) < s(i) < g_\delta(i)$ for each $i \in \mathrm{dom}\{s\}$. Since $s^* \supset s$, also $f_\gamma(i) < s^*(i) < g_\delta(i)$ for all $i \in \mathrm{dom}\{s\}$. This holds for all $k \in \omega$ and $\mathrm{dom}\{s\} \geq k$, what proves the claim. $\square$

Note that Claim 12 directly implies $f_\gamma \prec s^* \prec g_\delta$ for all $\gamma < \alpha, \delta < \beta$. If not, assume that we find $\gamma' < \alpha$ for which $f_{\gamma'} \not\prec s^*$. Let $\gamma' < \gamma$. Since $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is a pregap, $f_{\gamma'} \prec f_\gamma < s^*$, what is a contradiction. The case that there is δ' for which $s^* \not\prec g_{\delta'}$ is similar. $\square$

3.2.1 Destructibility of gaps

Theorem 24 states that given any gap, we can always find a forcing notion which destroys it. However, Theorem 24 does not provide any information about the properties of the respective Layers Interpolation Order. So we slightly switch our focus and consider the question:

> *Given a (pre-)gap, for which class of partially ordered sets there exists a forcing notion that destroys it?*

In this section we again follow [5], although in [5] it is suggested that many results have been known to Kunen before, see [7].

Definition 10. Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α, β)-gap. Let $\mathcal{C}$ be a class of partially ordered sets. If there is a forcing notion $(\mathbb{P}, <_\mathbb{P}, \mathbf{1}_\mathbb{P}) \in \mathcal{C}$ such that

$$\mathbf{1}_\mathbb{P} \Vdash \text{"}(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta}) \text{ is not a gap"},$$

then we say that $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is $\mathcal{C}$-destructible. If $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is not $\mathcal{C}$-destructible, it is called $\mathcal{C}$-indestructible.

Sometimes we also use a closely related notion, which is intuitively clear, still, we define it for completeness:

Definition 11. Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α, β)-gap and let $\mathbb{P}$ be a forcing notion. If

$$\mathbf{1}_\mathbb{P} \Vdash \text{"}(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta}) \text{ is a gap"},$$

then we say that the gap survives forcing with $\mathbb{P}$, otherwise we say that forcing with $\mathbb{P}$ destroys the gap.

To obtain results in connection with Definition 10 we investigate properties of Layers Interpolation Order for given gaps. We are interested in the following properties of partially ordered sets:

Definition 12. Let $(\mathbb{P}, <_\mathbb{P})$ be a partially ordered set.

1. We say a subset Q of $\mathbb{P}$ is centered if for every finite subset S of Q there exists a $<_\mathbb{P}$-minimal element of S in S. If $\mathbb{P}$ is the countable union of centered sets, we say that $\mathbb{P}$ is σ-centered. The class of σ-centered partially ordered sets is denoted by $\sigma - \mathcal{C}$.

2. We say a subset Q of $\mathbb{P}$ is linked, if any two elements of Q are compatible. If $\mathbb{P}$ is the countable union of linked sets, we say that $\mathbb{P}$ is σ-linked. We denote the class of σ-linked partial orders with $\sigma - \mathcal{L}$.

3. We say that $\mathbb{P}$ is Knaster, if every uncountable subset of $\mathbb{P}$ has an uncountable subset which is linked. We denote the class of Knaster partially ordered sets with $\mathcal{K}$.

4. For a regular uncountable cardinal λ, we say that $\mathbb{P}$ is λ-Knaster, if every ν-sized subset of $\mathbb{P}$ has a ν-sized linked subset for all regular uncountable cardinals $\nu \leq \lambda$. The class of λ-Knaster p.o. sets is denoted by $\lambda - \mathcal{K}$.

5. We say that $\mathbb{P}$ is strongly Knaster, if $\mathbb{P}$ is λ-Knaster for every regular uncountable cardinal λ and denote the class of such p.o. sets with $\mathcal{K}_\forall$.

Remark. Note that we the following:

- If a subset Q of $\mathbb{P}$ is centered, then it is linked. Thus if $\mathbb{P}$ is σ-centered, it is σ-linked.
- If $\mathbb{P}$ is σ-linked, it is strongly Knaster.
- If $\mathbb{P}$ is strongly Knaster, then $\mathbb{P}$ is λ-Knaster for any uncountable cardinal λ. If $\mathbb{P}$ is λ-Knaster, then $\mathbb{P}$ is Knaster.
- If $\mathbb{P}$ is Knaster, then it is ccc.

Proposition 25. *Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be a (α, β)-pregap such that one of the following holds:*

1. $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is not a gap,

2. α or β has cofinality ω.

Then the corresponding Layers Interpolation Order $\mathbf{L}_{\alpha,\beta}$ is σ-centered.

Proof. 1.: Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be a pregap and suppose that h interpolates it. For each $s \in {}^{<\omega}\omega$ and $n \in \omega$ we define

$L_{s,n} =$

$$\{(X, Y, s, n) \in \mathbf{L}_{\alpha,\beta} \mid \forall i > \mathrm{dom}\{s\} \colon \max_{\gamma \in X}\{f_\gamma(i)\} + n \leq h(i) \leq \min_{\delta \in Y}\{g_\delta(i)\} - n\}.$$

For any $s \in {}^{<\omega}\omega$ and $n \in \omega$, the set $L_{s,n}$ is centered. This holds since if $(X, Y, s, n), (X', Y', s, n) \in L_{s,n}$, then $(X \cup X', Y \cup Y', s, n) \in L_{s,n}$ and thus for each finite subset S there is an element in $\mathbf{L}_{\alpha,\beta}$ stronger than every $s \in S$.

Now let

$$L = \bigcup_{s \in {}^{<\omega}\omega, n \in \omega} L_{s,n}.$$

Note that there are only countable many $L_{s,n}$, so that L is σ-centered. Further L is a dense subset of $\mathbf{L}_{\alpha,\beta}$:

Let $(X, Y, s, n) \in \mathbf{L}_{\alpha,\beta}$. Then we know that for any $\gamma \in X, \delta \in Y$ we have that $f_\gamma(i) + n < g_\delta(i) - n$ for $i > \mathrm{dom}\{s\}$. Further we can find $m > n$ for which

$$\max_{\gamma \in X}\{f_\gamma(i)\} + n \leq h(i) \leq \min_{\delta \in Y}\{g_\delta(i)\} - n,$$

for all $i \geq m$. Thus we can find $t \in {}^{<\omega}\omega$ such that $\mathrm{dom}\{t\} = m$, $t \restriction \mathrm{dom}\{s\} = s$ and

$$\max_{\gamma \in X}\{f_\gamma(i)\} + n \leq t(i) \leq \min_{\delta \in Y}\{g_\delta(i)\} - n,$$

for all $i \in [\mathrm{dom}\{s\}, \mathrm{dom}\{t\})$. But now $(X, Y, t, n) \in L$ and $(X, Y, t, n) \lhd (X, Y, s, n)$, thus L is indeed dense.

We claim that in order to prove σ-centeredness of some partial order $\mathbb{P}$ it suffices to show that there is a dense subset of $\mathbb{P}$ which is σ-centered, concluding the proof of 1.

So let $D \subset \mathbb{P}$ be dense and σ-centered. Suppose $D = \bigcup_{n \in \omega} D_i$, where D_i is centered for each $i \in \omega$. Let $D_i' = \{p \in \mathbb{P} \mid \exists d \in D_i \colon d <_{\mathbb{P}} p\}$ and let $D' = \{D_i \cup D_i'\}_{i \in \omega}$. Now since D is dense, $\bigcup D' = \bigcup_{i \in \omega} D_i \cup D_i' = \mathbb{P}$. Note that each $D_i \cup D_i'$ is centered: If S is finite subset of $D_i \cup D_i'$, then $S = \{s_1, ..., s_k\}$. For all $s_l \notin D_i$, i.e. $s_l \in D_i'$, replace s_l with s_l' such that $s_l' <_{\mathbb{P}} s_l$ and obtain a new finite subset S'. We can do this by definition of D_i'. Now $S' \subset D_i$, thus there is an $s \in \mathbb{P}$ stronger than any element from S'. But this s is also stronger than any element of S, witnessing that S is centered.

2.: Without loss of generality we assume that $\alpha = \omega$ and $\beta \neq \omega$, because if both α, β are ω, $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ can not be a gap and by 1. the statement of 2. follows.

For $s \in {}^{<\omega}\omega$, $n \in \omega$ and $X \in [\alpha]^{<\aleph_0}$ we let $L_{s,n,X}$ be the collection of $(X, Y, s, n) \in \mathbf{L}_{\alpha,\beta}$ for some $Y \in [\beta]^{<\aleph_0}$, which is of countable size. But now each $L_{s,n,X}$ is centered and

$$\mathbf{L}_{\alpha,\beta} = \bigcup_{s\in{}^{<\omega}\omega, n\in\omega, X\in[\alpha]^{<\aleph_0}} L_{s,n,X}.$$

□

The last proposition immediately gives the important

Corollary 26. *Rothberger Gaps are destructible by σ-centered partially ordered sets.*

Proof. By Proposition 25, for any given Rothberger Gap $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$, the corresponding Layer Interpolation Order $\mathbf{L}_{\alpha,\beta}$ is σ-centered. □

For the next considerations we will make use of the following notions:

Definition 13. Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α, β)-pregap. We call the pregap symmetric if α and β are of the same cofinality and otherwise asymmetric.

Proposition 27. *Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an asymmetric pregap. Then the corresponding partial order $\mathbf{L}_{\alpha,\beta}$ is strongly Knaster.*

Proof. Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an asymmetric pregap. Since any σ-linked (and thus in particular any σ-centered) partial order $\mathbb{P}$ is strongly Knaster, by Proposition 25 we can assume that both α and β are regular uncountable cardinals. Without loss of generality, we assume that $\alpha < \beta$.

Let $A = \{(X_\xi, Y_\xi, s_\xi, n_\xi) \in \mathbf{L}_{\alpha,\beta} \mid \xi < \nu\}$ be given.

First, we suppose that $\nu < \beta$. Since all Y_ξ are finite, we have that $\bigcup_{\xi<\nu} Y_\xi \subsetneq \beta$, thus we can find $\rho < \beta$ such that in fact $\bigcup_{\xi<\nu} Y_\xi \subset \rho$. But then $A \subset \mathbf{L}_{\alpha,\rho}$ and g_ρ is an interpolating element of $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\rho})$. So by Proposition 25, we obtain that $\mathbf{L}_{\alpha,\rho}$ is σ-centered and therefore strongly Knaster. So we find a ν-sized linked subset of A in $\mathbf{L}_{\alpha,\rho}$, which is clearly also in $\mathbf{L}_{\alpha,\beta}$, witnessing that $\mathbf{L}_{\alpha,\beta}$ is strongly Knaster.

Now suppose that $\nu \geq \beta$. Because $\alpha < \nu$, we can find a set $I \subset \nu$ of size ν such that for all $i, j \in I$ we have $X_i = X_j := X$, $s_i = s_j := s$ and $n_i = n_j := n$. But then $\{(X, Y_i, s_i, n_i) \in A \mid i \in I\}$ is a linked subset of A of size ν. □

As a corollary, we obtain that asymmetric gaps are destructible by Knaster partial orders:

Corollary 28. *Let* $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ *be an asymmetric gap. Then it is destructible by a strongly Knaster partial order.*

Now we consider symmetric pregaps, for which there is a nice condition on the corresponding Layer Interpolation Order to decide whether it is a gap or not:

Proposition 29. *Let* α *be a regular uncountable cardinal number and suppose* $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\alpha})$ *is a symmetric pregap. Then the corresponding Layer Interpolation Order* $\mathbf{L}_{\alpha,\alpha}$ *is* α*-Knaster if and only if* $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\alpha})$ *is not a gap.*

Proof. $\Longrightarrow$: Suppose that the pregap is not a gap. Then, by Proposition 25, we know that $\mathbf{L}_{\alpha,\alpha}$ is σ-centered, what implies that it is α-Knaster.

$\Longleftarrow$: Now we suppose that $\mathbf{L}_{\alpha,\alpha}$ is α-Knaster. For each $\gamma < \alpha$ we can find $s_\gamma \in {}^{<\omega}\omega$ and $n_\gamma \in \omega$ for which $(\{\gamma\}, \{\gamma\}, s_\gamma, n_\gamma) \in \mathbf{L}_{\alpha,\alpha}$. This is because $f_\gamma \prec g_\gamma$. Consider the set $\{(\{\gamma\}, \{\gamma\}, s_\gamma, n_\gamma)\}_{\gamma<\alpha}$. This is a α-sized subset of $\mathbf{L}_{\alpha,\alpha}$, thus we can find an α-sized set $A \subseteq \alpha$ for which $\{(\{\gamma\}, \{\gamma\}, s_\gamma, n_\gamma)\}_{\gamma\in A}$ is linked, since $\mathbf{L}_{\alpha,\alpha}$ is α-Knaster.

Since α is uncountable, we can thin out A to a set $B \subseteq A$ of size α such that $s_b = s_{b'} := s$ and $n_b = n_{b'} := n$ for all $b, b' \in B$.

Note that $\{(\{\gamma\}, \{\gamma\}, s, n)\}_{\gamma\in B}$ is linked. Thus for $\gamma, \delta \in B$ such that $\gamma \neq \delta$, we obtain an $(X, Y, t, m) \in \mathbf{L}_{\alpha,\alpha}$ such that $(X, Y, t, m) \lhd (\{\gamma\}, \{\gamma\}, s, n)$ and $(X, Y, t, m) \lhd (\{\delta\}, \{\delta\}, s, n)$. By definition of $\lhd$ we obtain that $\gamma, \delta \in X \cap Y$, $s \subset t$ and $n < m$. Further, again by definition of $\lhd$, this implies that

1. $f_\gamma(i) + n < g_\delta(i) - n$ and $f_\delta(i) + n < g_\gamma(i) - n$
   for all $i \in [\mathrm{dom}\{s\}, \mathrm{dom}\{t\})$,

2. $f_\gamma(i) + m < g_\delta(i) - m$ and $f_\delta(i) + m < g_\gamma(i) - m$ for all $i \geq \mathrm{dom}\{t\}$.

Because $m > n$ we obtain $f_\gamma(i) + n < g_\delta(i) - n$ for all $i \geq \mathrm{dom}\{s\}$ for any two distinct $\gamma, \delta \in B$. This allows us to define

$$h(i) = \begin{cases} \max\{f_\gamma(i) \mid \gamma \in B\} & \text{if } i \geq \mathrm{dom}\{s\} \\ 1 & \text{otherwise.} \end{cases}$$

But then h is an interpolating real for the pregap $(\{f_\gamma\}_{\gamma\in B}, \{g_\delta\}_{\delta\in B})$ and since B is of size α, it is cofinal in α, what implies that h interpolates $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\alpha})$. □

Remark. For the special case $\alpha = \omega_1$, Proposition 29 gives us the following:

An (ω_1, ω_1)-pregap is a gap if and only if the corresponding Layer Interpolation Order $\mathbf{L}_{\omega_1,\omega_1}$ is not Knaster.

Corollary 30. *Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α, β)-gap. If at least one of α or β is not of cofinality ω_1, then the corresponding Layer Interpolation Order $\mathbf{L}_{\alpha,\beta}$ is Knaster.*

In other words, if at least one of α or β is not of cofinality ω_1, then $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is $\mathcal{K}$-destructible.

Proof. If $\alpha \neq \beta$, the statement follows from Proposition 27.

On the other hand, suppose $\alpha = \beta > \omega_1$. Note that Proposition 25 allows us to assume that both α and β are uncountable. Consider an uncountable subset L of $\mathbf{L}_{\alpha,\beta}$, of size λ for some uncountable λ, which we enumerate by $\{(X_\xi, Y_\xi, s_\xi, n_\xi)\}_{\xi<\lambda}$. We want to find an uncountable subset of L which is linked. Consider $\{X_\xi\}_{\xi<\omega_1}$ and note that $\bigcup_{\xi<\omega_1} X_\xi \subsetneq \alpha$, since each X_ξ is finite. Now define $\kappa = \bigcup_{\xi<\lambda} X_\xi$ and consider the pregap $(\{f_\gamma\}_{\gamma<\kappa}, \{g_\delta\}_{\delta<\beta})$. Then f_κ interpolates this pregap. This implies, again by Proposition 25, that the corresponding Layer Interpolation Order $\mathbf{L}_{\kappa,\beta}$ is σ-centered, thus a countable union of centered sets. Now note that $\{(X_\xi, Y_\xi, s_\xi, n_\xi)\}_{\xi<\kappa} \subseteq \mathbf{L}_{\kappa,\beta}$. Since κ is uncountable and $\mathbf{L}_{\kappa,\beta}$ is the countable union of centered sets, there is an uncountable subset of $\{(X_\xi, Y_\xi, s_\xi, n_\xi)\}_{\xi<\kappa}$ in one of these centered sets. But this set is linked, thus a witness for $\mathbf{L}_{\alpha,\beta}$ being Knaster. □

3.2.2 Destroying (ω_1, ω_1)-gaps

We now slightly switch our focus to (ω_1, ω_1)-gaps, as we did in Chapter 2 when we discussed Hausdorff Gaps and Special Gaps. Indeed, these notion will appear again and play a key role in the following considerations. We will make use of the class of forcing notions preserving ω_1, which we denote by Ω_1. Then we define a special kind of (ω_1, ω_1)-gaps as follows (see [11]):

Definition 14. Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be an (ω_1, ω_1)-gap. Then we say the gap is a strong gap if it Ω_1-indestructible.

For the rest of this section we follow [11] and will be able to provide two characterisations of strong gaps, one in terms of the corresponding Layer Interpolation Order. This is to some extent surprising, as in principle there are a lot of different forcing notions in Ω_1 which potentially could destroy the gap. However, it turns out that a combinatorial condition on $\mathbf{L}_{\omega_1,\omega_1}$ does the trick:

Proposition 31. *Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (ω_1, ω_1)-gap. Then the gap is a strong gap if and only if the corresponding Layer Interpolation Order $\boldsymbol{L}_{\omega_1,\omega_1}$ is not ccc.*

Proof. $\Longleftarrow$: Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be as in the statement and suppose that $\mathbf{L}_{\omega_1,\omega_1}$ is not ccc. Thus we can find an uncountable antichain A in $\mathbf{L}_{\omega_1,\omega_1}$.

We aim to show that A is still an antichain in any generic extension obtain by forcing with any $\mathbb{P} \in \Omega_1$. Then we obtain that $\mathbf{L}_{\omega_1,\omega_1}$ is not σ-linked in any generic extension and therefore in particular not σ-centered. This is because any two elements in A cannot lie in the same linked set, thus there are at least uncountable many subsets which are not linked. We can state that as

$$\mathbf{1}_\mathbb{P} \Vdash \text{"}\mathbf{L}_{\omega_1,\omega_1} \text{ is not } \sigma\text{-centered"}.$$

But then, by Proposition 25, we know that $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ has to be a gap in the generic extension, as otherwise $\mathbf{L}_{\omega_1,\omega_1}$ would be σ-centered, i.e.

$$\mathbf{1}_\mathbb{P} \Vdash \text{"}(\{f_\gamma\}_{\gamma<\omega_1}\check{,} \{g_\delta\}_{\delta<\omega_1}) \text{ is a gap"}.$$

To see that A is an antichain in any generic extension by $\mathbb{P}$, consider $(X, Y, s, n), (X', Y', s', n') \in \mathbf{L}_{\omega_1,\omega_1}$ for which

$$\mathbf{1}_\mathbb{P} \Vdash \text{"}(X, \check{Y}, s, n) \text{ and } (X', \check{Y'}, s', n') \text{ are compatible"}.$$

But then, since $\mathbb{P} \in \Omega_1$ and $X, Y, X', Y', s, s', n, n' \in \omega_1$, we obtain that (X, Y, s, n) and (X', Y', s', n') are compatible in the ground model as well. But this implies in turn that if two elements of $\mathbf{L}_{\omega_1,\omega_1}$ are incompatible in the ground model, they are incompatible in any generic extension by $\mathbb{P}$. Thus A remain an antichain after forcing with $\mathbb{P}$.

