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„Spin-3-Gravity in Three-Dimensional Flat Space"

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Abstract

The theory of spin-3-gravity on 3d Minkowski-space is analysed in the metric-like formalism. Starting from the cubic interaction vertices classified by K. Mkrtchyan in 2018, the first-order corrections to the gauge transformations induced by these vertices are calculated. Due to the existence of dimension-dependent-identities (DDI's) the construction of higher-spin theories in three dimensions differs significantly from the higher-dimensional case. The Noether-procedure hence needs to be adapted to account for the influence of DDI's. Making use of the first-order gauge deformations, the gauge brackets emerging from the non-abelian deformations are identified.

By imposing the Jacobi-identity for the gauge brackets and demanding closure of the extended gauge algebra the coupling constants of spin-3-gravity are fixed and the consistency of the theory is confirmed at the global symmetry level. Furthermore, the structure constants of the global symmetry algebra are determined.

After the consistency of spin-3-gravity has been verified and most free parameters have been fixed (up to a global factor), the coupling of matter in the form of massive scalar and Maxwell fields to the theory is investigated. In accordance with Prokushkin-Vasiliev-theory it is shown that once matter is introduced to the theory an infinite tower of higher-spin fields is needed in order to close the gauge algebra.

Zusammenfassung

In dieser Arbeit wird die Theorie der Spin-3-Gravitation in dreidimensionaler Minkowski-Raumzeit im metrischen Formalismus untersucht. Aufgrund der Existenz von sogenannten dimension-dependent-identities (DDI‘s) unterscheidet sich die Konstruktion von higher-spin-Theorien in drei Raumzeit-Dimensionen erheblich von den höherdimensionalen Fällen. Um den Einfluss dieser DDI’s zu berücksichtigen, muss die Noether-Prozedur für den vorliegenden Fall adaptiert werden. Als Ausgangspunkt für die Konstruktion der Theorie dienen die kubischen Vertizes, welche von K. Mkrtchyan im Jahr 2018 klassifiziert worden sind. Die von diesen Vertizes induzierten Deformationen der Eichtransformationen werden zur ersten Ordnung ermittelt. Anschließend werden die Kommutatoren dieser Eichdeformationen berechnet und die daraus resultierenden Eichklammern bestimmt.

Durch die Forderung, dass die Eichklammern die Jacobi-Identität erfüllen und die erweiterte Eichalgebra abgeschlossen ist, werden die Kopplungskonstanten der Spin-3-Gravitation fixiert und die kubische Konsistenz der Theorie auf dem Level der globalen Symmetrie-Algebra demonstriert. Des Weiteren werden die Strukturkonstanten dieser Algebra ermittelt.

Nachdem die Konsistenz der Spin-3-Gravitation gezeigt und der Großteil der freien Parameter fixiert ist, wird die Kopplung von Materie in Form von massiven Skalarfeldern und Maxwell-Feldern an die Theorie untersucht. In Übereinstimmung mit den Erkenntnissen aus der Prokushkin-Vasiliev-Theorie wird gezeigt, dass für die konsistente Kopplung von Materie ein unendlicher Turm von higher-spin-Feldern benötigt wird, um die Abgeschlossenheit der Eichalgebra zu gewährleisten.

Contents

Introduction

Since the first investigations of fields with spin greater than two in the first half of the 20th century [1], the subject of higher-spin theories has consistently attracted attention. One reason for this is that higher-spin-gravity theories (i.e. theories where the higher-spin fields are coupled to the gravitational field) are potential candidates for a consistent description of quantum gravity. Furthermore, such theories are expected to appear in the tensionless (high-energy) limit of string theory, so the study of higher-spin theories could lead to a better understanding of string theory itself.

The interest in these theories intesified at the beginning of the new millenium when Klebanov and Polyakov proposed a holographic duality relation between higher-spin theories on d-dimensional anti-deSitter-space (AdS) and a $(d-1)$-dimensional conformal field theory (CFT) on the boundary [2], resembling the string-theoretic AdS/CFT-duality conjectured by Maldacena in 1998 [3]. Since higher-spin theories are in general easier to analyse than string theory the higher-spin duality relation allows for the study of holographic principles in a more managable setting [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

The AdS/CFT duality becomes especially strong when the dual CFT is two-dimensional. In this case, the conformal symmetry algebra becomes an infinite-dimensional $\mathcal{W}$-algebra [17], whose powerful symmetry allows for many exact results. For this reason, three-dimensional higher-spin theories are particularly interesting. The higher-spin AdS_3/CFT_2-duality was first analyzed in [18], also see [19] for a review.

Over the years different formalisms in which higher-spin theories can be constructed have been developed. The metric-like formalism [20, 21, 22, 23, 24], based on the work of Fronsdal [25, 26] uses symmetric tensor fields as basic objects, in analogy to the standard formulation of general relativity, which is based on the metric tensor. In this formalism the action is constructed perturbatively in powers of the fields, making use of the so-called Noether-procedure [27]. While this process is quite tedious, it has some important advantages. First, if one wishes to construct a quantum theory of higher-spin fields, the metric-like action is best suited for quantization. Second, in contrast to any other formalism, the coupling of matter to the theory can be done in a straightforward way [28, 29, 30, 31, 27, 32, 33].

Starting from the free Fronsdal-Lagrangian for higher-spin fields [25], the next step of the Nother-procedure consists of the construction of cubic interaction vertices. These vertices contain the first non-trivial interaction terms between different fields. Even though cubic vertices are well-understood by now, much less is known about quartic or even higher-order vertices [34, 35, 36, 37, 38].

Cubic vertices in flat space have first been classified in the lightcone-gauge [39, 40, 41, 42, 43, 44, 45] for $d = 4$ [46] and in arbitrary spacetime dimension [47], then in the covariant form [48]. For AdS_d the cubic vertices have been classified in [49].

In three dimensions, little progress was made compared to higher dimensions, until recently the 3d cubic vertices have been classified by Mkrtchyan [50, 51]. Due to the huge influence of dimension-dependent-identities (DDI's) the construction of higher-spin-theories in three dimensions differs substantially from the higher-dimensional case.

In the same fashion as three dimensional general relativity [52], higher-spin gravity theories in three dimensions can also be formulated as a Chern-Simons (CS)-theory [53]. This description uses the so-called frame-like language [54, 55, 56, 57], in which the basic objects are the generalized vielbein and the spin-connection. The Lagrangian then consists of connection one-forms built out of these objects. An advantage of the CS-formulation is that it is very easy to analyze the gauge sector of the underlying theory. However, the coupling of matter turns out to be quite difficult, although some progress was made [58, 59, 60, 61]. In this setting, it is well-known how to construct spin-s-gravity theories [57] with symmetry algebra $\mathrm{sl}(s) \bigoplus \mathrm{sl}(s)$ (see [52] for the case of spin-3-gravity). Even though these theories can be described both in the frame-like and in the metric-like setting, not much is known about the relation of these formulations. A first step towards the connection of the two pictures was done in [62, 63], where the metric-like expressions for the Lagrangian and the gauge transformations have been extracted from the CS-theory up to a certain order. In [64] also the inverse direction was investigated.

Last, there is also the BRST-approach [65, 66, 67] coming from string-theory, which will not be relevant for this work.

When facing the task of constructing consistent higher-spin-gravity theories one also has to take various no-go results into consideration (see [68] for a review).

A prominent example would be the Aragone-Deser-problem [69], which constitutes of the fact that in Minkowski space of dimension greater than three higher-spin fields cannot couple to gravity in a consistent way. However, in three dimensions the no-go result of Aragone and Deser does not apply, as was shown by the same authors [70]. This is due to the fact that for $d = 3$ the Weyl-tensor vanishes and the problematic terms that lead to inconsistencies in higher dimensions can be canceled by appropriate counterterms. In fact, none of the known no-go results apply to three dimensions, even in the flat space case [71].

Another way to bypass the Aragone-Deser-problem is to switch from flat space to AdS. This consideration has led to the construction of Vasiliev-theory [72, 73, 74, 75, 76, 77, 78, 79], first for $d = 4$ in 1990 [80] and then for arbitrary spacetime dimension [81] (see [82] for a review). Vasiliev-theory (in 3d Prokushkin-Vasiliev-theory) is to date the only known example of a fully interacting consistent theory involving higher-spin fields together with matter. It consists of an infinite tower of higher-spin fields, one for each spin value, as well as some massive complex scalar fields. The global symmetry algebra of the theory is the higher-spin algebra hs(λ) [83] (also see [84] for a detailed analysis of higher-spin algebras) and the mass of the scalar fields depends on the parameter λ.

Although Vasiliev-theory is known for more than 30 years, there is still no standard Lagrangian description available. Instead, the theory is formulated in the so-called "unfolded-formalism" on the level of equations of motion. This circumstance makes it very difficult to extract information from the theory, although some steps in this direction have been done in [85, 86, 87]. However, it seems that for a full understanding of the theory a Lagrangian description is indispensable.

In this work we take the next step towards such a Lagrangian description of Prokushkin-Vasiliev theory: the construction of spin-3-gravity at the cubic level on three-dimensional Minkowski-space in the metric-like formalism. We work in flat space rather than AdS because the Noether-procedure is much easier to perform. As already mentioned, none of the various no-go results apply in this case. Also, as long as we do not include matter into the theory, the higher-spin algebra hs(λ) can be truncated to sl(3) $\bigoplus$ sl(3), so spin-3-gravity is expected to be consistent by itself. Once all the tools necessary to construct spin-3-gravity are developed, they can in principle be applied to spin-s-gravity for arbitrary s. In a final step, one can try to incorporate matter to the theory, resulting in a metric-like formulation of Prokushkin-Vasiliev-theory.

The outline of this thesis is as follows: chapter 1 introduces the notation and conventions used. In chapter 2 the cubic vertices for 3d flat space that have been classified in [50] are presented. Making use of these vertices, chapters 3 and 4 deal with the calculation of the first-order gauge deformations and the induced gauge brackets for spin-3-gravity, respectively. To do this, the method of [22] is generalised to three dimensions. In chapter 5 the consistency of the theory is verified by explicitly checking various consistency conditions. Also the coupling constants are fixed up to a global factor. Chapter 6 briefly discusses the structure constants of the global symmetry algebra. Finally, the coupling of matter in the form of massive scalar fields and Maxwell fields to the theory is investigated in chapter 7 and the need for an infinite higher-spin tower is recovered. We end with a discussion of our results and an outline of future research in chapter 8.

1 Notation and Definitions

Fields and gauge parameters

In this work, we will only deal with parity-even fields of integer spin. In the Fronsdal formulation, which will be discussed in the next section, such fields are described as fully symmetric tensors with rank equal to their respective spin. For these tensors it is convenient to introduce an index-free notation, where all indices are contracted with auxiliary vectors a_i

$$\Phi_i^{(s)}(x_i, a_i) \coloneqq \frac{1}{s!}\Phi^{(s)}_{i,\mu_1\ldots\mu_s}(x_i) a_i^{\mu_1}\ldots a_i^{\mu_s}. \tag{1.1}$$

Here i labels the different fields. All relevant operations on these fields can now be expressed as contractions of the a_i and a few partial derivatives

$$\nabla_{i,\mu} \coloneqq \frac{\partial}{\partial x_i^\mu}, \qquad \partial_{i,\mu} \coloneqq \frac{\partial}{\partial a_i^\mu}. \tag{1.2}$$

For example, the operations of gradient, divergence and trace can now be written as

$$Grad\ \Phi_i^{(s)}(x_i, a_i) \coloneqq (a_i \cdot \nabla_i)\Phi_i^{(s)}(x_i, a_i), \tag{1.3}$$

$$Div\ \Phi_i^{(s)}(x_i, a_i) \coloneqq (\partial_i \cdot \nabla_i)\Phi_i^{(s)}(x_i, a_i), \tag{1.4}$$

$$Tr\ \Phi_i^{(s)}(x_i, a_i) \coloneqq (\partial_i \cdot \partial_i)\Phi_i^{(s)}(x_i, a_i). \tag{1.5}$$

A gauge parameter of a massless spin-s field can be expressed in the same index-free way as the field

$$\epsilon_i^{(s-1)}(x_i, a_i) \coloneqq \frac{1}{(s-1)!}\epsilon^{(s-1)}_{i,\mu_1\ldots\mu_{s-1}}(x_i) a_i^{\mu_1}\ldots a_i^{\mu_{s-1}}. \tag{1.6}$$

For the sake of simplicity, the superscripts of the fields and parameters that indicate their respective spin, as well as their arguments, will be suppressed most of the time.

Free Fronsdal theory

A consistent description of free massless fields of arbitrary (half-)integer spin s has been found in 1978 by Fronsdal [25, 26]. The fields of integer spin are described as fully symmetric rank-s tensors obeying the so-called Fronsdal-equation

$$\mathcal{F}\Phi = \Box\Phi - (a \cdot \nabla)\mathcal{D}(\Phi) = 0, \tag{1.7}$$

where $\mathcal{D}$ is the deDonder-tensor

$$\mathcal{D}(\Phi) = (\partial \cdot \nabla)\Phi - \frac{1}{2}(a \cdot \nabla)(\partial \cdot \partial)\Phi. \tag{1.8}$$

In order for this equation to propagate the correct degrees of freedom for a massless spin-s field, we need to impose double-tracelessness of the field

$$(\partial \cdot \partial)^2\Phi = 0. \tag{1.9}$$

One can find a Lagrangian whose EOM are equivalent to the Fronsdal-equation above. This Fronsdal-Lagrangian is [51]

$$\mathcal{L}^{(s)} = \frac{1}{2}\Phi(\overleftarrow{\partial_a} \cdot \overrightarrow{\partial_a})^s\tilde{\mathcal{F}}, \tag{1.10}$$

where we have

$$\tilde{\mathcal{F}} = \mathcal{F} - \frac{1}{4}a^2\partial_a^2\mathcal{F}. \tag{1.11}$$

One can check that this Lagrangian is invariant under the gauge transformation

$$\delta\Phi = (a \cdot \nabla)\epsilon, \tag{1.12}$$

iff we choose the gauge parameter to be traceless.

$$(\partial \cdot \partial)\epsilon = 0 \tag{1.13}$$

In order to fix the gauge, we need to choose a gauge condition. One option is the deDonder-gauge, $D = 0$. However, this gauge condition still leaves some residual freedom, given by parameters that satisfy the so-called differential constraint

$$\Box\epsilon = 0. \tag{1.14}$$

In deDonder gauge, the simplified EOM of the Fronsdal field is

$$\Box\Phi = 0. \tag{1.15}$$

It is worth mentioning that one can always choose to set $D = 0$, even in the off-shell case. From now on, we will always work in deDonder-gauge.

Noether procedure

In order to calculate the deformations of the gauge transformations induced by cubic interaction vertices, we make use of the so-called Noether-procedure. First, we assume that the full Lagrangian of the theory, as well as all gauge transformations, can be expanded as a power series in some coupling contant g

$$\mathcal{S} = \mathcal{S}^{(2)} + g \cdot \mathcal{S}^{(3)} + \dots \tag{1.16}$$

$$\delta_\epsilon\Phi = \delta_\epsilon^{(0)}\Phi + g \cdot \delta_\epsilon^{(1)}\Phi + \dots \tag{1.17}$$

Here we made the power of the coupling constant explicit. The supersripts indicate the power of the fields in the respective terms.

We now demand gauge invariance of the action (1.16) under the transformations (1.17) at all orders in g. This condition gives a set of equations, the Noether-equations, which at lowest order reads

$$\delta_\epsilon^{(0)}\mathcal{S}^{(2)} = 0. \tag{1.18}$$

This is nothing but the familiar gauge invariance condition of the free action, which gives us the free EOM. At first order in the coupling constant we get

$$\delta_\epsilon^{(1)}\mathcal{S}^{(2)} + \delta_\epsilon^{(0)}\mathcal{S}^{(3)} = 0. \tag{1.19}$$

Since the first term on the LHS of (1.19) is again proportional to the free EOM, this equation implies a weaker condition on $\mathcal{S}^{(3)}$

$$\delta^{(0)}\mathcal{S}^{(3)} \approx 0. \tag{1.20}$$

Here $\approx$ means equivalence up to terms that vanish on-shell. One can now try to construct a cubic action that obeys (1.20). Making use of the Fronsdal-Lagrangian (1.10), equation (1.19) then allows us to calculate the first-order gauge deformations $\delta_\epsilon^{(1)}$.

When constructing cubic interactions it turns out to be beneficial to split the action into two parts: the TT-part, which does not contain any traces and divergences and the DT-part, which contains the remaining terms

$$\mathcal{S}^{(3)} = \mathcal{S}^{(3)}_{TT} + \mathcal{S}^{(3)}_{DT}. \tag{1.21}$$

Since the zero-order gauge variations of traces and divergences are proportional to themselves up to terms involving the differential constraint (1.14), equation (1.20) gives an independent condition on the TT-part of the cubic action [88]

$$[\delta^{(0)}\mathcal{S}^{(3)}]_{TT} \approx [\delta^{(0)}\mathcal{S}^{(3)}_{TT}]_{TT} \approx 0. \tag{1.22}$$

Therefore, the TT-part can be calculated without any knowledge of the DT-terms. It has also been shown [48] that for massless fields the DT-part of the cubic action is already completely determined by its TT-part when enforcing gauge invariance. After obtaining the TT-part of the action, one can systematically add the remaining traces and divergences by using the methods of [48].

Killing-tensors

A Killing-tensor $\bar{\epsilon}$ is defined as a solution of the generalized Killing-equation

$$(a \cdot \nabla)\bar{\epsilon} = 0. \tag{1.23}$$

From the above expression it is immediate that a zero-order gauge transformation with a Killing-tensor as its gauge parameter is a trivial symmetry of the action. Therefore, by using Killing-tensors as gauge parameters, one can investigate the global symmetries and the underlying algebra of a higher-spin theory. In chapter 5, we will make use of this in order to verify the consistency of spin-3-gravity on the global symmetry level.

On Minkowski-space, there is a particularly simple class of solutions to (1.23) that are polynomials in x [89]

$$\bar{\epsilon}^{(s-1)}_{(k)}(a,x) = \bar{\epsilon}^{(s-1)}_{(k),\mu_1\ldots\mu_{s-1},\nu_1\ldots\nu_k} a^{\mu_1}\ldots a^{\mu_{s-1}} x^{\nu_1}\ldots x^{\nu_k}, \qquad k \leq s-1. \tag{1.24}$$

Solutions of this type exist for all (non-negative) integers $s-1$. These Killing-tensors have symmetries corresponding to $(s-1,\, k)$ Young-diagrams. Note that in three dimensions, all Killing-tensors with $k > 1$ vanish, so we only have to deal with Killing-tensors that are at most linear in x. For spin-3-gravity, the relevant Killing-tensors are

$$\bar{\epsilon}^{(2)}_{(1),\alpha\beta} = -\bar{\epsilon}^{(2)}_{(1),\beta\alpha}, \tag{1.25}$$

$$\bar{\epsilon}^{(3)}_{(1),\alpha\beta\gamma} = \bar{\epsilon}^{(3)}_{(1),(\alpha\beta)\gamma}, \qquad \bar{\epsilon}^{(3)}_{(1),(\alpha\beta\gamma)} = 0. \tag{1.26}$$

Here we have supressed the constant solutions to the Killing-equation, since the calculations for this type of gauge parameters are mostly trivial and can easily be obtained from the calculations involving parameters with $k = 1$. In addition to the above symmetries, the Killing-equation also ensures that these tensors are traceless and divergence-free. Again, for the sake of simplicity the superscripts and subscripts will be suppressed most of the time.

Dimension-dependent-identities

When dealing with cubic interaction vertices, as well as the deformations of gauge transformations they induce, we also need to take Dimension-Dependent-Identities (DDI's, also often called Schouten-identities) into account. A DDI is an equation that arises from anti-symmetrizing over a set of indices larger than the spacetime dimension d, which is therefore identically zero. Naturally, the influence of these identities becomes stronger the lower the spacetime dimension is. For $d = 3$, the existence of DDI's already has a huge impact on the construction of higher-spin theories, as will be discussed later on.

One can systematically construct DDI's by contracting tensors with the so-called generalized Kronecker-Delta [50]

$$\delta^{\mu_1\ldots\mu_{d+1}}_{\nu_1\ldots\nu_{d+1}} := \delta^{[\mu_1}_{[\nu_1}\delta^{\mu_2}_{\nu_2}\ldots\delta^{\mu_{d+1}]}_{\nu_{d+1}]} \equiv 0. \tag{1.27}$$

Depending on the set of tensors we choose to contract with $\delta^{\mu_1\ldots\mu_{d+1}}_{\nu_1\ldots\nu_{d+1}}$, we get different

DDI's. For $d = 3$, we need a set of tensors with at least eight free indices to construct a DDI. As an example, if we look at an expression that contains three spin-2 fields, as well as two derivatives, one acting on the first field and one on the second field, the set of tensors we have to use for the corresponding DDI's is $\{\partial_{1,\mu}$, $\partial_{1,\mu}$, $\partial_{2,\mu}$, $\partial_{2,\mu}$, $\partial_{3,\mu}$, $\partial_{3,\mu}$, $\nabla_{1,\mu}$, $\nabla_{2,\mu}$ $\}$. Such a set cannot contain the same tensor more than two times, because otherwise one would have to contract two copies of the same tensor either to the upper set or the lower set of anti-symmetrized indices in $\delta^{\mu_1 \ldots \mu_{d+1}}_{\nu_1 \ldots \nu_{d+1}}$, rendering the whole expression trivial.

2 3D cubic vertices

In this chapter we present the TT cubic vertices that have been classified in [50]. Also, all DDI's that are relevant for the analysis of these vertices are calculated, including their off-shell parts.

2.1 Cubic vertices for spin-2 and spin-3

The full cubic Lagrangian of a higher-spin theory can be written as a sum of interaction terms for different spin triples

$$\mathcal{L}^{(3)} = \sum_{s_i,n} g^{(n)}_{s_1s_2s_3}\mathcal{L}^{(n)}_{s_1s_2s_3}. \tag{2.1}$$

Here $g^{(n)}_{s_1s_2s_3}$ are the coupling constants of the cubic interactions and n labels different interaction terms for the same spin triple. The coupling constants will be omitted from now on, unless they are absolutely necessary.

A single interaction part in the cubic Lagrangian can now be written in terms of a vertex operator $\mathcal{V}$ acting on the fields

$$\mathcal{L}^{(n)}_{s_1s_2s_3} = \mathcal{V}^{(n)}_{s_1s_2s_3}\Phi_1^{(s_1)}\Phi_2^{(s_2)}\Phi_3^{(s_3)}. \tag{2.2}$$

The vertex operator is built out of differential operators obtained by contracting the partial derivatives (1.2)

$$y_i \coloneqq \partial_i \cdot \nabla_{i+1}, \qquad z_i \coloneqq \partial_{i-1}\cdot\partial_{i+1}, \tag{2.3}$$

$$B_{ij} \coloneqq \nabla_i \cdot \nabla_j, \tag{2.4}$$

together with the trace- and divergence-operators (1.4)-(1.5). Here we use a cyclic numbering for which $i \equiv i+2$. Note that all these operators commute with each other. Since total derivative terms in the action will not affect the EOM, we will already discard them in the Lagrangian. Up to total derivatives, we then have the following relations for the above operators

$$y_i = -\partial_i \cdot \nabla_{i-1} - Div_i, \tag{2.5}$$

$$B_{ii+1} = \frac{1}{2}(\Box_{i-1} - \Box_i - \Box_{i+1}). \tag{2.6}$$

Using field redefinitions in the Fronsdal-Lagrangian (1.10), one can replace all Laplacians with trace- and divergence-terms, which can be dropped. The TT-part of a vertex-operator is thus a function of only six variables, $\mathcal{V} \equiv \mathcal{V}(y_i, z_i)$.

In order to calculate the zero-order gauge variation of cubic vertices, we need the commutators of the above operators with the gradient operator $(a \cdot \nabla)$

$$\begin{aligned} &[y_i, (a_i \cdot \nabla_i)] = B_{ii+1}, \qquad &&[y_{i\pm 1}, (a_i \cdot \nabla_i)] = 0, \\ &[z_i, (a_i \cdot \nabla_i)] = 0, \qquad &&[z_{i\pm 1}, (a_i \cdot \nabla_i)] \approx \pm y_{i\mp 1}. \end{aligned} \tag{2.7}$$

The zero-order gauge variation of a cubic interaction term is

$$\delta^{(0)}\mathcal{S}^{(3)} = \{\int dx \prod_j dx_j \delta(x - x_j) \sum_i \mathcal{V}(a_i \cdot \nabla_i)\epsilon_i \Phi_{i+1}\Phi_{i-1}\}\Big|_{a_i=0}. \tag{2.8}$$

Here we evaluate the expression at $a_i = 0$, since the gauge variation cannot depend on any auxilliary variables. Pulling out the gradient operator, this reads

$$\delta^{(0)}\mathcal{S}^{(3)} = \int dx \prod_j dx_j \delta(x - x_j) \sum_i \{(a_i \cdot \nabla_i)\mathcal{V}\epsilon_i \Phi_{i+1}\Phi_{i-1} + [\mathcal{V}, (a_i \cdot \nabla_i)]\epsilon_i \Phi_{i+1}\Phi_{i-1}\}\Big|_{a_i=0}. \tag{2.9}$$

The first term vanishes once the a_i are set to zero, whereas the second term can be rewritten making use of the commutators above

$$\delta^{(0)}\mathcal{S}^{(3)} = \int dx \prod_j dx_j \delta(x - x_j) \sum_i \tilde{\mathcal{G}}_i \mathcal{V}\epsilon_i \Phi_{i+1}\Phi_{i-1}. \tag{2.10}$$

From this we can see that the zero-order gauge variation of a vertex operator w.r.t. Φ_i is $\tilde{\mathcal{G}}_i\mathcal{V}$, with the operator of gauge variation

$$\tilde{\mathcal{G}}_i \coloneqq y_{i-1}\partial_{z_{i+1}} - y_{i+1}\partial_{z_{i-1}} + B_{ii+1}\partial_{y_i}. \tag{2.11}$$

Therefore, condition (1.20) reads

$$\mathcal{G}_i\mathcal{V} \approx 0. \tag{2.12}$$

where $\mathcal{G}_i$ now is the operator of on-shell gauge variation obtained from (2.11) by simply dropping the B_{ii+1}-term.

One can now try to construct all TT vertex operators obeying (2.12). For parity-even fields of integer spin this was done by Mkrtchyan in [50]. It turns out that in contrast to the higher-dimensional case, for $d = 3$ there exists at most one vertex for each triple of spins (the only exception being the case of three spin-1 fields, which will not play a role here). For fields of spins $s_i \geq 2$ there exist two different types of vertices, depending on whether the sum of the spins is even or odd. For an even sum of spins, a vertex for a given spin triple exists if the spins obey triangle inequalities $s_{i+1} + s_{i-1} \geq s_i + 2$. In this case, the vertex has two derivatives and reads

$$\mathcal{V}_{s_1s_2s_3} = y_3 z_1^{n_1} z_2^{n_2} z_3^{n_3+1}[(s_2+s_3-2)y_1z_1 + (s_1+s_3-2)y_2z_2 + (s_3-1)y_3z_3], \tag{2.13}$$

where

$$n_i = \frac{1}{2}(s_{i-1} + s_{i+1} - s_i) - 1. \tag{2.14}$$

For an odd sum of spins, for a vertex to exist the spins have to obey the triangle inequalities $s_i < s_{i+1} + s_{i-1}$. These three-derivative vertices are

$$\mathcal{V}_{s_1s_2s_3} = y_1y_2y_3 z_1^{n_1} z_2^{n_2} z_3^{n_3}, \tag{2.15}$$

where

$$n_{i-1} + n_{i+1} + 1 = s_i. \tag{2.16}$$

At this point it is alredy necessary to take DDI's into account. The first reason for this is that most of these vertices are only on-shell gauge invariant because they obey (2.12) up to some on-shell DDI's. As an example, the vertex that minimally couples spin-s to spin-2, which is inconsistent in higher-dimensional Minkowski-space, exists in $d = 3$ only because of DDI's. Second, one can directly use DDI's in order to rewrite the vertices themselves. Since all vertices that only differ by DDI's are equivalent, we are free to choose a representative that is most convenient for our calculations. The DDI's that are relevant for the construction of cubic vertices will be presented in the next section.

In this work, we will restrict ourselves to the analysis of vertices for spin-2 fields and spin-3 fields. The relevant vertices are

$$\mathcal{V}_{222} = (y_1 z_1 + y_2 z_2 + y_3 z_3)^2, \tag{2.17}$$

$$\mathcal{V}_{332} = y_3 z_3^2 (3y_1 z_1 + 3y_2 z_2 + y_3 z_3), \tag{2.18}$$

$$\mathcal{V}_{333} = y_1 y_2 y_3 z_1 z_2 z_3, \tag{2.19}$$

$$\mathcal{V}_{322} = y_1 y_2 y_3 z_2 z_3. \tag{2.20}$$

The 2-2-2 vertex coincides with the usual Einstein-Hilbert vertex in higher dimensions. In index-notation, these vertices read

$$\begin{aligned} \mathcal{V}_{222}[\Phi] = \; & +\Phi_1^{\alpha\beta}\partial_\alpha\partial_\beta\Phi_2^{\gamma\delta}\Phi_{3,\gamma\delta} + \Phi_1^{\gamma\delta}\Phi_2^{\alpha\beta}\partial_\alpha\partial_\beta\Phi_{3,\gamma\delta} + \partial_\alpha\partial_\beta\Phi_{1,\gamma\delta}\Phi_2^{\gamma\delta}\Phi_3^{\alpha\beta} \\ & +2\Phi_1^{\alpha\gamma}\partial_\alpha\Phi_2^{\beta\delta}\partial_\beta\Phi_{3,\gamma\delta} + 2\partial_\beta\Phi_1^{\alpha\gamma}\partial_\alpha\Phi_{2,\gamma\delta}\Phi_3^{\delta\beta} + 2\partial_\beta\Phi_{1,\gamma\delta}\Phi_2^{\alpha\gamma}\partial_\alpha\Phi_3^{\beta\delta}, \end{aligned} \tag{2.21}$$

$$\mathcal{V}_{332}[\Phi] = 3\Phi_1^{\alpha\beta}\partial_\alpha\partial_\beta\Phi_2^{\gamma\delta\eta}\Phi_{3,\gamma\delta\eta} + 3\partial_\beta\Phi_1^{\alpha\delta\eta}\partial_\alpha\Phi_{2,\delta\eta}^{\gamma}\Phi_{3,\gamma}^{\beta} + 3\partial_\beta\Phi_1^{\gamma\delta\eta}\Phi_{2,\delta\eta}^{\alpha}\partial_\alpha\Phi_{3,\gamma}^{\beta}, \tag{2.22}$$

$$\mathcal{V}_{333}[\Phi] = \partial^\delta\Phi_{1,\ \gamma}^{\alpha\kappa}\partial^\gamma\Phi_{2,\alpha\rho}^{\eta}\partial^\rho\Phi_{3,\kappa\eta\delta}, \tag{2.23}$$

$$\mathcal{V}_{322}[\Phi] = \partial_\gamma\Phi_1^{\alpha\delta\eta}\partial_\alpha\Phi_{2,\delta}^{\beta}\partial_\beta\Phi_{3,\eta}^{\gamma}. \tag{2.24}$$

From these expressions one can see that the two-derivative vertices are symmetric in their fields of equal spin and therefore only need a single field for each spin value. The three-derivative vertices however are anti-symmetric in their fields of equal spin and require more than one field to be non-trivial. More precisely, one needs three distinct spin-3 fields for the 3-3-3 vertex and two spin-2 fields for the 3-2-2 vertex.

2.2 DDI's for cubic vertices

We now come to the calculation of the DDI's that are relevant for analyzing cubic vertices. These DDI's are obtained by contracting the partial derivatives (1.2) with the generalized Kronecker-delta (1.27). The contractions will give us variables (2.3) and (2.4) as well as traces and divergences, which we discard right away. Since we can at most use two copies of ∂_i for each i, the DDI's need to have at least two derivatives. On the other hand, if we contract more than four derivatives with the generalized Kronecker-delta, we end up with a total derivative. Hence, the number of derivatives ranges from two to four.

In order to check the on-shell gauge invariance (2.12) of the vertices it would be sufficient to know the respective on-shell DDI's (i.e. discarding all B_{ij}-terms and Laplacians). However, in the next chapter we will have to calculate the off-shell variation of cubic vertices, so we also include the off-shell terms of the DDI's.

two derivatives:

$$y_1^2 z_1^2 + y_2^2 z_2^2 + 2y_1 y_2 z_1 z_2 + 2z_1 z_2 z_3 \Box_3 = 0 \tag{2.25}$$

$$y_1^2 z_1^2 + y_3^2 z_3^2 + 2y_1 y_3 z_1 z_3 + 2z_1 z_2 z_3 \Box_2 = 0 \tag{2.26}$$

$$y_2^2 z_2^2 + y_3^2 z_3^2 + 2y_2 y_3 z_2 z_3 + 2z_1 z_2 z_3 \Box_1 = 0 \tag{2.27}$$

$$y_1^2 z_1^2 + y_1 y_2 z_1 z_2 + y_1 y_3 z_1 z_3 - y_2 y_3 z_2 z_3 - 2z_1 z_2 z_3 B_{23} = 0 \tag{2.28}$$

$$y_2^2 z_2^2 + y_1 y_2 z_1 z_2 + y_2 y_3 z_2 z_3 - y_1 y_3 z_1 z_3 - 2z_1 z_2 z_3 B_{13} = 0 \tag{2.29}$$

$$y_3^2 z_3^2 + y_1 y_3 z_1 z_3 + y_2 y_3 z_2 z_3 - y_1 y_2 z_1 z_2 - 2z_1 z_2 z_3 B_{12} = 0 \tag{2.30}$$

three derivatives:

$$y_1y_3(y_2z_2+y_3z_3)+z_2(y_1z_1+\frac{1}{2}y_2z_2-\frac{1}{2}y_3z_3)\Box_1+\frac{1}{2}z_2(y_2z_2+y_3z_3)(\Box_2-\Box_3)=0 \quad (2.31)$$

$$y_1y_2(y_2z_2+y_3z_3)+z_3(y_1z_1-\frac{1}{2}y_2z_2+\frac{1}{2}y_3z_3)\Box_1+\frac{1}{2}z_3(y_2z_2+y_3z_3)(\Box_3-\Box_2)=0 \quad (2.32)$$

$$y_2y_3(y_1z_1+y_3z_3)+z_1(y_2z_2+\frac{1}{2}y_1z_1-\frac{1}{2}y_3z_3)\Box_2+\frac{1}{2}z_1(y_1z_1+y_3z_3)(\Box_1-\Box_3)=0 \quad (2.33)$$

$$y_1y_2(y_1z_1+y_3z_3)+z_3(y_2z_2-\frac{1}{2}y_1z_1+\frac{1}{2}y_3z_3)\Box_2+\frac{1}{2}z_3(y_1z_1+y_3z_3)(\Box_3-\Box_1)=0 \quad (2.34)$$

$$y_2y_3(y_1z_1+y_2z_2)+z_1(y_3z_3+\frac{1}{2}y_1z_1-\frac{1}{2}y_2z_2)\Box_3+\frac{1}{2}z_1(y_1z_1+y_2z_2)(\Box_1-\Box_2)=0 \quad (2.35)$$

$$y_1y_3(y_1z_1+y_2z_2)+z_2(y_3z_3-\frac{1}{2}y_1z_1+\frac{1}{2}y_2z_2)\Box_3+\frac{1}{2}z_2(y_1z_1+y_2z_2)(\Box_2-\Box_1)=0 \quad (2.36)$$

four derivatives:

$$y_1^2y_2^2+y_1y_2z_3(\Box_3-\Box_1-\Box_2)=0 \quad (2.37)$$

$$y_1^2y_3^2+y_1y_3z_2(\Box_2-\Box_1-\Box_3)=0 \quad (2.38)$$

$$y_2^2y_3^2+y_2y_3z_1(\Box_1-\Box_2-\Box_3)=0 \quad (2.39)$$

$$y_1^2y_2y_3+y_1(y_1z_1+\frac{1}{2}y_2z_2+\frac{1}{2}y_3z_3)\Box_1+\frac{1}{2}y_1(y_2z_2-y_3z_3)(\Box_2-\Box_3)=0 \quad (2.40)$$

$$y_1y_2^2y_3+y_2(y_2z_2+\frac{1}{2}y_1z_1+\frac{1}{2}y_3z_3)\Box_2+\frac{1}{2}y_2(y_1z_1-y_3z_3)(\Box_1-\Box_3)=0 \quad (2.41)$$

$$y_1y_2y_3^2+y_3(y_3z_3+\frac{1}{2}y_1z_1+\frac{1}{2}y_2z_2)\Box_3+\frac{1}{2}y_3(y_1z_1-y_2z_2)(\Box_1-\Box_2)=0 \quad (2.42)$$

We can summarize these DDI's as

$$(G - y_i z_i)^2 + 2 z_{i+1} z_{i-1} z_i \Box_i = 0, \tag{2.43}$$

$$y_i z_i G - y_{i-1} z_{i-1} y_{i+1} z_{i+1} - z_{i+1} z_{i-1} z_i (\Box_i - \Box_{i+1} - \Box_{i-1}) = 0, \tag{2.44}$$

$$(y_i y_{i\pm 1} \pm \frac{1}{2} z_{i\mp 1} (\Box_{i-1} - \Box_{i+1}))(G - y_i z_i) + z_{i\mp 1}(y_i z_i \pm \frac{1}{2} y_{i-1} z_{i-1} \mp y_{i+1} z_{i+1}) \Box_i = 0, \tag{2.45}$$

$$y_i^2 y_{i+1}^2 + y_i y_{i+1} z_{i-1} (\Box_{i-1} - \Box_i - \Box_{i+1}) = 0, \tag{2.46}$$

$$\begin{aligned} y_i^2 y_{i+1} y_{i-1} + & y_i (y_i z_i + \frac{1}{2} y_{i+1} z_{i+1} + \frac{1}{2} y_{i-1} z_{i-1}) \Box_i \\ & + \frac{1}{2} y_i (y_{i+1} z_{i+1} - y_{i-1} z_{i-1})(\Box_{i+1} - \Box_{i-1}) = 0, \end{aligned} \tag{2.47}$$

with $G = \sum_i y_i z_i$. These identities coincide with the DDI's presented in [50] in the on-shell case.

3 First-order deformations of gauge transformations

Having all possible cubic interactions of spin-2 fields and spin-3 fields at hand, we can now determine the first-order corrections to the gauge transformations they induce. In higher dimensions, this has already been done in [22]. However, in three dimensions this method needs to be adaped to account for the influence of DDI's. We will therefore first review the procedure of [22] and then generalize it to $d = 3$. Note that even though we restrict ourselves to the vertices (2.17)-(2.20) here, this method can be used for all cubic vertices.

3.1 Calculating first-order deformations from cubic vertices

We want to extract the first-order gauge deformations from the Noether-equation

$$\delta_{\epsilon}^{(1)} \mathcal{S}^{(2)} + \delta_{\epsilon}^{(0)} \mathcal{S}^{(3)} = 0. \tag{3.1}$$

The variation of the action corresponding to the Fronsdal-Lagrangian (1.10) is again proportional to the EOM. For simplicity, we restrict ourselves to the variation w.r.t. the first field

$$\delta_{\epsilon_1}^{(1)} \mathcal{S}^{(2)} = \int (\delta_{\epsilon_1}^{(1)} \Phi_2 \Box_2 \Phi_2 + \delta_{\epsilon_1}^{(1)} \Phi_3 \Box_3 \Phi_3). \tag{3.2}$$

Here we imply that the first-order gauge deformations are fully contracted with the respective EOM. In chapter 2 we have already calculated the zero-order gauge variation of the cubic action

$$\begin{aligned} \delta_{\epsilon_1}^{(0)} \mathcal{S}^{(3)} &= \int dx \prod_j dx_j \delta(x - x_j) \tilde{\mathcal{G}}_1 \mathcal{V} \epsilon_1 \Phi_2 \Phi_3 \\ &= \int dx \prod_j dx_j \delta(x - x_j) (\mathcal{G}_1 + \frac{1}{2}(\Box_3 - \Box_1 - \Box_2)\partial_{y_1}) \mathcal{V} \epsilon_1 \Phi_2 \Phi_3. \end{aligned} \tag{3.3}$$

The gauge parameter satisfies the differential constraint (1.14), so the term proportional to $\Box_1$ can be dropped. Also, if we do not take DDI's into account, we can directly set $\mathcal{G}_1\mathcal{V} = 0$ in the above. Equation (3.1) then reads

$$\int dx(\delta^{(1)}_{\epsilon_1}\Phi_2\Box_2\Phi_2 + \delta^{(1)}_{\epsilon_1}\Phi_3\Box_3\Phi_3) = -\int dx\frac{1}{2}(\Box_3 - \Box_2)\partial_{y_1}\mathcal{V}\epsilon_1\Phi_2\Phi_3. \tag{3.4}$$

From this we can read off the first-order gauge deformations

$$\delta^{(1)}_{\epsilon_1}\Phi_2 = \frac{1}{2}\partial_{y_1}\mathcal{V}\epsilon_1\Phi_3, \tag{3.5}$$

$$\delta^{(1)}_{\epsilon_1}\Phi_3 = -\frac{1}{2}\partial_{y_1}\mathcal{V}\epsilon_1\Phi_2. \tag{3.6}$$

Note that since on the LHS of (3.4) no further derivatives act on $\Box_2\Phi_2$ and $\Box_3\Phi_3$, the derivatives included in the variables of the vertex operator need to be partially integrated in such a way that the derivatives only act on the gauge parameter and the remaining field. For example, for the first-order gauge deformation (3.5) this means

$$\tilde{y}_1 \coloneqq -\partial_1 \cdot \nabla_3, \qquad \tilde{y}_2 \coloneqq c\ a \cdot \nabla_3 - (1-c)\ a \cdot \nabla_1, \qquad \tilde{y}_3 \coloneqq \partial_3 \cdot \nabla_1, \tag{3.7}$$

$$\tilde{z}_1 \coloneqq a \cdot \partial_3, \qquad \tilde{z}_2 \coloneqq \partial_1 \cdot \partial_3, \qquad \tilde{z}_3 \coloneqq \partial_1 \cdot a. \tag{3.8}$$

In $\tilde{y}_2$ the derivative can act on both the first and the third argument. The parameter c will be called a partial-integration-parameter. These parameters are expected to be completely fixed by demanding consistency of the underlying theory. We also replaced all ∂_2's by a's in order to have an expression for the gauge deformation that is not contracted to $\Box_2\Phi_2$.

An analogous calculation for the other gauge variations leads to similar expressions for the remaining gauge deformations

$$\delta^{(1)}_{\epsilon_2}\Phi_1 = -\frac{1}{2}\partial_{y_2}\mathcal{V}\epsilon_2\Phi_3, \tag{3.9}$$

$$\delta^{(1)}_{\epsilon_2}\Phi_3 = \frac{1}{2}\partial_{y_2}\mathcal{V}\epsilon_2\Phi_1, \tag{3.10}$$

$$\delta^{(1)}_{\epsilon_3}\Phi_1 = \frac{1}{2}\partial_{y_3}\mathcal{V}\epsilon_3\Phi_2, \tag{3.11}$$

$$\delta^{(1)}_{\epsilon_3}\Phi_2 = -\frac{1}{2}\partial_{y_3}\mathcal{V}\epsilon_3\Phi_1. \tag{3.12}$$

As already mentioned, this approach needs to be slightly modified in the presence of DDI's. Now the Noether-equation has to be fulfilled only up to an off-shell DDI that we call $\tilde{D}$

$$\delta^{(1)}_{\epsilon_1}\mathcal{S}^{(2)} + \delta^{(0)}_{\epsilon_1}\mathcal{S}^{(3)} = \tilde{D}. \tag{3.13}$$

Also, we cannot set $\mathcal{G}_1\mathcal{V} = 0$ in (3.3) anymore, since we potentially have $\mathcal{G}_1\mathcal{V} = D$, for some on-shell DDI D. In the on-shell variation we were able to set all on-shell DDI's to zero. However, in the off-shell variation on-shell DDI's do not vanish anymore and we get

$$\delta^{(0)}_{\epsilon_1}\mathcal{S}^{(3)} = \int dx[D + \frac{1}{2}(\Box_3 - \Box_2)\partial_{y_1}\mathcal{V}]\epsilon_1\Phi_2\Phi_3. \tag{3.14}$$

D only consists of on-shell terms, which cannot be canceled via terms in the first-order deformations, as they are not proportional to the EOM. Since we still have that $\delta^{(0)}_{\epsilon_1}\mathcal{S}^{(3)} \approx D$ on-shell, we see that $\tilde{D}$ in (3.13) actually has to be the off-shell completion of D. We can therefore complete D to its off-shell version by adding the appropriate off-shell terms, which we call Δ in the following. More precisely, we have $D + \Delta = \tilde{D}$. This leads to

$$\begin{aligned}\delta^{(0)}_{\epsilon_1}\mathcal{S}^{(3)} &= \int dx[D + \frac{1}{2}(\Box_3 - \Box_2)\partial_{y_1}\mathcal{V}]\epsilon_1\Phi_2\Phi_3 \\ &= \int dx[\tilde{D} - \Delta + \frac{1}{2}(\Box_3 - \Box_2)\partial_{y_1}\mathcal{V}]\epsilon_1\Phi_2\Phi_3.\end{aligned} \tag{3.15}$$

$\tilde{D}$ can now be dropped, since it is an off-shell DDI. This leaves us with

$$\delta^{(0)}_{\epsilon_1}\mathcal{S}^{(3)} = \int dx[-\Delta + \frac{1}{2}(\Box_3 - \Box_2)\partial_{y_1}\mathcal{V}]\epsilon_1\Phi_2\Phi_3. \tag{3.16}$$

Δ only consists of off-shell terms, which can be compensated by appropriate terms in the first-order deformations. The terms in Δ proportional to $\Box_1$ can again be dropped because of the differential constraint. We can then split Δ into two parts, proportional to $\Box_2$ and $\Box_3$ respectively

$$\Delta = \Delta_2\Box_2 + \Delta_3\Box_3. \tag{3.17}$$

The Noether-equation then reads

$$\begin{aligned}&\int dx(\delta^{(1)}_{\epsilon_1}\Phi_2\Box\Phi_2 + \delta^{(1)}_{\epsilon_1}\Phi_3\Box\Phi_3) = \\ -&\int dx[\Box_3(\frac{1}{2}\partial_{y_1}\mathcal{V}(y,z) - \Delta_3) + \Box_2(-\frac{1}{2}\partial_{y_1}\mathcal{V}(y,z) - \Delta_2)]\epsilon_1\Phi_2\Phi_3.\end{aligned} \tag{3.18}$$

From this, we can read off the corrected first-order deformations in the presence of DDI's

$$\delta^{(1)}_{\epsilon_1}\Phi_2 = (\frac{1}{2}\partial_{y_1}\mathcal{V} + \Delta_2)\epsilon_1\Phi_3, \tag{3.19}$$

$$\delta^{(1)}_{\epsilon_1}\Phi_3 = (-\frac{1}{2}\partial_{y_1}\mathcal{V} + \Delta_3)\epsilon_1\Phi_2. \tag{3.20}$$

An analogous calculation leads to

$$\delta^{(1)}_{\epsilon_2}\Phi_1 = (-\frac{1}{2}\partial_{y_2}\mathcal{V} + \tilde{\Delta}_1)\epsilon_1\Phi_3, \tag{3.21}$$

$$\delta^{(1)}_{\epsilon_2}\Phi_3 = (\frac{1}{2}\partial_{y_2}\mathcal{V} + \tilde{\Delta}_3)\epsilon_1\Phi_3, \tag{3.22}$$

$$\delta^{(1)}_{\epsilon_3}\Phi_2 = (-\frac{1}{2}\partial_{y_3}\mathcal{V} + \bar{\Delta}_2)\epsilon_3\Phi_1, \tag{3.23}$$

$$\delta^{(1)}_{\epsilon_3}\Phi_1 = (\frac{1}{2}\partial_{y_3}\mathcal{V} + \bar{\Delta}_1)\epsilon_3\Phi_2. \tag{3.24}$$

Here the distinction between Δ_i, $\tilde{\Delta}_i$ and $\bar{\Delta}_i$ is important: these terms come from different DDI's, since the gauge variations of a vertex w.r.t. the different fields do in general not produce the same DDI. Hence, the $\tilde{\Delta}_i$'s correspond to the off-shell part of the DDI in the gauge variation $\mathcal{G}_2\mathcal{V}$ and the $\bar{\Delta}_i$'s correspond to the off-shell part of the DDI in the gauge variation $\mathcal{G}_3\mathcal{V}$.

3.2 First-order deformations for spin-2 and spin-3

If we want to use equations (3.19)-(3.24) to calculate the first-order gauge deformations, we first need to calculate the Δ_i's, $\tilde{\Delta}_i$'s and $\bar{\Delta}_i$'s for the vertices (2.17)-(2.20). This is done by performing the on-shell variations of these vertices, identifying the DDI's that make them on-shell gauge invariant (in general linear combinations of the basic DDI's (2.25)-(2.42)) and reading off the off-shell parts of these DDI's. We do however not need all of the Δ_i's, $\tilde{\Delta}_i$'s and $\bar{\Delta}_i$'s in order to calculate the first-order deformations, since a lot of the deformations will coincide. As an example, the 3-3-3 vertex will only induce one first-order gauge deformation, so the expressions in (3.19)-(3.24) all agree and it suffices to know e.g. Δ_2.

The 2-2-2 vertex (2.17) is gauge invariant even without the use of DDI's, therefore all the Δ_i's, $\tilde{\Delta}_i$'s and $\bar{\Delta}_i$'s are zero. For the other vertices, the relevant expressions are:

3-3-2 vertex:

$$\Delta_2 = 3z_1z_3(y_1z_1 + y_2z_2) \tag{3.25}$$

$$\Delta_3 = -6z_1z_3(y_3z_3 + \frac{1}{2}y_1z_1 - \frac{1}{2}y_2z_2) \tag{3.26}$$

$$\bar{\Delta}_1 = 0 \tag{3.27}$$

3-3-3 vertex:

$$\Delta_2 = -z_1(y_2^2z_2^2 + \frac{1}{2}y_1y_3z_1z_3 + \frac{1}{2}y_1y_2z_1z_2) \tag{3.28}$$

3-2-2 vertex:

$$\Delta_2 = -y_2^2z_2^2 - \frac{1}{2}y_1y_2z_1z_2 - \frac{1}{2}y_1y_3z_1z_3 \tag{3.29}$$

$$\tilde{\Delta}_1 = y_1^2z_1z_2 + \frac{1}{2}y_1y_2z_2^2 + \frac{1}{2}y_1y_3z_2z_3 \tag{3.30}$$

$$\tilde{\Delta}_3 = -\frac{1}{2}y_1y_2z_2^2 + \frac{1}{2}y_1y_3z_2z_3 \tag{3.31}$$

Now we are able to calculate the first-order deformations:

2-2-2 vertex:

$$\delta^{(1)}_{\epsilon_2}\Phi_1 = -(y_2 z_2^2 + y_1 z_1 z_2 + y_3 z_2 z_3)\epsilon_2\Phi_3 \tag{3.32}$$

3-3-2 vertex:

$$\delta^{(1)}_{\epsilon_2}\Phi_1 = -z_2 z_3(3y_1 z_1 + 3y_2 z_2 + \frac{3}{2}y_3 z_3)\epsilon_2\Phi_3 \tag{3.33}$$

$$\delta^{(1)}_{\epsilon_3}\Phi_1 = z_3^2(\frac{3}{2}y_1 z_1 + \frac{3}{2}y_2 z_2 + y_3 z_3)\epsilon_3\Phi_2 \tag{3.34}$$

$$\delta^{(1)}_{\epsilon_1}\Phi_3 = -z_1 z_3(3y_1 z_1 - 3y_2 z_2 + \frac{15}{2}y_3 z_3)\epsilon_1\Phi_2 \tag{3.35}$$

3-3-3 vertex:

$$\delta^{(1)}_{\epsilon_2}\Phi_1 = z_2(y_1^2 z_1^2 + \frac{1}{2}y_1 y_2 z_1 z_2 + \frac{1}{2}y_2 y_3 z_2 z_3 - \frac{1}{2}y_1 y_3 z_1 z_3)\epsilon_2\Phi_3 \tag{3.36}$$

3-2-2 vertex:

$$\delta^{(1)}_{\epsilon_2}\Phi_1 = (y_1^2 z_1 z_2 + \frac{1}{2}y_1 y_2 z_2^2)\epsilon_2\Phi_3 \tag{3.37}$$

$$\delta^{(1)}_{\epsilon_1}\Phi_2 = (-y_2^2 z_2^2 - \frac{1}{2}y_1 y_2 z_1 z_2 - \frac{1}{2}y_1 y_3 z_1 z_3 + \frac{1}{2}y_2 y_3 z_2 z_3)\epsilon_1\Phi_3 \tag{3.38}$$

$$\delta^{(1)}_{\epsilon_3}\Phi_2 = (+\frac{1}{2}y_1 y_3 z_3^2 - y_1 y_2 z_2 z_3)\epsilon_3\Phi_1 \tag{3.39}$$

As can be seen from e.g. (3.19), the number of derivatives in the first-order gauge deformations is lowered by one compared to the vertex they stem from. Therefore, for the three-derivative vertices one can see that some of the deformations can be changed by using the two-derivative DDI's (2.25)-(2.30), whereas for the two-derivative vertices such ambiguities do not exist.

