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„Hyperbolicity or non-hyperbolicity of certain one-relator groups"

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Abstract

Das Hauptziel dieser Arbeit ist die Verifikation der Hyperbolizität oder Nicht-Hyperbolizität bestimmter Gruppen mit Präsentationen der Form $\langle a, b \mid r \rangle$ für ein reduziertes, zyklisch reduziertes Wort r der Form $ab^ka^{-1}b^lab^ma^{-1}b^n$ oder $ab^kab^la^{-1}b^ma^{-1}b^n$. Die verwendeten Methoden wurden aus Ivanov-Schupp und Buskin entnommen.

(English Version)
The main aim of this thesis is to verify hyperbolicity or non-hyperbolicity of certain groups with presentations of the form $\langle a, b \mid r \rangle$ for a freely reduced, cyclically reduced word r of the form $ab^ka^{-1}b^lab^ma^{-1}b^n$ or $ab^kab^la^{-1}b^ma^{-1}b^n$. The methods used have been taken from Ivanov-Schupp and Buskin.

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1 Introduction

The main purpose of this thesis is to discuss hyperbolicity or non-hyperbolicity of certain one-relator groups with presentations of the form $G = \langle a, b \mid r \rangle$ with r of the form $r = ab^k ab^l a^{-1} b^m a^{-1} b^n$ with $l, n \neq 0$ or $ab^k a^{-1} b^l ab^m a^{-1} b^n$ with $k, l, m, n \neq 0$. Henceforth, we also refer to relators of this type as relators of type $(1/1/{-1}/{-1}), (1/{-1}/1/{-1})$, respectively. Hyperbolicity is proved using the criterion for hyperbolicity of groups satisfying condition $C(p){+}T(q)$, for $(p, q) \in \{(3, 6), (6, 3), (4, 4)\}$, that is given in Ivanov-Schupp [6]. The concrete method is a method that is used by Buskin [2].
We identify some cases to which the criterion from Ivanov-Schupp does not apply because they satisfy $C(p) + T(q)$ with $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$. Moreover, for some groups, we prove that they satisfy $C(6) + T(3)$, but do not discuss hyperbolicity or non-hyperbolicity.
In this thesis, while the concept of imprimitivity rank is not used in the proofs of the main results, it is mentioned because the only possibly non-hyperbolic one-relator groups are, because of results of B. B. Newman [12] and Linton [8], those for which the defining relator has imprimitivity rank 2.
The cases studied are of interest because Ivanov-Schupp [6] decide the question of hyperbolicity or non-hyperbolicity for G, r of analogous form with at least one and at most three occurences of $a^{\pm 1}$, and Ivanov-Schupp [6] and Buskin [2] treat many of the cases with $a^{\pm 1}$ occurring 4 times in total and at least 3 occurrences of a.
We state the two main results (Proposition 1, Proposition 3) and two more results (Proposition 2, Proposition 4): The first proposition gives a family of hyperbolic one-relator groups of type $(1/{-1}/1/{-1})$. The second proposition consists of some non-hyperbolic examples of groups of type $(1/{-1}/1/{-1})$. The third one yields some hyperbolic examples of type $(1/1/{-1}/{-1})$ as well as families of groups of type $(1/1/{-1}/{-1})$ for which the sufficient direction of the hyperbolicity criterion in Ivanov-Schupp does not apply. The fourth gives families of non-hyperbolic groups of type $(1/1/{-1}/{-1})$:

Proposition 1 (Type $(1/{-1}/1/{-1})$, hyperbolic examples)**.** *Let $G = \langle a, b \mid r \rangle$ with $r = ab^k a^{-1} b^l ab^m a^{-1} b^n$ for $k, l, m, n \neq 0$. If $\frac{|m|}{|k|}, \frac{|l|}{|n|} \notin \{1, 2, \frac{1}{2}\}$ or $k = m$ and $l = n$, then G is hyperbolic.*

Proposition 2 (Type $(1/{-1}/1/{-1})$, non-hyperbolic examples)**.** *Consider the case $r = ab^k a^{-1} b^l ab^m a^{-1} b^n$ with $k, l, m, n \neq 0$. Suppose it is not the case that $(k, l) = (m, n)$. If $1 \in \{\frac{|k|}{|m|}, \frac{|n|}{|l|}\}$, then G is not hyperbolic.*

In Proposition 43 in section 3.2.2, we investigate groups of type $(1/-1/1/-1)$ that have $\frac{|k|}{|m|} \neq 1$ and $\frac{|n|}{|l|} \neq 1$, but for which one or both of $\frac{|k|}{|m|} \neq 1$ and $\frac{|n|}{|l|} \neq 1$ are in $\{2, \frac{1}{2}\}$.

Proposition 3 (Type $(1/1/-1/-1)$). *Consider $G = \langle a, b \mid r \rangle$ with r of the form $r = ab^k ab^l a^{-1} b^m a^{-1} b^n$, with $l, n \neq 0$ and $|k+m| \notin \{0, |l|, |n|\}$. If one of the following holds, then the group is hyperbolic:*

- $k + 2l + m = 0$ *and* $k + n + m - l = 0$
- $k - 2l + m = 0$ *and* $k + m + l - n = 0$

If $k + m - l - n = 0$, then the group is non-hyperbolic. If $k + l + m + n = 0$, then the sufficient criterion for hyperbolicity fails. The same is true if $l = n$. A common non-hyperbolic subcase of these situations is $l = n = -k = -m$. For $l = -n$, the sufficient criterion for hyperbolicity fails.

Proposition 4. *Suppose $G = \langle a, b \mid r \rangle$ with $r = ab^k ab^l a^{-1} b^m a^{-1} b^n$ for $l, n \neq 0$. In the following cases, G is not hyperbolic:*

- $k = -m$
- $k = m = -l = -n$
- $k + m = l + n \neq 0$ *and one of*
  1. $k = l$
  2. $k = n$
  3. $|k + m| \notin \{|n|, |l|\}$.

The aim for Propositions 1 and 3 is to apply the sufficient criterion for hyperbolicity from [6] that is based on the construction of diagrams, maps, respectively, with certain properties to the groups mentioned in these propositions, and give non-hyperbolic sub-examples for some of the cases for which the sufficient criterion fails.
For this paper, we do not decide cases using a computer. In particular, note that we treat some infinite families of one-relator groups.

2 Hyperbolicity, Small Cancellation Conditions $C(p) + T(q)$

In this section, we give definitions, results and ideas that are relevant for the rest of the paper.

Definition 5. A finitely presented group $G = \langle A \mid R \rangle$ is said to be *hyperbolic* if and only if it satisfies a linear isoperimetric inequality, i.e. there exists a constant $L \geq 0$ such that for every $w =_G 1$, one has that

$$\min\{d \mid w \text{ can be written as a product of } d \text{ conjugates of elements of } R^{\pm 1}\} \leq L|w|.$$

[6, p. 1851-1852]
Equivalently, one defines hyperbolicity as follows: A finitely generated group is said to be *hyperbolic* (in the sense of Gromov) if its Cayley graph is a $\delta-$hyperbolic metric space for some $\delta > 0$ [1, Definition III.Γ.2.1, p. 448]

In this thesis, small cancellation conditions of certain types will occur. We introduce them now.

Definition 6. Let F be a free group.

1. A subset $R \subseteq F$ is called *symmetrized* if every $r \in R$ is cyclically reduced and for any $r \in R$, R contains r^{-1} and R contains all cyclically reduced conjugates of r. For a symmetrized subset R of F, a word $b \in F$ is called a piece (with respect to R) if there exist distinct $r_1, r_2 \in R$ and $c_1, c_2 \in F$ such that $r_1 = bc_1$ and $r_2 = bc_2$ and each of bc_1 and bc_2 is freely and cyclically reduced. [10, p. 239-240]

2. A symmetrized subset R of F is said to satisfy *condition* $C(p)$ if no element of R is a product of fewer than p pieces. It satisfies *condition* $T(q)$ if and only if the following holds: if $3 \leq h < q$ and $r_1, ...r_h \in R$ such that none of the pairs $(r_1, r_2), ...(r_{h-1}, r_h), (r_h, r_1)$ are an inverse pair, then at least one of $r_1 r_2, ..., r_{h-1} r_h, r_h r_1$ is reduced without cancellation. [10, p. 240, 241]

Moreover, we define diagrams over groups.

Definition 7. 1. A *map* M is a finite simplicial 2-complex M that is embedded in the plane, connected and simply connected. We call 0-cells of a map *vertices*, 1-cells are called *edges*, 2-cells are called *faces*. The *degree* $d(v)$ of a vertex v is defined to be the number of edges incident with it, counting loops twice. The *degree* $d(\pi)$ of a face π of M is defined to be the number of vertices on the boundary of π that have degree at least 3. Denote by $|\partial\pi|$ the number of edges on $\partial\pi$, which satisfies $d(\pi) \leq |\partial\pi|$. A vertex v of M is said to be *interior* if $v \notin \partial M$, *exterior* otherwise. A face π is said to be *interior* if ∂M and $\partial\pi$ do not share a (non-oriented) edge, *exterior* otherwise. A *submap* of a map M is a subcomplex of M which is itself a map. [6, p. 1853-1854]

2. For a group $G = \langle A \mid R \rangle$ with set of generators A and set of cyclically reduced relators R, a *(van Kampen) diagram* Δ over the presentation $G = \langle A \mid R \rangle$ is a map that is equipped with a labeling function $\Phi : \{e \mid \text{e oriented edge of } \Delta\} \to A \cup A^{-1}$ such that

   a) $\Phi(e^{-1}) = \Phi(e)^{-1}$
   b) If Π is a face in Δ and $\partial\Pi = e_1 \cdots e_l$ is the boundary cycle of Π, then $\Phi(\partial\Pi) = \Phi(e_1) \cdots \Phi(e_l)$ is a cyclic permutation of some $r \in R^\epsilon$, for $\epsilon \in \{-1, 1\}$

   The boundary of a face is equipped either with positive (counterclockwise) or negative orientation.

   A pair Π_1, Π_2 of cells is called a *reducible pair* if their boundaries share a vertex v such that their boundary labels agree if they are read starting at v and according to opposite orientations. If a diagram Δ contains no reducible pairs, then it is said to be *reduced.* A diagram is said to be *minimal* if no diagram with the same boundary label has fewer faces.

See [6, p. 1858] and [2, p. 87].

The small cancellation conditions $C(p)$ and $T(q)$ have geometric consequences: In Definition 7, the number of vertices of degree at least 3 on the boundary of a face is called the degree of the face, while in [10], the degree of a face is defined to be the number of vertices on its boundary, in particular, vertices of degree 2 contribute to the degree. But without loss of generality, interior vertices of a diagram can be assumed to have degree greater or equal to 3. If v is a degree 2 interior vertex and e_1, e_2 are the edges incident with v, then make e_1, e_2 into one edge e, making v an interior point of e. Do this for every interior vertex of degree 2 and extend the labeling function as follows: Allow the labeling function to take values in the nontrivial elements of $F(A)$, set $\Phi(e) = \Phi(e_1)\Phi(e_2)$. [10, p. 242]

Lemma 8 ([10, Lemma V.2.2., p. 242])**.** *Let R be a symmetrized set of elements of a free group F, and let M be a reduced diagram over $G = F/N(R)$.*

1. *If R satisfies $C(p)$, then each interior face D of M has $d(D) \geq p$.*
2. *If R satisfies property $T(q)$, then each interior vertex v of M has $d(v) \geq q$.*

Proof. The first item follows from the fact that labels of interior edges are pieces. The latter can be proven as follows: Let e be an interior edge of M such that e

has label c. There are faces D_1, D_2 of M such that $e \in \partial D_1 \cap \partial D_2$. Thus, the boundaries of D_1, D_2 have labels $r_1, r_2 \in R$ of the form $r_1 = ca, r_2 = c^{-1}b$. As R is symmetrized, $r_2 \in R$ implies $bc^{-1} \in R$, and because M is a reduced diagram, one has that $a \neq b^{-1}$. Thus, c is a piece. If D is an interior face, then the label of ∂D is a product of $d(D)$ pieces. Hence, $C(p)$ implies that $d(D) \geq p$. [10, p. 242]
For the second part, consider an interior vertex v of M with $d(v) = h$, and let $e_1, ..., e_h$ be the oriented edges incident at v. Then for each i (indices modulo h) e_{i+1}, e_i are consecutive edges on the boundary of a region D_i of M. There is a path α_i such that ∂D_i is $e_i^{-1}\alpha_i e_{i+1}$. Let f_i be the label of e_i, a_i the label of α_i, then ∂D_i has label $r_i = f_i^{-1}a_i f_{i+1}$. As M is reduced, one never has $r_i = r_{i+1}^{-1}$, and since each $f_i \neq 1$, there is cancellation in each of $r_1r_2, ..., r_{h-1}r_h, r_hr_1$. Thus, $T(q)$ fails for $q > h$. Thus, $T(q)$ implies $d(v) \geq q$ for every interior vertex v of M. [10, p. 242] □

Now, we introduce the concept of (p, q)-maps:
In Ivanov-Schupp, the following definition is given:

Definition 9 ([6, p. 1854])**.** For $(p, q) \in \{(3, 6), (4, 4), (6, 3)\}$, a map M is called a *(p, q)-map* if and only if $d(\pi) \geq p$ for all interior faces of M and $d(v) \geq q$ for all interior vertices v of M with $d(v) > 2$, and there is no interior vertex of degree one. A diagram over a group G is called a *(p, q)-diagram* if the underlying map is a (p, q)-map.

One can ignore degree 2 interior vertices in this context, without performing surgery as above. Henceforth, we will stick to the convention of Definition 9. We will usually not apply surgery to remove degree 2 interior vertices of a diagram or map.
Accordingly, one has the following definition:

Definition 10 ([6, p. 1854])**.** A group is said to satisfy the *small cancellation condition* $C(p) + T(q)$ if every reduced diagram over the group is a (p, q)-diagram.