$\Longrightarrow$: Assume to the contrary that $\mathbf{L}_{\omega_1,\omega_1}$ is not ccc. Then the corresponding gap $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ is destructible by a ccc forcing notion, which preserves ω_1. Thus the gap is not a strong gap. $\square$

There is a nice theorem - again due to Woodin - which gives a second useful characterisation of strong gaps:

Theorem 32. *Let* $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ *be an* (ω_1, ω_1)*-gap. Then it is a strong gap if and only if it is equivalent to a Special Gap.*

Proof. $\Longrightarrow$: Suppose $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ is a strong gap. We aim to find an equivalent (ω_1, ω_1)-gap $(\{f'_\gamma\}_{\gamma<\omega_1}, \{g'_\delta\}_{\delta<\omega_1})$ such that there is a natural number m for which:

1. $f'_\gamma(j) \leq g'_\gamma(j)$ for all $j > m$,
2. for all pairs $\gamma, \delta < \omega_1$ we can find an $l > m$ for which $f'_\gamma(l) > g'_\delta(l)$ or $f'_\delta(l) > g'_\gamma(l)$.

First we make use of Proposition 31 and obtain an uncountable antichain A in $\mathbf{L}_{\omega_1,\omega_1}$, which we enumerate by $\{(X_\gamma, Y_\gamma, s_\gamma, n_\gamma)\}_{\gamma<\omega_1}$. Because there exist only countable many $s_\gamma \in {}^{<\omega}\omega$ and $n_\gamma \in \omega$, we can assume that for any $(X_\gamma, Y_\gamma, s_\gamma, n_\gamma)$, $(X_\delta, Y_\delta, s_\delta, n_\delta) \in A$ it holds that $s_\gamma = s_\delta := s$ and $n_\gamma = n_\delta := n$. Similarly, we assume that X_γ and Y_γ are all of the same (finite) cardinality.

Using the Δ-System Lemma, without loss of generality, we further assume that $\{X_\gamma\}_{\gamma<\omega_1}$ and $\{Y_\gamma\}_{\gamma<\omega_1}$ are Δ-systems with roots $X = \bigcap_{\gamma\in\omega_1} X_\gamma$ and $Y = \bigcap_{\gamma\in\omega_1} Y_\gamma$, respectively.

We conclude that for any two $\gamma, \delta < \omega_1$ it holds $X_\gamma \setminus X_\delta \neq \emptyset$ and $X_\delta \setminus X_\gamma \neq \emptyset$ as well as $Y_\gamma \setminus Y_\delta \neq \emptyset$ and $Y_\delta \setminus Y_\gamma \neq \emptyset$. If this would not be the case, assume without loss of generality that we could find γ, δ for which $X_\gamma \setminus X_\delta = \emptyset$, i.e. $X_\gamma \subseteq X_\delta$ and since X_γ and X_δ are of the same cardinality, even $X_\gamma = X_\delta$. But then if $Y_\gamma \setminus Y_\delta = \emptyset$ or $Y_\delta \setminus Y_\gamma = \emptyset$ we obtain that $(X_\gamma, Y_\gamma, s, n) \lhd (X_\delta, Y_\delta, s, n)$ or vice versa, contradicting the fact that A is an antichain.

It follows that both $\bigcup_{\gamma<\omega_1} X_\gamma$ and $\bigcup_{\gamma<\omega_1} Y_\gamma$ are cofinal in ω_1. Thus we can assume that $\max\{X_\gamma\} \in X_\gamma \setminus X$ and $\max\{Y_\gamma\} \in Y_\gamma \setminus Y$, because $\bigcup_{\gamma<\omega_1} X_\gamma$ and $\bigcup_{\gamma<\omega_1} Y_\gamma$ are cofinal in ω_1 and X, Y are not.

Now we define for any $\gamma < \omega_1$:

$$f'_\gamma(j) = \begin{cases} \max\{f_x(j) \mid x \in X_\gamma\} + 2 \cdot n & \text{if } j > \operatorname{dom}\{s\} \\ 1 & \text{otherwise.} \end{cases}$$

and

$$g'_\gamma(j) = \begin{cases} \min\{g_y(j) \mid y \in Y_\gamma\} & \text{if } j > \mathrm{dom}\{s\} \\ 1 & \text{otherwise.} \end{cases}$$

Then we claim that $(\{f'_\gamma\}_{\gamma<\omega_1}\}, \{g'_\gamma\}_{\gamma<\omega_1})$ is as desired and satisfies 1. and 2.

To see this, first note that since all X_γ and Y_γ are finite together with the fact that $\{f_\gamma\}_{\gamma<\omega_1}$ and $\{g_\gamma\}_{\gamma<\omega_1}$ are ordered with respect to $\prec$, we obtain that eventually $f'_\gamma = f_{\max\{X_\gamma\}} + 2 \cdot n$ and similarly $g'_\gamma = g_{\max\{Y_\gamma\}}$. Since we assumed $\max\{X_\gamma\} \notin X$ and $\max\{Y_\gamma\} \notin Y$ for any $\gamma < \omega_1$, we obtain that eventually $f'_\gamma \neq f'_\delta$ and $g'_\gamma \neq g'_\delta$ for distinct $\gamma, \delta < \omega_1$.

Further we get that $(\{f'_\gamma\}_{\gamma<\omega_1}\}, \{g'_\gamma\}_{\gamma<\omega_1})$ is equivalent to the given gap $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$, since both $\bigcup_{\gamma<\omega_1} X_\gamma$ and $\bigcup_{\gamma<\omega_1} Y_\gamma$ are cofinal in ω_1.

So we are left with showing that 1. and 2. hold. Therefore, let $m := \mathrm{dom}\{s\}$.

To see 1., consider $\gamma < \omega_1$ and observe that $f'_\gamma(j) = f_x(j) + 2 \cdot n$ for some $x \in X_\gamma$ and all $j > m$. Similarly, $g'_\gamma(j) = g_y(j)$ for $y \in Y_\gamma$ and all $j > m$. Since $(X_\gamma, Y_\gamma, s, n) \in \mathbf{L}_{\omega_1,\omega_1}$, we obtain that for $j > \mathrm{dom}\{s\} = m$ we have that $f_x(j) + 2 \cdot n < g_y(j)$ for any $x \in X_\gamma, y \in Y_\gamma$. But this implies 1.

For 2., let $\gamma \neq \delta$ be given. Consider $(X_\gamma, Y_\gamma, s, n)$, $(X_\delta, Y_\delta, s, n) \in A \subseteq \mathbf{L}_{\omega_1,\omega_1}$ and note that they are incompatible. But to ensure that $(X_\gamma, Y_\gamma, s, n)$ and $(X_\delta, Y_\delta, s, n)$ are incompatible, it must be the case that there are $x \in X_\gamma$, $y \in Y_\gamma$ and $j > \mathrm{dom}\{s\}$ for which either $f_\gamma(j) + 2 \cdot n > g_\delta(j)$ or $f_\delta(j) + 2 \cdot n > g_\gamma(j)$. This is exactly statement 2. as needed.

$\Longleftarrow$: Suppose that $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ is equivalent to a Special Gap. Consider $\mathbb{P} \in \Omega_1$ and let $V[G]$ be a generic extension obtained via forcing with $\mathbb{P}$. Then we use Proposition 13 in $V[G]$ to obtain that $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ is indeed a strong gap. □

Because Special Gaps are equivalent to Hausdorff Gaps, we obtain the

Corollary 33. *Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be a Hausdorff Gap. Then the gap is a strong gap.*

3.3 Gaps surviving forcing

It is a natural question to ask under which circumstances a gap survives a generic extension. We will establish results for both symmetric and asymmetric gaps, but also more general consideration are given. For the rest of this chapter, we follow [5].

We start with a general result stating that forcing with a small poset preserves gaps:

Theorem 34. *Suppose $\alpha \leq \beta$ are infinite regular cardinal numbers such that at least β is uncountable. Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α, β)-gap and $\mathbb{P}$ be a partial order of cardinality $\lambda < \beta$ for which α and β are regular cardinals in any generic extension. Then*

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{"}(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})\check{} \text{ is an } (\check{\alpha}, \check{\beta})\text{-gap."}$$

Proof. We prove by contradiction. If the statement of the proposition is false, we can use the maximal principle to find a $\mathbb{P}$-name for a real h such that

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{"}h \text{ interpolates } (\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})\check{}\text{"}.$$

For each $\delta < \beta$ we can find $p_\delta \in \mathbb{P}$ and $n_\delta < \omega$ so that $p_\delta \Vdash$ "$h(i) < \check{g}_\delta(i)$ for all $i > \check{n}_\delta$". Then we use that $\lambda < \beta$ and β is uncountable to find a cofinal subset $X \subseteq \beta$ such that $p_x = p_{x'} := p$ and $n_x = n_{x'} := n$ for all $x, x' \in X$.

Similarly, for any $\gamma < \alpha$ we can find $q_\gamma \in \mathbb{P}$, with $q_\gamma <_{\mathbb{P}} p$, and $m_\gamma < \omega$ such that $q_\gamma \Vdash$ "$\check{f}_\gamma(i) < h(i)$ for all $i > \check{m}_\gamma$".

We now have to distinct the cases that α is uncountable and α is countable. First we assume that α is uncountable, then we can find a cofinal subset $Y \subseteq \alpha$ for which $m_y = m_{y'} := m$ for all $y, y' \in Y$. Then $q_\gamma \Vdash$ "$\check{f}_\gamma(i) < h(i) < \check{g}_x(i)$ for all $i > \max\{\check{m}, \check{n}\}$" for any $x \in X$. This allows us to define, in the ground model, the real t as follows:

$$t(i) = \begin{cases} \max\{f_y(i) \mid y \in Y\} & \text{if } i > \max\{m, n\} \\ 1 & \text{otherwise.} \end{cases}$$

But then t interpolates $(\{f_x\}_{x\in X}, \{f_y\}_{y\in Y})$ which is equivalent to the original gap $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$, since X, Y are cofinal in α, β, respectively.

If α is countable, it is ω. We can choose m_γ for any $\gamma < \omega$ in a way such that $m_\gamma > m_\xi$ for all $\xi < \gamma$, i.e. the sequence of m_γ's is increasing. Now we are able to define t in $V[G]$ by

$$t(i) = \begin{cases} \max\{f_\xi(i) \mid \xi < \gamma\} & \text{if } i \in [m_\gamma, m_{\gamma+1}) \\ 1 & \text{otherwise} \end{cases}$$

Then t interpolates $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$. □

We can extend the previous theorem to forcing iterations. For a precise definition and further results regarding iterated forcing we refer to [9].

Corollary 35. *Suppose $\alpha < \beta$ are infinite regular cardinal numbers such that at least β is uncountable. Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α, β)-gap. Suppose there is a β-stage finite support iteration of forcing notions*

$$(\langle \mathbb{P}_\delta \mid \delta \leq \beta \rangle, \langle \dot{\mathbb{Q}}_\delta \mid \delta < \beta \rangle)$$

such that $|\mathbb{P}_\delta| < \beta$ for any $\delta < \beta$.

Then $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is preserved when forcing with the iteration, or otherwise stated

$$\mathbf{1}_{\mathbb{P}_\beta} \Vdash \text{"}(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})\check{} \textit{ is an } (\check{\alpha}, \check{\beta})\textit{-gap"}.$$

Proof. Suppose to the contrary that there is a $\mathbb{P}_\beta$-name h for a real that interpolates the gap. Then we can find a $p \in \mathbb{P}_\beta$ that forces this, i.e.

$$p \Vdash \text{"}\check{f}_\gamma \prec h \prec \check{g}_\delta \text{ for all } \gamma < \check{\alpha} \text{ and all } \delta < \check{\beta}\text{"}.$$

Now we use that $(\langle \mathbb{P}_\delta \mid \delta \leq \beta \rangle, \langle \dot{\mathbb{Q}}_\delta \mid \delta < \beta \rangle)$ has finite support and obtain that h is already a $\mathbb{P}_\xi$-name for some $\xi < \beta$. But this implies that the restriction of p to $\mathbb{P}_\xi$ already forces that h interpolates the gap, that means

$$p \upharpoonright \xi \Vdash \text{"}\check{f}_\gamma \prec h \prec \check{g}_\delta \text{ for all } \gamma < \check{\alpha} \text{ and all } \delta < \check{\beta}\text{"}.$$

Now we can use the assumption that $|\mathbb{P}_\xi| < \beta$ and Theorem 34 to obtain a contradiction. □

3.3.1 Indestructibility of symmetric gaps

In contrast to Theorem 34, where we did not assume anything except $\alpha \leq \beta$, we now distinct between asymmetric and symmetric gaps. Once again, we follow [5] and start with the result:

Theorem 36. *Let α be a regular uncountable cardinal number and let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\alpha})$ be a symmetric (α, α)-gap. Suppose $\mathbb{P}$ is a partially ordered set that is α-Knaster. Then $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\alpha})$ is $\alpha-\mathcal{K}$-indestructible.*

Proof. We aim to show that

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{"}(\{f_\gamma\}_{\gamma<\alpha} \check{,} \{g_\delta\}_{\delta<\alpha}) \text{ is a gap"}.$$

If this not the case, we can find a $\mathbb{P}$-name h and an element $p \in \mathbb{P}$ such that

$$p \Vdash \text{"}(\{f_\gamma\}_{\gamma<\alpha} \check{,} \{g_\delta\}_{\delta<\alpha}) \text{ is interpolated by } h\text{"}.$$

Now fix an $\delta < \alpha$. Then for any $\gamma < \alpha$, let $q_\gamma < p$ and $m_\gamma < \omega$ be such that

$$q_\gamma \Vdash \text{"}\check{f}_\gamma(i) < h(i) < \check{g}_\delta(i) \text{ for all } i > m_\gamma\text{"}.$$

Now because α is uncountable, we can without loss of generality assume that $m_\gamma = m'_\gamma =: m$ for any $\gamma, \gamma' < \omega$.

Now we use that $\mathbb{P}$ is α-Knaster and find a cofinal subset $X \subseteq \alpha$ such that $F = \{q_x\}_{x\in X}$ is linked. We can extend F to a $\mathbb{P}$-generic filter $G \subseteq \mathbb{P}$. But then in $V[G]$ we have that $f_x(i) < h(i) < g_\delta(i)$ whenever $i > m$ and for all $x \in X$. So this must hold in the ground model, which allows us to define

$$t(n) = \begin{cases} \max\{f_x(i) \mid x \in X\} & \text{if } i > m \\ 1 & \text{otherwise} \end{cases}$$

But then t interpolates $(\{f_x\}_{x\in X}, \{g_\delta\}_{\delta<\alpha})$, a contradiction. □

Theorem 36 immediately gives us the

Corollary 37. *Let α be a regular uncountable cardinal number and let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\alpha})$ be a symmetric (α, α)-gap. Then $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\alpha})$ is $\mathcal{K}_\forall$-indestructible.*

Now, as in some previous sections, we switch our focus to (ω_1, ω_1)-gaps, for which we state a special case of Corollary 37:

Corollary 38. *Let* $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ *be a symmetric* (ω_1, ω_1)*-gap. Then it is* $\mathcal{K}$*-indestructible.*

We also want to highlight a direct implication of Corollary 37:

Corollary 39. *Let* α *be a regular uncountable cardinal number and let* $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\alpha})$ *be a symmetric* (α, α)*-gap. Then* $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\alpha})$ *is* $\sigma - \mathcal{L}$*-indestructible.*

Proof. By Corollary 37, we know that the given gap is $\mathcal{K}_\forall$-indestructible. We show that any σ-linked partially ordered set is $\mathcal{K}_\forall$:

Let $\mathbb{P}$ be σ-linked. Thus we can write $\mathbb{P} = \bigcup_{n\in\omega} \mathbb{P}_n$ for linked sets $\mathbb{P}_n$. Now let λ be an uncountable cardinal and suppose we have an λ-sized subset of $\mathbb{P}$, $\{p_\xi\}_{\xi<\lambda}$. But then it must be the case that there exists a natural number m such that λ-many p_ξ's are in $\mathbb{P}_m$ - this gives the linked subset of size λ witnessing that $\mathbb{P}$ is λ-Knaster. □

For the case of (ω_1, ω_1)-gaps, we now consider a slightly different question as before, namely:

> *Given an* (ω_1, ω_1)*-gap, can we force with a partially ordered set such that the gap is* $\mathcal{C}$*-indestructible for some class of posets* $\mathcal{C}$ *after the forcing?*

At least for the special case of $\mathcal{C} = \Omega_1$ we will obtain that there is a forcing notion $\mathbb{P}$, depending on some given (ω_1, ω_1)-gap, such that the gap is equivalent to an Ω_1-indestructible gap in any generic extension by $\mathbb{P}$. This is the goal for the remaining part of this section. Originally this construction is due to Kunen [7], although it can be found also in [11] or [5]. In the notation we are using in this work, it is most reasonable to follow [5] for the rest of the section.

Definition 15. Let $f, g, f', g' \in {}^\omega\omega$, then we define an equivalence relation $\approx$ on ${}^\omega\omega \times {}^\omega\omega$ by letting

$$(f, g) \approx (f', g') \text{ if and only if } f =^* f' \text{ and } g =^* g'.$$

For $f, g \in {}^\omega\omega$, we denote with $[(f, g)]$ the equivalence class of (f, g) with respect to $\approx$.

Definition 16. Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be an (ω_1, ω_1)-pregap. Then $\mathbf{S}_{\omega_1,\omega_1}$ is the collection of finite sets $S \subseteq {}^\omega\omega \times {}^\omega\omega$ for which

1. S contains at most one element of $[(f_\gamma, g_\gamma)]$ and any element of S is in $[(f_\gamma, g_\gamma)]$ for an $\gamma < \omega_1$,
2. for $(f, g) \in S$ it holds that $f(i) \leq g(i)$ for any $i \in \omega$,
3. for any two $(f, g), (f', g') \in S$ there is a $j \in \omega$ for which $f(j) > g'(j)$ or $f'(j) > g(j)$.

For $S, S' \in \mathbf{S}_{\omega_1,\omega_1}$, we let $S \ll S'$ if and only if $S \supset S'$. The maximal element $\mathbf{1}_{\mathbf{S}_{\omega_1,\omega_1}}$ is the empty set $\emptyset$.

Remark (Remark and Definition)**.**

1. By property 1. in Definition 16, for any $S \in \mathbf{S}_{\omega_1,\omega_1}$ we have an index-set $I_S \subseteq \omega_1$ such that $\gamma \in I_S$ if and only if there is an $(f, g) \in S$ with $(f, g) \in [(f_\gamma, g_\gamma)]$. We will call I_S the support of S and denote it with support(S).
2. Note that given an equivalence class $[(f_\gamma, g_\gamma)]$, we can always find a pair $(f, g) \in [(f_\gamma, g_\gamma)]$ for which $f(i) \leq g(i)$ for all $i \in \omega$. This is because $f_\gamma \prec g_\gamma$, so we have an $k \in \omega$ such that $f(i) < g(i)$ for all $i > k$ and we can put $f(j) = 0$ and $g(j) = 1$ for $j < k$ and still preserve that $f =^* f_\gamma$ as well as $g =^* g_\gamma$.