In index-notation the above deformations read:

2-2-2 vertex:

$$\begin{aligned}\delta^{(1)}_{\epsilon_2}\Phi_1 = &-c\ \partial_\mu\epsilon_2^\alpha a^\mu\Phi_{3,\alpha\nu}a^\nu + (1-c)\ \epsilon_2^\alpha\partial_\mu\Phi_{3,\alpha\nu}a^\mu a^\nu \\ &-\epsilon_2^\alpha\partial_\alpha\Phi_{3,\mu\nu}a^\mu a^\nu + \partial_\alpha\epsilon_{2,\mu}a^\mu\Phi^\alpha_{3,\nu}a^\nu\end{aligned} \tag{3.40}$$

3-3-2 vertex:

$$\begin{aligned}\delta^{(1)}_{\epsilon_1}\Phi_3 = &-\frac{15}{2}d\ \partial_\mu\epsilon_1^{\alpha\beta}a^\mu\Phi_{2,\alpha\beta\nu}a^\nu + \frac{15}{2}(1-d)\ \epsilon_1^{\alpha\beta}\partial_\mu\Phi_{2,\alpha\beta\nu}a^\mu a^\nu \\ &-3\partial_\alpha\epsilon_{1,\beta\mu}a^\mu\Phi_{2,\ \nu}^{\alpha\beta}a^\nu - 3\epsilon_1^{\alpha\beta}\partial_\alpha\Phi_{2,\beta\mu\nu}a^\mu a^\nu\end{aligned} \tag{3.41}$$

$$\begin{aligned}\delta^{(1)}_{\epsilon_3}\Phi_1 = &+\frac{3}{2}e\ \epsilon_3^\alpha\partial_\mu\Phi_{2,\alpha\nu\lambda}a^\mu a^\nu a^\lambda - \frac{3}{2}(1-e)\ \partial_\mu\epsilon_3^\alpha a^\mu\Phi_{2,\alpha\nu\lambda}a^\nu a^\lambda \\ &+\frac{3}{2}\partial_\alpha\epsilon_{3,\mu}a^\mu\Phi^\alpha_{2,\nu\lambda}a^\nu a^\lambda - \epsilon_3^\alpha\partial_\alpha\Phi_{2,\mu\nu\lambda}a^\mu a^\nu a^\lambda\end{aligned} \tag{3.42}$$

$$\begin{aligned}\delta^{(1)}_{\epsilon_2}\Phi_1 = &+3f\ \epsilon^\alpha_{2,\nu}a^\nu\partial_\mu\Phi_{3,\alpha\lambda}a^\mu a^\lambda - 3(1-f)\ \partial_\mu\epsilon^\alpha_{2,\nu}a^\mu a^\nu\Phi_{3,\alpha\lambda}a^\lambda \\ &+\frac{3}{2}\partial_\alpha\epsilon_{2,\mu\nu}a^\mu a^\nu\Phi^\alpha_{3,\lambda}a^\lambda - 3\epsilon^\alpha_{2,\mu}a^\mu\partial_\alpha\Phi_{3,\nu\lambda}a^\nu a^\lambda\end{aligned} \tag{3.43}$$

3-3-3 vertex:

$$\begin{aligned}\delta^{(1)}_{\epsilon_2}\Phi_1 = &+j^2\ \partial_\mu\partial_\nu\epsilon_2^{\alpha\beta}a^\mu a^\nu\Phi_{3,\alpha\beta\lambda}a^\lambda - j(1-j)\ \partial_\nu\epsilon_2^{\alpha\beta}a^\nu\partial_\mu\Phi_{3,\alpha\beta\lambda}a^\mu a^\lambda \\ &-(1-j)j\ \partial_\mu\epsilon_2^{\alpha\beta}a^\mu\partial_\nu\Phi_{3,\alpha\beta\lambda}a^\nu a^\lambda + (1-j)^2\ \epsilon_2^{\alpha\beta}\partial_\mu\partial_\nu\Phi_{3,\alpha\beta\lambda}a^\mu a^\nu a^\lambda \\ &+\frac{1}{2}j\ \partial_\mu\partial_\alpha\epsilon^\beta_{2,\nu}a^\mu a^\nu\Phi^\alpha_{3,\beta\lambda}a^\lambda - \frac{1}{2}(1-j)\ \partial_\alpha\epsilon^\beta_{2,\nu}a^\nu\partial_\mu\Phi^\alpha_{3,\beta\lambda}a^\mu a^\lambda \\ &+\frac{1}{2}j\ \partial_\mu\epsilon_2^{\alpha\beta}a^\mu\partial_\alpha\Phi_{3,\beta\nu\lambda}a^\nu a^\lambda - \frac{1}{2}(1-j)\ \epsilon_2^{\alpha\beta}\partial_\mu\partial_\alpha\Phi_{3,\beta\nu\lambda}a^\mu a^\nu a^\lambda \\ &-\frac{1}{2}\partial_\alpha\epsilon^\beta_{2,\mu}a^\mu\partial_\beta\Phi^\alpha_{3,\nu\lambda}a^\nu a^\lambda\end{aligned} \tag{3.44}$$

3-2-2 vertex:

$$\begin{aligned}\delta^{(1)}_{\epsilon_2}\Phi_1 = &+k_1^2\ \partial_\mu\partial_\nu\epsilon_2^\alpha a^\mu a^\nu\Phi_{3,\alpha\lambda}a^\lambda - (1-k_1)k_1\ \partial_\mu\epsilon_2^\alpha a^\mu\partial_\nu\Phi_{3,\alpha\lambda}a^\nu a^\lambda \\ &-k_1(1-k_1)\ \partial_\nu\epsilon_2^\alpha a^\nu\partial_\mu\Phi_{3,\alpha\lambda}a^\mu a^\lambda + (1-k_1)^2\ \epsilon_2^\alpha\partial_\mu\partial_\nu\Phi_{3,\alpha\lambda}a^\mu a^\nu a^\lambda \\ &+\frac{1}{2}k_1\ \partial_\mu\epsilon_2^\alpha a^\mu\partial_\alpha\Phi_{3,\nu\lambda}a^\nu a^\lambda - \frac{1}{2}(1-k_1)\ \epsilon_2^\alpha\partial_\alpha\partial_\mu\Phi_{3,\nu\lambda}a^\mu a^\nu a^\lambda\end{aligned} \tag{3.45}$$

$$\begin{aligned}\delta^{(1)}_{\epsilon_1}\Phi_2 = &-k_2^2\ \epsilon_1^{\alpha\beta}\partial_\mu\partial_\nu\Phi_{3,\alpha\beta}a^\mu a^\nu + k_2(1-k_2)\ \partial_\nu\epsilon_1^{\alpha\beta}a^\nu\partial_\mu\Phi_{3,\alpha\beta}a^\mu \\ &+(1-k_2)k_2\ \partial_\mu\epsilon_1^{\alpha\beta}a^\mu\partial_\nu\Phi_{3,\alpha\beta}a^\nu - (1-k_2)^2\ \partial_\mu\partial_\nu\epsilon_1^{\alpha\beta}a^\mu a^\nu\Phi_{3,\alpha\beta} \\ &+\frac{1}{2}k_2\ \epsilon_1^{\alpha\beta}\partial_\alpha\partial_\mu\Phi_{3,\beta\nu}a^\mu a^\nu - \frac{1}{2}(1-k_2)\ \partial_\mu\epsilon_1^{\alpha\beta}a^\mu\partial_\alpha\Phi_{3,\beta\nu}a^\nu \\ &+\frac{1}{2}k_2\ \partial_\alpha\epsilon_{1,\beta\nu}a^\nu\partial_\mu\Phi_3^{\alpha\beta}a^\mu - \frac{1}{2}(1-k_2)\ \partial_\mu\partial_\alpha\epsilon_{1,\beta\nu}a^\mu a^\nu\Phi_3^{\alpha\beta} \\ &+\frac{1}{2}\partial_\alpha\epsilon^\beta_{1,\mu}a^\mu\partial_\beta\Phi^\alpha_{3,\nu}a^\nu\end{aligned} \tag{3.46}$$

$$\delta^{(1)}_{\epsilon_3}\Phi_2 = +k_3\ \partial_\mu\partial_\alpha\epsilon_3^\beta a^\mu\Phi^\alpha_{1,\beta\nu}a^\nu - (1-k_3)\ \partial_\alpha\epsilon_3^\beta\partial_\mu\Phi^\alpha_{1,\beta\nu}a^\mu a^\nu \\ -\frac{1}{2}\partial_\alpha\epsilon_3^\beta\partial_\beta\Phi^\alpha_{1,\mu\nu}a^\mu a^\nu \tag{3.47}$$

Here we used different letters for the deformations of the 2-2-2 vertex and the 3-3-2 vertex, since these are the deformations we will analyze further in chapter 5.

We have checked that for the two-derivative vertices the deformations related by an interchange of fields of equal spin indeed agree, whereas for the three-derivative vertices the deformations related by such an interchange differ by a sign.

It is important to note that even in cases where no DDI's can be used in the first-order gauge deformations, the above expressions are still not unambiguous, since we can make use of field redefinitions and gauge parameter redefinitions to change them. Lets say we have a field redefinition of the form

$$\Phi_3 \to \Phi_3 + \Omega_3(y_i, z_i, B_{ij})\Phi_1\Phi_2. \tag{3.48}$$

Normally, the zero-order gauge transformation of Φ_3 w.r.t. a gauge parameter of a different field, say Φ_1, is zero. However, if we perform the above field redefinition, the zero-order gauge transformation gets modified as

$$\delta^{(0)}_{\epsilon_1}\Phi_3 \to \delta^{(0)}_{\epsilon_1}\Phi_3 + \Omega_3(y_i, z_i, B_{ij})\delta^{(0)}_{\epsilon_1}\Phi_1\Phi_2 = \Omega_3(y_i, z_i, B_{ij})a_1\cdot\nabla_1\epsilon_1\Phi_2. \tag{3.49}$$

This term contributes to the first-order gauge deformation $\delta^{(1)}_{\epsilon_1}\Phi_3$. Similarly, for parameter redefinitions of the form

$$\epsilon_3 \to \tilde{\epsilon_3} = \epsilon_3 + \Delta_{12}(y_i, z_i, B_{ij})\epsilon_1\Phi_2 + \Delta_{21}(y_i, z_i, B_{ij})\epsilon_2\Phi_1, \tag{3.50}$$

the zero-order gauge transformation of Φ_3 gets modified as

$$(a_3\cdot\nabla_3)\epsilon_3 \to (a_3\cdot\nabla_3)\epsilon_3 + (a_3\cdot\nabla_3)\Delta_{12}(y_i, z_i, B_{ij})\epsilon_1\Phi_2 + (a_3\cdot\nabla_3)\Delta_{21}(y_i, z_i, B_{ij})\epsilon_2\Phi_1. \tag{3.51}$$

The term proportional to Δ_{12} contributes to $\delta^{(1)}_{\epsilon_1}\Phi_3$, whereas the term containing Δ_{21}

contributes to $\delta^{(1)}_{\epsilon_2}\Phi_3$. The most general form of the first-order gauge deformation (3.20) is thus

$$\delta^{(1)}_{\epsilon_1}\Phi_3 = (-\frac{1}{2}\partial_{y_1}\mathcal{V} + \Delta_3 + [\Omega_3, (a_1 \cdot \nabla_1)] + (a_3 \cdot \nabla_3)\Delta_{12})\epsilon_1\Phi_2. \qquad (3.52)$$

4 Commutators of gauge transformations

In a full interacting theory it is natural to assume that the gauge transformations form an algebra

$$\delta_{\epsilon_1}\delta_{\epsilon_2}\Phi - \delta_{\epsilon_2}\delta_{\epsilon_1}\Phi = \delta_{[[\epsilon_1,\epsilon_2]]}\Phi + \text{trivial}, \tag{4.1}$$

where $[[\epsilon_1, \epsilon_2]]$ is referred to as the gauge bracket. In this chapter we calculate the above commutator at the lowest order. The "trivial"-part does not play any role in this calculation and will be discussed in chapter 5 when it becomes relevant. We again follow [22] and generalize their method to three dimensions.

4.1 Calculating zero-order commutators of gauge transformations

For simplicity, we focus on the gauge bracket $[[\epsilon_1, \epsilon_2]]$, the other brackets can be calculated completely analogously. Expanding this bracket as a power series in g

$$[[\epsilon_1, \epsilon_2]] = g[[\epsilon_1, \epsilon_2]]^{(0)} + g^2[[\epsilon_1, \epsilon_2]]^{(1)} + \dots, \tag{4.2}$$

and using the expansion for the gauge transformation (1.17), equation (4.1) at lowest order can be written as

$$\delta^{(0)}_{\epsilon_1}\delta^{(1)}_{\epsilon_2}\Phi - \delta^{(0)}_{\epsilon_2}\delta^{(1)}_{\epsilon_1}\Phi = \delta^{(0)}_{[[\epsilon_1,\epsilon_2]]^{(0)}}\Phi = (a\cdot\nabla)[[\epsilon_1, \epsilon_2]]^{(0)}. \tag{4.3}$$

In order to extract the gauge bracket from the LHS of (4.3) we define new variables

$$A := \frac{1}{2}a\cdot\nabla \equiv \frac{1}{2}a(\nabla_1 + \nabla_2), \qquad B = B_{12} = \nabla_1\cdot\nabla_2, \tag{4.4}$$

$$\tilde{y_1} = \partial_1\cdot\nabla_2, \qquad \tilde{y_2} = -\partial_2\cdot\nabla_1, \qquad \tilde{y_3} = \frac{1}{2}a\cdot(\nabla_1 - \nabla_2), \tag{4.5}$$

$$\tilde{z_1} = a \cdot \partial_2, \qquad \tilde{z_2} = a \cdot \partial_1, \qquad \tilde{z_3} = \partial_1 \cdot \partial_2. \tag{4.6}$$

Note that this choice of $\tilde{y_3}$ already fixes the partial integration parameter to $\frac{1}{2}$. We will however still work with arbitrary partial integration parameters for the brackets in chapter 5, since the consistency conditions set these parameters to $\frac{1}{2}$ either way. For the variable A it holds that

$$a \cdot \nabla_1 = \frac{1}{2}a(\nabla_1 + \nabla_2) + \frac{1}{2}a(\nabla_1 - \nabla_2) = A + \tilde{y_3}, \tag{4.7}$$

$$a \cdot \nabla_2 = \frac{1}{2}a(\nabla_1 + \nabla_2) - \frac{1}{2}a(\nabla_1 - \nabla_2) = A - \tilde{y_3}. \tag{4.8}$$

The new variables satisfy the following commutation relations with the operators of zero-order gauge variation

$$\begin{aligned}
&[\tilde{y_1}, a_1 \cdot \nabla_1] = B, && [\tilde{y_2}, a_1 \cdot \nabla_1] = 0, && [\tilde{y_3}, a_1 \cdot \nabla_1] = 0,\\
&[\tilde{z_1}, a_1 \cdot \nabla_1] = 0, && [\tilde{z_2}, a_1 \cdot \nabla_1] = a \cdot \nabla_1 = \tilde{y_3} + A, && [\tilde{z_3}, a_1 \cdot \nabla_1] = -\tilde{y_2},\\
&[\tilde{y_1}, a_2 \cdot \nabla_2] = 0, && [\tilde{y_2}, a_2 \cdot \nabla_2] = B_{23} = -B - \Box_2, && [\tilde{y_3}, a_2 \cdot \nabla_2] = 0,\\
&[\tilde{z_1}, a_2 \cdot \nabla_2] = a \cdot \nabla_2 = -\tilde{y_3} + A, && [\tilde{z_2}, a_2 \cdot \nabla_2] = 0, && [\tilde{z_3}, a_2 \cdot \nabla_2] = \tilde{y_1}.
\end{aligned} \tag{4.9}$$

We will first calculate the zero-order bracket for the case where no DDI's need to be considered and do the generalization afterwards. We start by calculating the term $\delta^{(1)}_{\epsilon_1}\delta^{(0)}_{\epsilon_2}\Phi_3$ for the third field:

$$\delta^{(0)}_{\epsilon_1}\delta^{(1)}_{\epsilon_2}\Phi_3 = [\frac{1}{2}\partial_{y_2}\mathcal{V}(y,z), (a_1 \cdot \nabla_1)]\epsilon_1\epsilon_2 \tag{4.10}$$

Using the above commutation relations for the newly defined variables (and dropping the ˜ for simpler notation) we get

$$\delta^{(0)}_{\epsilon_1}\delta^{(1)}_{\epsilon_2}\Phi_3 = \frac{1}{2}(B\partial_{y_1}\partial_{y_2}\mathcal{V}(y,z) + A\partial_{z_2}\partial_{y_2}\mathcal{V}(y,z) + \mathcal{G}_1\partial_{y_2}\mathcal{V}(y,z))\epsilon_1\epsilon_2, \tag{4.11}$$

where all terms proportional to $\Box_1$ or $\Box_2$ can be dropped because of the differential constraint (1.14). For the second term we get

$$\delta^{(0)}_{\epsilon_2}\delta^{(1)}_{\epsilon_1}\Phi_3 = \frac{1}{2}(B\partial_{y_2}\partial_{y_1}\mathcal{V}(y,z) - A\partial_{z_1}\partial_{y_1}\mathcal{V}(y,z) - \mathcal{G}_2\partial_{y_1}\mathcal{V}(y,z))\epsilon_1\epsilon_2. \tag{4.12}$$

The commutator then is

$$\begin{aligned}\delta^{(0)}_{\epsilon_1}\delta^{(1)}_{\epsilon_2}\Phi_3 - \delta^{(0)}_{\epsilon_2}\delta^{(1)}_{\epsilon_1}\Phi_3 &= (\frac{1}{2}A(\partial_{z_2}\partial_{y_2} + \partial_{z_1}\partial_{y_1})\mathcal{V}(y,z)\\ &\quad+\mathcal{G}_1\partial_{y_2}\mathcal{V}(y,z) + \mathcal{G}_2\partial_{y_1}\mathcal{V}(y,z))\epsilon_1\epsilon_2\\ &= (\frac{1}{4}a\cdot\nabla(\partial_{z_2}\partial_{y_2} + \partial_{z_1}\partial_{y_1})\mathcal{V}(y,z)\\ &\quad+\mathcal{G}_1\partial_{y_2}\mathcal{V}(y,z) + \mathcal{G}_2\partial_{y_1}\mathcal{V}(y,z))\epsilon_1\epsilon_2.\end{aligned} \tag{4.13}$$

We can rewrite the terms containing the operators $\mathcal{G}_i$ as

$$\begin{aligned}\mathcal{G}_1\partial_{y_2}\mathcal{V}(y,z) + \mathcal{G}_2\partial_{y_1}\mathcal{V}(y,z) &= \partial_{y_2}\mathcal{G}_1\mathcal{V}(y,z) + \partial_{z_3}\mathcal{V}(y,z) + \partial_{y_1}\mathcal{G}_2\mathcal{V}(y,z) - \partial_{z_3}\mathcal{V}(y,z)\\ &= \partial_{y_2}\mathcal{G}_1\mathcal{V}(y,z) + \partial_{y_1}\mathcal{G}_2\mathcal{V}(y,z).\end{aligned} \tag{4.14}$$

Due to the on-shell gauge invariance of the vertex these terms can be dropped, since we are not considering DDI's at this point. The bracket therefore is

$$[[\epsilon_1,\epsilon_2]]^{(0)} = \frac{1}{4}(\partial_{y_1}\partial_{z_1} + \partial_{y_2}\partial_{z_2})\mathcal{V}(y,z)\epsilon_1\epsilon_2 = \frac{1}{2}(\partial_{z_2}\delta^{(1)}_{\epsilon_2}\Phi_3 - \partial_{z_1}\delta^{(1)}_{\epsilon_1}\Phi_3)\epsilon_1\epsilon_2. \tag{4.15}$$

Note that the deformations on the RHS of (4.15) are the ones which do not contain contributions from DDI's. Also, we work with the deformations as differential operators and writing out any arguments explicitly.

Coming now to the case where we consider DDI's we need to take care of two things: first, the first-order gauge deformations get corrected in the presence of DDI's, as we have seen in the last chapter. Second, we can no longer drop the terms proportional to $\mathcal{G}_i\mathcal{V}(y,z)$ in (4.14), since these terms could be proportional to some on-shell DDI, which no longer vanishes in the off-shell case. The first term in the commutator then is

$$\begin{aligned}\delta^{(0)}_{\epsilon_1}\delta^{(1)}_{\epsilon_2}\Phi_3 &= [\frac{1}{2}\partial_{y_2}\mathcal{V}(y,z)+\tilde{\Delta}_3, a_1\cdot\nabla_1]\epsilon_1\epsilon_2\\ &= (B\partial_{y_1}+A\partial_{z_2}+\mathcal{G}_1)(\frac{1}{2}\partial_{y_2}\mathcal{V}(y,z)+\tilde{\Delta}_3)\epsilon_1\epsilon_2,\end{aligned} \tag{4.16}$$

whereas the second term is

$$\delta^{(0)}_{\epsilon_2}\delta^{(1)}_{\epsilon_1}\Phi_3 = (-B\partial_{y_2}+A\partial_{z_1}+\mathcal{G}_2)(-\frac{1}{2}\partial_{y_1}\mathcal{V}(y,z)+\Delta_3)\epsilon_1\epsilon_2. \tag{4.17}$$

For the commutator we get

$$\begin{aligned}\delta^{(0)}_{\epsilon_1}\delta^{(1)}_{\epsilon_2}\Phi_3-\delta^{(0)}_{\epsilon_2}\delta^{(1)}_{\epsilon_1}\Phi_3 = [&A(\partial_{z_2}\delta^{(1)}_{\epsilon_2}\Phi_3-\partial_{z_1}\delta^{(1)}_{\epsilon_1}\Phi_3)+B(\partial_{y_1}\tilde{\Delta}_3+\partial_{y_2}\Delta_3)\\ &+\mathcal{G}_1\delta^{(1)}_{\epsilon_2}\Phi_3-\mathcal{G}_2\delta^{(1)}_{\epsilon_1}\Phi_3]\epsilon_1\epsilon_2.\end{aligned} \tag{4.18}$$

This can also be written as

$$\delta^{(0)}_{\epsilon_1}\delta^{(1)}_{\epsilon_2}\Phi_3-\delta^{(0)}_{\epsilon_2}\delta^{(1)}_{\epsilon_1}\Phi_3 = [A(\partial_{z_2}\delta^{(1)}_{\epsilon_2}\Phi_3-\partial_{z_1}\delta^{(1)}_{\epsilon_1}\Phi_3)+\tilde{\mathcal{G}}_1\delta^{(1)}_{\epsilon_2}\Phi_3-\tilde{\mathcal{G}}_2\delta^{(1)}_{\epsilon_1}\Phi_3]\epsilon_1\epsilon_2, \tag{4.19}$$

where the deformations are now the corrected deformations, already including contributions from DDI's. The terms that are proportional to A will give the corrected bracket, whereas the remaining terms need to vanish, in order for the gauge transformations to still form an algebra. So we need to have

$$(\tilde{\mathcal{G}}_1\delta^{(1)}_{\epsilon_2}\Phi_3-\tilde{\mathcal{G}}_2\delta^{(1)}_{\epsilon_1}\Phi_3)\epsilon_1\epsilon_2 = 0. \tag{4.20}$$

Writing this out leads to:

$$\frac{1}{2}\mathcal{G}_1\partial_{y_2}\mathcal{V}+\frac{1}{2}\mathcal{G}_2\partial_{y_1}\mathcal{V}+\mathcal{G}_1\tilde{\Delta}_3-\mathcal{G}_2\Delta_3+B\partial_{y_1}\tilde{\Delta}_3+B\partial_{y_2}\Delta_3. \tag{4.21}$$

Using (4.14), we can rewrite this as:

$$\frac{1}{2}\partial_{y_2}\mathcal{G}_1\mathcal{V}+\frac{1}{2}\partial_{y_1}\mathcal{G}_2\mathcal{V}+\mathcal{G}_1\tilde{\Delta}_3-\mathcal{G}_2\Delta_3+B\partial_{y_1}\tilde{\Delta}_3+B\partial_{y_2}\Delta_3. \tag{4.22}$$

Up to terms proportional to $\Box_1$ or $\Box_2$, which vanish due to the differential constraint, we have that

$$\mathcal{G}_1\mathcal{V} = -\Delta_3\Box_3, \qquad \mathcal{G}_2\mathcal{V} = -\tilde{\Delta}_3\Box_3, \tag{4.23}$$

$$B\partial_{y_1}\tilde{\Delta}_3 = \frac{1}{2}\partial_{y_1}\tilde{\Delta}_3\Box_3, \qquad B\partial_{y_2}\Delta_3 = \frac{1}{2}\partial_{y_1}\Delta_3\Box_3. \tag{4.24}$$

Plugging these expressions into (4.22) leaves us with

$$\mathcal{G}_1\tilde{\Delta}_3 - \mathcal{G}_2\Delta_3. \tag{4.25}$$

We can now multiply the above by $\Box_3$ and use (4.23) again to arrive at

$$\mathcal{G}_1\mathcal{G}_2\mathcal{V} - \mathcal{G}_2\mathcal{G}_1\mathcal{V}, \tag{4.26}$$

which vanishes since the operators of on-shell gauge variation commute. Since $\Box_3$ also commutes with the $\mathcal{G}_i$ we can infer that the extra terms in (4.19) indeed vanish.

From (4.19) we can now read off the corrected gauge bracket

$$[[\epsilon_1, \epsilon_2]]^{(0)} = \frac{1}{2}(\partial_{z_2}\delta^{(1)}_{\epsilon_2}\Phi_3 - \partial_{z_1}\delta^{(1)}_{\epsilon_1}\Phi_3)\epsilon_1\epsilon_2. \tag{4.27}$$

The formula is the same as in the case with no DDI's, but here the corrected deformations including contributions from DDI's have to be inserted. An analogous calculation for the other zero-order brackets yields

$$[[\epsilon_2, \epsilon_3]]^{(0)} = \frac{1}{2}(\partial_{z_3}\delta^{(1)}_{\epsilon_3}\Phi_1 - \partial_{z_2}\delta^{(1)}_{\epsilon_2}\Phi_1)\epsilon_2\epsilon_3, \tag{4.28}$$

$$[[\epsilon_3, \epsilon_1]]^{(0)} = \frac{1}{2}(\partial_{z_1}\delta^{(1)}_{\epsilon_1}\Phi_2 - \partial_{z_3}\delta^{(1)}_{\epsilon_3}\Phi_2)\epsilon_3\epsilon_1. \tag{4.29}$$

4.2 Zero-order commutators for spin-2 and spin-3

Making use of equations (4.27)-(4.29), we can now calculate the zero-order gauge brackets induced by the vertices (2.17)-(2.20):

2-2-2 vertex:

$$[[\epsilon_1, \epsilon_2]]^{(0)} = (\frac{3}{2}y_1 z_1 + \frac{3}{2}y_2 z_2 + y_3 z_3)\epsilon_1\epsilon_2 \tag{4.30}$$

3-3-2 vertex:

$$[[\epsilon_1, \epsilon_2]]^{(0)} = (\frac{3}{2}y_1 z_1 z_3 + \frac{3}{2}y_2 z_2 z_3 + \frac{15}{2}y_3 z_3^2)\epsilon_1\epsilon_2 \tag{4.31}$$

$$[[\epsilon_2, \epsilon_3]]^{(0)} = (3y_1 z_1 z_3 + \frac{9}{2}y_2 z_2 z_3 + \frac{9}{4}y_3 z_3^2)\epsilon_2\epsilon_3 \tag{4.32}$$

3-3-3 vertex:

$$[[\epsilon_1, \epsilon_2]]^{(0)} = (-y_3^2 z_3^2 - \frac{1}{4}y_1 y_3 z_1 z_3 - \frac{1}{4}y_2 y_3 z_2 z_3 - y_1 y_2 z_1 z_2)\epsilon_1\epsilon_2 \tag{4.33}$$

3-2-2 vertex:

$$[[\epsilon_1, \epsilon_2]]_3^{(0)} = (\frac{1}{4}y_1 y_3 z_3 - \frac{3}{4}y_1 y_2 z_2)\epsilon_1\epsilon_2 \tag{4.34}$$

$$[[\epsilon_2, \epsilon_3]]_1^{(0)} = (-y_1^2 z_1 - \frac{1}{2}y_1 y_3 z_3 - \frac{1}{2}y_1 y_2 z_2)\epsilon_2\epsilon_3 \tag{4.35}$$

One can see that the bracket induced by the 3-3-3 vertex can again be changed by using the two-derivative DDI's (2.25)-(2.30), whereas for the other brackets this is not possible.

In index-notation these brackets read:

2-2-2 vertex:

$$[[\epsilon_1, \epsilon_2]]^{(0)} = +g_1\ \partial_\mu \epsilon_1^\alpha a^\mu \epsilon_{2,\alpha} - (1-g_1)\ \epsilon_1^\alpha \partial_\mu \epsilon_{2,\alpha} a^\mu \tag{4.36}$$

$$+\frac{3}{2}\epsilon_1^\alpha \partial_\alpha \epsilon_{2,\mu} a^\mu - \frac{3}{2}\partial_\alpha \epsilon_{1,\mu} a^\mu \epsilon_2^\alpha \tag{4.37}$$

3-3-2 vertex:

$$[[\epsilon_1,\epsilon_2]]^{(0)} = +\frac{15}{2}g_2\ \partial_\mu\epsilon_1^{\alpha\beta}a^\mu\epsilon_{2,\alpha\beta} - \frac{15}{2}(1-g_2)\ \epsilon_1^{\alpha\beta}\partial_\mu\epsilon_{2,\alpha\beta}a^\mu + \frac{3}{2}\epsilon_1^{\alpha\beta}\partial_\alpha\epsilon_{2,\beta\mu}a^\mu - \frac{3}{2}\partial_\alpha\epsilon_{1,\beta\mu}a^\mu\epsilon_2^{\alpha\beta} \tag{4.38}$$

$$[[\epsilon_2,\epsilon_3]]^{(0)} = +3g_3\ \partial_\mu\epsilon_{2,\nu}^{\alpha}a^\mu a^\nu\epsilon_{3,\alpha} - 3(1-g_3)\ \epsilon_{2,\nu}^{\alpha}a^\nu\partial_\mu\epsilon_{3,\alpha}a^\mu + \frac{9}{2}\epsilon_{2,\mu}^{\alpha}a^\mu\partial_\alpha\epsilon_{3,\nu}a^\nu - \frac{9}{4}\partial_\alpha\epsilon_{2,\mu\nu}a^\mu a^\nu\epsilon_3^\alpha \tag{4.39}$$

3-3-3 vertex:

$$\begin{aligned}[[\epsilon_1,\epsilon_2]]^{(0)} = &-g_4^2\ \partial_\mu\partial_\nu\epsilon_1^{\alpha\beta}a^\mu a^\nu\epsilon_{2,\alpha\beta} + g_4(1-g_4)\ \partial_\mu\epsilon_1^{\alpha\beta}a^\mu\partial_\nu\epsilon_{2,\alpha\beta}a^\nu \\ &+(1-g_4)g_4\ \partial_\nu\epsilon_1^{\alpha\beta}a^\nu\partial_\mu\epsilon_{2,\alpha\beta}a^\mu - (1-g_4)^2\ \epsilon_1^{\alpha\beta}\partial_\mu\partial_\nu\epsilon_{2,\alpha\beta}a^\mu a^\nu \\ &-\frac{1}{4}g_4\ \partial_\mu\epsilon_1^{\alpha\beta}a^\mu\partial_\alpha\epsilon_{2,\beta\nu}a^\nu + \frac{1}{4}(1-g_4)\ \epsilon_1^{\alpha\beta}\partial_\alpha\partial_\mu\epsilon_{2,\beta\nu}a^\mu a^\nu \\ &+\frac{1}{4}g_4\ \partial_\alpha\partial_\mu\epsilon_{1,\beta\nu}a^\mu a^\nu\epsilon_2^{\alpha\beta} - \frac{1}{4}(1-g_4)\ \partial_\alpha\epsilon_{1,\beta\nu}a^\nu\partial_\mu\epsilon_2^{\alpha\beta}a^\mu \\ &+\partial_\alpha\epsilon_{1,\mu}^{\beta}a^\mu\partial_\beta\epsilon_{2,\nu}^{\alpha}a^\nu\end{aligned} \tag{4.40}$$

3-2-2 vertex:

$$[[\epsilon_1,\epsilon_2]]^{(0)} = +\frac{1}{4}g_5\ \partial_\mu\epsilon_1^{\alpha\beta}a^\mu\partial_\alpha\epsilon_{2,\beta} - \frac{1}{4}(1-g_5)\ \epsilon_1^{\alpha\beta}\partial_\alpha\partial_\mu\epsilon_{2,\beta}a^\mu \tag{4.41}$$

$$+\frac{3}{4}\partial_\alpha\epsilon_{1,\mu}^{\beta}a^\mu\partial_\beta\epsilon_2^\alpha \tag{4.42}$$

$$\begin{aligned}[[\epsilon_2,\epsilon_3]]^{(0)} = &-g_6^2\ \partial_\mu\partial_\nu\epsilon_2^{\alpha}a^\mu a^\nu\epsilon_{3,\alpha} + g_6(1-g_6)\ \partial_\mu\epsilon_2^{\alpha}a^\mu\partial_\nu\epsilon_{3,\alpha}a^\nu \\ &+(1-g_6)g_6\ \partial_\nu\epsilon_2^{\alpha}a^\nu\partial_\mu\epsilon_{3,\alpha}a^\mu - (1-g_6)^2\ \epsilon_2^{\alpha}\partial_\mu\partial_\nu\epsilon_{3,\alpha}a^\mu a^\nu \\ &+\frac{1}{2}g_6\ \partial_\alpha\partial_\mu\epsilon_{2,\nu}a^\mu a^\nu\epsilon_3^\alpha - \frac{1}{2}(1-g_6)\ \partial_\alpha\epsilon_{2,\nu}a^\nu\partial_\mu\epsilon_3^\alpha a^\mu \\ &-\frac{1}{2}g_6\ \partial_\mu\epsilon_2^{\alpha}a^\mu\partial_\alpha\epsilon_{3,\nu}a^\nu + \frac{1}{2}(1-g_6)\ \epsilon_2^{\alpha}\partial_\alpha\partial_\mu\epsilon_{3,\nu}a^\mu a^\nu\end{aligned} \tag{4.43}$$

Again we have checked that for the two-derivative vertices the brackets related by an interchange of fields of equal spin indeed agree, whereas for the three-derivative vertices the respective brackets differ by a sign.

Note that these brackets are not yet traceless, so the trace part still needs to be subtracted in order to obtain a traceless gauge parameter. Since we will only be using Killing tensors as gauge parameters from now on and the bracket of two Killing tensors is again a Killing tensor, the trace-part will not play any role here.

As for the first-order gauge deformations, the above expressions for the brackets are not unambiguous, since they can be changed by the means of redefinitions. For parameter redefinitions and field redefinitions (3.50) and (3.48) discussed at the end of chapter 3, the most general expression for the gauge bracket is rather lenghty and given in [22].

5 Consistency conditions for spin-3-gravity

From now on we will restrict ourselves to a specific higher-spin theory, which is spin-3-gravity. This theory consists of only one spin-2 field and one spin-3 field. Therefore, only the two-derivative vertices (2.17) and (2.18) need to be considered. We now want to check the consistency of this theory at the global symmetry level, which requires that we use Killing-tensors as gauge parameters. More precisely, we will use the Killing-tensors given in (1.25) and (1.26).

For Killing-tensors, two major simplifications for both the first-order deformations and the brackets occur. First, as can be seen from (3.49) and (3.51), all contributions coming from parameter redefinitions and field redefinitions vanish when the Killing-equation is satisfied. Second, since the Killing tensors (1.25), (1.26) are always traceless and divergence-free, our choice to only work with TT-vertices does not lead to any restrictions. This means that when using Killing-tensors as gauge parameters both the first-order deformations and the bracket become unambiguously determined. However, care needs to be taken when dealing with traces and divergences of first-order deformations, as we will see in 5.1.

We now come to the conditions that need to be satisfied for spin-3-gravity to be consistent. The first condition we want to check regards the EOM of the theory. In the full theory it is expected that the full EOM of the fields are invariant (up to on-shell terms) under the full gauge transformations. The gauge variation of the EOM, i.e. the Fronsdal-equation, can be written as

$$\delta_\epsilon \mathcal{F}\Phi = \delta_\epsilon^{(0)} \mathcal{F}\Phi + \delta_\epsilon^{(1)} \mathcal{F}\Phi + \cdots = 0. \tag{5.1}$$

Since the Fronsdal-equation is invariant under zero-order gauge transformations, the first term on the RHS is zero. At the next order, we are left with

$$\mathcal{F}\delta_\epsilon^{(1)}\Phi = 0, \tag{5.2}$$

where $\mathcal{F}$ is the Fronsdal-tensor, as given in (1.7). Therefore, we expect that the first-order gauge deformations fulfill the Fronsdal-equation, just like the field itself.

The second consistency condition involves the gauge brackets. Since gauge transformations are of the type

$$\Phi \to \Phi + \delta_\epsilon \Phi \tag{5.3}$$

and thus substitutions, they have to satisfy the Jacobi-identity

$$\sum_{cyc.} [\delta_{\epsilon_1}, [\delta_{\epsilon_2}, \delta_{\epsilon_3}]]\Phi = 0. \tag{5.4}$$

Expanding both the gauge transformations and the brackets as power series in g, the Jacobi-identity at lowest order reads [27]

$$\sum_{cyc.} \{\delta^{(0)}_{[[\epsilon_1,[[\epsilon_2,\epsilon_3]]^{(0)}]]^{(0)}}\Phi + \delta^{(0)}_{[[\epsilon_2,\epsilon_3]]^{(1)}(a_1\cdot\nabla_1\epsilon_1)}\Phi\} = 0. \tag{5.5}$$

This has to hold for arbitrary gauge parameters, so we have

$$\sum_{cyc.} \{[[\epsilon_1, [[\epsilon_2, \epsilon_3]]^{(0)}]]^{(0)} + [[\epsilon_2, \epsilon_3]]^{(1)}(a_1 \cdot \nabla_1 \epsilon_1)\} = 0. \tag{5.6}$$

For Killing-tensors the zero-order bracket is already the full bracket of the global symmetry algebra [ME3], so the higher-order corrections to the bracket are zero and the second term in the above vanishes. This leaves us with

$$\sum_{cyc.} [[\bar{\epsilon}_1, [[\bar{\epsilon}_2, \bar{\epsilon}_3]]^{(0)}]]^{(0)} = 0. \tag{5.7}$$

The last condition we need to check is related to the demand that gauge transformations form an algebra (4.1). The lowest-order part (4.3) of this equation was used to calculate the zero-order gauge brackets. At one order higher we have [MA5]

$$\begin{aligned}(a\cdot\nabla)[[\epsilon_1,\epsilon_2]]^{(1)}(\Phi) + \delta^{(1)}_{[[\epsilon_1,\epsilon_2]]^{(0)}}(\Phi)\tilde{\Phi} = {} & \delta^{(1)}_{\epsilon_2}(\delta^{(1)}_{\epsilon_1}(\Phi))\tilde{\Phi} - \delta^{(1)}_{\epsilon_1}(\delta^{(1)}_{\epsilon_2}(\Phi))\tilde{\Phi} \\ & + \delta^{(2)}_{\epsilon_2}((a_1\cdot\nabla_1)\epsilon_1,\Phi)\tilde{\Phi} - \delta^{(2)}_{\epsilon_1}((a_2\cdot\nabla_2)\epsilon_2,\Phi)\tilde{\Phi} \\ & + \delta^{(2)}_{\epsilon_2}(\Phi,(a_1\cdot\nabla_1)\epsilon_1)\tilde{\Phi} - \delta^{(2)}_{\epsilon_1}(\Phi,(a_2\cdot\nabla_2)\epsilon_2)\tilde{\Phi}.\end{aligned} \tag{5.8}$$

For Killing-tensors, the last four terms on the RHS and the first term on the LHS vanish, which gives

$$\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))\tilde{\Phi} - \delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi))\tilde{\Phi} + \delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]^{(0)}}(\Phi)\tilde{\Phi} = 0 + \text{trivial}. \tag{5.9}$$

At this order, the "trivial"-part can consist of terms that are either proportional to the free EOM (which is related to the demand that the gauge algebra only closes on-shell) or of the form of zero-order gauge transformations [84]. We will refer to this condition as first-order closure of the gauge algebra (FCGA). Only if all of the above conditions are fulfilled, the theory of spin-3-gravity is consistent at cubic order.

Before we start to check these conditions, there is another calculational tool we need, which is the concept of dualisation. In $d = 3$ a tensor $\bar{\epsilon}$ of mixed-symmetry type (2,1), like the spin-3 Killing tensors with $k = 1$ we are working with, is dual to a symmetric rank-2 tensor $\tilde{\epsilon}$ by the following relations:

$$\tilde{\epsilon}_{\alpha\beta} = \bar{\epsilon}_{\alpha ab}\epsilon^{ab}{}_{\beta} + \bar{\epsilon}_{\beta ab}\epsilon^{ab}{}_{\alpha}, \tag{5.10}$$

$$\bar{\epsilon}_{\alpha\beta\gamma} = \tilde{\epsilon}_{\alpha a}\epsilon^{a}{}_{\beta\gamma} + \tilde{\epsilon}_{\beta a}\epsilon^{a}{}_{\alpha\gamma}, \tag{5.11}$$

where ϵ is the Levi-Civita tensor. The symmetric rank-2 tensor defined this way is traceless by definition.

Using the dualised spin-3 Killing tensors has some advantages. First, it allows one to get rid of the Killing symmetry and replace it with a much easier symmetry. Second, in the case of the FCGA the dualisation reduces the number of indices we have to deal with, which greatly simplifies the calculation of relevant DDI's.

5.1 Fronsdal-equation for the gauge deformations

As already mentioned, from now on we will always use the Killing-tensors in (1.25) and (1.26) as gauge parameters. When checking the Fronsdal-equation for the first-order gauge deformations it is important to note that even though we work in the TT-setting we still need to keep the trace- and divergence terms in the Fronsdal-tensor, since the gauge deformations they act on are no longer traceless and divergence-free in general. This means that even if we drop all divergences and traces of the field itself, the DT-terms in the Fronsdal-tensor can still produce

TT-terms when acting on a gauge deformation. The condition we want to check therefore is

$$\mathcal{F}\delta_{\bar{\epsilon}}^{(1)}\Phi = \Box\delta_{\bar{\epsilon}}^{(1)}\Phi - (a\cdot\nabla)(\partial\cdot\nabla)\delta_{\bar{\epsilon}}^{(1)}\Phi - \frac{1}{2}(a\cdot\nabla)^2(\partial\cdot\partial)\delta_{\bar{\epsilon}}^{(1)}\Phi = 0. \tag{5.12}$$

Even though the gauge deformations themselves are not necessarily traceless and divergence-free any more, we can still drop traces and divergences of the gauge parameter and the field itself, as well as the TT-EOM for the field, $\Box\Phi = 0$. Equivalence up to these terms will be denoted by $\approx$.

2-2-deformation:

$$\begin{aligned}\mathcal{F}\delta_{\bar{\epsilon}}^{(1)}\Phi \approx &+ 2(1-c)\bar{\epsilon}^{\alpha\beta}\partial_\mu\partial_\beta\Phi_{\alpha\nu}a^\mu a^\nu + (1+c)\bar{\epsilon}^{\alpha\beta}\partial_\mu\partial_\beta\Phi_{\alpha\nu}a^\mu a^\nu\\ &- (1-c)\bar{\epsilon}^{\alpha\beta}\partial_\mu\partial_\beta\Phi_{\alpha\nu}a^\mu a^\nu + 2\bar{\epsilon}^{\alpha\beta}\partial_\mu\partial_\alpha\Phi_{\beta\nu}a^\mu a^\nu\\ =& \,0\end{aligned} \tag{5.13}$$

As we can see, the 2-2-deformation satisfies the Fronsdal-equation for arbitrary values of the partial integration parameter c.

2-3-deformation:

$$\begin{aligned}\mathcal{F}\delta_{\bar{\epsilon}}^{(1)}\Phi \approx &+ 3e\bar{\epsilon}^{\alpha\beta}\partial_\mu\partial_\beta\Phi_{\alpha\nu\lambda}a^\mu a^\nu a^\lambda - \frac{3}{2}\bar{\epsilon}^{\alpha\beta}\partial_\mu\partial_\beta\Phi_{\alpha\nu\lambda}a^\mu a^\nu a^\lambda\\ &+ 3\bar{\epsilon}^{\alpha\beta}\partial_\mu\partial_\alpha\Phi_{\beta\nu\lambda}a^\mu a^\nu a^\lambda + \frac{3}{2}(2-e)\bar{\epsilon}^{\alpha\beta}\partial_\mu\partial_\beta\Phi_{\alpha\nu\lambda}a^\mu a^\nu a^\lambda\\ =& \,0\end{aligned} \tag{5.14}$$

Again, the Fronsdal-equation is satisfied for arbitrary values of e.

3-2-deformation:

$$\begin{aligned}\mathcal{F}\delta_{\bar{\epsilon}}^{(1)}\Phi \approx &+ 15(1-d)\bar{\epsilon}^{\alpha\beta\gamma}\partial_\mu\partial_\gamma\Phi_{\alpha\beta\nu}a^\mu a^\nu + 3\bar{\epsilon}^{\alpha\gamma\beta}\partial_\alpha\partial_\gamma\Phi_{\beta\mu\nu}a^\mu a^\nu\\ &+ (\frac{15}{2}d - \frac{3}{2})\bar{\epsilon}^{\alpha\beta\gamma}\partial_\mu\partial_\gamma\Phi_{\alpha\beta\nu}a^\mu a^\nu - \frac{15}{2}(1-d)\bar{\epsilon}^{\alpha\beta\gamma}\partial_\mu\partial_\gamma\Phi_{\alpha\beta\nu}a^\mu a^\nu\\ &- 3\bar{\epsilon}^{\beta\gamma\alpha}\partial_\mu\partial_\alpha\Phi_{\beta\gamma\nu}a^\mu a^\nu\\ =&+ 3\bar{\epsilon}^{\alpha\beta\gamma}\partial_\mu\partial_\gamma\Phi_{\alpha\beta\nu}a^\mu a^\nu + 3\bar{\epsilon}^{\alpha\gamma\beta}\partial_\alpha\partial_\gamma\Phi_{\beta\mu\nu}a^\mu a^\nu\end{aligned} \tag{5.15}$$

In contrast to the previous cases, this does not vanish identically. However, the above expression is actually a DDI, so the Fronsdal-equation is again fulfilled. Since the partial integration parameter drops out of the calculation, this is the case for all values of d.

3-3-deformation:

$$\begin{aligned}\mathcal{F}\delta^{(1)}_{\bar{\epsilon}}\Phi \approx &+3f\bar{\epsilon}^{\alpha\beta}{}_{\rho}x^{\rho}\partial_{\mu}\partial_{\nu}\partial_{\lambda}\Phi_{\alpha\beta}a^{\mu}a^{\nu}a^{\lambda} + 3\bar{\epsilon}^{\alpha\beta}{}_{\rho}x^{\rho}\partial_{\mu}\partial_{\alpha}\partial_{\beta}\Phi_{\nu\lambda}a^{\mu}a^{\nu}a^{\lambda} \\ &-6\bar{\epsilon}^{\alpha\beta}{}_{\rho}x^{\rho}\partial_{\mu}\partial_{\nu}\partial_{\alpha}\Phi_{\beta\lambda}a^{\mu}a^{\nu}a^{\lambda} + (9f-3)\bar{\epsilon}^{\alpha\beta}{}_{\mu}a^{\mu}\partial_{\nu}\partial_{\lambda}\Phi_{\alpha\beta}a^{\nu}a^{\lambda} \\ &+6\bar{\epsilon}^{\alpha\beta}{}_{\mu}a^{\mu}\partial_{\alpha}\partial_{\beta}\Phi_{\nu\lambda}a^{\nu}a^{\lambda} - 12\bar{\epsilon}^{\alpha\beta}{}_{\mu}a^{\mu}\partial_{\nu}\partial_{\alpha}\Phi_{\beta\lambda}a^{\mu}a^{\nu}a^{\lambda}\end{aligned} \tag{5.16}$$

Here we can see that the partial integration parameter does not drop out of the calculation. In the above expression we have two different types of terms: those with two derivatives and those with three derivatives acting on Φ. For these terms we have different DDI's, so the two groups need to vanish individually. Both groups form a DDI only for $f = 1$. This means that the Fronsdal-equation is fulfilled for a unique value of the partial integration parameter.

To summarize some obervations gained in these calculations:

- the 2-2-deformation is traceless for all values of c, divergence-free for $c = 1$ and fulfills the Fronsdal-equation for all values of c
- the 2-3-deformation is traceless for all values of e, divergence-free for $e = 0$ and fulfills the Fronsdal-equation for all values of e
- the 3-2-deformation is traceless for all values of d, divergence-free for $d = \frac{4}{5}$ and fulfills the Fronsdal-equation for all values of d
- the 3-3-deformation is neither traceless nor divergence-free for any value of f and fulfills the Fronsdal-equation only for $f = 1$

5.2 Jacobi-identity

For the Jacobi-identity (5.7) we have many different cases that need to be distinguished, depending on the choice of gauge parameters. In the following we will calculate the Jacobi-identity for all possible triples of spin-2 parameters and spin-3 parameters, which covers all cases that are relevant for spin-3-gravity.

5.2.1 Three spin-2 parameters

We start with the case of three spin-2 parameters. Here all three terms in the Jacobi-identity consist of the same double-bracket, but with cyclically permuted labels for the Killing tensors, which means that we do not have to include any coupling constants in this calculation. We have to use the 2-2-bracket from the 2-2-2-vertex both as the inner bracket and the outer bracket. For Killing-tensors this bracket reads:

$$[[\bar{\epsilon}_1, \bar{\epsilon}_2]] = (g_1 + \frac{3}{2})\bar{\epsilon}^{\alpha}_{1,\mu} a^{\mu} \bar{\epsilon}_{2,\alpha\rho} x^{\rho} - (\frac{5}{2} - g_1)\bar{\epsilon}^{\alpha}_{1,\rho} x^{\rho} \bar{\epsilon}_{2,\alpha\mu} a^{\mu} \tag{5.17}$$

.

The double bracket then is:

$$\begin{aligned}[[\bar{\epsilon}_3, [[\bar{\epsilon}_1, \bar{\epsilon}_2]]]] = \; & +g_1 \partial_\mu \bar{\epsilon}^{\alpha}_3 [[\bar{\epsilon}_1, \bar{\epsilon}_2]]_\alpha - (1 - g_1)\bar{\epsilon}^{\alpha}_3 \partial_\mu [[\bar{\epsilon}_1, \bar{\epsilon}_2]]_\alpha \\ & + \frac{3}{2}\bar{\epsilon}^{\alpha}_3 \partial_\alpha [[\bar{\epsilon}_1, \bar{\epsilon}_2]]_\mu - \frac{3}{2}[[\bar{\epsilon}_1, \bar{\epsilon}_2]]^{\alpha} \partial_\alpha \bar{\epsilon}_{3,\mu}\end{aligned} \tag{5.18}$$

Plugging in the inner bracket this reads:

$$\begin{aligned}[[\bar{\epsilon}_3, [[\bar{\epsilon}_1, \bar{\epsilon}_2]]]] = \; & \bar{\epsilon}^{\alpha}_{3,\mu} a^{\mu} [+(g_1(g_1 + \frac{3}{2}) + \frac{3}{2}(g_1 + \frac{3}{2}))\bar{\epsilon}^{\beta}_{1,\alpha} \bar{\epsilon}_{2,\beta\rho} x^{\rho} \\ & -(g_1(\frac{5}{2} - g_1) + \frac{3}{2}(\frac{5}{2} - g_1))\bar{\epsilon}^{\beta}_{1,\rho} x^{\rho} \bar{\epsilon}_{2,\beta\alpha}] \\ & + \bar{\epsilon}^{\alpha}_{3,\rho} x^{\rho} [+(\frac{3}{2}(g_1 + \frac{3}{2}) + (1 - g_1)(\frac{5}{2} - g_1))\bar{\epsilon}^{\beta}_{1,\mu} a^{\mu} \bar{\epsilon}_{2,\beta\alpha} \\ & -(\frac{3}{2}(\frac{5}{2} - g_1) + (1 - g_1)(g_1 + \frac{3}{2}))\bar{\epsilon}^{\beta}_{1,\alpha} \bar{\epsilon}_{2,\beta\mu} a^{\mu}]\end{aligned} \tag{5.19}$$

The other two terms are obtained by cyclic permutation of the Killing tensors:

$$\begin{aligned}[[\bar{\epsilon}_1, [[\bar{\epsilon}_2, \bar{\epsilon}_3]]]] = \; & \bar{\epsilon}^{\alpha}_{1,\mu} a^{\mu} [+(g_1(g_1 + \frac{3}{2}) + \frac{3}{2}(g_1 + \frac{3}{2}))\bar{\epsilon}^{\beta}_{2,\alpha} \bar{\epsilon}_{3,\beta\rho} x^{\rho} \\ & -(g_1(\frac{5}{2} - g_1) + \frac{3}{2}(\frac{5}{2} - g_1))\bar{\epsilon}^{\beta}_{2,\rho} x^{\rho} \bar{\epsilon}_{3,\beta\alpha}] \\ & + \bar{\epsilon}^{\alpha}_{1,\rho} x^{\rho} [+(\frac{3}{2}(g_1 + \frac{3}{2}) + (1 - g_1)(\frac{5}{2} - g_1))\bar{\epsilon}^{\beta}_{2,\mu} a^{\mu} \bar{\epsilon}_{3,\beta\alpha} \\ & -(\frac{3}{2}(\frac{5}{2} - g_1) + (1 - g_1)(g_1 + \frac{3}{2}))\bar{\epsilon}^{\beta}_{2,\alpha} \bar{\epsilon}_{3,\beta\mu} a^{\mu}]\end{aligned} \tag{5.20}$$

$$
\begin{aligned}
[[\bar{\epsilon}_2, [[\bar{\epsilon}_3, \bar{\epsilon}_1]]]] = \bar{\epsilon}^{\alpha}_{2,\mu} a^{\mu} [&+(g_1(g_1 + \frac{3}{2}) + \frac{3}{2}(g_1 + \frac{3}{2}))\bar{\epsilon}^{\beta}_{3,\alpha}\bar{\epsilon}_{1,\beta\rho} x^{\rho} \\
&-(g_1(\frac{5}{2} - g_1) + \frac{3}{2}(\frac{5}{2} - g_1))\bar{\epsilon}^{\beta}_{3,\rho} x^{\rho}\bar{\epsilon}_{1,\beta\alpha}] \\
+\bar{\epsilon}^{\alpha}_{2,\rho} x^{\rho} [&+(\frac{3}{2}(g_1 + \frac{3}{2}) + (1 - g_1)(\frac{5}{2} - g_1))\bar{\epsilon}^{\beta}_{3,\mu} a^{\mu}\bar{\epsilon}_{1,\beta\alpha} \\
&-(\frac{3}{2}(\frac{5}{2} - g_1) + (1 - g_1)(g_1 + \frac{3}{2}))\bar{\epsilon}^{\beta}_{3,\alpha}\bar{\epsilon}_{1,\beta\mu} a^{\mu}]
\end{aligned}
\tag{5.21}
$$

Summing up these terms leads to:

$$
\begin{aligned}
&\sum_{cyc}[[\bar{\epsilon}_3, [[\bar{\epsilon}_1, \bar{\epsilon}_2]]]] = \\
&\quad \bar{\epsilon}_{1,\beta\alpha}\bar{\epsilon}^{\alpha}_{2,\rho} x^{\rho}\bar{\epsilon}^{\beta}_{3,\mu} a^{\mu}[+\frac{3}{2}(g_1 + \frac{3}{2}) + (1 - g_1)(\frac{5}{2} - g_1) - g_1(g_1 + \frac{3}{2}) - \frac{3}{2}(g_1 + \frac{3}{2})] \\
&+\bar{\epsilon}^{\alpha}_{1,\rho} x^{\rho}\bar{\epsilon}_{2,\beta\alpha}\bar{\epsilon}^{\beta}_{3,\mu} a^{\mu}[-\frac{3}{2}(\frac{5}{2} - g_1) - (1 - g_1)(g_1 + \frac{3}{2}) + g_1(\frac{5}{2} - g_1) + \frac{3}{2}(\frac{5}{2} - g_1)] \\
&+\bar{\epsilon}^{\alpha}_{1,\mu} a^{\mu}\bar{\epsilon}_{2,\beta\alpha}\bar{\epsilon}^{\beta}_{3,\rho} x^{\rho}[+g_1(g_1 + \frac{3}{2}) + \frac{3}{2}(g_1 + \frac{3}{2}) - \frac{3}{2}(g_1 + \frac{3}{2}) - (1 - g_1)(\frac{5}{2} - g_1)] \\
&+\bar{\epsilon}_{1,\beta\alpha}\bar{\epsilon}^{\alpha}_{2,\mu} a^{\mu}\bar{\epsilon}^{\beta}_{3,\rho} x^{\rho}[-g_1(\frac{5}{2} - g_1) - \frac{3}{2}(\frac{5}{2} - g_1) + \frac{3}{2}(\frac{5}{2} - g_1) + (1 - g_1)(g_1 + \frac{3}{2})] \\
&+\bar{\epsilon}^{\alpha}_{1,\mu} a^{\mu}\bar{\epsilon}^{\beta}_{2,\rho} x^{\rho}\bar{\epsilon}_{3,\beta\alpha}[-g_1(\frac{5}{2} - g_1) - \frac{3}{2}(\frac{5}{2} - g_1) + \frac{3}{2}(\frac{5}{2} - g_1) + (1 - g_1)(g_1 + \frac{3}{2})] \\
&+\bar{\epsilon}^{\alpha}_{1,\rho} x^{\rho}\bar{\epsilon}^{\beta}_{2,\mu} a^{\mu}\bar{\epsilon}_{3,\beta\alpha}[+\frac{3}{2}(g_1 + \frac{3}{2}) + (1 - g_1)(\frac{5}{2} - g_1) - g_1(g_1 + \frac{3}{2}) - \frac{3}{2}(g_1 + \frac{3}{2})]
\end{aligned}
\tag{5.22}
$$

Demanding that this expression vanishes (i.e. that the Jacobi-identity is satisfied) leads to a system of equations that can easily be solved. It turns out that by choosing $\frac{1}{2}$ for the partial integration parameter of the 2-2-bracket the Jacobi-identity can be fulfilled:

$$
\sum_{cyc}[[\bar{\epsilon}_3, [[\bar{\epsilon}_1, \bar{\epsilon}_2]]]] = 0 \tag{5.23}
$$

As mentioned in chapter 4, it is already necessary in the derivation of the bracket to choose $g_i = \frac{1}{2}$ for all partial integration parameters of the brackets, so this is an important conistency check. This choice is also the only one that makes the 2-2-bracket manifestly anti-symmetric.