Theorem 11 ([6], p. 1852, due to Gersten, Short)**.** *A finitely presented group with a presentation that satisfies $C(p) + T(q)$ with $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$ is hyperbolic.*

A condition $C(p)+T(q)$ for (p, q) one of $(6, 3), (3, 6), (4, 4)$ and no further condition does not yield hyperbolicity of the group: Consider the tesselations of the Euclidean plane $\mathbb{E}^2$ by equilateral triangles ($(3, 6)$), squares ($(4, 4)$) or regular hexagons ($(6, 3)$), respectively. Note that $\mathbb{E}^2$ is not hyperbolic. The tesselations described above have arbitrarily large submaps whose area grows quadratically with respect to the perimeter. (See [1, p. 384], [10, p. 246] for discussion of tesselations of $\mathbb{E}^2$).)
One needs another definition:

Definition 12 ([6, p. 1854])**.** A map M is said to be a *regular (p, q)-map* provided that $d(\pi) = p$ for every interior face of M, $d(v) = q$ for every interior vertex v of M with $d(v) > 2$, and M has no interior vertices of degree 1.

The following theorem is used in Ivanov-Schupp to give a condition that is sufficient for hyperbolicity of a group as in Definition 10:

Theorem 13 ([6, Theorem 1, p. 1854])**.** *Let M be a (p, q)-map, for (p, q) an element of $\{(3, 6), (4, 4), (6, 3)\}$, and let the radius of every regular (p, q)-submap of M be bounded by a constant K. Then there is a constant $L = L(p, q, K)$ such that $|\text{number of faces of } M| \leq L|\partial M|$.*

Idea of proof: First, one shows that without loss of generality:

- $(P1)$ No proper subpath of the boundary ∂M of M bounds a submap of M.
- $(P2)$ There are no vertices of degree less than or equal to 2 in M.

In the proof of the theorem, the number of vertices, edges and faces of M is denoted V, E, F, respectively. The proof starts with writing $2E$ first in terms of degrees of vertices, then in terms of degrees of faces: $2E = \sum_{v \text{ vertex of} M} d(v)$, but also $2E = \sum_{\pi \text{ face of} M} d(\pi) + |\partial M|$. Moreover, one uses the fact that $V - E + F = 1$. Note also that for (p, q) as given, $2\frac{p}{q} + 2 = p$. These 3 observations together with $(P1)$ and the fact that $\frac{p}{q}(q-3) - 1 \leq \frac{1}{2}$ if (p, q) is as in the statement yield a lower bound for $\frac{1}{2}|\partial M|$ in terms of (sums of) degrees of exterior and interior vertices and exterior and interior faces. Now, define a regular interior face of M to be an interior face of M that has degree p and such that all vertices on its boundary have degree q. Call an interior face irregular if it is not regular. The goal is to estimate the number of exterior faces, the number of regular interior faces and the number of irregular interior faces: Consider a regular face π and a shortest path from a vertex in $\partial\pi$ to a vertex that lies on the boundary of the map, on the boundary of an exterior face or on the boundary of an irregular face. The number of terminal vertices of these paths is thus bounded by the sum $\sum_{\pi \text{ exterior face of } M} d(\pi) + \sum_{\pi \text{ irregular face of } M} d(\pi) + |\partial M|$. The length of such a path is less or equal to K. Every vertex on such a path that is not the terminal vertex has degree q. Consider a map f that assigns π to v' for π a regular face and v' a terminal vertex as above. The number of regular faces that are mapped to such a v' is less than or equal to q^K. For exterior vertices, see that the number of exterior vertices is bounded from above by $|\partial M|$. Now, one estimates the number of irregular interior faces. For that, we define angles of faces and a weight function on angles and on faces. Only angles at vertices of degree $> q$ on the boundary of irregular faces are given positive weight. For other angles, the value of the weight function is zero. The weight of a face that is not irregular is zero. The weight of an irregular face is the sum of $d(\pi) - p$ and the sum over all angles of the face. This weight function yields an estimate for the number of irregular faces (use $(P2)$ and the fact that M is a (p, q)-map). These three estimates together give the desired inequality for F. See [6, Proof of Theorem 1, p. 1854-1858].
Ivanov-Schupp state a necessary and sufficient condition for hyperbolicity of a group satisfying a small cancellation condition as in Definition 10:

Theorem 14 ([6, Theorem 2, p. 1859])**.** *A finitely presented group $G = \langle A \mid R \rangle$ satisfying a small cancellation condition $C(p) + T(q)$ for $(p, q) \in \{(3, 6), (4, 4), (6, 3)\}$ is hyperbolic if and only if there is a constant K such that for every minimal diagram Δ over G, the radii of regular (p, q)-submaps of the map associated with Δ by forgetting labels do not exceed K.*

The fact that existence of such K is a condition sufficient for hyperbolicity is obtained directly from Theorem 13, for all minimal diagrams are reduced, and hence regular (p,q)-submaps of maps associated with minimal diagrams by forgetting labels are regular reduced (p,q)-maps. See [6, Proof of Theorem 2, p. 1859].
The following property is used in Ivanov-Schupp ([6, p. 1862]): Let $G = \langle A \mid r \rangle$, for $r = a^{\epsilon_0}B_0a^{\epsilon_1}B_1....a^{\epsilon_{k-1}}B_{k-1}$, where $\epsilon_i \in \{-1,1\}$ and $k \geq 2$, and let B_i, for $i = 0,1,...,k-1$, contain no a or a^{-1}. We say G has property $(P2)$ if the subgroup $\langle B_0, B_1, ..., B_{k-1} \rangle \leq F(A)$ is cyclic and $\langle B_0, B_1, ..., B_{k-1} \rangle \leq \langle B \rangle$, where B is a non-empty cyclically reduced word that is not a proper power in $F(A)$.
Ivanov-Schupp prove the following lemma (see [6, p. 1864-1867]): Assuming r as above has $(P2)$, put $B_i = B^{l_i}, i = 0,1,...,k-1$, and consider the word $\bar{r}$ that one obtains from r by replacing all B^{l_i} by b^{l_i}, where b is a new letter, $b \notin A$. Define $\bar{G} = \langle a,b \mid \bar{r} \rangle$ (Ivanov-Schupp: $\langle a, a^{-1}, b, b^{-1} \mid \bar{r} \rangle$).

Lemma 15 ([6, Lemma 3.3, p. 1864]). *Let $G = \langle A \mid r \rangle$ be a one-relator group such that $r = a^{\epsilon_0}B_0a^{\epsilon_1}B_1....a^{\epsilon_{k-1}}B_{k-1}$ as above. Let r have property $(P2)$. If $\bar{G}$ satisfies a linear isoperimetric inequality, then so does G.*

The following results will be used to prove non-hyperbolicity of a one-relator group in some cases. Note that a one-relator group is torsion-free if the relator is not a proper power. See [7, p. 58], [11, p. 266], and [10, p. 108].
The following lemma is given in Ivanov-Schupp:

Lemma 16 ([6, Lemma 4.1, p. 1867], Gromov). *For any non-trivial element g of a torsion-free hyperbolic group Γ there are unique g_0, m with $g_0 \in \Gamma, g_0$ not a proper power in Γ, $m > 0$, such that $g = g_0^m$.*

Lemma 16 / Gromov's result is used in Ivanov-Schupp to prove the following lemma:

Lemma 17 ([6, Lemma 4.3, p. 1867]). *Let x and y be elements of a torsion-free hyperbolic group Γ, $y \neq 1$, and $xy^kx^{-1} = y^l$ for $k \neq 0$. Then $xyx^{-1} = y$ (and $k = l$).*

see [6, Proof of Lemma 4.3, p. 1867-1868]. Let $x = x_0^{m_0}, y = y_0^{n_0}$ the unique representations as in Lemma 16. One has $(x_0^{m_0}y_0x_0^{-m_0})^{kn_0} = y_0^{ln_0}$, where $x_0^{m_0}y_0x_0^{-m_0}$ and y_0 are not proper powers. Lemma 16 yields one of $x_0^{m_0}y_0x_0^{-m_0} = y_0^{\pm 1}$. If $x_0^{m_0}y_0x_0^{-m_0} = y_0$, then another application of Lemma 16 yields that x_0 and y_0 commute, and $k = l$. If $x_0^{m_0}y_0x_0^{-m_0} = y_0^{-1}$, then $y_0x^2y_0^{-1}x^{-2} = 1$, whence x and y_0 commute by Lemma 16, and thus x_0 and y_0 commute by another application of the same lemma, i.e. $x_0^{m_0}y_0x_0^{-m_0} = y_0$, but in this case, $y_0 = y_0^{-1}$, a contradiction to torsion-freeness of Γ. $\square$

One also has:

Lemma 18 (See [6, Proof of Theorem 3, p. 1874].). *Let Γ be a torsion free hyperbolic group, and let $x, y \neq 1$ be elements of Γ such that there is $k > 0$ with $x^k = y^k$. Then $x = y$.*

Proof. See [6, Proof of Theorem 3, p. 1874, Lemma 4.3, p. 1867], [2, Proof of Lemma 2.7, p. 88].
One has $xy^kx^{-1} = xx^kx^{-1} = x^k = y^k$, hence $xyx^{-1} = y$ by Lemma 17. Thus, $1 = x^ky^{-k} = (xy^{-1})^k$. As the group is torsion-free, this implies that $x = y$. □

Lemma 19. *A hyperbolic group has no subgroup isomorphic to $\mathbb{Z}^2$. ([6, p. 1852, 1867], due to Gromov)*

The ideas from Lemmas 16 and 17 put together yield the following statement:

Lemma 20. *Let Γ be a torsion-free hyperbolic group. Then for every nontrivial element $y \in \Gamma$, there is a unique maximal cyclic subgroup $\langle y_0 \rangle$ of Γ such that any element of the commensurator of $\langle y_0 \rangle$ is an element of $\langle y_0 \rangle$, i.e., $xy_0^kx^{-1} = y_0^l$ for $k, l \neq 0$ implies $x \in \langle y_0 \rangle$, so that the subgroup $\langle y_0 \rangle$ equals its own commensurator.*

Proof. Let $y \in \Gamma, y \neq 1$. Existence of a unique maximal cyclic subgroup $\langle y_0 \rangle \leq \Gamma$ with $y \in \langle y_0 \rangle$ is Lemma 16. Let $x \in \Gamma$ be in the commensurator of $\langle y_0 \rangle$, i.e. $xy_0^kx^{-1} = y_0^l$ for $k, l \neq 0$. Then, by Lemma 17, $k = l$ and x and y_0 commute. Let $\langle x_0 \rangle$ be the maximal cyclic subgroup containing x. By Lemma 17, the fact that x and y_0 commute yields commutativity of x_0 and y_0, hence $\langle x_0, y_0 \rangle$ is an Abelian subgroup of Γ. The only torsion-free Abelian groups that can be generated by 2 elements are $\mathbb{Z}^2$ and $\mathbb{Z}$. By Lemma 19, the hyperbolic group Γ has no subgroup isomorphic to $\mathbb{Z}^2$, so $\langle x_0, y_0 \rangle \cong \mathbb{Z}$. Since $\langle y_0 \rangle$ was maximal cyclic, $\langle x_0, y_0 \rangle$ is a subgroup of $\langle y_0 \rangle$. This implies $x_0 \in \langle y_0 \rangle$, which yields $x \in \langle y_0 \rangle$. [3] □

3 Hyperbolicity or non-hyperbolicity of certain one-relator groups

3.0.1 Imprimitivity rank, Whitehead automorphisms and non-hyperbolic examples

In this section, we introduce the concept of imprimitivity rank.

Definition 21. Let F be a free group. A word w in F is called *primitive* if it belongs to a free generating set of F. Otherwise, it is said to be *imprimitive*. The *imprimitivity rank* $\pi(w)$ is defined to be the number $\min\{rk(H) \mid w \in H \leq F, \text{w not primitive in H}\}$ if such H exists, and ∞ otherwise. See [13, p. 5], [5].

Note [13] uses the term primitivity rank. In Definition 21, we use the convention of [5], who say imprimitivity rank.
This ocurs in the context of hyperbolicity or non-hyperbolicity of one-relator groups: A one-relator group $G = \langle A \mid r \rangle$ cannot be non-hyperbolic unless the defining relator r has imprimitivity rank 2:
An element w of a free group has imprimitivity rank 0 if and only if it is trivial in the free group. The quotient of a free group by the normal closure of an element that is trivial in it is the same free group and hence hyperbolic. [5]
An element of a free group has imprimitivity rank 1 if and only if it is a proper power. Hence, if r has imprimitivity rank 1, then G is hyperbolic by the B. B. Newman Spelling Theorem. See [12, p. 569], [13, p. 6] and [5].
An element of a free group has imprimitivity rank ∞ if and only if it is a primitive element. The quotient of a free group by the normal closure of an element of some free generating set is again a free group, of one lower rank. See [13, p. 6], [11, Theorem N3, p. 167] and [5].
When r has imprimitivity rank ≥ 3, G is hyperbolic. See [8, p. 1] and [9, p. 549].
For imprimitivity rank 2, G can be hyperbolic or non-hyperbolic. A non-hyperbolic example is $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$. See [6, p. 1852].

So if one considers $G = \langle a, b \mid r \rangle$ with r nontrivial in $F(a, b)$ and r not a proper power, then r has imprimitivity rank 2 if it is imprimitive, ∞ otherwise.

To check a word of this form for imprimitivity, first note that for F a free group of finite rank with basis X, $Aut(F)$ is generated by the Whitehead automorphisms. [10, p. 42]

Definition 22. The *Whitehead automorphisms of the first kind* are all automorphisms that are given by a permutation of $X^{\pm 1}$. The *Whitehead automorphisms of the second kind* are defined as follows: Fix x in $X^{\pm 1}$. For any subset $Z \subseteq X^{\pm 1} \backslash \{x, x^{-1}\}$, we define an automorphism as follows: for $y \in X^{\pm 1} \backslash \{x, x^{-1}\}$, y is mapped to itself if $y, y^{-1} \notin Z$, y is mapped to xy if $y \in Z, y^{-1} \notin Z$, y is mapped to yx^{-1} if $y \notin Z, y^{-1} \in Z$, and y is mapped to xyx^{-1} if $y, y^{-1} \in Z$. See [14]. For this formulation of the definition, see [5].