We start our investigations on $\mathbf{S}_{\omega_1,\omega_1}$ with a useful combinatorial property of $\mathbf{S}_{\omega_1,\omega_1}$, see [11]:

Theorem 40. *Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be an (ω_1, ω_1)-pregap and let $\mathbf{S}_{\omega_1,\omega_1}$ be the corresponding partially ordered set. Then $\mathbf{S}_{\omega_1,\omega_1}$ is ccc if and only if $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ is a gap.*

Proof. $\Longrightarrow$: We show that if $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ is not a gap, then $\mathbf{S}_{\omega_1,\omega_1}$ is not ccc. Thus let h be an interpolating function for $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$. For any $\gamma < \omega_1$ we can find a pair $(f'_\gamma, g'_\gamma) \in [(f_\gamma, g_\gamma)]$ such that $f'_\gamma(i) \leq g'_\gamma(i)$ for all naturals i.

For any $\gamma < \omega_1$ we find an $n_\gamma \in \omega$ such that $f'_\gamma(i) < h(i) < g'_\gamma(i)$ for all $i > n_\gamma$. We further find an uncountable subset $X \subseteq \omega_1$ for which $n_x = n_{x'} =: n$ for any two $x, x' \in X$. Since the set of initial segments

$\{f'_x(0), f'_x(1), ..., f'_x(n)\}$ is countable for all $x \in X$, we can also find an uncountable set $Y \subseteq X$ such that $f'_y(i) = f'_{y'}(i)$ for all $i \in \{0, 1, ..., n\}$ and all $y, y' \in Y$. But then for all $y, y' \in Y$ and all $i \in \omega$ we obtain that $f'_y(i) \leq g'_{y'}(i)$ as well as $f'_{y'}(i) \leq g'_y(i)$ for all $i \in \omega$.

This means that for $y \neq y'$, $y, y' \in Y$, $\{(f'_y, g'_y)\}$ and $\{(f'_{y'}, g'_{y'})\}$ are not compatible in $\mathbf{S}_{\omega_1,\omega_1}$: Any $S \ll \{(f'_y, g'_y)\}, \{(f'_{y'}, g'_{y'})\}$ must satisfy that $(f'_y, g'_y), (f'_{y'}, g'_{y'}) \in S$, thus there must be an $j \in \omega$ for which $f'_y(j) > g'_{y'}(j)$ or $f'_{y'}(j) > g'_y(j)$. But this can not be the case, as we have just shown.

This implies that $\{S_y\}_{y \in Y}$, where $S_y = \{(f'_y, g'_y)\}$, is an uncountable antichain in $\mathbf{S}_{\omega_1,\omega_1}$.

$\Longleftarrow$: Suppose to the contrary that $\mathbf{S}_{\omega_1,\omega_1}$ is not ccc and we can find an uncountable antichain $A \subseteq \mathbf{S}_{\omega_1,\omega_1}$. We show that in this case $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ is not a gap.

Instead of A itself, we first consider the set of supports of elements of A, i.e. $I = \{I_S \mid S \in A\}$. Without loss of generality, we can use the Δ-System Lemma to assume that I is a Δ-system with root R. For any $\gamma \in R$ and any $S \in A$ we know by definition of $\mathbf{S}_{\omega_1,\omega_1}$ that S consists of exactly one element of $[(f_\gamma, g_\gamma)]$. Therefore, we can without loss of generality assume that for any two $S, S' \in A$ and any $\gamma \in R$ it holds that the respective elements of $[(f_\gamma, g_\gamma)]$ are the same in S and S'; one can also express this as $S \cap [(f_\gamma, g_\gamma)] = S' \cap [(f_\gamma, g_\gamma)]$ for all $\gamma \in R$ and all $S, S' \in A$. This also implies that the indizes of the witnesses of the incompatibility of $S, S' \in A$ must lie in $I_S \setminus R$ and $I_{S'} \setminus R$, respectively.

We now aim to "thin out" A to get an (ω_1, ω_1)-pregap associated with it. We do this as follows:

For $S \in A$ let $\gamma_S \in I_S \setminus R$ be the index for which $f_{\gamma_S} \prec f_\gamma$ for any $\gamma \in I_S \setminus R$, $\gamma \neq \gamma_S$. Similarly, let $\delta_S \in I_S \setminus R$ be such that $g_\delta \prec g_{\delta_S}$ for any $\delta \in I_S \setminus R$, $\delta \neq \delta_S$. Since $\gamma_S, \delta_S \in I_S \setminus R$, for distinct $S, S' \in A$ also $\gamma_S, \gamma_{S'}$ and $\delta_S, \delta_{S'}$ are distinct, respectively. Thus we obtain a pregap $(\{f_{\gamma_S}\}_{S \in A}, \{g_{\delta_S}\}_{S \in A})$ which is by construction equivalent to the given pregap $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$.

Then, for any $S \in A$, we define the reals

$$f_S(i) = \min\{f'_\gamma(i) \mid \gamma \in I_S \setminus R\},$$

$$g_S(i) = \max\{g'_\gamma(i) \mid \gamma \in I_S \setminus R\},$$

where f'_γ and g'_γ are as in the $\Longrightarrow$-direction. We can define f_S and g_S this way since $I_S \setminus R$ is finite for any $S \in A$. By our choice of γ_S and δ_S, we obtain that $f_S =^* f_{\gamma_S}$ and $g_S =^* g_{\gamma_S}$. This implies that $(\{f_S\}_{S\in A}, \{g_S\}_{S\in A})$ is equivalent to $(\{f_{\gamma_S}\}_{S\in A}, \{g_{\delta_S}\}_{S\in A})$, and thus equivalent to the given pregap $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$.

Now pick any $S, S' \in A$ and observe the following: Since S and S' are incompatible, we must find indizes $\xi_S \in I_S$ and $\xi'_{S'} \in I_{S'}$ for which $f_{\xi_S}(i) \leq g_{\xi'_{S'}}(i)$ and $f_{\xi'_{S'}}(i) \leq g_{\xi_S}(i)$ for all $i \in \omega$. We conclude that this also holds for f'_{ξ_S}, g'_{ξ_S} and $f'_{\xi'_{S'}}, g'_{\xi'_{S'}}$, respectively, i.e. $f'_{\xi_S}(i) \leq g'_{\xi'_{S'}}(i)$ and $f'_{\xi'_{S'}}(i) \leq g'_{\xi_S}(i)$ for all $i \in \omega$. Since the incompatibility of S and S' happens on $I_S \setminus R$ and $I_{S'} \setminus R$, we obtain $\xi_S \in I_S \setminus R$, $\xi'_{S'} \in I_{S'} \setminus R$.

In particular these inequalities imply that $f_S(i) \leq g_{S'}(i)$ for all $i \in \omega$. But this holds for any $S \in A$, so that $g_{S'}(i)$ is an upper bound of $\max\{f_S(i) \mid S \in A\}$. Thus we can define $h(i) = \max\{f_S(i) \mid S \in A\}$, which interpolates $(\{f_S\}_{S\in A}, \{g_S\}_{S\in A})$ implying that $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ is not a gap. □

We are now going to state and prove the main result in our considerations of (ω_1, ω_1)-gaps - this is that forcing with $\mathbf{S}_{\omega_1,\omega_1}$ makes the corresponding (ω_1, ω_1)-gap a strong gap in every generic extension by $\mathbf{S}_{\omega_1,\omega_1}$, see [5].

Theorem 41. *Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be an (ω_1, ω_1)-gap and let $\boldsymbol{S}_{\omega_1,\omega_1}$ be the corresponding forcing notion. Then in any generic extension by $\boldsymbol{S}_{\omega_1,\omega_1}$ the gap is Ω_1-indestructible, or, otherwise stated*

$$\mathbf{1}_{\boldsymbol{S}_{\omega_1,\omega_1}} \Vdash \text{"}(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})\check{} \textit{ is a strong gap."}$$

We prove the theorem by showing that in any generic extension by $\mathbf{S}_{\omega_1,\omega_1}$ there is an equivalent gap that is a Special Gap - and thus the given gap is a strong gap in this generic extension by Theorem 32.

We show the existence of the equivalent gap by explicitly defining names for its elements:

To the end of this section, for a given (ω_1, ω_1)-gap $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$, define for any $\gamma < \omega_1$ the $\mathbf{S}_{\omega_1,\omega_1}$-names:

$$\dot{f}^\gamma = \{((m,n)\check{}, \{(f,g)\}) \mid (m,n) \in \omega \times \omega, (f,g) \in [(f_\gamma, g_\gamma)] \text{ and } f(m) = n\}$$

and

$$\dot{g}^\gamma = \{((m,n)\check{}, \{(f,g)\}) \mid (m,n) \in \omega \times \omega, (f,g) \in [(f_\gamma, g_\gamma)] \text{ and } g(m) = n\}.$$

We will need to make use of a genericity argument, for which the following remark will be useful:

Remark. Given a gap $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ and the corresponding forcing notion $\mathbf{S}_{\omega_1,\omega_1}$, the following sets are dense open for any $\gamma < \omega_1$:

$$D_\gamma = \{S \in \mathbf{S}_{\omega_1,\omega_1} \mid \gamma \in I_S\}$$

Proof of Remark. Let $\gamma < \omega_1$ be arbitrary and consider $S \in \mathbf{S}_{\omega_1,\omega_1}$. We can assume without loss of generality that $\gamma \notin I_S$. Pick any $(f, g) \in [(f_\gamma, g_\gamma)]$ for which $f(n) \leq g(n)$ for any $n \in \omega$. Now since S is finite, we can change f to f' such that the value $f(1)$ satisfies $f(1) > \max\{f_s(1) \mid s \in I_S\}$ and $f = f'$ elsewhere. Then $S \cup \{(f', g)\} \in \mathbf{S}_{\omega_1,\omega_1}$ and stronger than S. Thus D_γ is dense since $S \cup \{(f', g)\} \in D_\gamma$.

The fact that D_γ is open is clear since $S' \ll S$ if and only if $S' \supset S$, i.e. $\gamma \in I_S$ implies $\gamma \in I_{S'}$. $\square$

The first thing on our way to prove Theorem 41 is

Proposition 42. *Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be an (ω_1, ω_1)-gap. Let $\boldsymbol{S}_{\omega_1,\omega_1}$ be the corresponding forcing notion and let $\gamma < \omega_1$. Then*

1. $\mathbf{1}_{\boldsymbol{S}_{\omega_1,\omega_1}} \Vdash$ *"$\dot{f}^\gamma$ is a real."*

2. $\mathbf{1}_{\boldsymbol{S}_{\omega_1,\omega_1}} \Vdash$ *"$\dot{g}^\gamma$ is a real."*

Proof. We prove the Proposition only for 1., since the proof of 2. is nearly equal word by word.

Let $\gamma < \omega_1$ and $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be an (ω_1, ω_1)-gap. Let G be any $\mathbf{S}_{\omega_1,\omega_1}$-generic filter. Recall that the $\mathbf{S}_{\omega_1,\omega_1}$-name $\dot{f}^\gamma$ is defined as

$$\dot{f}^\gamma = \{((\check{m,n}), \{(f, g)\}) \mid (m, n) \in \omega \times \omega, (f, g) \in [(f_\gamma, g_\gamma)] \text{ and } f(m) = n\},$$

what implies that the evaluation of $\dot{f}^\gamma$ in G, ${\dot{f}^\gamma}^G$, is a set of pairs $(m, n) \in \omega \times \omega$.

Note further that if $(m, n) \in {\dot{f}^\gamma}^G$, there is $\{(f, g)\} \in G$ for which $(f, g) \in [(f_\gamma, g_\gamma)]$ and $f_\gamma(m) = n$. Since any two elements of G are compatible and any $S \in G$ can only consist of one element of $[(f_\gamma, g_\gamma)]$, we obtain that ${\dot{f}^\gamma}^G = f$. In particular, ${\dot{f}^\gamma}^G$ is a function.

Still, we need to show that such an $(f,g) \in [(f_\gamma, g_\gamma)]$ exists in G. But this is ensured by the density of the set D_γ together with the fact that G is upwards closed. $\square$

Remark. The proof of 42 gives us the following useful observations:

For any $\mathbf{S}_{\omega_1,\omega_1}$-generic filter G it holds that in $V[G]$:

1. $\dot{f^\gamma}^G = f$ for f such that $\{(f,g)\} \in G$ and $(f,g) \in [(f_\gamma, g_\gamma)]$,
2. $\dot{g^\delta}^G = g$ for g such that $\{(f,g)\} \in G$ and $(f,g) \in [(f_\delta, g_\delta)]$.

Proposition 43. *Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ be an (ω_1,ω_1)-gap. Let $\boldsymbol{S}_{\omega_1,\omega_1}$ be the corresponding forcing notion and let $\gamma < \omega_1$. Then*

$$\boldsymbol{1}_{\boldsymbol{S}_{\omega_1,\omega_1}} \Vdash \text{"}(\{\dot{f^\gamma}\}_{\gamma<\omega_1}, \{\dot{g^\delta}\}_{\delta<\omega_1}) \textit{ is a Special Gap."}$$

Proof. By Proposition 42, we are left with proving that

$$\mathbf{1}_{\mathbf{S}_{\omega_1,\omega_1}} \Vdash \text{"}(\{\dot{f^\gamma}\}_{\gamma<\omega_1}, \{\dot{g^\delta}\}_{\delta<\omega_1}) \text{ is a pregap."}$$

and that we have indeed a Special Gap.

But by the previous remark, we know that for any $\mathbf{S}_{\omega_1,\omega_1}$-generic G we have $\dot{f^\gamma}^G = f$ for any $\gamma < \omega_1$ where f is such that there is g for which $(f,g) \in [(f_\gamma, g_\gamma)]$. This implies that $f =^* f_\gamma$, thus $\dot{f^\gamma}^G =^* f_\gamma$ in $V[G]$. similarly, we obtain $\dot{g^\delta}^G =^* g_\delta$ for any $\delta < \omega_1$. But this implies that $(\{\dot{f^\gamma}^G\}_{\gamma<\omega_1}, \{\dot{g^\delta}^G\}_{\delta<\omega_1})$ is a pregap in $V[G]$.

Note that by Theorem 40 we know that forcing with $\mathbf{S}_{\omega_1,\omega_1}$ preserves ω_1, so that $(\{\dot{f^\gamma}^G\}_{\gamma<\omega_1}, \{\dot{g^\delta}^G\}_{\delta<\omega_1})$ is in fact an (ω_1,ω_1)-pregap in $V[G]$.

To see that $(\{\dot{f^\gamma}^G\}_{\gamma<\omega_1}, \{\dot{g^\delta}^G\}_{\delta<\omega_1})$ is a Special Gap, note that 2. and 3. of Definition 16 together with the previous remark gives this immediately. $\square$

It is now easy to show our main result of the section:

Proof of Theorem 41. Let $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ and $\mathbf{S}_{\omega_1,\omega_1}$ be as in the statement of Theorem 41. Note that by the previous remark we obtain

$$\mathbf{1}_{\mathbf{S}_{\omega_1,\omega_1}} \Vdash \text{"}(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1}) \text{ is equivalent to } (\{\dot{f^\gamma}\}_{\gamma<\omega_1}, \{\dot{g^\delta}\}_{\delta<\omega_1})\text{."}$$

But then we can apply Theorem 32 to obtain that

$$\mathbf{1}_{\mathbf{S}_{\omega_1,\omega_1}} \Vdash \text{"}(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1}) \text{ is a strong gap."},$$

as desired. $\square$

3.3.2 Indestructibility of asymmetric gaps

In contrast to the last section, we focus on asymmetric gaps and their properties in connection to indestructibility. Rothberger Gaps will play an important role in the process. We follow [5] here.

An important result is that asymmetric gaps of sufficiently large size survive forcing with $\sigma - \mathcal{L}$.

Theorem 44. *Let α, β be regular cardinals of size at least ω_1 such that $\alpha \leq \beta$. Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α, β)-gap and let $\mathbb{P} \in \sigma - \mathcal{L}$ be a partially ordered set with maximal element $\mathbf{1}_\mathbb{P}$. Then*

$$\mathbf{1}_\mathbb{P} \Vdash \text{"}(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})\check{} \text{ is a gap."}$$

In other words, if α, β are uncountable cardinals, then any (α, β)-gap is $\sigma - \mathcal{L}$-indestructible.

Proof. Let α, β and $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be as in the statement. Suppose the theorem is false, then we can find $p \in \mathbb{P}$ and a $\mathbb{P}$-name for a real h such that

$$p \Vdash \text{"}h \text{ interpolates } (\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})\check{}\text{"}.$$

Since $p \Vdash$ "$\check{f}_\gamma \prec h$" for any $\gamma < \alpha$ we can choose an $p_\gamma <_\mathbb{P} p$ and $n_\gamma \in \omega$ for which $p_\gamma \Vdash$ "$\check{f}_\gamma(i) < h(i)$ for all $i > n_\gamma$". By our usual argument considering the uncountability of α we can find a cofinal subset $A \subseteq \alpha$ such that for any $a, a' \in A$ it holds that $n_a = n_{a'} := n_A$.

Now we fix an $a \in A$ and note that, since $p \Vdash$ "$\check{f}_a \prec h \prec \check{g}_\delta$", for any $\delta < \beta$ we can find $p^a_\delta <_\mathbb{P} p_a$ and $n_\delta \geq n_A \in \omega$ so that $p^a_\delta \Vdash$ "$\check{f}_a(i) < h(i) < \check{g}_\delta(i)$ for all $i > n_\delta$".

By an cardinality argument, we can choose a cofinal $B^a \subseteq \beta$ for which $n_b = n_{b'} := n^a$ for all $b, b' \in B^a$, where B^a and n^a crucially depend on the fixed $a \in A$. Choose an α-sized cofinal set $A' \subseteq A$ for which $n^a = n^{a'} := n$ for any $a, a' \in A'$.

Using the fact that $\mathbb{P}$ is σ-linked, we obtain $\mathbb{P} = \bigcup_{k\in\omega} \mathbb{P}_k$, where each $\mathbb{P}_k$ is linked. For any $a \in A'$, we can find a natural number i^a and a cofinal $C^a \subseteq B^a$ such that $p^a_c \in \mathbb{P}_{i^a}$ for all $c \in C^a$.

We find further $A'' \subseteq A'$ cofinal such that $i^a = i^{a'} := i^*$ for any $a, a' \in A''$.

Now fix some $a^* \in A''$ and let $C = C^{a^*}$. Then we claim that for all $a \in A''$ and all $c \in C$ we have $f_a(i) < g_c(i)$ for all $i > n$:

This follows since for any $a \in A''$, p_a is compatible with p^a_c for all $c \in C$. This is because $p^a_{c'} \in \mathbb{P}_{i^*}$ for some $c' \in C^a$ and also $p^{a^*}_c \in \mathbb{P}_{i^*}$, together with the fact that $\mathbb{P}_{i^*}$ is linked and $p^a_{c'} <_{\mathbb{P}} p_a$. This allows us to define

$$t(i) = \begin{cases} 1 & \text{if } i < n \\ \max\{f_a(i) \mid a \in A''\} & \text{else} \end{cases}$$

But then t interpolates $\{(f_a)_{a \in A''}, (g_c)_{c \in C}\}$, what gives the theorem, since $A'' \subseteq \alpha$ and $C \subseteq \beta$ are cofinal. □

If we combine this result with results concerning symmetric gaps, we obtain that gaps "of at least uncountable size" are $\sigma - \mathcal{L}$-indestructible:

Corollary 45. *Let α, β be regular cardinals and $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be a (α, β)-gap. If both α and β are uncountable, then $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is $\sigma - \mathcal{L}$-indestructible.*

Proof. This is a combination of Corollary 39 and Theorem 44. □

We obtain another important result:

Corollary 46. *Let $\alpha \leq \beta$ be regular cardinals and $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α, β)-gap. Then it is equivalent*

1. *$\alpha = \omega$, i.e. the gap is a Rothberger Gap,*

2. *$(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is $\sigma - \mathcal{L}$-destructible,*

3. *$\mathbf{L}_{\alpha,\beta}$ is σ-centered.*

Proof. 1. $\implies$ 3.: By Proposition 25, we know that $\mathbf{L}_{\alpha,\beta}$ is σ-centered.