5.2.2 One spin-3 parameter, two spin-2 parameters

Next comes the case of two spin-2 parameters and one spin-3 parameter. This time, not all terms in the Jacobi-identity use the same brackets. Since the Killing-tensors are the same in all three terms, we can discard their prefactors. The first double bracket is of the form [[3,[[2,2]]]] and consists of the 2-2-bracket from 2-2-2 as the inner bracket and the 3-2-bracket from 3-3-2 as the outer bracket, whereas the other two brackets are of the form [[2,[[2,3]]]] and [[2,[[3,2]]]] respectively and consist of the 2-3-bracket from 3-3-2 as the inner bracket and the same bracket as the outer bracket. Since the first term ist the only one that contains a bracket from the 2-2-2 vertex we have to include a relative coupling constant $\tilde{g}_{222} = \frac{g_{222}}{g_{332}}$ in the first term. For this term, the inner bracket is:

$$[[\bar{\epsilon}_1, \bar{\epsilon}_2]] = \tilde{g}_{222}[(g_1 + \frac{3}{2})\bar{\epsilon}^{\alpha}_{1,\mu}a^{\mu}\bar{\epsilon}_{2,\alpha\rho}x^{\rho} - (\frac{5}{2} - g_1)\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\bar{\epsilon}_{2,\alpha\mu}a^{\mu}] \tag{5.24}$$

The double bracket then is:

$$\begin{aligned}[[\bar{\epsilon}_3, [[\bar{\epsilon}_1, \bar{\epsilon}_2]]]] = \; & +3g_3\partial_{\mu}\bar{\epsilon}^{\alpha}_{3,\nu}[[\bar{\epsilon}_1, \bar{\epsilon}_2]]_{\alpha} - 3(1-g_3)\bar{\epsilon}^{\alpha}_{3,\nu}\partial_{\mu}[[\bar{\epsilon}_1, \bar{\epsilon}_2]]_{\alpha} \\ & +\frac{9}{2}\bar{\epsilon}^{\alpha}_{3,\mu}\partial_{\alpha}[[\bar{\epsilon}_1, \bar{\epsilon}_2]]_{\nu} - \frac{9}{4}\partial_{\alpha}\bar{\epsilon}_{3,\mu\nu}[[\bar{\epsilon}_1, \bar{\epsilon}_2]]^{\alpha}\end{aligned} \tag{5.25}$$

Plugging in the inner bracket this reads:

$$\begin{aligned}[[\bar{\epsilon}_3, [[\bar{\epsilon}_1, \bar{\epsilon}_2]]]] = \; & \tilde{g}_{222}\bar{\epsilon}^{\alpha}_{3,\nu\rho}a^{\nu}x^{\rho}[+(6(1-g_3)(\frac{5}{2} - g_1) + 9(g_1 + \frac{3}{2}))\bar{\epsilon}^{\beta}_{1,\mu}a^{\mu}\bar{\epsilon}_{2,\beta\alpha} \\ & -(6(1-g_3)(g_1 + \frac{3}{2}) + 9(\frac{5}{2} - g_1))\bar{\epsilon}_{1,\beta\alpha}\bar{\epsilon}^{\beta}_{2,\mu}a^{\mu}] \\ & +\tilde{g}_{222}\bar{\epsilon}^{\alpha}_{3,\mu\nu}a^{\mu}a^{\nu}[+(6g_3+9)(g_1 + \frac{3}{2})\bar{\epsilon}_{1,\beta\alpha}\bar{\epsilon}^{\beta}_{2,\rho}x^{\rho} \\ & -(6g_3+9)(\frac{5}{2} - g_1)\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\bar{\epsilon}_{2,\beta\alpha}]\end{aligned} \tag{5.26}$$

For the second term the inner bracket is:

$$[[\bar{\epsilon}_2, \bar{\epsilon}_3]] = (6(1-g_3) + 9)\bar{\epsilon}^{\alpha}_{2,\mu}a^{\mu}\bar{\epsilon}_{3,\alpha\nu\rho}a^{\nu}x^{\rho} - (6g_3 + 9)\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\bar{\epsilon}_{3,\alpha\mu\nu}a^{\mu}a^{\nu} \tag{5.27}$$

The double bracket then is:

$$[[\bar{\epsilon}_1,[[\bar{\epsilon}_2,\bar{\epsilon}_3]]]] = -3g_3\partial_\mu[[\bar{\epsilon}_2,\bar{\epsilon}_3]]_{\alpha\nu}\bar{\epsilon}_1^\alpha + 3(1-g_3)[[\bar{\epsilon}_2,\bar{\epsilon}_3]]_{\alpha\nu}\partial_\mu\bar{\epsilon}_1^\alpha$$
$$-\frac{9}{2}[[\bar{\epsilon}_2,\bar{\epsilon}_3]]^\alpha_\mu\partial_\alpha\bar{\epsilon}_{1,\nu} + \frac{9}{4}\partial_\alpha[[\bar{\epsilon}_2,\bar{\epsilon}_3]]_{\mu\nu}\bar{\epsilon}_1^\alpha \quad (5.28)$$

Plugging in the inner bracket this reads:

$$\begin{aligned}[[\bar{\epsilon}_1,[[\bar{\epsilon}_2,\bar{\epsilon}_3]]]] = \bar{\epsilon}^\alpha_{1,\mu}a^\mu[&+((3(1-g_3)+\frac{9}{2})(6(1-g_3)+9))\bar{\epsilon}_{2,\beta\alpha}\bar{\epsilon}^\beta_{3,\nu\rho}a^\nu x^\rho\\
&+((3(1-g_3)+\frac{9}{2})(6(1-g_3)+9))\bar{\epsilon}_{2,\beta\nu}a^\nu\bar{\epsilon}^\beta_{3,\alpha\rho}x^\rho\\
&-((3(1-g_3)+\frac{9}{2})(6g_3+9))\bar{\epsilon}_{2,\beta\rho}x^\rho\bar{\epsilon}^\beta_{3,\alpha\nu}a^\nu\\
&-((3(1-g_3)+\frac{9}{2})(6g_3+9))\bar{\epsilon}_{2,\beta\rho}x^\rho\bar{\epsilon}^\beta_{3,\nu\alpha}a^\nu]\\
+\bar{\epsilon}^\alpha_{1,\rho}x^\rho[&+(3g_3(6g_3+9)+\frac{9}{2}(6(1-g_3)+9))\bar{\epsilon}_{2,\beta\mu}a^\mu\bar{\epsilon}^\beta_{3,\nu\alpha}a^\nu\\
&-(3g_3(6(1-g_3)+9)+\frac{9}{2}(6g_3+9))\bar{\epsilon}_{2,\beta\alpha}\bar{\epsilon}^\beta_{3,\mu\nu}a^\mu a^\nu\\
&+(3g_3((6g_3+9)-(6(1-g_3)+9)))\bar{\epsilon}_{2,\beta\mu}a^\mu\bar{\epsilon}^\beta_{3,\alpha\nu}a^\nu]\end{aligned} \quad (5.29)$$

The third term can be obtained from the second one by interchanging the labels of $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ and switching the overall sign:

$$\begin{aligned}[[\bar{\epsilon}_2,[[\bar{\epsilon}_3,\bar{\epsilon}_1]]]] = \bar{\epsilon}^\alpha_{2,\mu}a^\mu[&-((3(1-g_3)+\frac{9}{2})(6(1-g_3)+9))\bar{\epsilon}_{1,\beta\alpha}\bar{\epsilon}^\beta_{3,\nu\rho}a^\nu x^\rho\\
&-((3(1-g_3)+\frac{9}{2})(6(1-g_3)+9))\bar{\epsilon}_{1,\beta\nu}a^\nu\bar{\epsilon}^\beta_{3,\alpha\rho}x^\rho\\
&+((3(1-g_3)+\frac{9}{2})(6g_3+9))\bar{\epsilon}_{1,\beta\rho}x^\rho\bar{\epsilon}^\beta_{3,\alpha\nu}a^\nu\\
&+((3(1-g_3)+\frac{9}{2})(6g_3+9))\bar{\epsilon}_{1,\beta\rho}x^\rho\bar{\epsilon}^\beta_{3,\nu\alpha}a^\nu]\\
+\bar{\epsilon}^\alpha_{2,\rho}x^\rho[&-(3g_3(6g_3+9)+\frac{9}{2}(6(1-g_3)+9))\bar{\epsilon}_{1,\beta\mu}a^\mu\bar{\epsilon}^\beta_{3,\nu\alpha}a^\nu\\
&+(3g_3(6(1-g_3)+9)+\frac{9}{2}(6g_3+9))\bar{\epsilon}_{1,\beta\alpha}\bar{\epsilon}^\beta_{3,\mu\nu}a^\mu a^\nu\\
&-(3g_3((6g_3+9)-(6(1-g_3)+9)))\bar{\epsilon}_{1,\beta\mu}a^\mu\bar{\epsilon}^\beta_{3,\alpha\nu}a^\nu]\end{aligned} \quad (5.30)$$

Summing up these terms leads to:

$$\begin{aligned}
&\sum_{cyc}[[\bar\epsilon_3,[[\bar\epsilon_1,\bar\epsilon_2]]]] = \\
&\bar\epsilon_{1,\beta\alpha}\bar\epsilon^{\beta}_{2,\rho}x^\rho\bar\epsilon^{\alpha}_{3,\mu\nu}a^\mu a^\nu[+(6g_3+9)(g_1+\frac{3}{2})\tilde g_{222}-3g_3(6(1-g_3)+9)-\frac{9}{2}(6g_3+9)] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon_{2,\beta\alpha}\bar\epsilon^{\alpha}_{3,\mu\nu}a^\mu a^\nu[-(6g_3+9)(\frac{5}{2}-g_1)\tilde g_{222}+3g_3(6(1-g_3)+9)+\frac{9}{2}(6g_3+9)] \\
&+\bar\epsilon^{\beta}_{1,\mu}a^\mu\bar\epsilon_{2,\beta\alpha}\bar\epsilon^{\alpha}_{3,\nu\rho}a^\nu x^\rho[+6(1-g_3)(\frac{5}{2}-g_1)\tilde g_{222}+9(g_1+\frac{3}{2})\tilde g_{222} \\
&\qquad -(3(1-g_3)+\frac{9}{2})(6(1-g_3)+9)] \\
&+\bar\epsilon_{1,\beta\alpha}\bar\epsilon^{\beta}_{2,\mu}a^\mu\bar\epsilon^{\alpha}_{3,\nu\rho}a^\nu x^\rho[-6(1-g_3)(g_1+\frac{3}{2})\tilde g_{222}-9(\frac{5}{2}-g_1)\tilde g_{222} \\
&\qquad +(3(1-g_3)+\frac{9}{2})(6(1-g_3)+9)] \\
&+\bar\epsilon^{\beta}_{1,\nu}a^\nu\bar\epsilon^{\alpha}_{2,\mu}a^\mu\bar\epsilon_{3,\beta\alpha\rho}x^\rho[+(3(1-g_3)+\frac{9}{2})(6(1-g_3)+9) \\
&\qquad -(3(1-g_3)+\frac{9}{2})(6(1-g_3)+9)] \\
&+\bar\epsilon^{\alpha}_{1,\mu}a^\mu\bar\epsilon^{\beta}_{2,\rho}x^\rho\bar\epsilon_{3,\beta\nu\alpha}a^\nu[+(3(1-g_3)+\frac{9}{2})(6g_3+9)-3g_3(6g_3+9-(6(1-g_3)+9)) \\
&\qquad -(3(1-g_3)+\frac{9}{2})(6g_3+9)] \\
&+\bar\epsilon^{\alpha}_{1,\mu}a^\mu\bar\epsilon^{\beta}_{2,\rho}x^\rho\bar\epsilon_{3,\alpha\nu\beta}a^\nu[+(3(1-g_3)+\frac{9}{2})(6g_3+9)-3g_3(6g_3+9-(6(1-g_3)+9)) \\
&\qquad -(3g_3(6g_3+9)+\frac{9}{2}(6(1-g_3)+9))] \\
&+\bar\epsilon^{\alpha}_{1,\rho}x^\rho\bar\epsilon^{\beta}_{2,\mu}a^\mu\bar\epsilon_{3,\beta\nu\alpha}a^\nu[-3g_3(6g_3+9-(6(1-g_3)+9))-(3(1-g_3)+\frac{9}{2})(6g_3+9) \\
&\qquad +3g_3(6g_3+9)+\frac{9}{2}(6(1-g_3)+9)] \\
&+\bar\epsilon^{\alpha}_{1,\rho}x^\rho\bar\epsilon^{\beta}_{2,\mu}a^\mu\bar\epsilon_{3,\alpha\nu\beta}a^\nu[-3g_3(6g_3+9-(6(1-g_3)+9))-(3(1-g_3)+\frac{9}{2})(6g_3+9) \\
&\qquad +(3(1-g_3)+\frac{9}{2})(6g_3+9)]
\end{aligned} \tag{5.31}$$

This is solved by setting $g_1 = g_3 = \frac{1}{2}$ and $\tilde g_{222} = 3$. From this we have already obtained the value of the relative coupling constant of the 2-2-2 vertex and the 3-3-2 vertex, which is the only relevant coupling constant in spin-3-gravity. An important consistency check for the theory is that all further calculations yield the same value of this coupling constant.

5.2.3 Two spin-3 parameters, one spin-2 parameter

We come to the case of two spin-3 parameters and one spin-2 parameters. The first double bracket is of the form [[2,[[3,3]]]] and consists of the 3-3-bracket from 3-3-2 as the inner bracket and the 2-2-bracket from 2-2-2 as the outer bracket, whereas the other two double brackets are of the form [[3,[[3,2]]]] and [[3,[[2,3]]]] respectively and consist of the 3-2-bracket from 3-3-2 as the inner bracket and the 3-3 bracket from the same vertex as the outer bracket. As in the last case, we discard the prefactors of the Killing-tensors. Also, since the first term is the only one that contains a bracket from the 2-2-2 vertex, it recieves a relative coupling constant. For this term the inner bracket is:

$$[[\bar{\epsilon}_1, \bar{\epsilon}_2]] = 3(1 + 10g_2)\bar{\epsilon}_{1,\ \mu}^{\alpha\beta} a^\mu \bar{\epsilon}_{2,\alpha\beta\rho} x^\rho - 3(1 + 10(1 - g_2))\bar{\epsilon}_{1,\ \rho}^{\alpha\beta} x^\rho \bar{\epsilon}_{2,\alpha\beta\mu} a^\mu \tag{5.32}$$

The double bracket then is:

$$\begin{aligned}[[\bar{\epsilon}_3, [[\bar{\epsilon}_1, \bar{\epsilon}_2]]]] = \tilde{g}_{222}[&+g_1\partial_\mu \bar{\epsilon}_3^\alpha [[\bar{\epsilon}_1, \bar{\epsilon}_2]]_\alpha - (1 - g_1)\bar{\epsilon}_3^\alpha \partial_\mu [[\bar{\epsilon}_1, \bar{\epsilon}_2]]_\alpha \\ &+\frac{3}{2}\bar{\epsilon}_3^\alpha \partial_\alpha [[\bar{\epsilon}_1, \bar{\epsilon}_2]]_\mu - \frac{3}{2}\partial_\alpha \bar{\epsilon}_{3,\mu}[[\bar{\epsilon}_1, \bar{\epsilon}_2]]^\alpha]\end{aligned} \tag{5.33}$$

Plugging in the inner bracket this reads:

$$\begin{aligned}[[\bar{\epsilon}_3, [[\bar{\epsilon}_1, \bar{\epsilon}_2]]]] = \tilde{g}_{222}\bar{\epsilon}_{3,\rho}^\alpha x^\rho [&+(3(1 - g_1)(1 + 10(1 - g_2)) + \frac{9}{2}(1 + 10g_2))\bar{\epsilon}_{1,\ \mu}^{\beta\gamma} a^\mu \bar{\epsilon}_{2,\beta\gamma\alpha} \\ &-(3(1 - g_1)(1 + 10g_2) + \frac{9}{2}(1 + 10(1 - g_2)))\bar{\epsilon}_{1,\ \alpha}^{\beta\gamma} \bar{\epsilon}_{2,\beta\gamma\mu} a^\mu] \\ +\tilde{g}_{222}\bar{\epsilon}_{3,\mu}^\alpha a^\mu [&+(3(g_1 + \frac{3}{2})(1 + 10g_2))\bar{\epsilon}_{1,\ \alpha}^{\beta\gamma} \bar{\epsilon}_{2,\beta\gamma\rho} x^\rho \\ &-(3(g_1 + \frac{3}{2})(1 + 10(1 - g_2)))\bar{\epsilon}_{1,\ \rho}^{\beta\gamma} x^\rho \bar{\epsilon}_{2,\beta\gamma\alpha}]\end{aligned} \tag{5.34}$$

For the second term, the inner bracket is:

$$[[\bar{\epsilon}_2, \bar{\epsilon}_3]] = -3(g_3 + \frac{3}{2})\bar{\epsilon}_{2,\mu\nu}{}^\alpha a^\mu a^\nu \bar{\epsilon}_{3,\alpha\rho} x^\rho - 3(3 + 2(1 - g_3))\bar{\epsilon}_{2,\mu\rho}^\alpha a^\mu x^\rho \bar{\epsilon}_{3,\alpha\nu} a^\nu \tag{5.35}$$

The double bracket then is:

$$\begin{aligned}[[\bar\epsilon_1,[[\bar\epsilon_2,\bar\epsilon_3]]]] = &+\frac{15}{2}g_2\partial_\mu\bar\epsilon_1^{\alpha\beta}[[\bar\epsilon_2,\bar\epsilon_3]]_{\alpha\beta} - \frac{15}{2}(1-g_2)\bar\epsilon_1^{\alpha\beta}\partial_\mu[[\bar\epsilon_2,\bar\epsilon_3]]_{\alpha\beta}\\ &+\frac{3}{2}\bar\epsilon_1^{\alpha\beta}\partial_\alpha[[\bar\epsilon_2,\bar\epsilon_3]]_{\beta\mu} - \frac{3}{2}\partial_\alpha\bar\epsilon_{1,\beta\mu}[[\bar\epsilon_2,\bar\epsilon_3]]^{\alpha\beta}\end{aligned} \tag{5.36}$$

Plugging in the inner bracket this reads:

$$\begin{aligned}[[\bar\epsilon_1,[[\bar\epsilon_2,\bar\epsilon_3]]]] = \bar\epsilon^{\alpha\beta}_{1,\ \mu}a^\mu[&+(-6(\frac{3}{2}+15g_2)(g_3+\frac{3}{2}))\bar\epsilon_{2,\alpha\beta\gamma}\bar\epsilon^{\gamma}_{3,\rho}x^\rho\\ &-(6(\frac{3}{2}+15g_2)(3+2(1-g_3)))\bar\epsilon^{\gamma}_{2,\alpha\rho}x^\rho\bar\epsilon_{3,\gamma\beta}]\\ +\bar\epsilon^{\alpha\beta}_{1,\ \rho}x^\rho[&+(90(1-g_2)(g_3+\frac{3}{2})+\frac{9}{2}(3+2(1-g_3)))\bar\epsilon_{2,\alpha\beta\gamma}\bar\epsilon^{\gamma}_{3,\mu}a^\mu\\ &-(9(3+2(1-g_3))+90(1-g_2)(3+2(1-g_3)))\bar\epsilon^{\gamma}_{2,\mu\alpha}a^\mu\bar\epsilon_{3,\gamma\beta}\\ &-(18(g_3+\frac{3}{2})+90(1-g_2)(3+2(1-g_3)))\bar\epsilon_{2,\alpha\mu}{}^{\gamma}a^\mu\bar\epsilon_{3,\gamma\beta}]\end{aligned} \tag{5.37}$$

The third term can be obtained from the second one by interchanging the labels of $\bar\epsilon_1$ and $\bar\epsilon_2$ and switching the overall sign:

$$\begin{aligned}[[\bar\epsilon_2,[[\bar\epsilon_3,\bar\epsilon_1]]]] = \bar\epsilon^{\alpha\beta}_{2,\ \mu}a^\mu[&+(6(\frac{3}{2}+15g_2)(g_3+\frac{3}{2}))\bar\epsilon_{1,\alpha\beta\gamma}\bar\epsilon^{\gamma}_{3,\rho}x^\rho\\ &+(6(\frac{3}{2}+15g_2)(3+2(1-g_3)))\bar\epsilon^{\gamma}_{1,\alpha\rho}x^\rho\bar\epsilon_{3,\gamma\beta}]\\ +\bar\epsilon^{\alpha\beta}_{2,\ \rho}x^\rho[&-(90(1-g_2)(g_3+\frac{3}{2})+\frac{9}{2}(3+2(1-g_3)))\bar\epsilon_{1,\alpha\beta\gamma}\bar\epsilon^{\gamma}_{3,\mu}a^\mu\\ &+(9(3+2(1-g_3))+90(1-g_2)(3+2(1-g_3)))\bar\epsilon^{\gamma}_{1,\mu\alpha}a^\mu\bar\epsilon_{3,\gamma\beta}\\ &+(18(g_3+\frac{3}{2})+90(1-g_2)(3+2(1-g_3)))\bar\epsilon_{1,\alpha\mu}{}^{\gamma}a^\mu\bar\epsilon_{3,\gamma\beta}]\end{aligned} \tag{5.38}$$

Summing up these terms leads to:

$$
\begin{aligned}
&\sum_{cyc}[[\bar{\epsilon}_3,[[\bar{\epsilon}_1,\bar{\epsilon}_2]]]] = \\
&\bar{\epsilon}_{1,\alpha\beta\gamma}\bar{\epsilon}_{2,\ \mu}^{\alpha\beta}a^\mu\bar{\epsilon}_{3,\rho}^{\gamma}x^\rho[-3(1-g_1)(1+10g_2)\tilde{g}_{222}-\frac{9}{2}(1+10(1-g_2))\tilde{g}_{222} \\
&\qquad +6(\frac{3}{2}+15g_2)(g_3+\frac{3}{2})] \\
&+\bar{\epsilon}_{1,\alpha\beta\gamma}\bar{\epsilon}_{2,\ \rho}^{\alpha\beta}x^\rho\bar{\epsilon}_{3,\mu}^{\gamma}a^\mu[+3(g_1+\frac{3}{2})(1+10g_2)\tilde{g}_{222}-90(1-g_2)(g_3+\frac{3}{2}) \\
&\qquad -\frac{9}{2}(3+2(1-g_3))] \\
&+\bar{\epsilon}_{1,\ \rho}^{\alpha\beta}x^\rho\bar{\epsilon}_{2,\alpha\beta\gamma}\bar{\epsilon}_{3,\mu}^{\gamma}a^\mu[-3(g_1+\frac{3}{2})(1+10g_2)\tilde{g}_{222}+90(1-g_2)(g_3+\frac{3}{2}) \\
&\qquad +\frac{9}{2}(3+2(1-g_3))] \\
&+\bar{\epsilon}_{1,\ \mu}^{\alpha\beta}a^\mu\bar{\epsilon}_{2,\alpha\beta\gamma}\bar{\epsilon}_{3,\rho}^{\gamma}x^\rho[+3(1-g_1)(1+10(1-g_2))\tilde{g}_{222}+\frac{9}{2}(1+10g_2)\tilde{g}_{222} \\
&\qquad -6(\frac{3}{2}+15g_2)(g_3+\frac{3}{2})] \\
&+\bar{\epsilon}_{1,\alpha\mu\gamma}a^\mu\bar{\epsilon}_{2,\ \rho}^{\alpha\beta}x^\rho\bar{\epsilon}_{3,\beta}^{\gamma}[+18(g_3+\frac{3}{2})+90(1-g_2)(3+2(1-g_3)) \\
&\qquad -6(\frac{3}{2}+15g_2)(3+2(1-g_3))] \\
&+\bar{\epsilon}_{1,\gamma\mu\alpha}a^\mu\bar{\epsilon}_{2,\ \rho}^{\alpha\beta}x^\rho\bar{\epsilon}_{3,\beta}^{\gamma}[+9(3+2(1-g_3))+90(1-g_2)(3+2(1-g_3)) \\
&\qquad -6(\frac{3}{2}+15g_2)(3+2(1-g_3))] \\
&+\bar{\epsilon}_{1,\ \rho}^{\alpha\beta}x^\rho\bar{\epsilon}_{2,\mu\alpha}^{\gamma}a^\mu\bar{\epsilon}_{3,\gamma\beta}[-9(3+2(1-g_3))-90(1-g_2)(3+2(1-g_3)) \\
&\qquad +6(\frac{3}{2}+15g_2)(3+2(1-g_3))] \\
&+\bar{\epsilon}_{1,\ \rho}^{\alpha\beta}x^\rho\bar{\epsilon}_{2,\alpha\mu}{}^{\gamma}a^\mu\bar{\epsilon}_{3,\gamma\beta}[-18(g_3+\frac{3}{2})-90(1-g_2)(3+2(1-g_3)) \\
&\qquad +6(\frac{3}{2}+15g_2)(3+2(1-g_3))]
\end{aligned}
\tag{5.39}
$$

This is solved by $g_1 = g_2 = g_3 = \frac{1}{2}$ and $\tilde{g}_{222} = 3$, exactly as expected.

5.2.4 Three spin-3 parameters

The last case is the one of three spin-3 parameters. All double brackets are of the form [[3,[[3,3]]]] and consist of the 3-3-bracket from 3-3-2 as the inner bracket and the 3-2-bracket from the same vertex as the outer bracket. We again discard the prefactors of the Killing-tensors. Also, there is no need to include any coupling constants. For the first term, the inner bracket is:

$$[[\bar{\epsilon}_1, \bar{\epsilon}_2]] = 3(1+10g_2)\bar{\epsilon}_{1,\ \mu}^{\alpha\beta} a^\mu \bar{\epsilon}_{2,\alpha\beta\rho} x^\rho - 3(1+10(1-g_2))\bar{\epsilon}_{1,\ \rho}^{\alpha\beta} x^\rho \bar{\epsilon}_{2,\alpha\beta\mu} a^\mu \tag{5.40}$$

The double bracket then is:

$$\begin{aligned}[[\bar{\epsilon}_3, [[\bar{\epsilon}_1, \bar{\epsilon}_2]]]] = \ & +3g_3 \partial_\mu \bar{\epsilon}_{3,\nu}^{\alpha} [[\bar{\epsilon}_1, \bar{\epsilon}_2]]_\alpha - 3(1-g_3)\bar{\epsilon}_{3,\nu}^{\alpha} \partial_\mu [[\bar{\epsilon}_1, \bar{\epsilon}_2]]_\alpha \\ & + \frac{9}{2} \bar{\epsilon}_{3,\mu}^{\alpha} \partial_\alpha [[\bar{\epsilon}_1, \bar{\epsilon}_2]]_\nu - \frac{9}{4} \partial_\alpha \bar{\epsilon}_{3,\mu\nu} [[\bar{\epsilon}_1, \bar{\epsilon}_2]]^\alpha \end{aligned} \tag{5.41}$$

Plugging in the inner bracket this reads:

$$\begin{aligned}[[\bar{\epsilon}_3, [[\bar{\epsilon}_1, \bar{\epsilon}_2]]]] = \bar{\epsilon}_{3,\alpha\mu\nu} a^\mu a^\nu [& +(3(6g_3+9)(1+10g_2))\bar{\epsilon}_{1,\ \alpha}^{\beta\gamma} \bar{\epsilon}_{2,\beta\gamma\rho} x^\rho \\ & -(3(6g_3+9)(1+10(1-g_2)))\bar{\epsilon}_{1,\ \rho}^{\beta\gamma} x^\rho \bar{\epsilon}_{2,\beta\gamma\alpha}] \\ + \bar{\epsilon}_{3,\nu\rho}^{\alpha} a^\nu x^\rho [& -(27(1+10(1-g_2)) + 18(1-g_3)(1+10g_2))\bar{\epsilon}_{1,\ \alpha}^{\beta\gamma} \bar{\epsilon}_{2,\beta\gamma\nu} a^\nu \\ & +(18(1-g_3)(1+10(1-g_2)) + 27(1+10g_2))\bar{\epsilon}_{1,\ \nu}^{\beta\gamma} a^\nu \bar{\epsilon}_{2,\beta\gamma\alpha}] \end{aligned} \tag{5.42}$$

The other double brackets can be obtained from the first one by cyclic permutation of the labels of the Killing tensors:

$$\begin{aligned}[[\bar{\epsilon}_1, [[\bar{\epsilon}_2, \bar{\epsilon}_3]]]] = \bar{\epsilon}_{1,\alpha\mu\nu} a^\mu a^\nu [& +(3(6g_3+9)(1+10g_2))\bar{\epsilon}_{2,\ \alpha}^{\beta\gamma} \bar{\epsilon}_{3,\beta\gamma\rho} x^\rho \\ & -(3(6g_3+9)(1+10(1-g_2)))\bar{\epsilon}_{2,\ \rho}^{\beta\gamma} x^\rho \bar{\epsilon}_{3,\beta\gamma\alpha}] \\ + \bar{\epsilon}_{1,\nu\rho}^{\alpha} a^\nu x^\rho [& -(27(1+10(1-g_2)) + 18(1-g_3)(1+10g_2))\bar{\epsilon}_{2,\ \alpha}^{\beta\gamma} \bar{\epsilon}_{3,\beta\gamma\nu} a^\nu \\ & +(18(1-g_3)(1+10(1-g_2)) + 27(1+10g_2))\bar{\epsilon}_{2,\ \nu}^{\beta\gamma} a^\nu \bar{\epsilon}_{3,\beta\gamma\alpha}] \end{aligned} \tag{5.43}$$

$$\begin{aligned}[[\bar{\epsilon}_2, [[\bar{\epsilon}_3, \bar{\epsilon}_1]]]] = \bar{\epsilon}_{2,\alpha\mu\nu} a^\mu a^\nu [& +(3(6g_3+9)(1+10g_2))\bar{\epsilon}_{3,\ \alpha}^{\beta\gamma} \bar{\epsilon}_{1,\beta\gamma\rho} x^\rho \\ & -(3(6g_3+9)(1+10(1-g_2)))\bar{\epsilon}_{3,\ \rho}^{\beta\gamma} x^\rho \bar{\epsilon}_{1,\beta\gamma\alpha}] \\ + \bar{\epsilon}_{2,\nu\rho}^{\alpha} a^\nu x^\rho [& -(27(1+10(1-g_2)) + 18(1-g_3)(1+10g_2))\bar{\epsilon}_{3,\ \alpha}^{\beta\gamma} \bar{\epsilon}_{1,\beta\gamma\nu} a^\nu \\ & +(18(1-g_3)(1+10(1-g_2)) + 27(1+10g_2))\bar{\epsilon}_{3,\ \nu}^{\beta\gamma} a^\nu \bar{\epsilon}_{1,\beta\gamma\alpha}] \end{aligned} \tag{5.44}$$

All terms in the above double brackets are different from each other, so it seems that the Jacobi-identity is not satisfied. However, the sum of the three terms could be a linear combination of DDI's, in which case the Jacobi-identitiy would still be fulfilled. Since the calculation of DDI's for terms like the ones above is very tedious (mostly because of the Killing-symmetry of the parameters), we instead dualise the spin-3 parameters and calculate DDI's for these new terms. Setting the partial integration parameters to $g_2 = g_3 = \frac{1}{2}$, as dictated by the preceding calculations, the first double bracket is:

$$\begin{aligned}[[\bar{\epsilon}_3,[[\bar{\epsilon}_1,\bar{\epsilon}_2]]]] = &+\bar{\epsilon}_{3,\mu\nu}{}^{\alpha}a^{\mu}a^{\nu}[-108\bar{\epsilon}_{1,\ \alpha}^{\beta\gamma}\bar{\epsilon}_{2,\beta\gamma\rho}x^{\rho} + 108\bar{\epsilon}_{1,\ \rho}^{\beta\gamma}x^{\rho}\bar{\epsilon}_{2,\beta\gamma\alpha}] \\ &+\bar{\epsilon}_{3,\nu\rho}^{\alpha}a^{\nu}x^{\rho}[-216\bar{\epsilon}_{1,\ \alpha}^{\beta\gamma}\bar{\epsilon}_{2,\beta\gamma\nu}a^{\nu} + 216\bar{\epsilon}_{1,\ \nu}^{\beta\gamma}a^{\nu}\bar{\epsilon}_{2,\beta\gamma\alpha}]\end{aligned} \tag{5.45}$$

We replace the spin-3 Killing tensors by symmetric rank-2 tensors via (5.11) and use

$$\begin{aligned}\epsilon^{\alpha\beta\gamma}\epsilon^{\mu\nu\lambda} = &\, g^{\alpha\mu}g^{\beta\nu}g^{\gamma\lambda} + g^{\alpha\nu}g^{\beta\lambda}g^{\gamma\mu} + g^{\alpha\lambda}g^{\beta\mu}g^{\gamma\nu} \\ &-g^{\alpha\mu}g^{\beta\lambda}g^{\gamma\nu} - g^{\beta\nu}g^{\alpha\lambda}g^{\gamma\mu} - g^{\gamma\lambda}g^{\alpha\nu}g^{\beta\mu}\end{aligned} \tag{5.46}$$

on the Levi-Civita tensors coming from $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$. The dualisation of the first double bracket then is:

$$\begin{aligned}[[\bar{\epsilon}_3,[[\bar{\epsilon}_1,\bar{\epsilon}_2]]]]_{dual} = &+2\tilde{\epsilon}_{1,\beta}^{\alpha}\tilde{\epsilon}_{2,\rho}^{\beta}x^{\rho}\tilde{\epsilon}_{3,\mu}^{c}a^{\mu}\epsilon_{c\alpha\nu}a^{\nu} + 2\tilde{\epsilon}_{1,\nu}^{\beta}a^{\nu}\tilde{\epsilon}_{2,\beta}^{\alpha}\tilde{\epsilon}_{3,\mu}^{c}a^{\mu}\epsilon_{c\alpha\rho}x^{\rho} \\ &+2\tilde{\epsilon}_{1,\nu}^{\beta}a^{\nu}\tilde{\epsilon}_{2,\beta}^{\alpha}\tilde{\epsilon}_{3,\alpha}^{c}\epsilon_{c\mu\rho}a^{\mu}x^{\rho} - 2\tilde{\epsilon}_{1,\rho}^{\beta}x^{\rho}\tilde{\epsilon}_{2,\beta}^{\alpha}\tilde{\epsilon}_{3,\mu}^{c}a^{\mu}\epsilon_{c\alpha\nu}a^{\nu} \\ &-2\tilde{\epsilon}_{1,\beta}^{\alpha}\tilde{\epsilon}_{2,\nu}^{\beta}a^{\nu}\tilde{\epsilon}_{3,\alpha}^{c}\epsilon_{c\mu\rho}a^{\mu}x^{\rho} - 2\tilde{\epsilon}_{1,\beta}^{\alpha}\tilde{\epsilon}_{2,\nu}^{\beta}a^{\nu}\tilde{\epsilon}_{3,\mu}^{c}a^{\mu}\epsilon_{c\alpha\rho}x^{\rho}\end{aligned} \tag{5.47}$$

The other double brackets can again be obtained by cyclic permutation of the gauge parameters:

$$\begin{aligned}[[\bar{\epsilon}_1,[[\bar{\epsilon}_2,\bar{\epsilon}_3]]]]_{dual} = &+2\tilde{\epsilon}_{1,\mu}^{c}a^{\mu}\epsilon_{c\alpha\nu}a^{\nu}\tilde{\epsilon}_{2,\beta}^{\alpha}\tilde{\epsilon}_{3,\rho}^{\beta}x^{\rho} + 2\tilde{\epsilon}_{1,\mu}^{c}a^{\mu}\epsilon_{c\alpha\rho}x^{\rho}\tilde{\epsilon}_{2,\nu}^{\beta}a^{\nu}\tilde{\epsilon}_{3,\beta}^{\alpha} \\ &+2\tilde{\epsilon}_{1,\alpha}^{c}\epsilon_{c\mu\rho}a^{\mu}x^{\rho}\tilde{\epsilon}_{2,\nu}^{\beta}a^{\nu}\tilde{\epsilon}_{3,\beta}^{\alpha} - 2\tilde{\epsilon}_{1,\mu}^{c}a^{\mu}\epsilon_{c\alpha\nu}a^{\nu}\tilde{\epsilon}_{2,\rho}^{\beta}x^{\rho}\tilde{\epsilon}_{3,\beta}^{\alpha} \\ &-2\tilde{\epsilon}_{1,\alpha}^{c}\epsilon_{c\mu\rho}a^{\mu}x^{\rho}\tilde{\epsilon}_{2,\beta}^{\alpha}\tilde{\epsilon}_{3,\nu}^{\beta}a^{\nu} - 2\tilde{\epsilon}_{1,\mu}^{c}a^{\mu}\epsilon_{c\alpha\rho}x^{\rho}\tilde{\epsilon}_{2,\beta}^{\alpha}\tilde{\epsilon}_{3,\nu}^{\beta}a^{\nu}\end{aligned} \tag{5.48}$$

$$\begin{aligned}[[\bar{\epsilon}_2,[[\bar{\epsilon}_3,\bar{\epsilon}_1]]]]_{dual} = &+2\tilde{\epsilon}_{1,\rho}^{\beta}x^{\rho}\tilde{\epsilon}_{2,\mu}^{c}a^{\mu}\epsilon_{c\alpha\nu}a^{\nu}\tilde{\epsilon}_{3,\beta}^{\alpha} + 2\tilde{\epsilon}_{1,\beta}^{\alpha}\tilde{\epsilon}_{2,\mu}^{c}a^{\mu}\epsilon_{c\alpha\rho}x^{\rho}\tilde{\epsilon}_{3,\nu}^{\beta}a^{\nu} \\ &+2\tilde{\epsilon}_{1,\beta}^{\alpha}\tilde{\epsilon}_{2,\alpha}^{c}\epsilon_{c\mu\rho}a^{\mu}x^{\rho}\tilde{\epsilon}_{3,\nu}^{\beta}a^{\nu} - 2\tilde{\epsilon}_{1,\beta}^{\alpha}\tilde{\epsilon}_{2,\mu}^{c}a^{\mu}\epsilon_{c\alpha\nu}a^{\nu}\tilde{\epsilon}_{3,\rho}^{\beta}x^{\rho} \\ &-2\tilde{\epsilon}_{1,\nu}^{\beta}a^{\nu}\tilde{\epsilon}_{2,\alpha}^{c}\epsilon_{c\mu\rho}a^{\mu}x^{\rho}\tilde{\epsilon}_{3,\beta}^{\alpha} - 2\tilde{\epsilon}_{1,\nu}^{\beta}a^{\nu}\tilde{\epsilon}_{2,\mu}^{c}a^{\mu}\epsilon_{c\alpha\rho}x^{\rho}\tilde{\epsilon}_{3,\beta}^{\alpha}\end{aligned} \tag{5.49}$$

The full Jacobi-identity then is:

$$\begin{aligned}\sum_{cyc.}[[\bar{\epsilon}_3,[[\bar{\epsilon}_1,\bar{\epsilon}_2]]]]_{dual} = &+2\tilde{\epsilon}^{\alpha}_{1,\beta}\tilde{\epsilon}^{\beta}_{2,\rho}x^{\rho}\tilde{\epsilon}^{c}_{3,\mu}a^{\mu}\epsilon_{c\alpha\nu}a^{\nu} + 2\tilde{\epsilon}^{\beta}_{1,\nu}a^{\nu}\tilde{\epsilon}^{\alpha}_{2,\beta}\tilde{\epsilon}^{c}_{3,\mu}a^{\mu}\epsilon_{c\alpha\rho}x^{\rho}\\
&+2\tilde{\epsilon}^{\beta}_{1,\nu}a^{\nu}\tilde{\epsilon}^{\alpha}_{2,\beta}\tilde{\epsilon}^{c}_{3,\alpha}\epsilon_{c\mu\rho}a^{\mu}x^{\rho} - 2\tilde{\epsilon}^{\beta}_{1,\rho}x^{\rho}\tilde{\epsilon}^{\alpha}_{2,\beta}\tilde{\epsilon}^{c}_{3,\mu}a^{\mu}\epsilon_{c\alpha\nu}a^{\nu}\\
&-2\tilde{\epsilon}^{\alpha}_{1,\beta}\tilde{\epsilon}^{\beta}_{2,\nu}a^{\nu}\tilde{\epsilon}^{c}_{3,\alpha}\epsilon_{c\mu\rho}a^{\mu}x^{\rho} - 2\tilde{\epsilon}^{\alpha}_{1,\beta}\tilde{\epsilon}^{\beta}_{2,\nu}a^{\nu}\tilde{\epsilon}^{c}_{3,\mu}a^{\mu}\epsilon_{c\alpha\rho}x^{\rho}\\
&+2\tilde{\epsilon}^{\alpha}_{2,\beta}\tilde{\epsilon}^{\beta}_{3,\rho}x^{\rho}\tilde{\epsilon}^{c}_{1,\mu}a^{\mu}\epsilon_{c\alpha\nu}a^{\nu} + 2\tilde{\epsilon}^{\beta}_{2,\nu}a^{\nu}\tilde{\epsilon}^{\alpha}_{3,\beta}\tilde{\epsilon}^{c}_{1,\mu}a^{\mu}\epsilon_{c\alpha\rho}x^{\rho}\\
&+2\tilde{\epsilon}^{\beta}_{2,\nu}a^{\nu}\tilde{\epsilon}^{\alpha}_{3,\beta}\tilde{\epsilon}^{c}_{1,\alpha}\epsilon_{c\mu\rho}a^{\mu}x^{\rho} - 2\tilde{\epsilon}^{\beta}_{2,\rho}x^{\rho}\tilde{\epsilon}^{\alpha}_{3,\beta}\tilde{\epsilon}^{c}_{1,\mu}a^{\mu}\epsilon_{c\alpha\nu}a^{\nu}\\
&-2\tilde{\epsilon}^{\alpha}_{2,\beta}\tilde{\epsilon}^{\beta}_{3,\nu}a^{\nu}\tilde{\epsilon}^{c}_{1,\alpha}\epsilon_{c\mu\rho}a^{\mu}x^{\rho} - 2\tilde{\epsilon}^{\alpha}_{2,\beta}\tilde{\epsilon}^{\beta}_{3,\nu}a^{\nu}\tilde{\epsilon}^{c}_{1,\mu}a^{\mu}\epsilon_{c\alpha\rho}x^{\rho}\\
&+2\tilde{\epsilon}^{\alpha}_{3,\beta}\tilde{\epsilon}^{\beta}_{1,\rho}x^{\rho}\tilde{\epsilon}^{c}_{2,\mu}a^{\mu}\epsilon_{c\alpha\nu}a^{\nu} + 2\tilde{\epsilon}^{\beta}_{3,\nu}a^{\nu}\tilde{\epsilon}^{\alpha}_{1,\beta}\tilde{\epsilon}^{c}_{2,\mu}a^{\mu}\epsilon_{c\alpha\rho}x^{\rho}\\
&+2\tilde{\epsilon}^{\beta}_{3,\nu}a^{\nu}\tilde{\epsilon}^{\alpha}_{1,\beta}\tilde{\epsilon}^{c}_{2,\alpha}\epsilon_{c\mu\rho}a^{\mu}x^{\rho} - 2\tilde{\epsilon}^{\beta}_{3,\rho}x^{\rho}\tilde{\epsilon}^{\alpha}_{1,\beta}\tilde{\epsilon}^{c}_{2,\mu}a^{\mu}\epsilon_{c\alpha\nu}a^{\nu}\\
&-2\tilde{\epsilon}^{\alpha}_{3,\beta}\tilde{\epsilon}^{\beta}_{1,\nu}a^{\nu}\tilde{\epsilon}^{c}_{2,\alpha}\epsilon_{c\mu\rho}a^{\mu}x^{\rho} - 2\tilde{\epsilon}^{\alpha}_{3,\beta}\tilde{\epsilon}^{\beta}_{1,\nu}a^{\nu}\tilde{\epsilon}^{c}_{2,\mu}a^{\mu}\epsilon_{c\alpha\rho}x^{\rho}\end{aligned} \tag{5.50}$$

We now want to rewrite this in more convenient variables. In order to do so, we need to fix a few conventions regarding the Levi-Civita tensor. If an index of ϵ is contracted with $\tilde{\epsilon}_i$, we move this index to the i-th position of the indices of ϵ. Also, if any a's and x's are contracted with ϵ, we always move the a's to the left of the x's. The new variables are then defined as:

- $e_i \ldots$ $\tilde{\epsilon}_i$ contracted with ϵ (at the i-th position)
- $A_i \ldots$ a contracted with $\tilde{\epsilon}_i$
- $X_i \ldots$ x contracted with $\tilde{\epsilon}_i$
- $A_{\epsilon_i} \ldots$ a contracted with ϵ at the i-th position
- $X_{\epsilon_i} \ldots$ x contracted with ϵ at the i-th position
- z_i defined as usual

Note that the subscripts in A_{ϵ_i} and X_{ϵ_i} are strictly speaking obsolete, since the positions of these objects are already completely fixed by our conventions and the other variables, but writing them out makes it easier to check the consistency of all steps of the calculation.

In these variables, the Jacobi-identity (5.50) reads:

$$
\begin{aligned}
\sum_{cyc.}[[\bar{\epsilon}_3,[[\bar{\epsilon}_1,\bar{\epsilon}_2]]]]_{dual} = &+2e_2e_3A_3X_1A_{\epsilon_1}z_3 + 2e_1e_3A_3X_2A_{\epsilon_2}z_3 + 2e_3A_1A_{\epsilon_1}X_{\epsilon_2}z_1z_3 \\
&-2e_2e_3A_1A_3X_{\epsilon_1}z_3 - 2e_1e_3A_2A_3X_{\epsilon_2}z_3 - 2e_3A_2A_{\epsilon_1}X_{\epsilon_2}z_2z_3 \\
&+2e_1e_3A_1X_2A_{\epsilon_2}z_1 + 2e_1e_2A_1X_3A_{\epsilon_3}z_1 + 2e_1A_2A_{\epsilon_2}X_{\epsilon_3}z_1z_2 \\
&-2e_1e_3A_1A_2X_{\epsilon_2}z_1 - 2e_1e_2A_1A_3X_{\epsilon_3}z_1 - 2e_1A_3A_{\epsilon_2}X_{\epsilon_3}z_1z_3 \\
&+2e_1e_2A_2X_3A_{\epsilon_3}z_2 + 2e_2e_3A_2X_1A_{\epsilon_1}z_2 + 2e_2A_3A_{\epsilon_3}X_{\epsilon_1}z_2z_3 \\
&-2e_1e_2A_2A_3X_{\epsilon_3}z_2 - 2e_2e_3A_1A_2X_{\epsilon_1}z_2 - 2e_2A_1A_{\epsilon_3}X_{\epsilon_1}z_1z_2
\end{aligned} \tag{5.51}
$$

The DDI's that are relevant for these terms consist of the following objects:

$$\{\partial_{a_1},\partial_{a_1},\partial_{a_2},\partial_{a_2},\partial_{a_3},\partial_{a_3},a,a,x,\epsilon\} \tag{5.52}$$

It turns out that the only relevant DDI's are those where all three indices of the Levi-Civita tensor are contracted with the generalized Kronecker-delta. All other DDI's are linear combinations of the aforementioned ones, as has been explicitly checked. One needs six DDI's to see that the terms in the Jacobi-identity indeed vanish. These DDI's are:

$$(e_1e_2X_3A_{\epsilon_3} - e_1e_2A_3X_{\epsilon_3} + e_1A_{\epsilon_2}X_{\epsilon_3}z_1 + e_2A_{\epsilon_1}X_{\epsilon_3}z_2)A_1z_1 = 0 \tag{5.53}$$

$$(e_1e_2X_3A_{\epsilon_3} - e_1e_2A_3X_{\epsilon_3} + e_1A_{\epsilon_2}X_{\epsilon_3}z_1 + e_2A_{\epsilon_1}X_{\epsilon_3}z_2)A_2z_2 = 0 \tag{5.54}$$

$$(e_1e_3X_2A_{\epsilon_2} - e_1e_3A_2X_{\epsilon_2} - e_1A_{\epsilon_2}X_{\epsilon_3}z_1 + e_3A_{\epsilon_1}X_{\epsilon_2}z_3)A_1z_1 = 0 \tag{5.55}$$

$$(e_1e_3X_2A_{\epsilon_2} - e_1e_3A_2X_{\epsilon_2} - e_1A_{\epsilon_2}X_{\epsilon_3}z_1 + e_3A_{\epsilon_1}X_{\epsilon_2}z_3)A_3z_3 = 0 \tag{5.56}$$

$$(e_2e_3X_1A_{\epsilon_1} - e_2e_3A_1X_{\epsilon_1} - e_2A_{\epsilon_1}X_{\epsilon_3}z_2 - e_3A_{\epsilon_1}X_{\epsilon_2}z_3)A_2z_2 = 0 \tag{5.57}$$

$$(e_2e_3X_1A_{\epsilon_1} - e_2e_3A_1X_{\epsilon_1} - e_2A_{\epsilon_1}X_{\epsilon_3}z_2 - e_3A_{\epsilon_1}X_{\epsilon_2}z_3)A_3z_3 = 0 \tag{5.58}$$

The terms in (5.51) are then just the sum of these six DDI's. We therefore see that the Jacobi-identity is still fulfilled if DDI's are taken into account.

To sum up our results from this section, the Jacobi-identity is fullfilled in all cases that arise for spin-3-gravity. In the case of three spin-3 parameters, one needs to make use of DDI's to show that the Jacobi-identity holds. The partial integration parameters have been determined to be $\frac{1}{2}$ for all brackets, as expected. Also, the relative coupling constant between the 2-2-2 vertex and the 3-3-2 vertex has been determined to be $\tilde{g}_{222} = 3$.

5.3 First-order closure of gauge algebra

In the same fashion as for the Jacobi-identity, many different cases need to be distinguished when checking (5.9). Here we only have to specify the spin of two gauge parameters, but one can also choose different spins for the fields Φ and $\tilde{\Phi}$, which typically results in more than one calculation for each pair of gauge parameters.

5.3.1 Two spin-2 parameters

We start with the case of two spin-2 parameters. Here we have two different choices for the fields Φ and $\tilde{\Phi}$: they can either both be spin-2 fields or both be spin-3 fields, all other cases are trivial.

In the case where Φ and $\tilde{\Phi}$ are spin-2 fields, we use the 2-2-deformation of the 2-2-2 vertex both as the inner gauge deformation and the outer one in the first two terms of (5.9). The same gauge deformation is used for the third term, but with the 2-2-bracket from 2-2-2 as the gauge parameter. By the term "s1-s2-deformation" we refer to the first-order transformation of a spin-s2 field w.r.t. a spin-s1 parameter. Since the Killing-tensors in all terms are the same, we can discard their prefactors from the beginning. Also, there is no need to introduce coupling constants. For the first term, the inner deformation is:

$$\delta^{(1)}_{\bar{\epsilon}_1}(\Phi) = -2\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\partial_{\alpha}\Phi_{\mu\nu}a^{\mu}a^{\nu}+2(1-c)\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\partial_{\mu}\Phi_{\alpha\nu}a^{\mu}a^{\nu}+2(1+c)\bar{\epsilon}_{1,\mu\alpha}a^{\mu}\Phi^{\alpha}{}_{\nu}a^{\nu} \tag{5.59}$$

The double gauge deformation then is:

$$\begin{aligned}\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) = &-\bar{\epsilon}^{\alpha}_2\partial_{\alpha}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\mu\nu} - c\partial_{\mu}\bar{\epsilon}^{\alpha}_2(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\alpha\nu} \\ &+(1-c)\bar{\epsilon}^{\alpha}_2\partial_{\mu}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\alpha\nu} + \partial_{\alpha}\bar{\epsilon}_{2,\mu}\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)^{\alpha}{}_{\nu}\end{aligned} \tag{5.60}$$

Plugging in the inner gauge deformation from above this reads:

$$\begin{aligned}\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi)) = \bar\epsilon_{2,\mu}{}^{\alpha}a^{\mu}[&-4(1+c)\bar\epsilon^{\beta}_{1,\rho}x^{\rho}\partial_{\beta}\Phi_{\alpha\nu}a^{\nu}+2(1+c)(1-c)\bar\epsilon^{\beta}_{1,\rho}x^{\rho}\partial_{\alpha}\Phi_{\beta\nu}a^{\nu}\\ &+2(1+c)(1-c)\bar\epsilon^{\beta}_{1,\rho}x^{\rho}\partial_{\nu}\Phi_{\beta\alpha}a^{\nu}+2(1+c)(1+c)\bar\epsilon_{1,\alpha\beta}\Phi^{\beta}{}_{\nu}a^{\nu}\\ &+2(1+c)(1+c)\bar\epsilon_{1,\nu\beta}a^{\nu}\Phi^{\beta}{}_{\alpha}]\\ +\bar\epsilon^{\alpha}_{2,\rho}x^{\rho}[&+(-4(1-c)-2(1-c)(1+c))\bar\epsilon^{\beta}_{1,\alpha}\partial_{\mu}\Phi_{\beta\nu}a^{\mu}a^{\nu}\\ &+(-4(1-c)+2(1-c)(1-c))\bar\epsilon^{\beta}_{1,\sigma}x^{\sigma}\partial_{\alpha}\partial_{\mu}\Phi_{\beta\nu}a^{\mu}a^{\nu}\\ &+(-4(1+c)-2(1-c)(1-c))\bar\epsilon^{\beta}_{1,\mu}a^{\mu}\partial_{\alpha}\Phi_{\beta\nu}a^{\nu}\\ &+(2(1-c)(1-c)-2(1-c)(1+c))\bar\epsilon^{\beta}_{1,\mu}a^{\mu}\partial_{\nu}\Phi_{\beta\alpha}a^{\nu}\\ &+4\bar\epsilon^{\beta}_{1,\alpha}\partial_{\beta}\Phi_{\mu\nu}a^{\mu}a^{\nu}+4\bar\epsilon^{\beta}_{1,\sigma}x^{\sigma}\partial_{\beta}\partial_{\alpha}\Phi_{\mu\nu}a^{\mu}a^{\nu}\\ &-4(1-c)\bar\epsilon^{\beta}_{1,\mu}a^{\mu}\partial_{\beta}\Phi_{\alpha\nu}a^{\nu}-4(1-c)\bar\epsilon^{\beta}_{1,\sigma}x^{\sigma}\partial_{\beta}\partial_{\mu}\Phi_{\alpha\nu}a^{\mu}a^{\nu}\\ &+2(1-c)(1-c)\bar\epsilon^{\beta}_{1,\sigma}x^{\sigma}\partial_{\mu}\partial_{\nu}\Phi_{\beta\alpha}a^{\mu}a^{\nu}]\end{aligned} \tag{5.61}$$

The second double gauge tranformation consists of the same gauge deformations and gauge parameters as the first one, so it can be obtained by switching $\bar\epsilon_1$ and $\bar\epsilon_2$ in the above expression:

$$\begin{aligned}\delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) = \bar\epsilon_{1,\mu}{}^{\alpha}a^{\mu}[&-4(1+c)\bar\epsilon^{\beta}_{2,\rho}x^{\rho}\partial_{\beta}\Phi_{\alpha\nu}a^{\nu}+2(1+c)(1-c)\bar\epsilon^{\beta}_{2,\rho}x^{\rho}\partial_{\alpha}\Phi_{\beta\nu}a^{\nu}\\ &+2(1+c)(1-c)\bar\epsilon^{\beta}_{2,\rho}x^{\rho}\partial_{\nu}\Phi_{\beta\alpha}a^{\nu}+2(1+c)(1+c)\bar\epsilon_{2,\alpha\beta}\Phi^{\beta}{}_{\nu}a^{\nu}\\ &+2(1+c)(1+c)\bar\epsilon_{2,\nu\beta}a^{\nu}\Phi^{\beta}{}_{\alpha}]\\ +\bar\epsilon^{\alpha}_{1,\rho}x^{\rho}[&+(-4(1-c)-2(1-c)(1+c))\bar\epsilon^{\beta}_{2,\alpha}\partial_{\mu}\Phi_{\beta\nu}a^{\mu}a^{\nu}\\ &+(-4(1-c)+2(1-c)(1-c))\bar\epsilon^{\beta}_{2,\sigma}x^{\sigma}\partial_{\alpha}\partial_{\mu}\Phi_{\beta\nu}a^{\mu}a^{\nu}\\ &+(-4(1+c)-2(1-c)(1-c))\bar\epsilon^{\beta}_{2,\mu}a^{\mu}\partial_{\alpha}\Phi_{\beta\nu}a^{\nu}\\ &+(2(1-c)(1-c)-2(1-c)(1+c))\bar\epsilon^{\beta}_{2,\mu}a^{\mu}\partial_{\nu}\Phi_{\beta\alpha}a^{\nu}\\ &+4\bar\epsilon^{\beta}_{2,\alpha}\partial_{\beta}\Phi_{\mu\nu}a^{\mu}a^{\nu}+4\bar\epsilon^{\beta}_{2,\sigma}x^{\sigma}\partial_{\beta}\partial_{\alpha}\Phi_{\mu\nu}a^{\mu}a^{\nu}\\ &-4(1-c)\bar\epsilon^{\beta}_{2,\mu}a^{\mu}\partial_{\beta}\Phi_{\alpha\nu}a^{\nu}-4(1-c)\bar\epsilon^{\beta}_{2,\sigma}x^{\sigma}\partial_{\beta}\partial_{\mu}\Phi_{\alpha\nu}a^{\mu}a^{\nu}\\ &+2(1-c)(1-c)\bar\epsilon^{\beta}_{2,\sigma}x^{\sigma}\partial_{\mu}\partial_{\nu}\Phi_{\beta\alpha}a^{\mu}a^{\nu}]\end{aligned} \tag{5.62}$$

For the last term in the FCGA we need the 2-2-bracket from 2-2-2. We will still use arbitary partial integration parameters in the brackets, but since these calculations can only contain one bracket at a time, the parameters for the brackets will all be labeled g. The bracket is:

$$[[\bar{\epsilon}_2, \bar{\epsilon}_1]] = (g + \frac{3}{2})\bar{\epsilon}^{\alpha}_{2,\mu} a^{\mu} \bar{\epsilon}_{1,\alpha\rho} x^{\rho} - (\frac{5}{2} - g)\bar{\epsilon}^{\alpha}_{2,\rho} x^{\rho} \bar{\epsilon}_{1,\alpha\mu} a^{\mu} \tag{5.63}$$

The gauge deformation is:

$$\begin{aligned}\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = &-[[\bar{\epsilon}_2, \bar{\epsilon}_1]]^{\alpha}\partial_{\alpha}\Phi_{\mu\nu} - c\partial_{\mu}[[\bar{\epsilon}_2, \bar{\epsilon}_1]]^{\alpha}\Phi_{\alpha\nu} \\ &+(1-c)[[\bar{\epsilon}_2, \bar{\epsilon}_1]]^{\alpha}\partial_{\mu}\Phi_{\alpha\nu} + \partial_{\alpha}[[\bar{\epsilon}_2, \bar{\epsilon}_1]]_{\mu}\Phi^{\alpha}{}_{\nu}\end{aligned} \tag{5.64}$$

Plugging in the bracket this reads:

$$\begin{aligned}\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = &-2(c(g + \frac{3}{2}) + \frac{5}{2} - g)\bar{\epsilon}_{2,\beta\alpha}\bar{\epsilon}^{\beta}_{1,\mu} a^{\mu}\Phi^{\alpha}{}_{\nu} a^{\nu} \\ &+2(c(\frac{5}{2} - g) + g + \frac{3}{2})\bar{\epsilon}_{2,\beta\mu} a^{\mu}\bar{\epsilon}^{\beta}_{1,\alpha}\Phi^{\alpha}{}_{\nu} a^{\nu} \\ &-2(g + \frac{3}{2})\bar{\epsilon}^{\beta\alpha}_{2}\bar{\epsilon}_{1,\beta\rho} x^{\rho}\partial_{\alpha}\Phi_{\mu\nu} a^{\mu} a^{\nu} \\ &+2(\frac{5}{2} - g)\bar{\epsilon}_{2,\beta\rho} x^{\rho}\bar{\epsilon}^{\beta\alpha}_{1}\partial_{\alpha}\Phi_{\mu\nu} a^{\mu} a^{\nu} \\ &+2(1-c)(g + \frac{3}{2})\bar{\epsilon}^{\beta\alpha}_{2}\bar{\epsilon}_{1,\beta\rho} x^{\rho}\partial_{\mu}\Phi_{\alpha\nu} a^{\mu} a^{\nu} \\ &-2(1-c)(\frac{5}{2} - g)\bar{\epsilon}_{2,\beta\rho} x^{\rho}\bar{\epsilon}^{\beta\alpha}_{1}\partial_{\mu}\Phi_{\alpha\nu} a^{\mu} a^{\nu}\end{aligned} \tag{5.65}$$

The FCGA then reads:

$$
\begin{aligned}
&\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) - \delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) + \delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = \\
&\quad \bar{\epsilon}_{1,\alpha\beta}\bar{\epsilon}_{2,\mu}{}^{\alpha} a^{\mu}\Phi^{\beta}{}_{\nu}a^{\nu}[+2(1+c)(1+c) - 2(c(\frac{5}{2}-g)+g+\frac{3}{2})] \\
&\quad +\bar{\epsilon}_{1,\mu\beta}a^{\mu}\bar{\epsilon}_{2,\nu}{}^{\alpha}a^{\nu}\Phi^{\beta}{}_{\alpha}[+2(1+c)(1+c)-2(1+c)(1+c)] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\bar{\epsilon}_{2,\mu}{}^{\alpha}a^{\mu}\partial_{\beta}\Phi_{\alpha\nu}a^{\nu}[-4(1+c)+4(1+c)+2(1-c)(1-c)] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\bar{\epsilon}_{2,\mu}{}^{\alpha}a^{\mu}\partial_{\alpha}\Phi_{\beta\nu}a^{\nu}[+2(1+c)(1-c)-4(1-c)] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\bar{\epsilon}_{2,\mu}{}^{\alpha}a^{\mu}\partial_{\nu}\Phi_{\beta\alpha}a^{\nu}[+2(1+c)(1-c)+2(1-c)(1-c)-2(1-c)(1+c)] \\
&\quad +\bar{\epsilon}_{1,\mu}{}^{\alpha}a^{\mu}\bar{\epsilon}_{2,\alpha\beta}\Phi^{\beta}{}_{\nu}a^{\nu}[-2(1+c)(1+c)+2(c(g+\frac{3}{2})+\frac{5}{2}-g)] \\
&\quad +\bar{\epsilon}_{1,\mu}{}^{\alpha}a^{\mu}\bar{\epsilon}^{\beta}_{2,\rho}x^{\rho}\partial_{\beta}\Phi_{\alpha\nu}a^{\nu}[+4(1+c)-4(1+c)-2(1-c)(1-c)] \\
&\quad +\bar{\epsilon}_{1,\mu}{}^{\alpha}a^{\mu}\bar{\epsilon}^{\beta}_{2,\rho}x^{\rho}\partial_{\alpha}\Phi_{\beta\nu}a^{\nu}[-2(1+c)(1-c)+4(1-c)] \\
&\quad +\bar{\epsilon}_{1,\mu}{}^{\alpha}a^{\mu}\bar{\epsilon}^{\beta}_{2,\rho}x^{\rho}\partial_{\nu}\Phi_{\beta\alpha}a^{\nu}[-2(1+c)(1-c)-2(1-c)(1-c)+2(1-c)(1+c)] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\alpha}\bar{\epsilon}^{\alpha}_{2,\rho}\partial_{\mu}\Phi_{\beta\nu}a^{\mu}a^{\nu}[-4(1-c)-2(1-c)(1+c)+2(1-c)(\frac{5}{2}-g)] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\alpha}\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\partial_{\beta}\Phi_{\mu\nu}a^{\mu}a^{\nu}[+4-2(\frac{5}{2}-g)] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\sigma}x^{\sigma}\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\partial_{\beta}\partial_{\alpha}\Phi_{\mu\nu}a^{\mu}a^{\nu}[+4-4] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\sigma}x^{\sigma}\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\partial_{\mu}\partial_{\nu}\Phi_{\beta\alpha}a^{\mu}a^{\nu}[+2(1-c)(1-c)-2(1-c)(1-c)] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\sigma}x^{\sigma}\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\partial_{\mu}\partial_{\beta}\Phi_{\alpha\nu}a^{\mu}a^{\nu}[-4(1-c)+4(1-c)-2(1-c)(1-c)] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\sigma}x^{\sigma}\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\partial_{\alpha}\partial_{\mu}\Phi_{\beta\nu}a^{\mu}a^{\nu}[-4(1-c)+2(1-c)(1-c)+4(1-c)] \\
&\quad +\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\bar{\epsilon}^{\beta}_{2,\alpha}\partial_{\mu}\Phi_{\beta\nu}a^{\mu}a^{\nu}[+4(1-c)+2(1-c)(1+c)-2(1-c)(g+\frac{3}{2})] \\
&\quad +\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\bar{\epsilon}^{\beta}_{2,\alpha}\partial_{\beta}\Phi_{\mu\nu}a^{\mu}a^{\nu}[-4+2(g+\frac{3}{2})]
\end{aligned}
\tag{5.66}
$$

Setting $g = \frac{1}{2}$ and $c = 1$, the FCGA can be fulfilled:

$$\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) - \delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) + \delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = 0 \tag{5.67}$$

Together with our results from the Jacobi-identity, this shows that the 2-2-2 vertex is consistent in three dimensions, which is of course a well-known fact. This calculation has now also fixed the value of the partial integration parameter of the first-order deformation corresponding to 2-2-2 as $c = 1$, whereas the parameter of the bracket $g = \frac{1}{2}$ is in agreement with previous calculations.