Example 23. The set of Whitehead automorphisms of $F(a,b)$: First kind (I.): the Whitehead automorphisms of $F(a,b)$ of the first kind are the eight automorphisms that are given by a permutation of $\{a^{\pm 1}, b^{\pm 1}\}$.
Second kind (II.):

1. $a \mapsto bab^{-1}, b \mapsto b$
2. $a \mapsto b^{-1}ab, b \mapsto b$
3. $a \mapsto a, b \mapsto aba^{-1}$
4. $a \mapsto a, b \mapsto a^{-1}ba$
5. $a \mapsto ab, b \mapsto b$
6. $a \mapsto ab^{-1}, b \mapsto b$
7. $a \mapsto ba, b \mapsto b$
8. $a \mapsto b^{-1}a, b \mapsto b$
9. $a \mapsto a, b \mapsto ab$
10. $a \mapsto a, b \mapsto a^{-1}b$
11. $a \mapsto a, b \mapsto ba$
12. $a \mapsto a, b \mapsto ba^{-1}$

In order to decide whether or not $w \in F$ is primitive or imprimitive, one can apply Whitehead's algorithm, i.e., apply the Whitehead automorphisms of the seond kind to the word one by one (and always do cyclic reduction after applying a Whitehead automorphism), and if some Whitehead automorphism makes the word shorter, apply the same procedure to the resulting word, iterate until there is no word that is shortened by application of a Whitehead automorphism and cyclic reduction. The word $w \in F$ is primitive if and only if it Whitehead reduces to length 1. Compare [5] and [14].

Lemma 24. *The word $r = ab^k a^{-1} b^l ab^m a^{-1} b^n$ for $k, l, m, n \neq 0$ and $(k, l) \neq (m, n)$ is imprimitive in $F(a, b)$.*

Proof. Apply Whitehead's algorithm to r to show that it does not Whitehead reduce to length 1: Second kind: 1. – 8. Application of the automorphism and cyclic reduction gives r.

9. Application of the automorphism yields $a(ab)^k a^{-1}(ab)^l a(ab)^m a^{-1}(ab)^n$, i.e. $+|k|$ letters and no cancellations for k, $+|l|$ letters and one cancellation (=the cancellation of one pair) for l, $+|m|$ letters and no cancellation for m, and $|n|$ more letters and one cancellation for n. Hence the total change in the number of letters is: $+|k|+|l|+|m|+|n|-4 \geq 0$

10. The automorphism gives $a(a^{-1}b)^k a^{-1}(a^{-1}b)^l a(a^{-1}b)^m a^{-1}(a^{-1}b)^n$, which gives $+|k|$ letters and one cancellation for k, $|l|$ more letters and no cancellation for l, $|m|$ more letters and one cancellation for m, and $+|n|$ letters, but no cancellation for n. Hence as above, the word is not shorter after application of the automorphism and cyclic reduction.

11. This automorphism yields $a(ba)^k a^{-1}(ba)^l a(ba)^m a^{-1}(ba)^n$, which gives $+|k|$ letters and one cancellation for k, $+|l|$ letters and no cancellation for l, $+|m|$ letters and one cancellation for m, and $|n|$ more letters and no cancellation for n. Hence as above, the result of application of the automorphism and cyclic reduction is not shorter than the original word.

12. This automorphism gives $a(ba^{-1})^k a^{-1}(ba^{-1})^l a(ba^{-1})^m a^{-1}(ba^{-1})^n$, which yields $+|k|+|l|+|m|+|n|$ letters and two cancellations, one for l, one for n. As above, one sees that the application of the automorphism and cyclic reduction do not make the word shorter.

   Hence r is Whitehead minimal, and it is of length > 1, hence in particular, r does not Whitehead reduce to length 1.

□

Lemma 25. *The word $r = ab^k ab^l a^{-1} b^m a^{-1} b^n$ for $k, m, l, n \neq 0$ is imprimitive in $F(a, b)$.*

Proof. First, let the exponents k, m have the same sign. Applying the Whitehead automorphism of the first kind defined by $a \mapsto a, b \mapsto b^{-1}$ if necessary, we can assume $k > 0, m > 0$. After applying II. 6. k times, r has form $a^2 b^l a^{-1} b^{m+k} a^{-1} b^n$. Automorphisms II. 1.-8. do not make the word shorter. It remains to show that application of items II. 9.-12. do not make it shorter:
Automorphism II. 9. gives $a^2(ab)^l a^{-1}(ab)^{m+k} a^{-1}(ab)^n$, which gives one cancellation for $m+k$ and one for n, hence there are $m+k+|l|+|n|-4 \geq 0$ more elements.
Similarly, item II. 10. gives $a^2(a^{-1}b)^l a^{-1}(a^{-1}b)^{m+k} a^{-1}(a^{-1}b)^n$, which gives one cancellation for l and no further cancellations, hence the new word is not shorter.
The map II. 11. yields $a^2(ba)^l a^{-1}(ba)^{m+k} a^{-1}(ba)^n$, which gives one cancellation for l and one for the $m+k$ part, hence the word is not shorter after the application of 11.. Finally, II. 12. yields $a^2(ba^{-1})^l a^{-1}(ba^{-1})^{m+k} a^{-1}(ba^{-1})^n$, which yields one cancellation for n and no cancellation for the $m+k$ part (as $k+m > 0$).

Now, suppose k, m have distinct signs. Up to application of Whitehead automorphisms, we may suppose $m < 0$ and $k \geq |m|$, and applying II. 6 $|m|$ times, we obtain $ab^{k+m} ab^l a^{-2} b^n$.

No Whitehead automormphisms reduce the length of this word except in the cases that satisfy $k+m=|l|=|n|=1$, for which II. 10. and II. 12. yield reduction. Now let $k+m=1, |l|=1, |n|=1$. Item II. 10. gives $b^2a^{-3}b$ for $l=1, n=1$ and $b^2a^{-2}b^{-1}a$ for $l=1, n=-1$. For $l=-1, n=1$, it gives $bab^{-1}a^{-2}b$, which is the same as the latter word up to application of Whitehead automorphisms, and for $l=-1, n=-1$, it yields $bab^{-1}a^{-1}b^{-1}a$. All these are Whitehead minimal and do not have length one. Item II. 12. yields $b^2a^{-3}b$ for $l=n=1$, $ab^2a^{-2}b^{-1}$ for $l=1, n=-1$, the values $l=-1, n=1$ give $bab^{-1}a^{-2}b$, and $l=-1, n=-1$ gives $abab^{-1}a^{-1}b^{-1}$. Again, all these are Whitehead minimal and do not have length one. □

Lemma 26. *By similar arguments, we have: If $r=ab^kab^la^{-1}b^ma^{-1}b^n$ with $l,n\neq 0$, then r is Whitehead minimal if $k=m=0$.*

Corollary 27. *The words of Lemmas 24, 25, 26 have imprimitivity rank 2.*

For the imprimitivity rank 2 examples introduced in Lemmas 24, 25, one can detect at least one non-hyperbolic example each, using results from Ivanov-Schupp [6] (see also Buskin [2]):

Lemma 28. *Let $G=\langle a,b \mid r\rangle$ with the defining relator r having imprimitivity rank 2. Then the one-relator group G is not cyclic. In particular, $a,b\neq 1$ in G. Moreover, if G is hyperbolic, then a,b do not commute and do not belong to the same maximal cyclic subgroup.*

Proof. As the Abelianization of G is a one-relator quotient of $\mathbb{Z}^2$ and thus nontrivial, G is nontrivial. The group G does not have torsion because G has torsion if and only if r is a proper power if and only if r has imprimitivity rank 1. See [7, p. 58], [13, p. 6]. Moreover, the group G cannot be isomorphic to $\mathbb{Z}$ because it is free of rank 1 if and only if r is primitive if and only if r has imprimitivity rank ∞. See [10], [13, p. 6], [11, Theorem N3, p. 167].
If G is hyperbolic, then, by Lemma 20, a belongs to a maximal cyclic subgroup $\langle a_0\rangle$ that equals its own commensurator, and b belongs to a maximal cyclic subgroup $\langle b_0\rangle$ with the same property. Suppose a,b commute, then $a,b\in\langle a_0\rangle=\langle b_0\rangle$, and thus the one-relator group G is cyclic, contradiction. □

Lemma 29. *Let $G=\langle a,b \mid r\rangle$ with $r=ab^ka^{-1}b^lab^ma^{-1}b^n$ for $k,l,m,n\neq 0$.If $l=-n$ or $k=-m$, then G is not hyperbolic.*

Proof. If $l=-n$ and $k\neq -m$, then $r=ab^ka^{-1}b^lab^ma^{-1}b^{-l}$, which is conjugate to $b^ka^{-1}b^lab^ma^{-1}b^{-l}a$. Thus, $b^{-k}=a^{-1}b^lab^ma^{-1}b^{-l}a$. If G is hyperbolic, then, by Lemma 17, $b=a^{-1}b^laba^{-1}b^{-l}a$. In particular, $b^k=a^{-1}b^lab^ka^{-1}b^{-l}a$ and thus $1=b^kb^{-k}=a^{-1}b^lab^{k+m}a^{-1}b^{-l}a$, a contradiction to torsion-freeness as $k+m\neq 0$.
The case $l\neq -n$ and $k=-m$ can be handled analogously, and it can be reduced to the previous case by an automorphism of the one-relator group.
If $l=-n$ and $k=-m$, then one has $ab^{-m}a^{-1}b^lab^ma^{-1}=b^l$. If the one-relator group is hyperbolic and $b\in\langle b_0\rangle, b=b_0^{n_0}$ for the unique maximal cyclic subgroup $\langle b_0\rangle$ as in

Lemmas 16 and 20, we get that $ab^{-m}a^{-1} \in \langle b_0 \rangle$. Hence $ab^{-m}a^{-1} \in \langle b_0 \rangle \cap a\langle b_0 \rangle a^{-1}$. As both $\langle b_0 \rangle$ and $a\langle b_0 \rangle a^{-1}$ were maximal cyclic subgroups, one obtains $a\langle b_0 \rangle a^{-1} = \langle b_0 \rangle$. By Lemma 20, this implies $a \in \langle b_0 \rangle$, a contradiction to Lemma 28. $\square$

Lemma 30. *(Proposition 4) Suppose $G = \langle a, b \mid r \rangle$ with $r = ab^k ab^l a^{-1} b^m a^{-1} b^n$ for $l, n \neq 0$. In the following cases, G is not hyperbolic:*

- $k = -m$
- $k = m = -l = -n$
- $k + m = l + n \neq 0$ *and one of*
  1. $k = l$
  2. $k = n$
  3. $|k + m| \notin \{|n|, |l|\}$

The proof will be broken into separate cases.

Lemma 31. *Suppose $G = \langle a, b \mid r \rangle$ for $r = ab^k ab^{-k} a^{-1} b^k a^{-1} b^{-k}$ with $k \neq 0$. Then G is not hyperbolic.*

Proof. Get $b^{-k}ab^kab^{-k}a^{-1}b^ka^{-1} = 1$. If G is hyperbolic and $a \in \langle a_0 \rangle, a = a_0^{m_0}, b \in \langle b_0 \rangle$ as in Lemma 16, then one obtains, by Lemma 20, that $b^{-k}ab^k \in \langle a_0 \rangle$, i.e. $(b^{-k}a_0b^k)^{m_0} = a_0^{n_0}$, which implies commutativity of b^k and a_0 by Lemma 16. Then one obtains commutativity of a_0 and b_0 by another application of Lemma 16, and as $\langle a_0 \rangle$ equals its own commensurator, one obtains $b_0 \in \langle a_0 \rangle$, a contradiction to Lemma 28. $\square$

Lemma 32. *Suppose $G = \langle a, b \mid r \rangle$ for $r = ab^k ab^l a^{-1} b^m a^{-1} b^n, k = -m$ and $l, n \neq 0$. Then G is not hyperbolic.*

Proof. By application of the automorphism given by $a \mapsto ab^m$, this reduces to $a^2b^la^{-2}b^n$ for $l, n \neq 0$, which gives a non-hyperbolic group: If G were hyperbolic, then a and b would commute and belong to the same maximal cyclic subgroup of G, which is not possible by Lemma 28. $\square$

Lemma 33. *Let $G = \langle a, b \mid r \rangle$ for $r = ab^k ab^l a^{-1} b^m a^{-1} b^n$ with $l, n \neq 0, k+m-l-n = 0$ and k, m not both equal to 0 (The case $k = -m = 0$ is a subcase of Lemma 32). Assume that one of the following is satisfied:*

1. $|k + m| \notin \{0, |n|, |l|\}$
2. $k = l$
3. $k = n$

Then G is not hyperbolic.