3. $\implies$ 2.: Follows, since every σ-centered partial order is σ-linked.

2. $\implies$ 1.: We show $\neg 1 \implies \neg$ 2: Suppose our given gap is not a Rothberger Gap. Then we can apply Theorem 44 to obtain that the gap is not $\sigma - \mathcal{L}$-destructible. □

This corollary can be reformulated as follows:

Corollary 47. *A gap is $\sigma - \mathcal{L}$-destructible if and only if it is a Rothberger Gap.*

3.3.3 $\mathbf{F}_{\alpha,\beta}$-indestructible gaps

In section 3.1 we introduced the forcing notion $\mathbf{F}_{\alpha,\beta}$ which, for given cardinals α, β, forced an (α, β)-gap in generic extensions. Now we come back to this forcing notion and ask the question:

Which gaps survive extensions made by $\boldsymbol{F}_{\alpha,\beta}$?

This question is in particular interesting if we have some gaps at hand and want to introduce a new gap (using $\mathbf{F}_{\alpha,\beta}$) while preserving the gaps we have. We will see that a quite big class of gaps actually does survive forcing with $\mathbf{F}_{\alpha,\beta}$ and we have already done some work that will lead us to this. This section is again due to Scheepers, see [5].

Before we are able to apply one of the previous results, we need the following:

Proposition 48. *Let α, β be ordinals. Then $\boldsymbol{F}_{\alpha,\beta}$ is strongly Knaster.*

Proof. We need to show that for any regular uncountable cardinal λ and a λ-sized subset S of $\mathbf{F}_{\alpha,\beta}$ we can find a λ-sized subset $L \subseteq S$ which is linked.

So let λ be regular, uncountable and a subset $S \subseteq \mathbf{F}_{\alpha,\beta}$ of size λ be given. We enumerate S as $\{p_\nu\}_{\nu<\lambda}$. Recall that any condition p_ν is a finite partial function $[\phi_{\alpha,\beta}]^{<\aleph_0} \times \omega \to \omega$. We denote the domain of p_ν by $F_\nu \times n_\nu$.

Now we use that λ is uncountable and regular to obtain a λ-sized subset X such that for any $x, x' \in X$ we have $n_x = n_{x'} := n$. Consider the set $\{F_x\}_{x\in X}$ and note that we can apply the Δ-system lemma and get a λ-sized set $Y \subseteq X$ so that $\{F_y\}_{y\in Y}$ forms a Δ-system with root $F \in [\phi_{\alpha,\beta}]^{<\aleph_0}$. Since there are only countably many options for the finite sets $p_y(F \times n)$, we can find a λ-sized set $Z \subseteq Y$ for which $p_z \restriction F \times n = p_{z'} \restriction F \times n$ for any $z, z' \in Z$.

Now consider $\{p_z\}_{z\in Z} \subseteq \{p_\nu\}_{\nu<\lambda}$ and note that any two elements of $\{p_z\}_{z\in Z}$ are compatible, because for their domains $F_z \times n_z$ and $F_{z'} \times n_{z'}$ it holds that $n_z = n_{z'}$ (what implies that condition 2. in the ordering of $\mathbf{F}_{\alpha,\beta}$ is always satisfied) and they agree on their common domain (which is $F \times n$). But this is the same as saying that $\{p_z\}_{z\in Z}$ is linked. $\square$

Corollary 49. *Let α, β be uncountable cardinals, then any symmetric gap is indestructible by forcing with $\boldsymbol{F}_{\alpha,\beta}$.*

Proof. The result follows from Proposition 48 and Corollary 37. □

So the remaining question is: How do asymmetric gaps behave under forcing with $\mathbf{F}_{\alpha,\beta}$ for some cardinals α and β?

In fact, it will turn out that we have already established some results in connection with this question. However, what's left is the following result, stating that "small gaps" survive forcing with $\mathbf{F}_{\lambda,\kappa}$ for "large κ":

Theorem 50. *Suppose $\alpha < \beta$ are infinite regular cardinal numbers and let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α, β)-gap. Further let $\lambda \leq \kappa$ be infinite regular cardinals such that $\beta \leq \kappa$. Then $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ survives forcing with $\boldsymbol{F}_{\lambda,\kappa}$.*

Proof. We prove by contradiction. So let's suppose the statement of the theorem is false. Then, by the Maximal Principle, we find a condition $p \in \mathbf{F}_{\lambda,\kappa}$ and a $\mathbf{F}_{\lambda,\kappa}$-name h for a real, such that

$$p \Vdash \text{"}h \text{ interpolates } (\{f_\gamma\}_{\gamma<\alpha}\check{,} \{g_\delta\}_{\delta<\beta})\text{."}$$

For each $\delta < \beta$ we can pick a condition $p_\delta \ll p$ and a natural number m_δ for which

$$p_\delta \Vdash \text{"}h(i) < g_\delta(i) \text{ for all } i > m_\delta\text{."}$$

Now, as usual in this kind of arguments, we use that β is uncountable and can find a cofinal set $X \subseteq \beta$ (similar as we found Z in the proof of Proposition 48), for which $m_x = m_{x'} := m$ for $x, x' \in X$, the domain of p_x is $F_x \times n$, the domains $\{\text{dom}\{p_x\}\}_{x\in X}$ form a Δ-system with root $F \times n$ and all conditions agree on the root, i.e. $p_x \restriction F \times n = p_{x'} \restriction F \times n$ for $x, x' \in X$. We write $p = p_x \restriction F \times n$ for the restriction of some p_x (and thus all) to $F \times n$. Here, as it follows from the definition of $\mathbf{F}_{\lambda,\kappa}$, we have $F_\alpha, F \in [\phi_{\kappa,\lambda}]^{<\aleph_0} \times \omega$ and $n \in \omega$.

Now we have to distinct between the cases that α is countable and uncountable, as usual in such arguments. We start with the countable case:

So assume that α is countable, then, as it is regular, $\alpha = \omega$. For any $j \in \omega$ we pick $q_j \ll p$ and $l_j \in \omega$ such that

$$q_j \Vdash \text{"}f_j(i) < h(i) \text{ for all } i > l_j\text{."}$$

We can choose the q_j and l_j such that $l_0 \leq l_1 \leq l_2 \leq \dots$ and $q_0 \gg q_1 \gg q_2 \gg \dots$ inductively.

Denoting the domain of $q_i \in \mathbf{F}_{\lambda,\kappa}$ with $Q_i \times k_i$, we note that the set $Q = \bigcup_{j\in\omega} Q_i$ is countable as a countable union of finite sets. Thus so is $Q \times \omega$. On the other hand, the sets $\mathrm{dom}\{p_\delta\} \setminus F \times n$ are pairwise disjoint, thus their union is uncountable (in fact, of size β). Therefore, for each $x \in X$ we can pick a $z \in X$ such that $x < z$ and $(\mathrm{dom}\{p_z\} \setminus (F \times n)) \cap Q \times \omega = \emptyset$. In other words, there is a $z \in X$ such that its domain does not intersect with the domain of any q_j, except possibly on $F \times n$. Note that since $p_z \restriction F \times n = p$ and $q_j \ll p$ for any $j \in \omega$. Together with what we have just shown, this implies that p_ξ and q_j are compatible for all $i \in \omega$. Therefore, in some generic extension by $\mathbf{F}_{\lambda,\kappa}$, we have that $f_j(i) < h(i) < g_z(i)$ for all $j \in \omega$ and $i > l_j$. In particular, it must hold in the ground model that $f_j(i) < g_\xi(i)$ for all $j \in \omega$ and $i > l_j$.

Now we consider the set of $z \in X$ for z as above, i.e. $A = \{z \in X \mid (\mathrm{dom}\{p_z\} \setminus (F \times n)) \cap Q \times \omega = \emptyset\}$. Because we can find a $z \in A$ for which $x < z$ for any $x \in X$, we conclude that A is cofinal in β.

This allows us to define

$$t(i) = \begin{cases} f_j(i) & \text{if } i \in [l_j, l_{j+1}) \\ 1 & \text{otherwise} \end{cases}$$

Then we claim that t interpolates the gap $(\{f_j\}_{j\in\omega}, \{g_a\}_{a\in A})$. This is because for any $i > l_0$ and $i \in [l_j, l_{j+1})$ we have that $t(i) = f_j(i) < g_a(i)$ for all $a \in A$ by the way we have chosen l_j and A. Since t is piece-wise equal to some f_j it follows that $t \prec g_a$ for any $a \in A$. For a fixed $k \in \omega$, note that $t(i) = f_j(i)$ where $j > k$ whenever $i > l_{k+1}$, thus $f_k \prec t$. This implies that t interpolates $(\{f_j\}_{j\in\omega}, \{g_a\}_{a\in A})$, which is equivalent to the original gap.

If now α is uncountable, we proceed as follows: We aim to use a similar argument as for the countable case, so we pick, for any $\gamma < \alpha$, a condition $q_\alpha \ll p$ and a natural number l_α for which

$$q_\alpha \Vdash \text{"} f_\alpha(i) < h(i) \text{ for all } i > l_\alpha.\text{"}$$

Now we use that α is uncountable and obtain a cofinal subset $Y \subseteq \alpha$ such that $l_y = l_{y'} := l$, the domains $\mathrm{dom}\{p_y\}$ form a Δ-system with root $G \times u$ and $q_y \restriction G \times u = q_{y'} \restriction G \times u$ for $y, y' \in Y$.

Note that the set $B = \{x \in X \mid (\mathrm{dom}\{p_x\} \setminus F \times n) \cap G \times u = \emptyset\}$ is of cardinality β, because the sets $\mathrm{dom}\{p_x\} \setminus F \times n$ are pairwise disjoint and X is of size β.

We now prove the claim:

Claim 13. *Let $b \in B$ be given. Then there exists a $b' > b$ in B for which $(dom\{p_{b'}\} \setminus F \times n) \cap dom\{q_y\} = \emptyset$ for all $y \in Y$.*

Proof of claim. If the statement of the claim is false, we find a $b \in B$ witnessing this. Then for all $b' > b$ we find $y \in Y$ for which $(\mathrm{dom}\{p_{b'}\} \setminus F \times n) \cap \mathrm{dom}\{q_y\} \neq \emptyset$. For b' we denote the minimal such $y \in Y$ with $y_{b'}$.

Since $|Y| = \alpha < \beta = |B|$, we find a β-sized set $C \subseteq B$ such that $y_c = y_{c'} := y^*$ for any $c, c' \in C$. This means that for all $c \in C$ we have that $(\mathrm{dom}\{p_c\} \setminus F \times n) \cap \mathrm{dom}\{q_{y^*}\} \neq \emptyset$. But $\mathrm{dom}\{q_{y^*}\}$ is finite, what means that there must be at least two (in fact, β-many) elements of C, say $c_1 \neq c_2$ for which $(\mathrm{dom}\{p_{c_1}\} \setminus F \times n) \cap \mathrm{dom}\{q_{y^*}\} = (\mathrm{dom}\{p_{c_2}\} \setminus F \times n) \cap \mathrm{dom}\{q_{y^*}\}$. Now this cannot be the case, because the sets $\mathrm{dom}\{p_\delta\} \setminus F \times n$ are pairwise disjoint for $\delta < \beta$ by definition of $F \times n$. □

Claim 13 guarantees us that the set $B' = \{b \in B \mid (\mathrm{dom}\{p_b\} \setminus F \times n) \cap \mathrm{dom}\{q_y\} = \emptyset \text{ for all } y \in Y\}$ is of cardinality β, thus in particular cofinal.

Further, by the way we have chosen the sets B' and Y, we know that p_b is compatible with q_y for all $y \in Y$ and all $b \in B'$. Thus we obtain $f_y(i) < g_b(i)$ for all $i > \max\{m, l\}$. So we can define

$$t(i) = \begin{cases} \max\{f_y(i) \mid y \in Y\} & \text{if } i > \max\{m, l\} \\ 1 & \text{otherwise} \end{cases}$$

Then t interpolates $(\{f_y\}_{y \in Y}, \{g_b\}_{b \in B'})$, what gives the theorem. □

Now we can summarise the results we have derived:

Corollary 51. *Let α, β be ordinal numbers and $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be an (α, β)-gap. Then the gap survives forcing with $\boldsymbol{F}_{\lambda,\kappa}$ for any infinite regular cardinals λ, κ for which $\lambda \leq \kappa$.*

Proof. We can, without loss of generality, assume that α and β are regular cardinals, since only cofinality matters. Further, we can, without loss of generality, assume that $\alpha \leq \beta$.

Corollary 49 implies that if $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is symmetric, it survives forcing with $\mathbf{F}_{\lambda,\kappa}$.

If the gap is asymmetric, we have to distinct between cases:

First, assume that α, β are both infinite. Then, if $\beta \leq \kappa$ we can use Theorem 50 to see that the gap survives. On the other hand, if $\beta > \kappa$, we note the following: The partial order $\mathbf{F}_{\lambda,\kappa}$ is of size κ, and by Proposition 21 it is also ccc, thus preserves cardinals. Therefore, we can apply Theorem 34 to obtain that in this case the gap survives.

Now assume that $\alpha = 1$ (this includes all cases where α is finite). Then we have seen in Theorem 17, in the proof of the "3. $\implies$ 1." direction that we can construct an (ω, β)-pregap given a $(1, \beta)$-pregap and that the constructed pregap is a gap if and only if the given $(1, \beta)$-pregap is a gap. So if any $(1, \beta)$-gap is destroyed by forcing with $\mathbf{F}_{\lambda,\kappa}$, so is an (ω, β)-gap, which cannot be the case as we have just shown.

A similar argument can be applied for the case $\alpha = 0$ using the direction "2. $\implies$ 3." of the proof of Theorem 17. □

Remark. We can state the last result a bit more informal:

Any gap survives forcing with $\boldsymbol{F}_{\lambda,\kappa}$.

Chapter 4

Gaps in $\mathcal{P}(\omega)/\mathbf{Fin}$

We end our considerations regarding gaps in ${}^{\omega}\omega$ and consider the second important partially ordered set in this work: Gaps in $\mathcal{P}(\omega)/\text{Fin}$ with the almost inclusion order $\subset^*$.

We say that an infinite set $A \in \mathcal{P}(\omega)$ is *almost included* in $B \in \mathcal{P}(\omega)$ or *almost subset* of B and denote $A \subset^* B$ if and only if $A \setminus B$ is finite and $B \setminus A$ is infinite. We write $A =^* B$ to express that $A \setminus B$ and $B \setminus A$ are both finite.

Instead of writing $(\mathcal{P}(\omega), \subset^*)$ for the partially ordered set, we will write $\mathcal{P}(\omega)/\text{Fin}$ for simplicity and to highlight that we consider infinite subsets of ω and that we "do not care about finite subsets".

Main source for this chapter is [12].

4.1 Definitions and connections with gaps in ${}^{\omega}\omega$

Following [12], the notions of pregap and gap are defined as follows:

Definition 17 (Pregap)**.** Given two totally ordered sets $(I, <_I), (J, <_J)$ with minimal element and a pair $(\{A_i\}_{i\in I}, \{B_j\}_{j\in J})$ such that A_i and B_j are infinite subsets of ω, we say that the pair is an (I, J)-pregap if $A_{i_1} \subset^* A_{i_2}$, $B_{j_1} \subset^* B_{j_2}$ for $i_1 <_I i_2, j_1 <_J j_2$ and $A_i \cap B_j = \emptyset$ for all $i \in I, j \in J$.

Definition 18 (Gap)**.** Let $(\{A_i\}_{i\in I}, \{B_j\}_{j\in J})$ be a pregap. Then we say that the pregap is a gap if there exists no $A \in \mathcal{P}(\omega)$ for which $A_i \subset^* A$ for all $i \in I$ and $A \cap B_j =^* \emptyset$ for all $j \in J$. If the pregap is no gap, any A witnessing this is said to interpolate the pregap.

Remark. As in the Chapter 2 we will almost always consider ordinal or cardinal numbers as index sets I, J. However, we state the definition more general.

The definition we gave here for (pre-)gaps in $\mathcal{P}(\omega)/\mathrm{Fin}$ is slightly different from the definition of (pre-)gaps in ${}^{\omega}\omega$. So we give a second definition as used in [5]:

Definition 19 ((Pre-)Gap 2)**.** Given two totally ordered sets $(I, <_I), (J, <_J)$ with minimal element and a pair $(\{A_i\}_{i\in I}, \{B_j\}_{j\in J})$ such that A_i and B_j are infinite subsets of ω, we say that the pair is a pregap of second type if

$$A_{i_1} \subset^* A_{i_2} \subset^* B_{j_2} \subset^* B_{j_1}$$

for all $i_1 <_I i_2$ and $j_1 <_J j_2$.

We say that the pregap is a gap of second type if there is no $A \in \mathcal{P}(\omega)$ such that $A_i \subset^* A \subset^* B_j$ for any $i \in I, j \in J$.

Now the thing is that the definitions are closely related:

Proposition 52. *Let $(I, <_I), (J, <_J)$ be ordered sets and $(\{A_i\}_{i\in I}, \{B_j\}_{j\in J})$ be a pair of infinite subsets of ω. Then if $(\{A_i\}_{i\in I}, \{B_j\}_{j\in J})$ is a (pre-)gap there is a unique (pre-)gap of second type corresponding to it and vice versa.*

Proof. $\Longrightarrow$: Suppose $(\{A_i\}_{i\in I}, \{B_j\}_{j\in J})$ is a pregap. Put $B_j^* = B_j^c$ and $A_i^* = A_i$, then $(\{A_i^*\}_{i\in I}, \{B_j^*\}_{j\in J})$ is a pregap of second type.

Now suppose we can find an $A \in \mathcal{P}(\omega)$ that interpolates the pregap $(\{A_i^*\}_{i\in I}, \{B_j^*\}_{j\in J})$. Then $A_i = A_i^* \subset^* A$ for any $i \in I$. Let $j \in J$ and note that $B_j \cap A =^* \emptyset$, thus $A \subset^* B_j^c = B_j^*$. But then $(\{A_i\}_{i\in I}, \{B_j\}_{j\in J})$ is not a gap.

$\Longleftarrow$: If we have a gap of the second type $(\{A_i\}_{i\in I}, \{B_j\}_{j\in J})$ we again set $A_i^* = A_i$ and $B_j^* = B_j^c$ to obtain a pregap $(\{A_i^*\}_{i\in I}, \{B_j^*\}_{j\in J})$.

To see that this is indeed a gap, suppose we are given an interpolating A for $(\{A_i^*\}_{i\in I}, \{B_j^*\}_{j\in J})$, i.e. $A_i^* \subset^* A$ and $A \cap B_j^* = \emptyset$ for all $i \in I$ and all $j \in J$. Note that $A_i = A_i^* \subset^* A$. Further $A \cap B_j^* = \emptyset$ implies that $A \subset^* B_j$, so $(\{A_i\}_{i\in I}, \{B_j\}_{j\in J})$ can not be a gap of second type. □

Remark. This result enables us to switch between the two definitions of a gap, which we will do without explicitly mentioning.