We can now also choose Φ and $\tilde{\Phi}$ to be spin-3 fields. In this case we use the 2-3-deformation from the 3-3-2 vertex both as the inner gauge deformation and the outer one. The same gauge deformation is used for the third term and we still have to use the 2-2-bracket from 2-2-2 as the gauge parameter. We again discard the prefactors of the Killing-tensors. This time, the last term is the only one that contains an object from the 2-2-2 vertex, so we have to add a relative coupling constant. For the first term, the inner deformation is:

$$\delta^{(1)}_{\bar{\epsilon}_1}(\Phi) = -9(2-e)\bar{\epsilon}^{\alpha}_{1,\mu}a^{\mu}\Phi_{\alpha\nu\lambda}a^{\nu}a^{\lambda} + 9e\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\partial_{\mu}\Phi_{\alpha\nu\lambda}a^{\mu}a^{\nu}a^{\lambda} - 6\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\partial_{\alpha}\Phi_{\mu\nu\lambda}a^{\mu}a^{\nu}a^{\lambda} \tag{5.68}$$

The double gauge deformation then is:

$$\begin{aligned}\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) = &+\frac{3}{2}e\bar{\epsilon}^{\alpha}_{2}\partial_{\mu}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\alpha\nu\lambda} - \frac{3}{2}(2-e)\partial_{\mu}\bar{\epsilon}^{\alpha}_{2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\alpha\nu\lambda}\\ &-\bar{\epsilon}^{\alpha}_{2}\partial_{\alpha}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\mu\nu\lambda}\end{aligned} \tag{5.69}$$

Plugging in the inner gauge deformation from above this reads:

$$\begin{aligned}\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) = \bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}[&-(27e(2-e)+54e)\bar{\epsilon}^{\beta}_{1,\alpha}\partial_{\mu}\Phi_{\beta\nu\lambda}a^{\mu}a^{\nu}a^{\lambda}\\ &+(54e^2-54e(2-e))\bar{\epsilon}^{\beta}_{1,\nu}a^{\nu}\partial_{\mu}\Phi_{\beta\alpha\lambda}a^{\mu}a^{\lambda}\\ &+(27e^2+54(2-e))\bar{\epsilon}^{\beta}_{1,\mu}a^{\mu}\partial_{\alpha}\Phi_{\beta\nu\lambda}a^{\nu}a^{\lambda}\\ &+(27e^2-54e)\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\partial_{\alpha}\partial_{\mu}\Phi_{\beta\nu\lambda}a^{\mu}a^{\nu}a^{\lambda}\\ &+54e^2\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\partial_{\nu}\partial_{\mu}\Phi_{\beta\alpha\lambda}a^{\mu}a^{\nu}a^{\lambda} - 54e\bar{\epsilon}^{\beta}_{1,\mu}a^{\mu}\partial_{\beta}\Phi_{\alpha\nu\lambda}a^{\nu}a^{\lambda}\\ &-54e\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\partial_{\beta}\partial_{\mu}\Phi_{\alpha\nu\lambda}a^{\mu}a^{\nu}a^{\lambda} + 36\bar{\epsilon}^{\beta}_{1,\alpha}\partial_{\beta}\Phi_{\mu\nu\lambda}a^{\mu}a^{\nu}a^{\lambda}\\ &+36\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\partial_{\beta}\partial_{\alpha}\Phi_{\mu\nu\lambda}a^{\mu}a^{\nu}a^{\lambda}]\\ +\bar{\epsilon}^{\alpha}_{2,\mu}a^{\mu}[&+27(2-e)^2\bar{\epsilon}^{\beta}_{1,\alpha}\Phi_{\beta\nu\lambda}a^{\nu}a^{\lambda} + 54(2-e)^2\bar{\epsilon}^{\beta}_{1,\nu}a^{\nu}\Phi_{\beta\alpha\lambda}a^{\lambda}\\ &-27e(2-e)\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\partial_{\alpha}\Phi_{\beta\nu\lambda}a^{\nu}a^{\lambda}\\ &-54e(2-e)\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\partial_{\nu}\Phi_{\beta\alpha\lambda}a^{\nu}a^{\lambda}\\ &+54(2-e)\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\partial_{\beta}\Phi_{\alpha\nu\lambda}a^{\nu}a^{\lambda}]\end{aligned} \tag{5.70}$$

The second double gauge tranformation can be obtained by switching $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ in the above expression:

$$\begin{aligned}\delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) = \bar\epsilon^{\alpha}_{1,\rho}x^\rho[&-(27e(2-e)+54e)\bar\epsilon^{\beta}_{2,\alpha}\partial_\mu\Phi_{\beta\nu\lambda}a^\mu a^\nu a^\lambda \\ &+(54e^2-54e(2-e))\bar\epsilon^{\beta}_{2,\nu}a^\nu\partial_\mu\Phi_{\beta\alpha\lambda}a^\mu a^\lambda \\ &+(27e^2+54(2-e))\bar\epsilon^{\beta}_{2,\mu}a^\mu\partial_\alpha\Phi_{\beta\nu\lambda}a^\nu a^\lambda \\ &+(27e^2-54e)\bar\epsilon^{\beta}_{2,\rho}x^\rho\partial_\alpha\partial_\mu\Phi_{\beta\nu\lambda}a^\mu a^\nu a^\lambda \\ &+54e^2\bar\epsilon^{\beta}_{2,\rho}x^\rho\partial_\nu\partial_\mu\Phi_{\beta\alpha\lambda}a^\mu a^\nu a^\lambda - 54e\bar\epsilon^{\beta}_{2,\mu}a^\mu\partial_\beta\Phi_{\alpha\nu\lambda}a^\nu a^\lambda \\ &-54e\bar\epsilon^{\beta}_{2,\rho}x^\rho\partial_\beta\partial_\mu\Phi_{\alpha\nu\lambda}a^\mu a^\nu a^\lambda + 36\bar\epsilon^{\beta}_{2,\alpha}\partial_\beta\Phi_{\mu\nu\lambda}a^\mu a^\nu a^\lambda \\ &+36\bar\epsilon^{\beta}_{2,\rho}x^\rho\partial_\beta\partial_\alpha\Phi_{\mu\nu\lambda}a^\mu a^\nu a^\lambda] \\ +\bar\epsilon^{\alpha}_{1,\mu}a^\mu[&+27(2-e)^2\bar\epsilon^{\beta}_{2,\alpha}\Phi_{\beta\nu\lambda}a^\nu a^\lambda + 54(2-e)^2\bar\epsilon^{\beta}_{2,\nu}a^\nu\Phi_{\beta\alpha\lambda}a^\lambda \\ &-27e(2-e)\bar\epsilon^{\beta}_{2,\rho}x^\rho\partial_\alpha\Phi_{\beta\nu\lambda}a^\nu a^\lambda \\ &-54e(2-e)\bar\epsilon^{\beta}_{2,\rho}x^\rho\partial_\nu\Phi_{\beta\alpha\lambda}a^\nu a^\lambda \\ &+54(2-e)\bar\epsilon^{\beta}_{2,\rho}x^\rho\partial_\beta\Phi_{\alpha\nu\lambda}a^\nu a^\lambda]\end{aligned} \tag{5.71}$$

For the last term in the FCGA we again need the 2-2-bracket from 2-2-2:

$$[[\bar\epsilon_2,\bar\epsilon_1]] = [(g+\frac{3}{2})\bar\epsilon^{\alpha}_{2,\mu}a^\mu\bar\epsilon_{1,\alpha\rho}x^\rho - (\frac{5}{2}-g)\bar\epsilon^{\alpha}_{2,\rho}x^\rho\bar\epsilon_{1,\alpha\mu}a^\mu]\tilde g_{222} \tag{5.72}$$

The gauge deformation is:

$$\begin{aligned}\delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi) = &+\frac{3}{2}e[[\bar\epsilon_2,\bar\epsilon_1]]^\alpha\partial_\mu\Phi_{\alpha\nu\lambda} - \frac{3}{2}(1-e)\partial_\mu[[\bar\epsilon_2,\bar\epsilon_1]]^\alpha\Phi_{\alpha\nu\lambda} \\ &+\frac{3}{2}\partial_\alpha[[\bar\epsilon_2,\bar\epsilon_1]]_\mu\Phi^\alpha{}_{\nu\lambda} - [[\bar\epsilon_2,\bar\epsilon_1]]^\alpha\partial_\alpha\Phi_{\mu\nu\lambda}\end{aligned} \tag{5.73}$$

Plugging in the bracket this reads:

$$\begin{aligned}\delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi) = &\tilde g_{222}[+9e(g+\frac{3}{2})\bar\epsilon^{\beta}_{2,\alpha}\bar\epsilon_{1,\beta\rho}x^\rho - 9e(\frac{5}{2}-g)\bar\epsilon^{\beta}_{2,\rho}x^\rho\bar\epsilon_{1,\beta\alpha}]\partial_\mu\Phi^\alpha{}_{\nu\lambda}a^\mu a^\nu a^\lambda \\ &+\tilde g_{222}[-6(g+\frac{3}{2})\bar\epsilon^{\beta}_{2,\alpha}\bar\epsilon_{1,\beta\rho}x^\rho + 6(\frac{5}{2}-g)\bar\epsilon^{\beta}_{2,\rho}x^\rho\bar\epsilon_{1,\beta\alpha}]\partial_\alpha\Phi_{\mu\nu\lambda}a^\mu a^\nu a^\lambda \\ &+\tilde g_{222}[+(9(g+\frac{3}{2})+9(1-e)(\frac{5}{2}-g))\bar\epsilon^{\beta}_{2,\mu}a^\mu\bar\epsilon_{1,\beta\alpha} \\ &\quad -(9(1-e)(g+\frac{3}{2})+9(\frac{5}{2}-g))\bar\epsilon^{\beta}_{2,\alpha}\bar\epsilon_{1,\beta\mu}a^\mu]\Phi^\alpha{}_{\nu\lambda}a^\nu a^\lambda\end{aligned} \tag{5.74}$$

The FCGA then reads:

$$
\begin{aligned}
&\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi)) - \delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) + \delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi) = \\
&\bar\epsilon^{\beta}_{1,\alpha}\bar\epsilon^{\alpha}_{2,\rho}x^\rho\partial_\mu\Phi_{\beta\nu\lambda}a^\mu a^\nu a^\lambda[-27e(2-e)-54e+9e(\frac{5}{2}-g)\tilde g_{222}] \\
&+\bar\epsilon^{\beta}_{1,\alpha}\bar\epsilon^{\alpha}_{2,\rho}x^\rho\partial_\beta\Phi_{\mu\nu\lambda}a^\mu a^\nu a^\lambda[+36-6(\frac{5}{2}-g)\tilde g_{222}] \\
&+\bar\epsilon^{\beta}_{1,\nu}a^\nu\bar\epsilon^{\alpha}_{2,\rho}x^\rho\partial_\mu\Phi_{\beta\alpha\lambda}a^\mu a^\lambda[+54e^2] \\
&+\bar\epsilon^{\beta}_{1,\mu}a^\mu\bar\epsilon^{\alpha}_{2,\rho}x^\rho\partial_\alpha\Phi_{\beta\nu\lambda}a^\nu a^\lambda[+27e^2] \\
&+\bar\epsilon^{\beta}_{1,\mu}a^\mu\bar\epsilon^{\alpha}_{2,\rho}x^\rho\partial_\beta\Phi_{\alpha\nu\lambda}a^\nu a^\lambda[-54e+27e(2-e)] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon^{\alpha}_{2,\sigma}x^\sigma\partial_\alpha\partial_\beta\Phi_{\mu\nu\lambda}a^\mu a^\nu a^\lambda[+36-36] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon^{\alpha}_{2,\sigma}x^\sigma\partial_\mu\partial_\nu\Phi_{\beta\alpha\lambda}a^\mu a^\nu a^\lambda[+0] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon^{\alpha}_{2,\sigma}x^\sigma\partial_\mu\partial_\beta\Phi_{\alpha\nu\lambda}a^\mu a^\nu a^\lambda[-27e^2] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon^{\alpha}_{2,\sigma}x^\sigma\partial_\mu\partial_\alpha\Phi_{\beta\nu\lambda}a^\mu a^\nu a^\lambda[+27e^2] \\
&+\bar\epsilon^{\beta}_{1,\alpha}\bar\epsilon^{\alpha}_{2,\mu}a^\mu\Phi_{\beta\nu\lambda}a^\nu a^\lambda[+27(2-e)^2-9(g+\frac{3}{2})\tilde g_{222}-9(1-e)(\frac{5}{2}-g)\tilde g_{222}] \\
&+\bar\epsilon^{\beta}_{1,\nu}a^\nu\bar\epsilon^{\alpha}_{2,\mu}a^\mu\Phi_{\beta\alpha\lambda}a^\lambda[+0] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon^{\alpha}_{2,\mu}a^\mu\partial_\alpha\Phi_{\beta\nu\lambda}a^\nu a^\lambda[+27e^2] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon^{\alpha}_{2,\mu}a^\mu\partial_\nu\Phi_{\beta\alpha\lambda}a^\nu a^\lambda[-54e^2] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon^{\alpha}_{2,\mu}a^\mu\partial_\beta\Phi_{\alpha\nu\lambda}a^\nu a^\lambda[-27e^2] \\
&+\bar\epsilon^{\alpha}_{1,\rho}x^\rho\bar\epsilon^{\beta}_{2,\alpha}\partial_\mu\Phi_{\beta\nu\lambda}a^\mu a^\nu a^\lambda[+27e(2-e)+54e-9e(g+\frac{3}{2})\tilde g_{222}] \\
&+\bar\epsilon^{\alpha}_{1,\rho}x^\rho\bar\epsilon^{\beta}_{2,\alpha}\partial_\beta\Phi_{\mu\nu\lambda}a^\mu a^\nu a^\lambda[-36+6(g+\frac{3}{2})\tilde g_{222}] \\
&+\bar\epsilon^{\alpha}_{1,\mu}a^\mu\bar\epsilon^{\beta}_{2,\alpha}\Phi_{\beta\nu\lambda}a^\nu a^\lambda[-27(2-e)^2+9(1-e)(g+\frac{3}{2})\tilde g_{222}+9(\frac{5}{2}-g)\tilde g_{222}]
\end{aligned}
\tag{5.75}
$$

This is solved by setting $g = \frac{1}{2}$, $e = 0$ and $\tilde g_{222} = 3$, which perfectly agrees with our previous results. We again observe that by choosing $e = 0$ the term in the deformation where the derivative with the free index acts on the field vanishes, just like for the 2-2-deformation.

5.3.2 One spin-3 parameter, one spin-2 parameter

Now we come to the case of one spin-2 parameter, which will be $\bar\epsilon_1$, and one spin-3 parameter, which will be $\bar\epsilon_2$. Here we have again two possibilities to choose the deformations: we can either use the 2-2-deformation from 2-2-2 as the inner deformation and the 3-3-deformation from 3-3-2 as the outer deformation in the first term, as well as the 3-3-deformation from 3-3-2 as the inner deformation and the 2-3-deformation from the same vertex as the outer deformation in the second term. In this case, Φ is a spin-2 field, $\tilde{\Phi}$ is a spin-3 field and the deformation in the last term is the 3-3-deformation from 3-3-2.

The other possibililty is to use the 2-3-deformation from 3-3-2 as the inner deformation and the 3-2-deformation from the same vertex as the outer deformation in the first term, as well as the 3-2-deformation from 3-3-2 as the inner deformation and the one from 2-2-2 as the outer deformation in the second term. Then Φ is a spin-3 field, $\tilde{\Phi}$ is a spin-2 field and the deformation in the last term is the 3-2-deformation from 3-3-2. In both cases, we need to use the 3-2-bracket from 3-3-2 as the gauge parameter in the last term.

Starting with the case where Φ is a spin-2 field and $\tilde{\Phi}$ is a spin-3 field, the inner deformation of the first term is:

$$\delta^{(1)}_{\bar{\epsilon}_1}(\Phi) = \tilde{g}_{222}[-2(1+c)\bar{\epsilon}^{\alpha}_{1,\mu}a^{\mu}\Phi_{\alpha\nu}a^{\nu} + 2(1-c)\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\partial_{\mu}\Phi_{\alpha\nu}a^{\mu}a^{\nu} - 2\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\partial_{\alpha}\Phi_{\mu\nu}a^{\mu}a^{\nu}] \tag{5.76}$$

The double gauge deformation then is:

$$\begin{aligned}\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) = &+3f\bar{\epsilon}^{\alpha}_{2,\nu}\partial_{\mu}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\alpha\lambda} - 3(1-f)\partial_{\mu}\bar{\epsilon}^{\alpha}_{2,\nu}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\alpha\lambda} \\ &-3\bar{\epsilon}^{\alpha}_{2,\mu}\partial_{\alpha}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\nu\lambda} + \frac{3}{2}\partial_{\alpha}\bar{\epsilon}_{2,\mu\nu}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))^{\alpha}{}_{\lambda}\end{aligned} \tag{5.77}$$

Plugging in the inner gauge deformation from above this reads:

$$\begin{aligned}
\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi)) = \tilde g_{222}\bar\epsilon_{2,\mu\nu}{}^{\alpha}a^\mu a^\nu[&-6(2-f)(1+c)\bar\epsilon^{\beta}_{1,\alpha}\Phi_{\beta\lambda}a^\lambda\\
&-6(2-f)(1+c)\bar\epsilon^{\beta}_{1,\lambda}a^\lambda\Phi_{\beta\alpha}\\
&+6(2-f)(1-c)\bar\epsilon^{\beta}_{1,\rho}x^\rho\partial_\alpha\Phi_{\beta\lambda}a^\lambda\\
&+6(2-f)(1-c)\bar\epsilon^{\beta}_{1,\rho}x^\rho\partial_\lambda\Phi_{\beta\alpha}a^\lambda\\
&-12(2-f)\bar\epsilon^{\beta}_{1,\rho}x^\rho\partial_\beta\Phi_{\alpha\lambda}a^\lambda]\\
+\tilde g_{222}\bar\epsilon^{\alpha}_{2,\mu\rho}a^\mu x^\rho[&+24\bar\epsilon^{\beta}_{1,\alpha}\partial_\beta\Phi_{\nu\lambda}a^\nu a^\lambda+24\bar\epsilon^{\beta}_{1,\rho}x^\rho\partial_\alpha\partial_\beta\Phi_{\nu\lambda}a^\nu a^\lambda\\
&+12f(1-c)\bar\epsilon^{\beta}_{1,\rho}x^\rho\partial_\nu\partial_\lambda\Phi_{\alpha\beta}a^\nu a^\lambda-24f\bar\epsilon^{\beta}_{1,\nu}a^\nu\partial_\beta\Phi_{\alpha\lambda}a^\lambda\\
&-24f\bar\epsilon^{\beta}_{1,\rho}x^\rho\partial_\nu\partial_\beta\Phi_{\alpha\lambda}a^\nu a^\lambda\\
&+(24(1+c)+12f(1-c))\bar\epsilon^{\beta}_{1,\nu}a^\nu\partial_\alpha\Phi_{\beta\lambda}a^\lambda\\
&-(24(1-c)+12f(1+c))\bar\epsilon^{\beta}_{1,\alpha}\partial_\nu\Phi_{\beta\lambda}a^\nu a^\lambda\\
&+(-24(1-c)+12f(1-c))\bar\epsilon^{\beta}_{1,\rho}x^\rho\partial_\nu\partial_\alpha\Phi_{\beta\lambda}a^\nu a^\nu\\
&+(12f(1-c)-12f(1+c))\bar\epsilon^{\beta}_{1,\lambda}a^\lambda\partial_\nu\Phi_{\beta\alpha}a^\nu]
\end{aligned} \tag{5.78}$$

For the second term, the inner gauge deformation is:

$$\begin{aligned}
\delta^{(1)}_{\bar\epsilon_2}(\Phi) = &+6(2-f)\bar\epsilon_{2,\mu\nu}{}^{\alpha}a^\mu a^\nu\Phi_{\alpha\lambda}a^\lambda+12f\bar\epsilon^{\alpha}_{2,\nu\rho}a^\nu x^\rho\partial_\mu\Phi_{\alpha\lambda}a^\mu a^\lambda\\
&-12\bar\epsilon^{\alpha}_{2,\mu\rho}a^\mu x^\rho\partial_\alpha\Phi_{\nu\lambda}a^\nu a^\lambda
\end{aligned} \tag{5.79}$$

The double gauge deformation then is:

$$\begin{aligned}
\delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) = &+\frac{3}{2}e\bar\epsilon^{\alpha}_1\partial_\mu(\delta^{(1)}_{\bar\epsilon_2}(\Phi))_{\alpha\nu\lambda}-\frac{3}{2}(1-e)\partial_\mu\bar\epsilon^{\alpha}_1(\delta^{(1)}_{\bar\epsilon_2}(\Phi))_{\alpha\nu\lambda}\\
&+\frac{3}{2}\partial_\alpha\bar\epsilon_{1,\mu}(\delta^{(1)}_{\bar\epsilon_2}(\Phi))^{\alpha}{}_{\nu\lambda}-\bar\epsilon^{\alpha}_1\partial_\alpha(\delta^{(1)}_{\bar\epsilon_2}(\Phi))_{\mu\nu\lambda}
\end{aligned} \tag{5.80}$$

Plugging in the inner gauge deformation from above this reads:

$$
\begin{aligned}
\delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) = \bar\epsilon^{\alpha}_{1,\mu}a^\mu[&-18(2-e)(2-f)\bar\epsilon_{2,\nu\lambda}{}^{\beta}a^\nu a^\lambda\Phi_{\beta\alpha}\\
&-36(2-e)(2-f)\bar\epsilon_{2,\alpha\nu}{}^{\beta}a^\nu\Phi_{\beta\lambda}a^\lambda\\
&+36(2-e)\bar\epsilon^{\beta}_{2,\alpha\rho}x^\rho\partial_\beta\Phi_{\nu\lambda}a^\nu a^\lambda+72(2-e)\bar\epsilon^{\beta}_{2,\nu\rho}a^\nu x^\rho\partial_\beta\Phi_{\alpha\lambda}a^\lambda\\
&-36f(2-e)\bar\epsilon^{\beta}_{2,\alpha\rho}x^\rho\partial_\nu\Phi_{\beta\lambda}a^\nu a^\lambda-36f(2-e)\bar\epsilon^{\beta}_{2,\nu\rho}a^\nu x^\rho\partial_\alpha\Phi_{\beta\lambda}a^\lambda\\
&-36f(2-e)\bar\epsilon^{\beta}_{2,\nu\rho}a^\nu x^\rho\partial_\lambda\Phi_{\beta\alpha}a^\lambda]\\
+\bar\epsilon^{\alpha}_{1,\rho}x^\rho[&+72\bar\epsilon^{\beta}_{2,\mu\rho}a^\mu x^\rho\partial_\alpha\partial_\beta\Phi_{\nu\lambda}a^\nu a^\lambda-36e\bar\epsilon^{\beta}_{2,\alpha\rho}x^\rho\partial_\mu\partial_\beta\Phi_{\nu\lambda}a^\mu a^\nu a^\lambda\\
&-72e\bar\epsilon^{\beta}_{2,\nu\mu}a^\mu a^\nu\partial_\beta\Phi_{\alpha\lambda}a^\lambda-72e\bar\epsilon^{\beta}_{2,\nu\rho}a^\nu x^\rho\partial_\mu\partial_\beta\Phi_{\alpha\lambda}a^\mu a^\lambda\\
&+36f\cdot e\bar\epsilon^{\beta}_{2,\alpha\rho}x^\rho\partial_\mu\partial_\nu\Phi_{\beta\lambda}a^\mu a^\nu a^\lambda+36f\cdot e\bar\epsilon^{\beta}_{2,\nu\rho}a^\nu x^\rho\partial_\mu\partial_\lambda\Phi_{\alpha\beta}a^\mu a^\lambda\\
&-(36(2-f)+18f\cdot e)\bar\epsilon_{2,\mu\nu}{}^{\beta}a^\mu a^\nu\partial_\alpha\Phi_{\beta\lambda}a^\lambda\\
&+(72+36e)\bar\epsilon^{\beta}_{2,\mu\alpha}a^\mu\partial_\beta\Phi_{\nu\lambda}a^\nu a^\lambda\\
&+36e\bar\epsilon_{2,\alpha\mu}{}^{\beta}a^\mu\partial_\beta\Phi_{\nu\lambda}+(-72f+36f\cdot e)\bar\epsilon^{\beta}_{2,\nu\rho}a^\nu x^\rho\partial_\alpha\partial_\mu\Phi_{\beta\lambda}a^\mu a^\lambda\\
&+(18e(2-f)-18f\cdot e)\bar\epsilon_{2,\mu\nu}{}^{\beta}a^\mu a^\nu\partial_\lambda\Phi_{\beta\alpha}a^\lambda\\
&-(72f+36f\cdot e)\bar\epsilon^{\beta}_{2,\nu\alpha}a^\nu\partial_\mu\Phi_{\beta\lambda}a^\mu a^\lambda\\
&+(36e(2-f)-36f\cdot e)\bar\epsilon_{2,\alpha\nu}{}^{\beta}a^\nu\partial_\mu\Phi_{\beta\lambda}a^\mu a^\lambda]
\end{aligned}
\tag{5.81}
$$

For the last term we need the 3-2-bracket from 3-3-2:

$$
[[\bar\epsilon_2,\bar\epsilon_1]] = -3(g+\frac{3}{2})\bar\epsilon_{2,\mu\nu}{}^{\alpha}a^\mu a^\nu\bar\epsilon_{1,\alpha\rho}x^\rho-3(3+2(1-g))\bar\epsilon^{\alpha}_{2,\mu\rho}a^\mu x^\rho\bar\epsilon_{1,\alpha\nu}a^\nu \tag{5.82}
$$

The gauge deformation is:

$$
\begin{aligned}
\delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi) = &+3f[[\bar\epsilon_2,\bar\epsilon_1]]^{\alpha}{}_{\nu}\partial_\mu\Phi_{\alpha\lambda}-3(1-f)\partial_\mu[[\bar\epsilon_2,\bar\epsilon_1]]^{\alpha}{}_{\nu}\Phi_{\alpha\lambda}\\
&-3[[\bar\epsilon_2,\bar\epsilon_1]]^{\alpha}{}_{\mu}\partial_\alpha\Phi_{\nu\lambda}+\frac{3}{2}\partial_\alpha[[\bar\epsilon_2,\bar\epsilon_1]]_{\mu\nu}\Phi^{\alpha}{}_{\lambda}
\end{aligned}
\tag{5.83}
$$

Plugging in the bracket this reads:

$$
\begin{aligned}
\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = {} & [+36(g+\frac{3}{2})\bar{\epsilon}^{\alpha}_{2,\mu}{}^{\beta}a^{\mu}\bar{\epsilon}_{1,\beta\rho}x^{\rho} + 18(3+2(1-g))\bar{\epsilon}^{\beta\alpha}_{2,\ \rho}x^{\rho}\bar{\epsilon}_{1,\beta\mu}a^{\mu} \\
& +18(3+2(1-g))\bar{\epsilon}^{\beta}_{2,\mu\rho}a^{\mu}x^{\rho}\bar{\epsilon}_{1,\beta}{}^{\alpha}]\partial_{\alpha}\Phi_{\nu\lambda}a^{\nu}a^{\lambda} \\
& +[-36f(g+\frac{3}{2})\bar{\epsilon}^{\alpha}_{2,\nu}{}^{\beta}a^{\nu}\bar{\epsilon}_{1,\beta\rho}x^{\rho} - 18f(3+2(1-g))\bar{\epsilon}^{\beta\alpha}_{2,\ \rho}x^{\rho}\bar{\epsilon}_{1,\beta\nu}a^{\nu} \\
& -18f(3+2(1-g))\bar{\epsilon}^{\beta}_{2,\nu\rho}a^{\nu}x^{\rho}\bar{\epsilon}_{1,\beta}{}^{\alpha}]\partial_{\mu}\Phi_{\alpha\lambda}a^{\mu}a^{\lambda} \\
& +[-(18(g+\frac{3}{2})+9(1-f)(3+2(1-g)))\bar{\epsilon}_{2,\mu\nu}{}^{\beta}a^{\mu}a^{\nu}\bar{\epsilon}^{\ \alpha}_{1,\beta} \\
& -(18(3+2(1-g))+18(1-f)(3+2(1-g)))\bar{\epsilon}^{\beta}_{2,\mu}{}^{\alpha}a^{\mu}\bar{\epsilon}_{1,\beta\nu}a^{\nu} \\
& +(36(1-f)(g+\frac{3}{2})-18(1-f)(3+2(1-g)))\bar{\epsilon}^{\alpha}_{2,\mu}{}^{\beta}a^{\mu}\bar{\epsilon}_{1,\beta\nu}a^{\nu}]\Phi_{\alpha\lambda}a^{\lambda}
\end{aligned}
\tag{5.84}
$$

The FCGA then reads:

$$
\begin{aligned}
&\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi)) - \delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) + \delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi) = \\
&\bar\epsilon^{\beta}_{1,\alpha}\bar\epsilon_{2,\mu\nu}{}^{\alpha}a^\mu a^\nu \Phi_{\beta\lambda}a^\lambda[-6(2-f)(1+c)\tilde g_{222} + 18(g+\frac{3}{2}) \\
&\qquad +9(1-f)(3+2(1-g))] \\
&+\bar\epsilon^{\beta}_{1,\lambda}a^\lambda\bar\epsilon_{2,\mu\nu}{}^{\alpha}a^\mu a^\nu \Phi_{\beta\alpha}[-6(2-f)(1+c)\tilde g_{222} + 18(2-f)(2-e)] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon_{2,\mu\nu}{}^{\alpha}a^\mu a^\nu \partial_\lambda\Phi_{\beta\alpha}a^\lambda[+6(2-f)(1-c)\tilde g_{222} - 18e(2-f) + 18f\cdot e] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon_{2,\mu\nu}{}^{\alpha}a^\mu a^\nu \partial_\alpha\Phi_{\beta\lambda}a^\lambda[+6(2-f)(1-c)\tilde g_{222} - 36e] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon_{2,\mu\nu}{}^{\alpha}a^\mu a^\nu \partial_\beta\Phi_{\alpha\lambda}a^\lambda[-12(2-f)\tilde g_{222} + 36(2-f) + 18f\cdot e] \\
&+\bar\epsilon^{\beta}_{1,\alpha}\bar\epsilon^{\alpha}_{2,\mu\rho}a^\mu \partial_\beta\Phi_{\nu\lambda}a^\nu a^\lambda[+24\tilde g_{222} - 18(3+2(1-g))] \\
&+\bar\epsilon^{\beta}_{1,\alpha}\bar\epsilon^{\alpha}_{2,\nu\rho}a^\nu \partial_\mu\Phi_{\beta\lambda}a^\mu a^\lambda[-24(1-c)\tilde g_{222} - 12f(1+c)\tilde g_{222} \\
&\qquad +18f(3+2(1-g))] \\
&+\bar\epsilon^{\beta}_{1,\mu}a^\mu\bar\epsilon^{\alpha}_{2,\nu\rho}a^\nu x^\rho \partial_\beta\Phi_{\alpha\lambda}a^\lambda[-24f\cdot\tilde g_{222} + 36f(2-e)] \\
&+\bar\epsilon^{\beta}_{1,\mu}a^\mu\bar\epsilon^{\alpha}_{2,\nu\rho}a^\nu x^\rho \partial_\alpha\Phi_{\beta\lambda}a^\lambda[+24(1+c)\tilde g_{222} + 12f(1-c)\tilde g_{222} - 72(2-e)] \\
&+\bar\epsilon^{\beta}_{1,\lambda}a^\lambda\bar\epsilon^{\alpha}_{2,\nu\rho}a^\nu x^\rho \partial_\mu\Phi_{\beta\alpha}a^\mu[+12f(1-c)\tilde g_{222} - 12f(1+c)\tilde g_{222} + 36f(2-e)] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon^{\alpha}_{2,\nu\sigma}a^\nu x^\sigma \partial_\alpha\partial_\beta\Phi_{\mu\lambda}a^\mu a^\lambda[+24\tilde g_{222} - 72] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon^{\alpha}_{2,\nu\sigma}a^\nu x^\sigma \partial_\mu\partial_\lambda\Phi_{\beta\alpha}a^\mu a^\lambda[+12f(1-c)\tilde g_{222} - 36f\cdot e] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon^{\alpha}_{2,\nu\sigma}a^\nu x^\sigma \partial_\mu\partial_\alpha\Phi_{\beta\lambda}a^\mu a^\lambda[-24(1-c)\tilde g_{222} + 12f(1-c)\tilde g_{222} + 72e] \\
&+\bar\epsilon^{\beta}_{1,\rho}x^\rho\bar\epsilon^{\alpha}_{2,\nu\sigma}a^\nu x^\sigma \partial_\mu\partial_\beta\Phi_{\alpha\lambda}a^\mu a^\lambda[-24f\cdot\tilde g_{222} + 72f - 36f\cdot e] \\
&+\bar\epsilon^{\alpha}_{1,\mu}a^\mu\bar\epsilon^{\beta}_{2,\alpha\rho}a^\nu x^\rho \partial_\beta\Phi_{\nu\lambda}a^\nu a^\lambda[-36(2-e) + 18(3+2(1-g))] \\
&+\bar\epsilon^{\alpha}_{1,\mu}a^\mu\bar\epsilon^{\beta}_{2,\alpha\rho}x^\rho \partial_\nu\Phi_{\beta\lambda}a^\nu a^\lambda[+36f(2-e) - 18f(3+2(1-g))] \\
&+\bar\epsilon^{\alpha}_{1,\mu}a^\mu\bar\epsilon_{2,\alpha\nu}{}^{\beta}a^\nu \Phi_{\beta\lambda}a^\lambda[+36(2-e)(2-f) - 18(3+2(1-g)) \\
&\qquad -18(1-f)(3+2(1-g))] \\
&+\bar\epsilon^{\alpha}_{1,\rho}x^\rho\bar\epsilon^{\beta}_{2,\mu\alpha}a^\mu \partial_\beta\Phi_{\nu\lambda}a^\nu a^\lambda[-72 - 36e + 36(g+\frac{3}{2})] \\
&+\bar\epsilon^{\alpha}_{1,\rho}x^\rho\bar\epsilon^{\beta}_{2,\nu\alpha}a^\nu \partial_\mu\Phi_{\beta\lambda}a^\mu a^\lambda[+72f + 36f\cdot e - 36f(g+\frac{3}{2})]
\end{aligned}
\tag{5.85}
$$

This is solved by setting $c = 1$, $e = 0$, $g = \frac{1}{2}$ and $\tilde g_{222} = 3$, which is again in perfect agreement with our previous results. Interestingly, the parameter f is not fixed by this calculation. It is expected that it will be fixed by the other cases.

Coming now to the second case where Φ is a spin-3 field and $\tilde{\Phi}$ is a spin-2 field, the first term in the inner gauge deformation is:

$$\delta^{(1)}_{\bar{\epsilon}_1}(\Phi) = -9(2-e)\bar{\epsilon}^{\alpha}_{1,\mu}a^{\mu}\Phi_{\alpha\nu\lambda}a^{\nu}a^{\lambda} + 9e\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\partial_{\mu}\Phi_{\alpha\nu\lambda}a^{\mu}a^{\nu}a^{\lambda} - 6\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\partial_{\alpha}\Phi_{\mu\nu\lambda}a^{\mu}a^{\nu}a^{\lambda} \tag{5.86}$$

The double gauge deformation then is:

$$\begin{aligned}\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) = &-\frac{15}{2}d\partial_{\mu}\bar{\epsilon}_2^{\alpha\beta}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\alpha\beta\nu} + \frac{15}{2}(1-d)\bar{\epsilon}_2^{\alpha\beta}\partial_{\mu}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\alpha\beta\nu}\\ &-3\bar{\epsilon}_2^{\alpha\beta}\partial_{\alpha}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\beta\mu\nu} - 3\partial_{\alpha}\bar{\epsilon}_{2,\beta\mu}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))^{\alpha\beta}{}_{\nu}\end{aligned} \tag{5.87}$$

Plugging in the inner gauge deformation from above this reads:

$$\begin{aligned}\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) = \bar{\epsilon}^{\alpha\beta}_{2,\ \rho}x^{\rho}[&+270(e(1-d)-(2-e)(1-d))\bar{\epsilon}_{1,\gamma\nu}a^{\nu}\partial_{\mu}\Phi^{\gamma}{}_{\alpha\beta}a^{\mu}\\ &-(216e+540(1-d)(2-e))\bar{\epsilon}_{1,\gamma\alpha}\partial_{\mu}\Phi^{\gamma}{}_{\beta\nu}a^{\mu}a^{\nu}\\ &+(216(2-e)+540e(1-d))\bar{\epsilon}_{1,\gamma\mu}a^{\mu}\partial_{\alpha}\Phi^{\gamma}{}_{\beta\nu}a^{\nu}\\ &+270e(1-d)\bar{\epsilon}_{1,\gamma\rho}x^{\rho}\partial_{\mu}\partial_{\nu}\Phi^{\gamma}{}_{\alpha\beta}a^{\mu}a^{\nu}\\ &+(-216e+540e(1-d))d\bar{\epsilon}_{1,\gamma\rho}x^{\rho}\partial_{\mu}\partial_{\alpha}\Phi^{\gamma}{}_{\beta\nu}a^{\mu}a^{\nu}\\ &-540(1-d)\bar{\epsilon}^{\gamma}_{1,\mu}a^{\mu}\partial_{\gamma}\Phi_{\alpha\beta\nu}a^{\nu} - 540(1-d)\bar{\epsilon}^{\gamma}_{1,\rho}x^{\rho}\partial_{\mu}\partial_{\gamma}\Phi_{\alpha\beta\nu}a^{\mu}a^{\nu}\\ &+216(1-e)\bar{\epsilon}^{\gamma}_{1,\alpha}\partial_{\beta}\Phi_{\gamma\mu\nu}a^{\mu}a^{\nu} - 108e\bar{\epsilon}^{\gamma}_{1,\sigma}x^{\sigma}\partial_{\alpha}\partial_{\beta}\Phi_{\gamma\mu\nu}a^{\mu}a^{\nu}\\ &+216\bar{\epsilon}^{\gamma}_{1,\alpha}\partial_{\gamma}\Phi_{\beta\mu\nu}a^{\mu}a^{\nu} + 216\bar{\epsilon}^{\gamma}_{1,\sigma}x^{\sigma}\partial_{\alpha}\partial_{\gamma}\Phi_{\beta\mu\nu}a^{\mu}a^{\nu}]\\ +\bar{\epsilon}^{\alpha\beta}_{2,\ \mu}a^{\mu}[&-18(2-e)(3-15d)\bar{\epsilon}_{1,\gamma\nu}a^{\nu}\Phi^{\gamma}{}_{\alpha\beta}\\ &-36(2-e)(3-15d)\bar{\epsilon}_{1,\gamma\alpha}\Phi^{\gamma}{}_{\beta\nu}a^{\nu}\\ &+18e(3-15d)\bar{\epsilon}_{1,\gamma\rho}x^{\rho}\partial_{\nu}\Phi^{\gamma}{}_{\alpha\beta}a^{\nu}\\ &+36e(3-15d)\bar{\epsilon}_{1,\gamma\rho}x^{\rho}\partial_{\alpha}\Phi^{\gamma}{}_{\beta\nu}a^{\nu}\\ &-36(3-15d)\bar{\epsilon}^{\gamma}_{1,\rho}x^{\rho}\partial_{\gamma}\Phi_{\alpha\beta\nu}a^{\nu}]\end{aligned} \tag{5.88}$$

For the second term, the inner gauge deformation is:

$$\begin{aligned}\delta^{(1)}_{\bar{\epsilon}_2}(\Phi) = &+90(1-d)\bar{\epsilon}^{\alpha\beta}_{2,\ \rho}x^{\rho}\partial_{\mu}\Phi_{\alpha\beta\nu}a^{\mu}a^{\nu} + 9(2-10d)\bar{\epsilon}^{\alpha\beta}_{2,\ \mu}a^{\mu}\Phi_{\alpha\beta\nu}a^{\nu}\\ &-36\bar{\epsilon}^{\alpha\beta}_{2,\ \rho}x^{\rho}\partial_{\alpha}\Phi_{\beta\mu\nu}a^{\mu}a^{\nu}\end{aligned} \tag{5.89}$$

The double gauge deformation then is:

$$\delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) = \tilde{g}_{222}[-c\partial_\mu\bar{\epsilon}^\alpha_1(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi))_{\alpha\nu} + (1-c)\bar{\epsilon}^\alpha_1\partial_\mu(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi))_{\alpha\nu} \\ -\bar{\epsilon}^\alpha_1\partial_\alpha(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi))_{\mu\nu} + \partial_\alpha\bar{\epsilon}_{1,\mu}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi))^\alpha{}_\nu] \tag{5.90}$$

Plugging in the inner gauge deformation from above this reads:

$$\begin{aligned}\delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) = \tilde{g}_{222}\bar{\epsilon}^\alpha_{1,\mu}a^\mu[&-90(1+c)(1-d)\bar{\epsilon}^{\beta\gamma}_{2,\ \rho}x^\rho\partial_\alpha\Phi_{\beta\gamma\nu}a^\nu \\ &-90(1+c)(1-d)\bar{\epsilon}^{\beta\gamma}_{2,\ \rho}x^\rho\partial_\nu\Phi_{\beta\gamma\alpha}a^\nu \\ &-9(1+c)(2-10d)\bar{\epsilon}^{\beta\gamma}_{2,\ \alpha}\Phi_{\beta\gamma\nu}a^\nu \\ &-9(1+c)(2-10d)\bar{\epsilon}^{\beta\gamma}_{2,\ \nu}a^\nu\Phi_{\beta\gamma\alpha} \\ &+72(1+c)\bar{\epsilon}^{\beta\gamma}_{2,\ \rho}x^\rho\partial_\beta\Phi_{\gamma\alpha\nu}a^\nu] \\ +\tilde{g}_{222}\bar{\epsilon}^\alpha_{1,\rho}x^\rho[&+90(1-c)(1-d)\bar{\epsilon}^{\beta\gamma}_{2,\ \sigma}x^\sigma\partial_\mu\partial_\nu\Phi_{\beta\gamma\alpha}a^\mu a^\nu \\ &+(1-c)(9(2-10d)+90(1-d))\bar{\epsilon}^{\beta\gamma}_{2,\ \mu}a^\mu\partial_\nu\Phi_{\beta\gamma\alpha}a^\nu \\ &+(-180(1-d)+9(1-c)(2-10d))\bar{\epsilon}^{\beta\gamma}_{2,\ \alpha}\partial_\mu\Phi_{\beta\gamma\nu}a^\mu a^\nu \\ &+(-18(2-10d)+90(1-c)(1-d))\bar{\epsilon}^{\beta\gamma}_{2,\ \mu}a^\mu\partial_\alpha\Phi_{\beta\gamma\nu}a^\nu \\ &+(-180(1-d)+90(1-c)(1-d))\bar{\epsilon}^{\beta\gamma}_{2,\ \sigma}x^\sigma\partial_\alpha\partial_\mu\Phi_{\beta\gamma\nu}a^\mu a^\nu \\ &+72\bar{\epsilon}^{\beta\gamma}_{2,\ \alpha}\partial_\beta\Phi_{\gamma\mu\nu}a^\mu a^\nu + 72\bar{\epsilon}^{\beta\gamma}_{2,\ \sigma}x^\sigma\partial_\alpha\partial_\beta\Phi_{\gamma\mu\nu}a^\mu a^\nu \\ &-72(1-c)\bar{\epsilon}^{\beta\gamma}_{2,\ \mu}a^\mu\partial_\beta\Phi_{\gamma\alpha\nu}a^\nu \\ &-72(1-c)\bar{\epsilon}^{\beta\gamma}_{2,\ \sigma}x^\sigma\partial_\mu\partial_\beta\Phi_{\gamma\alpha\nu}a^\mu a^\nu]\end{aligned} \tag{5.91}$$

For the last term we again need the 3-2-bracket from 3-3-2:

$$[[\bar{\epsilon}_2, \bar{\epsilon}_1]] = -3(3+2(1-g))\bar{\epsilon}^\alpha_{2,\mu\rho}a^\mu x^\rho\bar{\epsilon}_{1,\alpha\nu}a^\nu - 3(g+\frac{3}{2})\bar{\epsilon}_{2,\mu\nu}{}^\alpha a^\mu a^\nu\bar{\epsilon}_{1,\alpha\rho}x^\rho \tag{5.92}$$

The gauge deformation is:

$$\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = -\frac{15}{2}d\partial_\mu[[\bar{\epsilon}_2, \bar{\epsilon}_1]]^{\alpha\beta}\Phi_{\alpha\beta\nu} + \frac{15}{2}(1-d)[[\bar{\epsilon}_2, \bar{\epsilon}_1]]^{\alpha\beta}\partial_\mu\Phi_{\alpha\beta\nu} \\ -3[[\bar{\epsilon}_2, \bar{\epsilon}_1]]^{\alpha\beta}\partial_\alpha\Phi_{\beta\mu\nu} - 3\partial_\alpha[[\bar{\epsilon}_2, \bar{\epsilon}_1]]_{\beta\mu}\Phi^{\alpha\beta}{}_\nu \tag{5.93}$$

Plugging in the bracket this reads:

$$\begin{aligned}\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = &[+108(g+\frac{3}{2})\bar{\epsilon}_2^{\alpha\beta\gamma}\bar{\epsilon}_{1,\gamma\rho}x^\rho + 54(3+2(1-g))\bar{\epsilon}^{\gamma\alpha}_{2,\ \rho}x^\rho\bar{\epsilon}_{1,\gamma}{}^\beta \\ &+54(3+2(1-g))\bar{\epsilon}^{\gamma\beta}_{2,\ \rho}x^\rho\bar{\epsilon}_{1,\gamma}{}^\alpha]\partial_\alpha\Phi_{\beta\mu\nu}a^\mu a^\nu \\ &+[-270(1-d)(3+2(1-g))\bar{\epsilon}^{\gamma\alpha}_{2,\ \rho}x^\rho\bar{\epsilon}_{1,\gamma}{}^\beta \\ &-270(1-d)(g+\frac{3}{2})\bar{\epsilon}_2^{\alpha\beta\gamma}\bar{\epsilon}_{1,\gamma\rho}x^\rho]\partial_\mu\Phi_{\alpha\beta\nu}a^\mu a^\nu \\ &+[+(-27(3+2(1-g))+270d(g+\frac{3}{2}))\bar{\epsilon}_2^{\alpha\beta\gamma}\bar{\epsilon}_{1,\gamma\mu}a^\mu \\ &+(108(g+\frac{3}{2})-270d(3+2(1-g)))\bar{\epsilon}_{2,\beta\mu}{}^\gamma a^\mu\bar{\epsilon}_{1,\gamma\alpha} \\ &+(54(3+2(1-g))-270d(3+2(1-g)))\bar{\epsilon}^{\gamma}_{2,\mu\beta}a^\mu\bar{\epsilon}_{1,\gamma\alpha}]\Phi_{\alpha\beta\nu}a^\nu\end{aligned} \tag{5.94}$$

The FCGA then reads:

$$\begin{aligned}
&\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi)) - \delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) + \delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi) = \\
&\bar\epsilon^{\gamma}_{1,\nu} a^\nu \bar\epsilon^{\alpha\beta}_{2,\ \rho} x^\rho \partial_\mu \Phi_{\gamma\alpha\beta} a^\mu [-270(1-d)(2-e) + 270(1-d)e \\
&\qquad +90(1+c)(1-d)\tilde g_{222}] \\
&+\bar\epsilon^{\gamma}_{1,\alpha} \bar\epsilon^{\alpha\beta}_{2,\ \rho} x^\rho \partial_\mu \Phi_{\gamma\beta\nu} a^\mu a^\nu [-216e - 540(1-d)(2-e) + 270(1-d)(3+2(1-g))] \\
&+\bar\epsilon^{\gamma}_{1,\mu} a^\mu \bar\epsilon^{\alpha\beta}_{2,\ \rho} x^\rho \partial_\gamma \Phi_{\alpha\beta\nu} a^\nu [-540(1-d) + 90(1+c)(1-d)\tilde g_{222}] \\
&+\bar\epsilon^{\gamma}_{1,\rho} x^\rho \bar\epsilon^{\alpha\beta}_{2,\ \sigma} x^\sigma \partial_\mu \partial_\nu \Phi_{\gamma\alpha\beta} a^\mu a^\nu [+270e(1-d) - 90(1-c)(1-d)\tilde g_{222}] \\
&+\bar\epsilon^{\gamma}_{1,\rho} x^\rho \bar\epsilon^{\alpha\beta}_{2,\ \sigma} x^\sigma \partial_\mu \partial_\gamma \Phi_{\alpha\beta\nu} a^\mu a^\nu [-540(1-d) + 180(1-d)\tilde g_{222} - 90(1-c)(1-d)\tilde g_{222}] \\
&+\bar\epsilon^{\gamma}_{1,\nu} a^\nu \bar\epsilon^{\alpha\beta}_{2,\ \mu} a^\mu \Phi_{\gamma\alpha\beta} [-18(2-e)(3-15d) + 9(1+c)(2-10d)\tilde g_{222}] \\
&+\bar\epsilon^{\gamma}_{1,\rho} x^\rho \bar\epsilon^{\alpha\beta}_{2,\ \mu} a^\mu \partial_\nu \Phi_{\gamma\alpha\beta} a^\nu [+18e(3-15d) - (1-c)(9(2-10d) + 90(1-d))\tilde g_{222}] \\
&+\bar\epsilon^{\gamma}_{1,\rho} x^\rho \bar\epsilon^{\alpha\beta}_{2,\ \mu} a^\mu \partial_\gamma \Phi_{\alpha\beta\nu} a^\nu [-36(3-15d) + 18(2-10d)\tilde g_{222} - 90(1-c)(1-d)\tilde g_{222}] \\
&+\bar\epsilon^{\alpha}_{1,\mu} a^\mu \bar\epsilon^{\beta\gamma}_{2,\ \alpha} \Phi_{\beta\gamma\nu} a^\nu [+9(1+c)(2-10d)\tilde g_{222} - 27(3+2(1-g)) \\
&\qquad +270d(g+\frac{3}{2})] \\
&+\bar\epsilon^{\alpha}_{1,\rho} x^\rho \bar\epsilon^{\beta\gamma}_{2,\ \alpha} \partial_\mu \Phi_{\beta\gamma\nu} a^\mu a^\nu [+180(1-d)\tilde g_{222} - 9(1-c)(2-10d)\tilde g_{222} \\
&\qquad -270(1-d)(g+\frac{3}{2})] \\
&+\bar\epsilon^{\alpha}_{1,\rho} x^\rho \bar\epsilon^{\beta\gamma}_{2,\ \alpha} \partial_\beta \Phi_{\gamma\mu\nu} a^\mu a^\nu [-72\tilde g_{222} + 108(g+\frac{3}{2})] \\
&+\bar\epsilon^{\gamma}_{1,\alpha} \bar\epsilon^{\alpha\ \ \beta}_{2,\mu} a^\mu \Phi_{\gamma\beta\nu} a^\nu [+36(3-15d)(2-e) - 54(3+2(1-g)) \\
&\qquad +270(3+2(1-g))] \\
&+\bar\epsilon^{\gamma}_{1,\alpha} \bar\epsilon^{\beta\ \ \alpha}_{2,\mu} a^\mu \Phi_{\gamma\beta\nu} a^\nu [+36(3-15d)(2-e) - 108(g+\frac{3}{2}) + 270(3+2(1-g))] \\
&+\bar\epsilon^{\gamma}_{1,\rho} x^\rho \bar\epsilon^{\alpha\beta}_{2,\ \mu} a^\mu \partial_\alpha \Phi_{\gamma\beta\nu} a^\nu [+36e(3-15d) + 72(1-c)\tilde g_{222}] \\
&+\bar\epsilon^{\gamma}_{1,\alpha} \bar\epsilon^{\alpha\beta}_{2,\ \rho} x^\rho \partial_\beta \Phi_{\gamma\mu\nu} a^\mu a^\nu [-108e + 108(2-e) - 54(3+2(1-g))] \\
&+\bar\epsilon^{\gamma}_{1,\alpha} \bar\epsilon^{\alpha\beta}_{2,\ \rho} x^\rho \partial_\gamma \Phi_{\beta\mu\nu} a^\mu a^\nu [+216 - 54(3+2(1-g))] \\
&+\bar\epsilon^{\gamma}_{1,\mu} a^\mu \bar\epsilon^{\alpha\beta}_{2,\ \rho} x^\rho \partial_\alpha \Phi_{\gamma\beta\nu} a^\nu [+216(2-e) + 540e(1-d) - 72(1+c)\tilde g_{222}] \\
&+\bar\epsilon^{\gamma}_{1,\sigma} x^\sigma \bar\epsilon^{\alpha\beta}_{2,\ \rho} x^\rho \partial_\alpha \partial_\gamma \Phi_{\beta\mu\nu} a^\mu a^\nu [+216 - 72\tilde g_{222}] \\
&+\bar\epsilon^{\gamma}_{1,\sigma} x^\sigma \bar\epsilon^{\alpha\beta}_{2,\ \rho} x^\rho \partial_\alpha \partial_\mu \Phi_{\gamma\beta\nu} a^\mu a^\nu [-216e + 540e(1-d) + 72(1-c)\tilde g_{222}] \\
&+\bar\epsilon^{\gamma}_{1,\rho} x^\rho \bar\epsilon^{\alpha\beta}_{2,\ \sigma} x^\sigma \partial_\alpha \partial_\beta \Phi_{\gamma\mu\nu} a^\mu a^\nu [-9e]
\end{aligned} \tag{5.95}$$

This is solved by setting $e = 0$, $c = 1$, $g = \frac{1}{2}$ and $\tilde g_{222} = 3$, which again perfectly agrees with our previous results. Much like f in the previous calculation the parameter d is not fixed by this.