Proof. First, consider item 1., i.e., assume $k+m=l+n\neq 0, |k+m|\notin \{|n|,|l|\}$. Then up to an automorphism, $r=a^2(b^la^{-1}b^n)^2$, which implies $a^2=(b^{-n}ab^{-l})^2$. Assume the group is hyperbolic. Then by Lemma 18, $a=b^{-n}ab^{-l}$, and Lemma 17 yields $a^{-1}ba=b$, i.e. a and b commute. As above, this would imply, by Lemma 20, that a,b were elements of the same maximal cyclic subgroup, a contradiction, see Lemma 28.
Now consider the subcase $k=l$ (item 2.), which is equivalent to $m=n$ here. This yields $(ab^k)^2=(b^{-m}a^{-1})^2$. Assume the one-relator group is hyperbolic, then one obtains $ab^k=b^{-m}a^{-1}$, which implies $ab^ka^{-1}=b^{-m}$. By Lemma 20, this yields that a and b are elements of the same maximal cyclic subgroup of G, which is a contradiction.
Perform the analogous steps for the subcase $k=n$, which is equivalent to the subcase $m=l$. Moreover, this can be obtained from the previous item by application of an automorphism of the one-relator group. □

3.0.2 Proof of hyperbolicity of some groups with a single defining relator using the sufficient criterion for hyperbolicity from Ivanov-Schupp and failure of this criterion for some one-relator groups

Ivanov-Schupp prove hyperbolicity of certain one-relator groups using their Theorem 1 (Theorem 13 in this paper). In this section, we discuss how the fact that all reduced maps are (p,q)-maps after performance of a certain surgery is proved for certain one-relator groups in Theorem 3 of Ivanov-Schupp and how a common bound on the radii of regular (p,q)-submaps is established there. We also treat some of the ideas stated by Buskin, who analyzes certain groups with presentations of the form $\langle a,b\mid r\rangle, r=ab^kab^lab^ma^{-1}b^n$.
Then, we pass to the proof of Propositions 2 and 1 in this paper, in which ideas from Ivanov-Schupp and Buskin are used. For $G=\langle a,b\mid r\rangle, r=ab^ka^{-1}b^lab^ma^{-1}b^n$, r freely and cyclically reduced, r not a proper power, we identify some cases to which the criterion from Ivanov-Schupp does not apply because they satisfy $C(p)+T(q)$ with $\frac{1}{p}+\frac{1}{q}>\frac{1}{2}$. Moreover, for some groups $G=\langle a,b\mid r\rangle, r=ab^ka^{-1}b^lab^ma^{-1}b^n$, with r freely and cyclically reduced, r not a proper power, we prove that they satisfy $C(6)+T(3)$, but do not discuss hyperbolicity or non-hyperbolicity.
Finally, we prove Proposition 3, using ideas from Ivanov-Schupp, Buskin. Proposition 4 is Lemma 32. Moreover, for $r=ab^kab^la^{-1}b^ma^{-1}b^n$, r freely and cyclically reduced, we give one example (up to symmetry) of a family of groups that have $C(5)+T(3)$, but not $C(6)$.

Definition 34. [6, p. 1860-1861] For a one-relator group $G=\langle A\mid r\rangle$ with r non-empty cyclically reduced word, $a\in A$ a letter such that a and/or its inverse occurs in r, and $r\equiv a^{\epsilon_0}B_0a^{\epsilon_1}B_1...a^{\epsilon_{k-1}}B_{k-1}$, $k\geq 2, \epsilon_i\in\{-1,1\}$, B_i not containing a or a^{-1} for $0\leq i\leq k-1$, and for Δ a diagram over the given presentation, an *a-edge* of Δ is an oriented edge of Δ with label a or a^{-1}.
Let e_0, f_0^{-1} be a-edges on the boundary $\partial\pi_0$ of a face π_0 of a diagram Δ such that the arc $e_0v_0f_0^{-1}$ has the property that there are no a-edges in v_0. Then the *a-star* $St(e_0,f_0)$ defined by the a-edges e_0,f_0 is defined to be the following sequence of a-edges: Assume e_0^{-1} belongs to $\partial\pi_1$ and consider the arc $e_1v_1e_0^{-1}$, v_1 containing no a-edges, of $\partial\pi_1$ (there

are at least two a-edges on $\partial\pi_1$ by the assumption that $k \geq 2$). Assuming $e_1^{-1} \in \partial\pi_2$, we proceed analogously, and after $l-1$ steps, either $e_{l-1} = f_0$ or $e_{l-1} \in \partial\Delta$. In the first case, the a-star $St(e_0, f_0)$ consisting of the oriented a-edges is said to be *interior* and its *label* $\Phi(St(e_0, f_0))$ is $\Phi(St(e_0, f_0)) = \Phi(v_{l-1}v_{l-2}...v_1v_0)$. In the second case, the construction is extended as follows: If $f_0 \in \partial\pi_{-1}$, then take the arc of $\partial\pi_{-1}$ of the form $f_0v_{-1}f_{-1}^{-1}$, for an a-edge f_{-1} and v_{-1} containing no a-edges, continue like that. After several steps, one obtains $f_{-(m-1)}^{-1} \in \partial\Delta$. Then the a-star $St(e_0, f_0)$ consisting of the oriented a-edges $f_{-(m-1)}, f_{-(m-2)}, ..., f_0, e_0, e_1..., e_{l-1}$ is called *exterior* and its *label* $\Phi(St(e_0, f_0))$ is the word $\Phi(St(e_0, f_0)) = \Phi(v_{l-1}...v_1v_0v_{-1}...v_{-(m-1)})$. (Remark: Note that Ivanov-Schupp write labels from r^*, but allow only positive orientation of boundaries of faces)

Ivanov-Schupp and Buskin both use the following technique: Given a reduced diagram Δ over G, contract all edges labelled $b^{\pm 1}$ to a point, omit labels of a-edges and treat the resulting map $\bar{\Delta}$. Then $|\Delta(2)| = |\bar{\Delta}|, |\partial\Delta| \leq |\partial\bar{\Delta}|$. Thus, if $|\bar{\Delta}(2)| \leq L|\partial\bar{\Delta}|$, then one has $|\Delta(2)| = |\bar{\Delta}(2)| \leq L|\partial\bar{\Delta}| \leq L|\partial\Delta|$. See [6, p. 1871] and [2, p. 88].

In this thesis, three cases from Ivanov- Schupp, Theorem 3 ([6, Theorem 3, p. 1852-1853, and Proof of Theorem 3, p. 1870-1879]), are mentioned: Consider the group $G = \langle A \mid r \rangle$, with A a finite alphabet, r a cyclically reduced word in A. We consider the following subcases:

- Part 3, Case 5: $r = aBaCaD$ with B, C, D having no occurrences of $a^{\pm 1}$. By applying an automorphism of the free group on A to r, pass to $a^2CB^{-1}aDB^{-1}$, assume $rank(\langle CB^{-1}, DB^{-1} \rangle) = 1, CB^{-1} = E^{n_1}$ and $DB^{-1} = E^{n_2}$, where $n_1, n_2 \neq 0$ and E is not a proper power in the free group. Without loss of generality, E can be assumed to be cyclically reduced. Let $|n_1| \neq |n_2|, |n_1| \neq |2n_2|, |n_2| \neq |2n_1|$. In this case, the group is hyperbolic.

Now, let $r = aBaCa^{-1}D$ with B, C, D having no occurrences of $a^{\pm 1}$. Pass to $a^2Ba^{-1}C$ via an automorphism of $F(A)$ (the new B, C are $B^{-1}CB, D$, respectively), assume $rank(\langle B, C \rangle) = 1, \langle B, C \rangle \leq \langle E \rangle$, for a nonempty, cyclically reduced word E that is not a proper power in the free group on A, and $B = E^{n_1}$ and $C = E^{n_2}$.

- Part 4, Case 4: Moreover, let $n_1 = 2n_2$ or $n_2 = 2n_1$. In this case, the group is hyperbolic.

- Part 4, Case 5: Let $|n_1| \neq |n_2|, |n_1| \neq |2n_2|, |n_2| \neq |2n_1|$. In this case, the group is hyperbolic.

Note that in particular, E has no occurrences of $a^{\pm 1}$ in the above cases.
Ivanov Schupp, Strategy of proof for Part 3, Case 5, and Part 4, Case 5, of Theorem 3:

1. Starting with a word of (e.g) the form $a^2E^{n_1}a^{-1}E^{n_2}$, E a word as above, reduce to a word of the form $a^2b^{n_1}a^{-1}b^{n_2}$, b a letter. This uses Lemma 16 [6, Lemma 3.3, p. 1864].

2. Consider a reduced diagram Δ over the new group, contract all b-edges and forget the remaining labels to get $\bar{\Delta}$.

3. The map after contraction is a $(3,6)$-map. The 3 part is immediate, as there are 3 occurrences of $a^{\pm 1}$ in the word $a^2 b^{n_1} a^{-1} b^{n_2}$. For the properties of interior vertices of $\bar{\Delta}$, consider a-stars in Δ corresponding to interior vertices in $\bar{\Delta}$. After proving that interior vertices of $\bar{\Delta}$ of degree other than 2 have degree ≥ 6, examine the vertices of degree $q = 6$ in $\bar{\Delta}$, consider angles at them. The case of a regular $(3,6)$-submap of radius $r \geq 2$ is reduced to a regular $(3,6)$-submap that takes the form of a hexagon consisting of $6r^2$ triangles. The analysis of the angles in this hexagon gives a (combinatorial) estimate for the radius. One obtains a common upper bound for the radii of regular (p,q)-maps, and thus, Theorem 13 [6, Theorem 1, p. 1854] and the (in-)equalities relating area and length of boundary of Δ and area and length of boundary of $\bar{\Delta}$ yield hyperbolicity.

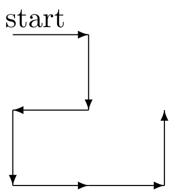
([6, p. 1870-1874, 1879])
Theorem 3, Part 4, case 4 in Ivanov Schupp: This is a hyperbolic case treated similarly to the above cases: The proof of hyperbolicity of the given group, G, is reduced (via Lemma 16 ([6, Lemma 3.3, p. 1864]) and [6, Lemma 3.4, p. 1867], Ivanov-Schupp) to proving hyperbolicity of a different group, G_1. Then, a HNN presentation of G_1 is given, and the proof is reduced to proving hyperbolicity of a subgroup G_0 of G_1. Then, they prove that any reduced diagram Δ over G_0 is a $(3,6)$-diagram up to some surgery that constitutes no loss of generality. (Angles are used in the process of proving $(3,6)$.) Then consider the map $M_{\Delta'}$ obtained from this diagram by omitting degree 2 interior vertices and labels, and consider the angles in it. Similarly to the above, we get an estimate for the radius of any regular submap of $M_{\Delta'}$, which yields the desired inequality for $M_{\Delta'}$ and thus for Δ. ([6, p. 1875-1879])
Details for Theorem 3, Part 4, Case 5: Without loss of generality, $r = a^2 B a^{-1} C$ with $B = E^{n_1}, C = E^{n_2}$, E a nonempty, cyclically reduced word, E not being a proper power, and E containing no $a^{\pm 1}$. By Lemma 16 ([6, Lemma 3.3, p. 1864]), it suffices to show that $\bar{G} = \langle a, b \mid a^2 b^{n_1} a^{-1} b^{n_2} \rangle$ is hyperbolic. Let W be a non-empty word over the generators of $\bar{G}$ such that W equals 1 in $\bar{G}$, but no proper subword of W does. Take a diagram Δ over $\bar{G}$ such that $|\partial\Delta| = W$ and $|\Delta(2)|$ is minimal. Contraction of all b-edges yields a map $\bar{\Delta}$ as above (disregard labels of remaining edges). The map $\bar{\Delta}$ is a $(3,6)$-map: Any face of $\bar{\Delta}$ has degree 3. Now take an interior vertex v in $\bar{\Delta}$, consider all consecutive (list edges in positive direction) oriented edges with terminal vertex v, $e_0, ...e_{l-1}$. Consider the preimages in Δ, denote them $e_0, ...e_{l-1}$ as well. The e_i form an interior a-star in Δ. Since Δ is reduced and the label of the interior $a-$star equals 1 in the free group, get $l \geq 4$: $l = 1$ contradicts the fact that the relator is cyclically reduced; $l = 2$ gives a contradiction to Δ reduced; $l = 3$ yields only impossible cases because $b^{\pm n_1}$, which lies between a and a^{-1} in (a conjugate of) r or its inverse, cannot be neighbor to $b^{\pm n_2}$, which lies between a^{-1} and a in (a conjugate of) r or its inverse, b^{n_i} cannot be neighbor to b^{-n_i} (because we construct reduced maps), and $n_i \neq 0$, i.e. $b^{n_i} \neq 1$.
Consider $l = 4$. This implies one of $n_1 + n_2 = 0$ or $n_1 - n_2 = 0$, $l = 5$ would imply one

of $2n_1 + n_2 = 0, -2n_1 + n_2 = 0, 2n_2 + n_1 = 0, -2n_2 + n_1 = 0$: Analyze interior a-stars for which $l = 4$ or 5. As above, note that $b^{\pm n_1}$ cannot be neighbor to $b^{\pm n_2}$ due to the structure of the relator. Recall that we are constructing reduced maps yields that b^{n_i} cannot be neighbor to b^{-n_i}, and note that $n_i \neq 0$, i.e. $b^{n_i} \neq 1$. Hence, one can prove the above implications by sketching a polygon with 4 or 5 sides, respectively, such that sides carry labels in $\{0, \pm n_1, \pm n_2\}$ (whose elements stand for $\{\pm b^0 = \pm 1, b^{\pm n_1}, b^{\pm n_2}\}$), $n_i, -n_i$ never label adjacent sides and a side labelled $\pm n_i$ is never adjacent to a side labelled $-n_j$ or n_j, $j \neq i$. For $l = 4$, such a polygon has two occurrences of 0, one occurrence of $\pm n_1$ and one occurrence of $\pm n_2$, either with both n_i carrying the same sign or with the n_i carrying opposite signs. For $l = 5$, one has two occurrences of 0, and the polygon has either two occurrences of n_i or two occurrences of $-i$ for one index $i \in \{1, 2\}$. Moreover, there is one occurrence of $\pm n_j, j \neq i$.
See that we have excluded the cases $|n_1| = |n_2|, |n_1| = |2n_2|, |n_2| = |2n_1|$.
So $\bar{\Delta}$ is a $(3, 6)$-map.
Now one proves, using the same ideas as above, that $l = 6$ gives one of $n_1 + 3n_2, n_1 - 3n_2, n_2 + 3n_1$ or $n_2 - 3n_1$. Assign labels to angles (0 for ± 1, i for $\pm n_i$, $i = 1, 2$). Use this to estimate the radius $r \geq 2$ of a regular $(3, 6)$-submap of M_Δ as above, passing to the hexagon, H_r, consisting of $6r^2$ triangles. E.g. for $3n_2 = n_1$: At every interior vertex of H_r, there are precisely 3 twos, 1 one and two zeros. Every triangle has one 0, one 1 and one 2. The number of angles in the hexagon with label 1 equals the number of faces. This, together with the fact that the number of boundary edges of the hexagon equals the number of exterior vertices, yields the estimate given in Ivaonv-Schupp. ([6, p. 1879])