Obviously, especially after reading chapters on gaps in ${}^{\omega}\omega$, the question arises how gaps in $\mathcal{P}(\omega)/\mathrm{Fin}$ and ${}^{\omega}\omega$ are connected. Our first observation is that whenever we have a gap in ${}^{\omega}\omega$ we can obtain one in $\mathcal{P}(\omega)/\mathrm{Fin}$:

Proposition 53. *Let α and β be regular cardinals. Then if there is an (α, β)-gap in $({}^{\omega}\omega, \prec)$ there is an (α, β)-gap in $\mathcal{P}(\omega)/Fin$.*

Proof. Suppose we have a gap $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ in $({}^{\omega}\omega, \prec)$. Let $A_\gamma = \{(m,n) \in \omega \mid n \leq f_\gamma(m)\}$ and $B_\delta = \{(m,n) \mid n \leq g_\delta(m)\}$. Because $f_\gamma \prec f_\rho$ whenever $\gamma < \rho < \alpha$ we obtain $A_\gamma \subset^* A_\rho$. For the same reason it holds that $A_\gamma \subset^* B_\delta$ and $B_\delta \subset^* B_\xi$ for $\gamma < \alpha$, $\delta < \rho < \beta$. This implies that $(\{A_\gamma\}_{\gamma<\alpha}, \{B_\delta\}_{\delta<\beta})$ is a pregap.

To see that it is a gap, suppose there is $A \in \mathcal{P}(\omega)$ for which $A_\gamma \subset^* A \subset^* B_\delta$ for any $\gamma < \alpha$ and $\delta < \beta$. Then we can define $f(m) = \max\{n \mid (m,n) \in A\}$ (note that we can do so only because $A \subset^* B_\delta$). Then f interpolates $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$, which is a contradiction.

We now use the well-known fact that there is a bijection between ω and $\omega \times \omega$ to conclude that there is a gap in $\mathcal{P}(\omega)/\mathrm{Fin}$. □

As in the case of gaps in $({}^{\omega}\omega, \prec)$ it is in fact enough to consider cofinal index sets:

Proposition 54. *Let $(\{A_\gamma\}_{\gamma<\alpha}, \{B_\delta\}_{\delta<\beta})$ be an (α, β)-pregap for ordinals α, β. Consider cofinal sets $A \subseteq \alpha, B \subseteq \beta$, then it is equivalent:*

1. *$(\{A_\gamma\}_{\gamma<\alpha}, \{B_\delta\}_{\delta<\beta})$ is a gap.*
2. *$(\{A_\gamma\}_{\gamma\in A}, \{B_\delta\}_{\delta\in B})$ is a gap.*

Proof. Using the second definition of a gap we gave above, the proof is exactly the same as for the case of gaps in $({}^{\omega}\omega, \prec)$ given in Proposition 2. □

We shortly switch our focus to the partially ordered set $({}^{\omega}\omega, <^*)$, where $f <^* g$ denotes that f eventually dominates g.

Then we have the result:

Proposition 55. *Suppose α and β are infinite regular cardinal numbers. Then there is an (α, β)-gap in $({}^{\omega}\omega, \prec)$ if and only if there is an (α, β)-gap in $({}^{\omega}\omega, <^*)$.*

Proof. Since $f \prec g$ implies $f <^* g$ it is clear that if we have a (α, β)-gap $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ in $({}^\omega\omega, \prec)$ it is also a pregap in $({}^\omega\omega, <^*)$. Now we have to show that it is actually a gap. Suppose the gap is interpolated by h in $({}^\omega\omega, <^*)$. Then $f_\gamma <^* h <^* g_\delta$ for any $\gamma < \alpha, \delta < \beta$. Since $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is a gap in $({}^\omega\omega, \prec)$, there is $\gamma < \alpha$ or $\delta < \beta$ (or both) for which $f_\gamma \not\prec h$ or $h \not\prec g_\delta$. We suppose that the first case holds, as the second case is similar. Now use that α is regular infinite, thus a limit ordinal. So we can find $\gamma' > \gamma$ and obtain $f_\gamma \prec f_{\gamma'}$. Further $f_\gamma <^* f_{\gamma'} <^* h$, a contradiction to $f_\gamma \not\prec h$. So $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is indeed a gap in $({}^\omega\omega, <^*)$.

Now assume we have a gap $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ in $({}^\omega\omega, <^*)$. Then define $f^*_\gamma(n) = n \cdot f_\gamma(n)$ so that $f^*_\xi(n) - f^*_\gamma(n) = n \cdot (f_\xi(n) - f_\gamma(n))$. This implies that $f^*_\gamma \prec f^*_\xi$ for $\gamma < \xi$. Similarly, with $g^*_\delta(n) = n \cdot g_\delta(n)$ we obtain a pregap $(\{f^*_\gamma\}_{\gamma<\alpha}, \{g^*_\delta\}_{\delta<\beta})$. To see that this is in fact a gap, note that given an interpolating function h, by $h'(n) = \lceil \frac{h(n)}{n} \rceil$ we obtain a function interpolating $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$. But this can not be the case by assumption. □

We now give a very important definition:

Definition 20. We denote with $\mathfrak{b}$ the minimal cardinality of an unbounded set in $({}^\omega\omega, <^*)$. Here, a set $U \subseteq {}^\omega\omega$ is unbounded if there is no $h \in {}^\omega\omega$ such that $f <^* h$ for all $f \in U$.

Then we can show the interesting result, which is originally due to Rothberger [4]:

Theorem 56. *The cardinal $\mathfrak{b}$ is the minimal cardinal κ for which there is a (ω, κ)-gap in $\mathcal{P}(\omega)/Fin$.*

Remark. As in the previous chapters, we will call a gap of the form (ω, κ) or (κ, ω) an (ω, κ)-Rothberger Gap.

Proof of Theorem 56. Suppose we have an (κ, ω)-gap $(\{A_\gamma\}_{\gamma<\kappa}, \{B_\delta\}_{\delta<\omega})$ such that κ is minimal with that property. We aim to find an unbounded sequence of length κ.

We first note the following: $A_\gamma \cap B_\delta \neq \emptyset$ for all but finitely many $\gamma < \kappa$, $\delta < \omega$ and $\gamma \neq \kappa$. This is because if this would not be the case we could just take $A = \bigcup_{\gamma<\kappa} A_\gamma$ and obtain an interpolating element for the given gap. Further, by the basic definition of a pregap, we know $A_\gamma \cap B_\delta$ is finite.

Now we can define

$$f_\gamma(i) = \max\{n \mid n \in A_\gamma \cap B_i\}.$$

We claim that the sequence of reals $(f_\gamma)_{\gamma<\kappa}$ is unbounded in $({}^\omega\omega, <^*)$.

So assume this is not the case, then there is $f \in {}^\omega\omega$ such that $f_\gamma <^* f$ for all $\gamma < \kappa$. Now let $A = \bigcup_{\gamma<\kappa} A_\gamma$, then obviously $A_\gamma \subset^* A$. Since $f_\gamma <^* f$ for all $\gamma < \kappa$ and $f_\gamma(i) = \max\{n \mid n \in A_\gamma \cap B_i\}$ we can conclude that $A \cap B_i \subseteq \{1, 2, \dots, f(i)\}$ for any $i \in \omega$. In particular $A \cap B_i$ is finite and this implies $A \cap B_i =^* \emptyset$. Therefore, A interpolates the given gap $(\{A_\gamma\}_{\gamma<\kappa}, \{B_\delta\}_{\delta<\omega})$, which is a contradiction.

We have shown $\mathfrak{b} \leq \kappa$. To see that also $\kappa \leq \mathfrak{b}$ holds, note that any unbounded family in $({}^\omega\omega, <^*)$ of size $\mathfrak{b}$ is a $(\mathfrak{b}, 0)$-gap. By proposition 55, we can without loss of generality assume that we have an unbounded family in $({}^\omega\omega, \prec)$, that is, a $(\kappa, 0)$-gap. Now we use Theorem 17 to obtain a κ-Rothberger Gap in $({}^\omega\omega, \prec)$. But now proposition 53 ensures us the existence of a κ-Rothberger Gap in $\mathcal{P}(\omega)/\text{Fin}$ and we obtain $\kappa \leq \mathfrak{b}$. $\square$

4.2 Hausdorff Gaps and Special Gaps in $\mathcal{P}(\omega)/\text{Fin}$

similarly to the notions of Hausdorff and Special Gaps in $({}^\omega\omega, \prec)$, we can define this notions in $\mathcal{P}(\omega)/\text{Fin}$ as well. We will mostly follow [12] in this section.

To simplify the language used when speaking about gaps in $\mathcal{P}(\omega)/\text{Fin}$, we introduce the following notion:

Definition 21 (Tower)**.** A family $\{T_i\}_{i\in I}$ of infinite subsets of ω for some ordered index-set $(I, <_I)$ is called a tower, if $T_{i_1} \subset^* T_{i_2}$ whenever $i_1 <_I i_2$.

The relation between gaps and tower is clear: Each (pre-)gap is a pair of towers.

Definition 22 (Hausdorff Gap)**.** Let $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ be a pregap. We say that the gap is a Hausdorff Gap if there is a cofinal set $C \subseteq \omega_1$ such that the set $\{\xi \in C \mid A_\xi \cap B_\delta \subseteq n\}$ is finite for any $n \in \omega$ and any $\delta < \omega_1$.

We denote the class of Hausdorff Gaps with $\mathcal{H}$.

Definition 23 (Subgap)**.** We call the gap $(\{A_c\}_{c\in C}, \{B_c\}_{c\in C})$ a subgap of $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$.

Remark. Given the definition of a subgap, one state the definition of a Hausdorff Gap using the *Hausdorff Property.* The Hausdorff Property for a gap $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ is the following statement:

The set $\{\xi \in \omega_1 \mid A_\xi \cap B_\delta \subseteq n\}$ *is finite for any* $n \in \omega$ *and all* $\delta < \omega_1$.

Then a Hausdorff Gap is a pregap having a subgap that satisfies the Hausdorff Property.

As in the case of $({}^\omega\omega, \prec)$ the first thing to note is that the name Hausdorff Gap in indeed satisfied:

Proposition 57. *A Hausdorff Gap is a gap.*

Proof. Without loss of generality, we assume we are given a gap that satisfies the Hausdorff Property. We may denote this gap by $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$. Then we continue similarly as in the proof of Proposition 10.

Suppose $(\{A_\gamma\}_{\gamma<\alpha}, \{B_\delta\}_{\delta<\beta})$ is not a gap. Then we find a subset of the naturals A that interpolates the pregap, i.e. $A_\gamma \subset^* A$ and $A \cap B_\gamma =^* \emptyset$ for any $\gamma < \omega_1$.

For any $\gamma < \omega_1$ we find a n_γ such that $A_\gamma \setminus A \subseteq n_\gamma$. So we find an uncountable $X \subseteq \omega_1$ such that for all $x_1, x_2 \in X$ we have $n_{x_1} = n_{x_2} := n$. Further we find, for any $x \in X$, a m_x for which $A \cap B_x \subseteq m_x$. So we find an uncountable $Y \subseteq X$ such that for any $y_1, y_2 \in Y$ it holds that $m_{y_1} = m_{y_2} := m$.

Now fix a $y \in Y$ that has infinitely predecessors in Y. We can find such a y, since Y is uncountable.

Now for any $\xi < y$ such that $\xi \in Y$ we obtain that $A_\xi \setminus A \subseteq n$ and $A \cap B_y \subseteq m$. Thus $A_\xi \cap B_y \subseteq \max\{n, m\} := k$. But then $\{\xi < y \mid A_\xi \cap B_y \subseteq k\}$ is infinite, which is a contradiction. □

Corollary 58. *There exists a Hausdorff Gap in* $\mathcal{P}(\omega)/Fin$.

Proof. This follows immediately from Theorem 12 and Proposition 53. The Hausdorff property follows from the construction of the gap in $\mathcal{P}(\omega)/\text{Fin}$ in the proof of Proposition 53. □

Definition 24 (Special Gap)**.** Let $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ be a pregap. We say that the pregap is a Special Gap if it has a cofinal subgap $(\{A_i\}_{i\in I}, \{B_i\}_{i\in I})$ for which $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for all $i \neq j \in I$.

We denote the class of Special Gaps by $\mathcal{SP}$.

Proposition 59. *Let* $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ *be a Special Gap. Then it is a gap in* $\mathcal{P}(\omega)/Fin$.

Proof. Let $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ be a Special Gap and suppose that it is not a gap. Since the pregap is special, we can find a cofinal (thus uncountable) $C \subseteq \omega_1$ for which $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for any $i < j \in C$.

Because the pregap is not a gap, we can find $A \in \mathcal{P}(\omega)$ such that $A_\gamma \subset^* A$ and $B_\gamma \cap A =^* \emptyset$ for all $\gamma < \omega_1$. This implies that there are $n_\gamma, m_\gamma \in \omega$ for which $A_\gamma \setminus A \subseteq n_\gamma$ and $B_\gamma \cap A \subseteq m_\gamma$ for any $\gamma < \omega_1$. In particular, this holds for all $\gamma \in C$. We obtain that there is a cofinal set $D \subseteq C$ for which $n_{d_1} = n_{d_2} := n$ and $m_{d_1} = m_{d_2} := m$ for $d_1, d_2 \in D$. Letting $l = \max\{m, n\}$, we obtain that $A_d \setminus A \subseteq l$ and $B_d \cap A \subseteq l$ for all $d \in D$. In particular, $A_{d_1} \cap B_{d_2} = A_{d_1} \cap B_{d_2} \cap l$ for any $d_1, d_2 \in D$.

Since the set D is uncountable and l is finite, we can find a cofinal $E \subseteq D$ such that $A_e \setminus A = a$ and $B_e \cap A = b$ for $a, b \subseteq l$ and all $e \in E$. From the fact that $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ is a pregap we know that $A_e \cap B_e = \emptyset$, thus $a \cap b = \emptyset$. But then for $e_1, e_2 \in E$ we obtain that $A_{e_1} \cap B_{e_2} = A_{e_2} \cap B_{e_1} = \emptyset$, since $E \subseteq D$. But $E \subseteq C$ is cofinal and this is a contradiction, because $E \subseteq C$. □

Proposition 60. *Let* $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ *be a Special Gap. Then the inverted gap* $(\{B\}_{\gamma<\omega_1}, \{A_\delta\}_{\delta<\omega_1})$ *is also a Special Gap.*

Proof. This follows immediately from the definition of a Special Gap in $\mathcal{P}(\omega)/\text{Fin}$. □

Definition 25 (Left-Oriented Gap.)**.** Let $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ be a pregap. Then we say that the gap is left-oriented if it contains a subgap that satisfies $A_\gamma \cap B_\delta \neq \emptyset$ for all $\gamma < \delta < \omega_1$.

We denote the collection of all left-oriented gaps by $\mathcal{LO}$.

The following proposition follows immediately from the respective definitions:

Proposition 61. *Every left-oriented gap is special.*

We establish a result that emphasizes the connection between Hausdorff Gaps and left-oriented gaps.

Proposition 62. *Let* $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ *be a Hausdorff Gap. Then it is left-oriented.*

Proof. The idea of the proof is basically the same as in the proof of Theorem 15. Define $f\colon \omega_1 \to [\omega_1]^{<\aleph_0}$ by

$$f(\gamma) = \{\delta < \gamma \mid A_\delta \cap B_\gamma = \emptyset\}.$$

It is obvious that $\gamma \notin f(\gamma)$. Further $f(\gamma)$ is countable for any γ, because $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ is a Hausdorff Gap. Therefore we can apply Proposition 16 to obtain a cofinal $X \subseteq \omega_1$ such that $x \notin f(y)$ for any $x \neq y \in X$. But then $(\{A_x\}_{x\in X}, \{B_x\}_{x\in X})$ is a subgap that witnessing $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ is a left-oriented gap. □

From Propositions 62 and 61 we get the

Corollary 63. $\mathcal{H} \subseteq \mathcal{LO} \subseteq \mathcal{SP}$.

Now it is an interesting question whether the classes of gaps coincide or not, i.e. if every Special Gap is left-oriented and if every left-oriented gap is Hausdorff. It will turn out that the answer is no in both cases. In the remainder of this section we will provide an example that proves this.

4.2.1 Gaps in $\mathcal{P}(\omega)$/Fin and towers

It is not surprising that there are deep connections between gaps in $\mathcal{P}(\omega)$/Fin and towers as each such gap is a pair of towers. To answer the questions of the end of last section, we have to establish two small results on this. The following definitions and results can be found in [12].

There are definitions of certain types of towers, similarly as we have for gaps. Note that we focus on ω_1-towers.

Definition 26 (Hausdorff Tower)**.** A tower $\{T_i\}_{i\in\omega_1}$ is called Hausdorff if it contains a cofinal subtower $\{T_i\}_{i\in X}$ such that $\{y < x \mid T_y \setminus T_x \subseteq n\}$ is finite for all $x \in X$ and all $n \in \omega$.

Definition 27 (Special Tower)**.** A tower $\{T_i\}_{i\in\omega_1}$ is called special if it contains a cofinal subtower $\{T_i\}_{i\in X}$ such that $T_y \nsubseteq T_x$ for all $x, y \in X$.

Definition 28 (Suslin Tower)**.** We say a tower is Suslin if it is not special.

We can state the important results, which also motivate the names for the towers we just defined:

Proposition 64. *Let* $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ *be a Hausdorff Gap. Then* $\{A_\gamma\}_{\gamma<\omega_1}$ *is a Hausdorff tower.*

Proof. Without loss of generality, we assume that $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ itself (and not some subgap) satisfies the Hausdorff Property. This means that $\{\xi < \gamma \mid A_\xi \cap B_\gamma \subseteq n\}$ is finite for any $\gamma < \omega_1$ and any $n \in \omega$.

Suppose that the proposition is false, then we find γ and n such that $X = \{\xi < \gamma \mid A_\xi \setminus A_\gamma \subseteq n\}$ is infinite. For any $\gamma < \omega_1$ we know that $A_\gamma \cap B_\gamma = \emptyset$. But then for $\xi \in X$ we obtain $A_\xi \cap B_\gamma = ((A_\xi \setminus A) \cup A) \cap B_\gamma = (A_\xi \setminus A) \cap B \subseteq n$. This contradicts the fact that the gap is a Hausdorff Gap. □

This result together with the fact that there are Hausdorff Gaps in $\mathcal{P}(\omega)/\mathrm{Fin}$ implies:

Proposition 65. *There exists a Hausdorff Tower.*

A very similar fact is true for Special Towers:

Proposition 66. *Let* $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ *be a left-oriented gap. Then* $\{A_\gamma\}_{\gamma<\omega_1}$ *is special.*

Proof. Because $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ is left-oriented, we may, without loss of generality, assume that $A_\gamma \cap B_\delta \neq \emptyset$ for all $\gamma < \delta \in \omega_1$. For $\gamma, \delta \in \omega_1$, w.l.o.g. $\gamma < \delta$, we have $A_\delta \nsubseteq A_\gamma$ since $A_\gamma \subset^* A_\delta$. But now $A_\gamma \cap B_\delta \neq \emptyset$ and $A_\delta \cap B_\delta = \emptyset$, thus $A_\gamma \nsubseteq A_\delta$. But this means $\{A_\gamma\}_{\gamma<\omega_1}$ is special. □

4.2.2 Special Gaps that are not Hausdorff

The goal of this subsection is to prove

Theorem 67. *There exists gap* $(\{A_\gamma\}_{\gamma<\omega_1}, \{B_\delta\}_{\delta<\omega_1})$ *in* $\mathcal{P}(\omega)/Fin$ *that is left-oriented, but not Hausdorff and such that* $(\{B_\delta\}_{\delta\in\omega_1}, \{A_\gamma\}_{\gamma\in\omega_1})$ *is special, but not left-oriented.*

We prove the theorem by constructing a gap that has the desired properties. We follow [12] in the process.