5.3.3 Two spin-3 parameters

The last case is the one of two spin-3 parameters. As we will see, this case differs fundamentally from the previous calculations for multiple reasons. First, in the same fashion as in the calculation of the Jacobi-identity for three spin-3 parameters it will become necessary to make use of DDI's. Second, we also need to account for possible terms in the "trivial" part on the RHS of (5.9), i.e. terms containing the EOM or zero-order gauge transformations.

The biggest difference to the other cases is that we can no longer ignore the fact that the first-order gauge deformations are no longer traceless and divergence-free in general. In the first two terms of (5.9) possible DT-terms in the outer deformations can produce TT-terms when acting on the inner deformations that will still be present after dropping all traces and divergences of the field itself.

For the values of the partial integration parameters c and e, corresponding to the 2-2- and 2-3-deformation respectively, that have been fixed by the preceding calculations, these deformations indeed become traceless and divergence-free. For that reason no DT-terms can contribute to any of the previous calculations. However, we know from 5.1 that the 3-3-deformation is neither traceless nor divergence-free for any values of the partial integration parameter f and that the 3-2 deformation, whose partial integration parameter d also remains unfixed, is only divergence-free for $d = \frac{4}{5}$, which is not necessarily the value that is required for the consistency of the theory.

Hence, much like in the calculations of the Fronsdal-equation for the gauge deformations, where we had to keep the DT-terms in the Fronsdal-tensor, we now need to consider all possible DT-terms that act on the 3-2-deformation and the 3-3-deformation, rather than on the field itself. In principle, all DT-terms in the vertices and the deformations can be systematically calculated once the TT-part of the cubic action is known [48], but in this work we take a different approach. For the case of two spin-3-parameters, the only deformations that contribute to the first two terms of (5.9) are the 3-2-deformation and the 3-3-deformation. We therefore build an ansatz for the DT-part of these deformations and fix the coefficients by demanding that (5.9) holds.

In addition to DT-terms in the outer deformations that produce TT-terms when acting on the inner deformations there is also another type of DT-term that produces TT-terms when contributing to the inner deformations. These terms are proportional to the metric $g_{\mu\nu}$ and therefore contain traces of the field in the EOM that is contracted to the deformation, as can be seen from (3.4). In the index-free setting, we have that $g_{\mu\nu}a^\mu a^\nu \equiv a^2$. Acting with an outer deformation on these terms can again produce TT-terms that will still be present after dropping all traces and divergences of the field, including terms that contain a^2. Due to the index-structure of the deformations, such terms can only be present in the 3-3-deformation.

Indeed, all of the above considerations are necessary in order to show that the FCGA indeed holds. As in the other cases, we encounter two different possibilities for choosing the deformations: we can either use the 3-3-deformation from 3-3-2 as the inner deformation and the 3-2-deformation from the same vertex as the outer deformation in the first term, as well as the same deformations in the second term. In this case, Φ and $\tilde{\Phi}$ are both spin-2 fields and the deformation in the last term is the one of the 2-2-2 vertex.

The other possibility is to use the 3-2-deformation from 3-3-2 as the inner deformation and the 3-3-deformation from 3-3-2 as the outer deformation in the first term, as well as the same deformations in the second term. Then Φ and $\tilde{\Phi}$ are both spin-3 fields and the deformation in the last term is the 2-3-deformation from the 3-3-2 vertex. In both cases we need to use the 3-3-bracket from 3-3-2 in the last term.

Starting with the case where Φ and $\tilde{\Phi}$ are spin-2 fields, the inner gauge deformation in the first term is:

$$
\begin{aligned}
\delta^{(1)}_{\bar{\epsilon}_1}(\Phi) = &+ 6(2-f)\bar{\epsilon}_{1,\mu\nu}{}^{\alpha} a^\mu a^\nu \Phi_{\alpha\lambda} a^\lambda - 12\bar{\epsilon}^{\alpha}_{1,\mu\rho} a^\mu x^\rho \partial_\alpha \Phi_{\nu\lambda} a^\nu a^\lambda \\
&+ 12 f \bar{\epsilon}^{\alpha}_{1,\nu\rho} a^\nu x^\rho \partial_\mu \Phi_{\alpha\lambda} a^\mu a^\lambda + 4w_1 g_{\mu\nu} a^\mu a^\nu \bar{\epsilon}^{\alpha\beta}_{1,\ \rho} x^\rho \partial_\lambda a^\lambda \Phi_{\alpha\beta} \\
&+ 4w_2 g_{\mu\nu} a^\mu a^\nu \bar{\epsilon}^{\alpha\beta}_{1,\ \rho} x^\rho \partial_\alpha \Phi_{\beta\lambda} a^\lambda + 4w_3 g_{\mu\nu} a^\mu a^\nu \bar{\epsilon}^{\alpha\beta}_{1,\ \lambda} a^\lambda \Phi_{\alpha\beta}
\end{aligned}
\tag{5.96}
$$

Here we have added all possible terms containing the metric with coefficients w_i to the 3-3-deformation. It is not necessary to add any other DT-terms to the inner deformations, since the trace- and divergence-operators would act directly on the field, producing no TT-terms. The double gauge deformation then is:

$$\begin{aligned}\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) = &-\frac{15}{2}d\partial_\mu\bar{\epsilon}_2^{\alpha\beta}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\alpha\beta\nu} + \frac{15}{2}(1-d)\bar{\epsilon}_2^{\alpha\beta}\partial_\mu(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\alpha\beta\nu} \\ &-3\bar{\epsilon}_2^{\alpha\beta}\partial_\alpha(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\beta\mu\nu} - 3\partial_\alpha\bar{\epsilon}_{2,\beta\mu}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))^{\alpha\beta}{}_\nu \\ &+d_1\bar{\epsilon}^\alpha_{2,\mu}\partial_\beta(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))^\beta{}_{\alpha\nu} + d_2\bar{\epsilon}^\alpha_{2,\mu}\partial_\alpha(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))^\beta{}_{\beta\nu} \\ &+d_3\bar{\epsilon}^\alpha_{2,\mu}\partial_\nu(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))^\beta{}_{\beta\alpha} + d_4\bar{\epsilon}_{2,\mu\nu}\partial_\alpha(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))^{\alpha\beta}{}_\beta \\ &+d_5\partial_\alpha\bar{\epsilon}_{2,\mu\nu}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))^{\alpha\beta}{}_\beta\end{aligned} \tag{5.97}$$

where the terms with coefficients d_i represent the ansatz for the DT-part of the deformation. Note that only those terms that are relevant for this calculation are added here. The result obtained after plugging in the inner gauge deformation from above and discarding all remaining DT-terms is rather lengthy and given as (A.1) in appendix A. The second double gauge tranformation consists of the same gauge deformations and gauge parameters as the first one, so it can be obtained by switching $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$. It is given as (A.2).

For the last term we need the 3-3-bracket from 3-3-2:

$$[[\bar{\epsilon}_2,\bar{\epsilon}_1]] = 3r(1+10g)\bar{\epsilon}^{\alpha\beta}_{2,\ \mu}a^\mu\bar{\epsilon}_{1,\alpha\beta\rho}x^\rho - 3r(1+10(1-g))\bar{\epsilon}^{\alpha\beta}_{2,\ \rho}x^\rho\bar{\epsilon}_{1,\alpha\beta\mu}a^\mu \tag{5.98}$$

Here we introduced a rescaling factor r in order to test whether the overall constant of the bracket needs to be rescaled. The gauge deformation is:

$$\begin{aligned}\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = \tilde{g}_{222}[&-c\partial_\mu[[\bar{\epsilon}_2,\bar{\epsilon}_1]]^\alpha\Phi_{\alpha\nu} + (1-c)[[\bar{\epsilon}_2,\bar{\epsilon}_1]]^\alpha\partial_\mu\Phi_{\alpha\nu} \\ &-[[\bar{\epsilon}_2,\bar{\epsilon}_1]]^\alpha\partial_\alpha\Phi_{\mu\nu} + \partial_\alpha[[\bar{\epsilon}_2,\bar{\epsilon}_1]]_\mu\Phi^\alpha{}_\nu]\end{aligned} \tag{5.99}$$

Plugging in the bracket this reads:

$$\begin{aligned}\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = \tilde{g}_{222}\cdot r[&+(6(1-c)(1+10g)\bar{\epsilon}_2^{\beta\gamma\alpha}\bar{\epsilon}_{1,\beta\gamma\rho}x^\rho \\ &-6(1-c)(1+10(1-g))\bar{\epsilon}^{\beta\gamma}_{2,\ \rho}x^\rho\bar{\epsilon}_{1,\beta\gamma}{}^\alpha)\partial_\mu\Phi_{\alpha\nu}a^\mu a^\nu \\ &+(-6(1+10g)\bar{\epsilon}_2^{\beta\gamma\alpha}\bar{\epsilon}_{1,\beta\gamma\rho}x^\rho \\ &+6(1+10(1-g))\bar{\epsilon}^{\beta\gamma}_{2,\ \rho}x^\rho\bar{\epsilon}_{1,\beta\gamma}{}^\alpha)\partial_\alpha\Phi_{\mu\nu}a^\mu a^\nu \\ &+(6(c(1+10(1-g))+1+10g)\bar{\epsilon}^{\beta\gamma}_{2,\ \mu}a^\mu\bar{\epsilon}_{1,\beta\gamma}{}^\alpha \\ &-(6(c(1+10g)+1+10(1-g)))\bar{\epsilon}_2^{\beta\gamma\alpha}\bar{\epsilon}_{1,\beta\gamma\mu}a^\mu)\Phi_{\alpha\nu}a^\nu]\end{aligned} \tag{5.100}$$

Adding these terms up and dropping an overall factor of 8, the full FCGA is again very lengthy and given as (A.3) in appendix A.

Similar to what we did for the Jacobi-identity of three spin-3 parameters, we now want to dualise this expression in order to simplify the calculation of DDI's. Since we only have two gauge parameters and the field does not need to be dualised, we can eliminate all Levi-Civita tensors, which simplifies the structure of the terms a lot. During the dualisation, we already discard terms proportional to the free EOM, since these are allowed on the RHS of (5.9). Also, in order to shorten the expressions we are dealing with, we rewrite the result of the dualisation in more convenient variables. These new variables are defined as (here we treat Φ as the object with label 3):

- $A_i \ldots$ a contracted with the i-th object
- $X_i \ldots$ x contracted with the i-th object
- z_i defined as usual
- $p_i \ldots$ derivative contracted with $\tilde{\epsilon}_i$
- $p_x \ldots$ derivative contracted with x
- $p_a \ldots$ derivative contracted with a

The FCGA then is given as (A.4).

We have three different types of terms in this expression, classified by the number of derivatives they contain, which coincides with the number of x's in these terms. Since these types of terms consist of different objects (or of a different amount of them), there will also be different types of DDI's corresponding to them. The terms in the smallest group, which have no derivative and no x, consist of the objects $\{\partial_{a_1}$, ∂_{a_1}, ∂_{a_2}, ∂_{a_2}, ∂_{a_3}, ∂_{a_3}, a, a $\}$. There is only one DDI that can be built out of these objects, which is:

$$A_1^2 z_1^2 - 2A_1A_2z_1z_2 + A_2^2z_2^2 - 2A_1A_3z_1z_3 + A_3^2z_3^2 - 2A_2A_3z_2z_3 = 0 \tag{5.101}$$

One can check that this DDI cannot be used to eliminate all terms in (A.4) of this type. Therefore, we see that we also need to allow terms that are of the form of a zero-order gauge transformation with a field-dependent parameter on the RHS of (5.9). This means we have another set of equations:

$$\delta^{(0)}_{E(\tilde{\epsilon_1},\tilde{\epsilon_2},\Phi)} \equiv (a^\mu \partial_\mu) E(\tilde{\epsilon_1},\tilde{\epsilon_2},\Phi) \approx 0 \tag{5.102}$$

We can now eliminate this freedom in the following way: since all expressions of the form $(a^\mu\partial_\mu)E(\tilde{\epsilon_1},\tilde{\epsilon_2},\Phi)$ will contain a term proportional to $p_a\Phi$, we can replace these terms with their "partial integrated" version, thus eliminating all terms where $(a^\mu\partial_\mu)$ acts on Φ. After eliminating these terms the FCGA is still very lengthy and given as (A.5).

When the terms containing p_a are partially integrated the parameter d drops out of the calculation, so it will not be fixed here.

Note that since we chose to eliminate all terms containing p_a in (A.4), we have to do the same for the DDI's that we are going to calculate. This leads to an interesting phenomenon: if we have a DDI consisting of n derivatives and n x's, the terms which contain a p_a will be replaced by terms which have $n-1$ derivatives and x's. For terms of the form $p_a p_x(\dots)$ the number of derivatives will be reduced even further, since performing one partial integration will transform p_x into another p_a, which again needs to be partially integrated. By this mechanism, a DDI originally consisting only of terms with a fixed number n of derivatives can now also contain terms with lower n, all the way down to $n = 0$. Therefore, in the new DDI's, terms of different types will be mixed.

The advantage of this is that the terms with $n = 0$ in (A.4), which did not cancel before by the means of the only DDI available, could now be canceled by DDI's with a higher number of derivatives. A big disadvantage of this mechanism is that it does not provide any cutoff for n. Even a DDI with $n = 500$ could contain terms with no derivatives. Even though (A.4) does not contain any terms with more than two derivatives, the DDI's with $n = 500$ could still be linearly combined in a way that cancels all higher order terms and leaves us with a probably new (i.e. linearly independent) DDI that only contains terms with a maximum of two derivatives. It can be speculated that all DDI's constructed this way can never be linearly independent, but there seems to be no way to be sure. However, it turns out that this does not lead to any problem and it suffices to calculate DDI's with at most two derivatives.

In order to see if the terms in (A.5) are a linear combination of DDI's all DDI's for n up to (and including) $n = 2$ have been calculated with Mathematica. We already set $g = \frac{1}{2}$ and $\tilde{g}_{222} = 3$ as dictated by the previous calculations. There are three solutions to the above system of equations:

solution 1:

$$
\begin{aligned}
d_1 &= -6 & d_2 &= \frac{6(3f+2w_2-6)}{3f+5w_2-6} \\
d_4 &= -\frac{3(3f-4w_2-6)}{2(3f+5w_2-6)} & d_5 &= \frac{-3d_3f-5d_3w_2+6d_3+18f-6w_2-36}{2(3f+5w_2-6)} \\
w_1 &= \frac{-3f^2-2fw_2+9f+5w_2-6}{3(f-2)} & w_3 &= \frac{-6f^2-7fw_2+18f+4w_2-12}{6(f-2)} \\
r &= 2
\end{aligned}
\tag{5.103}
$$

solution 2:

$$
\begin{aligned}
d_1 &= -6 & d_2 &= \frac{12(7w_2-6)}{35w_2-12} \\
d_5 &= \frac{-35d_3w_2+12d_3-42w_2-72}{2(35w_2-12)} & f &= \frac{10}{7} \\
w_1 &= \frac{1}{28}(-35w_2-12) & w_3 &= \frac{1}{28}(49w_2-12) \\
r &= 2
\end{aligned}
\tag{5.104}
$$

solution 3:

$$
\begin{aligned}
d_1 &= -6 & d_4 &= \frac{1}{4}(-3)(d_2-4) \\
d_5 &= \frac{1}{2}(2d_2-d_3-6) & f &= 2 \\
w_1 &= -\frac{6(d_2-3)}{5d_2-12} & w_2 &= 0 \\
w_3 &= -\frac{18}{5d_2-12} & r &= 2
\end{aligned}
\tag{5.105}
$$

All parameters that are not explicitly listed are understood to remain free. Interestingly, all solutions require a rescaling of the 3-3-bracket from 3-3-2 by two. As we will see, the next case also requires the exact same rescaling. The partial integration parameter of the 3-2 deformation d remains free since it drops out and the parameter of the 3-3-deformation f remains free in solution 1. Therefore, solution 1 is compatible with all previous calculations and the FCGA can be fulfilled.

Coming to the second case, where Φ and $\tilde{\Phi}$ are both spin-3 field, the inner gauge deformation in the first term is:

$$\begin{aligned}\delta^{(1)}_{\bar{\epsilon}_1}(\Phi) = &-36\bar{\epsilon}^{\alpha\beta}_{1,\ \rho}x^\rho\partial_\alpha\Phi_{\beta\mu\nu}a^\mu a^\nu + 90(1-d)\bar{\epsilon}^{\alpha\beta}_{1,\ \rho}x^\rho\partial_\mu\Phi_{\alpha\beta\nu}a^\mu a^\nu \\ &+9(2-10d)\bar{\epsilon}^{\alpha\beta}_{1,\ \mu}a^\mu\Phi_{\alpha\beta\nu}a^\nu\end{aligned} \tag{5.106}$$

Here it is not possible to introduce any terms proportional to the metric that do not contain any traces or divergences of the field or the gauge parameter. The double gauge deformation then is:

$$\begin{aligned}\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) = &+3f\bar{\epsilon}^{\alpha}_{2,\nu}\partial_\mu(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\alpha\lambda} - 3(1-f)\partial_\mu\bar{\epsilon}^{\alpha}_{2,\nu}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\alpha\lambda} \\ &-3\bar{\epsilon}^{\alpha}_{2,\mu}\partial_\alpha(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))_{\nu\lambda} + \frac{3}{2}\partial_\alpha\bar{\epsilon}_{2,\mu\nu}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))^{\alpha}_{\ \lambda} \\ &+h_1\bar{\epsilon}_{2,\mu\nu}\partial_\lambda(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))^{\alpha}_{\ \alpha} + h_2\bar{\epsilon}_{2,\mu\nu}\partial_\alpha(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))^{\alpha}_{\ \lambda}\end{aligned} \tag{5.107}$$

where the terms proportional to the h_i again represent the ansatz for the DT-part of the deformation. Plugging in the inner gauge deformation from above this reads:

$$
\begin{aligned}
\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi)) = \bar\epsilon_{2,\mu\nu}{}^{\alpha} a^\mu a^\nu [& -216(2-f)\bar\epsilon^{\beta\gamma}_{1,\ \rho} x^\rho \partial_\beta \Phi_{\gamma\alpha\lambda} a^\lambda \\
& +270(1-d)(2-f)\bar\epsilon^{\beta\gamma}_{1,\ \rho} x^\rho \partial_\alpha \Phi_{\gamma\beta\lambda} a^\lambda \\
& +270(1-d)(2-f)\bar\epsilon^{\beta\gamma}_{1,\ \rho} x^\rho \partial_\lambda \Phi_{\gamma\beta\alpha} a^\lambda \\
& +27(2-10d)(2-f)\bar\epsilon^{\beta\gamma}_{1,\ \alpha} \Phi_{\gamma\beta\lambda} a^\lambda \\
& +27(2-10d)(2-f)\bar\epsilon^{\beta\gamma}_{1,\ \lambda} a^\lambda \Phi_{\gamma\beta\alpha}] \\
+\bar\epsilon^{\alpha}_{2,\mu\rho} a^\mu x^\rho [& +432\bar\epsilon^{\beta\gamma}_{1,\ \alpha} \partial_\beta \Phi_{\gamma\nu\lambda} a^\nu a^\lambda + 432\bar\epsilon^{\beta\gamma}_{1,\ \sigma} x^\sigma \partial_\alpha \partial_\beta \Phi_{\gamma\nu\lambda} a^\nu a^\lambda \\
& -432f\bar\epsilon^{\beta\gamma}_{1,\ \nu} a^\nu \partial_\beta \Phi_{\gamma\alpha\lambda} a^\lambda - 432f\bar\epsilon^{\beta\gamma}_{1,\ \sigma} x^\sigma \partial_\nu \partial_\beta \Phi_{\gamma\alpha\lambda} a^\nu a^\lambda \\
& +540f(1-d)\bar\epsilon^{\beta\gamma}_{1,\ \sigma} x^\sigma \partial_\nu \partial_\lambda \Phi_{\beta\gamma\alpha} a^\nu a^\lambda \\
& +6f(90(1-d)+9(2-10d))\bar\epsilon^{\beta\gamma}_{1,\ \nu} a^\nu \partial_\lambda \Phi_{\beta\gamma\alpha} a^\lambda \\
& +(54f(2-10d)-1080(1-d))\bar\epsilon^{\beta\gamma}_{1,\ \alpha} \partial_\nu \Phi_{\beta\gamma\lambda} a^\nu a^\lambda \\
& +(540f(1-d)-1080(1-d))\bar\epsilon^{\beta\gamma}_{1,\ \sigma} x^\sigma \partial_\nu \partial_\alpha \Phi_{\beta\gamma\lambda} a^\nu a^\lambda \\
& +(540f(1-d)-108(2-10d))\bar\epsilon^{\beta\gamma}_{1,\ \nu} a^\nu \partial_\alpha \Phi_{\beta\gamma\lambda} a^\lambda] \\
+\bar\epsilon_{2,\mu\nu\rho} a^\mu a^\nu x^\rho [& +(72h_2 + 180h_2(1-d) + 18h_2(2-10d))\bar\epsilon^{\beta\gamma\alpha}_1 \partial_\alpha \Phi_{\beta\gamma\lambda} a^\lambda]
\end{aligned}
\tag{5.108}
$$

As one can see, all contributions from the h_1-term drop out. The second double gauge tranformation consists again of the same gauge deformations and gauge parameters as the first one, so it can be obtained by switching $\bar\epsilon_1$ and $\bar\epsilon_2$ in the above expression:

$$\begin{aligned}\delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) = \bar\epsilon_{1,\mu\nu}{}^{\alpha}a^\mu a^\nu[&-216(2-f)\bar\epsilon_{2,\ \rho}^{\beta\gamma}x^\rho\partial_\beta\Phi_{\gamma\alpha\lambda}a^\lambda\\
&+270(1-d)(2-f)\bar\epsilon_{2,\ \rho}^{\beta\gamma}x^\rho\partial_\alpha\Phi_{\gamma\beta\lambda}a^\lambda\\
&+270(1-d)(2-f)\bar\epsilon_{2,\ \rho}^{\beta\gamma}x^\rho\partial_\lambda\Phi_{\gamma\beta\alpha}a^\lambda\\
&+27(2-10d)(2-f)\bar\epsilon_{2,\ \alpha}^{\beta\gamma}\Phi_{\gamma\beta\lambda}a^\lambda\\
&+27(2-10d)(2-f)\bar\epsilon_{2,\ \lambda}^{\beta\gamma}a^\lambda\Phi_{\gamma\beta\alpha}]\\
+\bar\epsilon^{\alpha}_{1,\mu\rho}a^\mu x^\rho[&+432\bar\epsilon_{2,\ \alpha}^{\beta\gamma}\partial_\beta\Phi_{\gamma\nu\lambda}a^\nu a^\lambda+432\bar\epsilon_{2,\ \sigma}^{\beta\gamma}x^\sigma\partial_\alpha\partial_\beta\Phi_{\gamma\nu\lambda}a^\nu a^\lambda\\
&-432f\bar\epsilon_{2,\ \nu}^{\beta\gamma}a^\nu\partial_\beta\Phi_{\gamma\alpha\lambda}a^\lambda-432f\bar\epsilon_{2,\ \sigma}^{\beta\gamma}x^\sigma\partial_\nu\partial_\beta\Phi_{\gamma\alpha\lambda}a^\nu a^\lambda\\
&+540f(1-d)\bar\epsilon_{2,\ \sigma}^{\beta\gamma}x^\sigma\partial_\nu\partial_\lambda\Phi_{\beta\gamma\alpha}a^\nu a^\lambda\\
&+6f(90(1-d)+9(2-10d))\bar\epsilon_{2,\ \nu}^{\beta\gamma}a^\nu\partial_\lambda\Phi_{\beta\gamma\alpha}a^\lambda\\
&+(54f(2-10d)-1080(1-d))\bar\epsilon_{2,\ \alpha}^{\beta\gamma}\partial_\nu\Phi_{\beta\gamma\lambda}a^\nu a^\lambda\\
&+(540f(1-d)-1080(1-d))\bar\epsilon_{2,\ \sigma}^{\beta\gamma}x^\sigma\partial_\nu\partial_\alpha\Phi_{\beta\gamma\lambda}a^\nu a^\lambda\\
&+(540f(1-d)-108(2-10d))\bar\epsilon_{2,\ \nu}^{\beta\gamma}a^\nu\partial_\alpha\Phi_{\beta\gamma\lambda}a^\lambda]\\
+\bar\epsilon_{1,\mu\nu\rho}a^\mu a^\nu x^\rho[&+(72h_2+180h_2(1-d)+18h_2(2-10d))\bar\epsilon_2^{\beta\gamma\alpha}\partial_\alpha\Phi_{\beta\gamma\lambda}a^\lambda]\end{aligned} \tag{5.109}$$

For the last term we need the 3-3-bracket from 3-3-2:

$$[[\bar\epsilon_2,\bar\epsilon_1]] = 3r(1+10g)\bar\epsilon_{2,\ \mu}^{\alpha\beta}a^\mu\bar\epsilon_{1,\alpha\beta\rho}x^\rho - 3r(1+10(1-g))\bar\epsilon_{2,\ \rho}^{\alpha\beta}x^\rho\bar\epsilon_{1,\alpha\beta\mu}a^\mu \tag{5.110}$$

where we again introduced the rescaling factor r. The gauge deformation is:

$$\begin{aligned}\delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi) = &+\frac{3}{2}e[[\bar\epsilon_2,\bar\epsilon_1]]^\alpha\partial_\mu\Phi_{\alpha\nu\lambda}-\frac{3}{2}(1-e)\partial_\mu[[\bar\epsilon_2,\bar\epsilon_1]]^\alpha\Phi_{\alpha\nu\lambda}\\
&-[[\bar\epsilon_2,\bar\epsilon_1]]^\alpha\partial_\alpha\Phi_{\mu\nu\lambda}+\frac{3}{2}\partial_\alpha[[\bar\epsilon_2,\bar\epsilon_1]]_\mu\Phi^\alpha_{\ \nu\lambda}\end{aligned} \tag{5.111}$$

Plugging in the bracket this reads:

$$\begin{aligned}\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = &(+27e\cdot r(1+10g)\bar{\epsilon}_2^{\beta\gamma\alpha}\bar{\epsilon}_{1,\beta\gamma\rho}x^\rho \\ &\quad -27e\cdot r(1+10(1-g))\bar{\epsilon}_{2,\ \rho}^{\beta\gamma}x^\rho\bar{\epsilon}_{1,\beta\gamma}{}^{\alpha})\partial_\mu\Phi^\alpha{}_{\nu\lambda}a^\mu a^\nu a^\lambda \\ &+(-18r(1+10g)\bar{\epsilon}_2^{\beta\gamma\alpha}\bar{\epsilon}_{1,\beta\gamma\rho}x^\rho \\ &\quad +18r(1+10(1-g))\bar{\epsilon}_{2,\ \rho}^{\beta\gamma}x^\rho\bar{\epsilon}_{1,\beta\gamma}{}^{\alpha})\partial_\alpha\Phi_{\mu\nu\lambda}a^\mu a^\nu a^\lambda \\ &+(+(27r(1-e)(1+10(1-g))+27(1+10g))\bar{\epsilon}_{2,\ \mu}^{\beta\gamma}a^\mu\bar{\epsilon}_{1,\beta\gamma}{}^{\alpha} \\ &+(-27r(1-e)(1+10g)-27(1+10(1-g)))\bar{\epsilon}_2^{\beta\gamma\alpha}\bar{\epsilon}_{1,\beta\gamma\mu}a^\mu)\Phi_{\alpha\nu\lambda}a^\nu a^\lambda\end{aligned} \tag{5.112}$$

All terms in the above are different, so there is no chance that the FCGA can be fulfilled without the use of DDI's. Adding these terms up and cancelling an overall factor of 24, the FCGA is given as (A.6).

We now dualise this expression, again discarding terms proportional to the free EOM and using the new variables from the previous case. The result is (A.7). It is again necessary to account for zero-order gauge transformations on the RHS of (5.9). After replacing all terms containing p_a the dualised FCGA is (A.8).

Again, all DDI's with n up to (and including) $n = 2$ have been calculated with Mathematica. For this system of equations we have two solutions:

solution 1:

$$d = \frac{4}{5} \qquad r = 2 \tag{5.113}$$

solution 2:

$$h_2 = 0 \qquad r = 2 \tag{5.114}$$

As in the previous case, solutions only exist for a rescaling-factor of two. Since d remained unfixed in the previous calculations and f remains unfixed here, both solutions are compatible with our previous results. It is worth mentioning that for both $h_2 = 0$ and $d = \frac{4}{5}$ the contributions from the last remaining DT-term in the outer deformation vanish. Therefore, the DT-parts are not necessary for this calculation.

To sum up our results from this section, the FCGA (5.9) is fulfilled in all cases that arise for spin-3-gravity. Together with our calculations regarding the Jacobi-identity this proves the cubic consistency of the theory at the global symmetry level. Whereas the cases involving at least one spin-2 parameter work out without any additional structure, the cases with two spin-3 parameters make it necessary to allow for zero-order gauge transformations on the RHS of (5.9). The partial integration parameters of the first-order deformations are fixed as follows:

- For the parameter of the 2-2-deformation we have $c = 1$ and for the parameter of the 2-3-deformation we have $e = 0$. In both cases the term where the derivative ∂_μ acts on the field vanishes.
- For the parameter of the 3-2-deformation we have two solutions, one where $d = \frac{4}{5}$ and one where d remains unfixed. The second solution requires $h_2 = 0$. If it turns out that we need to have $h_2 \neq 0$ when calculating the DT-parts of the vertices and the first-order deformations, this would fix d to be $\frac{4}{5}$, which is also the only value of d for which the deformation becomes divergence-free.
- For the parameter of the 3-3-deformation we have three solutions, one for which $f = \frac{10}{7}$, which seems very unlikely, one for which $f = 2$ and one for which f remains free. As in the case of the parameter d, it is expected that the full knowledge of the DT-parts of the vertices will fix this parameter.

Both calculations with two spin-3 parameters require a rescaling of the 3-3-bracket by a factor of two. Such a rescaling is compatible with our calculations for the Jacobi-identity, since in all cases where it is present it appears in all three terms of the Jacobi-identity. Thus, a rescaling just gives a global factor. The calculations of the FCGA also confirmed that all bracket parameters need to be $\frac{1}{2}$ and that we also have $\tilde{g}_{222}$.

6 Global symmetry structure constants

In this chapter we will use the explicit expressions of the gauge brackets for the 2-2-2 vertex and the 3-3-2 vertex that we calculated in chapter 4 to analyze the global symmetry algebra underlying spin-3-gravity. This theory is expected to have a sl(3) $\bigoplus$ sl(3) algebra structure and we are interested in the structure constants of this algebra.

For a detailed description about the calculation of structure constants see [84]. In our case, the structure constants are obtained from the gauge bracket via

$$f^{s_3}_{\ \ s_1 s_2} := [[\bar{\epsilon}_{s_1}, \bar{\epsilon}_{s_2}]]_{s_3}, \tag{6.1}$$

where $\bar{\epsilon}_i$ are again Killing-tensors. In order to calculate from these the cyclic structure constants, we need the invariant bilinear form, which can be chosen as [90]

$$k_{s,s'} = \langle \bar{\epsilon}^k_s | \, |\bar{\epsilon}^{k'}_{s'} \rangle = \delta_{ss'} \delta_{kk'} b_{s,k} \frac{(\partial_1 \cdot \partial_2)^{s-1}}{(s-1)!} \frac{(\nabla_1 \cdot \nabla_2)^k}{k!} \bar{\epsilon}^{(k)}_1 \bar{\epsilon}^{(k')}_2 . \tag{6.2}$$

This expression is diagonal in both s and k. Using this bilinear form, we can calculate the cyclic structure constants:

$$f_{s_1 s_2 s_3} = \sum_s k_{s_1,s} f^s_{\ \ s_1 s_2} = \langle \bar{\epsilon}_{s_1} | \, | [[\bar{\epsilon}_{s_2}, \bar{\epsilon}_{s_3}]] \rangle \tag{6.3}$$

The constants b_s in the bi-linear form can be fixed (up to a normalisation) by demanding cyclicity of the cyclic structure constants

$$\langle \bar{\epsilon}^{(k)}_1 | \, | [[\bar{\epsilon}^{(k')}_2, \bar{\epsilon}^{(k'')}_3]] \rangle = \langle \bar{\epsilon}^{(\tilde{k})}_3 | \, | [[\bar{\epsilon}^{(\tilde{k'})}_1, \bar{\epsilon}^{(\tilde{k''})}_2]] \rangle = \langle \bar{\epsilon}^{(\bar{k})}_2 | \, | [[\bar{\epsilon}^{(\bar{k'})}_3, \bar{\epsilon}^{(\bar{k''})}_1]] \rangle \, . \tag{6.4}$$

This requirement already imposes strict limits on the possible values of the $b_{s,k}$ considering that in three dimensions k can only be zero or one. For $k = k' = k'' = 0$ all brackets vanish, so we get a trivial result. If two of the k's are one and the last one is zero, we also end up with trivial expressions, since the bi-linear form is diagonal. In the cases where one of the k's is one and the others are zero, the bracket vanishes if the parameter with $k = 1$ is outside of the bracket. However, if the parameter with $k = 1$ is inside the bracket, we get non-zero results for the other two structure constants. Cyclicity then demands that the $b_{s,k}$ in these expressions are zero. This fixes the $b_{s,0}$ to be

$$b_{2,0} = b_{3,0} = 0. \tag{6.5}$$

In the following, we will demand that $f_{332} = f_{233} = f_{323}$ for $k = k' = k'' = 1$ in order to fix $\frac{b_{2,1}}{b_{3,1}}$, which is the last remaining relevant factor for spin-3-gravity.

Using the brackets calculated in chapter 4 and plugging in the Killing-tensors (1.25) and (1.26), the cyclic structure constants of spin-3-gravity are

$$f_{222} = 4b_{2,1}\bar{\epsilon}_3^{\gamma\beta}\bar{\epsilon}_{1,\gamma}^{\alpha}\bar{\epsilon}_{2,\alpha\beta}, \tag{6.6}$$

$$f_{233} = 9b_{2,1}\bar{\epsilon}_3^{\gamma\delta}\bar{\epsilon}_{1,\ \gamma}^{\alpha\beta}\bar{\epsilon}_{2,\alpha\beta\delta}, \tag{6.7}$$

$$f_{323} = 3b_{3,1}\bar{\epsilon}_2^{\gamma\beta\delta}(\bar{\epsilon}_{1,\gamma b}{}^{\alpha}\bar{\epsilon}_{3,\alpha\delta} + 2\bar{\epsilon}_{1,\gamma\delta}^{\alpha}\bar{\epsilon}_{3,\alpha\beta}), \tag{6.8}$$

$$f_{332} = -3b_{3,1}\bar{\epsilon}_1^{\gamma\beta\delta}(\bar{\epsilon}_{2,\gamma\beta}{}^{\alpha}\bar{\epsilon}_{3,\alpha\delta} + 2\bar{\epsilon}_{2,\gamma\delta}^{\alpha}\bar{\epsilon}_{3,\alpha\beta}). \tag{6.9}$$

From this we can already see that the structure constants corresponding to the 2-2-2 vertex are indeed cyclic, since f_{222} is symmetric in the three Killing-tensors. For the structure constants of the 3-3-2 vertex the cyclicity is less obvious, it seems that we only have $f_{323} = f_{332}$. In order to make the cyclicity visible, we will dualise the spin-3 Killing-tensors using (5.11). The dualised structure constants then are

$$f_{233} = 18b_{2,1}\tilde{\epsilon}_1^{\beta\alpha}\tilde{\epsilon}_{2,\alpha}^{\gamma}\bar{\epsilon}_{3,\beta\gamma}, \tag{6.10}$$

$$f_{323} = 36b_{3,1}\tilde{\epsilon}_1^{\beta\alpha}\tilde{\epsilon}_{2,\alpha}^{\gamma}\bar{\epsilon}_{3,\beta\gamma}, \tag{6.11}$$

$$f_{332} = 36b_{3,1}\tilde{\epsilon}_1^{\beta\alpha}\tilde{\epsilon}_{2,\alpha}^{\gamma}\bar{\epsilon}_{3,\beta\gamma}. \tag{6.12}$$

This makes clear that choosing $\frac{b_{2,1}}{b_{3,1}} = 2$ results in cyclic structure constants.

7 Coupling matter to spin-3-gravity

After verifying the consistency of spin-3-gravity, we now turn to the task of coupling matter to the theory. Here the term matter refers to fields of spin-0 (scalar fields) and fields of spin-1 (Maxwell fields), since these fields carry propagating degrees of freedom in three dimensions. It is known that once matter is coupled to a higher-spin theory, one is forced to add an infinite tower of higher-spin fields to close the gauge algebra [50]. In the following, we want to see at which point it becomes necessary to consider fields of spin $s \geq 4$. Also, it is checked whether the coupling of massive matter fields works in the same way.

Another peculiarity of $d = 3$ is that massless scalar fields are dual to massless Maxwell fields [51]. Therefore, all calculations that work out for massless scalar fields should also work out for massless Maxwell fields and the coupling constants of the respective vertices are expected to be the same. For massive fields however, this duality does no longer hold and we need to analyze the two types of fields separately.

7.1 Massive scalar fields

A scalar field can be coupled to spin-2 and spin-3 via the vertices

$$\mathcal{V}_{200} = y_1^2, \tag{7.1}$$

$$\mathcal{V}_{300} = y_1^3. \tag{7.2}$$

Note that in contrast to higher dimensions there exist no vertices of the type s-s-0 in $d = 3$ [1]. In index-notation these vertices read

$$\mathcal{V}_{200}[\Phi] = \Phi_1^{\alpha\beta} \partial_\alpha \partial_\beta \Phi_2 \Phi_3, \tag{7.3}$$

$$\mathcal{V}_{300}[\Phi] = \Phi_1^{\alpha\beta\gamma} \partial_\alpha \partial_\beta \partial_\gamma \Phi_2 \Phi_3. \tag{7.4}$$

The 2-0-0 vertex is symmetric in Φ_2 and Φ_3, so a real scalar field would be sufficient. The 3-0-0 vertex however is anti-symmetric in these fields, so we need two different real scalar fields, or alternatively, a complex scalar field.

It is important to note that the above vertices have been constructed for massless fields, so we need to check whether they are still gauge invariant for massive scalar fields. Only the massless spin-2 field and the massless spin-3 field induce a gauge transformation, so we have to check

$$\begin{aligned}\delta^{(0)}_{\epsilon_1}\mathcal{V}_{s00} &= [\mathcal{V}_{s00},(a_1\cdot\nabla_1)]\epsilon_1\Phi_2\Phi_3\\ &= \partial_{y_1}\mathcal{V}_{s00}B_{12}\epsilon_1\Phi_2\Phi_3\\ &= \frac{1}{2}\partial_{y_1}\mathcal{V}_{s00}(\Box_3-\Box_1-\Box_2)\epsilon_1\Phi_2\Phi_3\\ &= \frac{1}{2}\partial_{y_1}\mathcal{V}_{s00}(m_3^2-m_2^2)\epsilon_1\Phi_2\Phi_3.\end{aligned} \tag{7.5}$$

In the above it has been used that for the vertices of the type s-0-0 we have that $\mathcal{G}_1\mathcal{V}_{s00}=0$, even without the use of DDI's. From this we can see that these vertices are indeed gauge invariant also in the massive case if we choose the masses of the two scalars to be the same.

The next thing we want to check is whether we can calculate the first-order gauge deformations in the same way as in the massless case. Since no DDI's are needed for the on-shell gauge invariance of the s-0-0 vertices, we do not have to consider any DDI-contributions. Again, only the first fields in the 2-0-0 vertex and the 3-0-0 vertex are massless, so the only relevant gauge deformations are $\delta^{(1)}_{\epsilon_1}\Phi_{2/3}$. The first-order Noether-equation then is

$$\delta^{(1)}_{\epsilon_1}\mathcal{S}^{(2)}+\delta^{(0)}_{\epsilon_1}\mathcal{S}^{(3)}=0. \tag{7.6}$$

We now have to include a mass term in the EOM of the scalar fields, so the variation of the quadratic action reads

$$\delta^{(1)}_{\epsilon_1}\mathcal{S}^{(2)}=\int[\delta^{(1)}_{\epsilon_1}\Phi_2(\Box\Phi_2+m^2\Phi_2)+\delta^{(1)}_{\epsilon_1}\Phi_3(\Box\Phi_3+m^2\Phi_3)]. \tag{7.7}$$

The off-shell variation of the cubic action is the same as in the massless case:

$$\delta^{(0)}_{\epsilon_1}\mathcal{S}^{(3)}=\int\partial_{y_1}\mathcal{V}\epsilon_1\frac{1}{2}(\Box_3-\Box_2)\Phi_2\Phi_3 \tag{7.8}$$

In the massless case this variation of the cubic action was already proportional to the EOM, so that the first-order gauge deformations could just be read off. In the massive case, we have additional mass terms in the EOM, which we also need to add in the cubic variation:

$$\delta^{(0)}_{\epsilon_1}\mathcal{S}^{(3)} = \int \partial_{y_1}\mathcal{V}\epsilon_1\frac{1}{2}(\Box_3 + m^2 - m^2 - \Box_2)\Phi_2\Phi_3 \tag{7.9}$$

Now this variation is again proportional to the EOM and by comparing this to the first-order gauge variation of the quadratic action we have obtained the first-order gauge deformations w.r.t. the first field

$$\delta^{(1)}_{\epsilon_1}\Phi_3 = -\frac{1}{2}\partial_{y_1}\mathcal{V}\epsilon_1\Phi_2, \tag{7.10}$$

$$\delta^{(1)}_{\epsilon_1}\Phi_2 = \frac{1}{2}\partial_{y_1}\mathcal{V}\epsilon_1\Phi_3. \tag{7.11}$$

These are exactly the same expressions as for the massless case, so we can still calculate first-order gauge deformations in the usual way. The explicit gauge deformations are:

2-0-0 vertex:

$$\delta^{(1)}_{\epsilon_1}\Phi_2 = y_1\epsilon_1\Phi_3 \tag{7.12}$$

3-0-0 vertex:

$$\delta^{(1)}_{\epsilon_1}\Phi_2 = \frac{3}{2}y_1^2\epsilon_1\Phi_3 \tag{7.13}$$

In index-notation these deformations read:

2-0-0 vertex:

$$\delta^{(1)}_{\epsilon_1}\Phi_2 = -\epsilon_1^{\alpha}\partial_{\alpha}\Phi_3 \tag{7.14}$$

3-0-0 vertex:

$$\delta^{(1)}_{\epsilon_1}\Phi_2 = \frac{3}{2}\epsilon_1^{\alpha\beta}\partial_\alpha\partial_\beta\Phi_3 \tag{7.15}$$

Since there are no gauge parameters for the scalar fields, no brackets are induced by the 2-0-0 vertex and the 3-0-0 vertex. This means that no contributions to the Jacobi-identity and no structure constants arise, as expected. Verifying the Fronsdal-equation is straightforward and will not be done here. Lastly, we still need to check the closure of the gauge algebra for the above deformations. This will be done in the following.

two spin-2 parameters

Starting with the case of two deformations w.r.t. spin-2 parameters, we have to choose the 2-2-bracket from 2-2-2 for the last term in (5.9) and then the deformation in this can again be w.r.t. a spin-2 parameter. In the same way as in chapter 5 we will use the Killing-tensors (1.25) and (1.26) as gauge parameters, but discard their prefactors. For the first term, the inner deformation is:

$$\delta^{(1)}_{\bar{\epsilon}_1}(\Phi) = -\bar{\epsilon}^{\alpha}_{1,\rho}x^\rho\partial_\alpha\Phi \tag{7.16}$$

The double deformation then is:

$$\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) = \bar{\epsilon}^{\alpha}_{2,\sigma}x^\sigma\bar{\epsilon}^{\beta}_{1,\alpha}\partial_\beta\Phi + \bar{\epsilon}^{\alpha}_{2,\sigma}x^\sigma\bar{\epsilon}^{\beta}_{1,\rho}x^\rho\partial_\alpha\partial_\beta\Phi \tag{7.17}$$

The second term can be calculated from the first one by interchanging $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$:

$$\delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) = \bar{\epsilon}^{\alpha}_{1,\sigma}x^\sigma\bar{\epsilon}^{\beta}_{2,\alpha}\partial_\beta\Phi + \bar{\epsilon}^{\alpha}_{1,\sigma}x^\sigma\bar{\epsilon}^{\beta}_{2,\rho}x^\rho\partial_\alpha\partial_\beta\Phi \tag{7.18}$$

For the third term we need the 2-2-bracket from 2-2-2. Since this term is the only one containing an object from the 2-2-2 vertex, we need to add a relative coupling constant $\frac{g_{222}}{g_{200}}$:

$$[[\bar{\epsilon}_2,\bar{\epsilon}_1]] = \frac{g_{222}}{g_{200}}[(g+\frac{3}{2})\bar{\epsilon}^{\alpha}_{2,\mu}a^\mu\bar{\epsilon}_{1,\alpha\rho}x^\rho - (\frac{5}{2}-g)\bar{\epsilon}^{\alpha}_{2,\rho}x^\rho\bar{\epsilon}_{1,\alpha\mu}a^\mu] \tag{7.19}$$

The third term then is:

$$\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = -[[\bar{\epsilon}_2,\bar{\epsilon}_1]]^\alpha\partial_\alpha\Phi \tag{7.20}$$

Plugging in the bracket from above this reads:

$$\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = \frac{g_{222}}{g_{200}}[-(g+\frac{3}{2})\bar{\epsilon}^{\beta}_{2,\alpha}\bar{\epsilon}_{1,\beta\rho}x^{\rho}\partial_{\alpha}\Phi + (\frac{5}{2}-g)\bar{\epsilon}^{\beta}_{2,\rho}x^{\rho}\bar{\epsilon}_{1,\beta}{}^{\alpha}\partial_{\alpha}\Phi] \tag{7.21}$$

The FCGA then reads:

$$\begin{aligned} &\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) - \delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) + \delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = \\ &\bar{\epsilon}^{\alpha}_{1,\sigma}x^{\sigma}\bar{\epsilon}^{\beta}_{2,\alpha}\partial_{\beta}\Phi[-1+\frac{g_{222}}{g_{200}}(g+\frac{3}{2})] \\ +&\bar{\epsilon}^{\beta}_{1,\alpha}\bar{\epsilon}^{\alpha}_{2,\sigma}x^{\sigma}\partial_{\beta}\Phi[+1-\frac{g_{222}}{g_{200}}(\frac{5}{2}-g)] \end{aligned} \tag{7.22}$$

This can be fulfilled by setting $g=\frac{1}{2}$ as usual and $\frac{g_{222}}{g_{200}}=\frac{1}{2}$. We see that for coupling a scalar field to gravity we do not need fields of higher spin, as expected.

one spin-2 parameter, one spin-3 parameter

Moving to the case of one spin-2 parameter, which will be $\bar{\epsilon}_1$ and one spin-3 parameter, which will be $\bar{\epsilon}_2$, the bracket in the last term is then the 3-2-bracket from 3-3-2 and the deformation in the last term is again w.r.t. a spin-3 parameter. For the first term, the inner deformation is:

$$\delta^{(1)}_{\bar{\epsilon}_1}(\Phi) = -\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\partial_{\alpha}\Phi \tag{7.23}$$

The double deformation then is:

$$\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) = -3\bar{\epsilon}^{\alpha\beta}_{2,\ \rho}x^{\rho}\bar{\epsilon}^{\gamma}_{1,\sigma}x^{\sigma}\partial_{\alpha}\partial_{\beta}\partial_{\gamma}\Phi - 6\bar{\epsilon}^{\alpha\beta}_{2,\ \rho}x^{\rho}\bar{\epsilon}^{\gamma}_{1,\alpha}\partial_{\beta}\partial_{\gamma}\Phi \tag{7.24}$$

For the second term, the inner deformation is:

$$\delta^{(1)}_{\bar{\epsilon}_2}(\Phi) = 3\bar{\epsilon}^{\alpha\beta}_{2,\ \rho}x^{\rho}\partial_{\alpha}\partial_{\beta}\Phi \tag{7.25}$$

The double deformation then is:

$$\delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) = -3\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\bar{\epsilon}^{\beta\gamma}_{2,\ \alpha}\partial_{\beta}\partial_{\gamma}\Phi - 3\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\bar{\epsilon}^{\beta\gamma}_{2,\ \sigma}x^{\sigma}\partial_{\alpha}\partial_{\beta}\partial_{\gamma}\Phi \tag{7.26}$$

For the third term we need the 3-2-bracket from 3-3-2. Again, we have to introduce a relative coupling constant $\frac{g_{332}}{g_{200}}$:

$$[[\bar{\epsilon}_2, \bar{\epsilon}_1]] = \frac{g_{332}}{g_{200}}[-3(g+\frac{3}{2})\bar{\epsilon}_{2,\mu\nu}{}^{\alpha}a^\mu a^\nu \bar{\epsilon}_{1,\alpha\rho}x^\rho - 3(3+2(1-g))\bar{\epsilon}^{\alpha}_{2,\mu\rho}a^\mu x^\rho \bar{\epsilon}_{1,\alpha\nu}a^\nu] \quad (7.27)$$

The third term then is:

$$\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = \frac{3}{2}[[\bar{\epsilon}_2, \bar{\epsilon}_1]]^{\alpha\beta}\partial_\alpha\partial_\beta\Phi \quad (7.28)$$

Plugging in the bracket from above this reads:

$$\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = \frac{g_{332}}{g_{200}}[-9(g+\frac{3}{2})\bar{\epsilon}_2^{\alpha\beta\gamma}\bar{\epsilon}_{1,\gamma\rho}x^\rho\partial_\alpha\partial_\beta\Phi + 9(3+2(1-g))\bar{\epsilon}^{\gamma\alpha}_{2,\ \rho}x^\rho\bar{\epsilon}^{\beta}_{1,\gamma}\partial_\alpha\partial_\beta\Phi] \quad (7.29)$$

The FCGA then reads:

$$\begin{aligned} &\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) - \delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) + \delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = \\ &\bar{\epsilon}^{\alpha}_{1,\rho}x^\rho\bar{\epsilon}^{\beta\gamma}_{2,\ \alpha}\partial_\beta\partial_\gamma\Phi[+3 - 9(g+\frac{3}{2})\frac{g_{332}}{g_{200}}] \\ +&\bar{\epsilon}^{\gamma}_{1,\alpha}\bar{\epsilon}^{\alpha\beta}_{2,\ \rho}x^\rho\partial_\beta\partial_\gamma\Phi[-6 + 9(3+2(1-g))\frac{g_{332}}{g_{200}}] \end{aligned} \quad (7.30)$$

This can be solved by setting $g = \frac{1}{2}$ and $\frac{g_{332}}{g_{200}} = \frac{1}{6}$. We can now use $\frac{g_{222}}{g_{200}} = \frac{1}{2}$ from the previous calculation and combine it with the results from here to get $\frac{g_{222}}{g_{332}} = 3$, which is in perfect agreement with our results from chapter 5.

Again we observe that this calculation works out without the need of introducing additional fields.

two spin-3 parameters

The last case is the one of spin-3 parameters. The bracket in the last term is then the 3-3-bracket from 3-3-2 and the deformation in this term is w.r.t. a spin-2 parameter. We already use the rescaling factor of two that we determined in chapter 5. For the first term, the inner deformation is:

$$\delta^{(1)}_{\bar{\epsilon}_1}(\Phi) = 3\bar{\epsilon}^{\alpha\beta}_{1,\rho} x^\rho \partial_\alpha \partial_\beta \Phi \tag{7.31}$$

The double deformation then is:

$$\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) = 9\bar{\epsilon}^{\alpha\beta}_{2,\rho} x^\rho [\bar{\epsilon}^{\gamma\delta}_{1,\sigma} x^\sigma \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \Phi + 2\bar{\epsilon}^{\gamma\delta}_{1,\alpha} \partial_\beta \partial_\gamma \partial_\delta \Phi] \tag{7.32}$$

The second term can be obtained from the first one by switching the gauge parameters:

$$\delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) = 9\bar{\epsilon}^{\alpha\beta}_{1,\rho} x^\rho [\bar{\epsilon}^{\gamma\delta}_{2,\sigma} x^\sigma \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \Phi + 2\bar{\epsilon}^{\gamma\delta}_{2,\alpha} \partial_\beta \partial_\gamma \partial_\delta \Phi] \tag{7.33}$$

For the last term we need the 3-3-bracket from 3-3-2 and a relative coupling constant $\frac{g_{332}}{g_{300}}$:

$$[[\bar{\epsilon}_2, \bar{\epsilon}_1]] = \frac{g_{332}}{g_{300}} [6(1+10g)\bar{\epsilon}^{\alpha\beta}_{2,\ \mu} a^\mu \bar{\epsilon}_{1,\alpha\beta\rho} x^\rho - 6(1+10(1-g))\bar{\epsilon}^{\alpha\beta}_{2,\ \rho} x^\rho \bar{\epsilon}_{1,\alpha\beta\mu} a^\mu] \tag{7.34}$$

The third term then is:

$$\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = [[\bar{\epsilon}_2, \bar{\epsilon}_1]]^\alpha \partial_\alpha \Phi \tag{7.35}$$

Plugging in the bracket from above this reads:

$$\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = \frac{g_{332}}{g_{300}} [6(1+10g)\bar{\epsilon}^{\alpha\beta\gamma}_2 \bar{\epsilon}_{1,\alpha\beta\rho} x^\rho \partial_\gamma \Phi - 6(1+10(1-g))\bar{\epsilon}^{\alpha\beta}_{2,\ \rho} x^\rho \bar{\epsilon}_{1,\alpha\beta}{}^{\gamma} \partial_\gamma \Phi] \tag{7.36}$$

Both the first term and the second term contain a contribution where four derivatives act on the field Φ. These two terms cancel in the commutator. The remaining terms have three derivatives acting on Φ and need to be canceled by contributions from the third term. But as one can see, the contributions from the third term do not have enough derivatives acting on Φ. This is exactly the point at which one is forced to introduce fields of higher spin: if we had a 4-3-3 vertex, we would also have a 3-3-bracket from this vertex, which gives back a spin-4 parameter. The corresponding gauge deformation then has three derivatives and could give the contributions necessary to cancel the remaining terms. In 7.3 we show that this indeed works out.

Another interesting feature of the above calculation is that no contributions with two derivatives acting on Φ appear in the first two terms. Cancelling these contributions would require the 3-3-bracket from the 3-3-3 vertex, which is known to be inconsistent. So the absence of such contributions is an important consistency check for the theory.

7.2 Massive Maxwell fields

Analogously to the scalar case, we can couple Maxwell fields to spin-2 and spin-3 via the vertices

$$\mathcal{V}_{211} = y_1^2 z_1 + y_1 y_2 z_2 + y_1 y_3 z_3, \tag{7.37}$$

$$\mathcal{V}_{311} = y_1^3 z_1 + y_1^2 y_2 z_2 + y_1^2 y_3 z_3. \tag{7.38}$$

In index-notation these vertices read

$$\mathcal{V}_{211}[\Phi] = -\Phi_1^{\alpha\beta}\partial_\alpha\Phi_2^\gamma\partial_\beta\Phi_{3,\gamma} + \Phi_{1,\gamma}^\alpha\partial_\alpha\Phi_2^\beta\partial_\beta\Phi_3^\gamma + \partial_\alpha\Phi_{1,\gamma}^\beta\partial_\beta\Phi_2^\gamma\Phi_3^\alpha, \tag{7.39}$$

$$\mathcal{V}_{311}[\Phi] = -\Phi_1^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\Phi_2^\delta\partial_\gamma\Phi_{3,\delta} + \Phi_1^{\alpha\beta\delta}\partial_\alpha\partial_\beta\Phi_2^\gamma\partial_\gamma\Phi_{3,\delta} + \partial_\gamma\Phi_1^{\alpha\beta\delta}\partial_\alpha\partial_\beta\Phi_{2,\delta}\Phi_3^\gamma. \tag{7.40}$$

Again, the first vertex is symmetric in the spin-1 fields, whereas the second one is anti-symmetric w.r.t. those fields. As opposed to the scalar case, for Maxwell fields we also have vertices of the form s-s-1

$$\mathcal{V}_{221} = y_1 y_2 y_3 z_3, \tag{7.41}$$

$$\mathcal{V}_{331} = y_1 y_2 y_3 z_3^2, \tag{7.42}$$

as well as two self-interaction vertices

$$\mathcal{V}_{111} = y_1 z_1 y_2 z_2 y_3 z_3, \tag{7.43}$$

$$\tilde{\mathcal{V}}_{111} = y_1 y_2 y_3. \tag{7.44}$$

In index-notation these vertices read

$$\mathcal{V}_{221}[\Phi] = \partial_\alpha \Phi_1^{\beta\delta} \partial_\beta \Phi_{2,\delta}^{\gamma} \partial_\gamma \Phi_3^{\alpha}, \tag{7.45}$$

$$\mathcal{V}_{331}[\Phi] = \partial_\alpha \Phi_1^{\beta\delta\eta} \partial_\beta \Phi_{2,\delta\eta}^{\gamma} \partial_\gamma \Phi_3^{\alpha}, \tag{7.46}$$

$$\mathcal{V}_{111}[\Phi] = \Phi_1^{\alpha} \partial_\alpha \Phi_2^{\beta} \Phi_{3,\beta} + \Phi_1^{\beta} \Phi_2^{\alpha} \partial_\alpha \Phi_{3,\beta} + \partial_\alpha \Phi_1^{\beta} \Phi_{2,\beta} \Phi_3^{\alpha}, \tag{7.47}$$

$$\tilde{\mathcal{V}}_{111}[\Phi] = \partial_\gamma \Phi_1^{\alpha} \partial_\alpha \Phi_2^{\beta} \partial_\beta \Phi_3^{\gamma}. \tag{7.48}$$

Both vertices of the type s-s-1 are anti-symmetric in the respective massless fields. The first one of the two self-interaction vertices is the well-known Yang-Mills vertex.