Buskin, Theorem 1.2 [2, Theorem 1.2 and 1.2' (equivalent), p. 86]: Contraction of b-edges is used in some hyperbolic cases (see [2, Proof of Theorem 1.2,,p. 91-102, in particular, p. 94-102]):
Buskin shows that the results of contraction are $(4, 4)$-maps for reduced diagrams over $G = \langle a, b \mid a^{-1}b^{n_0}ab^{n_1}ab^{n_2}ab^{n_3} \rangle$ for $n_0, n_2, n_3 \neq 0, n_1 = 0, |n_2| \neq |n_0|$ and $|n_2| \neq |n_3|$. The process that constitutes the rest of the proof can be described as follows: For any possible degree 4 interior vertex of the contracted map, draw the 4 tiles that would be surrounding it in a regular $(4, 4)$-submap (note that these settings are considered up to rotation and reflection) and analyze the equations resulting from the boundaries of these tiles. Note that squares (tiles) result from faces of the original map and are obtained by contraction of b-edges, so boundary edges of tiles in regular $(4, 4)$-maps are oriented edges coming from oriented a-edges of the original diagram, and each of the 4 corners of a square corresponds to the corresponding power of b.
Get equation systems describing relations between exponents on b's. Now for any such equation system, one wants to see whether the radii of regular $(4, 4)$-maps are commonly bounded. For any such system, one can start with any of the equations in it or any of the consequences of these equations. For any such choice, draw the vertex corresponding to the chosen starting equation, add new tiles according to the snake rule given in Buskin: ([2, p. 94-98])

Definition 35 (*snake rule, snake method*, [2, p. 97-98])**.** Consider a one-relator group

$G = \langle a, b \mid r \rangle$ with r of the form $r = ab^kab^lab^ma^{-1}b^n$, $r = ab^kab^la^{-1}b^ma^{-1}b^n$ or $r = ab^ka^{-1}b^lab^ma^{-1}b^n$, and r freely and cyclically reduced, r such that for every reduced map over G, the result of contracting all b-edges to points and omitting the remaining labels is a $(4,4)$-map. We describe a way of building regular reduced $(4,4)$-maps that occur as submaps of contracted maps: Start with a vertex that can occur as a valence 4 interior vertex of a contracted map. Draw the 4 tiles surrounding it. Add tiles according to the rule indicated by the picture below. New tiles have to be labeled in such a way that the situation at new vertices does not lead to a contradiction together with the system and the assumptions on the exponents. If the contrary is the case, i.e., if one arrives at a contradiction, stop. For the contradiction means that it is not possible for the current vertex to occur as an interior vertex of degree 4 of a regular reduced $(4,4)$-submap of a result of contraction. The rule indicated by this picture is called the *snake rule*. We will call this method the *snake method*. If the method terminates for one of the starting situations, we will say that the snake method terminates or the snake terminates (for this situation).



Start with the top left arrow, proceed in the way the arrows show, then analogously (see [2, p.98]).
If, for some system, any possible starting situation yields bounded radius, then the corresponding system describes a case where there is a common bound on the radii of regular $(4,4)$-maps, and thus G is hyperbolic. In the cases from Buskin described above, the snake method yields hyperbolicity. See [2, p. 94-102].
The snake method is based on the sufficient criterion for hyperbolicity seen in Theorem 13 (Theorem 1 Ivanov-Schupp, [6, p. 1854]) (s.a. SUFFICIENCY direction of Theorem 14 (Theorem 2 in Ivanov-Schupp,[6, p. 1859])). See [2, p. 88].
Buskin, Theorem 1.1 ([2, Theorem 1.1 and 1.1' (equivalent), p. 86]), hyperbolic case: In this case, there is no contraction. Buskin shows $C(6) + T(3)$ and then builds regular reduced $(6,3)$-diagrams, showing that this process terminates for all subcases: For any subcase, first, partitions of the relator word into precisely 6 pieces are listed. Then, the process of successively attaching hexagonal cells is described. [2, p. 89-91]

The next two subsubsections will treat $(4,4)$-tilings as in Buskin for new groups. For $G = \langle a, b \mid r \rangle$ with r of the form $r = ab^k ab^l a^{-1} b^m a^{-1} b^n$ or $r = ab^k a^{-1} b^l ab^m a^{-1} b^n$, we use the strategies that are used in Buskin's paper: Analyze interior vertices of an arbitrary reduced diagram over G. Check when the contracted map of any reduced diagram is a $(4,4)$-map and use Buskin's method.
Next, we prove some results that say the group has $C(6)+T(3)$ or a small cancellation condition $C(p)+T(q)$ with $1/p+1/q > 1/2$ in a certain situation. As for hyperbolicity and non-hyperbolicity, we treat groups for which the contracted map of any reduced map is a $(4,4)$-map.

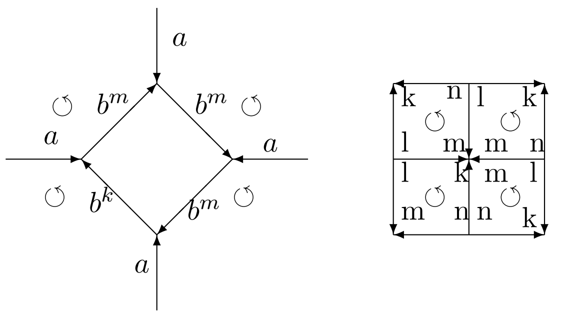
Application of Buskin's snake method to groups of type $(1/-1/1/-1)$

Example 36. Let $G = \langle A \mid r \rangle$ for $r = aw^k a^{-1} w^l aw^m a^{-1} w^n$ with r freely and cyclically reduced, r not a proper power, w a word with no occurrences of a or its inverse. This implies $|k|, |l|, |m|, |n| \geq 1$ and w freely reduced. We may exclude the case $k = m$ and $l = n$, i.e., the case where $r = (ab^k a^{-1} b^l)^2$, because we know that one-relator groups with torsion are hyperbolic. Up to an automorphism of the free group on the generators, one can assume w is cyclically reduced. Without loss of generality, w is not a proper power. So G satisfies the conditions from Lemma 15. Consider $\bar{G} = \langle a, b \mid ab^k a^{-1} b^l ab^m a^{-1} b^n \rangle$. Then hyperbolicity of $\bar{G}$ implies hyperbolicity of G, see 15 ([6, Lemma 3.3]). To $\bar{G}$, one can apply contraction of b-edges as described above. From now on, focus on $\bar{G}$. First, we consider the cases $k = -m$ and $l = -n$. If at least one of $k = -m$ and $l = -n$ holds, then $\bar{G}$ is non-hyperbolic, see Lemma 29.
Next, we would like to see when the result of contraction obtained from a reduced diagram is a $(4,4)$-map. The contracted map has no valence 1 vertices. We see that a degree 2 interior vertex in this contracted map requires $k = \pm m$ or $l = \pm n$ by the form of r: A degree 2 interior vertex of the result of contraction arises from a pair of faces sharing an interior arc with label of the form $ab^{\cdots} a^{-1}$ or $a^{-1} b^{\cdots} a$.
If we have a valence 3 interior vertex in the contracted setting, this vertex yields an equation that implies $|2k| = |m|$, $|2m| = |k|$, $|2n| = |l|$, or $|2l| = |n|$. This follows from the situation with respect to interior a-stars in reduced diagrams over the group, see below.
Now, consider cases where the map after contraction is a $(4,4)-$map. That is, we demand $\frac{|k|}{|m|} \notin \{\frac{1}{2}, 1, 2\}$, $\frac{|l|}{|n|} \notin \{\frac{1}{2}, 1, 2\}$.
First, consider valence 4 interior vertices of the contracted map (respectively, the a-stars corresponding to them). Note that both $+k, -k$ can be neighbors of both $+m, -m$: One cannot obtain reducible pairs that way because $|k| \neq |m|$. Both $l, -l$ can be neighbor of both $+n, -n$ as one has $|l| \neq |n|$. k can be neighbor of itself, $-k$ can be neighbor of itself, analogously for l, m, n. For in an element of the symmetrization $\{r\}^*$, the exponents $\pm m, \pm k$ occur in the subwords $ab^{\pm k} a^{-1}$ and $ab^{\pm m} a^{-1}$, respectively, whereas $\pm l, \pm n$ occur in subwords of the form $a^{-1} b^{\pm l} a$ and $a^{-1} b^{\pm n} a$, respectively. In other words, the successor of an a is $b^{\pm k}$ or $b^{\pm m}$, and so is the power of b preceding an a^{-1}. The successor of an a^{-1} is $b^{\pm n}$ or $b^{\pm l}$, and the same is true for the power of b preceding an a. Hence in the

construction of an interior a-star, the possible labels for v_i's belonging to the boundaries of neighboring faces sharing an a-edge are as described above.
Thus, we get:

- vertices/ a-stars corresponding to equations $3m \pm k = 0, -3m \pm k = 0, 3k \pm m = 0, -3k \pm m = 0$, analogously for l and n
- cases that give $0 = 0$ and hence no new information ($k+m-k-m = 0, l-n+n-l = 0$)
- cases that yield a contradiction to $k = 0$, to $l = 0$, to $m = 0$ or to $n = 0$ (cases that yield equations of the form $k + m - k + m = 0$)
- Cases that yield $|k| = |m|$ or $|l| = |n|$, contradiction: $2k \pm 2m = 0, -2k \pm 2m = 0$ and the analogs for l and n

Then, apply Buskin's starting map and snake method. Example: starting situation $3m + k = 0$: Situation before contracting all b-edges (left), situation in the result of contraction (right):



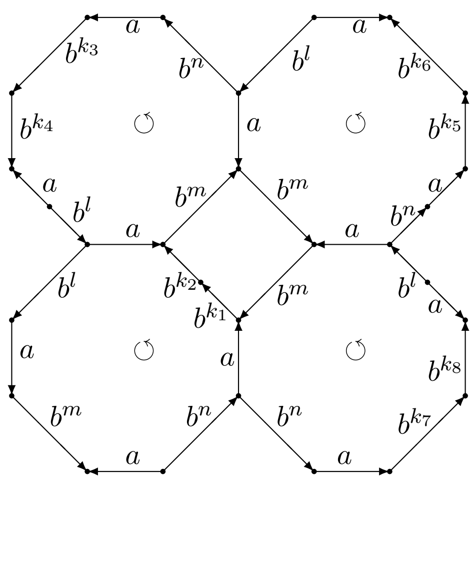
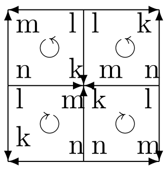


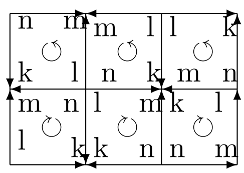
Figure: The faces in the above figure are assumed to be interior. Interior a-star for the setting $k = -3m, |3m| > |l|, |n|$.

System $3m + k = 0$ and $3l + n = 0$ gives a hyperbolic group: For starting situation $3m + k = 0$, the vertex with two $n's$ will make any snake terminate: In a regular $(4, 4)$-submap of the map obtained from a reduced diagram by contracting all b-edges and omitting the remaining labels, one cannot have an interior vertex whose equation contains two positive n's, since none of the equations in the system has two positive n's.

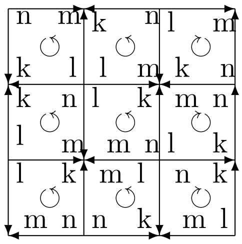
Now, consider the situations $k - m - k + m = 0$ and $l - n - l + n = 0$: Start with

$k + m - k - m = 0$: 

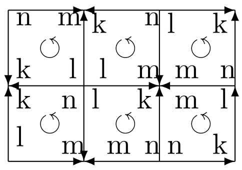
Again, n, n makes the snake terminate.

Now, consider $l + n - l - n = 0$: 

This snake terminates at the point where one has the situation containing n, n. Finally, consider starting situation $3l + n = 0$:



terminates



terminates

We first turn to the proof of Proposition 2.

Proof. (of Proposition 2, broken in parts)

Remark 37. (on the case $k = m$ and the case $l = n$) Consider the case $k = m, |l| \neq |n|$ and the case $l = n, |k| \neq |m|$. (In either of these cases, a reduced diagram over G could have a degree 2 interior vertex.) Note that $a, b \neq 1$ in the one-relator group by Lemma 28. First, consider the case $k = m, l \neq n, l \neq -n$. Then $r = ab^m a^{-1} b^l ab^m a^{-1} b^n$, which yields $ab^m a^{-1} = b^{-n} ab^{-m} a^{-1} b^{-l}$ and $ab^m a^{-1} = b^{-l} ab^{-m} a^{-1} b^{-n}$. Hence, one has $b^{-n} ab^{-m} a^{-1} b^{-l} = b^{-l} ab^{-m} a^{-1} b^{-n}$, or, equivalently, $b^{-n} ab^{-m} a^{-1} b^{-l} b^n ab^m a^{-1} b^l = 1$, which is equivalent to
$b^{l-n} ab^{-m} a^{-1} b^{-(l-n)} ab^m a^{-1} = 1$. Note that $l - n \neq 0$ and $m \neq 0$. By Lemma 17, if the group is hyperbolic, then b and $ab^m a^{-1}$ commute. Then r gives $ab^{2m} a^{-1} b^{l+n} = 1$, another application of Lemma 17 yields that a and b commute. This is a contradiction by Lemma 20. Analogously, for $l = n, k \neq m, k \neq -m$, the group is non-hyperbolic. Pass from this case to the first case by applying the automorphism of the one-relator group that interchanges the roles of k and m and those of l and n.