Proof of Theorem 67. We will define a forcing notion that will add a gap with the desired properties. We will denote this forcing notion with $\mathbb{P}$.

Conditions in $\mathbb{P}$ are of the form $p = (I_p, n_p, (A_{i,p}, B_{i,p})_{i \in I_p})$ for $I_p \in [\omega_1]^{<\aleph_0}$, n_p a natural number and $A_{i,p}, B_{i,p} \subseteq n_p$.

The idea is that for a generic $G \subseteq \mathbb{P}$ we obtain a gap (A_γ, B_γ) by letting $A_\gamma = \bigcup_{p \in G} A_{\gamma,p}$ and $B_\gamma = \bigcup_{p \in G} B_{\gamma,p}$. To ensure that this works, we require that $A_{i,p} \cap B_{i,p} = \emptyset$ for $i \in I_p$ and any $p \in \mathbb{P}$. Because we want to obtain a left-oriented gap, we additionally require that $A_{i,p} \cap B_{j,p} \neq \emptyset$ for any $i < j$ in I_p.

The idea as just described also inspires the definition of strengthening a condition. We say that $q < p$ if

1. $I_p \subseteq I_q$,
2. $n_p \leq n_q$,
3. $A_{i,q} \cap n_p = A_{i,p}$ and $B_{i,q} \cap n_p = B_{i,p}$ for all $i \in I_p$,
4. for $i < j \in I_p$ and $n_p \leq k < n_q$, if $k \in A_{i,q}$ then also $k \in A_{j,q}$ and similarly if $k \in B_{i,q}$ then $k \in B_{j,q}$.

Claim 14. *The sets $\{p \in \mathbb{P} \mid \gamma \in I_p\}$ are dense for every $\gamma < \omega_1$.*

Proof. Let $\gamma < \omega_1$ be given and $(I_p, n_p, (A_{i,p}, B_{i,p})_{i \in I_p})$ be an arbitrary condition in $\mathbb{P}$. If $\gamma \in I_p$ there is nothing to prove, so we assume $\gamma \notin I_p$. Now let $I' = I_p \cup \{\gamma\}$ and $n' = n_p$. Then for $A_{\gamma,p}, B_{\gamma,p}$ arbitrary subsets of n' we obtain with $p' := (I', n', (A_{i,p}, B_{i,p})_{i \in I'})$ that $p' < p$. □

Now suppose we have a generic G at hand and let $A_\gamma = \bigcup_{p \in G} A_{\gamma,p}$ and $B_\gamma = \bigcup_{p \in G} B_{\gamma,p}$. We can do that due to conditions 1. to 3. in the definition of strengthening in $\mathbb{P}$.

Note that due to Claim 14 we obtain that $I = \bigcup_{p \in G} I_p$ is cofinal. So without loss of generality, we can denote the pair $(\{A_\gamma\}_{\gamma \in I}, \{B_\gamma\}_{\gamma \in I})$ (that is in $V[G]$) by $(\{A_\gamma\}_{\gamma \in \omega_1}, \{B_\gamma\}_{\gamma \in \omega_1})$.

Claim 15. *$(\{A_\gamma\}_{\gamma \in \omega_1}, \{B_\gamma\}_{\gamma \in \omega_1})$ is a left-oriented gap in $V[G]$.*

Proof. We first show that it is a pregap. For that, we first have to show that $A_\gamma \cap B_\gamma = \emptyset$ for any $\gamma < \omega_1$. But this is ensured by the requirement on conditions $p \in \mathbb{P}$ that $A_{i,p} \cap B_{i,p} = \emptyset$ for all $i \in I_p$.

Further we need to verify that $A_\gamma \subset^* A_\delta$ and $B_\gamma \subset^* B_\delta$ for $\gamma < \delta$. But this follows from condition 4. in the definition of strengthening in $\mathbb{P}$ as any counterexample, together with the fact that two elements of a generic filter are compatible, would yield a contradiction to condition 4.

So we obtain that $(\{A_\gamma\}_{\gamma\in\omega_1}, \{B_\gamma\}_{\gamma\in\omega_1})$ is indeed a pregap. Since for conditions $p \in \mathbb{P}$ we require that $A_{i,p} \cap B_{i,q} \neq \emptyset$ whenever $i < j$, we obtain that the pregap is in fact a left-oriented gap.

Also, the forcing notion $\mathbb{P}$ also preserves ω_1, because it is ccc. We do not prove this here, since the proof is somewhat technical and lengthy, however, it can be found in [12], Theorem 41. □

Now the point is:

Claim 16. *The gap* $(\{A_\gamma\}_{\gamma\in\omega_1}, \{B_\gamma\}_{\gamma\in\omega_1})$ *is not Hausdorff.*

Proof. By Proposition 64, it is enough to obtain a contradiction from the assumption that $\{A_\gamma\}_{\gamma<\omega_1}$ is Hausdorff (as a tower). So assume that this tower is Hausdorff. Then we can find a condition $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\check{X}$ for an uncountable subset $X \subseteq \omega_1$ such that

$$p \Vdash \text{"}\{A_x\}_{x\in\check{X}} \text{ is Hausdorff."}$$

Because X is uncountable, we can find an uncountable set $Y \subseteq \omega_1$ such that

$$Y = \{\gamma < \omega_1 \mid \exists q_\gamma < p\colon q_\gamma \in G,\ \gamma \in I_{q_\gamma} \text{ and } q_\gamma \Vdash \gamma \in \check{X}\}.$$

We can find Y because of Claim 14. We will now define a condition that will force that the left part of the given gap, the tower $\{B_\gamma\}_{\gamma<\omega_1}$, is Suslin.

Since I_{q_γ} is finite for all $\gamma \in Y$, we may assume that the I_{q_γ} are all of the same (finite) cardinality.

Apply the Δ-system lemma to the collection $\{I_{q_\gamma}\}_{\gamma\in Y}$ to obtain a root R, i.e. $I_{q_\gamma} \cap I_{q_\delta} = R$ for all q_γ, q_δ as in Y and $\gamma, \delta \in Y$. By Claim 14, we can assume that $I_{q_\gamma} \setminus R < I_{q_\delta} \setminus R$ (pointwise) for $\gamma < \delta$. Also, we can assume that $\gamma \notin R$ for all $\gamma \in Y$, because Y is uncountable and R is finite.

Now consider q_γ for $\gamma \in Y$. Then q_γ consists of the finite set $I_{q_\gamma} \supseteq R$, n_{q_γ} and finitely many pairs $(A_{i,q_\gamma}, B_{i,q_\gamma})_{i\in I_{q_\gamma}}$ of subsets of ω. Since Y is uncountable, we can assume n_{q_γ} be to equal for all $\gamma \in Y$; we denote this number by n. Also, for the same reason, we may assume that $A_{i,q_\gamma} = A_{i,q_\delta}$ and $B_{i,q_\gamma} = B_{i,q_\delta}$ for all $i \in R$ and all $\gamma, \delta \in Y$.

Because the I_{q_γ} are all of the same finite cardinality, we can find $h \in \omega$ such that $|I_{q_\gamma} \setminus R| = h$. Enumerate the set $I_{q_\gamma} \setminus R$ by $r_1^\gamma, ..., r_h^\gamma$. We can assume that $B_{r_l^\gamma, q_\gamma} = B_{r_l^\delta, q_\delta}$ for any $\gamma, \delta \in Y$ and $l \leq h$. We can do this, since we again consider countably many finite sets over an uncountable index set, which is Y. Also, since $\gamma \notin R$, note that $\gamma = r_l^\gamma$ for some $l \leq h$.

Because Y is uncountable, we can find $\gamma^* \in Y$ that has infinitely many predecessors in Y. Then we define a special conditions q^* as follows:

Find γ_0, the minimal element of Y, which we can find since $Y \subseteq \omega_1$. Define $q^* = (I_{q^*}, n^*, (A_{i,q^*} B_{i,q^*}))$ by

1. $I_{q^*} = I_{q_{\gamma^*}} \cup I_{q_{\gamma_0}}$,
2. $n^* = n + 1$,
3. $A_{i,q^*} = \begin{cases} A_{i,q_{\gamma^*}} & \text{if } i \in I_{\gamma^*} \\ A_{i,q_{\gamma_0}} \cup \{n\} & \text{if } i \in I_{\gamma_0} \setminus R, \end{cases}$
4. $B_{i,q^*} = \begin{cases} B_{i,q_{\gamma_0}} & \text{if } i \in I_{\gamma_0} \\ B_{i,q_{\gamma^*}} \cup \{n\} & \text{if } i \in I_{\gamma^*} \setminus R. \end{cases}$

Then it is clear that $q^* < q_{\gamma_0}$ and $q^* < q_{\gamma^*}$. Since $q_{\gamma_0} \Vdash \gamma_0 \in \check{X}$ and $q_{\gamma^*} \Vdash \gamma^* \in \check{X}$, we obtain $q^* \Vdash \gamma_0, \gamma^* \in \check{X}$. Because γ_0 is minimal in Y and condition 4. in the definition of strengthening of the forcing at hand, together with the fact that $B_{\gamma_0, q^*} \subseteq B_{\gamma^*, q^*}$, we obtain that $q^* \Vdash B_{\gamma_0} \subseteq B_{\gamma^*}$. Note that this means that $q^* \Vdash$ "$\{B_\gamma\}_{\gamma \in \omega_1}$ is Suslin."

Up to this point, we did not make use of the assumption on X that $p \Vdash$ "$\{A_x\}_{x \in \check{X}}$ is Hausdorff.". Using the assumption together with $q^* < p$, we obtain an $r < q^*$ and $m < \omega$ such that

$$r \Vdash \text{"}|\{\gamma < \gamma^* \mid \gamma \in \check{X} \text{ and } A_\gamma \setminus A_{\gamma^*} \subseteq n+1\}| < m.\text{"}$$

Since I_r is finite and γ^* has infinitely many predecessors in Y, we can find m-many $\{\xi_1, \xi_2, ..., \xi_m\}$ in Y, $\xi_i < \xi_{i+1}$, such that $I_r \cap [\min\{I_{q_{\xi_1}} \setminus R\}, \max\{I_{q_{\xi_m}} \setminus R\}] = \emptyset$. Here the q_{ξ_i} are as in the definition of Y. Now we define a new condition s as follows:

1. $I_s = I_r \cup \bigcup_{1 \leq i \leq m} I_{q_{\xi_i}}$,
2. $n_s = n_r + m$,

3. $A_{i,s} = \begin{cases} A_{i,r} & \text{if } i \in I_r \text{ and } i \leq \max\{R\} \\ A_{i,r} \cup \{n_r\} & \text{if } i \in I_r \text{ and } \max\{R\} < i < \min\{I_{q_{\xi_1}} \setminus R\} \\ A_{i,r} \cup [n_r, n_r + m) & \text{if } i \in I_r \text{ and } \max\{I_{q_{\xi_m}} \setminus R\} < i \\ (A_{r_l^{\gamma_0},r} \cup \{n_r + j\}) \cap n_s & \text{if } i \in I_{q_{\xi_j}} \setminus R \text{ and } i = r_l^{\xi_j},\ l \leq h \end{cases}$

4. $B_{i,s} = \begin{cases} B_{i,r} & \text{if } i \in I_r \\ B_{i,q_{\xi_j}} \cup [n_r, n_r + j) & \text{if } i \in I_{q_{\xi_j}} \setminus R \end{cases}$

Now we claim that $s < r$ and $s < q_{\xi_i}$ for $i = 1, ..., m$:

It is clear that $I_s \supseteq I_r, I_{q_{\xi_i}}$ and $n_s \geq n_r, n_{q_{\xi_i}}$.

Also, for $i \in I_r$ we have $A_{i,s} \cap n_r = A_{i,r}$, because we always add natural number greater than n_r to $A_{i,r}$ to obtain $A_{s,i}$, if $i \in I_r$. For $i \in I_{\xi_j}$ for some $j \in [1, m]$ we distinct two cases: First, if $i \in R$, by our assumption on Y, $A_{i,q_{\xi_j}} = A_{i,r} = A_{i,s} \cap n_r = A_{i,s} \cap n_{q_{\xi_j}}$, where the last equality holds because $r < q^*$ and $q^* \in Y$. If now $i \in I_{q_{\xi_j}} \setminus R$, we use that $A_{r_l^{\gamma},q_\gamma} = A_{r_l^{\delta},q_\delta}$ for all $\gamma, \delta \in Y$ and $l \leq h$. Using a very similar argument we obtain that $B_{i,s} \cap n_r = B_{i,r}$ and $B_{i,s} \cap n_{q_{\xi_j}} = B_{i,q_{\xi_j}}$.

Finally, if $i < i' \in I_r$ and $n_r \leq l < n_s$, if $l \in A_{i,s}$ then $l \in A_{i',s}$. similarly, if $i < i' \in I_{q_{\xi_j}}$ and $n_{q_{\xi_j}} \leq l < n_s$, if $l \in A_{i,s}$ then $l \in A_{i',s}$. The same holds for $B_{i,s}$.

Since $s < r, q_{\xi_1}, ..., q_{\xi_m}$, we obtain (with $r < q^* < q_{\xi_0}$) that

$$s \Vdash \{\xi_0, \xi_1, ..., \xi_m\} \subseteq \check{X}.$$

Now consider any ξ_j for $j \leq m$. Then $\xi_i = r_l^{\xi_j}$ for some $l \leq h$. So $A_{\xi_i,s} = A_{r_l^{\xi_i},s} = (A_{r_l^{\gamma_0},r} \cup \{n_r + i\}) \cap n_s$. Note that $\gamma^* \in I_r$ and since $I_{\gamma^*} \setminus R > I_{\xi_j} \setminus R$ pointwise, together with $\gamma^* \in I_{\gamma^*} \setminus R$, we get that $A_{\gamma^*,s} = A_{\gamma^*,r} \cup [n_r, n_r + m)$. Thus $A_{\xi_j,s} \setminus A_{\gamma^*,s} = A_{r_l^{\gamma_0},r} \setminus A_{\gamma^*,r} \subseteq n + 1$ by definition of the conditions r and q^*. But from this we obtain that $s \Vdash A_{\xi_j} \setminus A_{\gamma^*} \subseteq n + 1$ for any $j \leq m$, what implies that

$$s \Vdash \text{"}|\{\gamma < \gamma^* \mid A_\gamma \setminus A_{\gamma^*} \subseteq n + 1\}| \geq m.\text{"}$$

which is a contradiction to $s < r$.

We conclude that the assumption that $\{A_x\}_{x \in \check{X}}$ is Hausdorff (as a tower) is false. This proves the claim.

□

The proof of the previous claim has an important side-result:

Claim 17. *The tower* $\{B_\gamma\}_{\gamma\in\omega_1}$ *is Suslin.*

Now Claim 16, together with Proposition 64, implies that the generic gap $(\{A_\gamma\}_{\gamma\in\omega_1}, \{B_\gamma\}_{\gamma\in\omega_1})$ is not Hausdorff, but left-oriented by Claim 15. Thus it is also special.

On the other hand, the gap $(\{B_\gamma\}_{\gamma\in\omega_1}, \{A_\gamma\}_{\gamma\in\omega_1})$ is Special by Proposition 60. But, by Proposition 66 and Claim 17, this gap is not left-oriented. □

We obtain as a corollary:

Corollary 68. $\mathcal{H} \subsetneq \mathcal{LO} \subsetneq \mathcal{SP}$.

Chapter 5

Gaps under additional axioms

It is a typical set-theoretical question to ask what can be stated about certain objects under ZFC plus some additional axioms, such as CH or MA. We will follow this approach here as well and derive some results of the gap structure of $({}^{\omega}\omega, \prec)$ under such additional axioms and ZFC. We will mostly consider gaps in $({}^{\omega}\omega, \prec)$, because by Proposition 53 we already know that the existence of a gap in $({}^{\omega}\omega, \prec)$ implies the existence of a gap in $\mathcal{P}(\omega)/\mathrm{Fin}$.

It shall be noted here that many of the results mentioned are not explicitly proved, since often side-results and lengthy arguments are required and the goal of this chapter is to provide an overview of the most important results. However, references are provided for the interested reader. Again, most of the material can be found in [5], although often the original ideas appeared in different works, such as [13] or [7].

5.1 Gaps under CH

As a starting point we consider gaps and CH. It turns out that under CH it is pretty easy to establish the existence of (ω_1, ω_1)-gaps, which can be found in [13]:

Theorem 69 (CH). *Assume CH. Then there exists an (ω_1, ω_1)-gap in $({}^{\omega}\omega, \prec)$.*

Proof. Use CH to enumerate ${}^\omega\omega$ and denote this enumeration by $\{f_\gamma\}_{\gamma<\omega_1}$. We build a gap $(\{g_\gamma\}_{\gamma<\omega_1}, \{h_\gamma\}_{\gamma<\omega_1})$ inductively.

Start with g_0 and h_0 as an arbitrary pair for which $g_0 \prec h_0$. Then for $\gamma < \omega_1$ pick g_γ and h_γ such that

1. $g_\gamma \prec h_\gamma$,
2. $g_\xi \prec g_\gamma \prec h_\gamma \prec h_\xi$ for all $\xi < \gamma$,
3. there is no $\xi < \gamma$ for which $g_\gamma \prec f_\xi \prec h_\gamma$, i.e. the pair (g_γ, h_γ) cannot be interpolated by an element from $\{f_\xi\}_{\xi<\gamma}$.

We can perform this induction, because if would not be possible to find such a pair at a step below ω_1 we would have a (γ,γ)-gap for γ countable, which is not possible by Theorem 3.

The object $(\{g_\gamma\}_{\gamma<\omega_1}, \{h_\gamma\}_{\gamma<\omega_1})$ is obviously a pregap, since it satisfies conditions 1. and 2. To see that it is indeed a gap, assume that there would be $f \in {}^\omega\omega$ interpolating it. Then $f = f_\xi$ for $\xi < \omega_1$. Pick an $\gamma > \xi$ and note that by condition 3. on the pair (g_γ, h_γ) it is impossible that f interpolates (g_γ, h_γ) so in particular it can not interpolate the pregap. Thus the pregap is a gap. □

Another interesting result is that in presence of GCH there are symmetric gaps of arbitrary large size. We will not give a proof for this fact, an explanation on how to prove it can be found in [5].

Theorem 70 (GCH)**.** *Assume GCH. Then there exist $2^{\aleph_\alpha}$ many non-equivalent $(\omega_\alpha, \omega_\alpha)$-gaps in $({}^\omega\omega, \prec)$. The same holds for gaps in $\mathcal{P}(\omega)/Fin$.*

5.2 Gaps under Martin's Axiom

Another interesting axiom is *Martin's Axiom*, most often abbreviated as MA. There are different versions of MA, often related to Definition 12. They are of different strength, in the sense that often one is implied by the other. We will give them below and state their relations. The goal when investigating the influence of these axioms on the gaps in $({}^\omega\omega, \prec)$ is to use "as less assumptions as possible", i.e., reformulated in our situation, to use the weakest version of MA possible.

5.2.1 Versions of Martin's Axiom

Martin's Axiom in it's different versions states that for any partial order satisfying certain properties and any collection of less than $2^{\aleph_0}$ many dense sets, there exists a filter that intersects every element of the collection. The different variations of MA arise when one varies the properties of the partial orders that are considered. The versions of MA used here can be found in [5].