Moving on to the question of gauge invariance, we can first note that the self-interaction vertices are trivially gauge invariant since they do not contain any gauge fields. The vertices of the type s-1-1 can be made gauge invariant in the same way as the corresponding scalar vertices, i.e. by choosing the masses of the Maxwell fields to be the same. Here it is important to note that analogous to the scalar case we have $\mathcal{G}_1 \mathcal{V}_{s11} = 0$ without the use of DDI's. For the vertices of the type s-s-1 however we encounter a problem:

$$\begin{aligned} \delta_{\epsilon_1}^{(0)} \mathcal{V}_{ss1} &= [\mathcal{V}_{ss1}, a_1 \cdot \nabla_1] \epsilon_1 \Phi_2 \Phi_3 \\ &= \partial_{y_1} \mathcal{V}_{ss1} B_{12} \epsilon_1 \Phi_2 \Phi_3 \\ &= \frac{1}{2} \partial_{y_1} \mathcal{V}_{ss1} (\Box_3 - \Box_1 - \Box_2) \epsilon_1 \Phi_2 \Phi_3 \\ &= \frac{1}{2} m_3^2 \partial_{y_1} \mathcal{V}_{ss1} \epsilon_1 \Phi_2 \Phi_3 \neq 0, \end{aligned} \tag{7.49}$$

and similarly for the variation w.r.t. the second field. There is no chance that we can make this vertex gauge invariant. It seems that for the coupling of massive matter fields to spin-3-gravity one needs vertices with at least two matter fields such that the mass terms can cancel each other. Of course, one can still analyze the s-s-1 vertices for massless Maxwell fields, but we will not do this here.

We now calculate the first-order gauge deformations induced by the vertices of the type s-1-1 in the usual way:

2-1-1 vertex:

$$\delta_{\epsilon_1}^{(1)} \Phi_2 = (y_1 z_1 + \frac{1}{2} y_2 z_2 + \frac{1}{2} y_3 z_3) \epsilon_1 \Phi_3 \tag{7.50}$$

3-1-1 vertex:

$$\delta^{(1)}_{\epsilon_1}\Phi_2 = (\frac{3}{2}y_1^2 z_1 + y_1 y_2 z_2 + y_1 y_3 z_3)\epsilon_1\Phi_3 \tag{7.51}$$

In index-notation these deformations read:

2-1-1 vertex:

$$\begin{aligned}\delta^{(1)}_{\epsilon_1}\Phi_2 = &+\frac{1}{2}u\ \epsilon_1^\alpha\partial_\mu\Phi_{3,\alpha}a^\mu - \frac{1}{2}(1-u)\ \partial_\mu\epsilon_1^\alpha a^\mu\Phi_{3,\alpha}\\ &+\frac{1}{2}\partial_\alpha\epsilon_{1,\mu}a^\mu\Phi_3^\alpha - \epsilon_1^\alpha\partial_\alpha\Phi_{3,\mu}a^\mu\end{aligned} \tag{7.52}$$

3-1-1 vertex:

$$\begin{aligned}\delta^{(1)}_{\epsilon_1}\Phi_2 = &-v\ \epsilon_1^{\alpha\beta}\partial_\alpha\partial_\mu\Phi_{3,\beta}a^\mu + (1-v)\ \partial_\mu\epsilon_1^{\alpha\beta}a^\mu\partial_\alpha\Phi_{3,\beta}\\ &-\partial_\alpha\epsilon^\beta_{1,\mu}a^\mu\partial_\beta\Phi_3^\alpha + \frac{3}{2}\epsilon_1^{\alpha\beta}\partial_\alpha\partial_\beta\Phi_{3,\mu}a^\mu\end{aligned} \tag{7.53}$$

As in the scalar case, no new brackets and therefore no structure constants or contributions to the Jacobi-identity emerge. The Fronsdal-equation is fulfilled for arbitrary values of u and v. We again have to check the closure of the gauge algebra for the above deformations.

two spin-2 parameters

The first case is the one of two spin-2 parameters. For the bracket we again have to choose the 2-2-bracket from 2-2-2 and then the deformation in the third term is again w.r.t. a spin-2 parameter. For the first term, the inner deformation is:

$$\delta^{(1)}_{\bar{\epsilon}_1}(\Phi) = -\bar{\epsilon}^\alpha_{1,\rho}x^\rho\partial_\alpha\Phi_\mu a^\mu + \frac{1}{2}u\bar{\epsilon}^\alpha_{1,\rho}x^\rho\partial_\mu\Phi_\alpha a^\mu - \frac{1}{2}(2-u)\bar{\epsilon}^\alpha_{1,\mu}a^\mu\Phi_\alpha \tag{7.54}$$

The double deformation then is:

$$\begin{aligned}
\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) = \bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}[&+\bar{\epsilon}^{\beta}_{1,\alpha}\partial_{\beta}\Phi_{\mu}a^{\mu} + \bar{\epsilon}^{\beta}_{1,\sigma}x^{\sigma}\partial_{\alpha}\partial_{\beta}\Phi_{\mu}a^{\mu} \\
&-\frac{1}{2}u\bar{\epsilon}^{\beta}_{1,\mu}a^{\mu}\partial_{\beta}\Phi_{\alpha} - \frac{1}{2}u\bar{\epsilon}^{\beta}_{1,\sigma}x^{\sigma}\partial_{\mu}\partial_{\beta}\Phi_{\alpha}a^{\mu} \\
&+(\frac{1}{4}u^2 + \frac{1}{2}(2-u))\bar{\epsilon}^{\beta}_{1,\mu}a^{\mu}\partial_{\alpha}\Phi_{\beta} \\
&-(\frac{1}{2}u + \frac{1}{4}u(2-u))\bar{\epsilon}^{\beta}_{1,\alpha}\partial_{\mu}\Phi_{\beta}a^{\mu} \\
&+(\frac{1}{4}u^2 - \frac{1}{2}u)\bar{\epsilon}^{\beta}_{1,\sigma}x^{\sigma}\partial_{\alpha}\partial_{\mu}\Phi_{\beta}a^{\mu}] \\
+\bar{\epsilon}^{\alpha}_{2,\mu}a^{\mu}[&+\frac{1}{2}(2-u)\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\partial_{\beta}\Phi_{\alpha} - \frac{1}{4}u(2-u)\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\partial_{\alpha}\Phi_{\beta} \\
&+\frac{1}{4}(2-u)^2\bar{\epsilon}^{\beta}_{1,\alpha}\Phi_{\beta}]
\end{aligned} \tag{7.55}$$

The second term can be calculated from the first one by interchanging $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$:

$$\begin{aligned}
\delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) = \bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}[&+\bar{\epsilon}^{\beta}_{2,\alpha}\partial_{\beta}\Phi_{\mu}a^{\mu} + \bar{\epsilon}^{\beta}_{2,\sigma}x^{\sigma}\partial_{\alpha}\partial_{\beta}\Phi_{\mu}a^{\mu} \\
&-\frac{1}{2}u\bar{\epsilon}^{\beta}_{2,\mu}a^{\mu}\partial_{\beta}\Phi_{\alpha} - \frac{1}{2}u\bar{\epsilon}^{\beta}_{2,\sigma}x^{\sigma}\partial_{\mu}\partial_{\beta}\Phi_{\alpha}a^{\mu} \\
&+(\frac{1}{4}u^2 + \frac{1}{2}(2-u))\bar{\epsilon}^{\beta}_{2,\mu}a^{\mu}\partial_{\alpha}\Phi_{\beta} \\
&-(\frac{1}{2}u + \frac{1}{4}u(2-u))\bar{\epsilon}^{\beta}_{2,\alpha}\partial_{\mu}\Phi_{\beta}a^{\mu} \\
&+(\frac{1}{4}u^2 - \frac{1}{2}u)\bar{\epsilon}^{\beta}_{2,\sigma}x^{\sigma}\partial_{\alpha}\partial_{\mu}\Phi_{\beta}a^{\mu}] \\
+\bar{\epsilon}^{\alpha}_{1,\mu}a^{\mu}[&+\frac{1}{2}(2-u)\bar{\epsilon}^{\beta}_{2,\rho}x^{\rho}\partial_{\beta}\Phi_{\alpha} - \frac{1}{4}u(2-u)\bar{\epsilon}^{\beta}_{2,\rho}x^{\rho}\partial_{\alpha}\Phi_{\beta} \\
&+\frac{1}{4}(2-u)^2\bar{\epsilon}^{\beta}_{2,\alpha}\Phi_{\beta}]
\end{aligned} \tag{7.56}$$

For the third term we need the 2-2-bracket from 2-2-2 and a relative coupling constant $\frac{g_{222}}{g_{211}}$:

$$[[\bar{\epsilon}_2, \bar{\epsilon}_1]] = \frac{g_{222}}{g_{211}}[(g + \frac{3}{2})\bar{\epsilon}^{\alpha}_{2,\mu}a^{\mu}\bar{\epsilon}_{1,\alpha\rho}x^{\rho} - (\frac{5}{2} - g)\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\bar{\epsilon}_{1,\alpha\mu}a^{\mu}] \tag{7.57}$$

The third term then is:

$$\begin{aligned}\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = &-[[\bar{\epsilon}_2,\bar{\epsilon}_1]]^\alpha\partial_\alpha\Phi_\mu + \frac{1}{2}u[[\bar{\epsilon}_2,\bar{\epsilon}_1]]^\alpha\partial_\mu\Phi_\alpha \\ &-\frac{1}{2}(1-u)\partial_\mu[[\bar{\epsilon}_2,\bar{\epsilon}_1]]^\alpha\Phi_\alpha + \frac{1}{2}\partial_\alpha[[\bar{\epsilon}_2,\bar{\epsilon}_1]]_\mu\Phi^\alpha\end{aligned} \tag{7.58}$$

Plugging in the bracket from above this reads:

$$\begin{aligned}\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = &\frac{g_{222}}{g_{211}}[-(g+\frac{3}{2})\bar{\epsilon}_2^{\beta\alpha}\bar{\epsilon}_{1,\beta\rho}x^\rho + (\frac{5}{2}-g)\bar{\epsilon}^\beta_{2,\rho}x^\rho\bar{\epsilon}_{1,\beta}{}^\alpha]\partial_\alpha\Phi_\mu a^\mu \\ &+\frac{g_{222}}{g_{211}}[+\frac{1}{2}u(g+\frac{3}{2})\bar{\epsilon}_2^{\beta\alpha}\bar{\epsilon}_{1,\beta\rho}x^\rho - \frac{1}{2}u(\frac{5}{2}-g)\bar{\epsilon}^\beta_{2,\rho}x^\rho\bar{\epsilon}_{1,\beta}{}^\alpha]\partial_\mu\Phi_\alpha a^\mu \\ &+\frac{g_{222}}{g_{211}}[-\frac{1}{2}((1-u)(g+\frac{3}{2})+\frac{5}{2}-g)\bar{\epsilon}^\beta_{2,\alpha}\bar{\epsilon}_{1,\beta\mu}a^\mu \\ &\quad+\frac{1}{2}((1-u)(\frac{5}{2}-g)+g+\frac{3}{2})\bar{\epsilon}^\beta_{2,\mu}a^\mu\bar{\epsilon}_{1,\beta\alpha}]\Phi^\alpha\end{aligned} \tag{7.59}$$

The FCGA then reads:

$$
\begin{aligned}
&\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) - \delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) + \delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = \\
&+\bar{\epsilon}^{\beta}_{1,\sigma}x^{\sigma}\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\partial_{\alpha}\partial_{\beta}\Phi_{\mu}a^{\mu}[+1-1] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\mu}a^{\mu}\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\partial_{\alpha}\Phi_{\beta}[+\frac{1}{4}u^2] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\mu}a^{\mu}\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\partial_{\beta}\Phi_{\alpha}[-\frac{1}{2}u + \frac{1}{4}u(2-u)] \\
&\qquad +\bar{\epsilon}^{\alpha}_{1,\mu}a^{\mu}\bar{\epsilon}^{\beta}_{2,\alpha}\Phi_{\beta}[-\frac{1}{4}(2-u)^2 + \frac{1}{2}\frac{g_{222}}{g_{211}}((1-u)(g+\frac{3}{2}) + \frac{5}{2} - g)] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\bar{\epsilon}^{\alpha}_{2,\mu}a^{\mu}\partial_{\beta}\Phi_{\alpha}[-\frac{1}{4}u^2] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\rho}x^{\rho}\bar{\epsilon}^{\alpha}_{2,\mu}a^{\mu}\partial_{\alpha}\Phi_{\beta}[+\frac{1}{2}u - \frac{1}{4}u(2-u)] \\
&\qquad +\bar{\epsilon}^{\beta}_{1,\alpha}\bar{\epsilon}^{\alpha}_{2,\mu}a^{\mu}\Phi_{\beta}[+\frac{1}{4}(2-u)^2 - \frac{1}{2}\frac{g_{222}}{g_{211}}(g+\frac{3}{2} + (1-u)(\frac{5}{2} - g))] \\
&+\bar{\epsilon}^{\beta}_{1,\sigma}x^{\sigma}\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\partial_{\mu}\partial_{\alpha}\Phi_{\beta}a^{\mu}[+\frac{1}{4}u^2] \\
&+\bar{\epsilon}^{\beta}_{1,\sigma}x^{\sigma}\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\partial_{\mu}\partial_{\beta}\Phi_{\alpha}a^{\mu}[-\frac{1}{4}u^2] \\
&\quad +\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\bar{\epsilon}^{\beta}_{2,\alpha}\partial_{\beta}\Phi_{\mu}a^{\mu}[-1 + \frac{g_{222}}{g_{211}}(g+\frac{3}{2})] \\
&\quad +\bar{\epsilon}^{\alpha}_{1,\rho}x^{\rho}\bar{\epsilon}^{\beta}_{2,\alpha}\partial_{\mu}\Phi_{\beta}a^{\mu}[+\frac{1}{2}u + \frac{1}{4}u(2-u) - \frac{1}{2}u(g+\frac{3}{2})] \\
&\quad +\bar{\epsilon}^{\beta}_{1,\alpha}\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\partial_{\beta}\Phi_{\mu}a^{\mu}[+1 - \frac{g_{222}}{g_{211}}(\frac{5}{2} - g)] \\
&\quad\ +\bar{\epsilon}^{\beta}_{1,\alpha}\bar{\epsilon}^{\alpha}_{2,\rho}x^{\rho}\partial_{\mu}\Phi_{\beta}a^{\mu}[-\frac{1}{2}u - \frac{1}{4}u(2-u) + \frac{1}{2}u(\frac{5}{2} - g)]
\end{aligned}
\tag{7.60}
$$

This can be solved by setting $u = 0$, $g = \frac{1}{2}$ and $\frac{g_{222}}{g_{211}} = \frac{1}{2}$. Setting $u = 0$ sets the term where the derivative acts on the field to zero in the deformation used above. This is similar to what we observe for the other spin-2-deformations of spin-3-gravity. Also, the relative coupling constant $\frac{g_{222}}{g_{211}}$ agrees with $\frac{g_{222}}{g_{200}}$, which was calculated in the scalar case. This is somewhat expected, since the above calculation also works for massless Maxwell fields. In this case the vertices $\mathcal{V}_{200}$ and $\mathcal{V}_{211}$ are also related by the duality transformation of the fields and the coupling constants agree.

one spin-2 parameter, one spin-3 parameter

Next we come to the case of one spin-2 parameter and one spin-3 parameter. The bracket is then the 3-2-bracket from 3-3-2 and the deformation in the third terms is again w.r.t. a spin-3 parameter. For the first term, the inner deformation is:

$$\delta^{(1)}_{\bar\epsilon_1}(\Phi) = -\bar\epsilon^{\alpha}_{1,\rho}x^\rho\partial_\alpha\Phi_\mu a^\mu + \frac{1}{2}u\bar\epsilon^{\alpha}_{1,\rho}x^\rho\partial_\mu\Phi_\alpha a^\mu - \frac{1}{2}(2-u)\bar\epsilon^{\alpha}_{1,\mu}a^\mu\Phi_\alpha \tag{7.61}$$

The double deformation then is:

$$\begin{aligned}
\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi)) = \bar\epsilon^{\alpha\beta}_{2,\ \rho}x^\rho[&-6\bar\epsilon^{\gamma}_{1,\alpha}\partial_\beta\partial_\gamma\Phi_\mu a^\mu - 3\bar\epsilon^{\gamma}_{1,\sigma}x^\sigma\partial_\alpha\partial_\beta\partial_\gamma\Phi_\mu a^\mu\\
&+3u\bar\epsilon^{\gamma}_{1,\alpha}\partial_\beta\partial_\mu\Phi_\gamma a^\mu + 2v\bar\epsilon^{\gamma}_{1,\alpha}\partial_\gamma\partial_\mu\Phi_\beta a^\mu\\
&+2v\bar\epsilon^{\gamma}_{1,\mu}a^\mu\partial_\gamma\partial_\alpha\Phi_\beta + 2v\bar\epsilon^{\gamma}_{1,\sigma}x^\sigma\partial_\gamma\partial_\alpha\partial_\mu\Phi_\beta a^\mu\\
&-2v(u-1)\bar\epsilon^{\gamma}_{1,\alpha}\partial_\beta\partial_\mu\Phi_\gamma a^\mu + u(\frac{3}{2}-v)\bar\epsilon^{\gamma}_{1,\sigma}x^\sigma\partial_\alpha\partial_\beta\partial_\mu\Phi_\gamma a^\mu\\
&-(\frac{3}{2}(2-u) + u\cdot v)\bar\epsilon^{\gamma}_{1,\mu}a^\mu\partial_\alpha\partial_\beta\Phi_\gamma]\\
&+\bar\epsilon^{\alpha\beta}_{2,\ \mu}a^\mu[-2(1-v)\bar\epsilon^{\gamma}_{1,\alpha}\partial_\gamma\Phi_\beta - 2(1-v)\bar\epsilon^{\gamma}_{1,\rho}x^\rho\partial_\alpha\partial_\gamma\Phi_\beta\\
&+(u(1-v)+\frac{1}{2}u)\bar\epsilon^{\gamma}_{1,\rho}x^\rho\partial_\alpha\partial_\beta\Phi_\gamma]\\
&+\bar\epsilon^{\beta}_{2,\mu\alpha}a^\mu[2\bar\epsilon^{\gamma}_{1,\beta}\partial_\gamma\Phi_\alpha + 2\bar\epsilon^{\gamma}_{1,\rho}x^\rho\partial_\beta\partial_\gamma\Phi_\alpha\\
&+(-u-u(1-v)+(2-u)(1-v))\bar\epsilon^{\gamma}_{1,\beta}\partial_\alpha\Phi_\gamma\\
&+(2-u-u(1-v)+(2-u)(1-v))\bar\epsilon^{\gamma}_{1,\alpha}\partial_\beta\Phi_\gamma]
\end{aligned} \tag{7.62}$$

For the second term, the inner deformation is:

$$\begin{aligned}
\delta^{(1)}_{\bar\epsilon_2}(\Phi) = &+3\bar\epsilon^{\alpha\beta}_{2,\ \rho}x^\rho\partial_\alpha\partial_\beta\Phi_\mu a^\mu - 2v\bar\epsilon^{\alpha\beta}_{2,\ \rho}x^\rho\partial_\alpha\partial_\mu\Phi_\beta a^\mu\\
&+2(1-v)\bar\epsilon^{\alpha\beta}_{2,\ \mu}a^\mu\partial_\alpha\Phi_\beta - 2\bar\epsilon^{\beta}_{2,\mu\alpha}a^\mu\partial_\beta\Phi_\alpha
\end{aligned} \tag{7.63}$$

The double deformation then is:

$$\begin{aligned}
\delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) = \bar\epsilon^{\alpha}_{1,\mu}a^\mu[&-\frac{3}{2}(2-u)\bar\epsilon^{\beta\gamma}_{2,\ \rho}x^\rho\partial_\beta\partial_\gamma\Phi_\alpha + v(2-u)\bar\epsilon^{\beta\gamma}_{2,\ \rho}x^\rho\partial_\beta\partial_\alpha\Phi_\gamma \\
&-(1-v)(2-u)\bar\epsilon^{\beta\gamma}_{2,\ \alpha}\partial_\beta\Phi_\gamma + (2-u)\bar\epsilon^{\gamma}_{2,\alpha\beta}\partial_\gamma\Phi^\beta] \\
&+\bar\epsilon^{\alpha}_{1,\rho}x^\rho[-3\bar\epsilon^{\beta\gamma}_{2,\ \alpha}\partial_\beta\partial_\gamma\Phi_\mu a^\mu - 3\bar\epsilon^{\beta\gamma}_{2,\ \sigma}x^\sigma\partial_\alpha\partial_\beta\partial_\gamma\Phi_\mu a^\mu \\
&+\frac{3}{2}u\bar\epsilon^{\beta\gamma}_{2,\ \mu}a^\mu\partial_\beta\partial_\gamma\Phi_\alpha + \frac{3}{2}u\bar\epsilon^{\beta\gamma}_{2,\ \sigma}x^\sigma\partial_\beta\partial_\gamma\partial_\mu\Phi_\alpha a^\mu \\
&+2v(1-\frac{1}{2}u)\bar\epsilon^{\beta\gamma}_{2,\ \sigma}x^\sigma\partial_\beta\partial_\alpha\partial_\mu\Phi_\gamma a^\mu \\
&-(2v+u+u(1-v))\bar\epsilon^{\gamma}_{2,\alpha\beta}\partial_\mu\partial_\gamma\Phi_\beta a^\mu \\
&-(2v+u(1-v))\bar\epsilon^{\beta}_{2,\alpha\gamma}\partial_\mu\partial_\gamma\Phi_\beta a^\mu \\
&+(2(1-v)+u\cdot v+2)\bar\epsilon^{\gamma}_{2,\mu\beta}a^\mu\partial_\mu\partial_\gamma\partial_\alpha\Phi_\beta \\
&+(2(1-v)+u\cdot v)\bar\epsilon^{\beta}_{2,\mu\gamma}a^\mu\partial_\mu\partial_\gamma\partial_\alpha\Phi_\beta]
\end{aligned} \tag{7.64}$$

For the third term we need the 3-2-bracket from 3-3-2 and a relative coupling constant $\frac{g_{332}}{g_{211}}$:

$$[[\bar\epsilon_2,\bar\epsilon_1]] = \frac{g_{332}}{g_{211}}[-3(g+\frac{3}{2})\bar\epsilon_{2,\mu\nu}{}^{\alpha}a^\mu a^\nu\bar\epsilon_{1,\alpha\rho}x^\rho - 3(3+2(1-g))\bar\epsilon^{\alpha}_{2,\mu\rho}a^\mu x^\rho\bar\epsilon_{1,\alpha\nu}a^\nu] \tag{7.65}$$

The third term then is:

$$\begin{aligned}
\delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi) = &+\frac{3}{2}[[\bar\epsilon_2,\bar\epsilon_1]]^{\alpha\beta}\partial_\alpha\partial_\beta\Phi_\mu - v[[\bar\epsilon_2,\bar\epsilon_1]]^{\alpha\beta}\partial_\alpha\partial_\mu\Phi_\beta \\
&+(1-v)\partial_\mu[[\bar\epsilon_2,\bar\epsilon_1]]^{\alpha\beta}\partial_\alpha\Phi_\beta - \partial_\alpha[[\bar\epsilon_2,\bar\epsilon_1]]^{\beta}{}_{\mu}\partial_\beta\Phi^\alpha
\end{aligned} \tag{7.66}$$

Plugging in the bracket from above this reads:

$$
\begin{aligned}
\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = &\frac{g_{332}}{g_{211}}[+9(2g+3)\bar{\epsilon}_2^{\gamma\alpha\beta}\bar{\epsilon}_{1,\gamma\rho}x^\rho - 9(3+2(1-g))\bar{\epsilon}^{\gamma}_{2,\alpha\rho}x^\rho\bar{\epsilon}_{1,\gamma\beta}]\partial_\alpha\partial_\beta\Phi_\mu a^\mu \\
&+\frac{g_{332}}{g_{211}}[-3v(2g+3)\bar{\epsilon}^{\gamma}_{2,\alpha\beta}\bar{\epsilon}_{1,\gamma\rho}x^\rho - 3v(2g+3)\bar{\epsilon}^{\gamma}_{2,\beta\alpha}\bar{\epsilon}_{1,\gamma\rho}x^\rho \\
&\qquad +3v(3+2(1-g))\bar{\epsilon}^{\gamma}_{2,\alpha\rho}x^\rho\bar{\epsilon}_{1,\gamma\beta} \\
&\qquad +3v(3+2(1-g))\bar{\epsilon}^{\gamma}_{2,\beta\rho}x^\rho\bar{\epsilon}_{1,\gamma\alpha}]\partial_\alpha\partial_\mu\Phi_\beta a^\mu \\
&+\frac{g_{332}}{g_{211}}[+3(1-v)(2g+3)\bar{\epsilon}^{\gamma\alpha}_{2,\ \beta}\bar{\epsilon}_{1,\gamma\mu}a^\mu + 3(1-v)(2g+3)\bar{\epsilon}^{\gamma\ \ \alpha}_{2,\beta}\bar{\epsilon}_{1,\gamma\mu}a^\mu \\
&\qquad -3(1-v)(3+2(1-g))\bar{\epsilon}^{\gamma\alpha}_{2,\ \mu}a^\mu\bar{\epsilon}_{1,\gamma\beta} \\
&\qquad -3(1-v)(3+2(1-g))\bar{\epsilon}^{\gamma}_{2,\beta\mu}a^\mu\bar{\epsilon}_{1,\gamma}{}^{\alpha}]\partial_\alpha\Phi_\beta \\
&+\frac{g_{332}}{g_{211}}[-3(2g+3)\bar{\epsilon}^{\gamma}_{2,\beta\mu}a^\mu\bar{\epsilon}_{1,\gamma\alpha} - 3(2g+3)\bar{\epsilon}^{\gamma}_{2,\mu\beta}a^\mu\bar{\epsilon}_{1,\gamma\alpha} \\
&\qquad +3(3+2(1-g))\bar{\epsilon}^{\gamma}_{2,\beta\alpha}\bar{\epsilon}_{1,\gamma\mu}a^\mu + 3(3+2(1-g))\bar{\epsilon}^{\gamma}_{2,\mu\alpha}a^\mu\bar{\epsilon}_{1,\gamma\beta}]\partial_\beta\Phi_\alpha
\end{aligned}
\tag{7.67}
$$

The FCGA then reads:

$$
\begin{aligned}
&\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi)) - \delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) + \delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi) = \\
&\quad \bar\epsilon^{\alpha}_{1,\mu} a^{\mu} \bar\epsilon^{\beta\gamma}_{2,\ \rho} x^{\rho} \partial_\beta \partial_\gamma \Phi_\alpha [-u \cdot v] \\
&\quad +\bar\epsilon^{\alpha}_{1,\mu} a^{\mu} \bar\epsilon^{\beta\gamma}_{2,\ \rho} x^{\rho} \partial_\beta \partial_\alpha \Phi_\gamma [-v(2-u) + 2v] \\
&\qquad +\bar\epsilon^{\alpha}_{1,\mu} a^{\mu} \bar\epsilon_{2,\alpha}{}^{\beta\gamma} \partial_\beta \Phi_\gamma [-2 + u - (1-v)(2-u) + 3(3 + 2(1-g))\frac{g_{332}}{g_{211}} \\
&\qquad\qquad +3(1-v)(2g+3)\frac{g_{332}}{g_{211}}] \\
&\qquad +\bar\epsilon^{\alpha}_{1,\mu} a^{\mu} \bar\epsilon_{2,\alpha}{}^{\gamma\beta} \partial_\beta \Phi_\gamma [-(1-v)(2-u) + 3(1-v)(2g+3)\frac{g_{332}}{g_{211}}] \\
&+\bar\epsilon^{\alpha}_{1,\rho} x^{\rho} \bar\epsilon^{\beta\gamma}_{2,\ \sigma} x^{\sigma} \partial_\alpha \partial_\beta \partial_\gamma \Phi_\mu a^{\mu} [+3 - 3] \\
&\quad +\bar\epsilon^{\alpha}_{1,\rho} x^{\rho} \bar\epsilon^{\beta\gamma}_{2,\ \alpha} \partial_\beta \partial_\gamma \Phi_\mu a^{\mu} [+3 - \frac{9}{2}(2g+3)\frac{g_{332}}{g_{211}}] \\
&\quad +\bar\epsilon^{\alpha}_{1,\rho} x^{\rho} \bar\epsilon_{2,\alpha}{}^{\gamma\beta} \partial_\gamma \partial_\mu \Phi_\beta a^{\mu} [+2v + u(1-v) + u - 3v(2g+3)\frac{g_{332}}{g_{211}}] \\
&\quad +\bar\epsilon^{\alpha}_{1,\rho} x^{\rho} \bar\epsilon_{2,\alpha}{}^{\beta\gamma} \partial_\gamma \partial_\mu \Phi_\beta a^{\mu} [+2v + u(1-v) - 3v(2g+3)\frac{g_{332}}{g_{211}}] \\
&+\bar\epsilon^{\alpha}_{1,\rho} x^{\rho} \bar\epsilon^{\beta\gamma}_{2,\ \sigma} x^{\sigma} \partial_\mu \partial_\beta \partial_\gamma \Phi_\alpha a^{\mu} [-u \cdot v] \\
&+\bar\epsilon^{\alpha}_{1,\rho} x^{\rho} \bar\epsilon^{\beta\gamma}_{2,\ \sigma} x^{\sigma} \partial_\mu \partial_\beta \partial_\alpha \Phi_\gamma a^{\mu} [+u \cdot v] \\
&\quad +\bar\epsilon^{\alpha}_{1,\rho} x^{\rho} \bar\epsilon^{\beta\gamma}_{2,\ \mu} a^{\mu} \partial_\beta \partial_\gamma \Phi_\alpha [-u + u(1-v)] \\
&\quad +\bar\epsilon^{\alpha}_{1,\rho} x^{\rho} \bar\epsilon^{\gamma}_{2,\mu\beta} a^{\mu} \partial_\alpha \partial_\gamma \Phi_\beta [-u \cdot v] \\
&\quad +\bar\epsilon^{\alpha}_{1,\rho} x^{\rho} \bar\epsilon^{\beta}_{2,\mu\gamma} a^{\mu} \partial_\alpha \partial_\gamma \Phi_\beta [-u \cdot v] \\
&\quad +\bar\epsilon^{\gamma}_{1,\alpha} \bar\epsilon^{\alpha\beta}_{2,\ \rho} x^{\rho} \partial_\beta \partial_\gamma \Phi_\mu a^{\mu} [-6 + 9(3 + 2(1-g))\frac{g_{332}}{g_{211}}] \\
&\quad +\bar\epsilon^{\gamma}_{1,\alpha} \bar\epsilon^{\alpha\beta}_{2,\ \rho} x^{\rho} \partial_\mu \partial_\gamma \Phi_\beta a^{\mu} [+2v - 3v(3 + 2(1-g))\frac{g_{332}}{g_{211}}] \\
&\quad +\bar\epsilon^{\gamma}_{1,\alpha} \bar\epsilon^{\alpha\beta}_{2,\ \rho} x^{\rho} \partial_\mu \partial_\beta \Phi_\gamma a^{\mu} [+3u - 2v(u-1) - 3v(3 + 2(1-g))\frac{g_{332}}{g_{211}}] \\
&\qquad +\bar\epsilon^{\gamma}_{1,\alpha} \bar\epsilon^{\alpha}_{2,\mu}{}^{\beta} a^{\mu} \partial_\gamma \Phi_\beta [+2(1-v) + 2 - 3(3 + 2(1-g))\frac{g_{332}}{g_{211}} \\
&\qquad\qquad -3(1-v)(3 + 2(1-g))\frac{g_{332}}{g_{211}}] \\
&\qquad +\bar\epsilon^{\gamma}_{1,\alpha} \bar\epsilon^{\beta}_{2,\mu}{}^{\alpha} a^{\mu} \partial_\gamma \Phi_\beta [+2(1-v) - 3(1-v)(3 + 2(1-g))\frac{g_{332}}{g_{211}}] \\
&\qquad +\bar\epsilon^{\gamma}_{1,\beta} \bar\epsilon^{\beta}_{2,\mu}{}^{\alpha} a^{\mu} \partial_\alpha \Phi_\gamma [-u - u(1-v) + (2-u)(1-v) \\
&\qquad\qquad -3(1-v)(3 + 2(1-g))\frac{g_{332}}{g_{211}}] \\
&\qquad +\bar\epsilon^{\gamma}_{1,\beta} \bar\epsilon^{\alpha}_{2,\mu}{}^{\beta} a^{\mu} \partial_\alpha \Phi_\gamma [+2 - u - u(1-v) + (2-u)(1-v) - 3(2g+3)\frac{g_{332}}{g_{211}} \\
&\qquad\qquad -3(1-v)(3 + 2(1-g))\frac{g_{332}}{g_{211}}]
\end{aligned} \tag{7.68}
$$

This can be solved by setting $u = 0$, $g = \frac{1}{2}$ and $\frac{g_{332}}{g_{211}} = \frac{1}{6}$, which in turn also gives $\frac{g_{222}}{g_{332}} = 3$. Again, these results coincide with the ones obtained in previous calculations. Similar to the cases in 5.3 the parameter of the spin-3-deformation v remains unfixed by this.

two spin-3 parameters

Lastly, we check the case of two spin-3 parameters. The bracket is then the 3-3-bracket from 3-3-2 and the deformation in the third term is w.r.t. a spin-2 parameter. For the first term, the inner deformation is:

$$\begin{aligned}\delta^{(1)}_{\bar{\epsilon}_1}(\Phi) = &+3\epsilon^{\alpha\beta}_{1,\ \rho}x^\rho\partial_\alpha\partial_\beta\Phi_\mu a^\mu - 2v\epsilon^{\alpha\beta}_{1,\ \rho}x^\rho\partial_\alpha\partial_\mu\Phi_\beta a^\mu \\ &+2(1-v)\epsilon^{\alpha\beta}_{1,\ \mu}a^\mu\partial_\alpha\Phi_\beta - 2\epsilon^{\beta}_{1,\mu\alpha}a^\mu\partial_\beta\Phi^\alpha\end{aligned} \tag{7.69}$$

The double deformation then is:

$$\begin{aligned}\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) = &\bar{\epsilon}^{\alpha\beta}_{2,\ \rho}x^\rho[+18\bar{\epsilon}^{\gamma\delta}_{1,\ \alpha}\partial_\beta\partial_\gamma\partial_\delta\Phi_\mu a^\mu + 9\bar{\epsilon}^{\gamma\delta}_{1,\ \sigma}x^\sigma\partial_\alpha\partial_\beta\partial_\gamma\partial_\delta\Phi_\mu a^\mu \\ &-6v\bar{\epsilon}^{\gamma\delta}_{1,\ \alpha}\partial_\mu\partial_\gamma\partial_\delta\Phi_\beta a^\mu - 6v\bar{\epsilon}^{\gamma\delta}_{1,\ \mu}a^\mu\partial_\alpha\partial_\gamma\partial_\delta\Phi_\beta \\ &-6v\bar{\epsilon}^{\gamma\delta}_{1,\ \sigma}x^\sigma\partial_\mu\partial_\alpha\partial_\gamma\partial_\delta\Phi_\beta a^\mu - 6\bar{\epsilon}^{\delta}_{1,\mu\gamma}a^\mu\partial_\alpha\partial_\beta\partial_\delta\Phi^\gamma \\ &+4v\bar{\epsilon}^{\delta}_{1,\beta\gamma}\partial_\mu\partial_\alpha\partial_\delta\Phi^\gamma a^\mu + (4v^2-6v)\bar{\epsilon}^{\gamma\delta}_{1,\ \sigma}x^\sigma\partial_\mu\partial_\alpha\partial_\beta\partial_\gamma\Phi_\delta a^\mu \\ &+(4v^2-12v-4v(1-v))\bar{\epsilon}^{\gamma\delta}_{1,\ \beta}\partial_\mu\partial_\alpha\partial_\gamma\Phi_\delta a^\mu \\ &+(6(1-v)+4v^2)\bar{\epsilon}^{\gamma\delta}_{1,\ \mu}a^\mu\partial_\alpha\partial_\beta\partial_\gamma\Phi_\delta] \\ &+\bar{\epsilon}^{\alpha\beta}_{2,\ \mu}a^\mu[+6(1-v)\bar{\epsilon}^{\gamma\delta}_{1,\ \alpha}\partial_\gamma\partial_\delta\Phi_\beta + 6(1-v)\bar{\epsilon}^{\gamma\delta}_{1,\ \rho}x^\rho\partial_\alpha\partial_\gamma\partial_\delta\Phi_\beta \\ &-4v(1-v)\bar{\epsilon}^{\gamma\delta}_{1,\ \alpha}\partial_\gamma\partial_\beta\Phi_\delta - 4(1-v)\bar{\epsilon}^{\delta}_{1,\beta\gamma}\partial_\alpha\partial_\delta\Phi^\gamma \\ &+4(1-v)^2\bar{\epsilon}^{\gamma\delta}_{1,\ \beta}\partial_\alpha\partial_\gamma\Phi_\delta - (4v(1-v)+2v)\bar{\epsilon}^{\gamma\delta}_{1,\ \rho}x^\rho\partial_\alpha\partial_\gamma\partial_\beta\Phi_\delta] \\ &+\bar{\epsilon}^{\beta}_{2,\mu\alpha}a^\mu[-6\bar{\epsilon}^{\gamma\delta}_{1,\ \beta}\partial_\gamma\partial_\delta\Phi_\alpha - 6\bar{\epsilon}^{\gamma\delta}_{1,\ \rho}x^\rho\partial_\beta\partial_\gamma\partial_\delta\Phi_\alpha \\ &+4v\bar{\epsilon}^{\gamma\delta}_{1,\ \beta}\partial_\gamma\partial_\alpha\Phi_\delta - 4(1-v)\bar{\epsilon}^{\gamma\delta}_{1,\ \alpha}\partial_\gamma\partial_\beta\Phi_\delta \\ &+4\bar{\epsilon}^{\delta}_{1,\alpha\gamma}\partial_\beta\partial_\delta\Phi^\gamma]\end{aligned} \tag{7.70}$$

The second term can be obtained from the first one by switching the gauge parameters:

$$
\begin{aligned}
\delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) = \bar\epsilon^{\alpha\beta}_{1,\ \rho}x^\rho[&+18\bar\epsilon^{\gamma\delta}_{2,\ \alpha}\partial_\beta\partial_\gamma\partial_\delta\Phi_\mu a^\mu + 9\bar\epsilon^{\gamma\delta}_{2,\ \sigma}x^\sigma\partial_\alpha\partial_\beta\partial_\gamma\partial_\delta\Phi_\mu a^\mu\\
&-6v\bar\epsilon^{\gamma\delta}_{2,\ \alpha}\partial_\mu\partial_\gamma\partial_\delta\Phi_\beta a^\mu - 6v\bar\epsilon^{\gamma\delta}_{2,\ \mu}a^\mu\partial_\alpha\partial_\gamma\partial_\delta\Phi_\beta\\
&-6v\bar\epsilon^{\gamma\delta}_{2,\ \sigma}x^\sigma\partial_\mu\partial_\alpha\partial_\gamma\partial_\delta\Phi_\beta a^\mu - 6\bar\epsilon^{\delta}_{2,\mu\gamma}a^\mu\partial_\alpha\partial_\beta\partial_\delta\Phi^\gamma\\
&+4v\bar\epsilon^{\delta}_{2,\beta\gamma}\partial_\mu\partial_\alpha\partial_\delta\Phi^\gamma a^\mu + (4v^2-6v)\bar\epsilon^{\gamma\delta}_{2,\ \sigma}x^\sigma\partial_\mu\partial_\alpha\partial_\beta\partial_\gamma\Phi_\delta a^\mu\\
&+(4v^2-12v-4v(1-v))\bar\epsilon^{\gamma\delta}_{2,\ \beta}\partial_\mu\partial_\alpha\partial_\gamma\Phi_\delta a^\mu\\
&+(6(1-v)+4v^2)\bar\epsilon^{\gamma\delta}_{2,\ \mu}a^\mu\partial_\alpha\partial_\beta\partial_\gamma\Phi_\delta]\\
+\bar\epsilon^{\alpha\beta}_{1,\ \mu}a^\mu[&+6(1-v)\bar\epsilon^{\gamma\delta}_{2,\ \alpha}\partial_\gamma\partial_\delta\Phi_\beta + 6(1-v)\bar\epsilon^{\gamma\delta}_{2,\ \rho}x^\rho\partial_\alpha\partial_\gamma\partial_\delta\Phi_\beta\\
&-4v(1-v)\bar\epsilon^{\gamma\delta}_{2,\ \alpha}\partial_\gamma\partial_\beta\Phi_\delta - 4(1-v)\bar\epsilon^{\delta}_{2,\beta\gamma}\partial_\alpha\partial_\delta\Phi^\gamma\\
&+4(1-v)^2\bar\epsilon^{\gamma\delta}_{2,\ \beta}\partial_\alpha\partial_\gamma\Phi_\delta - (4v(1-v)+2v)\bar\epsilon^{\gamma\delta}_{2,\ \rho}x^\rho\partial_\alpha\partial_\gamma\partial_\beta\Phi_\delta]\\
+\bar\epsilon^{\beta}_{1,\mu\alpha}a^\mu[&-6\bar\epsilon^{\gamma\delta}_{2,\ \beta}\partial_\gamma\partial_\delta\Phi_\alpha - 6\bar\epsilon^{\gamma\delta}_{2,\ \rho}x^\rho\partial_\beta\partial_\gamma\partial_\delta\Phi_\alpha\\
&+4v\bar\epsilon^{\gamma\delta}_{2,\ \beta}\partial_\gamma\partial_\alpha\Phi_\delta - 4(1-v)\bar\epsilon^{\gamma\delta}_{2,\ \alpha}\partial_\gamma\partial_\beta\Phi_\delta\\
&+4\bar\epsilon^{\delta}_{2,\alpha\gamma}\partial_\beta\partial_\delta\Phi^\gamma]
\end{aligned}
\tag{7.71}
$$

For the third term we need the 3-3-bracket from 3-3-2 and a relative coupling constant $\frac{g_{332}}{g_{311}}$. We will again use the rescaling factor of two for the bracket:

$$
[[\bar\epsilon_2,\bar\epsilon_1]] = \frac{g_{332}}{g_{311}}[6(1+10g)\bar\epsilon^{\alpha\beta}_{2,\ \mu}a^\mu\bar\epsilon_{1,\alpha\beta\rho}x^\rho - 6(1+10(1-g))\bar\epsilon^{\alpha\beta}_{2,\ \rho}x^\rho\bar\epsilon_{1,\alpha\beta\mu}a^\mu] \tag{7.72}
$$

The third term then is:

$$
\begin{aligned}
\delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi) = &-[[\bar\epsilon_2,\bar\epsilon_1]]^\alpha\partial_\alpha\Phi_\mu + \frac{1}{2}u[[\bar\epsilon_2,\bar\epsilon_1]]^\alpha\partial_\mu\Phi_\alpha\\
&-\frac{1}{2}(1-u)\partial_\mu[[\bar\epsilon_2,\bar\epsilon_1]]^\alpha\Phi_\alpha + \frac{1}{2}\partial_\alpha[[\bar\epsilon_2,\bar\epsilon_1]]_\mu\Phi^\alpha
\end{aligned}
\tag{7.73}
$$

Plugging in the bracket from above this reads:

$$\begin{aligned}\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = &\frac{g_{332}}{g_{311}}[-6(1+10g)\bar{\epsilon}_2^{\beta\gamma\alpha}\bar{\epsilon}_{1,\beta\gamma\rho}x^\rho + 6(1+10(1-g))\bar{\epsilon}_{2,\ \rho}^{\beta\gamma}x^\rho\bar{\epsilon}_{1,\beta\gamma}{}^{\alpha}]\partial_\alpha\Phi_\mu a^\mu \\ &+\frac{g_{332}}{g_{311}}[+3u(1+10g)\bar{\epsilon}_2^{\beta\gamma\alpha}\bar{\epsilon}_{1,\beta\gamma\rho}x^\rho - 3u(1+10(1-g))\bar{\epsilon}_{2,\ \rho}^{\beta\gamma}x^\rho\bar{\epsilon}_{1,\beta\gamma}{}^{\alpha}]\partial_\mu\Phi_\alpha a^\mu \\ &+\frac{g_{332}}{g_{311}}[-3((1-u)(1+10g)+1+10(1-g))\bar{\epsilon}_{2,\ \alpha}^{\beta\gamma}\bar{\epsilon}_{1,\beta\gamma\mu}a^\mu \\ &\qquad +3((1-u)(1+10(1-g))+1+10g)\bar{\epsilon}_{2,\ \mu}^{\beta\gamma}a^\mu\bar{\epsilon}_{1,\beta\gamma\alpha}]\Phi^\alpha\end{aligned} \tag{7.74}$$

The third term only contains terms with zero derivatives or one derivative acting on Φ, whereas the first two terms only contain contributions with more than one derivative on Φ. As in the scalar case, we observe that we need to introduce a spin-4 field to close the gauge algebra.

7.3 Spin-4 and beyond

As we have seen in the calculations of the FCGA for two spin-3 parameters, in the presence of matter it is necessary to introduce fields of even higher spin in order to achieve consistency. More precisely, it is expected that the problems with the cases involving two spin-3 parameters can be resolved by introducing a spin-4 field together with a 4-3-3 vertex and use the corresponding 3-3-bracket in the last term. This bracket gives back a spin-4 parameter, which allows use to use deformations induced by the 4-0-0 vertex in the scalar case and the 4-1-1 vertex in the Maxwell case respectively.

Using the results from [50], we can see that a 4-3-3 vertex indeed exists. It is symmetric in the spin-3 fields, so it will also be present in Prokushkin-Vasiliev theory (since this theory only includes one field for each spin).

The 4-3-3 vertex is

$$\begin{aligned}\mathcal{V}_{433} = &+3y_1^2z_1^2z_2z_3 + 2y_2^2z_2^3z_3 + 2y_3^2z_2z_3^3 \\ &+4y_2y_3z_2^2z_3^2 + 5y_1y_3z_1z_2z_3^2 + 5y_1y_2z_1z_2^2z_3.\end{aligned} \tag{7.75}$$

In index-notation this reads

$$\begin{aligned}\mathcal{V}_{433}[\Phi] = &+3\Phi_1^{\alpha\beta\rho\sigma}\partial_\alpha\partial_\beta\Phi_{2,\ \sigma}^{\gamma\delta}\Phi_{3,\gamma\delta\rho} + 2\Phi_1^{\gamma\delta\rho\sigma}\Phi_{2,\ \gamma}^{\alpha\beta}\partial_\alpha\partial_\beta\Phi_{3,\delta\rho\sigma} + 2\Phi_1^{\gamma\delta\rho\sigma}\Phi_{2,\delta\rho\sigma}\Phi_{3,\ \gamma}^{\alpha\beta} \\ &+4\partial_\alpha\Phi_1^{\gamma\delta\rho\sigma}\Phi_{2,\rho\sigma}^{\beta}\partial_\beta\Phi_{3,\gamma\delta}^{\alpha} + 5\partial_\alpha\Phi_1^{\beta\delta\rho\sigma}\partial_\beta\Phi_{2,\rho\sigma}^{\gamma}\Phi_{3,\gamma\delta}^{\alpha} + 5\Phi_1^{\alpha\delta\rho\sigma}\partial_\alpha\Phi_{2,\gamma\delta}^{\beta}\Phi_{3,\rho\sigma}^{\gamma}.\end{aligned} \tag{7.76}$$

The corresponding 3-3-bracket is given by

$$[[\epsilon_2, \epsilon_3]] = z_2 z_3(8y_1 z_1 + 7y_2 z_2 + 7y_3 z_3)\epsilon_2\epsilon_3. \tag{7.77}$$

In index-notation this bracket reads

$$\begin{aligned}[[\epsilon_2, \epsilon_3]] = &+8g_7\ \partial_\mu\epsilon_{2,\nu}^{\alpha}a^\mu a^\nu\epsilon_{3,\alpha\lambda}a^\lambda - 8(1-g_7)\ \epsilon_{2,\nu}^{\alpha}a^\nu\partial_\mu\epsilon_{3,\alpha\lambda}a^\mu a^\lambda \\ &+7\epsilon_{2,\mu}^{\alpha}a^\mu\partial_\alpha\epsilon_{3,\nu\lambda}a^\nu a^\lambda - 7\partial_\alpha\epsilon_{2,\mu\nu}a^\mu a^\nu\epsilon_{3,\lambda}^{\alpha}a^\lambda.\end{aligned} \tag{7.78}$$

For the deformations in the last term of the additional condition, we also need the 4-0-0 vertex and the 4-1-1 vertex. These read

$$\mathcal{V}_{400} = y_1^4, \tag{7.79}$$

$$\mathcal{V}_{411} = y_1^4 z_1 + y_1^3 y_2 z_2 + y_1^3 y_3 z_3. \tag{7.80}$$

In index-notation they are given by

$$\mathcal{V}_{400}[\Phi] = \Phi_1^{\alpha\beta\gamma\delta}\partial_\alpha\partial_\beta\partial_\gamma\partial_\delta\Phi_2\Phi_3, \tag{7.81}$$

$$\mathcal{V}_{411}[\Phi] = \Phi_1^{\alpha\beta\gamma\delta}\partial_\alpha\partial_\beta\partial_\gamma\partial_\delta\Phi_2^{\rho}\Phi_{3,\rho} + \Phi_1^{\alpha\beta\gamma\rho}\partial_\alpha\partial_\beta\partial_\gamma\Phi_2^{\delta}\partial_\delta\Phi_{3,\rho} + \partial_\delta\Phi_1^{\alpha\beta\gamma\rho}\partial_\alpha\partial_\beta\partial_\gamma\Phi_{2,\rho}\Phi_3^{\delta}. \tag{7.82}$$

The induced gauge deformations then are:

4-0-0 vertex:

$$\delta_{\epsilon_1}^{(1)}\Phi_2 = 2y_1^3\epsilon_1\Phi_3 \tag{7.83}$$

4-1-1 vertex:

$$\delta^{(1)}_{\epsilon_1}\Phi_2 = (2y_1^3 z_1 + \frac{3}{2}y_1^2 y_2 z_2 + \frac{3}{2}y_1^2 y_3 z_3)\epsilon_1\Phi_3 \tag{7.84}$$

In index-notation these read:

4-0-0 vertex:

$$\delta^{(1)}_{\epsilon_1}\Phi_2 = -2\epsilon_1^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma\Phi_3 \tag{7.85}$$

4-1-1 vertex:

$$\begin{aligned}\delta^{(1)}_{\epsilon_1}\Phi_2 = &+\frac{3}{2}t\epsilon_1^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\mu\Phi_{3,\gamma}a^\mu - \frac{3}{2}(1-t)\partial_\mu\epsilon_1^{\alpha\beta\gamma}a^\mu\partial_\alpha\partial_\beta\Phi_{3,\gamma}\\ &-2\epsilon_1^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma\Phi_{3,\mu}a^\mu + \frac{3}{2}\partial_\gamma\epsilon^{\alpha\beta}_{1,\ \mu}a^\mu\partial_\alpha\partial_\beta\Phi_3^\gamma\end{aligned} \tag{7.86}$$

We can now use these objects to check whether the gauge algebra can be closed in this way. Starting with the scalar case, recall that the first two terms of the FCGA are:

$$\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi)) = 9\bar\epsilon^{\alpha\beta}_{2,\rho}x^\rho[\bar\epsilon^{\gamma\delta}_{1,\sigma}x^\sigma\partial_\alpha\partial_\beta\partial_\gamma\partial_\delta\Phi + 2\bar\epsilon^{\gamma\delta}_{1,\alpha}\partial_\beta\partial_\gamma\partial_\delta\Phi] \tag{7.87}$$

$$\delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) = 9\bar\epsilon^{\alpha\beta}_{1,\rho}x^\rho[\bar\epsilon^{\gamma\delta}_{2,\sigma}x^\sigma\partial_\alpha\partial_\beta\partial_\gamma\partial_\delta\Phi + 2\bar\epsilon^{\gamma\delta}_{2,\alpha}\partial_\beta\partial_\gamma\partial_\delta\Phi] \tag{7.88}$$

For the last term we now use the 3-3-bracket from the 4-3-3 vertex and the 4-0-deformation from the 4-0-0 vertex. Introducing two relative coupling constants $\frac{g_{433}}{g_{300}}$ and $\frac{g_{400}}{g_{300}}$ we then have:

$$\delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi) = \frac{g_{433}}{g_{300}}\frac{g_{400}}{g_{300}}[48(4g+7)\bar\epsilon^{\beta\gamma}_{2,\ \alpha}\bar\epsilon^{\alpha\delta}_{1,\rho}x^\rho - 48(4(1-g)+7)\bar\epsilon^{\alpha\delta}_{2,\rho}x^\rho\bar\epsilon^{\beta\gamma}_{1,\ \alpha}]\partial_\beta\partial_\gamma\partial_\delta\Phi \tag{7.89}$$

One can check that the contributions from this new term indeed cancel the remaining contributions from the first two terms by setting $g = \frac{1}{2}$, as demanded by the anti-symmetry of the bracket, and $\frac{g_{433}}{g_{300}}\frac{g_{400}}{g_{300}} = \frac{1}{24}$.