The above Remark and Example 29 prove Proposition 2. □

Now, we pass to Proposition 1. In Example 36, we saw that:

Lemma 38. *Let $G = \langle a, b \mid r \rangle$ with $r = ab^k a^{-1} b^l ab^m a^{-1} b^n$ for $k, l, m, n \neq 0$ and $\frac{|k|}{|m|}, \frac{|l|}{|n|} \notin \{1, 2, \frac{1}{2}\}$. For any reduced diagram over G, the result of contraction of b-edges to points and removal of remaining labels is a $(4,4)$-map.*

Lemma 39. *For G as in Lemma 38, consider valence 4 interior vertices of the results of contraction of all b-edges. Up to $Aut(F(a,b))$ and inversion of r, the roles of k and m can be interchanged, and those of l and n can be interchanged separately. Up to interchanging the roles of k and m or those of l and n and multiplication with -1, the possible equations on interior vertices of degree 4 in the result of contraction are:*

- $3m - k = 0$
- $3m + k = 0$
- $3l + n = 0$
- $3l - n = 0$
- $k - m - k + m = 0,\ l - n - l + n = 0$

Now, we draw the starting maps / starting situations corresponding to the equations in Lemma 39 one by one and analyze the boundary to find equation systems / systems of equations and consequences. The result of this process is

Lemma 40. *For a group as in Lemmas 38 and 39, one obtains the following equation systems ($k-m-k+m = 0$, $l-n-l+n = 0$ are not listed in systems) up to $Aut(F(a,b))$ and inversion of r:*

- $3m + k = 0$
- $3m - k = 0$
- $3m + k = 0, 3l + n = 0$
- $3m + k = 0, 3l - n = 0$
- $3m - k = 0, 3l - n = 0$

Moreover, one can have a group satisfying only the trivial equations.

In Example 36, we see a hyperbolic example. More generally, we have the following result:

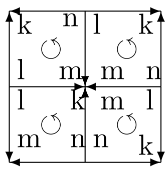
Lemma 41. *For each of the systems given in Lemma 40, one obtains a hyperbolic group by Buskin's snake tiling method: For all starting situations (i.e., the ones listed in the system AND $k + m - k - m = 0, l - n + n - l = 0$) and all choices that can be made in the snake tiling procedure, the snake method terminates. For groups satisfying only the trivial equations, all snakes terminate.*

Proof. (of Lemma 41) The proof is done using Buskin's snake rule [2]. We are going to prove hyperbolicity for

- system $3m + k = 0, 3l + n = 0$,

- system $3m + k = 0, 3l - n = 0$,
- system $3m - k = 0, 3l - n = 0$,
- system $3m - k = 0$ and
- system $3m + k = 0$.

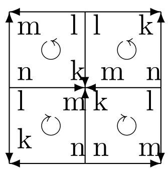
Moreover, we see that a group for which one has only the trivial equations is hyperbolic. First, we look at starting situation $3m + k = 0$ (compare Example 36)



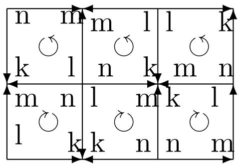
The vertex with two n's (note that one can move this vertex to the position of the first new vertex by rotation) makes the snake terminate for the following systems (the trivial equations /situations that belong to all the systems are not listed in systems)

1. $3m + k = 0$
2. $3m + k = 0, 3l - n = 0$
3. $3m + k = 0, 3l + n = 0$ (the system discussed in Example 36).

Now, consider the trivial situations $k - m - k + m = 0$ and $l - n - l + n = 0$: $k + m - k - m = 0$

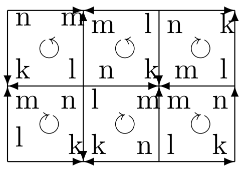


Again, n, n makes the snake terminate for systems 1. – 3. above. Moreover, it makes the snake terminate for systems $3m - k = 0$ and $3m - k = 0, 3l - n = 0$. This snake terminates for the system consisting only of the trivial equations.

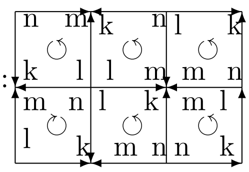
Now, consider $l + n - l - n = 0$: 

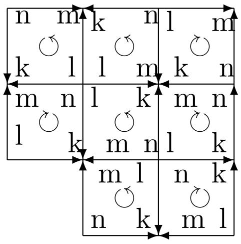
For items 1. – 3. above, this snake terminates at the point where one has the situation containing n, n. Moreover, this snake terminates at n, n for $3m - k = 0$ as well as for $3m - k = 0, 3l - n = 0$ and the system consisting only of the trivial equations.

For systems $3m - k = 0$ and $3m - k = 0, 3l - n = 0$, one has the option:

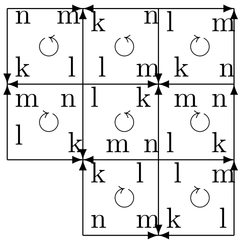


This snake terminates at the situation with k, k.

For system $3m+k = 0, 3l-n = 0$, consider starting map $3l-n = 0$: 
terminates at n, n

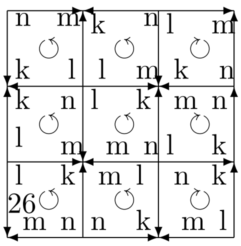


terminates at $+m+m+?-k=0$

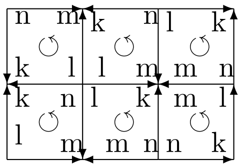


terminates (moreover, this terminates for system $3m-k=0, 3l-n=0$)

For system $3m+k=0, 3l+n=0$, consider starting situation $3l+n=0$ (see Example 36)



terminates



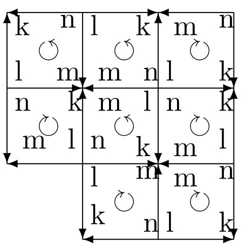
terminates

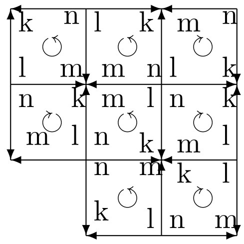
The figures seen up to now prove that the following systems give hyperbolic groups:

- $3m + k = 0$
- $3m + k = 0, 3l + n = 0$ (discussed in Example 36)
- $3m + k = 0, 3l - n = 0$

For system $3m - k = 0$, we have already done the trivial situations $k + m - k - m = 0, l + n - l - n = 0$. For starting situation $3m - k = 0$, one can only have $n + l - n - l = 0$ at the first new valence 4 vertex that one creates by adding tiles, at the second new vertex, there are the possibilities $3m - k = 0$ and $k - m - k + m = 0$, but in either of these two cases, the snake terminates at the third potential new vertex (order of tiling as in Buskin). Hence the system $3m - k = 0$ yields a hyperbolic group as well.

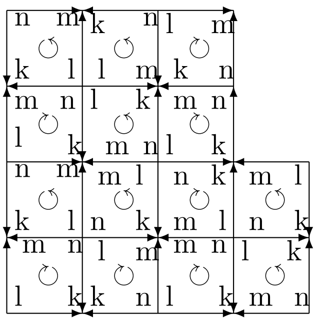
Snake $3m - k = 0, n + l - n - l = 0, 3m - k = 0$ terminates:



Snake $3m - k = 0, n + l - n - l, k - m - k + m = 0$ terminates: 

Continue the discussion by considering the rest of the starting situations for $3m - k = 0, 3l - n = 0$:

First, discuss starting situation $3l - n = 0$:

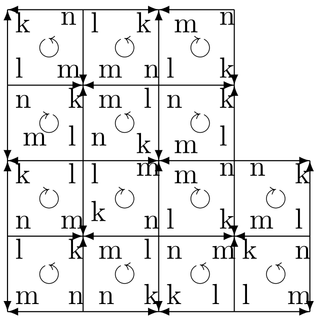


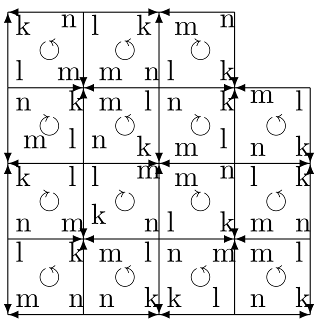
terminates

It remains to consider starting situation $3m - k = 0$:
First option (vertex 0= starting situation, proceed as in Buskin): $n - l + n + l = 0$ at vertex 1, $k - m - k + m = 0$ at vertex 2, snake terminates at vertex 3 (the same snake can be found above)
Second option $n - l - n + l = 0$ at vertex 1, $k - 3m = 0$ at vertex 2, $3l - n = 0$ at vertex 3, $-m - k + m + k = 0$ at vertex 4, $-n - l + n + l = 0$ vertex 5, $k - m + k + m = 0$ at vertex 6, then, the snake terminates.
Third option $n - l - n + l = 0$ at vertex 1, $k - 3m = 0$ at vertex 2, $3l - n = 0$ at vertex

3, $-m-k+m+k=0$ at vertex 4, $-n-l+n+l=0$ vertex 5, $3m-k=0$ at vertex 6, $l+n-l-n=0$ at the next vertex, situation terminates at vertex 8 because there are 2 k's.

Second option



third option 

Hence, the system $3m-k=0, 3l-n=0$, too, yields a hyperbolic group.

□

Proof. (of Proposition 1) Let $G = \langle a, b \mid r \rangle$ with $r = ab^k a^{-1} b^l ab^m a^{-1} b^n$ for $k, l, m, n \neq 0$. Firstly, if $(k, m) = (l, n)$, then r has imprimitivity rank 1 in $F(a, b)$, and thus, the one-relator group G is hyperbolic. If $\frac{|m|}{|k|}, \frac{|l|}{|n|} \notin \{1, 2, \frac{1}{2}\}$, then Lemma 38 shows that the result of contraction of b-edges and omitting the remaining labels is a $(4,4)$-map for any reduced diagram. An interior vertex of degree 4 in such a result of contraction can only occur for a trivial equation (e.g. $k+m-k-m=0$) or in the case that at least one of $\frac{|m|}{|k|}$ and $\frac{|l|}{|n|}$ is an element of $\{3, \frac{1}{3}\}$, see Lemma 39. In any of these cases, the radii of regular reduced $(4,4)$-maps are commonly bounded, this was proven in Lemma 41. Thus, in these cases, the group is hyperbolic by Theorem 13. □

Examples of families of groups of type $(1/-1/1/-1)$ satisfying condition $C(4)+T(3), C(5)+T(3)$ or $C(6)+T(3)$

Now, we discuss certain subcases of the cases described in Lemma 42, in which case interior vertices of degree 3 can occur after contraction of all b-edges. We check for $C(p)$. We identify some groups that have $C(6)+T(3)$, which means that in particular, the criterion stated in Theorem 14 (Theorem 2 of Ivanov-Schupp) can be applied to them, and some that have $C(4)+T(3)$ or $C(5)+T(3)$, neither of which is one of the conditions for which Theorem 14 can be applied.

For cases which have $|k| \neq |m|$ and $|l| \neq |n|$, but are such that the contracted map of a reduced diagram is not (necessarily) a $(4,4)$-map, we prove a lemma that describes these cases up to automorphisms of $F(a,b)$, inversion of r and renaming of exponents.

Lemma 42. *Up to $Aut(F(a,b))$, inversion of r and renaming of exponents, the case where $G = \langle a, b \mid r \rangle$ with $r = ab^k a^{-1} b^l ab^m a^{-1} b^n$ with $k, l, m, n \neq 0, |k| \neq |m|$ and $|l| \neq |n|$ and the contracted map is not (necessarily) a $(4,4)$-map gives the following (systems of) equations:*

- $2m + k = 0.$
- $2m - k = 0$
- $2m + k = 0, 2l + n = 0$
- $2m - k = 0, 2l - n = 0$
- $2m + k = 0, 2l - n = 0$

For the first two items, assume without loss of generality that $|n| \geq |l|$.

Since there is no piece with two occurrences of a, a^{-1} in it (total number of a, a^{-1}'s is 4) in the cases in Lemma 42 (note that $k \neq m, k \neq -m, l \neq n, l \neq -n$ in Lemma 42), every partition has at least 4 pieces (one might have $C(p)$ with $p > 4$).
We state some results for groups as in Lemma 42, summarizing them into one proposition. For the proof, the proposition will be broken into lemmas.

Proposition 43.
- *Let G be as in the subcase $(\frac{k}{m}, \frac{n}{l}) = (2,2)$ of Lemma 42. Then G satisfies the small cancellation condition $C(4)$.*
- *If G is a one-relator group as in one of the following subcases of Lemma 42, then G satisfies condition $C(5)$:*
  - $\frac{|k|}{|m|} = 2 = \frac{|n|}{|l|}$ *and* $(\frac{k}{m}, \frac{n}{l}) \neq (2,2)$
  - $\frac{k}{m} = 2, \frac{n}{l} < -2$ *and* $|m| = |n|$
  - $\frac{k}{m} = -2, \frac{n}{l} < -2$ *and* $|2m| \neq |n|$
  - $\frac{k}{m} = -2, \frac{n}{l} > 2$ *and* $|2m| = |n| > |l| > |m|$ *or* $|2m| > |n| > |l|$

– $\frac{k}{m} = -2, \frac{n}{l} < -2$ *and* $|m| = |n|$.

- *Another non-*$C(6)$ *subcase of Lemma 42 is* $\frac{k}{m} = 2, \frac{n}{l} > 2$.
- *If* G *is as in one of the following subcases of Lemma 42, then the group satisfies condition* $C(6)$*:*
  – $\frac{k}{m} = -2$, $|n| > |2m| > |m| = |l|$ *and* $\frac{n}{l} < -2$ *or* $\frac{n}{l} > 2$
  – $\frac{k}{m} = 2, \frac{n}{l} < -2$ *and* $|n| > |2m| > |l| = |m|$.

Lemma 44. *The subcase* $\frac{|k|}{|m|} = \frac{|n|}{|l|} = 2$ *from Lemma 42 has* $C(p)+T(q)$ *with* $1/p+1/q > 1/2$.