Definition 29. ($\mathrm{MA}_{\sigma-\mathcal{C}}$) $\mathrm{MA}_{\sigma-\mathcal{C}}$ is the following statement:
Let $\mathbb{P}$ be a partially ordered set that is σ-centered. Then for every collection $\{D_i\}_{i\in I} \subset \mathcal{P}(\mathbb{P})$ of dense sets of size $< 2^{\aleph_0}$ there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_i \neq \emptyset$ for any $i \in I$.

The other versions of MA are absolutely analogous to $\mathrm{MA}_{\sigma-\mathcal{C}}$, still we state them for the sake of completeness.

Definition 30. ($\mathrm{MA}_{\sigma-\mathcal{L}}$) $\mathrm{MA}_{\sigma-\mathcal{L}}$ is the following statement:
Let $\mathbb{P}$ be a partially ordered set that is σ-linked. Then for every collection $\{D_i\}_{i\in I} \subset \mathcal{P}(\mathbb{P})$ of dense sets of size $< 2^{\aleph_0}$ there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_i \neq \emptyset$ for any $i \in I$.

Definition 31. ($\mathrm{MA}_{\mathcal{K}_\forall}$) $\mathrm{MA}_{\mathcal{K}_\forall}$ is the following statement:
Let $\mathbb{P}$ be a partially ordered set that is strongly Knaster. Then for every collection $\{D_i\}_{i\in I} \subset \mathcal{P}(\mathbb{P})$ of dense sets of size $< 2^{\aleph_0}$ there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_i \neq \emptyset$ for any $i \in I$.

Definition 32. ($\mathrm{MA}_{\mathcal{K}}$) $\mathrm{MA}_{\mathcal{K}}$ is the following statement:
Let $\mathbb{P}$ be a partially ordered set which is Knaster. Then for every collection $\{D_i\}_{i\in I} \subset \mathcal{P}(\mathbb{P})$ of dense sets of size $< 2^{\aleph_0}$ there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_i \neq \emptyset$ for any $i \in I$.

Definition 33. (MA_{ccc}) MA_{ccc} is the following statement:
Let $\mathbb{P}$ be a partially ordered set which is ccc. Then for every collection $\{D_i\}_{i\in I} \subset \mathcal{P}(\mathbb{P})$ of dense sets of size $< 2^{\aleph_0}$ there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_i \neq \emptyset$ for any $i \in I$.

By the remark given after Definition 12 the following statement is clear:

Theorem 71. $MA_{\omega_1} \implies MA_{\mathcal{K}} \implies MA_{\mathcal{K}_\forall} \implies MA_{\sigma-\mathcal{L}} \implies MA_{\sigma-\mathcal{C}}$.

5.2.2 Consequences of MA to gaps in $({}^\omega\omega, \prec)$

We will now derive consequences from the presence of the different versions of MA to gaps in $({}^\omega\omega, \prec)$. Inspired by Theorem 71 we start with the weakest axiom, which implies certain requirements on existing Rothberger Gaps (see [14] or [5]):

Proposition 72. *Assume $MA_{\sigma-\mathcal{C}}$ holds and α, β are infinite regular cardinal numbers. Then for an (α,β)-gap $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ in ${}^\omega\omega$ either $\omega < \min\{\alpha,\beta\}$ or $2^{\aleph_0} = \max\{\alpha,\beta\}$.*

Proof. Consider Layers Interpolation Order $\mathbf{L}_{\alpha,\beta}$. We may assume, without loss of generality that $\alpha \leq \beta$.

If $\alpha = \omega$, then by Proposition 25 we obtain that $\mathbf{L}_{\omega,\beta}$ is σ-centered. For $k, l \in \omega$ and $\delta < \beta$ let $S_{k,l,\delta} = \{(X,Y,s,n) \in \mathbf{L}_{\omega,\beta} \mid k \in X,\ \delta \in Y,\ |s| \geq l\}$. Then every such set is dense open in $\mathbf{L}_{\omega,\beta}$, as we have seen in the proof of Claim 12 in the proof of Theorem 24, because the sets $S_{k,l,\delta}$ are precisely the intersection of the dense open sets B_γ, B_δ and A_k in the proof of the Claim, so this intersection is dense open itself.

By the definition of $\mathbf{L}_{\omega,\beta}$, if $\beta < 2^{\aleph_0}$, then there less than $2^{\aleph_0}$ such sets $S_{k,l,\delta}$. But then, by $\text{MA}_{\sigma-\mathcal{C}}$, we find a generic G that intersects all sets $S_{k,l,\delta}$. Defining

$$h = \bigcup_{(X,Y,s,n)\in G} s$$

then gives us an interpolating element of $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$.

Thus if $\alpha = \omega$ it must hold that $\beta < 2^{\aleph_0}$, which proves the proposition since our assumption $\alpha \leq \beta$ implies that if $\alpha > \omega$ also $\min\{\alpha,\beta\} > \omega$. □

Together with the following little proposition, we will be able to derive an interesting existence-result on gaps in $({}^\omega\omega, \prec)$ (and thus in $\mathcal{P}(\omega)/\text{Fin}$).

Proposition 73. *There exists an (ω,α)-Rothberger Gap for some uncountable cardinal α in $({}^\omega\omega, \prec)$.*

Proof. The proof will make use of Lemma of Zorn. We start with a increasing sequence of reals $(f_n)_{n\in\omega}$, i.e. $f_n \prec f_m$ for $n < m$. Now we aim to find a decreasing sequence $(g_\gamma)_{\gamma<\alpha}$ such that $(\{f_n\}_{n\in\omega}, \{g_\gamma\}_{\gamma<\alpha})$ is a gap. We are only interested in existence, so we use Lemma of Zorn.

Let $\mathcal{C} \subseteq \mathcal{P}({}^\omega\omega)$ be the collection of sets X such that X is linearly ordered by $\prec$ and we have $f_n \prec g$ for all $g \in X$, $n \in \omega$. We know that $\mathcal{C}$ is not empty, because if it were, we would obtain the $(\omega, 0)$-gap $(\{f_n\}_{n\in\omega}, \emptyset)$, a contradiction to Theorem 3.

Now we can define a partial order on $\mathcal{C}$ as follows: For $X, Y \in \mathcal{C}$ let $X < Y$ if and only if $X \subsetneq Y$ and $f \prec f'$ for all $f \in X, f' \in Y$. Note that every increasing chain in $(\mathcal{C}, <)$ has an upper bound, namely the union of the chain elements. This allows to apply Lemma of Zorn.

Thus we find a maximal element C of $\mathcal{C}$. Let α be the cofinality of C and order C with respect to $\prec$. Denote the elements of C by g_γ for $\gamma < \alpha$, where the index respects the ordering of C, i.e. $g_\gamma \prec g_\delta$ for $\gamma < \delta$. Then $(\{f_n\}_{n\in\omega}, \{g_\gamma\}_{\gamma<\alpha})$ is a gap, because otherwise C would not be maximal. Finally, Theorem 3 implies that α is uncountable. □

Note that the previous proposition did not make use of $\mathrm{MA}_{\sigma-\mathcal{C}}$. However, combining Proposition 72 and Proposition 73 immediately gives us:

Corollary 74. *Assume $MA_{\sigma-\mathcal{C}}$ holds. Then there exists an $(\omega, 2^{\aleph_0})$-Rothberger Gap in $({}^\omega\omega, \prec)$.*

Later it will be useful to have a more universal version of Proposition 73:

Proposition 75. *Consider an infinite regular cardinal number $\alpha < 2^{\aleph_0}$ and assume the following:*

1. *there is an increasing sequence of reals $(f_\alpha)_{\gamma<\alpha}$ that has an upper bound in $({}^\omega\omega, \prec)$,*

2. *for any $\xi < \alpha$ there is no (ξ, α)-gap in $({}^\omega\omega, \prec)$.*

Then there exists some regular cardinal $\beta \geq \alpha$ for which there is an (α, β)-gap.

Proof. In the proof of Proposition 73 we only used Theorem 3 two times. Now we can perform the exact same proof, but with α replacing ω and condition 1 and condition 2 replacing the use of Theorem 3 in the first and second place, respectively. □

It is very interesting that it is even possible to express $\mathrm{MA}_{\sigma-\mathcal{C}}$ as a statement on the existence of gaps (for a proof, consult [5], [15] and [3]):

Theorem 76. *Under the assumption that $2^{\aleph_0} = \aleph_2$, the two statements*

(i) $MA_{\sigma-\mathcal{C}}$,

(ii) There is no $(\omega_1, 1)$-gap in $\mathcal{P}(\omega)/Fin$,

are equivalent.

Now we switch our focus to $\mathrm{MA}_{\mathcal{K}_\forall}$. The assumption of $\mathrm{MA}_{\sigma-\mathcal{L}}$ does not imply interesting consequences on gaps in $({}^{\omega}\omega, \prec)$ so we leave it out for the moment. Under $\mathrm{MA}_{\mathcal{K}_\forall}$, there are certain requirements on asymmetric gaps, as shown by the following result (see [5]):

Theorem 77. *Assume $MA_{\mathcal{K}_\forall}$. Then for an (α, β)-gap $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ in ${}^{\omega}\omega$, the gap is either symmetric or $2^{\aleph_0} \in \{\alpha, \beta\}$.*

Proof. Without loss of generality assume $\alpha \leq \beta$. Consider Layers Interpolation Order $\mathbf{L}_{\alpha,\beta}$. Let $S_{\gamma,l,\delta} = \{(X, Y, s, n) \in \mathbf{L}_{\alpha,\beta} \mid \gamma \in X, \delta \in Y, \ |s| \geq l\}$, then, as in Proposition 72, all such sets are open dense for any $\gamma < \alpha, \ \delta < \beta, \ l \in \omega$.

Now if $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is asymmetric, then by Proposition 27 we obtain that $\mathbf{L}_{\alpha,\beta}$ is in $\mathcal{K}_\forall$. If we additionally assume $\beta < 2^{\aleph_0}$, we can use $\mathrm{MA}_{\mathcal{K}_\forall}$ to obtain a generic G that intersects all sets $S_{\gamma,l,\delta}$. We can then use G, as in Proposition 72, to obtain an interpolating element for the gap $((\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$, a contradiction. ☐

We end our short considerations about $\mathrm{MA}_{\mathcal{K}_\forall}$ with an Independence result. Therefore, recall that, by Proposition 72, under $\mathrm{MA}_{\mathcal{K}_\forall}$ only (α, β)-gaps can appear for which $\max\{\alpha, \beta\} \leq 2^{\aleph_0}$. By Theorem 77 we can thin out the space of gaps in $({}^{\omega}\omega, \prec)$ even further and are left with two possible types of gaps:

- symmetric (α, α)-gaps for $\omega < \alpha \leq 2^{\aleph_0}$,
- asymmetric $(\alpha, 2^{\aleph_0})$-gaps for $\omega \leq \alpha < 2^{\aleph_0}$.

Now we consult [5] once again and consider Theorems 85 and 87, which are summarized as follows:

Theorem 78. *Let β be a regular uncountable number.*

1. *There is a model of ZFC + $MA_{\mathcal{K}_\forall}$ for which*
   (a) $\beta < 2^{\aleph_0}$,
   (b) *there is an (α, α)-gap for any regular uncountable $\alpha \leq 2^{\aleph_0}$,*
   (c) *there is an $(\alpha, 2^{\aleph_0})$-gap for any regular uncountable $\alpha < 2^{\aleph_0}$.*
2. *There is a model of ZFC + $MA_{\mathcal{K}_\forall}$ for which*
   (a) $\beta < 2^{\aleph_0}$,
   (b) *there is no (α, α)-gap for any regular uncountable $\alpha \leq 2^{\aleph_0}$,*
   (c) *there is no $(\alpha, 2^{\aleph_0})$-gap for any regular uncountable $\alpha < 2^{\aleph_0}$.*

The statement of this theorem can be rephrased as follows:

Corollary 79. *The existence of (α, β)-gaps in $({}^{\omega}\omega, \prec)$ for regular uncountable α, β is independent from ZFC + $MA_{\mathcal{K}_\forall}$.*

Again a result found in [5], it is in particular interesting that the stronger version of MA, namely $\mathrm{MA}_{\mathcal{K}}$, has a big influence on the possible types of gaps:

Theorem 80. *Assume $MA_{\mathcal{K}}$ holds and α, β are infinite regular cardinal numbers. Then for an (α, β)-gap $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ either $\alpha = \beta = \omega_1$ or $2^{\aleph_0} \in \{\alpha, \beta\}$.*

Proof. The proof is very similar to those of Proposition 72 and Theorem 77, respectively.

We may assume that the statement of the theorem is false, i.e. assume we have a gap $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ at hand, such that it is not the case that $\alpha = \beta = \omega_1$ and $\alpha, \beta < 2^{\aleph_0}$. We use Corollary 30 to obtain that $\mathbf{L}_{\alpha,\beta}$ is Knaster. Then with $S_{\gamma,l,\delta}$ as in the proof of Theorem 77 we can proceed exactly as in the proof of Theorem 72, using $\mathrm{MA}_{\mathcal{K}}$, to derive a contradiction. □

Corollary 81. *Assume $MA_{\mathcal{K}}$. Then for any $\alpha < 2^{\aleph_0}$, $\alpha \neq \omega_1$, there exists an $(\alpha, 2^{\aleph_0})$-gap.*

Proof. We start proving the little

Claim 18. *Assume $MA_{\sigma-\mathcal{C}}$. Then every sequence of reals of length less than $2^{\aleph_0}$ has an upper bound.*

Proof. Let $\xi < 2^{\aleph_0}$ and $(f_\rho)_{\rho<\xi}$ be given. If it were not be bounded, then $(\{f_\rho\}_{\rho<\xi}, \emptyset)$ would be an $(\xi, 0)$-gap, which can not be the case by Proposition 72. □

Now we use the claim (since $\mathrm{MA}_{\mathcal{K}}$ is stronger than $\mathrm{MA}_{\sigma-\mathcal{C}}$) and with assumption $\alpha \neq \omega_1$ combined with Theorem 80 we obtain the two conditions we need to apply Proposition 75. This gives us a $\beta \geq \alpha$ for which there is an (α, β)-gap. Another use of Theorem 80 finishes the proof. □

Note that we can use Theorem 79 and Corollary 81 to distinguish $\mathrm{MA}_{\mathcal{K}}$ and $\mathrm{MA}_{\mathcal{K}_\forall}$, i.e. we have shown the following result, which is a priori completely independent of gaps:

Corollary 82. *$MA_{\mathcal{K}_\forall} \not\Rightarrow MA_{\mathcal{K}}$.*

We are now in the situation that the existence of gaps is not forbidden or required by some version of MA only for $(\omega_1, 2^{\aleph_0})$- and $(2^{\aleph_0}, 2^{\aleph_0})$-gaps. There is little influence of $\mathrm{MA}_{\mathcal{K}}$ on gaps of this kind, which will be clear after the considerations of MA_{ccc} and gaps.

Proposition 83. *Assume MA_{ccc}. If $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ is an (ω_1, ω_1)-gap, then it is equivalent to a Special Gap.*

Proof. Suppose $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ is an (ω_1, ω_1)-gap and consider the corresponding partially ordered set $\mathbf{S}_{\omega_1,\omega_1}$ from Definition 16. We can use Theorem 40 and obtain that $\mathbf{S}_{\omega_1,\omega_1}$ is ccc. In the remark after Theorem 41 we have shown that the sets $D_\gamma = \{S \in \mathbf{S}_{\omega_1,\omega_1} \mid \gamma \in I_S\}$ are dense open. Now we can apply MA_{ccc} to $\{D_\gamma\}_{\gamma<\omega_1}$ and get a filter G that intersects with each of the D_γ's.

But then we find the desired Special Gap as follows: For $\gamma < \omega_1$ pick $S_\gamma \in G \cap D_\gamma$. Since $\gamma \in I_{S_\gamma}$, we can find $(a_\gamma, b_\gamma) \in [(f_\gamma, g_\gamma)]$ in S_γ. Then by the definition of $\mathbf{S}_{\omega_1,\omega_1}$ the gap $(\{a_\gamma\}_{\gamma<\omega_1}, \{b_\gamma\}_{\gamma<\omega_1})$ is equivalent to $(\{f_\gamma\}_{\gamma<\omega_1}, \{g_\delta\}_{\delta<\omega_1})$ and is a Special Gap. □

What was left undecided by $\mathrm{MA}_{\mathcal{K}}$ is the question on the existence of $(\omega_1, 2^{\aleph_0})$- and $(2^{\aleph_0}, 2^{\aleph_0})$-gaps. This is independent of MA_{ccc} (and so in particular also independent form $\mathrm{MA}_{\mathcal{K}}$). The proof of the following can be found in [5].

Theorem 84. *For both of the two statements below there is a model for ZFC + MA_{ccc} in which the respective statement holds.*

1. *There exists no $(\omega_1, 2^{\aleph_0})$-gap and no $(2^{\aleph_0}, 2^{\aleph_0})$-gap in $({}^{\omega}\omega, \prec)$,*
2. *There exists an $(\omega_1, 2^{\aleph_0})$-gap and a $(2^{\aleph_0}, 2^{\aleph_0})$-gap in $({}^{\omega}\omega, \prec)$.*

Thus the existence of $(\omega_1, 2^{\aleph_0})$-gaps and $(2^{\aleph_0}, 2^{\aleph_0})$-gaps is independent from MA_{ccc}.

We will now end our considerations on gaps and versions of MA and switch the focus to another axiom.

5.3 Gaps and OCA/PFA

In this section we consider the influence of the *Open Coloring Axiom* - abbreviated OCA - and the *Proper Forcing Axiom* - or abbreviated PFA - and their effects on gaps in ${}^{\omega}\omega$.

5.3.1 Gaps and OCA

We start with OCA. The Open Coloring Axiom in the version we consider was introduced in [14] by Todorcevic. In fact, many statements in this section are originally due to him.

The statement OCA requires some topology on the real numbers. We will consider the standard topology on the reals, i.e. the topology that has the open intervals as a basis. Then we obtain the standard topology on $\mathbb{R}^2$ by taking the product topology. A topology on $[\mathbb{R}]^2$ - the collection of two-elementary subsets of $\mathbb{R}$ - arises in a natural way: We can identify every two-elementary subset $\{x, y\} \in [\mathbb{R}]^2$ with (x, y) if $x < y$ or (y, x) if $y < x$. So any $M \subseteq [\mathbb{R}]^2$ is open if and only if the corresponding set in $\mathbb{R}^2$ is open.

The following definition is due to Todorcevic [14]:

Definition 34 (OCA)**.** The Open Coloring Axiom is the following statement:
Let $X \subseteq \mathbb{R}$. Then for each partition $[X]^2 = P_1 \cup P_2$ for which P_1 is open in $[X]^2$, exactly one of the following holds:

1. *there exists an uncountable subset $Y \subseteq X$ such that $[Y]^2 \subseteq P_1$,*
2. *there are $J_i \subseteq X$, $i \geq 1$, for which $X = \bigcup_{i=1}^{\infty} J_i$ and $[J_i]^2 \subseteq P_2$ for all $i \geq 1$.*

In order to apply OCA in the light of gaps, we have the below crucial connection (Proposition 86, for which we only give a sketchy proof. For what follows, endow ω with the discrete topology, i.e. every subset is open, and endow ${}^{\omega}\omega$ with the corresponding Tychonoff product topology. Then in ${}^{\omega}\omega$ the basic open sets are sets of sequences of natural numbers, were one element in the sequence is fixed and the other elements vary over ω; such sets are of the form $\omega \times ... \times \omega \times \{n\} \times \omega \times ...$ for some natural n. This in particular also means that sets of the form $[n_1] \times [n_2] \times ... \times [n_m] \times \omega \times ...$ are open, where $[n_i] = \{1, 2, ..., n_i\}$. Note that these sets also form a basis for the topology on ${}^{\omega}\omega$.