For the Maxwell case, the first two terms of the FCGA are:

$$\begin{aligned}
\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi)) = \bar\epsilon^{\alpha\beta}_{2,\ \rho}x^\rho[&+18\bar\epsilon^{\gamma\delta}_{1,\ \alpha}\partial_\beta\partial_\gamma\partial_\delta\Phi_\mu a^\mu + 9\bar\epsilon^{\gamma\delta}_{1,\ \sigma}x^\sigma\partial_\alpha\partial_\beta\partial_\gamma\partial_\delta\Phi_\mu a^\mu \\
&-6v\bar\epsilon^{\gamma\delta}_{1,\ \alpha}\partial_\mu\partial_\gamma\partial_\delta\Phi_\beta a^\mu - 6v\bar\epsilon^{\gamma\delta}_{1,\ \mu}a^\mu\partial_\alpha\partial_\gamma\partial_\delta\Phi_\beta \\
&-6v\bar\epsilon^{\gamma\delta}_{1,\ \sigma}x^\sigma\partial_\mu\partial_\alpha\partial_\gamma\partial_\delta\Phi_\beta a^\mu - 6\bar\epsilon^{\delta}_{1,\mu\gamma}a^\mu\partial_\alpha\partial_\beta\partial_\delta\Phi^\gamma \\
&+4v\bar\epsilon^{\delta}_{1,\beta\gamma}\partial_\mu\partial_\alpha\partial_\delta\Phi^\gamma a^\mu + (4v^2-6v)\bar\epsilon^{\gamma\delta}_{1,\ \sigma}x^\sigma\partial_\mu\partial_\alpha\partial_\beta\partial_\gamma\Phi_\delta a^\mu \\
&+(4v^2-12v-4v(1-v))\bar\epsilon^{\gamma\delta}_{1,\ \beta}\partial_\mu\partial_\alpha\partial_\gamma\Phi_\delta a^\mu \\
&+(6(1-v)+4v^2)\bar\epsilon^{\gamma\delta}_{1,\ \mu}a^\mu\partial_\alpha\partial_\beta\partial_\gamma\Phi_\delta] \\
+\bar\epsilon^{\alpha\beta}_{2,\ \mu}a^\mu[&+6(1-v)\bar\epsilon^{\gamma\delta}_{1,\ \alpha}\partial_\gamma\partial_\delta\Phi_\beta + 6(1-v)\bar\epsilon^{\gamma\delta}_{1,\ \rho}x^\rho\partial_\alpha\partial_\gamma\partial_\delta\Phi_\beta \\
&-4v(1-v)\bar\epsilon^{\gamma\delta}_{1,\ \alpha}\partial_\gamma\partial_\beta\Phi_\delta - 4(1-v)\bar\epsilon^{\delta}_{1,\beta\gamma}\partial_\alpha\partial_\delta\Phi^\gamma \\
&+4(1-v)^2\bar\epsilon^{\gamma\delta}_{1,\ \beta}\partial_\alpha\partial_\gamma\Phi_\delta - (4v(1-v)+2v)\bar\epsilon^{\gamma\delta}_{1,\ \rho}x^\rho\partial_\alpha\partial_\gamma\partial_\beta\Phi_\delta] \\
+\bar\epsilon^{\beta}_{2,\mu\alpha}a^\mu[&-6\bar\epsilon^{\gamma\delta}_{1,\ \beta}\partial_\gamma\partial_\delta\Phi_\alpha - 6\bar\epsilon^{\gamma\delta}_{1,\ \rho}x^\rho\partial_\beta\partial_\gamma\partial_\delta\Phi_\alpha \\
&+4v\bar\epsilon^{\gamma\delta}_{1,\ \beta}\partial_\gamma\partial_\alpha\Phi_\delta - 4(1-v)\bar\epsilon^{\gamma\delta}_{1,\ \alpha}\partial_\gamma\partial_\beta\Phi_\delta \\
&+4\bar\epsilon^{\delta}_{1,\alpha\gamma}\partial_\beta\partial_\delta\Phi^\gamma]
\end{aligned} \tag{7.90}$$

$$\begin{aligned}
\delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) = \bar\epsilon^{\alpha\beta}_{1,\ \rho}x^\rho[&+18\bar\epsilon^{\gamma\delta}_{2,\ \alpha}\partial_\beta\partial_\gamma\partial_\delta\Phi_\mu a^\mu + 9\bar\epsilon^{\gamma\delta}_{2,\ \sigma}x^\sigma\partial_\alpha\partial_\beta\partial_\gamma\partial_\delta\Phi_\mu a^\mu \\
&-6v\bar\epsilon^{\gamma\delta}_{2,\ \alpha}\partial_\mu\partial_\gamma\partial_\delta\Phi_\beta a^\mu - 6v\bar\epsilon^{\gamma\delta}_{2,\ \mu}a^\mu\partial_\alpha\partial_\gamma\partial_\delta\Phi_\beta \\
&-6v\bar\epsilon^{\gamma\delta}_{2,\ \sigma}x^\sigma\partial_\mu\partial_\alpha\partial_\gamma\partial_\delta\Phi_\beta a^\mu - 6\bar\epsilon^{\delta}_{2,\mu\gamma}a^\mu\partial_\alpha\partial_\beta\partial_\delta\Phi^\gamma \\
&+4v\bar\epsilon^{\delta}_{2,\beta\gamma}\partial_\mu\partial_\alpha\partial_\delta\Phi^\gamma a^\mu + (4v^2-6v)\bar\epsilon^{\gamma\delta}_{2,\ \sigma}x^\sigma\partial_\mu\partial_\alpha\partial_\beta\partial_\gamma\Phi_\delta a^\mu \\
&+(4v^2-12v-4v(1-v))\bar\epsilon^{\gamma\delta}_{2,\ \beta}\partial_\mu\partial_\alpha\partial_\gamma\Phi_\delta a^\mu \\
&+(6(1-v)+4v^2)\bar\epsilon^{\gamma\delta}_{2,\ \mu}a^\mu\partial_\alpha\partial_\beta\partial_\gamma\Phi_\delta] \\
+\bar\epsilon^{\alpha\beta}_{1,\ \mu}a^\mu[&+6(1-v)\bar\epsilon^{\gamma\delta}_{2,\ \alpha}\partial_\gamma\partial_\delta\Phi_\beta + 6(1-v)\bar\epsilon^{\gamma\delta}_{2,\ \rho}x^\rho\partial_\alpha\partial_\gamma\partial_\delta\Phi_\beta \\
&-4v(1-v)\bar\epsilon^{\gamma\delta}_{2,\ \alpha}\partial_\gamma\partial_\beta\Phi_\delta - 4(1-v)\bar\epsilon^{\delta}_{2,\beta\gamma}\partial_\alpha\partial_\delta\Phi^\gamma \\
&+4(1-v)^2\bar\epsilon^{\gamma\delta}_{2,\ \beta}\partial_\alpha\partial_\gamma\Phi_\delta - (4v(1-v)+2v)\bar\epsilon^{\gamma\delta}_{2,\ \rho}x^\rho\partial_\alpha\partial_\gamma\partial_\beta\Phi_\delta] \\
+\bar\epsilon^{\beta}_{1,\mu\alpha}a^\mu[&-6\bar\epsilon^{\gamma\delta}_{2,\ \beta}\partial_\gamma\partial_\delta\Phi_\alpha - 6\bar\epsilon^{\gamma\delta}_{2,\ \rho}x^\rho\partial_\beta\partial_\gamma\partial_\delta\Phi_\alpha \\
&+4v\bar\epsilon^{\gamma\delta}_{2,\ \beta}\partial_\gamma\partial_\alpha\Phi_\delta - 4(1-v)\bar\epsilon^{\gamma\delta}_{2,\ \alpha}\partial_\gamma\partial_\beta\Phi_\delta \\
&+4\bar\epsilon^{\delta}_{2,\alpha\gamma}\partial_\beta\partial_\delta\Phi^\gamma]
\end{aligned} \tag{7.91}$$

Using again the 3-3-bracket from 4-3-3, the 4-1-deformation from 4-1-1 and introducing two relative coupling constants $\frac{g_{433}}{g_{311}}$ and $\frac{g_{411}}{g_{311}}$ we have:

$$
\begin{aligned}
\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi) = \frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}&[+48(4g+7)\bar{\epsilon}_{2,\ \delta}^{\alpha\beta}\bar{\epsilon}_{1,\ \rho}^{\delta\gamma}x^{\rho} \\
&-48(4(1-g)+7)\bar{\epsilon}_{2,\ \rho}^{\delta\alpha}x^{\rho}\bar{\epsilon}_{1,\ \delta}^{\beta\gamma}]\partial_\alpha\partial_\beta\partial_\gamma\Phi_\mu a^\mu \\
+\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}&[-12t(4g+7)\bar{\epsilon}_{2,\ \delta}^{\alpha\beta}\bar{\epsilon}_{1,\ \rho}^{\delta\gamma}x^{\rho}-24t(4g+7)\bar{\epsilon}_{2,\ \delta}^{\gamma\alpha}\bar{\epsilon}_{1,\ \rho}^{\delta\beta}x^{\rho} \\
&+12t(4(1-g)+7)\bar{\epsilon}_{2,\ \rho}^{\delta\gamma}x^{\rho}\bar{\epsilon}_{1,\ \delta}^{\alpha\beta} \\
&+24t(4(1-g)+7)\bar{\epsilon}_{2,\ \rho}^{\delta\alpha}x^{\rho}\bar{\epsilon}_{1,\ \delta}^{\beta\gamma}]\partial_\alpha\partial_\beta\partial_\mu\Phi_\gamma a^\mu \\
+\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}&[+12(1-t)(4g+7)\bar{\epsilon}_{2,\ \delta}^{\alpha\beta}\bar{\epsilon}_{1,\ \mu}^{\delta\gamma}a^{\mu}+24(1-t)(4g+7)\bar{\epsilon}_{2,\ \delta}^{\gamma\alpha}\bar{\epsilon}_{1,\ \mu}^{\delta\beta}a^{\mu} \\
&-12(1-t)(4(1-g)+7)\bar{\epsilon}_{2,\ \mu}^{\delta\gamma}a^{\mu}\bar{\epsilon}_{1,\ \delta}^{\alpha\beta} \\
&-24(1-t)(4(1-g)+7)\bar{\epsilon}_{2,\ \mu}^{\delta\alpha}a^{\mu}\bar{\epsilon}_{1,\ \delta}^{\beta\gamma} \\
&-12(4g+7)\bar{\epsilon}_{2,\ \delta}^{\alpha\beta}\bar{\epsilon}_{1,\mu\gamma}^{\delta}a^{\mu}-24(4g+7)\bar{\epsilon}_{2,\mu\delta}^{\alpha}a^{\mu}\bar{\epsilon}_{1,\ \gamma}^{\delta\beta} \\
&+12(4(1-g)+7)\bar{\epsilon}_{2,\mu\gamma}^{\delta}a^{\mu}\bar{\epsilon}_{1,\ \delta}^{\alpha\beta} \\
&+24(4(1-g)+7)\bar{\epsilon}_{2,\ \gamma}^{\delta\alpha}\bar{\epsilon}_{1,\mu\delta}^{\beta}a^{\mu}]\partial_\alpha\partial_\beta\Phi^\gamma
\end{aligned} \tag{7.92}
$$

In this case, the FCGA is not identically fulfilled. Analogously to the calculations of 5.3 we dualise the spin-3 parameters using (5.11) and rewrite the result in the new variables. Note that the Laplacians created during the dualisation cannot be dropped anymore, since they produce terms proportional to the mass of the Maxwell fields. Adding these terms up and dropping a factor of 4, the FCGA is again rather lengthy and given as (B.1) in appendix B.

There is no solution to the above system of equations, even if we set $m^2 = 0$. We therefore see that it is again necessary to allow for terms of the form of zero-order gauge transformations on the RHS of (5.9). After replacing all terms containing p_a the FCGA is:

$$
\begin{aligned}
\{\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) - \delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) + \delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi)\}_{dual} = \\
+A_1 z_1 z_3[+108m^2\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}] \\
+A_1 p_1 p_2 z_1[+648\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}} - 27] \\
+A_2 z_2 z_3[-108m^2\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}] \\
+A_2 p_1 p_2 z_2[+27 - 648\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}] \\
+A_3 p_1 X_2 z_3[+24m^2(4g-11)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}} + 3] \\
+A_3 p_2 X_1 z_3[+24m^2(4g+7)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}} - 3] \\
+A_3 p_1 p_2 p_x z_3[+192(2g-1)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}] \\
+A_3 p_1 p_2^2 X_1[+(696-96g)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}} - 27] \\
+A_3 p_1^2 p_2 X_2[+27 - 24(4g+25)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}]
\end{aligned}
\tag{7.93}
$$

We can see that the parameter v drops out of the calculation and will therefore not be fixed by this. There is still no solution to the above system of equations. Setting $g = \frac{1}{2}$, as dictated by the anti-symmetry of the bracket and $\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}} = \frac{1}{24}$ in accordance with the corresponding scalar calculation, we get:

$$
\{\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) - \delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) + \delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi)\}_{dual} = \frac{9}{2}m^2 A_1 z_1 z_3 - \frac{9}{2}m^2 A_2 z_2 z_3 \tag{7.94}
$$

From this we can see that the FCGA is fulfilled in the massless case, but not for massive Maxwell fields. This means that we do not observe any contradiction with the calculations for the scalar fields, since the only deviation occurs in the massive case, for which the dualisation relation between the scalar field and the Maxwell field no longer exists.

8 Discussions

summary of results

The theory of spin-3-gravity has been successfully constructed on three-dimensional Minkowski space at the cubic TT level. Taking into account the influence of DDI's, the methods of [22] regarding the calculation of the first-order gauge deformations and the zero-order gauge brackets have been generalised to the three-dimensional case. By explicitly checking the Jacobi-identity for the gauge brackets and the closure of the gauge algebra at first order, the consistency of the theory has been verified for the global symmetry algebra.

In a final step, matter in the form of scalar and Maxwell fields has been introduced to spin-3-gravity. Whereas scalar fields can be coupled both as massless and massive fields, the Maxwell fields need to be chosen to be massless. It has also been observed that in the presence of matter, the algebra of spin-3-gravity can no longer close and it becomes necessary to include fields of even higher spin, up to infinity. This is best illustrated in the scalar case: the scalar coupling vertex s-0-0 has s derivatives, so the gauge deformation it induces has $s-1$ derivatives. Hence, the commutator of two spin-s deformations will contain $2s-2$ derivatives. If we have only one copy for each higher-spin field, the only brackets available have one derivative, so the deformation in the last term of (5.9) needs to have $2s-3$ derivatives to match the number in the first two terms. This deformation then has to come from the (2s-2)-0-0 vertex, which requires a field with spin $2s-2$. An example of this would be the commutator of two spin-3-deformations, which gives back a spin-4-deformation. Since $2s-2 > s$ for $s > 2$, we can see that as soon as we have a field of spin $s > 2$ coupled to the scalar field, closing the gauge algebra requires also a field of spin $s' = 2s-2 > s$. This leads to the familiar infinite tower of higher-spin fields, which is also present in Prokushkin-Vasiliev-theory.

Using multiple copies of each higher-spin field would give us access to brackets with two derivatives instead of one. Considering again the commutator of two spin-s-deformations, the deformation in the last term of (5.9) would then need to have $2s-3$ derivatives. We have that $2s-3 > s$ only for $s > 3$, so in the case of spin-3-gravity this could still work out without the need for additional fields.

However, in the case of two spin-3-deformations the only available bracket with two derivatives comes from the 3-3-3 vertex, which is known to be inconsistent [91].

An easy way to avoid the emergence of an infinite tower of higher spins in spin-3-gravity is using a real scalar field instead of a complex one: the 3-0-0 vertex requires two copies of the scalar field, therefore it cannot couple to a real scalar field and the problematic commutator of two spin-3-deformation does not arise [50].

In this work only the coupling of massive matter fields to spin-3-gravity has been investigated. Using massless matter fields instead, new vertices, such as the s-s-1 vertices, that are not gauge invariant for massive fields, and the s-1-0 vertices become available. One can then check whether these vertices can be successfully implemented into the theory.

All partial integration parameters of the brackets have been fixed to be $\frac{1}{2}$, as demanded by the anti-symmetry of the commutator. For the spin-2-deformations the parameters are fixed in such a way that the terms where the free derivative ∂_μ acts on the field vanish. It can be conjectured that this is the case for all spin-2-deformations induced by the minimal coupling vertices $\mathcal{V}_{ss2}$. On the other hand, the parameters of the spin-3-deformations cannot be fixed at the current level. It is expected that these parameters are fixed once the full vertices, including traces and divergences, are known.

For now, it is still possible to draw some conclusions about the parameters of the spin-3-deformations. In [62] the 3-2-deformation of 3d spin-3-gravity has been calculated in terms of the metric-like fields from the frame-like theory. Comparing the deformation in (3.41) to the result from this paper and using a Killing-tensor as gauge parameter in order to avoid ambiguities related to redefinitions, one can see that the two expressions agree (up to a numerical factor) for $d = \frac{1}{5}$. The 3-2-deformation then takes the form:

$$\delta^{(1)}_{\bar{\epsilon}_1}\Phi_3 = +6\bar{\epsilon}_1^{\alpha\beta}\partial_\mu\Phi_{2,\alpha\beta\nu}a^\mu a^\nu - 3\bar{\epsilon}_1^{\alpha\beta}\partial_\alpha\Phi_{2,\beta\mu\nu}a^\mu a^\nu \tag{8.1}$$

For the 3-3-deformation the Fronsdal-equation demands $f = 1$. However, if the traceless constraint of the gauge parameters gets deformed by higher-order interactions, this will also result in a change in the Fronsdal-equation, possibly leading to a different solution for the parameter f.

Consistency of the theory also requires that the relative coupling constant of the

2-2-2 vertex and the 3-3-2 vertex is chosen to be $\frac{g_{222}}{g_{332}} = 3$. The calculations involving matter predict this value as well.

It turns out that in the commutator of two spin-3-deformations it is necessary to allow for zero-order gauge transformations with a field-dependent parameter on the RHS of (5.9). This phenomenon also appears in the matter calculations. It would be interesting to know if this structure is also present in the cases involving fields with higher spin values.

further research and applications

An immediate application of the methods developed in this work is the construction of Prokushkin-Vasiliev-theory [83] in the metric-like formalism. This theory contains one field for each higher-spin value and will be built out of the vertices in (2.13). One can then use the formulae (3.19)-(3.24) and (4.27)-(4.29) for these vertices to calculate the first-order gauge deformations and the zero-order gauge brackets. The consistency of the theory is then proven in the exact same way as was done for spin-3-gravity in this work.

When investigating the global symmetry algebra, one can again use the Killing-tensors (1.24). While the spin-indices will increase, in three dimensions these tensors can still be at most linear in x. Hence, the generalisation of the dualisation (5.11) can be used to replace the parameters of symmetry type (s-1,1) by fully symmetric rank-(s-1) tensors, which significantly simplifies the analysis of the consistency conditions. Also, since one can use at most two copies of a building block in the construction of DDI's, increasing the number of spin-indices does not lead to any new identities. Therefore the DDI's relevant for the anlysis of the higher-spin cases are the same that have been calculated in this work, multiplied by some contractions of the partial derivatives (1.2) and the auxilliary vectors a_i. The only novelty that is expected to appear in the construction of Prokushkin-Vasiliev-theory is that the mass m of the scalar field has to be fixed by the parameter λ of the higher-spin algebra $hs(\lambda)$.

Examples of other theories that can now be constructed using the same methods are parity-odd theories built out of the vertices given in [51] and colored higher-spin theories such as [92, 93], containing more than one copy of each higher-spin field, therefore allowing for vertices of the type (2.15).

Appendices

Appendix A

FCGA for two spin-3 parameters, $\Phi, \tilde{\Phi}$ spin-2

The first double deformation:

$$
\begin{aligned}
\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi)) = \bar\epsilon_{2,\ \mu}^{\alpha\beta} a^\mu [& +12(3-15d)(2-f)\bar\epsilon_{1,\alpha\beta}{}^{\gamma}\Phi_{\gamma\nu}a^\nu \\
& +24(3-15d)(2-f)\bar\epsilon_{1,\alpha\nu}{}^{\gamma}a^\nu\Phi_{\gamma\beta} \\
& -24(3-15d)\bar\epsilon^{\gamma}_{1,\nu\rho}a^\nu x^\rho\partial_\gamma\Phi_{\alpha\beta} - 48(3-15d)\bar\epsilon^{\gamma}_{1,\alpha\rho}x^\rho\partial_\gamma\Phi_{\beta\nu}a^\nu \\
& +24f(3-15d)\bar\epsilon^{\gamma}_{1,\alpha\rho}x^\rho\partial_\beta\Phi_{\gamma\nu}a^\nu + 24f(3-15d)\bar\epsilon^{\gamma}_{1,\alpha\rho}x^\rho\partial_\nu\Phi_{\beta\gamma}a^\nu \\
& +24f(3-15d)\bar\epsilon^{\gamma}_{1,\nu\rho}a^\nu x^\rho\partial_\alpha\Phi_{\beta\gamma}] \\
+\bar\epsilon_{2,\ \rho}^{\alpha\beta} x^\rho [& +(180(2-f)(1-d)+72f)\bar\epsilon_{1,\alpha\beta}{}^{\gamma}\partial_\mu\Phi_{\gamma\nu}a^\mu a^\nu \\
& -(180f(1-d)+72(2-f))\bar\epsilon_{1,\mu\nu}{}^{\gamma}a^\mu a^\nu\partial_\alpha\Phi_{\beta\gamma} \\
& +(360f(1-d)-144f)\bar\epsilon^{\gamma}_{1,\nu\sigma}a^\nu x^\sigma\partial_\alpha\partial_\mu\Phi_{\beta\gamma}a^\mu \\
& +(360f(1-d)-144f)\bar\epsilon^{\gamma}_{1,\alpha\sigma}x^\sigma\partial_\beta\partial_\mu\Phi_{\gamma\nu}a^\mu a^\nu \\
& +360(2-f)(1-d)\bar\epsilon_{1,\alpha\nu}{}^{\gamma}a^\nu\partial_\mu\Phi_{\beta\gamma}a^\mu \\
& -360(1-d)\bar\epsilon^{\gamma}_{1,\nu\mu}a^\nu a^\mu\partial_\gamma\Phi_{\beta\alpha} \\
& -360(1-d)\bar\epsilon^{\gamma}_{1,\nu\sigma}a^\nu x^\sigma\partial_\mu\partial_\gamma\Phi_{\beta\alpha}a^\mu - 720(1-d)\bar\epsilon^{\gamma}_{1,\alpha\mu}a^\mu\partial_\gamma\Phi_{\beta\nu}a^\nu \\
& -720(1-d)\bar\epsilon^{\gamma}_{1,\alpha\sigma}x^\sigma\partial_\mu\partial_\gamma\Phi_{\beta\nu}a^\mu a^\nu \\
& +360f(1-d)\bar\epsilon^{\gamma}_{1,\alpha\mu}a^\mu\partial_\beta\Phi_{\gamma\nu}a^\nu \\
& +360f(1-d)\bar\epsilon^{\gamma}_{1,\alpha\mu}a^\mu\partial_\nu\Phi_{\gamma\beta}a^\nu + 360f(1-d)\bar\epsilon^{\gamma}_{1,\alpha\sigma}x^\sigma\partial_\mu\partial_\nu\Phi_{\gamma\beta}a^\mu a^\nu \\
& -144(2-f)\bar\epsilon_{1,\beta\mu}{}^{\gamma}a^\mu\partial_\alpha\Phi_{\gamma\nu}a^\nu + 144\bar\epsilon^{\gamma}_{1,\alpha\beta}\partial_\gamma\Phi_{\mu\nu}a^\mu a^\nu \\
& +288\bar\epsilon^{\gamma}_{1,\mu\alpha}a^\mu\partial_\gamma\Phi_{\beta\nu}a^\nu + 144\bar\epsilon^{\gamma}_{1,\beta\sigma}x^\sigma\partial_\alpha\partial_\gamma\Phi_{\mu\nu}a^\mu a^\nu \\
& +288\bar\epsilon^{\gamma}_{1,\mu\sigma}a^\mu x^\sigma\partial_\alpha\partial_\gamma\Phi_{\beta\nu}a^\nu - 144f\bar\epsilon^{\gamma}_{1,\mu\alpha}a^\mu\partial_\beta\Phi_{\gamma\nu}a^\nu \\
& -144f\bar\epsilon^{\gamma}_{1,\mu\alpha}a^\mu\partial_\nu\Phi_{\gamma\beta}a^\nu - 144f\bar\epsilon^{\gamma}_{1,\mu\sigma}a^\mu x^\sigma\partial_\alpha\partial_\beta\Phi_{\gamma\nu}a^\nu] \\
+\bar\epsilon_{2,\mu\nu}{}^{\alpha}[& +(48f\cdot d_5 + 80w_1\cdot d_5 - 16w_1(\frac{3}{2}-\frac{15}{2}d))\bar\epsilon_{1,\ \rho}^{\gamma\beta}x^\rho\partial_\alpha\Phi_{\gamma\beta}
\end{aligned}
$$

$$
\begin{aligned}
&+(48f\cdot d_5-96d_5+80w_2\cdot d_5-16w_2(\frac{3}{2}-\frac{15}{2}d))\bar{\epsilon}_{1,\ \rho}^{\gamma\delta}x^\rho\partial_\gamma\Phi_{\delta\alpha}\\
&+(-24d_5(2-f)+80w_3\cdot d_5-16w_3(\frac{3}{2}-\frac{15}{2}d))\bar{\epsilon}_{1,\ \alpha}^{\gamma\beta}\Phi_{\gamma\beta}]\\
+\bar{\epsilon}_{2,\mu\nu\rho}[&+(24f\cdot d_4-24d_4(2-f)+48d_4+80w_1\cdot d_4\\
&-40w_2\cdot d_4+80w_3\cdot d_4+16w_1\cdot d_1\\
&-8w_2\cdot d_1+16w_3\cdot d_1)\bar{\epsilon}_1^{\beta\gamma\alpha}\partial_\alpha\Phi_{\beta\gamma}]\\
+\bar{\epsilon}_{2,\mu\rho}^{\alpha}a^\mu x^\rho[&+(240w_1(1-d)-96w_1+48f\cdot d_3\\
&+80w_1\cdot d_3+6f\cdot d_2+80w_1\cdot d_2+4w_1\cdot d_1)\bar{\epsilon}_{1,\ \sigma}^{\gamma\delta}x^\sigma\partial_\alpha\partial_\nu\Phi_{\gamma\delta}a^\nu\\
&-48d_1\bar{\epsilon}_{1,\ \sigma}^{\gamma\beta}x^\sigma\partial_\beta\partial_\gamma\Phi_{\alpha\nu}a^\nu\\
&+(48f\cdot d_2-96d_2+80w_2\cdot d_2\\
&-96w_2+24f\cdot d_1+16w_2\cdot d_1)\bar{\epsilon}_{1,\ \sigma}^{\gamma\delta}x^\sigma\partial_\alpha\partial_\gamma\Phi_{\delta\nu}a^\nu\\
&+(240w_2(1-d)+48f\cdot d_3-96d_3\\
&+80w_2\cdot d_3+24f\cdot d_1+16w_2\cdot d_1)\bar{\epsilon}_{1,\ \sigma}^{\gamma\delta}x^\sigma\partial_\gamma\partial_\nu\Phi_{\alpha\delta}a^\nu\\
&+(48f\cdot d_2+80w_1\cdot d_2-24d_3(2-f)+80w_3\cdot d_3\\
&-96w_1+240w_3(1-d)-12f\cdot d_1\\
&+16w_1\cdot d_1+2w_3\cdot d_1)\bar{\epsilon}_{1,\ \alpha}^{\gamma\delta}\partial_\nu\Phi_{\gamma\delta}a^\nu\\
&+(48f\cdot d_2-96d_2+80w_2\cdot d_2-96w_2+16w_2\cdot d_1)\bar{\epsilon}_{1,\ \beta}^{\gamma\delta}\partial_\gamma\Phi_{\delta\nu}a^\nu\\
&+24f\cdot d_1\bar{\epsilon}_{1,\alpha}^{\gamma\ \ \beta}\partial_\beta\Phi_{\gamma\nu}a^\nu+(24d_1(2-f)-48d_1)\bar{\epsilon}_{1,\alpha}^{\gamma\ \ \beta}\partial_\gamma\Phi_{\beta\nu}a^\nu\\
&+(48f\cdot d_3+80w_1\cdot d_3-24d_2(2-f)+80w_3\cdot d_2\\
&+240w_1(1-d)-96w_3-12f\cdot d_1+16w_1\cdot d_1\\
&+16w_3\cdot d_1)\bar{\epsilon}_{1,\ \nu}^{\gamma\delta}a^\nu\partial_\alpha\Phi_{\gamma\delta}\\
&+(240w_2(1-d)-96d_3+80w_2\cdot d_3\\
&+16w_2\cdot d_1+48f\cdot d_3)\bar{\epsilon}_{1,\ \nu}^{\gamma\delta}a^\nu\partial_\gamma\Phi_{\delta\alpha}\\
&+24f\cdot d_1\bar{\epsilon}_{1,\nu}^{\gamma\ \ \beta}a^\nu\partial_\beta\Phi_{\gamma\alpha}+(24d_1(2-f)-48d_1)\bar{\epsilon}_{1,\nu}^{\gamma\ \ \beta}a^\nu\partial_\gamma\Phi_{\beta\alpha}]
\end{aligned}
\tag{A.1}
$$

The second double deformation:

$$
\begin{aligned}
\delta_{\bar{\epsilon}_1}^{(1)}(\delta_{\bar{\epsilon}_2}^{(1)}(\Phi))=\bar{\epsilon}_{1,\ \mu}^{\alpha\beta}a^\mu[&+12(3-15d)(2-f)\bar{\epsilon}_{2,\alpha\beta}^{\ \ \ \ \gamma}\Phi_{\gamma\nu}a^\nu\\
&+24(3-15d)(2-f)\bar{\epsilon}_{2,\alpha\nu}^{\ \ \ \ \gamma}a^\nu\Phi_{\gamma\beta}\\
&-24(3-15d)\bar{\epsilon}_{2,\nu\rho}^{\gamma}a^\nu x^\rho\partial_\gamma\Phi_{\alpha\beta}-48(3-15d)\bar{\epsilon}_{2,\alpha\rho}^{\gamma}x^\rho\partial_\gamma\Phi_{\beta\nu}a^\nu\\
&+24f(3-15d)\bar{\epsilon}_{2,\alpha\rho}^{\gamma}x^\rho\partial_\beta\Phi_{\gamma\nu}a^\nu+24f(3-15d)\bar{\epsilon}_{2,\alpha\rho}^{\gamma}x^\rho\partial_\nu\Phi_{\beta\gamma}a^\nu
\end{aligned}
$$

$$
\begin{aligned}
&+24f(3-15d)\bar{\epsilon}^{\gamma}_{2,\nu\rho}a^{\nu}x^{\rho}\partial_{\alpha}\Phi_{\beta\gamma}] \\
+\bar{\epsilon}_{1,\ \rho}^{\alpha\beta}x^{\rho}[&+(180(2-f)(1-d)+72f)\bar{\epsilon}_{2,\alpha\beta}{}^{\gamma}\partial_{\mu}\Phi_{\gamma\nu}a^{\mu}a^{\nu} \\
&-(180f(1-d)+72(2-f))\bar{\epsilon}_{2,\mu\nu}{}^{\gamma}a^{\mu}a^{\nu}\partial_{\alpha}\Phi_{\beta\gamma} \\
&+(360f(1-d)-144f)\bar{\epsilon}^{\gamma}_{2,\nu\sigma}a^{\nu}x^{\sigma}\partial_{\alpha}\partial_{\mu}\Phi_{\beta\gamma}a^{\mu} \\
&+(360f(1-d)-144f)\bar{\epsilon}^{\gamma}_{2,\alpha\sigma}x^{\sigma}\partial_{\beta}\partial_{\mu}\Phi_{\gamma\nu}a^{\mu}a^{\nu} \\
&+360(2-f)(1-d)\bar{\epsilon}_{2,\alpha\nu}{}^{\gamma}a^{\nu}\partial_{\mu}\Phi_{\beta\gamma}a^{\mu} \\
&-360(1-d)\bar{\epsilon}^{\gamma}_{2,\nu\mu}a^{\nu}a^{\mu}\partial_{\gamma}\Phi_{\beta\alpha} \\
&-360(1-d)\bar{\epsilon}^{\gamma}_{2,\nu\sigma}a^{\nu}x^{\sigma}\partial_{\mu}\partial_{\gamma}\Phi_{\beta\alpha}a^{\mu}-720(1-d)\bar{\epsilon}^{\gamma}_{2,\alpha\mu}a^{\mu}\partial_{\gamma}\Phi_{\beta\nu}a^{\nu} \\
&-720(1-d)\bar{\epsilon}^{\gamma}_{2,\alpha\sigma}x^{\sigma}\partial_{\mu}\partial_{\gamma}\Phi_{\beta\nu}a^{\mu}a^{\nu} \\
&+360f(1-d)\bar{\epsilon}^{\gamma}_{2,\alpha\mu}a^{\mu}\partial_{\beta}\Phi_{\gamma\nu}a^{\nu} \\
&+360f(1-d)\bar{\epsilon}^{\gamma}_{2,\alpha\mu}a^{\mu}\partial_{\nu}\Phi_{\gamma\beta}a^{\nu}+360f(1-d)\bar{\epsilon}^{\gamma}_{2,\alpha\sigma}x^{\sigma}\partial_{\mu}\partial_{\nu}\Phi_{\gamma\beta}a^{\mu}a^{\nu} \\
&-144(2-f)\bar{\epsilon}_{2,\beta\mu}{}^{\gamma}a^{\mu}\partial_{\alpha}\Phi_{\gamma\nu}a^{\nu}+144\bar{\epsilon}^{\gamma}_{2,\alpha\beta}\partial_{\gamma}\Phi_{\mu\nu}a^{\mu}a^{\nu} \\
&+288\bar{\epsilon}^{\gamma}_{2,\mu\alpha}a^{\mu}\partial_{\gamma}\Phi_{\beta\nu}a^{\nu}+144\bar{\epsilon}^{\gamma}_{2,\beta\sigma}x^{\sigma}\partial_{\alpha}\partial_{\gamma}\Phi_{\mu\nu}a^{\mu}a^{\nu} \\
&+288\bar{\epsilon}^{\gamma}_{2,\mu\sigma}a^{\mu}x^{\sigma}\partial_{\alpha}\partial_{\gamma}\Phi_{\beta\nu}a^{\nu}-144f\bar{\epsilon}^{\gamma}_{2,\mu\alpha}a^{\mu}\partial_{\beta}\Phi_{\gamma\nu}a^{\nu} \\
&-144f\bar{\epsilon}^{\gamma}_{2,\mu\alpha}a^{\mu}\partial_{\nu}\Phi_{\gamma\beta}a^{\nu}-144f\bar{\epsilon}^{\gamma}_{2,\mu\sigma}a^{\mu}x^{\sigma}\partial_{\alpha}\partial_{\beta}\Phi_{\gamma\nu}a^{\nu}] \\
+\bar{\epsilon}_{1,\mu\nu}{}^{\alpha}[&+(48f\cdot d_5+80w_1\cdot d_5-16w_1(\frac{3}{2}-\frac{15}{2}d))\bar{\epsilon}_{2,\ \rho}^{\gamma\beta}x^{\rho}\partial_{\alpha}\Phi_{\gamma\beta} \\
&+(48f\cdot d_5-96d_5+80w_2\cdot d_5-16w_2(\frac{3}{2}-\frac{15}{2}d))\bar{\epsilon}_{2,\ \rho}^{\gamma\delta}x^{\rho}\partial_{\gamma}\Phi_{\delta\alpha} \\
&+(-24d_5(2-f)+80w_3\cdot d_5-16w_3(\frac{3}{2}-\frac{15}{2}d))\bar{\epsilon}_{2,\ \alpha}^{\gamma\beta}\Phi_{\gamma\beta}] \\
+\bar{\epsilon}_{1,\mu\nu\rho}[&+(24f\cdot d_4-24d_4(2-f)+48d_4+80w_1\cdot d_4 \\
&-40w_2\cdot d_4+80w_3\cdot d_4+16w_1\cdot d_1 \\
&-8w_2\cdot d_1+16w_3\cdot d_1)\bar{\epsilon}_2^{\beta\gamma\alpha}\partial_{\alpha}\Phi_{\beta\gamma}] \\
+\bar{\epsilon}^{\alpha}_{1,\mu\rho}a^{\mu}x^{\rho}[&+(240w_1(1-d)-96w_1+48f\cdot d_3 \\
&+80w_1\cdot d_3+6f\cdot d_2+80w_1\cdot d_2+4w_1\cdot d_1)\bar{\epsilon}_{2,\ \sigma}^{\gamma\delta}x^{\sigma}\partial_{\alpha}\partial_{\nu}\Phi_{\gamma\delta}a^{\nu} \\
&-48d_1\bar{\epsilon}_{2,\ \sigma}^{\gamma\beta}x^{\sigma}\partial_{\beta}\partial_{\gamma}\Phi_{\alpha\nu}a^{\nu} \\
&+(48f\cdot d_2-96d_2+80w_2\cdot d_2 \\
&-96w_2+24f\cdot d_1+16w_2\cdot d_1)\bar{\epsilon}_{2,\ \sigma}^{\gamma\delta}x^{\sigma}\partial_{\alpha}\partial_{\gamma}\Phi_{\delta\nu}a^{\nu} \\
&+(240w_2(1-d)+48f\cdot d_3-96d_3 \\
&+80w_2\cdot d_3+24f\cdot d_1+16w_2\cdot d_1)\bar{\epsilon}_{2,\ \sigma}^{\gamma\delta}x^{\sigma}\partial_{\gamma}\partial_{\nu}\Phi_{\alpha\delta}a^{\nu} \\
&+(48f\cdot d_2+80w_1\cdot d_2-24d_3(2-f)+80w_3\cdot d_3 \\
&-96w_1+240w_3(1-d)-12f\cdot d_1
\end{aligned}
$$

$$
\begin{aligned}
&+16w_1\cdot d_1+2w_3\cdot d_1)\bar{\epsilon}_{2,\ \alpha}^{\gamma\delta}\partial_\nu\Phi_{\gamma\delta}a^\nu\\
&+(48f\cdot d_2-96d_2+80w_2\cdot d_2-96w_2+16w_2\cdot d_1)\bar{\epsilon}_{2,\ \beta}^{\gamma\delta}\partial_\gamma\Phi_{\delta\nu}a^\nu\\
&+24f\cdot d_1\bar{\epsilon}_{2,\alpha}^{\gamma\ \ \beta}\partial_\beta\Phi_{\gamma\nu}a^\nu+(24d_1(2-f)-48d_1)\bar{\epsilon}_{2,\alpha}^{\gamma\ \ \beta}\partial_\gamma\Phi_{\beta\nu}a^\nu\\
&+(48f\cdot d_3+80w_1\cdot d_3-24d_2(2-f)+80w_3\cdot d_2\\
&+240w_1(1-d)-96w_3-12f\cdot d_1+16w_1\cdot d_1\\
&+16w_3\cdot d_1)\bar{\epsilon}_{2,\ \nu}^{\gamma\delta}a^\nu\partial_\alpha\Phi_{\gamma\delta}\\
&+(240w_2(1-d)-96d_3+80w_2\cdot d_3\\
&+16w_2\cdot d_1+48f\cdot d_3)\bar{\epsilon}_{2,\ \nu}^{\gamma\delta}a^\nu\partial_\gamma\Phi_{\delta\alpha}\\
&+24f\cdot d_1\bar{\epsilon}_{2,\nu}^{\gamma\ \ \beta}a^\nu\partial_\beta\Phi_{\gamma\alpha}+(24d_1(2-f)-48d_1)\bar{\epsilon}_{2,\nu}^{\gamma\ \ \beta}a^\nu\partial_\gamma\Phi_{\beta\alpha}]
\end{aligned}
\tag{A.2}
$$

The full FCGA:

$$
\begin{aligned}
&\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi))-\delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi))+\delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi)=\\
&+\bar{\epsilon}^{\gamma}_{1,\beta\sigma}x^\sigma\bar{\epsilon}_{2,\ \rho}^{\alpha\beta}x^\rho\partial_\alpha\partial_\mu\Phi_{\gamma\nu}a^\mu a^\nu[+45(1-d)f-18f+90(1-d)]\\
&+\bar{\epsilon}^{\gamma}_{1,\mu\sigma}a^\mu x^\sigma\bar{\epsilon}_{2,\ \rho}^{\alpha\beta}x^\rho\partial_\alpha\partial_\beta\Phi_{\gamma\nu}a^\nu[+6d_1-18f]\\
&+\bar{\epsilon}^{\gamma\delta}_{1,\ \sigma}x^\sigma\bar{\epsilon}^{\alpha}_{2,\mu\rho}a^\mu x^\rho\partial_\alpha\partial_\nu\Phi_{\gamma\delta}a^\nu[+6d_2f+6d_3f+45(1-d)+4d_1w_1\\
&\qquad+10d_2w_1+10d_3w_1+30(1-d)w_1-12w_1]\\
&+\bar{\epsilon}^{\gamma\delta}_{1,\ \sigma}x^\sigma\bar{\epsilon}^{\alpha}_{2,\mu\rho}a^\mu x^\rho\partial_\alpha\partial_\gamma\Phi_{\delta\nu}a^\nu[+6fd_2+10w_2d_2-12d_2+3d_1f\\
&\qquad+2d_1w_2-12w_2-36]\\
&+\bar{\epsilon}^{\gamma\delta}_{1,\ \sigma}x^\sigma\bar{\epsilon}^{\alpha}_{2,\mu\rho}a^\mu x^\rho\partial_\gamma\partial_\nu\Phi_{\delta\alpha}a^\nu[+6fd_3+10w_2d_3-12d_3+3d_1f\\
&\qquad+18f-45f(1-d)+2d_1w_2+30(1-d)w_2]\\
&+\bar{\epsilon}^{\gamma}_{1,\alpha\sigma}x^\sigma\bar{\epsilon}_{2,\ \rho}^{\alpha\beta}x^\rho\partial_\gamma\partial_\mu\Phi_{\beta\nu}a^\mu a^\nu[-45(1-d)f+18f-90(1-d)]\\
&+\bar{\epsilon}^{\gamma}_{1,\mu\sigma}a^\mu x^\sigma\bar{\epsilon}_{2,\ \rho}^{\alpha\beta}x^\rho\partial_\alpha\partial_\gamma\Phi_{\beta\nu}a^\nu[-6fd_2-10w_2d_2+12d_2-3d_1f\\
&\qquad-2d_1w_2+12w_2+36]\\
&+\bar{\epsilon}^{\gamma\beta}_{1,\ \sigma}x^\sigma\bar{\epsilon}^{\alpha}_{2,\mu\rho}a^\mu x^\rho\partial_\beta\partial_\gamma\Phi_{\alpha\nu}a^\nu[+18f-6d_1]\\
&+\bar{\epsilon}^{\gamma}_{1,\nu\sigma}a^\nu x^\sigma\bar{\epsilon}_{2,\ \rho}^{\alpha\beta}x^\rho\partial_\gamma\partial_\mu\Phi_{\alpha\beta}a^\mu[-6d_2f-6d_3f-45(1-d)-4d_1w_1\\
&\qquad-10d_2w_1-10d_3w_1-30(1-d)w_1+12w_1]\\
&+\bar{\epsilon}_{1,\alpha\beta}^{\quad\ \gamma}\bar{\epsilon}_{2,\ \rho}^{\alpha\beta}x^\rho\partial_\gamma\Phi_{\mu\nu}a^\mu a^\nu[+\frac{9}{2}r\tilde{g}_{222}-9]\\
&+\bar{\epsilon}_{1,\ \rho}^{\alpha\beta}x^\rho\bar{\epsilon}_{2,\alpha\beta}^{\quad\ \gamma}\partial_\gamma\Phi_{\mu\nu}a^\mu a^\nu[+9-\frac{9}{2}r\tilde{g}_{222}]
\end{aligned}
$$

$$
\begin{aligned}
&+\bar\epsilon_{1,\alpha\beta}{}^{\gamma}\bar\epsilon_{2,\ \rho}^{\alpha\beta}x^\rho\partial_\mu\Phi_{\gamma\nu}a^\mu a^\nu[+9f+\frac{45}{2}(2-f)(1-d)-\frac{9}{2}r(1-c)\tilde g_{222}]\\
&+\bar\epsilon_{1,\ \rho}^{\alpha\beta}x^\rho\bar\epsilon_{2,\alpha\beta}{}^{\gamma}\partial_\mu\Phi_{\gamma\nu}a^\mu a^\nu[-9f-\frac{45}{2}(2-f)(1-d)+\frac{9}{2}r(1-c)\tilde g_{222}]\\
&+\bar\epsilon_{1,\ \rho}^{\alpha\beta}x^\rho\bar\epsilon_{2,\mu\alpha}^{\gamma}a^\mu\partial_\beta\Phi_{\gamma\nu}a^\nu[+18f]\\
&+\bar\epsilon_{1,\mu\rho}^{\alpha}a^\mu x^\rho\bar\epsilon_{2,\alpha}^{\gamma\ \ \beta}\partial_\beta\Phi_{\gamma\nu}a^\nu[-3d_1f]\\
&+\bar\epsilon_{1,\alpha}^{\gamma\ \ \beta}\bar\epsilon_{2,\mu\rho}^{\alpha}a^\mu x^\rho\partial_\beta\Phi_{\gamma\nu}a^\nu[+3d_1f]\\
&+\bar\epsilon_{1,\mu\alpha}^{\gamma}a^\mu\bar\epsilon_{2,\ \rho}^{\alpha\beta}x^\rho\partial_\beta\Phi_{\gamma\nu}a^\nu[-18f]\\
&+\bar\epsilon_{1,\alpha\mu}^{\gamma}a^\mu\bar\epsilon_{2,\ \rho}^{\alpha\beta}x^\rho\partial_\beta\Phi_{\gamma\nu}a^\nu[+12(\frac{3}{2}-\frac{15}{2}d)+45f(1-d)]\\
&+\bar\epsilon_{1,\alpha\rho}^{\gamma}x^\rho\bar\epsilon_{2,\mu}^{\alpha\beta}a^\mu\partial_\beta\Phi_{\gamma\nu}a^\nu[+6(\frac{3}{2}-\frac{15}{2}d)f+90(1-d)]\\
&+\bar\epsilon_{1,\ \rho}^{\alpha\beta}x^\rho\bar\epsilon_{2,\beta\mu}{}^{\gamma}a^\mu\partial_\alpha\Phi_{\gamma\nu}a^\nu[+18(2-f)]\\
&+\bar\epsilon_{1,\beta\mu}{}^{\gamma}a^\mu\bar\epsilon_{2,\ \rho}^{\alpha\beta}x^\rho\partial_\alpha\Phi_{\gamma\nu}a^\nu[-18(2-f)]\\
&+\bar\epsilon_{1,\mu\rho}^{\alpha}a^\mu x^\rho\bar\epsilon_{2,\ \alpha}^{\gamma\delta}\partial_\nu\Phi_{\gamma\delta}a^\nu[+3d_3(2-f)+\frac{3}{2}d_1f-6d_2f-2d_1w_1\\
&\qquad -10d_2w_1+12w_1-2d_1w_3-10d_3w_3-30(1-d)w_3]\\
&+\bar\epsilon_{1,\ \alpha}^{\gamma\delta}\bar\epsilon_{2,\mu\rho}^{\alpha}a^\mu x^\rho\partial_\nu\Phi_{\gamma\delta}a^\nu[-3d_3(2-f)+6d_2f+2d_1w_1+10d_2w_1\\
&\qquad -12w_1+2d_1w_3+10d_3w_3+30(1-d)w_3-\frac{3}{2}d_1f]\\
&+\bar\epsilon_{1,\ \nu}^{\gamma\delta}a^\nu\bar\epsilon_{2,\mu\rho}^{\alpha}a^\mu x^\rho\partial_\alpha\Phi_{\gamma\delta}[+6(\frac{3}{2}-\frac{15}{2}d)-3d_2(2-f)+6d_3f+2d_1w_1\\
&\qquad +10d_3w_1+30(1-d)w_1+2d_1w_3\\
&\qquad +10d_2w_3-12w_3-\frac{3}{2}d_1f]\\
&+\bar\epsilon_{1,\alpha\mu}^{\gamma}a^\mu\bar\epsilon_{2,\ \rho}^{\alpha\beta}x^\rho\partial_\nu\Phi_{\gamma\beta}a^\nu[+45f(1-d)-6f(\frac{3}{2}-\frac{15}{2}d)]\\
&+\bar\epsilon_{1,\ \rho}^{\gamma\beta}x^\rho\bar\epsilon_{2,\mu\nu}{}^{\alpha}a^\mu a^\nu\partial_\alpha\Phi_{\gamma\beta}[+6d_5f-2(\frac{3}{2}-\frac{15}{2}d)w_1+10d_5w_1-\frac{45}{2}(1-d)]\\
&+\bar\epsilon_{1,\nu\rho}^{\gamma}a^\nu x^\rho\bar\epsilon_{2,\mu}^{\alpha\beta}a^\mu\partial_\alpha\Phi_{\gamma\beta}[-6fd_3-10w_2d_3+12d_3+6(\frac{3}{2}-\frac{15}{2}d)f\\
&\qquad -2d_1w_2-30(1-d)w_2]\\
&+\bar\epsilon_{1,\mu\rho}^{\alpha}a^\mu x^\rho\bar\epsilon_{2,\nu}^{\gamma\ \ \beta}a^\nu\partial_\beta\Phi_{\gamma\alpha}[-3d_1f]\\
&+\bar\epsilon_{1,\nu}^{\gamma\ \ \beta}a^\nu\bar\epsilon_{2,\mu\rho}^{\alpha}a^\mu x^\rho\partial_\beta\Phi_{\gamma\alpha}[+3d_1f]\\
&+\bar\epsilon_{1,\mu\rho}^{\alpha}a^\mu x^\rho\bar\epsilon_{2,\ \alpha}^{\gamma\delta}\partial_\gamma\Phi_{\delta\nu}a^\nu[-6fd_2-10w_2d_2+12d_2-2d_1w_2+12w_2]\\
&+\bar\epsilon_{1,\ \alpha}^{\gamma\delta}\bar\epsilon_{2,\mu\rho}^{\alpha}a^\mu x^\rho\partial_\gamma\Phi_{\delta\nu}a^\nu[+6fd_2+10w_2d_2-12d_2+2d_1w_2-12w_2]
\end{aligned}
$$

$$
\begin{aligned}
&+\bar\epsilon_{1,\ \rho}^{\gamma\delta} x^\rho \bar\epsilon_{2,\mu\nu}{}^{\alpha} a^\mu a^\nu \partial_\gamma \Phi_{\delta\alpha}[+6fd_5+10w_2d_5-12d_5+9(2-f) \\
&\qquad +\frac{45}{2}f(1-d)-2(\frac{3}{2}-\frac{15}{2}d)w_2] \\
&+\bar\epsilon_{1,\ \nu}^{\gamma\delta} a^\nu \bar\epsilon_{2,\mu\rho}^{\alpha} a^\mu x^\rho \partial_\gamma \Phi_{\delta\alpha}[+6fd_3+10w_2d_3-12d_3-6f(\frac{3}{2}-\frac{15}{2}d) \\
&\qquad +2d_1w_2+30(1-d)w_2] \\
&+\bar\epsilon_{1,\ \rho}^{\alpha\beta} x^\rho \bar\epsilon_{2,\mu\alpha}^{\gamma} a^\mu \partial_\gamma \Phi_{\beta\nu} a^\nu[-36] \\
&+\bar\epsilon_{1,\mu\rho}^{\alpha} a^\mu x^\rho \bar\epsilon_{2,\alpha}^{\gamma\ \ \beta} \partial_\gamma \Phi_{\beta\nu} a^\nu[+6d_1-3d_1(2-f)] \\
&+\bar\epsilon_{1,\alpha}^{\gamma\ \ \beta} \bar\epsilon_{2,\mu\rho}^{\alpha} a^\mu x^\rho \partial_\gamma \Phi_{\beta\nu} a^\nu[+3d_1(2-f)-6d_1] \\
&+\bar\epsilon_{1,\mu\alpha}^{\gamma} a^\mu \bar\epsilon_{2,\ \rho}^{\alpha\beta} x^\rho \partial_\gamma \Phi_{\beta\nu} a^\nu[+36] \\
&+\bar\epsilon_{1,\alpha\mu}^{\gamma} a^\mu \bar\epsilon_{2,\ \rho}^{\alpha\beta} x^\rho \partial_\gamma \Phi_{\beta\nu} a^\nu[-6(\frac{3}{2}-\frac{15}{2}d)f-90(1-d)] \\
&+\bar\epsilon_{1,\alpha\rho}^{\gamma} x^\rho \bar\epsilon_{2,\mu}^{\alpha\beta} a^\mu \partial_\gamma \Phi_{\beta\nu} a^\nu[-12(\frac{3}{2}-\frac{15}{2}d)-45f(1-d)] \\
&+\bar\epsilon_{1,\mu\rho}^{\alpha} a^\mu x^\rho \bar\epsilon_{2,\nu}^{\gamma\ \ \beta} a^\nu \partial_\gamma \Phi_{\beta\alpha}[+6d_1-3d_1(2-f)] \\
&+\bar\epsilon_{1,\nu}^{\gamma\ \ \beta} a^\nu \bar\epsilon_{2,\mu\rho}^{\alpha} a^\mu x^\rho \partial_\gamma \Phi_{\beta\alpha}[+3d_1(2-f)-6d_1] \\
&+\bar\epsilon_{1,\mu\nu}{}^{\gamma} a^\mu a^\nu \bar\epsilon_{2,\ \rho}^{\alpha\beta} x^\rho \partial_\gamma \Phi_{\alpha\beta}[-6d_5f+\frac{45}{2}(1-d)+2(\frac{3}{2}-\frac{15}{2}d)w_1-10d_5w_1] \\
&+\bar\epsilon_{1,\nu\rho}^{\gamma} a^\nu x^\rho \bar\epsilon_{2,\mu}^{\alpha\beta} a^\mu \partial_\gamma \Phi_{\alpha\beta}[-6(\frac{3}{2}-\frac{15}{2}d)+3d_2(2-f)+\frac{3}{2}d_1f-6d_3f-2d_1w_1 \\
&\qquad -10d_3w_1-30(1-d)w_1-2d_1w_3-10d_2w_3+12w_3] \\
&+\bar\epsilon_{1,\alpha\beta}{}^{\gamma} \bar\epsilon_{2,\mu}^{\alpha\beta} a^\mu \Phi_{\gamma\nu} a^\nu[+3(\frac{3}{2}-\frac{15}{2}d)(2-f)+\frac{9}{2}r(c+1)\tilde g_{222}] \\
&+\bar\epsilon_{1,\mu}^{\alpha\beta} a^\mu \bar\epsilon_{2,\alpha\beta}{}^{\gamma} \Phi_{\gamma\nu} a^\nu[-3(\frac{3}{2}-\frac{15}{2}d)(2-f)-\frac{9}{2}r(c+1)\tilde g_{222}] \\
&+\bar\epsilon_{1,\alpha\nu}{}^{\gamma} a^\nu \bar\epsilon_{2,\mu}^{\alpha\beta} a^\mu \Phi_{\gamma\beta}[+6(\frac{3}{2}-\frac{15}{2}d)(2-f)] \\
&+\bar\epsilon_{1,\mu}^{\alpha\beta} a^\mu \bar\epsilon_{2,\alpha\nu}{}^{\gamma} a^\nu \Phi_{\gamma\beta}[-6(2-f)(\frac{3}{2}-\frac{15}{2}d)] \\
&+\bar\epsilon_{1,\mu\sigma}^{\gamma} a^\mu x^\sigma \bar\epsilon_{2,\ \rho}^{\alpha\beta} x^\rho \partial_\alpha \partial_\nu \Phi_{\beta\gamma} a^\nu[-6fd_3-10w_2d_3+12d_3-3d_1f \\
&\qquad -18f+45f(1-d)-2d_1w_2-30(1-d)w_2] \\
&+\bar\epsilon_{1,\ \rho}^{\alpha\beta} x^\rho \bar\epsilon_{2,\mu\alpha}^{\gamma} a^\mu \partial_\nu \Phi_{\beta\gamma} a^\nu[+18f] \\
&+\bar\epsilon_{1,\mu\alpha}^{\gamma} a^\mu \bar\epsilon_{2,\ \rho}^{\alpha\beta} x^\rho \partial_\nu \Phi_{\beta\gamma} a^\nu[-18f] \\
&+\bar\epsilon_{1,\alpha\rho}^{\gamma} x^\rho \bar\epsilon_{2,\mu}^{\alpha\beta} a^\mu \partial_\nu \Phi_{\beta\gamma} a^\nu[+6f(\frac{3}{2}-\frac{15}{2}d)-45f(1-d)] \\
&+\bar\epsilon_{1,\alpha\nu}{}^{\gamma} a^\nu \bar\epsilon_{2,\ \rho}^{\alpha\beta} x^\rho \partial_\mu \Phi_{\beta\gamma} a^\mu[+45(2-f)(1-d)]
\end{aligned}
$$

$$
\begin{aligned}
&+\bar{\epsilon}_{1,\ \rho}^{\alpha\beta} x^\rho \bar{\epsilon}_{2,\alpha\nu}{}^{\gamma} a^\nu \partial_\mu \Phi_{\beta\gamma} a^\mu [-45(2-f)(1-d)] \\
&+\bar{\epsilon}_1^{\beta\gamma\alpha} \bar{\epsilon}_{2,\mu\nu\rho} a^\mu a^\nu x^\rho \partial_\alpha \Phi_{\beta\gamma}[-3(2-f)d_4 + 3fd_4 + 10w_1 d_4 - 5w_2 d_4 \\
&\qquad +10w_3 d_4 + 6d_4 + 2d_1 w_1 - d_1 w_2 + 2d_1 w_3] \\
&+\bar{\epsilon}_{1,\mu\nu\rho} a^\mu a^\nu x^\rho \bar{\epsilon}_2^{\beta\gamma\alpha} \partial_\alpha \Phi_{\beta\gamma}[+3(2-f)d_4 - 3fd_4 - 10w_1 d_4 + 5w_2 d_4 - 10w_3 d_4 \\
&\qquad -6d_4 - 2d_1 w_1 + d_1 w_2 - 2d_1 w_3] \\
&+\bar{\epsilon}_{1,\mu\nu}{}^{\gamma} a^\mu a^\nu \bar{\epsilon}_{2,\ \rho}^{\alpha\beta} x^\rho \partial_\alpha \Phi_{\beta\gamma}[-6fd_5 - 10w_2 d_5 + 12d_5 - 9(2-f) \\
&\qquad -\frac{45}{2}f(1-d) + 2(\frac{3}{2} - \frac{15}{2}d)w_2] \\
&+\bar{\epsilon}_{1,\ \alpha}^{\beta\gamma} \bar{\epsilon}_{2,\mu\nu}{}^{\alpha} a^\mu a^\nu \Phi_{\beta\gamma}[-3d_5(2-f) - 2(\frac{3}{2} - \frac{15}{2}d)w_3 + 10d_5 w_3] \\
&+\bar{\epsilon}_{1,\mu\nu}{}^{\alpha} a^\mu a^\nu \bar{\epsilon}_{2,\ \alpha}^{\beta\gamma} \Phi_{\beta\gamma}[+3d_5(2-f) + 2(\frac{3}{2} - \frac{15}{2}d)w_3 - 10d_5 w_3]
\end{aligned} \tag{A.3}
$$