Proof. Firstly $k = 2m$ and $n = 2l$ yields a $C(4)$ group: $[b^l ab^m][b^m a^{-1}][b^l ab^m][a^{-1}b^l]$ is a partition of a conjugate of r into 4 pieces. Hence the group has $C(4) + T(3)$, and $\frac{1}{4} + \frac{1}{3} = \frac{7}{12} > \frac{1}{2}$.
Now consider the case where $k = -2m$ and $n = -2l$. There is no partition of an element of the symmetrization of the set $\{r\}$ into 4 pieces, but there are partitions of r or a conjugate into 5 pieces for all such groups: If $|2m| > |2l|$, then take $[ab^{-m}][b^{-m}a^{-1}][b^l a][b^m a^{-1}][b^{-2l}]$. If $|2l| > |2m|$, then take $[b^{-2m}][a^{-1}b^l][ab^m][a^{-1}b^{-l}][b^{-l}a]$. So in the case $k = -2m, n = -2l$, the group is $C(5) + T(3)$, and $\frac{1}{5} + \frac{1}{3} = \frac{8}{15} > \frac{1}{2}$.
Consider the case where $k = -2m$ and $n = 2l$. Again, the group is $T(3)$ and $C(5)$: If $|2m| > |2l|$, then take $[ab^{-m}][b^{-m}a^{-1}][b^l a][b^m a^{-1}][b^{2l}]$. If $|2l| > |2m|$, consider the partition $[b^{-2m}][a^{-1}b^l][ab^m][a^{-1}b^l][b^l a]$. □

Lemma 45. *If we are in the subcase of Lemma 42 with* $\frac{k}{m} = 2, \frac{n}{l} > 2$*,then the group is not* $C(6)$.

Proof. The partition $[ab^m][b^m a^{-1}b^l][a][b^m a^{-1}b^l][b^{n-l}]$ is a partition into pieces that proves the claim. □

Lemma 46. *If one is in the subcase* $\frac{k}{m} = 2, \frac{n}{l} < -2$ *of Lemma 42 and moreover* $|m| = |n| > |l|$*, then the group is* $C(5)$*, but not* $C(6)$.

Proof. There exists no element of $\{r\}^*$ that admits a partition into 4 or fewer pieces. We give a partition of r into 5 pieces. Take $[ab^m][b^m a^{-1}][b^l a][b^m a^{-1}][b^n]$. □

Note that in Lemma 46, $|m| \geq 2$.

Lemma 47. *If we are in subcase* $\frac{k}{m} = -2, \frac{n}{l} < -2$ *in Lemma 42 and* $|m| > |n| > |l|$*, then the group is* $C(5)$*, but not* $C(6)$.

Proof. There are no partitions of elements of $\{r\}^*$ into 4 or fewer pieces. Thus, naming a partition of an element of the symmetrization $\{r\}^*$ of $\{r\}$ into 5 pieces proves the claim: Take $[b^{-m}][b^{-m}a^{-1}][b^l a][b^m a^{-1}][b^n]$ (for any choice of signs that agrees with the assumptions). □

In Lemma 47, one has $|m| \geq 3$.

Lemma 48. *If one is in the subcase $\frac{k}{m} = -2, \frac{n}{l} < -2$ of Lemma 42 and one of the following holds, then the group has condition $C(5)$, but it does not have $C(6)$:*

- $|n| > |l| > |m|$
- $|n| > |2m| > |m| > |l|$
- $|2m| > |n| > |m| > |l|$

Proof. Partitions into four or fewer pieces are not possible. If $|2m| > |n| > |m| > |l|$, consider $[ab^{-m}][b^{-m}a^{-1}][b^{l}a][b^{m}a^{-1}][b^{n}]$.
If $|n| > |l| > |m|$ or $|n| > |2m| > |m| > |l|$, take $[b^{-2m}][a^{-1}b^{l}][ab^{m}][a^{-1}b^{l+n}][b^{-l}a]$. □

Lemma 49. *Let G be a one-relator group as in subcase $\frac{k}{m} = -2, \frac{n}{l} > 2$ of Lemma 42. If one of the following additional conditions holds, then the group is $C(5)$, but not $C(6)$:*

- $|n| = |2m| > |l| > |m|$
- $|n| > |l|$ *and* $|2m| > |n|$

Proof. Partitions into four or fewer pieces are not possible. For the first of the two items listed above, take $[ab^{-m}][b^{-m}a^{-1}][b^{l}a][b^{m}a^{-1}][b^{n}]$.
For the second item, take $[ab^{-m}][b^{-m}a^{-1}][b^{l}a][b^{m}a^{-1}][b^{n}]$. □

Also, consider the following example:

Example 50. If we are in subcase $\frac{k}{m} = -2$ of Lemma 42, l, n, m have the same sign, $|n| > |2m| > |l| > |m|$ and $n - l = m$, then the group is $C(5)$. There are no partitions of elements of $\{r\}^*$ into 4 pieces, but one has $[b^{-2m}][a^{-1}b^{l}][ab^{m}][a^{-1}b^{l}][b^{n-l}a]$. More generally, this works if $l, m, n > 0$ and $0 < n - l \leq m$. Analogously, this partition works if $l, m, n < 0$ and $0 > n - l \geq m$.

Lemma 51. *If one is in the subcase $\frac{k}{m} = -2, \frac{n}{l} < -2$ of Lemma 42 and $|m| = |n|$, then the group is $C(5)$, and it is not $C(6)$.*

Proof. The group is $C(5)$. For proving that it is not $C(6)$, we consider the following partition into pieces (In each of the four subcases

- $m > 0, n > 0, l < 0$
- $m < 0, n > 0, l < 0$
- $m > 0, n < 0, l > 0$
- $m < 0, n < 0, l > 0$,

one can take the same partition): $[ab^{-m}][b^{-m}a^{-1}][b^{l}a][b^{m}a^{-1}][b^{n}]$. □

Now, one would like to analyze cases where $|m| = |n|$ and for which the situation $|m| = |n| = 1$ is possible. (The fact that one has $C(6)$ examples with $|m| = |n| = 1$, see [4], was the motivation for checking families for which $|m| = |n| = 1$ is possible.)

Lemma 52. *Let G be a group with r as in subcase $\frac{k}{m} = 2, \frac{n}{l} < -2$ of Lemma 42. Moreover, let $|n| > |2m| > |l| = |m|$. Then the group has property $C(6)$.*

Proof. As above, we would like to prove that there is no partition of r, its inverse or a conjugate of one of them into 4 or 5 pieces. A partition into 4 pieces is impossible. To see that a partition into 5 pieces is impossible as well, consider the two subcases $m = l$ and $m = -l$.

A partition into 5 blocks would consist of 4 pieces containing one a or a^{-1} and some power of b each and one block containing only a power of b. For $m = -l$, we have that $r = ab^{2m}a^{-1}b^{-m}ab^{m}a^{-1}b^{n}$ and $r^{-1} = b^{-n}ab^{-m}a^{-1}b^{m}ab^{-2m}a^{-1}$. Note that m, n have the same sign in this case. If the process of splitting up b^n or b^{-n} between pieces involves a single power of b piece, then the partition has at least 6 blocks. So b^n or b^{-n} must be divided up between a and a^{-1}. See that if one tries to split $a^{-1}b^{n}a$ or $a^{-1}b^{-n}a$ into two pieces in the following way: $[a^{-1}b^{\cdots}][b^{\cdots}a]$ $(*)$, one gets a contradiction because the only pieces that could occur in such a partition are $[a^{-1}b^{i}], [a^{-1}b^{-i}], [b^{i}a], [b^{-i}a], 1 \leq i \leq |m|$, but $|n| > |2m|$. For $m = l$, one has $r = ab^{2m}a^{-1}b^{m}ab^{m}a^{-1}b^{n}$. Again, in a partition of r, r^{-1} or a conjugate of one of those into five pieces, the b^n or b^{-n} part would be divided up between its neighbor a and its neighbor a^{-1}, which would result in two blocks of the form $(*)$. But again, the only pieces that could be used to form such a partition are $[a^{-1}b^{i}], [a^{-1}b^{-i}], [b^{i}a], [b^{-i}a], 1 \leq i \leq |m|$. Again, the process fails because $|n| > |2m|$. □

Lemma 53. *Let G be a group as in case $\frac{k}{m} = -2, \frac{n}{l} < -2$ of Lemma 42, and let $|n| > |2m| > |m| = |l|$. Then the group is $C(6)$.*

Proof. If $m = l, n > 0, m, l < 0$, then dividing up $a^{-1}b^{n}a$ between pieces would give $0 < m_0 \leq -m$ with $n - m_0 \leq -m$, hence $n \leq -2m$, contradiction. In the inverse or one of its conjugates, $-n < 0, -m, -l > 0$ occur. Get $m_0 < 0$ with $-n - m_0 \geq m, m_0 \geq m$, and hence a contradiction. The case $m = l, n < 0, m, l > 0$ is obtained from the latter case via the map given by $b \mapsto b^{-1}$.

Now consider $m = -l, n, m > 0, l < 0$ and obtain $0 < m_0 \leq m$ with $n - m_0 \leq m$ and thus $n \leq 2m$, contradiction. For the inverse, consider $-m = -(-l), -n, -m < 0, -l > 0$, and get $0 > m_0 \geq -m$ with $-n - m_0 \geq -m$, hence $-n \geq -2m$. The case $m = -l, n, m < 0, l > 0$ is obtained from $m = -l, n, m > 0, l < 0$ via the automorphism of the free group given by $a \mapsto a, b \mapsto b^{-1}$. □

Lemma 54. *Consider subcase $\frac{k}{m} = -2$ in Lemma 42 with $\frac{n}{l} > 2$ and $|n| > |2m| > |m| = |l|$. Then the group is $C(6)$.*

Proof. First, consider the subcase $m = l$. In this case l, m, n have the same sign. There exists no partition of an element of $\{r\}^*$ into 4 pieces (or fewer pieces). A partition into 5 pieces would consist of four blocks that contain one a or a^{-1} each, and one block that consists of a power of b. The term b^n and the term b^{-n} are not pieces. Hence in a partition into 5 pieces, either of these is divided up between its neighbor a and its neighbor a^{-1}.

If $l, m, n > 0$, then this process of dividing b^n up between pieces would look like

$[a^{-1}b^{n-m_0}][b^{m_0}a]$ with $m_0 > 0, n - m_0 \leq m, m_0 \leq m$, which yields $n \leq m_0 + m \leq 2m$, contradiction to the assumption. For the inverse or its conjugates, we have that $-l, -m, -n < 0$, which would yield an n_0 with $n_0 < 0, -n - n_0 \geq -m, n_0 \geq -m$, which gives $-n \geq -2m$, and hence $|n| \leq |2m|$, contradiction.
The case $m = l, l, m, n < 0$ is the same as the latter case up to application of the automorphism of the free group given by $a \mapsto a, b \mapsto b^{-1}$.
Hence any partition into pieces has at least 6 blocks.

Now consider $m = -l$. Take $l, n > 0, m < 0$ first. Then if we divide b^n up as above, we get $m_0 > 0$ with $m_0 \leq -m, n - m_0 \leq -m$, which gives $n \leq -2m$, contradiction. For the inverse and its conjugates, consider $-l, -n < 0, -m > 0$. Dividing $a^{-1}b^{-n}a$ into two pieces gives $n_0 < 0$ with $n_0 \geq -m$ and $-n - n_0 \geq -m$, which gives $-n \geq -2m$, which implies $n \leq 2m$, contradiction.
From the case $l, n < 0, m > 0$, one can pass to $l, n > 0, m < 0$ via $a \mapsto a, b \mapsto b^{-1}$. $\square$

We pass to words of a different form:

Application of Buskin's snake method to groups of type $(1/1/-1/-1)$

Consider $r = ab^kab^la^{-1}b^ma^{-1}b^n$, with $l, n \neq 0, k \neq -m$, and $|k+m| \neq |n|, |k+m| \neq |l|$. Then the contracted map of any reduced diagram is a $(4,4)$-map. By the form of r, a vertex of degree two in a result of contraction requires $k = -m$. A degree 3 vertex in the result of contraction can only result from an interior a-star with one of the v_i labelled b^k, another one labelled b^m and the third one either $b^{\pm n}$ or $b^{\pm l}$, or from an interior $a-$star with one v_i labelled b^{-k}, another one labelled b^{-m} and the third one labelled either $b^{\pm n}$ or $b^{\pm l}$.
Now, describe all possible degree 4 interior vertices in results of contraction:

Lemma 55. *Take $r = ab^kab^la^{-1}b^ma^{-1}b^n$, with $k, m, l, n \neq 0$ and $|k+m| \notin \{0, |n|, |l|\}$. Up to exchange of (k,l) and (m,n) and the automorphism defined by $a \mapsto a, b \mapsto b^{-1}$, there are the following degree 4 situations (degree 4 interior vertices in contracted maps):*

1. $k + m + 2l = 0$
2. $k + m - 2l = 0$
3. $k + l + m + n = 0$
4. $k + m - l - n = 0$
5. $k + m + l - n = 0$
6. $m + l + n - m = 0$ *and* $k + l + n - k = 0$*, which are equivalent*
7. $m + l - n - m = 0$ *and* $k + l - n - k = 0$*, which are equivalent*

Lemma 56. *Let $G = \langle a, b \mid r \rangle$ with $r = ab^k ab^l a^{-1} b^m a^{-1} b^n$ for $l, n \neq 0$. Up to automorphism of G induced by $Aut(F(a,b))$ and inversion of r, one can interchange the roles of k and m and/or those of l and n. In particular, without loss of generality, one may assume $|n| \geq |l| > 0$, $k \geq 0$ and $m \in \{k, k-1\}$,*

Note that in items 1., 2. in Lemma 55, $|k+m| = |2l|$. Hence one obtains, up to $Aut(F(a,b))$, that $r = a^2 b^l a^{-1} b^{-2l} a^{-1} b^l$, $r = a^2 b^l a^{-1} b^{2l} a^{-1} b^l$, respectively. So by this remark and Lemma 56, in item 1, one can assume without loss of generality that $k = m = -l$, while in item 2, one can assume without loss of generality that $k = m = l$.