Proposition 85. *The topological space ${}^{\omega}\omega$ is hereditarily separable.*

Proof. We aim to show that for each subset $X \subseteq {}^{\omega}\omega$ there exists a countable dense subset of X. Note that if we have an uncountable subset $X \subseteq {}^{\omega}\omega$, the initial segments (which are finite) of all elements of X are countably many. Take for each such initial segment one element of X, then we obtain a countable dense subset in X. If X is not uncountable, the statement is trivially true. □

Proposition 86. *The space ${}^{\omega}\omega$ is homeomorphic to the irrational numbers with the inherited topology from $\mathbb{R}$.*

Sketch of proof. The idea is to use continued fractions to identify sequences of naturals with irrational numbers. It is a well-known fact from number theory that we can represent any irrational number as a continued fraction expansion (see for example [16], Theorem 170).

Conversely, any continued fraction is irrational. So see this, consider a rational number r. Then write $r = r_0 + \frac{1}{r'}$ for some rational r'. Then the denominator of r' is smaller in absolute value than the denominator of r. But the denominator of r is of finite integer, so r is no infinite continued fraction.

So in a natural way we have a isomorphic function from ${}^{\omega}\omega$ to the positive irrationals, which we may, for the sake of this proof, denote by $\mathbb{I}$. Namely, the mapping ${}^{\omega}\omega \to \mathbb{I}$ can be stated as follows:

$$(n_1, n_2, n_3, n_4, ...) \mapsto n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \dots}}}$$

Now given a basic open set in $\mathbb{I}$, we know it is of the form $(a, b) \cap \mathbb{I}$ for real numbers a, b. But then the preimage of $(a, b) \cap \mathbb{I}$ under the mapping defined above is of the form $[n_1] \times [n_2] \times ... \times [n_m] \times \omega \times ...$, thus open.

Similarly, the image of any set of the form $[n_1] \times [n_2] \times ... \times [n_m] \times \omega \times ...$ is a bounded set of irrationals, thus open in $\mathbb{I}$.

This shows that the positive irrationals are homeomorphic to ${}^{\omega}\omega$. Since we can split up ${}^{\omega}\omega$ into two parts in a homeomorphic way, e.g. by taking the odd and even elements of some sequence in ${}^{\omega}\omega$, we obtain the desired result. □

Now we are ready to use OCA when considering gaps in ${}^{\omega}\omega$. OCA is strong enough to forbid the existence of many type of gaps in ${}^{\omega}\omega$. In fact, the only allowed gaps are certain Rothberger Gaps. This result is due to Todorcevic [14].

Theorem 87. *Assume OCA. For any two regular uncountable cardinal numbers $\alpha \leq \beta$ such that $\beta > \omega_1$ there is no (α, β)-gap in $({}^{\omega}\omega, \prec)$.*

Proof. Let $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ be a pregap for α, β as in the statement of the theorem. We will show that it is not a gap.

To be able to apply OCA, we define a partition.

For each $\gamma < \alpha$ we know that $f_\gamma \prec g_\delta$ for any $\delta < \beta$. For each such δ there is an n_δ^γ such that $f_\gamma(l) < g_\delta(l)$ for all $l > n_\delta^\gamma$. Because β is uncountable, there is a minimal n_γ such that $n_\delta^\gamma = n_\gamma$ for uncountably many $\delta < \beta$; we denote the set of such δ by I_γ. Then I_γ is actually cofinal in β. By the usual argument on cardinalities, we may assume that $n_{\gamma_1} = n_{\gamma_2} =: n$ for $\gamma_1, \gamma_2 < \alpha$.

Now we define the set where we want to use OCA to be $X = \{(f_\gamma, g_\delta) \mid \gamma < \alpha,\ \delta \in I_\gamma\}$. The product ${}^\omega\omega \times {}^\omega\omega$ is homeomorphic to ${}^\omega\omega$, so we can identify X with some subset of ${}^\omega\omega$ and can use OCA on it. To do so, we need a partition $[X]^2 = P_1 \cup P_2$, where P_1 is open.

Let P_1 be the set $\{\{(f_\gamma, g_\delta), (f_\xi, g_\rho)\} \mid \exists m > n \colon f_\gamma(m) > g_\rho(m)$ or $f_\xi(m) > g_\delta(m)\}$.

We show that P_1 is open: Consider some $x = \{(f_\gamma, g_\delta), (f_\xi, g_\rho)\}$ in P_1. By definition of P_1 we can find $m > n$ for which $f_\gamma(m) > g_\rho(m)$ or $f_\xi(m) > g_\delta(m)$. But then we can consider the initial segments $f_\gamma \restriction m$, $g_\delta \restriction m$, $f_\xi \restriction m$ and $g_\rho \restriction m$. Then the set $O \subset P_1$ of all 2-elementary subsets of pairs of reals that extend these initial segments is an open neighbourhood of x, what implies that P_1 is open.

This allows to apply OCA to $X = P_1 \cup P_2$. This means that one the follow must hold:

1. there exists an uncountable subset $Y \subseteq X$ such that $[Y]^2 \subseteq P_1$,
2. there are $J_i \subseteq X$, $i \geq 1$, for which $X = \bigcup_{i=1}^{\infty} J_i$ and $[J_i]^2 \subseteq P_2$ for all $i \geq 1$.

We show that 1. is not possible:

Assume there would be such a $Y \subseteq X$. Suppose $Y = \{(f^y_\gamma, g^y_\gamma)\}_{\gamma\in\omega_1}$ is listed in a way such that $f^y_\gamma \prec f^y_\delta$ and $g^y_\delta \prec g^y_\gamma$ whenever $\gamma < \delta$. The reason that this is possible is that for two elements in Y, (f^y_γ, g^y_γ) and (f^y_δ, g^y_δ), we have that $f^y_\gamma \neq f^y_\delta$ and $g^y_\gamma \neq g^y_\delta$. This follows because $\{(f^y_\gamma, g^y_\gamma), (f^y_\delta, g^y_\delta)\} \in P_1$ and if $f^y_\gamma = f^y_\delta$ or $g^y_\gamma = g^y_\delta$ this cannot be the case by definition of P_1 (and in particular the way we have chosen n). Then we obtain that $(\{f^y_\gamma\}_{\gamma<\omega_1}, \{g^y_\delta\}_{\delta<\omega_1})$ is an (ω_1, ω_1)-pregap. Now we use the assumption that $\beta > \omega_1$ to find a f_ξ for which $f^y_\gamma \prec f_\xi$ for all $\gamma < \omega_1$, i.e. this f_ξ interpolates $(\{f^y_\gamma\}_{\gamma<\omega_1}, \{g^y_\delta\}_{\delta<\omega_1})$. So for all $\gamma < \omega_1$ we can find a minimal natural number i_γ for which $f^y_\gamma(j) < f_\xi(j) < g^y_\gamma(j)$ for all $j > i_\gamma$. Since this holds for all $\gamma \in \omega_1$, we find an uncountable subset of ω_1 such that the respective i_γ's are equal for each of it's elements; we may denote that number by i^*. Then by the minimality of the number n we know that $n \leq i^*$. This means that for all $j > i^*$ we have $f^y_\gamma(j) < g^y_\delta(j)$ for all $\gamma, \delta \in \omega_1$.

Now we consider the sets of initial segments of the form $f^y_\gamma \restriction i^*$ and $g^y_\gamma \restriction i^*$. There are uncountably many of them and each is a finite sequence of naturals; thus there are at most countably many different such initial segments. This means that we can find uncountably many elements $\{(f^{y*}_\gamma, g^{y*}_\gamma), (f^{y*}_\delta, g^{y*}_\delta)\}$ in Y for which $f^{y*}_\gamma \restriction i^* = f^{y*}_\delta \restriction i^*$ and $g^{y*}_\gamma \restriction i^* = g^{y*}_\delta \restriction i^*$ for any such $\gamma, \delta \in I$, where I is some uncountable index-set. But then no pair $\{(f^{y*}_\gamma, g^{y*}_\gamma), (f^{y*}_\delta, g^{y*}_\delta)\}$ can be in P_1: For l such that $n \leq l < i^*$ it holds that $f^{y*}_\gamma(l) = f^{y*}_\delta(l) < g^{y*}_\delta(l) = g^{y*}_\gamma$. For $l > i^*$ we even know $f^y_\gamma(j) < g^y_\delta(j)$ for all $\gamma, \delta \in \omega_1$ and all elements of Y. But then no pair $\{(f^{y*}_\gamma, g^{y*}_\gamma), (f^{y*}_\delta, g^{y*}_\delta)\}$ satisfies the condition to be in P_1.

Thus we know that condition 2. must hold. Let $X = \bigcup_{i=1}^\infty J_i$ and $[J_i]^2 \subseteq P_2$.

We show that the pregap $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$ is not a gap in this case:

For that we fix an $\gamma < \alpha$. Consider I_γ as in the definition of X and for each $\delta \in I_\gamma$ chose some index $k^\gamma_\delta \in \omega$ such that $(f_\gamma, g_\delta) \in J_{k^\gamma_\delta}$. We can find k^γ_δ due to the assumption that 2. holds. Because I_γ is uncountable, there exists an uncountable set $I^*_\gamma \subseteq I_\gamma$ for which $k^\gamma_\delta = k^\gamma_\xi =: k_\gamma$ for $\delta, \xi \in I^*_\gamma$. We can find I^*_γ such that it is cofinal in β. Further, there is a cofinal $A \subseteq \alpha$ such that $k_\gamma = k_\rho =: k$ for any $\gamma, \rho \in A$.

Now for two elements $\gamma, \rho \in A$ we note the following: If $\delta \in I^*_\rho$, then $f_\gamma(l) < g_\delta(l)$ for all $l > n$. This is because otherwise we would obtain a contradiction to $\{(f_\gamma, g_\xi), (f_\rho, g_\delta)\} \notin P_1$ for some ξ for which $(f_\gamma, g_\xi), (f_\rho, g_\delta) \in J_k$.

Finally, for a fixed $\gamma \in A$ we define a function

$$s(l) = \begin{cases} 1 & \text{if } l \leq n \\ \min\{g_\delta(l) \mid \delta \in I^*_\gamma\} & \text{otherwise} \end{cases}$$

Then, since I^*_γ is cofinal, the function s interpolates the pregap $(\{f_\gamma\}_{\gamma<\alpha}, \{g_\delta\}_{\delta<\beta})$, what proves the theorem. □

The next result has a priori nothing to do with gaps, still it is very interesting and will be immediately useful (see [14] or [5]):

Theorem 88. *Assume OCA. Then the minimal cardinality of an unbounded subset of* $({}^\omega\omega, \prec)$ *is greater than* $\aleph_1$.

Proof. Consider a subset of ${}^{\omega}\omega$, $S = \{f_\gamma\}_{\gamma<\omega_1}$. Without loss of generality we may assume that the set is ordered, i.e. $f_\gamma \prec f_\delta$ for any $\gamma < \delta$ and further that each f_γ is increasing.

We can define a partition $[S]^2 = P_1 \cup P_2$ as follows: We let $\{f_\gamma, f_\delta\} \in P_1$ if and only if $\gamma < \delta$ and there exists a $m \in \omega$ for which $f_\gamma(m) > f_\delta(m)$. As described earlier, we can map $[S]^2$ in a unique to the set of pairs $S^2 = \{(f_\gamma, f_\delta) \mid \{f_\gamma, f_\delta\} \in [S]^2 \text{ and } \gamma < \delta\}$. We do similarly with P_1 and P_2, but will not explicitly denote this (since it is clear from context which object we refer to).

Consider $(f_\gamma, f_\delta) \in P_1$. Then by definition of P_1 we know that we find some $m \in \omega$ such that $f_\gamma(m) > f_\delta(m)$. Now let $f_\gamma \upharpoonright m =: r_\gamma$ and $f_\delta \upharpoonright m =: r_\delta$ denote the first m elements of the sequences f_γ and f_δ, respectively. Put $O = \{(f, g) \in S^2 \mid f \upharpoonright m = r_\gamma \text{ and } g \upharpoonright m = r_\delta\}$. But then $O \subseteq P_1$, O is open and $(f_\gamma, f_\delta) \in O$. Thus P_1 is open.

Now we can apply OCA. It is impossible to write $S = \bigcup_{i=1}^{\infty} J_i$ for $[J_i]^2 \subseteq P_2$. Because if it would be, there would be an index j such that J_j is uncountable. Then for all $f_\gamma, f_\delta \in J_j$ for which $\gamma < \delta$, we have $f_\gamma(n) \leq f_\delta(n)$, because otherwise we obtain a contradiction to $[J_j]^2 \subseteq P_2$. Now for each $\gamma < \omega_1$, let $X_\gamma = \{(n, m) \mid m \leq f_\gamma(n)\} \subset \omega \times \omega$. So the X_γ are basically the "values below the graph of f_γ". Since $f_\gamma \prec f_\delta$ whenever $\gamma < \delta$, we obtain $X_\gamma \subsetneq X_\delta \subsetneq \omega \times \omega$. So the collection $\{X_\gamma\}_{f_\gamma \in J_j}$ would be an uncountable strictly increasing sequence in $\omega \times \omega$ and this is impossible.

So it must be the case that there exists an uncountable $S' \subseteq S$ that satisfies $[S']^2 \subseteq P_1$.

We now need a little side-result, which can be proven using topological properties of $({}^{\omega}\omega, \prec)$:

Claim 19. *Let $\{g_\gamma\}_{\gamma<\omega_1}$ be unbounded and assume $g_\gamma \prec g_\delta$ and $g_\gamma(n) \leq g_\gamma(n+1)$ for any $\gamma < \delta < \omega_1$. Then there are $\alpha < \beta < \omega_1$ for which $g_\alpha(n) \leq g_\beta(n)$ for all natural numbers n.*

Proof. Because ${}^{\omega}\omega$ is hereditarily separable by Proposition 85, we find a dense countable subset $\{g_{\gamma_n}\}_{n\in\omega}$. Now we pick a ξ that is bigger than all γ_n.

Since $g_\xi \prec g_\gamma$ for all $\gamma > \xi$, there is n_γ such that $g_\xi(k) < g_\gamma(k)$ for $k > n_\gamma$. Without loss of generality we can assume that $n_\gamma = n_\delta =: n$ for

$\gamma, \delta > \xi$.

Now we use the fact that $\{g_\gamma\}_{\gamma<\omega_1}$ is unbounded and obtain that there must be a minimal $m > n$ such that the set of values $\{g_\gamma(m) \mid \gamma > \xi\}$ is unbounded in ω. Further assume that the initial segments of length m coincide for all g_γ with $\gamma > \xi$, i.e. $g_\gamma \restriction m = g_\delta \restriction m := s$ for $\xi < \gamma, \delta < \omega_1$.

Using that $\{g_{\gamma_n}\}_{n\in\omega}$ is dense, we find $l \in \omega$ such that $s \subset g_{\gamma_l}$. This holds, since for $\gamma > \xi$ we have that $g_\gamma \in s \times \prod_{i\in\omega} \omega$ and $s \times \prod_{i\in\omega} \omega$ is open, so there must be some element of the dense set in the open set containing g_γ.

Because $\xi > \gamma_l$ we find $h > m$ for which $g_{\gamma_l}(k) < g_\xi(k)$ for $k > h$.

Now fix some ρ such that $\xi < \rho < \omega_1$ and such that $g_\rho(m) > \max\{f_{\gamma_l}(k) \mid k \leq h\} = z$. This is possible because $\{g_\gamma(m) \mid \gamma > \xi\}$ is unbounded.

Then $g_\rho \restriction m = g_{\gamma_l} \restriction m$, i.e. $g_\rho(k) = g_{\gamma_l}(k)$ for $k < m$. Further, from the fact that g_ρ is increasing and $g_\rho(m) > z$, we obtain that $g_\rho(k) > g_{\gamma_l}(k)$ for $m \leq k \leq h$. But for $k > h$ (so also $k > n$), we know that $g_\rho(k) > g_\xi(k) > g_{\gamma_l}(k)$. So we set $\alpha = \gamma_l$ and $\beta = \rho$ and obtain the claim. □

The claim can be used to see that S' must be bounded. Otherwise, we would have a contradiction to the claim, since S' is of size ω_1 and the statement of the claim does not hold for S'. □

As an immediate corollary we obtain that OCA forbids some Rothberger Gaps:

Corollary 89. *Assume OCA. Then there exists no (ω, ω_1)-Rothberger Gaps.*

Proof. If the corollary was false, by Theorem 17 we obtain that there would be an $(\omega_1, 0)$-Rothberger Gap. But this contradicts Theorem 88. □

The following result by Scheepers [5] will lead to an interesting Corollary on cardinal characteristics:

Theorem 90. *Assume OCA. Then there exists an (ω, ω_2)-Rothberger Gap in $({}^\omega\omega, \prec)$.*

Proof. By Theorem 88 the minimal cardinality of an unbounded family in $({}^\omega\omega, \prec)$ is at least $\aleph_2$. We distinguish two cases: If there exists an unbounded family of size $\aleph_2$ in $({}^\omega\omega, \prec)$ or not.

If there is one, we obtain an $(\omega_2, 0)$-gap. Thus we can use Theorem 17 to obtain the statement of the theorem.

If now there is no unbounded family of size $\aleph_2$, we can apply Proposition 75 for $\alpha = \aleph_2$: Condition 1. in the proposition is fulfilled since we have no unbounded $\aleph_2$-sized sequence. Condition 2. of Proposition 75 states that there are not (ξ, α)-gaps for $\xi < \alpha$. By Theorem 87, this is true for $\xi = \aleph_1$. If this was not true for $\xi = \aleph_0$, there would be an (ω, ω_2)-gap, so the statement of the theorem holds. So assume that there is no such gap. Then the proposition gives an (β, α)-gap for $\beta \geq \alpha$. But this is impossible by Theorem 87. □

Corollary 91. *Assume OCA. Then* $\mathfrak{b} = \aleph_2$.

Proof. By Theorem 90 there is an (ω, ω_2)-Rothberger Gap in $({}^{\omega}\omega, \prec)$. Then by Theorem 17, there is an $(\omega_2, 0)$-gap. So Theorem 88 gives the statement of the corollary. □

5.3.2 Gaps and PFA

The last axiom we consider is the well known Proper Forcing Axiom - PFA. We will not explicitly consider it in connection with gaps, since we already established many some interesting results. This is because PFA implies OCA, what gives us all the statements we have shown in Section 5.3.1. For the sake of completness, we state PFA:

Definition 35 (PFA)**.** The Proper Forcing Axiom is the following statement:
Let $\mathbb{P}$ *be a proper poset and let* $\{D_\gamma\}_{\gamma<\omega_1}$ *be a family of dense sets in* $\mathbb{P}$. *Then there exists a filter that has non-empty intersection with each* D_γ.

As stated above, the crucial result is:

Theorem 92 (Todorcevic)**.** *PFA* $\implies$ *OCA.*

The prove is rather technical and has nothing to do with gaps, so we will not prove the statement here. However, a proof can be found in [14].

Remark. Theorem 92 immediately implies that all results we established in Section 5.3.1 still hold in presence of PFA.

We summarise the most important results:

Corollary 93. *Assume PFA. Then each of the following statements holds:*

1. *For any two regular uncountable cardinal numbers $\alpha \leq \beta$ such that $\beta > \omega_1$ there is no (α, β)-gap in $({}^\omega\omega, \prec)$.*

2. *There exists no (ω, ω_1)-Rothberger Gaps.*

3. *There exists an (ω, ω_2)-Rothberger Gap in $({}^\omega\omega, \prec)$.*

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