The FCGA, dualised:

$$
\begin{aligned}
&\{\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) - \delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) + \delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi)\}_{dual} = \\
&+A_3 p_2 p_a x^2 z_2 z_3[-6fd_3 - 20w_1 d_3 + 10w_2 d_3 - 12d_3 - 99f + 225fd + 450d \\
&\qquad +60dw_1 - 36w_1 - 4d_1(2w_1+3) - 30dw_2 \\
&\qquad +42w_2 - 2d_2(9f + 10w_1 + 5w_2 - 6) - 414] \\
&+A_2 p_a p_x X_3 z_2 z_3[-60w_1 d - 30w_2 d - 90d + 3f(d_1 + 4d_2 + 15d - 9) + 8d_1 w_1 \\
&\qquad +20d_2 w_1 + 36w_1 + 2d_1 w_2 + 30w_2 \\
&\qquad +2d_3(9f + 10w_1 + 5w_2 - 6) + 90] \\
&+a\cdot x p_2 p_a X_3 z_2 z_3[-60w_1 d + 30w_2 d - 90d - 3f(d_1 - 4d_2 + 15d - 9) + 8d_1 w_1 \\
&\qquad +20d_2 w_1 + 36w_1 - 2d_1 w_2 - 30w_2 \\
&\qquad +2d_3(3f + 10w_1 - 5w_2 + 6) + 90] \\
&+A_2 p_a X_3 z_2 z_3[-12fd_5 - 20w_2 d_5 + 24d_5 - 15d_1 f + 24d_4 f - 243f + 225fd \\
&\qquad -450d + 20d_1 w_1 + 40d_4 w_1 - 60dw_1 + 12w_1 - 4d_1 w_2 - 20d_4 w_2 \\
&\qquad -30dw_2 + 6w_2 + 20d_1 w_3 + 40d_4 w_3 - 120dw_3 + 96w_3 \\
&\qquad +4d_3(6f + 5w_1 + 10w_3 - 6) + 2d_2(15f + 20w_1 + 10w_3 - 6) + 342] \\
&+A_3 p_a p_x X_2 z_2 z_3[+18(-5df + 3f - 10d + 10)] \\
&+A_3 p_a X_2 z_2 z_3[+12d_1 f - 180df + 252f + 9r\tilde{g}_{222} - 9rc\tilde{g}_{222} - 4d_1 w_1 + 24w_1 \\
&\qquad -2d_1 w_2 + 12w_2 - 2d_2(9f + 10w_1 + 5w_2 - 6) - 4d_1 w_3 + 60dw_3 \\
&\qquad -60w_3 - 2d_3(3f + 10w_3 - 6) - 108] \\
&+A_3 a\cdot x p_2 p_x z_2 z_3[+3d_1 f + 2d_1(w_2 + 6) - 12(3f + w_2 + 3) + 2d_2(3f + 5w_2 - 6)]
\end{aligned}
$$

$$
\begin{aligned}
&+A_2A_3p_xz_2z_3[-15d_1f-24d_4f-24d_5f-45df+9f-270d-8d_1w_1\\
&\quad-40d_4w_1-40d_5w_1-60dw_1+12w_1+10d_1w_2+20d_4w_2\\
&\quad-30dw_2+6w_2+4d_2(3f+5w_2-6)+2d_3(3f+5w_2-6)\\
&\quad-8d_1w_3-40d_4w_3+198]\\
&+A_3a\cdot xp_2z_2z_3[+6d_1f+405df-225f+810d-4d_1w_1+60dw_1-60w_1\\
&\quad+2d_1w_2-30dw_2+30w_2-2d_3(3f+10w_1-5w_2+6)\\
&\quad-4d_1w_3+24w_3-2d_2(3f+10w_3-6)-594]\\
&+A_2A_3z_2z_3[+18f(5d-1)-9r\tilde{g}_{222}-9rc\tilde{g}_{222}+12w_3-60d(w_3+3)\\
&\quad-4d_5(3f+10w_3-6)+36]\\
&+A_2A_3p_x^2z_2z_3[-12(w_2+3)+d_1(3f+2w_2)+2d_2(3f+5w_2-6)]\\
&+A_3p_1p_ax^2z_1z_3[+6fd_3+20w_1d_3-10w_2d_3+12d_3+99f-225fd-450d\\
&\quad-60dw_1+36w_1+4d_1(2w_1+3)+30dw_2-42w_2\\
&\quad+2d_2(9f+10w_1+5w_2-6)+414]\\
&+a\cdot xp_1p_aX_3z_1z_3[+60w_1d-30w_2d+90d+3f(d_1-4d_2+15d-9)\\
&\quad-8d_1w_1-20d_2w_1-36w_1+2d_1w_2+30w_2\\
&\quad-2d_3(3f+10w_1-5w_2+6)-90]\\
&+A_3p_ap_xX_1z_1z_3[+18(10(d-1)+f(5d-3))]\\
&+A_3p_aX_1z_1z_3[-12d_1f+180df-252f-9r\tilde{g}_{222}+9rc\tilde{g}_{222}+4d_1w_1\\
&\quad-24w_1+2d_1w_2-12w_2+2d_2(9f+10w_1+5w_2-6)\\
&\quad+4d_1w_3-60dw_3+60w_3+2d_3(3f+10w_3-6)+108]\\
&+A_3a\cdot xp_1p_xz_1z_3[-3d_1f-2d_1(w_2+6)+12(3f+w_2+3)-2d_2(3f+5w_2-6)]\\
&+A_3a\cdot xp_1z_1z_3[-6d_1f-405df+225f-810d+4d_1w_1-60dw_1+60w_1\\
&\quad-2d_1w_2+30dw_2-30w_2+2d_3(3f+10w_1-5w_2+6)\\
&\quad+4d_1w_3-24w_3+2d_2(3f+10w_3-6)+594]\\
&+A_3p_1p_aX_2X_3z_3[+18(10(d-1)+f(5d-3)]\\
&+A_3p_2p_aX_1X_3z_3[+18(-5df+3f-10d+10)]\\
&+A_2A_3p_1p_xX_3z_3[+3d_1f+12(3f-w_2-3)+2d_1(w_2-6)+2d_2(3f+5w_2-6)]\\
&+A_2A_3p_1X_3z_3[-12fd_5-20w_2d_5+24d_5+21d_1f-18f+180fd+180d\\
&\quad+6d_1w_2-60dw_2+12w_2\\
&\quad+4d_2(3f+5w_2-6)+2d_3(3f+5w_2-6)-216]\\
&+A_3^2p_1X_2z_3[+180d-9f(d_1-10d+2)-9r\tilde{g}_{222}-2d_1w_2+12w_2\\
&\quad-2d_2(3f+5w_2-6)-198]\\
&+A_3^2p_2X_1z_3[-180d+9f(d_1-10d+2)+9r\tilde{g}_{222}+2d_1w_2-12w_2\\
&\quad+2d_2(3f+5w_2-6)+198]
\end{aligned}
$$

$$
\begin{aligned}
+A_2p_1p_aX_3^2z_3[&+3f(d_1+15d-9)+2(d_1-15d+15)w_2+2d_3(3f+5w_2-6)]\\
+a\cdot xp_1p_aX_2z_1z_2[&-60w_1d-30w_2d-90d+3f(d_1+4d_2+15d-9)+8d_1w_1\\
&+20d_2w_1+36w_1+2d_1w_2+30w_2\\
&+2d_3(9f+10w_1+5w_2-6)+90]\\
+A_2p_ap_xX_1z_1z_2[&+60w_1d-30w_2d+90d+3f(d_1-4d_2+15d-9)-8d_1w_1\\
&-20d_2w_1-36w_1+2d_1w_2+30w_2\\
&-2d_3(3f+10w_1-5w_2+6)-90]\\
+a\cdot xp_2p_aX_1z_1z_2[&-3f(d_1+4d_2+15d-9)-2d_3(9f+10w_1+5w_2-6)\\
&-2(2(2d_1+5d_2+9)w_1+d_1w_2+15w_2\\
&-15d(2w_1+w_2+3)+45)]\\
+A_2p_aX_1z_1z_2[&+9d_1f-24d_4f+45df+153f+90d-24d_1w_1-40d_4w_1\\
&+60dw_1+12w_1+6d_1w_2+20d_4w_2-30dw_2+30w_2-24d_1w_3\\
&-40d_4w_3+180dw_3-156w_3-2d_2(21f+30w_1+10w_3-6)\\
&-2d_3(12f+10w_1-5w_2+30w_3-12)-18]\\
+A_2a\cdot xp_1z_1z_2[&+3f(2d_1+15d-9)-4d_5(9f+10w_1+5w_2-6)\\
&+6(2w_1+w_2-5d(2w_1+w_2+3)+9)]\\
+A_2p_1p_aX_2X_3z_2[&-3f(d_1+4d_2+15d-9)-2d_3(9f+10w_1+5w_2-6)\\
&-2(2(2d_1+5d_2+9)w_1+d_1w_2\\
&+15w_2-15d(2w_1+w_2+3)+45)]\\
+A_2^2p_1X_3z_2[&+6d_1f-45df+9f+90d-4d_1w_1+60dw_1-60w_1-2d_1w_2\\
&+30dw_2-30w_2-2d_3(9f+10w_1+5w_2-6)-4d_1w_3+24w_3\\
&-2d_2(3f+10w_3-6)-18]\\
+A_3p_2p_aX_1X_2z_2[&+2(3fd_3+10w_1d_3-5w_2d_3+6d_3+36f-90fd-180d\\
&-30dw_1+18w_1+d_1(4w_1+6)+15dw_2-21w_2\\
&+d_2(9f+10w_1+5w_2-6)+162)]\\
+A_2A_3p_1p_xX_2z_2[&+12(-3f+w_2+3)-2d_2(3f+5w_2-6)+d_1(-3f-2w_2+12)]\\
+A_3a\cdot xp_1p_2X_2z_2[&+3d_1f+12(3f-w_2-3)+2d_1(w_2-6)+2d_2(3f+5w_2-6)]\\
+A_2A_3p_1X_2z_2[&-3f(2d_1+15d+27)+4d_5(9f+10w_1+5w_2-6)\\
&-6(2w_1+w_2-5d(2w_1+w_2+3)-27)]\\
+A_2A_3p_2p_xX_1z_2[&-3d_1f-2d_1(w_2+6)+12(3f+w_2+3)-2d_2(3f+5w_2-6)]\\
+A_2A_3p_2X_1z_2[&+9d_1f+24d_4f-180df+198f-360d+12d_1w_1+40d_4w_1\\
&-60dw_1+60w_1-12d_1w_2-20d_4w_2+30dw_2+6w_2\\
&+2d_3(3f+10w_1-5w_2+6)+12d_1w_3+40d_4w_3-24w_3\\
&+d_2(-12f-30w_2+20w_3+24)+72]
\end{aligned}
$$

$$
\begin{aligned}
&+A_3 a\cdot x p_2^2 X_1 z_2[-3d_1 f-2d_1w_2+12w_2-2d_2(3f+5w_2-6)+36]\\
&+A_2p_1p_aX_1X_3z_1[-60w_1d+30w_2d-90d-3f(d_1-4d_2+15d-9)+8d_1w_1\\
&\qquad+20d_2w_1+36w_1-2d_1w_2-30w_2\\
&\qquad+2d_3(3f+10w_1-5w_2+6)+90]\\
&+A_3p_1p_aX_1X_2z_1[-2(3fd_3+10w_1d_3-5w_2d_3+6d_3+36f-90fd-180d\\
&\qquad-30dw_1+18w_1+d_1(4w_1+6)+15dw_2-21w_2\\
&\qquad+d_2(9f+10w_1+5w_2-6)+162)]\\
&+A_3a\cdot xp_1^2X_2z_1[-12(w_2+3)+d_1(3f+2w_2)+2d_2(3f+5w_2-6)]\\
&+A_3a\cdot xp_1p_2X_1z_1[+12(-3f+w_2+3)-2d_2(3f+5w_2-6)+d_1(-3f-2w_2+12)]\\
&+A_2A_3p_1X_1z_1[-3(-90d+3f(d_1-15d+3)+2(d_1-6)w_2\\
&\qquad+2d_2(3f+5w_2-6)+126)]\\
&+A_2A_3p_1^2X_2X_3[-3d_1f-2d_1w_2+12w_2-2d_2(3f+5w_2-6)+36]\\
&+A_2A_3p_1p_2X_1X_3[+3d_1f+2d_1(w_2+6)-12(3f+w_2+3)+2d_2(3f+5w_2-6)]\\
&+A_2p_ap_xX_2z_2^2[-3f(d_1+15d-9)-2(d_1-15d+15)w_2-2d_3(3f+5w_2-6)]\\
&+a\cdot xp_2p_aX_2z_2^2[+3f(d_1+15d-9)+2(d_1-15d+15)w_2+2d_3(3f+5w_2-6)]\\
&+A_2p_aX_2z_2^2[+4d_5(3f+5w_2-6)+6(f(d_1-30d+18)\\
&\qquad-w_2+5d(w_2+9)-39)]\\
&+A_2^2p_xz_2^2[-3f(d_1+15d-3)-2(d_1-15d+15)w_2-2d_3(3f+5w_2-6)]\\
&+A_2a\cdot xp_2z_2^2[-27(f-2)(5d-1)A_2^2z_2^2+(-3f(2d_1-15d+9)\\
&\qquad-4d_5(3f+5w_2-6)-6((5d-1)w_2+6))]\\
&+a\cdot xp_1p_aX_1z_1^2[-3f(d_1+15d-9)-2(d_1-15d+15)w_2-2d_3(3f+5w_2-6)]\\
&+p_ap_xX_3z_1z_3A_1[-3f(d_1+4d_2+15d-9)-2d_3(9f+10w_1+5w_2-6)\\
&\qquad-2(2(2d_1+5d_2+9)w_1+d_1w_2+15w_2\\
&\qquad-15d(2w_1+w_2+3)+45)]\\
&+A_1p_aX_3z_1z_3[+12fd_5+20w_2d_5-24d_5+15d_1f-24d_4f+243f-225fd\\
&\qquad+450d-20d_1w_1-40d_4w_1+60dw_1-12w_1+4d_1w_2\\
&\qquad+20d_4w_2+30dw_2-6w_2-20d_1w_3-40d_4w_3+120dw_3-96w_3\\
&\qquad-4d_3(6f+5w_1+10w_3-6)\\
&\qquad-2d_2(15f+20w_1+10w_3-6)-342]\\
&+A_1A_3p_xz_1z_3[+15d_1f+24d_4f+24d_5f+45df-9f+270d+8d_1w_1\\
&\qquad+40d_4w_1+40d_5w_1+60dw_1-12w_1-10d_1w_2-20d_4w_2\\
&\qquad+30dw_2-6w_2-4d_2(3f+5w_2-6)-2d_3(3f+5w_2-6)\\
&\qquad+8d_1w_3+40d_4w_3-198]\\
&+A_1A_3z_1z_3[+3f(6-30d)+9r\tilde{g}_{222}+9rc\tilde{g}_{222}-12w_3+60d(w_3+3)
\end{aligned}
$$

$$
\begin{aligned}
&+4d_5(3f+10w_3-6)-36] \\
+A_1A_3p_x^2z_1z_3[&-3d_1f-2d_1w_2+12w_2-2d_2(3f+5w_2-6)+36] \\
+A_1A_3p_2p_xX_3z_3[&+12(-3f+w_2+3)-2d_2(3f+5w_2-6)+d_1(-3f-2w_2+12)] \\
+A_1A_3p_2X_3z_3[&+12fd_5+20w_2d_5-24d_5-21d_1f+18f-180fd-180d \\
&-6d_1w_2+60dw_2-12w_2-4d_2(3f+5w_2-6) \\
&-2d_3(3f+5w_2-6)+216] \\
+A_1p_2p_aX_3^2z_3[&-3f(d_1+15d-9)-2(d_1-15d+15)w_2-2d_3(3f+5w_2-6)] \\
+A_1p_ap_xX_2z_1z_2[&-60w_1d+30w_2d-90d-3f(d_1-4d_2+15d-9)+8d_1w_1 \\
&+20d_2w_1+36w_1-2d_1w_2-30w_2 \\
&+2d_3(3f+10w_1-5w_2+6)+90] \\
+A_1p_aX_2z_1z_2[&-9d_1f+24d_4f-45df-153f-90d+24d_1w_1+40d_4w_1 \\
&-60dw_1-12w_1-6d_1w_2-20d_4w_2+30dw_2-30w_2+24d_1w_3 \\
&+40d_4w_3-180dw_3+156w_3+2d_2(21f+30w_1+10w_3-6) \\
&+2d_3(12f+10w_1-5w_2+30w_3-12)+18] \\
+A_1a\cdot xp_2z_1z_2[&-3f(2d_1+15d-9)+4d_5(9f+10w_1+5w_2-6) \\
&-6(2w_1+w_2-5d(2w_1+w_2+3)+9)] \\
+A_1p_2p_aX_2X_3z_2[&+60w_1d-30w_2d+90d+3f(d_1-4d_2+15d-9)-8d_1w_1 \\
&-20d_2w_1-36w_1+2d_1w_2+30w_2 \\
&-2d_3(3f+10w_1-5w_2+6)-90] \\
+A_1A_2p_2X_3z_2[&-24d_4f+45df-9f+90d-12d_1w_1-40d_4w_1+60dw_1 \\
&-60w_1+6d_1w_2+20d_4w_2-30dw_2+30w_2 \\
&-2d_3(3f+10w_1-5w_2+6)-12d_1w_3-40d_4w_3+24w_3 \\
&-2d_2(3f+10w_3-6)-18] \\
+A_1A_3p_2X_2z_2[&-270d+9f(d_1-15d+3)+6(d_1-6)w_2 \\
&+6d_2(3f+5w_2-6)+378] \\
+A_1p_2p_aX_1X_3z_1[&-60w_1d-30w_2d-90d+3f(d_1+4d_2+15d-9)+8d_1w_1 \\
&+20d_2w_1+36w_1+2d_1w_2+30w_2 \\
&+2d_3(9f+10w_1+5w_2-6)+90] \\
+A_1A_2p_1X_3z_1[&+24d_4f-45df+9f-90d+12d_1w_1+40d_4w_1-60dw_1 \\
&+60w_1-6d_1w_2-20d_4w_2+30dw_2-30w_2 \\
&+2d_3(3f+10w_1-5w_2+6)+12d_1w_3+40d_4w_3-24w_3 \\
&+2d_2(3f+10w_3-6)+18] \\
+A_1A_3p_1p_xX_2z_1[&+3d_1f+2d_1(w_2+6)-12(3f+w_2+3)+2d_2(3f+5w_2-6)] \\
+A_1A_3p_1X_2z_1[&-9d_1f-24d_4f+180df-198f+360d-12d_1w_1-40d_4w_1
\end{aligned}
$$

$$
\begin{aligned}
&+60dw_1 - 60w_1 + 12d_1w_2 + 20d_4w_2 - 30dw_2 - 6w_2 \\
&-2d_3(3f + 10w_1 - 5w_2 + 6) - 12d_1w_3 - 40d_4w_3 + 24w_3 \\
&+2d_2(6f + 15w_2 - 10w_3 - 12) - 72] \\
+A_1A_3p_2p_xX_1z_1&[+3d_1f + 12(3f - w_2 - 3) + 2d_1(w_2 - 6) + 2d_2(3f + 5w_2 - 6)] \\
+A_1A_3p_2X_1z_1&[+3f(2d_1 + 15d + 27) - 4d_5(9f + 10w_1 + 5w_2 - 6) \\
&+6(2w_1 + w_2 - 5d(2w_1 + w_2 + 3) - 27)] \\
+A_1A_3p_1p_2X_2X_3&[-3d_1f - 2d_1(w_2 + 6) + 12(3f + w_2 + 3) - 2d_2(3f + 5w_2 - 6)] \\
+A_1A_3p_2^2X_1X_3&[-12(w_2 + 3) + d_1(3f + 2w_2) + 2d_2(3f + 5w_2 - 6)] \\
+A_1p_ap_xX_1z_1^2&[+3f(d_1 + 15d - 9) + 2(d_1 - 15d + 15)w_2 + 2d_3(3f + 5w_2 - 6)] \\
+p_aX_1z_1^2A_1&[-6f(d_1 - 30d + 18) + 6w_2 - 30d(w_2 + 9) \\
&-4d_5(3f + 5w_2 - 6) + 234] \\
+A_1a \cdot xp_1z_1^2&[+3f(2d_1 - 15d + 9) + 4d_5(3f + 5w_2 - 6) + 6(5dw_2 - w_2 + 6)] \\
+A_1^2p_2X_3z_1&[-6d_1f + 45df - 9f - 90d + 4d_1w_1 - 60dw_1 + 60w_1 + 2d_1w_2 \\
&-30dw_2 + 30w_2 + 2d_3(9f + 10w_1 + 5w_2 - 6) + 4d_1w_3 - 24w_3 \\
&+2d_2(3f + 10w_3 - 6) + 18] \\
+A_1^2p_xz_1^2&[+3f(d_1 + 15d - 3) + 2(d_1 - 15d + 15)w_2 + 2d_3(3f + 5w_2 - 6)] \\
+A_1^2z_1^2&[+27(f - 2)(5d - 1)]
\end{aligned}
\tag{A.4}
$$

The FCGA, after rewriting terms containing p_a:

$$
\begin{aligned}
\{\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi)) - \delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi)) + \delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi)\}_{dual} &= \\
+A_3a \cdot xp_2p_xz_2z_3&[+3d_1f + 2d_1(w_2 + 6) - 12(3f + w_2 + 3) + 2d_2(3f + 5w_2 - 6)] \\
+A_2A_3p_xz_2z_3&[-2(3(3d_1 + 2d_3 + 4d_4 + 4d_5 + 3)f \\
&+2(4d_1 + 5d_3 + 10d_4 + 10d_5 + 6)w_1 - 2(2d_1 + 5d_4 - 6)w_2 \\
&+2d_2(5w_1 - 5w_2 + 6) + 4(d_1w_3 + 5d_4w_3 + 9))] \\
+A_3a \cdot xp_2z_2z_3&[+9d_1f + 18d_2f - 54f - 24(w_1 + w_2 - w_3 - 6) \\
&+4d_1(w_1 + w_2 - w_3 + 6) + 4d_2(5w_1 + 5w_2 - 5w_3 - 3)] \\
+A_2A_3z_2z_3&[+2(-9r\tilde g_{222} + 5d_2w_2 + 5d_3w_2 + 10d_5w_2 + 6w_2 - 10d_2w_3 \\
&-10d_3w_3 - 20d_5w_3 - 12w_3 - 2d_4(6f + 10w_1 - 5w_2 + 10w_3) \\
&+d_1(3f - 4w_1 + 4w_2 - 8w_3) + 36)] \\
+A_2A_3p_x^2z_2z_3&[-12(w_2 + 3) + d_1(3f + 2w_2) + 2d_2(3f + 5w_2 - 6)] \\
+A_3a \cdot xp_1p_xz_1z_3&[-3d_1f - 2d_1(w_2 + 6) + 12(3f + w_2 + 3) - 2d_2(3f + 5w_2 - 6)] \\
+A_3a \cdot xp_1z_1z_3&[-9d_1f - 18d_2f + 54f + 24(w_1 + w_2 - w_3 - 6)
\end{aligned}
$$

$$
\begin{aligned}
&-4d_1(w_1+w_2-w_3+6)+4d_2(-5w_1-5w_2+5w_3+3)]\\
+A_2A_3p_1p_xX_3z_3&[+3d_1f+12(3f-w_2-3)+2d_1(w_2-6)+2d_2(3f+5w_2-6)]\\
+A_2A_3p_1X_3z_3&[-12fd_5-20w_2d_5+24d_5+15d_1f+90f\\
&+2d_1w_2-48w_2+4d_2(3f+5w_2-6)-2d_3(3f+5w_2-6)-36]\\
+A_3^2p_1X_2z_3&[-9(d_1-4)f-9r\tilde{g}_{222}-2d_1w_2+12w_2-2d_2(3f+5w_2-6)-18]\\
+A_3^2p_2X_1z_3&[+9(d_1-4)f+9r\tilde{g}_{222}+2d_1w_2-12w_2\\
&+2d_2(3f+5w_2-6)+18]\\
+A_2a\cdot xp_1z_1z_2&[+3d_1f-12d_2f-8d_1w_1-20d_2w_1-24w_1\\
&-2d_1w_2-24w_2-2d_3(9f+10w_1+5w_2-6)\\
&-4d_5(9f+10w_1+5w_2-6)-36]\\
+A_2^2p_1X_3z_2&[+9(d_1-2)f+4(d_1-6)w_1-4d_1w_3+24w_3\\
&+2d_2(3f+10w_1-10w_3+6)+72]\\
+A_2A_3p_1p_xX_2z_2&[+12(-3f+w_2+3)-2d_2(3f+5w_2-6)+d_1(-3f-2w_2+12)]\\
+A_3a\cdot xp_1p_2X_2z_2&[+3d_1f+12(3f-w_2-3)+2d_1(w_2-6)+2d_2(3f+5w_2-6)]\\
+A_2A_3p_1X_2z_2&[-3d_1f+12d_2f-108f+8d_1w_1+20d_2w_1\\
&+24w_1+2d_1w_2+24w_2+2d_3(9f+10w_1+5w_2-6)\\
&+4d_5(9f+10w_1+5w_2-6)+252]\\
+A_2A_3p_2p_xX_1z_2&[-3d_1f-2d_1(w_2+6)+12(3f+w_2+3)-2d_2(3f+5w_2-6)]\\
+A_2A_3p_2X_1z_2&[+9d_1f-30d_2f+24d_4f+126f+40d_4w_1+24w_1-20d_4w_2\\
&+48w_2+40d_4w_3-24w_3+4d_1(w_1-3w_2+3w_3-3)\\
&+4d_2(-5w_1-10w_2+5w_3+9)-252]\\
+A_3a\cdot xp_2^2X_1z_2&[-3d_1f-2d_1w_2+12w_2-2d_2(3f+5w_2-6)+36]\\
+A_3a\cdot xp_1^2X_2z_1&[-12(w_2+3)+d_1(3f+2w_2)+2d_2(3f+5w_2-6)]\\
+A_3a\cdot xp_1p_2X_1z_1&[+12(-3f+w_2+3)-2d_2(3f+5w_2-6)+d_1(-3f-2w_2+12)]\\
+A_2A_3p_1X_1z_1&[-2(-9f-12w_2+d_1(3f+2w_2-6)+2d_2(3f+5w_2-6)+72)]\\
+A_2A_3p_1^2X_2X_3&[-3d_1f-2d_1w_2+12w_2-2d_2(3f+5w_2-6)+36]\\
+A_2A_3p_1p_2X_1X_3&[+3d_1f+2d_1(w_2+6)-12(3f+w_2+3)+2d_2(3f+5w_2-6)]\\
+A_2^2p_xz_2^2&[-18f]\\
+A_2a\cdot xp_2z_2^2&[-9d_1f-2d_1w_2-24w_2-2d_3(3f+5w_2-6)\\
&-4d_5(3f+5w_2-6)-36]\\
+A_2^2z_2^2&[-9d_1f-54f-2d_1w_2-24w_2-2d_3(3f+5w_2-6)\\
&-4d_5(3f+5w_2-6)+180]\\
+A_1A_3p_xz_1z_3&[+6(3d_1+2d_3+4d_4+4d_5+3)f\\
&+4(4d_1+5d_3+10d_4+10d_5+6)w_1-4(2d_1+5d_4-6)w_2
\end{aligned}
$$

$$
\begin{aligned}
&+4d_2(5w_1-5w_2+6)+8(d_1w_3+5d_4w_3+9)]\\
+A_1A_3z_1z_3&[+2(9r\tilde{g}_{222}-5d_2w_2-5d_3w_2-10d_5w_2-6w_2+10d_2w_3+10d_3w_3\\
&+20d_5w_3+12w_3+d_1(-3f+4w_1-4w_2+8w_3)\\
&+2d_4(6f+10w_1-5w_2+10w_3)-36)]\\
+A_1A_3p_x^2z_1z_3&[-3d_1f-2d_1w_2+12w_2-2d_2(3f+5w_2-6)+36]\\
+A_1A_3p_2p_xX_3z_3&[+12(-3f+w_2+3)-2d_2(3f+5w_2-6)+d_1(-3f-2w_2+12)]\\
+A_1A_3p_2X_3z_3&[+12fd_5+20w_2d_5-24d_5-15d_1f-90f\\
&-2d_1w_2+48w_2-4d_2(3f+5w_2-6)+2d_3(3f+5w_2-6)+36]\\
+A_1a\cdot xp_2z_1z_2&[-3d_1f+12d_2f+8d_1w_1+20d_2w_1\\
&+24w_1+2d_1w_2+24w_2+2d_3(9f+10w_1+5w_2-6)\\
&+4d_5(9f+10w_1+5w_2-6)+36]\\
+A_1A_2p_2X_3z_2&[-3(d_1+8d_4-6)f-4(d_1+10d_4+6)w_1+4d_1w_2\\
&+20d_4w_2-12d_1w_3-40d_4w_3+24w_3\\
&+2d_2(3f+10w_1-10w_3+6)+72]\\
+A_1A_3p_2X_2z_2&[+2(-9f-12w_2+d_1(3f+2w_2-6)+2d_2(3f+5w_2-6)+72)]\\
+A_1A_2p_1X_3z_1&[+3(d_1+8d_4-6)f+4((d_1+10d_4+6)w_1-d_1w_2\\
&-5d_4w_2+3d_1w_3+10d_4w_3-6w_3-18)\\
&-2d_2(3f+10w_1-10w_3+6)]\\
+A_1A_3p_1p_xX_2z_1&[+3d_1f+2d_1(w_2+6)-12(3f+w_2+3)+2d_2(3f+5w_2-6)]\\
+A_1A_3p_1X_2z_1&[-9d_1f+30d_2f-6(4d_4+21)f-8(5d_4+3)w_1+20d_4w_2\\
&-48w_2-40d_4w_3+24w_3-4d_1(w_1-3w_2+3w_3-3)\\
&+4d_2(5w_1+10w_2-5w_3-9)+252]\\
+A_1A_3p_2p_xX_1z_1&[+3d_1f+12(3f-w_2-3)+2d_1(w_2-6)+2d_2(3f+5w_2-6)]\\
+A_1A_3p_2X_1z_1&[+3d_1f-12d_2f+108f-8d_1w_1-20d_2w_1-24w_1-2d_1w_2\\
&-24w_2-2d_3(9f+10w_1+5w_2-6)\\
&-4d_5(9f+10w_1+5w_2-6)-252]\\
+A_1A_3p_1p_2X_2X_3&[-3d_1f-2d_1(w_2+6)+12(3f+w_2+3)-2d_2(3f+5w_2-6)]\\
+A_1A_3p_2^2X_1X_3&[-12(w_2+3)+d_1(3f+2w_2)+2d_2(3f+5w_2-6)]\\
+A_1a\cdot xp_1z_1^2&[+9d_1f+2d_1w_2+24w_2+2d_3(3f+5w_2-6)\\
&+4d_5(3f+5w_2-6)+36]\\
+A_1^2p_2X_3z_1&[-9(d_1-2)f-4(d_1-6)w_1+4(d_1-6)w_3\\
&-2d_2(3f+10w_1-10w_3+6)-72]\\
+A_1^2p_xz_1^2&[+18f]\\
+A_1^2z_1^2&[+9d_1f+54f+2d_1w_2+24w_2+2d_3(3f+5w_2-6)
\end{aligned}
$$

$$+4d_5(3f+5w_2-6)-180] \tag{A.5}$$

FCGA for two spin-3 parameters, $\Phi, \tilde{\Phi}$ spin-3

The full FCGA:

$$\begin{aligned}
&\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi))-\delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi))+\delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi)=\\
&+\bar\epsilon^{\alpha}_{1,\mu\rho}a^\mu x^\rho\bar\epsilon^{\beta\gamma}_{2,\ \sigma}x^\sigma\partial_\alpha\partial_\beta\Phi_{\gamma\nu\lambda}a^\nu a^\lambda[-18]+\bar\epsilon^{\beta\gamma}_{1,\ \sigma}x^\sigma\bar\epsilon^{\alpha}_{2,\mu\rho}a^\mu x^\rho\partial_\alpha\partial_\beta\Phi_{\gamma\nu\lambda}a^\nu a^\lambda[+18]\\
&+\bar\epsilon^{\alpha}_{1,\mu\rho}a^\mu x^\rho\bar\epsilon^{\beta\gamma}_{2,\ \sigma}x^\sigma\partial_\beta\partial_\nu\Phi_{\gamma\alpha\lambda}a^\nu a^\lambda[+18f]+\bar\epsilon^{\beta\gamma}_{1,\ \sigma}x^\sigma\bar\epsilon^{\alpha}_{2,\mu\rho}a^\mu x^\rho\partial_\beta\partial_\nu\Phi_{\gamma\alpha\lambda}a^\nu a^\lambda[-18f]\\
&+\bar\epsilon^{\alpha}_{1,\mu\rho}a^\mu x^\rho\bar\epsilon^{\beta\gamma}_{2,\ \sigma}x^\sigma\partial_\alpha\partial_\nu\Phi_{\beta\gamma\lambda}a^\nu a^\lambda[+45(1-d)-\frac{45}{2}f(1-d)]\\
&+\bar\epsilon^{\beta\gamma}_{1,\ \sigma}x^\sigma\bar\epsilon^{\alpha}_{2,\mu\rho}a^\mu x^\rho\partial_\alpha\partial_\nu\Phi_{\beta\gamma\lambda}a^\nu a^\lambda[+\frac{45}{2}f(1-d)-45(1-d)]\\
&+\bar\epsilon^{\alpha}_{1,\mu\rho}a^\mu x^\rho\bar\epsilon^{\beta\gamma}_{2,\ \sigma}x^\sigma\partial_\lambda\partial_\nu\Phi_{\beta\gamma\alpha}a^\nu a^\lambda[-\frac{45}{2}f(1-d)]\\
&+\bar\epsilon^{\beta\gamma}_{1,\ \sigma}x^\sigma\bar\epsilon^{\alpha}_{2,\mu\rho}a^\mu x^\rho\partial_\lambda\partial_\nu\Phi_{\beta\gamma\alpha}a^\nu a^\lambda[+\frac{45}{2}f(1-d)]\\
&\quad+\bar\epsilon_{1,\beta\gamma\rho}x^\rho\bar\epsilon^{\beta\gamma\alpha}_{2}\partial_\alpha\Phi_{\mu\nu\lambda}a^\mu a^\nu a^\lambda[-\frac{9}{2}r]+\bar\epsilon^{\beta\gamma\alpha}_{1}\bar\epsilon_{2,\beta\gamma\rho}x^\rho\partial_\alpha\Phi_{\mu\nu\lambda}a^\mu a^\nu a^\lambda[+\frac{9}{2}r]\\
&\quad+\bar\epsilon^{\alpha}_{1,\mu\rho}a^\mu x^\rho\bar\epsilon^{\beta\gamma}_{2,\ \alpha}\partial_\beta\Phi_{\gamma\nu\lambda}a^\nu a^\lambda[-18]+\bar\epsilon^{\beta\gamma}_{1,\ \alpha}\bar\epsilon^{\alpha}_{2,\mu\rho}a^\mu x^\rho\partial_\beta\Phi_{\gamma\nu\lambda}a^\nu a^\lambda[+18]\\
&\quad+\bar\epsilon^{\beta\gamma}_{1,\ \rho}x^\rho\bar\epsilon_{2,\mu\nu}{}^{\alpha}a^\mu a^\nu\partial_\beta\Phi_{\gamma\alpha\lambda}a^\lambda[-9(2-f)]+\bar\epsilon_{1,\mu\nu}{}^{\alpha}a^\mu a^\nu\bar\epsilon^{\beta\gamma}_{2,\ \rho}x^\rho\partial_\beta\Phi_{\gamma\alpha\lambda}a^\lambda[+9(2-f)]\\
&\quad+\bar\epsilon^{\alpha}_{1,\mu\rho}a^\mu x^\rho\bar\epsilon^{\beta\gamma}_{2,\ \nu}a^\nu\partial_\beta\Phi_{\gamma\alpha\lambda}a^\lambda[+18f]+\bar\epsilon^{\beta\gamma}_{1,\ \nu}a^\nu\bar\epsilon^{\alpha}_{2,\mu\rho}a^\mu x^\rho\partial_\beta\Phi_{\gamma\alpha\lambda}a^\lambda[-18f]\\
&\quad+\bar\epsilon^{\alpha}_{1,\mu\rho}a^\mu x^\rho\bar\epsilon^{\beta\gamma}_{2,\ \alpha}\partial_\nu\Phi_{\beta\gamma\lambda}a^\nu a^\lambda[+45(1-d)-3f(\frac{3}{2}-\frac{15}{2}d)]\\
&\quad+\bar\epsilon^{\beta\gamma}_{1,\ \alpha}\bar\epsilon^{\alpha}_{2,\mu\rho}a^\mu x^\rho\partial_\nu\Phi_{\beta\gamma\lambda}a^\nu a^\lambda[+3f(\frac{3}{2}-\frac{15}{2}d)-45(1-d)]\\
&\quad+\bar\epsilon^{\beta\gamma\alpha}_{1}\bar\epsilon_{2,\mu\nu\rho}a^\mu a^\nu x^\rho\partial_\alpha\Phi_{\beta\gamma\lambda}a^\lambda[+(\frac{3}{2}-\frac{15}{2}d)h_2+\frac{15}{2}(1-d)h_2+3h_2]\\
&\quad+\bar\epsilon^{\beta\gamma}_{1,\ \rho}x^\rho\bar\epsilon_{2,\mu\nu}{}^{\alpha}a^\mu a^\nu\partial_\alpha\Phi_{\beta\gamma\lambda}a^\lambda[+\frac{45}{4}(2-f)(1-d)]\\
&\quad+\bar\epsilon_{1,\mu\nu}{}^{\alpha}a^\mu a^\nu\bar\epsilon^{\beta\gamma}_{2,\ \rho}x^\rho\partial_\alpha\Phi_{\beta\gamma\lambda}a^\lambda[-\frac{45}{4}(2-f)(1-d)]\\
&\quad+\bar\epsilon^{\alpha}_{1,\mu\rho}a^\mu x^\rho\bar\epsilon^{\beta\gamma}_{2,\ \nu}a^\nu\partial_\alpha\Phi_{\beta\gamma\lambda}a^\lambda[+6(\frac{3}{2}-\frac{15}{2}d)-\frac{45}{2}f(1-d)]\\
&\quad+\bar\epsilon_{1,\mu\nu\rho}a^\mu a^\nu x^\rho\bar\epsilon^{\beta\gamma\alpha}_{2}\partial_\alpha\Phi_{\beta\gamma\lambda}a^\lambda[-(\frac{3}{2}-\frac{15}{2}d)h_2-\frac{15}{2}(1-d)h_2-3h_2]\\
&\quad+\bar\epsilon^{\beta\gamma}_{1,\ \nu}a^\nu\bar\epsilon^{\alpha}_{2,\mu\rho}a^\mu x^\rho\partial_\alpha\Phi_{\beta\gamma\lambda}a^\lambda[+\frac{45}{2}f(1-d)-6(\frac{3}{2}-\frac{15}{2}d)]
\end{aligned}$$

$$
\begin{aligned}
&+\bar{\epsilon}_{1,\ \rho}^{\beta\gamma} x^\rho \bar{\epsilon}_{2,\mu\nu}{}^{\alpha} a^\mu a^\nu \partial_\lambda \Phi_{\beta\gamma\alpha} a^\lambda [+\frac{45}{4}(2-f)(1-d)] \\
&+\bar{\epsilon}_{1,\mu\nu}{}^{\alpha} a^\mu a^\nu \bar{\epsilon}_{2,\ \rho}^{\beta\gamma} x^\rho \partial_\lambda \Phi_{\beta\gamma\alpha} a^\lambda [-\frac{45}{4}(2-f)(1-d)] \\
&+\bar{\epsilon}_{1,\mu\rho}^{\alpha} a^\mu x^\rho \bar{\epsilon}_{2,\ \nu}^{\beta\gamma} a^\nu \partial_\lambda \Phi_{\beta\gamma\alpha} a^\lambda [-3(\frac{3}{2}-\frac{15}{2}d)f-\frac{45}{2}(1-d)f] \\
&+\bar{\epsilon}_{1,\ \nu}^{\beta\gamma} a^\nu \bar{\epsilon}_{2,\mu\rho}^{\alpha} a^\mu x^\rho \partial_\lambda \Phi_{\beta\gamma\alpha} a^\lambda [+3(\frac{3}{2}-\frac{15}{2}d)f+\frac{45}{2}(1-d)f] \\
&+\bar{\epsilon}_{1,\ \alpha}^{\beta\gamma} \bar{\epsilon}_{2,\mu\nu}{}^{\alpha} a^\mu a^\nu \Phi_{\beta\gamma\lambda} a^\lambda [+\frac{3}{2}(\frac{3}{2}-\frac{15}{2}d)(2-f)] \\
&+\bar{\epsilon}_{1,\mu\nu}{}^{\alpha} a^\mu a^\nu \bar{\epsilon}_{2,\ \alpha}^{\beta\gamma} \Phi_{\beta\gamma\lambda} a^\lambda [-\frac{3}{2}(2-f)(\frac{3}{2}-\frac{15}{2}d)] \\
&+\bar{\epsilon}_{1,\ \lambda}^{\beta\gamma} a^\lambda \bar{\epsilon}_{2,\mu\nu}{}^{\alpha} a^\mu a^\nu \Phi_{\beta\gamma\alpha} [+\frac{3}{2}(\frac{3}{2}-\frac{15}{2}d)(2-f)] \\
&+\bar{\epsilon}_{1,\mu\nu}{}^{\alpha} a^\mu a^\nu \bar{\epsilon}_{2,\ \lambda}^{\beta\gamma} a^\lambda \Phi_{\beta\gamma\alpha} [-\frac{3}{2}(2-f)(\frac{3}{2}-\frac{15}{2}d)] \\
&+\bar{\epsilon}_{1,\beta\gamma\rho} x^\rho \bar{\epsilon}_2^{\beta\gamma\alpha} \partial_\mu \Phi_{\alpha\nu\lambda} a^\mu a^\nu a^\lambda [+\frac{27}{4}re] + \bar{\epsilon}_1^{\beta\gamma\alpha} \bar{\epsilon}_{2,\beta\gamma\rho} x^\rho \partial_\mu \Phi_{\alpha\nu\lambda} a^\mu a^\nu a^\lambda [-\frac{27}{4}re] \\
&+\bar{\epsilon}_{1,\beta\gamma\mu} a^\mu \bar{\epsilon}_2^{\beta\gamma\alpha} \Phi_{\alpha\nu\lambda} a^\nu a^\lambda [-\frac{27}{4}r(2-e)] + \bar{\epsilon}_1^{\beta\gamma\alpha} \bar{\epsilon}_{2,\beta\gamma\mu} a^\mu \Phi_{\alpha\nu\lambda} a^\nu a^\lambda [+\frac{27}{4}r(2-e)]
\end{aligned}
\tag{A.6}
$$

The FCGA, dualised:

$$
\begin{aligned}
&\{\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) - \delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) + \delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi)\}_{dual} = \\
&+A_3^3 p_2 X_1 z_3[+9(r+2)] + A_3^3 p_1 X_2 z_3[-9(r+2)] \\
&+A_1 A_3^2 p_2^2 X_1 X_3[+18] + A_2 A_3^2 p_1 p_2 X_1 X_3[+18] \\
&+A_2 A_3^2 p_1^2 X_2 X_3[-18] + A_1 A_3^2 p_1 p_2 X_2 X_3[-18] \\
&+A_2 A_3^2 p_1 X_1 z_1[-54] + A_1 A_3^2 p_2 X_1 z_1[-9(f-2)(5d-3)] \\
&+A_3^2 a\cdot x p_1 p_2 X_1 z_1[-18] + A_1 A_3^2 p_2 p_x X_1 z_1[+18] \\
&+A_3^2 a\cdot x p_1^2 X_2 z_1[+18] + A_1 A_3^2 p_1 X_2 z_1[+9f(5d-7)+6(5d-4)(2h_2-3)] \\
&+A_1 A_3^2 p_1 p_x X_2 z_1[+18] + A_3^2 p_1 p_a X_1 X_2 z_1[+9(-10d+f(5d-7)+8)] \\
&+A_3^2 a\cdot x p_2^2 X_1 z_2[-18] + A_2 A_3^2 p_2 X_1 z_2[-9f(5d-7)-6(5d-4)(2h_2-3)] \\
&+A_2 A_3^2 p_2 p_x X_1 z_2[-18] + A_2 A_3^2 p_1 X_2 z_2[+9(f-2)(5d-3)] \\
&+A_1 A_3^2 p_2 X_2 z_2[+54] + A_3^2 a\cdot x p_1 p_2 X_2 z_2[+18] \\
&+A_2 A_3^2 p_1 p_x X_2 z_2[-18] + A_3^2 p_2 p_a X_1 X_2 z_2[+9(10d+f(7-5d)-8)] \\
&+A_2 A_3^2 p_1 X_3 z_3[-36(f-2)] + A_1 A_3^2 p_2 X_3 z_3[+36(f-2)] \\
&+A_2 A_3^2 p_1 p_x X_3 z_3[+18] + A_1 A_3^2 p_2 p_x X_3 z_3[-18]
\end{aligned}
$$

$$+A_1A_3^2p_x^2z_1z_3[-18] + A_1A_3^2z_1z_3[+\frac{9}{2}(-40d + 4f(5d-1) + 6r - 3re + 8)]$$
$$+A_3^2a \cdot xp_1z_1z_3[+9(10d + f(7-5d) - 2)]$$
$$+A_1A_3^2p_xz_1z_3[+9f(5d-3) + 6(8h_2 - 5d(2h_2+3) + 9)]$$
$$+A_3^2a \cdot xp_1p_xz_1z_3[-18] + A_3^2p_aX_1z_1z_3[+90d + f(9-45d) - \frac{27re}{2} - 72]$$
$$+A_3^2p_1p_ax^2z_1z_3[+9(10d + f(7-5d) - 8)] + A_2A_3^2p_x^2z_2z_3[+18]$$
$$+A_2A_3^2z_2z_3[-\frac{9}{2}(-40d + 4f(5d-1) + 6r - 3re + 8)]$$
$$+A_3^2a \cdot xp_2z_2z_3[+9(-10d + f(5d-7) + 2)]$$
$$+A_2A_3^2p_xz_2z_3[+6(-8h_2 + 5d(2h_2+3) - 9) - 9f(5d-3)]$$
$$+A_3^2a \cdot xp_2p_xz_2z_3[+18]$$
$$+A_3^2p_aX_2z_2z_3[-90d + 9f(5d-1) + \frac{27re}{2} + 72]$$
$$+A_3^2p_2p_ax^2z_2z_3[+9(-10d + f(5d-7) + 8)]$$
$$+A_1^2A_3z_1^2[-9(f-2)(5d-1)] + A_1A_3a \cdot xp_1z_1^2[+18(f-2)]$$
$$+fA_1^2A_3p_xz_1^2[-18] + A_1A_3p_aX_1z_1^2[-9(f-2)(5d-3)]$$
$$+A_3a \cdot xp_1p_aX_1z_1^2[+18f] + A_1A_3p_ap_xX_1z_1^2[-18f]$$
$$+A_2^2A_3z_2^2[+9(f-2)(5d-1)] + A_2A_3a \cdot xp_2z_2^2[-18(f-2)]$$
$$+A_2^2A_3p_xz_2^2[+18f] + A_2A_3p_aX_2z_2^2[+9(f-2)(5d-3)]$$
$$+A_3a \cdot xp_2p_aX_2z_2^2[-18f] + A_2A_3p_ap_xX_2z_2^2[+18f]$$
$$+A_1A_2A_3p_1X_3z_1[-60h_2d + 90d - 9f(5d-7) + 48h_2 - 18]$$
$$+A_1^2A_3p_2X_3z_1[+9(10d + f(3-5d) - 2)]$$
$$+A_2A_3p_1p_aX_1X_3z_1[+90(d-1) - 9f(5d-7)]$$
$$+A_1A_3p_2p_aX_1X_3z_1[+90(d-1) - 9f(5d-3)]$$
$$+A_2^2A_3p_1X_3z_2[+9(-10d + f(5d-3) + 2)]$$
$$+A_1A_2A_3p_2X_3z_2[+9f(5d-7) + 6(-8h_2 + 5d(2h_2-3) + 3)]$$
$$+A_2A_3p_1p_aX_2X_3z_2[+9(5df - 3f - 10d + 10)]$$
$$+A_1A_3p_2p_aX_2X_3z_2[+9(5df - 7f - 10d + 10)]$$
$$+A_2A_3a \cdot xp_1z_1z_2[-9(f-2)(5d-3)]$$
$$+A_1A_3a \cdot xp_2z_1z_2[+9(f-2)(5d-3)]$$
$$+A_2A_3p_aX_1z_1z_2[+90f(2d-1) + 12(5d-4)(h_2-6)]$$
$$+A_3a \cdot xp_2p_aX_1z_1z_2[+9(5df - 3f - 10d + 10)]$$
$$+A_2A_3p_ap_xX_1z_1z_2[+9(5df - 7f - 10d + 10)]$$
$$+A_1A_3p_aX_2z_1z_2[-6(15f(2d-1) + 2(5d-4)(h_2-6))]$$

$$
\begin{aligned}
&+A_3 a\cdot x p_1 p_a X_2 z_1 z_2[+90(d-1)-9f(5d-3)]\\
&+A_1 A_3 p_a p_x X_2 z_1 z_2[+90(d-1)-9f(5d-7)]\\
&+A_2 A_3 p_1 p_a X_3^2 z_3[-18f]+A_1 A_3 p_2 p_a X_3^2 z_3[+18f]\\
&+A_1 A_3 p_a X_3 z_1 z_3[+6(3f(15d-8)+10d(h_2-6)-8h_2+42)]\\
&+A_3 a\cdot x p_1 p_a X_3 z_1 z_3[+9(5df-7f-10d+10)]\\
&+A_1 A_3 p_a p_x X_3 z_1 z_3[+9(5df-3f-10d+10)]\\
&+A_2 A_3 p_a X_3 z_2 z_3[-6(3f(15d-8)+10d(h_2-6)-8h_2+42)]\\
&+A_3 a\cdot x p_2 p_a X_3 z_2 z_3[+90(d-1)-9f(5d-7)]\\
&+A_2 A_3 p_a p_x X_3 z_2 z_3[+90(d-1)-9f(5d-3)]\\
&+A_1^2 p_a X_3 z_1^2[+18f(3-5d)]+A_1 p_a^2 X_1 X_3 z_1^2[-45f(d-1)]\\
&+A_2^2 p_a X_3 z_2^2[+18f(5d-3)]+A_2 p_a^2 X_2 X_3 z_2^2[+45f(d-1)]\\
&+A_2 a\cdot x p_a z_1 z_2^2[-45(f-2)(d-1)]+a\cdot x p_a^2 X_2 z_1 z_2^2[-45f(d-1)]\\
&+A_1 a\cdot x p_a z_1^2 z_2[+45(f-2)(d-1)]+a\cdot x p_a^2 X_1 z_1^2 z_2[+45f(d-1)]\\
&+A_1 p_a^2 X_3^2 z_1 z_3[+45f(d-1)]+A_2 p_a^2 X_3^2 z_2 z_3[-45f(d-1)]
\end{aligned}
\tag{A.7}
$$

The FCGA, after rewriting terms containing p_a:

$$
\begin{aligned}
&\{\delta^{(1)}_{\bar\epsilon_2}(\delta^{(1)}_{\bar\epsilon_1}(\Phi))-\delta^{(1)}_{\bar\epsilon_1}(\delta^{(1)}_{\bar\epsilon_2}(\Phi))+\delta^{(1)}_{[[\bar\epsilon_2,\bar\epsilon_1]]}(\Phi)\}_{dual}=\\
&+A_1 A_2 A_3 p_1 X_3 z_1[+(48-60d)h_2+72]+A_1 A_2 A_3 p_2 X_3 z_2[+12((5d-4)h_2-6)]\\
&+A_1 A_3 a\cdot x p_1 z_1^2[-36]+A_1 A_3 a\cdot x p_2 z_1 z_2[-36]\\
&+A_1 A_3^2 z_1 z_3[-60dh_2+27r+48h_2-54]+A_1 A_3^2 p_1 X_2 z_1[+12(5d-4)h_2]\\
&+A_1 A_3^2 p_x z_1 z_3[-12((5d-4)h_2+3)]+A_1 A_3^2 p_1 p_2 X_2 X_3[-18]\\
&+A_1 A_3^2 p_1 p_x X_2 z_1[+18]+A_1 A_3^2 p_2 p_x X_1 z_1[+18]\\
&+A_1 A_3^2 p_2 p_x X_3 z_3[-18]+A_1 A_3^2 p_2^2 X_1 X_3[+18]\\
&+A_1 A_3^2 p_2 X_1 z_1[+36]+A_1 A_3^2 p_2 X_2 z_2[+36]\\
&+A_1^2 A_3 p_2 X_3 z_1[+72]+A_1 A_3^2 p_2 X_3 z_3[-72]\\
&+A_1 A_3^2 p_x^2 z_1 z_3[-18]+A_1^2 A_3 z_1^2[+36]\\
&+A_2 A_3 a\cdot x p_1 z_1 z_2[+36]+A_2 A_3 a\cdot x p_2 z_2^2[+36]\\
&+A_2 A_3^2 z_2 z_3[+60dh_2-27r-48h_2+54]+A_2 A_3^2 p_2 X_1 z_2[+12(4-5d)h_2]\\
&+A_2 A_3^2 p_x z_2 z_3[+12((5d-4)h_2+3)]+A_2 A_3^2 p_1 p_2 X_1 X_3[+18]\\
&+A_2 A_3^2 p_1 p_x X_2 z_2[-18]+A_2 A_3^2 p_1 p_x X_3 z_3[+18]\\
&+A_2 A_3^2 p_1 X_1 z_1[-36]+A_2 A_3^2 p_1^2 X_2 X_3[-18]\\
&+A_2 A_3^2 p_1 X_2 z_2[-36]+A_2^2 A_3 p_1 X_3 z_2[-72]
\end{aligned}
$$

$$
\begin{gathered}
+A_2A_3^2p_1X_3z_3[+72] + A_2A_3^2p_2p_xX_1z_2[-18] \\
+A_2A_3^2p_x^2z_2z_3[+18] + A_2^2A_3z_2^2[-36] \\
+A_3^2a\cdot xp_1p_2X_1z_1[-18] + A_3^2a\cdot xp_1p_2X_2z_2[+18] \\
+A_3^2a\cdot xp_1p_xz_1z_3[-18] + A_3^2a\cdot xp_1^2X_2z_1[+18] \\
+A_3^2a\cdot xp_1z_1z_3[+36] + A_3^2a\cdot xp_2p_xz_2z_3[+18] \\
+A_3^2a\cdot xp_2^2X_1z_2[-18] + A_3^2a\cdot xp_2z_2z_3[-36] \\
+A_3^3p_1X_2z_3[-9(r+2)] + A_3^3p_2X_1z_3[+9(r+2)]
\end{gathered} \tag{A.8}
$$

Appendix B

FCGA for two spin-3 parameters, $\Phi, \tilde{\Phi}$ Maxwell fields

The FCGA, dualised:

$$
\begin{gathered}
\{\delta^{(1)}_{\bar{\epsilon}_2}(\delta^{(1)}_{\bar{\epsilon}_1}(\Phi)) - \delta^{(1)}_{\bar{\epsilon}_1}(\delta^{(1)}_{\bar{\epsilon}_2}(\Phi)) + \delta^{(1)}_{[[\bar{\epsilon}_2,\bar{\epsilon}_1]]}(\Phi)\}_{dual} = \\
+A_2p_1X_1z_1[+2v^2m^2] + A_1p_1X_2z_1[+2v^2m^2] \\
+p_1p_aX_1X_2z_1[+2v^2m^2] + A_2p_2X_1z_2[-2v^2m^2] \\
+A_1p_2X_2z_2[-2v^2m^2] + p_2p_aX_1X_2z_2[-2v^2m^2] \\
+A_3p_2X_1z_3[+3m^2(8(4g+7)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}} - 3)] \\
+A_3p_1X_2z_3[+3m^2(8(4g-11)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}} + 3)] \\
+A_1z_1z_3[+m^2(-2v^2+3v-6((4g+7)t-18)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}})] \\
+a\cdot xp_1z_1z_3[-4v^2m^2] \\
+p_aX_1z_1z_3[-m^2(2v^2-3v+6(4g+7)t\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}})] \\
+p_1p_ax^2z_1z_3[-2v^2m^2] \\
+A_2z_2z_3[+m^2(2v^2-3v-6((4g-11)t+18)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}})] \\
+a\cdot xp_2z_2z_3[+4v^2m^2] \\
+p_aX_2z_2z_3[+m^2(2v^2-3v+6(11-4g)t\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}})] \\
+p_2p_ax^2z_2z_3[+2v^2m^2] \\
+A_3p_1p_2^2X_1[+2v^2+(696-96g)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}} - 27]
\end{gathered}
$$

$$
\begin{aligned}
&+A_3p_1^2p_2X_2[-2v^2-24(4g+25)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}+27]\\
&+A_1p_1p_2^2X_3[+2v^2]+A_2p_1^2p_2X_3[-2v^2]\\
&+p_1p_2^2p_aX_1X_3[+2v^2]+p_1^2p_2p_aX_2X_3[-2v^2]\\
&+A_2p_1^2z_1[+6(v^2-2v+(4g+25)t\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}})]\\
&+A_1p_1p_2z_1[+3(-2v^2+8v+4(4gt-29t+54)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}-9)]\\
&+A_2p_1^2p_xz_1[+2v^2]\\
&+p_1p_2p_aX_1z_1[-6(v^2-4v+2(29-4g)t\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}})]\\
&+p_1^2p_aX_2z_1[+8v^2-12v+6(4g+25)t\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}]\\
&+p_1^2p_ap_xX_2z_1[+2v^2]\\
&+A_1p_2^2z_2[-6(v^2-2v+(29-4g)t\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}})]\\
&+A_2p_1p_2z_2[+3(2v^2-8v+4(4gt+25t-54)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}+9)]\\
&+A_1p_2^2p_xz_2[-2v^2]\\
&+p_2^2p_aX_1z_2[-8v^2+12v+6(4g-29)t\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}]\\
&+p_2^2p_ap_xX_1z_2[-2v^2]\\
&+p_1p_2p_aX_2z_2[+6(v^2-4v+2(4g+25)t\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}})]\\
&+A_3p_1p_2z_3[+48(1-2g)t\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}]\\
&+A_3p_1p_2p_xz_3[+192(2g-1)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}]\\
&+p_1p_2p_aX_3z_3[+48(1-2g)t\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}]\\
&+p_1p_ap_x^2z_1z_3[-2v^2]\\
&+p_1p_az_1z_3[-4v^2-12(8tg-12g-4t+15)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}+9]\\
&+p_1p_ap_xz_1z_3[-8(v^2+6(2g-1)t\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}})]\\
&+p_2p_ap_x^2z_2z_3[+2v^2]\\
&+p_2p_az_2z_3[+4v^2+12(-8tg+12g+4t+3)\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}}-9]\\
&+p_2p_ap_xz_2z_3[+8(v^2+6(1-2g)t\frac{g_{433}}{g_{311}}\frac{g_{411}}{g_{311}})]
\end{aligned}
\tag{B.1}
$$

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