Lemma 57. *Take $r = ab^k ab^l a^{-1} b^m a^{-1} b^n$, with $l, n \neq 0, k \neq -m$ and $|k+m| \neq |l|, |k+m| \neq |n|$. Assuming without loss of generality that $|n| \geq |l| > 0$, $k \geq 0$ and $m \in \{k, k-1\}$, one obtains the following valence 4 situations (valence 4 interior vertices in contracted maps):*
$k = m$

1. $2m + 2l = 0$
2. $2m - 2l = 0$
3. $2m + l + n = 0$
4. $2m - l - n = 0$
5. $2m + l - n = 0$
6. $m + l + n - m = 0$
7. $m + l - n - m = 0$

$k = m + 1$

3. $2m + 1 + l + n = 0$
4. $2m + 1 - l - n = 0$
5. $2m + 1 + l - n = 0$
6. $m + l + n - m = 0$ *and* $k + l + n - k = 0$*, which are equivalent*
7. $m + l - n - m = 0$ *and* $k + l - n - k = 0$*, which are equivalent*

The proofs of Propositions 3 and 4 consist of the proofs of several lemmas. We will see that

Lemma 58. *Let G be as in Lemma 57. If one makes one of the additional assumptions*

- $k = m = -l = -n$
- $k + m = l + n \neq 0$ *and one of* $(k, m) = (n, l), (k, m) = (l, n)$

- $k+m=l+n\neq 0$ *and* $|k+m|\neq\{|l|,|n|\}$

then the group is non-hyperbolic.
If one has $k=m=l$ *and* $n=3l$ *or* $k=m=-l$ *and* $n=3l$*, then the group is hyperbolic.*
The other subcases of Lemma 57 are not decided here.
In particular, the subcase $l=-n$ *of Lemmas 57 and 55 is not decided here. Also, note that each of items* $1., 2., 5.$ *in Lemma 57/ Lemma 55 has a hyperbolic subcase by the above, while each of* $1., 2., 3., 7.$ *is proved to have at least one non-hyperbolic subcase.*
Moreover, if G *is as in Lemma 56 with* $k=m=0$*, then the group is non-hyperbolic.*

Example 59. Consider $ab^kab^la^{-1}b^ma^{-1}b^n$, with $k,l,m,n\neq 0$ and $|k+m|\notin\{0,|n|,|l|\}$. Then the contracted map obtained from any reduced diagram is a $(4,4)$-map. Take the starting situation $k+2l+m=0$. Analysis of the 4 tiles in this starting situation yields that this starting map allows 3 possible systems of equations with consequences:

1. *a)* $k+2l+m=0$, *b)* $k+l+m+n=0$, *c)* $-m+n+m-l=0$, *d)* $2n+m+k=0$, *e)* $n-k-l+k=0$,
2. *a), f)* $-m+n+m+l=0$, *g)* $-2n+k+m=0$, *h)* $n-k+l+k=0$,
3. *a), i)* $k+n+m-l=0$.

Assuming without loss of generality that $k+m=2m, k\neq 0$, we get

1. $k=m=-l, l=n$ (In this case, the group is non-hyperbolic by the subcase of Lemma 30 that is seen in Lemma 31.),
2. $k=m=n, l=-n$ and
3. $k=m=-l, n=3l=-3k$ (hyperbolic, see below)

System 1. allows an unbounded radius regular reduced $(4,4)$-map. Moreover, we have already seen that these groups are non-hyperbolic. System 2. allows an unbounded radius regular reduced $(4,4)$-map as well. For System 3., the snake method terminates for both starting situations. Hence, the group is hyperbolic by Theorem 13. The proof of hyperbolicity for System 3. using the snake method can be found below. (Proof of Lemma 61). An infinite radius regular reduced $(4,4)$-map for each of Systems 1. and 2. will be seen in the proof of Lemma 61.

Example 60. The case $k-2l+m=0$: This case has 3 subcases:

1. $-2l+k+m=0, l+k+n-k=0, l+m+n-m=0, k-l+m+n=0, k+2n+m=0$
2. $k-2l+m=0, k-l-n+m=0, k+l-n-k=0, m+l-n-m=0, k-2n+m=0$
3. $k-2l+m=0, -k+n-l-m=0.$

Assuming without loss of generality that $k=l=m\neq 0$ yields

1. $k = l = m, l = -n$. This is the same as $k = m = n, l = -n$ (which occurs in Example 59) up to automorphism of the one-relator group.

2. $k = l = m = n$ (This implies that $k + m - l - n = 0$, hence, in this case, the group is non-hyperbolic by the subcase of Lemma 30 covered in Lemma 33.)

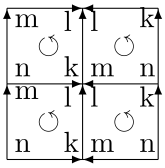
3. $k = l = m, n = 3l$ (hyperbolic, see below).

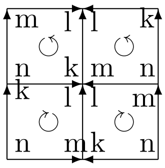
For each of Systems 1. and 2., once can build an infinite radius regular reduced $(4, 4)$-map, see proof of Lemma 61. Note that System 2. always yields a non-hyperbolic group, as seen above. System 3. gives a hyperbolic group by Theorem 13 as Buskin's snake method terminates for both starting situations. See proof of Lemma 61.

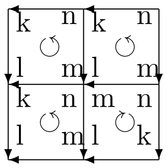
Lemma 61. *Items 3.,4.,6.,7. in Lemma 55 always yield groups that allow an infinite radius regular reduced* $(4, 4)$*-map. Item 1. yields two systems that allow infinite radius regular reduced* $(4, 4)$*-maps and one that gives hyperbolic groups, see Example 59. Item 2. yields* 2 *systems that allow an infinite radius regular reduced* $(4, 4)$*-map and one system that gives hyperbolic groups, see Example 60. Item 5. gives one system that yields an infinite radius regular reduced* $(4, 4)$*-map and a hyperbolic one that corresponds to system 3. of Example 60.*

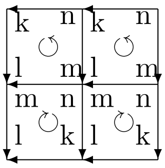
Proof. First, we prove that a group as in item 6. or item 7. of Lemma 55 yields a situation for which Buskin's snake method does not terminate: Case $l + n = 0$: Show this using Buskin's snake method on the starting situation $-m + n + m + l = 0$. This situation allows an infinite radius regular reduced $(4, 4)$-map that uses only the situations $-m + n - m + l = 0, -k + l - k + n = 0$, up to multiplication of the equation with -1. Analogously, for $l - n = 0$, the starting situation $-l + k + n - k = 0$ allows an infinite radius regular reduced $(4, 4)$-map using only the equations $-l + k + n - k = 0, l - m - n + m = 0$, up to multiplication by -1.
For item 3., one can build a regular reduced $(4, 4)$-map of infinite radius using only the situation $k + l + m + n = 0$.
For item 4., one can build such a map using situation $k + m - l - n = 0$, up to multiplication of the equation with -1.
Analysis of the starting situation described by item 5 gives one case that allows an infinite radius regular reduced $(4, 4)$-map:

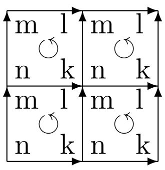
$k + m + l - n = 0, k + l + n - k = 0, m + l + n - m = 0, k + 2l + m = 0, k - 2n + m = 0$
Moreover, we have a hyperbolic case, $k + m + l - n = 0, k - 2l + m = 0$, compare system 3. in Example 60.
Item 1. is Example 59.
Item 2. is Example 60.
We now give the maps that prove the claims. We first give all valence 4 starting situations from Lemma 55 (up to rotation/reflection), then give an infinite radius regular reduced $(4, 4)$-map for those examples that allow such a map, and we prove hyperbolicity using Buskin's snake rule for the systems that give hyperbolic groups.

Starting situation $k+m+2l=0$: 

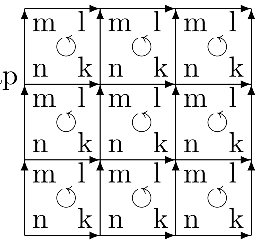
Starting situation $k+m-2l=0$: 

Now, consider 6./m ($l+n=0$) and 7./m ($l-n=0$): 

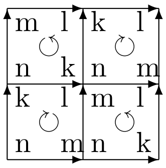


The picture for 3. is: 

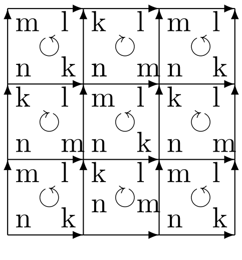
For starting situation 3., build infinite radius regular reduced $(4,4)$-map continuing like this.

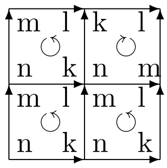


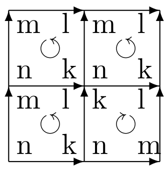
The picture for 4. is:

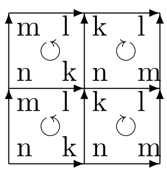


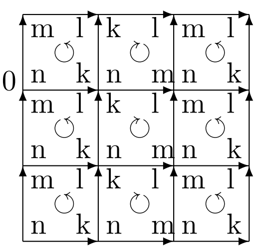
To obtain an infinite radius regular reduced $(4,4)$-map for 4., continue like this.



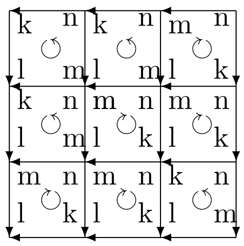
The picture for 5. is: 

Moreover, there is 6./$k + l + n - k = 0$: 

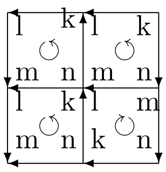
Analogously, draw 7./$k + l - n - k = 0$: 

Get infinite radius regular reduced $(4, 4)$-map for 7./$k+l-n-k = 0$ continuing like this. 

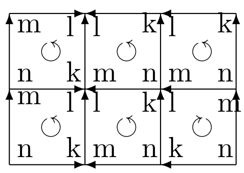
For an infinite radius regular reduced $(4,4)$-map for $6.m$, continue like this.



For the case $k+2l+m=0, k+n+m-l=0/\ k+n-l+m=0$, the snake method terminates:

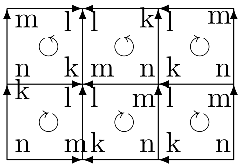


$k+2l+m=0, k+n+m-l=0/\ 2l+k+m=0$

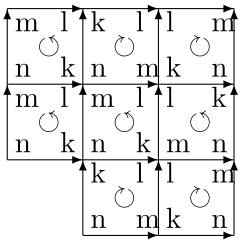


The snake method terminates after that.

Now apply the snake method to the system consisting of item 2. and item 5., $k+m-2l=0, k+m+l-n=0$:



terminates



terminates

□

Lemma 62. *If* $r = ab^kab^la^{-1}b^ma^{-1}b^n$*, with* $k, l, n \neq 0, m = 0$ *and* $|k| = |k+m| \neq |n|, |k| = |k+m| \neq |l|$*, then all valence 4 situations from Lemma 55 are possible. Passing to Lemma 57, one has* $k = m + 1 = 1$*.*

Items $k+2l+m = 0$, $k+l-n-k = m+l-n-l = 0$ and $k+l+m+n = 0$ in Lemma 55 share the non-hypberbolic subcase $k = m, l = n = -k$, Lemma 31.
Item $k+m-l-n = 0$ in Lemma 55 or 62 gives a non-hyperbolic group (Lemma 33), a subcase of this is the $k+m-l-n = 0, k+m-2l = 0, l = n$ subcase of Lemma 55 or 62.
By the latter remark and the above remark on Lemma 31 ($k = m, l = n = -k$), we have seen that items $1., 2., 3., 4., 7.$ in Lemma 55 have non-hyperbolic subcases, and so have items $2., 4., 7.$ if one passes to Lemma 62.

Proof. (of Proposition 3) Combining Lemmas 61 and 62 and the above remarks proves Proposition 3. □

Proof. (of Proposition 4) Proposition 4 is Lemma 30. □

Now, we describe a situation in which a contracted map can have valence 3 interior vertices. This is a $C(5)+T(3)$ case, hence a case to which Theorems 13, 14 do not apply.

Lemma 63. *Let $r = ab^k ab^l a^{-1} b^m a^{-1} b^n$, with $k, l, m, n \neq 0, k \neq -m$. First, let $|k+m| = |l|$. If b^n is a piece, then the group is $C(5)$, but not $C(6)$. Analogously /by symmetry, consider r as in the first sentence with $|k+m| = |n|$ and b^l a piece, then the group is $C(5)$, but not $C(6)$.*

Proof. We prove the first sub-statement, that is, we take an r as in the first sentence of the statement such that $|k+m| = |l|$ and b^n is a piece. As $k \neq -m$, there is no decomposition into 4 or fewer pieces. If $l = k+m$, then $[ab^k][ab^k][b^m a^{-1}][b^m a^{-1}][b^n]$ is a decomposition into 5 pieces. If $l = -k-m$, then $[ab^k][ab^{-m}][b^{-k} a^{-1}][b^m a^{-1}][b^n]$ proves that the group is not $C(6)$. □

We proved Propositions 1- 2 (on groups of type $(1/-1/1/-1)$) and Proposition 3 (on type $(1/1/-1/-1)$). Proving Propositions 1 and 3 was the main aim of this thesis. Moreover, for some words of type $(1/-1/1/-1)$, we proved $C(6)+T(3)$ or $C(p)+T(q)$, $\frac{1}{p}+\frac{1}{q} > \frac{1}{2}$ without discussing hyperbolicity or non-hyperbolicity. For words of type $(1/1/-1/-1)$, we gave one family of groups (up to symmetry) that have $C(5)+T(3)$, without discussing hyperbolicity or non-hyperbolicity. Moreover, we proved Proposition 4.

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